Tight Lower Bounds for Problems Parameterized by **Rank-width**

Virtual Discrete Math Colloquium
IBS, February 1

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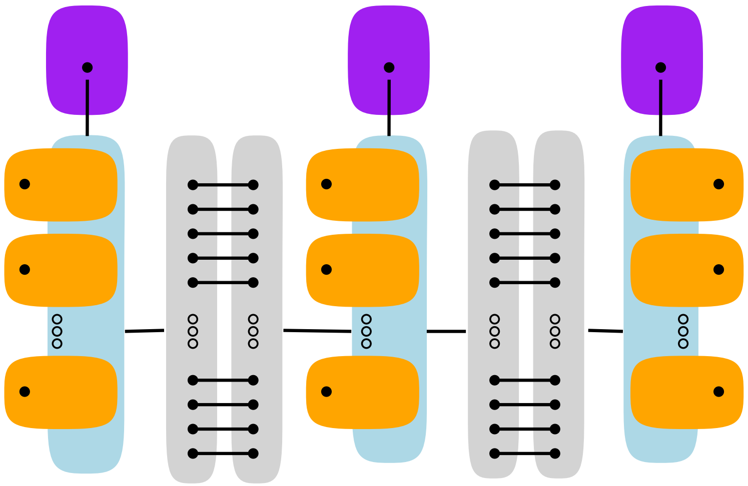
University of Warsaw

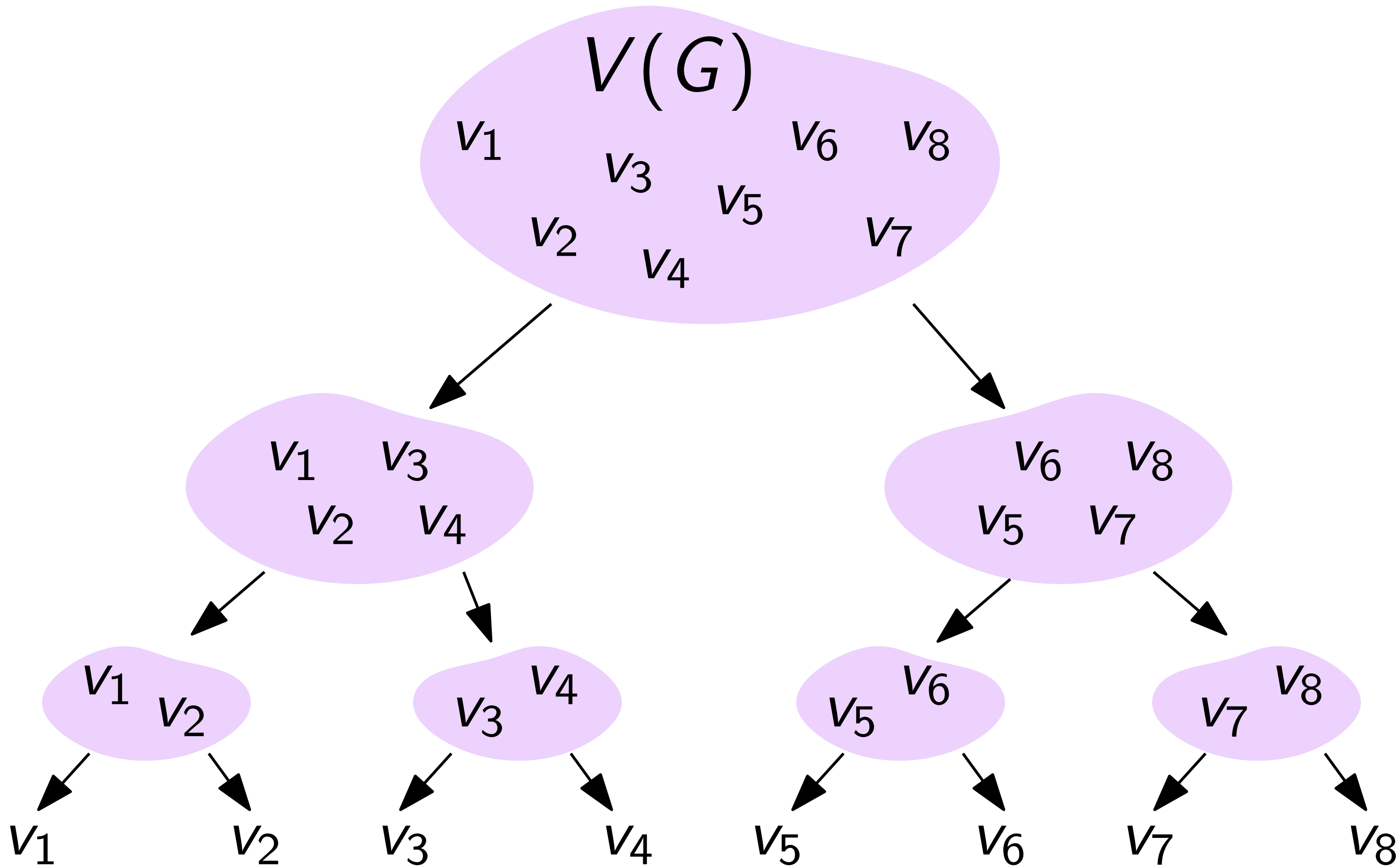


Feat **Tuukka Korhonen**
University of Bergen

Jesper Nederlof
University of Eindhoven

Graph width parameters

Recursively decompose a graph into **simple cuts**



Layout: recursively **cut** the **vertex set** in **two**

Graph width parameters

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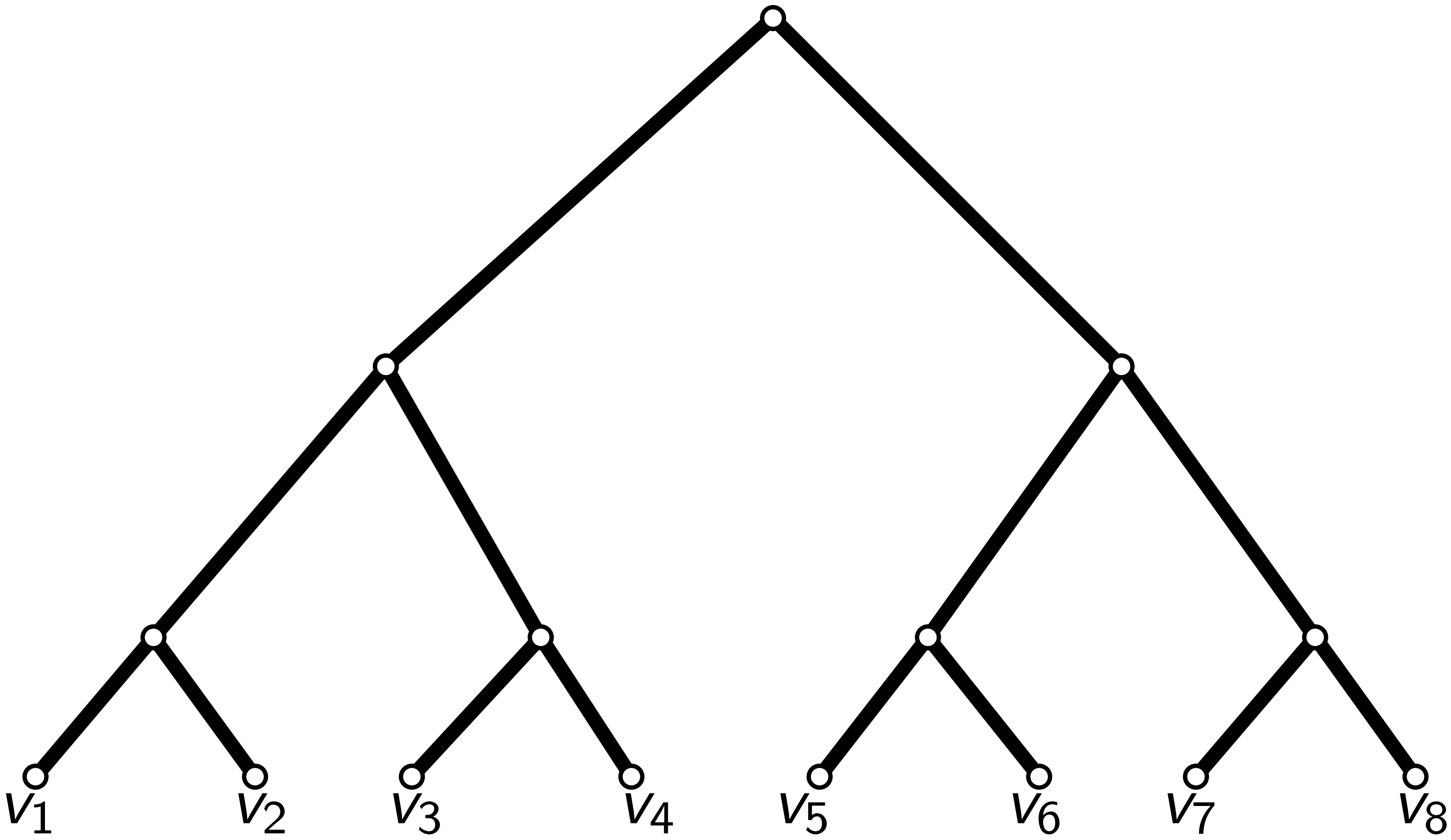

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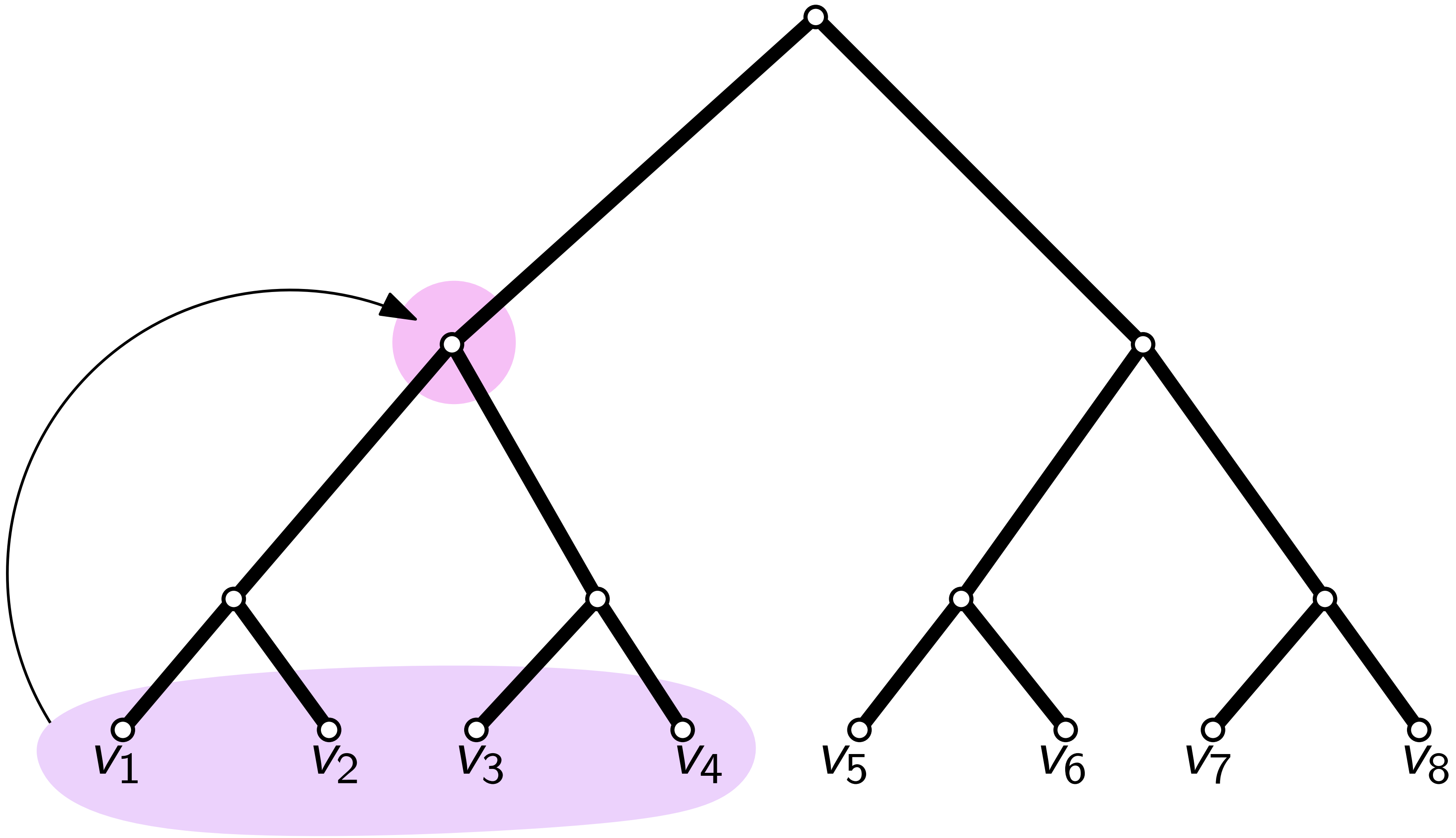
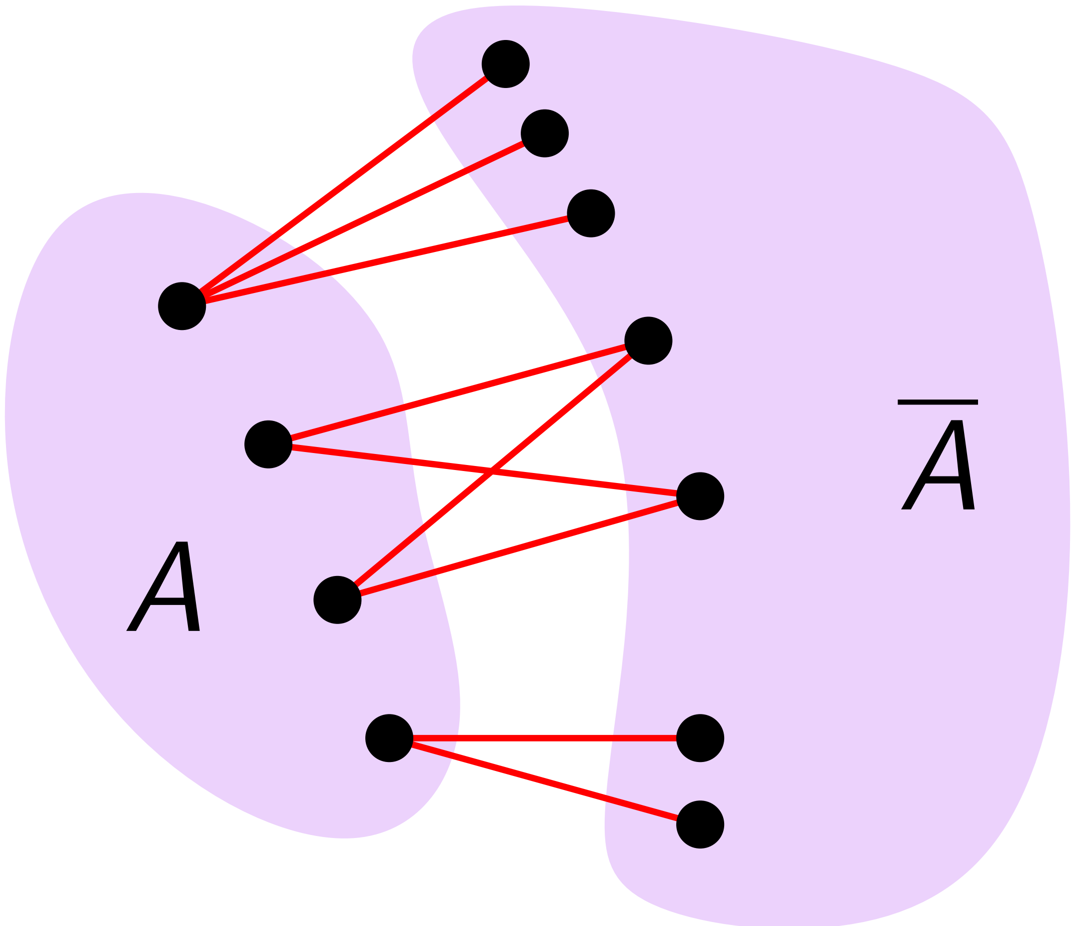


