Tactical Problems Involving Several Actions

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[*Editor's note:* This paper seems to have no digital record, despite being cited many times in the game theory literature. I was able to obtain a printed copy from *Contributions to the Theory of Games, Vol. III*, 1957, which was originally in the library of Bell Laboratories. It was checked out three times. I have slightly modified the notation, which was originally typeset on a typewriter, to better fit modern LaTeX. I have also added Editor's notes (signified as this one) throughout the text to clarify some claims and express my own misunderstanding of certain bits. I have tried to do so in a way that adds to the paper.]

1 Introduction

A zero-sum, two-person game can be defined by a tripler (X, Y, Ψ) where X and Y are two closed sets, and Ψ is a real-valued, measurable function defined on $X \times Y$; Ψ is called the *pay-off* or *utility function.* The elements $x \in X$ and $y \in Y$ are called *pure strategies*; the positive measures with total measure 1 defined over X and Y are called (*mixed*) *strategies.* The game has a solution if there exist two strategies $F(x)$ and $G(y)$ such that for all $y \in Y$,

$$\int \Psi(x,y)dF(x) \geq v,$$

and for all $x \in X$,

$$\int \Psi(x,y)dG(y) \leq v.$$

F adn G are called *optimal strategies*; v is the *value* of the game.

We shall consider a class of games that can be interpreted as tactical problems. Each game will represent a context between two players who are trying to achieve the same objective. When one of them succeeds, he wins one unit; his opponent loses the same amount, and the contest is over. Each player has limited resources and can only make a fixed number of attempts to reach his goal; these attempts must be made during the interval $0 \leq t \leq 1$, and each attempt may fail or succeed. At $t = 0$, every attempt fails; at $t = 1$, every attempt succeeds; at any other time t, an attempt made by Player 1 will be successful with probability $P(t)$, and will fail with probability $1 - P(t)$; an attempt made by Player 2 succeeds with probability $Q(t)$, fails with probability $1 - Q(t)$; the functions P and Q increase continuously. Each player knows these functions and the total number of attempts that his opponent can make; however, after the contest begins each player is unable to find out how many unsuccessful attempts have been made by his opponent. This description can be specialized to a combat between two airplanes: P and Q describe the accuracy of firing machinery, and the initial resources correspond to the total amount of ammunition that each plane can carry; since it is assumed that each pilot is unable to find out how many times his opponent has fired and missed, this problem is often called a *silent duel*. In the formal description of the game, x and y will be vectors that describe the times when the attempts are made; $\Psi(x, y)$ will be the expected gain for Player 1.

A special silent duel was solved by L. S. Shapley. A class of similar problems (called Games of Timing) has been studied by M Shiffman and S. Karlin; the problems considered in Karlin include a larger class of utility functions, but allow only one action by each player. It should be pointed out that the silent duels are essentially different from the noisy duels considered by D. Blackwell and M. A. Girshick; apparently, the recursive method they used cannot be adapted for the solution of our problems.

2 Description of the optimal strategies

Each game is characterized by two numbers m and n that denote the total number of attempts that each player can make. The solutions are characterized by two sets of numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_,$ that depend only on P, Q, m, and n. In every optimal strategoy, the i-th action of Player 1 and the j-th action of Player 2 must be carried out during the intervals $[a_i, a_{i+1}], [b_j, b_{j+1}]$; in every game, $a_1 = b_1$, and $a_{n+1} = b_{m+1} = 1$.

The optimal strategies are not necessarily unique, but each player has precisely one in which all attempts are made independently. In this special strategy, Player 1 will make his i-th attempt at time x_i, chosen at random by means of a probability distribution $F_i(x_i)$; $F_n(x_n)$ may have a discrete mass α at $x_n = 1$; away from 1, eah F_i is absolutely continuous and has a piecewise continuous density. The discontinuities of the densities occur at the points $b_1, \ldots b_m$; precisely,

$$dF_i(x_i) = \begin{cases} h_i f^*(x_i) dx_i & \text{if } a_i < x_i < a_{i+1}, \\ 0 & \text{if } x_i \notin [a_i, a_{i+1}], \end{cases}$$

where

$$f^*(t) = \prod_{b_j > t} [1 - Q(b_j)] \frac{Q'(t)}{Q^2(t) P(t)}.$$

The constants h_i and h_{i+1} are related by the equation

$$h_i = [1 - D_i]h_{i+1},$$

where

$$D_i = \int_{a_i}^{a_{i+1}} P(t)dF_i(t).$$

Player 2 has a similar strategy that can be described in an analogous way. It is sufficient to interchange the roles of P, Q, a_i and b_j in the previous description.

