Minimization of Sequential Co-occurrences under Capacity Constraints

Abstract

Back when I was a boy scout, at our big group gatherings, we'd often play games in teams. There'd be, like, N game stations, and twice as many groups. We'd go from station to station, and at each one, two groups would face off. Then we'd rotate: half the groups would go clockwise, the other half counter-clockwise, so we'd hit up every single station, one after the other.

But some years, we'd end up playing against the same darn groups multiple times, and I remember thinking, 'Hey, there are tons of other groups we haven't even seen! Why do we face the same groups again and again?' I wondered why that was happening, and if it was possible to find better schedules.

This paper investigates the problem of scheduling group visits to workshops while minimizing repeated interactions between groups under capacity constraints. The goal is to determine an optimal sequence that ensures each group visits all workshops while limiting repeated pairwise encounters.

1 Introduction

In various scheduling and assignment problems, minimizing repeated encounters between entities is essential for achieving fairness, privacy, or efficiency. Consider a scenario in which $2N$ groups must visit N workshops in a structured manner. At each time step, each workshop accommodates exactly two groups, and each group must visit all workshops exactly once.

A key objective is to reduce the number of repeated encounters between any two groups across all time steps. This ensures a distribution that minimizes unnecessary interactions and enhances diversity within the scheduling framework. The problem can be framed as an optimization model, aiming to assign sequences that satisfy these constraints while minimizing the cumulative pairwise encounters.

Note: To the author's knowledge, this result is novel. However, the possibility of a prior solution cannot be entirely excluded due to the limitations of the literature search. This problem may already have been introduced, maybe with another name and a different formulation.

2 Problem Formalization

Let's now define formally our problem, that we will call the group meeting minimization problem (GMM).

Let $A_{i,t}$ represent the workshop assigned to group i at time step t, where $A_{i,t} \in \{0, 1, \ldots, N-1\}$. Define $c_{i,j,t}$ as a binary variable indicating whether groups i and j share the same workshop at time t:

$$c_{i,j,t} = \begin{cases} 1, & \text{if } A_{i,t} = A_{j,t} \\ 0, & \text{otherwise.} \end{cases}$$

Define $r_{i,j}$ as the cumulative number of times groups i and j have shared a workshop **after having already met** up to time t by induction on t:

$$r_{i,j}^{(t)} = \begin{cases} r_{i,j}^{(t-1)} + c_{i,j,t}, & \text{if } \exists t' < t \text{ s.t. } c_{i,j,t'} = 1 \\ r_{i,j}^{(t-1)}, & \text{otherwise.} \end{cases}, \quad \text{with } r_{i,j}^{(0)} = 0.$$

We define $G \coloneqq [\![0, 2N-1]\!]$ and $N \coloneqq [\![0, N-1]\!]$.

(note N stands at the same time for an integer and for the set containing all natural numbers smaller than $N-1$, which is the same thing using the set-theoretic definition of natural numbers, so everything is fine)

The objective function to minimize is:

$$\min \sum_{1 \leq i < j \leq 2N} r_{i,j}.$$

The constraints are:

- Each group visits each workshop exactly once.

$$\forall g \in G, \forall a \in N, \#\{t \in N \mid A_{g,t} = a\} = 1$$

  ie $\forall g \in G, t \mapsto A_{g,t}$ is a bijection (from N to N).

- Each workshop hosts exactly two groups at each time step.

$$\forall a \in N, \forall t \in N, \#\{g \in G \mid A_{g,t} = a\} = 2$$

This formulation presents a structured approach to optimizing group allocations while minimizing repeated encounters, offering potential applications in scheduling, event organization, and combinatorial optimization.

In the following, the event "$\exists t' < t, c_{i,j,t'} = 1$ and $c_{i,j,t} = 1$" (ie the groups i and j meet again) will be called either a repeated co-occurrence, a recurring encounter, a reunion event or a recurrent workshop match between i and j at time t.

