Recognition of Linear and Star Variants of Leaf Powers is in P

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Leaf Powers

Graph theoretical approach

[Nishimura, Ragde and Thilikos 2000]

A graph G is the **k-leaf power** of a tree T if

- $V(G)$ is the set of leaves of T and
- for every $u, v \in V(G)$, $uv \in E(G) \iff \mathsf{dist}_T(u, v) \leqslant k$.

$\textsc{Leaf Powers} = \bigcup_{k\in\mathbb{N}} k\text{-}\textsc{Leaf Powers}$.

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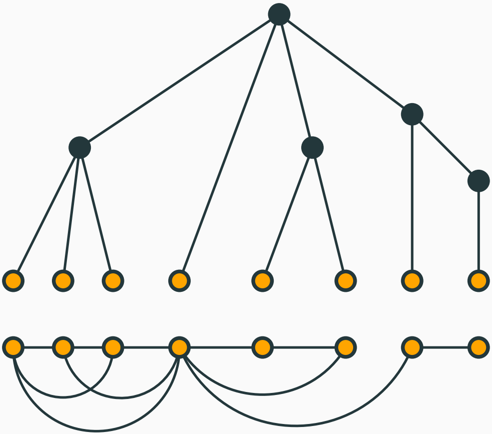
- T is called a **leaf root** of G.

Example of 3-leaf power



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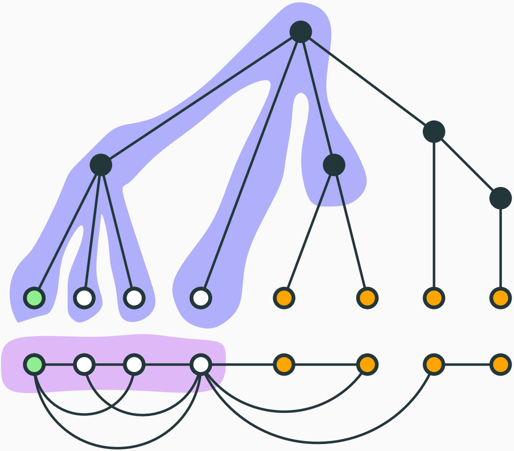

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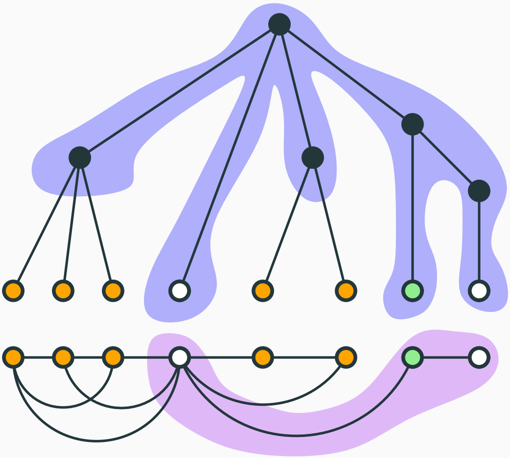
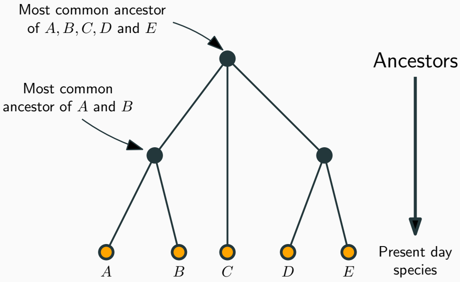


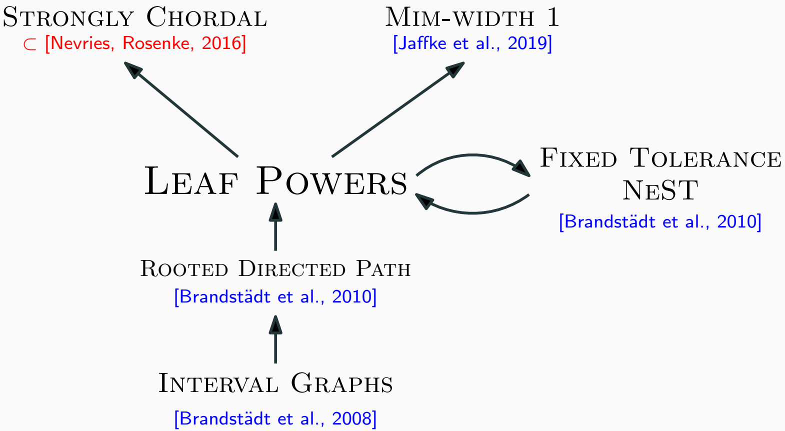
Phylogenetics



Classical phylogeny problem

- Input: **phylogenetic distance matrix** D of a **set** X of organisms.
- Output: a **tree** T with set of **leaves** X such that, for every $x, y \in X$, the distance between x, y in T is **close to** $D[x, y]$.