EXAMPLE. Symmetric Game: $P(t) = Q(t)$, and $n = m$. In this case the two players have the same optimal strategies; $\alpha = 0$, and $a_k = b_k, k = 1, \ldots, n$. Furthermore

$$\begin{aligned} P(a_n - k) &= \frac{1}{2k-3} & k &= 0, 1, \ldots, n-1, \\ dF_{n-k}(t) &= \frac{1}{4(k+1)} \frac{P'(t)}{P^3(t)} dt & a_{n-k} &< t < a_{n-k+1}. \end{aligned}$$

3 Definitions of X, Y, P, Q and Ψ

X and Y are defined as

$$\begin{aligned} X &= \{x \in E^n \mid 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\} \\ Y &= \{y \in E^m \mid 0 \leq y_1 \leq y_2 \leq \cdots \leq y_m \leq 1\}. \end{aligned}$$

E^n and E^m denote the n and m-dimensional Euclidean spaces.

P and Q will be real-valued, continuously differentiable functions defined on the interval $[0, 1]$. They must satisfy the following conditions, stated just for P:

$$\begin{aligned} &P(0) = 0 \\ &P(1) = 1 \\ &P'(t) > 0 \text{ for } 0 < t < 1 \end{aligned}$$

The symbols $z, r(z_k), s(z_k), \Psi(z)$ are defined only when $x \in X$ and $y \in Y$ are two vectors such that $x_i \neq y_j$, all i, j. Then z denotes the vector whose components are the numbers $x_1, \ldots, y_m$, re-arranged in increasing order; for $k = 1, \ldots, n+m$, either $z_k = x_i$ (for some i), or $z_k = y_j$ (for some j), but not both. Thus

$$r(z_k) = \begin{cases} P(x_i) & \text{if } z_k = x_i, \\ -Q(y_j) & \text{if } z_k = y_j; \end{cases}$$

and

$$s(z_k) = \begin{cases} P(x_i) & \text{if } z_k = x_i, \\ Q(y_j) & \text{if } z_k = y_j; \end{cases}$$

are well defined functions. Finally, $\Psi(z)$ is defined recursively as follows:

$$\begin{aligned}\Psi(z_1,\dots,z_k) &= r(z_1) + [1 - s(z_1)]\Psi(z_2,\dots,z_k)\\ \Psi(z_2,\dots,z_k) &= r(z_2) + [1 - s(z_2)]\Psi(z_3,\dots,z_k)\\ &\vdots\\ \Psi(z_k) &= r(z_k).\end{aligned}$$

[*Editor's note:* The formulas above literally encode the logical formula "The player who acts at time z_1 succeeds at z_1 OR (That player acts but misses at z_1 AND the next player to act succeeds at time z_2 OR(...))." The alternating between r and s is to ensure that all successful actions pay (expected) utility to Player 1. If Player 2 wins, it's a negative payoff to Player 1. The s is needed to ensure failed actions represent probabilities.]

The number of components in x and y is not essential, and the definition can be applied to a single vector $x \in X$, or to a single vector $y \in Y$. In this case one may say that the other vector has no components.

The pay-off function $\Psi(x, y)$ is defined as follows: if each component of x is different from each component of y, then

$$\Psi(x, y) = \Psi(z)$$

where $\Psi(z)$ is the number defined above. If the given condition is not satisfied, then

$$\Psi(\overline{x}, \overline{y}) = \frac{1}{2}[\Psi(\overline{x+0}, \overline{y-0}) + \Psi(\overline{x-0}, \overline{y+0})]$$

[*Editor's note:* Restrepo uses overlines for vectors (which I have omitted except in the above formula). I have not see his notation $\overline{x \pm 0}$ before. I am tempted to interpret that the author means to say, "When x and y contain the same value, $\Psi(x, y)$ is the expected payoff when constructing z by breaking ordering ties randomly."]