Layout: recursively **cut** the **vertex set** in **two**

Width parameters

Simplicity of cuts is measured with a **function** f: $2^{V(G)} \to \mathbb{N}$

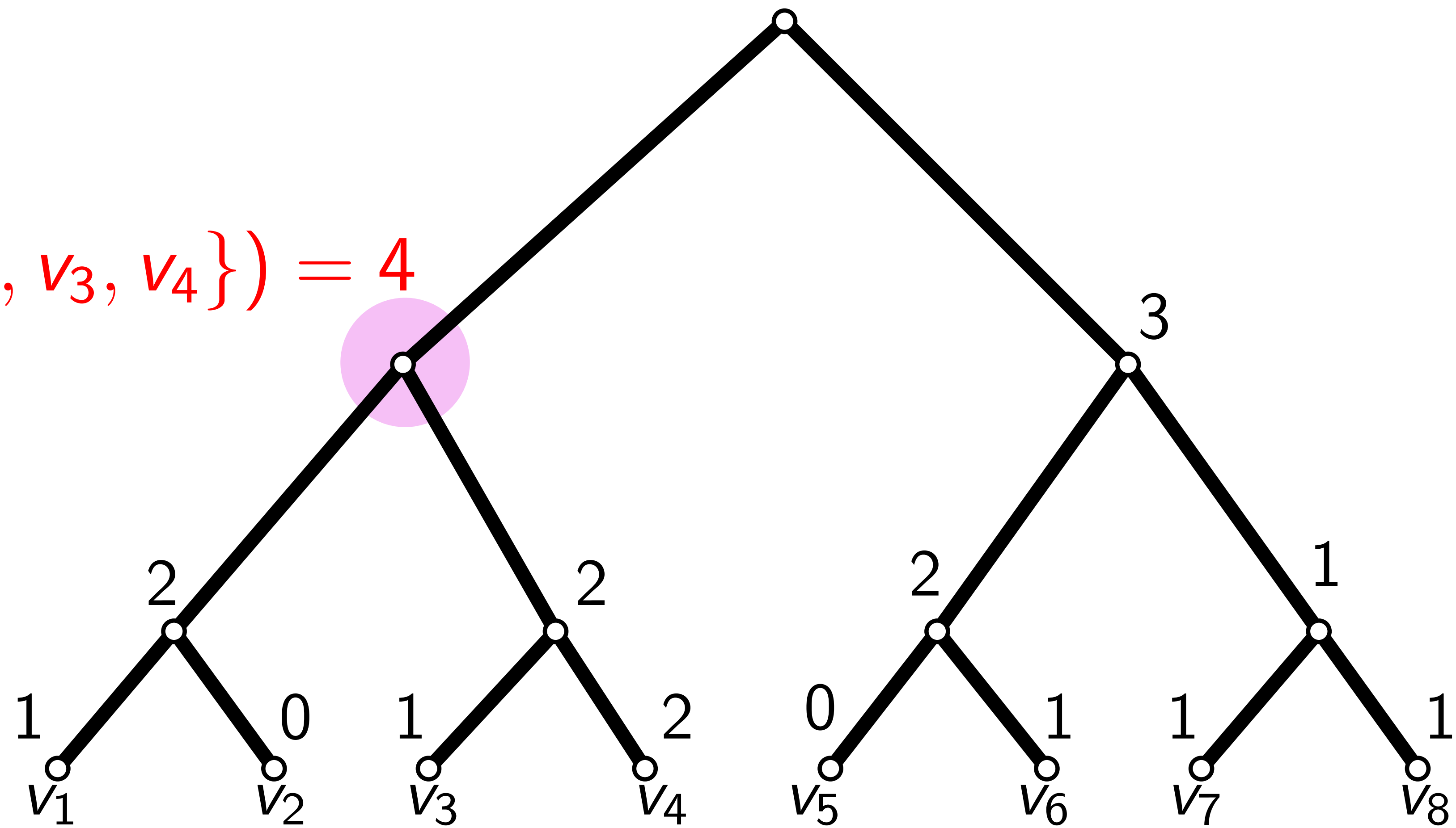


Different notions of **simplicity** = different **width parameters**

Width parameters

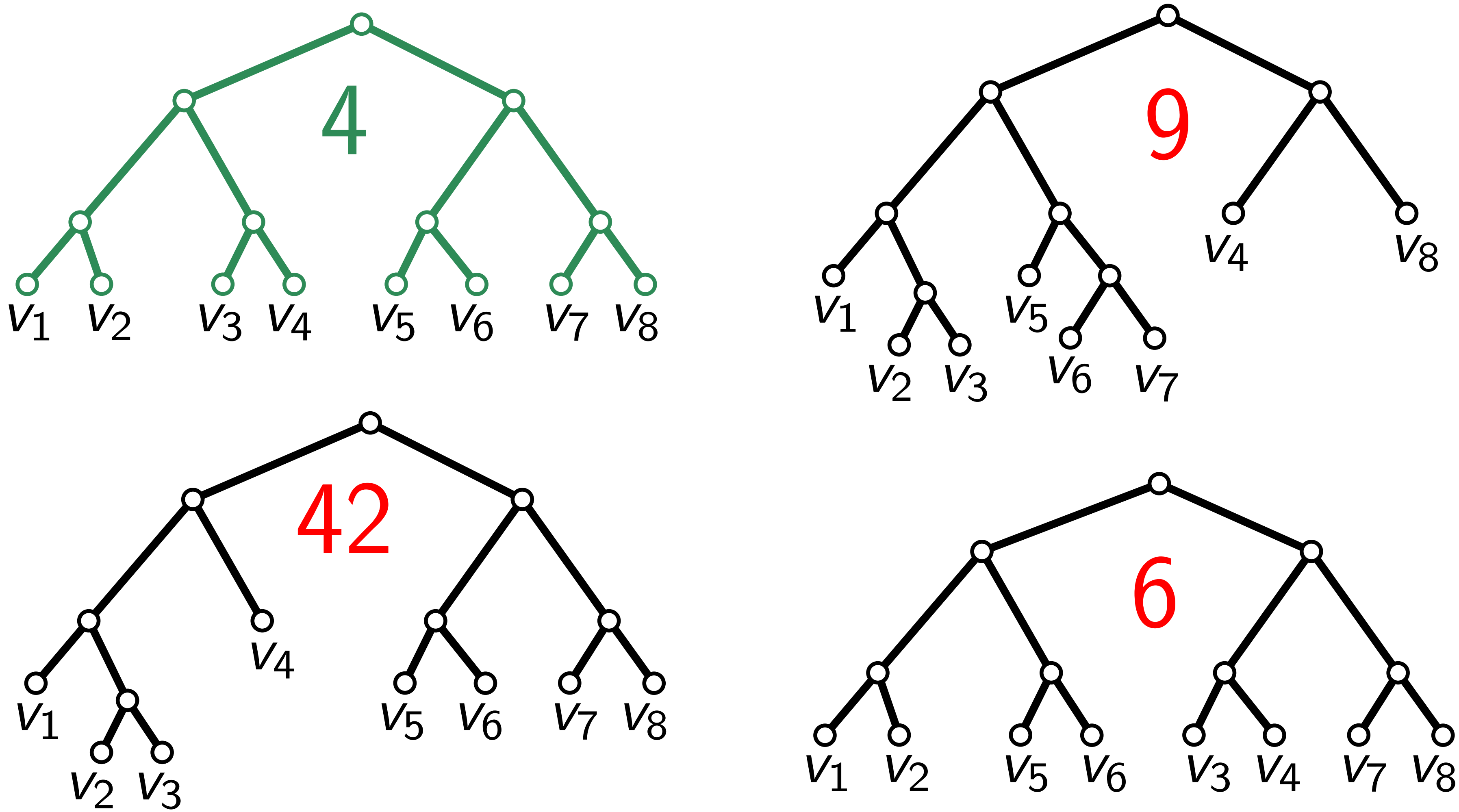
Width of a **decomposition** D = **max** f(cut) over the **cuts** of D

$f(\{v_1, v_2, v_3, v_4\}) = 4$



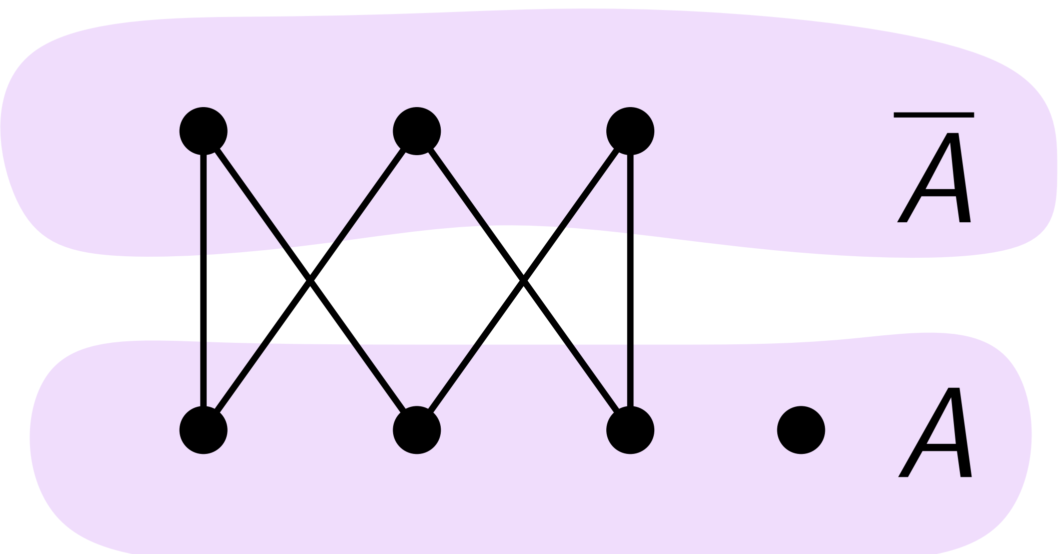
Width parameters

Width of a **graph** = **min widths** of its **decompositions**



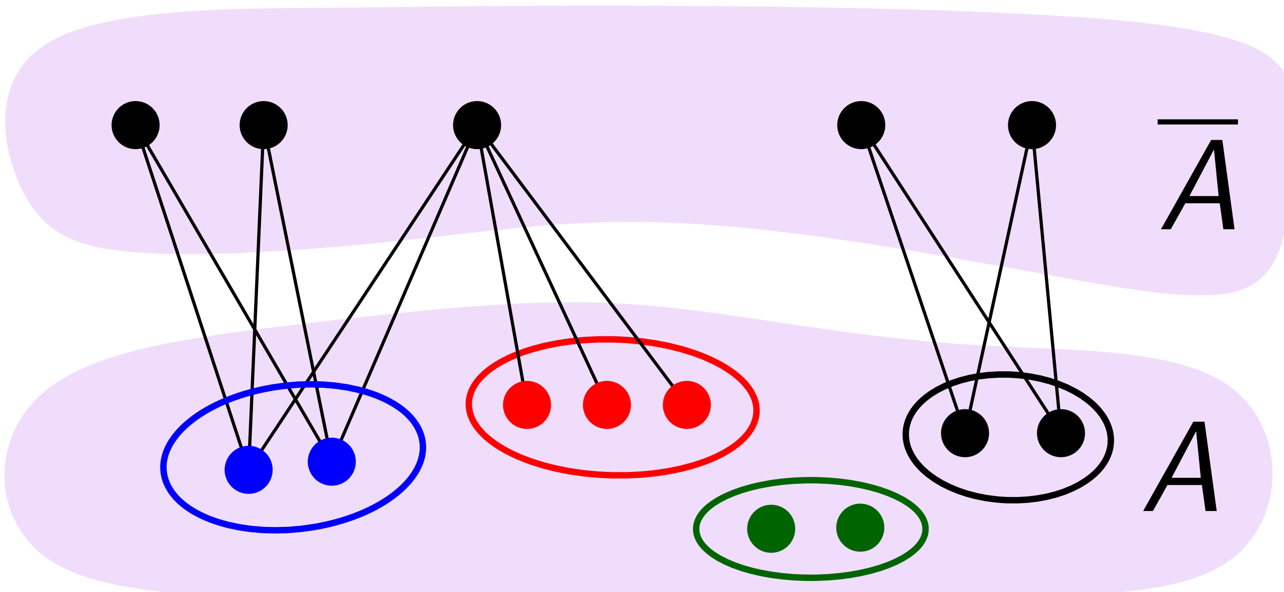
Width of a graph **class** = **max widths** of its **graphs**

Rank-width [Oum, 2006]
Defined from $\mathrm{rw}(A) :=$ the **rank of the adjacency matrix** between A and $\overline{A}$ over the binary field.



$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Module-width [Rao, 2006]
Defined from $\mathrm{cw}^\star(A) :=$ **nb. of different rows** in the adjacency matrix between A and $\overline{A}$.



Linearly equivalent to clique-width!

Theorem [Rao, 2006]
For all graphs G, we have $\mathrm{cw}^\star(G) \leqslant \mathrm{cw}(G) \leqslant 2\mathrm{cw}^\star(G)$.

Module-width [Rao, 2006]

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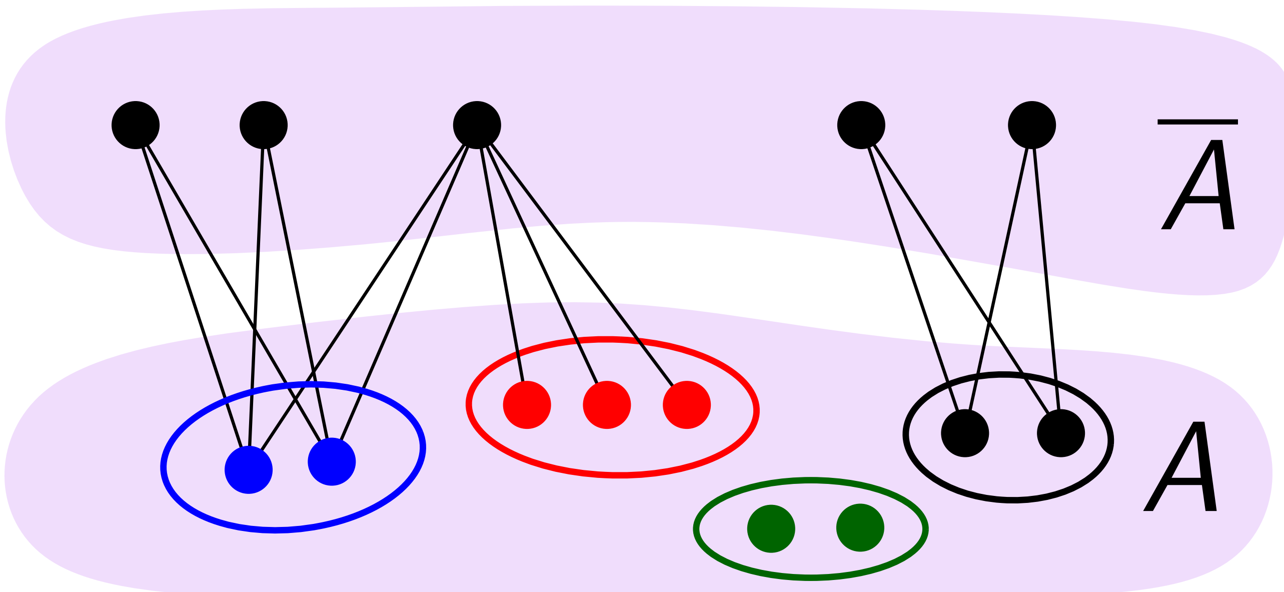
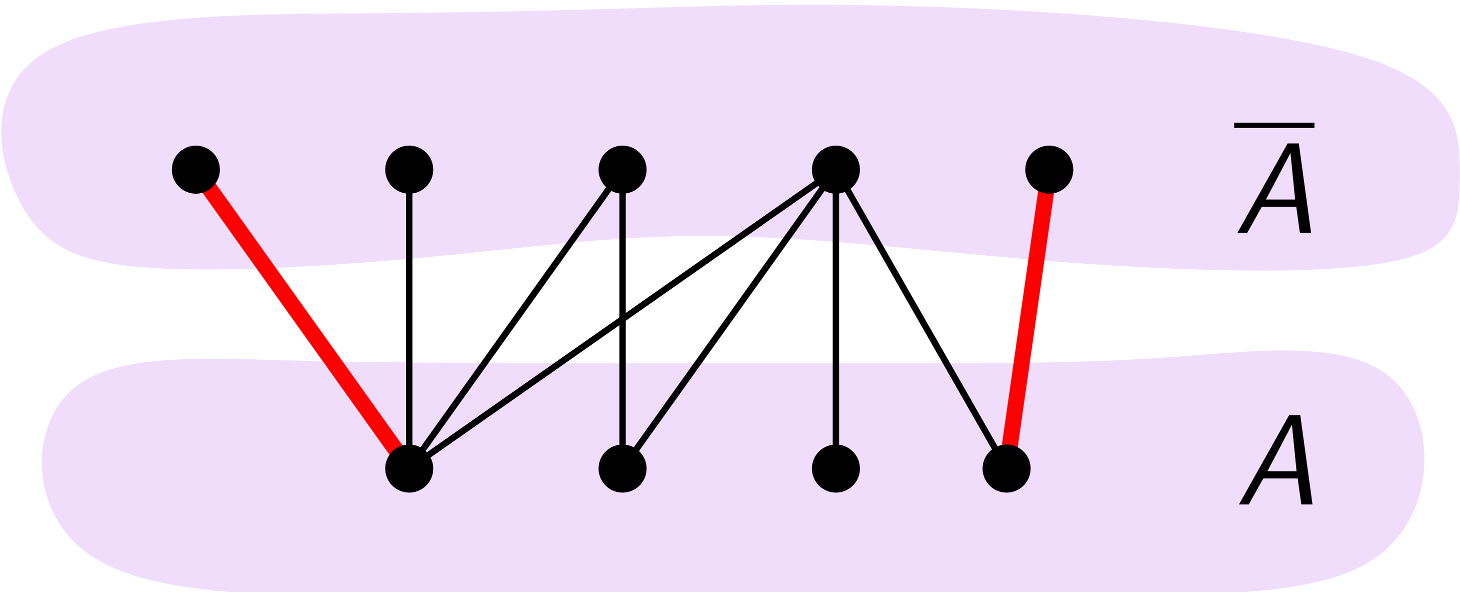


Equivalent to rank-width!