3 Equivalent Formulations

This problem can be expressed in terms of functions:

Observation 3.1. *$(A_{g,t})_{g\in G,t\in N}$ is a valid schedule if and only if $\forall g \in G, t \mapsto A_{g,t}$ is a bijection from N to itself, and it exists a unique $f : G \times N \to G$ such that $A_{f(g,t),t} = A_{g,t}$ and $g \neq f(g,t)$*

Proof. By double implication:

- $\implies$: clear (f will map a group with the only one it is competing against: $\forall a \in N, \forall t \in N, \#\{g \in G \mid A_{g,t} = a\} = 2$: a such f clearly exist and is unique).

- $\impliedby$: also clear: f exist so for all $t, f(g,t) \neq g$ are at the same workshop so $\forall a \in N, \forall t \in N, \#\{g \in G \mid A_{g,t} = a\} \geq 2$ or this set is empty, and as f is unique we have $\forall a \in N, \forall t \in N, \#\{g \in G \mid A_{g,t} = a\} \leq 2$ (only one possible choice). The two previous statements (and the fact that we have $2N$ groups) clearly implies that $\forall a \in N, \forall t \in N, \#\{g \in G \mid A_{g,t} = a\} = 2$ (for instant fix t first and then observe all the inequalities have to be saturated).

$\square$

Observation 3.2. *A schedule is free from recurring encounters if and only if $\forall g \in G, f(g,.) : N \to G$ is injective.*

Proof. Clear: the list of groups g will face is $[f(g,t) \mid t \in N]$. So the numbers of recurring encounters in a schedule will be the number of duplicates in this list (here we define the number of duplicate as the minimum number of elements one should remove in order to only have distinct values).

In particular: a schedule is free from recurring encounters $\iff$ $[f(g,t) \mid t \in N]$ has no duplicates $\iff$ $\forall g \in G, f(g,.) : N \to G$ is injective. $\square$

This problem can also be expressed using graph and more specifically edge coloring:

Definition 3.3. A graph G is said to admit a N-quadratic-edge-coloring if:

- G has $2N$ vertices
- G has N^2 edges, all indexed (ie colored) by a tuple:
  $E = \{e_{a,t} \mid a \in N, t \in N\}$
- $\forall a, t1 \neq t2 : e_{a,t1} \cap e_{a,t2} = \emptyset$: edges with same first colors are independent.
- $\forall a1 \neq a2, t : e_{a1,t} \cap e_{a2,t} = \emptyset$: edges with same second colors are independent.

Observation 3.4. *A graph G with k 2-cycles admitting a N-quadratic-edge-coloring has the following properties:*

- *for all $a \in N$, considering only the edges with first color a will give a perfect matching.*
- *for all $t \in N$, considering only the edges with second color t will give a perfect matching.*

Proof. This is quite obvious (for both statements): for all $a \in N$, we have N edges with this color ($e_{a,t}$ for all $t \in N$) and they are independent. We have N independent edges and $2N$ vertices, thus we have a perfect matching. $\square$

Proposition 3.5. *There exist a valid schedule for N workshop ($N > 0$) with k recurring encounters if and only if there exist a graph G with k 2-cycles admitting a N-quadratic-edge-coloring.*

Remark 3.6. Here we say a graph has k 2-cycles if k is the minimum number of edges one has to remove in order to make all 2-cycles in G disappear.

Let us introduce an important tool before writing the proof of our proposition:

Definition 3.7. Given $(A_{g,t})_{g\in G,t\in N}$, define G_E the encounter graph as follow:

- $V = G(= 2N)$
- $E = \{e_{A_{g,t},t} := (g, g') \mid \exists t, A_{g,t} = A_{g',t}, g \neq g'\}$

Similarly define $G_{E,t}$ the encounter graph at time t as follow:

- $V_t = G(= 2N)$
- $E_t = \{(g, g') \mid A_{g,t} = A_{g',t}, g \neq g'\}$

Proof. (of the proposition above)

- $\Longrightarrow$: Let $(A_{g,t})_{g\in G,t\in N}$ a valid schedule for N workshop ($N > 0$) with k recurring encounters. Consider G_E the encounter graph. We will show G_E has k 2-cycles and admit a N-quadratic-edge-coloring.

  Clearly G_E has $2N$ vertices. As $\forall a \in N, \forall t \in N, \#\{g \in G \mid A_{g,t} = a\} = 2$, G_E clearly has N^2 edges, naturally indexed by $N \times N$.