4 Some properties of $\Psi(x, y)$

$\Psi(x, y)$ is continuous as long as the relative order of the components of x and y does not change. Furthermore, $\Psi(x, y)$ is skew-symmtric in the following sense: if the roles of X and Y are interchanged, $\Psi(y, x) = -\Psi(x, y)$. Other simple properties of Ψ are given in the next three lemmas.

Lemma 1. *If $z = (z_1, \dots, z_t, z_{t+1}, \dots, z_k)$, then*

$$\Psi(z) = \Psi(z_1, \dots, z_t) + \prod_{i=1}^{t}[1 - s(z_i)]\Psi(z_{t+1}, \dots, z_k)$$

Proof. If $t = 1$, the formula reduces to the definition of $\Psi(z)$. For $t > 1$, the formula is proved by induction. The details are omitted.

[*Editor's note:* While the above follows from induction and some algebra, it is perhaps easier to prove it in words: the payoff of the whole game is the payoff of the first t steps, provided some action taken in those steps succeeds, OR, the payoff in the last $k - t$ steps, provided *no* action taken in the first t steps succeeds.]

$\square$

Lemma 2. *If* $z = (z_1, \ldots, z_{t-1}, z_t, z_{t+1}, \ldots, z_k)$, *then*

$$\begin{aligned}\Psi(z) &= \Psi(z_1, \ldots, z_{t-1}, z_{t+1}, \ldots, z_k)\\ &\quad + \prod_{i=1}^{t-1}[1 - s(z_i)][r(z_t) - s(z_t)\Psi(z_{t+1}, \ldots, z_k)]\end{aligned}$$

Proof. By Lemma 1,

$$\Psi(z) = \Psi(z_1, \ldots, z_{t-1}) + \prod_{i=1}^{t-1}[1 - s(z_i)]\Psi(z_t, \ldots, z_k),$$

and

$$\Psi(z_t, \ldots, z_k) + r(_t) + [1 - s(z_i)]\Psi(z_{t+1}, \ldots, z_k).$$

Therefore,

$$\begin{aligned}\Psi(z) &= \Psi(z_1, \ldots, z_{t-1}) + \prod_{i=1}^{t-1}[1 - s(z_i)]\Psi(z_{t+1}, \ldots, z_k)\\ &\quad + \prod_{i=1}^{t-1}[1 - s(z_i)][r(z_t) - s(z_t)\Psi(z_{t+1}, \ldots, z_k)]\end{aligned}$$

Lemma 1 shows that the first two terms can be replaced by
$\Psi(z_1, \ldots, z_{t-1}, z_{t+1}, \ldots, z_k)$.

$\square$

Lemma 3. *For any fixed y, $\Psi(x, y)$ is a monotone increasing function of each component x_i of x as long as x_i ranges over an open interval that does not contain any components of y. Similarly, for any fixed x, $\Psi(x, y)$ is a monotone decreasing function of each component y_j of y as long as y_j ranges over an open interval that does not contain any components of x.*

*[*Editor's note: *Restated, fixing Player 2's strategy, in considering the action time for action i, it is better for Player 1 to wait as long as possible before acting, as their probability of success increases over time. This is not true if Player 1 waits so long that Player 2 gets to act, because Player 2 may succeed. But so long as that doesn't happen, Player 1's expected utility increases.]*

Proof. Since $\Psi(x, y)$ is skew-symmetric in x and y, it is sufficient to prove only the first half of the lemma. The result is established by means of Lemma 2. Indeed, any component of x_i appears only in the term

$$\prod_{k=1}^{i-1}[1 - P(x_k)] \prod_{y_j < x_i} [1 - Q(y_j)][P(x_i) - P(x_i)\Psi(z^*)],$$

where z^* is the vector constructed from those components of x and y that are larger than x_i. It is clear that the first two factors are positive; furthermore $-1 \leq \Psi(z) \leq 1$, for all z. Thus the coefficient of $P(x_i)$ is positive, and $\Psi(x, y)$ is a monotone increasing function fo x_i.

