A new stability analysis of variable time step central difference method for transient dynamics viscoelastic problems.

ENOC24, Delft, Netherlands

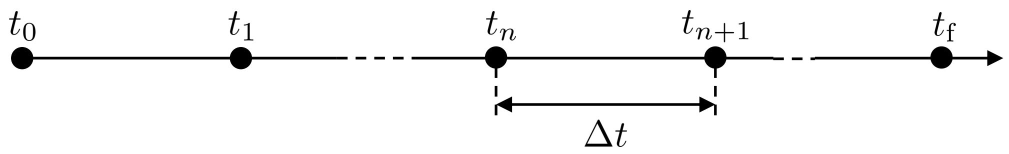
Benjamin GEORGETTE (benjamin.georgette@insa-lyon.fr)
Co-authors : Pr. Anthony GRAVOUIL and Pr. David DUREISSEIX

Introduction

Explicit integration schemes

Explicit time integration calculates the current state using only past information.

Equation of dynamics at time $t_n = n \times \Delta t$



$$m a_n = f(u_n)$$

u_n Displacement at time t_n

v_n Velocity at time t_n

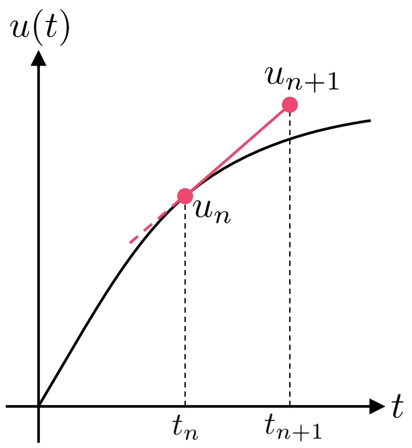
a_n Acceleration at time t_n

m Mass

Forward Euler method (explicit)

$$u_{n+1} = u_n + \Delta t v_n$$

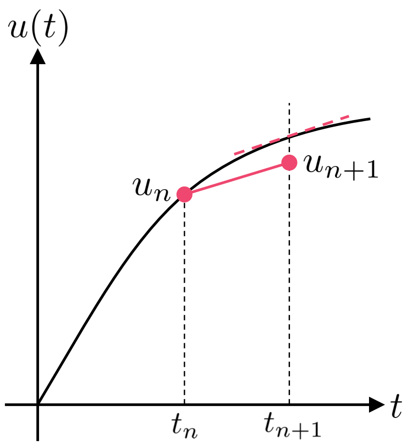
$$u_{n+2} - 2u_{n+1} + u_n = \frac{\Delta t^2}{m} f(u_n)$$



Backward Euler method (implicit)

$$u_{n+1} = u_n + \Delta t v_{n+1}$$

$$u_n - 2u_{n-1} + u_{n-2} = \frac{\Delta t^2}{m} f(u_n)$$

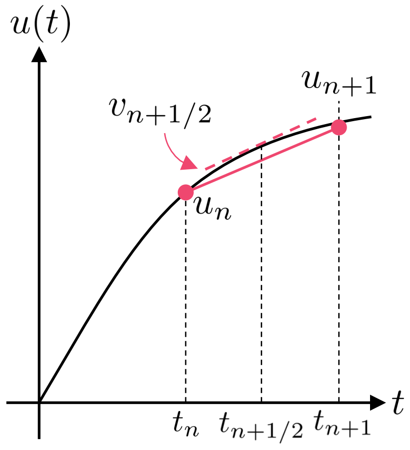


Introduction

Explicit integration schemes

Explicit time integration calculates the current state using only past information.

- Advantage : fast computation
- Drawback : stability limit



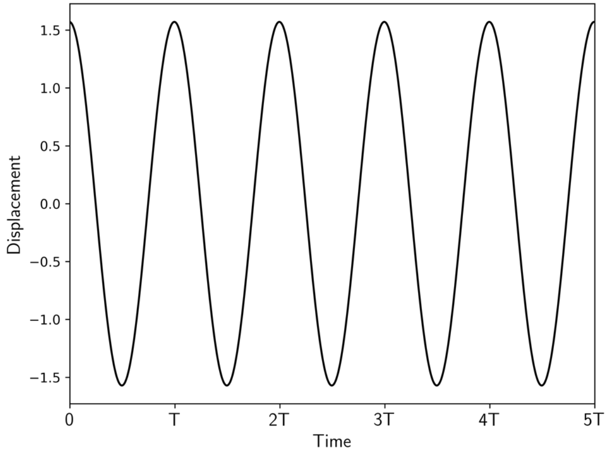
Central difference scheme

- Explicit
- Structure preserving scheme (symplectic)[1]
  → exact conservation of angular momentum

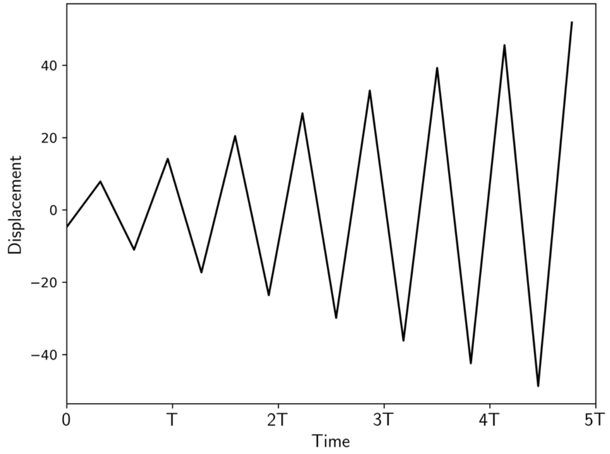
$$u_{n+1} = u_n + \Delta t v_{n+1/2}$$
$$\frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta t^2} = a_n$$

$$\ddot{u}(t) + \omega^2 u(t) = 0$$
$$(\omega \Delta t) \leq 2 \Rightarrow \Delta t_{\mathrm{cr}}^{\mathrm{e}} = 2/\omega$$

$\Delta t \leq \Delta t_{\mathrm{cr}}^{\mathrm{e}}$



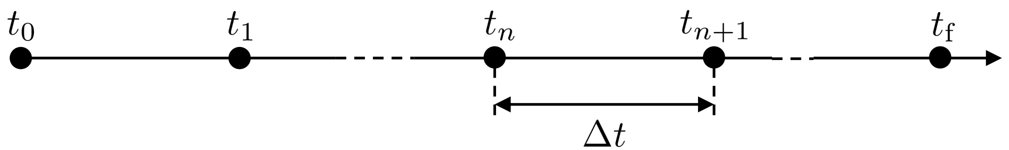
$\Delta t > \Delta t_{\mathrm{cr}}^{\mathrm{e}}$



1. Simo, J.C., N. Tarnow, and K.K. Wong. 'Exact Energy-Momentum Conserving Algorithms and Symplectic Schemes for Nonlinear Dynamics'. *Computer Methods in Applied Mechanics and Engineering* 100, no. 1 (October 1992): 63–116.

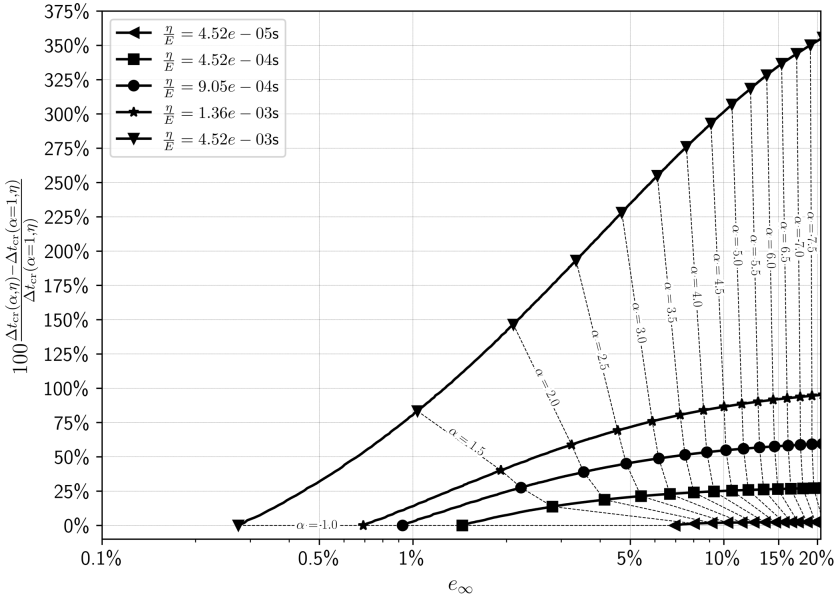
Outline of the presentation – main results

Constant time step



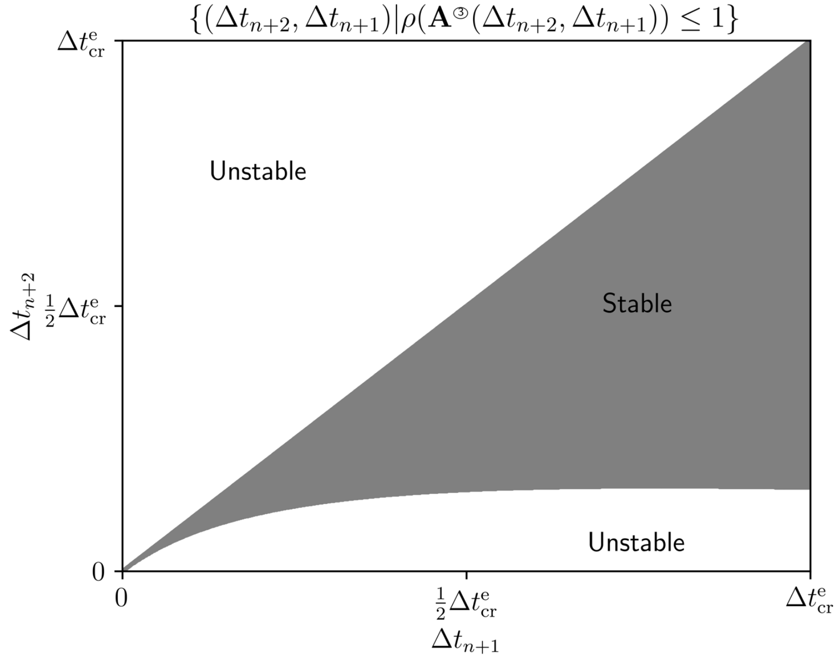
→ New integration method for the viscous stress-strain relationship with great stability properties.

$$(1-\alpha)\boldsymbol{\sigma}_n^{\mathrm{v}} + \alpha\boldsymbol{\sigma}_{n+1}^{\mathrm{v}} = \eta\mathbb{I} : \boldsymbol{\varepsilon}(\boldsymbol{v}_{n+1/2}), \alpha \in \mathbb{R}^+$$



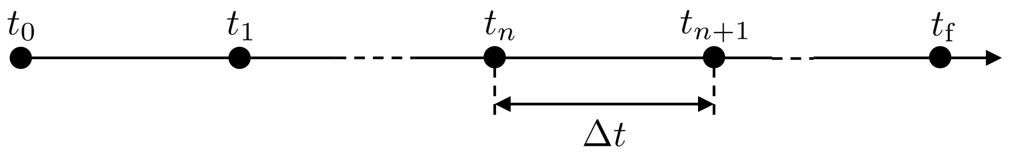
Variable time step

→ Apparition of instability areas below the critical time step.
→ Impossibility to augment the time step within the simulation for a non dissipative system.
→ Only the strong stability analysis based on the formulation of the scheme as a multistep scheme is relevant.



