

INTERVAL GAMES

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Let I_1 and I_2 be intervals of real numbers. An interval game is played in this way: player 1 secretly selects $x \in I_1$ and player 2 secretly and independently selects $y \in I_2$. After x and y are revealed, payoffs are given by some predetermined function with domain $I_1 \times I_2$.

The payoff function for a zero sum interval game is a function $A : I_1 \times I_2 \rightarrow \mathbb{R}$. This is interpreted to mean that the second player gives the first player a payoff of $A(x, y)$. Therefore player 1 wants to make A as large as possible while player 2 wants to make A as small as possible.

Example 1. The game played on $[0, 1] \times [0, 1]$ with payoffs given by $A(x, y) = 2x - 4xy + y^2$ is a zero sum interval game. If player 1 selects $x = \frac{1}{2}$ and player 2 selects 1, then player 2 would give player 1 a payoff of $A(\frac{1}{2}, 1) = 0$.

A solution to an interval game consists of three things: an optimal strategy for player 1, an optimal strategy for player 2, and the value of the game (the value of the game is the expected payoff when both players employ optimal strategies). However, not every interval game has a solution!

Example 2. The game “who can select the biggest number?” is an interval game played on $\mathbb{R} \times \mathbb{R}$ with payoff $A(x, y) = 1$ if $x > y$ and $A(x, y) = -1$ if $x \leq y$. It has no optimal strategy since there is no biggest real number.

Fortunately, some interval games do have solutions. If I_1 and I_2 are both of the form $[a, b]$ for $a, b \in \mathbb{R}$ and if the payoff function $A(x, y)$ is continuous, then a solution was proved to exist by Ville in 1938. This was done by approximating interval games with large matrix games. Even when a solution is known to exist, there are no known efficient techniques for finding it analytically. Finding a general method for solving interval games without resorting to approximations using matrix games remains an open problem in mathematical game theory.

1. SADDLE POINTS

How can we determine if an interval game has pure strategy solutions for both players? In other words, under which conditions are there numbers $x_0 \in I_1$ and $y_0 \in I_2$ such that player 1 always selects x_0 , player 2 always selects y_0 , and the value of the game is $A(x_0, y_0)$? This situation is analogous to the discrete problem of determining which matrix games have saddle points.

Suppose x_0 and y_0 are optimal strategies for the first and second players and suppose that it is possible to take two partial derivatives of $A(x, y)$. Since player 2 always plays y_0 , the number x_0 must maximize $A(x, y_0)$ for $x \in I_1$. Remembering calculus, this means that $A_x(x_0, y_0) = 0$ and that $A_{xx}(x_0, y_0) < 0$. Similarly, since player 1 always plays x_0 , the number y_0 must minimize $A(x_0, y)$ for $y \in I_2$. This means that $A_y(x_0, y_0) = 0$ and that $A_{yy}(x_0, y_0) > 0$. This gives us a way to check if $A(x, y)$ has a saddle point solution:

- (1) First, find a point (x_0, y_0) with $A_x(x_0, y_0) = A_y(x_0, y_0) = 0$.
- (2) Then, check to see if $A_{xx}(x_0, y_0) < 0$ and $A_{yy}(x_0, y_0) > 0$.

If such a point exists and is found within $I_1 \times I_2$, then player 1 should always select x_0 , player 2 should always select y_0 , and the value of the game is $A(x_0, y_0)$.

Example 3. Consider the game on $[0, 1] \times [0, 1]$ with payoffs given by $A(x, y) = -2x^2 + 2x - 3xy + y^2 + 2$. Looking for a saddle point, the solution to

$$\begin{cases} 0 = A_x(x, y) = -4x + 2 - 3y \\ 0 = A_y(x, y) = -3x + 2y \end{cases}$$

is the point $(\frac{4}{17}, \frac{6}{17})$. We notice that $A_{xx}(\frac{4}{17}, \frac{6}{17}) = -4 < 0$ and $A_{yy}(\frac{4}{17}, \frac{6}{17}) = 2 > 0$, so we have found the optimal strategies for player 1 and player 2. The value of the game is $A(\frac{4}{17}, \frac{6}{17}) = \frac{38}{17}$. The point $(\frac{4}{17}, \frac{6}{17})$ is a maximum when x is allowed to vary and a minimum if y is allowed to vary.

