A new notion of Representative Sets for Graph Coloring

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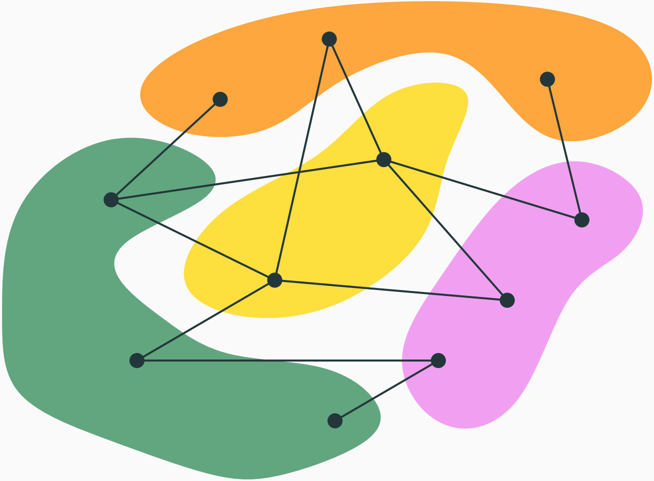
GRAA Seminar, January 27, 2022

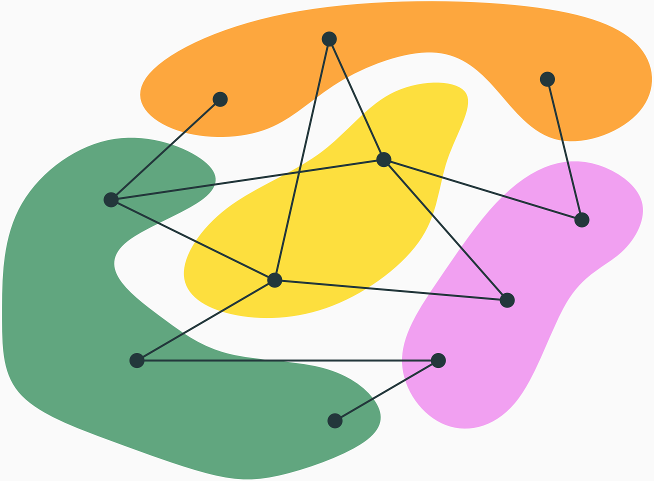
Graph Coloring

A **coloring** of G is a **partition** of $V(G)$ into **independent sets**.



Graph Coloring

A **coloring** of $A \subseteq V(G)$ is a **coloring** of $G[A]$.



Graph Coloring

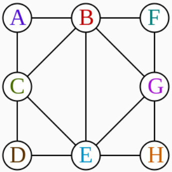
q-Coloring

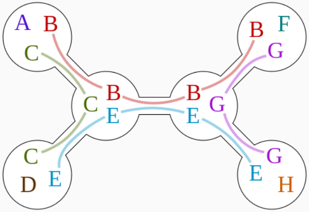
- Input: a graph G and a non-negative integer q.
- Question: Does G a **coloration** with at most q **colors**?

Graph Coloring

- Input: a graph G.
- Output: A **coloration of minimum size**.

Width measures

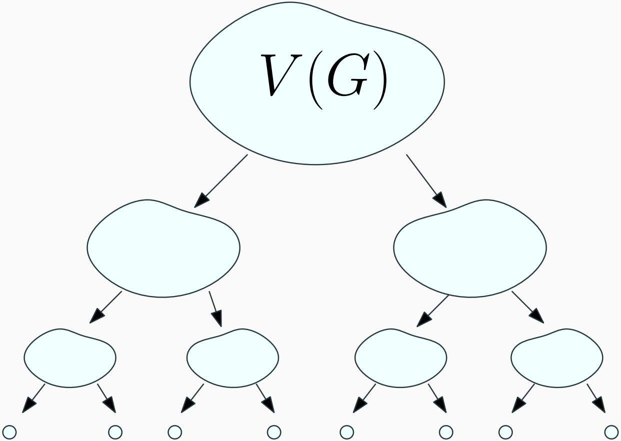




- Tree-width, clique-width, rank-width, mim-width,...
- Measure the **structural complexity** of graphs.
- Gives **efficient** algorithms for many **NP-hard** problems.

Divide a graph

Recursively decompose your graph...

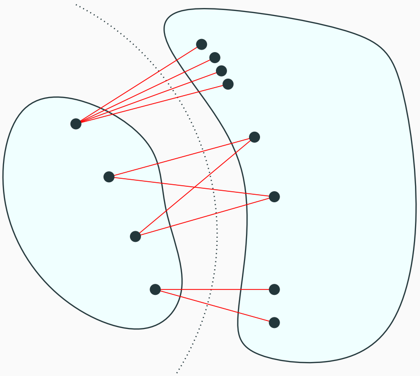


Branch-decomposition: recursively **cut** **the vertex set** in **two**

Divide a graph

Recursively decompose your graph... into **simple** cuts

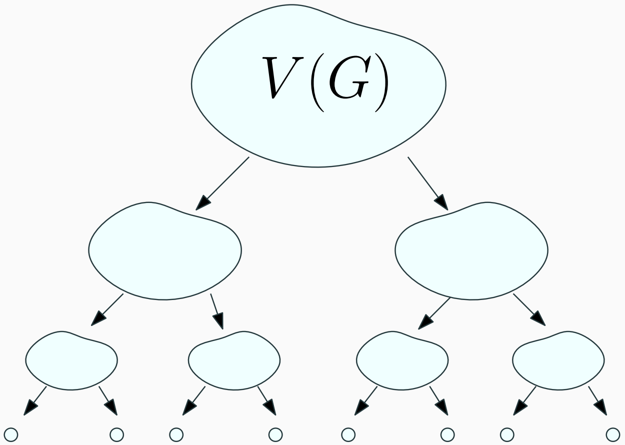
Simplicity of a cut is measured with a **function** f: **cut** $\rightarrow \mathbb{N}$.



Different notions of **simplicity** = different **width measures**.

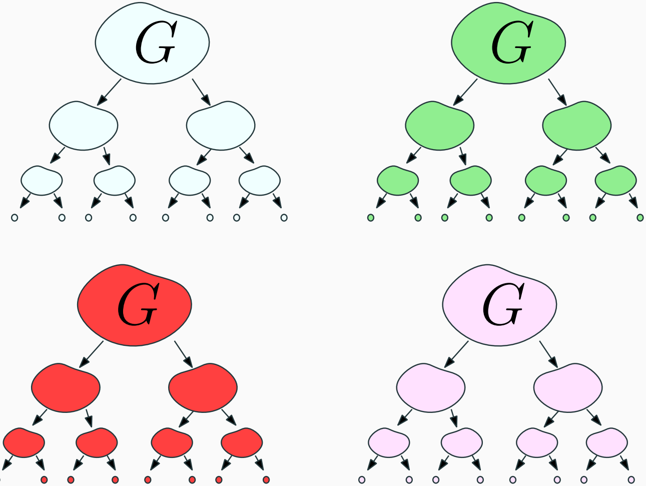
Divide a graph

Width of a **decomposition** D = **max** f(cut) over the **cuts** of D.



