MATH/CMSC 27700 FALL 2020: HW 6

Due: November 13th at 9PM.

Please Fill out the Mid Semester Survey if you have not already done so. The link is on the main Canvas Page.

Note 1: As noted, the "Take Home Test" Question (in red) must be worked on individually.

Note 2: Answers must be fully explained (unless otherwise specified). Follow the Homework policy (see syllabus) and remember to *write up all solutions on your own.*

(1) Give axioms for a first order theory of fields[1].

(2) (Larger Portion of Marks than other questions) Let M_1 and M_2 be models of a first order theory (L, Γ). Let $g : M_1 \to M_2$ be a morphism.

Consider the following properties:

(I) g is an elementary morphism.

(II) g is an injection[2] between the sets M_1 and M_2.

(III) g is a bijection between the sets M_1 and M_2.

(IV) $g(\phi_1(R)) = \phi_2(R)$ for each relation[3] R in the language L.

(V) g is an bijection between the sets M_1 and M_2, and $g(\phi_1(R)) = \phi_2(R)$ for each relation R.

For each of the following statements $(a) - (e)$ say which of the assumptions $(I) - (V)$ imply the result. Justify both that the properties specified imply the results, and the remaining properties do not imply the given result (For many of these a single sentence is fine. However some of them require significantly more.):

(a) Let χ be a sentence of L not containing the symbols $\forall$ or $\exists$ (we call this quantifier free). Then $M_1 \vDash \chi$ if and only if $M_2 \vDash \chi$.

(b) Take Home Test Q: Let χ be a sentence not containing the symbol R, for any relationship[4] R of L. Then $M_1 \vDash \chi$ if and only if $M_2 \vDash \chi$.

(c) Let χ be a sentence of the form $\exists x \varphi$, where φ is a quantifier free[5] formula of the language L. Then

$$M_1 \vDash \chi \Rightarrow M_2 \vDash \chi.$$

That is to say $M_1 \vDash \chi$ implies $M_2 \vDash \chi$.

(d) For any formula $\chi(x)$ in which only the variable x appears freely, the morphism g gives a bijection between the sets $S_1 = \{a_1 \in M_1 | M_1 \vDash \chi(a_1) = 1\}$ and $S_2 = \{a_2 \in M_2 | M_2 \vDash \chi(a_2)\}$ (informally speaking S_i is the subset of M_i of elements a such that in the model M_i the statement $\chi(a)$ is true).

(3) Take Home Test Question: Show that every ultrafilter on a finite set is principal.

(4) Let (L, Γ) be a first order theory. Let M be a model of (L, Γ). Let $A \subset M$ be a subset. Define the new language $L_A := L \cup \{c_a | a \in A\}$, where we consider c_a to be a constant of L_A (informally we have added one constant to L for each element of A). We can consider M as a model of (L_A, Γ) (where the constant c_a is mapped to a).

Consider the set of formulas $\psi \in Form(L_A)$, such that the variables appearing freely in ψ are in the set $x_1, ..., x_n$, call this set of formulas $Form_{\vec{x}}(L_A)$. We define an equivalence relation of

[1] If you are unfamiliar with the definition of a field, see e.g. `https://en.wikipedia.org/wiki/Field_(mathematics)#Classic_definition` for an informal definition.

[2] By this we mean the underlying map of sets $g : S_1 \to S_2$ is an injection.

[3] Here we are denoting by ϕ_i the map $\phi_i : Relations^n(L) \to \mathcal{P}(M_i^n)$.

[4] Recall that in the convention of this class equality symbol $=$ is a logical symbol rather than a relationship.

[5] Does not contain the symbols $\forall$ or $\exists$

$Form_{\vec{x}}(L_A)$ by $\psi(\vec{x}) \sim_M \chi(\vec{x})$ if and only if

$$M \vDash \forall \vec{x}(\psi(\vec{x}) \leftrightarrow \chi(\vec{x})).$$

(a) Show that $Form_{\vec{x}}(L_A)/\sim_M$ has the structure of a Boolean algebra, with the Boolean operations $\wedge$, $\vee$, $\neg$ descending from the operations denoted by the same symbol on $Form(L_A)$. You may not assume that these operations descend – this needs to be proved.

(b) (Bonus/Challenge) Show that ultrafilters of $Form_{\vec{x}}(L_A)/\sim_M$ correspond to (the image under quotienting by $\sim_M$) of a maximal[6] set $p(\vec{x}) \subset Form_{\vec{x}}(L_A)$, with the property that for any finite subset $p_{fin}(\vec{x}) \subset p(\vec{x})$ we have there exists $\vec{m} \in M^n$ (depending on the subset $p_{fin}(\vec{x})$ chosen) such that $M \vDash p_{fin}(\vec{m})$.

[6] I.e. maximal with respect to inclusion, among sets with the given property.