In multivariate calculus, a point satisfying conditions 1 and 2 above is also called a saddle point. However, calculus saddle points may not be game theory saddle points. A calculus saddle point is a point which is a maximum when looking in one direction and a minimum in another direction. A calculus saddle point could, say, minimize $A(x, y_0)$ and maximize $A(x_0, y)$ instead of maximizing $A(x, y_0)$ and minimizing $A(x_0, y)$.

Example 4. Consider the game on $[0, 1] \times [0, 1]$ with payoff function $A(x, y) = 4x^2 + 4y^2 + 2x + 2y - 12xy - 1$. Looking for a saddle point, the solution to

$$\begin{cases} 0 = A_x(x, y) = 8x + 2 - 12y \\ 0 = A_y(x, y) = 8y + 2 - 12x \end{cases}$$

is $x = y = \frac{1}{2}$. This is not a game theory saddle point because $A_{xx}(\frac{1}{2}, \frac{1}{2}) = 8 \not< 0$. However, it is a calculus saddle point.

It turns out that the solution for this game is this: player 1 should select 0 with probability $\frac{1}{2}$ and 1 with probability $\frac{1}{2}$, player 2 should always select $\frac{1}{2}$, and the value of the game is 1. To verify that this is indeed a solution, we need to check that x 's strategy maximizes $A(x, \frac{1}{2})$ and that y 's strategy minimizes the expected payoff when x employs his strategy. Indeed, if the second player selects $\frac{1}{2}$, then $A(x, \frac{1}{2}) = 4(x - \frac{1}{2})^2$ is maximized on $[0, 1]$ at either 0 or 1. If the first player selects 0 with probability $\frac{1}{2}$ and 1 with probability $\frac{1}{2}$, then $\frac{1}{2}A(0, y) + \frac{1}{2}A(1, y) = 4y^2 - 4y + 2$ is minimized at $y = \frac{1}{2}$. This verifies that we have reached an equilibrium point and thus found a solution to the game.

2. CUMULATIVE DISTRIBUTION FUNCTIONS AND RIEMANN-STIELTJES INTEGRATION

Solutions to interval games involve cumulative distribution functions. Let X be a random variable. The cumulative distribution function for X is defined to be the function $F : \mathbb{R} \rightarrow [0, 1]$ such that

$$F(x) = (\text{the probability that the random variable } X \text{ is } \leq x).$$

Immediately from the definition, we can see that $F(x)$ is nondecreasing, $\lim_{x \rightarrow -\infty} F(x) = 0$, $\lim_{x \rightarrow \infty} F(x) = 1$, and the probability that the random variable X lies in $(a, b]$ is $F(b) - F(a)$.

Example 5. Suppose that we decide to select $x = \frac{1}{3}$ with probability $\frac{1}{2}$ and $x = \frac{2}{3}$ with probability $\frac{1}{2}$ when playing an interval game. The distribution describing this solution is given by

$$F(x) = \begin{cases} 0 & \text{if } x \in (-\infty, \frac{1}{3}) \\ \frac{1}{2} & \text{if } x \in [\frac{1}{3}, \frac{2}{3}) \\ 1 & \text{if } x \in [\frac{2}{3}, \infty). \end{cases}$$

Example 6. Some distribution functions are continuous. We may decide to play a game on $[0, 1] \times [0, 1]$ according to the distribution $F(x)$ given by

$$F(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0] \\ x & \text{if } x \in (0, 1] \\ 1 & \text{if } x \in (1, \infty). \end{cases}$$

Here, the probability that $x \in (\frac{1}{3}, \frac{2}{3}]$ is selected is equal to $F(\frac{2}{3}) - F(\frac{1}{3}) = \frac{1}{3}$.

