More applications of the d-neighbor equivalence: acyclic and connectivity constraints

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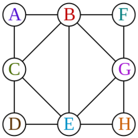
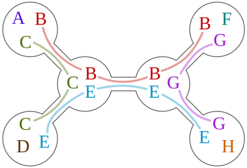
Width measures

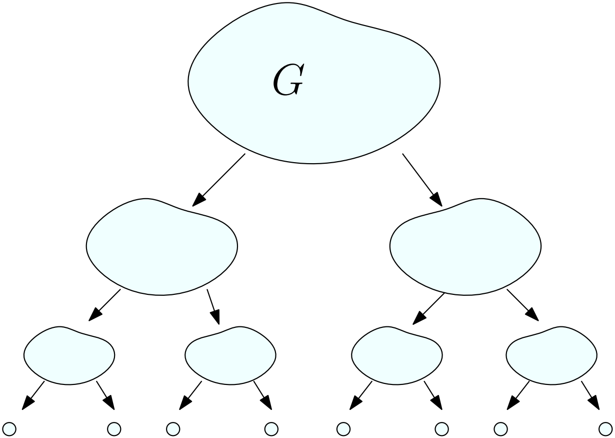




- Tree-width is nice but unbounded in any dense graph class.
- Many NP-hard problems are tractable on some dense graph classes.
  → Explainable with clique-width, rank-width, mim width.

Divide a graph

Recursively decompose your graph...

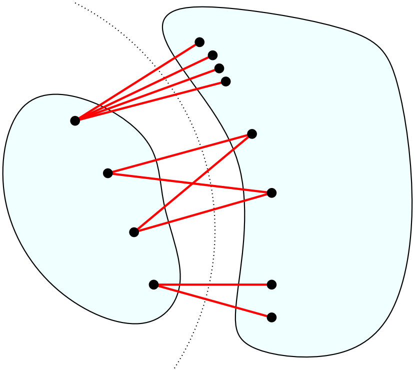


We recursively cut the vertex set in two

Divide a graph

Recursively decompose your graph… into simple cuts.

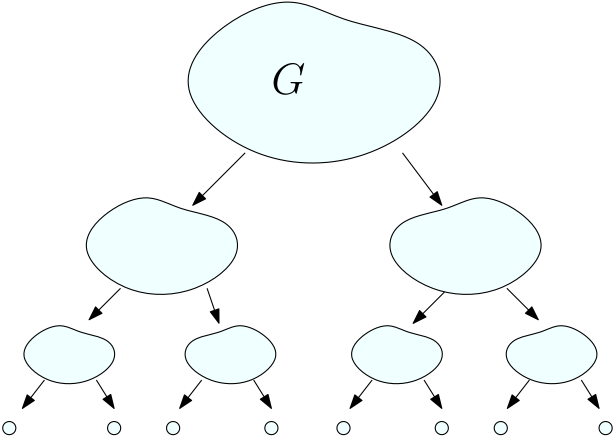
→ We describe the simplicity of a cut with a function f: cut → $\mathbb{N}$.



Different notions of simplicity = different width measures.

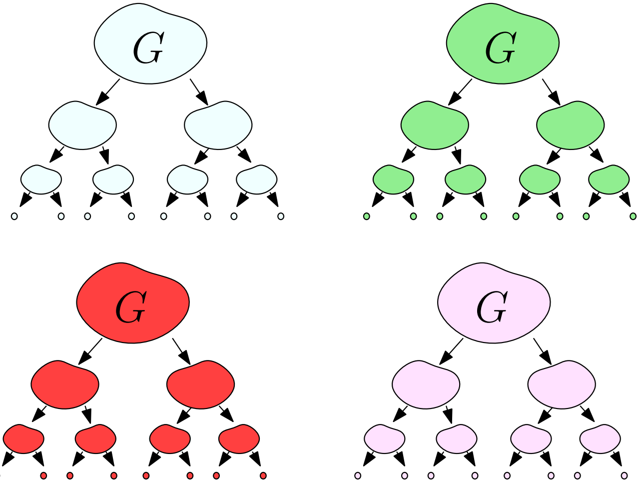
Divide a graph

Width of a decomposition $D := \max \mathsf{f}(\text{cut})$ among the cuts of D.



