A new stability analysis of variable time step central difference method for transient dynamics viscoelastic problems

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Summary. We develop a new explicit integration method for transient dynamics computation of viscoelastic materials, surpassing the stability limits of Belytschko's widely used half-lagged velocity approximation. Based on the central difference (CD) scheme, our method integrates the viscous stress strain law, while keeping an explicit scheme. Moreover, we provide a new stability analysis of the variable time step central difference method. We prove that CD's stability can be ensured only thanks to the zero-stability criteria, based on the multistep formulation of the method. We perform analytical developments on a single degree of freedom problem. Thus, we show that, without viscous damping, ensuring stability requires to decrease, or at least to maintain constant the time step during the time integration. Furthermore, we prove that, with viscous damping, there exists a slight possibility of increasing the time step during the time integration.

Extended abstract

Among many methods, explicit integration schemes are attractive in transient dynamic problem solving, due to their potential computing speed (growing with the use of parallel computing), thanks to a possible matrix-free resolution. The central difference method retains our attention in this communication [3, 1].

The α viscous stress averaging scheme

Transient viscoelastic problems pose a challenge when integrating the viscous stress-strain law while keeping an explicit scheme. Specifically, for a classic Kelvin material, a direct integration of the time-discrete viscous strain-stress relationship $\boldsymbol{\sigma}^{\mathrm{v}}_{n+1} = (\eta\mathbb{I}) : \dot{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{n+1})$, where $\boldsymbol{\sigma}^{\mathrm{v}}_{n+1}$ is the discretized viscous stress tensor at t_{n+1}, η is the viscosity parameter, and $\dot{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{n+1})$ is the time derivative of the strain tensor calculated from the displacement $\boldsymbol{u}_{n+1}$ at t_{n+1}, results in an implicit scheme. To adress this issue, Belytschko [1] introduces a lag of a half time step within velocities in the calculation the viscous strain stress law, which leads to the classical central difference scheme for transient dynamics of viscoelastic materials. To improve this approximation, we propose a viscous stress-averaging method around $t_{n+1/2}$, by introducing a new parameter $\alpha \in \mathbb{R}^+$ (with $\boldsymbol{v}_{n+1/2}$ the velocity at time $t_{n+1/2}$):

$$(1-\alpha)\boldsymbol{\sigma}^{\mathrm{v}}_{n} + \alpha\boldsymbol{\sigma}^{\mathrm{v}}_{n+1} = \eta\mathbb{I} : \boldsymbol{\varepsilon}(\boldsymbol{v}_{n+1/2}) \tag{1}$$

This integration method of the viscous stress strain laws keeps the scheme explicit. By setting $\alpha = 1$, we come back to the previous approximation designed by Belytschko. $\alpha = 0$, leads to an implicit scheme and is excluded from the analysis. When computing the critical time step of the method, based on a spectral stability criteria, we find that this new integration method increases stability boundaries. Computations were made on a bar, with different viscosity parameters. For a given material, and a spatial discretization, we varied the parameter α, and observed the impact on the critical time step. Figure 1 represents, for the bar with different viscosities, the time step gain as a function of the error e_∞, for different values of α. Dotted lines represent iso-α curves.