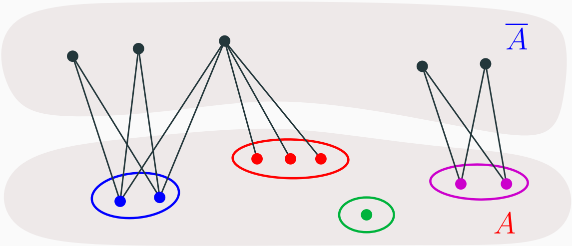
Divide a graph

Width of a **graph** $G =$ **min** $\{$ **widths** of its **decompositions** $\}$.



Module-width

Defined from the function $\mathrm{mw}(A) := |\{N(v) \cap \overline{A} \,:\, v \in A\}|$.



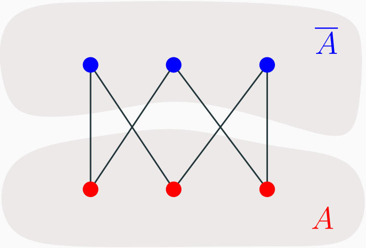
Linearly equivalent to clique-width:

Rao 2006

For all graphs G, we have $\mathrm{mw}(G) \leq \mathrm{cw}(G) \leq 2\mathrm{mw}(G)$.

Rank-width

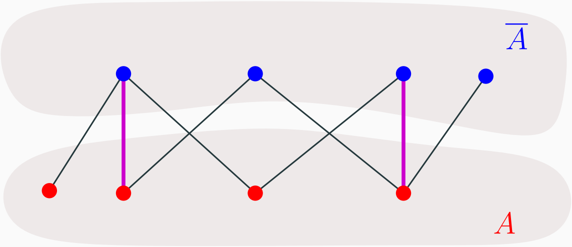
Defined from the function $\mathrm{rw}(A) :=$ the rank of adjacency matrix between A and $\overline{A}$ over $GF(2)$.



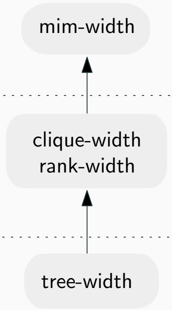
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Maximum Induced Matching width (mim-width)

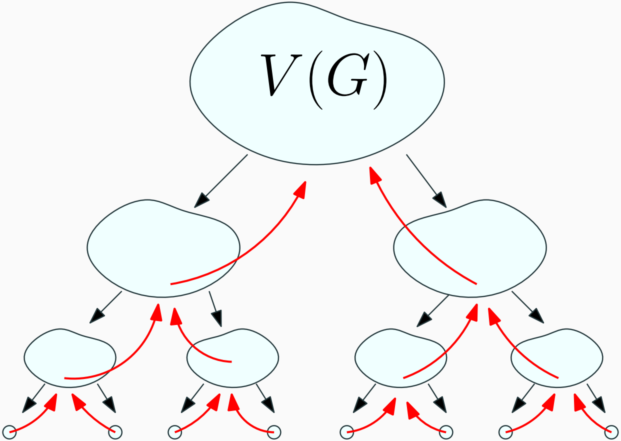
Defined from the function $\mathsf{mim}(A) :=$ the size of a maximum induced matching in the bipartite graph between A and $\overline{A}$.



Modeling Power		Meta-algorithmic applications
Interval, permutation, k-polygon, Dilworth-k, convex graphs,...	mim-width	Neighborhood Logic (TBA)
Cographs, hereditary distance	clique-width rank-width	MSO_1
Tree, partial k-tree	tree-width	MSO_2



Conquer Graph Coloring



Tree-width and Graph Coloring

Folklore

q-Coloring is solvable in time $O(q^{\mathsf{tw}} \cdot n)$.

Lokshtanov, Marx, Saurabh, SODA 2011

For any $\epsilon > 0$, q-Coloring **cannot be solved** in time $O((q-\epsilon)^{\mathsf{tw}} \cdot n^{O(1)})$ unless **SETH** fails.

Clique-width and Graph Coloring

Courcelle, Durand, Raskin, ArXiv 2021

Given a decomposition of clique-width cw, q-Coloring is solvable in time

$$O(m \cdot \min\{q^{O(2^{\mathsf{cw}})}, 2^{O(q \cdot \mathsf{cw})}\}).$$

Lampis, ICALP 2018

q-Coloring **cannot be solved** in time $O((2^q - 2 - \epsilon)^{\mathsf{cw}} \cdot n^{O(1)})$ for any $\epsilon > 0$, unless **SETH** fails.

Fomin, Golovach, Lokshtanov, Saurabh, Zehavi, SODA2018

Graph Coloring **cannot be solved** in time $O(n^{2^{o(\mathsf{cw})}} \cdot n^{O(1)})$ unless **ETH** fails.

Other-widths and Graph Coloring

Ganian, Hliněný, Obdržálek, EJC 2013

Graph Coloring is solvable in time $O(n^{2^{O(\mathrm{rw}^2)}})$.

Vatshelle, PhD thesis 2012

- Graphs of **mim-width 1** are **perfect** $\Rightarrow$ Polytime solvable.
- **Circular arc graphs** have **mim-width at most 2** and a decomposition can be computed in polytime $\Rightarrow$ NP-hard.

Meta-algorithm for q-Coloring

Bui-Xuan, Telle, Vatshelle, TCS 2013

Given a **branch-decomposition** $\mathcal{L}$, there is an algorithm for **LCVP** problems whose running time for q-**Coloring** is upper bounded by:

- $2^{O(q\cdot\mathsf{cw}(\mathcal{L}))}\cdot n^{O(1)}$.
- $2^{O(q\cdot\mathsf{rw}(\mathcal{L})^2)}\cdot n^{O(1)}$.
- $n^{O(q\cdot\mathsf{mim}(\mathcal{L}))}$.

Our Results

Theorem

There exists a **greedy** algorithm that, given a decomposition of mim-width 1, solves Graph Coloring in time $O(n^2)$.

Theorem

Given a **branch-decomposition** $\mathcal{L}$, there is an algorithm for q-**Coloring** whose running time is upper bounded by:

- $q^{O(2^{\mathsf{cw}(\mathcal{L})})} \cdot n^{O(1)}$ and $2^{O(q \cdot \mathsf{cw}(\mathcal{L}))} \cdot n^{O(1)}$.
- $q^{O(2^{\mathsf{rw}(\mathcal{L})^2})} \cdot n^{O(1)}$ and $2^{O(q \cdot \mathsf{rw}(\mathcal{L})^2)} \cdot n^{O(1)}$.
- $n^{O(q \cdot \mathsf{mim}(\mathcal{L}))}$.

Similar upper bounds for the variant of rank-width in $\mathbb{Q}$.