  We have $\forall g \in G, \forall a \in N, \#\{t \in N \mid A_{g,t} = a\} = 1$. So $\forall a, t1 \neq t2$: $e_{a,t1} \cap e_{a,t2} = \emptyset$: clear as $g \in e_{a,t1} \cap e_{a,t2} \implies A_{g,t1} = a = A_{g,t2}$.

  Moreover, $\forall a1 \neq a2, t : e_{a1,t} \cap e_{a2,t} = \emptyset$ as $g \in e_{a1,t} \cap e_{a2,t} \implies A_{g,t} = a1 \neq a2 = A_{g,t}$.

  Having a recurring encounter is equivalent to have $A_{g1,t1} = A_{g2,t1}$ and $A_{g1,t2} = A_{g2,t2}$, so it is quite clear that with k recurring encounter one will have k 2-cycles: removing all the 2-cycles is equivalent to remove all edges with $A_{g1,t} = A_{g2,t}$ except at most one (where t varies), ie reducing $\{t \in N \mid A_{g1,t} = A_{g2,t}\}$ to a set with at most one element (for all $g1, g2$), ie removing k edges (the one that corresponds to recurring encounters).

- $\Longleftarrow$: Let G be a graph with k 2-cycles and admitting a N-quadratic-edge-coloring. Without loss of generality, one can assume that the vertex are indexed by integers in $2N$.

  For all g, for all t, define $A_{g,t} = a$ where a is such that $g \in e_{a,t}$. A such a exist and is unique as, for a fixed t, considering only the edges with second color t will give a perfect matching (cf previous observation). So it exist a unique edge with second color t in which g lies.

  For all $a \in N$, considering only the edges with second color a will give a perfect matching (cf previous observation). So clearly $\#\{t \in N \mid A_{g,t} = a\} = \{t \in N \mid g \in e_{a,t}\} = 1$. Moreover, for all $a \in N, t \in N$, we have $\#\{g \in G \mid A_{g,t} = a\} = \#\{g \in G \mid g \in e_{a,t}\} = 2$: clear: $e_{a,t}$ is fixed and an edge contains exactly two vertices.

  Finally, with the same reasoning as above, here each 2-cycle corresponds to a recurring encounter: an edge represent an encounter, so if $m > 0$ edges are linking two vertices, then it means there are m encounters between the groups, so $m - 1$ recurring encounter (the first one is not recurrent, all the others are).

$\square$

Observation 3.8. *A graph G with k 2-cycles admitting a N-quadratic-edge-coloring has the following property:*
if G is a complete bipartite graph, then $k = 0$.

Remark 3.9. We will see later on that the converse does not hold: one can have N-quadratic-edge-coloring graphs that contains odd cycles and no cycle of length 2.

Proof. Here, we don't have anything to prove: we can derive this immediately from the proposition above as a complete bipartite graph has only cycles of length 4 up to $2N$ (if we only consider path that do not go trough the same vertices twice). Just note that it makes sense to speak about complete bipartite graphs: if we have two sets of size N V_1 and V_2 then the complete bipartite graph between these two will have $2N$ vertices and N^2 edges. $\square$

To summarize, we have the following (which is a immediate consequence of the above, the condition $k = \sum_g N - |f(g, N)|$ do not appear in the observations above but is immediately derived from the proof):

Proposition 3.10. *The three following statements are equivalent:*

- *There exist a valid schedule for N workshop ($N > 0$) with k recurring encounters.*

- *$\forall g \in G, t \mapsto A_{g,t}$ is a bijection from N to itself, it exists a unique $f : G \times N \to G$ such that $A_{f(g,t),t} = A_{g,t}$ and $g \neq f(g,t)$, and the sum of the number of collision of the $(f(g,.) : N \to G)_{g \in G}$ is k: $k = \sum_g N - |f(g, N)|$.*

- *there exist a graph G with k 2-cycles admitting a N-quadratic-edge-coloring.*

4 Solution when N is Odd

Observation 4.1. *if N is odd, then one can find a scheduling without any recurring encounter.*

Proof. Just consider the following assignment:

$$A_{g,t} = \begin{cases} \frac{g}{2} + t \mod N, & \text{if } g \text{ is even} \\ \frac{g-1}{2} - t \mod N, & \text{otherwise.} \end{cases}$$

Note: we are identifying N and $\mathbb{Z}/N\mathbb{Z}$

- this assignment is correct:

  Clearly all group visit each workshop once: If g even: $\{t \in N \mid A_{g,t} = a\} = \{t \in N \mid \frac{g}{2} + t \mod N = a\} = \{a - \frac{g}{2} \mod N\}$: of cardinal one.

