2.4 Proofs of the Necessary Conditions and of Their Invariant Character

A. Proofs of the Necessary Conditions

2.4.i (Lemma). *Let $h(t)$, $a \le t \le b$, $h \in L_1[a,b]$, be a given real valued function such that*

$$\int_a^b h(t)\eta'(t)\,dt = 0 \tag{2.4.1}$$

for all AC real valued functions $\eta(t)$, $a \le t \le b$, with $\eta(a) = 0$, $\eta(b) = 0$ and $\eta'(t)$ essentially bounded. Then $h(t) = C$, a constant, $t \in [a,b]$ (a.e.).

Proof.

(a) First assume that $h(t)$ be essentially bounded. For every real constant C we have $C\int_a^b \eta'(t)\,dt = C[\eta(b) - \eta(a)] = 0$ and thus

$$\int_a^b [h(t) - C]\eta'(t)\,dt = 0 \tag{2.4.2}$$

for all η as in the lemma and any constant C. If we take

$$\eta(t) = \int_a^t [h(\alpha) - C]\,d\alpha, \qquad C = (b-a)^{-1}\int_a^b h(t)\,dt, \tag{2.4.3}$$

then η is AC in $[a,b]$ with $\eta(a) = \eta(b) = 0$, with $\eta'(t) = h(t) - C$ essentially bounded, and by substitution in (2.4.2) we have $\int_a^b [h(t) - C]^2\,dt = 0$. Thus, $h(t) = C$ in $[a,b]$ (a.e.).

(b) If we only know that h is L_1 integrable, the above proof does not apply. The following proof can be traced back to work of DuBois-Reymond [1]. First we know that for almost all $t \in [a,b]$, $h(t)$ is finite and the derivative of its indefinite integral $\int_{t_1}^t h(s)\,ds$. If E_0 denote the complementary set, then meas $E_0 = 0$. Let τ_1, τ_2 be any two points in $[t_1, t_2] - E_0$, $\tau_1 < \tau_2$, and let δ be any number $0 < \delta < 2^{-1}(\tau_2 - \tau_1)$. Let us define $\eta(t)$, $a \le t \le b$, by taking $\eta(t) = 0$ for $t \in [a, \tau_1]$ and for $t \in [\tau_2, b]$; $\eta(t) = \delta$ for $t \in [\tau_1 + \delta, \tau_2 - \delta]$; $\eta(t) = t - \tau_1$ for $t \in [\tau_1, \tau_1 + \delta]$; $\eta(t) = \tau_2 - t$ for $t \in [\tau_2 - \delta, \tau_2]$. Then (2.4.1) yields

$$\int_{\tau_1}^{\tau_1+\delta} h(t)\,dt = \int_{\tau-\delta}^{\tau_2} h(t)\,dt,$$

and by division by δ and passage to the limit as $\delta \to 0+$, we derive $h(\tau_1) = h(\tau_2)$. Since τ_1 and τ_2 are any two points of $[a,b]$, we conclude that $h(t) = C$ for $t \in [a,b]$ (a.e.). □

Note that it is enough to require that (2.4.1) is true for all η continuous with sectionally continuous derivative (and $\eta(t_1) = \eta(t_2) = 0$).

2.4.ii (LEMMA). *Let $h(t)$, $a \le t \le b$, $h \in L_1[a,b]$, be a given real valued L-integrable function such that*

$$\int_a^b h(t)\eta(t)\,dt = 0 \tag{2.4.4}$$

for all real valued functions $\eta(t)$, $a \le t \le b$, of class C^∞ in $[a,b]$ with $\eta(a) = \eta(b) = 0$. Then $h(t) = 0$ in $[a,b]$ (a.e.).

Proof.

(a) First, let us assume that h is continuous in $[a,b]$ and that (2.4.4) holds for all functions $\eta(t)$, $a \le t \le b$, continuous in $[a,b]$ with $\eta(a) = \eta(b) = 0$ [alternatively, for all η of some class C^m, $m \ge 1$]. Then, $h(t)$ must be identically zero in $[a,b]$. Indeed, in the contrary case, there would be some $t_0 \in (a,b)$ with $h(t_0) \ne 0$, say $h(t_0) > 0$, and thus an interval $[t_0 - c, t_0 + c] \subset [a,b]$ where $h(t) > 0$. Then, we take $\eta(t) = 0$ for $|t - t_0| \ge c$, and $\eta(t) = c - |t - t_0|$ for $|t - t_0| \le c$ [alternatively, we take, for $|t - t_0| \le c$, either $\eta(t) = (c^2 - (t - t_0)^2)^m$, or $\eta(t) = \exp[-(c^2 - (t - t_0)^2)^{-1}]$. In any case, we have

$$0 = \int_a^b h(t)\eta(t)\,dt = \int_{t_0 - c}^{t_0 + c} h(t)\eta(t)\,dt > 0,$$

a contradiction. The last inequality follows from the fact that both $h(t)$ and $\eta(t)$ are positive in $(t_0 - c, t_0 + c)$.

