1. Suppose that $y_t = \phi y_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}$, where ε_t is i.i.d. with mean zero and variance σ^2, and $\theta, \phi \in (-1, 1)$. Consider the instrumental variable estimator

$$\widetilde{\phi} = \frac{\sum_{t=3}^{T} y_{t-2}y_t}{\sum_{t=3}^{T} y_{t-2}y_{t-1}}$$

Derive the limiting distribution of $\widetilde{\phi}$, stating clearly any additional assumptions you need. Then let $\widetilde{\varepsilon}_t = y_t - \widetilde{\phi}y_{t-1}$, and propose an estimator of θ, σ^2.

Consider $y_t = \phi y_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}$

let $u_t = \varepsilon_t - \theta\varepsilon_{t-1}$

$\Rightarrow y_t = \phi y_{t-1} + u_t$.

It is easy to see that $E(y_{t-1}u_t) \neq 0$ since $E(y_{t-1}\varepsilon_{t-1}) = \sigma^2\theta \neq 0$.

As such choose y_{t-2} as an instrument for y_{t-1}. Using the 2-step regression method:

1) Reg. y_{t-1} on y_{t-2} i.e. $y_{t-1} = \delta y_{t-2} + e_t$

$$\Rightarrow \hat{\delta} = \frac{\sum_{t=3}^{T} y_{t-2}y_{t-1}}{\sum_{t=3}^{T} y_{t-2}^2}$$

(2) Now we have $\begin{cases} y_t = \lambda y_{t-2} + \phi e_t + u_t & : \lambda = \phi\delta \\ u_t = \varepsilon_t - \theta\varepsilon_{t-1} \end{cases}$

Following Hamilton; we can iteratively compute $\hat{\lambda}$ and $\hat{\theta}$ such that $\hat{\lambda}$ is close to the OLS estimate i.e.

$$\hat{\lambda} = \frac{\sum_{t=3}^{T} y_{t-2}y_t}{\sum_{t=3}^{T} y_{t-2}^2} \quad \text{then} \quad \hat{\phi} = \frac{\hat{\lambda}}{\hat{\delta}} = \frac{\sum_{t=3}^{T} y_{t-2}y_t}{\sum_{t=3}^{T} y_{t-2}y_{t-1}}$$

Now

$$(\hat{\tilde{\phi}} - \phi) = \frac{\sum_{t=3}^{T} y_{t-2}u_t}{\sum_{t=3}^{T} y_{t-2}y_{t-1}}$$

$$\sqrt{T}\left(\hat{\phi}-\phi\right) = \frac{\frac{1}{\sqrt{T}}\sum_{t=3}^{T} y_{t-2}u_t}{\frac{1}{T}\sum_{t=3}^{T} y_{t-2}y_{t-1}}$$

To apply Asymptotic Properties; we need some assumptions

for Law of Large Numbers:

If X_t is Strictly stationary and ergodic with $\mathbb{E}\|Y\| < \infty$

Then $\bar{X} \xrightarrow[p]{} \mu$. from Hansen.

Assume Y_t is Strictly stationary and Ergodic then

$$\frac{1}{(T-3)}\sum_{t=3}^{T} Y_{t-2}Y_{t-1} \xrightarrow[p]{} \mathbb{E}\left(Y_{t-2}Y_{t-1}\right) = \gamma_1$$

But $\gamma_1 = \mathbb{E}(y_{t-1}y_{t-2}) = \mathbb{E}\left(y_{t-2}\left(\phi y_{t-2} + u_{t-1}\right)\right) = \phi\,\mathbb{E}(y_{t-2}^2) + \mathbb{E}(y_{t-2}u_{t-1})$

But $\mathbb{E}(y_{t-2}u_{t-1}) = 0$

Since y_t is Covariance Stationary, then

$$Var(y_t) = \phi Var(y_{t-1}) + Var(u_t) \Leftrightarrow (1-\phi)\gamma_0 = \sigma^2 - \theta^2\sigma^2$$

$$\therefore \mathbb{E}(y_{t-2}^2) = \gamma_0 = \frac{\sigma^2(1-\theta^2)}{1-\phi}.$$

Hence $\mathbb{E}\left(y_{t-2}y_{t-1}\right) = \phi\gamma_0 = \dfrac{\phi\sigma^2(1-\theta^2)}{1-\phi}$

Hence $\frac{1}{T}\sum y_{t-2}y_{t-1} \xrightarrow[p]{} \frac{\phi}{1-\phi}\sigma^2(1-\theta^2)$. —— (i)

Now to discuss Asymptotic dist; we need to show that u_t is a MDS.