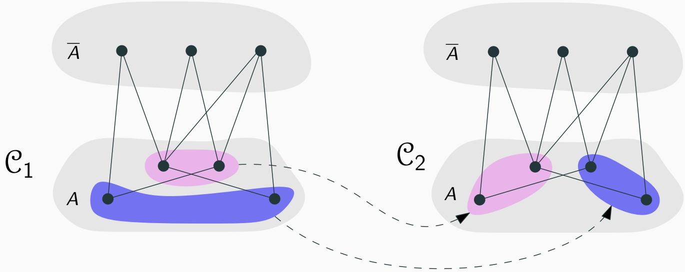
A New Notion of Representativity

Between partial colorings

Given two **colorings $\mathcal{C}_1, \mathcal{C}_2$ of A** and $f : \mathcal{C}_1 \to \mathcal{C}_2$, we say that $\mathcal{C}_1$ **is f-better than** $\mathcal{C}_2$ if f is **injective** and for every $I \in \mathcal{C}_1$:

$$N(I) \cap \overline{A} \subseteq N(f(I)) \cap \overline{A}.$$

$\mathcal{C}_1$ **is better than** $\mathcal{C}_2$ if $\mathcal{C}_1$ is f-better than $\mathcal{C}_2$ for a function f.



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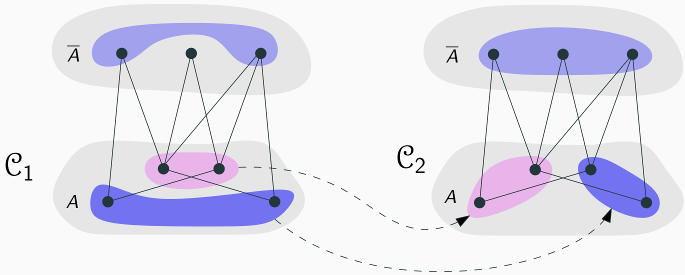

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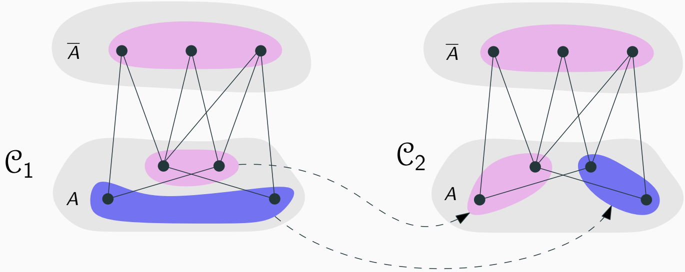

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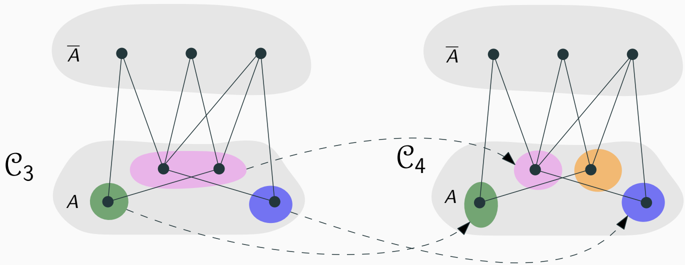

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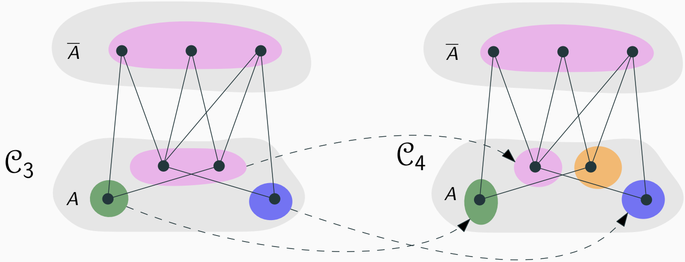
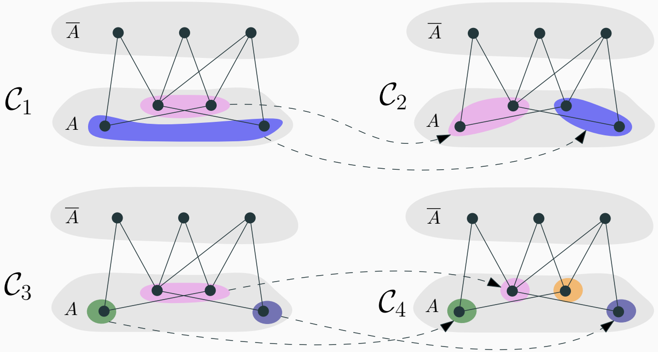


New notion of representativity

Between collections of partial colorings

Given two **collections $\mathbf{K}_1, \mathbf{K}_2$ of colorings of A**, we say that **$\mathbf{K}_1$ represents $\mathbf{K}_2$**, if, for every $\mathcal{C}_2 \in \mathbf{K}_2$, there exists $\mathcal{C}_1 \in \mathbf{K}_1$ that is **better** than $\mathcal{C}_2$.



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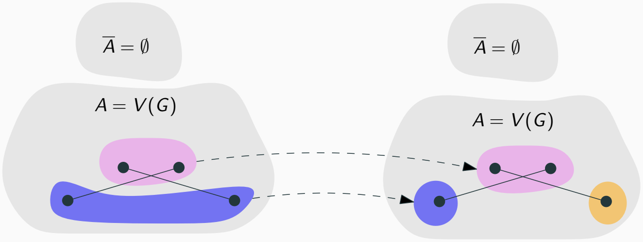
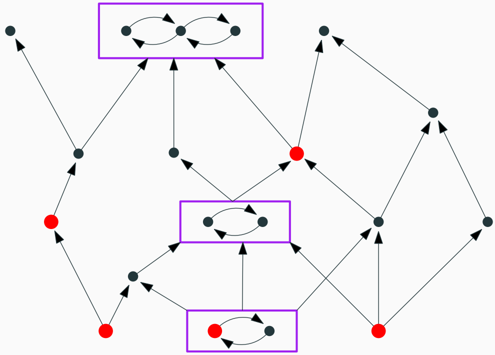
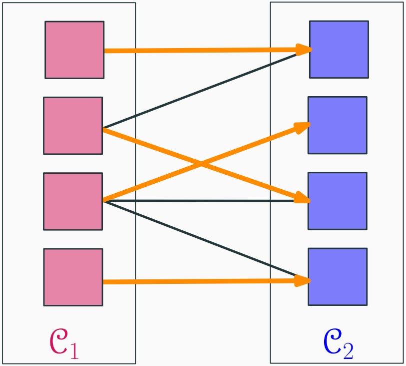


For any optimal coloring $\mathcal{C}$ of G, $\{\mathcal{C}\}$ represents the set of all colorings of G



Fact

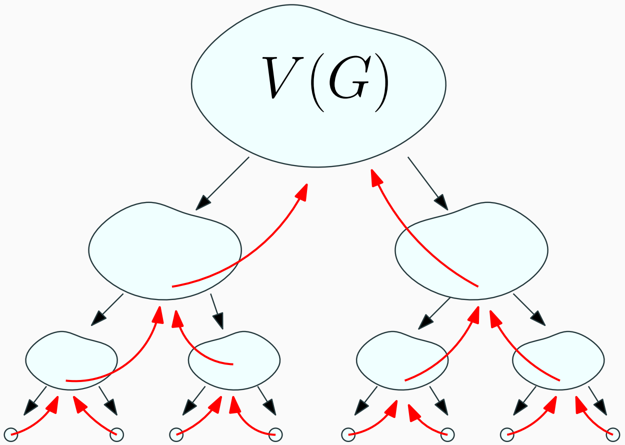
The relation “is better” is a **preorder** (quasiorder), it is reflexive and **transitive**.



