

1 Question 1

- 1.1 Design an R function (or use another programme if you prefer) to evaluate the empirical ACF for a given series up to a specified maximum lag and plots it along with a data plot.

```
my.acf <- function(y, lag) {  
  
  N <- length(y)  
  y.mean <- mean(y)  
  
  out <- numeric(lag + 1)  
  out[1] <- var(y)  
  
  # Evaluating ACF for each lag  
  for(h in 2:(lag + 1)) {  
    temp <- 0.0  
    for(t in 1:(N-(h-1))) {  
      temp <- temp + (y[t] - y.mean) * (y[t+(h-1)] - y.mean)  
    }  
    out[h] <- temp/N  
  }  
  # Rescale to form autocorrelation function  
  out <- out/out[1]  
  # Plotting ACF and time series  
  par(mfrow = c(2, 1))  
  grid <- seq(from = 0, to = lag, len = length(out))  
  plot.ts(y, type = "l", ylab = expression(Y[t]), main = "Time Series")  
  plot(grid, out, type = "h", main = "ACF", xlab = "Lag", ylab = "ACF")  
  # Plotting x axis  
  abline(h = 0)  
  # Plotting confidence intervals  
  abline(h = 2/sqrt(N), lty = 2, col = "red")  
  abline(h = -2/sqrt(N), lty = 2, col = "red")  
  par(mfrow = c(1, 1))  
}
```

- 1.2 Take $Y_t = \epsilon_t$ where ϵ_t is Gaussian white noise.

```
T <- 200  
Y <- rnorm(T, 0, 1)  
my.acf(Y)
```

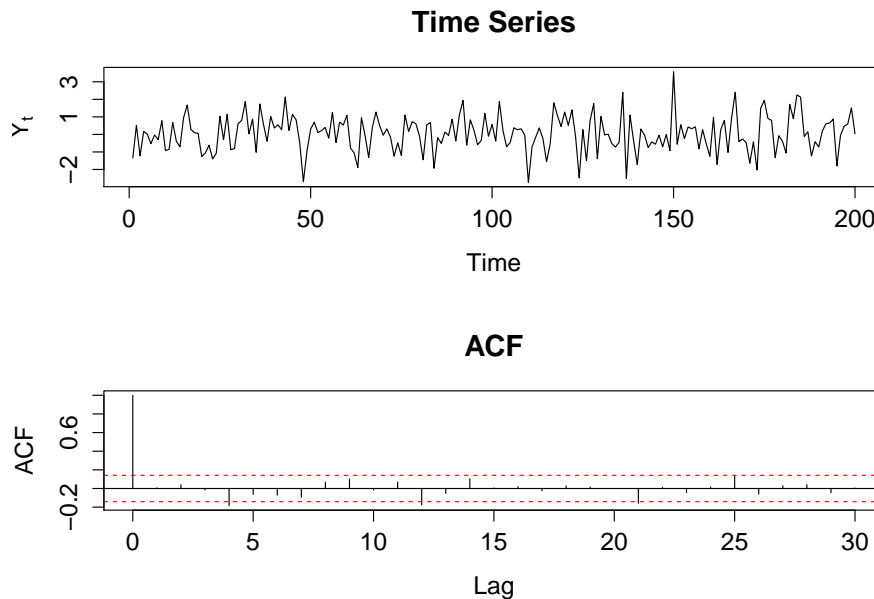


Figure 1: Time series and ACF plot for series where $Y_t = \epsilon_t$ showing no significant autocorrelation at any lag > 0

1.3 Take Y_t to be white noise with outliers. Generate a white noise process as in P2 and randomly choose a number of time points at which you simulate from a Gaussian distribution with a much larger variance. Identify how the ACF deviates from that in P2 and relate these differences to the form of the simulated series.

```
T <- 200
Y.outliers <- rnorm(T, 0, 1)

# Percentege of outliers
n.outliers <- 20
outliers.indexes <- sample(T, n.outliers)

# Simulating outliers from normal (0, 10)
Y.outliers[outliers.indexes] <- replicate(n.outliers, rnorm(1, 0, 20))

par(mfrow = c(2, 1))
plot(Y)
plot(Y.outliers)
par(mfrow = c(1, 1))

#ACF
acf(Y)
```

```
acf(Y.outliers)
```

