

GSERM - St. Gallen 2022

Analyzing Panel Data

June 7, 2022

Two-Way Variation

Two-way variation:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \gamma V_i + \delta W_t + u_{it}$$

where V_i don't vary over time (within a unit), and W_t don't vary across units (for a given time point).

Note that we can write:

$$\alpha_i = \sum(\gamma V_i)$$

and

$$\eta_t = \sum(\delta W_t).$$

So:

$$\begin{aligned} Y_{it} &= \mathbf{X}_{it}\boldsymbol{\beta} + \gamma V_i + \delta W_t + u_{it} \\ &= \mathbf{X}_{it}\boldsymbol{\beta} + \alpha_i + \eta_t + u_{it} \end{aligned}$$

One- and Two-Way “Unit Effects”

“Two-way” unit effects:

$$Y_{it} = \mathbf{X}_{it}\beta + \alpha_i + \eta_t + u_{it}$$

“One-way” effects:

- Assuming $\alpha_i = 0$ (w.l.o.g):

$$Y_{it} = \mathbf{X}_{it}\beta + \eta_t + u_{it} \quad (\text{time})$$

- Assuming $\eta_t = 0$ (w.l.o.g):

$$Y_{it} = \mathbf{X}_{it}\beta + \alpha_i + u_{it} \quad (\text{units})$$

(One-Way) “Fixed” Effects

“Brute force” model fits:

$$\begin{aligned} Y_{it} &= \mathbf{X}_{it}\beta_{FE} + \alpha_i + u_{it} \\ &= \mathbf{X}_{it}\beta_{FE} + \alpha_1 I(i=1)_i + \alpha_2 I(i=2)_i + \dots + u_{it} \end{aligned}$$

Alternatively, decompose:

$$\bar{\mathbf{X}}_i = \frac{\sum_{N_i} \mathbf{X}_{it}}{N_i}$$

and

$$\tilde{\mathbf{X}}_{it} = \mathbf{X}_{it} - \bar{\mathbf{X}}_i.$$

Yields:

$$Y_{it} = \bar{\mathbf{X}}_i\beta_B + \tilde{\mathbf{X}}_{it}\beta_W + \alpha_i + u_{it}$$

But!

$$\text{corr}(\bar{\mathbf{X}}_i\beta_B, \alpha_i) = 1.0$$

Means that:

$$\begin{aligned}Y_{it}^* &= Y_{it} - \bar{Y}_i \\ \mathbf{X}_{it}^* &= \mathbf{X}_{it} - \bar{\mathbf{X}}_i\end{aligned}$$

gives:

$$Y_{it}^* = \mathbf{X}_{it}^* \beta_{FE} + u_{it}.$$

→ **A “Fixed Effects” Model is actually a “Within-Effects” Model.**

Standard F -test for

$$H_0 : \alpha_i = \alpha_j \forall i \neq j$$

versus

$$H_A : \alpha_i \neq \alpha_j \text{ for some } i \neq j$$

is $\sim F_{N-1, NT-(N-1)}$.

Running Example Data: WDI, 1960-2021

The World Development Indicators

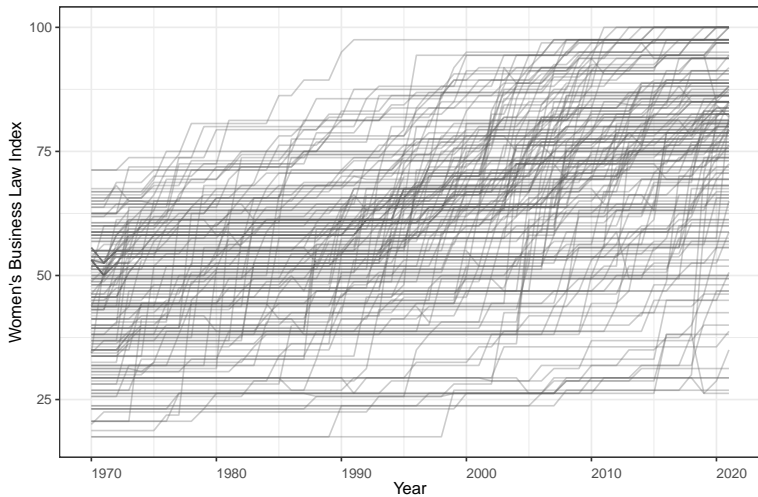
- Cross-national country-level time series data
- $N = 215$ countries, $T = 72$ years (1960-2021) + missingness
- Variables:
 - Geography: land area, arable land
 - Population indicators
 - Demographics: Birth rates, life expectancy, etc.
 - Economics: GDP, inflation, trade, FDI, etc.
 - Governments: expenditures, policies, etc.
- Full descriptions are listed in the Github repo [here](#)

Data Summary

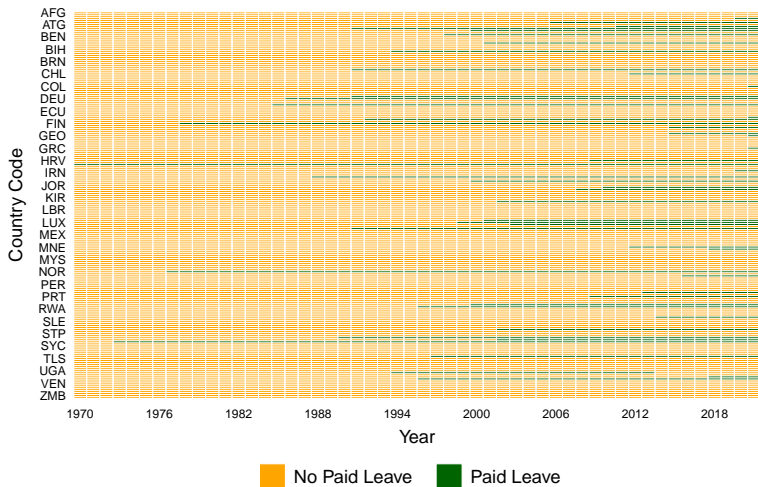
```
> describe(wdi,fast=TRUE,ranges=FALSE,check=TRUE)
```

	vars	n	mean	sd	se
ISO3	1	13330	NaN	NA	NA
Year	2	13330	1990.50	17.90	0.16
Region	3	13330	NaN	NA	NA
country	4	13330	NaN	NA	NA
LandArea	5	12906	613525.38	1766486.19	15549.43
ArablePercent	6	10935	13.51	13.49	0.13
Population	7	13073	24638671.91	103129756.53	901978.87
PopGrowth	8	12856	1.79	1.68	0.01
RuralPopulation	9	13045	48.61	25.74	0.23
UrbanPopulation	10	13045	51.39	25.74	0.23
BirthRatePer1K	11	12112	28.32	13.10	0.12
FertilityRate	12	11847	3.97	2.01	0.02
PrimarySchoolAge	13	10696	6.14	0.62	0.01
LifeExpectancy	14	11829	64.37	11.46	0.11
AgeDepRatioOld	15	11731	10.34	6.36	0.06
CO2Emissions	16	5535	4.28	5.41	0.07
GDP	17	9585	242308268086.15	1101606170189.79	11252014966.83
GDPPerCapita	18	9582	11685.74	18675.05	190.78
GDPPerCapGrowth	19	9598	1.89	6.21	0.06
Inflation	20	8275	23.88	332.39	3.65
TotalTrade	21	8363	78.18	54.14	0.59
Exports	22	8363	36.41	28.85	0.32
Imports	23	8372	41.78	27.87	0.30
FDIIn	24	8195	5.57	45.42	0.50
AgriEmployment	25	5394	29.67	24.35	0.33
NetAidReceived	26	8633	453209476.19	867754118.55	9339331.98
MobileCellSubscriptions	27	9849	33.70	50.29	0.51
NaturalResourceRents	28	8745	6.61	10.87	0.12
MilitaryExpenditures	29	7333	2.74	3.23	0.04
GovtExpenditures	30	8012	16.20	8.19	0.09
HIVDeaths	31	4041	7221.50	19857.71	312.38
WomenBusLawIndex	32	9776	59.23	18.46	0.19
PaidParentalLeave	33	9776	0.10	0.30	0.00
ColdWar	34	13330	0.48	0.50	0.00

Visualization (using panelView)



Categorical Variable Visualization



WDI's Women, Business and the Law Index (WBLI)

The basis for a 2021 World Bank [report](#)...

