

Tactical Problems Involving Several Actions

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[*Editor's note:* This paper seems to have no digital record, despite being cited many times in the game theory literature. Many years after first learning about games of timing from Anthony Mendes, a colleague posed a version of this problem to me, which spurred me to find the origin. I obtained a printed copy of *Contributions to the Theory of Games, Vol. III*, 1957, which was originally in the library of Bell Labs. It was checked out three times. I have slightly modified the notation, which was originally typeset on a typewriter. I also added notes (signified as this one) to clarify some claims. I have tried to do so in a way that adds to the paper.]

1 Introduction

A zero-sum, two-person game can be defined by a triplet (X, Y, Ψ) where X and Y are two closed sets, and Ψ is a real-valued, measurable function defined on $X \times Y$; Ψ is called the *pay-off* or *utility function*. The elements $x \in X$ and $y \in Y$ are called *pure strategies*; the positive measures with total measure 1 defined over X and Y are called (*mixed*) *strategies*. The game has a solution if there exist two strategies $F(x)$ and $G(y)$ such that for all $y \in Y$,

$$\int \Psi(x, y) dF(x) \geq v,$$

and for all $x \in X$,

$$\int \Psi(x, y) dG(y) \leq v.$$

F and G are called *optimal strategies*; v is the *value* of the game.

We shall consider a class of games that can be interpreted as tactical problems. Each game will represent a context between two players who are trying to achieve the same objective. When one of them succeeds, he wins one unit; his opponent loses the same amount, and the contest is over. Each player has limited resources and can only make a fixed number of attempts to reach his goal; these attempts must be made during the interval $0 \leq t \leq 1$, and each attempt may fail or succeed. At $t = 0$, every attempt fails; at $t = 1$, every attempt

succeeds; at any other time t , an attempt made by Player 1 will be successful with probability $P(t)$, and will fail with probability $1 - P(t)$; an attempt made by Player 2 succeeds with probability $Q(t)$, fails with probability $1 - Q(t)$; the functions P and Q increase continuously. Each player knows these functions and the total number of attempts that his opponent can make; however, after the contest begins each player is unable to find out how many unsuccessful attempts have been made by his opponent. This description can be specialized to a combat between two airplanes: P and Q describe the accuracy of firing machinery, and the initial resources correspond to the total amount of ammunition that each plane can carry; since it is assumed that each pilot is unable to find out how many times his opponent has fired and missed, this problem is often called a *silent duel*. In the formal description of the game, x and y will be vectors that describe the times when the attempts are made; $\Psi(x, y)$ will be the expected gain for Player 1.

A special silent duel was solved by L. S. Shapley. A class of similar problems (called Games of Timing) has been studied by M. Shiffman and S. Karlin; the problems considered in Karlin include a larger class of utility functions, but allow only one action by each player. It should be pointed out that the silent duels are essentially different from the noisy duels considered by D. Blackwell and M. A. Girshick; apparently, the recursive method they used cannot be adapted for the solution of our problems.

2 Description of the optimal strategies

Each game is characterized by two numbers m and n that denote the total number of attempts that each player can make. The solutions are characterized by two sets of numbers a_1, \dots, a_n and b_1, \dots, b_m that depend only on P, Q, m , and n . In every optimal strategy, the i -th action of Player 1 and the j -th action of Player 2 must be carried out during the intervals $[a_i, a_{i+1}]$, $[b_j, b_{j+1}]$; in every game, $a_1 = b_1$, and $a_{n+1} = b_{m+1} = 1$.

The optimal strategies are not necessarily unique, but each player has precisely one in which all attempts are made independently. In this special strategy, Player 1 will make his i -th attempt at time x_i , chosen at random by means of a probability distribution $F_i(x_i)$; $F_n(x_n)$ may have a discrete mass α at $x_n = 1$; away from 1, each F_i is absolutely continuous and has a piecewise continuous density. The discontinuities of the densities occur at the points b_1, \dots, b_m ; precisely,

$$dF_i(x_i) = \begin{cases} h_i f^*(x_i) dx_i & \text{if } a_i < x_i < a_{i+1} \\ 0 & \text{if } x_i \notin [a_i, a_{i+1}] \end{cases}$$

where

$$f^*(t) = \prod_{b_j > t} [1 - Q(b_j)] \frac{Q'(t)}{Q^2(t)P(t)}.$$

The constants h_i and h_{i+1} are related by the equation

$$h_i = [1 - D_i]h_{i+1},$$

where

$$D_i = \int_{a_i}^{a_{i+1}} P(t) dF_i(t).$$

Player 2 has a similar strategy that can be described in an analogous way. It is sufficient to interchange the roles of P, Q, a_i and b_j in the previous description.

EXAMPLE. Symmetric Game: $P(t) = Q(t)$, and $n = m$. In this case the two players have the same optimal strategies; $\alpha = 0$, and $a_k = b_k, k = 1, \dots, n$. Furthermore

$$\begin{aligned} P(a_{n-k}) &= \frac{1}{2k+3} & k &= 0, 1, \dots, n-1, \\ dF_{n-k}(t) &= \frac{1}{4(k+1)} \frac{P'(t)}{P^3(t)} dt & a_{n-k} < t < a_{n-k+1}. \end{aligned}$$

3 Definitions of X, Y, P, Q and Ψ

X and Y are defined as

$$\begin{aligned} X &= \{x \in E^n \mid 0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq 1\} \\ Y &= \{y \in E^m \mid 0 \leq y_1 \leq y_2 \leq \dots \leq y_m \leq 1\}. \end{aligned}$$

E^n and E^m denote the n and m -dimensional Euclidean spaces.