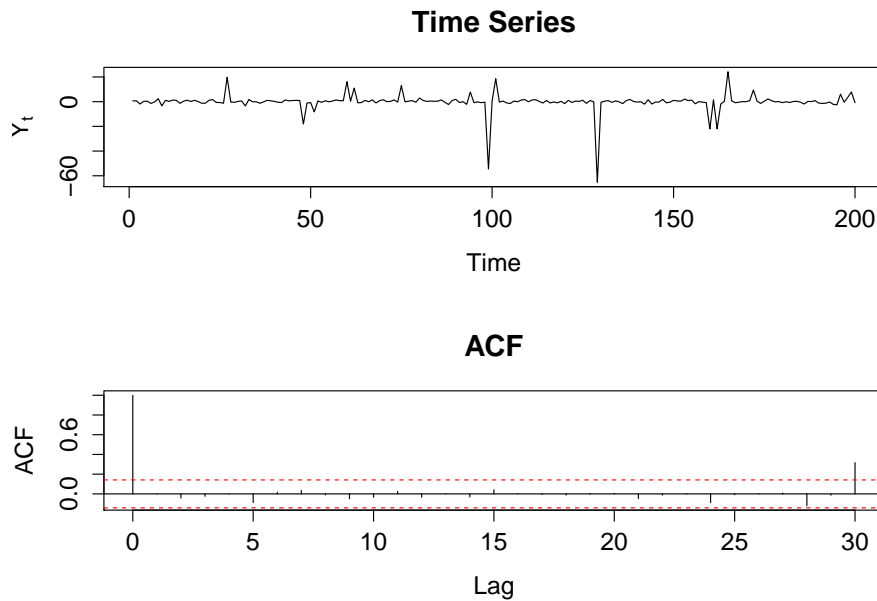


Figure 2: Time series and ACF plot for series where $Y_t = \epsilon_t$ with observations randomly simulated with larger variance. The ACF plot shows a significant autocorrelation at lag 30.

1.4 Take Y_t to be a Normal Random walk: $Y_t = Y_{t-1} + \epsilon_t$ and $Y_1 = \epsilon_1$

```
T <- 200
random.walk <- numeric(T)
errors <- rnorm(T, 0, 1)

random.walk <- errors[1]
for(t in 2:T) {
  random.walk[t] <- random.walk[t-1] + errors[t]
}

my.acf(random.walk, lag = 30)
```

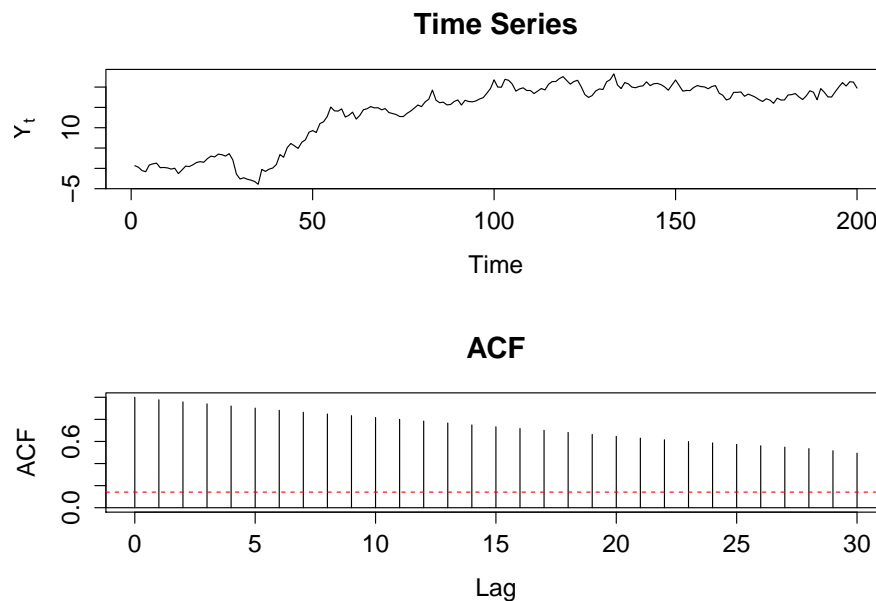


Figure 3: Time series and ACF plot for random walk.