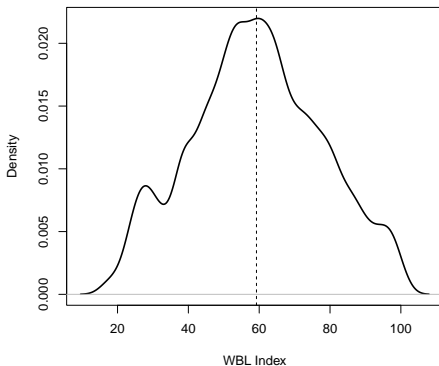
- Examines “the laws and regulations that affect women’s economic opportunity in 190 economies.”
- An index comprising eight indicators “structured around women’s interactions with the law as they move through their careers: *Mobility, Workplace, Pay, Marriage, Parenthood, Entrepreneurship, Assets, and Pension.*”
- The WBL Index:
 - Theoretically ranges from 0 - 100
 - In practice: Lowest values \approx 20
 - Higher values correspond to higher levels of women’s empowerment and greater opportunities and support for women, particularly in business
- “Better performance in the areas measured by the Women, Business and the Law index is associated with a more narrow gender gap in development outcomes, higher female labor force participation, lower vulnerable employment, and greater representation of women in national parliaments.”

WBLI: Total Variation

```
> WDI<-pdata.frame(wdi)
> WBLI<-WDI$WomenBusLawIndex
> class(WBLI)
[1] "pseries" "numeric"

> describe(WBLI,na.rm=TRUE) # all variation
```

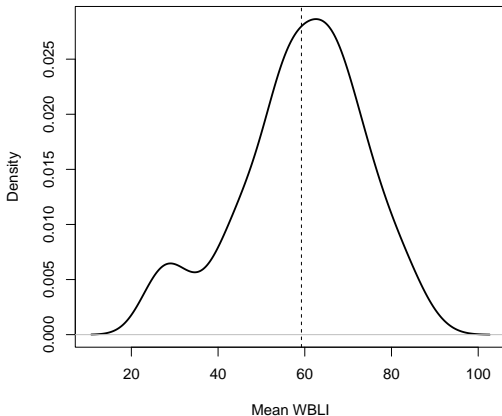
	vars	n	mean	sd	median	trimmed	mad	min	max	range	skew	kurtosis	se
X1	1	9776	59.2	18.5	58.8	59.2	18.5	17.5	100	82.5	0.04	-0.55	0.19



WBLI: "Between" Variation

```
> describe(plm::between(WBLI,effect="individual",na.rm=TRUE)) # "between" variation
```

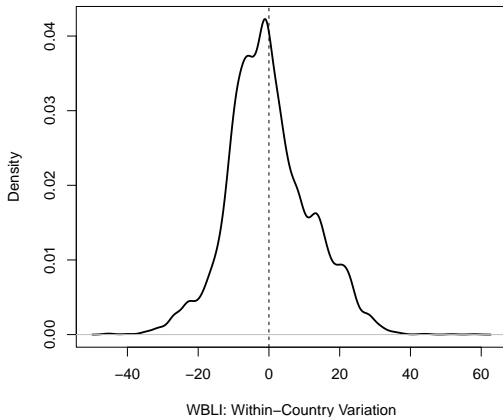
	vars	n	mean	sd	median	trimmed	mad	min	max	range	skew	kurtosis	se
X1	1	188	59.2	14.4	60.5	60.1	13.3	23.4	90	66.6	-0.47	-0.13	1.05



WBLI: "Within" Variation

```
> describe(Within(WBLI,na.rm=TRUE)) # "within" variation
```

	vars	n	mean	sd	median	trimmed	mad	min	max	range	skew	kurtosis	se
X1	1	9776	0	11.5	-0.93	-0.33	10.4	-45.3	58	103	0.26	0.45	0.12



A Regression Model

Regression model:

$$\text{WBLI}_{it} = \beta_0 + \beta_1 \text{Population Growth}_{it} + \beta_2 \text{Urban Population}_{it} + \beta_3 \text{Fertility Rate}_{it} + \beta_4 \ln(\text{GDP Per Capita})_{it} + \beta_5 \text{Natural Resource Rents}_{it} + \beta_6 \text{Cold War}_t + u_{it}$$

Descriptive Statistics:

	vars	n	mean	sd	min	max	range	se
WomenBusLawIndex	1	7566	59.99	18.79	17.50	100.00	82.50	0.22
PopGrowth	2	7566	1.70	1.46	-6.77	17.51	24.28	0.02
UrbanPopulation	3	7566	51.19	23.90	2.85	100.00	97.16	0.27
FertilityRate	4	7566	3.67	1.91	0.90	8.61	7.70	0.02
NaturalResourceRents	5	7566	6.82	10.45	0.00	87.51	87.51	0.12
ColdWar	6	7566	0.32	0.47	0.00	1.00	1.00	0.01
lnGDPPerCap	7	7566	8.28	1.44	5.32	11.63	6.31	0.02

Regression: Pooled OLS

```
> OLS<-plm(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+          log(GDPPerCapita)+NaturalResourceRents+ColdWar,
+          data=WDI,model="pooling")
```

```
> summary(OLS)
Pooling Model
```

Call:

```
plm(formula = WomenBusLawIndex ~ PopGrowth + UrbanPopulation +
    FertilityRate + log(GDPPerCapita) + NaturalResourceRents +
    ColdWar, data = WDI, model = "pooling")
```

Unbalanced Panel: n = 186, T = 1-50, N = 7566

Residuals:

Min.	1st Qu.	Median	3rd Qu.	Max.
-46.94	-8.02	1.05	9.30	49.73

Coefficients:

	Estimate	Std. Error	t-value	Pr(> t)
(Intercept)	54.8395	1.7261	31.77	< 2e-16 ***
PopGrowth	-3.1926	0.1437	-22.21	< 2e-16 ***
UrbanPopulation	-0.0584	0.0109	-5.37	0.000000083 ***
FertilityRate	-1.7928	0.1652	-10.85	< 2e-16 ***
log(GDPPerCapita)	3.1544	0.1993	15.83	< 2e-16 ***
NaturalResourceRents	-0.3486	0.0162	-21.54	< 2e-16 ***
ColdWar	-11.3437	0.3716	-30.53	< 2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Total Sum of Squares: 2670000

Residual Sum of Squares: 1280000

R-Squared: 0.519

Adj. R-Squared: 0.519

F-statistic: 1361.01 on 6 and 7559 DF, p-value: <2e-16

“Fixed” (Within) Effects

```
> FE<-plm(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+         log(GDPPerCapita)+NaturalResourceRents+ColdWar,data=WDI,
+         effect="individual",model="within")
```

```
> summary(FE)
Oneway (individual) effect Within Model
```

```
Call:
plm(formula = WomenBusLawIndex ~ PopGrowth + UrbanPopulation +
      FertilityRate + log(GDPPerCapita) + NaturalResourceRents +
      ColdWar, data = WDI, effect = "individual", model = "within")
```