P and Q will be real-valued, continuously differentiable functions defined on the interval $[0, 1]$. They must satisfy the following conditions, stated just for P :

$$\begin{aligned} P(0) &= 0 \\ P(1) &= 1 \\ P'(t) &> 0 \text{ for } 0 < t < 1 \end{aligned}$$

The symbols $z, r(z_k), s(z_k), \Psi(z)$ are defined only when $x \in X$ and $y \in Y$ are two vectors such that $x_i \neq y_j$, all i, j . Then z denotes the vector whose components are the numbers x_1, \dots, y_m , re-arranged in increasing order; for $k = 1, \dots, n+m$, either $z_k = x_i$ (for some i), or $z_k = y_j$ (for some j), but not both. Thus

$$r(z_k) = \begin{cases} P(x_i) & \text{if } z_k = x_i, \\ -Q(y_j) & \text{if } z_k = y_j; \end{cases}$$

and

$$s(z_k) = \begin{cases} P(x_i) & \text{if } z_k = x_i, \\ Q(y_j) & \text{if } z_k = y_j; \end{cases}$$

are well defined functions. Finally, $\Psi(z)$ is defined recursively as follows:

$$\begin{aligned}\Psi(z_1, \dots, z_k) &= r(z_1) + [1 - s(z_1)]\Psi(z_2, \dots, z_k) \\ \Psi(z_2, \dots, z_k) &= r(z_2) + [1 - s(z_2)]\Psi(z_3, \dots, z_k) \\ &\vdots \\ \Psi(z_k) &= r(z_k).\end{aligned}$$

[*Editor's note:* The formulas above literally encode the logical formula “The player who acts at time z_1 succeeds at z_1 OR (That player acts but misses at z_1 AND the next player to act succeeds at time z_2 OR (...)).” The usage of r makes Ψ encode the expected utility to Player 1. If Player 2 wins, it's a negative payoff to Player 1. The s is needed to ensure failed actions represent probabilities.]

The number of components in x and y is not essential, and the definition can be applied to a single vector $x \in X$, or to a single vector $y \in Y$. In this case one may say that the other vector has no components.

The pay-off function $\Psi(x, y)$ is defined as follows: if each component of x is different from each component of y , then

$$\Psi(x, y) = \Psi(z)$$

where $\Psi(z)$ is the number defined above. If the given condition is not satisfied, then

$$\Psi(\bar{x}, \bar{y}) = \frac{1}{2}[\Psi(\overline{x + 0}, \overline{y - 0}) + \Psi(\overline{x - 0}, \overline{y + 0})]$$

[*Editor's note:* Restrepo uses overlines for vectors (which I have omitted except in the above formula). I have not seen his notation $\bar{x} \pm \bar{0}$ before. I believe it should be interpreted as a left or right limit by some arbitrarily small perturbation, i.e., $\Psi(\bar{x} + \bar{0}) = \lim_{t \rightarrow 0} \Psi(\bar{x} + t\bar{1})$. If correct, this seems like a somewhat complicated way to say “break ties randomly.”]

4 Some properties of $\Psi(x, y)$

$\Psi(x, y)$ is continuous as long as the relative order of the components of x and y does not change. Furthermore, $\Psi(x, y)$ is skew-symmetric in the following sense: if the roles of X and Y are interchanged, $\Psi(y, x) = -\Psi(x, y)$. Other simple properties of Ψ are given in the next three lemmas.

Lemma 1. *If $z = (z_1, \dots, z_t, z_{t+1}, \dots, z_k)$, then*

$$\Psi(z) = \Psi(z_1, \dots, z_t) + \prod_{i=1}^t [1 - s(z_i)]\Psi(z_{t+1}, \dots, z_k)$$

Proof. If $t = 1$, the formula reduces to the definition of $\Psi(z)$. For $t > 1$, the formula is proved by induction. The details are omitted.

[*Editor's note:* While the above follows from induction and some algebra, it is perhaps easier to prove it in words: the payoff of the whole game is the payoff of the first t steps, provided some action taken in those steps succeeds, OR, the payoff in the last $k - t$ steps, provided *no* action taken in the first t steps succeeds.] □

Lemma 2. *If $z = (z_1, \dots, z_{t-1}, z_t, z_{t+1}, \dots, z_k)$, then*

$$\begin{aligned} \Psi(z) &= \Psi(z_1, \dots, z_{t-1}, z_{t+1}, \dots, z_k) \\ &\quad + \prod_{i=1}^{t-1} [1 - s(z_i)] [r(z_t) - s(z_t) \Psi(z_{t+1}, \dots, z_k)] \end{aligned}$$

Proof. By Lemma 1,

$$\Psi(z) = \Psi(z_1, \dots, z_{t-1}) + \prod_{i=1}^{t-1} [1 - s(z_i)] \Psi(z_t, \dots, z_k),$$

and

$$\Psi(z_t, \dots, z_k) = r(z_t) + [1 - s(z_t)] \Psi(z_{t+1}, \dots, z_k).$$

Therefore,

$$\begin{aligned} \Psi(z) &= \Psi(z_1, \dots, z_{t-1}) + \prod_{i=1}^{t-1} [1 - s(z_i)] \Psi(z_{t+1}, \dots, z_k) \\ &\quad + \prod_{i=1}^{t-1} [1 - s(z_i)] [r(z_t) - s(z_t) \Psi(z_{t+1}, \dots, z_k)] \end{aligned}$$

Lemma 1 shows that the first two terms can be replaced by $\Psi(z_1, \dots, z_{t-1}, z_{t+1}, \dots, z_k)$. □

Lemma 3. *For any fixed y , $\Psi(x, y)$ is a monotone increasing function of each component x_i of x as long as x_i ranges over an open interval that does not contain any components of y . Similarly, for any fixed x , $\Psi(x, y)$ is a monotone decreasing function of each component y_j of y as long as y_j ranges over an open interval that does not contain any components of x .*

[*Editor's note:* Restated, fixing Player 2's strategy, in considering the action time for action i , it is better for Player 1 to wait as long as possible before acting, as their probability of success increases over time. This is not true if Player 1 waits so long that Player 2 acts, because Player 2 may succeed. But so long as that doesn't happen, Player 1's expected utility increases.]