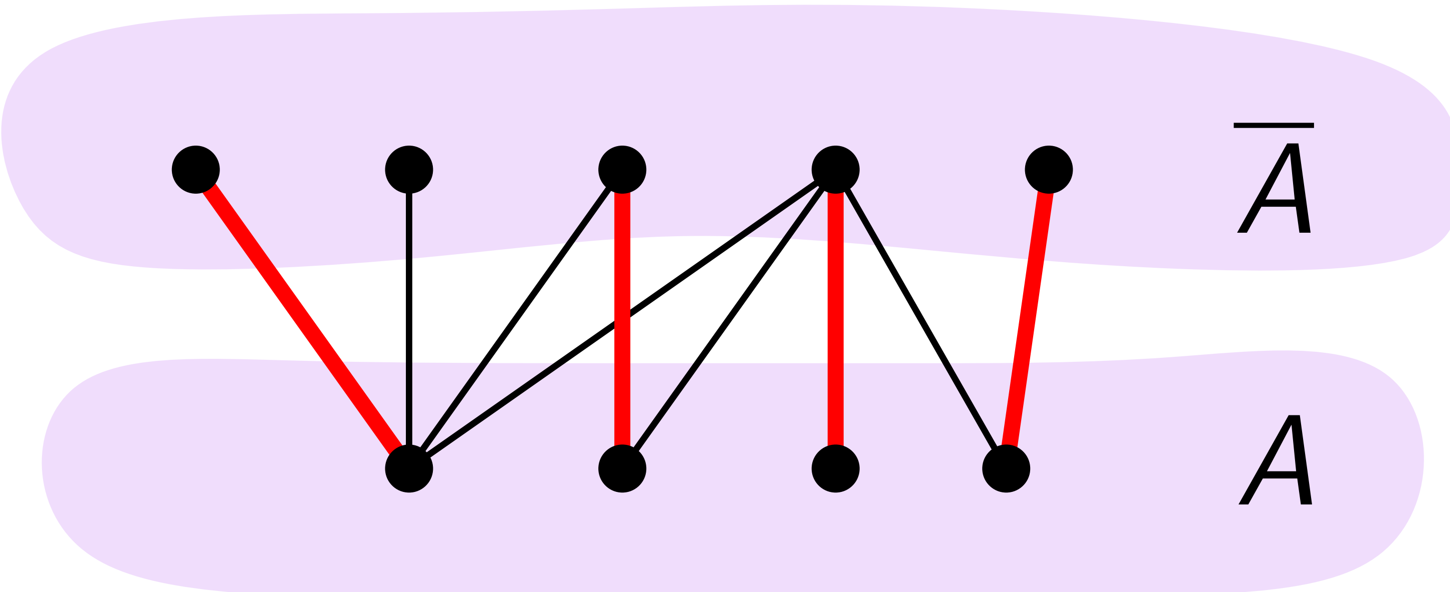
Fact

For all cut $(A, \overline{A})$, we have $\mathrm{rw}(A) \leq \mathrm{cw}^\star(A) \leq 2^{\mathrm{rw}(A)}$.

Mim-width [Vatshelle, 2012]
Defined from $\mathrm{mim}(A) :=$ size of a **maximum induced matching** in the bipartite graph between A and $\overline{A}$.



Maximum matching width [Vatshelle, 2012]
Defined from $\mathrm{tw}^\star(A) :=$ size of a **maximum matching** in the bipartite graph between A and $\overline{A}$.



Linearly equivalent to tree-width!

Theorem [Vatshelle, 2012]
For all graph G, we have $\frac{1}{3}\mathrm{tw}(G) + 1 \leq \mathrm{tw}^\star(G) \leqslant \mathrm{tw}(G)$.

Maximum matching width [Vatshelle, 2012]
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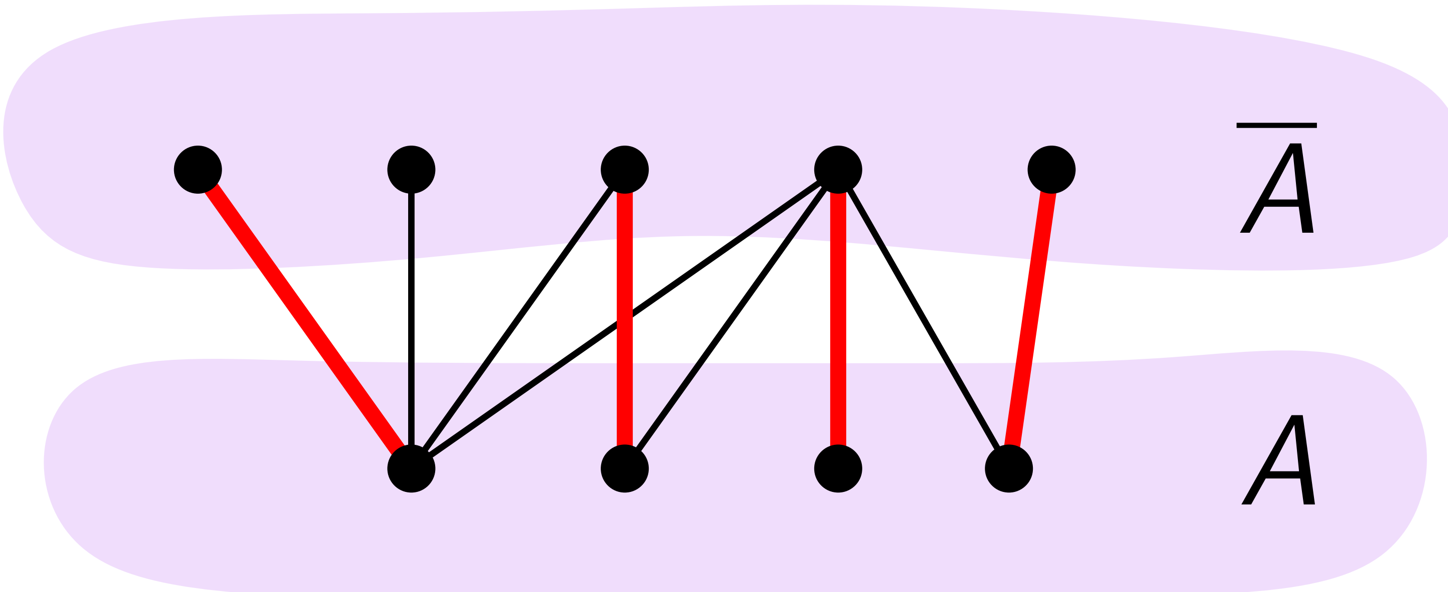


Theorem [Vatshelle, 2012]
For all cut $(A, \overline{A})$, we have $\mathrm{mim}(A) \leq \mathrm{rw}(A) \leqslant \mathrm{tw}^\star(A)$.

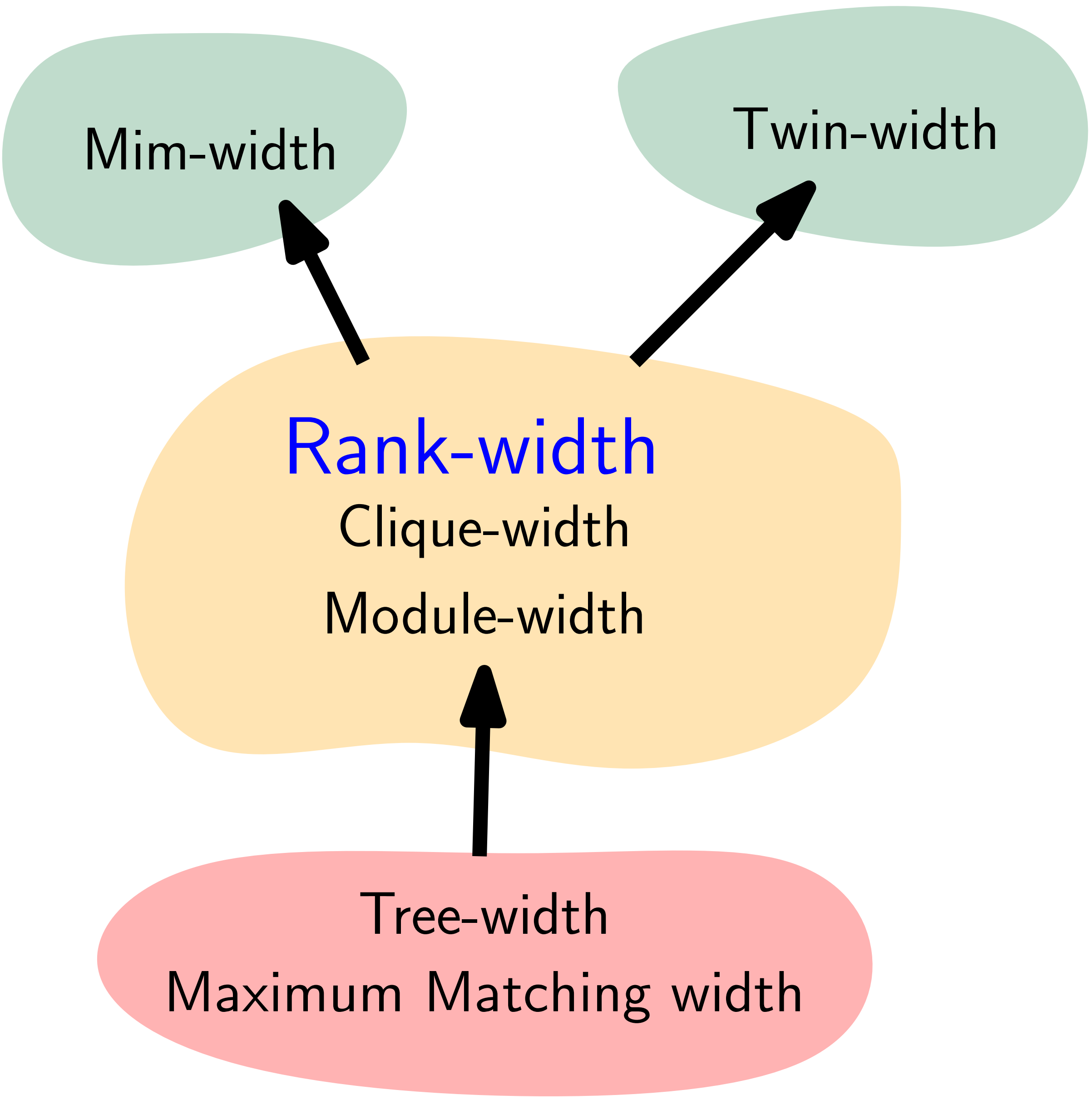
Comparing widths

- **Modeling power**

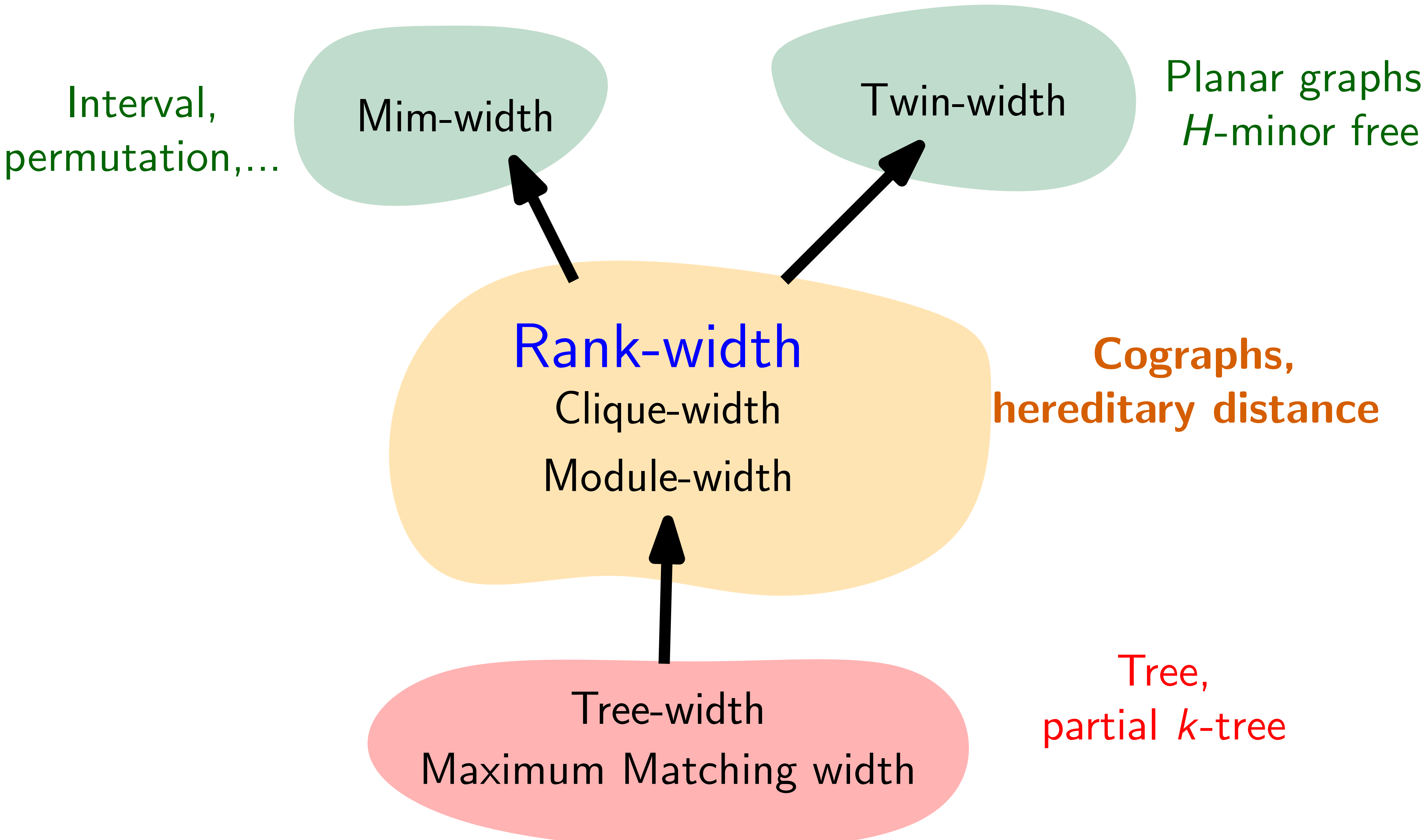
- **Complexity of computing a good decomposition**
  - NP-hardness everywhere
  - We know **efficient FPT** approximation algorithms for **tree-width** and **rank-width**

- **Algorithmic applications**

Modeling Power



Modeling Power



Computing Good Decomposition

Theorem [Oum, 2009]
Rank-width can be **3-approximated** in time $8^{\mathrm{rw}} n^4$.

Theorem [Korhonen and Fomin, 2021]
Rank-width can be **2-approximated** in time $2^{2^{O(\mathrm{rw})}} n^2$.

$\mathrm{rw}(A)$ is **symmetric** and **submodular**

$$\mathrm{rw}(X) + \mathrm{rw}(Y) \geq \mathrm{rw}(X \cap Y) + \mathrm{rw}(X \cup Y)$$

Algorithmic Meta-Theorems

Theorem [Courcelle, Makowsky, Rotics, 2000]
All problems expressible in **$\mathbf{MSO_1}$** are solvable in time $f(\mathrm{cw})n$ given a decomposition of bounded **clique-width**.

- Cons: $f(\mathrm{cw}) = 2^{2^{\cdot^{\cdot^{\cdot^{\mathrm{cw}}}}}}$

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Theorem [B., Dreier, Jaffke, 2023]
All problems expressible in **A&C DN** are solvable in time $2^{O(\mathrm{rw}^4)}n^{O(1)}$.

- k-Independent Set: $\exists X,\ |X| \geq k \wedge X \cap N(X) = \emptyset$

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Efficient Algorithms

Theorem [Oum, 2006]
For all cut $(A, \overline{A})$, we have $\mathrm{cw}^\star(A) \leq 2^{\mathrm{rw}(A)} + 1$.

- $2^{O(\mathrm{cw})} n^{O(1)}$ time algo. $\Rightarrow 2^{2^{O(\mathrm{rw})}} n^{O(1)}$ time algo.

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Theorem [Bui-Xuan, Telle and Vatshelle, 2010]
Independent Set and **Dominating Set** can be solved in time $2^{O(\mathrm{rw}^2)} n^{O(1)}$.

Theorem [Ganian and Hliněný 2010]
Feedback Vertex Set can be solved in time $2^{O(\mathrm{rw}^2)} n^{O(1)}$.

Generalizations

Problems that can be solved in time $2^{O(\mathrm{rw}^2)} n^{O(1)}$

Locally Checkable Vertex Subset (LCVS)

Independent Set Max. Induced Matching
Dominating Set Perfect Code
Induced Matching Total Dominating Set

[Bui-Xuan, Telle and Vatshelle, 2013]

Locally Checkable Vertex Partitioning (LCVP)

k-Coloring Odd Cycle Transversal
H-Homomorphism Perfect Matching Cut
H-Covering $\cdots$

[Bui-Xuan, Telle and Vatshelle, 2013]

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Induced Matching — Total Dominating Set

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Locally Checkable Vertex Partitioning (LCVP)

k-Coloring — Odd Cycle Transversal
H-Homomorphism — Perfect Matching Cut
H-Covering — ...