(b) Now let h be only of class $L_1[a,b]$. Then, for almost all $t_0 \in (a,b)$, $h(t_0)$ is the derivative of $H_0(t) = \int_a^t h(\tau)\,d\tau$ at t_0. Let t_0 be any such point, and assume, if possible, that $h(t_0) = m \ne 0$, say $m > 0$. Then, we can take $c > 0$ sufficiently small so that $[t_0 - c, t_0 + c] \subset [a,b]$ and $H(t) = \int_{t_0}^t h(\tau)\,d\tau$ lies between $2^{-1}m(t - t_0)$ and $2^{-1}(3m)(t - t_0)$ for $|t - t_0| \le c$. We take now η of class C^∞ with $\eta(t) = 0$ for $|t - t_0| \ge c$, $\eta(t_0) = 1$, and $-\eta'$ of the same sign of $t - t_0$ for $0 < |t - t_0| < c$. Then, by integration by parts, we have

$$0 = \int_a^b h(t)\eta(t)\,dt = -\int_{t_0 - c}^{t_0 + c} H(t)\eta'(t)\,dt > 0,$$

a contradiction. Thus, $h(t_0) = 0$, and we have proved that h is zero a.e. in $[a,b]$. □

Proof of Euler's equations for a weak local minimum and x' essentially bounded. We prove here Euler's equations (E_i) not only for a strong local minimum as stated in (2.2.i), but also for any weak local minimum. Thus, we assume here that the vector function $x(t) = (x^1, \dots, x^n)$, $t_1 \le t \le t_2$, is AC with x' essentially bounded, and that x is a weak local minimum for the functional. We do not repeat the assumptions of (2.2.i). Let $\eta(t) = (\eta^1, \dots, \eta^n)$, $t_1 \le t \le t_2$, be any given AC vector function with $\eta(t_1) = \eta(t_2) = 0$ and η' essentially bounded. Then there is some $a_0 > 0$ such that $y(t) = x(t) + a\eta(t)$, $t_1 \le t \le t_2$, has graph in A for all $|a| \le a_0$, and there is an a_1, $0 < a_1 \le a_0$, such that $I[y] \ge I[x]$ for all $|a| \le a_1$. Then $\psi(a) \ge \psi(0)$ for $|a| \le a_1$, where

ψ is the function defined in (2.3.1) and finally $\psi'(0) = 0$, or $J_1[\eta] = 0$ as stated in (2.3.ii).

If, for fixed i, we take $\eta(t) = (\eta^1, \ldots, \eta^n)$ with $\eta^j = 0$ for all $j \neq i$, then the relation $J_1[\eta] = 0$ reduces to

$$\int_{t_1}^{t_2} [f_{0x^i}(t, x(t), x'(t))\eta^i(t) + f_{0x'^i}(t, x(t), x'(t))\eta'^i(t)]\, dt = 0$$

for all AC real valued functions $\eta^i(t)$ with $\eta^i(t_1) = \eta^i(t_2) = 0$ and η'^i essentially bounded. By integration by parts (McShane [I, 36.1, p. 209]) we derive

$$\int_{t_1}^{t_2} \left[f_{0x'^i}(t, x(t), x'(t)) - \int_{t_1}^{t} f_{0x^i}(\tau, x(\tau), x'(\tau))\, d\tau \right] \eta'^i(t)\, dt = 0,$$

and by (2.4.i), therefore,

$$f_{0x'^i}(t, x(t), x'(t)) - \int_{t_1}^{t} f_{0x^i}(\tau, x(\tau), x'(\tau))\, d\tau = C_i, \qquad t \in [t_1, t_2] \text{ (a.e.)}, \tag{2.4.5}$$