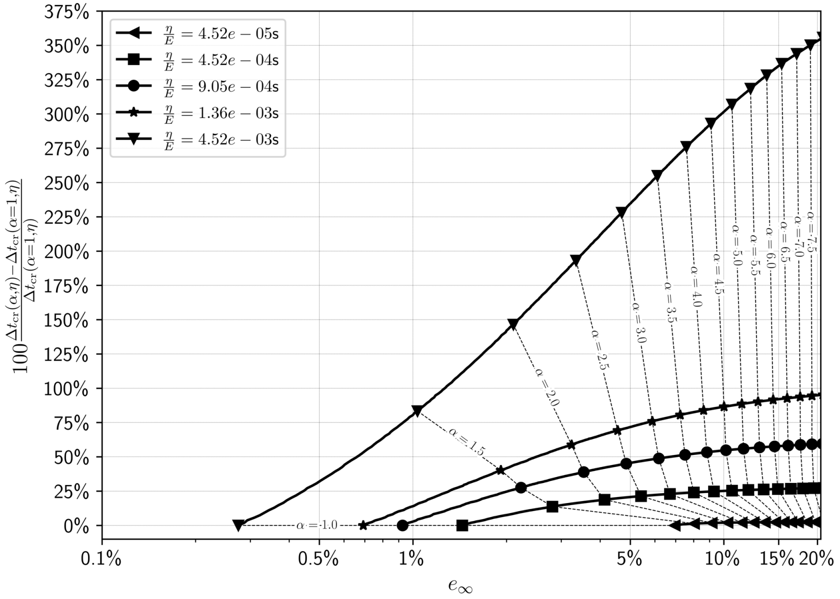
Outline of the presentation – main results

Constant time step



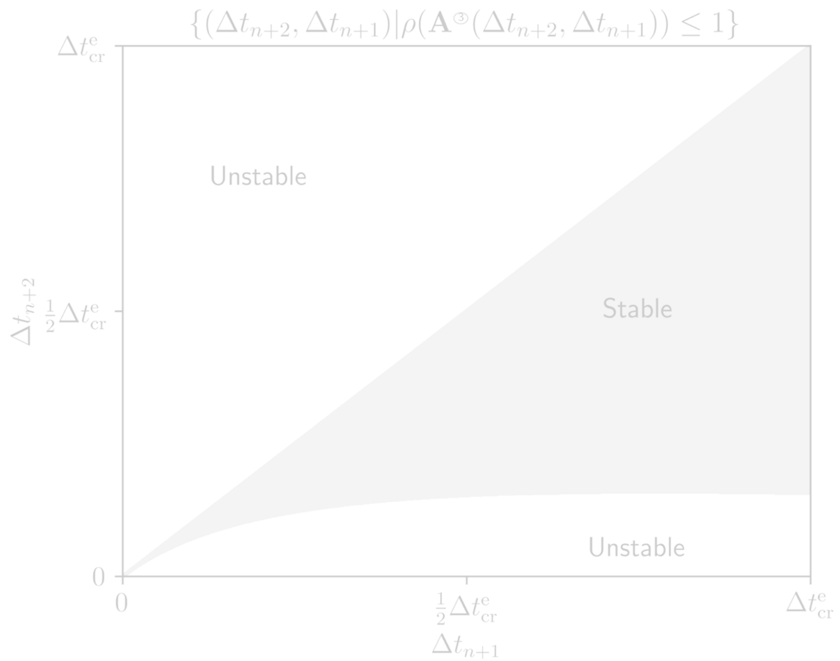
→ New integration method for the viscous stress-strain relationship with great stability properties.

$$(1-\alpha)\boldsymbol{\sigma}_n^\mathrm{v} + \alpha\boldsymbol{\sigma}_{n+1}^\mathrm{v} = \eta\mathbb{I} : \boldsymbol{\varepsilon}(\boldsymbol{v}_{n+1/2}), \alpha \in \mathbb{R}^+$$



Variable time step

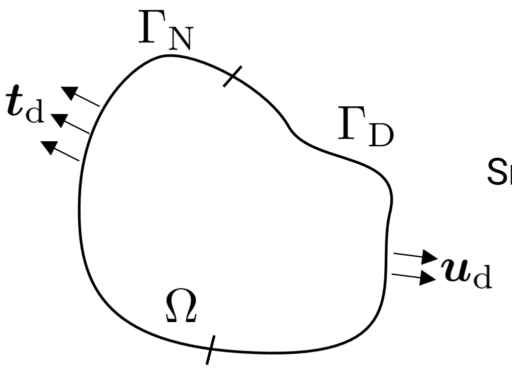
→ Apparition of instability areas below the critical time step.
→ Impossibility to augment the time step within the simulation for a non dissipative system.
→ Only the strong stability analysis based on the formulation of the scheme as a multistep scheme is relevant.



Constant time step

The semi-discrete equations of structural dynamics

For a continuum deformable body (in small strains), the **strong form** of the problem is



Newton's second law	$\mathrm{div}(\boldsymbol{\sigma}) + \boldsymbol{f}_{\mathrm{d}} = \dfrac{\partial \boldsymbol{p}}{\partial t}$	$\forall \boldsymbol{x} \in \Omega$
Neumann's condition	$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t}_{\mathrm{d}}$	$\forall \boldsymbol{x} \in \Gamma_{\mathrm{N}}$
Small strains approximation	$\boldsymbol{\varepsilon} = \dfrac{1}{2}\left(\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} + \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}^{\mathsf{T}}\right)$	$\forall \boldsymbol{x} \in \Omega$
	$\boldsymbol{v} = \dfrac{\partial \boldsymbol{u}}{\partial t}$	$\forall \boldsymbol{x} \in \Omega$
Dirichlet condition	$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{d}}$	$\forall \boldsymbol{x} \in \Gamma_{\mathrm{D}}$
Stress-strain behaviour law	$\boldsymbol{\sigma} = f(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}})$	$\forall \boldsymbol{x} \in \Omega$
	$\boldsymbol{p} = \rho \boldsymbol{v}$	$\forall \boldsymbol{x} \in \Omega$

$\boldsymbol{\sigma}$	stress
$\boldsymbol{\varepsilon}$	strain
$\boldsymbol{f}_{\mathrm{d}}$	volumic force
$\boldsymbol{u}$	displacement
$\boldsymbol{v}$	velocity
$\boldsymbol{p}$	linear momentum
ρ	density
$\boldsymbol{x}$	position

The strong form can be expressed in the **weak form** with $\mathscr{U}_0^{[\boldsymbol{x}]} := \left\{\boldsymbol{u} \in \mathscr{H}^1(\Omega) | \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0} \ \forall \boldsymbol{x} \in \Gamma_{\mathrm{D}}\right\}$

$$\forall \boldsymbol{u}^\star \in \mathscr{U}_0^{[\boldsymbol{x}]}, \forall t \in \mathscr{T},$$

$$-\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{u}^\star)\mathrm{d}\Omega + \int_{\Gamma_{\mathrm{N}}} \boldsymbol{t}_{\mathrm{d}}.\boldsymbol{u}^\star \mathrm{d}\Gamma_{\mathrm{N}} + \int_\Omega \boldsymbol{f}_{\mathrm{d}}.\boldsymbol{u}^\star \mathrm{d}\Omega - \int_\Omega \rho \frac{\partial \boldsymbol{v}}{\partial t}.\boldsymbol{u}^\star \mathrm{d}\Omega = 0$$

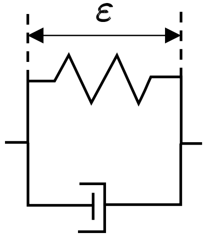
Constant time step

The semi-discrete equations of structural dynamics

$$\forall \boldsymbol{u}^\star \in \mathscr{U}_0^{[\boldsymbol{x}]}, \forall t \in \mathscr{T},$$

$$-\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{u}^\star)\mathrm{d}\Omega + \int_{\Gamma_\mathrm{N}} \boldsymbol{t}_\mathrm{d}.\boldsymbol{u}^\star \mathrm{d}\Gamma_\mathrm{N} + \int_\Omega \boldsymbol{f}_\mathrm{d}.\boldsymbol{u}^\star \mathrm{d}\Omega - \int_\Omega \rho \frac{\partial \boldsymbol{v}}{\partial t}.\boldsymbol{u}^\star \mathrm{d}\Omega = 0$$

Consider a **Kelvin viscoelastic material**



Linear damping !

$$\boldsymbol{\sigma} = \boxed{\boldsymbol{\sigma}^\mathrm{e}} + \boxed{\boldsymbol{\sigma}^\mathrm{v}}, \boldsymbol{\sigma} = \boxed{\mathbb{H} : \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x},t))} + \boxed{\eta \mathbb{I} : \dot{\boldsymbol{\varepsilon}}(\boldsymbol{u}(\boldsymbol{x},t))}$$

$\mathbb{H}$ Fourth order elasticity tensor

$\mathbb{I}$ Fourth order identity tensor

η Viscosity parameter

Consider a **finite element approximation on displacements** $\boldsymbol{u}(\boldsymbol{x},t) = \sum_{i=0}^{N_\mathrm{nodes}} \boldsymbol{s}^i(\boldsymbol{x}) u^i(t)$

$\boldsymbol{s}^i(\boldsymbol{x})$ Shape functions

$$\boldsymbol{V}(t) = \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t}(t)$$

$$\mathbf{M}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}(t) = \boldsymbol{F}^\mathrm{ext}(t) - \boldsymbol{F}^\mathrm{int}(t)$$

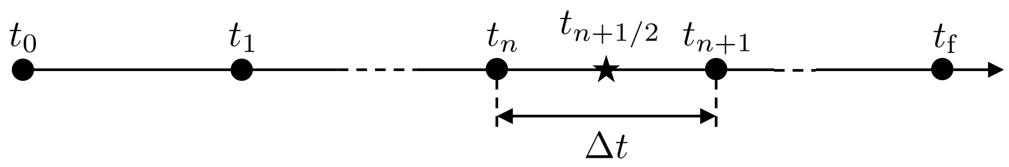
$$\boldsymbol{F}^\mathrm{ext}(t) = \left(\cdots (F^\mathrm{ext}(t))^i \cdots\right)^\mathsf{T}, \text{with } (F^\mathrm{ext}(t))^i = \int_{\Gamma_\mathrm{N}} \boldsymbol{t}_\mathrm{d}(\boldsymbol{x},t).\boldsymbol{s}^i(\boldsymbol{x})\mathrm{d}\Gamma_\mathrm{N} + \int_\Omega \boldsymbol{f}_\mathrm{d}(\boldsymbol{x},t).\boldsymbol{s}^i(\boldsymbol{x})\mathrm{d}\Omega$$

$$\boldsymbol{F}^\mathrm{int}(t) = (\cdots (F^\mathrm{int}(t))^i \cdots)^\mathsf{T}, \text{with } (F^\mathrm{int}(t))^i = \boxed{\Sigma_{j=0}^{N_\mathrm{nodes}} K^{ij} u^j(t)} + \boxed{u^i(t) \int_\Omega \boldsymbol{\sigma}^\mathrm{v} : \boldsymbol{\varepsilon}(\boldsymbol{s}^i(\boldsymbol{x}))\mathrm{d}\Omega}$$

Constant time step

Time discretisation

Central difference scheme



$$\boldsymbol{V}(t) = \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t}(t) \quad \longrightarrow \quad \boldsymbol{U}_{n+1} = \boldsymbol{U}_n + \Delta t \boldsymbol{V}_{n+1/2}$$

$$\mathbf{M}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}(t) = \boldsymbol{F}^{\mathrm{ext}}(t) - \boldsymbol{F}^{\mathrm{int}}(t) \quad \longrightarrow \quad \mathbf{M}(\boldsymbol{V}_{n+3/2} - \boldsymbol{V}_{n+1/2}) = \Delta t(\boldsymbol{F}^{\mathrm{ext}}(t_{n+1}) - \boldsymbol{F}^{\mathrm{int}}_{n+1})$$

$$\boldsymbol{F}^{\mathrm{int}}(t) = (\cdots (F^{\mathrm{int}}(t))^i \cdots)^{\mathsf{T}}, \text{with } (F^{\mathrm{int}}(t))^i = \Sigma_{j=0}^{N_{\mathrm{nodes}}} K^{ij} u^j(t) + u^i(t) \int_\Omega \boldsymbol{\sigma}^{\mathrm{v}} : \boldsymbol{\varepsilon}(\boldsymbol{s}^i(\boldsymbol{x}))\mathrm{d}\Omega$$

$$\boldsymbol{\sigma}^{\mathrm{v}} = \eta \mathbb{I} : \dot{\boldsymbol{\varepsilon}}(\boldsymbol{u}(\boldsymbol{x}, t))$$

How to express the internal forces, while keeping an explicit scheme ?

Constant time step

The semi-discrete equations of structural dynamics

Direct integration of the viscous stress-strain law

$$\boldsymbol{\sigma}^{\mathrm{v}}_{n+1} = (\eta \mathbb{I}) : \dot{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{n+1}), \ \frac{d}{dt}\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}\left(\frac{d\boldsymbol{u}}{dt}\right)$$

$$(F^{\mathrm{int}})^i_{n+1} = \sum_{j=0}^{N_{\mathrm{nodes}}} \left(K^{ij} u^j_{n+1} + v^j_{n+1} \underbrace{\int_\Omega \eta \mathbb{I} : \boldsymbol{\varepsilon}(\boldsymbol{s}^i(\boldsymbol{x})) : \boldsymbol{\varepsilon}(\boldsymbol{s}^j(\boldsymbol{x})) \mathrm{d}\Omega}_{C^{ij}} \right)$$

$$\Rightarrow \boldsymbol{F}^{\mathrm{int}}_{n+1} = \mathbf{K}\boldsymbol{U}_{n+1} + \mathbf{C}\boldsymbol{V}_{n+1} \ , \text{ with } \boldsymbol{V}_{n+1} = \frac{1}{2}\left(\boldsymbol{V}_{n+3/2} + \boldsymbol{V}_{n+1/2}\right)$$

$$\mathbf{M}(\boldsymbol{V}_{n+3/2} - \boldsymbol{V}_{n+1/2}) = \Delta t \left(\boldsymbol{F}^{\mathrm{ext}}(t_{n+1}) - \mathbf{K}\boldsymbol{U}_{n+1} - \frac{1}{2}\mathbf{C}\left(\boldsymbol{V}_{n+3/2} + \boldsymbol{V}_{n+1/2}\right)\right)$$

Implicit scheme !

Approximation by Belytschko[1]

$$\boldsymbol{\sigma}^{\mathrm{v}}_{n+1} = \eta \mathbb{I} : \boldsymbol{\varepsilon}(\boldsymbol{v}_{n+1/2})$$

$$\mathbf{M}(\boldsymbol{V}_{n+3/2} - \boldsymbol{V}_{n+1/2}) = \Delta t \left(\boldsymbol{F}^{\mathrm{ext}}(t_{n+1}) - \mathbf{K}\boldsymbol{U}_{n+1} - \mathbf{C}\boldsymbol{V}_{n+1/2}\right)$$

1. Belytschko, Ted, W. K. Liu, B. Moran, and Khalil I. Elkhodary. *Nonlinear Finite Elements for Continua and Structures*. Second edition. Chichester, West Sussex, United Kingdon: Wiley, 2014.