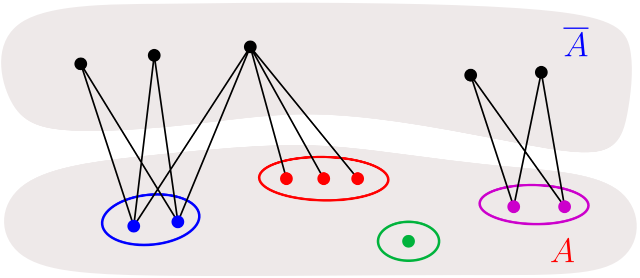
Divide a graph

Width of a graph G := $\min$ width of the decompositions of G.



Module-width

Defined from the function mw$(A) := |\{N(v) \cap \overline{A} : v \in A\}|$.



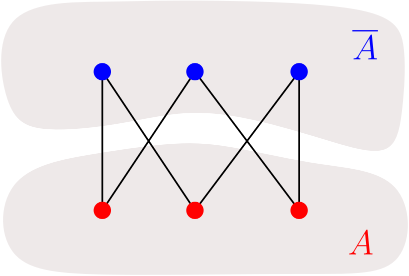
Linearly equivalent to clique-width:

[Rao 2006]

For all graphs G, we have mw$(G) \le$ cw$(G) \le 2$mw(G).

Rank-width

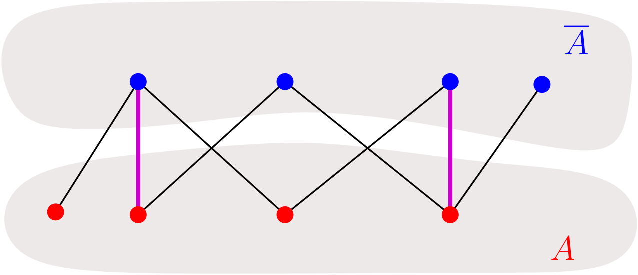
Defined from the function rw(A) := the rank of adjacency matrix between A and $\overline{A}$ over $GF(2)$.



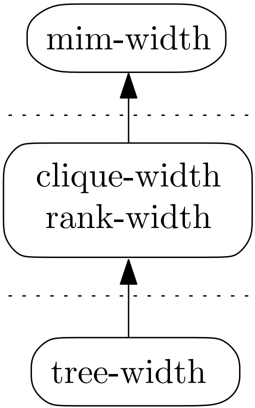
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Maximum Induced Matching width (mim-width)

Defined from the function mim(A) := the size of a maximum induced matching in the bipartite graph between A and $\overline{A}$.



Generality / Modeling power



permutation, interval, k-polygon, ...