□

5 Strategies of the class O

Definition 1. *A strategy $F(x)$ belongs to the class O if it satisfies the following conditions:*

1. *F is separate, i.e.*

$$F(x) = \prod_{i=1}^{n} F_i(x_i).$$

2. *Each F_i is a positive measure with total mass 1; furthermore, the support of each F_i (i.e., the complement of the largest open set on which F_i vanishes) is a non-degenerate interval $[a_i, a_{i+1}]$; also, $a_1 > 0$, and $a_{n+1} = 1$.*

3. *The first $n - 1$ measures are continuous; the last measure F_n may have only one discontinuity with mass α at $x_n = 1$.*

Equivalent conditions are used when dealing with measures $G(y)$ defined over Y.

Notation 1. *In the following sections, D_i and $R(y)$ will always denote the expected values of P and Ψ. That is,*

$$D_i = \int_{a_i}^{a_{i+1}} P(t) dF_i(t)$$
$$R(y) = \int \Psi(x, y) dF(x).$$

The vector D^k and the function $\phi(D^k)$ are defined as follows:

$$D^k = (D_{k+1}, D_{k+2}, \ldots, D_n)$$
$$\phi(D^{k-1}) = D_k + [1 - D_k]\phi(D^k)$$
$$\ldots$$
$$\phi(D^n) = 0.$$

Using the last definition, it is easy to see that

$$\int \Psi(x_{k+1}, \ldots, x_n) dF_{k+1}(x_{k+1}) \ldots dF_n(s_n) = \phi(D^k).$$

Lemma 4. *Let $F(x)$ be in the class O, and let $y \in Y$ be a vector whose last component y_m is contained in the open interval (a_k, a_{k+1}). Then*

$$\begin{aligned}
&R(y_1, \ldots, y_{m-1}, y_m) - R(y_1, \ldots, y_{m-1}) \\
&= \prod_{i=1}^{k-1} [1 - D_i] \prod_{j=1}^{m-1} [1 - Q(y_j)][-Q(y_m)] \cdot \\
&\cdot \left\{ 2 \int_{y_m}^{a_{k+1}} P(x_k) dF_k(x_k) + [1 - D_k][1 + \phi(D^k)] \right\}.
\end{aligned}$$

Proof. Let y be the given vector and let $x = (x_1, \ldots, x_n)$ be any vector contained in the support of $F(x)$. Lemma 2 can be used to separate all the terms of $\Psi(x, y)$ that depend on y_m; thus, if $a_k \leq x_k < y_m$,

$$\begin{aligned}
&\Psi(x, (y_1, \ldots, y_{m-1}, y_m)) - \Psi(x, (y_1, \ldots, y_{m-1})) \\
&= \prod_{i=1}^{k} [1 - P(x_i)] \prod_{j=1}^{m-1} [1 - Q(y_j)][-Q(y_m)][1 + \Psi(x_{k+1}, \ldots, x_n)].
\end{aligned}$$

But if $y_m < x_k \leq a_{k+1}$, [Editor's note: the only difference is the very last term includes x_k.]

$$\begin{aligned}
&\Psi(x, (y_1, \ldots, y_{m-1}, y_m)) - \Psi(x, (y_1, \ldots, y_{m-1})) \\
&= \prod_{i=1}^{k} [1 - P(x_i)] \prod_{j=1}^{m-1} [1 - Q(y_j)][-Q(y_m)][1 + \Psi(x_k, x_{k+1}, \ldots, x_n)].
\end{aligned}$$

The vectors x with $x_k = y_m$ have F-measure zero. Therefore,
[For brevity, define:

$$A = \prod_{i=1}^{k-1} (1 - D_i) \prod_{j=1}^{m-1} (1 - Q(y_j))(-Q(y_m)).$$

Then:]

$$\begin{aligned}
&R(y_1, \ldots, y_{m-1}, y_m) - R(y_1, \ldots, y_{m-1}) \\
&= \int [\Psi(x, (y_1, \ldots, y_{m-1}, y_m)) - \Psi(x, (y_1, \ldots, y_{m-1}))] dF(x) \\
&= A \int_{a_k}^{y_m} [1 - P(x_k)][1 + \phi(D^k)] dF_k(x_k) + A \int_{y_m}^{a_{k+1}} [1 + P(x_k) + (1 - P(x_k))\phi(D^k)] dF_k(x_k) \\
&= 2A \int_{y_m}^{a_{k+1}} \left[P(x_k) dF_k(x_k) + [1 - D_k][1 + \phi(D^k)] \right].
\end{aligned}$$

