Poroelasticity

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1 Introduction

Soil, rock, and sponges are examples of porous materials where a solid skeleton contains pores filled with a fluid like air or water. The solid skeleton is often elastic and may deform because of the movement of the fluid. We shall refer to such fluid-solid mixtures as *porous media*. The physics of porous media is essential in numerous application areas. Three primary examples are listed below.

- *Hydrocarbon recovery*: rock is a porous medium where oil, water, and gas flow during oil and gas production.
- *Buildings on soil*: soil is a porous medium containing air and water, and the weight of a heavy building will cause the soil to deform (consolidate).
- *Biomedical processes*: the spinal chord and the brain are examples on porous materials that deform due to movement of a fluid (cerebrospinal fluid, often known as CSF). through

In the following, we shall present the partial differential equations (PDEs) modeling flow and deformation in porous media. These equations are commonly known as the *Biot equations*.

2 Mathematical Model

The Biot equations we work with here are based on a series of physical assumptions. The porosity of the material must be small (thus leaving out sponges as porous media described by the Biot model). We also assume that there is only one fluid filling *all* the pores inside the medium. The deformation of the solid skeleton is small and elastic such that standard linear elasticity theory applies. We shall also assume that the fluid is incompressible or at most slightly compressible. Temperature effects are neglected, but can relatively easily be incorporated. Regarding time scales, we leave out wave motion and consider slow or stationary processes.

The Biot equations reduce to one vector PDE governing the deformation of the solid skeleton and one scalar PDE governing the pressure (and hence the flow) of the fluid.

We introduce the following mathematical quantities, which will frequently appear in the equations:

TABLE!

- $\boldsymbol{u}$: displacement of the solid skeleton,
- $\boldsymbol{q}$: relative velocity of the fluid and the solid skeleton,
- p: pressure in the fluid,
- ϱ_s: solid density,
- ϱ_f: fluid density,
- μ: fluid viscosity (dynamic),
- K: isotropic (scalar) permeability,
- $\boldsymbol{K}$: anisotropic permeability tensor,
- $\boldsymbol{\sigma}$: total stress tensor in the solid skeleton,
- $\boldsymbol{\sigma}'$: effective stress tensor due to deformation,
- $\boldsymbol{\varepsilon}$: strain tensor corresponding to $\boldsymbol{\sigma}$ through Hooke's law,
- Ω: spatial domain occupied by the porous medium,
- $[0, T]$: time interval under consideration.

2.1 Solid phase

The equations in the solid are based on Newton's second law, which for a continuum with stress tensor $\boldsymbol{\sigma}$, displacement (or deformation) field $\boldsymbol{u}$, and external body forces $\boldsymbol{f}$ reads

$$\varrho_s \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma} + \varrho_s \boldsymbol{f}\,. \tag{1}$$

The acceleration term on the left-hand side is important for wave motion, but for the slow flow and deformation processes considere here the term can be neglected. The most common form of $\boldsymbol{f}$ is the gravity force, here denoted by $\boldsymbol{g}$.

The concept of *effective stress* is important in porous media. The effective stress $\boldsymbol{\sigma}'$ represents the stress carried by the solid skeleton and is related to the deformations of the solid. The total stress $\boldsymbol{\sigma}$ is the sum of the effective stress and the stress induced by the fluid in the skeleton. The latter is expressed through the isotropic stress tensor $-\alpha p \boldsymbol{I}$. We then have

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - \alpha p \boldsymbol{I}, \tag{2}$$

where $\boldsymbol{I}$ is the identity tensor and α is the Biot-Willis parameter, which equals unity for incompressible fluids (see below). Hooke's generalized law for three-dimensional elasticity in an isotriopic elastic medium is

$$\boldsymbol{\sigma}' = \lambda \mathrm{tr}\boldsymbol{\varepsilon}\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}, \tag{3}$$

where $\boldsymbol{\varepsilon}$ is the strain tensor related to the displacements $\boldsymbol{u}$ by

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right). \tag{4}$$

The parameters λ and μ reflect elastic properties of the medium and are called the *Lame* parameters. They can be related to other widely used elastic parameters such as the Young's modulus, Poisson's ratio, the bulk modulus, and the shear modulus. For example,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}, \tag{5}$$

where E is Young's modulus and ν is Poisson's ratio.