Unbalanced Panel: n = 186, T = 1-50, N = 7566

Residuals:

	Min.	1st Qu.	Median	3rd Qu.	Max.
	-33.144	-4.832	-0.406	4.802	41.184

Coefficients:

	Estimate	Std. Error	t-value	Pr(> t)
PopGrowth	0.0735	0.1194	0.62	0.538
UrbanPopulation	0.2482	0.0205	12.09	<2e-16 ***
FertilityRate	-2.0657	0.1657	-12.47	<2e-16 ***
log(GDPPerCapita)	9.1607	0.3104	29.51	<2e-16 ***
NaturalResourceRents	0.0353	0.0182	1.94	0.052 .
ColdWar	-7.1920	0.2951	-24.37	<2e-16 ***

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Total Sum of Squares: 934000

Residual Sum of Squares: 434000

R-Squared: 0.535

Adj. R-Squared: 0.523

F-statistic: 1414.37 on 6 and 7374 DF, p-value: <2e-16

A Nicer Table

Table: Models of WBLI

	OLS	FE
Population Growth	-3.190*** (0.144)	0.073 (0.119)
Urban Population	-0.058*** (0.011)	0.248*** (0.021)
Fertility Rate	-1.790*** (0.165)	-2.070*** (0.166)
ln(GDP Per Capita)	3.150*** (0.199)	9.160*** (0.310)
Natural Resource Rents	-0.349*** (0.016)	0.035* (0.018)
Cold War	-11.300*** (0.372)	-7.190*** (0.295)
Constant	54.800*** (1.730)	
Observations	7,566	7,566
R ²	0.519	0.535
Adjusted R ²	0.519	0.523
F Statistic	1,361.000*** (df = 6; 7559)	1,414.000*** (df = 6; 7374)

*p<0.1; **p<0.05; ***p<0.01

Time-Period Fixed Effects

The model is:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \eta_t + u_{it}$$

which is estimated via:

$$\begin{aligned} Y_{it}^{**} &= Y_{it} - \bar{Y}_t \\ \mathbf{X}_{it}^{**} &= \mathbf{X}_{it} - \bar{\mathbf{X}}_t \end{aligned}$$

$$Y_{it}^{**} = \beta_{FE} \mathbf{X}_{it}^{**} + u_{it}.$$

Comparison: Unit vs. Time Fixed Effects

Table: FE Models of WBLI (Units vs. Time)

	FE.Units	FE.Time
Population Growth	0.073 (0.119)	-3.630*** (0.135)
Urban Population	0.248*** (0.021)	-0.054*** (0.010)
Fertility Rate	-2.070*** (0.166)	-0.657*** (0.159)
ln(GDP Per Capita)	9.160*** (0.310)	3.550*** (0.187)
Natural Resource Rents	0.035* (0.018)	-0.400*** (0.015)
Cold War	-7.190*** (0.295)	
Observations	7,566	7,566
R ²	0.535	0.439
Adjusted R ²	0.523	0.435
F Statistic	1,414.000*** (df = 6; 7374)	1,175.000*** (df = 5; 7511)

*p<0.1; **p<0.05; ***p<0.01

The specification:

$$Y_{it} = \mathbf{X}_{it}\beta + \alpha_i + u_{it}$$

...suggests that we can use an F -test to examine the hypothesis:

$$H_0 : \alpha_i = 0 \ \forall \ i$$

(and a similar test for $\eta_t = 0$ in the time-centered case).

Arguably better are Lagrange multiplier-based tests:

- Breusch-Pagan (1980)
- King and Wu (1997)
- See (e.g.) Croissant and Millo (2018, §4.1) for details

FE (Country) Model Tests

```
> pFtest(FE,OLS)
```

F test for individual effects

```
data: WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate + ...  
F = 78, df1 = 185, df2 = 7374, p-value <2e-16  
alternative hypothesis: significant effects
```

```
> plmtest(FE,effect=c("individual"),type=c("bp"))
```

Lagrange Multiplier Test - (Breusch-Pagan) for unbalanced panels

```
data: WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate + ...  
chisq = 44635, df = 1, p-value <2e-16  
alternative hypothesis: significant effects
```

```
> plmtest(FE,effect=c("individual"),type=c("kw"))
```

Lagrange Multiplier Test - (King and Wu) for unbalanced panels

```
data: WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate + ...  
normal = 211, p-value <2e-16  
alternative hypothesis: significant effects
```

Same For Time Effects

```
> pFtest(FE.Time,OLS)
```

F test for time effects

```
data:  WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate + ...  
F = 25, df1 = 48, df2 = 7511, p-value <2e-16  
alternative hypothesis: significant effects
```

```
> plmtest(FE.Time,effect=c("time"),type=c("bp"))
```

Lagrange Multiplier Test - time effects (Breusch-Pagan) for unbalanced panels

```
data:  WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate + ...  
chisq = 9920, df = 1, p-value <2e-16  
alternative hypothesis: significant effects
```

```
> plmtest(FE.Time,effect=c("time"),type=c("kw"))
```

Lagrange Multiplier Test - time effects (King and Wu) for unbalanced panels

```
data:  WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate + ...  
normal = 100, p-value <2e-16  
alternative hypothesis: significant effects
```

Fixed Effects: Interpretation

FE models are *subject-specific* models, that rely entirely on *within-unit* variability to estimate variable effects.

- This means that:

$$\hat{\beta}_k = \frac{\partial E(Y|\hat{\alpha})}{\partial X_k}$$

- That is, $\hat{\beta}_k$ is *the expected change in $E(Y)$ associated with a one-unit increase in observation i 's value of X_k*
- Key: *within-unit* changes in **X** are associated with *within-unit* expected changes in Y .
- In a linear model, the value of $\hat{\alpha}$ doesn't affect the value of that partial derivative...

Fixed Effects: Interpretation

Mummolo and Peterson (2018) note that:

“...because the within-unit variation is always smaller (or at least, no larger) than the overall variation in the independent variable, researchers should use within-unit variation to motivate counterfactuals when discussing the substantive impact of a treatment” (2018, 829).

Significance:

- Predictors **X** in FE models typically have both cross-sectional and temporal variation
- FE models only consider *within-unit* variation in **X** and *Y*
- As a result, the degree of actual/observed (within-unit) variation in predictors is almost always less – and sometimes significantly less – than if cross-sectional variation were also considered

Interpretation Example: Urban Population

UrbanPopulation – All Variation:

```
> with(WDI, sd(UrbanPopulation,na.rm=TRUE)) # all variation  
[1] 25.7
```

UrbanPopulation – “Within” Variation:

```
> WDI<-ddply(WDI, .(ISO3), mutate,  
+           UPMean = mean(UrbanPopulation,na.rm=TRUE))  
> WDI$UPWithin<-with(WDI, UrbanPopulation-UPMean)  
>  
> with(WDI, sd(UPWithin,na.rm=TRUE)) # "within" variation  
[1] 8.86
```

“While the overall variation in the independent variable may be large, the within-unit variation used to estimate β may be much smaller” (M & P 2018, 830).