Relations with other graph classes



Recognition of k-leaf power

Open question

Can we recognize k-**leaf powers** in **polynomial time**?

Value of k	Polynomial Time Algorithms
2	Trivial: disjoint union of cliques
3	[Nishimura, Ragde, Thilikos, 2000] [Brandstädt, Le, 2006]
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5	[Chang, Ko, 2007]
6	[Ducoffe, 2019]

Parameterized Complexity

[Eppstein and Havvaei, IPEC 2018]

k-**leaf powers** of **bounded degeneracy** are recognizable in **FPT time** ($O(f(k) \cdot n^{O(1)})$).

[Lafond, SODA 2022]

k-**leaf powers** are recognizable in **XP time** ($O(n^{f(k)})$).

Drop the k

$\textsc{Leaf Powers} = \bigcup_{k \in \mathbb{N}} k\text{-}\textsc{Leaf Powers}$.

Open question

Can we recognize **leaf powers** in **polynomial time**?

It might be better to not focus on k **the distance value**:

- Recognizing **k-leaf powers** might be **harder** than leaf powers!
- Algorithms for $k \leqslant 6$ does not seem to **generalize**.
- **XP algorithm** of [Lafond, 2022] **relies heavily** on k.

Tree Structural Approach

Open question

Given a class $\mathcal{C}$ of trees. Can we recognize **leaf powers with a leaf root in $\mathcal{C}$** in polynomial time?

- No **distance bound** allows to use **alternative models**.
- Might lead to interesting results!

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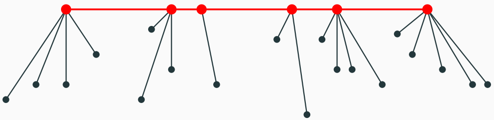
[Brandstädt, Hundt, 2008]

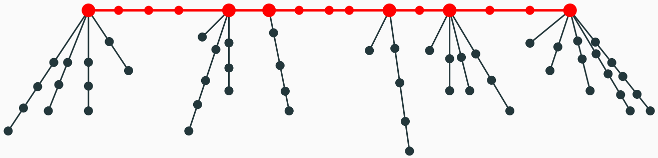
Unit interval graphs are exactly leaf powers with a **caterpillar leaf root**.



Our results

Linear Leaf Powers

Linear leaf powers are graphs with a leaf root that is a **subdivided caterpillar**.



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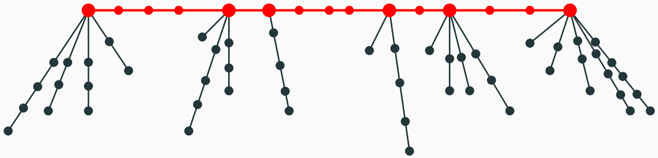


[B., Høgemo, Telle, Vatshelle, 2022]

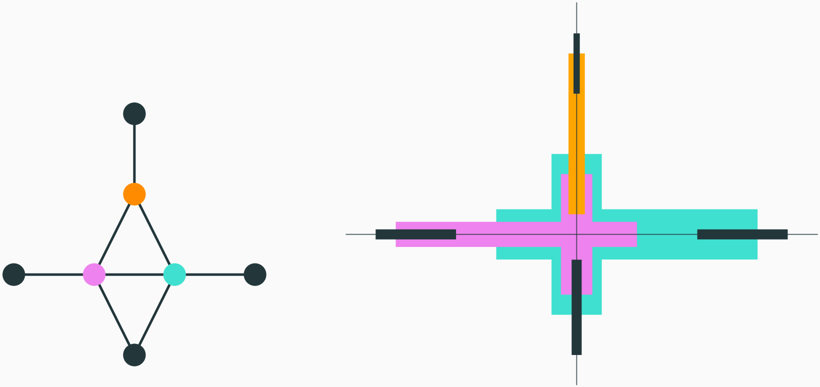
Linear Leaf Powers are equivalent to **Co-TT graphs**.

[Golovach, Heggernes, Lindzey, McConnell, Dos Santos, Spinrad, Szwarcfiter, 2014]

Co-TT graphs can be recognized in $O(n^2)$ **time**.