[Bui-Xuan, Telle and Vatshelle, 2013]

Connected, Acyclic LCVS

Connected, Acyclic LCVP

Connected Dominating Set — Feedback Vertex Set
Connected Vertex Cover — Longest Induced Path

[Bergougnoux and Kante, 2019]

Generalizations

Problems that can be solved in time $2^{O(\mathrm{rw}^2)} n^{O(1)}$

Locally Checkable Vertex Subset (LCVS)

Independent Set, Max. Induced Matching, Dominating Set, Perfect Code, Induced Matching, Total Dominating Set

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A&C DN

[Bergougnoux, Dreier and Jaffke, 2023]

Connected, Acyclic LCVS

Connected, Acyclic LCVP

Connected Dominating Set, Feedback Vertex Set, Connected Vertex Cover, Longest Induced Path

[Bergougnoux and Kante, 2019]

Exceptions

Theorem [Belmonte and Sau, 2021]
Some problems based on **parity** can be solved in time $2^{O(\mathrm{rw})} n^{O(1)}$ (e.g. finding large odd induced subgraphs and odd colorings).

Theorem [B., Papadopoulos and Telle 2020]
Subset Feedback Vertex Set and **Node Multiway Cut** can be solved in time $2^{O(\mathrm{rw}^3)} n^4$.

Lower Bounds

ETH (weaker form) [Impagliazzo and Paturi, 2001]
There is no $2^{o(n)}(n+m)^{O(1)}$ time algorithm for **3-CNF SAT**.

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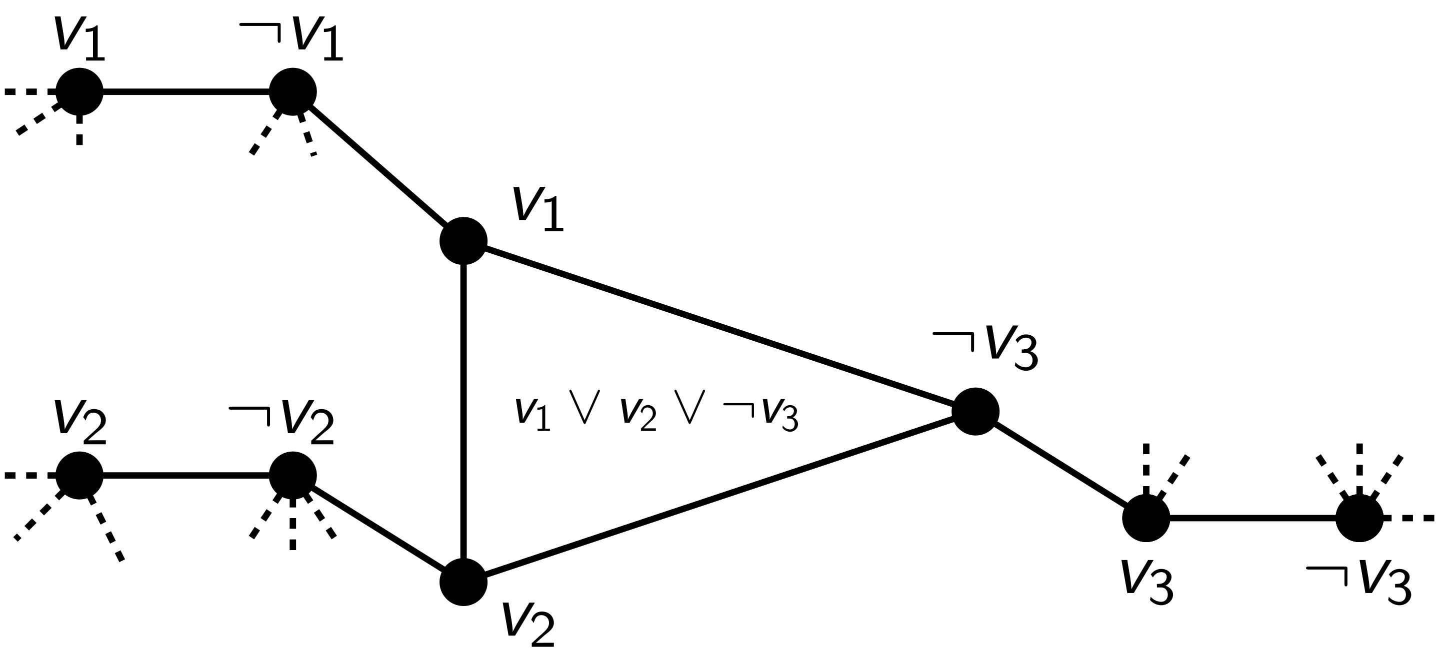

A $2^{o(|V(G)|)}n^{O(1)}$ time algo. for **IS** $\overset{\boxed{|V(G)| = 2n+3m}}{\Longrightarrow}$ A $2^{o(n+m)}(n+m)^{O(1)}$ time algo. for **3-CNF-SAT**

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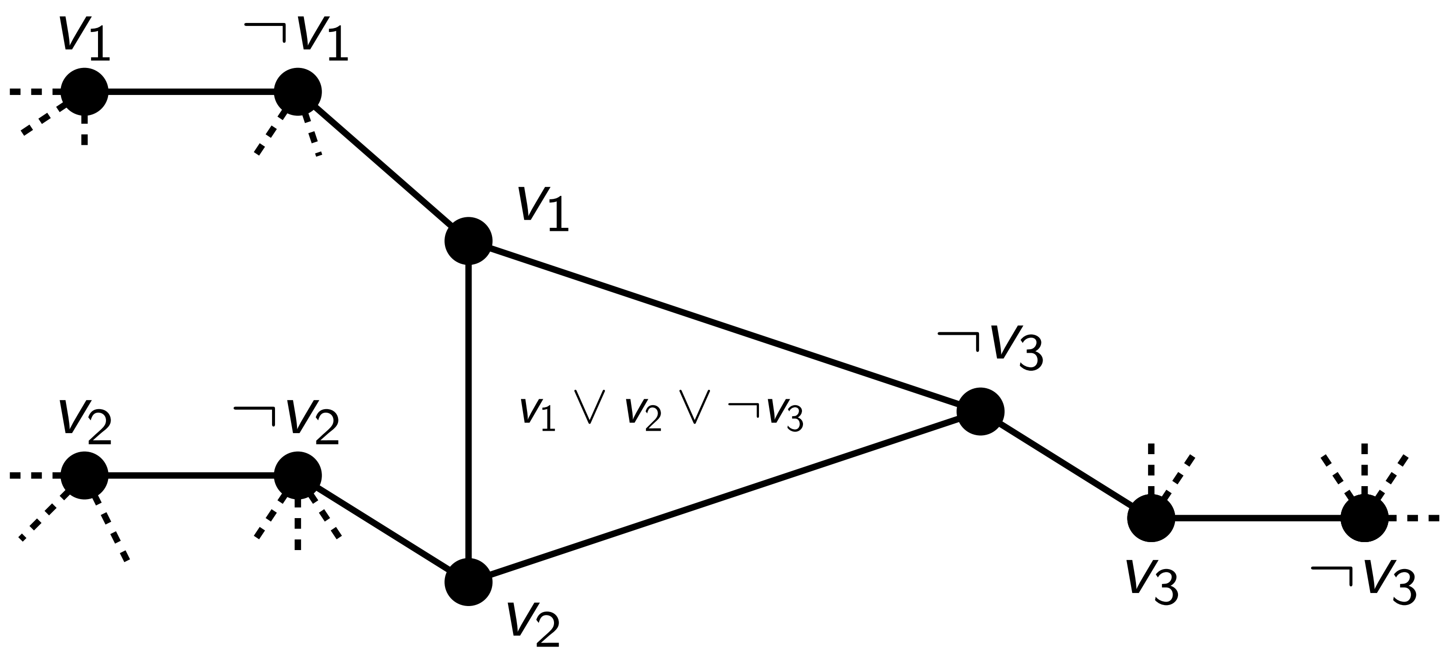


$|V(G)| = 2n + 3m$

Sparsification Lemma

A $2^{o(|V(G)|)} n^{O(1)}$ time algo. for **IS** $\Longrightarrow$ A $2^{o(n+m)}(n+m)^{O(1)}$ time algo. for **3-CNF-SAT** $\Longrightarrow$ **ETH** is false

Linear reductions [Folklore]

Under ETH, there is no $2^{o(|V(G)|)}n^{O(1)}$ time algorithm for

- Independent Set
- Dominating Set
- Maximum Induced Matching
- Feedback Vertex Set

$$\mathrm{tw}, \mathrm{cw}, \mathrm{rw} \leqslant |V(G)|$$

Corrolary

For each $k \in \{\mathrm{tw}, \mathrm{cw}, \mathrm{rw}\}$, under ETH, there is no $2^{o(k)}n^{O(1)}$ time algorithm for these problems.

Our results

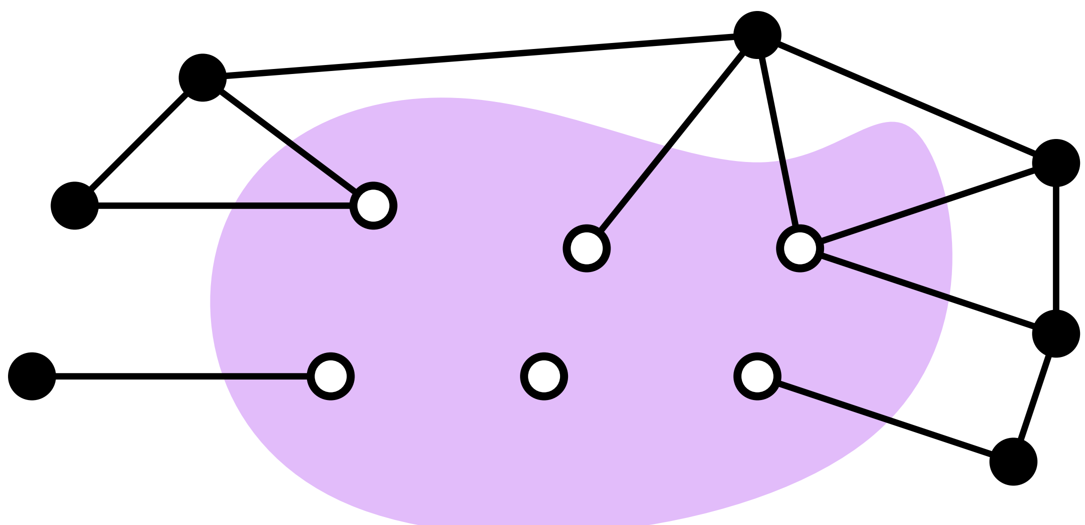
Theorem [B., Korhonen and Nederlof, 2022+]
Under ETH, there are no $2^{o(\mathrm{rw}^2)} n^{O(1)}$ time algorithms for

- Independent Set
- **Weighted** Dominating Set
- Maximum Induced Matching
- Feedback Vertex Set

$\Rightarrow$ The best known algorithms for these problems are **optimal** under ETH.

Holds also for **linear** rank-width

Algorithm and lower-bound for Independent Set

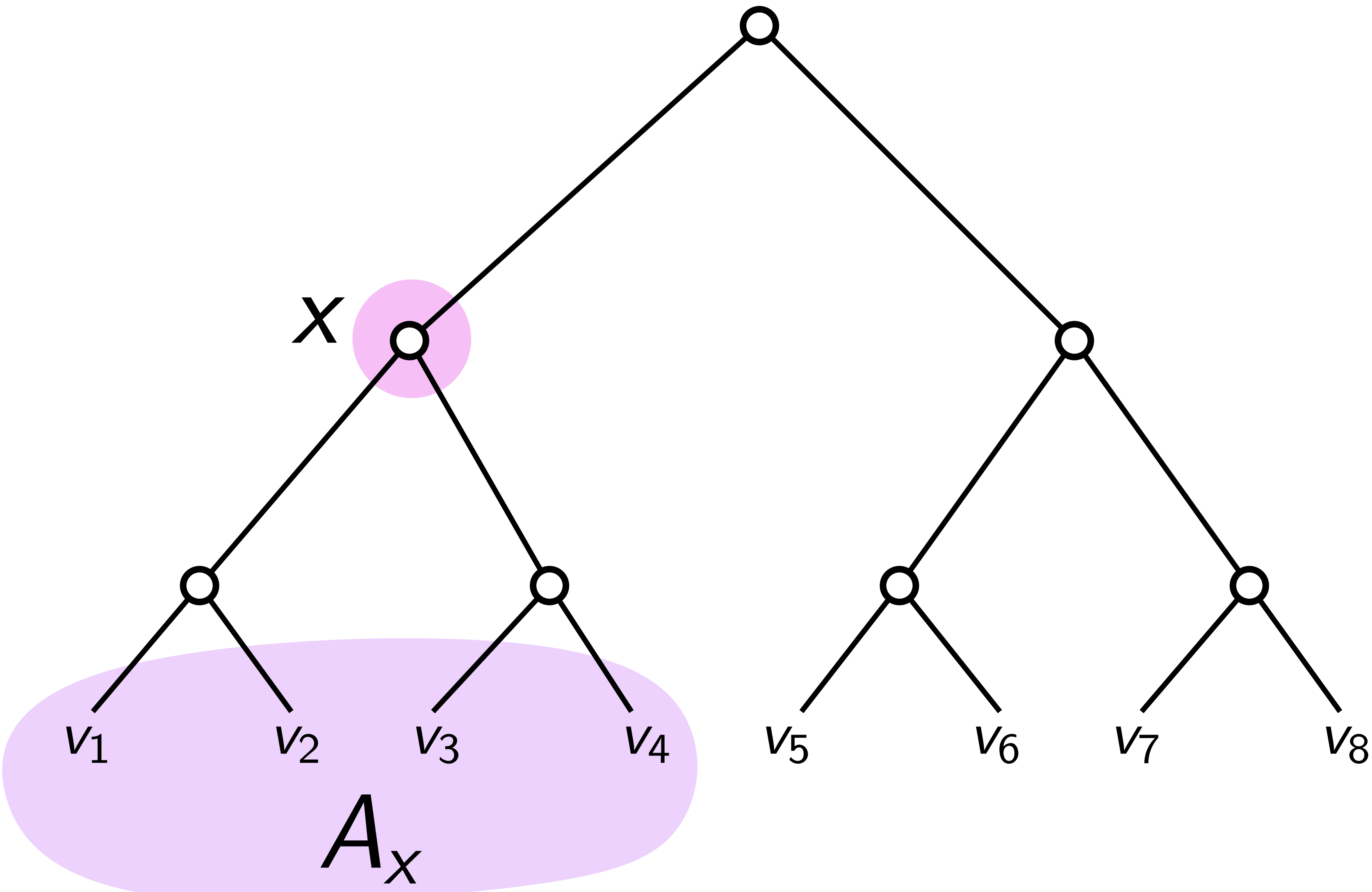


Theorem [Bui-Xuan, Telle and Vatshelle, 2010]
Independent Set can be solved in time $2^{O(\mathrm{rw}^2)} n^{O(1)}$.