Constant time step

The semi-discrete equations of structural dynamics

New approximation proposed $\boxed{(1-\alpha)\boldsymbol{\sigma}_n^{\mathrm{v}} + \alpha\boldsymbol{\sigma}_{n+1}^{\mathrm{v}} = \eta\mathbb{I} : \boldsymbol{\varepsilon}(\boldsymbol{v}_{n+1/2}), \alpha \in \mathbb{R}^+}$

- Generalisation of the Belytschko's approximation $\boldsymbol{\sigma}_{n+1}^{\mathrm{v}} = \eta\mathbb{I} : \boldsymbol{\varepsilon}(\boldsymbol{v}_{n+1/2}), \alpha = 1$
- Does not add any non-physical term $\lim\limits_{\Delta t \to 0} (1-\alpha)\boldsymbol{\sigma}_n^{\mathrm{v}} + \alpha\boldsymbol{\sigma}_{n+1}^{\mathrm{v}} - \eta\mathbb{I} : \boldsymbol{\varepsilon}(\boldsymbol{v}_{n+1/2}) = \boldsymbol{\sigma}^{\mathrm{v}} - \eta\mathbb{I} : \boldsymbol{\varepsilon}(\boldsymbol{v})$

$$(F^{\mathrm{int}})_{n+1}^i = \sum_{j=0}^{N_{\mathrm{nodes}}} \left(K^{ij} u_{n+1}^j + \alpha^{-1} C^{ij} v_{n+1/2}^j - \alpha^{-1}(1-\alpha) \int_\Omega \boldsymbol{\sigma}_n^{\mathrm{v}} : \boldsymbol{\varepsilon}(\boldsymbol{s}^i(\boldsymbol{x}))\mathrm{d}\Omega \right)$$

$$\Rightarrow \boldsymbol{F}_{n+1}^{\mathrm{int}} = \mathbf{K}\boldsymbol{U}_{n+1} + \alpha^{-1}\mathbf{C}\boldsymbol{V}_{n+1/2} - \alpha^{-1}(1-\alpha)\boldsymbol{F}^{\mathrm{v}}(\boldsymbol{\sigma}_n^{\mathrm{v}})$$

Algorithm 1 Central difference scheme with α viscous stress averaging procedure

Input : $\mathbf{M}, \mathbf{C}, \mathbf{K}, \boldsymbol{F}^{\mathrm{ext}}(t)$, $\boldsymbol{U}_0$, $\boldsymbol{V}_0$, $\boldsymbol{\sigma}_0^{\mathrm{v}}$, α
$\boldsymbol{F}_0^{\mathrm{int}} = \mathbf{K}\boldsymbol{U}_0 + \alpha^{-1}(\mathbf{C}\boldsymbol{V}_0 - \Delta t(1-\alpha)\boldsymbol{F}^{\mathrm{v}}(\boldsymbol{\sigma}_0^{\mathrm{v}}))$
Solve $\mathbf{M}(\boldsymbol{V}_{1/2} - \boldsymbol{V}_0) = \Delta t(\boldsymbol{F}^{\mathrm{ext}}(t_0) - \boldsymbol{F}_0^{\mathrm{int}})$ to get $\boldsymbol{V}_{1/2}$
while $n+1 \leq N_{\mathrm{t}}$ **do**
  $\boldsymbol{U}_{n+1} = \boldsymbol{U}_n + \Delta t\boldsymbol{V}_{n+1/2}$
  $\boldsymbol{\sigma}_{n+1}^{\mathrm{v}} = \alpha^{-1}\eta\mathbb{I} : \boldsymbol{\varepsilon}(\boldsymbol{v}_{n+1/2}) - \alpha^{-1}(1-\alpha)\boldsymbol{\sigma}_n^{\mathrm{v}}$
  $\boldsymbol{F}_{n+1}^{\mathrm{int}} = \mathbf{K}\boldsymbol{U}_{n+1} + \boldsymbol{F}^{\mathrm{v}}(\boldsymbol{\sigma}_{n+1}^{\mathrm{v}})$
  Solve $\mathbf{M}(\boldsymbol{V}_{n+3/2} - \boldsymbol{V}_{n+1/2}) = \Delta t\left(\boldsymbol{F}^{\mathrm{ext}}(t_{n+1}) - \boldsymbol{F}_{n+1}^{\mathrm{int}}\right)$ to get $\boldsymbol{V}_{n+3/2}$
end while

Constant time step

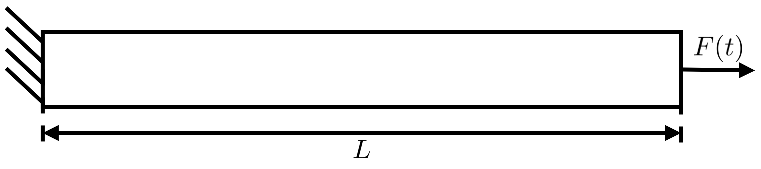
Stability analysis

Spectral stability analysis[1]

$$[\text{Spectral radius}]\,(\mathbf{A}(\Delta t,\alpha)) \leq 1$$

$$\begin{pmatrix} \boldsymbol{V}_{n+3/2} \\ \boldsymbol{U}_{n+1} \\ \boldsymbol{\sigma}_{n+1} \end{pmatrix} = \mathbf{A}(\Delta t,\alpha) \begin{pmatrix} \boldsymbol{V}_{n+1/2} \\ \boldsymbol{U}_{n} \\ \boldsymbol{\sigma}_{n} \end{pmatrix}$$

Computations on a beam



Material parameters[2]

Length L	Section S	Density ρ	Young Modulus E	Number of elements N
1000 mm	100 mm^2	1.05×10^4 kg m^{-3}	880 MPa	20

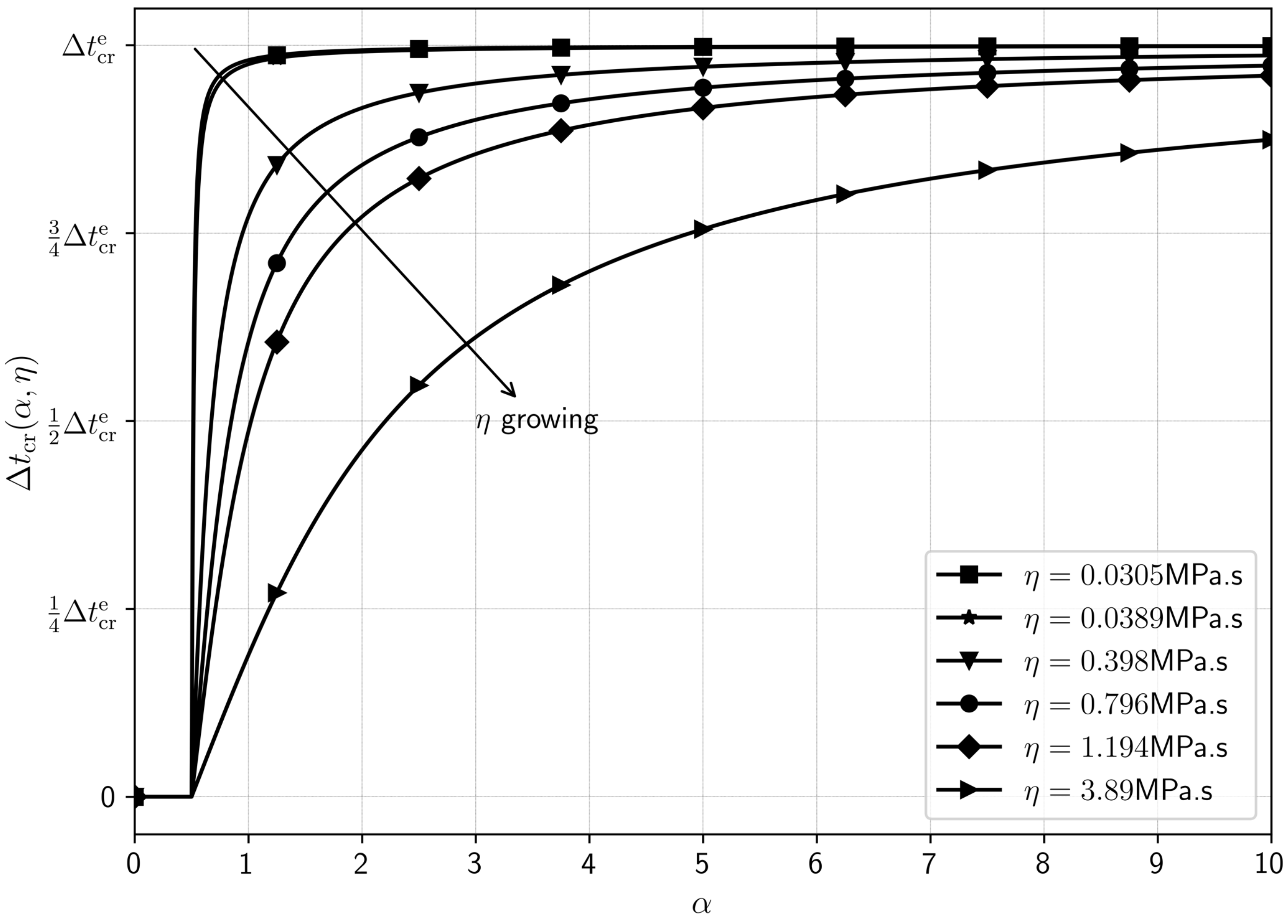
$$\mathbf{A}(\Delta t,\alpha) = \begin{pmatrix} \mathbf{M} & \Delta t\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{Id} & \mathbf{0} \\ \mathbf{0} & -E\mathbf{B}_\varepsilon & \mathbf{Id} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{M}-\Delta t\alpha^{-1}\mathbf{C} & -\Delta t\alpha^{-1}(1-\alpha)\mathbf{K} & \Delta t\alpha^{-1}(1-\alpha)\mathbf{B}_\sigma^{\mathsf{T}} \\ \Delta t\mathbf{Id} & \mathbf{Id} & \mathbf{0} \\ \alpha^{-1}\eta\mathbf{B}_\varepsilon & \alpha^{-1}(1-\alpha)E\mathbf{B}_\varepsilon & -\alpha^{-1}(1-\alpha)\mathbf{Id} \end{pmatrix}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{U}_{n+1}) = \mathbf{B}_\epsilon\boldsymbol{U}_{n+1}, \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{u}^\star)\mathrm{d}\Omega = \sum_{i=0}^{N_{\text{nodes}}} (u^\star)^i(t) \int_\Omega \boldsymbol{B}_\varepsilon^{i\mathsf{T}}\boldsymbol{\sigma}\mathrm{d}\Omega = \boldsymbol{U}^*.(\mathbf{B}_\sigma^{\mathsf{T}}\boldsymbol{\sigma})$$

1. Géradin, Michel, and Daniel Rixen. Mechanical Vibrations: Theory and Application to Structural Dynamics. Third edition. Chichester, West Sussex, United Kingdom: Wiley, 2015.
2. Hwang, Seyeon, Edgar I. Reyes, Kyoung-sik Moon, Raymond C. Rumpf, and Nam Soo Kim. 'Thermo-Mechanical Characterization of Metal/Polymer Composite Filaments and Printing Parameter Study for Fused Deposition Modeling in the 3D Printing Process'. Journal of Electronic Materials 44, no. 3 (March 2015): 771–77.