Lemma

We can decide whether $\mathcal{C}_1$ is better than $\mathcal{C}_2$ in time $O(n^3)$.

Merging



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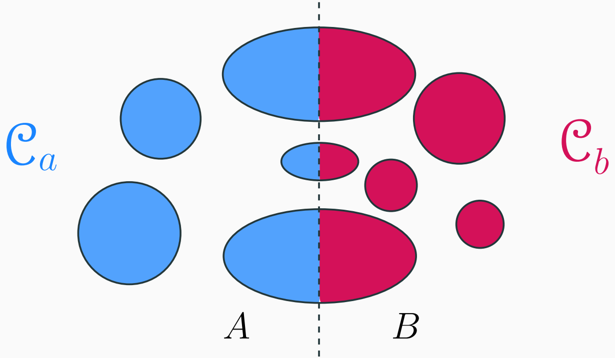
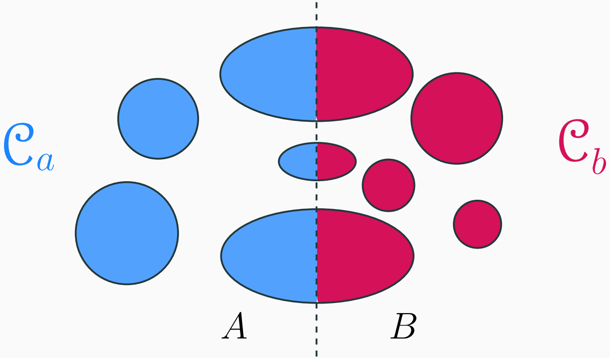


A coloring obtained from $\mathcal{C}_a$ and $\mathcal{C}_b$.

$\mathcal{C}_a \otimes_q \mathcal{C}_b = \{$ all the colorings obtained from $\mathcal{C}_a$ and $\mathcal{C}_b$ with at least q-colors $\}$.

Merging

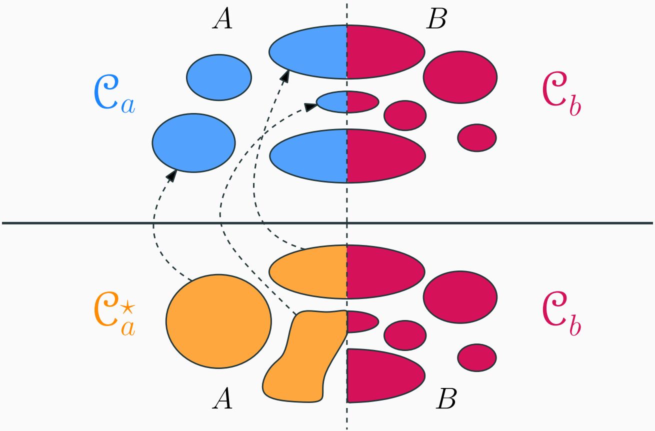


Given a collection of colorings $\mathbf{K}_a$ of A and $\mathbf{K}_b$ of B.

$$\mathbf{K}_a \otimes_q \mathbf{K}_b = \bigcup_{\mathcal{C}_a \in \mathbf{K}_a, \mathcal{C}_b \in \mathbf{K}_b} \mathcal{C}_a \otimes \mathcal{C}_b.$$

Lemma

If $\mathcal{C}_a^\star$ **is f-better than** $\mathcal{C}_a$, then for every $\mathcal{C}_b$, $\mathcal{C}_a^\star \otimes_q \mathcal{C}_b$ **represents** $\mathcal{C}_a \otimes_q \mathcal{C}_b$.



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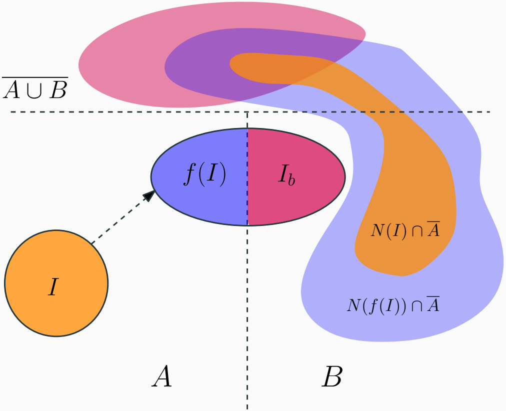

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Corollary

If $\mathbf{K}_a^\star$ **is represents** $\mathbf{K}_a$, then for every $\mathbf{K}_b$, $\mathbf{K}_a^\star \otimes_q \mathbf{K}_b$ **represents** $\mathbf{K}_a \otimes_q \mathbf{K}_b$.

Using **Representative sets** is **safe**!

Reduction

q-Coloring Merging

- Input: **two collections of colorings** $\mathbf{K}_a$ of A and $\mathbf{K}_b$ of B with $A, B, A \cup B$ being cuts induced by a decomposition $\mathcal{L}$.
- Output: **a representative set** of $\mathbf{K}_a \otimes_q \mathbf{K}_b$.

Theorem

If there exists an **algorithm** that for q-Coloring Merging whose **runtime** is $|\mathbf{K}_a| \cdot |\mathbf{K}_b| \cdot f(\mathcal{L})$ and **output size** is at most $g(\mathcal{L})$, then q-Coloring is solvable time $O(g(\mathcal{L})^2 \cdot \mathsf{f}(\mathcal{L}) \cdot n)$. given a decomposition $\mathcal{L}$.