Inserting (4) in (3) and combining with (2) gives

$$\boldsymbol{\sigma} = \lambda\nabla\cdot\boldsymbol{u}\,\boldsymbol{I} + \mu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right) - \alpha p\,\boldsymbol{I}\,. \tag{6}$$

This equation together with (1), neglecting accelerations and restricting body forces to gravity only, are then the two governing equations for elastic deformations in the skeleton. We can easily substitute (6) in (1) to get one vector PDE for $\boldsymbol{u}$, but it is coupled to the unknown fluid pressure p. Various specific models are discussed later.

2.2 Fluid phase

The PDEs governing the fluid motion consists of a mass balance equation and an average momentum equation famously known as Darcy's law. The mass balance equation is

$$S\frac{\partial p}{\partial t} + \nabla\cdot\left(\boldsymbol{q} - \alpha\frac{\partial\boldsymbol{u}}{\partial t}\right) = 0, \tag{7}$$

where the first term models accumulation of fluid in a small volume, while the other models the net outflow of the fluid from this volume. In the latter term, it is the relative velocity between the fluid ($\boldsymbol{q}$) and the solid skeleton ($\partial\boldsymbol{u}/\partial t$) that is essential. Darcy's law represents an average of the viscous fluid flow in a large number of pores. As the pores are very small, the local Reynolds number is small, and one can omit the inertia terms in the Navier-Stokes equations. The resulting Darcy's laws relates the fluid velocity $\boldsymbol{q}$ to the pressure and gravity forces:

$$\boldsymbol{q} = -\frac{K}{\mu}(\nabla p - \varrho_f\boldsymbol{g})\,. \tag{8}$$

The permeability K models how easily a presure gradient can move the fluid through the pores. Very often, especially in geological media, the premeability is larger in the horizontal directions than in the vertical direction, because the medium has through the geological history been severely compacted in the vertical direction due to gravity. When the permeability is not equal in all space directions, one introduces a permeability tensor and writes (8) as

$$\boldsymbol{q} = \frac{\boldsymbol{K}}{\mu} \cdot (\nabla p - \varrho_f \boldsymbol{g}) \,. \tag{9}$$

For example, a perfectly horizontal geological layer will have $\boldsymbol{K}$ diagonal with $K_{xx} = K_{yy}$ equal to the horizontal permeability and K_{zz} as the vertical permeability (usually much smaller than $K_{xx} = K_{yy}$).

We realize that (8) or (9) can be substituted in (7) to yield a diffusion-like PDE for p, but this PDE is coupled to the displacement $\boldsymbol{u}$ through the term $\alpha(\nabla \cdot \boldsymbol{u})_t$. Specific forms of the diffusion PDE for p are given later.

The parameter S is known as the *fluid storage coefficient* and measures how much more fluid volume that can be stored in the porous medium when the pressure increases. The S parameter can be related to the porosity and the compressibility of the fluid and solid through

$$S = \frac{\phi}{k_f} + \frac{\alpha - \phi}{k_s}, \tag{10}$$

where ϕ is the porosity (the fraction of fluid per unit volume of the medium),

$$k_f = \varrho_f \frac{\partial p}{\partial \varrho_f},$$

is the bulk modulus of the fluid, and $k_s = \lambda + \frac{2}{3}\mu$ is the bulk modulus of the solid. Frequently, the fluid is considered incompressible and then $k_f \to \infty$. The Biot-Willis coefficient α is closely related to the bulk moduli,

$$\alpha \approx 1 - \frac{k_f}{k_s} \,.$$

For an incompressible fluid, $S \approx (1-\phi)k_s^{-1}$. If also the solid skeleton is incompressible, $S = 0$.