Proof. Since $\Psi(x, y)$ is skew-symmetric in x and y , it is sufficient to prove only the first half of the lemma. The result is established by means of Lemma 2. Indeed, any component of x_i appears only in the term

$$\prod_{k=1}^{i-1} [1 - P(x_k)] \prod_{y_j < x_i} [1 - Q(y_j)] [P(x_i) - P(x_i)\Psi(z^*)],$$

where z^* is the vector constructed from those components of x and y that are larger than x_i . It is clear that the first two factors are positive; furthermore $-1 \leq \Psi(z) \leq 1$, for all z . Thus the coefficient of $P(x_i)$ is positive, and $\Psi(x, y)$ is a monotone increasing function of x_i . □

5 Strategies of the class O

Definition 1. A strategy $F(x)$ belongs to the class O if it satisfies the following conditions:

1. F is separate, i.e.

$$F(x) = \prod_{i=1}^n F_i(x_i).$$

2. Each F_i is a positive measure with total mass 1; furthermore, the support of each F_i (i.e., the complement of the largest open set on which F_i vanishes) is a non-degenerate interval $[a_i, a_{i+1}]$; also, $a_1 > 0$, and $a_{n+1} = 1$.
3. The first $n - 1$ measures are continuous; the last measure F_n may have only one discontinuity with mass α at $x_n = 1$.

Equivalent conditions are used when dealing with measures $G(y)$ defined over Y .

Notation 1. In the following sections, D_i and $R(y)$ will always denote the expected values of P and Ψ . That is,

$$D_i = \int_{a_i}^{a_{i+1}} P(t) dF_i(t)$$

$$R(y) = \int \Psi(x, y) dF(x).$$

The vector D^k and the function $\phi(D^k)$ are defined as follows:

$$D^k = (D_{k+1}, D_{k+2}, \dots, D_n)$$

$$\phi(D^{k-1}) = D_k + [1 - D_k]\phi(D^k)$$

$$\dots$$

$$\phi(D^n) = 0.$$

Using the last definition, it is easy to see that

$$\int \Psi(x_{k+1}, \dots, x_n) dF_{k+1}(x_{k+1}) \dots dF_n(x_n) = \phi(D^k).$$

Lemma 4. *Let $F(x)$ be in the class O , and let $y \in Y$ be a vector whose last component y_m is contained in the open interval (a_k, a_{k+1}) . Then*

$$\begin{aligned} & R(y_1, \dots, y_{m-1}, y_m) - R(y_1, \dots, y_{m-1}) \\ &= \prod_{i=1}^{k-1} [1 - D_i] \prod_{j=1}^{m-1} [1 - Q(y_j)] [-Q(y_m)] \cdot \\ & \cdot \left\{ 2 \int_{y_m}^{a_{k+1}} P(x_k) dF_k(x_k) + [1 - D_k] [1 + \phi(D^k)] \right\}. \end{aligned}$$

Proof. Let y be the given vector and let $x = (x_1, \dots, x_n)$ be any vector contained in the support of $F(x)$. Lemma 2 can be used to separate all the terms of $\Psi(x, y)$ that depend on y_m ; thus, if $a_k \leq x_k < y_m$,

$$\begin{aligned} & \Psi(x, (y_1, \dots, y_{m-1}, y_m)) - \Psi(x, (y_1, \dots, y_{m-1})) \\ &= \prod_{i=1}^k [1 - P(x_i)] \prod_{j=1}^{m-1} [1 - Q(y_j)] [-Q(y_m)] [1 + \Psi(x_{k+1}, \dots, x_n)]. \end{aligned}$$

But if $y_m < x_k \leq a_{k+1}$, [*Editor's note:* the only difference is the very last term includes x_k .]

$$\begin{aligned} & \Psi(x, (y_1, \dots, y_{m-1}, y_m)) - \Psi(x, (y_1, \dots, y_{m-1})) \\ &= \prod_{i=1}^k [1 - P(x_i)] \prod_{j=1}^{m-1} [1 - Q(y_j)] [-Q(y_m)] [1 + \Psi(x_k, x_{k+1}, \dots, x_n)]. \end{aligned}$$

The vectors x with $x_k = y_m$ have F -measure zero. Therefore, [*Editor's note:* For brevity, define:

$$A = \prod_{i=1}^{k-1} (1 - D_i) \prod_{j=1}^{m-1} (1 - Q(y_j)) (-Q(y_m)).$$

Then:]

$$\begin{aligned} & R(y_1, \dots, y_{m-1}, y_m) - R(y_1, \dots, y_{m-1}) \\ &= \int [\Psi(x, (y_1, \dots, y_{m-1}, y_m)) - \Psi(x, (y_1, \dots, y_{m-1}))] dF(x) \\ &= A \int_{a_k}^{y_m} [1 - P(x_k)] [1 + \phi(D^k)] dF_k(x_k) + A \int_{y_m}^{a_{k+1}} [1 + P(x_k) + (1 - P(x_k)) \phi(D^k)] dF_k(x_k) \\ &= 2A \int_{y_m}^{a_{k+1}} [P(x_k) dF_k(x_k) + [1 - D_k] [1 + \phi(D^k)]] . \end{aligned}$$

□

Lemma 5. *If F belongs to the class O , $R(y)$ is a continuous function of y , except when $y_m = 1$.*

Proof. The result follows from the fact that $\Psi(x, y)$ has only simple discontinuities located on the diagonals $x_i = y_j$; F is continuous except at $x_n = 1$. An alternate proof may be based on the fact that the right-hand side of the last equation varies continuously as y_m increases from $a_k - \varepsilon$ to $a_k + \varepsilon$.

□

6 Corresponding strategies

Definition 2. *Let $F(x)$ and $G(y)$ be two measures contained in the class O , and let S_f and S_G denote the supports of their measures. F and G form a pair of corresponding strategies if*

$$\int \Psi(x, y) dF(x) = v, \quad \text{for all } y \in S_G, y_m \neq 1$$

and

$$\int \Psi(x, y) dG(y) = v, \quad \text{for all } x \in S_F, x_n \neq 1$$

Notation 2. Since F and G belong to the class O ,

$$F(x) = \prod_{i=1}^n F_i(x_i), \quad G(y) = \prod_{j=1}^m G_j(y_j),$$

and the supports of F_i and G_j are intervals $[a_i, a_{i+1}]$, $[b_j, b_{j+1}]$. The numbers b_1, \dots, b_m can be rearranged into subsets, one subset for each interval $[a_i, a_{i+1}]$; the resulting array can be written in the form

$$\begin{aligned} a_1 &\leq b_{1,1} < b_{1,2} < \dots < b_{1,r_1} < a_2 \\ a_2 &\leq b_{2,1} < b_{2,2} < \dots < b_{2,r_2} < a_3 \\ &\vdots \\ a_n &\leq b_{n,1} < b_{n,2} < \dots < b_{n,r_n} < a_{n+1} = 1. \end{aligned}$$

It is possible that there are no b 's between two adjacent a 's.