  Similar if g odd.

  Each workshop hosts exactly two groups at each time step. More specifically, at time t the workshop w will host $2w - 2t \mod 2N$ and $2w + 1 + 2t \mod 2N$ (clear, by induction on t).

- This assignment is free from recurring encounter:

  It suffices to show that each group sees at least N different groups (it will imply that no group sees the same group twice, as they each see N different groups and there are N steps here).

  We will show that every even groups meet every odd group. Let $g = 2k$ be an even group and $g' = 2k' + 1$ be an odd group.

  We assumed N odd, let $N = 2K + 1$.

  We have $A_{g,t} = k + t \mod N$, $A_{g',t} = k' - t \mod N$.

  - if $k' - k$ is even: take $t = \frac{k'-k}{2} \mod N$:

  $$\begin{aligned} A_{g,t} &= k + t \mod N = k + \frac{k'-k}{2} \mod N \\ &= k' - \frac{k'-k}{2} \mod N = A_{g',t} \end{aligned}$$

  - if $k' - k$ is odd: take $t = K + \frac{k'-k+1}{2} \mod N$:

  $$\begin{aligned} A_{g,t} &= k + t \mod N = k + K + \frac{k'-k+1}{2} \mod N \\ &= k' - K - \frac{k'-k+1}{2} \mod N, \quad \text{as } K = -K - 1 \mod N \\ &= A_{g',t} \end{aligned}$$

Thus the above indeed defines a valid assignment without any recurring encounter.

□

5 Case where N is Divisible by 4

Assume for this section that $N = 4k$ with $k \geq 1$.

We show there exist a scheduling with no recurring encounter here: we build a valid scheduling without any recurring encounter.

Let us consider the following scheduling:

$$A_{g,t} = \begin{cases} \frac{g}{2} - t \mod N, & \text{if } g = 0 \mod 4 \\ \frac{g-1}{2} + t \mod N, & \text{if } g = 1 \mod 4 \text{ and } t < 2k \\ \frac{g-1}{2} + 2k - 1 - t \mod N, & \text{if } g = 1 \mod 4 \text{ and } t \geq 2k \\ \frac{g}{2} + t \mod N, & \text{if } g = 2 \mod 4 \\ \frac{g-1}{2} - t \mod N, & \text{if } g = 3 \mod 4 \text{ and } t < 2k \\ \frac{g-1}{2} + 2k + 1 + t \mod N, & \text{if } g = 3 \mod 4 \text{ and } t \geq 2k \end{cases}$$

Let's show this scheduling is valid and has no recurring encounter.

- $\forall g \in G, \forall a \in N, \#\{t \in N \mid A_{g,t} = a\} = 1$:
  - if $g = 0 \mod 4$ or $g = 2 \mod 4$: clear: moves by one at each step: after N steps, it will have visited all workshop.
  - if $g = 1 \mod 4$: we want to show $\{A_{g,t} \mid t \in N\} = N$ here (it implies that g visits all workshop at least once, thus exactly once as there are N step and only one workshop can be visited at each step).
    Let $t, t' \in N$ ie $0 \leq t, t' \leq 4k - 1$. If $0 \leq t, t' \leq 2k - 1$ or $2k \leq t, t' \leq 4k - 1$, it is easy to show $t \neq t' \implies A_{g,t} \neq A_{g,t'}$. So, assume $0 \leq t \leq 2k - 1$ and $2k \leq t' \leq 4k - 1$ (by symmetry).

    $$\begin{aligned} A_{g,t} = A_{g,t'} &\implies \frac{g-1}{2} + t = \frac{g-1}{2} + 2k - 1 - t' \mod N \\ &\implies t = 2k - 1 - t' \mod N \\ &\implies N \mid t + t' - 2k + 1 \end{aligned}$$