C_i a constant; and this relation holds for $t \in [t_1, t_2]$, and thus $f_{0x'^i}(t, x(t), x'(t))$ must coincide a.e. in $[t_1, t_2]$ with an AC function, say $-\lambda_i(t)$. Moreover, by identifying $f_{0x'^i}(t, x(t), x'(t))$ with $-\lambda_i(t)$, and differentiation, (2.4.5) yields $(d/dt) f_{0x'^i}(t, x(t), x'(t)) = f_{0x^i}(t, x(t), x'(t))$ a.e. in $[t_1, t_2]$. We have proved Euler's equations (E_i), $i = 1, \ldots, n$. □

Remark 1. Note that the hypothesis that x' be essentially bounded can be removed under the assumptions used in Section 2.3, Remark 3, namely that $x' \in L_p[t_1, t_2]$ for some $p \geq 1$, and that in a neighborhood $\Gamma_\delta \subset A$ of the graph Γ of x we have $|f_0|$, $|f_{0x}|$, $|f_{0x'}| \leq M|x'|^p + m$ for some constants $M, m \geq 0$ and all $(t, x, x') \in \Gamma_\delta \times R^n$. Indeed, as we have seen in Section 2.3, Remark 2, the functions $f_{0x}(t, x(t), x'(t))$, $f_{0x'}(t, x(t), x'(t))$ are L_1-integrable in $[t_1, t_2]$, and we still have $J_1[\eta; x] = 0$ for all AC n-vector functions $\eta(t)$, $t_1 \leq t \leq t_2$, with $\eta(t_1) = \eta(t_2) = 0$ and η' essentially bounded. The proof of the Euler equation above is still valid. Indeed, we still can integrate by parts, and we can still use the lemma (2.4.i) because the first member of (2.4.5) is certainly L_1-integrable in $[t_1, t_2]$.

Two examples are needed here. More examples will be discussed in Chapter 3.

Example 1. $I[x] = \int_{t_1}^{t_2} [a(t)x'^2 + 2b(t)xx' + c(t)x^2]\, dt$, x scalar, with $A = [t_1, t_2] \times R$, $n = 1$, and a, b, c constants or given bounded measurable functions in $[t_1, t_2]$. If $x(t)$, $t_1 \leq t \leq t_2$, is AC with essentially bounded derivative, and if $I[y] \geq I[x]$, say for all $y(t)$, $t_1 \leq t \leq t_2$, AC with essentially bounded derivative and $y(t_1) = x(t_1)$, $y(t_2) = x(t_2)$, then we know from the above that $2^{-1} f_{0x'} = ax' + bx$ is AC (or coincides with such a function a.e. in $[t_1, t_2]$), and the Euler equation $(d/dt) f_{0x'} = f_{0x}$ yields $(ax' + bx)' = bx' + cx$ a.e. in $[t_1, t_2]$. Thus, for the integral $I[x] = \int_{t_1}^{t_2} x'^2\, dt$, the Euler equation is $x'' = 0$. Let us take constants $\delta > 0$, m, $m_0 \geq 0$, such that $\Gamma_\delta \subset A$, $m_0 = \delta + \max x(t)$, and such that a, b, c are in absolute value less than m. If $a(t) \geq \mu > 0$ for some constant $\mu > 0$, then $f_0(t, x, x') = ax'^2 + 2bxx' + cx^2 \geq x'^2 - 2mm_0 x' - mm_0^2$, and for some constant C we also have $f_0 \geq (\mu/2)x'^2 - C$ for $(t, x) \in \Gamma_\delta$ and all $x' \in R$. Thus, the mere existence and finiteness of $I[x]$ implies that $x'(t)$ is L_2-integrable. For some constants

2.6 Smoothness Properties of Optimal Solutions

A. Existence and Continuity of the First Derivative

Let $f_0(t, x, x')$ be of class C^1 on $A \times R^n$, A closed, and let $x(t)$, $a \le t \le b$, be an AC n-vector function with graph in A. We shall need the simple hypothesis:

2.6.1. For each t, $a \le t \le b$, the n-vector function of u

$$[f_{0x'^i}(t, x(t), u),\ i = 1, \ldots, n]$$

never takes twice the same value as u describes R^n. In other words, $a \le t \le b$, $u, v \in R^n$, $u \ne v$ implies

$$[f_{0x'^i}(t, x(t), u),\ i = 1, \ldots, n] \ne [f_{0x'^i}(t, x(t), v),\ i = 1, \ldots, n].$$

