Problem 1 - Multiples of 3 or 5

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1 Query

If we list all the natural numbers below 10 that are multiples of 3 or 5, we get 3, 5, 6 and 9. The sum of these multiples is 23.

Find the sum of all the multiples of 3 or 5 below 1000.

2 Theory

Where's the fun in coding a brute force solution?

When counting elements of two sets A and B with a nonempty intersection, we have:

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{1}$$

The sum S of the first n natural numbers is given by the below formula:

$$\begin{aligned} S &= \sum^{n} i \\ &= \frac{n \cdot (n-1)}{2} \end{aligned} \tag{2}$$

3 Working Out a Solution

First, we find the set A of multiples of 3 below 1000, then the set B of multiples of 5 below 1000. The tricky part of this problem is recognizing that these sets have elements in common- a non-empty intersection. Thus, the solution is the sum of the elements of set A plus a similar sum of B minus the sum of the intersection, which is the set C of multiples of 15 below 1000.

When summing the first n multiples of some number, say 3, we quickly discover that the this sum S is the same as 3 multiplied by the sum of the first n natural numbers.

$$
\begin{aligned}
S &= 3, 6, 9, 12, ... \\
&= \sum^{n} 3 \cdot i \\
&= 3 \cdot \sum^{n} i \\
&= 3 \cdot \frac{n \cdot (n-1)}{2}
\end{aligned} \tag{3}
$$

Trivially, this is the case for the multiples of 5 and 15 as well. To count how many multiples are below a certain number, we less by one the ceiling of the quotient of the number by the multiplier. For 1000, we see that there are $\lfloor 1000/3 \rfloor - 1 = 333$ multiples of 3 below 1000; for 5, there are 199; and, for 15, there are 66.

Using (2) from the theory, we find partial sums of $166, 833$, $99, 500$ and $33, 165$ for the multiples of 3, 5, and 15, respectively, below 1000.

Thus, our answer to the query is:

$$
\begin{aligned}
S &= 166, 833 + 99, 500 - 33, 165 \\
&= 233, 168
\end{aligned} \tag{4}
$$

Now, to code this!