Pros and Cons of “Fixed” Effects

Pros:

- Mitigates (Some) Specification Bias
- Simple + Intuitive
- Widely Used/Understood

Cons (see e.g. [Collischon and Eberl 2020](#)):

- Can't Estimate β_B
- Slowly-Changing \mathbf{X} s
- (In)Efficiency / Inconsistency (Incidental Parameters)
- Cannot Control for (e.g.) Time-Varying Heterogeneity
- Sensitivity to Measurement Error

“Between” Effects

From the equation above:

$$Y_{it} = \bar{\mathbf{X}}_i \beta_B + \tilde{\mathbf{X}}_{it} \beta_W + \alpha_i + u_{it}$$

...we can derive a “Between Effects” model:

$$\bar{Y}_i = \bar{\mathbf{X}}_i \beta_B + u_{it}$$

This model:

- is essentially cross-sectional,
- is based on N observations,
- considers *only* between-unit (average) differences
- Interpretation:

$\hat{\beta}_B$ is the expected difference in Y between two units whose values on \bar{X} differ by a value of 1.0.

“Between” Effects

```
> BE<-plm(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+         log(GDPPerCapita)+NaturalResourceRents+ColdWar,data=WDI,
+         effect="individual",model="between")
```

```
> summary(BE)
```

Oneway (individual) effect Between Model

Call:

```
plm(formula = WomenBusLawIndex ~ PopGrowth + UrbanPopulation +
      FertilityRate + log(GDPPerCapita) + NaturalResourceRents +
      ColdWar, data = WDI, effect = "individual", model = "between")
```

Unbalanced Panel: n = 186, T = 1-50, N = 7566

Observations used in estimation: 186

Residuals:

	Min.	1st Qu.	Median	3rd Qu.	Max.
	-30.423	-6.319	0.332	8.067	22.183

Coefficients:

	Estimate	Std. Error	t-value	Pr(> t)
(Intercept)	50.3106	10.1621	4.95	0.0000016996 ***
PopGrowth	-5.8648	0.9173	-6.39	0.0000000014 ***
UrbanPopulation	-0.0497	0.0549	-0.91	0.366
FertilityRate	0.0633	1.0766	0.06	0.953
log(GDPPerCapita)	3.4052	1.1308	3.01	0.003 **
NaturalResourceRents	-0.3579	0.0888	-4.03	0.0000823194 ***
ColdWar	-12.5095	4.8879	-2.56	0.011 *

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Total Sum of Squares: 44700

Residual Sum of Squares: 17600

R-Squared: 0.605

Adj. R-Squared: 0.592

F-statistic: 45.7424 on 6 and 179 DF, p-value: <2e-16

A Nicer Table (Again)

Table: Models of WBLI

	OLS	FE	BE
Population Growth	-3.190*** (0.144)	0.073 (0.119)	-5.870*** (0.917)
Urban Population	-0.058*** (0.011)	0.248*** (0.021)	-0.050 (0.055)
Fertility Rate	-1.790*** (0.165)	-2.070*** (0.166)	0.063 (1.080)
ln(GDP Per Capita)	3.150*** (0.199)	9.160*** (0.310)	3.400*** (1.130)
Natural Resource Rents	-0.349*** (0.016)	0.035* (0.018)	-0.358*** (0.089)
Cold War	-11.300*** (0.372)	-7.190*** (0.295)	-12.500** (4.890)
Constant	54.800*** (1.730)		50.300*** (10.200)
Observations	7,566	7,566	186
R ²	0.519	0.535	0.605
Adjusted R ²	0.519	0.523	0.592
F Statistic	1,361.000*** (df = 6; 7559)	1,414.000*** (df = 6; 7374)	45.700*** (df = 6; 179)

*p<0.1; **p<0.05; ***p<0.01

“Random” Effects

Model:

$$Y_{it} = \mathbf{X}_{it}\beta + u_{it}$$

with:

$$u_{it} = \alpha_i + \lambda_t + \eta_{it}$$

and

$$\begin{aligned} E(\alpha_i) = E(\lambda_t) = E(\eta_{it}) &= 0, \\ E(\alpha_i \lambda_t) = E(\alpha_i \eta_{it}) = E(\lambda_t \eta_{it}) &= 0, \\ E(\alpha_i \alpha_j) &= \sigma_\alpha^2 \text{ if } i = j, 0 \text{ otherwise,} \\ E(\lambda_t \lambda_s) &= \sigma_\lambda^2 \text{ if } t = s, 0 \text{ otherwise,} \\ E(\eta_{it} \eta_{js}) &= \sigma_\eta^2 \text{ if } i = j, t = s, 0 \text{ otherwise,} \\ E(\alpha_i \mathbf{X}_{it}) = E(\lambda_t \mathbf{X}_{it}) = E(\eta_{it} \mathbf{X}_{it}) &= 0. \end{aligned}$$

If those assumptions are met, we can consider the “two-way variance components” model where:

$$\begin{aligned} \text{Var}(u_{it}) &= \text{Var}(Y_{it}|\mathbf{X}_{it}) \\ &= \sigma_{\alpha}^2 + \sigma_{\lambda}^2 + \sigma_{\eta}^2 \end{aligned}$$

If we assume $\lambda_t = 0$, then we get a model like:

$$Y_{it} = \mathbf{X}_{it}\beta + \alpha_i + \eta_{it}$$

with total error variance:

$$\sigma_u^2 = \sigma_{\alpha}^2 + \sigma_{\eta}^2.$$

“Random” Effects: Estimation

The model above will violate the standard OLS assumptions of uncorrelated errors, because the (compound) “errors” u_{it} within each unit share a common component α_i .

Consider the within- i variance-covariance matrix of the errors \mathbf{u} :

$$\begin{aligned} E(\mathbf{u}_i \mathbf{u}_i') \equiv \mathbf{\Sigma}_i &= \sigma_\eta^2 \mathbf{I}_T + \sigma_\alpha^2 \mathbf{1}\mathbf{1}' \\ &= \begin{pmatrix} \sigma_\eta^2 + \sigma_\alpha^2 & \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 \\ \sigma_\alpha^2 & \sigma_\eta^2 + \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_\alpha^2 & \sigma_\alpha^2 & \cdots & \sigma_\eta^2 + \sigma_\alpha^2 \end{pmatrix} \end{aligned}$$

Assuming conditional independence across units, we then have:

$$\text{Var}(\mathbf{u}) \equiv \mathbf{\Omega} = \begin{pmatrix} \mathbf{\Sigma}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{\Sigma}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{\Sigma}_N \end{pmatrix}$$

“Random” Effects: Estimation

We can then show that:

$$\Sigma^{-1/2} = \frac{1}{\sigma_\eta} \left[\mathbf{I}_T - \left(\frac{\theta}{T} \mathbf{\ddot{u}}' \right) \right]$$

where

$$\theta = 1 - \sqrt{\frac{\sigma_\eta^2}{T\sigma_\alpha^2 + \sigma_\eta^2}}$$

is an unknown quantity to be estimated.

Starting with an estimate of $\hat{\theta}$, calculate:

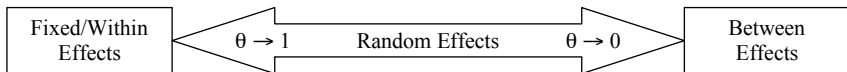
$$\begin{aligned} Y_{it}^* &= Y_{it} - \hat{\theta} \bar{Y}_i \\ X_{it}^* &= X_{it} - \hat{\theta} \bar{X}_i, \end{aligned}$$

then estimate:

$$Y_{it}^* = (1 - \hat{\theta})\alpha + X_{it}^* \beta_{RE} + [(1 - \hat{\theta})\alpha_i + (\eta_{it} - \hat{\theta} \bar{\eta}_i)]$$

and iterate between the two processes until convergence.

