MATH 27700 / CMSC 27700, Autumn 2020, Assignment 1

Due Friday, Oct. 9th, by 5:00 PM CDT on Canvas

1. Cantor's first paper on set theory was published in 1874. In addition to a proof of the uncountability of the reals, it also contained a proof of the existence of transcendental numbers. This problem gives a proof of this theorem. Recall that a complex number is *algebraic* if it is the root of a polynomial in one variable with rational coefficients, and is otherwise *transcendental*. To simplify things, you may assume that "countable" means "countably infinite".

- **a.** Show that the union of countably many countable sets is countable.
- **b.** Show that the set of pairs of natural numbers is countable.
- **c.** Show that for each $k \in \mathbb{N}$, the set of k-tuples of natural numbers is countable.
- **d.** Show that the set of polynomials in the single variable x with rational coefficients is countable.
- **e.** Show that the set of algebraic numbers is countable.
- **f.** Conclude that there is a transcendental number.

Notice that this proof is nonconstructive, in the sense that it proves the existence of a transcendental without actually producing a particular example. Cantor's original published proof was constructive (although he was aware of this nonconstructive argument), and in fact his diagonalization method can be used to construct a transcendental from a list of all algebraic numbers, which can also be constructed.

2. Let $A \subseteq [0,1]$ be a set, and consider the following two-player game. Players I and II alternate, selecting digits (i.e., numbers from 0 to 9) $x_0, x_1, \ldots$ and $y_0, y_1, \ldots$, respectively. Player I wins if the number $0.x_0y_0x_1y_1x_2y_2\ldots$ is in A. Otherwise, Player II wins. Show that if A is countable, then Player II has a winning strategy, i.e., a way of playing that guarantees a win no matter how Player I plays. [Hint: Consider Cantor's diagonalization method.]

3. Let $A \subseteq \mathbb{R}$ be uncountable. Show that there is an $a \in A$ such that both $A \cap (-\infty, a)$ and $A \cap (a, \infty)$ are uncountable. [This problem might be a bit more difficult. You will likely want to use problem 1.a here.]