MATH/CMSC 27700 FALL 2020: HW 7

Due: November 27th at 9PM.

Note 1: As noted, the "Take Home Test" Question (in red) must be worked on individually.

Note 2: Answers must be fully explained (unless otherwise specified). Follow the Homework policy (see syllabus) and remember to *write up all solutions on your own.*

Note: 3 There are several references to the notes. Due to the fact that the notes may be updated, these are all to the version of Noon Friday Nov 13. There is a separate file uploaded to canvas with the name "Notes Version Noon 13," for which the numbers appearing in references will be accurate.

Reminders: This is the second to last homework. There will be an update to the selected solutions to include some solutions to HW5 today, and there will be an update to include solutions to HW 6 next week.

(1) (Exercise 4.43 of the Notes [November 13th Noon version]): Use the compactness theorem for first order logic to prove the following:

Proposition 0.1. *Let M be a positive integer. Show that there exists $N = N(M)$, such that if we colour all edges of a graph*[1] *G with N vertices (and at most one edge between every pair of vertices, and no edges from a vertice to itself) from a set of 2 colours, then there exists a subset $V' \subset V$ of the vertices V, such that $|V'| = M$, and all edges between vertices in V' are the same colour.*

It may be helpful to see the discussion in notes surrounding this exercise. You may in particular wish to assume Proposition 4.38 of the notes.

(2) Let $S \subset \mathbb{N}$ be the set of prime numbers. Pick a non-principal ultrafilter $\mathcal{U}$ on S.

(a) Show that

$$\prod_{\mathcal{U}} \overline{\mathbb{F}}_p$$

is a field of characteristic zero[2] (with the field operations inherited from $\prod_{p \in S} \overline{\mathbb{F}}_p$). Here $\overline{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$ (i.e. it is the smallest field containing $\mathbb{F}_p$, and containing the roots of any polynomial). Recall that $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, with multiplication descending from multiplication on $\mathbb{Z}$.

(b) Show that the cardinality of this ultraproduct satisfies

$$\aleph_0 \leq |\prod_{\mathcal{U}} \overline{\mathbb{F}}_p| \leq 2^{\aleph_0},$$

where we recall that $\aleph_0$ was the cardinality of $\mathbb{N}$. You may want to look back to exercise 1. (e) of homework 1.

(c) Show that $\prod_{\mathcal{U}} \overline{\mathbb{F}}_p$ is algebraically closed.

(d) (Hard/Bonus) What is the cardinality of $\prod_{\mathcal{U}} \overline{\mathbb{F}}_p$?

(3) Fill in the following details in the definition of the model structure on $\prod_{\mathcal{U}} M_s$ from class.

[1] See e.g. `https://en.wikipedia.org/wiki/Graph_theory#Graph` if you are not familiar with the definition.

[2] We say that a field k is of characteristic q (q prime) if $1_k + ... + 1_k = 0_k$ (where we are adding q_k copies of 1_k). If a field is not of characteristic q for any prime q then we say that it is of characteristic zero.

(a) Take Home Test Question: Show that for a function f of rank n, there is a function $\phi_{\mathcal{U}}(f) : (\prod_{\mathcal{U}} M_s)^n \to \prod_{\mathcal{U}} M_s$ such that the diagram below commutes:

$$\begin{array}{ccc} (\prod_{s\in S} M_s)^n & \xrightarrow{\prod_{s\in S}\phi_s(f)} & \prod_{s\in S} M_s \\ \downarrow & & \downarrow \\ (\prod_{\mathcal{U}} M_s)^n & \xrightarrow{\phi_{\mathcal{U}}(f)} & \prod_{\mathcal{U}} M_s. \end{array}$$

(b) For a relation R or rank r we defined the relation $\phi_{\mathcal{U}}(R) \in \mathcal{P}((\prod_{\mathcal{U}} M_s)^n)$ as the set

$$\{([(a_{1,s})_{s\in S}], ..., [(a_{r,s})_{s\in S}]) | \{s \in S | \phi_s(R) \ni (a_{1,s}, ..., a_{r,s})\} \in \mathcal{U}\}.$$

Check that $\phi_{\mathcal{U}}(R)$ is well defined for a relation R of the language L. Note that this is necessary because we made a choice of representatives of the equivalence classes of $\sim_{\mathcal{U}}$, a. priori. the above definition could depend on this choice.

