Counting minimal transversals in β-acyclic Hypergraph

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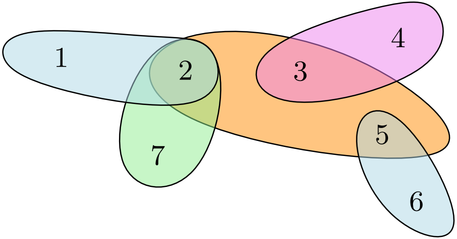
Hypergraphs

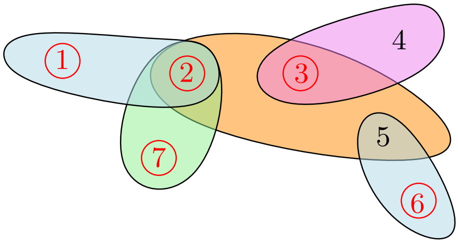


Definitions

Transversal

A set of vertices intersecting all hyperedges.

- Transversals of graph: Vertex Cover.
- Minimal *w.r.t.* inclusion.
- Every vertex has a private hyperedge.



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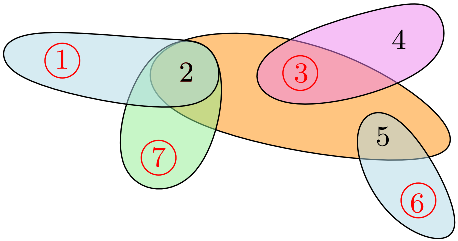


Motivations

- Graphs: vertex covers, dominating sets (transversals of closed neighborhoods),...
- AI: models of monotone CNF-formula.

Counting and enumerating them have a lot of applications

- Graphs Theory, A.I. (robustness), Datamining, Model-checking,...

Counting

Goal

Find the number of solutions to some problem.

- An analogous hierarchy to $P \subseteq NP$.
  - Polynomial
  - $\#P$
  - $\#P$-hard / complete
- Counting can be much harder than finding:
  - Minimal Transversal, Perfect Matching, 2-SAT,...
  - k-path, k-cycle!

Minimality matters

Theorem [Okamoto, Uno, Uehara 2005]

In chordal graphs:

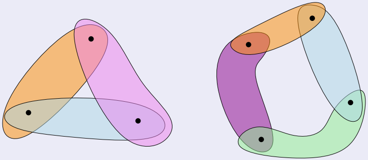
- Counting transversals is doable in polynomial time.
- Counting minimal transversals is $\#P$-complete.

- Chordal graph: no induced cycle of size ≤ 4.

β-acyclic

Definition

A hypergraph is β-acyclic if it does not have β-cycles.



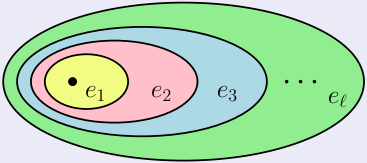
- The incidence graph (vertices/hyperedges) is Bipartite Chordal.

β-acyclic

Definition bis

A hypergraph is β-acyclic if

- Strong Elimination ordering on $V(\mathcal{H})$: β-leaf



$$e_1 \subseteq e_2 \subseteq e_3 \subseteq \ldots \subseteq e_\ell$$

- Not related to the x-width of the incidence graph.

Results on β-acyclic hypergraph

Theorem [Brault-Baron, Capelli, Mengel 2017]

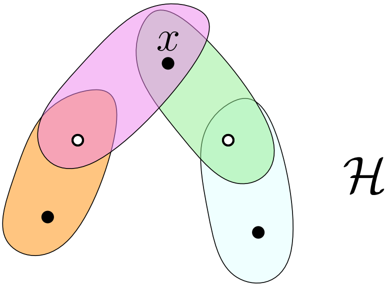
We can count the transversals of a β-acyclic hypergraph in polynomial time.

Theorem [B., Capelli, Kanté 2017]

We can count the minimal transversals of a β-acyclic hypergraph in polynomial time.

- The ingredients: recursive decomposition and "blocked transversals".

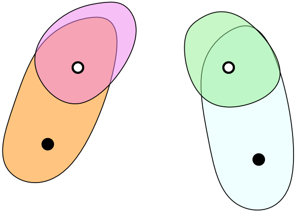
Basic facts



Fact

The nb. of minimal transversals of $\mathcal{H}$ not containing x is the nb. of minimal transversals of $\mathcal{H} - x$.

