Rotation Series

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Abstract

Rotation Series

To Do: Make graphics to explain the rotation of a function, check that we can ignore the additivity... The inverse of the power series is not necessarily the series of the inverses of the elements. Need a visualisation of the series, break it apart into individual terms, rotate each term and sum them back up! Should work.

1 Abstract

Consider an operation that rotates a function on the x and y axes, we can reduce this to two functions

$$X(x,t) = x\cos(t) + f(x)\sin(t)$$

and

$$F(x,t) = x\sin(t) + f(x)\cos(t)$$

then we consider a parametric plot of $F(X)$ at a given t. When $t=0$ or 2π, this gives a plot of $f(x)$, when $t=\pi$ we have $-f(-x)$, but when $t=\pi/2$, we have $x(f)$, which is essentially the inverse $f^{-1}(x)$. Then t allows us to interpolate between these.

We then realise there is a whole spectrum of inverse like functions. Consider a power series representation for $f(x)$. We can use series reversion to invert the series as an expansion for the inverse function $f^{-1}(x)$. What may no be immediately obvious is that we can rotate each coefficient individually and reclaim the rotated power series.

When we rotate a coefficient times a power qx^k we get a new generalised basis function $\phi_k(t,q)$, such that $\phi_k(n2\pi,q) = qx^k$ for integer n. We solve for the series reversion inverses of X and F.

Using q to represent the coefficient of the power term we have:

$$q \to A^{-1}(x) = \sec(t)(x - q\sin(t))$$

$$qx \to A^{-1}(x) = \frac{x}{\cos(t) + q\sin(t)}$$

$$qx^2 \to A^{-1}(x) = x\sec(t) - qx^2\tan(t)\sec^2(t) + 2q^2x^3\tan^2(t)\sec^3(t) - 5x^4\left(q^3\tan^3(t)\sec^4(t)\right) + 14q^4x^5\tan^4(t)\sec^5(t) + O\left(x^6\right)$$

which we can see is

$$A_2^{-1}(x) = \sec(t)x\sum_{k=0}^{\infty}\frac{(-1)^k}{1+k}\binom{2k}{k}q^k\tan^k(t)\sec^k(t)x^k$$

with Catalan number coefficients

$$qx^3 \to A^{-1}(x) = x\sec(t) - qx^3\tan(t)\sec^3(t) + 3q^2x^5\tan^2(t)\sec^5(t) - 12q^3\tan^3(t)x^7\sec^7(t)$$

$$A_3^{-1}(x) = \sec(t)x\sum_{k=0}^{\infty}\frac{(-1)^k}{1+2k}\binom{3k}{k}q^k\tan^k(t)\sec^{2k}(t)x^{2k}$$

we guess then that

$$A_n^{-1}(x) = \sec(t)x\sum_{k=0}^{\infty}\frac{(-1)^k}{1+(n-1)k}\binom{nk}{k}q^k\tan^k(t)\sec^{(n-1)k}(t)x^{(n-1)k}$$

$$A_4(x) = \sec(t)x\,{}_3F_2\left(\frac{1}{4},\frac{1}{2},\frac{3}{4};\frac{2}{3},\frac{4}{3};-\frac{4^4}{3^3}qx^3\tan(t)\sec^3(t)\right)$$

these are unweildy expressions for large n as the number of arguments in the hypergeometric function grow. We can however consider the Mellin transform of the function for example

$$\mathcal{M}[A_4(x)](s) = \frac{2^{\frac{2}{3}(-s-1)-\frac{4s}{3}-\frac{5}{6}}3^s\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)\Gamma\left(-\frac{2s}{3}-\frac{1}{6}\right)\Gamma\left(\frac{1}{6}-\frac{s}{3}\right)\Gamma\left(\frac{s}{3}+\frac{1}{3}\right)\sec(t)\left(q\tan(t)\sec^3(t)\right)^{\frac{1}{3}(-s-1)}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{3}-\frac{s}{3}\right)\Gamma\left(1-\frac{s}{3}\right)}$$

in general we have

$$I_x[x+ax^m] \to \frac{a^{\frac{s}{m-1}}\Gamma(1-\frac{ms}{m-1})\Gamma(\frac{s}{m-1})}{(m-1)\Gamma(2-s)}$$

This means we can smoothly parametrise the coefficients of the series expansion of a function as it rotates around the $x-y$ axis. In terms of the Mellin-transform, this reports the coefficient (with negative sign) through the RMT. This may interpolate the coefficients of the analytic function, and (the reflection of) its inverse.

2 Operators

Consider a clockwise operator R_θ which rotates a function by θ. We clearly have the examples $R_{\frac{\pi}{2}}[x^2] = \pm\sqrt{x}$, $R_\pi[x^2] = -x^2$.