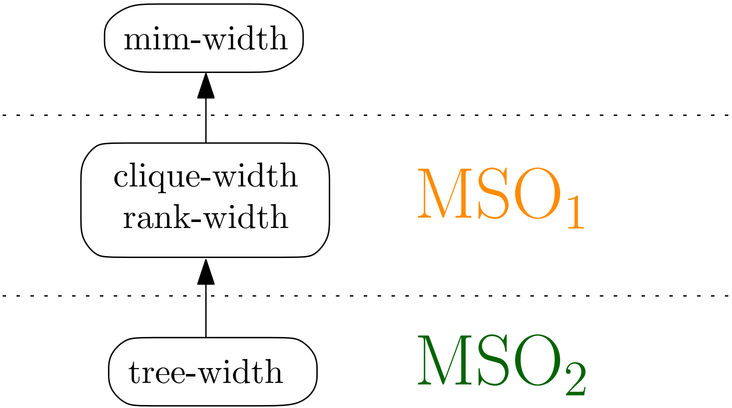
Cliques, hereditary distance

Trees, partial k-trees

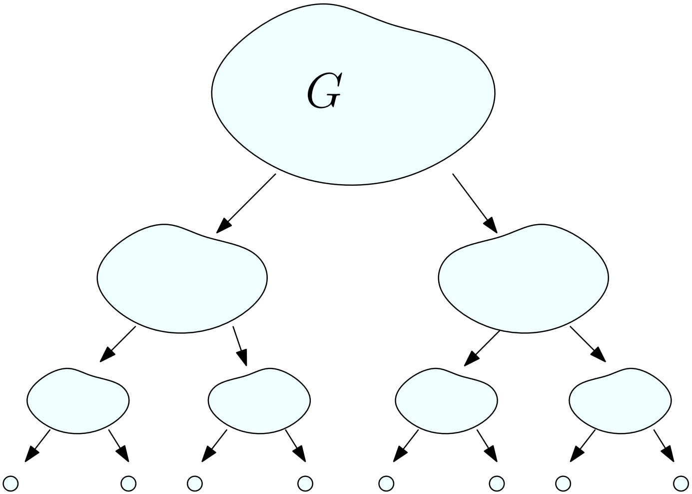
Computation complexity

- NP-hard for all these widths measures.
- Efficient algorithms for tree-width and rank-width.
  → Running time: $2^{O(k)} \cdot n^{O(1)}$.
- Tough open questions for clique-width and mim-width.

Algorithmic applications



Intuition: conquer



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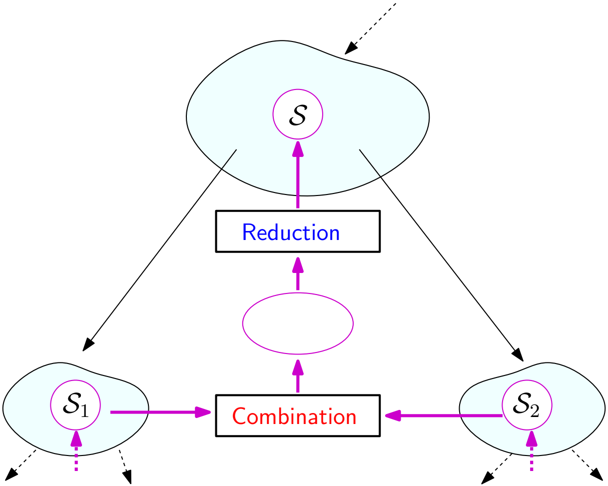

Intuition: conquer

In general

If it is enough to keep $\mathcal{N}$ partial solutions at each step, then we can solve the problem in time $\mathcal{N}^{O(1)} \cdot n^{O(1)}$.

One algorithm to rule them all

Theorem [B., Kanté 2019]

We have a meta-algorithm for the connected and acyclic variants of (σ, ρ)-dominating set problems.

- Connected dominating set,
- Connected vertex cover,
- Node-weighted Steiner tree,
- Feedback vertex set,
- Maximum induced tree,
- Longest induced path,
- Maximum induced linear forest,
- Max. induced tree of $\Delta \leq 42$,...

One algorithm to rule them all

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We have a meta-algorithm for the connected and acyclic variants of (σ, ρ)-dominating set problems.

Corollary [B., Kanté 2019]

These problems are solvable in time:

tree-width	$2^{O(\mathsf{tw})} \cdot n^{O(1)}$
clique-width	$2^{O(\mathsf{cw})} \cdot n^{O(1)}$
rank-width	$2^{O(\mathsf{rw}^2)} \cdot n^{O(1)}$
mim-width	$n^{O(\mathsf{mim})}$

$\Longrightarrow$ Polytime in interval graphs, permutations graphs, k-trapezoid,...

Key: d-neighbor equivalence

$\text{nec}_d(A)$: # of equivalence classes of the d-neighbor equivalence over A.

Theorem [Bui-Xuan, Telle, Vatshelle 2013]

It is enough to keep $\text{nec}_d(A) \cdot \text{nec}_d(\overline{A})$ partial solutions at each cut $(A, \overline{A})$, for any (σ, ρ)-Dominating Set problem.

Lemma [Vatshelle 2012]

$\text{nec}_d(A)$ is upper bounded by:

tree-width	$2^{d \cdot \text{tw}} \cdot n^{O(1)}$
clique-width	$2^{d \cdot \text{cw}} \cdot n^{O(1)}$
rank-width	$2^{d \cdot \text{rw}^2} \cdot n^{O(1)}$
mim-width	$n^{d \cdot \text{mim}}$

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Our result

Theorem [B., Kanté 2019]

It is enough to keep $\mathsf{nec}_d(A) \cdot \mathsf{nec}_d(\overline{A}) \cdot \mathsf{nec}_1(A)^2$ partial solutions at each cut $(A, \overline{A})$, for any connected variant of a (σ, ρ)-Dominating Set problem.