Constant time step

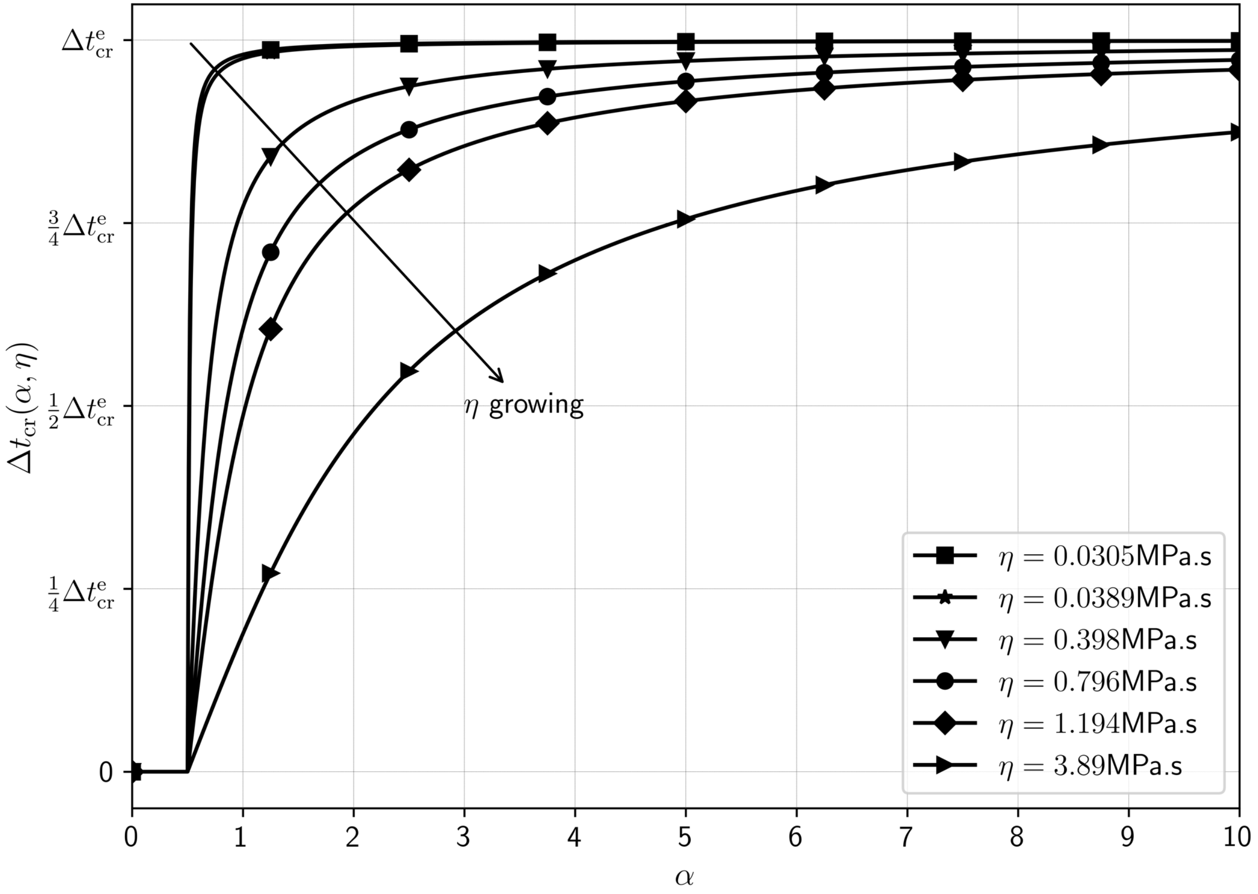
Stability analysis

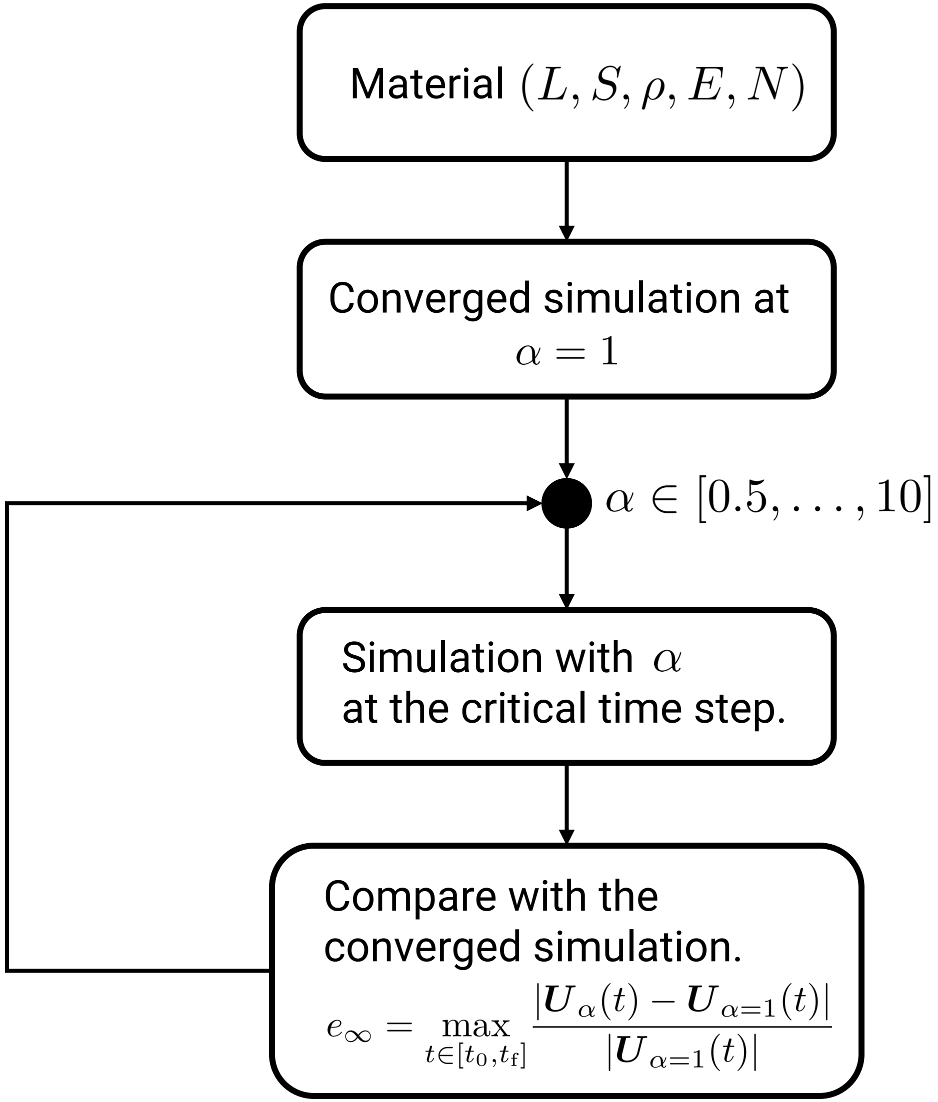


What is the impact of this approximation on the accuracy of the method ?

Constant time step

Stability analysis

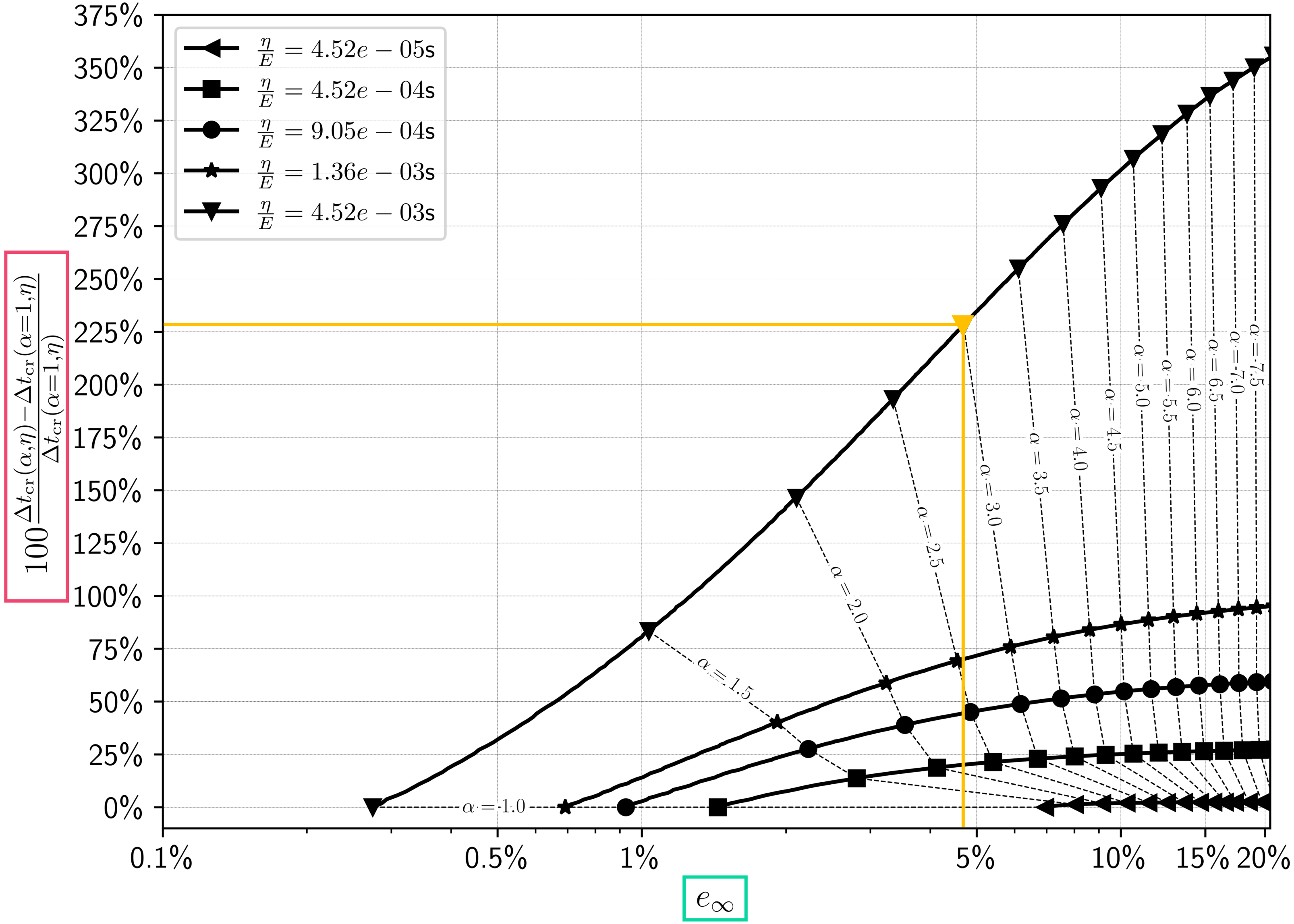




Constant time step

Stability analysis

Gain in critical time step with respect to the critical time step of Belytschko's approximation ($\alpha = 1$)

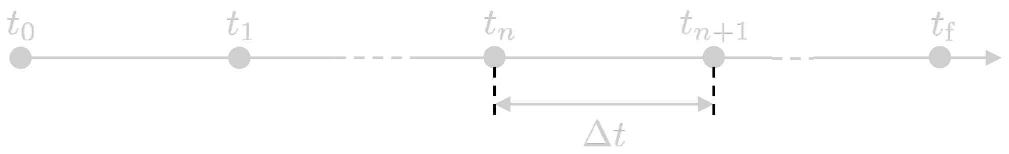


Comparison with the converged simulation

$$e_\infty = \max_{t\in[t_0,t_f]} \frac{|\boldsymbol{U}_\alpha(t) - \boldsymbol{U}_{\alpha=1}(t)|}{|\boldsymbol{U}_{\alpha=1}(t)|}$$

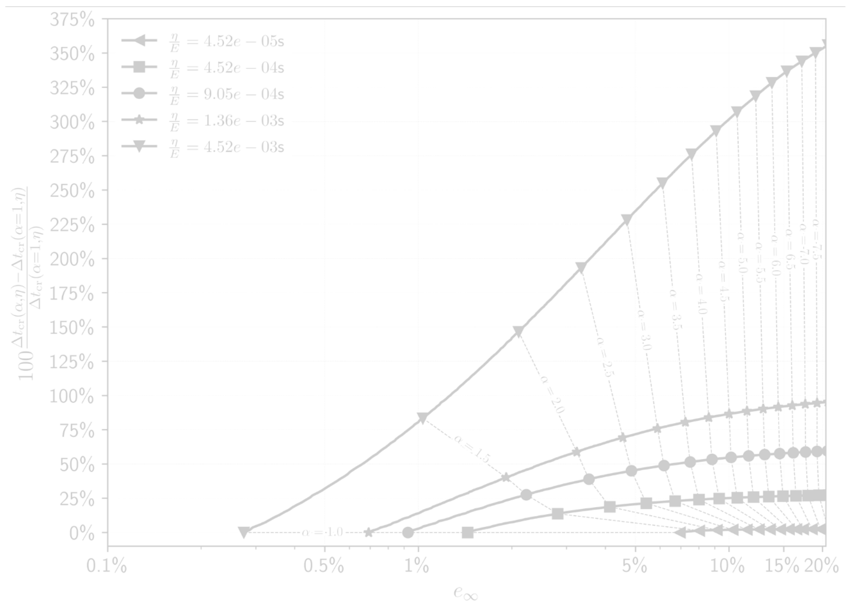
Outline of the presentation – main results

Constant time step



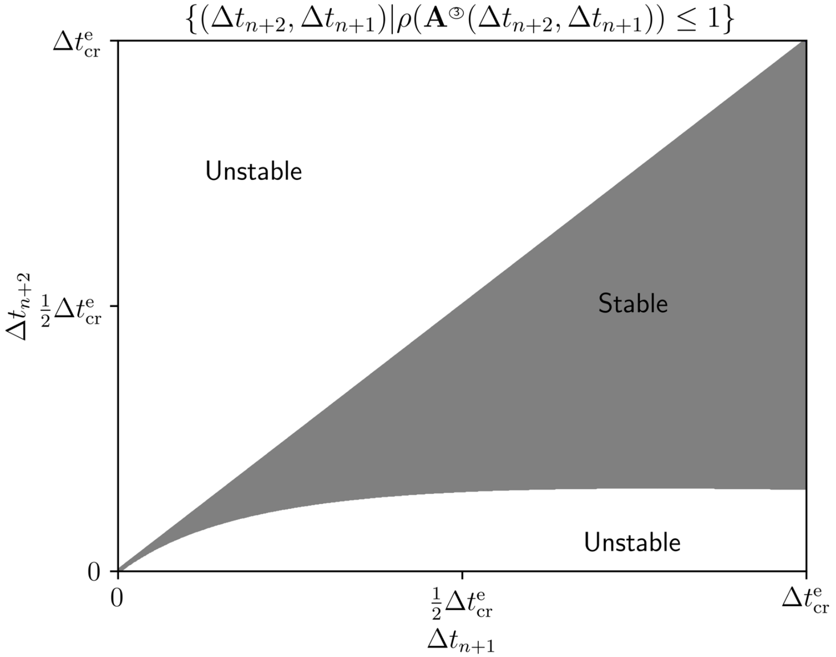
→ New integration method for the viscous stress-strain relationship with great stability properties.

$$(1-\alpha)\boldsymbol{\sigma}^{\mathrm{v}}_n + \alpha\boldsymbol{\sigma}^{\mathrm{v}}_{n+1} = \eta\mathbb{I} : \boldsymbol{\varepsilon}(\boldsymbol{v}_{n+1/2}), \alpha \in \mathbb{R}^+$$



Variable time step

→ Apparition of instability areas below the critical time step.
→ Impossibility to augment the time step within the simulation for a non dissipative system.
→ Only the strong stability analysis based on the formulation of the scheme as a multistep scheme is relevant.