Theorem [Bui-Xuan, Telle and Vatshelle, 2013]
There is an algorithm for IS whose runtime is upper bounded by

- $2^{O(\mathrm{tw}^\star)} n^{O(1)}$
- $2^{O(\mathrm{cw}^\star)} n^{O(1)}$
- $2^{O(\mathrm{rw}^2)} n^{O(1)}$
- $n^{O(\mathrm{mim})}$

For every node x of the layout, we compute a **small** set of partial solutions = independent sets included in A_x



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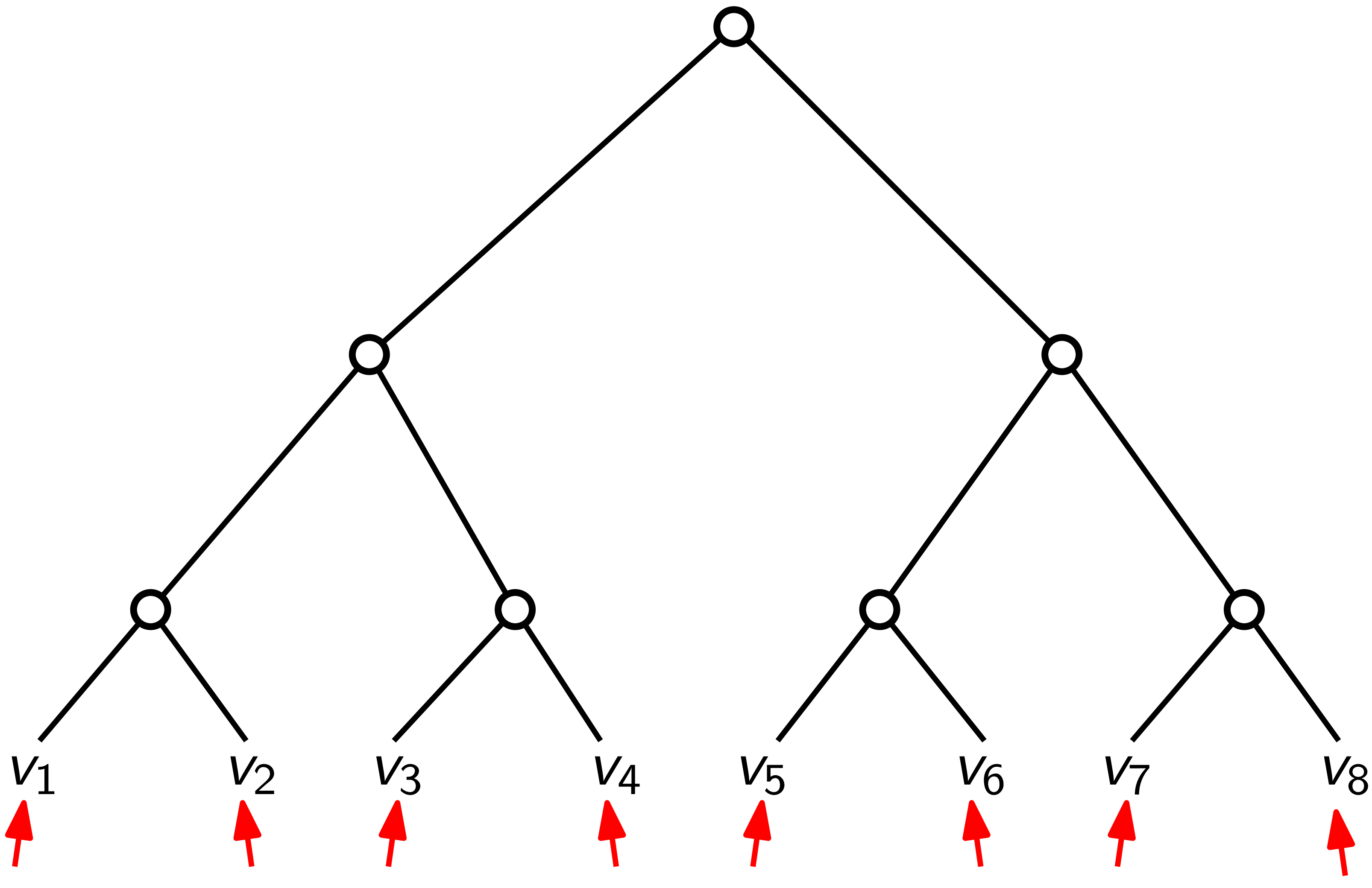

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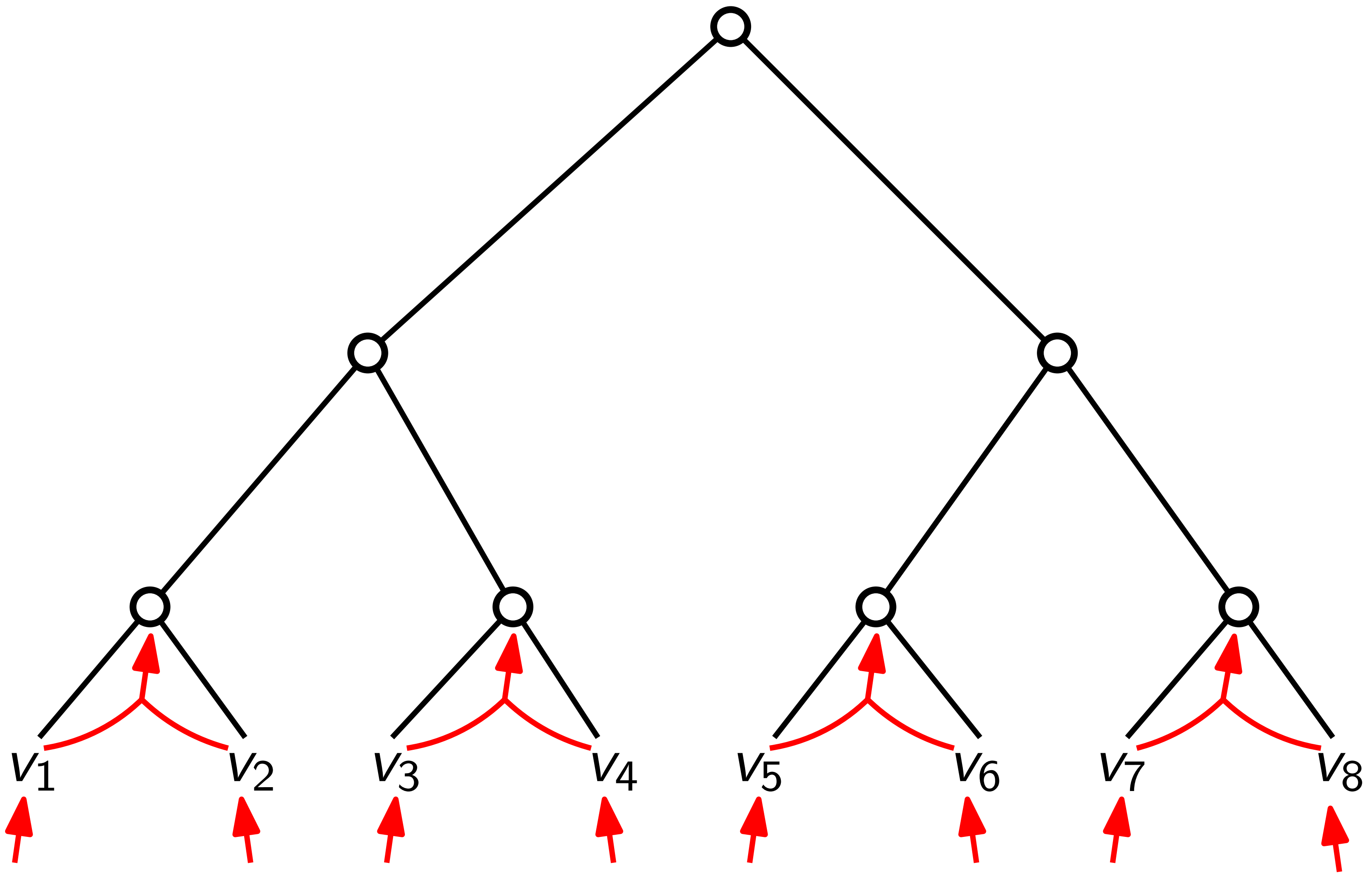

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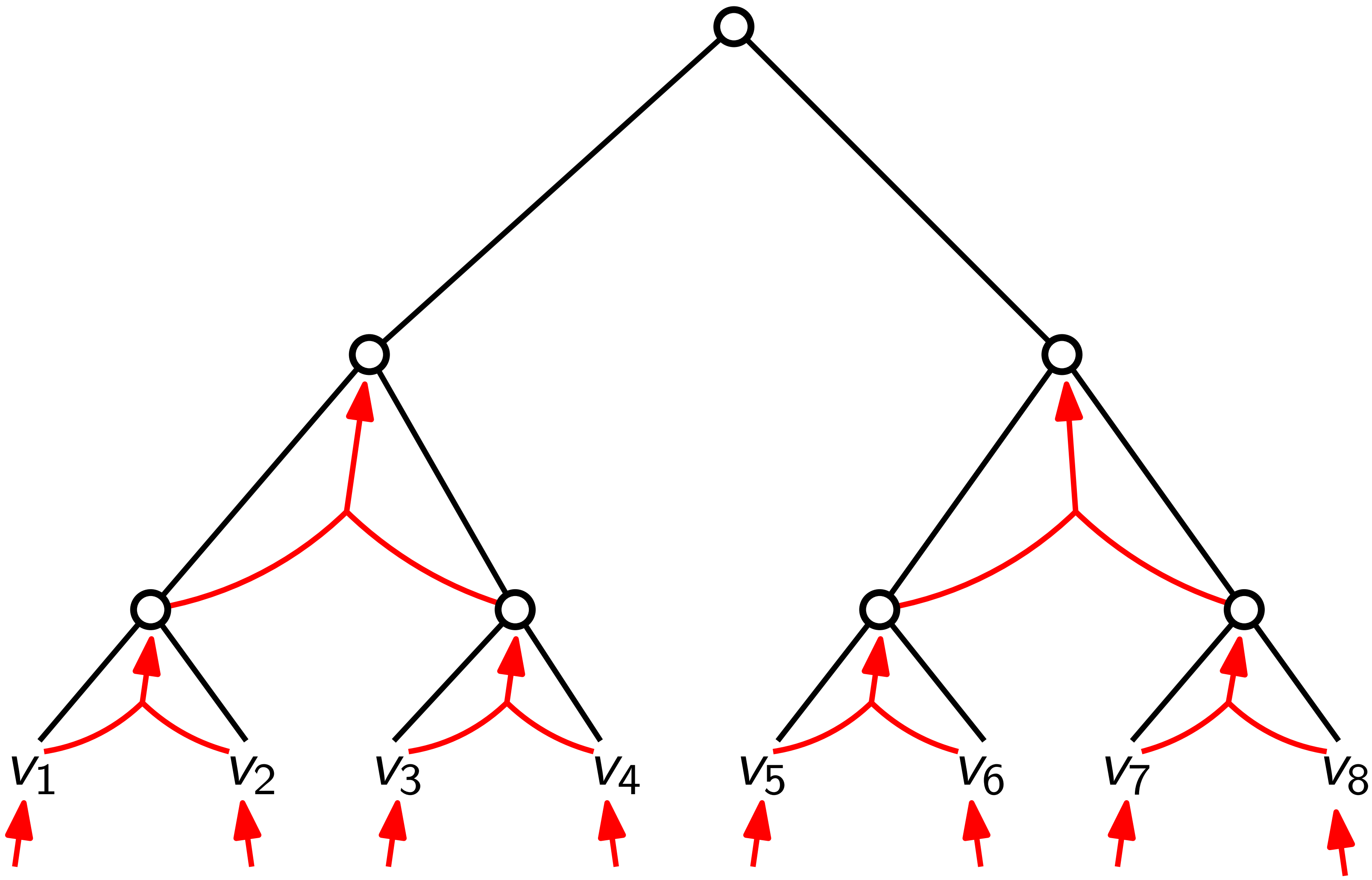

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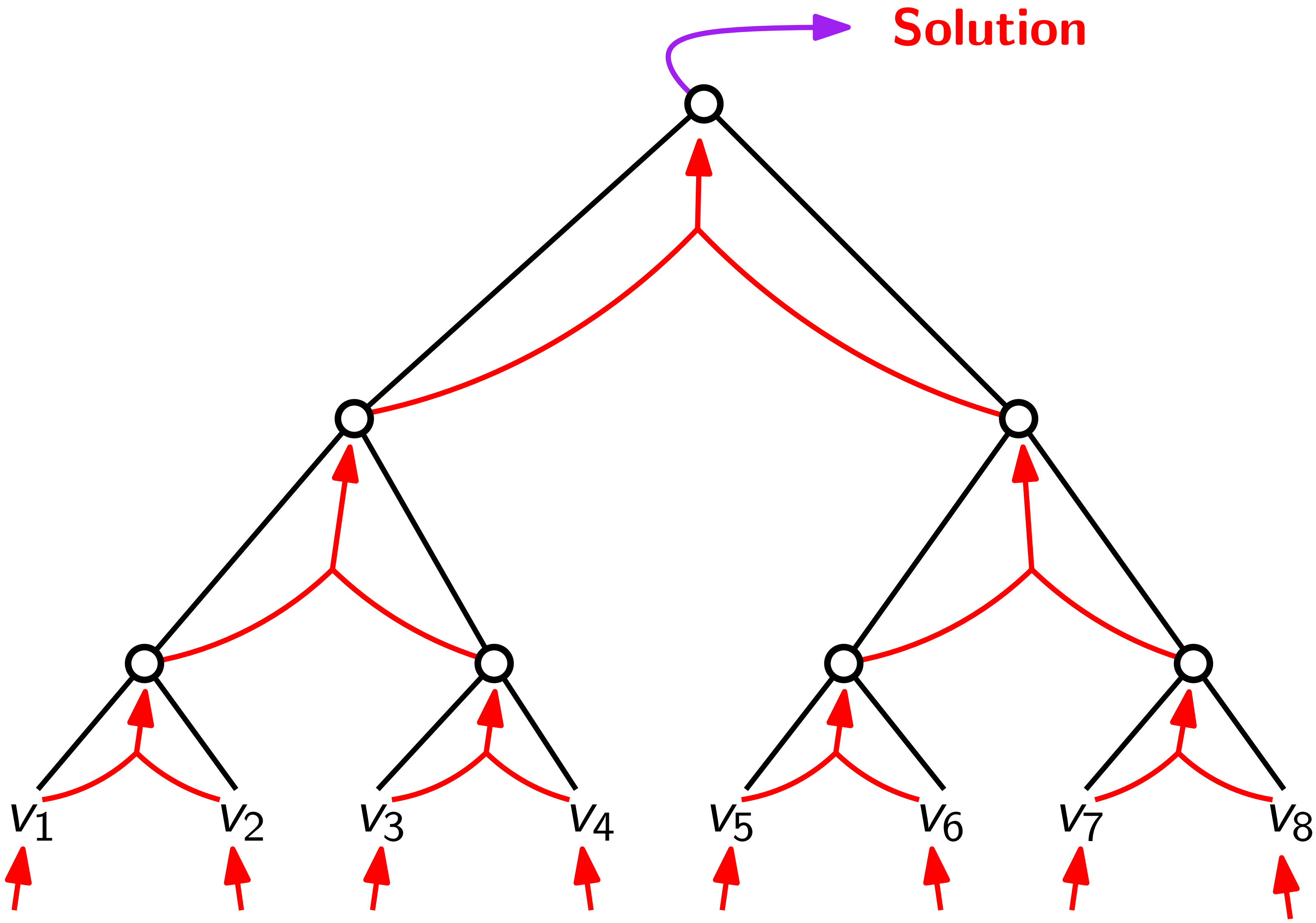
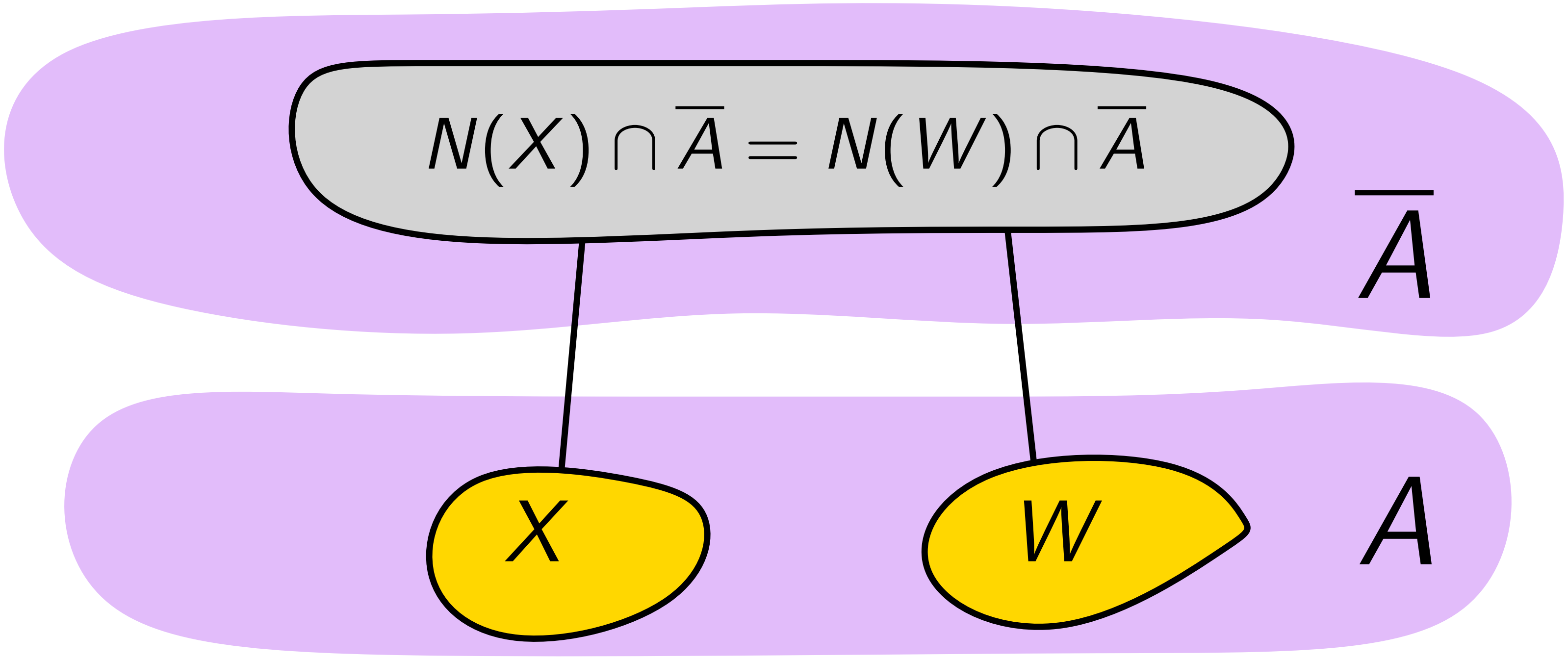


