Econ 722: Time Series
Midterm 1

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1. Suppose that $y_t = \phi y_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}$, where ε_t is i.i.d. with mean zero and variance σ^2, and $\theta, \phi \in (-1, 1)$. Consider the instrumental variable estimator

$$\widetilde{\phi} = \frac{\sum_{t=3}^{T} y_{t-2}y_t}{\sum_{t=3}^{T} y_{t-2}y_{t-1}}$$

Derive the limiting distribution of $\widetilde{\phi}$, stating clearly any additional assumptions you need. Then let $\widetilde{\varepsilon}_t = y_t - \widetilde{\phi}y_{t-1}$, and propose an estimator of θ, σ^2.

Consider $y_t = \phi y_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}$

let $u_t = \varepsilon_t - \theta\varepsilon_{t-1}$

$\Rightarrow y_t = \phi y_{t-1} + u_t$.

It is easy to see that $E(y_{t-1}u_t) \neq 0$ since $E(y_{t-1}\varepsilon_{t-1}) = \sigma^2\theta \neq 0$.

As such choose y_{t-2} as an instrument for y_{t-1}. Using the 2-step regression method;

1) Reg. y_{t-1} on y_{t-2} i.e. $y_{t-1} = \delta y_{t-2} + e_t$

$$\Rightarrow \hat{\delta} = \frac{\sum_{t=3}^{T} y_{t-2}y_{t-1}}{\sum_{t=3}^{T} y_{t-2}^2}.$$

2) Now we have $\begin{cases} y_t = \lambda y_{t-2} + \phi e_t + u_t \; ; \; \lambda = \phi\delta \\ u_t = \varepsilon_t - \theta\varepsilon_{t-1} \end{cases}$

Following Hamilton; we can iteratively compute $\hat{\lambda}$ and $\hat{\theta}$ such that $\hat{\lambda}$ is close to the OLS estimate i.e.

$$\hat{\lambda} = \frac{\sum_{t=3}^{T} y_{t-2}y_t}{\sum_{t=3}^{T} y_{t-2}^2} \quad \text{then} \quad \hat{\phi} = \frac{\hat{\lambda}}{\hat{\delta}} = \frac{\sum_{t=3}^{T} y_{t-2}y_t}{\sum_{t=3}^{T} y_{t-2}y_{t-1}}$$

$$\text{Now } (\hat{\widetilde{\phi}} - \phi) = \frac{\sum_{t=3}^{T} y_{t-2}u_t}{\sum_{t=3}^{T} y_{t-2}y_{t-1}}$$

$$\sqrt{T}\left(\hat{\phi}-\phi\right)=\frac{\frac{1}{\sqrt{T}}\sum_{t=3}^{T} y_{t-2}u_t}{\frac{1}{T}\sum_{t=3}^{T} y_{t-2}y_{t-1}}$$

To apply Asymptotic properties; we need some assumptions

for Law of Large Numbers:

If X_t is Strictly stationary and ergodic with $E\|y\|<\infty$

Then $\bar{X} \xrightarrow[p]{} \mu$. from Hansen.

Assume Y_t is Strictly stationary and Ergodic then

$$\frac{1}{(T-3)}\sum_{t=3}^{T} Y_{t-2}Y_{t-1} \xrightarrow[p]{} \mathbb{E}\left(Y_{t-2}Y_{t-1}\right)=\gamma_1$$

But $\gamma_1=\mathbb{E}(y_{t-1}y_{t-2})=\mathbb{E}\left(y_{t-2}\left(\phi y_{t-2}+u_{t-1}\right)\right)=\phi\,\mathbb{E}(y_{t-2}^2)+\mathbb{E}(y_{t-2}u_{t-1})$

But $\mathbb{E}(y_{t-2}u_{t-1})=0$

Since y_t is Covariance Stationary, then

$$Var(y_t)=\phi\,Var(y_{t-1})+Var(u_t) \Leftrightarrow (1-\phi)\gamma_0=\sigma^2-\theta^2\sigma^2$$

$$\therefore\ \mathbb{E}(y_{t-2}^2)=\gamma_0=\frac{\sigma^2(1-\theta^2)}{1-\phi}.$$

Hence $\mathbb{E}(y_{t-2}y_{t-1})=\phi\gamma_0=\dfrac{\phi\sigma^2(1-\theta^2)}{1-\phi}$

Hence $\frac{1}{T}\sum y_{t-2}y_{t-1} \xrightarrow[p]{} \frac{\phi}{1-\phi}\sigma^2(1-\theta^2).$ —— (1)

Now to discuss Asymptotic dist; we need to show that u_t is a MDS.