Corollary [B., Kanté 2019]

We can solve these problems in time:

tree-width	$2^{O(\mathsf{tw})} \cdot n^{O(1)}$
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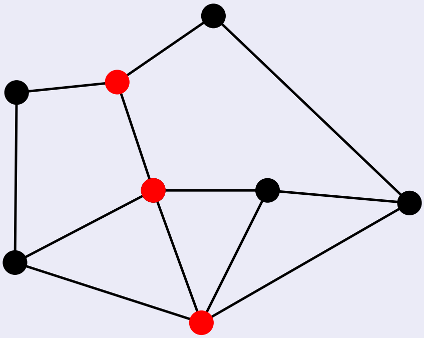
Connected Dominating set and 1-neighbor equivalence

Connected Dominating set and 1-neighbor width

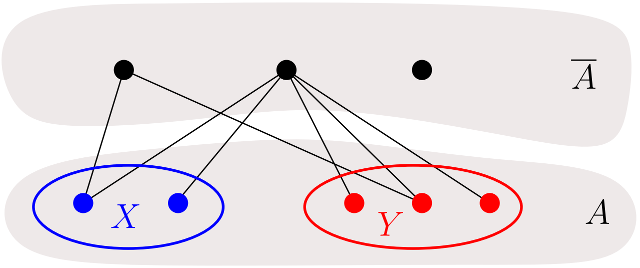
Connected dominating set

Finding a vertex set D of minimum weight which dominates all the vertices and which induces a connected graph.



1-neighbor equivalence relation

$X, Y \subseteq A$ are 1-neighbor equivalent in A if $N(X) \cap \overline{A} = N(Y) \cap \overline{A}$.



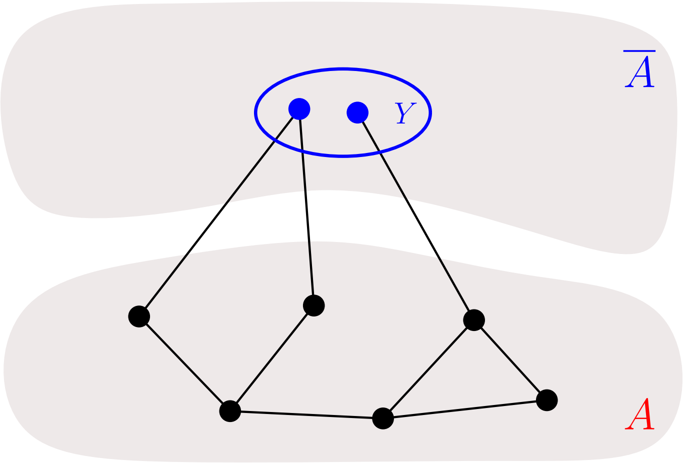
$\mathsf{nec}_1(A) :=$ # of equivalence classes over A.

Theorem [B. and Kanté 2018]

It is enough to keep $\mathsf{nec}_1(A)^3\mathsf{nec}_1(\overline{A})$ partial solutions for each cut $(A, \overline{A})$ to solve Connected dominating set.

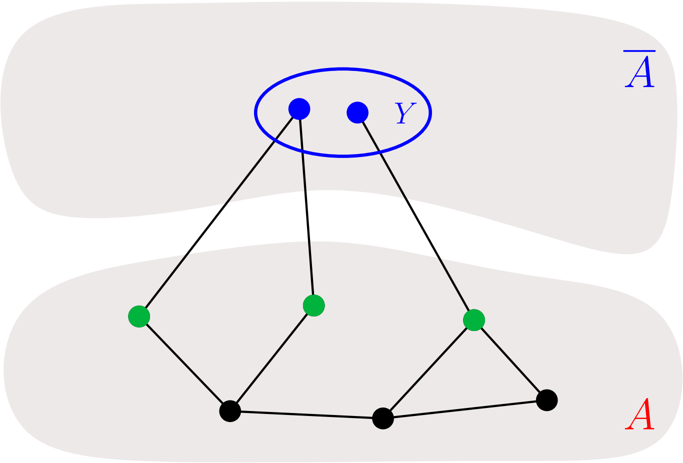
Dealing with domination

A set of partial solutions for each equivalence class R' of the 1-neighbor equivalence over $\overline{A}$.



Dealing with domination

The sets $Y \in R'$ have the same neighborhood in A.



