MATH 27700 / CMSC 27700, Autumn 2020, Assignment 2

Due Friday, Oct. 16th, by 5:00 PM CDT on Canvas

1. Give a definition by transfinite recursion of ordinal exponentiation, based on the ones for ordinal addition and ordinal multiplication. Then prove that for all ordinals α, β, and γ, we have $(\alpha^\beta)^\gamma = \alpha^{(\beta\cdot\gamma)}$.

2. A subset X of a limit ordinal γ is *unbounded* if for every $\alpha < \gamma$, there is a $\beta \in X$ such that $\alpha < \beta$.

- **a.** Show that if α is an ordinal, γ is a limit ordinal, and $X \subseteq \gamma$ is unbounded, then $\alpha + \gamma = \sup_{\beta \in X}(\alpha + \beta)$.
- **b.** Use transfinite induction to show that if α and δ are ordinals, then there is an ordinal β such that $\alpha + \beta = \delta$.
- **c.** Show that there are ordinals $\beta < \delta$ such that there is no ordinal α with $\alpha + \beta = \delta$.

3. Let $\aleph_0 = \omega$, and for an ordinal $\alpha > 0$, let $\aleph_\alpha$ be the least cardinal greater than $\aleph_\beta$ for all $\beta < \alpha$. (When thinking of these cardinals as ordinals, we often write ω_α instead of $\aleph_\alpha$.) Prove that if γ is a limit ordinal, then $\aleph_\gamma = \bigcup_{\alpha<\gamma} \aleph_\alpha$.

4. Let A be an infinite set. Show that A can be partitioned into two subsets of the same size as A. (That is, there exist $B, C \subset A$ such that $B \cap C = \emptyset$ and $B \cup C = A$, and $|B| = |C| = |A|$.) There are several ways to do this problem, but one idea is to first assume that A is an ordinal and think of how to generalize a splitting of ω into two infinite subsets. [In this problem, you should definitely assume the Axiom of Choice. Without it, there can even exist an *amorphous set*, which is an infinite set that is not the disjoint union of two infinite subsets.]