MATH 27700: MATHEMATICAL LOGIC 1 NOTES FALL 2020

BENEDICT MORRISSEY

Contents

1. Ordinal and Cardinal Arithmetic

Omitted.

2. Propositional Logic

Note: This is sometimes called Sentential Logic.

This is a simple form of logic. While it is limited in what it can talk about it is useful to cover first because it is a special case of and includes some of the machinery of the significantly more applicable first order logic (or second, third, ... order logic). Hence we can both use it to introduce some of the machinery, and prove analogous theorems to those about first order logic. The analogues for propositional logic are usually much easier to prove!

Idea: We have a set of symbols $\{A_i\}_{i\in I}$. Informally we should see A_i as referring to something (such as a statement) which can either be true or false.

We want to be able to assemble sentences which are finite strings consisting of these symbols, together with logical operators such as and ($\wedge$), or ($\vee$) and not ($\neg$).

- Goal 1: Say which strings are allowed. Check that each string has a unique interpretation.

We want to be able to complete reasoning of the form; If A_1, A_2 are true, then so is $A_1 \wedge A_2$.

- Goal 2: Describe a form of reasoning using sentences in propositional logic. This is a *syntactic notion of proof/truth*. We will denote this by $\Gamma \vdash \phi$, where under the assumptions Γ being a set of sentences we assume are true, and ϕ being a sentence we have proven is true using these assumptions.

There is another notion of truth, which we call the semantic notion of truth.

Namely if we have assigned each A_i as being true or false, this should also assign every sentence in the associated propositional logic as true or false. We will call a *model* of a set of sentences Γ to be a map t from sentences in the propositional logic to the set $\{0, 1\}$ (1 denotes true, while 0 denotes false), satisfying certain compatibility conditions (e.g. if $t(A_1) = t(A_2) = 1$ then $t(A_1 \wedge A_2) = 1$), and such that for $A \in \Gamma$

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we have $t(A) = 1$. We then say that $\Gamma \vDash B$ if for all such models $t(B) = 1$. We call this the *semantic notion of truth.*

- Goal 3: Describe the relation between the syntactic notion of truth, and the semantic notion of truth. We will show

$$(\Gamma \vdash B) \Leftrightarrow (\Gamma \vDash B).$$

This is the *Completeness Theorem* for propositional logic.

We will also show a closely related result, the *Compactness Theorem* for first order logic which can be seen as an abstract version of the semantic part of the Completeness Theorem.

Finally we will describe the set (category) of theories (pairs of the set of symbols $\{A_i\}_{i\in I}$) together with a set of sentences Γ [that we can see as axioms – we are assuming these to be true]), in an algebraic fashion in terms of Boolean algebras. We can then describe models in terms of certain morphisms of Boolean algebras.

If there is time we will consider Stone duality from the viewpoint of propositional logic, where it forms a deep link between models of a theory and the theory itself.

2.1. **Informal Propositional Logic.** We have a set of logical variables $A, B, C, ...$ each of which can be true (1) or false (0).

We have a set of logical connectives, whose truth values are defined as in table 1:

A	B	$A \wedge B$	$A \vee B$	$\neg A$	$A \to B$	$A \leftrightarrow B$	$\top$	$\bot$
1	0	0	1	0	0	0	1	0
1	1	1	1	0	1	1	1	0
0	0	0	0	1	1	1	1	0
0	1	0	1	1	1	0	1	0

TABLE 1. Definition of logical connectives

Remark 2.1. The equivalence of various logical connectives in the classical propositional logic we work in is not the case in various non-classical logics. One example of this is relevance logic which does not use that $\phi \to \psi$ is equivalent to $(\neg\phi) \vee \psi$, but requires some link between ϕ and ψ, in a similar way to the standard meaning of implies in non-mathematical language.

The reason for our definition of implies is primarily a matter of mathematical equivalence.

Definition 2.2. We call a formula a *tautology* if it is true for all values of the propositional variables.

Exercise 2.3. Show that the three formulas/sentences in definition 2.12 are tautologies.

2.2. **Sentences and Uniqueness of interpretation.**

Definition 2.4 (Language for Propositional Logic)**.** A Language $\mathcal{L}$ for propositional logic consists of

- A set $L = \{A_i\}_{i\in I}$. We call elements in this set *atomic formula* or *propositional variables.* This set must not contain the symbols appearing in the following bullet points, or the left or right parenthesis.
- A set of logical connectives[1], we will use $\leftrightarrow$, $\to$, $\vee$, $\wedge$, $\top$, $\bot$, $\neg$.
- Parentheses (,).

Definition 2.5 (Formulas/Sentences)**.** Consider the set of all finite strings of sequences of elements of L, logical connectives, and parentheses.

The set $Form(L)$ is the smallest subset of the above set, with the property that

- $A_i \in Form(L)$ for $A_i \in L$.

[1]We have some freedom to modify what these are.

- $\top, \bot \in Form(L)$.
- If $A, B \in Form(L)$ then so are:

$$(A) \leftrightarrow (B), (A) \to (B), (A) \vee (B), (A) \wedge (B), \neg(A) \tag{2.1}$$

We call an element of the set $Form(L)$ a *formula* of $\mathcal{L}$ or a *sentence* of $\mathcal{L}$.

It is not clear that $Form(L)$ is well defined. This is because it is not clear that there is a set with the required properties that is a subset of all other sets with the required properties.

One approach to resolve this is to define a function LC from the power set of the set of finite strings of elements of L, logical connectives, and parentheses to itself.

We define $LC(S) := S \cup \{(A) \leftrightarrow (B), (A) \to (B), (A) \vee (B), (A) \wedge (B), \neg(A) | A, B \in S\}$.

It is clear that

$$\cup_{i=1}^{\infty} LC^i(L \cup \{\top, \bot\}) \tag{2.2}$$

is the desired smallest such set. One could also use this as a definition. This description will be useful in proofs.

An alternative definition of this smallest set is as the set of strings C such that there does not exist a set S satisfying the conditions of the definition with $C \notin S$.

Variant 2.6 (Language using fewer logical connectives)**.** *We could clearly use a subset of the logical connectives, and only introduce the atomic formulas, and those formed from formulas by the smaller set of logical connectives considered.*

It is clear that the unique reading lemma (lemma 2.8) will still hold for this variant.

Definition 2.7. We define a *theory in propositional logic* to be a pair (L, Γ) of a language L and a subset $\Gamma \subset Form(L)$.

