Particle differential equation explanation

Bendik Dørum Nygaard

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A problem I had when creating the particle explosion was getting the particles to precisely reach the edge of the screen. I thought about it for a while, before realizing that the particles could be described by a simple differential equation. The particles start with a set speed, and that speed gets multiplied with a number $0 < n < 1$, that number is $1 - f$, where f is the friction acting on the particle. The particle therefore has an acceleration equal to $-v \cdot (1 - f)$, where v is the particle's speed and f is the friction. As both acceleration and velocity are a result of deriving position, I could create a differential equation: (p is the particle's position)

$$\begin{aligned} a &= -v \cdot (1 - f) \\ p'' &= -p' \cdot (1 - f) \end{aligned}$$

The solution to this differential equation is

$$p(t) = \frac{v \cdot e^{t(f-1)} - v}{f - 1}$$

This function describes the particle's position at a given time t. What we're interested in is the total distance traveled, the limit of the position as t approaches ∞. Logically, this limit only exists if the friction is greater than 0 and less than 1, so it's somewhat hard to make a computer compute the limit. Luckily it's easy to find the solution manually:

$$\begin{aligned} d &= \lim_{t\to\infty} \frac{v \cdot e^{t(f-1)} - v}{f - 1} \\ &= \frac{v \cdot \lim_{t\to\infty}(e^{t(f-1)}) - v}{f - 1} \end{aligned}$$

$$\lim_{t\to\infty} e^{t(f-1)} = e^{-\infty} = 0 \qquad | \quad 0 < f < 1$$

$$\begin{aligned} d &= \lim_{t\to\infty} \frac{v \cdot e^{t(f-1)} - v}{f - 1} \\ &= \frac{v \cdot 0 - v}{f - 1} \\ &= -\frac{v}{f - 1} \end{aligned}$$

This gives an equation for the distance traveled given velocity and friction, the only step remaining is solving the equation for the velocity, and then we're done:

$$
\begin{aligned}
&d = -\frac{v}{f-1} \\
\implies &v = -d(f-1) \\
\implies &v = d(1-f)
\end{aligned}
$$

This simple and elegant equation is what you can find in the code. Given how simple it is, I assume there is a much simpler way of finding it, but I think my way was quite fun.