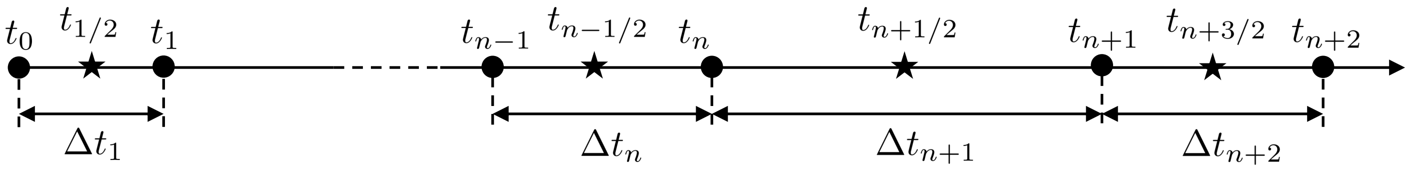


Variable time step

Motivations

How to build a variable time step grid such that the scheme remains stable ?

$$\Delta t_{n+1} = t_{n+1} - t_n$$



Robert D. Skeel, Joseph P. Wright[1,2,3]

- Study lead on the harmonic oscillator.

$$\ddot{u}(t) + \omega^2 u(t) = 0$$

- Unstable areas under the critical time step of constant time step because of the variable time step grid.

$$(\omega \Delta t) \leq 2 \Rightarrow \Delta t^{\mathrm{e}}_{\mathrm{cr}} = 2/\omega$$

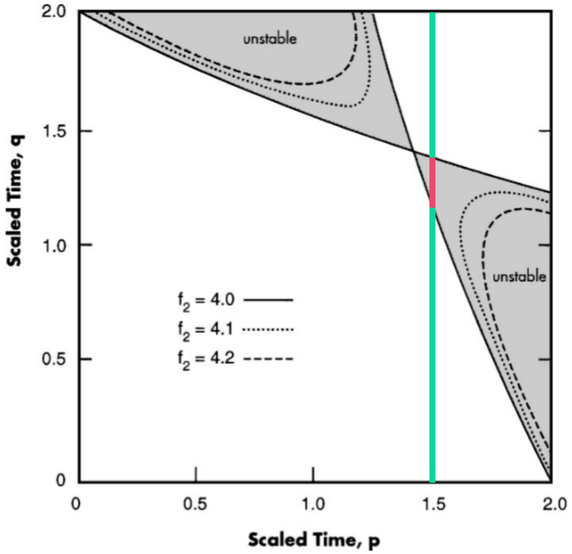


Figure from Joseph P. Wright (1998)[2]

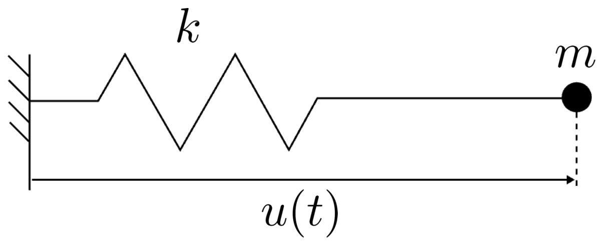
$$\mathbf{M}(\boldsymbol{V}_{n+3/2} - \boldsymbol{V}_{n+1/2}) = \Delta t(\boldsymbol{F}^{\mathrm{ext}}(t_{n+1}) - \boldsymbol{F}^{\mathrm{int}}_{n+1}) \rightsquigarrow m(v_{n+3/2} - v_{n+1/2}) = \frac{1}{2}(\Delta t_{n+2} + \Delta t_{n+1})(F^{\mathrm{ext}}_{n+1} - F^{\mathrm{int}}_{n+1})$$

1. Skeel, Robert D. 'Variable Step Size Destabilizes the Störmer/Leapfrog/Verlet Method'. *BIT* 33, no. 1 (March 1993): 172–75.
2. Wright, Joseph P. 'Numerical Instability Due to Varying Time Steps in Explicit Wave Propagation and Mechanics Calculations'. *Journal of Computational Physics* 140, no. 2 (March 1998): 421–31.
3. Skeel, Robert D. 'Comments on "Numerical Instability Due to Varying Time Steps in Explicit Wave Propagation and Mechanics Calculations" by Joseph P. Wright'. *Journal of Computational Physics* 145, no. 2 (September 1998): 758–59.

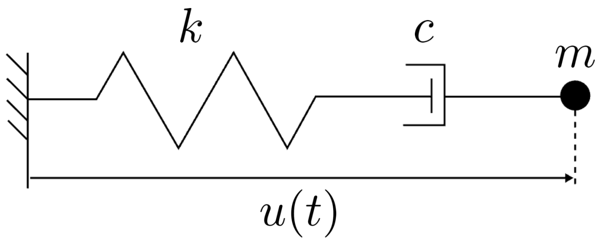
Variable time step

Motivations

Consider a mass-spring-damper system.



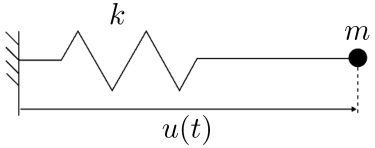
$$F^{\text{int}}_{n+1} = ku_{n+1}$$



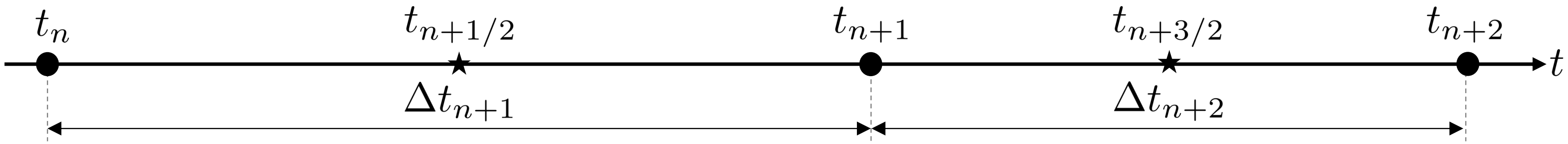
$$F^{\text{int}}_{n+1} = ku_{n+1} = cv_{n+1/2}$$

Variable time step

Elasticity : stability analysis



$$a_{n+1} = -\frac{k}{m} u_{n+1}$$



$$v_{n+1} = v_n + \frac{1}{2}\Delta t_{n+1}(a_n + a_{n+1})$$

$$u_{n+1} = u_n + \Delta t_{n+1} v_n + \frac{1}{2}\Delta t_{n+1}^2 a_n$$

$$\begin{pmatrix} v_{n+1} \\ u_{n+1} \end{pmatrix} = \mathbf{A}^{②}(\Delta t_{n+1}) \begin{pmatrix} v_n \\ u_n \end{pmatrix} \qquad \begin{pmatrix} v_{n+2} \\ u_{n+2} \end{pmatrix} = \mathbf{A}^{②}(\Delta t_{n+2}) \begin{pmatrix} v_{n+1} \\ u_{n+1} \end{pmatrix}$$

$$u_{n+1} = u_n + \Delta t_{n+1} v_{n+1/2}$$

$$m(v_{n+3/2} - v_{n+1/2}) = -k \frac{\Delta t_{n+2} + \Delta t_{n+1}}{2} u_{n+1}$$

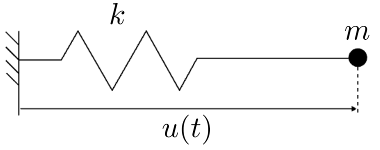
$$\begin{pmatrix} v_{n+3/2} \\ u_{n+1} \end{pmatrix} = \mathbf{A}^{①}(\Delta t_{n+2}, \Delta t_{n+1}) \begin{pmatrix} v_{n+1/2} \\ u_n \end{pmatrix}$$

$$u_{n+2} - \frac{\Delta t_{n+2} + \Delta t_{n+1}}{\Delta t_{n+1}} u_{n+1} + \frac{\Delta t_{n+2}}{\Delta t_{n+1}} u_n = -\frac{1}{2}(\Delta t_{n+2} + \Delta t_{n+1})\Delta t_{n+2} \frac{k}{m} u_{n+1}$$

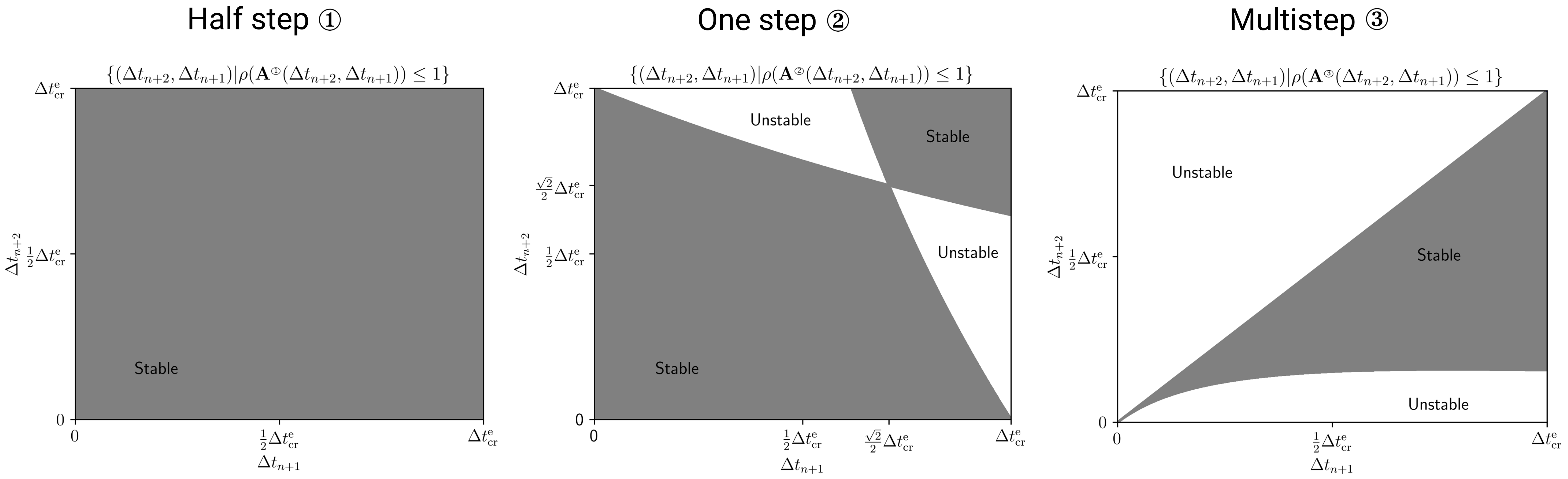
$$\begin{pmatrix} u_{n+2} \\ u_{n+1} \end{pmatrix} = \mathbf{A}^{③}(\Delta t_{n+2}, \Delta t_{n+1}) \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}$$

Variable time step

Elasticity : stability analysis



$F^{\mathrm{int}} = ku_{n+1}$



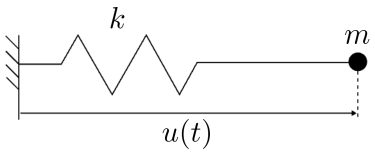
Why do these analyses differ ?

→ The amplification matrices from the three different formulations are not spectrally similar.

How can we know which analysis leads to a stable scheme ?

Variable time step

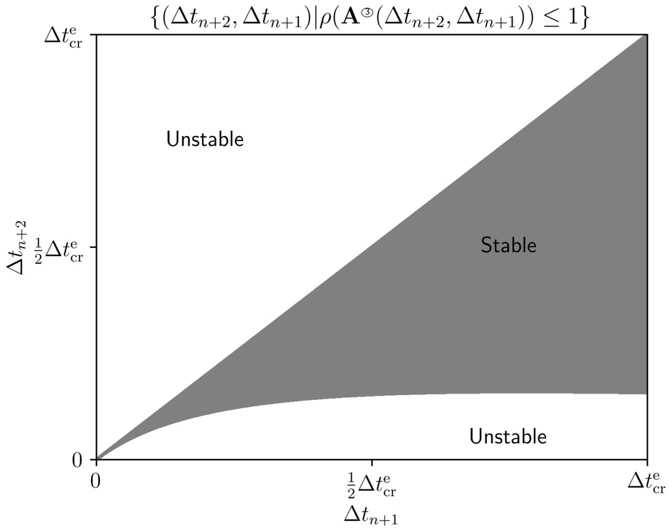
Elasticity : stability analysis



$F^{\text{int}} = k u_{n+1}$

Only the stability analysis from the multistep formulation can predict stability for the variable time step central difference.