    Which is absurd as $1 \leq t + t' - 2k + 1 \leq 4k - 1 = N - 1$.
    So indeed $t \neq t' \implies A_{g,t} \neq A_{g,t'}$: all workshops are visited.
  - if $g = 3 \mod 4$: similar to the previous case.
- $\forall a \in N, \forall t \in N, \#\{g \in G \mid A_{g,t} = a\} = 2$:
  Let $t \in N, a \in N$. Show $\#\{g \in G \mid A_{g,t} = a\} \geq 2$ (which is sufficient as we have N workshops and $2N$ groups).
  - if $t < 2k$ and $a + t = 0 \mod 2$: $A_{2(a+t),t} = A_{2(a-t)+1,t} = a$
  - if $t < 2k$ and $a + t = 1 \mod 2$: $A_{2(a-t),t} = A_{2(a+t)+1,t} = a$
  - if $t \geq 2k$ and $a + t = 0 \mod 2$: $A_{2(a+t),t} = A_{2(a-t)-4k-1,t} = a$
  - if $t \geq 2k$ and $a + t = 1 \mod 2$: $A_{2(a-t),t} = A_{2(a+t)-4k+3,t} = a$

- $\sum_{1\leq i<j\leq 2N} r_{i,j} = 0$, for this we show that each group sees at least N different groups. Let $g \in G$ a group:
  Note $G_i \coloneqq \{g \in G \mid g = i \mod 4\}$
  - if $g = 0 \mod 4$: g sees all groups in G_1 and G_3:
    We already know from the above (and it is clear) that g sees only group from G_1 and G_3, so it suffices to show that it doesn't see the same twice (it will implies that it sees exactly each group of $G_1 \cup G_3$ exactly once as it sees N groups without repetition in $G_1 \cup G_3$, of size N):
    * if $g' \in G_1$: from the above we have $\forall a \in N, \forall t \in N, \#\{g \in G \mid A_{g,t} = a\} = 2$, and we listed the list of all encounter at each workshop. So we know $G \in G_0$ and $g' \in G_1$ can only meet in the case $t < 2k$ and $a + t = 0 \mod 2$.

      $$\begin{aligned} g \text{ sees } g' \text{ twice } &\iff \exists t \neq t', \begin{cases} \frac{g}{2} - t = \frac{g'-1}{2} + t \mod N \\ \frac{g}{2} - t' = \frac{g'-1}{2} + t' \mod N \end{cases} \\ &\implies t - t' = t' - t \mod N \\ &\implies N \mid 2(t - t') \\ &\implies t = t' \end{aligned}$$

      Where the last implication holds because $0 \leq t, t' < 2k$ so $0 \leq 2(t - t') < 4k = N$.
    * similar if $g' \in G_3$:
      We know $G \in G_0$ and $g' \in G_1$ can only meet in the case $t \geq 2k$ and $a + t = 0 \mod 2$.

      $$\begin{aligned} g \text{ sees } g' \text{ twice } &\iff \exists t \neq t', \begin{cases} \frac{g}{2} - t = \frac{g'-1}{2} + 2k + 1 + t \mod N \\ \frac{g}{2} - t' = \frac{g'-1}{2} + 2k + 1 + t' \mod N \end{cases} \\ &\implies t - t' = t' - t \mod N \implies t = t' \end{aligned}$$

      Where the last implication holds because $2k \leq t, t' < 4k = N$ so $0 \leq 2(t - t') < 4k = N$.
  - if $g = 2 \mod 4$: g sees all groups in G_1 and G_3: similar to the previous case: for $0 \leq t < 2k$, g will meet G_3 and for $2k \leq t < 4k$, g will see G_1.
  - if $g = 1 \mod 4$: the above immediately implies g sees all groups in G_2 and G_4.
  - if $g = 3 \mod 4$: the above immediately implies g sees all groups in G_2 and G_4.

  So, each group sees at least N different groups. Thus we can't have any recurring encounter.

6 Case where N is Even but not Divisible by 4

For $N = 2$: clearly, there is only one valid schedule, up to renaming the groups.

Without loss of generality, assume $A_{0,0} = A_{1,0} = 0$ and $A_{2,0} = A_{3,0} = 1$ (ie we name the groups such that this holds).