(4) Note that an easy modification of our proof of the compactness theorem for first order logic also gives a proof of the compactness theorem for propositional logic. Namely give a propositional theory (L, Γ) a set of models $\{t_s\}_{s\in S}$ and an ultrafilter $\mathcal{U}$ on S we can define a model $\prod_{\mathcal{U}} t_s$ by:

$$\left(\prod_{\mathcal{U}} t_s\right)(\phi) = \begin{cases} 1, & \text{if } \{s | t_s(\phi) = 1\} \in \mathcal{U} \\ 0, & \text{otherwise,} \end{cases} \tag{0.1}$$

for any $\phi \in Form(L)$. We can then proceed to prove the compactness theorem in a way analogous to the first order case.

(a) Show that $\prod_{\mathcal{U}} t_s$ is a model of (L, Γ).

(b) Why does using the above construction to prove compactness (as was done in the first order case) not work if we work with the version of propositional logic where we allow infinite disjunctions? (See remark 3.43 in course notes of November 13) For this question you must not only identify which step doesn't work, but explain why it doesn't work/prove it doesn't work.

(5) Consider the set $\mathcal{M}$ of all models of L, modulo elementary equivalence[3].

For each sentence ψ define the set $U_\psi := \{[M] | M \vDash \psi\}$.

Consider the smallest set $\tau \subset \mathcal{P}(\mathcal{M})$, such that $\tau \ni U_\psi$ for each sentence ψ, τ is closed under finite intersection, and arbitrary unions[4].

(a) Show that for any subset $C \subset \tau$ such that $\cup_{U\in C} U = \mathcal{M}$, there is a finite subset $C_{fin} \subset C$ such that $\cup_{U\in C_{fin}} U = \mathcal{M}$.

(b) (Hard/Bonus) We define the category Top to consist of pairs of objects (S, τ), where S is a set, and $\tau \subset \mathcal{P}(S)$ is a set closed under finite unions and arbitrary intersections, such that $S \in \tau$. We define a morphism of this category $f \in Hom((S_1, \tau_1), (S_1, \tau_2))$ to be a morphisms of sets $f : S_1 \to S_2$ such that for any $U_2 \in \tau_2$ we have $f^{-1}(U_2) \in \tau_1$.

Consider the functor $Prop \to Set^{op}$ which on objects takes a propositonal theory (L, Γ) to the set of models of (L, Γ).

Show that this (i) is a functor[5], (ii) lifts to a functor $Prop \xrightarrow{Mod} Top^{op}$ (where we use the structure τ considered in part (a)), and (iii) factors as a functor $Mod = U \circ Lind$ for some functor $U : Bool \to Top^{op}$.

(6) Definability:

(a) Is every set $S \subset \mathbb{N}$ definable, where $\mathbb{N}$ is considered as a model for the first order theory of Peano arithmetic?

[3]Ie. there is an equivalence relation on models, defined by saying that $M_1 \sim M_2$ if there exist inverse maps $f : M_1 \to M_2$ and $g : M_2 \to M_1$ that are elementary maps of models. We take the equivalence classes of this relation.

[4]Including unions over the empty set.

[5]Recall that morphisms of the category $Prop$ were defined to be interpretations.

Hint: Consider how many sets there are, and how many definable sets there are.

(b) (Hard[6]) Is the set of even numbers in $\mathbb{N}$ definable, where we are considering $\mathbb{N}$ as a model of the theory of ordered sets[7].

Hint: In one approach to this question we replace this model with an elementarily equivalent[8] one, which has automorphisms.

[6]But not a bonus.

[7]The first order theory of ordered sets has a single relation $<$, no constants, and the axioms $\forall x, y(\neg((x < y) \wedge (y < x)))$, $\forall x, y((x < y) \vee (y < x))$, and $\forall x, y, z(((x < y) \wedge (y < z)) \rightarrow (x < z))$.

[8]That is to say one where all sentences have the same truth value.