Basic facts



$$\mathcal{H} - x$$

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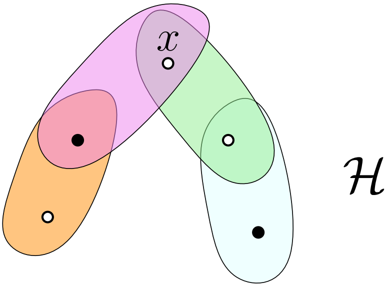

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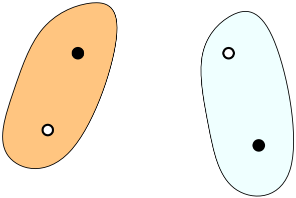
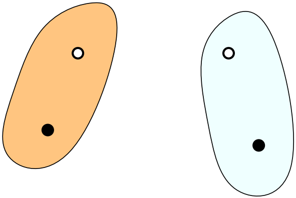


$$\mathcal{H} \setminus \mathcal{H}(x)$$

Basic facts



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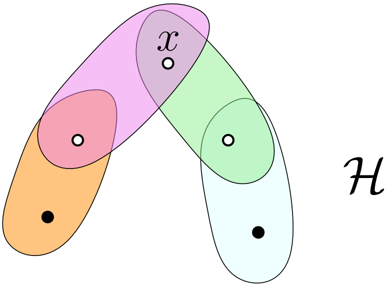

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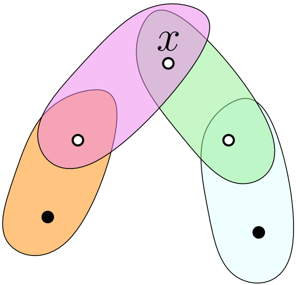


$\mathcal{H}$

Lemma

$T \cup x$ is a min. transversal of $\mathcal{H}$ iff:

- T is a min. transversal of $\mathcal{H} \setminus \mathcal{H}(x)$
- T is not a transversal of $\mathcal{H}$

Formula

Corollary

The following values are equal:

- The nb. of minimal transversals of $\mathcal{H}$ containing x.
- The nb. of minimal transversals of $\mathcal{H} - \mathcal{H}(x)$ that are not transversals of $\mathcal{H}$.
- The nb. of minimal transversals of $\mathcal{H} - \mathcal{H}(x)$ minus the nb. of minimal transversals of $\mathcal{H} - \mathcal{H}(x)$ that are transversals of $\mathcal{H}$.

Blocked-Transversals

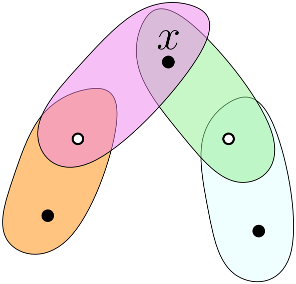
S: a subset of vertices.

Definition

S-blocked transversal of $\mathcal{H}$ (denoted by $S\text{-btr}(\mathcal{H})$):

- Minimal transversals of $\mathcal{H}$ and of $\mathcal{H} \setminus \mathcal{H}(S)$

- $\varnothing\text{-btr}(\mathcal{H})$ = min. transversals of $\mathcal{H}$.
- Private hyperedges cannot contain a vertex of S.



$\mathcal{H}$

Blocked Transversal

Lemma (reformulation)

The nb. of minimal transversals of $\mathcal{H}$ $=$

nb. of min. transversals of $\mathcal{H}$ excluding x

$+$ nb. of min. transversals of $\mathcal{H} \setminus \mathcal{H}(x)$

$-$ nb. of min. transversals of $\mathcal{H} \setminus \mathcal{H}(x)$ and of $\mathcal{H}$.

Blocked Transversal

Lemma (reformulation)

The nb. of minimal transversals of $\mathcal{H}$ =

$\#\varnothing\text{-btr}(\mathcal{H} - x)$

$+$ nb. of min. transversals of $\mathcal{H} \setminus \mathcal{H}(x)$

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Blocked Transversal

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The nb. of minimal transversals of $\mathcal{H}$ =

$$\#\varnothing\text{-btr}(\mathcal{H} - x)$$

$$+\ \#\varnothing\text{-btr}(\mathcal{H} \setminus \mathcal{H}(x))$$

$$-\ \#x\text{-btr}(\mathcal{H}).$$

Blocked Transversal

Lemma (reformulation)

The nb. of minimal transversals of $\mathcal{H}$ =

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$$+\ \#\varnothing\text{-btr}(\mathcal{H} \setminus \mathcal{H}(x))$$

$$-\ \#x\text{-btr}(\mathcal{H}).$$

Lemma (generalization)

The nb. of S-blocked transversals of $\mathcal{H}$ =

$$\#S\text{-btr}(\mathcal{H} - x)$$

$$+\ \#S\text{-btr}(\mathcal{H} \setminus \mathcal{H}(x))$$

$$-\ \#S \cup \{x\}\text{-btr}(\mathcal{H} \setminus\ (\mathcal{H}(x) \cap \mathcal{H}(S))).$$

Blocked Transversal

Using the formula leads to a combinatorial explosion in general!