Our results

NeS Model: **intersection model** for leaf powers



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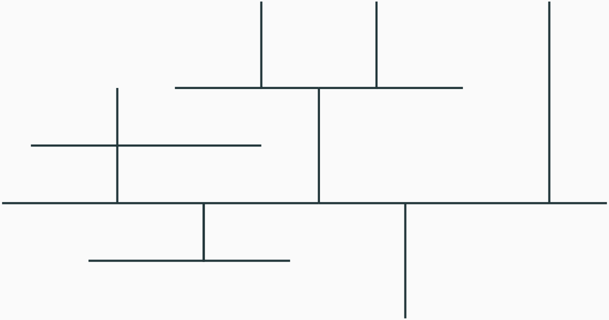
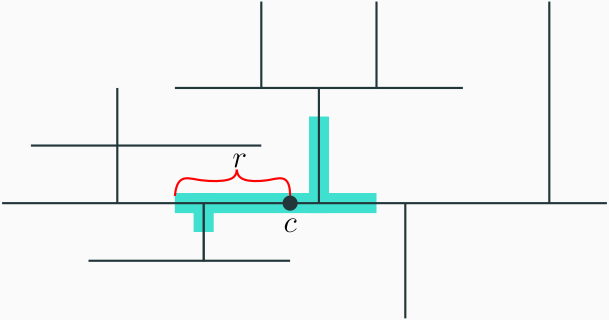
We can recognize in **polynomial time** **leaf powers** with a **Star NeS model**.

Star NeS model

NeS model

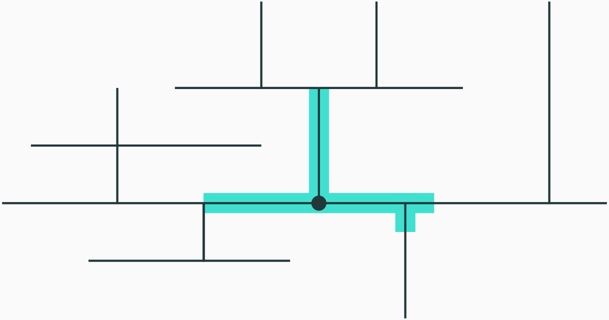


NeS model



Neighborhood Subtree with center c and radius r

NeS model



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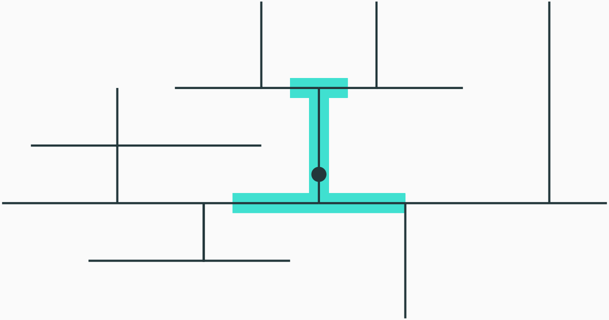

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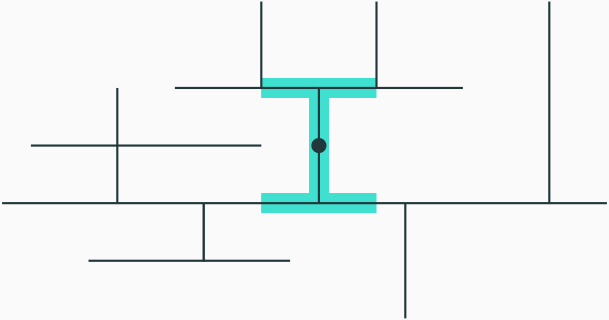

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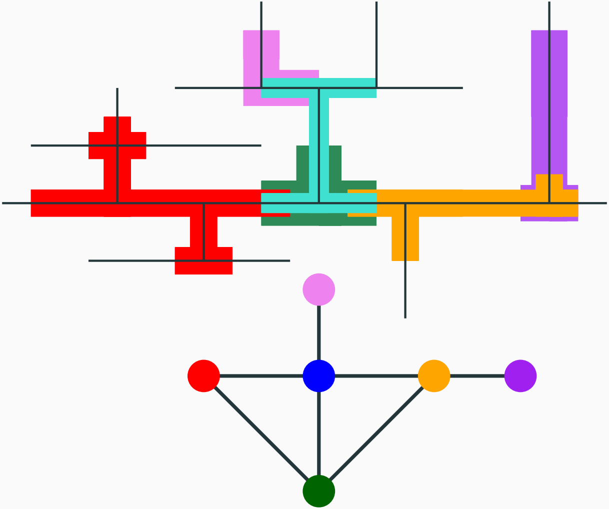