Every interval bounded by two adjacent b 's must contain precisely one component of each vector $y \in S_G$. It is possible to identify the component by means of the interval over which it ranges; for instance, $y_{i,j}$ will denote the component of y that is contained in the interval $[b_{i,j}, b_{i,j+1}]$. When two adjacent b 's are separated by one of the a 's, say a_i , it is convenient to write

$$a_i = b_{i,0}.$$

In this case, $y_{i-1,r_{i-1}}$ and $y_{i,0}$ denote the same component of y .

In the following lemma, α denotes the discrete mass that $F_n(x_n)$ may have at $x_n = 1$.

Lemma 6. *Let $F(x)$ and $G(y)$ be two strategies contained in the class O . Then*

$$\int \Psi(x, y) dF(x) = v, \quad \text{for all } y \in S_G, y_m \neq 1$$

if and only if the following conditions hold simultaneously:

1. *In the open interval $(b_{i,j}, b_{i,j+1})$, the measure $F_i(x_i)$ is absolutely continuous and*

$$dF_i(x_i) = h_{i,j} \frac{Q'(x_i)}{Q^2(x_i)P(x_i)} dx_i.$$

2. *The coefficients $h_{i,j}$ satisfy the equations*

$$\begin{aligned} 1 + 2\alpha &= D_n + 2h_{n,r_n} \\ h_{i,j-1} &= [1 - Q(b_{i,j})]h_{i,j}, \quad j = 1, \dots, r_i; i = 1, \dots, n \\ h_{i,r_i} &= [1 - D_i]h_{i+1,0}, \quad i = 1, \dots, n. \end{aligned}$$

The first condition requires that $a_1 \leq b_1$.

Proof. Let $y = (y_{1,1}, \dots, y_{n,r_n})$ be any vector in S_G , and define

$$K_{i,j} = \prod_{s=1}^{i-1} [1 - D_s] \prod_{y_{s,t} < y_{i,j}} [1 - Q(y_{s,t})],$$

where the second product is taken over all the components of y that precede $y_{i,j}$. By Lemma 4,

$$\begin{aligned} R(y) &= R(y_{1,1}, \dots, y_{n,r_n-1}) \\ &\quad - K_{n,r_n} Q(y_{n,r_n}) \left[2 \int_{y_{n,r_n}}^1 P(x_n) dF_n(x_n) + (1 - D_n) \right]. \end{aligned} \tag{1}$$

Therefore, $R(y)$ is independent of y_{n,r_n} if and only if

$$Q(y_{n,r_n}) \left[2 \int_{y_{n,r_n}}^1 P(x_n) dF_n(x_n) + (1 - D_n) \right] = 2h_{n,r_n}, \tag{2}$$

for all $b_{n,r_n} < y_{n,r_n} < 1$,

for some constant h_{n,r_n} ; furthermore, if (2) holds, equation (1) becomes

$$R(y) = R(y_{1,1}, \dots, y_{n,r_n-1}) - 2K_{n,r_n} h_{n,r_n}. \tag{3}$$

The dependence of $R(y)$ on the component $y_{i,j}$ is studied by successive applications of Lemma 4; two cases must be considered.

Case 1. $j \neq r_i$. Let us assume that if $R(y)$ is independent of all the components of y that lie beyond $y_{i,j}$, then

$$R(y) = R(y_{1,1}, \dots, y_{i,j}) - 2K_{i,j+1}\gamma_{i,j+1} \quad (4)$$

for some constant $\gamma_{i,j+1}$. Then, by definition,

$$K_{i,j+1} = [1 - Q(y_{i,j})]K_{i,j}.$$

Furthermore, Lemma 4 can be applied to $R(y_{1,1}, \dots, y_{i,j})$. Then

$$\begin{aligned} R(y) &= R(y_{1,1}, \dots, y_{i,j-1}) - 2K_{i,j}\gamma_{i,j+1} \\ &\quad - K_{i,j}Q(y_{i,j}) \left[2 \int_{y_{i,j}}^{a_{i+1}} P(x_i) dF_i(x_i) + (1 - D_i)(1 + \phi(D^i)) - 2\gamma_{i,j+1} \right]. \end{aligned}$$

Therefore, $R(y)$ is also independent of $y_{i,j}$ if and only if

$$\begin{aligned} Q(y_{i,j}) &\left[2 \int_{y_{i,j}}^{a_{i+1}} P(x_i) dF_i(x_i) + (1 - D_i)(1 + \phi(D^i)) - 2\gamma_{i,j+1} \right] \\ &= 2h_{i,j}, \text{ for all } b_{i,j} < y_{i,j} < b_{i,j+1}, \end{aligned} \quad (5)$$

for some constant $h_{i,j}$; furthermore, if (4) is valid for the indices $i, j+1$, and (5) is valid for the indices i, j , then

$$R(y) = R(y_{1,1}, \dots, y_{i,j-1}) - 2K_{i,j}\gamma_{i,j}$$

where

$$\gamma_{i,j} = h_{i,j} + \gamma_{i,j+1}. \quad (6)$$

This result justifies the assumption (4), provided that i is fixed.

