Econ 722: Time Series Assignment 1

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1. Why would you be surprised if I told you that y_t was MA(1) but that $corr(y_t, y_{t-1}) = 0.7$.

   Answer

   Let y_t be a stochastic process that admits an MA(1) process. $\Rightarrow y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$: where $\varepsilon_t \sim WN\left(0, v^2\right)$.
   It is straightforward to see that

$$\gamma_\theta = \sigma^2\left(1+\theta^2\right) \text{ and}$$
$$\gamma_j = \begin{cases} \theta\sigma^2; & j=1 \\ 0; & j>1 \end{cases}$$
$$\Rightarrow \text{cor}\left(y_t, y_{t-1}\right) = \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta}{1+\theta^2}$$

   Let θ_m represents the value of θ that maximizes ρ_1

$$\Rightarrow \theta_m = \underset{\theta}{\text{argmax}}\frac{\theta}{1+\theta^2}$$
$$\Rightarrow \theta_m = \pm 1$$

   Hence the maximum value that rho_1 can take is

$$rho_1 = \frac{1}{2}; \text{ where } \theta_m = 1$$

   So if y_t admits an $MA(1)$ process, then having Corr $\left(y_t, y_{t-1}\right) = 0.7$ is not possible. In other words, if $y_t \sim MA(1)$ process, the impulse response function has a short impact.

2. If x_t and y_t are each white noise, is $x_t + y_t$ white noise? Is this true in general?

   Answer:
   Let x_t and y_t be a white noise then by definition:
   $\mathbb{E}\left(x_t\right) = \mathbb{E}\left(y_t\right) = 0 \quad \forall t$,
   Var $\left(x_t\right) = \sigma_1^2$, Var $\left(y_t\right) = \sigma_2^2$, and
   cov $\left(x_t, x_{t-j}\right) = 0$, cov $\left(y_t, y_{t-j}\right) = 0 \quad \forall j \neq 0$.

Let $z_t = x_t + y_t$. Even tho $\mathbb{E}(z_t) = \mathbb{E}(x_t + y_t) = 0$, the $\text{cov}(z_t, z_{t-j})$ is given as:
$\mathbb{E}(\left(z_t z_{t-j}\right) = \mathbb{E}\left\{(x_t + y_t)\left(x_{t-j} + y_{t-j}\right)\right\}$
$= \mathbb{E}\left\{x_j x_{t-j} + x_t y_{t-j} + y_t x_{t-j} + y_t y_{t-j}\right\}$
$= \mathbb{E}\left(x_t y_{t-j}\right) + \mathbb{E}\left(y_t x_{t-j}\right) \neq 0$ unless x_t and y_t are independent.

Consider a simple counter-example:

Let $x_t = \varepsilon_t \backsim WN\left(0, \sigma^2\right)$ and $y_t = \varepsilon_{t-1}$.

$$\begin{aligned}
\Rightarrow z_t &= x_t + y_t = \varepsilon_t + \varepsilon_{t-1} \\
\text{cov}(z_t, z_{t-1}) &= \mathbb{E}(z_t - \mu)(z_{t-1} - \mu) \\
&= \mathbb{E}(\varepsilon_t + \varepsilon_{t-1})(\varepsilon_{t-1} + \varepsilon_{t-2}) \qquad \text{since } \mathbb{E}(z_t) = 0 \\
&= \mathbb{E}\left[\varepsilon_t\varepsilon_{t-1} + \varepsilon_t\varepsilon_{t-2} + \varepsilon_{t-1}^2 + \varepsilon_{t-1}\varepsilon_{t-2}\right] \\
&= \mathbb{E}\left(\varepsilon_{t-1}^2\right) = \sigma^2 \neq 0
\end{aligned}$$

3. Consider the AR(2) model $Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t$

   (a) Compute the mean and variance of Y_t.
   Answer:

   Consider $y_t \sim AR(2)$ process i.e.

$$\begin{aligned}
y_t &= c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t \\
\left(1 - \phi_1 L - \phi_2 L^2\right) y_t &= c + \varepsilon_t
\end{aligned}$$

   For some λ_1, λ_2, we have that

$$\begin{gathered}
\left(1 - \phi_1 \alpha - \phi_2 L^2\right) = (1 - \lambda_1 L)(1 - \lambda_2 L), \\
\phi_1 = \lambda_1 + \lambda_2 \text{ and } \phi_2 = -\lambda_1\lambda_2
\end{gathered}$$