For continuous distributions F , the answer to the question "what is the probability that you will play $x \in (a, b]?$ " is " $F(b) - F(a)$ ". However, although it may seem strange at first, the answer to the question "what is the probability that you will play $x = a?$ " is always "0". To see why this is true, consider selecting a number in $[0, 1]$ by randomly selecting each digit in the number's decimal expansion. What is the probability of selecting $1/3 = 0.3333 \dots$ using the distribution in example 6? The probability of randomly selecting the first 3 in this decimal expansion is $1/10$. The probability of randomly selecting both the first and second 3's

is $1/10^2$. More generally, the probability of randomly selecting the first k 3's is $1/10^k$. Since $1/10^k \rightarrow 0$ as $k \rightarrow \infty$, the probability of getting exactly $1/3$ is indeed 0.

Suppose that when playing an interval game with payoff $A(x, y)$, player 1 uses $F(x)$ while player 2 always selects y_0 . In order to check that the strategy for player 1 is optimal, we need to check that F maximizes the expected payoff when y_0 is used by player 2. To do this, we first need to understand how to calculate such an expected payoff.

To find the expected payoff, break I_1 into n subintervals $(x_0, x_1], (x_1, x_2], \dots, (x_{n-1}, x_n]$ just like when defining the Riemann integral.¹ The expected payoff on interval from $(x_i, x_{i+1}]$ is approximately

$$(\text{the payoff at } x_i, y_0)(\text{the probability that } x \text{ is in } (x_i, x_{i+1}]) = A(x_i, y_0)(F(x_{i+1}) - F(x_i)).$$

Adding the probabilities from all of these intervals together and taking the limit as $n \rightarrow \infty$, the expected payoff is

$$(1) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i, y_0)(F(x_{i+1}) - F(x_i)).$$

This is known as a Riemann–Stieltjes integral and it will be denoted by

$$\int A(x, y_0) dF(x).$$

The Riemann–Stieltjes integral in (1) is much the definition of the regular Riemann integral of $A(x, y_0)$ on the interval I_1 ; that is, (1) is much like

$$\int_{I_1} A(x, y_0) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i, y_0)(x_{i+1} - x_i).$$

The only difference between the regular Riemann integral and the Riemann–Stieltjes integral is that the “ $(x_{i+1} - x_i)$ ” has been replaced with “ $(F(x_{i+1}) - F(x_i))$ ”.

Theorem 1. Suppose that $F'(x)$ exists and is bounded I_1 . Then

$$\int A(x, y) dF(x) = \int_{I_1} A(x, y) F'(x) dx.$$

Proof. We have

$$\begin{aligned} \int A(x, y) dF(x) &= \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i, y)(F(x_{i+1}) - F(x_i)) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i, y) \frac{F(x_{i+1}) - F(x_i)}{x_{i+1} - x_i} (x_{i+1} - x_i) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i, y) F'(x^*) (x_{i+1} - x_i) \end{aligned}$$

where the last line used the mean value theorem to turn $(F(x_{i+1}) - F(x_i))/(x_{i+1} - x_i)$ into $F'(x^*)$ for some $x^* \in (x_{i+1}, x_i)$. This last line is the desired Riemann integral. \square

Now we are ready to formulate our definition of solution. Suppose that when playing the interval game $A(x, y)$, player 1 uses the cumulative distribution function $F(x)$ and player 2 uses the cumulative distribution function $G(y)$. This is a solution provided there is a $v \in \mathbb{R}$ such that

$$\int A(x, y_0) dF(x) \geq v \quad \text{and} \quad \int A(x_0, y) dG(y) \leq v$$

for all $x_0 \in I_1$ and $y_0 \in I_2$. In other words, player 1 can guarantee an expected payoff of at least v using F and player 2 can guarantee an expected payoff of at most v using G . These cumulative distribution functions F and G are called optimal.

¹If I_1 is not bounded, we can change the first interval to be $(-\infty, x_1)$ and the last interval to be (x_{n-1}, ∞) as needed.

