

# Further Examples for: A Natural Representation of Functions that Facilitates 'Exact Learning'

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## Abstract

Equations can often be hard to solve analytically, but numeric solutions are relatively easy to get, or advanced methods have been developed to ... . Exact learning is a method that attempts to find analytic solutions without solving equations, but only by looking at the data.

## 1 Introduction

Exact learning as introduced by Irwin [?] is a method for attempting to fit the analytic structure of a function by taking a Mellin transform. The Mellin space is ideal because many functions in  $[0, \infty)$  are expressed in terms of gamma functions and simple powers.

In this work we apply exact learning to a number of interesting problems.

## 2 Further Examples

BesselK functions, and powers of Bessel K functions

## 3 RDF Type functions

Consider  $1 - \text{Bessel}J[x]$  and the same with argument  $\sqrt{x}$  for an analogy to a function that extends to infinity

## 4 Training/Test Set

We should hold out set of points which are in a different space completely.

Line and combination searches. For some parameters, we might want to try rational numbers. For others ... This might ease the combinatorics a little. Otherwise advanced solvers will be good.

## 5 Exact Learning

## 6 Advanced Fitting Techniques

If we have exact data to fit to, then we can exploit known properties of the gamma function. Presume we have a solution as

$$\phi(s) = \Gamma(as + b)$$

for  $Re[\Gamma(ix)]$  the first root is  $x \approx \pm 1.80554707160510691987636662... = r_0$ . If we solve the point where  $\phi(s) = 0$ , starting for  $s = 0$ , we may get a root  $\kappa$ . Then we could posit that  $\kappa = Ir_0$ . If we assume  $a$  and  $b$  are real because they will likely exist as coefficients in the expansion.

If we keep a large list of close zeroes for the complex gamma function, we can attempt to use this information to match up the zeroes in our fingerprint/expression. If we add in a series expansion in terms of powers of  $s$ , and or a series of  $s^k \log \Gamma(s)$  terms, we may be able to partition the fingerprint.

This will give a set of linear equations to fit the gamma functions.

Because the moments are in a product format, a zero in one function will make the whole moment zero (unless there is an infinity).

In this sense we can drop a load of markers randomly, get each of them to approach the nearest zero, mark out the locations, look for a lattice type structure by finding a scale and shift.

We can equally do this for the reciprocal of the fingerprint, which converts the poles to zeroes and lets us solve for lower gamma functions.

For simple powers, we can use autocorrelation to detect oscillation in  $Re[\phi(it)]$  and  $Im[\phi(it)]$ .

Each time we detect a strong coefficient, we can remove it and note down the factor.

Assuming  $N$  gamma functions on top we

## 7 Schrodinger Equation

We set up the following numerical routines:

1. A 1-D time independent Schrödinger equation solver.
2. Interpolation
3. Numeric integration
4. Exact learning

For exact learning we choose a 7-coefficient fingerprint as:

$$\phi(s, \alpha) = \alpha_0 \alpha_1^s P_2(s) \Gamma(\alpha_2 s + \alpha_3)$$

where

$$P_2(s) = \alpha_4 + \alpha_5 s + \alpha_6 s^2$$

we run the BFGS routine fitting the complex log-moments obtained from the numeric integration to the fingerprint for all samples  $s \in S$  and receive the coefficient vector

$$\alpha = [0.34648851, 1.18917214, 0.50001699, 0.50004459, 3.99839001, -2.28486871, 2.28477497]$$

with a remarkably small loss  $L = 4.415494070864413e - 10$ . This can be mapped by eye onto the following rules

$$\alpha_2 = \alpha_3 = \frac{1}{2} \tag{1}$$

$$\alpha_4 = 4 \tag{2}$$

$$\alpha_5 = -\alpha_6 \equiv \pm\beta \tag{3}$$

in addition to this  $\alpha_1^2 \approx \sqrt{2}$ , guessing

$$\phi(s, \alpha) = \alpha_0 2^{s/2} (4 - \beta s + \beta s^2) \Gamma\left(\frac{s}{2} + \frac{1}{2}\right)$$

By the Ramanujan Master theorem we can write

$$\psi_{HO4}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \frac{\phi(-s)}{\Gamma(-s)} x^s =$$

## 8 Hydrogen Wavefunction

Mellin Transforms

$$\mathcal{M}[\psi_{100}(r, \theta, \phi)](s) = \frac{\sqrt{\frac{1}{a^3}} 2^{-s} \left(\frac{1}{a}\right)^{-s} \Gamma(s)}{\sqrt{\pi}}$$

$$\lambda_{nlm}(s, \theta, \phi) = \mathcal{M}[\psi_{nlm}(r, \theta, \phi)](s, \theta, \phi) = \frac{2^{1-s} (an)^s \Gamma(l+n+1) \Gamma(l+s)}{a^{3/2} n^2 \Gamma(l-s+2) \Gamma(n+s)} \sqrt{\frac{\Gamma(n-l)}{\Gamma(l+n+1)}} {}_2F_1(-l+s-1, l+s; n+s; 1) Y_l^m(t, p)$$

Harmonic Oscillator double Mellin transform:

$$-\frac{a^{\frac{1}{4}-\frac{s}{2}} \sin(\pi n) \sqrt{2^n \Gamma(n+1)} \Gamma(4t) 2^{-\frac{3n}{2}+s-10t-2} \Gamma\left(-\frac{n}{4}-t+\frac{1}{2}\right) \Gamma(-s-t+1) \Gamma\left(\frac{s}{4}-t\right)}{\pi^{9/4}}$$