   Claim: If $|\lambda_i| < 1 \forall i = 1, 2$ then y_t as stationary and invertibility holds, i.e. $\left(1 - \phi_1 L - \phi_2 2^2\right)^{-1}$ exists. In other words, y_t is stationary iff the roots (λ_1, λ_2) lies outside the unit circle $\left(1 - \phi_1 z - \phi_2 z^2\right)$.

$$\begin{aligned}
\Rightarrow Y_t &= \left(1 - \phi_1 L - \phi_2 L^2\right)^{-1}(c + \varepsilon_t) \\
Y_t &= (1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1}(c + \varepsilon_t)
\end{aligned}$$

So Given tat $|\lambda_i| < 1$ then we know that

$$(1-\lambda_i L)^{-1} = \sum_{k=0}^{\infty} \lambda_i^k L^k$$

Then with a little algebra, we obtain that:

$$Y_t = \frac{c}{1-\phi_1-\phi_2} + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \tag{1}$$

eqn where $\psi_j = \frac{\lambda_1^{j+1}-\lambda_2^{j+1}}{\lambda_1-\lambda_2}$.

Using equation 1, we can see that $Y_t \backsim MA(\infty)$ process and so the mean $\mu = \frac{c}{1-\phi_1-\phi_2}$.

Now using the fact that Y_t is stationary, we can compute the variance as:

$$\begin{aligned}
& y_t = (1-\phi_1-\phi_2)\,\mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t \\
& \Rightarrow y_t - \mu = (y_{t-1}-\mu)\,\phi_1 + (y_{t-2}-\mu)\,\phi_2 + \varepsilon_t \\
& \mathbb{E}\,(Y_t-\mu)^2 = \phi_1 \mathbb{E}\left[(y_t-\mu)\,(y_{t-1}-\mu)\right] + \phi_2 \mathbb{E}\left[(y_t-\mu)\,(y_{t-2}-\mu)\right] + \mathbb{E}\left[(y_t-\mu)\,\varepsilon_t\right] \\
& \gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \mathbb{E}\left((y_t-\mu)\,\varepsilon_t\right)
\end{aligned}$$

$$\begin{aligned}
& \text{But } \mathbb{E}\,(Y_t-\mu)\,(\varepsilon_t) = \mathbb{E}\left(\varepsilon_t\left(\varepsilon_t + \psi_1\varepsilon_{t-1} + \cdot\cdot\right)\right) = \sigma^2 \\
& \Longrightarrow \gamma_0 = \phi_1\gamma_0\rho_1 + \phi_2\gamma_0\rho_2 + \sigma^2 \\
& \gamma_0 = \frac{\sigma^2}{1-\phi_1\rho_1-\phi_2\rho_2}
\end{aligned}$$

(b) Analytically, compute the auto-correlation function for Y_t.
Answer:

$$\begin{aligned}
y_t - \mu &= (y_{t-1}-\mu)\,\phi_1 + (y_{t-2}-\mu)\,\phi_2 + \varepsilon_t \\
\mathbb{E}\,(y_t-\mu)\left(y_{t-j}-\mu\right) &= \phi_1 E\,(y_{t-1}-\mu)\left(y_{t-j}-\mu\right) + \mathbb{E}\,(y_{t-2}-\mu)\left(y_{t-j}-\mu\right)\phi_2 + \\
& \mathbb{E}\,(\varepsilon_t)\left(y_{t-j}-\mu\right) \\
& \gamma_j = \phi_1\gamma_{j-1} + \phi_2\gamma_{j-2} \quad \text{for } j = 1,2,3,\cdots \\
& \Rightarrow \rho_j = \phi_1\rho_{j-1} + \phi_2\rho_{j-2} \quad \text{for } j = 1,2,\ldots
\end{aligned}$$

Hence we obtain a second difference equation:

$$\rho_j = \phi_1\rho_{j-1} + \phi_2\rho_{j-2} \tag{2}$$

If $j = 1; \rho_1 = \phi_1 + \phi_2\rho_1$: Since $\rho_1 = \rho_{-1}$ and $\rho_0 = 1$

$$\Rightarrow \quad \rho_1 = \frac{\phi_1}{1-\phi_2}$$

For $j = 2$;

$$\rho_2 = \phi_1 p_1 + \phi_2 \Rightarrow \rho_2 = \frac{\phi_1^2}{1-\phi_2} + \phi_2$$

From equation 2, we see that the autocorrelation function ρ_j is a second-order difference equation, whose general solution is of the form

$$\phi_j = K_1 \lambda^j + K_2 \lambda^j$$

where λ is the root of the characteristic equation defined in part (a).