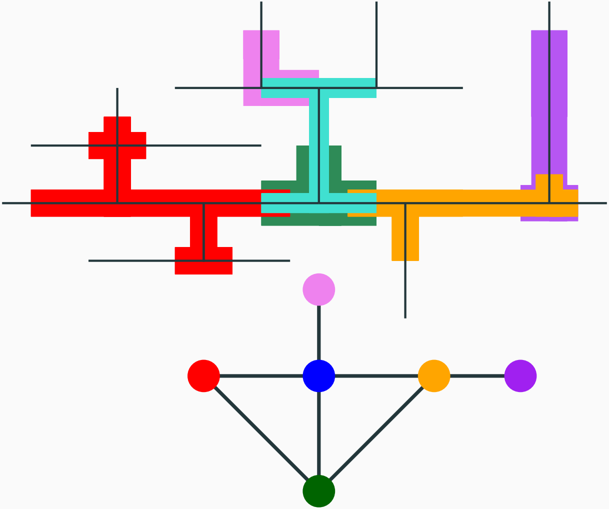
Powerful model

[Bibelnieks, Dearing, 1993] + [Brandstädt, Hundt, Mancini, Wagner, 2010]

A graph is a **leaf power** iff it admits a **NeS model**.

- **NeS models** are a nice generalization of **Interval models**!
- Many properties on leaf powers are **easy to prove** thanks to NeS models.

Adding a universal vertex



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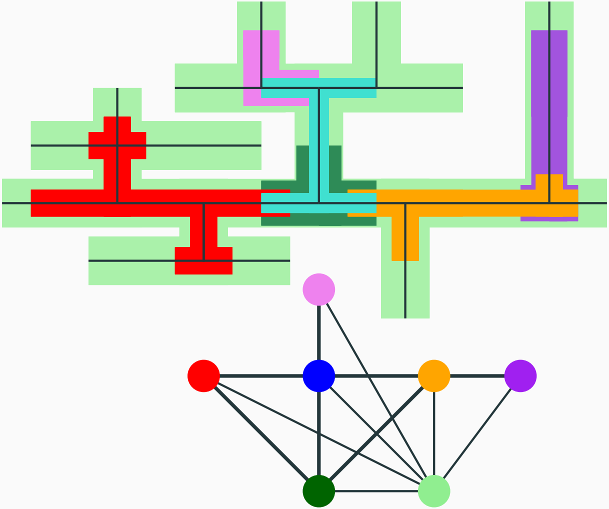
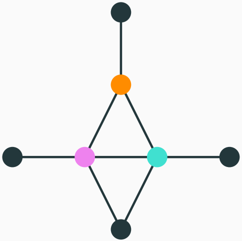
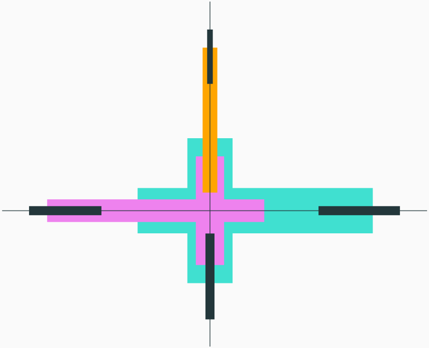


Star NeS model

Star NeS model: NeS model whose **embedded tree** is a **star**.

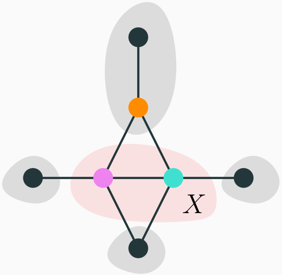


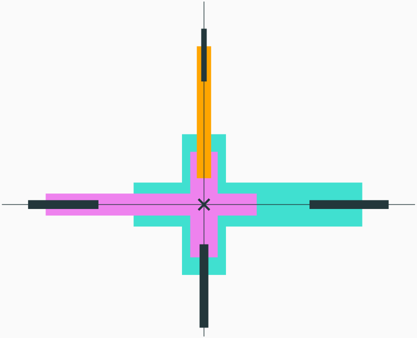


[B., Høgemo, Telle, Vatshelle, 2022]

We can recognize in **polynomial time leaf powers** with a **Star NeS model**.

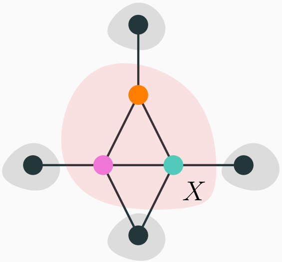
Some Observations

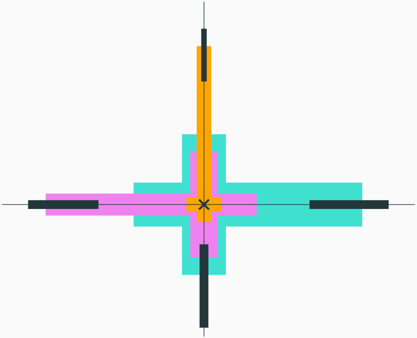




- $X = \{v \in V(G) \mid c \in T_v\}$ is a **clique**.
- The **lines of the star** induces a **partition** $\mathcal{B}$ of $V(G) \setminus X$.