At the step $t = 1$, the groups 0 and 1 have no choice but to visit the workshop 1, and similarly 2 and 3 will visit 0. So $A_{0,1} = A_{1,1} = 1$ and $A_{2,1} = A_{3,1} = 0$.

So this schedule is indeed the only valid one. It is clearly valid and each group makes a recurring encounter, so in total we have 4 recurring encounters.

Now, we assume $N = 4k + 2$ with $k \geq 1$.

We show there exist a scheduling with no recurring encounter here by building one.

before this, introduce the notion of difference sequence:

Definition 6.1. A difference sequence is an (ordered) list of $N - 1$ elements in $\mathbb{Z}/N\mathbb{Z}$. We will use python-like syntax to define them.

A difference sequence l defines unambiguously a schedule for a given group g with the following conventions:

- $A_{g,0} = \lfloor \frac{g}{2} \rfloor$
- $A_{g,t+1} = A_{g,t} + l[t]$ (again, using python-like syntax for lists)

So, instead of giving a schedule, one can associate a difference sequence with each group.

Remark 6.2. For the lists, we will use the python syntax, except for list concatenation, that will be denoted by `@`. I decide to not use `+` here to denote a non-commutative operation (and nobody should).

Let then consider the scheduling induced by the following difference sequences: the difference sequences associated with g is

$$\begin{cases} \texttt{[-1 for \_ in range(n-4)] @ [-3, 1, 1]}, & \text{if } g = 0 \mod 4 \\ \texttt{[1 for \_ in range(n//2-2)] @ [n//2+1] @} & \\ \quad \texttt{[-1 for \_ in range(n//2-3)] @ [-3, 1, 1]}, & \text{if } g = 1 \mod 4 \\ \texttt{[1 for \_ in range(n-4)] @ [1, 2, -1]}, & \text{if } g = 2 \mod 4 \\ \texttt{[-1 for \_ in range(n//2-2)] @ [-n//2-1] @} & \\ \quad \texttt{[1 for \_ in range(n//2-3)] @ [1, 2, -1]}, & \text{if } g = 3 \mod 4 \end{cases}$$

Let us now prove that our scheduling is valid and free from recurring encounters.

- We prove $\forall g \in G, t \mapsto A_{g,t}$ is a bijection (from N to N) by showing it is a surjection: each workshop is seen at least once (sufficient by equality of cardinals). Let $g \in G$ (all cases are clear, but we write them down explicitely as this will make the rest of the proof smother):

- if $g = 0 \mod 4$ or $g = 2 \mod 4$: clear: moves by one at each step: after N steps, it will have visited all workshop. g will visit $\frac{g}{2}, \frac{g}{2} - 1, \dots, \frac{g}{2} - N + 4$ and then $\frac{g}{2} - N + 1, \frac{g}{2} - N + 2, \frac{g}{2} - N + 3$. So, by reordering these, g will visit $\frac{g}{2}, \frac{g}{2} - 1, \dots, \frac{g}{2} - N + 1$ ie all workshop in N.
- if $g = 1 \mod 4$: similarly, g will visit $\frac{g-1}{2}, \frac{g-1}{2} + 1, \dots, \frac{g-1}{2} + \frac{N}{2} - 2$, then $\frac{g-1}{2} - 1$, then $\frac{g-1}{2} - 2, \dots \frac{g-1}{2} - \frac{N}{2} + 2$ then $\frac{g-1}{2} - \frac{N}{2} - 1, \frac{g-1}{2} - \frac{N}{2}, \frac{g-1}{2} - \frac{N}{2} + 1$.
- if $g = 2 \mod 4$: g/2, ... g/2 + n-4 then g/2 + n-3, g/2 + n-1, g/2 + n-2
- if $g = 3 \mod 4$: (g-1)/2, ... (g-1)/2 - n/2 + 2 then (g-1)/2 + 1 then (g-1)/2 + 2, ..., (g-1)/2 + n/2 -2, then (g-1)/2 + n/2 -1, (g-1)/2 + n/2 +1, (g-1)/2 + n/2