Lemma 2.8 (Unique Reading Lemma propositional logic)**.** *Let $\phi \in Form(L)$, then ϕ has exactly one of the following forms (equation 2.3), and can be represented in that form in a unique way:*

$$(A) \leftrightarrow (B), (A) \to (B), (A) \vee (B), (A) \wedge (B), A, \top, \bot. \tag{2.3}$$

We base our proof off that in [2]. We will prove this by induction on the length of the formula (number of symbols in the formula). For the base case where the formula contains a single symbol then it must by $A_i \in L$, or $\top$, or $\bot$, and the Lemma is true.

To prove the induction step we are going to use the following unique bracketing lemma.

Lemma 2.9 (Unique bracketing lemma)**.** *Let $c_+, c_i \subset \{1, ..., n\}$ be two disjoint subsets.*

There is at most one bijection $cl : c_+ \to c_-$ (which we will refer to as a bracketing) with the properties that:

- *We have $i \leq cl(i)$ for all $i \in c_+$.*
- *If $i, j \in c_+$ and $i < j < cl(i)$, then $i < j < cl(j) < cl(i)$.*

Suppose we have $\phi \in Form(L)$. Let n be the number of symbols in ϕ, and number these symbols left to right with the numbers 1 to n. Let c_+, c_- be the numbers corresponding to the open parentheses, and the closed parentheses respectively. Then:

Exercise 2.10. Show that there is a such a function cl given by mapping (the number of) a left parenthesis to the (number of the) associated right parenthesis.

Proof of lemma 2.9. We proceed by induction on n. If $n = 1$ then we must have $c_+ = c_- = \varnothing$ and the result follows.

Assume the result is true for all $n < k$, then for $n = k$:

If $1 \notin c_+$ we can "move everything down by one" and use the inductive hypothesis for $n = k - 1$. We can hence assume $1 \in c_+$.

Let $g : \{1, ..., n\} \to \mathbb{Z}$ be given by

$$g(c) = \begin{cases} 1 & \text{if } c \in c_+ \\ -1 & \text{if } c \in c_- \\ 0 & \text{else.} \end{cases}$$

Let $f(m) \sum_{i=1}^{m} g(i)$.

We claim that if there is such a function cl, then $cl(1)$ is the smallest value of m (positive integer) such that $f(m) = 0$.

Note that if there is no such m then $|c_+| \neq |c_-|$ and as such there is no bijection.

Note that by the two conditions cl gives a bijection between $c_+ \cap \{1, , , cl(1)\}$ and $c_- \cap \{1, ..., cl(1)\}$. This means that we must have $f(cl(1)) = 0$.

Suppose that m is the positive integer such that $f(m) = 0$. Then by the first condition cl^{-1} must map $c_- \cap \{1, .., m\}$ injectively to $c_+ \cap \{1, .., m\}$. As $|c_- \cap \{1, .., m\}| = |c_+ \cap \{1, .., m\}|$ it must be a bijection. Hence $cl(1) \in \{1, ..., m\}$, and hence $cl(1) = m$.

Considering the setting where we are assigning left brackets and right brackets in a formula ϕ we can write $\phi = (\psi)\chi$ and apply the inductive hypothesis. We can in fact apply the inductive hypothesis by considering the bracketing restricting to $\{2, ..., m-1\}$ and $\{m+1, ..., n\}$. The result follows. □

We will now use lemma 2.9 to prove the unique reading lemma.

Proof of unique reading lemma (lemma 2.8). Let $\psi \in Form(L)$. By exercise 2.10 there is at least one way to pair the brackets in ψ. By the unique bracketing lemma this pairing of brackets is unique. The result follows. □

Remark 2.11. Having proven the unique reading lemma using careful and pedantic placing of brackets we will regress to using brackets when we feel they are needed as is done in standard human readable mathematics.

2.2.1. *Choice of Logical Connectives.* Note that the set of logical connectives we specified above is more than necessary. This is because we have the following equivalences

- $\neg\phi$ is equivalent to $\phi \to \bot$.
- $\phi \vee \psi$ is equivalent to $(\neg\phi) \to \psi$. This is in turn equivalent to $(\phi \to \bot) \to \psi$
- $\top$ is equivalent to $\neg\bot$, which by the above is equivalent to $\bot \to \bot$.
- $\phi \wedge \psi$ is equivalent to $\neg(\neg\phi \vee \neg\psi)$. Using the above we can then write this as a (long) sentence involving only the logical connectives $\to$ and $\bot$. However we will slightly simplify things by seeing it as equivalent to $\neg(\phi \to \neg\psi)$.
- $\phi \leftrightarrow \psi$ is equivalent to $(\phi \to \psi) \wedge (\psi \to \phi)$.

There are several other choices of logical connectives we could make. Adding logical connectives makes working inside propositional logic easier. Using a slimmer set of logical connectives makes proving things about propositional logic easy.

2.3. **Syntactic Proof.** There are many choices for what we mean by a proof. Choices include natural deduction, and sequent calculus. We will use a version of Hilbert calculus. The version we use is easy to prove things about, but difficult to prove things with.

We first introduce some axioms, that we will use in writing out proofs.

We first use section 2.2.1 to work in propositional logic using only the logical symbols $\to$ and $\bot$.

If we write propositional logic using only the logical connectives $\to$ and $\bot$ we can then write the axioms of propositional logic as:

Definition 2.12 (Axioms of Propositional Logic, using only the logical connectives $\to, \bot$)**.** We will refer to the following sentences as axioms of propositional logic:

- $\phi \to (\psi \to \phi)$
- $((\phi \to \bot) \to \bot) \to \phi$.
- $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$

for any sentences ϕ, ψ, χ.

The idea is that we will be able to use these axioms together with deduction rules to deduce all tautologies/statements that are always true for any truth valuations.

You can check using truth tables that these are statements that are true for any truth valuations. It is not a. priori clear that we might not require additional axioms.

Remark 2.13. There are many other choices of axioms we could have made, that would give us the same result.

To include additional logical symbols we would need to add axioms describing their relationship to $\to$ and $\bot$.

It is not entirely clear what the axioms we would need to add are. For example to add the logical symbol $\neg$ we could add the axioms:

- $(\neg\phi) \to (\phi \to \bot)$
- $(\phi \to \bot) \to (\neg\phi)$.

It is however not immediately clear whether or not these axioms would be sufficient.

