PROJECT 1
COMPUTATIONAL METHODS IN FINANCIAL ENGINEERING

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1. Introduction

Conditional Value at risk, also referred to as CVaR, is a metric commonly used in finance and in risk management. Indeed, it is a convenient way of evaluating linear and non-linear derivatives, market risks, credit risks and many others.. It was first introduced in the late 1990's and popularized by the authors R. Tyrrell Rockafellar and Stanislav Uryasev in their paper titled "Optimization of Conditional Value-at-Risk".

In simple terms, the CVaR represents the maximum amount a portfolio is expected to lose, with a certain confidence level and over a certain period. The goal of the approach is to limit the risk of high losses. The authors chose to explore that method as an alternative to VaR since they deemed that it had undesirable mathematical characteristics.

2. Methodology

With this paper, the authors are presenting a practical technique where one can optimize CVaR and calculate VaR at the same time.

To achieve their results the authors based their approach on the β-VaR and β-CVaR. The latest being the conditional expectation of losses above an amount α with probability β. It is interesting to note that most methods of calculating the CVaR will either rely on linear models or use a Monte Carlo model.

They posed $f(x,y)$ as the loss associated with the decision vector X. In our case, X can be represented as a set of available portfolios subject to certain constraints. Y is a random vector that can be used to describe uncertainties that can affect the loss. For each X, $f(x,y)$ is a random variable. The variable y has a density denoted as $p(y)$. In order to replicate table 4, we will have to generate random samples from $p(y)$.

(see written notes)

Based on that probability $(p(y))$, we can express the probability of $f(x,y)$ not exceeding a threshold as;

$\Psi(x,\alpha) = \int_{f(x,y) \leq \alpha} p(y)\,dy$, where Ψ is the cumulative distribution function for x. This formula assumes that $\Psi(x,\alpha)$ is continuous with respect to α.

In this model the values associated with x and the probability level β $(\beta \in [0,1])$ are respectively denoted as $\alpha_\beta(x)$ and $\phi_\beta(x)$ where;

$\alpha_\beta(x) = \min\{\alpha \in \mathbb{R} : \Psi(x,\alpha) \geq \beta\}$ and

$\phi_\beta(x) = (1-\beta)^{-1} \int_{f(x,y) \geq \alpha_\beta(x)} f(x,y)\,p(y)\,dy$

where $\phi_\beta(x)$ is the conditional expectation of the loss associated with x in relation of that loss being "$\alpha_\beta(x)$" or greater. Then $\phi_\beta(x)$ and $\alpha_\beta(x)$ will be caracterized as the function F_β on $X \times \mathbb{R}$. This new function will be of great use to calculate the β-CVaR. It also has interesting properties such as being convex and continuously differentiable which will make optimisation much easier. That said function can be expressed as the followings

$F_\beta(x,\alpha) = \alpha + (1-\beta)^{-1} \int_{y \in \mathbb{R}^m} [f(x,y)-\alpha]^+ p(y)\,dy$ where $[t]^+ = \begin{cases} t & \text{when } t > 0 \\ 0 & \text{when } t \leq 0 \end{cases}$

The theorem 1 describes how we can find the β-CVaR of the loss associated with any $x \in X$ with the formula $\phi_\beta(x) = \min_{\alpha \in \mathbb{R}} F_\beta(x,\alpha)$. Note that we are minimizing with respect to α as we are trying to solve for the smallest loss possible. The set for the value of α in which a minimum is attained can be expressed as $A_\beta(x) = \underset{\alpha \in \mathbb{R}}{\arg\min}\, F_\beta(x,\alpha)$. The set is a non empty set with close bounded intervals. The left end point of $A_\beta(x)$ is the β-VaR of the loss ($\Leftrightarrow$ $\alpha_\beta(x)$ = left endpoint of $A_\beta(x)$) where one has always;

$\alpha_\beta(x) \in \underset{\alpha \in \mathbb{R}}{\arg\min}\, F_\beta(x,\alpha)$ and $\phi_\beta(x) = F_\beta(x, \alpha_\beta(x))$.

as we mentionned previously, the function of theorem 1 is really interesting due to its properties. The theorem also presented how we can obtain the β-CVaR whithout having to calculate the β-VaR first. It also shows that it's possible to approximate $F_\beta(x,\alpha)$ with the help of the probability distribution of y according to its density $(p(y))$. The estimation is done by generating a collection of vectors $\in Y$ and then their corresponding approximation. The formula reads as;

$\tilde{F}_\beta(x,\alpha) = \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} [f(x,y_k)-\alpha]^+$. This approch will still be convex but only piecewise linear in respect to α. Even if the function is not differendiable in respect to α anymore, it will still be fairly easy to minimize.