- $\forall a \in N, \forall t \in N, \#\{g \in G \mid A_{g,t} = a\} = 2$:
  Note $N_0 \coloneqq \{0, 2, \dots, N-2\}$ and $N_1 \coloneqq \{1, 3, \dots, N-1\}$.
  Note $G_i \coloneqq \{g \in G \mid g = i \mod 4\}$ By induction on t, we will show: that at time t, we have the following repartition:

$$\begin{cases} G_0 \text{ and } G_1 \text{ dwell in } N_{t \mod 2}, G_2 \text{ and } G_3 \text{ in } N_{t+1 \mod 2} & \text{if } t < \frac{N}{2} - 1 \\ G_0 \text{ and } G_3 \text{ dwell in } N_{t \mod 2}, G_1 \text{ and } G_2 \text{ in } N_{t+1 \mod 2} & \text{if } \frac{N}{2} - 1 \leq t < N - 2 \\ G_0 \text{ and } G_2 \text{ dwell in } N_{t \mod 2}, G_1 \text{ and } G_3 \text{ in } N_{t+1 \mod 2} & \text{if } N - 2 \leq t < N \end{cases}$$

  (by G_i dwell in N_* we mean that for each workshop of N_* there is exactly one group from G_i)
  The induction is straightforward (at time $t = 0$, clear by initialization, for $t > 0$ just observe that adding the same even number to all group in some G_i won't change the places where G_i dwell in, adding an odd one will change it, so a simple disjunction on t suffices).
  It immediately follows that at each time t, each workshop hosts two groups.

- $\sum_{1 \leq i < j \leq 2N} r_{i,j} = 0$, for this we show that two groups never meet more than once:
  We already from the above that groups of G_i do not meet other groups of G_i. So it suffices to show $\forall 0 \leq i < j \leq 3, g \in G_i$ and $g' \in G_j$ do not meet more than once. We have six cases (depending on i, j):

  - if $i = 0, j = 1$: assume g and g' meet at time t and t'. We want to show $t = t'$.
    According to the above it clearly implies $0 \leq t, t' < \frac{N}{2}$ (only with these t can groups from G_0 and G_1 meet). So $A_{g,t} = \frac{g}{2} - t, A_{g,t'} = \frac{g}{2} - t', A_{g',t} = \frac{g'-1}{2} + t$ and $A_{g',t} = \frac{g'-1}{2} + t'$ (easy to compute with our hypothesis on t, t'). Thus $\frac{g}{2} - t = \frac{g'-1}{2} + t$ and $\frac{g}{2} - t' = \frac{g'-1}{2} + t'$. ie $2t = 2t' = \frac{g-g'+1}{2}$ (modulo N). So t and t' are equal modulo $\frac{N}{2}$. As $0 \leq t, t' < \frac{N}{2}$, it implies $t = t'$.

- if $i = 2, j = 3$: similar to the previous case.
- if $i = 0, j = 3$: assume g and g' meet at time t and t'. By symmetry, assume $t \leq t'$. Now by contradiction (technically, we are not using reduction ab absurdum here) assume $t \neq t'$.
  According to the above, we have $\frac{N}{2} \leq t < t' < N - 2$.
  * if $t' = N - 3$: $A_{g,t'} = \frac{g}{2} - N + 1$ and $A_{g',t'} = \frac{g'-1}{2} + \frac{N}{2} - 1$, so $\frac{g-g'+1}{2} = \frac{N}{2} - 2$. Moreover $\frac{N}{2} \leq t < N - 3$. $A_{g,t} = \frac{g}{2} - t$ and $A_{g',t} = \frac{g'-1}{2} + 1 + (t - \frac{N}{2})$. So $\frac{g}{2} - t = \frac{g'-1}{2} + 1 + t - \frac{N}{2}$, ie $\frac{g-g'+1}{2} + \frac{N}{2} - 1 = 2t$. So, using the above $\frac{N}{2} - 2 + \frac{N}{2} - 1 = 2t$, ie $-3 = 2t$ (modulo N). This is obviously impossible (with parity argument for instance).
  * otherwise: $\frac{N}{2} \leq t, t' < N - 3$, so $A_{g,t} = \frac{g}{2} - t, A_{g,t'} = \frac{g}{2} - t', A_{g',t} = \frac{g'-1}{2} + 1 + (t - \frac{N}{2})$ and $A_{g',t'} = \frac{g'-1}{2} + 1 + (t' - \frac{N}{2})$. So we have $\frac{g}{2} - t = \frac{g'-1}{2} + 1 + (t - \frac{N}{2})$ and $\frac{g}{2} - t' = \frac{g'-1}{2} + 1 + (t' - \frac{N}{2})$. Therefore $t' - t = t - t'$ (by subtracting the two) ie $2t = 2t'$ (modulo N). So $t = t'$ modulo $\frac{N}{2}$. And as we have $\frac{N}{2} \leq t, t' < N - 3$, it implies indeed $t = t'$.
- if $i = 1, j = 2$: similar to the previous case.
- if $i = 0, j = 2$: By symmetry, assume $t \leq t'$. Now by contradiction assume $t \neq t'$. This implies $t = N - 2$ and $t' = N - 1$. We have $A_{g,t'} = A_{g,t+1} = A_{g,t} + 1$ and $A_{g',t'} = A_{g',t+1} = A_{g',t} - 1$. So $A_{g,t} = A_{g',t}$ and $A_{g,t'} = A_{g',t'}$ implies $1 = -1$ (modulo N). And this is obviously an absurdity.
- if $i = 1, j = 3$: similar to the previous case.