“Random” Effects: An Alternative View



Random Effects

```
> RE<-plm(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+          log(GDPPerCapita)+NaturalResourceRents+ColdWar,data=WDI,
+          effect="individual",model="random")
```

```
> summary(RE)
Oneway (individual) effect Random Effect Model
(Swamy-Arora's transformation)
```

```
Call:
plm(formula = WomenBusLawIndex ~ PopGrowth + UrbanPopulation +
      FertilityRate + log(GDPPerCapita) + NaturalResourceRents +
      ColdWar, data = WDI, effect = "individual", model = "random")
```

Unbalanced Panel: n = 186, T = 1-50, N = 7566

Effects:

	var	std.dev	share
idiosyncratic	58.91	7.68	0.4
individual	88.75	9.42	0.6

theta:

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.368	0.872	0.886	0.876	0.886	0.886

Residuals:

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
-33.3	-5.2	-0.4	0.0	5.5	40.9

Coefficients:

	Estimate	Std. Error	z-value	Pr(> z)
(Intercept)	2.56506	2.58528	0.99	0.32
PopGrowth	-0.07305	0.12134	-0.60	0.55
UrbanPopulation	0.12838	0.01920	6.69	0.000000000023 ***
FertilityRate	-2.37359	0.16536	-14.35	< 2e-16 ***
log(GDPPerCapita)	7.51273	0.29275	25.66	< 2e-16 ***
NaturalResourceRents	0.00581	0.01820	0.32	0.75
ColdWar	-8.48014	0.28934	-29.31	< 2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Total Sum of Squares: 982000

Residual Sum of Squares: 471000

R-Squared: 0.52

Adj. R-Squared: 0.52

A Nicer Table (Yet Again)

Table: Models of WBLI

	OLS	FE	BE	RE
Population Growth	-3.190*** (0.144)	0.073 (0.119)	-5.870*** (0.917)	-0.073 (0.121)
Urban Population	-0.058*** (0.011)	0.248*** (0.021)	-0.050 (0.055)	0.128*** (0.019)
Fertility Rate	-1.790*** (0.165)	-2.070*** (0.166)	0.063 (1.080)	-2.370*** (0.165)
ln(GDP Per Capita)	3.150*** (0.199)	9.160*** (0.310)	3.400*** (1.130)	7.510*** (0.293)
Natural Resource Rents	-0.349*** (0.016)	0.035* (0.018)	-0.358*** (0.089)	0.006 (0.018)
Cold War	-11.300*** (0.372)	-7.190*** (0.295)	-12.500** (4.890)	-8.480*** (0.289)
Constant	54.800*** (1.730)		50.300*** (10.200)	2.560 (2.580)
Observations	7,566	7,566	186	7,566
R ²	0.519	0.535	0.605	0.520
Adjusted R ²	0.519	0.523	0.592	0.520
F Statistic	1,361.000*** (df = 6; 7559)	1,414.000*** (df = 6; 7374)	45.700*** (df = 6; 179)	7,910.000***

* p<0.1; ** p<0.05; *** p<0.01

“Random” Effects: Testing

Intuition:

- RE models require that $\text{Cov}(X_{it}, \alpha_i) = 0$.
- FE models do not.

This means that:

Model	Reality	
	$\text{Cov}(X_{it}, \alpha_i) = 0$	$\text{Cov}(X_{it}, \alpha_i) \neq 0$
Fixed Effects	Consistent, Inefficient	Consistent, Efficient
Random Effects	Consistent, Efficient	Inconsistent

The Hausman Test

Hausman test (FE vs. RE):

$$\hat{W} = (\hat{\beta}_{\text{FE}} - \hat{\beta}_{\text{RE}})'(\hat{\mathbf{V}}_{\text{FE}} - \hat{\mathbf{V}}_{\text{RE}})^{-1}(\hat{\beta}_{\text{FE}} - \hat{\beta}_{\text{RE}})$$

$$W \sim \chi_k^2$$

Null: The RE model is consistent ($\text{Cov}(X_{it}, \alpha_i) = 0$).

Issues:

- Asymptotic
- No guarantee $(\hat{\mathbf{V}}_{\text{FE}} - \hat{\mathbf{V}}_{\text{RE}})^{-1}$ is positive definite
- A general specification test...

Hausman Test Results

Hausman test (FE vs. RE):

```
> phtest(FE, RE)
```

Hausman Test

```
data:  WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate + ...  
chisq = 946, df = 6, p-value <2e-16
```

alternative hypothesis: one model is inconsistent

Practical “Fixed” vs. “Random” Effects

Factors to consider:

- “Panel” vs. “TSCS” Data
- Nature of the Data-Generating Processes
- Importance of Non-Time-Varying Covariate Effects
- Dimension/Location of the Variance in the Data (N vs. T)

Connections: Hierarchical Linear Models

HLM Starting Points

Begin by considering a two-level “nested” data structure, with:

$$\begin{aligned} i &\in \{1, 2, \dots, N\} \text{ indexing first-level units, and} \\ j &\in \{1, 2, \dots, J\} \text{ indexing second-level groups.} \end{aligned}$$

A general two-level HLM is an equation of the form:

$$Y_{ij} = \beta_{0j} + \mathbf{X}_{ij}\beta_j + u_{ij} \tag{1}$$

where β_{0j} is a “constant” term, \mathbf{X}_{ij} is a $NJ \times K$ matrix of K covariates, β_j is a $K \times 1$ vector of parameters, and $u_{ij} \sim \text{i.i.d. } N(0, \sigma_u^2)$ is the usual random-disturbance assumption.

Each of the $K + 1$ “level-one” parameters is then allowed to vary across Q “level-two” variables \mathbf{Z}_j , so that:

$$\beta_{0j} = \gamma_{00} + \mathbf{Z}_j\gamma_0 + \varepsilon_{0j} \quad (2)$$

for the “intercept” and

$$\beta_{kj} = \gamma_{k0} + \mathbf{Z}_j\gamma_k + \varepsilon_{kj} \quad (3)$$

for the “slopes” of \mathbf{X} . The ε s are typically assumed to be distributed multivariate Normal, with parameters for the variances and covariances selected by the analyst. Substitution of (2) and (3) into (1) yields:

$$Y_{ij} = \gamma_{00} + \mathbf{Z}_j\gamma_0 + \mathbf{X}_{ij}\gamma_{k0} + \mathbf{X}_{ij}\mathbf{Z}_j\gamma_k + \mathbf{X}_{ij}\varepsilon_{kj} + \varepsilon_{0j} + u_{ij} \quad (4)$$

The form is essentially a model with “saturated” interaction effects across the various levels, as well as “errors” which are multivariate Normal.

Model Assumptions

- Linearity / Additivity
- Normality of u_s
- Homoscedasticity
- Residual Independence:
 - $\text{Cov}(\varepsilon_{\cdot j}, u_{ij}) = 0$
 - $\text{Cov}(u_{ij}, u_{i\ell}) = 0$

Model Fitting

- MLE
- “Restricted” MLE (“RMLE”)
- Choosing:
 - MLE is biased in small samples, especially for estimating variances
 - RMLE is not, but prevents use of LR tests when the models do not have identical fixed effects
 - In general: RMLE is better with small sample sizes, but MLE is fine in larger ones

Note that if we specify:

$$\beta_{0j} = \gamma_{00} + \varepsilon_{0j}$$

and

$$\beta_{kj} = \gamma_{k0}$$

we get:

$$Y_{ij} = \gamma_{00} + \mathbf{X}_{ij}\gamma_{k0} + \varepsilon_{0j} + u_{ij}$$

which is the RE model above (formally, a “one-level random-intercept” HLM).

