Borel–Weil Theory

Understand representations of a reductive algebraic group over field char 0. ⟷ In terms of geometry

$$\left(\begin{array}{c} GL(n,\mathbb{C}) \\ SL(n,\mathbb{C}) \\ SO(2n,\mathbb{C}) \\ Sp(2n,\mathbb{C}) \end{array}\right)$$

$\underbrace{\phantom{GL(n,\mathbb{C})}}_{\text{reductive alg group.}}$

group struct + structure of complex manifold

th $G\times G \to G$, $(g,h) \mapsto gh$, $G \to G$, $g \mapsto g^{-1}$ holomorphic diffeomorphisms.

$G = GL(n)$ matrix multiplication.

Representations

$\Phi : G \longrightarrow GL(V)$

→ group morphisms & complex diffeomorphisms

→ Do computations in G in terms of matrices (if $\mathrm{Ker}(\Phi) = \mathrm{Id}_G$).

→ Particles of a gauge theory ⟶ Representations of Forces group ⋉ Gauge group

→ Recover G from its category of representations

G finite group. $G \curvearrowright X = \{x_1, \dots, x_n\}$.

$\Leftrightarrow$ $G \xrightarrow[\text{group homomorphism.}]{} \mathrm{Aut}(X) \cong S_n$

$$\mathbb{C}[X] := \{f : X \to \mathbb{C}\} \cong \mathbb{C}^n$$

$$f \longmapsto (f(x_1), f(x_2), \dots, f(x_n)).$$

$G \curvearrowright \mathbb{C}[X]$

$$\boxed{(gf)(x) = f(g^{-1}x)}$$

$$h(gf)(x) = gf(h^{-1}x) = f(g^{-1}h^{-1}x) = f((hg)^{-1}x)$$

Can also use this formula if $G = GL(n,\mathbb{C})$ & $G \circlearrowright \underset{\sim}{X}$ complex manifold
action ~~map~~ is holomorphic.
$G \times X \longrightarrow X$ must be complex diffeomorphism

Group action.

$G \to \mathrm{Aut}(X)$
Group homomorphism

Defn 2.
$G \times X \xrightarrow{act} X.$

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mathrm{Id}_G \times act"] \arrow[d, "mult_G \times \mathrm{Id}"'] \arrow[dr, phantom, "\circlearrowright"] & G \times X \arrow[d] \\ G \times X \arrow[r, "act"] & X \end{tikzcd}$$

$(g,h,x) \mapsto (g, h\cdot x)$

$g\cdot(hx) = (gh)\cdot x$

$\Leftrightarrow$

Could consider
$\to$ holomorphic functions
$\to$ meromorphic functions $\to$

Sections of a vector bdle.

On $\mathbb{C}$ $f(x) = x$ holomorphic
$f(x) = \frac{1}{x}$ is meromorphic

$$s \curvearrowright \begin{array}{c} V \\ \downarrow p \\ X \end{array}$$ is a vector bdle.
$X \leftarrow$ complex manifold

$U_1 \cong \mathbb{C}$

glue along intersections.

~~Over each~~ $\forall\, U \in \cdot$ $V|_U = \{v \in V, p(v) \in U\}$
$\cong$
$U \times \mathbb{C}^n$

gluing map $U_1 \cap U_2 \xrightarrow{g} GL_n(\mathbb{C})$

identify $U_1 \times \mathbb{C}^n|_{U_1 \cap U_2}$ w/ $U_2 \times \mathbb{C}^n|_{U_1 \cap U_2}$
using g

Does G act on sections of a vector bundle $\begin{array}{c} V \\ s \uparrow\downarrow \\ X \end{array}$.

$G \times X \to X$.
action.

s.

Try: $\underbrace{(gs)(x)}_{\text{want something in } V|_x} = \rho(s(g^{-1}x)) \in V|_{g^{-1}x} \xrightarrow{\rho_g} V|_x$

What you need for this to work out

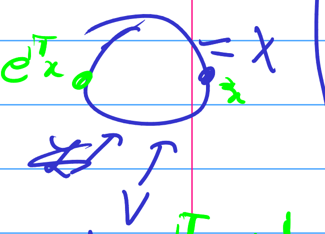
$$\begin{array}{ccc} & \circlearrowright & V \\ & & \downarrow \rho \\ G & \circlearrowright & X \end{array}$$

Choose this to be action of g on the vector bdle.