Mim-width one

Mim-width

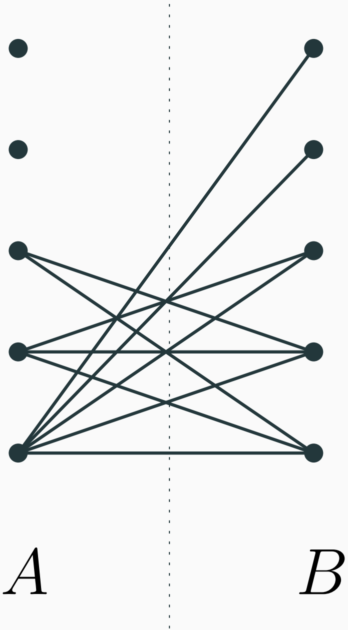
Lemma

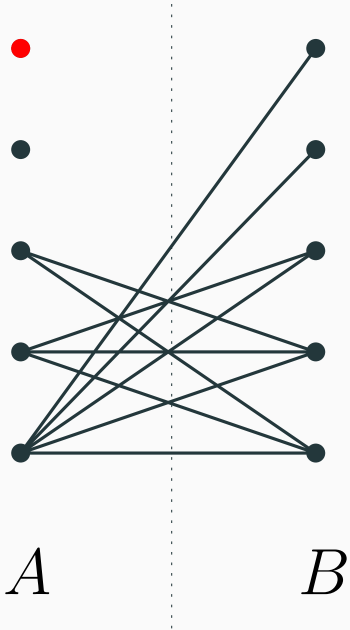
If $\text{mim}(\mathcal{L}) = |\mathbf{K}_a| = |\mathbf{K}_b| = 1$, then q-Coloring Merging can be solved in time $O(n)$ and **the output size in 1**.

- We represent **independent sets** with one vertex,
- We do a $O(n^2)$ **preprocessing** on the branch-decomposition.

Theorem

There exists a **greedy** algorithm that, given a decomposition of mim-width 1, outputs an optimal coloring in time $O(n^2)$.


A
B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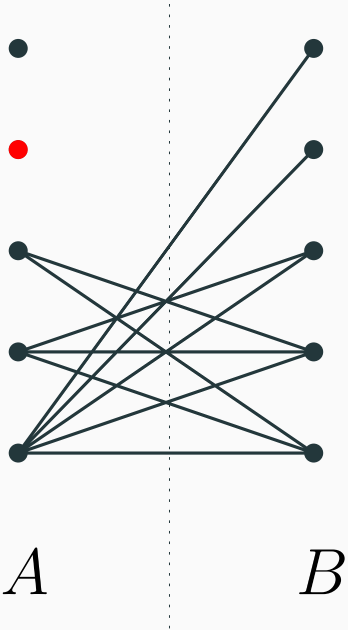
A
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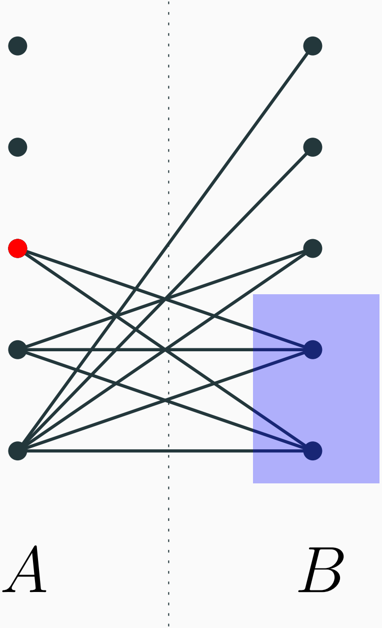
A
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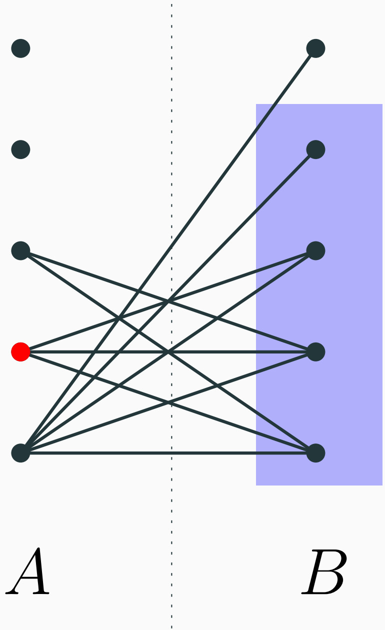
A
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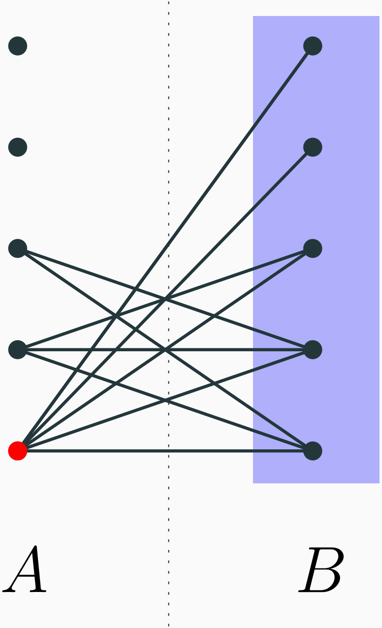
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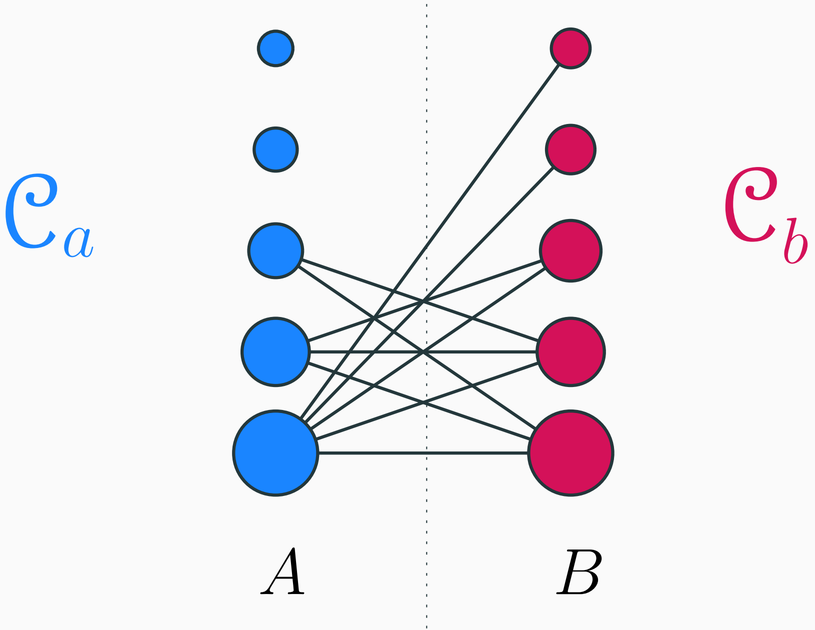
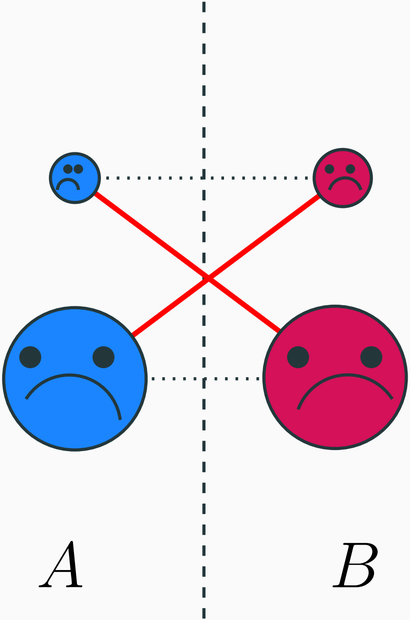
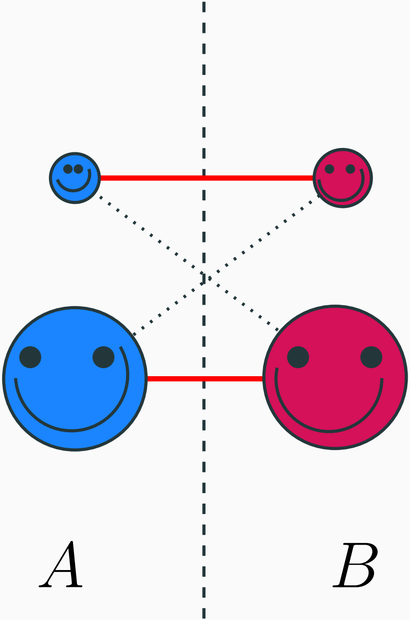
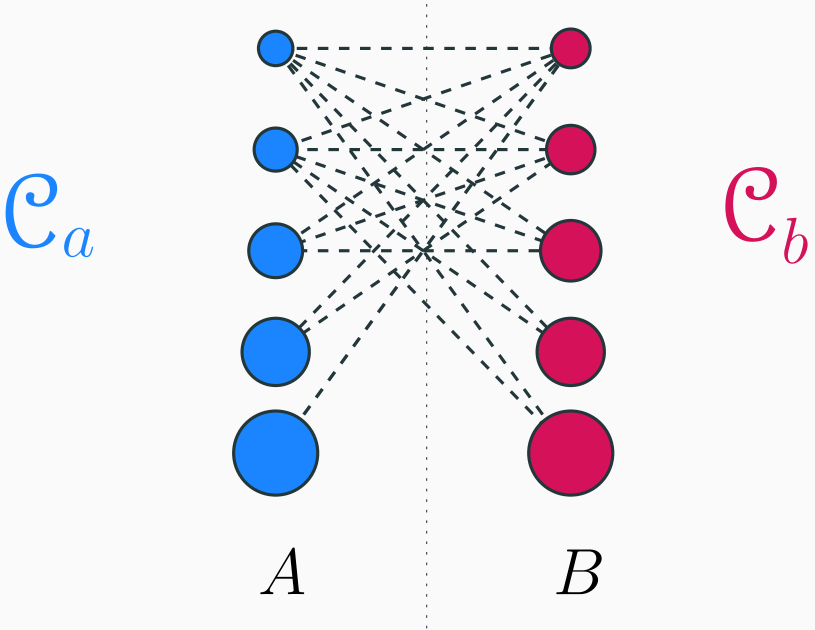