The theorem 2 discusses the downside of working directly with $\alpha_\beta(x)$ (since it has the mathematical properties of the β-VaR) and how we can operate with the expression $F_\beta(x,\alpha)$ in a much more convenient way. The theorem reads as such: $\min_{x \in X} \phi_\beta(x) = \min_{(x,\alpha) \in X \times \mathbb{R}} F_\beta(x,\alpha)$ which translate to saying that minimizing the β-CVaR of the loss associated with $x \; \forall x \in X$ is equivalent to minimizing $F_\beta(x,\alpha)$ over all $(x,\alpha) \in X \times \mathbb{R}$. In that case $F_\beta(x,\alpha)$ will be convex with respect to (x,α).

It is possible to use the approach we previously disscussed in order to optimize a portfolio of different financial instruments. First we set $x_j \geq 0$, for $j = 1, \dots, n$. with $\sum_{j=1}^{n} x_j = 1$, meaning that the total weights of the j securities must sum up to 1 (100%). Then we denote the return by y_j (in that case y is still a vector composed of different returns). The loss can then be expressed as $f(x,y) = -x^T y$. (We are multiplying the returns by the weights, since we are looking for the loss, vector x is negative).

It is important to note that in the portfolio approach, the VaR and CVaR are defined as percentages and not as a monetary value. With that in mind, we can obtain a performance function where the β-VaR and the β-CVaR are connected to the problem; $F_\beta(x,\alpha) = \alpha + (1-\beta)^{-1} \int_{y \in \mathbb{R}^n} [-x^T y - \alpha]^+ p(y)\,dy$. As in section 2 of the paper, the function will have properties (convex on x and α) making it appropriate for optimization. The mean and VaR of the loss associated with the portfolio x can be presented as $\mu(x) = -x^T m$ and $\sigma^2(x) = x^T V x$. To improve the programming, a return constraint "R" will be added to the model. The linear constraint is the following: $\mu(x) \leq -R$. with that new constraint the feasable set is $X = \{x : x\}$. We can also estimate $F_\beta(x,\alpha)$ as we did previously with a sample prod. dist. of y_j; $\tilde{F}_\beta(x,\alpha) = \alpha \frac{1}{q(1-\beta)} \sum_{k=1}^{q} [-x^T y_k - \alpha]^+$. This problem can be solved with convex programming.

The authors are proposing 3 problems to solve in order to (P1) minimize the β-CVaR: $\min \phi_\beta(x)$ over $x \in X$, (P2) min the β-VaR: $\min \alpha_\beta(x)$ over $x \in X$ and (P3) min variance: $\min \sigma^2(x)$ over $x \in X$. The P3 is quite interesting because it uses quadratic prog. The authors conclude with proposition 1 stating that any optimal portfolio x^* will be optimal for any two of P1, P2, P3 and that the solution to the two will be the same if $\beta \geq 0.5$, $\mu(x) \leq -R$, loss associated with $x \sim N$, $y \sim N$. We sadly won't go into the detail of the proof as it is of limited use to us.

3. Implementation

In order to obtain the results of Table 3, we must first define the optimization problem, which is like follows:

$$\min_{x} \quad \sigma^2(x)$$
$$\text{s.t.} \quad \begin{cases} \mu(x) \leq -R \\ x_i \geq 0 \\ \sum x_i = 1 \end{cases}$$

where $\mu(x)$ and $\sigma^2(x)$ denote the mean and variance of the loss associated with portfolio x, which minimizes variance. Using the results given in Tables 1 and 2, we have the column vector of mean returns $\mathbf{m}$, and the covariance matrix $\mathbf{V}$, with which we can solve for:

$$\mu(x) = -x^T m \qquad \text{and} \qquad \sigma^2(x) = x^T V x$$

Optimizing $\sigma^2(x)$ is therefore a quadratic programming problem, which can be solved by using "quadprod()":

$$\min_{x} \quad \frac{1}{2} x^T H x + f^T x$$
$$\text{s.t.} \quad \begin{cases} A \cdot x \leq b \\ Aeq \cdot x = beq \\ lb \leq x \leq ub \end{cases}$$

to solve for the asset weights that result in the minimum-variance optimal portfolio.

In order for the output to be correct, we must first transform our covariance matrix $\mathbf{V}$ into matrix H, by multiplying all values of V by 2. Other inputs to "quadprod" needed are the constraints which are the bounds (0,1) and the sum of weights (=1). As for constraint $\mu(x) \leq -R$, given the proof shown in Proposition 1 of the article, we optimize over a smaller set of $X \to X'$ so that we can use the equality $\mu(x) = -R$ instead.