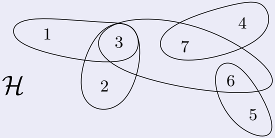
Decomposition of β-acyclic hypergraph

- Strong elimination ordering on $V(\mathcal{H})$: $x_1, \ldots, x_n$.
- Induced order on the hyperedges : $e_1, \ldots, e_m$.

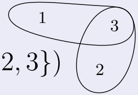
Definition

$\mathcal{H}(x, e) \Rightarrow$ the hyperedges $\leqslant e$ connected to e through vertices $\leqslant x$.

$\mathcal{H}$



$\mathcal{H}(3, \{2, 3\})$



- $\mathcal{H}(x_n, e_m) = \mathcal{H}$.
- Goal is to compute $\#\varnothing\text{-btr}(\mathcal{H}(x_n, e_m))$ with the help of:
  - The formula: $\#S\text{-btr}(\mathcal{H}) =$

    $\#S\text{-btr}(\mathcal{H} - x) + \#S\text{-btr}(\mathcal{H} \setminus \mathcal{H}(x)) - \#S \cup \{x\}\text{-btr}(\mathcal{H} \setminus (\mathcal{H}(x) \cap \mathcal{H}(S)))$
  - Structural properties of the hypergraphs $\mathcal{H}(x, e)$.

Avoid the explosion

Combinatorial explosion !

To compute $\#\{w\}\text{-btr}(\mathcal{H})$ we need to compute:

- $\#\{x, w\}\text{-btr}(\mathcal{H} \setminus (\mathcal{H}(x) \cap \mathcal{H}(w)))$,...

But in β-acyclic hypergraphs

We can compute $\#\{x, w\}\text{-btr}(\mathcal{H} \setminus (\mathcal{H}(x) \cap \mathcal{H}(w)))$ from

- $\#x\text{-btr}(\mathcal{H}')$.
- $\#w\text{-btr}(\mathcal{H}')$.

We can express each $\mathcal{H}'$ as a $\mathcal{H}(y, f)$ where $y < x$.

Dynamic programming

5 $\#\emptyset\text{-btr}(\mathcal{H}(5, e_5))$

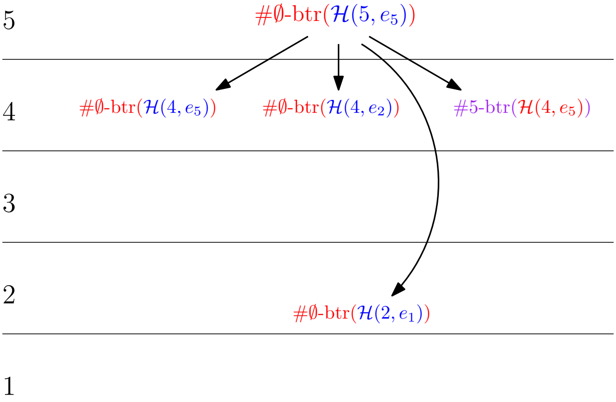
4

3

2

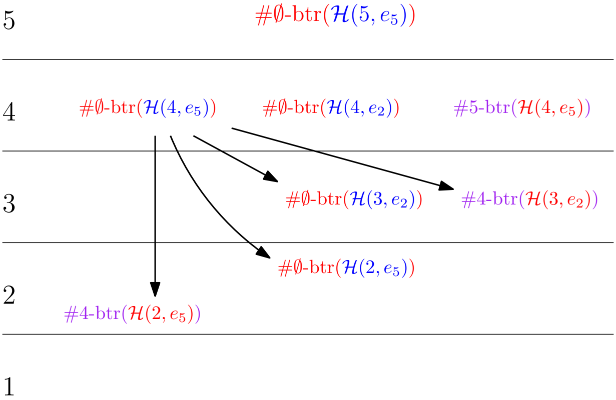
1

Dynamic programming



Each term is computable from smaller terms.

Dynamic programming



Each term is computable from smaller terms.