Some Observations





- We can assume w.l.o.g. that X is a **maximal clique**.
- We have $\mathrm{cc}(G - X) \sqsubseteq \mathcal{B}$.

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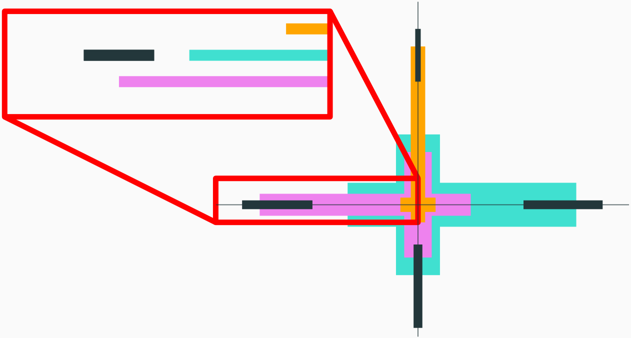

- For each $B \in \mathcal{B}$, $G[X \cup B]$ is an X**-interval graph**.

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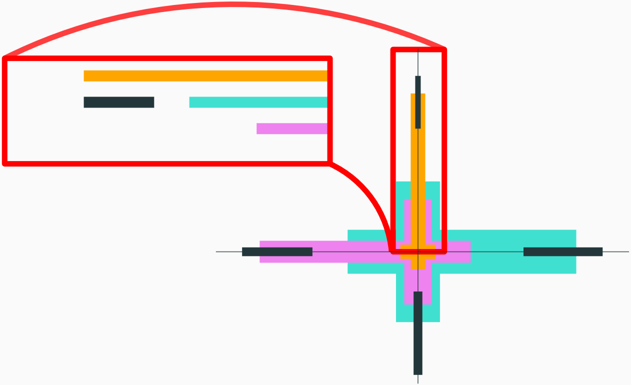

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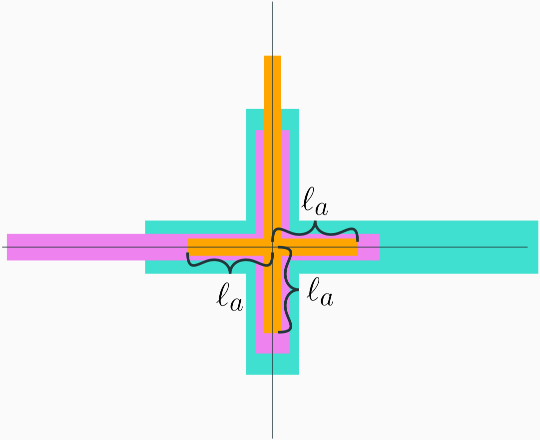

- There is a **natural order** on the vertices of X: $a < b < c$.

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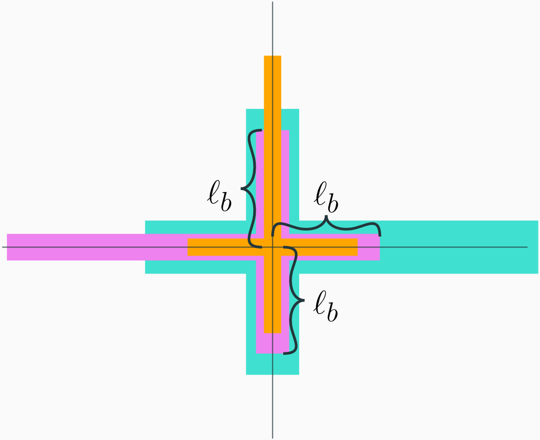

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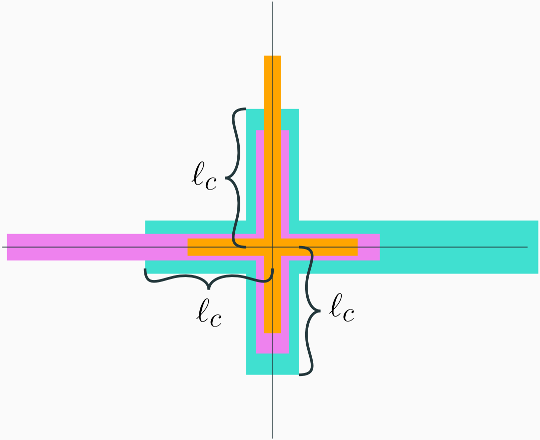