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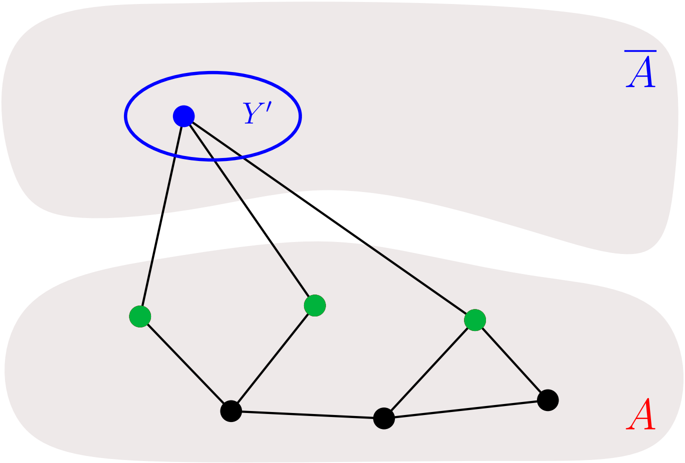
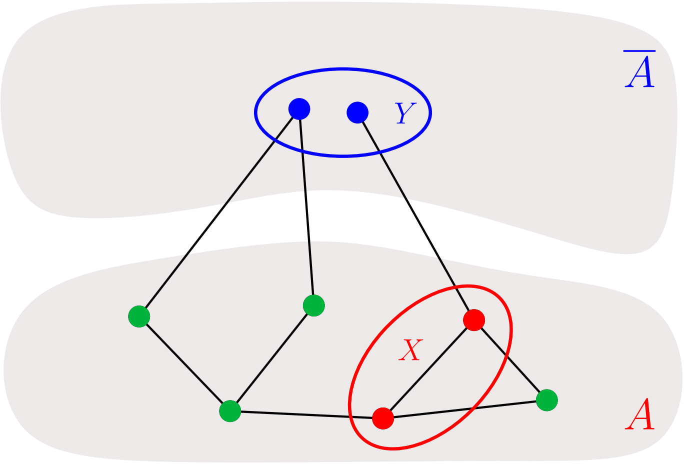


Dealing with domination

Partial solutions associated with R' :

$\rightarrow X \subseteq A$ such that $X \cup Y$ dominates A, for (all) $Y \in R'$



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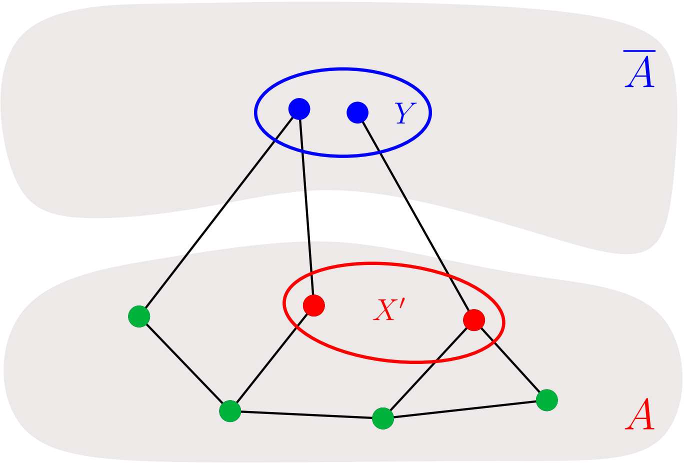
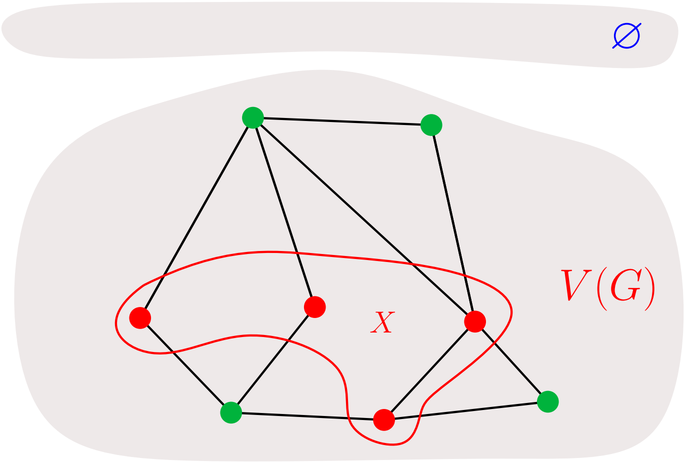


