MATH/CS 27700 FALL 2020: HW 3

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Due: October 23rd at 9PM.

Note 1: Questions 3,5 may be postponed to the next homework if we have not covered the relative material by October 20th. If so this will be announced on October 20th.

Note 2: As noted, the "Take Home Test" Question (in red) must be worked on individually.

Note 3: Answers must be fully explained (unless otherwise specified). Follow the Homework policy (see syllabus) and remember to *write up all solutions on your own.*

(1) This question aims to prove that if κ, η are non-zero cardinals, such that at least one is infinite, then

$$\kappa \oplus \eta = \kappa \otimes \eta = \max\{\kappa, \eta\} \tag{0.1}$$

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  (a) First deal with the case where either $\kappa = 1$ or $\eta = 1$.

  (b) Assume that $\kappa, \eta \geq 2$, prove

$$\max\{\kappa, \eta\} \leq \kappa \oplus \eta \leq \kappa \otimes \eta \leq \max\{\kappa, \eta\} \otimes \max\{\kappa, \eta\}.$$

  Conclude from the result of question 2 (Equation 0.2) that Equation 0.1 holds.

(2) (This question is worth significantly more points than other questions) In this question we will prove that for an infinite cardinal κ we have

$$\kappa = \kappa \otimes \kappa. \tag{0.2}$$

  (a) Show that $\kappa \leq \kappa \otimes \kappa$ for a non-zero cardinal κ.

We will prove equation 0.2 by contradiction[1]. Suppose that exists a cardinal η such that $\eta \neq \eta \otimes \eta$. Consider the set of cardinals less than or equal to η such that equation 0.2 does not hold. Then this set must have a minimal element λ, as it is a nonempty set of ordinals.

Note from Homework 1 Q 1 (b) $\lambda \neq Card(\mathbb{N}) = \aleph_0$. You may need this fact in part (d).

  (b) Let β be an ordinal. Consider the relation $\preceq$ on $\beta \times \beta$ defined by

$$(a_1, a_2) \preceq (b_1, b_2)$$

  if $\max\{a_1, a_2\} \leq \max\{b_1, b_2\}$ ($\leq$ is here the standard order on ordinals) or if $\max\{a_1, a_2\} = \max\{b_1, b_2\}$ and $(a_1, a_2) \leq_{lex} (b_1, b_2)$, where $\leq_{lex}$ refers to the lexicographic ordering on a product of two ordered sets (we use the standard order [by inclusion] on ordinals to give an order on κ).

  Show that $\preceq$ is a well ordering on $\beta \times \beta$.

  (c) Show that if α is an infinite cardinal then it is a limit ordinal.

  (d) Consider the order from part (b) for the ordinal $\beta = \lambda$ defined after part (a). Let $(a, b) \in \lambda \times \lambda$ (a, b are ordinals), and define the well ordered (by $\preceq$) set

$$I_{\preceq}((a, b)) := \{(c, d) \in \lambda \times \lambda | (c, d) \preceq (a, b)\}.$$

  Let $\epsilon = \max\{a, b\} + 1$. Show that

$$|I_{\preceq}((a, b))| \leq |\epsilon \times \epsilon| = |\epsilon| < \lambda, \tag{0.3}$$

  (You need to show each of the three equalities/inequalities in equation 0.3 holds).

[1] With a few minor changes one can change this proof to a proof by transfinite induction on ordinals.

(e) From equation 0.3 conclude that if γ is the ordinal corresponding to the well ordered set $I_{\preceq}((a,b))$ then

$$\gamma < \lambda \tag{0.4}$$

(here $\leq$ refers to the standard order on ordinals).

(f) From equation 0.4 conclude that $\lambda \otimes \lambda \leq \lambda$ obtaining a contradiction.

The result follows.

(3) We work in propositional logic with a finite language $|L| = n$ (i.e. n propositional variables). Let $|\Gamma| = 1$ (that is to say $|\Gamma|$ consists of a single formula). Let m be the number of models of Γ. What are the possible values m can take?

(4) "Take Home Test Question" Individual work only. Let κ, η, λ be cardinals. Show that $(\kappa^{\eta})^{\lambda} = \kappa^{\eta \otimes \lambda}$.

(5) "Take Home Test Question" Individual work only. Show that

$$(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$$

is a tautology.