Running the optimization, we obtain the minimum-variance portfolio; the function returns the optimal weights, as well as the optimal returns and variance. This information is used to create Table 3

Then, using the results from Table 3, we move to calculate the VaR and CVaR values, for a set of betas $\beta = (0.90, 0.95, 0.99)$. To do so, we use the standard deviation $\sigma(x)$ of the optimal portfolio to solve:

$$\alpha_\beta(x) = \mu(x) + C_1(\beta)\sigma(x) \quad \text{and} \quad \phi_\beta(x) = \mu(x) + C_2(\beta)\sigma(x)$$

defined in the article as (18) and (19), which gives the VaR and CVaR values, respectively. For the computation of $\alpha_\beta(x)$, the function "norminv" is used to calculate the normal CDF used for $C_1(\beta)$. To solve for the VaR values, the calculation was repeated over the list of betas. A similar process was used to produce the values of CVaR as well, except instead of simply using the normal CDF, the normal pdf was used through the function "normpdf". The pdf function was also multiplied by $-\left(\frac{1}{1-\beta}\right)$ to account for the derivation of the CDF. Finally, the VaR and CVaR values were displayed in Table 4.

Lastly, we follow the methodology proposed in Theorem 2 to minimize the β-CVaR, as it is shown to be very useful in that it is equivalent to minimizing $F_\beta(x, \alpha)$, which becomes a linear problem.

We start by defining the objective function, containing $\mu(x)$, α and u_k:

$$\min \ \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} u_k$$

$$\text{s.t.} \begin{cases} u_k \geq 0 \\ -x^T y - \alpha - u_k \leq 0 \\ \mu(x) \leq -R \\ x_i \geq 0 \\ \sum x_i = 1 \end{cases}$$

To define "y", we use the function "mvnrnd" to generate the same amount "q" of normal random numbers as the sample size in our simulations.

Given that the minimization of the objective function is a linear programming problem, "linprog" was used. Two options of "linprog" were changed using "optiset": a maximum number of iterations was set to avoid potential infinite loops, and the algorithm was specifically set to inter-point. We included the appropriate definitions and constraints, along with the minimization problem, inside of the function RiskCalculater.

Running RiskCalculater outputs almost all of the data we are interested in to produce Table 5: β's, sample size, weights, VaR, CVaR, iterations and time. The only calculation left is to determine VaR diff & CVaR diff, which is done by simply using our VaR & CVaR values from the simulation.

Multiple simulations are done by looping over a set of iteration counts.

4. Results

Following the methodology presented in the article, we were able to reproduce the same results for Tables 3 and 4. For Table 5, however, our results differed slightly. This is most likely due to the nature of the randomness associated with the simulation process as a whole.

Table 3*: Minimum-variance optimal portfolio*

SP500	GovBond	SmallCap
0.45201	0.11557	0.43242

Table 4*: VaR & CVaR obtained from minimum-variance approach*

Risk Level (B)	VaR	CVaR
{'0.90'}	0.067847	0.096975
{'0.95'}	0.090199	0.11591
{'0.99'}	0.13213	0.15298

Table 5: *Simulation-generated portfolios, VaR, CVaR*

beta	Sample Size	SP500	GovBond	SmallCap	VaR	VaR diff. (%)	CVaR	CVaR diff. (%)
0.9	1000	0.40696	0.13289	0.46015	0.070935	-4.353	0.096536	0.45435
0.9	3000	0.34511	0.15666	0.49823	0.066629	1.8276	0.095569	1.4707
0.9	5000	0.46584	0.11026	0.4239	0.068338	-0.71784	0.097815	-0.85905
0.9	10000	0.50592	0.094854	0.39923	0.06674	1.6588	0.094699	2.4036
0.9	20000	0.40681	0.13295	0.46024	0.068068	-0.32396	0.096746	0.23653
0.95	1000	0.45595	0.11406	0.42999	0.08921	1.1083	0.11121	4.2215
0.95	3000	0.64297	0.042177	0.31486	0.090479	-0.30984	0.11617	-0.22143
0.95	5000	0.49338	0.099671	0.40695	0.092186	-2.1554	0.11605	-0.12051
0.95	10000	0.45066	0.11609	0.43325	0.089377	0.9202	0.11597	-0.057196
0.95	20000	0.42091	0.12753	0.45156	0.089462	0.82394	0.115	0.78952
0.99	1000	0.7527	1.3072e-10	0.2473	0.14241	-7.2188	0.16343	-6.3943
0.99	3000	0.55394	0.076396	0.36966	0.13952	-5.2988	0.16065	-4.7764
0.99	5000	0.38113	0.14282	0.47605	0.12877	2.6062	0.14764	3.6114
0.99	10000	0.4633	0.11123	0.42546	0.13259	-0.35102	0.15059	1.5819
0.99	20000	0.39754	0.13651	0.46595	0.13233	-0.15565	0.15385	-0.56732

6. Bibliography

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7. Joint evaluation of contribution

We deem that our contributions to the project were balanced.