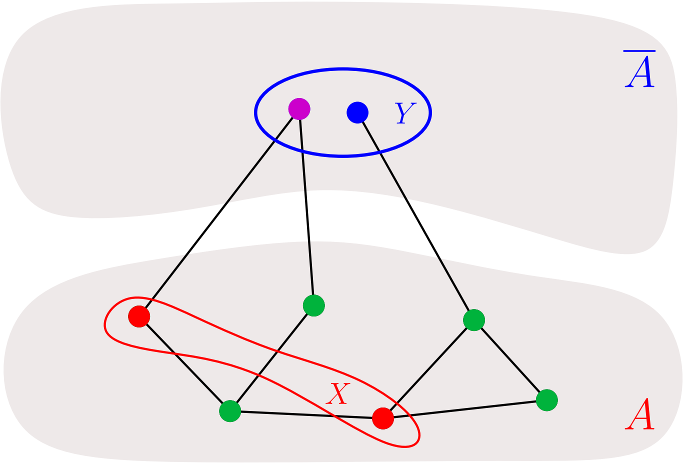
Dealing with domination

At the root of the decomposition, the cut is $(V(G), \varnothing)$:
$\rightarrow$ A partial solution is a dominating set.



Dealing with domination

Two partial solutions associated with R' are equivalent for the domination if they are 1-neighbor equivalent!



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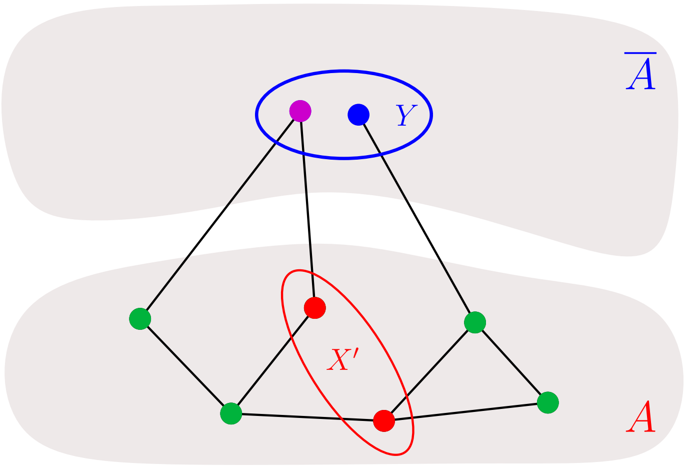

Lemma [Bui-Xuan, Telle and Vatshelle, 2013]

For Dominating set, it is enough to keep $\mathsf{nec}_1(A) \cdot \mathsf{nec}_1(\overline{A})$ partial solutions at each step.

One partial solution:

- for each 1-neighbor equivalence class R' of $\overline{A}$, and
- for each 1-neighbor equivalence class R of A

Dealing with connectivity

- We need an equivalence relation between sets of partial solutions.

Dealing with connectivity

- We need an equivalence relation between sets of partial solutions.

- For all $Y \in R'$ and for all sets of partial solutions $\mathcal{S}$, we define:

$$\text{best}(\mathcal{S}, Y) := \min\{\text{weight}(X) : X \in \mathcal{S} \text{ and } G[X \cup Y] \text{ is connected}\}.$$

R'-representativity

We say that $\mathcal{S}^\star$ R'-represents $\mathcal{S}$ if, for all $Y \in R'$, we have

$$\text{best}(\mathcal{S}, Y) = \text{best}(\mathcal{S}^\star, Y).$$

Representative set

- At the root $(V(G), \varnothing)$: an $\{\varnothing\}$-representative set of the set of all partial solutions must contain an optimal solution.

  $\rightarrow$ best$(\mathcal{S}, \varnothing) := \min\{$weigth$(X)$: $X \in \mathcal{S}$ and $G[X]$ is connected$\}$.

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Theorem [B., Kanté 2019]

There exists a function reduce:

- Input: a set of partial solutions $\mathcal{S} \subseteq 2^A$.
- Output: $\mathcal{S}^\star \subseteq \mathcal{S}$ such that $|\mathcal{S}^\star| \leq \mathsf{nec}_1(A)^2$ and $\mathcal{S}^\star$ R'-represents $\mathcal{S}$.
- Running time: $|\mathcal{S}| \cdot \mathsf{nec}_1(A)^{O(1)} \cdot n^2$.