Example 7. Let

$$A(x, y) = \begin{cases} 1 - x & \text{if } x < y \\ -y & \text{if } x \geq y \end{cases}$$

be a game on $[0, \infty) \times [0, \infty)$. Take

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - e^{-x} & \text{if } x \geq 0, \end{cases} \quad \text{and} \quad G(y) = \begin{cases} 0 & \text{if } y < 0, \\ 1 & \text{if } y \geq 0. \end{cases}$$

We claim that a solution is given by $F(x)$, $G(y)$ and the value of the game is 0. To verify that $F(x)$ is optimal for player 1, we calculate

$$\int A(x, y) dF(x) = \int_0^y (1 - x)e^{-x} dx + \int_y^\infty (-y)e^{-x} dx = 0.$$

Therefore, the least that player 2 can achieve when player 1 uses $F(x)$ is 0. As for player 2, the Riemann-Stieltjes integral $\int A(x, y) dG(y)$ is really just $A(x, 0)$. Since $A(x, 0) = 0$, this means that the most that player 1 can achieve when player 2 uses $G(x)$ is 0. We have verified that we indeed have a solution.

3. A WAY TO SOLVE SOME ZERO SUM SYMMETRIC GAMES

Here is a problem similar to exercise 3: Two men start running towards each other with loaded pistols drawn starting at time $t = 0$. Each pistol has one shot. At time $t = 1$, unless one man is already dead, the two men will be standing face to face. Each man has a probability of t of killing his opponent if he waits t seconds to fire. The duel is silent, meaning that each man does not know if their opponent has already fired (unless they are hit by their opponent's bullet). When should each man fire his pistol?

Let $A(x, y)$ be the function giving the payoffs for this game on the square. If both players miss or if both players fire and hit each other at the same instant, the payoff will be 0. Otherwise, we will assign a payoff 1 for the successful participant. Suppose player 1 selects x and player 2 selects y with $x < y$. Player 1 can win instantly with a successful shot; this happens with probability x . If player 1 misses, which happens with probability $1 - x$, then player 2 will win with probability y . Therefore, if $x < y$, $A(x, y) = x + (1 - x)y(-1)$. Similar reasoning when $y < x$ can be employed to find that

$$A(x, y) = \begin{cases} x - y(1 - x) & \text{if } x < y \\ 0 & \text{if } x = y \\ -y + x(1 - y) & \text{if } y < x. \end{cases}$$

This game is symmetric (meaning that the game is the same for either player) and so the value should be 0. There should also be one strategy given by a distribution F which is optimal for both players. Our goal is to find this function F .

We will assume F is constant except on some interval $(a, b]$, $F'(x)$ exists on (a, b) , and $F'(x) > 0$ for all $x \in (a, b)$. Hopefully we will be able to find a F under these assumptions; if not, then maybe these assumptions are too strong.

Suppose the expected payoff when player 1 uses F is a constant v regardless of the strategy employed by player 2. This would mean that the value of the game is at least v . But, since this game is symmetric, player 2 can also use F to guarantee a value of at most v ; implying that F is optimal for both players and the value of the game is indeed v . So, to solve this game, we will search for a strategy F which makes the expected payoff when player 1 uses F and player 2 uses any $y \in [0, 1]$ a constant. In symbols, this says

$$\begin{aligned} 0 = v &= \int_0^1 A(x, y) F'(x) dx \\ &= \int_0^y (x - y(1 - x)) F'(x) dx + \int_y^1 (-y + x(1 - y)) F'(x) dx. \end{aligned}$$