Multistep ③



$$u_{n+2} - \frac{\Delta t_{n+2} + \Delta t_{n+1}}{\Delta t_{n+1}} u_{n+1} + \frac{\Delta t_{n+2}}{\Delta t_{n+1}} u_n = -\frac{1}{2}(\Delta t_{n+2} + \Delta t_{n+1})\Delta t_{n+2}\frac{k}{m} u_{n+1}$$

Generator polynomial of the method $\theta(\zeta)$

Strong stability[1]

$$\theta(\zeta) = \zeta^2 + \frac{\Delta t_{n+2} + \Delta t_{n+1}}{\Delta t_{n+1}}\left(-1 + \frac{1}{2}\Delta t_{n+2}\Delta t_{n+1}\frac{k}{m}\right)\zeta + \frac{\Delta t_{n+2}}{\Delta t_{n+1}}$$

Spectral stability

$$\begin{pmatrix} u_{n+2} \\ u_{n+1} \end{pmatrix} = \mathbf{A}^{③}(\Delta t_{n+2}, \Delta t_{n+1}) \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}$$

The generator polynomial of the method[1] is equal to the characteristic polynomial of the multistep amplification matrix.

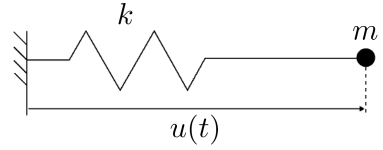
$$\theta(\zeta) = \det(\mathbf{A}^{③} - \zeta \mathbf{Id})$$

Within constant time step, the spectral stability analysis of the problem is exactly equivalent to the strong stability analysis of the problem.

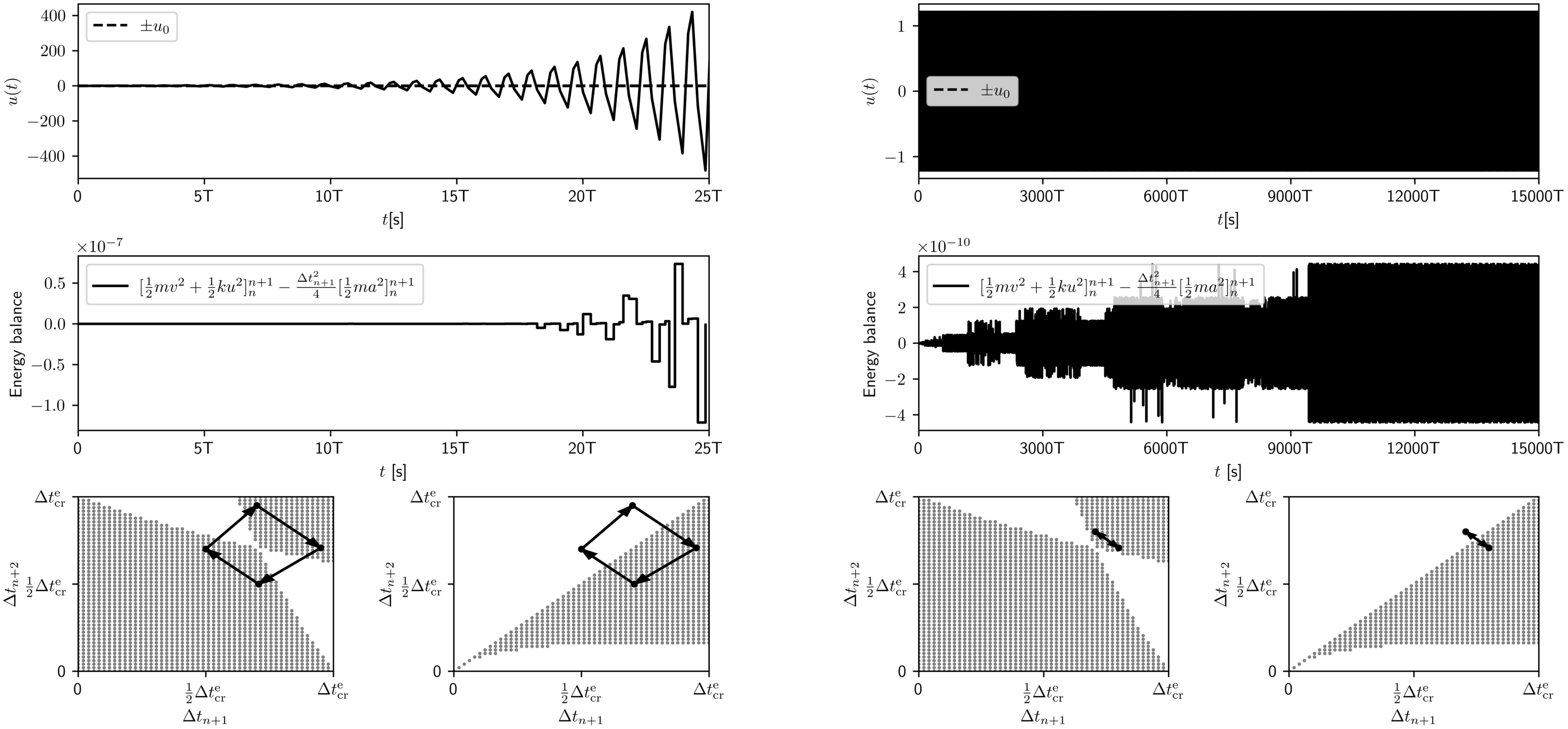
1. Dahlquist, Germund. 'Convergence and Stability in the Numerical Integration of Ordinary Differential Equations'. *Mathematica Scandinavica* 4, no. 1 (1956): 33–53.

Variable time step

Elasticity : examples

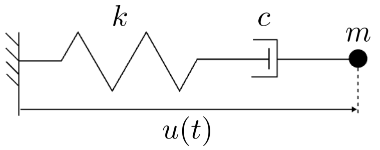


$F^{\mathrm{int}} = k u_{n+1}$



Variable time step

Viscoelasticity : stability analysis



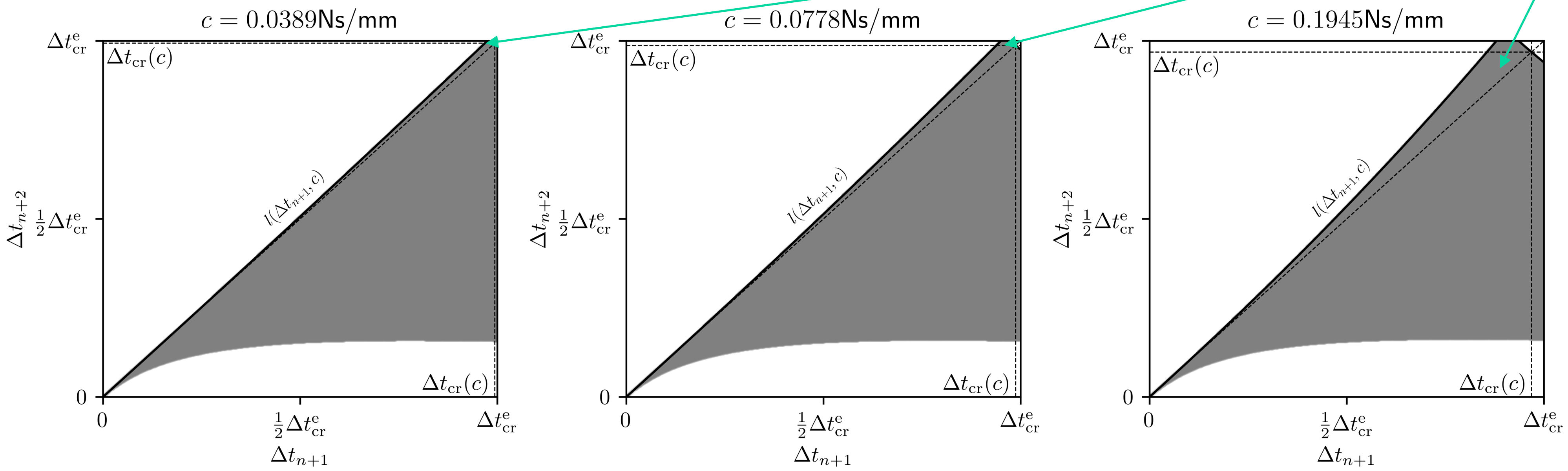
$$F^{\text{int}} = ku_{n+1} + cv_{n+1/2}$$

$$u_{n+1} = u_n + \Delta t_{n+1} v_{n+1/2}$$

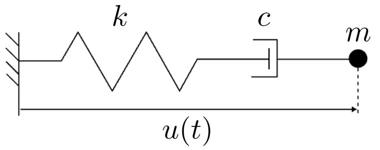
$$mu_{n+2} + \frac{\Delta t_{n+2} + \Delta t_{n+1}}{\Delta t_{n+1}} \left[-m + \frac{1}{2} \Delta t_{n+2} \Delta t_{n+1} (k + \frac{1}{\Delta t_{n+1}} c) \right] u_{n+1} + \frac{\Delta t_{n+2}}{\Delta t_{n+1}} \left[m - \frac{\Delta t_{n+2} + \Delta t_{n+1}}{2} c \right] u_n = 0$$

Perform a strong stability analysis (amplification matrix in the annex)

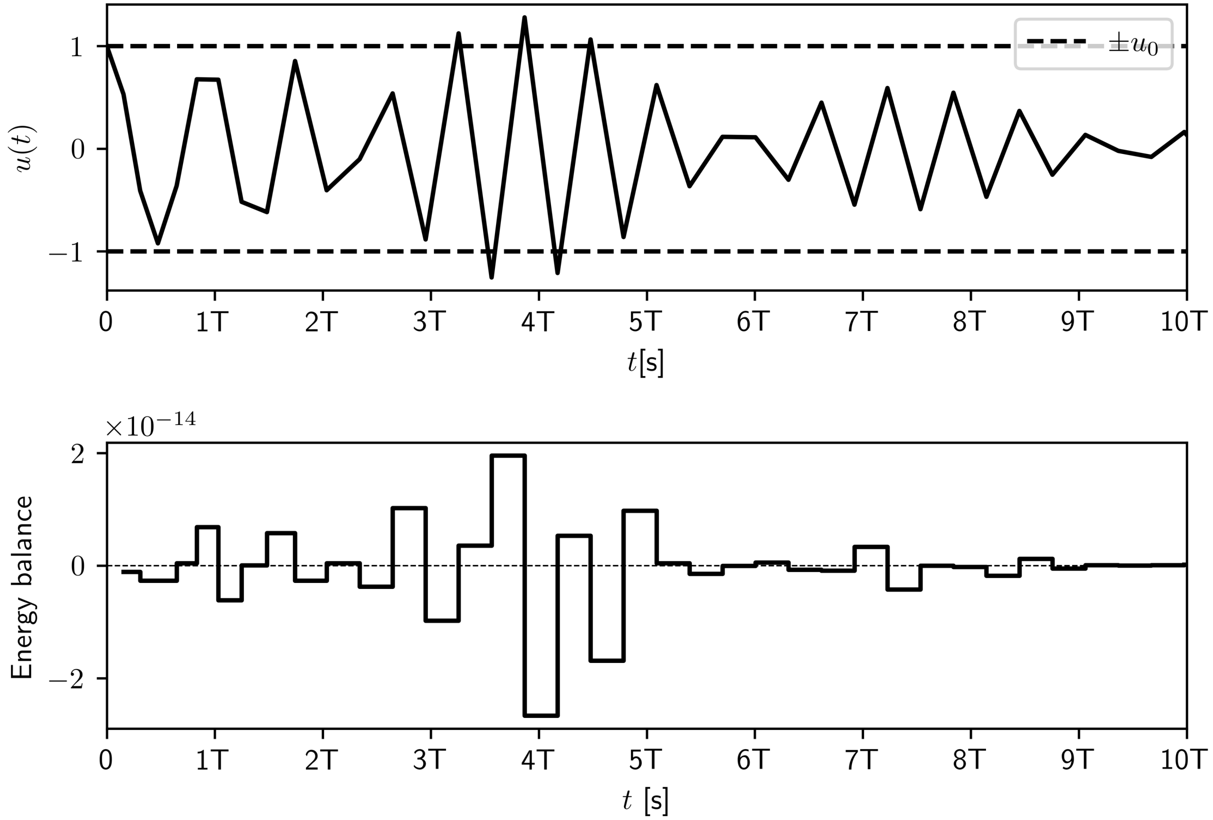
Possibility to increase the time step within a computation