In addition:

- HLMs can be expanded to 3- and 4- and higher-level models
- One can include cross-level interactions...

HLMs are widely used in education, psychology, ecology, etc. (less so in economics, political science). Also, there are many, many excellent [books](#), [websites](#), etc. that address HLMs

Random Effects Remix (using lmer)

```
> library(lme4)
> AltRE<-lmer(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+             log(GDPPerCapita)+NaturalResourceRents+ColdWar+(1|IS03),
+             data=WDI)

> summary(AltRE)
Linear mixed model fit by REML ['lmerMod']
Formula: WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +
        log(GDPPerCapita) + NaturalResourceRents + ColdWar + (1 | IS03)
Data: WDI
```

REML criterion at convergence: 53322

Scaled residuals:

Min	1Q	Median	3Q	Max
-4.321	-0.632	-0.058	0.636	5.367

Random effects:

Groups	Name	Variance	Std.Dev.
IS03	(Intercept)	323	17.97
Residual		59	7.68

Number of obs: 7566, groups: IS03, 186

Fixed effects:

	Estimate	Std. Error	t value
(Intercept)	-11.5141	2.9322	-3.93
PopGrowth	0.0323	0.1191	0.27
UrbanPopulation	0.2056	0.0199	10.31
FertilityRate	-2.1709	0.1643	-13.21
log(GDPPerCapita)	8.5722	0.3022	28.37
NaturalResourceRents	0.0269	0.0181	1.49
ColdWar	-7.6610	0.2906	-26.36

Correlation of Fixed Effects:

	(Intr)	PpGrwt	UrbnPop	FrtltR	l(GDPP	NtrlRR
PopGrowth	0.051					
UrbanPopltn	-0.183	-0.014				
FertilityRt	-0.399	-0.288	0.436			
lg(GDPPrCp)	-0.764	-0.059	-0.305	0.093		
NtrlRsrcRnt	-0.010	-0.113	-0.012	-0.053	-0.015	
ColdWar	-0.109	0.013	0.219	-0.422	0.101	0.055

Q: Are They The Same? [A: More Or Less]

Table: RE and HLM Models of WBLI

	RE	AltRE
Population Growth	-0.073 (0.121)	0.032 (0.119)
Urban Population	0.128*** (0.019)	0.206*** (0.020)
Fertility Rate	-2.370*** (0.165)	-2.170*** (0.164)
ln(GDP Per Capita)	7.510*** (0.293)	8.570*** (0.302)
Natural Resource Rents	0.006 (0.018)	0.027 (0.018)
Cold War	-8.480*** (0.289)	-7.660*** (0.291)
Constant	2.560 (2.580)	-11.500*** (2.930)
Observations	7,566	7,566
R ²	0.520	
Adjusted R ²	0.520	
Log Likelihood		-26,661.000
Akaike Inf. Crit.		53,340.000
Bayesian Inf. Crit.		53,402.000
F Statistic	7,910.000***	

* p<0.1; ** p<0.05; *** p<0.01

For more, see [here](#).

HLM with Country-Level Random β s for ColdWar

```
> HLM1<-lmer(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+           log(GDPPerCapita)+NaturalResourceRents+ColdWar+(ColdWar|IS03),
+           data=WDI,control=lmerControl(optimizer="bobyqa"))

> summary(HLM1)
Linear mixed model fit by REML ['lmerMod']
Formula: WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +
          log(GDPPerCapita) + NaturalResourceRents + ColdWar + (ColdWar | IS03)
Data: WDI
Control: lmerControl(optimizer = "bobyqa")

REML criterion at convergence: 50165

Random effects:
Groups   Name              Variance Std.Dev. Corr
IS03     (Intercept) 560.7      23.68
          ColdWar    131.6      11.47   -0.20
Residual              35.4       5.95
Number of obs: 7566, groups: IS03, 186

Fixed effects:
              Estimate Std. Error t value
(Intercept)   -29.248209   3.374962   -8.67
PopGrowth      -0.137795   0.099032   -1.39
UrbanPopulation  0.285673   0.023264   12.28
FertilityRate   -4.130147   0.177531  -23.26
log(GDPPerCapita) 10.946306   0.339078   32.28
NaturalResourceRents -0.000932   0.015744   -0.06
ColdWar        -2.647474   0.995578   -2.66

Correlation of Fixed Effects:
              (Intr) PpGrwt UrbnPp FrtltR l(GDPP NtrlRR
PopGrowth      0.052
UrbanPopltn   -0.132 -0.037
FertilityRt    -0.479 -0.197  0.480
lg(GDPPPrCp)  -0.722 -0.056 -0.384  0.168
NtrlRsrcRnt    0.031 -0.079  0.058 -0.011 -0.095
ColdWar        -0.089 -0.012  0.056 -0.118 -0.003  0.011
```

```
> anova(AltRE,HLM1)
```

```
refitting model(s) with ML (instead of REML)
```

```
Data: WDI
```

```
Models:
```

```
AltRE: WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +  
      log(GDPPerCapita) + NaturalResourceRents + ColdWar + (1 | IS03)
```

```
HLM1: WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate +  
      log(GDPPerCapita) + NaturalResourceRents + ColdWar + (ColdWar | IS03)
```

	npars	AIC	BIC	logLik	deviance	Chisq	Df	Pr(>Chisq)
AltRE	9	53323	53386	-26653	53305			
HLM1	11	50174	50250	-25076	50152	3154	2	<2e-16 ***

```
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
> VarCorr(HLM1)
```

Groups	Name	Std.Dev.	Corr
IS03	(Intercept)	23.68	
	ColdWar	11.47	-0.20
Residual		5.95	

Random Coefficients

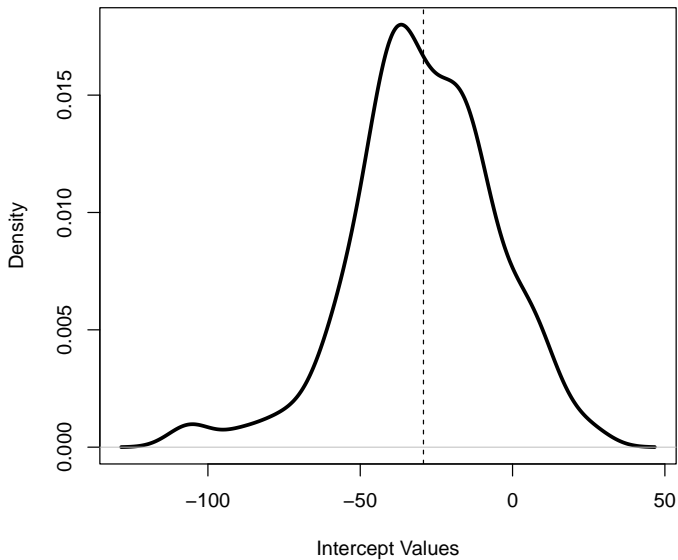
```
> Bs<-data.frame(coef(HLM1)[1])

> head(Bs)
  IS03..Intercept. IS03.PopGrowth IS03.UrbanPopulation IS03.FertilityRate
AFG           -19.2          -0.138              0.286             -4.13
AGO           -13.5          -0.138              0.286             -4.13
ALB           -11.3          -0.138              0.286             -4.13
ARE          -105.3          -0.138              0.286             -4.13
ARG           -49.2          -0.138              0.286             -4.13
ARM           -24.6          -0.138              0.286             -4.13
  IS03.log.GDPPerCapita. IS03.NaturalResourceRents IS03.ColdWar
AFG                10.9             -0.000932         -3.632
AGO                10.9             -0.000932        -14.877
ALB                10.9             -0.000932         -7.589
ARE                10.9             -0.000932         -0.729
ARG                10.9             -0.000932        -22.651
ARM                10.9             -0.000932         -3.106

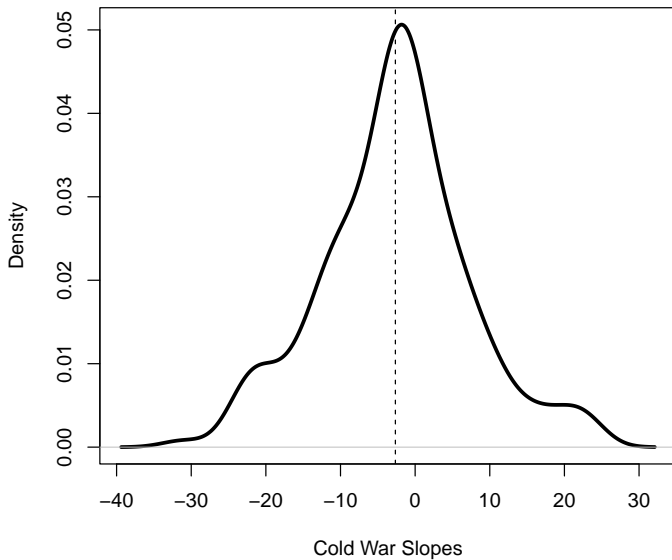
> mean(Bs$IS03..Intercept.)
[1] -29.2

> mean(Bs$IS03.ColdWar)
[1] -2.65
```