Recall that $u_t = \varepsilon_t - \theta\,\varepsilon_{t-1}$, where ε_t is iid with mean zero and variance σ^2.

So clearly ε_t is a MDS.

If the Info. set includes y_{t-1}, then u_t will not be a MDS since $E(u_t \mid y_{t-1}) = -\theta\,\varepsilon_{t-1} \neq 0$.

However if we can show that u_t satisfy strong mixing (α-mixing) then we can use the CLT for strong-mixing.

Since $u_t = \varepsilon_t - \theta\,\varepsilon_{t-1}$. Assume that ε_t has strong mixing or ε_t is a white noise.

Then since u_t a MA(1), u_t satisfy strong mixing then

$$\frac{1}{\sqrt{T}} \sum_{t=3}^{T} y_{t-2}\, u_t \sim N(0, \Omega);$$ where Ω = Longrun variance.

$$\text{i.e. } \Omega = \sum_{l=-\infty}^{\infty} E\left(y_{t-2-l}\, y_{t-2}\, u_t\, u_{t-l}\right)$$

$$= \gamma(0) + 2\sum_{l=1}^{\infty} \gamma(l)$$

where $\gamma(l) = E\left(y_{t-2-l}\, y_{t-2}\, u_t\, u_{t-l}\right)$ and

$$\gamma(0) = E\left(y_{t-2}^2\, u_{t-2}^2\right) = E\left(E(u_t^2 \mid y_{t-2})\, y_{t-2}^2\right)$$

But $E(u_t^2 \mid y_{t-2}) = E\left(\varepsilon_t^2 - 2\theta\,\varepsilon_t\varepsilon_{t-1} + \theta^2\varepsilon_{t-1}^2 \mid y_{t-2}\right) = (1+\theta^2)\,\sigma^2$

$$\Rightarrow E\left(y_{t-2}^2 u_t^2\right) = (1+\theta^2)\,\sigma^2\, E\left(y_{t-2}^2\right) = (1+\theta^2)\,\sigma^2\,\gamma_0 = \sigma^2(1+\theta^2)\frac{\sigma^2(1-\theta^2)}{1-\phi} = \frac{\sigma^4(1-\theta^4)}{1-\phi}$$

Now $$\gamma(1) = E\left(y_{t-3}\, y_{t-2}\, E(u_t u_{t-1} \mid y_{t-3})\right) = E\left(y_{t-3}\, y_{t-2}\, E\left(-\theta\,\varepsilon_{t-1}^2 \mid y_{t-3}\right)\right)$$

$$-\theta\sigma^2 E\left(y_{t-3}\, y_{t-2}\right) = -\theta\sigma^2 E\left(\phi\, y_{t-3}^2 + y_{t-3}\, u_{t-2}\right) = -\frac{\sigma^4\theta\phi\,(1-\theta^2)}{1-\phi}$$

But for $l \geq 2$: Since $\mathbb{E}(u_t u_{t-l}) = 0$ Then $\gamma(l) = 0 \ \forall\, l \geq 2$.

Hence
$$\Omega = \gamma(0) + 2\gamma(1)$$
$$= \frac{\sigma^4(1-\theta^4)}{1-\phi} - \frac{2\sigma^4\theta(1-\theta^2)\phi}{1-\phi}$$
$$= \frac{\sigma^4(1-\theta^2)(1+\theta^2) - 2\sigma^4(1-\theta^2)\theta\phi}{1-\phi}$$
$$\Omega = \frac{\sigma^4(1-\theta^2)}{1-\phi}\left[1+\theta^2 - 2\theta\phi\right]$$

Hence $\frac{1}{\sqrt{T}}\sum y_{t-2} u_t \sim N(0, \Omega)$ ——— (2)

and $\frac{1}{T}\sum y_{t-2} y_{t-1} \xrightarrow{p} \frac{\phi}{1-\phi}\sigma^2(1-\theta^2) = A$