This hypothesis (2.6.1) is certainly satisfied if

$$E(t, x(t), u, v) > 0 \tag{2.6.2}$$

for all $a \le t \le b$ and all $u, v \in R^n$, $u \ne v$. Indeed, assume, if possible, that for given $u, v \in R^n$, $u \ne v$, we have

$$l_v = f_{0x'^i}(t, x(i), u) = f_{0x'^i}(t, x(t), v), \qquad i = 1, \ldots, n.$$

Then

$$f_0(t, x(t), v) - f_0(t, x(t), u) - \sum_i l_i(v^i - u^i) = E(t, x(t), u, v) > 0,$$
$$f_0(t, x(t), u) - f_0(t, x(t), v) - \sum_i l_i(u^i - v^i) = E(t, x(t), v, u) > 0,$$

and by addition we obtain $0 > 0$, a contradiction. We have proved that (2.6.2) implies (2.6.1).

In turn (2.6.2) is certainly satisfied if f_0 has continuous second order partial derivatives $f_{0x'^ix'^j}$ and

$$Q(t, x(t), u, \xi) = \sum_{i,j=1}^{n} f_{0x'^ix'^j}(t, x(t), u)\xi_i\xi_j > 0 \tag{2.6.3}$$

for all t, u, ξ with $a \le t \le b$, $u, \xi \in R^n$, $\xi \ne 0$. To prove that (2.6.3) implies (2.6.2) we note that, by definition (2.1.2) and Taylor's formula, we have

$$E(t, x(t), u, v) = \int_0^1 \sum_i (v^i - u^i)[f_{0x'^i}(t, x(t), u + \alpha(v - u)) - f_{0x'^i}(t, x(t), u)]\, d\alpha$$
$$= \int_0^1 \int_0^1 \alpha \sum_{i,j} (v^i - u^i)(v^j - u^j) f_{0x'^ix'^j}(t, x(t), u + \alpha\beta(v - u))\, d\alpha\, d\beta.$$

For the statement and proof of the theorems below it is convenient to denote by $f_{0x'}$, f_{0x} the following n-vector functions on $A \times R^n$:

$$f_{0x'}(t,x,u) = [f_{0x'^i}(t,x,u),\ i = 1,\dots,n],$$
$$f_{0x}(t,x,u) = [f_{0x^i}(t,x,u),\ i = 1,\dots,n].$$

We may also need the further hypothesis:

2.6.4. For $u \in R^n$, $|u| \to +\infty$, we have $|f_{0x'}(t,x(t),u)| \to +\infty$ uniformly in $[a,b]$. In other words, we assume that, given $N > 0$, there is another constant $R \geq 0$ such that $t \in [a,b]$, $u \in R^n$, $|u| \geq R$ implies $|f_{0x'}(t,x(t),u)| \geq N$.

Note that (2.6.4) is not a consequence of (2.6.1). For instance, for $n = 1$, $f_0 = (1 + x'^2)^{1/2}$, we have $f_{0x'} = x'(1 + x'^2)^{-1/2}$. This is a strictly increasing bounded function of x' in $(-\infty, +\infty)$, and thus f_0 satisfies (2.6.1) but not (2.6.4).

On the other hand, $f_0 = -x'^2 + x'^4$ satisfies (2.6.4), but does not satisfy (2.6.1).

We shall now assume that the AC trajectory $x(t)$, $a \leq t \leq b$, satisfies the Euler equations (E_i), $i = 1,\dots,n$, of (2.2.i). Namely, we need to express this requirement as precisely as in Section 2.2:

2.6.5. There is an AC n-vector function $\phi(t) = (\phi_1,\dots,\phi_n)$, $a \leq t \leq b$, such that almost everywhere in $[a,b]$ we have

$$f_{0x'}(t,x(t),x'(t)) = \phi(t), \qquad (d/dt)\phi(t) = f_{0x}(t,x(t),x'(t)).$$

We begin with the following statement:

2.6.i (Boundedness of the First Derivatives). *If f_0 is of class C^1 in $A \times R^n$ and satisfies (2.6.4), and if $x(t)$, $a \leq t \leq b$, is any AC n-vector function satisfying (2.6.5), then x' is essentially bounded and x is Lipschitzian in $[a,b]$.*