1-Neighbor Equivalence

Two partial solutions $X, W \subseteq A$ are **equivalent** if



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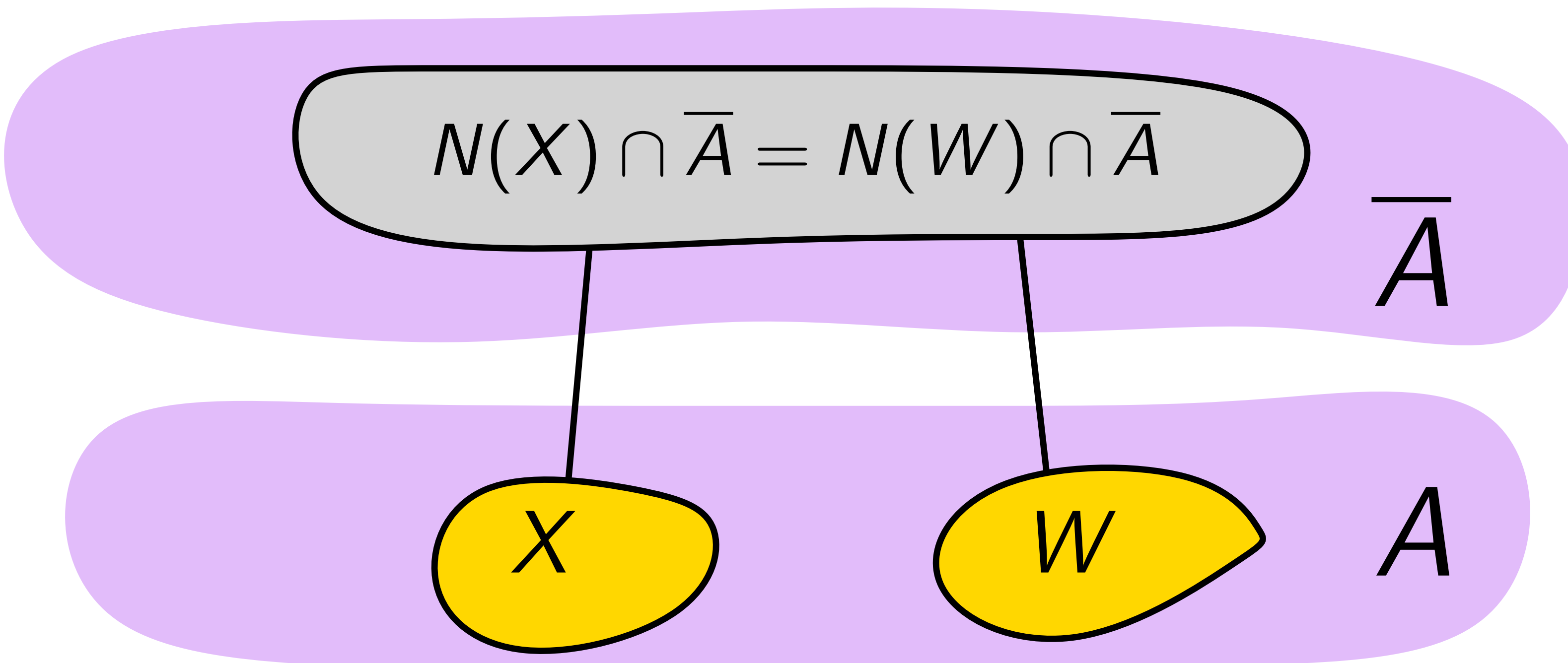

For every pair X, W of **equivalent** partial solutions and $Y \subseteq \overline{A}$

$$X \cup Y \text{ is a solution} \iff W \cup Y \text{ is a solution}$$

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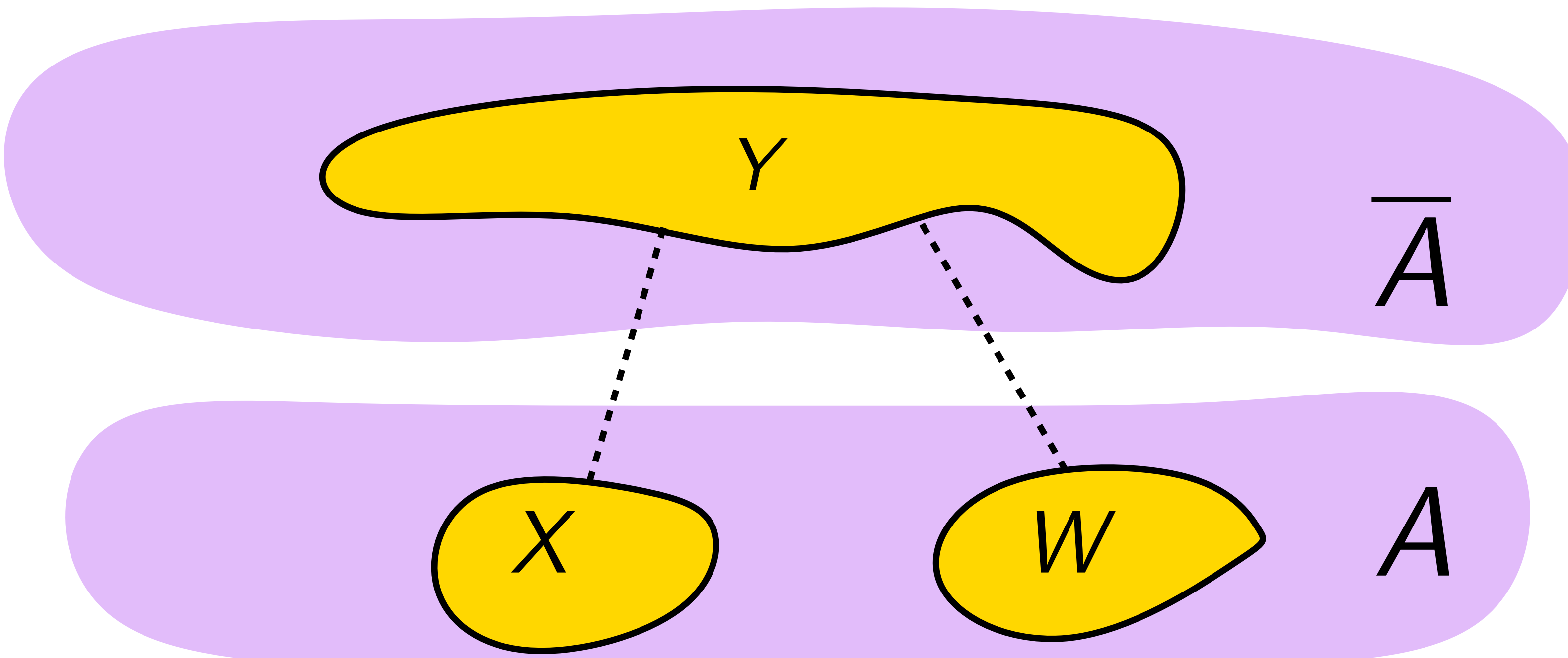

For every pair X, W of **equivalent** partial solutions and $Y \subseteq \overline{A}$

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For every node x and each **equivalence class** $\mathcal{C}$ over A_x
we compute one independent set $X \in \mathcal{C}$ of **maximum size**.

Theorem [Vatshelle, 2013]

The nb. of **eq. classes** $|N(X) \cap \overline{A} \mid X \subseteq A\}|$ is at most

- $2^{\mathrm{tw}^\star(A)}$
- $2^{\mathrm{cw}^\star(A)}$
- $2^{\mathrm{rw}(A)^2}$
- $n^{\mathrm{mim}(A)}$

For every node x and each **equivalence class** $\mathcal{C}$ over A_x
we compute one independent set $X \in \mathcal{C}$ of **maximum size**.

Theorem [Bui-Xuan, Telle and Vatshelle, 2013]
The running time of this algorithm is upper bounded by

- $2^{O(\mathrm{tw})} n^{O(1)}$
- $2^{O(\mathrm{cw})} n^{O(1)}$
- $2^{O(\mathrm{rw}^2)} n^{O(1)}$
- $n^{O(\mathrm{mim})}$

This is tight under ETH for each width parameter!

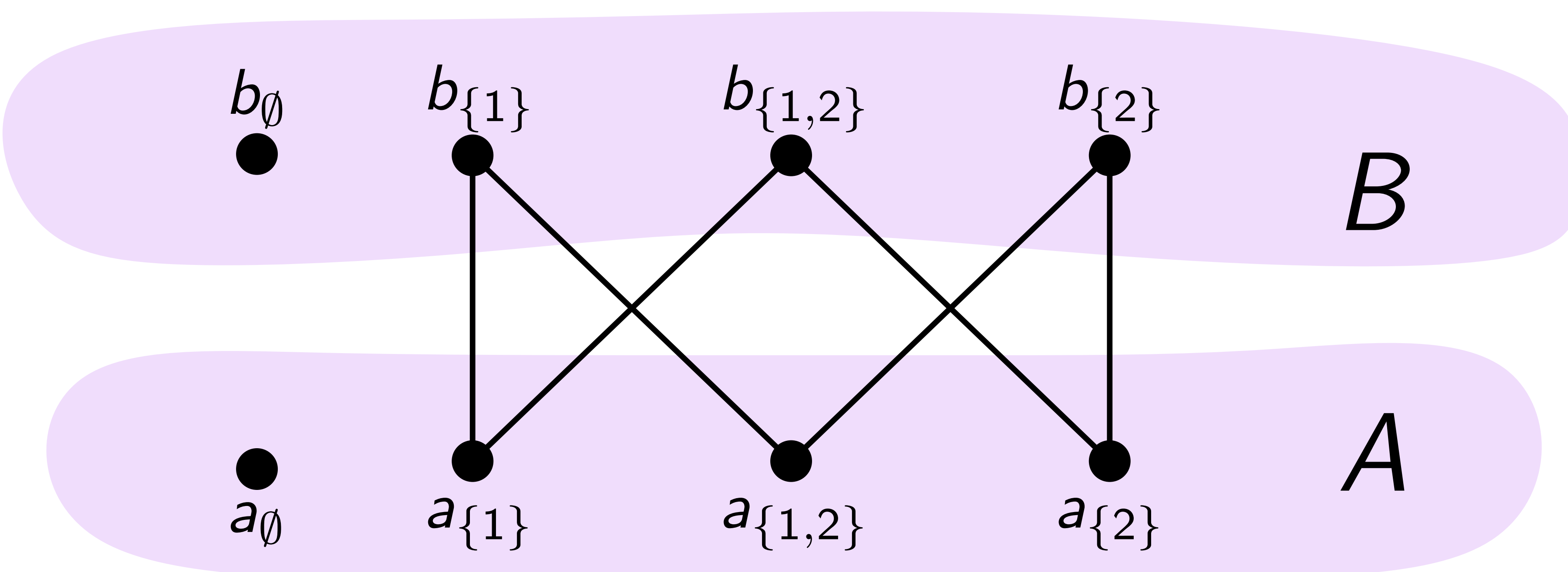
Lower bound

Theorem [B., Korhonen and Nederlof, 2022+]
Under ETH, there are no $2^{o(\mathrm{rw}^2)} n^{O(1)}$ time algorithms for Independent Set.

Universal rank cuts

Universal k-rank cut

- $A := \{a_s \mid s \subseteq [1,k]\}$
- $B := \{b_s \mid s \subseteq [1,k]\}$
- a_s and b_t are adjacent if and only if $|s \cap t|$ is **odd**



Universal rank cuts

The universal k-rank cut has **rank-width k**

Theorem [Bui-Xuan, Telle and Vatshelle, 2010]
Every twin-free cut of rank k is an induced subgraph of the **universal k-rank cut**.

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Theorem [Bui-Xuan, Telle and Vatshelle, 2011]

$$|\{N(X) \cap B \mid X \subseteq A\}| = 2^{\Omega(k^2)}$$

Overview

Reduction from 3-CNF SAT with k^2 **variables**

Lemma

Under ETH, there is no $2^{o(k^2)}(k+m)^{O(1)}$ time algorithm for 3-CNF SAT with k^2 variables