□

Lemma 5. *If F belongs to the class O, $R(y)$ is a continuous function of y, except when $y_m = 1$.*

Proof. The result follows from the fact that $\Psi(x, y)$ has only simple discontinuities located on the diagonals $x_i = y_j$; F is continuous except at $x_n = 1$. An alternate proof may be based on the fact that the right-hand side of the last equation varies continuously as y_m increases from $a_k - \varepsilon$ to $a_k + \varepsilon$.

□

6 Corresponding strategies

Definition 2. *Let $F(x)$ and $G(y)$ be two measures contained in the class O, and let S_f and S_G denote the supports of their measures. F and G form a pair of corresponding strategies if*

$$\int \Psi(x, y) dF(x) = v, \qquad \text{for all } y \in S_G, y_m \neq 1$$

and

$$\int \Psi(x, y) dG(x) = v, \qquad \text{for all } x \in S_F, x_n \neq 1$$

Notation 2. *Since F and G belong to the class O,*

$$F(x) = \prod_{i=1}^{n} F_i(x_i), \qquad G(y) = \prod_{j=1}^{m} G_j(y_j),$$

and the supports of F_i and G_j are intervals $[a_i, a_{i+1}], [b_j, b_{j+1}]$. The numbers $b_1, \ldots, b_m$ can be rearranged into subsets, one subset for each interval $[a_i, a_{i+1}]$; the resulting array can be written in the form

$$\begin{aligned}
&a_1 \leq b_{1,1} < b_{1,2} < \cdots < b_{1,r_1} < a_2 \\
&a_2 \leq b_{2,1} < b_{2,2} < \cdots < b_{2,r_2} < a_3 \\
&\quad\vdots \\
&a_n \leq b_{n,1} < b_{n,2} < \cdots < b_{n,r_n} < a_{n+1} = 1.
\end{aligned}$$

It is possible that there are no b's between two adjacent a's.

Every interval bounded by two adjacent b's must contain precisely one component of each vector $y \in S_G$. It is possible to identify the component by means of the interval over which it ranges; for instance, $y_{i,j}$ will denote the component of y that is contained in the interval $[b_{i,j}, b_{i,j+1}]$. When two adjaceny b's are separated by one of the a's, say a_i, it is convenient to write

$$a_i = b_{i,0}.$$

In this case, $y_{i-1,r_{i-1}}$ and $y_{i,0}$ denote the same component of y.

In the following lemma, α denotes the discrete mass that $F_n(x_n)$ may have at $x_n = 1$.

Lemma 6. *Let $F(x)$ and $G(y)$ be two strategies contained in the class O. Then*

$$\int \Psi(x,y)dF(x) = v, \qquad \text{for all } y \in S_G, y_m \neq 1$$

if and only if the following conditions hold simultaneously:

1. *In the open interval $(b_{i,j}, b_{i,j+1})$, the measure $F_i(x_i)$ is absolutely continuous and*

$$dF_i(x_i) = h_{i,j}\frac{Q'(x_i)}{Q^2(x_i)P(x_i)}dx_i.$$

2. *The coefficients $h_{i,j}$ satisfy the equations*

$$\begin{aligned} 1+2\alpha &= D_n + 2h_{n,r_n} \\ h_{i,j-1} &= [1-Q(b_{i,j})]h_{i,j}, \qquad j=1,\dots,r_i; i=1,\dots,n \\ h_{i,r_i} &= [1-D_i]h_{i+1,0}, \qquad i=1,\dots,n. \end{aligned}$$

The first condition requires that $a_1 \leq b_1$.