- 1.5 Take Y_t to be an autoregressive process of order 1, **AR(1)**: $Y_t = \alpha Y_{t-1} + \epsilon_t$ where $|\alpha| < 1$. When simulating the series take Y_1 to be sample from the unconditional or stationary distribution of the AR(1) process, namely $Y_t \sim N\left(0, \frac{\sigma_e^2}{1-\alpha^2}\right)$ where $\sigma_e^2 = \text{Var}(\epsilon_t)$. Determine the effect of α on the behaviour of the series and its ACF. Derive the true ACF of the AR(1) process and compare with your plots of empirical ACF. Verify that the estimated ACF converges towards the true ACF as N is increased.

```
ar1 <- function(alpha, T = 200) {
  errors <- rnorm(T)
  Y <- numeric(T)
  Y[1] <- rnorm(1, 0, sd = sqrt((1/(1-alpha^2))))
  for(t in 2:T) {
    Y[t] <- alpha * Y[t-1] + errors[t]
  }
  Y
}
```

```
my.acf(ar1(alpha = 0, T = 200), lag = 30)
my.acf(ar1(alpha = 0.5, T = 200), lag = 30)
```

```

my.acf(ar1(alpha = -0.5, T = 200), lag = 30)
my.acf(ar1(alpha = 0.999, T = 200), lag = 30)

acf.ar1 <- function(alpha, h) {
  (alpha^h)
}

acf(ar1(alpha = -0.8, T = 200))
lines(x = 0:30 + 0.5, y = acf.ar1(alpha = -0.8, 0:30), type = "h", lty =
  2, col = "red")

acf(ar1(alpha = -0.8, T = 1e5))
lines(x = 0:30 + 0.5, y = acf.ar1(alpha = -0.8, 0:30), type = "h", lty =
  2, col = "red")

```

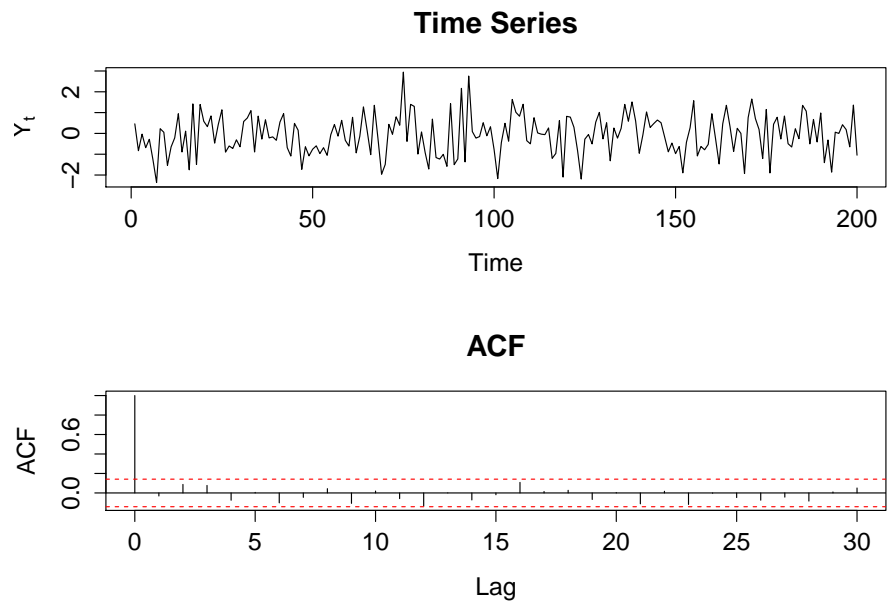


Figure 4: Time series and ACF plot for AR(1) with $\alpha = 0$.