Rewriting this equation, and using $\int_0^1 F'(x) dx = 1$ (this is true for any cumulative distribution function), we find

$$0 = \int_0^1 x F'(x) dx + y \int_0^y x F'(x) dx + y \int_1^y x F'(x) dx - y.$$

Taking $\partial/\partial y$ on both sides of this equation and simplifying, we have

$$0 = 2y^2 F'(y) + \int_0^y x F'(x) dx + \int_1^y x F'(x) dx - 1.$$

Taking $\partial/\partial y$ again and simplifying, we finally arrive at $0 = 6yF'(y) + 2y^2 F''(y)$. Students who have taken an elementary course in differential equations may recognize that we have found a Cauchy-Euler differential equation. The solutions are functions of the form $F(y) = y^m$ for some number m . This means $F'(y) = my^{m-1}$ and $F''(y) = m(m-1)y^{m-2}$. Plugging these functions into the differential equation and solving for m ,

$$\begin{aligned} 0 &= 6y(my^{m-1}) + 2y^2(m(m-1)y^{m-2}) \\ &= (2m^2 + 4m)y^m, \end{aligned}$$

and so $2m^2 + 4m = 0$. This means $m = 0$ or $m = -2$ and therefore, on $(a, b]$, $F(y) = C_1 + C_2 y^{-2}$ for some constants C_1, C_2 .

At this point, we know that

$$F(x) = \begin{cases} 0 & \text{if } x \leq a \\ C_1 + C_2 x^{-2} & \text{if } x \in (a, b] \\ 1 & \text{if } x > b. \end{cases}$$

To finish, we need to find a, b, C_1 , and C_2 .

Now,

$$\begin{aligned} E(F, b) &= \int_a^b (x - b(1-x))F'(x) dx \\ &\geq \int_a^b (x - 1(1-x))F'(x) dx \\ &= E(F, 1) \end{aligned}$$

where the middle inequality between integrals is because $b \leq 1$. We know $E(F, b) = 0$ and $E(F, 1) \geq 0$, the inequalities in the above calculation must be equalities. This implies $b = 1$.

Since $E(F, 1) = 0$,

$$\begin{aligned} 0 &= E(F, 1) = \int_a^1 (x - (1-x))F'(x) dx \\ &= \int_a^1 (2x - 1)(-2C_2 x^{-3}) dx \\ &= (-2C_1) \frac{-3a^2 + 4a - 1}{2a^2}. \end{aligned}$$

Therefore, $-3a^2 + 4a - 1 = 0$. This gives $a = \frac{1}{3}$ or $a = 1$. Since the solution $a = 1$ does not make sense here, we have found that $a = \frac{1}{3}$.

We now have that $F(\frac{1}{3}) = 0$ and $F(1) = 1$. Therefore,

$$1 = F(1) - F(1/3) = \int_{1/3}^1 F'(x) dx = \int_{1/3}^1 -2C_2 x^{-3} dx = -8C_2$$

and we have found $C_2 = -\frac{1}{8}$. Finally, using $1 = F(1) = C_1 - \frac{1}{8}$, we find that $C_1 = \frac{9}{8}$.

Putting everything together, the solution to the duel problem that both players should fire their weapon according to the distribution F given by

$$F(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{3} \\ \frac{9}{8} - \frac{1}{8}x^{-2} & \text{if } x \in (\frac{1}{3}, 1] \\ 1 & \text{if } x > 1. \end{cases}$$

So, for example, unless player 1 is dead, he will shoot before time $\frac{1}{2}$ with probability $F(\frac{1}{2}) = \frac{5}{8} = .625$.

Here what we just did in order to solve the duel game:

- (1) Found an explicit expression for $A(x, y)$.

- (2) Argued that if the game had a solution, then it has a value of 0.
- (3) Assumed there was a common cumulative distribution function $F(x)$ which was optimal for both players such that $F'(x)$ exists and is nonzero only on some interval $(a, b]$.
- (4) Found an equation formed from the expected payoff when player 1 does F and player 2 does y .
- (5) Differentiated (twice, in this example) the equation in found in 4.
- (6) Solved the differential equation to find the form of $F(x)$.
- (7) Solved for the constants a, b and those coming from the solution of the differential equation using properties of cumulative distribution functions.

This approach works for other symmetric games, like those found in exercises 9 and 10.

