Notes on nonholonomic systems

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1 Three-wheel robot

Three-wheel robot with dynamic actuators is also known as an "extended nonholonomic double integrator" (ENDI) – an extension of its basis called just "nonholonomic integrator" (NI), which is the same but with static actuators [1, 3, 2, 4, 5].

Here is a summary of models:

Cartesian coordinates	Nonholonomic coordinates
$\begin{aligned} \dot{x} &= v \cos\theta \\ \dot{y} &= v \sin\theta \\ \dot{\alpha} &= \omega \\ \dot{v} &= \frac{1}{m} F \\ \dot{\omega} &= \frac{1}{I} M \end{aligned} \qquad \text{(ENDICart)}$ State: $x_{\text{Cart}} := (x, y, \alpha, v, \omega)^\top$ Input: $u_{\text{Cart}} := (F, M)^\top$	$\begin{aligned} \dot{x}_1 &= \eta_1 \\ \dot{x}_2 &= \eta_2 \\ \dot{x}_3 &= x_2 \eta_1 - x_1 \eta_2 \\ \dot{\eta}_1 &= u_1 \\ \dot{\eta}_2 &= u_2 \end{aligned} \qquad \text{(ENDI)}$ State: $\begin{aligned} x_{\text{NH}} &:= (x_1, x_2, x_3, x_4, x_5)^\top \\ x_{\text{NI}} &:= (x_1, x_2, x_3)^\top \\ \eta &:= (\eta_1, \eta_2)^\top \end{aligned}$ Input: $u_{\text{NH}} := (u_1, u_2)^\top$
Transformations	
$\xleftarrow{\Phi_u(x_{\text{NH}}, u_{\text{NH}})}$ $\begin{aligned} F = &\quad m \left(u_2 + x_1 \eta_1 \eta_2 - \tfrac{1}{2} u_1 x_3 + \tfrac{1}{2} u_1 x_1 x_2 \right) \\ M = &\quad u_1 I \end{aligned} \qquad \text{(uNH2uCart)}$	$\xrightarrow{\Psi_x(x)}$ $x_{\text{NH}} = \Psi_x(x_{\text{Cart}}) = \begin{pmatrix} \alpha \\ x \cos\alpha + y \sin\alpha \\ -2(y \cos\alpha - x \sin\alpha) - \alpha(x \cos\alpha + y \sin\alpha) \\ \omega \\ (y \cos\alpha - x \sin\alpha)\omega + v \end{pmatrix} \qquad \text{(Cart2NH)}$

In the above, "Cart" and "NH" refers to Cartesian, respectively, nonholonomic coordinates, whereas "NI" is the nonholonomic integrator part of the state vector. In (ENDICart), x, y are the coordinates of the robot on the plane. The turning angle is α. The variable v denotes the velocity (in m/s), whereas ω is the turning speed (in rad/s). The parameters are the robot's mass m, and the moment of inertia around the vertical axis I. The inputs are the pulling force F and the turning torque M.

1.1 Derivation of transformations

Define

$$\xi := \begin{pmatrix} \alpha \\ x \cos\alpha + y \sin\alpha \\ y \cos\alpha - x \sin\alpha \end{pmatrix}$$

We can show that the transformation

$$x_{\mathrm{NI}} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ -2\xi_3 - \xi_1\xi_2 \end{pmatrix}, \qquad \eta = \begin{pmatrix} \omega \\ \xi_3\omega + v \end{pmatrix}$$

yields (ENDI).

To get the input for (ENDICart), we use the η-part of the transformation above, differentiate it

$$\begin{aligned} \dot{\eta}_1 = &\quad \dot{\omega} = \tfrac{1}{I}M \\ \dot{\eta}_2 = &\quad \dot{\xi}_3\omega + \dot{\omega}\xi_3 + \dot{v} = \dot{\xi}_3\omega + \tfrac{M}{I}\xi_3 + \tfrac{F}{m} \end{aligned}$$

Observe that

$$\begin{aligned} \dot{\eta}_1 = &\quad u_1 \\ \dot{\eta}_2 = &\quad u_2 \end{aligned}$$

From this, we immediately get

$$M = u_1 I$$

Further, plug

$$\begin{aligned} \omega = &\quad \eta_1 \\ \xi_3 = &\quad \tfrac{1}{2}\left(-x_3 - \xi_1\xi_2\right) \end{aligned}$$

and the related

$$\dot{\xi}_3 = -x_2\eta_1$$

into the expression for $\dot{\eta}_2$ to get

$$u_2 = -\eta_1^2 x_2 - \frac{1}{2}\left(x_1x_2u_1 + x_3u_1\right) + \frac{F}{m}$$

From this, we obtain

$$F = m\left(u_2 + x_2\eta_1^2 + \frac{1}{2}\left(x_1x_2u_1 + x_3u_1\right)\right)$$

References

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