Case 2. $j = r_i$. In the present notation y_{i,r_i} and $y_{i+1,0}$ denote the same component of y ; as a matter of fact, if two adjacent b 's are separated by $a_{i+1}, \dots, a_{i+k'}$, the component of y that follows y_{i,r_i} is $y_{i+k,1}$. The process outlined in Case 1 shows that if $R(y)$ is independent of the components of y that lie beyond y_{i,r_i} , then

$$R(y) = R(y_{1,1}, \dots, y_{i,r_i}) - 2K_{i+k,1}\gamma_{i+k,1}.$$

The arguments that were used in the previous case can be applied again, but in this case

$$K_{i+k,1} = \prod_{t=i}^{i+k-1} (1 - D_t)(1 - Q(y_{i,r_i}))K_{i,r_i}$$

and $R(y)$ is independent of y_{i,r_i} , if and only if [for all y_{i,r_i} between b_{i,r_i} and a_{i+1} , and letting $\varphi_i = (1 - D_i)(1 + \phi(D^i))$,]

$$2h_{i,r_i} = Q(y_{i,r_i}) \left[2 \int_{y_{i,r_i}}^{a_{i+1}} P(x_i) dF_i(x_i) + \varphi_i - 2 \prod_{t=i}^{i+k-1} (1 - D_t) \gamma_{i+k,1} \right] \quad (7)$$

for some constant h_{i,r_i} ; when equation 6 holds, $R(y)$ can be written in the form

$$R(y) = R(y_{1,1}, \dots, y_{i,r_i-1}) - 2K_{i,r_i} \gamma_{i,r_i} \quad (8)$$

with

$$\gamma_{i,r_i} = h_{i,r_i} + \prod_{t=i}^{i+k-1} (1 - D_t) \gamma_{i+k,1}. \quad (9)$$

These considerations show that $R(y)$ is independent of y if and only if condition 6 holds for every pair of indices i, j . In the special case $i = 1, j = r_n$, the given condition can also be written in the form

$$Q(y_{n,r_n}) \left[2 \int_{y_{n,r_n}}^1 P(x_n) dF_n^*(x_n) + 2\alpha + 1 - 2D_n \right] = 2h_{n,r_n},$$

where F_n^* is the continuous part of F_n ; every one of these conditions is of the form

$$2 \int_t^{a_{i+1}} P(x) dF_i(x) + C = \frac{2h_{i,j}}{Q(t)}, \quad b_{i,j} < t < b_{i,j+1},$$

with F_i continuous and C constant. An integration by parts shows that F_i must be absolutely continuous in the given interval, and then, by differentiation,

$$dF_i(t) = h_{i,j} \frac{Q'(t)}{Q^2(t)P(t)} dt, \quad b_{i,j} < t < b_{i,j+1}. \quad (10)$$

The integral that appears in each one of those conditions can be evaluated explicitly by means of ; it is then seen that holds if and only if holds and also

$$\begin{aligned} 2h_{n,r_n} &= 2\alpha + 1 - D_n \\ \frac{2h_{i,j}}{Q(b_{i,j+1})} &= 2 \int_{b_{i,j+1}}^{a_{i+1}} P(x_i) dF_i(x_i) \\ &\quad + (1 - D_i)(1 + \phi(D^i)) - 2\gamma_{i,j+1}, \quad j \neq r_i \\ \frac{2h_{i,r_i}}{Q(a_{i+1})} &= (1 - D_i)(1 + \phi(D^i)) - 2 \prod_{t=i}^{i+k-1} (1 - d_t) \gamma_{i+k,1}, \quad j = r_i. \end{aligned} \quad (11)$$

In order to complete the proof of the lemma it is sufficient to replace this system of equations by the equivalent system that is obtained by subtracting

from each equation the preceding one; for instance, if the equation that contains $h_{i,j+1}$ is subtracted from the equation that contains $h_{i,j}$, then

$$\begin{aligned} \frac{2h_{i,j}}{Q(b_{i,j+1})} - \frac{2h_{i,j+1}}{Q(b_{i,j+2})} &= \\ &= 2 \int_{b_{i,j+1}}^{b_{i,j+2}} P(x_i) dF_i(x_i) + 2\gamma_{i,j+2} - 2\gamma_{i,j+1} \\ &= 2h_{i,j+1} \left[\frac{1}{Q(b_{i,j+1})} - \frac{1}{Q(b_{i,j+2})} \right] - 2h_{i,j+1}. \end{aligned}$$

Simplifying,

$$h_{i,j} = [1 - Q(b_{i,j+1})]h_{i,j+1}.$$

Similarly, the equations that contain h_{i,r_i} and $h_{i+1,0}$ show that

$$h_{i,r_i} = (1 - D_i)h_{i+1,0}.$$

It may happen that one of the original a_i 's coincides with one of the original b_j 's, say $a_{i+1} = b_{i+1,1}$. In this case, the system 11 does not contain an equation with $h_{i+1,0}$ and the last derivation is not valid. However, subtracting in this case the equation with h_{i,r_i} from the equation with $h_{i+1,1}$, one can see that

$$h_{i,r_i} = (1 - D_i)[1 - Q(a_{i+1})]h_{i+1,1}.$$

In this case it is convenient to introduce the unnecessary parameter $h_{i+1,0}$ defined by

$$h_{i+1,0} = [1 - Q(a_{i+1})]h_{i+1,1} = [1 - Q(b_{i+1,1})]h_{i+1,1},$$

if $a_{i+1} = b_{i+1,1}$. With this definition, h_{i,r_i} , $h_{i+1,0}$, and $h_{i+1,1}$ are always related by the equations stated in the lemma. □

Lemma 7. *Let $F(x)$ and $G(y)$ be contained in the class O . If*

$$\int \Psi(x, y) dF(x) = v_1 \quad \text{for all } y \in S_G, y_m \neq 1,$$

then

$$\int \Psi(x, y) dF(x) \geq v_1 \quad \text{for all } y \in Y.$$

If

$$\int \Psi(x, y) dG(y) = v_2 \quad \text{for all } x \in S_F, x_n \neq 1,$$

then

$$\int \Psi(x, y) dG(x) \geq v_2 \quad \text{for all } x \in X.$$

Proof. First inequality. The definition of $\Psi(x, y)$ implies that

$$\Psi(x, y) \geq \Psi(x, \overline{y-0}), \quad \text{for all } x, y.$$

Therefore, it is sufficient to consider only those vectors y with $y_m \neq 1$. The inequality is proved by induction, and we may assume that it has been established for all the vectors y whose last $m - q$ components coincide with the last $m - q$ components of some vector in the support of $G(y)$.