2.3 Boundary conditions

For the solid, two types of boundary conditions are relevant: either prescribed stress vector $\boldsymbol{\sigma} \cdot \boldsymbol{n}$ or prescribed displacement vector $\boldsymbol{u}$. At each point of the boundary, three scalar conditions must be applied (two for 2D problems). A combination of prescribed stress and deformation state at a point is possible, for example, zero normal displacement and zero shear stress. We let $\partial\Omega_{D,u}$ denote the part of the boundary $\partial\Omega$ where $\boldsymbol{u}$ is known (Dirichlet conditions), and $\partial\Omega_{N,\sigma}$ is the part where the stress is known.

For the fluid, a scalar condition is needed at each point of the boundary. Either p has to be prescribed or the normal flux $\boldsymbol{q} \cdot \boldsymbol{n}$ must be known. We let

$\partial\Omega_{D,p}$ be the part of the boundary where Dirichlet conditions (p known) applies, and $\partial\Omega_{N,q}$ the part where the normal flux conditions are set.

2.4 Summary of key equations

The four key equations governing the mixture of a porous solid filled with a fluid are then the momentum balance for the solid,

$$\nabla \cdot \boldsymbol{\sigma} = -\varrho_s \boldsymbol{g}, \tag{11}$$

the constitutive law for the solid (6), the mass balance equation (7), and Darcy's law. When speak about the *coupled problem* when these four equations must be solved simultaneously.

Sometimes the velocity of the solid is small enough to justify dropping the term $\alpha\nabla \cdot \boldsymbol{u}_t$ in (7). The result is that (7) becomes independent of $\boldsymbol{u}$, and we get an *uncoupled problem* where we first can solve for p and then for $\boldsymbol{u}$. The equation for p arises from eliminating $\boldsymbol{q}$ in (7) with the aid of Darcy's law:

$$S\frac{\partial p}{\partial t} = \nabla \cdot \left(\frac{K}{\mu}(\nabla p - \varrho_f \boldsymbol{g})\right). \tag{12}$$

The governing equation for $\boldsymbol{u}$ arises by inserting (6) in (11):

$$\nabla \cdot \left(\lambda\nabla \cdot \boldsymbol{u}\,\boldsymbol{I} + \mu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)\right) = \alpha\nabla p - \varrho_s \boldsymbol{g}\,. \tag{13}$$

2.5 Variational formulation of the decoupled problem

We now consider discretizing the decoupled problem where the p equation (12) can be solved independently of the $\boldsymbol{u}$ equation (13).

The discretization of (12) can be done either prior to the variational formulation or after. Here we perform the variational formulation first. Let $v^{(p)} \in V^{(p)}$ be a test function in the function space $V^{(p)}$. This is the space where we seek p, or more precisely, $p - p_0$, where p_0 is a function to model nonzero Dirichlet conditions on $\partial\Omega_{D,p}$. Formally, $v^{(p)} = 0$ on $\partial\Omega_{D,p}$, but in practice this condition and p_0 are dropped and other methods used to enforce Dirichlet conditions in finite element impelementations.

The derivation of a variational formulation follows the usual steps of multiplying the governing equation by the test function and integrate second-order derivatives, or more precisely, all the divergence terms, by parts. The result becomes

$$\int_\Omega \left(S\frac{\partial p}{\partial t}v^{(p)} + \frac{K}{\mu}\nabla p \cdot \nabla v^{(p)} - \varrho_f \boldsymbol{g} \cdot \nabla v^{(p)}\right) \mathrm{d}x = 0\,. \tag{14}$$

Introducing a Backward Euler scheme in time, and letting p now denote the fluid pressure at the new time level and p_1 the pressure at the previous time level, we can write (14) as

$$\int_\Omega \left(Sp v^{(p)} + \Delta t \frac{K}{\mu} \nabla p \cdot \nabla v^{(p)} \right) \,\mathrm{d}x = \int_\Omega \left(Sp_1 - \Delta t \varrho_f \boldsymbol{g} \cdot \nabla v^{(p)} \right) \,\mathrm{d}x$$
$$+ \int_{\partial\Omega_{N,q}} \boldsymbol{q} \cdot \boldsymbol{n}\, v^{(p)} \,\mathrm{d}s,\ \forall v^{(p)} \in V^{(p)}, \tag{15}$$

where Δt is the time step, and we have placed all unknown terms on the left-hand side.