Recall that $u_t = \varepsilon_t - \theta \varepsilon_{t-1}$, where ε_t is iid with mean zero and variance σ^2.

So clearly ε_t is a MDS.

If the Info. Set includes y_{t-1}, then u_t will not be a MDS since $\mathbb{E}(u_t | y_{t-1}) = -\theta \varepsilon_{t-1} \neq 0$.

However if we can show that u_t satisfy strong mixing (α-mixing) then we can use the CLT for strong-mixing.

Since $u_t = \varepsilon_t - \theta \varepsilon_{t-1}$. Assume that ε_t has strong mixing or ε_t is a white noise.

Then since u_t a MA(1), u_t satisfy strong mixing then

$$\frac{1}{\sqrt{T}} \sum_{t=3}^{T} y_{t-2} u_t \sim N(0, \Omega) ;$$ where Ω = Longrun variance.

$$\text{i.e. } \Omega = \sum_{l=-\infty}^{\infty} \mathbb{E}\left(y_{t-2-l}\, y_{t-2}\, u_t u_{t-l}\right)$$

$$= \gamma(0) + 2 \sum_{l=1}^{\infty} \gamma(l)$$

where $\gamma(l) = \mathbb{E}\left(y_{t-2-l}\, y_{t-2}\, u_t u_{t-l}\right)$ and

$$\gamma(0) = \mathbb{E}\left(y_{t-2}^2 u_{t-2}^2\right) = \mathbb{E}\left(\mathbb{E}(u_t^2 | y_{t-2})\, y_{t-2}^2\right)$$

But $\mathbb{E}(u_t^2 | y_{t-2}) = \mathbb{E}\left(\varepsilon_t^2 - 2\theta \varepsilon_t \varepsilon_{t-1} + \theta^2 \varepsilon_{t-1}^2 \,|\, y_{t-2}\right) = (1+\theta^2)\sigma^2$

$$\Rightarrow \mathbb{E}(y_{t-2}^2 u_t^2) = (1+\theta^2)\sigma^2 \mathbb{E}(y_{t-2}^2) = (1+\theta^2)\sigma^2 \gamma_0 = \sigma^2(1+\theta^2)\frac{\sigma^2(1-\theta^2)}{1-\phi} = \frac{\sigma^4(1-\theta^4)}{1-\phi}$$

Now $\gamma(1) = \mathbb{E}\left(y_{t-3} y_{t-2}\, \mathbb{E}(u_t u_{t-1} | y_{t-3})\right) = \mathbb{E}\left(y_{t-3} y_{t-2}\, \mathbb{E}(-\theta \varepsilon_{t-1}^2 | y_{t-3})\right)$

$$= -\theta\sigma^2 \mathbb{E}(y_{t-3} y_{t-2}) = -\theta\sigma^2 \mathbb{E}\left(\phi y_{t-3}^2 + y_{t-3} u_{t-2}\right) = -\frac{\sigma^4 \theta \phi (1-\theta^2)}{1-\phi}$$

But for $l \geq 2$: Since $E(u_t u_{t-l}) = 0$ Then $\gamma(l) = 0 \ \forall\, l \geq 2$.

Hence
$$\Omega = \gamma(0) + 2\gamma(1)$$
$$= \frac{\sigma^4(1-\theta^4)}{1-\phi} - \frac{2\sigma^4\theta(1-\theta^2)\phi}{1-\phi}$$
$$= \frac{\sigma^4(1-\theta^2)(1+\theta^2) - 2\sigma^4(1-\theta^2)\theta\phi}{1-\phi}$$

$$\Omega = \frac{\sigma^4(1-\theta^2)}{1-\phi}\left[1+\theta^2 - 2\theta\phi\right]$$

Hence $\frac{1}{\sqrt{T}}\sum y_{t-2} u_t \sim N(0, \Omega)$ —— (2)

and $\frac{1}{T}\sum y_{t-2} y_{t-1} \xrightarrow{p} \frac{\phi}{1-\phi}\sigma^2(1-\theta^2) = A$

Then by slutsky; we have

$$\sqrt{T}(\hat{\phi} - \phi) \sim N(0, V)$$

where
$$V = \left[\frac{\phi^2}{(1-\phi)^2}\sigma^4(1-\theta^2)^2\right]\left[\frac{\sigma^4(1-\theta^2)}{(1-\phi)}(1-2\phi\theta+\theta^2)\right]$$

$$V = \frac{\phi^2}{(1-\phi)^3}(1-\theta^2)^3\sigma^8(1-2\phi\theta+\theta^2).$$

(b) Let $\hat{\varepsilon}_t = y_t - \hat{\phi} y_{t-1}$ i.e. $\hat{\varepsilon}_t$ are the residual from the regression $y_t = \phi y_{t-1} + u_t$ i.e. $\hat{u}_t = \hat{\varepsilon}_t$.