So, each group sees groups at most once. Thus we don't have any recurring encounter.

Remark 6.3. The encounter graph of the above schedule is not bipartite: there are odd cycles:

- let $g \in G_0$ and $g' \in G_2$ such that g and g' meet (such group exist: take any $g \in G_0$ and the group g' it's meeting at time $t = N - 1$ for instance).
- g met all group in G_1 (as G_0 and G_1 dwell in the same workshops for $\frac{N}{2} = |G_0| = |G_1|$ time steps).
- g' met $\frac{N}{2} - 2 > 0$ (as $N = 4k + 2 \geq 6$ here) groups in G_1.

So g and g' meet and meet a common group in G_1. Thus the encounter graph has cycles of length 3. Thus is not bipartite.

7 Conclusion

So, in the end, we have the following proposition (directly derived from the above) that solves our problem for all integers.

Proposition 7.1. *For all integer $N \geq 3$, one can find a valid scheduling without any recurring encounter.*

So our problem is solved (since the cases where $N < 3$ are kind of trivial; for $N < 2$ they don't really make sense, for $N = 2$ we will have exactly one recurring encounter per group).

One can easily compute and visualize the optimal solutions presented here using this webpage. It suffices to input an integer and an optimal scheduling is computed and printed. This is just a simple implementation in JavaScript of the solutions presented above.

A GitHub repository has been created for this problem. You can find that repository here. It contains this paper, a rocq file formalizing the proof of the property above and a README.

Now that this problem has been addressed, one could ask another question: we might also want to minimize the "total distance traveled by the groups between the meetings" (sum of the distances of each group) or the "time taken by the transitions" (maximum of the distances for each transition). One could try to find among the optimal schedules (optimal in terms of recurring encounter) one that also minimize one of the above objective function.

While our solution given here for odd integers also minimize these quantities (for the natural distance on $\mathbb{Z}/N\mathbb{Z}$ at least), it is not clear our solutions for even integers are also optimal in that matter. So it could be interesting to look in that direction.

Before optimizing the distances, one has to define a distance. A natural choice could be the natural distance on $\mathbb{Z}/N\mathbb{Z}$. One could also choose the natural distance on $\mathbb{N}$ (if the workshops form a line). Or if the workshops form a circle, one could choose the geometric distance (identifying the workshop with the nth roots of unity in $\mathbb{C}$ and take the canonical norm on $\mathbb{C}$).

So it can lead to a lot of problem that *could* be interesting.

One can also look at generalization of the problem.

For instance, one can add time and activities: for $N < T$, we look at graphs with $2N$ vertices and NT edges (indexed in $T \times T$). It corresponds to having $2N$ groups, T workshops and T activities. We allow workshop to host zero group or two group at each time step. We still want each group to visit each workshop.

Maybe it could also be interesting to look at cubic edge coloring or generalization in this direction, and see if this can be interpreted.