For the variational formulation of the elasticity problem for $\boldsymbol{u}$, we introduce a test vector function $\boldsymbol{v}^{(u)} \in V^{(u)}$. The space $V^{(u)}$ is where we seek $\boldsymbol{u}$ (or more precisely, the part of $\boldsymbol{u}$ with vanishing Dirichlet conditions). We require, formally, $\boldsymbol{v}^{(u)} = 0$ at $\partial\Omega_{D,u}$. The variational formulation is most easily derived by taking the inner product of $\boldsymbol{v}^{(u)}$ and the original equation $\nabla \cdot \boldsymbol{\sigma} = -\varrho_s \boldsymbol{g}$, and integrating by parts:

$$\int_\Omega \boldsymbol{\sigma} : \nabla \boldsymbol{v}^{(u)} \,\mathrm{d}x = \int_{\Omega_{N,\sigma}} \boldsymbol{v}^{(u)} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \,\mathrm{d}s - \int_\Omega \varrho_s \boldsymbol{g} \cdot \boldsymbol{v}^{(u)} \,\mathrm{d}x, \quad \forall \boldsymbol{v}^{(u)} \in V^{(u)}\,. \tag{16}$$

The surface integral contains the normal stress $\boldsymbol{\sigma} \cdot \boldsymbol{n}$, which is prescribed on $\partial\Omega_{N,\sigma}$ (mixed stress and displacement conditions at the same boundary point makes this a bit more complicated).

Since one can show that

$$\boldsymbol{\sigma} : \nabla \boldsymbol{v}^{(u)} = \boldsymbol{\sigma} : \frac{1}{2}(\nabla \boldsymbol{v}^{(u)} + (\nabla \boldsymbol{v}^{(u)})^T),$$

it is common to use the latter inner tensor product instead. That form is what naturally arises when deriving the variational form from the principle of minimum potential elastic energy. Nevertheless, we use the notationally more compact for $\boldsymbol{\sigma} : \nabla \boldsymbol{v}^{(u)}$ in the following.

Inserting the constitutive law (6) in (16) yields the final variational form for the deformation problem:

$$\int_\Omega \left(\lambda \nabla \cdot \boldsymbol{u}\, \boldsymbol{I} + \mu \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T \right) \right) : \nabla \boldsymbol{v}^{(u)} \,\mathrm{d}x = \int_{\Omega_{N,\sigma}} \boldsymbol{v}^{(u)} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \,\mathrm{d}s$$
$$- \int_\Omega \left(\varrho_s \boldsymbol{g} \cdot \boldsymbol{v}^{(u)} - \alpha p \nabla \cdot \boldsymbol{v}^{(u)} \right) \,\mathrm{d}x,\ \forall \boldsymbol{v}^{(u)} \in V^{(u)}\,. \tag{17}$$

Variational formulations of linear problems are often expressed in an abstract form $a(\cdot,\cdot) = L(\cdot)$. Here the abstract notation reads

$$a_p(p, v^{(p)}) = L(v^{(p)}), \quad \forall v^{(p)} \in V^{(p)},$$
$$a_u(\boldsymbol{u}, \boldsymbol{v}^{(u)}) = L(\boldsymbol{v}^{(u)}), \quad \forall \boldsymbol{v}^{(u)} \in V^{(u)}$$

with

$$
\begin{aligned}
a_p(p, v^{(p)}) &= \int_\Omega \left(S p v^{(p)} + \Delta t \frac{K}{\mu} \nabla p \cdot \nabla v^{(p)} \right) \mathrm{d}x, \\
L(v^{(p)}) &= \int_\Omega \left(S p_1 - \Delta t \varrho_f \boldsymbol{g} \cdot \nabla v^{(p)} \right) \mathrm{d}x + \int_{\partial\Omega_{N,q}} \boldsymbol{q} \cdot \boldsymbol{n}\, v^{(p)} \, \mathrm{d}s, \\
a_u(\boldsymbol{u}, \boldsymbol{v}^{(u)}) &= \int_\Omega \boldsymbol{\sigma} : \nabla \boldsymbol{v}^{(u)} \, \mathrm{d}x = \int_\Omega \left(\lambda \nabla \cdot \boldsymbol{u}\, \boldsymbol{I} + \mu \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T \right) \right) : \nabla \boldsymbol{v}^{(u)} \, \mathrm{d}x, \\
L(\boldsymbol{v}^{(u)}) &= \int_{\Omega_{N,\sigma}} \boldsymbol{v}^{(u)} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}s - \int_\Omega \left(\varrho_s \boldsymbol{g} \cdot \boldsymbol{v}^{(u)} - \alpha p \nabla \cdot \boldsymbol{v}^{(u)} \right) \mathrm{d}x \,.
\end{aligned}
$$