Proof. The n-vector function ϕ in (2.6.5) is AC, and hence continuous and bounded in $[a,b]$, say $|\phi(t)| \leq N$. By (2.6.4) there is $R \geq 0$ such that $t \in [a,b]$, $u \in R^n$, $|u| \geq R$ implies $|f_{0x'}(t,x(t),u)| \geq N + 1$. Since $|f_{0x'}(t,x(t),x'(t))| = |\phi(t)| \leq N$ a.e. in $[a,b]$, we conclude that $|x'(t)| \leq R$ also a.e. in $[a,b]$. Since x is AC, we derive that x is Lipschitzian of constant R in $[a,b]$. □

We are now in a position to state and prove the first main theorem concerning the smoothness of trajectories:

2.6.ii (Theorem (Tonelli): Continuity of the First Derivative). *If $x(t)$, $a \leq t \leq b$, is AC with graph in A and essentially bounded derivative x' in $[a,b]$, if f_0 is of class C^1 in $A \times R^n$, and (2.6.1), (2.6.5) hold, then x' exists everywhere in $[a,b]$ and is continuous in $[a,b]$, that is, x is of class C^1.*

If it is not known that x' is essentially bounded, then the conclusion of (2.6.ii) is still valid under the additional hypothesis (2.6.4) concerning f_0.

Proof. If $x(t)$ is known to be continuous in $[a, b]$ with sectionally continuous derivative $x'(t)$, we have only to prove that x has no corner point. This is simply a consequence of the Erdmann corner condition (f) of (2.2.i). As mentioned in Remark 1 in Section 2.2 under conditions (2.6.1) there cannot be corner points.

If x is AC in $[a, b]$ with x' essentially bounded, then $f_0(t, x(t), x'(t))$ is also essentially bounded, measurable, and L-integrable in $[a, b]$. In this situation, to prove (2.6.ii) we have to prove that (α) $x'(t)$ exists everywhere in $[a, b]$, and (β) $x'(t)$ is continuous in $[a, b]$.

Since x is AC in $[a, b]$, the derivative x' exists almost everywhere in $[a, b]$. First we assume that x' is essentially bounded, say $|x'(t)| \leq m$ for almost all $t \in [a, b]$.

Let S denote the set of all $t \in [a, b]$ where $x'(t)$, $\phi'(t)$ are defined, where $|x'(t)| \leq m$, where each of the n functions $f_{0x'^i}(t, x(t), x'(t))$ coincides with $\phi_i(t)$, $i = 1, \ldots, n$, and where the relations $(d/dt)\phi_i(t) = f_{0x^i}(t, x(t), x'(t))$ hold. Then $S \subset [a, b]$, meas $S = b - a$, and hence S is everywhere dense in $[a, b]$. If t_0 is any point of $[a, b]$, then t_0 is a point of accumulation of points $t \in S$ with $t \neq t_0$. Let us prove first that $x'(t)$ has a limit as $t \to t_0$ with $t \in S$. Suppose this is not true. Then there are sequences $[t_k]$, $[t_k']$ of points of S with $t_k \to t_0$, $t_k' \to t_0$, and such that $[x'(t_k)]$, $[x'(t_k')]$ have distinct limits. Since x' is bounded in S, then we can assume that both limits are finite, say $x'(t_k) \to u$, $x'(t_k') \to v$, $u \neq v$, u, v finite, $u, v \in R^n$. Since the relations $f_{0x'^i}(t, x(t), x'(t)) = \phi_i(t)$, $i = 1, \ldots, n$, hold at every point $t \in S$, in particular at $t = t_k$ and $t = t_k'$, then as $k \to \infty$ we obtain two relations which by comparison yield

$$f_{0x'^i}(t_0, x_0, u) = f_{0x'^i}(t_0, x_0, v), \qquad i = 1, \ldots, n,$$

where $x_0 = x(t_0)$. The hypothesis (2.6.1) implies $u = v$, a contradiction. This proves that $u(t_0) = \lim x'(t)$ exists and is finite as $t \to t_0$ along points of S, and this holds at every $t_0 \in [a, b]$. The same argument shows that u is a continuous function on $[a, b]$.