We could instead modify the below notion of Syntactic proof, by allowing us to replace any logical sentence by the version that only involves $\bot, \to$. Or we could add the axioms that for any sentence ϕ, state that ϕ is equivalent to a sentence (use the recursive definition of formulas to define a way to recursively replace all logical connectives in a formula with the connectives $\to$, $\bot$ to gain an equivalent formula) only using the logical connectives $\to$, $\bot$. Or we could just interpret any formula using other logical connectives as the equivalent formula which only uses $\to$, $\bot$.

Let Γ be a set of sentences of our language, and let ϕ be a sentence in our language.

Definition 2.14 (Syntactic Proof in propositional Logic)**.** In propositional logic a syntactic proof $\Gamma \vdash \phi$ is a finite sequence of sentences $S_1, ..., S_n$ $(n \in \mathbb{N})$ of our language, with the properties that $S_n = \phi$, and that for each $1 \leq k \leq n$ either:

- $S_k \in \Gamma$.
- S_k is one of the axioms listed in definition 2.12.
- $\exists\ i, j < k$ with the properties that S_j is $S_i \to S_k$. (Modus Ponens)

Remark 2.15. It is clear that we should consider such a series of sentences a proof. It is perhaps less clear that we should *only* consider such a series a proof. We might ask if we should have included additional axioms, and additional deduction methods. Essentially the completeness theorem for propositional logic tells us we can get all semantic consequences the above definition is sufficient.

This definition is useful in that it is relatively easy to reason about what we can prove, it is not so good if one actually wants to prove something! In this case one could use other definitions of proof, such as natural deduction.

Example 2.16. Suppose we want to prove the statement $(\neg A) \vee A$. We realize this is (in disguised form) the axiom $((A \to \bot) \to \bot) \to A$.

If on the other hand we start with $A \vee (\neg A)$ we can instead have to prove

$$B \to B$$

for $B = (A \to \bot)$.

A proof follows

$$B \to ((B \to B) \to B)$$
$$(B \to ((B \to B) \to B)) \to ((B \to (B \to B)) \to (B \to B))$$
$$(B \to (B \to B)) \to (B \to B) \text{ (Modus Ponens)}$$
$$B \to (B \to B)$$
$$B \to B \text{ (Modus Ponens)}$$

Exercise 2.17. Find a proof of the statement $A \to (\bot \to \bot)$.

To prove the completeness theorem we are going to give some properties of the set $\{\phi | \Gamma \vdash \phi\}$.
We firstly show the *deduction* theorem:

Theorem 2.18 (Deduction Theorem). *We have $\Gamma, \phi \vdash \psi$ if and only if $\Gamma \vdash \phi \to \psi$.*

Proof. The if direction is clear. We will hence concentrate on the only if.
By definition of a proof, we only need (see remark 2.19) to show this for:

- ψ an axiom or an element of $\Gamma \cup \{\phi\}$.
- ψ is such that we have already show $\Gamma \vdash (\phi \to \chi)$, and $\Gamma \vdash (\phi \to (\chi \to \psi))$.

In the first case if ψ is an axiom or in Γ a proof $\Gamma \vdash (\phi \to \psi)$ is given by

$$\psi \to (\phi \to \psi)$$
$$\psi$$
$$(\phi \to \psi) \text{ (Modus Ponens).}$$

In the first case if $\psi = \phi$, then we have example 2.16 gives a proof $\Gamma \vdash \psi \to \psi$.
In the second case a proof is given by concatenating the proofs of $\phi \to \chi$ and $\phi \to (\chi \to \psi)$ with

$$\phi \to \chi$$
$$\phi \to (\chi \to \psi)$$
$$(\phi \to (\chi \to \psi)) \to ((\phi \to \chi) \to (\phi \to \psi))$$
$$(\phi \to \chi) \to (\phi \to \psi)$$
$$\phi \to \psi.$$

The result follows. □

Remark 2.19. The purpose of this remark is to explain the reduction we made. Let $MP : \mathcal{P}(Form(L)) \to \mathcal{P}(Form(L))$ be the operation that takes a set S to the set S union with all $\phi \in Form(L)$ such that there exists $\chi \in Form(L)$ such that $\chi, \chi \to \phi \in S$.
Then by the definition of proof we have that

$$\{\psi | \Gamma \vdash \psi\} = \cup_{n \in \mathbb{N}} MP^n(\Gamma \cup \{\text{Axioms}\}).$$

Let $Pd(S) := \{\psi | S \vdash \phi\}$ for any set $S \subset Form(L)$. Let $Pd_{\phi\to}(S) := \{\psi | S \vdash \phi \to \psi\}$.
We are showing that $Pd(\Gamma \cup \{\phi\}) \subset Pd_{\phi\to}(\Gamma)$. We are doing this by inductively (on n) arguing that

$$MP^n(\Gamma \cup \{\phi\} \cup \{\text{axioms}\}) \subset Pd_{\phi\to}(\Gamma).$$

Example 2.20 (Explosion). We have that $\bot \to \phi$ for any sentence ϕ.
By the first axiom we have $\vdash (\bot \to (\phi \to \bot) \to \bot)$. We hence have that

$$\bot \vdash (\phi \to \bot) \to \bot.$$

The second axiom gives that $\bot \vdash \phi$.

Theorem 2.21. *The following hold true:*

(1) *For any Γ, and any $\phi \in Form(L)$, we have $\Gamma \vdash (\neg\phi) \vee \phi$.*
(2) *If $\Gamma_1 \subset \Gamma_2 \subset \{\phi | \Gamma \vdash \phi\}$, then $\Gamma_1 \vdash \phi$ if and only if $\Gamma_2 \vdash \phi$.*
(3) *If $\Gamma \cup \{\phi\} \vdash \chi$, and $\Gamma \cup \{\psi\} \vdash \chi$, then $\Gamma \cup \{\phi \vee \psi\} \vdash \chi$.*
(4) *$\Gamma \vdash \top$.*
(5) *$\neg(\phi \to \psi) \vdash \neg\psi, \phi$.*
(6) *$(\neg\phi) \wedge \phi \vdash \bot$.*
(7) *$\{\phi, \phi \to \psi\} \vdash \psi$.*
(8) *If $\Gamma_1 \subset \Gamma_2$ then if $\Gamma_1 \vdash \phi$ then $\Gamma_2 \vdash \phi$.*
(9) *If $\{\Gamma_i\}_{i \in I}$ is well ordered by inclusion then if $\cup_{i \in I}\Gamma_i \vdash \phi$, then there exists i such that $\Gamma_i \vdash \phi$.*

Proof. (1) In example 2.16 we showed that $\vdash (\neg\phi) \vee \phi$. This first observation under point (2) shows that we hence have $\Gamma \vdash (\neg\phi) \vee \phi$.