Let $y = (y_1, \dots, y_{q+1}, y_{q+2}^*, \dots, y_m^*)$ be a vector whose last $m - (q + 1)$ components coincide with the last $m - (q + 1)$ components of some vector in the support of G . The assumptions of this lemma imply that $R(y)$ must satisfy all the equations derived in the proof of Lemma 6. In particular, $R(y)$ must be independent of all the starred components, and (by 3),

$$R(y) = R(y_1, \dots, y_q, y_{q+1}) - 2 \prod_{s=1}^{i-1} (1 - D_s) \prod_{t=1}^{q+1} [1 - Q(y_t)] \gamma_{i,j+1},$$

where $b_{i,j+1} \leq y_{q+2}^* \leq b_{i,j+2}$. Consider four cases.

Case 1: $y_{q+1} > b_{i,j+1}$. In this case we must have $b_{i,j+1} < y_{q+1} \leq b_{i,j+2}$. Then, by Lemma 4,

$$\begin{aligned} R(y) &= R(y_1, \dots, y_q) - 2 \prod_{s=1}^{i-1} (1 - D_s) \prod_{t=1}^{q+1} [1 - Q(y_t)] \gamma_{i,j+1} \\ &\quad - \prod_{s=1}^{i-1} (1 - D_s) \prod_{t=1}^q [1 - Q(y_t)] \cdot \\ &\quad \cdot Q(y_{q+1}) \left[2 \int_{y_{q+1}}^{a_{i+1}} P(t) dF_i(t) + (1 - D_i)(1 + \phi(D^i)) \right]. \end{aligned}$$

Furthermore, by condition 2 from Lemma 6, if $b_{i,j+1} < y < b_{i,j+2}$,

$$2h_{i,j+1} = Q(y) \left[2 \int_y^{a_{i+1}} P(t) dF_i(t) + (1 - D_i)(1 + \phi(D^i)) - 2\gamma_{i,j+2} \right].$$

Hence,

$$\begin{aligned} R(y) &= R(y_1, \dots, y_q) - 2 \prod_{s=1}^{i-1} (1 - D_s) \prod_{t=1}^{q+1} [1 - Q(y_t)] \gamma_{i,j+1} \\ &\quad - \prod_{s=1}^{i-1} (1 - D_s) \prod_{t=1}^q [1 - Q(y_t)] \cdot \\ &\quad \cdot [2h_{i,j+1} + 2(\gamma_{i,j+2} - \gamma_{i,j+1})Q(y_{q+1})]. \end{aligned}$$

By equation 6, $\gamma_{i,j+1} - \gamma_{i,j} = -h_{i,j+1}$; clearly, $R(y)$ is a monotone increasing function of y_{q+1} . If the components y_p, y_{p+1}, \dots, y_q also lie in the interval $[b_{i,j+1}, b_{i,j+2}]$, the same argument will show that $R(y)$ is a monotone increasing function of these components, and $R(y)$ achieves a smaller value if all of these components are replaced by $b_{i,j+1}$; we can write $y_{q+1}^* = b_{i,j+1} \in [b_{i,j}, b_{i,j+1}]$. Then y_{q+1}^* coincides with the $(q+1)$ -th component of some vector in the support of $G(y)$ and

$$R(y) \geq R(y_1, \dots, y_{p-1}, b_{i,j+1}, \dots, b_{i,j+1}, y_{q+1}^*, y_{q+2}^*, \dots, y_m^*) \geq v.$$

Case 2: $b_{i,j} \leq y_{q+1} \leq b_{i,j+1}$. In this case we simply choose $y_{q+1}^* = y_{q+1}$.

Case 3: $a_1 \leq y_{q+1} < b_{i,j}$. For definiteness, assume that $b_{k,l} \leq y_{q+1} < b_{k,l+1}$. Applying Lemma 4 to $R(y)$ and simplifying the result by means of the appropriate condition 2 from Lemma 6,

$$\begin{aligned} R(y) &= R(y_1, \dots, y_q) - \prod_{s=1}^{i-1} (1 - D_s) \prod_{t=1}^q (1 - Q(y_t)) \gamma_{i,j+1} \\ &\quad - \prod_{s=1}^{k-1} (1 - D_s) \prod_{t=1}^q (1 - Q(y_t)) \\ &\quad \cdot \left[2h_{k,l} + 2Q(y_{q+1}) \left(\gamma_{k,l+1} - \prod_{\sigma=k}^{i-1} (1 - D_\sigma) \gamma_{i,j+1} \right) \right]. \end{aligned}$$

The definition of the constant $\gamma_{k,l+1}$ (equations 6 and 9) shows that $R(y)$ is a monotone decreasing function of y_{q+1} . In this case we may take $y_{q+1}^* = b_{i,j} \in [b_{i,j}, b_{i,j+1}]$.

Case 4: $y_{q+1} < a_1$. By Lemma 3, if $x \in S_F$, $\Psi(x, y)$ is a monotone decreasing function of y_{q+1} . Then $R(y)$ has the same property, and

$$R(y) \geq R(y_1, \dots, y_q, a_1, y_{q+2}^*, \dots, y_m^*).$$

The problem is now reduced to Case 3.

Second inequality. It is sufficient to interchange the roles of X and Y and to recall that $\Psi(y, x) = -\Psi(x, y)$. Then, by assumption

$$\int \Psi(x, y) dG(y) = v \quad \text{for all } x \in S_F, x_n \neq 1.$$

Then, by the first inequality (applied to $\Psi(y, x)$)

$$\int \Psi(y, x) dG(y) \geq -v.$$

□

Corollary 1. *If y is not contained in the support of $G(y)$, then*

$$\int \Psi(x, y) dF(x) \geq v.$$

Proof. This result follows from the strict monotonicity of the functions that appear in the proof of Lemma 7. □

Lemma 8. *Let $F(x)$ and $G(y)$ be two corresponding strategies such that at least one of them is continuous at 1. Then $v_1 = v_2$ and F and G are optimal.*

