MATH/CMSC 27700 FALL 2020: HW 5

Due: November 6th at 9PM.

Note 1: As noted, the "Take Home Test" Question (in red) must be worked on individually.

Note 2: Answers must be fully explained (unless otherwise specified). Follow the Homework policy (see syllabus) and remember to *write up all solutions on your own.*

Note 3: Later parts of Q4 may be difficult. I don't necessarily expect people to finish it.

(1) Let $f : B_1 \to B_2$ be a morphism[1] of Boolean algebras. Show that for all $b \in B_1$, we have:

$$f(\neg_{B_1} b) = \neg_{B_2} f(b)$$

Hint: In one approach to this problem you might want to show that complements in distributive lattices are unique.

(2) Let B be a Boolean algebra, and F be a filter. Recall we defined the equivalence relation $\sim_F$ on B by $b_1 \sim_F b_2$ is $((\neg b_1) \vee b_2) \wedge ((\neg b_2) \vee b_1)$.

We defined the operations

$$\vee_{B_F} : (B/\sim_F) \times (B/\sim_F) \to B/\sim_F,$$

$$\neg_{B_F} : B/\sim_F \to B/\sim_F$$

by $[b_1] \vee_{B_F} [b_2] := [b_1 \vee b_2]$. and $\neg_{B_F}[b_1] := [\neg b_1]$.

(a) Take Home Test Question: Show that $\neg_{B_F}$ is well defined.

(b) Show that $\vee_{B_F}$ is well defined, that is to say the above definition does not depend on the chosen representatives of the conjugacy class.

(3) (a) Take Home Test Question: Let $f : B_1 \to B_2$ be a morphism of Boolean algebras. Let $\leq_1, \leq_2$ be the corresponding partial orders under the equivalence between Boolean algebras and complemented distributive lattices. Show that if $b \leq_1 a$ then $f(b) \leq_2 f(a)$.

(b) Is the converse true? That is to say let $(L_1, \leq_1)$, $(L_2, \leq_2)$ be two complemented, distributive lattices. Let $g : L_1 \to L_2$ is a map of sets, with the property that if $b \leq_1 a$ then $g(b) \leq_2 g(a)$. Does g necessarily give a morphism of Boolean algebras (with respect to the Boolean algebra structures associated to the lattice structures)?

(4) (See note 3) Suppose that B is a boolean algebra.

(a) Show that if $(L_1, \Gamma_1) \xrightarrow{s} (L_2, \Gamma_2)$ is an *interpretation*[2] (see footnote), then this induces a morphism which we denote $Lind(s)$,

$$Lind(L_1, \Gamma_1) \xrightarrow{Lind(s)} Lind(L_2, \Gamma_2).$$

Show that this is compatible with composition of morphisms (i.e. for interpretations $(L_1, \Gamma_1) \xrightarrow{s_1} (L_2, \Gamma_2) \xrightarrow{s_2} (L_3, \Gamma_3)$, we have that $Lind(s_2 \circ s_1) = Lind(s_2) \circ Lind(s_1)$).

Note: This shows that $Lind$ is a *functor* from propositional theories to Boolean algebras.

[1]That is to say $f(b_1 \wedge_B b_2) = f(b_1) \wedge_C f(b_2)$, $f(b_1 \vee_B b_2) = f(b_1) \vee_C f(b_2)$, $f(0_B) = 0_C$, and $f(1_B) = 1_C$.

[2]We say that an *interpretation* of (L_1, Γ_1) in (L_2, Γ_2) is a morphism $s : L_1 \to Form(L_2)$, with the below detailed properties.

Firstly note that s naturally extends to a function (which we also denote s) $s : Form(L_1) \to Form(L_2)$.

We require that if $\Gamma_1 \vdash \phi$, then $\Gamma_2 \vdash s(\phi)$.

We denote an interpretation by

$$(L_1, \Gamma_1) \xrightarrow{s} (L_2, \Gamma_2).$$

Note that interpretations can be composed.

(b) Show that there is a propositional theory $Th(B)$ such that the associated Lindenström–Tarski algebra is isomorphic[3] to B, that is to say: $Lind(Th(B)) \cong B$.
Hint: Start by constructing a propositional theory (L', Γ') with a surjective morphism of sets

$$Lind(L', \Gamma') \to B,$$

and then modify as needed.

(c) Show that we can define Th in part (b) of this question such that for every morphism of boolean algebras $B_1 \xrightarrow{f} B_2$ we have an interpretation of propositional theories $Th(B_1) \xrightarrow{Th(f)} Th(B_2)$, and this is compatible with composition of morphisms (that is to say for $B_1 \xrightarrow{f} B_2 \xrightarrow{g} B_3$, we have that $Th(f \circ g) = Th(f) \circ Th(g)$, or abstractly $Th : Bool \to Prop$ is a functor [see notes Appendix B]).

(d) Bonus/Challenge: Show that we can define Th such that there is a *natural transformation* [in the sense given in the appendix to the notes] $Th \circ Lind \Leftarrow Id_{Prop}$., where $Prop$ is the category where objects are theories in propositional logic, and morphisms are interpretations, and Id_{Prop} is the identity functor on this category.

[3] Two Boolean algebras B_1, B_2 are isomorphic if there are inverse morphisms of Boolean algebras $f : B_1 \rightleftarrows B_2 : g$.