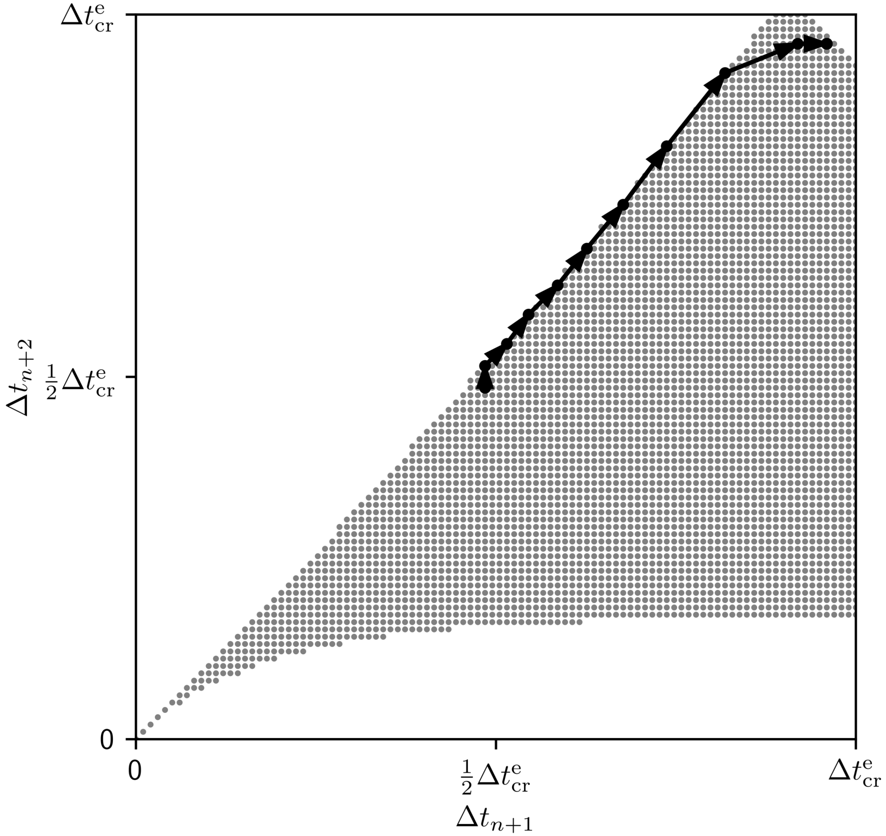


Variable time step
Viscoelasticity : stability analysis



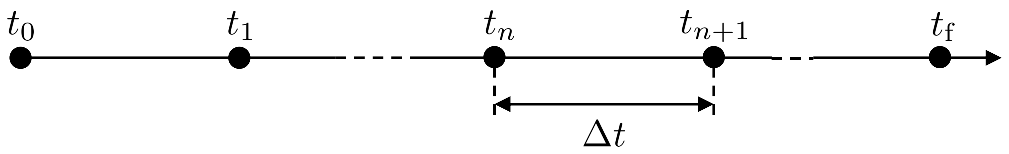
$$F^{\text{int}} = k u_{n+1} + c v_{n+1/2}$$





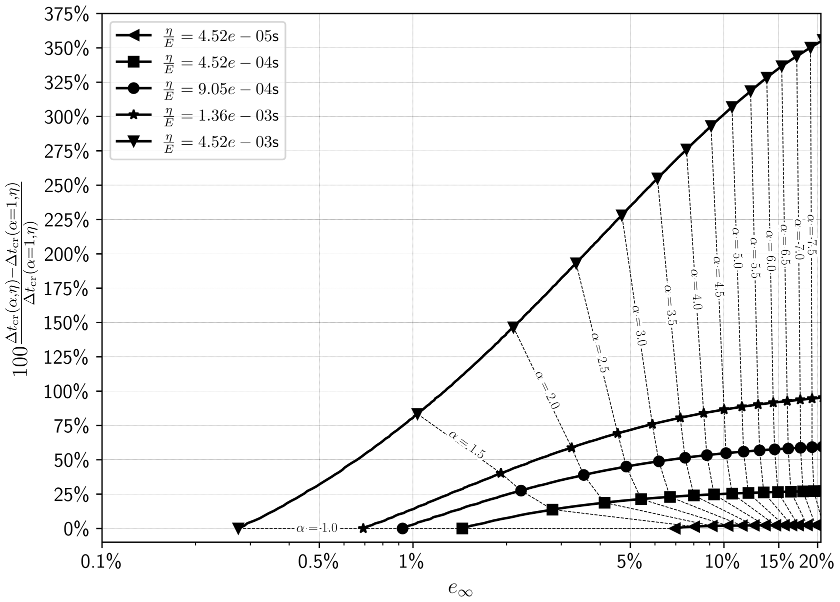
Summary - conclusion

Constant time step



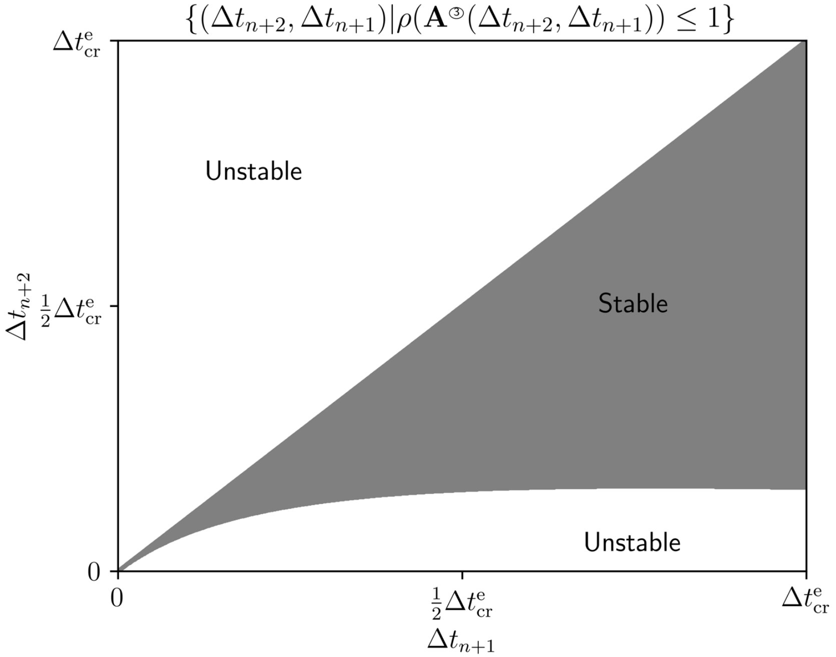
→ New integration method for the viscous stress-strain relationship with great stability properties.

$$(1-\alpha)\boldsymbol{\sigma}^{\mathrm{v}}_{n} + \alpha\boldsymbol{\sigma}^{\mathrm{v}}_{n+1} = \eta\mathbb{I} : \boldsymbol{\varepsilon}(\boldsymbol{v}_{n+1/2}), \alpha \in \mathbb{R}^{+}$$



Variable time step

→ Apparition of instability areas below the critical time step.
→ Impossibility to augment the time step within the simulation for a non dissipative system.
→ Only the strong stability analysis based on the formulation of the scheme as a multistep scheme is relevant.



Annex

Energy balance of the variable time step central difference method

The energy balance of the central difference method, in the case of a viscoelastic 0D problem, with the half lagged velocity approximation is, with $[u]_n^{n+1} = u_{n+1} - u_n$:

$$\left[\frac{1}{2}mv^2 + \frac{1}{2}ku^2\right]_n^{n+1} - \frac{\Delta t_{n+1}^2}{4}\left[\frac{1}{2}ma^2\right]_n^{n+1} = -\frac{1}{2}\frac{1}{\Delta t_{n+1}}c(u_{n+1} - u_n)^2 - \frac{1}{2}\frac{1}{\Delta t_n}c(u_{n+1} - u_n)(u_n - u_{n-1}) \tag{1}$$

Between t_n and t_{n+2}, the energy balance is:

$$\begin{aligned}\left[\frac{1}{2}mv^2 + \frac{1}{2}ku^2\right]_n^{n+2} - \frac{\Delta t_{n+1}^2}{4}\left[\frac{1}{2}ma^2\right]_n^{n+1} - \frac{\Delta t_{n+2}^2}{4}\left[\frac{1}{2}ma^2\right]_{n+1}^{n+2} \\ = -\frac{1}{2}\frac{1}{\Delta t_{n+2}}c(u_{n+2} - u_{n+1})^2 \\ -\frac{1}{2}\frac{1}{\Delta t_{n+1}}c(u_{n+1} - u_n)^2 - \frac{1}{2}\frac{1}{\Delta t_{n+1}}c(u_{n+2} - u_{n+1})(u_{n+1} - u_n) \\ -\frac{1}{2}\frac{1}{\Delta t_n}c(u_{n+1} - u_n)(u_n - u_{n-1})\end{aligned} \tag{2}$$

Annex

Energy balance of the variable time step central difference method - proof

We first express the central difference equation as introduced by Newmark, defining the mean $\langle u \rangle_n^{n+1} = \frac{1}{2}(u_{n+1} + u_n)$ and difference operator $[u]_n^{n+1} = (u_{n+1} - u_n)$:

$$[v]_n^{n+1} = \Delta t_{n+1} \langle a \rangle_n^{n+1} \tag{1}$$

$$[u]_n^{n+1} = \Delta t_{n+1} \langle v \rangle_n^{n+1} - \tfrac{1}{4}\Delta t_{n+1}^2 [a]_n^{n+1} \tag{2}$$

Then, the kinetic energy balance equation leads to, knowing that $ma_{n+1} = -ku_{n+1} - cv_{n+1/2}$:

$$\begin{aligned} \left[\tfrac{1}{2}mv^2\right]_n^{n+1} &= \langle v \rangle_n^{n+1} m [v]_n^{n+1} \\ &= \langle a \rangle_n^{n+1} m([u]_n^{n+1} + \tfrac{1}{4}\Delta t_{n+1}^2 [a]_n^{n+1}) \\ &= - \langle u \rangle_n^{n+1} k [u]_n^{n+1} + \tfrac{1}{4}\Delta t_{n+1}^2 \langle a \rangle_n^{n+1} m [a]_n^{n+1} \\ &\quad - \tfrac{1}{2}\tfrac{1}{\Delta t_{n+1}} c(u_{n+1} - u_n)(u_{n+1} - u_n) - \tfrac{1}{2}\tfrac{1}{\Delta t_n} c(u_{n+1} - u_n)(u_n - u_{n-1}) \end{aligned} \tag{3}$$

When summed with the kinetic energy balance between t_{n+2} and t_{n+1}, one obtains the equation stated in the previous slide.

Annex

Amplification matrix of the half step formulation

$$\mathbf{A}^{①}(\Delta t_{n+2}, \Delta t_{n+1}) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{1}$$

$$A_{11} = 1 - \frac{1}{2}\Delta t_{n+1}(\Delta t_{n+2} + \Delta t_{n+1})\frac{k}{m} \tag{2}$$

$$A_{12} = -\frac{1}{2}(\Delta t_{n+2} + \Delta t_{n+1})\frac{k}{m} \tag{3}$$

$$A_{21} = \Delta t_{n+1} \tag{4}$$

$$A_{22} = 1 \tag{5}$$

The analysis of the amplification matrix defined by the half step formulation of the central difference method leads to the following condition on time steps:

$$4 - \frac{1}{2}\Delta t_{n+1}(\Delta t_{n+2} + \Delta t_{n+1})\frac{k}{m} \geq 0 \tag{6}$$

Only the condition $1 + \det(\mathbf{A}^{①}) + \mathrm{tr}(\mathbf{A}^{①}) \geq 0$ is not true for all $\Delta t_{n+2}, \Delta t_{n+1}$, giving the condition stated above.

The condition is always true below the critical time step $\Delta t^{\mathrm{e}}_{\mathrm{cr}} = 2/\omega$ with $\omega = \sqrt{k/m}$. Then, this stability analysis states that there is no condition on the evolution of time steps below this critical time step.

Annex

Amplification matrix of the one step formulation

$$\mathbf{A}^{②}(\Delta t_{n+2}, \Delta t_{n+1}) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$\begin{aligned}
A_{11} &= (1 - \frac{1}{2}\Delta t_{n+1}^2 \frac{k}{m})(1 - \frac{1}{2}\Delta t_{n+2}^2 \frac{k}{m}) + \Delta t_{n+1}\Delta t_{n+2}\frac{k}{m}(-1 + \frac{1}{4}\Delta t_{n+2}^2 \frac{k}{m}) \\
A_{12} &= (1 - \frac{1}{2}\Delta t_{n+2}^2 \frac{k}{m})\Delta t_{n+1}\frac{k}{m}(-1 + \frac{1}{4}\Delta t_{n+1}^2 \frac{k}{m}) + \\
&+ (1 - \frac{1}{2}\Delta t_{n+1}^2 \frac{k}{m})\Delta t_{n+2}\frac{k}{m}(-1 + \frac{1}{4}\Delta t_{n+2}^2 \frac{k}{m}) \\
A_{21} &= \Delta t_{n+1}(1 - \frac{1}{2}\Delta t_{n+2}^2 \frac{k}{m}) + \Delta t_{n+2}(1 - \frac{1}{2}\Delta t_{n+1}^2 \frac{k}{m}) \\
A_{22} &= (1 - \frac{1}{2}\Delta t_{n+1}^2 \frac{k}{m})(1 - \frac{1}{2}\Delta t_{n+2}^2 \frac{k}{m}) + \Delta t_{n+1}\Delta t_{n+2}\frac{k}{m}(-1 + \frac{1}{4}\Delta t_{n+1}^2 \frac{k}{m})
\end{aligned}$$

Studying the invariants of the amplification matrix leads to the two following conditions:

$$(\Delta t_{n+2} + \Delta t_{n+1})^2 (1 - \frac{1}{4}\frac{k}{m}\Delta t_{n+2}\Delta t_{n+1}) \geq 0 \tag{1}$$

$$2 - (\Delta t_{n+2} + \Delta t_{n+1})^2 (1 - \frac{1}{4}\frac{k}{m}\Delta t_{n+2}\Delta t_{n+1}) \geq 0 \tag{2}$$