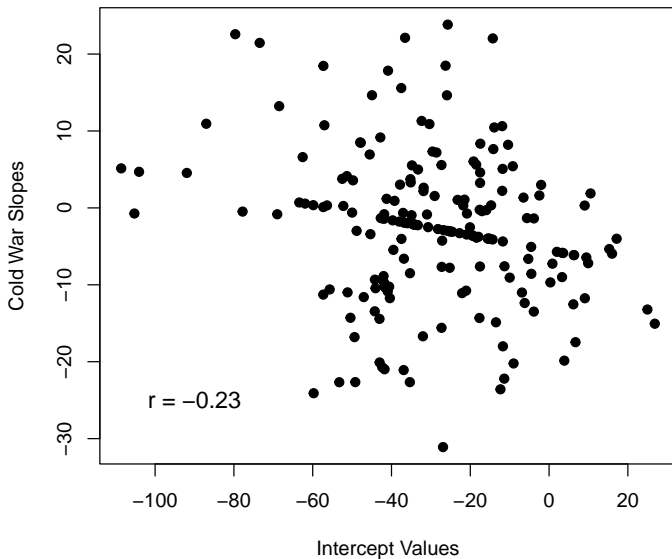
Random Intercepts (Plotted)



Random Slopes for CoIdWar (Plotted)



Scatterplot: Random Intercepts and Slopes



Separating Within and Between Effects

Recall that we can decompose the variance of our regression model as:

$$Y_{it} = \bar{\mathbf{X}}_i \beta_B + (\mathbf{X}_{it} - \bar{\mathbf{X}}_i) \beta_W + u_{it} \quad (5)$$

This raises the possibility of fitting the model in (5) directly...

- Simple to estimate (via OLS or other)
- Relatively easy interpretation
- Easy to test $\hat{\beta}_B = \hat{\beta}_W$

Example data: Separate effects for within- and between-country *Natural Resource Rents*...

Combining Within- and Between-Effects

Table: BE + WE Model of WBLI

	WEBE.OLS
Population Growth	−3.000*** (0.141)
Urban Population	−0.046*** (0.011)
Fertility Rate	−1.470*** (0.163)
ln(GDP Per Capita)	3.210*** (0.195)
Within-Country Nat. Resource Rents	0.090*** (0.030)
Between-Country Nat. Resource Rents	−0.515*** (0.019)
Cold War	−11.800*** (0.365)
Constant	53.500*** (1.700)
Observations	7,566
R ²	0.538
Adjusted R ²	0.537
Residual Std. Error	12.800 (df = 7558)
F Statistic	1,256.000*** (df = 7; 7558)

* p<0.1; ** p<0.05; *** p<0.01

Two-Way Unit Effects

Our original decomposition considered “two-way” effects:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \alpha_i + \eta_t + u_{it}$$

This implies that we can use (e.g.) an F -test to examine the hypothesis:

$$H_0 : \alpha_i = \eta_t = 0 \ \forall \ i, \ t$$

...that is, whether adding the (two-way) effects improves the model's fit.

We can also consider the partial hypotheses:

$$H_0 : \alpha_i = 0 \ \forall \ i$$

and

$$H_0 : \eta_t = 0 \ \forall \ t$$

separately.

Two-Way Effects: Good & Bad

The Good:

- Addresses both time-invariant, unit-level heterogeneity and cross-sectionally-invariant, time-point-specific heterogeneity
- May be “fixed” or “random” ...
- Two-way FE is equivalent to differences-in-differences when $X \in \{0, 1\}$ and $T = 2$ (more on that later)

The (Potentially) Bad:

- Ability to control for unobserved heterogeneity depends on (probably invalid) functional form assumptions (Imai and Kim 2020)
- FE *requires* predictors that vary both within and between both units and time points
- RE requires the (additional) assumption that $\text{Cov}(\mathbf{X}_{it}, \eta_t) = \text{Cov}(\alpha_i, \eta_t) = 0$
- Two-way effects models ask a *lot* of your data (effectively fits $N + T + k$ parameters using NT observations)

Example: Two-Way Fixed Effects

```
> TwoWayFE<-plm(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+               log(GDPPerCapita)+NaturalResourceRents+ColdWar,data=WDI,
+               effect="twoway",model="within")
```

```
> summary(TwoWayFE)
Twoways effects Within Model
```

```
Call:
plm(formula = WomenBusLawIndex ~ PopGrowth + UrbanPopulation +
     FertilityRate + log(GDPPerCapita) + NaturalResourceRents +
     ColdWar, data = WDI, effect = "twoway", model = "within")
```

```
Unbalanced Panel: n = 186, T = 1-50, N = 7566
```

```
Residuals:
    Min. 1st Qu.  Median 3rd Qu.    Max.
-31.756  -3.866   0.251   4.100  29.715
```

```
Coefficients:
                Estimate Std. Error t-value Pr(>|t|)
PopGrowth        -0.26475   0.10023   -2.64  0.0083 **
UrbanPopulation   -0.00333   0.01798   -0.19  0.8529
FertilityRate      1.03517   0.15072    6.87 7e-12 ***
log(GDPPerCapita)  2.50015   0.28544    8.76 <2e-16 ***
NaturalResourceRents -0.00420   0.01588   -0.26  0.7913
```

```
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
Total Sum of Squares:    306000
Residual Sum of Squares: 300000
R-Squared:                0.0179
Adj. R-Squared:          -0.0141
F-statistic: 26.7123 on 5 and 7326 DF, p-value: <2e-16
```