Then by Slutsky; we have

$$\sqrt{T}(\hat{\phi}-\phi) \sim N(0, V)$$

where
$$V = \left[\frac{\phi^2}{(1-\phi)^2}\sigma^4(1-\theta^2)^2\right]\left[\frac{\sigma^4(1-\theta^2)}{(1-\phi)}(1-2\phi\theta+\theta^2)\right]$$

$$V = \frac{\phi^2}{(1-\phi)^3}(1-\theta^2)^3\sigma^8(1-2\phi\theta+\theta^2).$$

(b) Let $\hat{\varepsilon}_t = y_t - \hat{\phi} y_{t-1}$ i.e. $\hat{\varepsilon}_t$ are the residual from the regression $y_t = \phi y_{t-1} + u_t$ i.e. $\hat{u}_t = \hat{\varepsilon}_t$.

Since $\hat{\varepsilon}_t = \varepsilon_t - \theta \varepsilon_{t-1}$ is a MA(1) model. The best way to estimate θ will MoM estimators i.e.

$$\gamma_0 = E(u_t^2) = (1+\theta^2)\sigma^2 = \frac{1}{T}\sum_{t=2}^{T} \hat{\varepsilon}_t^2 \longrightarrow (1)$$

$$\gamma_1 = E(u_t u_{t-1}) = -\theta\sigma^2 = \frac{1}{T}\sum \hat{\varepsilon}_t \hat{\varepsilon}_{t-1}^2 \longrightarrow (2)$$

Solving this 2 system of moment equations gives us an estimator for $\hat{\theta}$ and $\hat{\sigma}^2$.

2. Suppose that $y_s^* = \phi y_{s-1}^* + \varepsilon_s$, where ε_s is i.i.d. with mean zero and variance σ^2, while time is daily. Suppose that we only observe weekly observations (five days in a week) $y_1 = \sum_{s=1}^{5} y_s^*$, $y_2 = \sum_{s=6}^{10} y_s^*$, and so on. Derive a representation for y_t as an ARMA process.

Let $y_s^* = \phi y_{s-1}^* + \varepsilon_s$; ε_s is iid $(0, \sigma^2)$

So we only observe weekly observations such that

$$y_1 = \sum_{s=1}^{5} y_s^*, \quad y_2 = \sum_{s=6}^{10} y_s^*, \quad y_3 = \sum_{s=11}^{15} y_s^*, \dots$$

$$\Rightarrow y_2 = \sum_{s=5(2-1)+1}^{5(2)} y_s^*, \quad y_3 = \sum_{s=5(3-1)+1}^{5(3)} y_s^*, \quad y_4 = \sum_{5(4-1)+1}^{5(4)} y_s^*$$

$$\Rightarrow y_t = \sum_{s=5(t-1)+1}^{5(t)} y_s^*.$$

Start with Basic; we know ~~that~~ i.e. $y_1^* = y_0^* + \varepsilon_1$

$$y_2^* = \phi y_1^* + \varepsilon_2 = \phi y_0^* + \phi\varepsilon_1 + \varepsilon_2$$

$$y_3^* = \phi^2 y_1^* + \phi\varepsilon_2 + \varepsilon_3 = \phi^2 y_0^* + \phi^2\varepsilon_1 + \phi\varepsilon_2 + \varepsilon_3$$

$$y_4^* = \phi^3 y_0^* + \phi^3\varepsilon_1 + \phi^2\varepsilon_2 + \phi\varepsilon_3 + \varepsilon_4$$

$$y_5^* = \phi^4 y_0^* + \phi^4\varepsilon_1 + \phi^3\varepsilon_2 + \phi^2\varepsilon_3 + \phi\varepsilon_4 + \varepsilon_5$$

$$\Rightarrow y_1 = \sum_{s=1}^{5} y_s^* = (1+\phi+\phi^2+\phi^3+\phi^4)y_0^* + (1+\phi+\phi^2+\phi^3+\phi_4)\varepsilon_1$$
$$(1+\phi+\phi^2+\phi^3)\varepsilon_2 + (1+\phi+\phi^2)\varepsilon_3 + (1+\phi)\varepsilon_4 + \varepsilon_5$$

$$\Rightarrow y_1 = \psi_1 y_0^* + \theta_1\varepsilon_1 + \theta_2\varepsilon_2 + \theta_3\varepsilon_3 + \theta_4\varepsilon_4 + \varepsilon_5$$

hence $y_1 \sim ARMA(1, 5)$

By extrapolation, we can see that $y_t \sim ARMA$.