Benjamin Bemis Numerical Method's HW 8 AME 60614

10. Consider the two-dimensional Burgers equation, which is a non-linear model of the convection–diffusion process

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right).$$

We are interested in the steady state solution in the unit square, $0 \le x \le 1$, $0 \le y \le 1$ with the following boundary conditions

$$u(0, y) = u(1, y) = v(x, 1) = 0, \quad v(x, 0) = 1$$
$$u(x, 0) = u(x, 1) = \sin 2\pi x, \quad v(0, y) = v(1, y) = 1 - y.$$

The solutions of the Burgers equation usually develop steep gradients like those encountered in shock waves. Let $\nu = 0.015$.

(a) Solve this problem using an explicit method. Integrate the equations until steady state is achieved (to plotting accuracy). Plot the steady state velocities u, v. (If you have access to a surface plotter such as in MATLAB, use it. If not, plot the velocities along the two lines: $x = 0.5$ and $y = 0.5$.) Make sure that you can stand behind the accuracy of your solution. Note that since we seek only the steady state solution, the choice of the initial condition should be irrelevant.

(b) Formulate the problem using a second-order ADI scheme for the diffusion terms and an explicit scheme for the convection terms. Give the details including the matrices involved.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

BC.

$u(0,y) = u(1,y) = v(x,1) = 0 \qquad v(x,0) = 1$

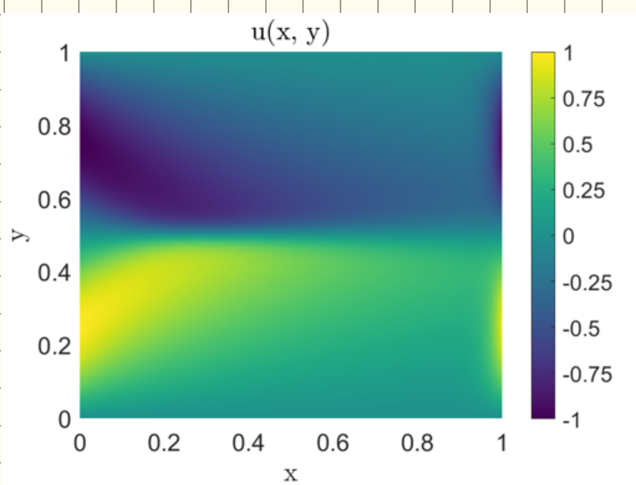
$u(x,0) = u(x,1) = \sin \pi x \qquad v(0,y) = v(1,y) = 1-y$

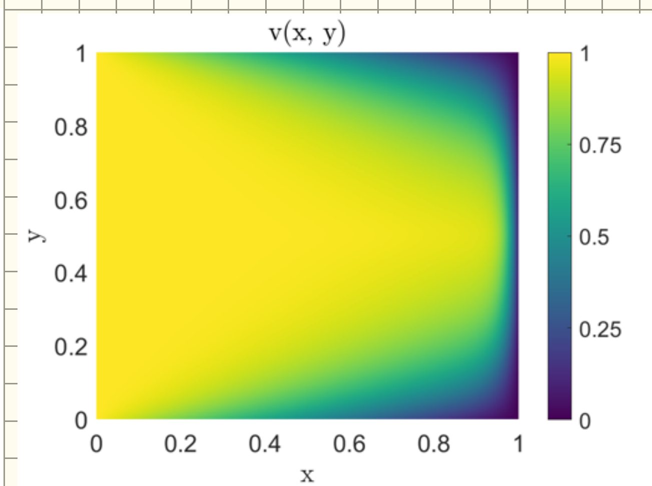
a) $\Delta x = \Delta y$ + Δt Explicit time step & 2nd order central differencing.

$$\frac{u^{n+1}_{i,j} - u^n_{ij}}{\Delta t} + u^n_{ij}\left(\frac{u^n_{i+1,j} - u^n_{i-1,j}}{2\Delta x}\right) + v^n_{ij}\left(\frac{u^n_{i+1,j} - u^n_{i-1,j}}{2\Delta y}\right) = \nu\left(\frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{\Delta x^2} + \frac{u^n_{i,j+1} - 2u^n_{ij} + u_{i,j-1}}{\Delta y^2}\right)$$

$$u^{n+1}_{ij} = u^n_{ij} - \Delta t\left\{u^n_{ij}\left(\frac{u^n_{i+1,j} - u^n_{i-1,j}}{2\Delta x}\right) + v^n_{ij}\left(\frac{u^n_{i+1,j} - u^n_{i-1,j}}{2\Delta y}\right)\right\} + \Delta t\,\nu\left(\frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{\Delta x^2} + \frac{u^n_{i,j+1} - 2u^n_{ij} + u_{i,j-1}}{\Delta y^2}\right)$$

Similarly for V instead of U.