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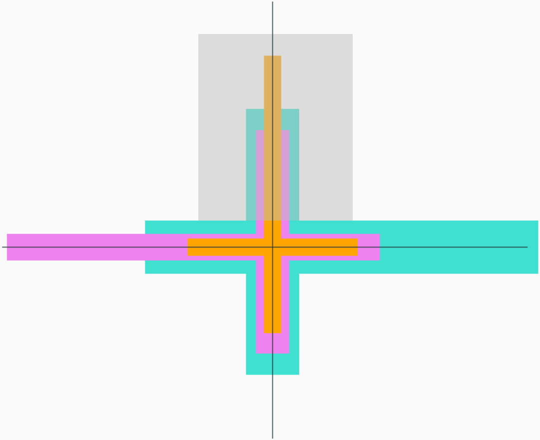

- $a < b < c$ is an **elimination order**...

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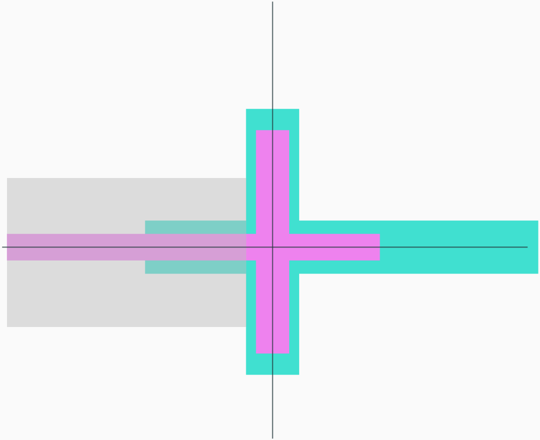

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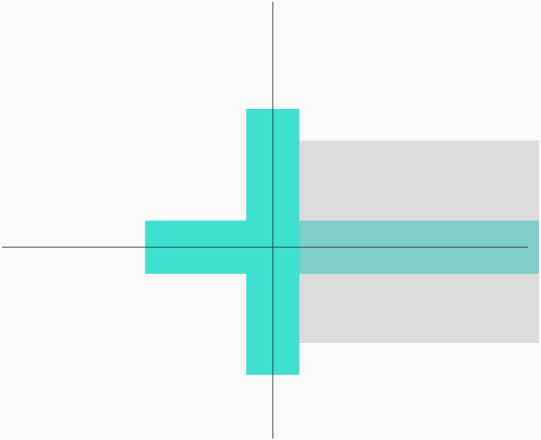

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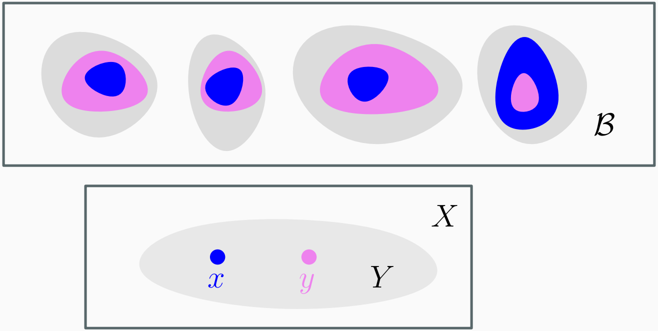
Removable vertex

x is **$\mathcal{B}$-removable** from $Y \subseteq X$ if $N(x)$ is not **minimal** in at most 1 **block** of $\mathcal{B}$.



Good Partition

A **good partition** of a graph G is a pair $(X, \mathcal{B})$ with:

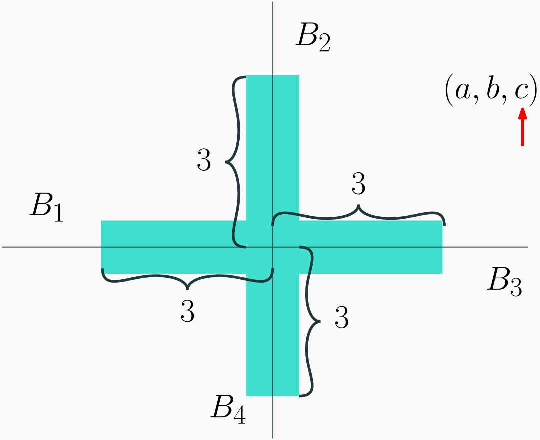
- X a **maximal clique**.
- $\mathcal{B}$ a partition of $V(G) \setminus X$.

with the following properties:

1. $\mathsf{cc}(G - X) \sqsubseteq \mathcal{B}$
2. For each $B \in \mathcal{B}$, $G[X \cup B]$ is an **X-interval** graph.
3. There exists an **elimination order** $(x_1, \ldots, x_t)$ of X such that $\forall i \in [t]$, x_i is $\mathcal{B}$-removable from $\{x_i, \ldots, x_t\}$.