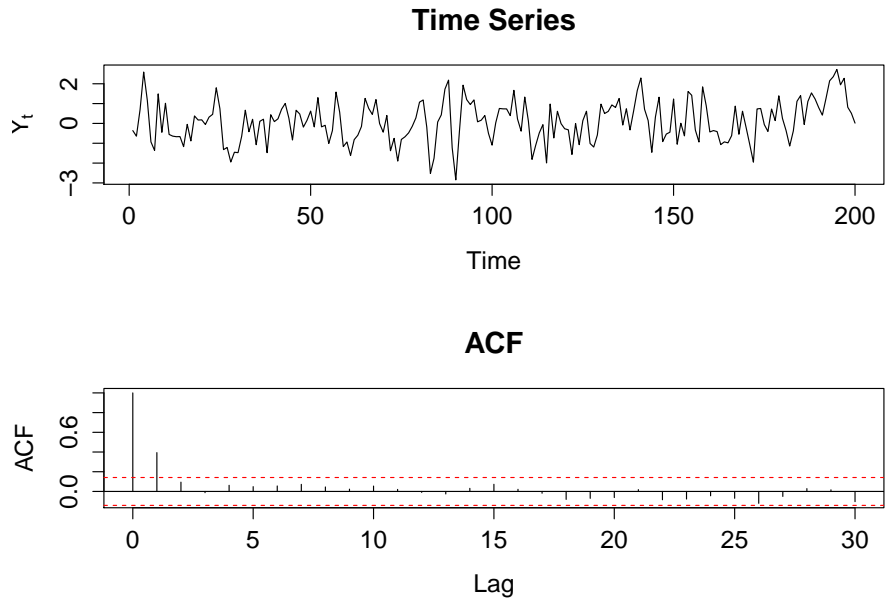


Figure 5: Time series and ACF plot for AR(1) with $\alpha = 0.5$.

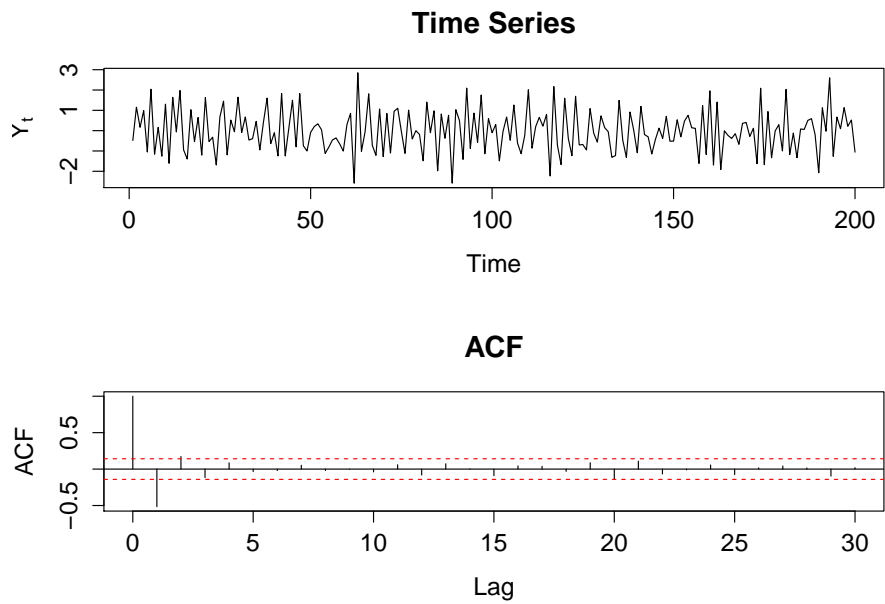


Figure 6: Time series and ACF plot for AR(1) with $\alpha = -0.5$.