Dynamic programming

5		$\#\emptyset\text{-btr}(\mathcal{H}(5, e_5))$	
4	$\#\emptyset\text{-btr}(\mathcal{H}(4, e_5))$	$\#\emptyset\text{-btr}(\mathcal{H}(4, e_2))$	$\#5\text{-btr}(\mathcal{H}(4, e_5))$
3		$\#\emptyset\text{-btr}(\mathcal{H}(3, e_2))$	$\#4\text{-btr}(\mathcal{H}(3, e_2))$
2			
1			

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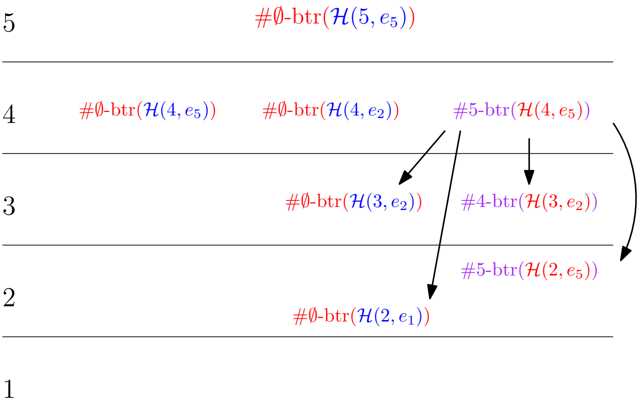
Dynamic programming



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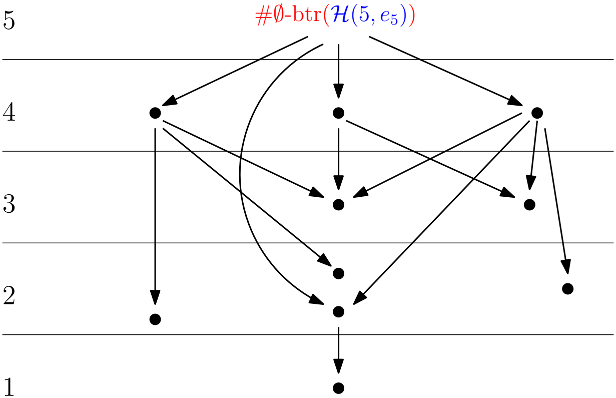
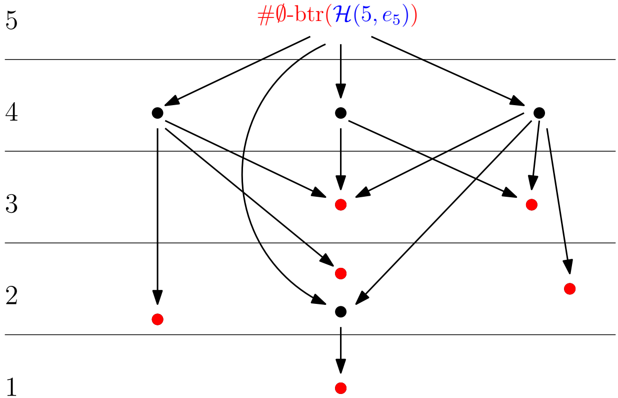


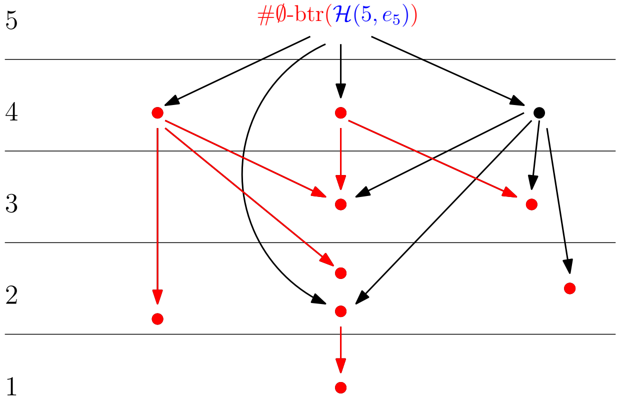
We end up with a DAG.

Dynamic programming



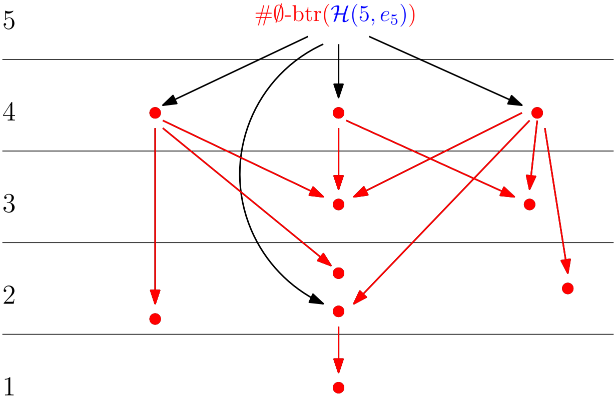
Sink are easily computable.