2.6 Variational formulation of the coupled problem

For the coupled problem we introduce the test functions $v^{(p)} \in V^{(p)}$ and $\boldsymbol{v}^{(u)} \in V^{(u)}$, which can be collected in a vector $(v^{(p)}, \boldsymbol{v}^{(u)}) \in V^{(p)} \times V^{(u)}$. By introducing $\mathcal{L}_p = 0$ for the flow PDE (7) and $\mathcal{L}_u = 0$ for the deformation PDE (11), we take the inner product of ($\mathcal{L}_p$, $\mathcal{L}_u$ and $(v^{(p)}, \boldsymbol{v}^{(u)})$ and integrate over Ω:

$$
\int_\Omega \left(\mathcal{L}_p v^{(p)} + \mathcal{L}_u \cdot \boldsymbol{v}^{(u)} \right) \mathrm{d}x = 0 \,. \tag{18}
$$

We realize that the various terms involved in the integral equal those in the uncoupled problem, modulo the term $\alpha \nabla \cdot \boldsymbol{u}_t$ that was not present in the uncoupled problem. All divergence terms are integrated by parts. We can also introduce a Backward Euler scheme to discretize p_t and $\nabla \cdot \boldsymbol{u}_t$ as done in the uncoupled problem. The variational form becomes

$$
\begin{aligned}
&\int_\Omega \left(S p v^{(p)} + \Delta t \boldsymbol{q} \cdot \nabla v^{(p)} \right) \mathrm{d}x - \int_\Omega \alpha \nabla v^{(p)} \cdot \boldsymbol{u} \, \mathrm{d}x \int_\Omega S p_1 \, \mathrm{d}x - \\
&\int_\Omega \alpha \nabla v^{(p)} \cdot \boldsymbol{u}_1 \, \mathrm{d}x - \int_{\partial\Omega_{N,q}} \boldsymbol{q} \cdot \boldsymbol{n}\, v^{(p)} \, \mathrm{d}s + \int_\Omega \boldsymbol{\sigma} : \nabla \boldsymbol{v}^{(u)} \, \mathrm{d}x - \\
&\int_\Omega \left(\varrho_s \boldsymbol{g} \cdot \boldsymbol{v}^{(u)} - \alpha p \nabla \cdot \boldsymbol{v}^{(u)} \right) \mathrm{d}x - \int_{\Omega_{N,\sigma}} \boldsymbol{v}^{(u)} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}s, \\
&\qquad \forall (v^{(p)}, \boldsymbol{v}^{(u)}) \in V^{(p)} \times V^{(u)},
\end{aligned} \tag{19}
$$

with

$$
\boldsymbol{q} = -\frac{K}{\mu} (\nabla p - \varrho_f \boldsymbol{g}), \tag{20}
$$

or

$$
\boldsymbol{q} = -\frac{\boldsymbol{K}}{\mu} (\nabla p - \varrho_f \boldsymbol{g}), \tag{21}
$$

and

$$\boldsymbol{\sigma} = \lambda\nabla\cdot\boldsymbol{u}\,\boldsymbol{I} + \mu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right) - \alpha p\,\boldsymbol{I}\,. \tag{22}$$