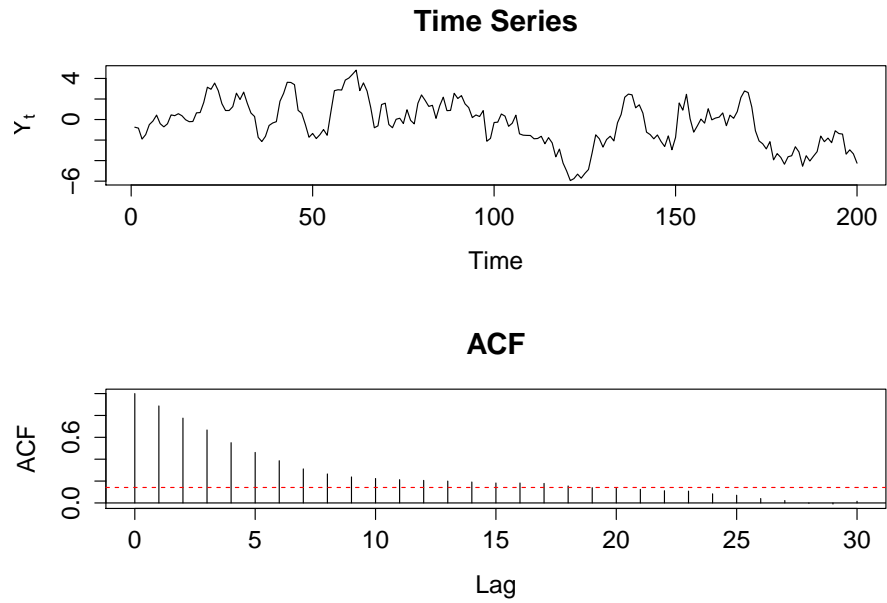


Figure 7: Time series and ACF plot for AR(1) with $\alpha = 0.999$.

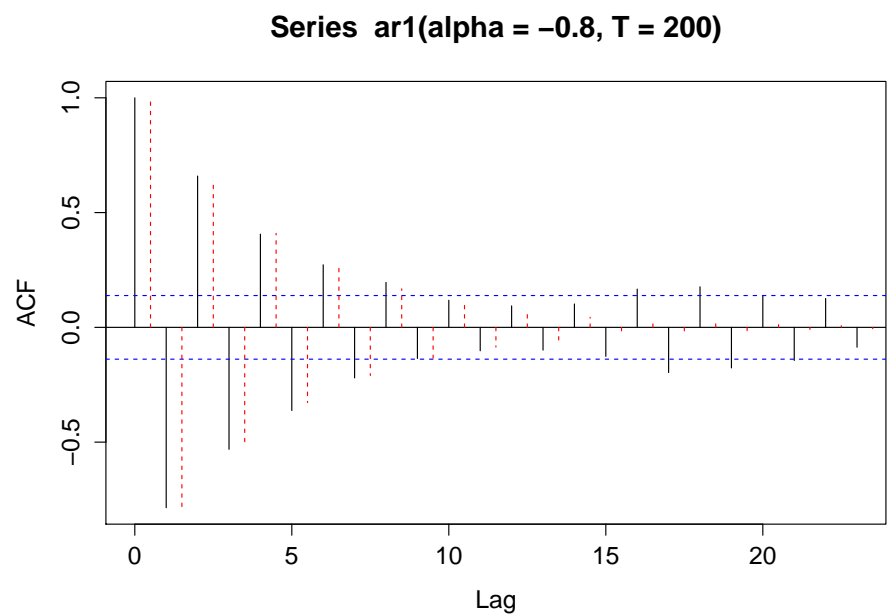


Figure 8: The empirical (solid) and true (dashed) ACF for a time series of length 200.