Proof. For definiteness, we may assume that G is continuous. By definition of v_1 ,

$$\int \Psi(x, y) dF(x) = v_1 \quad \text{for all } y \in S_G, y_m \neq 1.$$

Since the vectors with $y_m = 1$ have G -measure zero,

$$\iint \Psi(x, y) dF(x) dG(y) = \int v_1 dG(y) = v_1. \quad (12)$$

Similarly, by definition of v_2 ,

$$\int \Psi(x, y) dG(y) = v_2 \quad \text{for all } x \in S_F, x_n \neq 1.$$

Since G is continuous, the integral is continuous and the equation is valid for all $x \in S_F$. Therefore,

$$\iint \Psi(x, y) dF(x) dG(y) = \int v_2 dF(x) = v_2. \quad (13)$$

Equations 12 and 13 imply that $v_1 = v_2$; this number is denoted by v . Lemma 7 asserts that v is the value of the game, and that F and G are optimal strategies. □

Remark 1. It is well known that if F and G are optimal, then

$$\int \Psi(x, y) dF(x) = v \quad \text{for all } y \in S_G,$$

provided that y is a point of continuity of the integral; a similar result is valid when F and G are interchanged. In particular, any pair of optimal strategies that belong to the class O must be a pair of corresponding strategies, and it is easy to show that one of them must be continuous at $t = 1$. This remark may be taken as the converse of Lemma 8.

6.1 Characterization of corresponding strategies

Let $F(x)$ and $G(y)$ be two strategies contained in the class O , and let α and β denote the discrete masses that F_n and G_m may have at $x_n = 1, y_m = 1$. Let

$$D_i = \int_{a_i}^{a_{i+1}} P(t) dF_i(t)$$

$$E_j = \int_{b_j}^{b_{j+1}} Q(t) dG_j(t),$$

and let $f^*(x)$ and $g^*(y)$ be two discontinuous functions defined by

$$f^*(x) = \prod_{b_j > x} (1 - Q(b_j)) \frac{Q'(x)}{Q^2(x)P(x)} \quad (14)$$

$$g^*(y) = \prod_{a_i > y} (1 - P(a_i)) \frac{P'(y)}{P^2(y)Q(y)} \quad (15)$$

The results of Lemma 6 can be stated as follows: $F(x)$ and $G(y)$ form a pair of corresponding strategies if and only if the following conditions hold:

$$dF_i(x_i) = h_i f^*(x_i) dx_i, \quad a_i < x_i < a_{i+1} \quad i = 1, \dots, n \quad (16)$$

$$dG_j(y_j) = k_j g^*(y_j) dy_j, \quad b_j < y_j < b_{j+1} \quad j = 1, \dots, m \quad (17)$$

$$1 + 2\alpha = D_n + 2h_n \quad (18)$$

$$1 + 2\beta = E_m + 2k_m \quad (19)$$

$$h_i = (1 - D_i) h_{i+1} \quad (20)$$

$$k_j = (1 - E_j) k_{j+1} \quad (21)$$

$$a_1 = b_1 \quad (22)$$

If these equations are used to characterize the optimal strategies, it is necessary to include the continuity hypothesis ($\alpha\beta = 0$) of Lemma 8 and the normalizing equations

$$\int dF_i = \int dG_j = 1, \quad i = 1, \dots, n \quad j = 1, \dots, m.$$

7 Existence of a solution

We have shown that in order to find two optimal strategies it is sufficient to find a set of numbers $a_1, \dots, b_m, h_1, \dots, k_m, \alpha$, and β that satisfy the previous set of equations. In equations 18, 19, 20, and 21 it is convenient to write D_i and E_j in terms of f^* and g^* , and to eliminate h_n, k_m, h_i, k_j by means of the normalizing equations. In this form, the complete system is as follows:

Normalizing Equations

$$h_n \int_{a_n}^1 f^*(t) dt + \alpha = 1 \quad (23)$$

$$k_m \int_{b_m}^1 g^*(t) dt + \beta = 1 \quad (24)$$

$$h_i \int_{a_i}^{a_{i+1}} f^*(t) dt = 1, \quad i = 1, \dots, n-1 \quad (25)$$

$$k_j \int_{b_j}^{b_{j+1}} g^*(t) dt = 1, \quad j = 1, \dots, m-1 \quad (26)$$

Equations from Lemma 6 and Lemma 8

$$\int_{a_n}^1 [(1 + \alpha) - (1 - \alpha)P(t)] f^*(t) dt = 2(1 - \alpha) \quad (27)$$

$$\int_{b_m}^1 [(1 + \beta) - (1 - \beta)Q(t)] g^*(t) dt = 2(1 - \beta) \quad (28)$$

$$\int_{a_i}^{a_{i+1}} (1 - P(t)) f^*(t) dt = \frac{1}{h_{i+1}} \quad i = 1, \dots, n-1 \quad (29)$$

$$\int_{b_j}^{b_{j+1}} (1 - Q(t)) g^*(t) dt = \frac{1}{k_{j+1}} \quad j = 1, \dots, m-1 \quad (30)$$

$$a_1 = b_1 \quad (31)$$

$$\alpha\beta = 0. \quad (32)$$

Lemma 9. *Let a be any number in the interval $(0, 1]$, and let b_1, \dots, b_m be any set of parameters, subject to the restriction $0 < b_1 < \dots < b_m < 1$; let f^* be defined by equation 14. Then*

$$\lim_{x \rightarrow 0} \int_x^a (1 - P(t)) f^*(t) dt = \pm \infty.$$

Proof. Let

$$M = \prod_{j=1}^m (1 - Q(b_j)),$$

and choose c such that for $t \leq c$, $1 - P(t) \geq P(t)$. Then, for $x \leq c$,

$$\begin{aligned} \int_x^a (1 - P(t)) f^*(t) dt &\geq \int_x^c M \frac{Q'(t)}{Q^2(t)} dt + \int_c^a (1 - P(t)) f^*(t) dt \\ &= M \left[\frac{1}{Q(x)} - \frac{1}{Q(c)} \right] + \int_c^a (1 - P(t)) f^*(t) dt. \end{aligned}$$