It is easy in FEniCS implementations to work with mathematical relations between mathematical quantities entering variational form, so (19) is a convenient way to state the variational form, with the p-$\boldsymbol{q}$ and $\boldsymbol{\sigma}$-$\boldsymbol{u}$ relations available for substitution into the form. For mathematical writing it is common to perform such substitutions directly in the form:

$$\begin{aligned}
&\int_\Omega \left(Spv^{(p)} + \Delta t\frac{K}{\mu}\nabla p\cdot\nabla v^{(p)}\right)\,\mathrm{d}x - \int_\Omega \alpha\nabla v^{(p)}\cdot\boldsymbol{u}\,\mathrm{d}x - \\
&\int_\Omega \left(Sp_1 - \Delta t\varrho_f\boldsymbol{g}\cdot\nabla v^{(p)}\right)\,\mathrm{d}x - \int_\Omega \alpha\nabla v^{(p)}\cdot\boldsymbol{u}_1\,\mathrm{d}x - \int_{\partial\Omega_{N,q}} \boldsymbol{q}\cdot\boldsymbol{n}\,v^{(p)}\,\mathrm{d}s + \\
&\int_\Omega \lambda\nabla\cdot\boldsymbol{u}\,\boldsymbol{I} + \mu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right) : \nabla\boldsymbol{v}^{(u)}\,\mathrm{d}x - \\
&\int_\Omega \left(\varrho_s\boldsymbol{g}\cdot\boldsymbol{v}^{(u)} - \alpha p\nabla\cdot\boldsymbol{v}^{(u)}\right)\,\mathrm{d}x - \int_{\Omega_{N,\sigma}} \boldsymbol{v}^{(u)}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}s, \\
&\qquad \forall (v^{(p)}, \boldsymbol{v}^{(u)}) \in V^{(p)}\times V^{(u)}\,.
\end{aligned} \tag{23}$$

An abstract variational form of the coupled problem can be written as

$$a((p,\boldsymbol{u}),(v^{(p)},\boldsymbol{v}^{(u)})) = L((v^{(p)},\boldsymbol{v}^{(u)})),$$

with

$$\begin{aligned}
a((p,\boldsymbol{u}),(v^{(p)},\boldsymbol{v}^{(u)})) &= \int_\Omega \left(Spv^{(p)} + \Delta t\boldsymbol{q}\cdot\nabla v^{(p)}\right)\,\mathrm{d}x - \int_\Omega \alpha\nabla v^{(p)}\cdot\boldsymbol{u}\,\mathrm{d}x + \\
&\quad \int_\Omega \boldsymbol{\sigma} : \nabla\boldsymbol{v}^{(u)}\,\mathrm{d}x + \\
L((v^{(p)},\boldsymbol{v}^{(u)})) &= \int_\Omega Sp_1\,\mathrm{d}x - \int_\Omega \alpha\nabla v^{(p)}\cdot\boldsymbol{u}_1\,\mathrm{d}x - \int_{\partial\Omega_{N,q}} \boldsymbol{q}\cdot\boldsymbol{n}\,v^{(p)}\,\mathrm{d}s + \\
&\quad \int_\Omega \left(\varrho_s\boldsymbol{g}\cdot\boldsymbol{v}^{(u)} - \alpha p\nabla\cdot\boldsymbol{v}^{(u)}\right)\,\mathrm{d}x - \int_{\Omega_{N,\sigma}} \boldsymbol{v}^{(u)}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}s\,.
\end{aligned}$$

2.7 Reductions

Familiar equations: diffusion, Laplace, std elasticity, thermo-elasticity.

A Derivations

For now this is just a rough copy of the MekIT'13 paper. Note that the derivation is for incompressible media while the equations above are for compressible too.

A.1 Poroelasticity versus Porous Media Theory

There are two different theories in use today, which both describe wave propagation in porous materials. The first is *poroelasticity*, also known as Biot's theory, which builds on the work by Terzaghi. The second is the *theory of porous media*, which is derived from first principles in mechanics. The main difference between these theories lies in the way the solid-fluid interaction is handled. However, it turns out that for individually incompressible fluid and solid phase, the two approaches are identical as long as we assume that all pores are interconnected and that there is no exchange of mass (see [2]).

The fluid phase, consisting of the interstitial fluid, is a water-like incompressible fluid. The solid phase also contains a large amount of water and is nearly incompressible. The only problem with assuming an incompressible solid phase is that in the current approach the vasculature is part of the solid. The vasculature is compressible and expands during systole. However, since it only accounts for 3 total volume the errors we make is relatively small. In the following we will derive the equations from first principles based on [1] and [2].