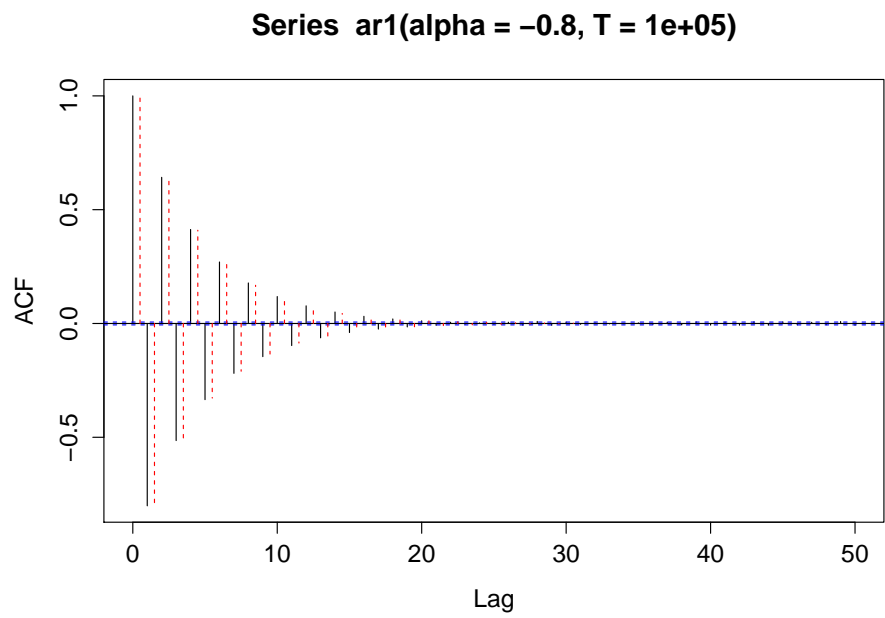


Figure 9: The empirical (solid) and true (dashed) ACF for a time series of length 100,000.

1.6 Take Y_t to be the deterministic logistic process defined by

$$Y_t = \lambda Y_{t-1}(1 - Y_{t-1})$$

where $Y_1 \in (0, 1)$ and $0 < \lambda \leq 4$

```
T <- 200
Y <- numeric(T)
Y[1] <- runif(1)
lambda <- 0.5

for(t in 2:T) {
  Y[t] <- lambda*Y[t-1]*(1 - Y[t-1])
}

my.acf(Y, 30)
```

1.7 Write a function to generate an AR(2) process for given r_1 and r_2 (write α_1 and α_2 as functions of r_1, r_2). Although it is possible to simulate Y_1 and Y_2 directly from the unconditional distribution it is simpler to first generate an AR(2) process from arbitrary starting values and then take Y_1 and Y_2 from the end of this series. Let $R_1 = 1/r_1$ and $R_2 = 1/r_2$. The true ACF of the AR(2) process for $h \geq 0$ is

$$\rho(h) = \left(\frac{R_1}{1-R_1^2} - \frac{R_2}{1-R_2^2} \right)^{-1} \left(\frac{R_1^{h+1}}{1-R_1^2} - \frac{R_2^{h+1}}{1-R_2^2} \right)$$

Examine the effect of r_1 and r_2 on the autocorrelation structure of the AR(2) process.

```
ar2_sim <- function(r1, r2, len = 200) {
  out <- rep(0, len)
  a1 <- (1 / r1) + (1 / r2)
  a2 <- - 1 / (r1 * r2)
  out <- arima.sim(n = len, list(ar = c(a1, a2)))
  out
}

true_acf <- function(r1, r2, h) {
  R1 <- 1 / r1
  R2 <- 1 / r2
  rho <- ((R1 / (1 - R1 ^ 2) - R2 / (1 - R2 ^ 2)) ^ - 1) *
    ((R1 ^ (h + 1)) / (1 - R1 ^ 2) - ((R2 ^ (h + 1)) / (1 - R2 ^
      2)))
  rho
}
```