Let us prove that $u(t) = x'(t)$ a.e. in $[t_1, t_2]$. Indeed, x' is measurable, and hence continuous on certain closed subsets K_s of $[a, b]$ with meas $K_s > b - a - s^{-1}$, and we know that almost every point of K_s is a point of density one for K_s. Hence, for every fixed s and $t_0 \in K_s$, there is a sequence $[t_k]$ of points $t_k \in S \cap K_s$ with $t_k \to t_0$, $x'(t_0) = \lim x'(t_k) = u(t_0)$ as $k \to \infty$. Thus, $x'(t) = u(t)$ a.e. on each K_s, and also $x'(t) = u(t)$ a.e. in $[a, b]$. Finally, for every $t \in [a, b]$ we have $x(t) - x(a) = \int_a^t x'(\tau)\, d\tau = \int_a^t u(\tau)\, d\tau$; hence x is continuously differentiable in $[a, b]$, $x' = u$ everywhere, and x is of class C^1 as stated. □

If x' is not known to be essentially bounded, but (2.6.4) holds, then from (2.6.i) we derive that x' is essentially bounded, and the argument above applies.

Remark 1. (A counterexample for theorem (2.6.i)). As we shall see in Section 3.10, the absolute minimum of the functional $I[x] = \int_{t_1}^{t_2} xx'^2\,dt$ with $x \geq 0$, $x(t_1) = 0$, $x(t_2) = x_2 > 0$, $t_1 < t_2$, is of the form $x(t) = k(t - t_1)^{2/3}$, $t_1 \leq t \leq t_2$, and x' is unbounded. Here $f_0 = xx'^2$ does not satisfy (2.6.1) along $x(t)$.

Remark 2. (A counterexample for theorem (2.6.ii)). The absolute minimum of the functional $I[x] = \int_{-1}^{1} x^2(1 - x')^2\,dt$ with $x(-1) = 0$, $x(1) = 1$, is certainly given by the trajectory x defined by $x(t) = 0$ for $-1 \leq t \leq 0$, $x(t) = t$ for $0 \leq t \leq 1$, and x is AC and x' is discontinuous at $t = 0$. Here $f_0(t, x, u) = x^2(1 - u)^2$ does not satisfy (2.6.1) along $x(t)$.

B. Existence and Continuity of the Second and Higher Derivatives

We shall prove here that under mild hypotheses, any arc of class C^1 satisfying the Euler equation is actually of class C^2 or higher. Precisely, we shall prove the statement

2.6.iii (THEOREM (WEIERSTRASS): CONTINUITY OF THE SECOND DERIVATIVE). *If $x(t)$, $a \leq t \leq b$, is of class C^1 with graph in A, if f_0 is of class C^2 $[C^m, m \geq 2]$, if (2.6.5) holds and if* $\det f_{0x'x'}$ *is never zero along x, that is,*

(2.6.6) $\det(f_{0x'^ix'^j}(t, x(t), x'(t)),\ i, j = 1, \ldots, n) \neq 0$ *for all* $a \leq t \leq b$,

then x is of class C^2 $[C^m]$ in $[a, b]$.

Condition (2.6.6) is certainly satisfied if

$$Q = \sum_{ij} f_{0x'^ix'^j}(t, x(t), x'(t))\xi_i\xi_j > 0 \tag{2.6.7}$$

for all $\xi = (\xi_1, \ldots, \xi_n) \neq 0$, $\xi \in R^n$, and all $t \in [a, b]$.

Proof of (2.6.iii). Let t be any point of $[a, b]$. If t is replaced by some $t + \Delta t$ also in $[a, b]$, $\Delta t \neq 0$, then the vectors $x = x(t)$, $x' = x'(t)$ are replaced by certain vectors $x + \Delta x = x(t + \Delta t)$, $x' + \Delta x' = x'(t + \Delta t)$, where $\Delta x, \Delta x' \to 0$ as $\Delta t \to 0$, since $x(t)$, $x'(t)$ are continuous at t by hypothesis. Also, $\Delta x/\Delta t \to x' = x'(t)$ as $\Delta t \to 0$. Finally, $f_{0x'}(t, x(t), x'(t))$, which we shall denote simply by $f_{0x'}(t, x, x')$, is replaced by $f_{0x'}(t + \Delta t, x + \Delta x, x' + \Delta x')$. By Taylor's formula we have

$$\begin{aligned} \Delta f_{0x'^i} &= f_{0x'^i}(t + \Delta t, x + \Delta x, x' + \Delta x') - f_{0x'^i}(t, x, x') \\ &= f_{0x'^it}\,\Delta t + \sum_j f_{0x'^ix^j}\,\Delta x^j + \sum_j f_{0x'^ix'^j}\,\Delta x'^j, \qquad i = 1, \ldots, n, \end{aligned} \tag{2.6.8}$$