Sketch of proof

Inspiration [Bodlaender, Cygan, Kratsch, Nederlof 2013]

Rank based approach: technique to obtained $2^{O(\mathsf{tw})} \cdot n$ time algorithms for many connectivity problems.

Let $\mathcal{M}$ be the $(\mathcal{S}, R')$-matrix:

$$\mathcal{M}[X, Y] := \begin{cases} 1 & \text{if } G[X \cup Y] \text{ is connected,} \\ 0 & \text{otherwise.} \end{cases}$$

- In $GF(2)$: a basis of the row space of $\mathcal{M}$ of minimum weight R'-represents $\mathcal{S}$.
- But $\mathcal{M}$ is too big to be computed.

Sketch of proof

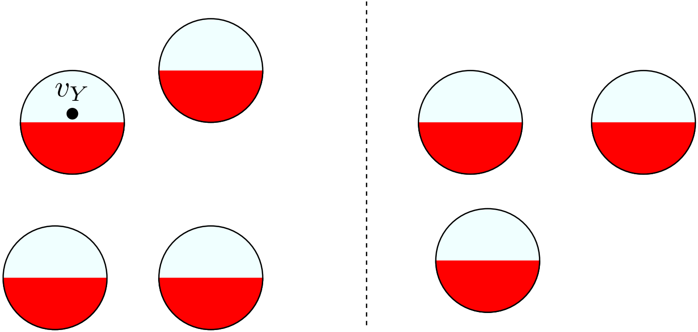
$\mathcal{P}$: the set of pairs (R'_1, R'_2) of 1-neighbor equivalence classes in $\overline{A}$.

$$\mathcal{S}\left\{\overbrace{\boxed{\quad\mathcal{M}\quad}}^{R'}\right. \overset{GF(2)}{=} \mathcal{S}\left\{\overbrace{\boxed{\mathcal{C}}}^{\mathcal{P}}\right. \bullet \mathcal{P}\left\{\overbrace{\boxed{\quad\overline{\mathcal{C}}\quad}}^{R'}\right.$$

- A basis of $\mathcal{C}$ is also a basis of $\mathcal{M}$.
- $|\mathcal{P}| = \mathsf{nec}_1(A)^2$.
- $\mathcal{C}$ is computable in time $|\mathcal{S}| \cdot |\mathcal{P}| \cdot n^2$.

Trick

There is $2^{|CC(X \cup Y)|-1}$ ways of dividing $X \cup Y$ such that v_Y is on one side.



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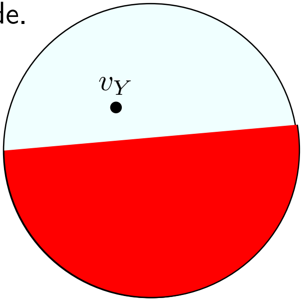
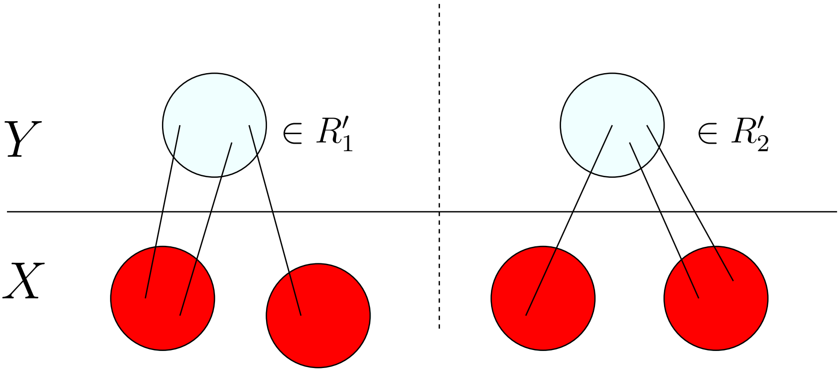