4. NONZERO SUM GAMES

A nonzero sum interval game has a payoff function $A : I_1 \times I_2 \rightarrow \mathbb{R} \times \mathbb{R}$. The first coordinate of $A(x, y)$ is interpreted as the payoff to player 1 and the second coordinate is the payoff two player 2. As usual, both players want to maximize their respective payoffs.

The concept of a solution for a nonzero sum game is difficult to define. However, it still makes sense to discuss equilibria. Suppose that player 1 uses the cumulative distribution function F and player 2 uses G . If the expected payoff to player 2 when player 1 uses F is constant and the expected payoff to player 1 when player 2 uses G is also (a possibly different) constant, then F and G give an equilibrium point.

If A is a symmetric nonzero sum game, then there may be one strategy F which can be used by both players to produce an equilibrium point. If the expected payoff to player 1 when this F is used against \hat{F} is strictly greater than the expected payoff when \hat{F} is used against itself for all strategies $\hat{F} \neq F$, then F is called an evolutionarily stable strategy (ESS). Evolutionarily stable strategies have been used to explain the behavior of organisms which evolve under the forces of natural selection since, when adopted by a population of players each playing two person games, an ESS cannot be invaded by any alternative strategy.

According to the definition, checking to see if a equilibria strategy F is an ESS requires calculating the expected payoffs of F versus \hat{F} and \hat{F} versus \hat{F} for all strategies \hat{F} . This is daunting since there are an infinite number of strategies \hat{F} to check! Luckily the next theorem, which we state without proof, provides a shortcut which cuts down the work considerably.

Theorem 2. Suppose A is a symmetric nonzero sum game and F , when used by both player 1 and player 2, is an equilibrium point. Then F is also an ESS if the expected payoff when F is used against y_0 is strictly greater than the expected payoff when y_0 is used against itself for all points y_0 .

Example 8 (The war of attrition). Starting at time $t = 0$, two players try to intimidate their opponent until one retreats, leaving a reward of utility r behind. Both players incur a cost depending on the length of the standoff.

The payoff function defined on $[0, \infty) \times [0, \infty)$ is given by

$$A(x, y) = \begin{cases} (-x, r - x) & \text{if } x < y \\ (r/2 - x, r/2 - x) & \text{if } x = y \\ (r - y, -y) & \text{if } x > y. \end{cases}$$

There is a pure-strategy asymmetric equilibria for this game: player 1 selects $t = r$ and player 2 selects $t = 0$. If player 1 uses $t = r$, then player 2 would have no incentive to deviate from the strategy of playing $t = 0$ since

$$A(r, y) = \begin{cases} (-r, 0) & \text{if } r < y \\ (-r/2, -r/2) & \text{if } r = y \\ (r - y, -y) & \text{if } r > y \end{cases}$$

never provides player 2 an opportunity to earn more than 0. A similar calculation shows that player 1 would have no reason to change from the strategy r when player 2 uses $y = 0$. This equilibria cannot be an ESS because both players are not using the same strategy.

In search for an ESS, let us suppose there is a differentiable distribution function F which, when used by both players, is an equilibrium point. When player 1 uses F , the expected payoff to player 2 is some

constant. Therefore,

$$\begin{aligned} \text{constant} &= \int_0^y (r-x)F'(x) dx + \int_y^\infty (-y)F'(x) dx \\ &= \int_0^y (r-x)F'(x) dx + (-y)(1-F(y)). \end{aligned}$$

Taking $\partial/\partial y$ on both sides of this equation and simplifying,

$$0 = rF'(y) + F(y) - 1.$$

This is relatively easy differential equation to solve since it is a first order linear differential equation. It may be solved using the integrating factor or by separating the variables and integrating. When this is done, we find $F(y) = 1 + Ke^{-y/r}$ where K is a constant. Since $F(0) = 0$, the constant $K = -1$. Therefore, the desired function F is

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-x/r} & \text{if } x \geq 0. \end{cases}$$

Is this function F an ESS? Take $y \in [0, \infty)$. The expected payoff to player 1 when player 1 uses F and player 2 uses y is

$$\int_0^y (-x) \left(\frac{1}{r} e^{-x/r} \right) dx + \int_y^\infty (r-y) \left(\frac{1}{r} e^{-x/r} \right) dx = 2re^{-y/r} - r.$$