Universal $2k$-rank cut

- $A := \{a_s \mid s \subseteq [1, 2k]\}$
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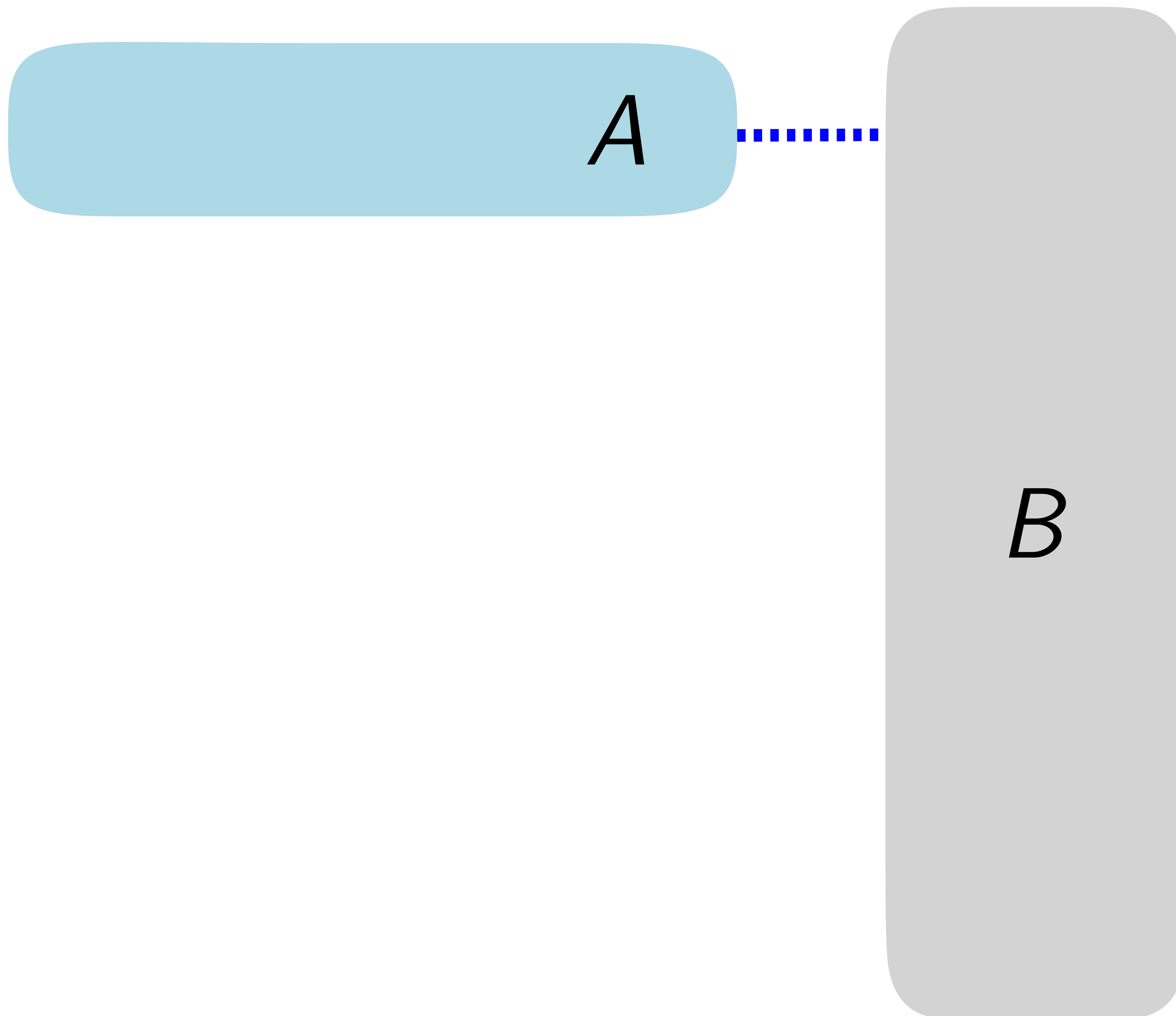
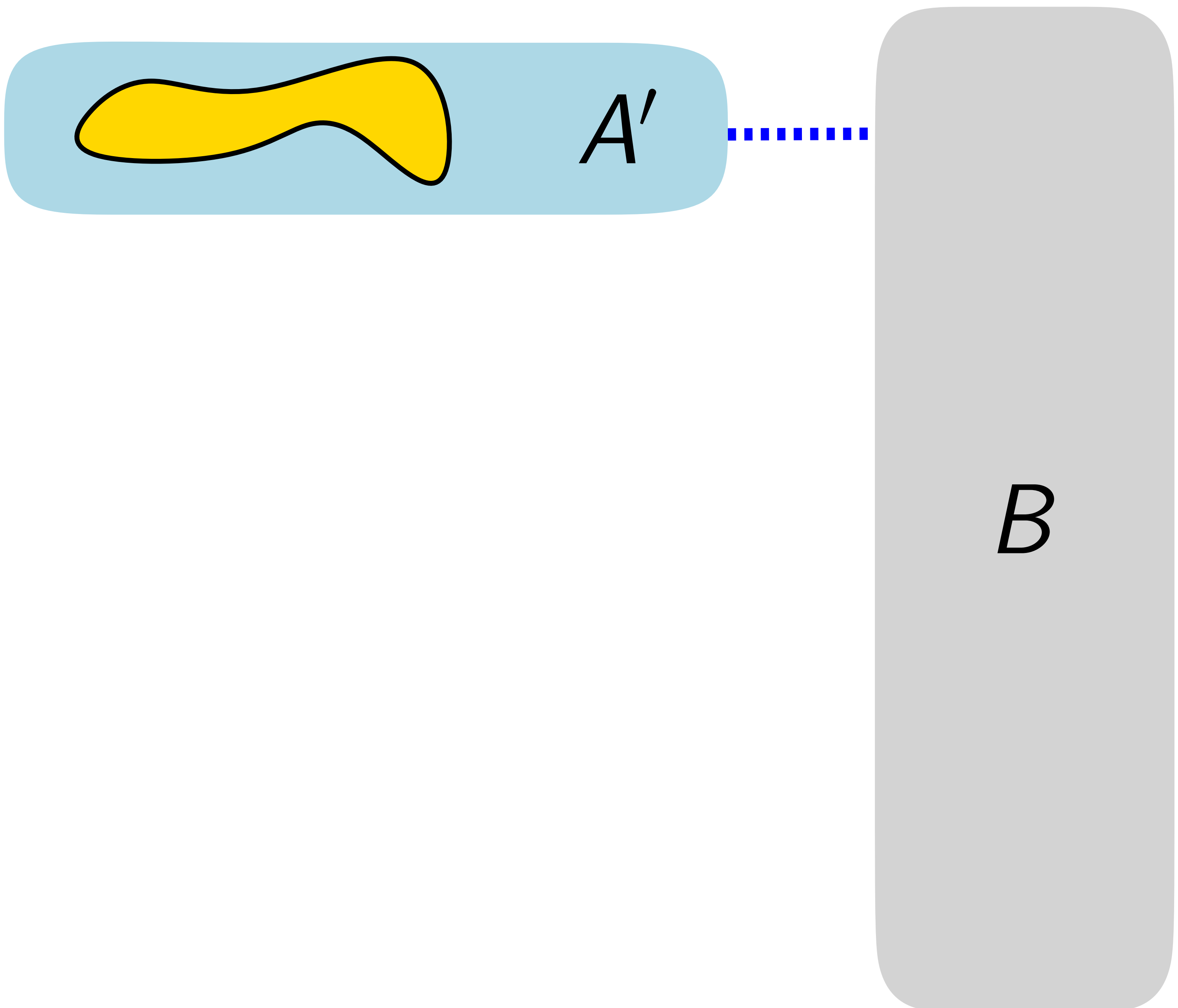
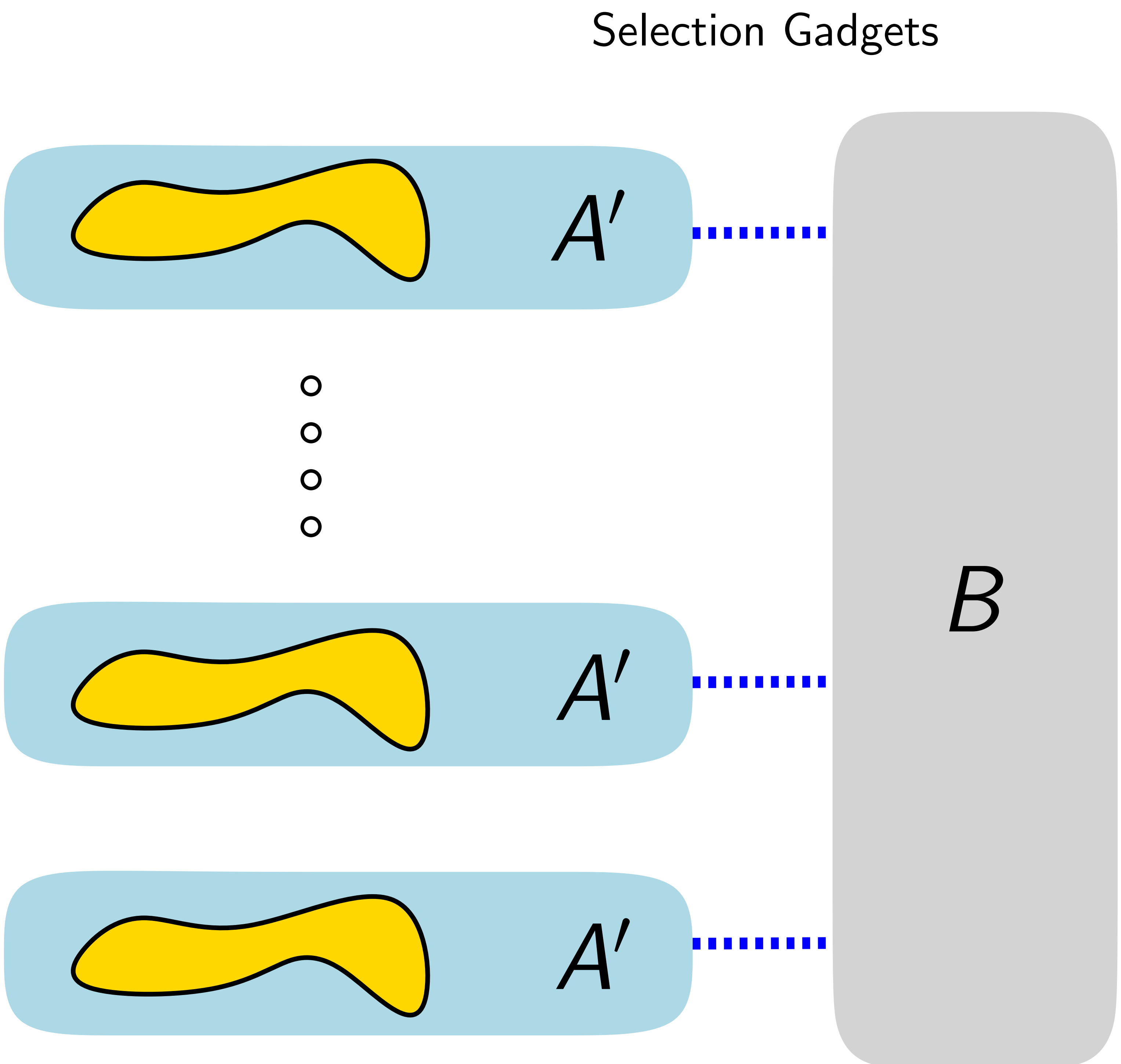
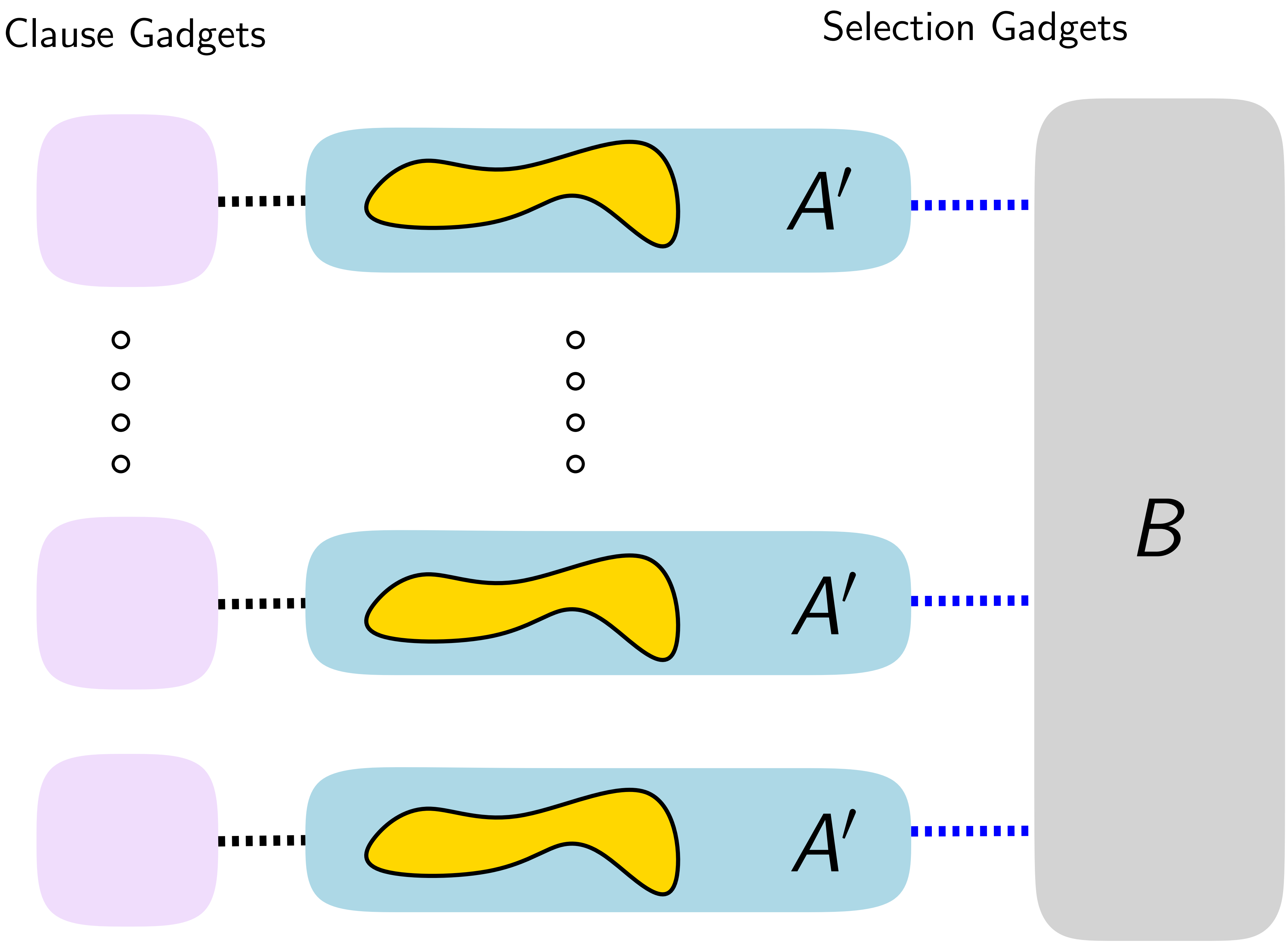

A
B



A'
B



Selection Gadgets
A'
A'
A'
B


Clause Gadgets
Selection Gadgets
A'
A'
A'
B

Selection Gadget

$$\mathrm{var}(\varphi) := \{v_{i,j} \mid i \in [1,k] \wedge j \in [k+1, 2k]\}$$

with $k = 3$

$v_{1,4}$	$v_{1,5}$	$v_{1,6}$
$v_{2,4}$	$v_{2,5}$	$v_{2,6}$
$v_{3,4}$	$v_{3,5}$	$v_{3,6}$

Every **interpretation** $f : \mathrm{var}(\varphi) \rightarrow \{0,1\}$ is associated with $X_f \subseteq A$

$$X_f = \{a_{s_1}, \ldots, a_{s_k}\}$$

$$s_i = \{i\} \cup \{j \in [k+1, 2k] \mid f(v_{i,j}) = 1\}$$

Selection Gadget

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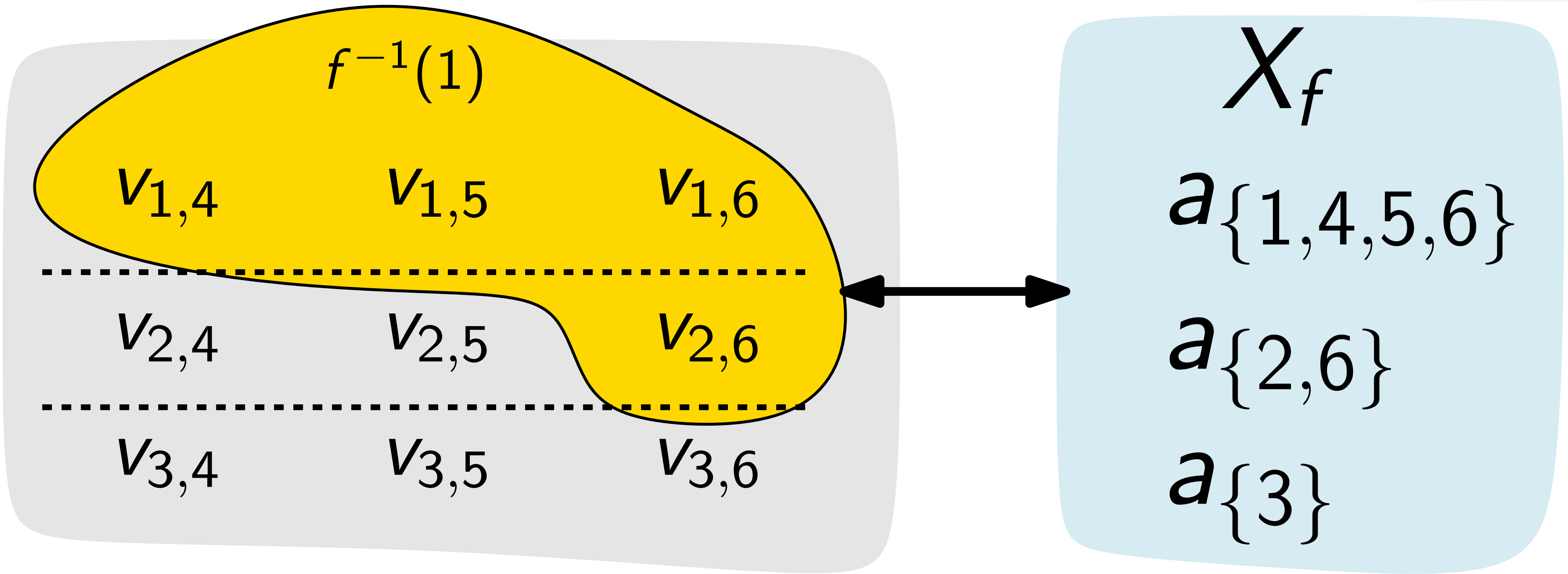


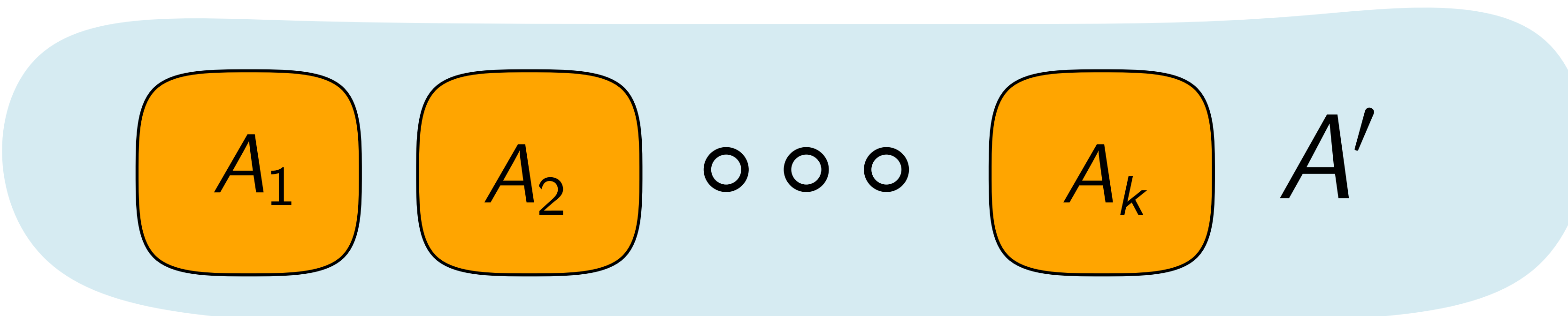
Every **interpretation** $f : \mathrm{var}(\varphi) \to \{0,1\}$ is associated with $X_f \subseteq A$

$$X_f = \{a_{s_1}, \ldots, a_{s_k}\}$$
$$s_i = \{i\} \cup \{j \in [k+1, 2k] \mid f(v_{i,j}) = 1\}$$

Selection Gadget

For every $i \in [1,k]$, let $A_i = \{a_s \in A \mid s \cap [1,k] = \{i\}\}$

and let $A' = A_1 \cup \cdots \cup A_k$.