This diagram commutes

$$g(\rho(v)) = \rho(g(v))$$
$$v \in V$$

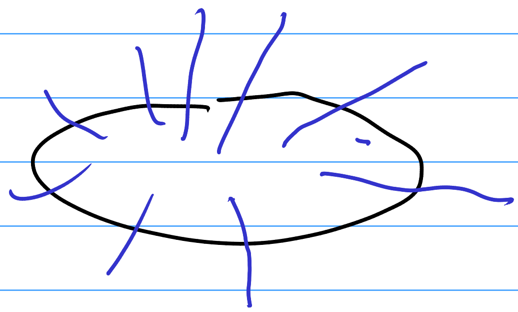
$e^{i\pi} \in G = S^1$ $\circlearrowright$ by rotation

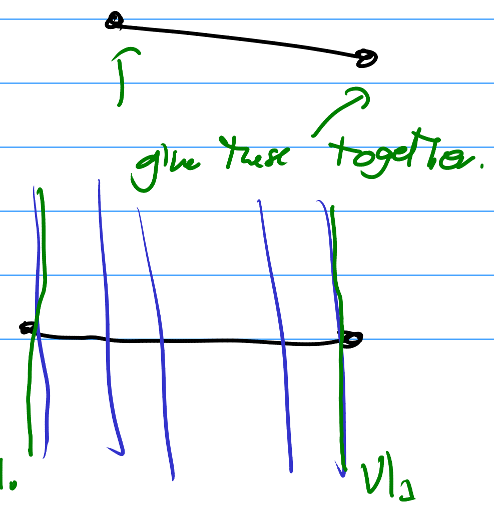


$V|_x \xrightarrow{e^{i\pi}} V|_{e^{i\pi}x}$ extra data.

Equivariant Vector bdle. $\Longrightarrow$ G acts on $\Gamma(X, V)$ space of sections of equivariant V.B.

Example V.B.
Möbius strip over ○.





Identify $V|_0$ w/ $V|_1$
give V B on circle.

$\mathbb{C} \times S^1$
$\downarrow$
S^1

$\mathbb{C} \times I / \{(z, \{0\}) \sim (-z, \{1\})\}$

$I = [0,1]$

could use $\mathbb{R}$ also

$G = GL(n)$ or $SL(n)$

Pick $X = \{$ Space of full filtrations of $\mathbb{C}^n \}$

$= \{ \{0\} \subset F^1 \subset F^2 \subset F^3 \subset \dots \subset F^n = \mathbb{C}^n \}$

Example $GL(2)$

$\{0\} \subset F \subset \mathbb{C}^2$

$\mathbb{C}^2 \setminus \{0\} \longrightarrow X =: \mathbb{P}^1$

$(a,b) \xmapsto{\text{proj.}} [\text{Line through } (a,b)]$

$= \{(ax, bx) \mid x \in \mathbb{C}\}$.

$m((a,b)) = m((d,e))$

if. $(d,e) = (ax, bx)$ for some $x \in \mathbb{C}$ (in fact $x \in \mathbb{C}^\times$)

View $\mathbb{P}^1 = (\mathbb{C}^2 \setminus \{0\}) / \mathbb{C}^\times$

Viewpt 2 is charts.

$[a,b]$ If $b \neq 0$

$\wr$

$[\frac{a}{b}, 1]$. I'm getting a copy $U_1 = \mathbb{C}$.

pts in $\mathbb{P}^1$, $b \neq 0$

If $a \neq 0$. $[a,b] \sim [1, \frac{b}{a}]$.

$U_2 = \mathbb{C}$

pts in $\mathbb{P}^1$ $a \neq 0$.

$U_1 \cup U_2 = \mathbb{P}^1$

$\mathbb{P}^1 = \{ \underset{z}{U_1} \text{ \& } \underset{w}{U_2}$ where we glue together $z \sim \frac{1}{w} \}$

$GL(3)$

$$0 \subset F_1 \subset F_2 \subset \mathbb{C}^3$$

Start by describ

$$0 \subset F^2 \subset \mathbb{C}^3$$

$F^2 \subset \mathbb{C}^3$ $\qquad\qquad$ $ax+by+cz=0.$

~~Correspond~~ $\qquad\qquad$ $(a,b,c) \in (\mathbb{C}^3)^*$

$$\{\text{Plane in } \mathbb{C}^3\} \longleftrightarrow \{\underline{\text{Line}} \text{ in } (\mathbb{C}^3)^*\}$$

$\not\in$

Could have used $(\lambda a, \lambda b, \lambda c)$ instead.

$$\cancel{\mathbb{C}^3}/ ((\mathbb{C}^3)^* \setminus \{\vec{0}\}) / \mathbb{C}^\times = \mathbb{P}^2((\mathbb{C}^3)^*)$$

vector bdle over this

$$\begin{array}{c} V \\ \downarrow \\ \mathbb{P}((\mathbb{C}^3)^*) \end{array}$$

fiber over $[a,b,c]$

is subspace $ax+by+cz=0$ inside $\mathbb{C}^3$.

$\rightarrow$ ~~Apply the proj constr~~

In each fiber were going to do the same thing.