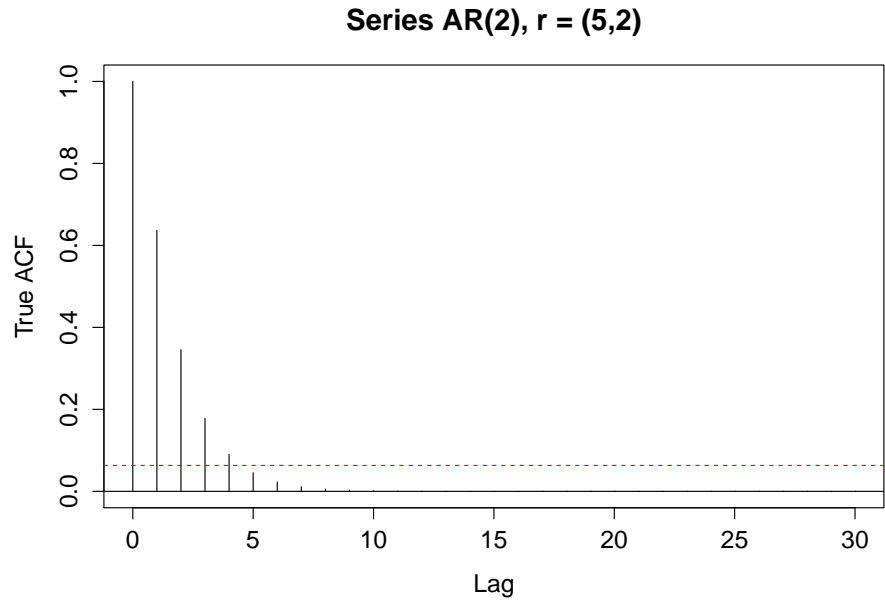


Figure 10: Time series and ACF plot for $r = (5, 2)$.

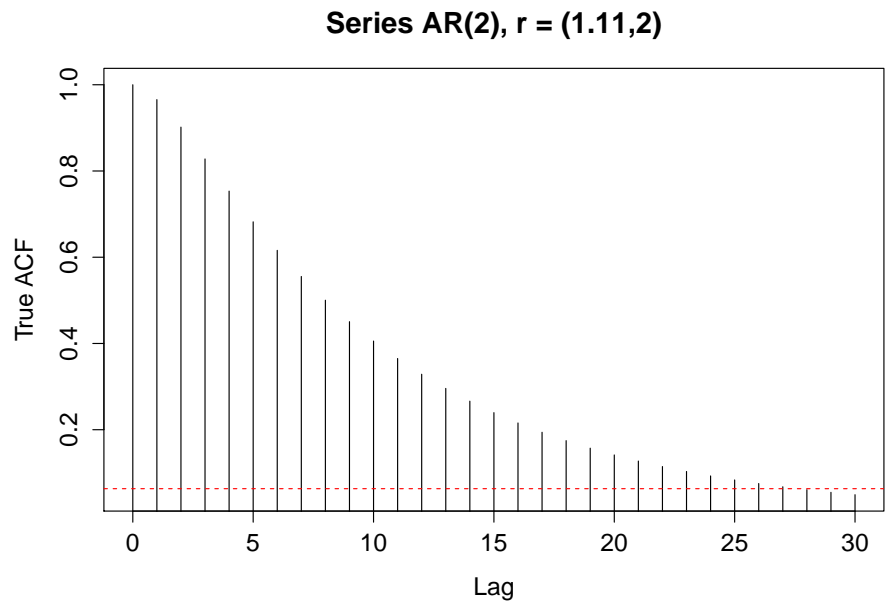


Figure 11: Time series and ACF plot for $r = (10/9, 2)$.

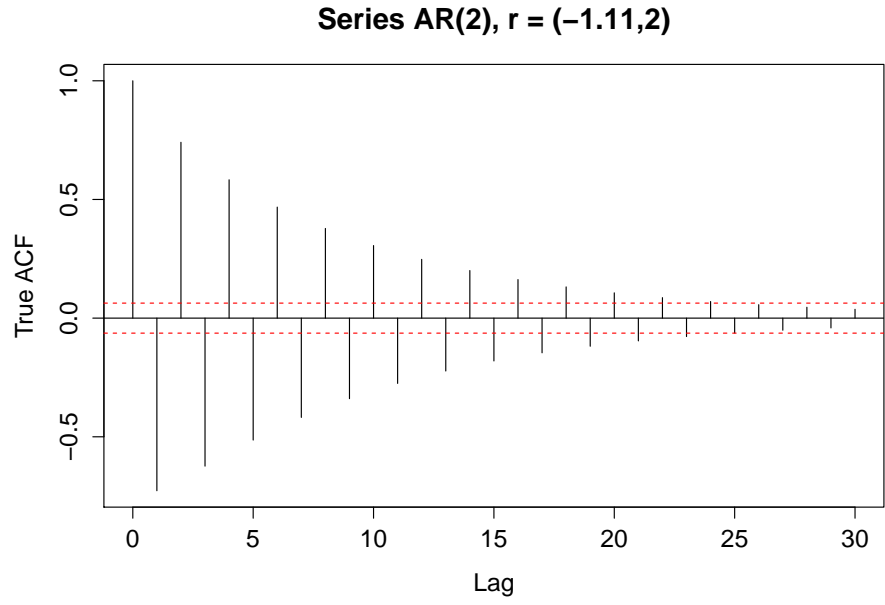


Figure 12: Time series and ACF plot for $r = (-10/9, 2)$.