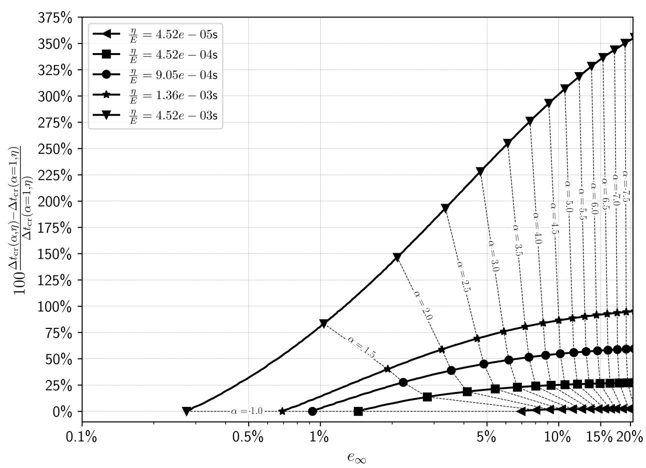
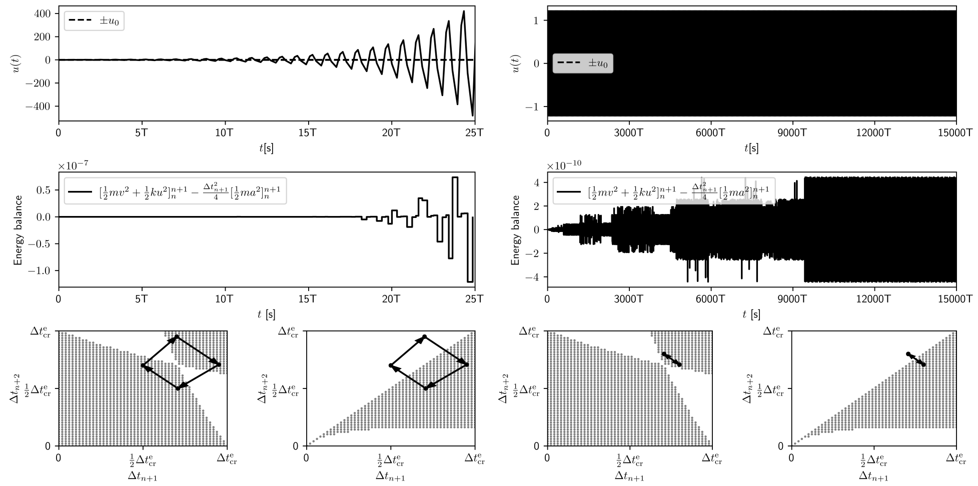


Figure 1: Critical time step gain versus error for different values of η and α.

For highly viscous materials, e.g., for $\eta/E = 4.52 \times 10^{-3}\,\mathrm{s}$ and $\alpha = 3$, the gain in critical time step equals to 225% compared to previous known approximation ($\alpha = 1$), with the infinite norm error remaining below 4%.

New and comprehensive analysis of the variable time step central difference scheme

Variable time step schemes may behave differently from constant time step scheme, thus requiring further studies. Skeel, and Wright afterwards [4, 5], have reported that variable time step central difference method inherits unstable areas under the classical critical time step. Thus, bounds for the evolution of the time step are calculated by Wright [5], conducting a spectral stability analysis of the one step formulation of the central difference method. A single degree of freedom spring-damper-mass system is considered. Therefore, we prove that only the zero-stability analysis [2], based on the formulation of the central difference method as a linear multistep method, leads to a stable scheme. Moreover, in the elastic case, the analysis points out the impossibility of increasing the time step during the simuation. Figures 2a and 2b depict the computation of displacement, energy balance, and the stable domains of time step evolution for both one-step [5] and multistep formulations with increasing time steps, as per [5]. Furthermore, we show that, with viscous damping, the time step can increase under an analytic limit, as shown in Figure 3.



(a)

(b) 62773 time steps.

Figure 2: Displacement, energy balance, time steps in stable zone predicted thanks to the one step and multistep formulations.

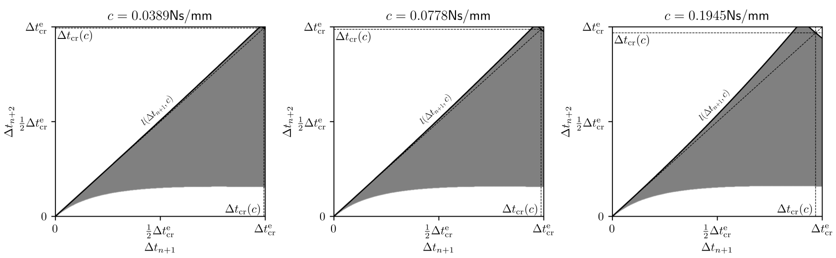


Figure 3: Stability area (in grey) for different viscous damping parameters.

Summary

In summary, our research provides a new explicit integration method for transient viscoelastic problems, and presents a new comprehensive stability analysis of the variable time step central difference method.

References

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