The first term tends to $+\infty$; the second one is finite. □

Lemma 10. *Let α be any number in the interval $[0, 1)$, and let b_1, \dots, b_m be any set of parameters, subject to the restriction $0 < b_1 < \dots < b_m < 1$. Under these assumptions, equations 23, 25, 27, and 29 have a unique solution $a_1, \dots, a_n, h_1, \dots, h_n$. If the b 's are fixed, a_n is a monotone increasing function of α , and $a_n \rightarrow 1$ as $\alpha \rightarrow 1$. If α is fixed, a_n is a decreasing function of b_m , and $a_n \rightarrow 0$ as $b_m \rightarrow 1$.*

Proof. Lemma 9 shows that equation 27 has a solution $a_n \in (0, 1]$; since the integrand is positive, the solution is unique, and it is clearly a continuous, monotone increasing function of α ; setting $\alpha = 1$, one obtains $a_n = 1$; but if $\alpha < 1$, a_n must be in $(0, 1)$, and h_n can be computed by means of equation 23. When a_n and h_n are known, a_{n-1} can be obtained from equation 29, and h_{n-1} from equation 25. The process can be continued until all the a 's and the h 's are found. Finally, it is necessary to consider a_n as a function of b_m . Let α and d be fixed, and consider

$$\int_d^{b_m} [(1 + \alpha) - (1 - \alpha)P(t)]f^*(t)dt + \int_{b_m}^1 [(1 + \alpha) - (1 - \alpha)P(t)]f^*(t)dt.$$

In the first term, the integrand contains the factor $(1 - Q(b_m))$ and tends to zero uniformly as $b_m \rightarrow 1$; in the second term the integrand is bounded, and the range of integration can be made arbitrarily small. Thus, as $b_m \rightarrow 1$, the solution a_n of equation 27 must approach 0. □

Remark 2. The value of each a_i depends only on those parameters b_j that are larger than a_i . The remaining parameters can be changed, and new parameters may be added. Also, the process is symmetric: if a_1, \dots, a_n, β are given parameters, equations 24, 26, 28, and 30 have unique solutions b_1, \dots, b_m . Results like those of Lemma 10 will hold.

Theorem 1. *The system of equations 23–32 has a unique solution. This solution determines two strategies $F(x)$ and $G(y)$ that are optimal, and these are the only optimal strategies that belong to the class O .*

Proof. It is sufficient to show that the system has a unique solution $a_1, \dots, a_n, b_1, \dots, b_m, h_1, \dots, h_n, k_1, \dots, k_m, \alpha, \beta$. Then $F(x)$ and $G(y)$ can be defined by equations 16 and 17. The normalizing equations show that F and G are strategies, and the remaining equations show that F and G satisfy the hypotheses of Lemma 8. The remark that follows that lemma shows that no other strategies in the class O can be optimal.

Existence. Consider two numbers α and β such that

$$0 \leq \alpha < 1, \quad \alpha\beta = 0, \quad 0 \leq \beta < 1,$$

and construct a set of numbers $a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^*$ as follows: first, compute two numbers a_n and b_m by means of equations 27 and 28 using no parameters in the definitions of f^* and g^* ; the resulting numbers are compared and the larger one is kept as a parameter; for definiteness, we assume that $a_n > b_m$, and define $a_n^* = a_n$; the smaller number is neglected. In the next step a number a_{n-1} is computed by means of equation 29 (without parameters), and a new b_m is computed from equation 28, using the single parameter a_n^* . It is clear that the new b_m is smaller than the one computed in the previous step, and therefore $b_m < a_n^*$; the two numbers a_{n-1} and b_m are compared, and the larger one is kept as a parameter. For definiteness, one may assume that $a_{n-1} > b_m$, and define $a_{n-1}^* = a_{n-1}$. The process is continued in this manner: at each step, a new a_i and a new b_j are computed, using as parameters the previously starred a 's and b 's; the two numbers are compared, and the larger one is kept as a parameter, denoted by a_i^* (or b_j^*); the corresponding h_i^* (or k_j^*) is computed by means of the appropriate normalizing equation. Since each parameter is smaller than those computed previously, it is clear that the resulting set is self-consistent: that is, if b_1^*, \dots, b_m^* are used as parameters, one obtains again a_1^*, \dots, a_n^* , and conversely. Therefore, these numbers satisfy the system of equations, *except for equation 31*.

Now, consider a_1^* and b_1^* as functions of α and β . If the previous construction is carried out with $\alpha = \beta = 0$, it may happen that $a_1^* = b_1^*$, and all equations are satisfied; otherwise, either $a_1^* < b_1^*$ or $b_1^* < a_1^*$; for definiteness *it may be assumed that $b_1^* < a_1^*$* . In this case, the same construction is applied with $\alpha = 0, \beta = 1 - \varepsilon$, and ε is chosen so small that in the first step $a_n < b_m$, and $b_m^* = b_m$ is arbitrarily close to 1. In the remaining steps a_n is always computed with the parameter b_m , and $a_n \rightarrow 0$; thus all the b^* 's are computed without parameters and remain bounded away from zero; then $a_1^* < a_n^* < b_1^*$; the inequality has been reversed as β increased from 0 to 1. Since the a 's and b 's are limits of integration of strictly positive densities, it is clear that a_1^* and b_1^* are continuous functions of β , and there exists a positive β for which $a_1^* = b_1^*$. If the construction with $\alpha = \beta = 0$ leads to the inequality $a_1^* < b_1^*$, the system of equations has a solution with $\alpha > 0, \beta = 0$.

Uniqueness. The solution is unique. Indeed, suppose that $a_1, \dots, a_n, b_1, \dots, b_m$ and $a_1^*, \dots, a_n^*, b_1^*, \dots, b_m^*$ are two solutions. These solutions give rise to two pairs of optimal strategies $F(x), G(y)$ and $F^*(x), G^*(y)$. Since G and G^* are optimal

$$\int \Psi(x, y) dF(x) = v, \quad \text{for all } y \in S_{G^*}, y_m \neq 1.$$

The corollary to Lemma 7 shows that $S_{G^*} \subset S_G$. Since the argument is symmetric, the two measures must have the same supports, i.e., $b_j^* = b_j$, for all j . Similarly, $a_i^* = a_i$. □

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