(2) Firstly suppose that $(S_1, ..., S_n)$ is a proof $\Gamma_1 \vdash \phi$. Then by definition it is also a proof $\Gamma_2 \vdash \phi$.

Secondly suppose that $(S_1, ...S_m)$ is a proof $\Gamma_2 \vdash \phi$. Let $S_{j_1}, ..., S_{j_n} \in \Gamma_2 \backslash \Gamma_1$. We then have that for each $1 \leq i \leq n$ there is a proof $(C_{i,1}), ..., C_{i,l_i})$ of S_{j_i}. Then

$$(C_{1,1}, ..., C_{1,l(1)-1}, C_{2,1}, ..., C_{2,l(2)-1}, ..., C_{n,l(n)-1}, S_1, ..., S_m)$$

is a proof $\Gamma_1 \vdash \phi$.

(3) We firstly reduce this to the case where ψ is $\neg\phi$.

Suppose that this is true for ϕ and $\neg\phi$. Assume (using the deduction theorem) $\Gamma \vdash \phi \to \chi, \psi \to \chi$. Let $\Gamma' := \Gamma \cup \{(\neg\phi) \to \chi\}$. Note that $\Gamma', \phi \vdash \chi$ and $\Gamma', \neg\phi \vdash \chi$, because a proof is given by (appending proofs of the necessary statements to the beginning of) $\neg\phi, \neg\phi \to \psi, \psi, \psi \to \chi, \chi$. Hence by part (1) of this theorem $\Gamma, \phi \vee \psi \vdash \chi$. Hence it is enough to show that if $\Gamma, \phi \vdash \chi$, and $\Gamma, \neg\phi \vdash \chi$, then $\Gamma \vdash \chi$. Equivalently it is enough to show there is a proof of:

$$(\phi \to \chi) \to (((\neg\phi) \to \chi) \to \chi).$$

Lemma (a): If $A \to B$, $B \to C$ we can prove $A \to C$.

Proof: By the deduction theorem it is enough to show that $A, A \to B, B \to C \vdash C$, which is clear.

Lemma (b): $(\phi \to \chi) \to (((\neg\chi) \to \phi) \to \chi)$.

By the deduction theorem it is enough to show that

$$\phi \to \chi, (\neg\chi) \to \phi \vdash \chi.$$

By lemma (a) and the deduction theory it is thus sufficient to show that $(\neg\chi \to \chi) \to \chi$.

An outline of a proof is given by

$$\neg\chi \to (\neg\neg(\delta \to \delta)) \to \neg\chi) \text{ (axiom)}$$
$$(\neg\neg(\delta \to \delta)) \to \neg\chi) \to (\chi \to \neg(\delta \to \delta)) \text{ HW4} + \text{DeductionTheorem.}$$
$$\neg\chi \to (\chi \to \neg(\delta \to \delta)) \text{ Lemma(a)}$$
$$(\neg\chi \to (\chi \to \neg(\delta \to \delta))) \to ((\neg\chi \to \chi) \to (\neg\chi \to \neg(\delta \to \delta))) \text{ Axiom}$$
$$((\neg\chi \to \chi) \to (\neg\chi \to \neg(\delta \to \delta))) \text{ MP}$$
$$(\neg\chi \to \neg(\delta \to \delta)) \to ((\delta \to \delta) \to \chi) \text{ HW4}$$
$$(\neg\chi \to \chi) \to ((\delta \to \delta) \to \chi)$$

We can now conclude the proof by using the deduction theorem and noting example 2.16.

Lemma (c) We have that

$$(((\neg\chi) \to \phi) \to \chi) \to (((\neg\phi \to \chi)) \to \chi).$$

From HW4 and the deduction theorem we have that$((\neg\chi) \to \phi) \leftrightarrow (\neg\phi) \to \chi)$. Hence we can use this and the deduction theorem to prove this.

By applying lemma (a) to Lemma (b) and lemma (c) we get the desired result.

(4) We set $\top = (\perp \to \perp)$. Specializing the proof of $B \to B$ in example 2.16 to $B = \perp$ shows this is true.

(5) Firstly we note that unpacking this sentence we get $(\phi \to \psi) \to \perp$. We first show that $\psi, ((\phi \to \psi) \to \perp) \vdash \perp$ showing by the deduction theorem that $((\phi \to \psi) \to \perp) \vdash (\psi \to \perp)$.

A proof is given by

$$\psi \to (\phi \to \psi)$$

$$\psi$$

$$(\phi \to \psi) \text{ (Modus Ponens)}$$

$$(\phi \to \psi) \to \perp$$

$$\perp \text{ (Modus Ponens).}$$

For ϕ we will first show that $((\phi \to \psi) \to \perp), \phi \to \perp \vdash \phi \to \psi$. By the deduction theorem it is equivalent to show that $((\phi \to \psi) \to \perp), \phi \to \perp, \phi \vdash \psi$. A "proof" is given by

$$\phi$$

$$\phi \to \perp$$

$$\perp$$

$$\perp \to \psi$$

$$\psi$$

where we are using $\perp \vdash \psi$ from example 2.20 (note: that we are abusing the notation of a proof slightly).

We hence have by modus ponens that $((\phi \to \psi) \to \perp), \phi \to \perp \vdash \perp$. By the deduction theorem we hence have that

$$((\phi \to \psi) \to \perp) \vdash ((\phi \to \perp) \to \perp).$$

Using modus ponens and the second axiom we hence have that

$$((\phi \to \psi) \to \perp) \vdash \phi.$$

(6) Unpacking $(\neg\phi) \wedge \phi$ we realize this is the sentence $(\phi \to \phi) \to \perp$. We use example 2.16 to show $(\neg\phi) \to (\neg\phi)$ and then apply modus ponens to derive $\perp$.

(7) A proof is given by the sequence $(\phi, \phi \to \psi, \psi)$.

(8) This was in fact proved under point (2).

(9) Because of the finite length of a proof, any proof must only contain finitely many sentences which are in $\cup_{i \in I}\Gamma_i$. Hence there must be some Γ_i which contains all these sentences. Hence any given proof of ϕ is in fact a proof $\Gamma_i \vdash \phi$ for some $i \in I$.