Dynamic programming



As a salmon, we go back to the source.

Dynamic programming



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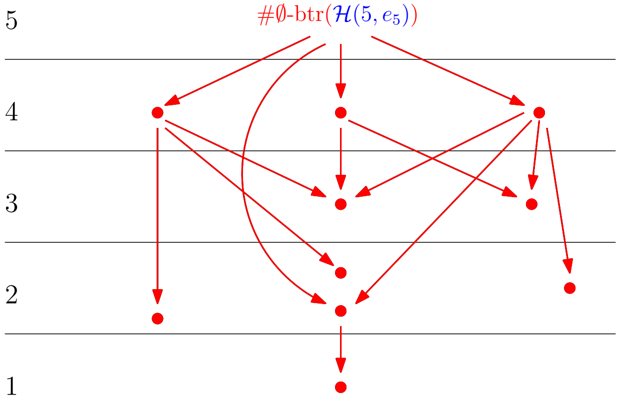


Number of terms is polynomial $\Rightarrow$ Algorithm polynomial.

Result and Consequences

Theorem [B. , Capelli, Kanté 2017]

We can count the number of minimal transversals of a β-acyclic hypergraph in polynomial time.

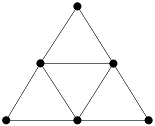
- Min. dominating sets $\Rightarrow$ min. transversals of closed neighborhoods.
- Strongly chordal graphs $\Leftrightarrow$ $\{N[x] \mid x \in V(G)\}$ is β-acyclic.

Corollary

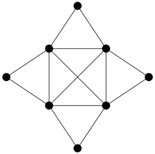
We can count the number of min. dominating sets of Strongly Chordal graph in polynomial time.

Minimal dominating sets

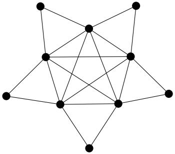
- *Stongly Chordal* graphs $\Leftrightarrow$ Chordal graphs $+$ k-sun free, for $k \geqslant 3$.



3-sun



4-sun



5-sun

[Kanté, Uno 2017]

Counting min. dominating sets of Chordal graphs is $\#P$-complete.

Minimal dominating sets

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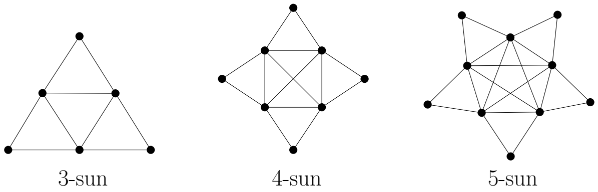


[Kanté, Uno 2017]

Counting min. dominating sets of Chordal graphs is $\#P$-complete.

Conjecture [Kanté, Uno 2017]

Counting min. dominating sets of a subclass of Chordal graphs is

- doable in polynomial time if this class is k-sun free, for $k \geqslant 4$,
- $\#P$-complet otherwise.

Possible generalizations

Theorem [Brault-Baron, Capelli, Mengel 2015]

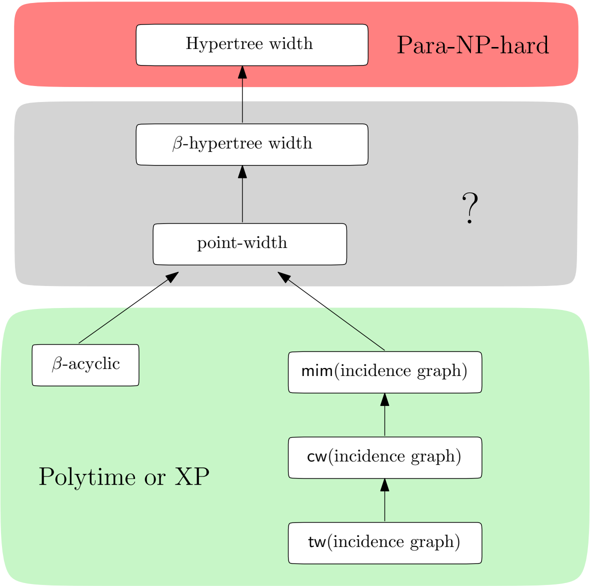
We can count the Models of a β-acyclic CNF-formula in polynomial time.

Corollary of our result

We can count the minimal models of a monotone β-acyclic CNF-formula in polynomial time.

- Can we do it for non-monotone formulas?

Beyond β-acyclicity



Thank you!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