[B., Høgemo, Telle, Vatshelle, 2022]

A graph admits a **star NeS model** iff it admits a **good partition**.



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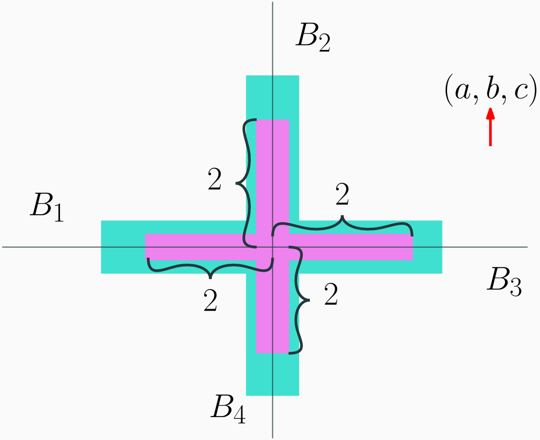

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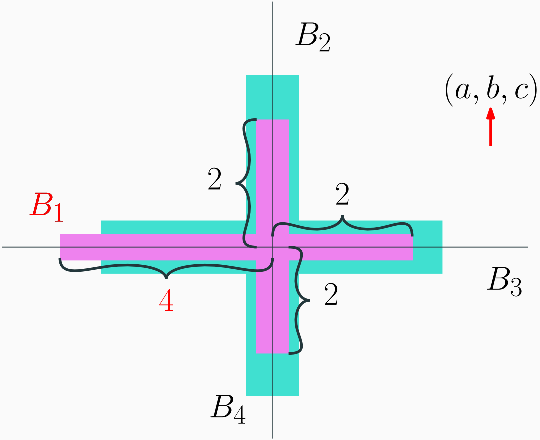

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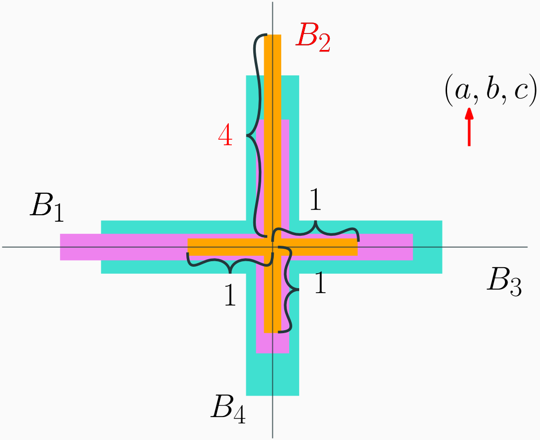

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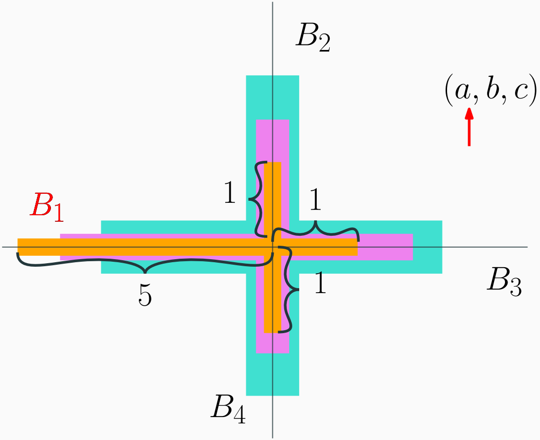


Algorithm

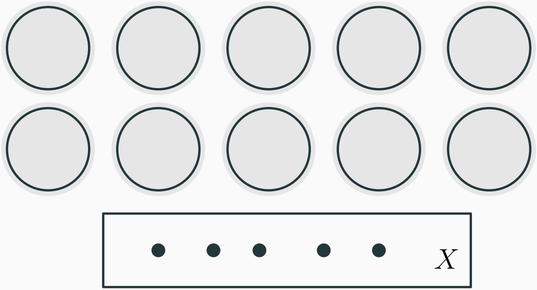
[B., Høgemo, Telle, Vatshelle, 2022]

We can decide whether a graph admits a **good partition** in **polynomial time**.

