MATH/CMSC 27700 FALL 2020: HW 8

Due: December 4th at 9PM.

Note 1: As noted, the "Take Home Test" Question (in red) must be worked on individually.

Note 2: Answers must be fully explained (unless otherwise specified). Follow the Homework policy (see syllabus) and remember to *write up all solutions on your own.*

Note 3: All content with the exception of the Löwenheim–Skolem Theorems (4.69 in Notes version Sunday 22nd), which may be useful for some questions, has already been covered in class.

Note 4: All references to the notes refer to the version of November 22nd, which have been uploaded to files in Canvas under a different name.

Reminders: This is the final Homework of this course.

(1) Details from the definability proof and the construction of the (Henkin) Model for Lindenbaum's Lemma. As such **do not use Gödel's Completeness Theorem or Lindenbaum's Lemma in the solutions to this question**.

Let (L, Γ) be a theory in first order logic.

(a) Show that if $\Gamma \vdash c_1 = c_2$, and $\Gamma \vdash c_2 = c_3$ then $\Gamma \vdash c_1 = c_3$.

(b) Take Home Test Question: Show that if $\Gamma \vdash \psi(c)$ (where ψ is a formula with one free variable x, and $\psi(c)$ is where we have substituted x with a constant c) then $\Gamma \vdash \neg\forall x(\neg\psi)$.

(c) For this part we assume that Γ has witnesses. Show that $\phi(f)$ in the proof of Lindenbaum's lemma is well defined. That is to say suppose that $c_i \sim_\Gamma c'_i$ for $1 \leq i \leq n$. Show that if d and d' are witnesses for the formulas

$$x = f(c_1, ..., c_n),$$

and

$$x = f(c'_1, ..., c'_n)$$

respectively then

$$\Gamma \vdash d = d'.$$

(2) More on Ultraproducts. Let X and J be sets. For each $j \in J$ let $\mathcal{U}_j$ be an ultrafilter on X. Let $\mathcal{V}$ be an ultrafilter on J.

Define the ultrafilter $\prod_{\mathcal{V}} \mathcal{U}_j$ by $X \supset S \in \prod_{\mathcal{V}} \mathcal{U}_j$ if and only if $\{j \in J | S \in \mathcal{U}_j\} \in \mathcal{V}$. Define $\mathcal{W} := \prod_{\mathcal{V}} \mathcal{U}_j$.

Let t_x be a model of a propositional theory (L, Γ) for each $x \in X$. Recall the definition of ultraproducts of models of theories in propositional logic from HW7 Q4.

(a) Show that

$$\prod_{\mathcal{W}} t_x = \prod_{\mathcal{V}} \left(\prod_{\mathcal{U}_j} t_x \right).$$

(b) Suppose that $(L_1, \Gamma_1) \xrightarrow{s} (L_2, \Gamma_2)$ is an interpretation of propositional theories. Let $\Psi_s : Mod(L_2, \Gamma_2) \to Mod(L_1, \Gamma_1)$ be the induced map from the set of models of (L_2, Γ_2) to the set of models of (L_1, Γ_1).

Show that for some ultrafilter $\mathcal{U}$ on a set X, and a set of of models t_x of (L_2, Γ_2) for each $x \in X$, we have that

$$\Psi_s(\prod_{\mathcal{U}} t_x) = \prod_{\mathcal{U}} \Psi_s(t_x).$$

Informally we can describe this as "ultraproducts commute with interpretations."

(c) (Bonus, Hard) Let (L_1, Γ_1) be a propositional theory. Let $\tau \subset \mathcal{P}(Mod(L_1, \Gamma_1))$ be the set defined in question 5 of HW7[1].
Let X be a set and let t_x be a model of (L_1, Γ_1) for each $x \in X$. Let $cl(X) \subset Mod(L_1, \Gamma_1)$ be the set of models t such that such that $t = \prod_{\mathcal{U}} t_x$ for some ultraproduct $\mathcal{U}$ on a subset $Y \subset X$. Show that $Mod(L_1, \Gamma_1) \backslash cl(X) \in \tau$ (ie. the complement of $cl(X)$ is in τ).

(3) (Note: Most parts of this question are independent of each other. This question is worth significantly more than Q 1,2) In this question we will prove a version of the Ax–Gröthendieck Theorem (Proposition 0.1 below). This essentially follows S4.6.4 in the course notes.

(a) Write down a countable set of axioms, that when added to the axioms of a field (HW6 Q1) together state that the field is of characteristic zero (see last homework for definition if necessary).

A version of the Ax–Göthendieck Theorem follows:

Proposition 0.1 (Simple case of Ax–Gröthendieck Theorem)**.** *Let k be an algebraically closed field Let $P : k^n \to k^n$ be a polynomial map. Then if P is injective it is surjective.*

(b) Take Home Test Question: Write down a set of first order sentences in the first order theory of fields that expresses proposition 0.1.

We call a theory (L, Γ) complete, if for every sentence (formula with no free variables) ψ, either $\Gamma \vDash \psi$ or $\Gamma \vDash \neg\psi$.

(c) Using the Löwenheim–Skolem theorems (or otherwise) show that if for each infinite cardinal $\kappa > |L|$ (recall that we defined a language L to contain countably many variable symbols) there is a unique model of a theory (L, Γ), and (L, Γ) has no finite models, then (L, Γ) is complete.

Steinitz' Theorem states that any two algebraically closed fields of the same characteristic, and the same cardinality, with cardinality strictly greater than $\aleph_0$, are isomorphic. Hence the above exercise shows that the theory of algebraically closed fields of a given characteristic is complete.

(d) Use the fact that the theory of algebraically closed fields of a given characteristic (0 or p) is complete to prove that the following are equivalent:
- The theory of algebraically closed fields of characteristic zero models a sentence ψ.
- The theory of algebraically closed fields of characteristic p models the sentence ψ for all but finitely many primes p.

Hint: Look at Homework 7.

(e) Show that the sentences you wrote in exercise 3b are satisfied in the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of characteristic p.
Argue that this means they are satisfied in the algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$. Hint: Reduce to a finite subfield of $\overline{\mathbb{F}_p}$ and use the same argument as used for $\mathbb{F}_p$.
Conclude that proposition 0.1 is true for a field of characteristic zero, and hence in all fields of characteristic zero.

(f) (Bonus: Not difficult, but it's really an algebra question) Prove Steinitz' Theorem, that is to say that any two algebraically closed fields of the same characteristic, and of the same uncountable cardinality, are isomorphic.

(4) For this question I am looking for around (the equivalent of in whatever writing style is used) 1-2 typed pages, however if more is *needed* you can write more.

Pick a particularly interesting idea/theorem[2]/application/construction/technique from this course or from mathematics directly relevant to this course.

[1]Strictly speaking this was defined for first order theories. Note that nothing changes in the case of propositional theories, except that we can ignore elementary equivalences, as all such equivalences are actually equalities in this setting.

[2]Other than the Ax–Gröthendieck Theorem.

Write your own explanation of it, you may wish to include some subset of definitions/examples/proofs/applications.