Proof: A 2x2 Matrix is a Vector Space

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$\mathbb{M}_{2x2}(\mathbb{R}) = \{\mathbb{M}_{2x2} : a_{ij} \in \mathbb{R}\}$ with

$A + B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$, and

$cA = c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$

Closure:

1. Let $A, B \in \mathbb{M}_{2x2}(\mathbb{R})$.

$A + B = C = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$.

$a_{11}, b_{11}, a_{12}, b_{12}, a_{21}, b_{21}, a_{22}, b_{22} \in \mathbb{R} \implies a_{11}+b_{11}, a_{12}+b_{12}, a_{21}+b_{21}, a_{22}+b_{22} \in \mathbb{R}$,

and $C \in \mathbb{M}_{2x2}$, so $A + B \in \mathbb{M}_{2x2}(\mathbb{R})$

That is, $\mathbb{M}_{2x2}(\mathbb{R})$ is closed under vector addition.

2. Let $A \in \mathbb{M}_{2x2}(\mathbb{R}), c \in \mathbb{R}$.

$cA = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$.

$c, a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \implies ca_{11}, ca_{12}, ca_{21}, ca_{22} \in \mathbb{R}$, so $cA \in \mathbb{M}_{2x2}(\mathbb{R})$.

That is, $\mathbb{M}_{2x2}(\mathbb{R})$ is closed under scalar multiplication.

Addition:

3. Commutativity - $A + B = B + A$ for every $A, B \in \mathbb{M}_{2x2}(\mathbb{R})$

$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} = \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22} \end{bmatrix}$ by commutativity of addition in $\mathbb{R}$.

$\begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = B + A$.

4. Associativity - $(A + B) + C = A + (B + C)$ for every $A, B, C \in \mathbb{M}_{2x2}(\mathbb{R})$.

$(A + B) + C = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} (a_{11}+b_{11})+c_{11} & (a_{12}+b_{12})+c_{12} \\ (a_{21}+b_{21})+c_{21} & (a_{22}+b_{22})+c_{22} \end{bmatrix} =$
$\begin{bmatrix} a_{11}+(b_{11}+c_{11}) & a_{12}+(b_{12}+c_{12}) \\ a_{21}+(b_{21}+c_{21}) & a_{22}+(b_{22}+c_{22}) \end{bmatrix}$ by associativity of addition in $\mathbb{R}$.

$\begin{bmatrix} a_{11}+(b_{11}+c_{11}) & a_{12}+(b_{12}+c_{12}) \\ a_{21}+(b_{21}+c_{21}) & a_{22}+(b_{22}+c_{22}) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{bmatrix} = A + (B + C)$.

5. Exisistence of an additive identity ($\vec{0}$), i.e., there must exist a $\vec{0}$ such that $A + \vec{0} = \vec{0} + A = A$ for every $A \in \mathbb{M}_{2x2}(\mathbb{R})$.

$A + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+c_{11} & a_{12}+c_{12} \\ a_{21}+c_{21} & a_{22}+c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \implies \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, i.e., $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the additive identity of $\mathbb{M}_{2x2}(\mathbb{R})$.

6. Existence of additive inverses, i.e., for every $A \in \mathbb{M}_{2x2}(\mathbb{R})$ there must exist some B such that $A + B = B + A = \vec{0}$.

$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies b_{11} = -a_{11}, b_{12} = -a_{12}, b_{21} = -a_{21}, and b_{22} = -a_{22}$, so $B = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix}$.

Multiplication:

7. Distribution of a scalar onto the sum of two vectors, i.e., for every $c \in \mathbb{R}$ and every $A, B \in \mathbb{M}_{2x2}(\mathbb{R})$, $c(A + B) = cA + cB$.

$c(A+B) = c\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} = \begin{bmatrix} c(a_{11}+b_{11}) & c(a_{12}+b_{12}) \\ c(a_{21}+b_{21}) & c(a_{22}+b_{22}) \end{bmatrix} = \begin{bmatrix} ca_{11}+cb_{11} & ca_{12}+cb_{12} \\ ca_{21}+cb_{21} & ca_{22}+cb_{22} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix} + \begin{bmatrix} cb_{11} & cb_{12} \\ cb_{21} & cb_{22} \end{bmatrix} = c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + c\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = cA + cB.$

8. Distribution of a vector onto the sum of two scalars, i.e., for every $c, d \in \mathbb{R}$ and every $A \in \mathbb{M}_{2x2}(\mathbb{R})$, $(c + d)A = cA + dA$.

$(c+d)A = \begin{bmatrix} (c+d)a_{11} & (c+d)a_{12} \\ (c+d)a_{21} & (c+d)a_{22} \end{bmatrix} = \begin{bmatrix} ca_{11}+da_{11} & ca_{12}+da_{12} \\ ca_{21}+da_{21} & ca_{22}+da_{22} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix} + \begin{bmatrix} da_{11} & da_{12} \\ da_{21} & da_{22} \end{bmatrix} = cA + dA.$

9. Scalar multiplication of a scalar multiplication, i.e., for every $c, d \in \mathbb{R}$ and $A \in \mathbb{M}_{2x2}(\mathbb{R})$, $c(dA) = (cd)A$.

$c(dA) = c\begin{bmatrix} da_{11} & da_{12} \\ da_{21} & da_{22} \end{bmatrix} = \begin{bmatrix} cda_{11} & cda_{12} \\ cda_{21} & cda_{22} \end{bmatrix} = \begin{bmatrix} (cd)a_{11} & (cd)a_{12} \\ (cd)a_{21} & (cd)a_{22} \end{bmatrix} = (cd)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = (cd)A.$

10. Scalar multiplication with the multiplicative identity of the field ($\mathbb{R}$), i.e., for every $A \in \mathbb{M}_{2x2}(\mathbb{R}), 1A = A$.

$1A = 1\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1a_{11} & 1a_{12} \\ 1a_{21} & 1a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A.$