Since $\hat{\varepsilon}_t = \varepsilon_t - \theta \varepsilon_{t-1}$ is a MA(1) model. The best way to estimate θ with MOM estimators i.e.

$$\gamma_0 = E(u_t^2) = (1+\theta^2)\sigma^2 = \frac{1}{T}\sum_{t=2}^{T} \hat{\varepsilon}_t^2 \longrightarrow (1)$$

$$\gamma_1 = E(u_t u_{t-1}) = -\theta\sigma^2 = \frac{1}{T}\sum \hat{\varepsilon}_t \hat{\varepsilon}_{t-1}^2 \longrightarrow (2)$$

Solving this 2 system of moment equations gives us an estimator for $\hat{\theta}$ and $\hat{\sigma}^2$.

2. Suppose that $y_s^* = \phi y_{s-1}^* + \varepsilon_s$, where ε_s is i.i.d. with mean zero and variance σ^2, while time is daily. Suppose that we only observe weekly observations (five days in a week) $y_1 = \sum_{s=1}^{5} y_s^*$, $y_2 = \sum_{s=6}^{10} y_s^*$, and so on. Derive a representation for y_t as an ARMA process.

let $y_s^* = \phi y_{s-1}^* + \varepsilon_s$; ε_s is iid $(0, \sigma^2)$

So we only observe weekly observations such that

$$y_1 = \sum_{s=1}^{5} y_s^*, \quad y_2 = \sum_{s=6}^{10} y_s^*, \quad y_3 = \sum_{s=11}^{15} y_s^*, \ldots$$

$$\Rightarrow y_2 = \sum_{s=5(2-1)+1}^{5(2)} y_s^*, \quad y_3 = \sum_{s=5(3-1)+1}^{5(3)} y_s^*, \quad y_4 = \sum_{s=5(4-1)+1}^{5(4)} y_s^*$$

$$\Rightarrow y_t = \sum_{s=5(t-1)+1}^{5(t)} y_s^*.$$

Start with Basic; we know ~~that~~ i.e. $y_1^* = y_0^* + \varepsilon_1$

$$y_2^* = \phi y_1^* + \varepsilon_2 = \phi y_0^* + \phi \varepsilon_1 + \varepsilon_2$$

$$y_3^* = \phi^2 y_1^* + \phi \varepsilon_2 + \varepsilon_3 = \phi^2 y_0^* + \phi^2 \varepsilon_1 + \phi \varepsilon_2 + \varepsilon_3$$

$$y_4^* = \phi^3 y_0^* + \phi^3 \varepsilon_1 + \phi^2 \varepsilon_2 + \phi \varepsilon_3 + \varepsilon_4$$

$$y_5^* = \phi^4 y_0^* + \phi^4 \varepsilon_1 + \phi^3 \varepsilon_2 + \phi^2 \varepsilon_3 + \phi \varepsilon_4 + \varepsilon_5$$

$$\Rightarrow y_1 = \sum_{s=1}^{5} y_s^* = (1 + \phi + \phi^2 + \phi^3 + \phi^4) y_0^* + (1 + \phi + \phi^2 + \phi^3 + \phi_4)\varepsilon_1$$
$$(1 + \phi + \phi^2 + \phi^3)\varepsilon_2 + (1 + \phi + \phi^2)\varepsilon_3 + (1+\phi)\varepsilon_4 + \varepsilon_5$$

$$\Rightarrow y_1 = \psi_1 y_0^* + \theta_1 \varepsilon_1 + \theta_2 \varepsilon_2 + \theta_3 \varepsilon_3 + \theta_4 \varepsilon_4 + \varepsilon_5$$

hence $y_1 \sim ARMA(1, 5)$

By extrapolation, we can see that $y_t \sim ARMA$

1. Using the US Dollar to Euro spot exchange not seasonally adjusted and at a monthly frequency:

   (a) To optimally choose the best ARMA model that best fits the data, use the information criteria for model selection. This allows you to select the model that best fits data by running the ARMA model for different combinations of (p,q) and selecting the model that minimizes the criterion(MSE) error. In my case, I plotted the ACF and PACF to get a good sense of the maximum number of lags to choose i.e. (p_{max}, q_{max}). So choosing the maximum lags to be (5,5), I run an ARMA model for different combinations of p and q, compute the BIC value for each model, and then select the model with the minimum BIC which turns out to be (p,q) = (1,1).
   We then run an ARMA(1,1) model which is the optimal model whose results are represented in table (1).

