MATH/CS 27700 FALL 2020: HW 4

BENEDICT MORRISSEY

Due: October 30th at 9PM.

Note 1: As noted, the "Take Home Test" Question (in red) must be worked on individually.

Note 2: Answers must be fully explained (unless otherwise specified). Follow the Homework policy (see syllabus) and remember to *write up all solutions on your own.*

(1) Suppose that we have theories in propositional logic [ie. a pair of a language L and a subset $\Gamma \subset Form(L)$] (L_1, Γ_1), (L_2, Γ_2), with L_1, L_2 not necessarily disjoint. You may assume L_1 and L_2 are finite, however this actually isn't needed. [For my original approach I wrote that your approach to HW3 problem 3 may be useful, however for the better approach this is not true.]

   (a) Take Home Test Question: Suppose in the language $L_1 \cup L_2$ we have that $\Gamma_1 \vDash \Gamma_2$. Show that there there exists $\Gamma_{12} \subset Form(L_1 \cap L_2)$ such that $\Gamma_1 \vDash \Gamma_{12}$ and $\Gamma_{12} \vDash \Gamma_2$.

   (b) Show that if $\Gamma_1 = \{\phi\}$, and $\Gamma_2 = \{\psi\}$, we can pick $\Gamma_{12} = \{\theta\}$ for some $\theta \in Form(L_1 \cap L_2)$

   Bonus: Remove the assumption that L_1 and L_2 are finite in Q1b), Either by using Q2b) [note this doesn't actually prove 1(b) unless we provide an alternative proof of 2(b) which is possible], or otherwise [better].

(2) In this question assume the result of 1 (b) *without the finiteness assumption.* That is to say assume for $\phi \in Form(L_1)$, $\psi \in Form(L_2)$, such that $\phi \vDash \psi$ (for the language $L_1 \cup L_2$) there is $\theta \in Form(L_1 \cap L_2)$ such that $\phi \vDash \theta$ and $\theta \vDash \psi$.

   (a) Suppose that we have consistent theories $(L_1, \{\phi\})$ and $(L_2, \{\chi\})$ as before. Suppose that $(L_1 \cup L_2, \{\phi, \chi\})$ is inconsistent. Show that there exists θ in $Form(L_1 \cap L_2)$ such that $\phi \vDash \theta$, and $\chi \vDash \neg\theta$.

   (b) Suppose that we have consistent theories (L_1, Γ_1) and (L_2, Γ_2) as before. Suppose that $(L_1 \cup L_2, \Gamma_1 \cup \Gamma_2)$ is inconsistent. Show that there exists θ in $Form(L_1 \cap L_2)$ such that $\Gamma_1 \vDash \theta$, and $\Gamma_2 \vDash \neg\theta$.

(3) Suppose $L' \supset L$. Let $\phi \in Form(L')$. Suppose that for any two truth functions t_1, t_2 (for L') such that $t_1|_{Form(L)} = t_2|_{Form(L)}$, we have that $t_1(\phi) = t_2(\phi)$.

   Show that[1] $\vDash \phi \leftrightarrow \psi$ for some $\psi \in Form(L)$.

(4) Take Home Test Question. In this question do not assume Lindebaum's lemma, or Gödel's completeness Theorem. Show that[2] $\vdash \alpha \to ((\neg\alpha) \to \beta)$.

(5) In this question do not assume Lindebaum's lemma, or Gödel's completeness Theorem. Show that if $\Gamma, \phi \vdash \psi$ then $\Gamma, \neg\psi \vdash \neg\phi$.

[1] By $\phi \leftrightarrow \psi$ I mean $(\phi \to \psi) \wedge (\psi \to \phi)$

[2] Note that this formula is simply $\alpha \to (\alpha \vee \beta)$. Note also that by $\vdash \phi$ we mean that $\varnothing \vdash \phi$.