□

2.4. **Semantic Truth/Proof.** The idea behind propositional logic is that each symbol $P_i \in L$ can either be true or False.

Consider such a map of assignments

$$\mathcal{P} \xrightarrow{t} \{\perp, \top\} \cong \{0, 1\}.$$

Construction 2.22. There exists a unique extension of t as above to a as above extends to a function:

$$Form(L) \xrightarrow{t} \{0, 1\}.$$

via the recursive definition:

- $t(\neg\phi) = 1 - t(\phi)$.
- $t(\phi \vee \psi) = \max(t(\phi), t(\psi))$.
- $t(\phi \wedge \psi) = \min(t(\phi), t(\psi))$.
- $t(\bot) = 0$.
- $t(\top) = 1$.
- $t(\phi \to \psi) = \max(1 - t(\phi), t(\psi))$.

Note that we are using the unique reading lemma.

Definition 2.23 (Truth function)**.** We call a function $t : Form(L) \to \{1, 0\}$ a truth function if

- $t(\neg\phi) = 1 - t(\phi)$.
- $t(\phi \vee \psi) = \max(t(\phi), t(\psi))$.
- $t(\phi \wedge \psi) = \min(t(\phi), t(\psi))$.
- $t(\bot) = 0$.
- $t(\top) = 1$.
- $t(\phi \to \psi) = \max(1 - t(\phi), t(\psi))$.

Proposition 2.24. *Any truth function $t : Form(L) \to \{0, 1\}$ is constructed from $t|_L : L \to \{0, 1\}$ by construction 2.22.*

Proof. Clear. □

Definition 2.25 (Model)**.** Let Γ be a set of sentences of $Form(L)$. We call a truth function t a *model* of Γ, if $t(\phi) = 1$ for all $\phi \in \Gamma$.

Definition 2.26 (Semantic notion of Truth)**.** We write $\Gamma \vDash \phi$ if for all models t of Γ, $t(\phi) = 1$.

People sometimes also write $t \vDash \phi$, if $t(\phi) = 1$.

This is the *semantic* version of the notion that "The sentences of Γ imply the statement ϕ."

Theorem 2.27 (Soundness of propositional logic)**.** *If $\Gamma \vdash \phi$ then $\Gamma \vDash \phi$.*

Proof. Let $\Gamma \vdash \phi$, and let $t : Form(L) \to \{0, 1\}$ be an arbitrary truth function.

Let $(S_k)_{k=1}^n$ be a proof of ϕ. We prove by induction that $t(S_k)$ is true for all $1 \le k \le n$. Recall that there were three possibilities for S_k, we could have any of:

- $S_k \in \Gamma$, in which case $t(S_k) = 1$ by definition of a model.
- S_k is an axiom, in which case $t(S_k) = 1$ (e.g. by checking truth tables [exercise]).
- $\exists\, i, j < k$ with the properties that S_j is $S_i \to S_k$. Hence by the induction hypothesis $t(S_i) = 1$, and
$$1 = t(S_i \to S_k) = \max\{1 - t(S_i), S_k\} = \max\{0, t(S_k)\},$$
hence $t(S_k) = 1$.

Hence $t(\phi) = t(S_n) = 1$.

As t was an arbitrary model, we hence have $\Gamma \vDash \phi$. □

Exercise 2.28. At first glance it looks like we did not include a base case in the induction proof inside Theorem 2.27. When did we cover the base case?

2.5. **Completeness Theorem.** An abstract interpretation of a proof system is as a map $\mathcal{P}(Form(L)) \xrightarrow{Pd} \mathcal{P}(Form(L))$. This map is given by

$$\mathcal{P}(Form(L)) \ni \Gamma \mapsto \{\phi | \phi \in Form(L), \Gamma \vdash \phi\}.$$

We will say that we can derive a contradiction from Γ if $\bot \in Pd(\Gamma)$.

The function Pd has the following properties, and these properties will be sufficient to provide a proof of the completeness Theorem for propositional logic.

Theorem 2.29. *The function Pd has the following properties:*

(1) *For any Γ, and any $\phi \in Form(L)$, we have $(\neg\phi) \vee \phi \in Pd(\Gamma)$.*
(2) *If $\Gamma \subset S \subset Pd(\Gamma)$, then $Pd(S) = Pd(\Gamma)$.*
(3) *$Pd(\Gamma \cup \{\phi \vee \psi\}) \supset Pd(\Gamma \cup \phi) \cap Pd(\Gamma \cup \psi)$ (i.e. if $\Gamma \cup \{\phi\} \vdash \chi$, and $\Gamma \cup \{\psi\} \vdash \chi$, then $\Gamma \cup \{\phi \vee \psi\} \vdash \chi$).*
(4) *$\top \in Pd(\Gamma)$ (i.e. $\Gamma \vdash \top$)*
(5) *$Pd(\neg(\phi \to \psi)) \ni \neg\psi, \phi$. (i.e. $\neg(\phi \to \psi) \vdash \neg\psi, \phi$).*
(6) *$\perp \in Pd(\phi, \neg\phi)$. (i.e. $(\neg\phi) \vee \phi \vdash \perp$).*
(7) *$Pd(\phi, \phi \to \psi) \ni \psi$. (i.e. $\phi, \phi \to \psi \vdash \psi$.)*
(8) *If $\Gamma_1 \subset \Gamma_2$ then $Pd(\Gamma_1) \subset Pd(\Gamma_2)$.*
(9) *If $\{\Gamma_i\}_{i \in I}$ is well ordered by inclusion then we have that $Pd(\cup_{i \in I}\Gamma_i) = \cup_{i \in I} Pd(\Gamma_i)$. (Equivalently if $\cup_{i \in I}\Gamma_i \vdash \phi$, then there exists i such that $\Gamma_i \vdash \phi$).*

This is just a rewrite of Theorem 2.21.

Let us first consider the existence of models for propositional logic. In particular we will prove proposition 2.6. We first introduce the following definition:

Definition 2.30 (Consistency)**.** We say that a set $\Gamma \subset Form(L)$ is not consistent if $\Gamma \vdash \perp$.

Otherwise we call Γ consistent.

Proposition 2.31 (Lindenbaum's Lemma)**.** *A set Γ is consistent if and only if it has a model.*

Remark 2.32. In fact we will prove that for any function $Pd : \mathcal{P}(Form(L)) \to \mathcal{P}(Form(L))$ satisfying the properties of Theorem 2.29, if Γ is such that $Pd(\Gamma) \ni \perp$, then Γ has a model.