$$\mathcal{C} \cdot \overline{\mathcal{C}}[X, Y] = 2^{|CC(X \cup Y)|-1}$$

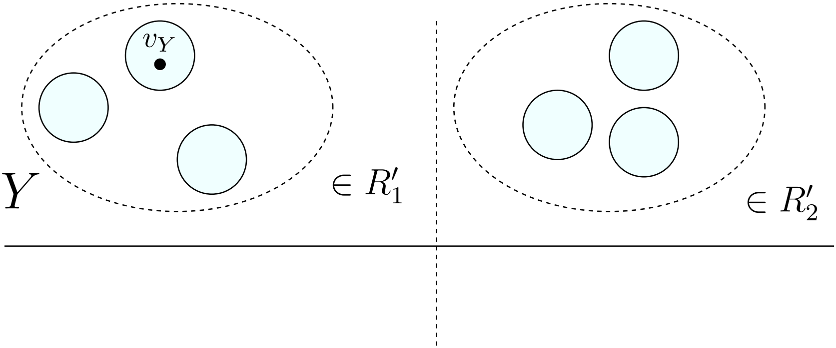
Trick

$\color{red}{\mathcal{C}}[X, \color{green}{(R'_1, R'_2)}]$=1 if and only if



Trick

$\overline{\mathcal{C}}[(R'_1, R'_2), Y]$=1 if and only if



Trick

$$\color{red}{\mathcal{C}} \cdot \color{blue}{\overline{\mathcal{C}}}\color{black}[X, Y] = 2^{|CC(X \cup Y)|-1}$$

$$\color{red}{\mathcal{C}} \cdot \color{blue}{\overline{\mathcal{C}}} \color{black}\overset{GF(2)}{=} \mathcal{M}$$

Acyclicity

Theorem [B., Kanté 2019]

It is enough to keep $\mathsf{nec}_d(A) \cdot \mathsf{nec}_d(\overline{A}) \cdot \mathsf{nec}_1(A)^2 \cdot \mathcal{X}$ partial solutions at each cut $(A, \overline{A})$, for any acyclic and acyclic+connected variant of a (σ, ρ)-Dominating Set problem.

Corollary [B., Kanté 2019]

We can solve these problems in time:

tree-width	$2^{O(\mathsf{tw})} \cdot n^{O(1)}$
clique-width	$2^{O(\mathsf{cw})} \cdot n^{O(1)}$
rank-width	$2^{O(\mathsf{rw}^2)} \cdot n^{O(1)}$
mim-width	$n^{O(\mathsf{mim})}$

Overview

Thanks to the d-neighbor equivalence, we obtain the best algorithms:

- for many problems
  → (σ, ρ)-Dominating Set problems and their variants.
- for many width-measures
  → tree-width, clique-width, rank-width, mim-width, $\mathbb{Q}$-rank-width.

The algorithms for clique-width and tree-width are optimal under ETH.
→ What about rank-width: is $2^{O(\mathsf{rw}^2)} \cdot n^{O(1)}$ optimal?

Hamiltonian cycle, Max Cut and Edge Dominating Set

Can we use the d-neighbor equivalence for W$[1]$-hard problems parameterized by clique-width?

- We can solve these 3 problems in time $2^{O(\mathsf{tw})} \cdot n$ and $n^{O(\mathsf{cw})}$.
  $\rightarrow$ Optimal under ETH.
- Using the d-neighbor equivalence width d a constant is useless.

Theorem [B., Kanté 19]

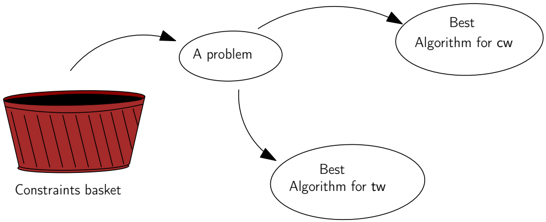
For Max Cut, it is enough to keep $\mathsf{nec}_n(A)$ partial solutions at each cut $(A, \overline{A})$.

Corollary [B., Kanté 19]

We can solve Max Cut in time $n^{O(\mathsf{cw})}$, $n^{O(\mathsf{rw}_{\mathbb{Q}})}$ and $n^{2^{O(\mathsf{rw})}}$.

The dream

Towards a "Courcelle's theorem" which gives efficient algorithms?



Example:

We want a set of vertices X of minimum weight satisfying:

$$\mathsf{DominatingSet}(X) \wedge \mathsf{Acyclic}(X) \wedge \mathsf{Connected}(X).$$

Thm $\Rightarrow$ We can find one in time: $2^{O(\mathsf{tw})} \cdot n$, $2^{O(\mathsf{cw})} \cdot n$, $2^{O(\mathsf{rw}^2)} \cdot n^3$.