(c) What is the condition for stationarity for an AR(2).
Answer:
Recall from Part (a) that the System is Stable if $|\lambda_i| < 1 \forall i = 1, 2$ and

$$\lambda_i = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4}\phi_2}{2}$$

Suppose that λ_1, λ_2 are 2 distinct real's then:

$$\phi_1^2 + 4\phi_2 \geq 0 \tag{3}$$

Secondly $|\lambda_i| < 1 \Rightarrow |\phi_1| < 2$ and

$$\begin{aligned}
&\phi_1 + \sqrt{\phi_1^2 + 4\phi_2} < 2 \text{ and } -2 < \phi_1 - \sqrt{\phi_1^2 + 4\phi_2} \\
&\Rightarrow \phi_1^2 + 4\phi_2 < (2 - \phi_1)^2 = 4 - 4\phi_1 + \phi_1^2 \\
&\phi_2 < 1 - \phi_1
\end{aligned}$$

So we have that:

$$\phi_2 < 1 - \phi_1 \tag{4}$$

Similarly: $\phi_1^2 + 4\phi_2 < (\phi_1 + 2)^2 = \phi^2 + 4\phi_1 + 4$

$$\Rightarrow \phi_2 < 1 + \phi_1$$

$$\phi_2 \geq -1 \tag{5}$$

Combining equations 3, 4, and 5: The conditions for stationarity:

$$\begin{aligned}
&\phi_1 + \phi_2 < 1 \\
&\phi_2 - \phi_1 < 1 \quad \text{and} \\
&\phi_2 \geq -1
\end{aligned}$$

4. Financial asset returns often have non-constant volatility or heteroskedasticity. In the examples in class, we have been looking at the series level, often of detrended or deseasonalized data. Because financial returns often display little systematic variation, we will look at returns directly.

   (a) Find a high-frequency data (e.g. daily) financial asset return series (call it t it and discuss the results.
   Answer:
   Figure 1 shows the return series of an Apple stock and one can observe that this

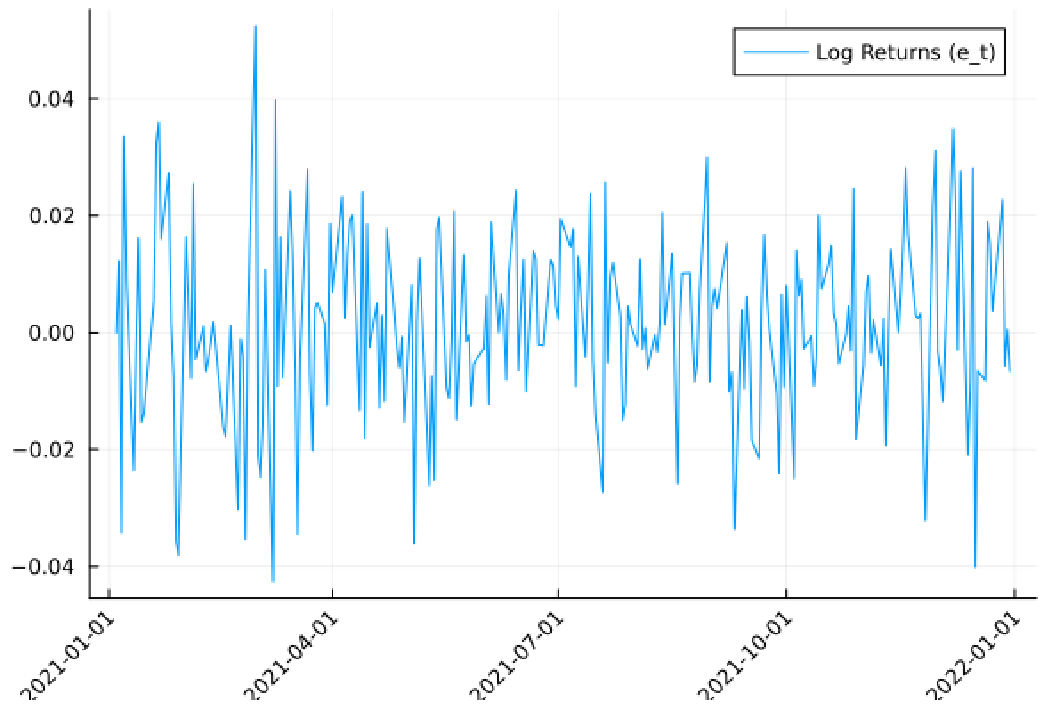
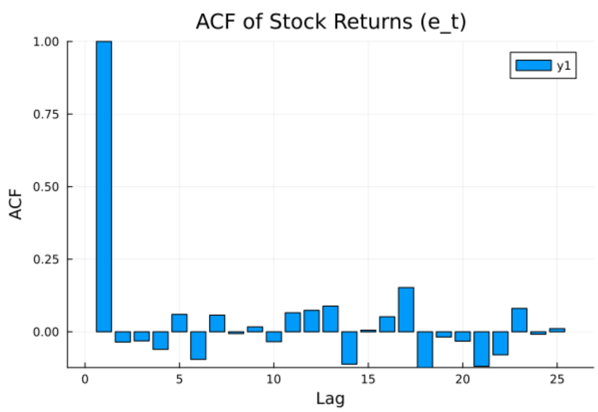