One direction is straightforward: If Γ is not consistent, then $\Gamma \vdash \perp$. By soundness this implies that $t(\perp) = 1$ for any model $t : Form(L) \to \{0, 1\}$. However by definition there is no such model.

We are now left to prove the harder direction: If Γ is consistent we need to construct/find/prove the existence of a model for Γ. Recall that specifying a model of t is equivalent to finding the set $\{\phi \in Form(L) | t(\phi) = 1\}$ or indeed to finding the $\{A_i \in L | t(A_i) = 1\}$. In the finite case we can try to go through the A_i (or the ϕ) and either add ϕ or $\neg\phi$ to Γ (having chosen an ordering on the ϕ).

We start by considering the case where L is finite.

If Γ does not give us a contradiction, then we need that at least of $\Gamma \cup \{\phi\}$ or $\Gamma \cup \{\neg\phi\}$ does not gives us a contradiction. This follows from properties (1-3) of Theorem 2.29. We choose one of the subset of $\{\phi, \neg\phi\}$ that does not give us a contradiction and add this to Γ.

Let $\Gamma' \supset \Gamma$ be the set we create after applying this process to all sentences of L.

Claim: Define $t : Form(L) \to \{0, 1\}$ by

$$t(\phi) = \begin{cases} 1, & \text{if } \phi \in \Gamma' \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Then t is a model of Γ in the sense of Definition 2.25.

Proof: We need to check the conditions in the definition of a truth function:

- $t(\neg\phi) = 1 - t(\phi)$. This follows because exactly one (Exercise: why exactly one, use property 6 of Theorem 2.29) of ϕ and $\neg\phi$ is in Γ'.
- $t(\phi \vee \psi) = \max(t(\phi), t(\psi))$. Using subsection 2.2.1 this is equivalent to the last point in this list.
- $t(\phi \wedge \psi) = \min(t(\phi), t(\psi))$. Using subsection 2.2.1 this is equivalent to the last point in this list.
- $t(\perp) = 0$. By construction Γ' is consistent, so this follows by the definition of consistency.
- $t(\top) = 1$. This follows by property 4 of Theorem 2.29.
- $t(\phi \to \psi) = \max(1 - t(\phi), t(\psi))$. Note that exactly one of each of the sets $\{(\phi \to \psi), \neg(\phi \to \psi)\}$, $\{\phi, \neg\phi\}$ and $\{\psi, \neg\psi\}$ is in Γ'. If $\neg(\phi \to \psi) \in \Gamma'$ then by property 5 of Theorem 2.29 we have $\phi, \neg\psi \in \Gamma'$, and this relation is satisfied. If $\phi \to \psi, \phi \in \Gamma'$ then $\psi \in \Gamma'$ by property 7 of Theorem

2.29, and this relation is satisfied. If $\phi \to \psi, \neg\phi \in \Gamma'$ then this relation is satisfied regardless of which of $\{\psi, \neg\psi\} \in \Gamma'$.

This construction works in the case L (and hence $Form(L)$) is finite. In the case where L is infinite we have to be more clever. One way that we can describe t is by $t^{-1}(1)$, which we can describe as a maximal consistent subset of $Form(L)$. We can use this description to define a model t in the general case

Recall:

Let $(P, \leq_P)$ be a partially ordered set. We call a totally ordered subset of P a *chain.* We call an element $x \in P$ a *maximal element* if there is no $y \in P$ with $y \geq_P x$, and $y \neq x$.

Lemma 2.33 (Zorn's Lemma)**.** *Let P be a partially ordered set such that every chain has an upper bound in P (ie. for a chain $C \subset P$, $\exists x \in P$ such that $x \geq y$, $\forall y \in C$). Then the set of maximal elements of P is nonempty.*

Remark 2.34. Note that P is non-empty by considering the empty set as a chain.

Proof. Assume that there is no maximal element in P. We use the axiom of choice to define a function g from well ordered subsets of P to P, with the property that for $S \subset P$ well ordered, $g(S) \notin S$, and $g(S) > s$ for all $s \in S$ (we here use the existence of an upper bound, and the assumption that there is no maximal element).

We say a set S satisfies property g if for all $s \in S$ $s = g(\{s' \in S | s' \leq s\})$.

Let S_1 and S_2 satisfy property g then either[2] $S_1 \subset S_2$ or $S_2 \subset S_1$. (This should remind you of ordinals, and the proof will indeed proceed similarly). Consider the maximal subset[3] S_3 which is an initial sequence[4] of S_1 and S_2. Then as S_1 and S_2 satisfy property g, we have either that $g(S_3) \in S_i$ $(i = 1, 2)$ or $S_i = S_3$. The result follows by the assumption of maximality of S_3.

Let G be the union of all sets satisfying property g. Then G satisfies property g . Furthermore $G \cup \{g(G)\}$ satisfies property g. Contradiction. The result follows. ☐

Exercise 2.35. Use Zorn's lemma to show that every (not necessarily finite dimensional) vector space has a basis.

Proof of Proposition 2.6. Consider the set P of consistent sets of formula of our language, containing Γ.

We claim that this satisfies the hypotheses of Zorn's lemma if Γ is consistent (Exercise).

There is hence a[5] maximal such set S. We claim that t defined by

$$t(\phi) = \begin{cases} 1, & \text{if } \phi \in S \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

is a model of Γ.

We need to check the conditions of a truth function. This is essentially the same as the discussion in the finite case. If we do not have either ϕ or $\neg\phi$ in S this would contradict maximality. If it did not satisfy one of the other properties it would contradict the consistency of S. ☐

We can now approach Gödel's completeness Theorem for propositional logic.

Theorem 2.36 (Gödel's completeness Theorem for propositional logic)**.** *If $\Gamma \vDash \phi$ then $\Gamma \vdash \phi$.*

Remark 2.37. We hence have that $\Gamma \vdash \phi$ if and only if $\Gamma \vDash \phi$.

[2]By $\subset$ we include the possibility of equality.

[3]This exists by taking the union of all such subsets.

[4]I.e. is equal to all elements less that a certain element.

[5]Not necessarily unique.

Proof. Suppose that $\Gamma \vDash \phi$. Then there is no model of $\Gamma \cup \{\neg\phi\}$. Hence by proposition 2.6 we must have that $\Gamma \cup \{\neg\phi\}$ is inconsistent.