B). $U \rightarrow \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \nu\left(\dfrac{\partial^2 u}{x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \qquad V \rightarrow \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right)$

Using Explicit Euler for convection and Crank-Nicholson for Diffusion then 2nd order central for x,y

$$\frac{u^{n+1} - u^n}{\Delta t} = -u^n\frac{\partial u^n}{\partial x} + v^n\frac{\partial u^n}{\partial y} + \frac{\nu}{2}\left(\frac{\partial^2 u^{n+1}}{\partial x^2} + \frac{\partial^2 u^n}{\partial x^2} + \frac{\partial^2 u^{n+1}}{\partial y^2} + \frac{\partial^2 u^n}{\partial y^2}\right) + O(\Delta t^2, \Delta x^2, \Delta y^2)$$

$$u^{n+1} - \frac{\nu\Delta t}{2}\left(\frac{\partial^2 u^{n+1}}{\partial x^2} + \frac{\partial^2 u^{n+1}}{\partial y^2}\right) = u_n - \Delta t\left[u^n\frac{\partial u^n}{\partial x} + v^n\frac{\partial u^n}{\partial y}\right] + \frac{\nu\Delta t}{2}\left(\frac{\partial^2 u^n}{\partial x^2} + \frac{\partial^2 u^n}{\partial y^2}\right) + \Delta t\, O(\Delta t^2, \Delta x^2, \Delta y^2)$$

$$\left(I - \frac{\nu\Delta t}{2}A_x - \frac{\nu\Delta t}{2}A_y\right)u^{n+1} = \left(I + \frac{\nu\Delta t}{2}A_x + \frac{\nu\Delta t}{2}A_y\right)u^n - \Delta t\left(u^n B_x + v^n B_y\right)u^n + TE$$

$$\left(I - \frac{\nu\Delta t}{2}A_x\right)\left(I - \frac{\nu\Delta t}{2}A_y\right)u^{n+1} = \left(I + \frac{\nu\Delta t}{2}A_x + \frac{\nu\Delta t}{2}A_y\right)u^n - \Delta t\left(u^n B_x + v^n B_y\right)u^n \underbrace{- \frac{\nu^2\Delta t^2}{4}A_x A_y\left(u^{n+1} - u^n\right) + TE\, O(\Delta t^3)}_{\text{combined into } TE\, O(\Delta t^3)}$$

Looking @ LHS

$\psi^{n+1} = \left(I - \frac{\nu\Delta t}{2}A_y\right)u^{n+1}$ LHS becomes $\left(I - \frac{\nu\Delta t}{2}A_x\right)\psi^{n+1} = RHS$

$$\psi_{i,j}^{n+1} - \frac{\nu\Delta t}{2\Delta x^2}\left(\psi_{i+1,j}^{n+1} - 2\psi_{i,j}^{n+1} + \psi_{i-1,j}^{n+1}\right) = RHS_{i,j}$$

Solve for $i = 1,2,3,\cdots N-1$
$j = 1,2,3, M-1$

$$\psi_{ij}^{n+1} = u_{ij}^{n+1} - \frac{\nu\Delta t}{2\Delta y^2}\left(u_{i,j+1}^{n+1} - 2u_{ij}^{n+1} + u_{i-1,j}^{n+1}\right)$$

Process follows for v_{ij}^{n+1} in the same manner.

Matrix setup: $\alpha = \frac{\nu\Delta t}{\Delta x^2}$

$$A = \begin{bmatrix} 1+2\alpha & -\alpha & \cdot & & \\ -\alpha & 1+2\alpha & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\alpha \\ 0 & & & -\alpha & 1+2\alpha \end{bmatrix}^{n+1} \quad RHS$$

$$1+2\alpha\,\psi_{i,j}^{n+1} - \alpha\,\psi_{i+1,j}^{n+1} - \alpha\,\psi_{i-1,j}^{n+1} = RHS \qquad \begin{aligned} i &= 1,2,3,\cdots N-1 \\ j &= 1,2,3,\cdots M-1 \end{aligned}$$

Applying the Periodic BC

$$\begin{bmatrix} 1 & 0 & \cdots & & & \cdots & 0 \\ 0 & 1+2\alpha & -\alpha & \cdot & & & \vdots \\ & -\alpha & 1+2\alpha & \ddots & 0 & & \vdots \\ & & \ddots & \ddots & \ddots & & \vdots \\ & 0 & & \ddots & \ddots & -\alpha & \vdots \\ & & & & -\alpha & 1+2\alpha & 0 \\ 0 & \cdots & & & \cdots & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} RHS_1 - 1+2\alpha\,\psi_{0,j} \\ RHS_2 \\ \vdots \\ RHS_{m-1} \\ RHS_m - (1+2\alpha)\psi_{m,j} \end{bmatrix}$$

$u_{i,j}^{n+1}$ where $\gamma = \frac{\nu\Delta t}{2\Delta y^2}$

$$\begin{bmatrix} 1+2\gamma & -\gamma & \cdot & & \\ -\gamma & 1+2\gamma & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \\ 0 & & & \ddots & 1+2\gamma \end{bmatrix} \qquad RHS = \psi_{ij}$$

$$1+2\gamma\,u_{i,j}^{n+1} - \gamma\,u_{i,j+1}^{n+1} - \gamma\,u_{i,j-1}^{n+1} = RHS \qquad BC \quad \sin(2\pi x) = u_{i,0} = u_{i,N}$$

$$\begin{bmatrix} 1 & 0 & \cdots & & & \cdots & 0 \\ 0 & 1+2\gamma & -\gamma & \cdot & & & \vdots \\ & -\gamma & 1+2\gamma & \ddots & 0 & & \vdots \\ & & \ddots & \ddots & \ddots & & \vdots \\ & 0 & & \ddots & \ddots & -\gamma & \vdots \\ & & & & -\gamma & 1+2\gamma & 0 \\ 0 & \cdots & & & \cdots & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \psi_1 - (1+2\gamma)u_{i,0} \\ \psi_2 \\ \vdots \\ \psi_{m-1} \\ \psi_N - (1+2\gamma)u_{i,N} \end{bmatrix}$$

The same will apply to V.