The expected payoff to player 1 when both opponents use y is $1/2 - y$. Using Theorem 2, we need to see whether or not $2re^{-y/r} - r > 1/2 - y$ for all y . To do this, we consider the function $2re^{-y/r} - r - (1/2 - y)$. Differentiating this function and setting equal to 0, the minimum of this function occurs at $r \ln 2$. The actual minimum of this function is $-1/2 + r \ln 2$. When this quantity is positive, F is an ESS; otherwise F is not an ESS. Therefore, F is an ESS provided $r > 1/(2 \ln 2) \approx 0.721$; otherwise F is not an ESS.

5. EXERCISES

1. Take $a, b \in \mathbb{R}$. Show that the game on $\mathbb{R} \times \mathbb{R}$ with payoffs given by $A(x, y) = (x - a)^2 - (y - b)^2$ has a no solution.
2. Take $a, b \in \mathbb{R}$. Show that the game on $\mathbb{R} \times \mathbb{R}$ with payoffs given by $A(x, y) = (y - b)^2 - (x - a)^2$ has a saddle point solution.
3. Solve the noisy duel problem: Starting at $t = 0$, two men start running towards each other with loaded pistols. Each man has one bullet to fire at their opponent. At time $t = 1$, unless one man is dead by then, the two men will be standing face to face. Each man has a probability of t of killing his opponent if he waits t seconds to fire. The duel is noisy, meaning that each man knows if their opponent has already fired and missed. When should each man fire his pistol?
4. Consider a zero sum interval game with payoff function $A(x, y)$ which value v . Take $a, b \in \mathbb{R}$ such that $a \geq 0$. Show that an optimal strategy for $A(x, y)$ is still optimal for $aA(x, y) + b$ and the value of $aA(x, y) + b$ is $av + b$.
5. Show the set of optimal strategies is convex; that is, show that if $F_1(x)$ and $F_2(x)$ are two optimal strategies for player 1 in an interval game, then so is $\lambda F_1(x) + (1 - \lambda)F_2(x)$ for all $\lambda \in [0, 1]$.
6. Let $A(x, y)$ be an interval game on $I \times I$ for some interval I such that there is one cumulative distribution function F which is optimal for both players. Solve the interval game on $I \times I$ with payoffs given by $A(x, y) - A(y, x)$.
7. Two players independently select numbers in $(0, \infty)$. The player who selected the smaller number, say t , pays e^{2t} to his opponent (there is no payoff if they both select the same number). By using the strategy $F(x) = 1 - e^{-x}$ for $x > 0$, how much can a player guarantee for himself? Does this game have a value?

8. Player 1 selects $x \in [0, 3]$ while player 2 selects $y \in [0, 2]$. Then the second player pays the first player \$1 if $x \in (y, y + 1)$ and \$0 otherwise. Show that an optimal strategy for the first player is to use the cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ x/3 & \text{if } x \in [0, 3], \text{ and} \\ 1 & \text{if } x > 3 \end{cases}$$

and an optimal strategy for the second player is to select each of the numbers 0, 1 and 2 with probability $1/3$. What is the value of the game?

9. Two people try to guess a random number found using the cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } x \in [0, 1], \\ 1 & \text{if } x > 1. \end{cases}$$

The person coming closest without guessing too high wins. Solve. As an extra challenge, solve when $F(x)$ is replaced with an arbitrary differentiable cumulative distribution function.

10. A beautiful woman will arrive at the airport at some time in the interval $[0, 1]$. The probability that she will arrive at or before time $t \in [0, 1]$ is t . Two handsome suitors will each visit the airport looking for the woman in order to give her a ride home. If a suitor arrives when the woman is not there, he will immediately leave under the assumption that she has already been picked up. The suitor who successfully picks up the woman wins 1 from the other suitor (if there is a tie or both suitors arrive before the woman, both suitors receive 0). When should each suitor arrive?