That is to say $\Gamma \cup \{\neg\phi\} \vdash \bot$. By the Deduction Theorem 2.18 we hence have that $\Gamma \vdash (\neg\phi) \to \bot$, that is to say $\Gamma \vdash ((\phi \to \bot) \to \bot)$. Applying the second axiom and modus ponens shows that $\Gamma \vdash \phi$. □

2.6. **Compactness Theorem.** We here prove a corollary of Proposition . Note that because any proof by definition only uses finitely many axioms, we have that every

Theorem 2.38 (Compactness Theorem for Propositional Logic)**.** *Suppose that every finite subset of* Γ *admits a model. Then* Γ *admits a model.*

Proof. If every finite subset of Γ admits a model. Then by proposition 2.6 shows that every finite subset of is consistent. However because any proof $\Gamma \vdash \bot$ must by definition of proof use only finitely many elements of Γ, we have that if every finite subset of Γ is consistent then Γ is consistent. Applying proposition 2.6 again shows that Γ has a model. □

Remark 2.39*.* This proof is rather disappointing in that given a model for every finite subset of Γ we have no idea how to construct a model for Γ. We will solve this problem later using ultrafilters.

Remark 2.40*.* In some ways the Compactness Theorem (2.38) can be viewed as more fundamental that Gödel's theorem (or proposition 2.6). This is because it doesn't depend on which notion of proof you use – instead it is purely about semantics.

2.7. **Boolean Algebras.** We want to give truth functions/models a more algebraic structure.

Definition 2.41 (Boolean Algebra)**.** A *Boolean Algebra* is a sextuple $(B, \wedge, \vee, \neg, 0, 1)$, where B is a set, $0, 1 \in B$, $\neg : B \to B$, and $\wedge, \vee : B \times B \to B$ are associative and commutative binary operations.

These must satisfy the boolean relations (for all $a, b, c \in B$):

$$\begin{aligned} a \vee (a \wedge b) &= a \\ a \wedge (a \vee b) &= a \\ a \vee 0 &= a \\ a \wedge 1 &= a \\ a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c) \\ a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c) \\ a \vee \neg a &= 1 \\ a \wedge \neg a &= 0. \end{aligned}$$

Example 2.42*.* Take $B = \mathcal{P}(S)$ (the power set of S) for some set S. For $A, B, C \in \mathcal{P}(S)$ define:

$$\begin{aligned} A \wedge B &:= A \cap B, \\ A \vee B &:= A \cup B, \\ \neg A &:= \{s \in S | s \notin A\}, \\ 1 &= S, \\ 0 &= \varnothing. \end{aligned}$$

Exercise 2.43. Show that example 2.42 satisfies the axioms (boolean relations) of a boolean algebra.

Example 2.44*.* Consider the case of example 2.42 where $|S| = 1$. Then $B = \{\varnothing, S\}$. This gives a two element boolean algebra that we also denote by $\{0, 1\}$, where operations agree with logical operations.

Definition 2.45. A morphism of boolean algebras from $(B, \wedge_B, \vee_B, \neg_B, 0_B, 1_B)$ to $(C, \wedge_C, \vee_C, \neg_C, 0_C, 1_C)$ is a morphism $f : B \to C$ such that for all $b_1, b_2 \in B$:

- $f(b_1 \wedge_B b_2) = f(b_1) \wedge_C f(b_2)$.
- $f(b_1 \vee_B b_2) = f(b_1) \vee_C f(b_2)$.
- $f(0_B) = 0_C$.
- $f(1_B) = 1_C$.

Exercise 2.46. Show that if f is a morphism of boolean algebras as above we have $f(\neg_B b_1) = \neg_C f(b_1)$.

We now want to give another way to describe boolean algebras in terms of lattices.

Definition 2.47. A *lattice* is a partially ordered set S such that any two elements have a unique supremum and infimum.

Remark 2.48. This gives two binary operations $S \times S \to S$ which we denote by $a \vee b$ and $a \wedge b$ $(a, b \in S)$, taking two elements to their supremum and infimum respectively.

Example 2.49. Consider the lattice given by $\mathcal{P}(A)$ for some set A, with the partial order given by inclusion.

Definition 2.50. A *distributive* lattice S is a lattice in which the operations $\vee$ and $\wedge$ distribute over each other. That is to say:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

for all $a, b, c \in S$.

Definition 2.51. Let S be a lattice with a greatest element, which we will denote 1, and a least element that we will denote 0.

A *complement* of $x \in S$ is an element $y \in S$, such that $x \wedge y = 0$, $x \vee y = 1$.

Exercise 2.52. Show that for a distributive lattice a complement is unique if it exists. In this case we write the complement of x as $\neg x$.

Definition 2.53. A *complemented* lattice is a lattice, with greatest element, and least element, such that every element has a complement.

Proposition 2.54. *Boolean Algebras are equivalent to complemented, distributive lattices.*

Proof. Let $(B, \wedge_B, \vee_B, \neg_B, 0_B, 1_B)$ be a boolean algebra. We define a partial order $\leq$ on B as follows:

Define $a \leq b$ $(a, b \in B)$ if $b = a \vee c$ for some $c \in B$.

Exercise: Show that this is a complemented, distributive lattice (with the complement of $a \in B$ given by $\neg a$).

Exercise: Show the converse, that is given a complemented, distributive lattice S, we have previously defined operations $\wedge, \vee, \neg$, and elements $0, 1$. Show that these satisfy the axioms of a boolean algebra. □

Construction 2.55 (Lindensbaum–Tarski Algebra for Propositional Logic). Let Γ be consistent. Consider the equivalent relation on $Form(L)$ given by $\phi \sim_\Gamma \psi$ if $\Gamma \vdash \phi \leftrightarrow \psi$.

Consider the partial order on $Form_L / \sim_\Gamma$ given by $\phi \preceq_\Gamma \psi$ if $\Gamma \vdash \phi \to \psi$.

Theorem 2.56. *The set with partial order defined in construction 2.55 is a complemented, distributive lattice, and hence a boolean algebra.*

We denote this boolean algebra by $Lind(L, \Gamma)$

Proof. It is clear that we have a partially ordered set. The least upper bound of two elements $[\phi]$, $[\psi]$ is $[\phi \vee \psi]$ (exercise: check this is well defined – that is to say we would have got the same equivalence class if we had chosen different representations of $[\phi]$ and $[\psi]$). The greatest lower bound is $[\phi \wedge \psi]$. hence $Lind(L, \Gamma)$ is a lattice.