Two-Way Effects: Testing

```
> # Two-way effects:
```

```
> pFtest(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+  
+       log(GDPPerCapita)+NaturalResourceRents+ColdWar,data=WDI,  
+       effect="twoway",model="within")
```

```
F test for twoways effects
```

```
data: WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate + ...  
F = 103, df1 = 233, df2 = 7326, p-value <2e-16  
alternative hypothesis: significant effects
```

```
> plmtest(TwoWayFE,c("twoways"),type=("kw"))
```

```
Lagrange Multiplier Test - two-ways effects (King and Wu) for unbalanced panels
```

```
data: WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate + ...  
normal = 187, p-value <2e-16  
alternative hypothesis: significant effects
```

```
> # One-way effects in the two-way model:
```

```
>
```

```
> plmtest(TwoWayFE,c("individual"),type=("kw"))
```

```
Lagrange Multiplier Test - (King and Wu) for unbalanced panels
```

```
data: WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate + ...  
normal = 211, p-value <2e-16  
alternative hypothesis: significant effects
```

```
> plmtest(TwoWayFE,c("time"),type=("kw"))
```

```
Lagrange Multiplier Test - time effects (King and Wu) for unbalanced panels
```

```
data: WomenBusLawIndex ~ PopGrowth + UrbanPopulation + FertilityRate + ...  
normal = 100, p-value <2e-16  
alternative hypothesis: significant effects
```

Two-Way Fixed Effects via 1m

```
> TwoWayFE.BF<-lm(WomenBusLawIndex~PopGrowth+UrbanPopulation+
+ FertilityRate+log(GDPPerCapita)+NaturalResourceRents+
+ factor(ISO3)+factor(Year),data=WDI)
```

```
> summary(TwoWayFE.BF)
```

Call:

```
lm(formula = WomenBusLawIndex ~ PopGrowth + UrbanPopulation +
    FertilityRate + log(GDPPerCapita) + NaturalResourceRents +
    factor(ISO3) + factor(Year), data = WDI)
```

Residuals:

Min	1Q	Median	3Q	Max
-31.76	-3.87	0.25	4.10	29.72

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	-17.26971	2.50039	-6.91	5.4e-12 ***
PopGrowth	-0.26475	0.10023	-2.64	0.0083 **
UrbanPopulation	-0.00333	0.01798	-0.19	0.8529
FertilityRate	1.03517	0.15072	6.87	7.0e-12 ***
log(GDPPerCapita)	2.50015	0.28544	8.76	< 2e-16 ***
NaturalResourceRents	-0.00420	0.01588	-0.26	0.7913
factor(ISO3)AGO	29.30071	2.01487	14.54	< 2e-16 ***
factor(ISO3)ALB	51.72646	2.02286	25.57	< 2e-16 ***
factor(ISO3)ARE	-5.95970	2.54027	-2.35	0.0190 *
.				
.				
.				
factor(Year)1978	4.82791	0.88259	5.47	4.6e-08 ***
factor(Year)1979	5.23708	0.88928	5.89	4.1e-09 ***

```
[ reached getOption("max.print") -- omitted 40 rows ]
```

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1

Residual standard error: 6.4 on 7326 degrees of freedom

(5764 observations deleted due to missingness)

Multiple R-squared: 0.888, Adjusted R-squared: 0.884

F-statistic: 242 on 239 and 7326 DF, p-value: <2e-16

Example: Two-Way Random Effects

```
> TwoWayRE<-plm(WomenBusLawIndex~PopGrowth+UrbanPopulation+FertilityRate+
+               log(GDPPerCapita)+NaturalResourceRents+ColdWar,data=WDI,
+               effect="twoway",model="random")
```

```
> summary(TwoWayRE)
```

Twoways effects Random Effect Model
(Swamy-Arora's transformation)

Call:
plm(formula = WomenBusLawIndex ~ PopGrowth + UrbanPopulation +
 FertilityRate + log(GDPPerCapita) + NaturalResourceRents +
 ColdWar, data = WDI, effect = "twoway", model = "random")

Unbalanced Panel: n = 186, T = 1-50, N = 7566

Effects:

	var	std.dev	share
idiosyncratic	40.993	6.403	0.31
individual	89.185	9.444	0.68
time	0.612	0.782	0.00

theta:

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
id	0.439	0.893	0.905	0.896	0.905	0.905
time	0.372	0.423	0.469	0.449	0.477	0.479
total	0.330	0.422	0.468	0.448	0.476	0.478

Residuals:

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
	-61.3	-6.4	2.4	0.4	10.6	32.2

Coefficients:

	Estimate	Std. Error	z-value	Pr(> z)
(Intercept)	21.29333	0.35391	60.17	< 2e-16 ***
PopGrowth	-0.23814	0.01549	-15.37	< 2e-16 ***
UrbanPopulation	0.06588	0.00254	25.97	< 2e-16 ***
FertilityRate	-0.58746	0.02216	-26.51	< 2e-16 ***
log(GDPPerCapita)	5.02484	0.03939	127.55	< 2e-16 ***
NaturalResourceRents	-0.01087	0.00239	-4.54	0.0000056 ***
ColdWar	-12.66453	0.05391	-234.91	< 2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Total Sum of Squares: 2670000

Residual Sum of Squares: 1740000

R-Squared: 0.351

Adj. R-Squared: 0.351

Chisq: 161011 on 6 DF, p-value: <2e-16

Table: Models of WBLI

	OLS	FE	BE	RE	TwoWayFE	TwoWayRE
Population Growth	-3.190*** (0.144)	0.073 (0.119)	-5.870*** (0.917)	-0.073 (0.121)	-0.265*** (0.100)	-0.238*** (0.015)
Urban Population	-0.058*** (0.011)	0.248*** (0.021)	-0.050 (0.055)	0.128*** (0.019)	-0.003 (0.018)	0.066*** (0.003)
Fertility Rate	-1.790*** (0.165)	-2.070*** (0.166)	0.063 (1.080)	-2.370*** (0.165)	1.030*** (0.151)	-0.587*** (0.022)
ln(GDP Per Capita)	3.150*** (0.199)	9.160*** (0.310)	3.400*** (1.130)	7.510*** (0.293)	2.500*** (0.285)	5.030*** (0.039)
Natural Resource Rents	-0.349*** (0.016)	0.035* (0.018)	-0.358*** (0.089)	0.006 (0.018)	-0.004 (0.016)	-0.011*** (0.002)
Cold War	-11.300*** (0.372)	-7.190*** (0.295)	-12.500** (4.890)	-8.480*** (0.289)		-12.700*** (0.054)
Constant	54.800*** (1.730)		50.300*** (10.200)	2.560 (2.580)		21.300*** (0.354)
Observations	7,566	7,566	186	7,566	7,566	7,566
R ²	0.519	0.535	0.605	0.520	0.018	0.351
Adjusted R ²	0.519	0.523	0.592	0.520	-0.014	0.351

* p<0.1; ** p<0.05; *** p<0.01

“Fixed Effects Individual Slope” models

- Cite: Bruederl, Josef, and Volker Ludwig. 2015. “Fixed-Effects Panel Regression.” In *The Sage Handbook of Regression Analysis and Causal Inference*, Eds. Henning Best and Christof Wolf. Los Angeles: Sage, pp. 327-357.
- FE + unit-level slopes for (some / all) predictor variables
- Equivalent to including $N - 1$ interactions between a predictor \mathbf{X} and each of the α_i s
- Also can test for homogeneity of estimated slopes (Hausman-like test)
- See the `feisr` R package, and its accompanying vignette, or `xtfeis` in Stata

Unit Effects Models: Software

R :

- the `plm` package; `plm` command
 - Fits one- and two-way FE, BE, RE models
 - Also fits first difference (FD) and instrumental variable (IV) models
- the `fixest` package; fast/scalable FE estimation for OLS and GLMs
- the `lme4` package; command is `lmer`
- the `nlme` package; command `lme`
- the `Paneldata` package

Stata : `xtreg`

- option `re` (the default) = random effects
- option `fe` = fixed (within) effects
- option `be` = between-effects
- Stata `package` `fect` = two-way models