Proof. Let $y = (y_{1,1},\dots,y_{n,r_n})$ be any vector in S_G, and define

$$K_{i,j} = \prod_{s=1}^{i-1}[1-D_s]\prod_{y_{s,t}<y_{i,j}}[1-Q(y_{s,t})],$$

where the second product is taken over all the components of y that precede $y_{i,j}$. By Lemma 4,

$$\begin{aligned} R(y) &= R(y_{1,1},\dots,y_{n,r_n-1}) \\ &\quad - K_{n,r_n}Q(y_{n,r_n})\left[2\int_{y_{n,r_n}}^1 P(x_n)dF_n(x_n) + (1-D_n)\right]. \end{aligned} \tag{1}$$

Therefore, $R(y)$ is independent of y_{n,r_n} if and only if

$$Q(y_{n,r_n})\left[2\int_{y_{n,r_n}}^1 P(x_n)dF_n(x_n) + (1-D_n)\right] = 2h_{n,r_n}, \qquad \text{for all } b_{n,r_n} < y_{n,r_n} < 1, \tag{2}$$

for some constant h_{n,r_n}; furthermore, if (2) holds, equation (1) becomes

$$R(y) = R(y_{1,1},\dots,y_{n,r_n-1}) - 2K_{n,r_n}h_{n,r_n}. \tag{3}$$

The dependence of $R(y)$ on the component $y_{i,j}$ is studied by successive applications of Lemma 4; two cases must be considered.

Case 1. $j \neq r_i$. Let us assume that if $R(y)$ is independent of all the components of y that lie beyond $y_{i,j}$, then

$$R(y) = R(y_{1,1}, \ldots, y_{i,j}) - 2K_{i,j+1}\gamma_{i,j+1} \tag{4}$$

for some constant $\gamma_{i,j+1}$. Then, by definition,

$$K_{i,j+1} = [1 - Q(y_{i,j})]K_{i,j}.$$

Furthermore, Lemma 4 can be applied to $R(y_{1,1}, \ldots, y_{i,j})$. Then

$$\begin{aligned} R(y) = {} & R(y_{1,1}, \ldots, y_{i,j-1}) - 2K_{i,j}\gamma_{i,j+1} \\ & - K_{i,j}Q(y_{i,k})\left[2\int_{y_{i,j}}^{a_{i+1}} P(x_i)dF_i(x_i) + (1 - D_i)(1 + \phi(D^1)) - 2\gamma_{i,j+1}\right]. \end{aligned}$$

Therefore, $R(y)$ is also independent of $y_{i,j}$ if and only if

$$\begin{aligned} Q(y_{i,j}) & \left[2\int_{y_{i,j}}^{a_{i+1}} P(x_i)dF_i(x_i) + (1 - D_i)(1 + \phi(D^i)) - 2\gamma_{i,j+1}\right] \\ & = 2h_{i,j}, \ \text{for all } b_{i,j} < y_{i,j} < b_{i,j+1}, \end{aligned} \tag{5}$$

for some constant $h_{i,j}$; furthermore, if (4) is valid for the indices $i, j+1$, and (5) is valid for the indices i, j, then

$$R(y) = R(y_{1,1}, \ldots, y_{i,j-1}) - 2K_{i,j}\gamma_{i,j}$$

where

$$\gamma_{i,j} = h_{i,j} + \gamma_{i,j+1}. \tag{6}$$

This result justifies the assumption (4), provided that i is fixed.

Case 2. $j = r_i$. In the present notation y_{i,r_i} and $y_{i+1,0}$ denote the same component of y; as a matter of fact, if two adjacent b's are separated by $a_{i+1}, \ldots, a_{i+k'}$, the component of y that follows y_{i,r_i} is $y_{i+k,1}$. The process outlined in Case 1 shows that if $R(y)$ is independent of the components of y that lie beyond y_{i,r_i}, then

$$R(y) = R(y_{1,1}, \ldots, y_{i,r_i}) - 2K_{i+k,1}\gamma_{i+k,1}.$$

The arguments that were used in the previous case can be applied again, but in this case

$$K_{i+k,1} = \prod_{t=i}^{i+k-1} (1 - D_t)(1 - Q(y_{i,r_i}))K_{i,r_i}$$

and $R(y)$ is independent of y_{i,r_i}, if and only if

$\square$