A.2 Momentum balance

To set up a balance of forces in a porous medium, both the fluid and the solid phase have to be considered. To distinguish between the fluid and solid components the subscript f and s are used respectively. For the fluid phase we have the momentum balance equation

$$\int_V \phi\rho_f \frac{D\boldsymbol{v}_f}{Dt}\mathrm{d}V = \int_V \phi\rho_f \boldsymbol{g}\mathrm{d}V + \int_S \boldsymbol{n}\cdot(\phi\boldsymbol{T}_f)\mathrm{d}S + \int_V \boldsymbol{F}_{fs}\mathrm{d}V \tag{24}$$

and for the solid phase

$$\int_V (1-\phi)\rho_s \frac{D\boldsymbol{v}_s}{Dt}\mathrm{d}V = \int_V (1-\phi)\rho_s \boldsymbol{g}\mathrm{d}V - \int_S \boldsymbol{n}\cdot((1-\phi)\boldsymbol{T}_s)\,\mathrm{d}S + \int_V \boldsymbol{F}_{sf}\mathrm{d}V. \tag{25}$$

The porosity ϕ enters the integrals since we are to integrate over the fluid or the solid, and the fluid occupies a fraction ϕ of the volume or surface, while the solid occupies the remaining fraction $1-\phi$. From the integral balance equations above we apply Gauss' theorem to the surface integrals and derive the corresponding partial differential equations

$$\phi\rho_f \frac{D\boldsymbol{v}_f}{Dt} = \phi\rho_f \boldsymbol{g} + \nabla\cdot(\phi\boldsymbol{T}_f) + \boldsymbol{F}_{fs} \tag{26}$$

$$(1-\phi)\rho_s \frac{D\boldsymbol{v}_s}{Dt} = (1-\phi)\rho_s \boldsymbol{g} - \nabla\cdot((1-\phi)\boldsymbol{T}_s) + \boldsymbol{F}_{sf} \tag{27}$$

with $\boldsymbol{v}$ as velocities, ρ partial densities, ϕ porosity, and $\boldsymbol{T}$ the stress tensors. The operator $\frac{D}{Dt}$ denotes the material time derivative following either a fluid or solid particle (depending on the equation). The gravity force is given by $\boldsymbol{g}$ term, but will be neglected in the following. The friction force between the fluid and solid

phase are represented by $\boldsymbol{F}_{sf}$ and $\boldsymbol{F_{fs}}$ and according to Newton's 3rd law these forces cancel each other out.

Adding equations (27) and (26), neglecting acceleration of the fluid and gravity forces result in the momentum balance for the fluid-solid mixture

$$(1-\phi)\rho_s \frac{D\boldsymbol{v}_s}{Dt} = -\nabla \cdot ((1-\phi)\boldsymbol{T}_s + \phi \boldsymbol{T}_f)\,. \tag{28}$$

For the fluid phase the stress tensor is governed by the hydrostatic pressure and reduces to $\boldsymbol{T}_f = -p\boldsymbol{I}$. That is, the extra stress $\boldsymbol{\sigma}_f^E$ due to viscous forces are negligible in the divergence term, but the friction between the fluid and solid because of viscous effects is substantial. However, this effect must be averaged and is modeled by the exchange terms $\mathbf{F}_{sf}$ and $\mathbf{F}_{fs}$. For the solid phase we need to take the internal stresses $\boldsymbol{\sigma}_s^E$ into account and set $\boldsymbol{T}_s = \boldsymbol{\sigma}_E - p\boldsymbol{I}$. Now the total stress tensor is given by

$$\boldsymbol{T}_T = (1-\phi)\boldsymbol{T}_s + \phi \boldsymbol{T}_f = (1-\phi)(\boldsymbol{\sigma}_s^E - p\boldsymbol{I}) - \phi p\mathbf{I} \approx \boldsymbol{\sigma} - p\mathbf{I}. \tag{29}$$

For linear elasticity the displacement vector $\boldsymbol{u}$ can be related to strain tensor field,

$$\boldsymbol{e} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}). \tag{30}$$

Assuming an isotropic, linear elastic medium, it can be shown that the following linear stress-strain relation holds:

$$\boldsymbol{\sigma} = 2\mu \boldsymbol{e} + \lambda(\mathrm{tr}\boldsymbol{e})\boldsymbol{I} \tag{31}$$

where μ and λ represent Lamé parameters.