Parameter	**Estimate**	**Std. Error**	**z value**	**Pr(>|z|)**
c	0.00317336	0.00260903	1.2163	0.2239
ϕ_1	0.979333	0.013225	74.0516	$< 1e{-}99$
θ_1	0.316292	0.0458835	6.89337	$< 1e{-}11$

Table 1: Estimation of ARMA(1,1) model

   (b) Compute one, two, and three steps ahead forecast for the exchange rate using your model in (a).
   Answer:

Step	**Forecast**
1	0.715422
2	0.00269936
3	0.00269936

Table 2: Forecasts for Each Step

   (c) Run a regression of y_t on y_{t-1}. To an approximation, this can be considered an estimate of the largest root of y_t. What is your estimate? Comment.
   Answer:
   Consider the regression

$$y_t = \alpha + \beta y_{t-1} + \varepsilon_t \tag{1}$$

   We obtain the results in table (**??**). We can observe that the coefficient or the root is around 0.986 which is quite close to 1. Since this estimate is not exactly 1, one can argue that our data is not a unit root process, so modeling the level data with an ARMA model seems consistent with the model's assumptions.

Parameter	Coef.	Std. Error	t	Pr(>|t|)	Lower 95%	Upper 95%
α	0.00195433	0.00196396	1.00	0.3205	-0.00191029	0.00581896
β	0.986909	0.00936303	105.40	$< 1 \times 10^{-99}$	0.968485	1.00533

(d) Test the data for a unit root. Choose carefully which test to use and which option. Defend your choices and comment on the results.

Answer:

We test the data for a unit root using the Augmented Dickey-Fuller test. Although the standard Dickey-Fuller test is an option. it assumes no autocorrelation in the error terms of the regression. This can make our results biased.

Additionally, the Augment Dickey Fully test allows the inclusion of deterministic trends as the plot of the Exchange rates exhibits some trending behavior.

A much more powerful test will be the ADF-GLS test which actually detrends the series before testing for the unit root and hence improves upon the precision of the test. However, I stick with the Augmented Dickey-Fuller test since it is the standard test that is widely used and available in the JULIA programming language.

$$\mathbb{H}_0 : \rho = 1$$
$$\mathbb{H}_1 : \rho < 1$$

Table (3) represents the test for the unit root using the ADF. The p-value recorded is about 0,65 showing that we fail to reject the null. So the test shows that the Exchange rate series is actually a unit root process and so our estimating the series with an ARMA model is not the correct specification. One way to solve this problem is to model the growth of the EXCHANGE rate series.

(e) What does your result in d) and e) say about your model in a)? Comment.

Since the US-euro Exchange rate follows a unit root process, then it means that our process is not stationary which is the required assumption for the ARMA model. So by running an ARMA model, we violate the assumption of the model and the results might be spurious. One way around this is to model the growth of the exchange rate series that is we compute the first difference. This transformation converts a unit root process into a stationary one, making the series suitable for ARMA modeling or other stationary time series models.

(f) Go to FRED database and download the dollar to the euro exchange rate at a daily frequency. Estimate the best ARMA model for the daily exchange rate. How does

Test Summary	
Population details:	
Parameter of interest:	Coefficient on lagged non-differenced variable
Value under H_0:	0
Point estimate:	-0.0172752
Test summary:	
Outcome with 95% confidence:	Fail to reject H_0
p-value:	0.6454
Details:	
Sample size in regression:	305
Number of lags:	1
ADF statistic:	-1.91772
Critical values at 1%, 5%, and 10%:	[-3.98857, -3.42489, -3.13551]

Table 3: Augmented Dickey-Fuller Unit Root Test Results

this model compare with the model for the monthly data? Comment.

Table 4 shows the optimal ARMA model which is an ARMA(1,0) model for the daily exchange rate series. That is by the BIC criterion, the best model for the daily exchange rate series is an AR(1) model. This makes sense since the current past value contains all the information of all the past value of the prices and so that alone can explain the behavior of the series. The estimate is close to the ARMA(1,1) model of the monthly series.

Parameter	Estimate	Std. Error	z value	Pr(>|z|)
c	0.000435259	0.000363129	1.19864	0.2307
ϕ_1	0.995934	0.00288781	344.876	$< 1e{-99}$

Table 4: Mean Equation Parameters