$\mathfrak{C}_a$
$\mathfrak{C}_b$
A
B


A
B


A
B



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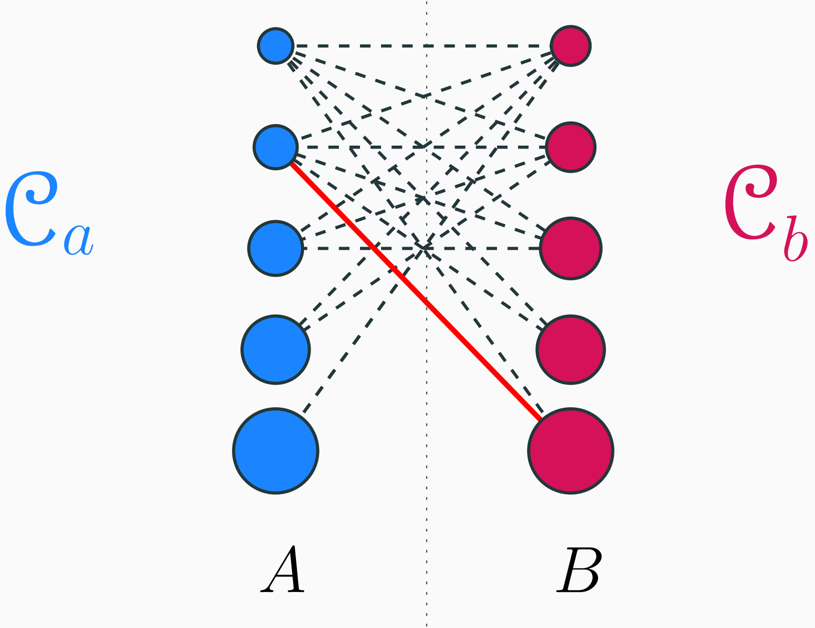