For small deformations,

$$\frac{D\boldsymbol{v}_s}{Dt} \approx \frac{\partial^2 \boldsymbol{u}}{\partial^2 t}.$$

Substituting (31) and (30) into (28) and rearranging yields:

$$(1-\phi)\rho_s \frac{\partial^2 \boldsymbol{u}}{\partial^2 t} = \nabla \cdot \left(\mu(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}) + \lambda(\nabla \cdot \boldsymbol{u})\boldsymbol{I}\right) - \nabla p \tag{32}$$

It is common to neglect the acceleration term on the left-hand side when we do not want to include fast sound waves in the model. From now on we therefore omit this term.

The Lamé parameters are used to characterize the elasticity of a medium and are related to the Poisson ratio ν and Young's modulus E as follows:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{33}$$

High values of μ and λ implies a stiff material.

A.3 Volume balance

Assuming the density of the solid and the fluid phase to be constant the mass balance can be transformed to a volume balance: For the solid phase we have

$$\frac{\partial(1-\phi)}{\partial t} + \nabla \cdot ((1-\phi) \cdot \boldsymbol{v}_s) = 0 \tag{34}$$

and for the fluid phase we have

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \boldsymbol{v}_f) \pm \Omega_F(\boldsymbol{x}, t) = 0. \tag{35}$$

Adding (34) and (35) yields:

$$\nabla \cdot (\phi \boldsymbol{v}_f + (1-\phi)\boldsymbol{v}_s) \pm \Omega_F(\boldsymbol{x}, t) = 0 \tag{36}$$

In the case of infinitesimal deformation,

$$\mathbf{v}_s = \frac{\partial \boldsymbol{u}}{\partial t} \tag{37}$$

The fluid velocity $\boldsymbol{v}_f$ can be derived from the momentum balance of the fluid (27), and again by neglecting acceleration of the fluid we obtain

$$-\nabla \cdot (\phi p \boldsymbol{I}) + \boldsymbol{F}_{fs} = 0 \tag{38}$$

As mentioned, $\boldsymbol{F}_{sf}$ represents the friction forces between the solid and fluid phases, which is modeled by the linear relation:

$$\boldsymbol{F}_{sf} = p\nabla\phi - \boldsymbol{R}(\boldsymbol{v}_f - \boldsymbol{v}_s) \tag{39}$$

where $\boldsymbol{R}$ is a second order tensor arising in the linearization of $\boldsymbol{F}_{sf}$. Replacing $\boldsymbol{F}_{sf}$ in (38) yields

$$-\nabla \cdot (\phi p \boldsymbol{I}) + p\nabla\phi - \boldsymbol{R}(\boldsymbol{v}_f - \boldsymbol{v}_s) = 0 \tag{40}$$

$$\boldsymbol{v}_f = -\phi \boldsymbol{R}^{-1} \nabla p + \boldsymbol{v}_s \tag{41}$$

Finally, $\boldsymbol{R}$ can be related to the conductivity of the porous medium [1]

$$\boldsymbol{R}^{-1} := \frac{\boldsymbol{K}}{\mu_w \phi}. \tag{42}$$

Substituting (37) and (41) in (36) and rearranging yields the final form of the fluid-solid mixture volume balance:

$$\nabla \cdot \left(\frac{\partial \boldsymbol{u}}{\partial t} - \frac{\boldsymbol{K}}{\mu_w} \nabla p \right) = \Omega_F(\boldsymbol{x}, t) \tag{43}$$

References

[1] S. M. Hassanizadeh and W. G. Gray. General conservation equations for multi-phase systems: 3. constitutive theory of porous media flow. *Advances in Water Resources*, 3:25–40, 1980.

[2] M. Schanz and S. Diebels. A comparative study of biot's theory and the linear theory of. *Acta Mechanica*, 161:213–235, 2003.