Figure 1: The graph of returns of Apple stock from 01/04/2021 to 01/01/2022.

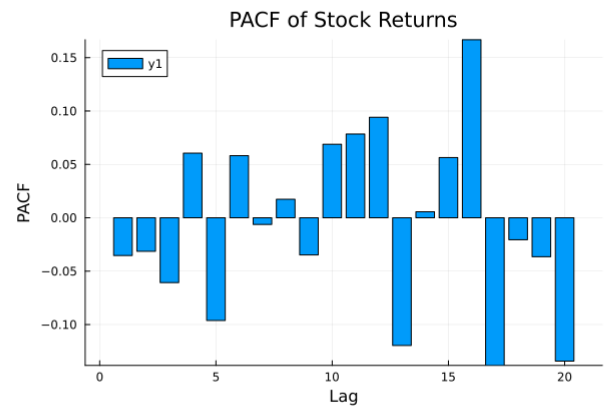
   series is highly volatile and shows clusters of volatility i.e. high volatility is followed by high volatility. This series has no trends or seasonalities and no cycles either which makes sense because this is a returns series.

   (b) Perform a correlogram analysis (ACF and PACF) of the e_t and discuss.
   Answer:
   The ACF tells us how past returns can affect or influence future returns. Figure 2a represents the ACF function of the returns of the Apple stock. We can observe that the ACF exhibits persistent slow decay, which demonstrates that the returns process follows an AR process. One can see that there lag 20 of the returns is correlated with with the current return. Figure 2b on the other hand depicts the

Partial Autocorrelation function is which is the regression of past values of Yt on the current process, holding other lags constant. The PACF function shows the predictive nature of each lag and from the graph, we can see that all lags till lag 20 have a predictive power except for lags 7, 14, and maybe 18. This helps us in model selection.



(a) The ACF of the returns of Apple stock.



(b) The PACF of the returns with a max lag of 20..

Figure 2: The ACF and PACF of the returns of Apple stock.

(c) Plot e_t^2 and discuss.
Answer:
The square of the returns might be a good way to model the second moment of the returns series. The plot of the returns square in Figure 3 shows that this data is still volatile but not as much as e_t. This makes sense since the negative values of this series is now positive. There seem to be some spikes around March and April 2021, showing that this period was highly volatile. Additionally, there seems to be seasonality in the volatility around May there was a spike, which was observed in September and January 2022. So there might be some seasonality of volatility in these periods. This could be that these periods are when they release their quarterly reports and so the markets become volatile in these periods.

(d) Examining a correlogram of e_t^2 often proves informative for assessing volatility persistence. Why might that be so? Perform a correlogram analysis of e_t^2 and discuss.
Answer:
From the ACF plot in Figure 4a, one can see that lags 6 to lag 9 is positively correlated with the current e_t^2 process. This shows that volatility in these past lags seems to have a high impact on the current volatility. However, after this lag, the effect of past volatility seems to decay. This observation seems to be heightened by the PACF plot in Figure 4b, also showing that current past lags of volatility have a high effect on the current volatility with expressly high effect coming from lags 5 to lag 9.