Furthermore this lattice is distributive, and has complements $[\phi] = [\neg\phi]$.

The result follows. □

Proposition 2.57. *Models of* Γ *correspond to morphisms of boolean algebras*

$$Lind(L,\Gamma) \xrightarrow{t'} \{0,1\}.$$

Proof. Firstly suppose that we have a morphism of boolean algebras t'. We get a truth function t by the composition

$$Form(L) \to Form(L)/\sim_\Gamma \xrightarrow{t'} \{0,1\}.$$

Exercise: Show that this is a model of Γ as in definition 2.25.

Secondly suppose that we have a model $t : Form(L) \to \{0,1\}$.

By the definition of a model it descends to $t' : Form(L) \to \{0,1\}$.

Exercise: Show that the definition of a truth function shows that this induces a morphism of boolean algebras. □

It is now worth considering the preimage of 1 under a morphism of Boolean algebras.

Definition 2.58. A *filter* of B is a subset $F \subset B$, with the property that for all $f_1, f_2 \in F$, $b \in B$ we have $f_1 \wedge f_2 \in F$, and $b \vee f_1 \in F$.

Proposition 2.59. *Let* $B \xrightarrow{f} C$ *be a morphism of Boolean algebras. We then have that* $f^{-1}(1_C)$ *is a filter.*

Exercise 2.60. Prove proposition 2.59.

Exercise 2.61. Conversely show that if $F \subset B$ is a filter, then there is a morphism of Boolean algebras $B \xrightarrow{f} C$ such that $f^{-1}(1) = F$.

Hint: Construct $C = B/\sim_F$, for the equivalence relation $b_1 \sim_F b_2$ if

$((\neg b_1) \vee b_2) \wedge (b_1 \vee (\neg b_2)) \in F$

Note: This may appear unmotivated, however it should seem significantly more motivated when you see the definition of the Lindenbaum–Tarski algebra and see that this formula is precisely $b_1 \leftrightarrow b_2$.

Definition 2.62. We say a filter $F \subset B$ is *proper* if $B \neq F$.

Definition 2.63. An *ultrafilter* is a filter F with the property that for all $b \in B$ either $b \in F$ or $\neg b \in F$.

Exercise 2.64. If F is an ultrafilter, show that there is no proper filter F' with $F \subsetneq F'$.

Exercise 2.65. Show that if $B \xrightarrow{f} \{0,1\}$ is a morphism of Boolean algebras, then $f^{-1}(1)$ is an ultrafilter.

Exercise 2.66. Conversely show that if $F \subset B$ is an ultrafilter then there is a morphism $B \xrightarrow{f} \{0,1\}$, with the property that $f^{-1}(1) = F$.

Definition 2.67. For convenience we restate the definition of a filter in the case where the Boolean algebra is a special case of example 2.42.

An *ultrafilter on a set* S is a set $F \subset \mathcal{P}(S)$, such that for all $A \subset S$, either $A \in F$ or $S \backslash A \in S$, and such that F is closed under union, and closed under intersection with sets in $\mathcal{P}(S)$.

Definition 2.68. An *ultrafilter on a set* S is a set $F \subset \mathcal{P}(S)$, such that:

- $\emptyset \notin F$.
- If $A, B \in \mathcal{P}(S)$, $A \subset B$, $A \in F$, then $B \in F$.
- If $A, B \in F$ then $A \cap B \in F$.
- If $A \in \mathcal{P}(X)$ then either A or $S \backslash A$ is in F.

Exercise 2.69. Show that definitions 2.67 and 2.68 are equivalent.

Exercise 2.70. Use Zorn's lemma to show ultrafilters exist. Note that this can be seen as a modification of our proof of one direction of lemma 2.6.

It is also worth asking what the precise relationship between theories in propositional logic (recall definition 2.7). More precisely we can ask:

Question 2.71. Which Boolean algebras are of the form $Lind(L, \Gamma)$ for some theory in propositional logic (L, Γ)?

Exercise 2.72. Let B be a boolean algebra. Show that there is a theory in propositional logic (L, Γ) such that $Lind(L, \Gamma) \cong B$.

Hint: Set $L = B \backslash \{0_B, 1_B\}$. Add propositions to Γ as needed.

Call this theory $Th(B)$.

Question 2.73. If there is a morphism $Lind(L_1, \Gamma_1) \to Lind(L_2, \Gamma_2)$ then what is (if any) the corresponding relationship between the theories (L_1, Γ_1) and (L_2, Γ_2)?

In particular if $Lind(L_1, \Gamma_1) = Lind(L_2, \Gamma_2)$, what is the relationship between (L_1, Γ_1) and $(L_2, \Gamma 2)$?

We will not fully answer this question, but will make a start at suggesting the maps Th and $Lind$ are *functorial*, in the sense that they are compatible with morphisms of boolean algebras, and interpretations of theories in propositional logic.

Definition 2.74. We say that an *interpretation* of (L_1, Γ_1) in (L_2, Γ_2) is a morphism $s : L_1 \to Form(L_2)$, with the below detailed properties.

Firstly note that s naturally extends to a function, which we also denote s, $s : Form(L_1) \to Form(L_2)$.

We require that if $\Gamma_1 \vdash \phi$, then $\Gamma_2 \vdash s(\phi)$.

We denote an interpretation by

$$(L_1, \Gamma_1) \xrightarrow{s} (L_2, \Gamma_2).$$

Exercise 2.75. Show that given an interpretation $(L_1, \Gamma_1) \xrightarrow{s} (L_2, \Gamma_2)$ we get a morphism of Boolean algebras

$$Lind(L_1, B_1) \to Lind(L_2, B_2).$$

Remark 2.76. We can go significantly further, alas probably beyond the scope of this course. The state of the art statement is that Th, and $Lind$ are an *adjoint functor pair*[6] between the categories of theories in propositional logic (morphisms being interpretations) and the category of boolean algebras. See [1] for further details.

The perhaps important point, is that ultimately one can consider the category of boolean algebras, and that of first order theories to be essentially the same.

References

[1] John Dougherty. Categorical logic. `http://www.johnedougherty.com/misc/catlog.pdf`, 2020. Course Notes.
[2] Yu I Manin. *A course in mathematical logic for mathematicians*, volume 53. Springer Science & Business Media, 2009.

[6] Or in fact weak inverses.