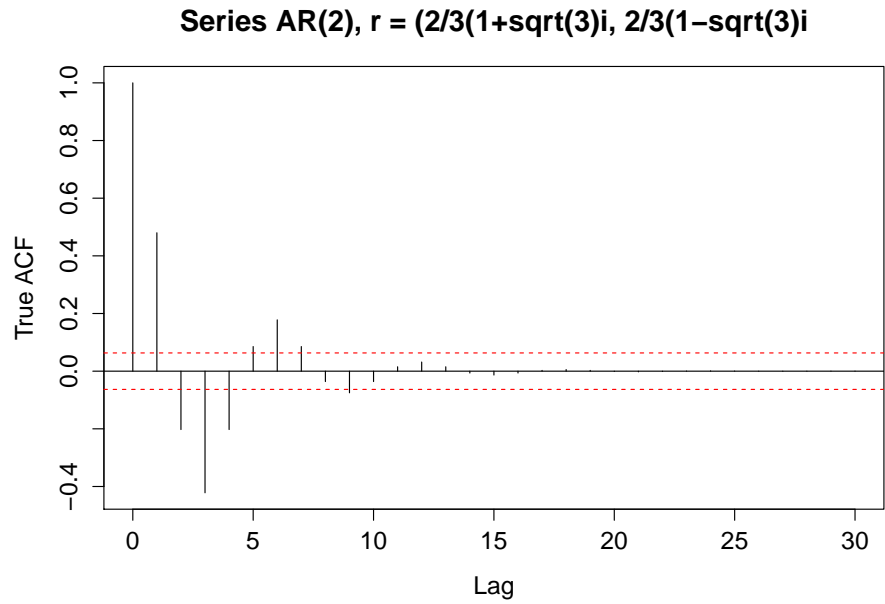


Figure 13: Time series and ACF plot for $r = (\frac{2}{3}(1 \pm \sqrt{3}i))$.

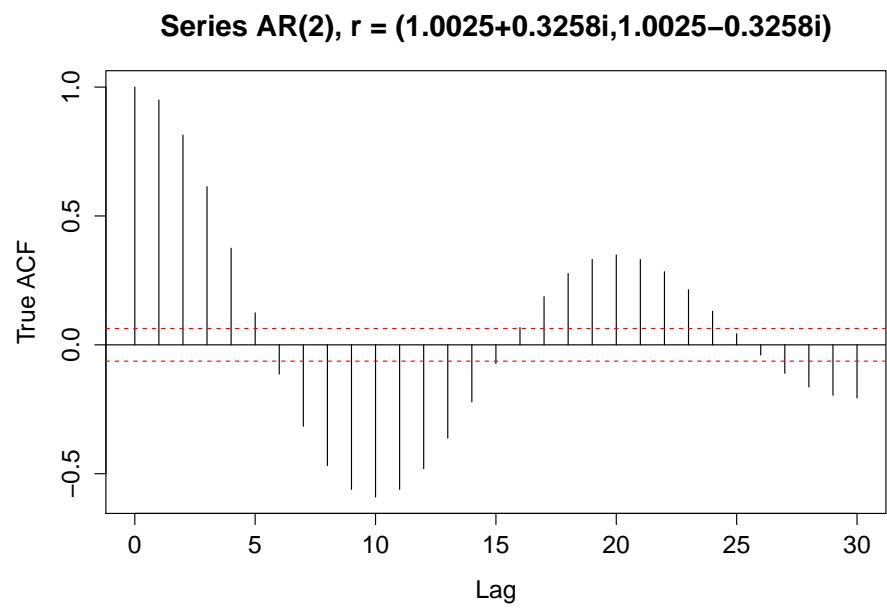


Figure 14: Time series and ACF plot for $r = (1.0025 \pm 0.3258i)$.