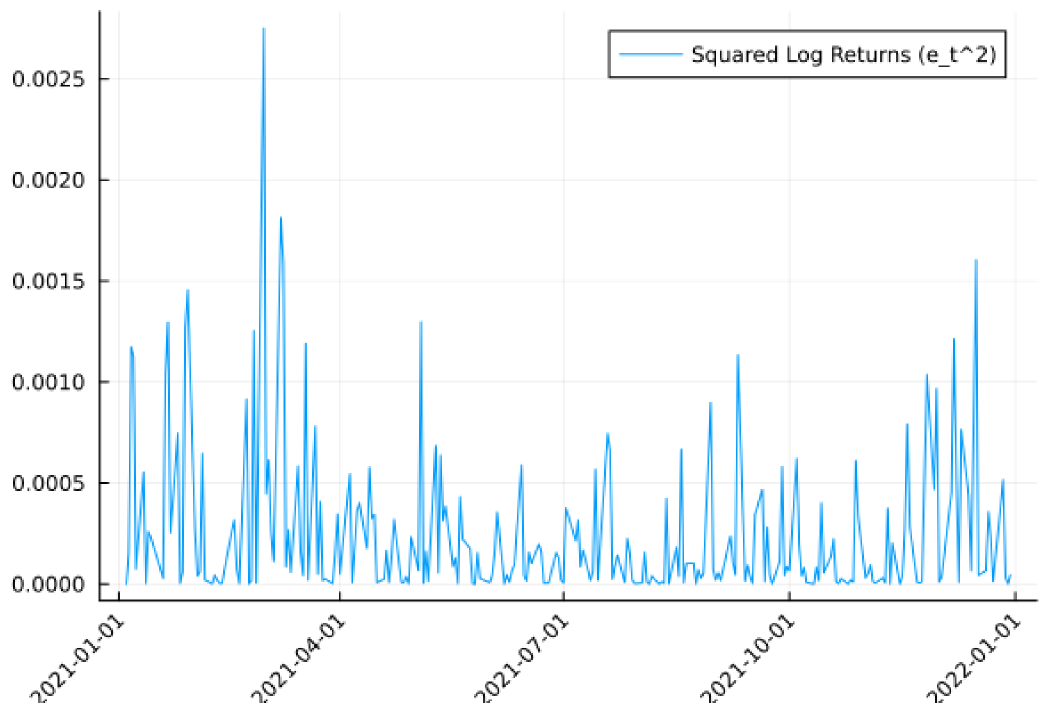
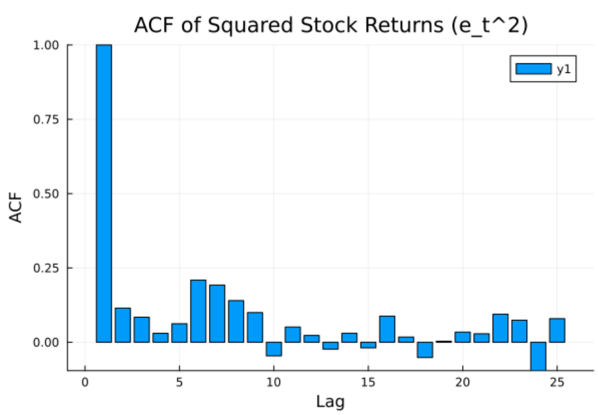
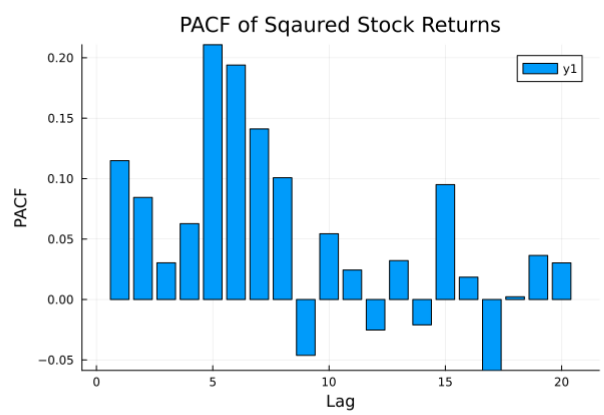


Figure 3: The graph of returns of Apple stock from 01/04/2021 to 01/01/2022.



(a) The ACF of the returns square of Apple stock.



(b) The PACF of the returns square with a max lag of 20

Figure 4: The ACF and PACF of the squared returns of Apple stock.