1. Check if G is **chordal**.
2. For every **maximal clique** X, try to construct a good partition **greedily**.

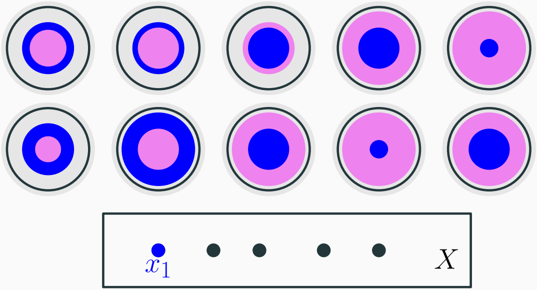
Greedy

Star with $\mathcal{A} = \mathsf{CC}(G - X)$ and check if every $B \in \mathcal{A}$ induces an X-interval with X.



Greedy

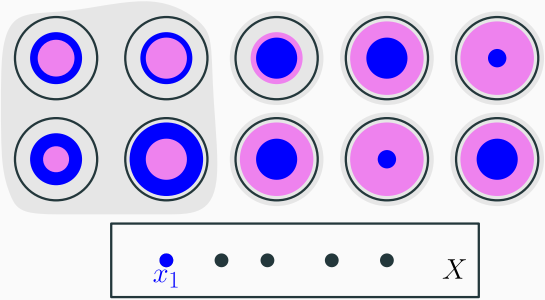
Find $x_1 \in X$ such that $X \cup \mathrm{notmin}(x_1, X, \mathcal{A})$ induces an X-interval.



x_1 is now $\mathcal{A}$-removable from X

Greedy

Merge the blocks of $\mathcal{A}$ in notmin$(x_1, X, \mathcal{A})$



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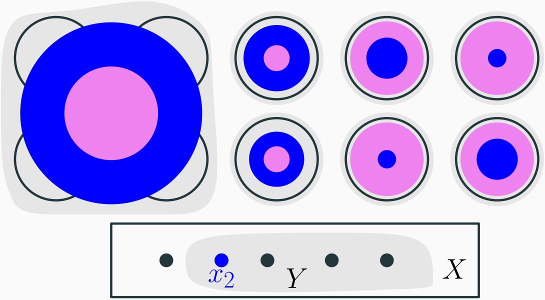
Find $x_2 \in Y$ such that $X \cup \mathrm{notmin}(x_2, Y, \mathcal{A})$ induces an X-interval.



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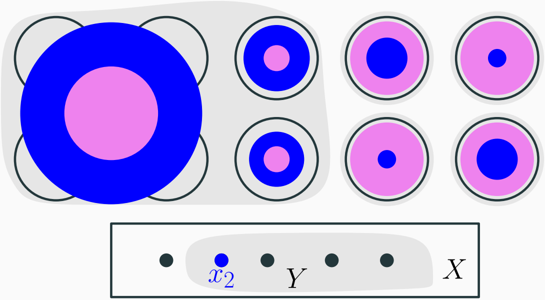
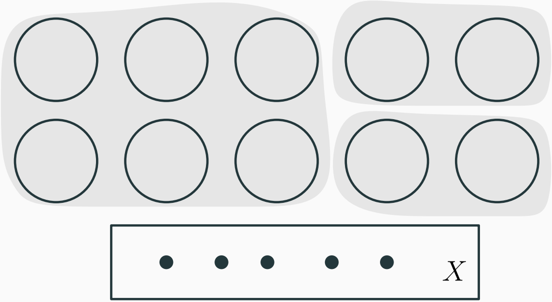


x_2 is now $\mathcal{A}$-removable from Y

Greedy

When $Y = \varnothing$, we return $(X, \mathcal{A})$.



$(x_1, x_2, \ldots, x_t)$ is an elimination order!

x_i is $\mathcal{A}$-removable from $\{x_i, x_{i+1}, \ldots, x_t\}$.

Correctness

[B., Høgemo, Telle, Vatshelle, 2022]

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  - (**Crux**) If the algo. stops following $(x_1, \ldots, x_t)$, then it follows the elimination order of **another good partition**.
- Everything can be done in **polynomial time**.

Conclusion

[B., Høgemo, Telle, Vatshelle, 2022]

We can recognize in polynomial time **linear leaf powers** and **leaf powers** with a **Star NeS model**.

- Can we recognize **leaf powers** in polynomial time?
- Continue the **tree structural approach**!
  - Embedded tree with **two internal nodes**.
  - Vertices with **non-trivial subtrees** induce a **clique**.

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- Leaf-Rank$(G) = \min\{k \in \mathbb{N} \mid G$ is a k-leaf power $\}$.

Conjecture

For every leaf power G, we have Leaf-Rank$(G) \leqslant n^{O(1)}$.

Thank you