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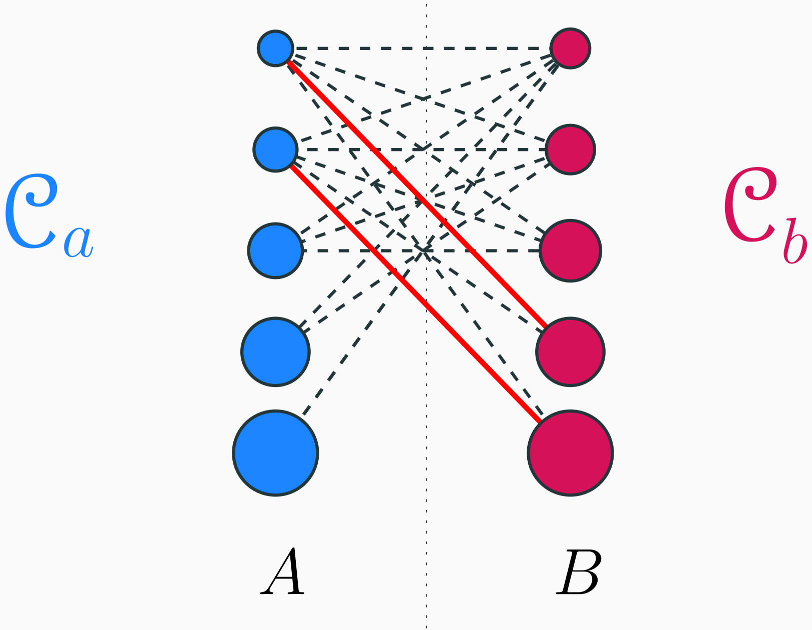
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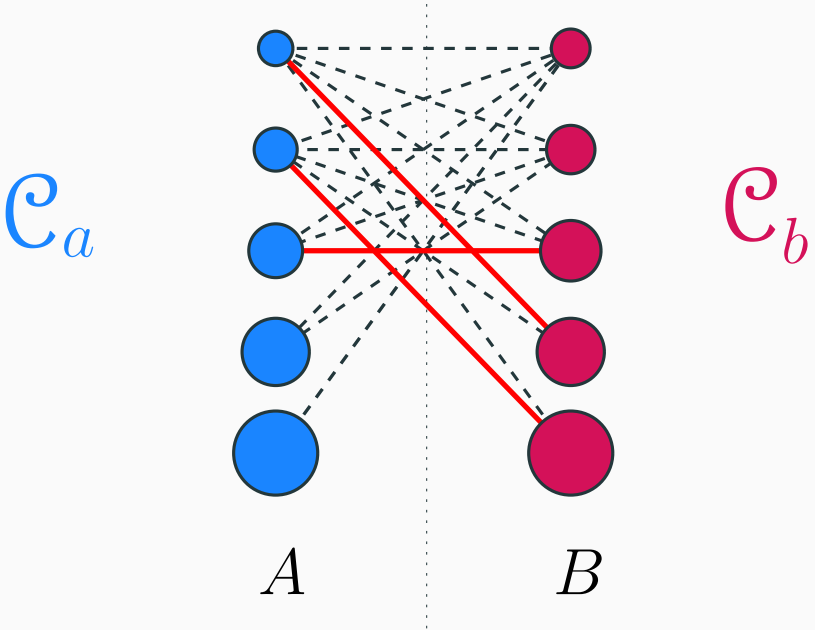
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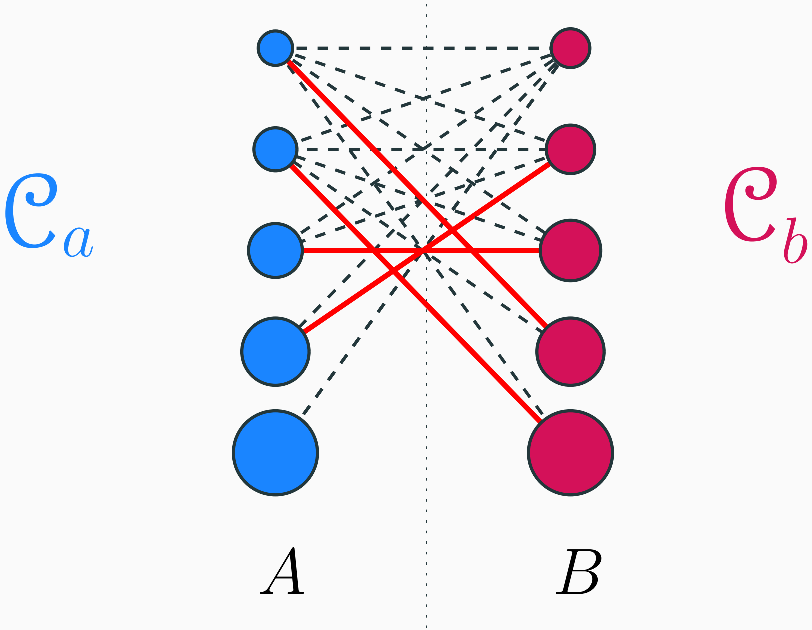
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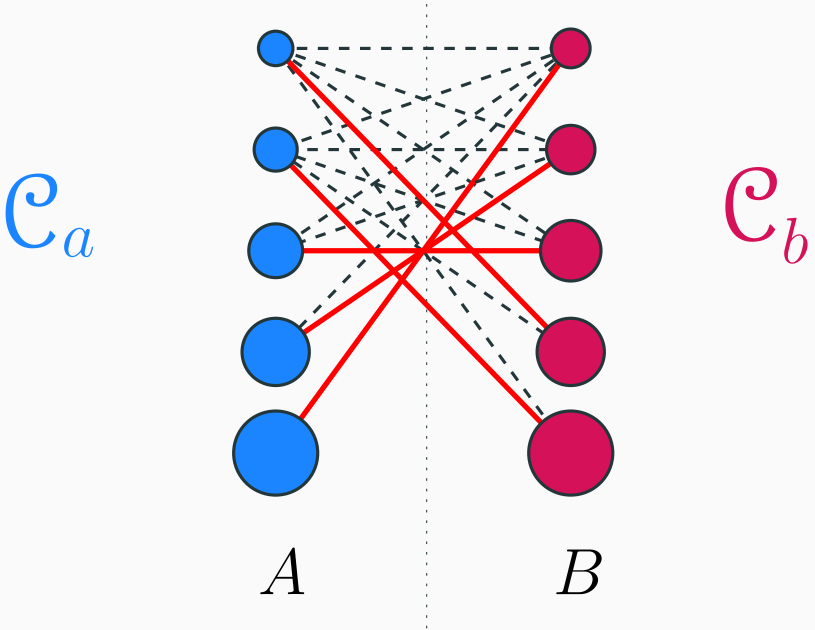

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$\mathfrak{C}_b$
A
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Clique-width, rank-width, mim-width,...

Clique-width, rank-width,...

$\mathsf{nec}(A) = |\{N(X) \cap \overline{A} : X \subseteq A| = |\text{neighborhoods across } A|$.

Lemma

q-Coloring Merging can be solved in time $O(\min(q^{O(\mathsf{nec}(\mathcal{L}))}, \mathsf{nec}(\mathcal{L})^{O(q)} \cdot q^{O(q)}) \cdot n^{O(1)})$ and the output size is at most $\min((q+1)^{\mathsf{nec}(\mathcal{L})}, \mathsf{nec}(\mathcal{L})^{q})$.

Theorem

Given a decomposition $\mathcal{L}$, q-Coloring is solvable in time $O(\min(q^{O(\mathsf{nec}(\mathcal{L}))}, \mathsf{nec}(\mathcal{L})^{O(q)} \cdot q^{O(q)}) \cdot n^{O(1)})$.

Clique-width, rank-width,...

Vatshelle, PhD thesis 2012

$\mathsf{nec}(A)$ is at most $2^{\mathsf{cw}(A)}$, $2^{\mathsf{rw}(A)^2}$ and $n^{\mathsf{mim}(A)}$.

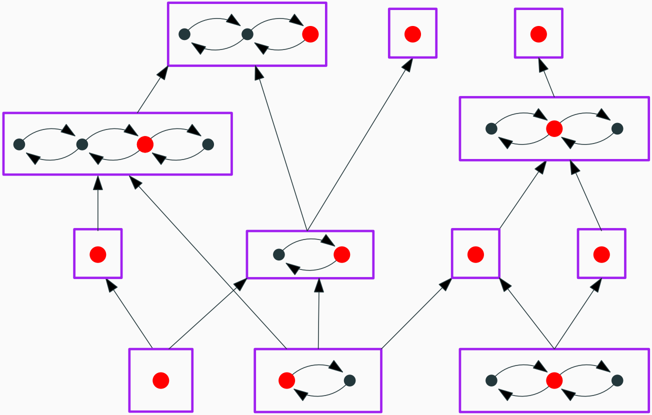
Theorem

We have an algorithm, taking a **branch-decomposition** $\mathcal{L}$ as input, for q-**Coloring** whose running time is upper bounded by:

- $q^{O(2^{\mathsf{cw}(\mathcal{L})})} \cdot n^{O(1)}$ and $2^{O(q \cdot \mathsf{cw}(\mathcal{L}))} \cdot n^{O(1)}$.
- $q^{O(2^{\mathsf{rw}(\mathcal{L})^2})} \cdot n^{O(1)}$ and $2^{O(q \cdot \mathsf{rw}(\mathcal{L})^2)} \cdot n^{O(1)}$.
- $n^{O(q \cdot \mathsf{mim}(\mathcal{L}))}$.