Annex

Amplification matrix of the multistep formulation

$$\mathbf{A}^{③}(\Delta t_{n+2}, \Delta t_{n+1}) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$A_{11} = \frac{1}{\Delta t_{n+1}}(\Delta t_{n+2} + \Delta t_{n+1})(1 - \frac{1}{2}\Delta t_{n+2}\Delta t_{n+1})$$

$$A_{12} = -\frac{\Delta t_{n+2}}{\Delta t_{n+1}}$$

$$A_{21} = 1$$

$$A_{22} = 0$$

Annex

Amplification matrix of the multistep formulation

The analysis of the amplification matrix defined by the multistep formulation leads to the following conditions:

$$\Delta t_{n+2}\Delta t_{n+1} \leq (\Delta t^{\mathrm{e}}_{\mathrm{cr}})^2 \tag{1}$$

$$\Delta t_{n+2} \leq \Delta t_{n+1} \tag{2}$$

$$(\Delta t_{n+2} + \Delta t_{n+1})^2(1 - \tfrac{1}{2}\Delta t_{n+2}\Delta t_{n+1}\frac{k}{m})^2 - 4\Delta t_{n+1}\Delta t_n < 0 \tag{3}$$

The first condition is given by :

$$\begin{aligned} 1 + \det(\mathbf{A}^{③}) + \mathrm{tr}(\mathbf{A}^{③}) \geq 0 &\Leftrightarrow (\Delta t_{n+2} + \Delta t_{n+1})(2 - \tfrac{1}{2}\Delta t_{n+2}\Delta t_{n+1}\frac{k}{m}) \geq 0 && (4)\\ &\Leftrightarrow \Delta t_{n+2}\Delta t_{n+1} \leq (\Delta t^{\mathrm{e}}_{\mathrm{cr}})^2 && (5)\end{aligned}$$

The second condition is given by $1 - \det(\mathbf{A}^{③}) \geq 0$. Finally, the last condition is obtained through $\mathrm{tr}^2(\mathbf{A}^{③}) - 4\det(\mathbf{A}^{③}) < 0$. Other conditions $(1 + \det(\mathbf{A}^{③}) \geq 0, 1 + \det(\mathbf{A}^{③}) - \mathrm{tr}(\mathbf{A}^{③}) \geq 0)$ lead to inequalities always true for any Δt_{n+2}, Δt_{n+1}.

Annex

Spectral similarity – the three formulations of the variable time step CD scheme are not spectrally similar

The study must be lead on two consecutive time steps. Each state vector is expressed as a function of the other formulations, thanks to the following matrices:

$$\begin{pmatrix} v_{n+3/2} \\ u_{n+1} \end{pmatrix} = \mathbf{B}_{n+2} \begin{pmatrix} u_{n+2} \\ u_{n+1} \end{pmatrix} \text{ with } \mathbf{B}_{n+2} = \frac{1}{\Delta t_{n+2}} \begin{pmatrix} 1 & -1 \\ 0 & \Delta t_{n+2} \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} v_{n+1} \\ u_{n+1} \end{pmatrix} = \mathbf{D}_{n+2} \begin{pmatrix} u_{n+2} \\ u_{n+1} \end{pmatrix} \text{ with } \mathbf{D}_{n+2} = \frac{1}{\Delta t_{n+2}} \begin{pmatrix} 1 & -(1 - \frac{1}{2}\Delta t_{n+2}^2 \frac{k}{m}) \\ 0 & \Delta t_{n+2} \end{pmatrix} \tag{2}$$

Since $\det(\mathbf{B}_{n+2}) = \det(\mathbf{D}_{n+2}) = 1$, the matrices can be inverted, and:

$$\begin{pmatrix} v_{n+1} \\ u_{n+1} \end{pmatrix} = \mathbf{D}_{n+2}\mathbf{B}_{n+2}^{-1} \begin{pmatrix} v_{n+3/2} \\ u_{n+1} \end{pmatrix} \tag{3}$$

We then compute the relationships between amplification matrices:

$$② \to ① \begin{pmatrix} v_{n+3/2} \\ u_{n+1} \end{pmatrix} = (\mathbf{D}_{n+2}\mathbf{B}_{n+2}^{-1})^{-1}\mathbf{A}^{②}(\Delta t_{n+2}, \Delta t_{n+1})(\mathbf{D}_{n+1}\mathbf{B}_{n+1}^{-1}) \begin{pmatrix} v_{n+1/2} \\ u_n \end{pmatrix} \tag{4}$$

$$③ \to ② \begin{pmatrix} v_{n+1} \\ u_{n+1} \end{pmatrix} = \mathbf{D}_{n+2}\mathbf{A}^{③}(\Delta t_{n+2}, \Delta t_{n+1})\mathbf{D}_{n+1}^{-1} \begin{pmatrix} v_n \\ u_n \end{pmatrix} \tag{5}$$

$$① \to ③ \begin{pmatrix} u_{n+2} \\ u_{n+1} \end{pmatrix} = \mathbf{B}_{n+2}^{-1}\mathbf{A}^{①}(\Delta t_{n+2}, \Delta t_{n+1})\mathbf{B}_{n+1} \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} \tag{6}$$

which lead to the following equations:

$$\mathbf{A}^{①} = (\mathbf{D}_{n+2}\mathbf{B}_{n+2}^{-1})^{-1}\mathbf{A}^{②}(\Delta t_{n+2}, \Delta t_{n+1})(\mathbf{D}_{n+1}\mathbf{B}_{n+1}^{-1}) \tag{7}$$

$$\mathbf{A}^{②} = \mathbf{D}_{n+2}\mathbf{A}^{③}(\Delta t_{n+2}, \Delta t_{n+1})\mathbf{D}_{n+1}^{-1} \tag{8}$$

$$\mathbf{A}^{③} = \mathbf{B}_{n+2}^{-1}\mathbf{A}^{①}(\Delta t_{n+2}, \Delta t_{n+1})\mathbf{B}_{n+1} \tag{9}$$

With constant time steps, $\mathbf{B}_{n+2} = \mathbf{B}_{n+1}$, $\mathbf{D}_{n+2} = \mathbf{D}_{n+1}$ and the amplification matrices are similar. However, in variable time step, the previous relationships show that they are only equivalent.

Annex

Viscoelastic amplification matrix (based on the multistep formulation)

The multistep formulation of this method is

$$m u_{n+2} + \frac{\Delta t_{n+2} + \Delta t_{n+1}}{\Delta t_{n+1}} \left[-m + \frac{1}{2} \Delta t_{n+2} \Delta t_{n+1} (k + \frac{1}{\Delta t_{n+1}} c) \right] u_{n+1} + \\ + \frac{\Delta t_{n+2}}{\Delta t_{n+1}} \left[m - \frac{\Delta t_{n+2} + \Delta t_{n+1}}{2} c \right] u_n = \frac{1}{2} (\Delta t_{n+2} + \Delta t_{n+1}) \Delta t_{n+2} F^{\mathrm{ext}}_{n+1} \tag{1}$$

The amplification matrix is then:

$$\mathbf{A}(\Delta t_{n+2}, \Delta t_{n+1}) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{2}$$

$$A_{11} = \frac{\Delta t_{n+2} + \Delta t_{n+1}}{\Delta t_{n+1}} \left(1 - \frac{1}{2} \Delta t_{n+2} \Delta t_{n+1} (\frac{k}{m} + \frac{1}{\Delta t_{n+1}} \frac{c}{m}) \right) \tag{3}$$

$$A_{12} = -\frac{\Delta t_{n+2}}{\Delta t_{n+1}} \left(1 - \frac{\Delta t_{n+2} + \Delta t_{n+1}}{2} \frac{c}{m} \right) \tag{4}$$

$$A_{21} = 1 \tag{5}$$

$$A_{22} = 0 \tag{6}$$

Annex

Viscoelastic amplification matrix (based on the multistep formulation)

The stability of the variable time step central difference scheme for transient viscoelastic 0D problem leads to the following conditions:

$$\Delta t_{n+2} \leq \frac{2}{\dfrac{c}{m} + \dfrac{1}{2}\dfrac{k}{m}\Delta t_{n+1}} \tag{1}$$

$$\begin{aligned}(\Delta t_{n+2} + \Delta t_{n+1})^2 \left(1 - \frac{1}{2}\Delta t_{n+2}\Delta t_{n+1}(\frac{k}{m} + \frac{1}{\Delta t_{n+1}}\frac{c}{m})\right)^2 + \\ - 4\Delta t_{n+2}\Delta t_{n+1}(1 - \frac{1}{2}(\Delta t_{n+2} + \Delta t_{n+1})\frac{c}{m}) < 0\end{aligned} \tag{2}$$

$$\forall c \neq 0,$$

$$\left|\begin{array}{l} \forall \Delta t_{n+1} \in \left[2(3-2\sqrt{2})\frac{m}{c}; 2(3+2\sqrt{2})\frac{m}{c}\right] \\ \quad \left| \ \Delta t_{n+2} \in \mathbb{R}^+ \right. \\ \forall \Delta t_{n+1} \in \left]0, 2(3-2\sqrt{2})\frac{m}{c}\right] \cup \left[2(3+2\sqrt{2})\frac{m}{c}, +\infty\right[\\ \quad \left|\begin{array}{l} \Delta t_{n+2} \in \left]0, l(\Delta t_{n+1})\right] \cup \left[g(\Delta t_{n+1}), +\infty\right[\\ \text{with} \begin{cases} l(\Delta t_{n+1}) = \frac{m}{c} - \frac{1}{2}\Delta t_{n+1} - \frac{m}{c}\sqrt{\frac{1}{4}\left(\frac{c}{m}\right)^2 \Delta t_{n+1}^2 - 3\frac{c}{m}\Delta t_{n+1} + 1} \\ r(\Delta t_{n+1}) = \frac{m}{c} - \frac{1}{2}\Delta t_{n+1} + \frac{m}{c}\sqrt{\frac{1}{4}\left(\frac{c}{m}\right)^2 \Delta t_{n+1}^2 - 3\frac{c}{m}\Delta t_{n+1} + 1} \end{cases} \end{array}\right. \end{array}\right. \tag{3}$$

Annex

Viscoelastic amplification matrix (based on the multistep formulation)

The first condition is given by:

$$1 + \det(\mathbf{A}) + \mathrm{tr}(\mathbf{A}) \geq 0 \Leftrightarrow (\Delta t_{n+2} + \Delta t_{n+1})(2 - \frac{1}{2}\Delta t_{n+2}\Delta t_{n+1}\frac{k}{m} - \Delta t_{n+2}\frac{c}{m}) \geq 0 \tag{1}$$

Since $\Delta t_{n+1}, \Delta t_{n+2} \in \mathbb{R}^+$, we obtain:

$$(2 - \frac{1}{2}\Delta t_{n+2}\Delta t_{n+1}\frac{k}{m} - \Delta t_{n+2}\frac{c}{m}) \geq 0 \Leftrightarrow \Delta t_{n+2} \leq \frac{2}{\frac{c}{m} + \frac{1}{2}\Delta t_{n+1}\frac{k}{m}} \tag{2}$$

The second condition is given by:

$$\mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A}) < 0 \tag{3}$$

Annex

Viscoelastic amplification matrix (based on the multistep formulation)

The final third condition is given by:

$$
\begin{aligned}
&1 - \det(\mathbf{A}(\Delta t_{n+2}, \Delta t_{n+1})) \geq 0 \\
&\Leftrightarrow \underbrace{\frac{1}{2}\frac{c}{m}\Delta t_{n+2}^2 + \left(-1 + \frac{1}{2}\frac{c}{m}\Delta t_{n+1}\right)\Delta t_{n+2} + \Delta t_{n+1}}_{\mathcal{P}(\Delta t_{n+2})} \geq 0.
\end{aligned} \tag{1}
$$

For $c = 0$ the condition $1 - \det(\mathbf{A})$ leads to $\Delta t_{n+2} \leq \Delta t_{n+1}$. The following developments are true only for $c \neq 0$. The polynomial $\mathcal{P}(\Delta t_{n+2})$ is positive for certain values of the discriminant:

$$
\delta^{[\Delta t_{n+2}]}(\Delta t_{n+1}) = \frac{1}{4}\left(\frac{c}{m}\right)^2 \Delta t_{n+1}^2 - 3\frac{c}{m}\Delta t_{n+1} + 1 \tag{2}
$$

For:

$$
\Delta t_{n+1} \in \left[2(3-2\sqrt{2})\frac{m}{c}; 2(3+2\sqrt{2})\frac{m}{c}\right] \tag{3}
$$

the discriminant $\delta^{[\Delta t_{n+2}]}(\Delta t_{n+1})$ is negative, and the polynomial $\mathcal{P}(\Delta t_{n+2})$ remains positive for all Δt_{n+2}. For:

$$
\Delta t_{n+1} \in \left]0, 2(3-2\sqrt{2})\frac{m}{c}\right] \cup \left[2(3+2\sqrt{2})\frac{m}{c}, +\infty\right[\tag{4}
$$

$\delta^{[\Delta t_{n+2}]}(\Delta t_{n+1})$ is positive, then the polynomial $\mathcal{P}(\Delta t_{n+2})$ is positive for certain values of Δt_{n+2}, determined by its left and right roots, defined in the proposition respectively by $l(\Delta t_{n+1})$ and $r(\Delta t_{n+1})$, which leads to the conditions (**??**) stated in the proposition.