where the arguments of all $f_{0x'^it}, f_{0x'^ix^j}, f_{0x'^ix'^j}$ are $t + \theta\Delta t$, $x + \theta\Delta x$, $x' + \theta\Delta x'$ for some θ, $0 < \theta < 1$, which depends on i, t, and Δt. Dividing the equation (2.6.8) by Δt we obtain

$$\frac{\Delta f_{0x'^i}}{\Delta t} = f_{0x'^it} + \sum_j f_{0x'^ix^j}\frac{\Delta x^j}{\Delta t} + \sum_j f_{0x'^ix'^j}\frac{\Delta x'^j}{\Delta t}, \qquad i = 1, \ldots, n, \tag{2.6.9}$$

and we interpret these equations as a linear algebraic system in the n unknowns $\Delta x'^j/\Delta t, j = 1, \ldots, n$. The determinant D of such a system has limit $D_0 \neq 0$ as $\Delta t \to 0$ with $D_0 = \det f_{0x'^i x'^j}(t, x(t), x'(t))$. Thus, for Δt sufficiently small, $D \neq 0$, and we can solve (2.6.9) with respect to the n quotients $\Delta x'^j/\Delta t$, $j = 1, \ldots, n$. As we know from Cramer's rule, each $\Delta x'^j/\Delta t$ is then the quotient of two determinants, the one in the denominator being D. We do not need their explicit expressions. We need only know that the n quotients above are of the form

$$\frac{\Delta x'^i}{\Delta t} = D^{-1} R_i\left(\frac{\Delta f_{0x'^j}}{\Delta t}, f_{0x'^j t}, f_{0x'^j x^s}, f_{0x'^j x'^s}, \frac{\Delta x^j}{\Delta t}\right), \tag{2.6.10}$$

where R_i is a polynomial with constant coefficients in the arguments listed in parentheses. In (2.6.10) we have $D \to D_0 \neq 0$ as $\Delta t \to 0$, and also $\Delta f_{0x'^j}/\Delta t \to (d/dt) f_{0x'^j}$, and these last derivatives exist and equal f_{0x^j} by virtue of (2.6.5). Also, $\Delta x^j/\Delta t \to x'^j(t)$ as $\Delta t \to 0$, and $f_{0x'^j t}, f_{0x'^j x^s}, f_{0x'^j x'^s}$ converge as $\Delta t \to 0$ to the same expressions with arguments $t, x(t), x'(t)$. This proves that $\Delta x'^j/\Delta t$ has a finite limit as $\Delta t \to 0$, that is, $x''^i(t)$ exists and is finite, and

$$x''^i(t) = D_0^{-1} R_i(f_{0x^j}, f_{0x'^j t}, f_{0x'^j x^s}, f_{0x'^j x^s}, x), \qquad i = 1, \ldots, n.$$

Since $D_0 \neq 0$ in $[a, b]$, we conclude that x''^i too is a continuous function of t in $[a, b]$. Note that we can perform now the same limit as $\Delta t \to 0$ in (2.6.9), and we obtain

$$(d/dt) f_{0x'^i} = f_{0x'^i t} + \sum_j f_{0x'^i x^j} x'^j + \sum_j f_{0x'^i x'^j} x''^j, \qquad i = 1, \ldots, n.$$

Using (2.6.5) we obtain the Euler equations in their explicit form (2.2.15). $\square$

Remark 3 (A COUNTEREXAMPLE FOR THEOREM (2.6.iii)). The absolute minimum of the functional $I[x] = \int_{-1}^{1} x^2(2t - x')^2\, dt$ with $x(-1) = 0$, $x(1) = 1$, is certainly given by the trajectory x defined by $x(t) = 0$ for $-1 \leq t \leq 0$, $x(t) = t^2$ for $0 \leq t \leq 1$, and x, x' are AC, but x'' is discontinuous at $t = 0$. Here $f_0(t, x, u) = x^2(2t - u)^2$ does not satisfy (2.6.6) along $x(t)$.

2.7 Proof of the Euler and DuBois-Reymond Conditions in the Unbounded Case

A. The Condition (S)

We shall use the same notation and general hypotheses as in Section 2.2, but now we allow the optimal AC solution $x(t)$, $t_1 \leq t \leq t_2$, to have unbounded derivative $x'(t)$. We shall need further requirements on the function $f_0(t, x, u)$. Namely, we shall assume