Similar upper bounds for the variant of rank-width in $\mathbb{Q}$.

We consider the **equivalence classes** induced by “is better than”.



Signature of $\mathcal{C} = \{N(I) \cap \overline{A} \, : \, I \in \mathcal{C}\}$.

Lemma

- $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent iff they have the same signature.
- There is at most $(q+1)^{\mathsf{nec}(A)}$ and $\mathsf{nec}(A)^q$ signatures for the q-colorings of A.

Conclusion

Results

Theorem

There exists a **greedy** algorithm that, given a decomposition of mim-width 1, solves Graph Coloring in time $O(n^2)$.

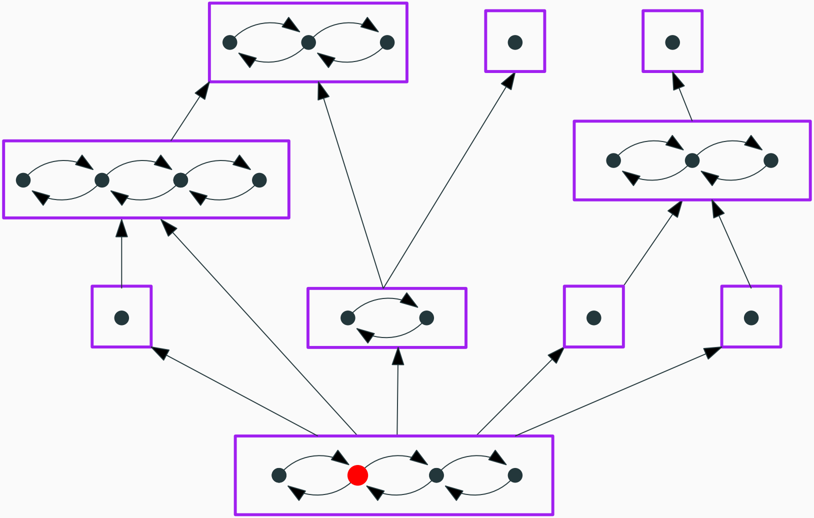
Theorem

Given a **branch-decomposition** $\mathcal{L}$, we have an algorithm for q-**Coloring** whose running time is upper bounded by:

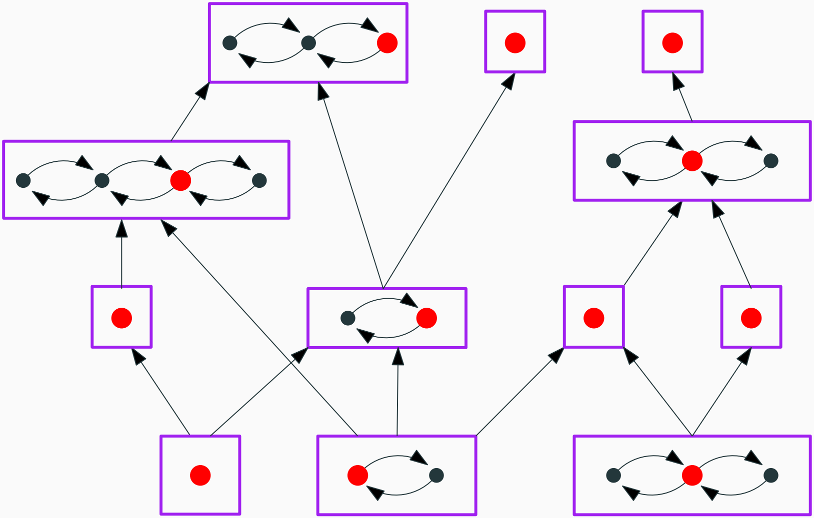
- $q^{O(2^{\mathsf{cw}(\mathcal{L})})} \cdot n^{O(1)}$ and $2^{O(q \cdot \mathsf{cw}(\mathcal{L}))} \cdot n^{O(1)}$.
- $q^{O(2^{\mathsf{rw}(\mathcal{L})^2})} \cdot n^{O(1)}$ and $2^{O(q \cdot \mathsf{rw}(\mathcal{L})^2)} \cdot n^{O(1)}$.
- $n^{O(q \cdot \mathsf{mim}(\mathcal{L}))}$.

Similar upper bounds for the variant of rank-width in $\mathbb{Q}$.

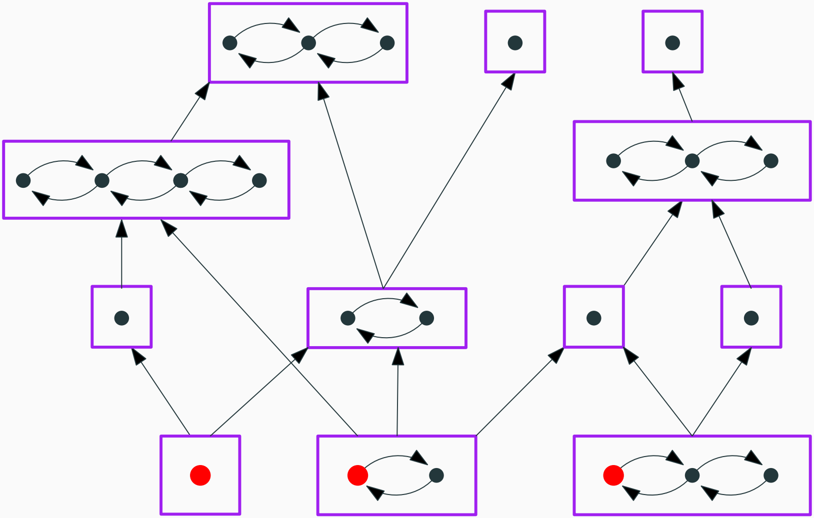
Representative set for mim-width one



Representative set use by the second algorithm



Representative set of minimum size



Representative of minimum size

Lemma

Minimum size and **minimal for the inclusion** is the same for the representative sets of the (q-)colorings of A.

Open Question

- Can you compute in **polynomial time** a coloring in $\mathcal{C}_a \otimes_q \mathcal{C}_b$ that is not **strictly worse** than another coloring in $\mathcal{C}_a \otimes_q \mathcal{C}_b$?
- Can you compute **a minimal representative set** S of $\mathcal{C}_a \otimes_q \mathcal{C}_b$ in time $(|S| + n)^{O(1)}$?

Open Question

Can we use this notion of **representativity** in other settings?

- Other graph classes? Other parameters?
- Other problems?

Thank you