$\mathbb{C}^\times$ acts on fiber. [as each fib is vector space]

$$\begin{array}{c} (V \setminus \{0\text{-section}\})/\mathbb{C}^\times \\ \downarrow \\ \mathbb{P}((\mathbb{C}^3)^*) \end{array}$$

~~Viewpt 2~~

$GL(n) \circlearrowright \left\{ \begin{array}{c} \text{Set of} \\ \text{filtrations} \end{array} \right\}$

$$0 \subset F^1 \subset \mathbb{C}^2$$
$$\downarrow g \in GL(n)$$
$$0 \subset g(F^1) \subset \mathbb{C}^2$$

This is a <u>transitive</u> action.

$$0 \subset F^1 \subset F^2 \subset \dots \subset F^n = \mathbb{C}^n$$

pick a compatible basis $f_1, f_2, f_3 \dots f_n \in \mathbb{C}^n$

$f_1 \in F_1$
$f_2 \in F_2$
$\vdots$ $f_i \in F^i$,

To an orderd basis $\rightsquigarrow$ assoc a filtration
$F_1 = \mathrm{Span}(f_1)$
$F_2 = \mathrm{Span}(f_1, f_2)$
$\vdots$

$GL(n)$ acts transitively on ordered bases

$$GL(n) \circlearrowright \{\text{ordered bases}\}$$
$$\downarrow$$
$$\{\text{filtrations}\}$$

$\Rightarrow GL(n)$ acts transitively on filtrations.

$$0 \subset F^1 \subset F^2 \subset \dots \subset F^n$$

ask what is the stabiliser of the filtration.

$\rightarrow$ Pick a basis ~~$f_1, f_2, \dots, f_n$~~ compatible w/ filtration as before.

$f_n, f_{n-1}, \dots, f_1$

$$B := \begin{pmatrix} * & & & \\ & * & & \star \\ & & * & \\ & 0 & & * \\ & & & & * \end{pmatrix}$$

On digal or above,

$f_1 \mapsto a f_1$

$f_2 \mapsto b f_2 + c f_1$

$f_3 \mapsto d f_3 + e f_2 + g f_1$

$\vdots$

$GL(n) \twoheadrightarrow X$

$g \longmapsto g \cdot [\text{chosen filtration}]$

$X \cong GL(n) / B.$

Equivat Line bdles on X.

B = upper Δ matrices in some basis.

$B \twoheadrightarrow T$ = diagonal matrices in $GL(n) = (\mathbb{C}^\times)^n$

Send everyth above diagonal to zero

$T \xrightarrow{\chi} \mathbb{C}^\times$

Character

$\chi \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n}$

$m_i \in \mathbb{Z}.$

Charact of $GL(n)$ $\longleftrightarrow \mathbb{Z}^n$

$$\begin{tikzcd} GL(n) \times \mathbb{C} / B \arrow[d] & =: L_\chi \arrow[dl] \\ GL(n) / B = X. & \end{tikzcd}$$

B acts on $\mathbb{C}$ via $B \to T \xrightarrow[\chi]{\text{chart}} \mathbb{C}^\times$

$$\left\{\begin{matrix}\text{Equivariant}\\ \text{vector bds}\\ \text{on } X\end{matrix}\right\} \xleftarrow{\cong} \left\{\begin{matrix}\text{Characters}\\ \text{of } T\end{matrix}\right\}$$

Equivs as T has to act on fibres by some character,

All represents this way.

$L_\lambda \longmapsto$ ~~Characters~~ Represent V_λ.

λ (has restricts) "dominant".

T-eigenspaces in V_λ wrt an order the largest T-eigenspace will ~~be~~ have T act by λ.

Borel–Weil Thm.

SL_2.

$$X = \mathbb{P}^1$$

$U_1 = \mathbb{C}_z$ $\qquad$ U_2 $\mathbb{C}_w$

$$z \sim \frac{1}{w}.$$

$U_1 \times \mathbb{C}$ $\qquad$ $U_2 \times \mathbb{C}$

(z, x) $\qquad$ (w, y)

$$(z, x) \sim \left(\frac{1}{z}, z^{\pm n} y\right).$$

$L_n = ((\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}) / \mathbb{C}^\times$

$z \in \mathbb{C}^\times$ acts on $\mathbb{C}$ by mult by z^n

$\downarrow$

$\mathbb{P}^1 = (\mathbb{C}^2 \setminus \{0\}) / \mathbb{C}^\times$

Chars

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \longmapsto a^n$$

Sections of line bdls L_k

~~$a_k z^k$~~ $\quad a_k z^k + a_{k-1} z^{k-1} w + a_{k-2} z^{k-2} w^2 + \dots + a_0 w^k$

homogeneous poly's $\quad$ scale by λ^k when scale

$$[z, w] \sim [\lambda z, \lambda w].$$

The $SL(2)$ action is

$$\begin{pmatrix} z \\ w \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix}$$

replace z, w in homogeneous poly by

$$SL_2 \circlearrowright \mathbb{C}^2.$$

induces $SL_2 \circlearrowright \mathrm{Sym}^k(\mathbb{C}^2)$.