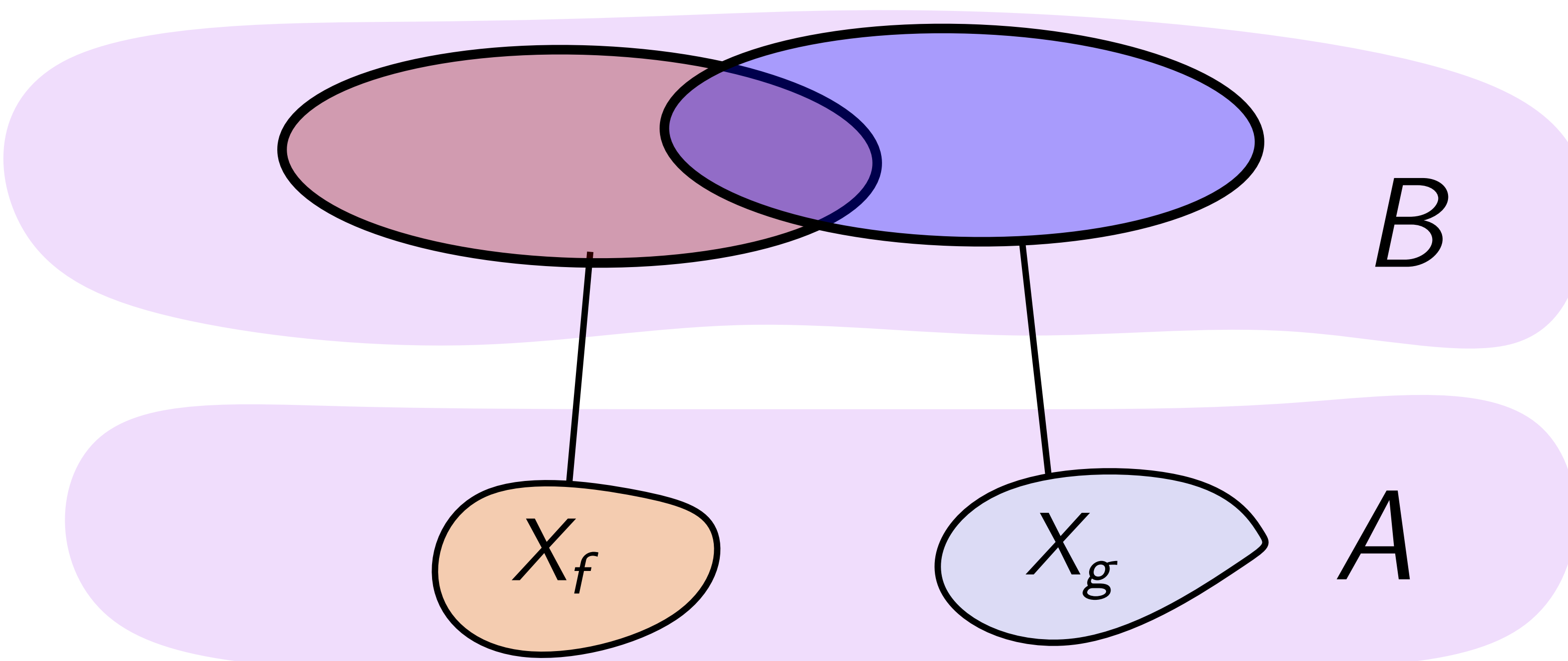
Lemma
Every maximal independent set of $G[A']$ is of the form X_f with f an **interpretation**

Selection Gadget

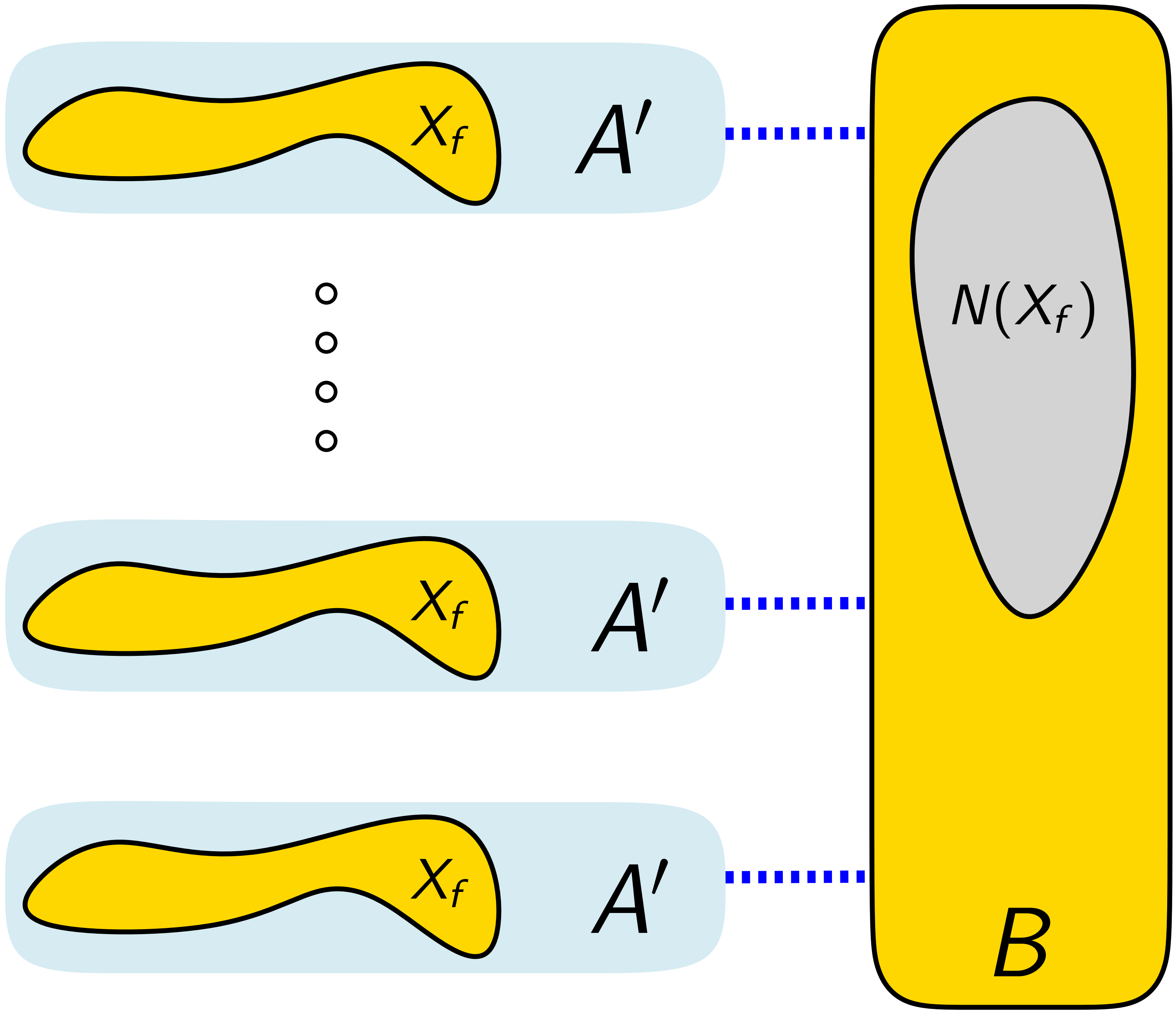
Lemma

For every pair of **distinct interpretations** f, g, the **neighborhoods** of X_f and X_g

- are different
- have the same size ($2^{2k} - 2^k$)



Selection Gadgets



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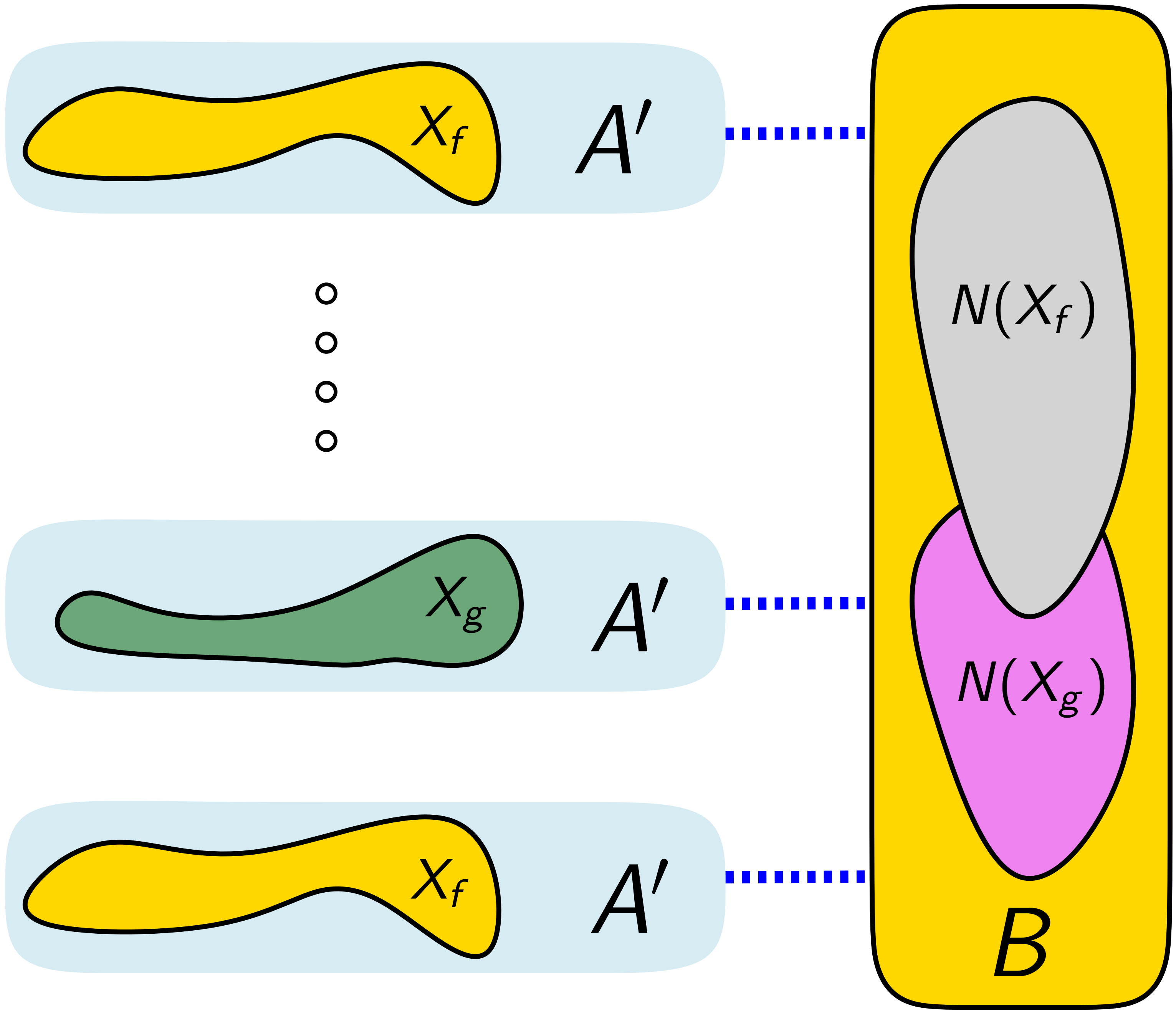
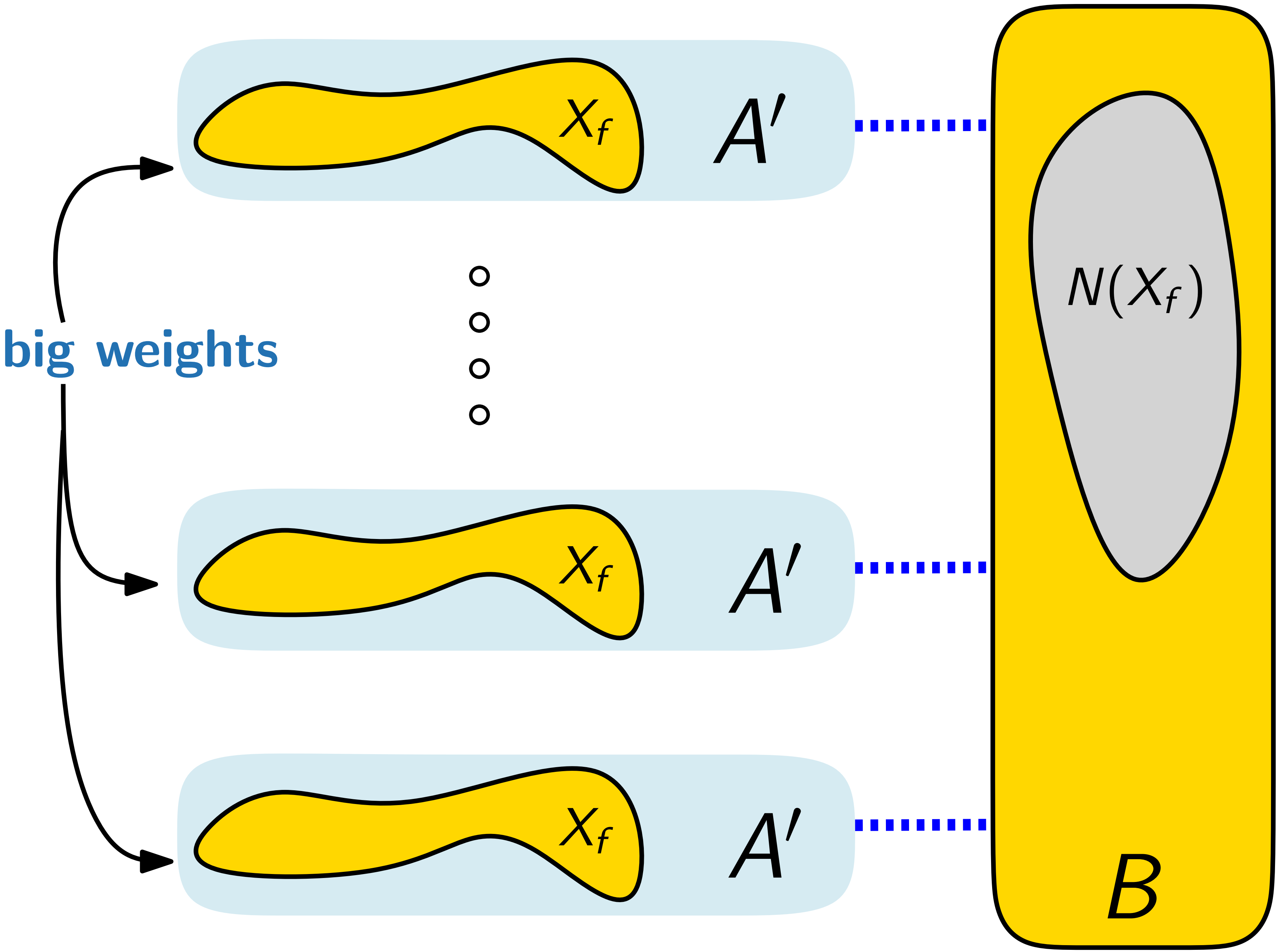
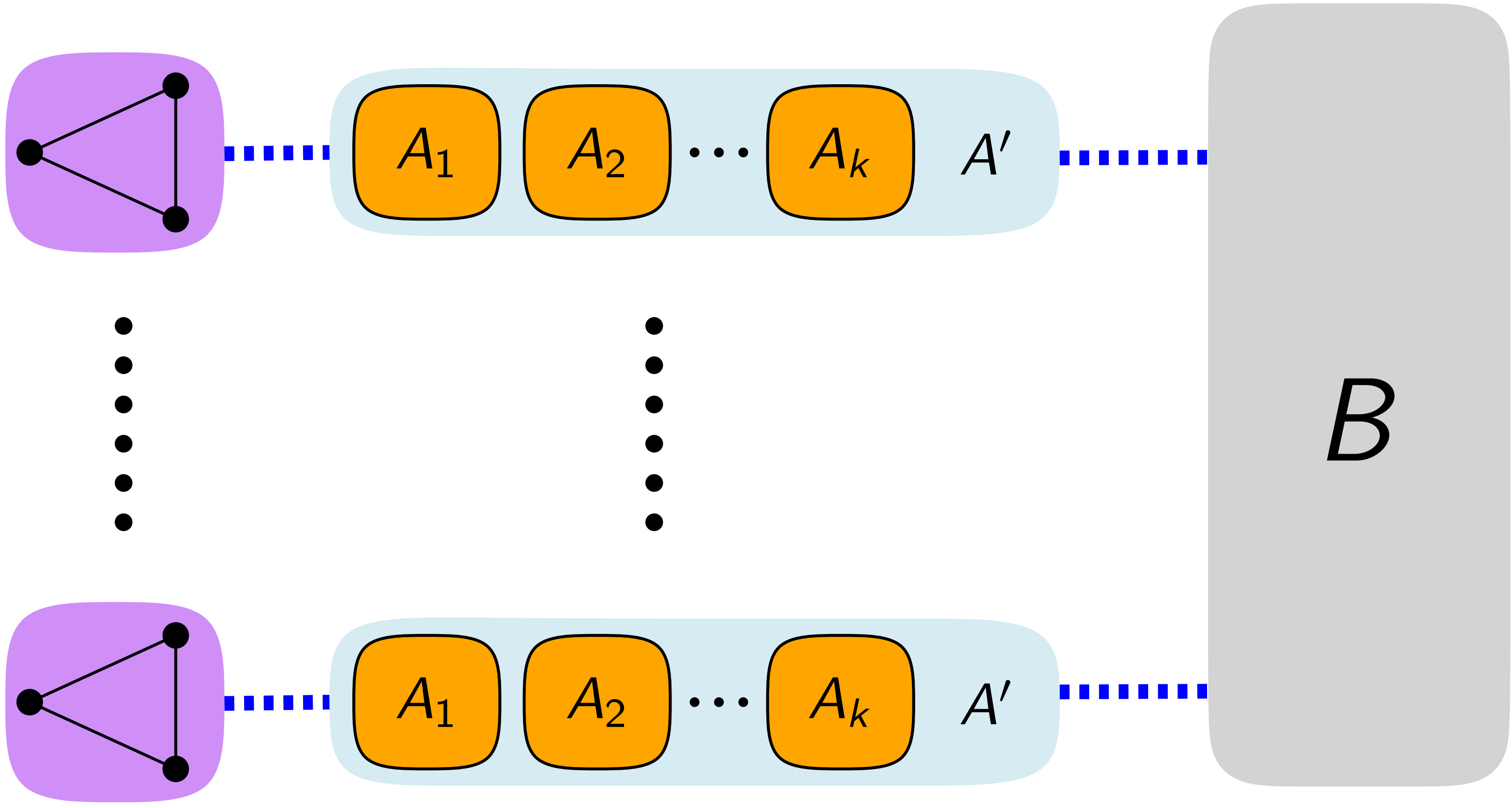


Selection Gadgets



Clause Gadgets

Clause gadget = a **triangle** and some edges with a copy of A'



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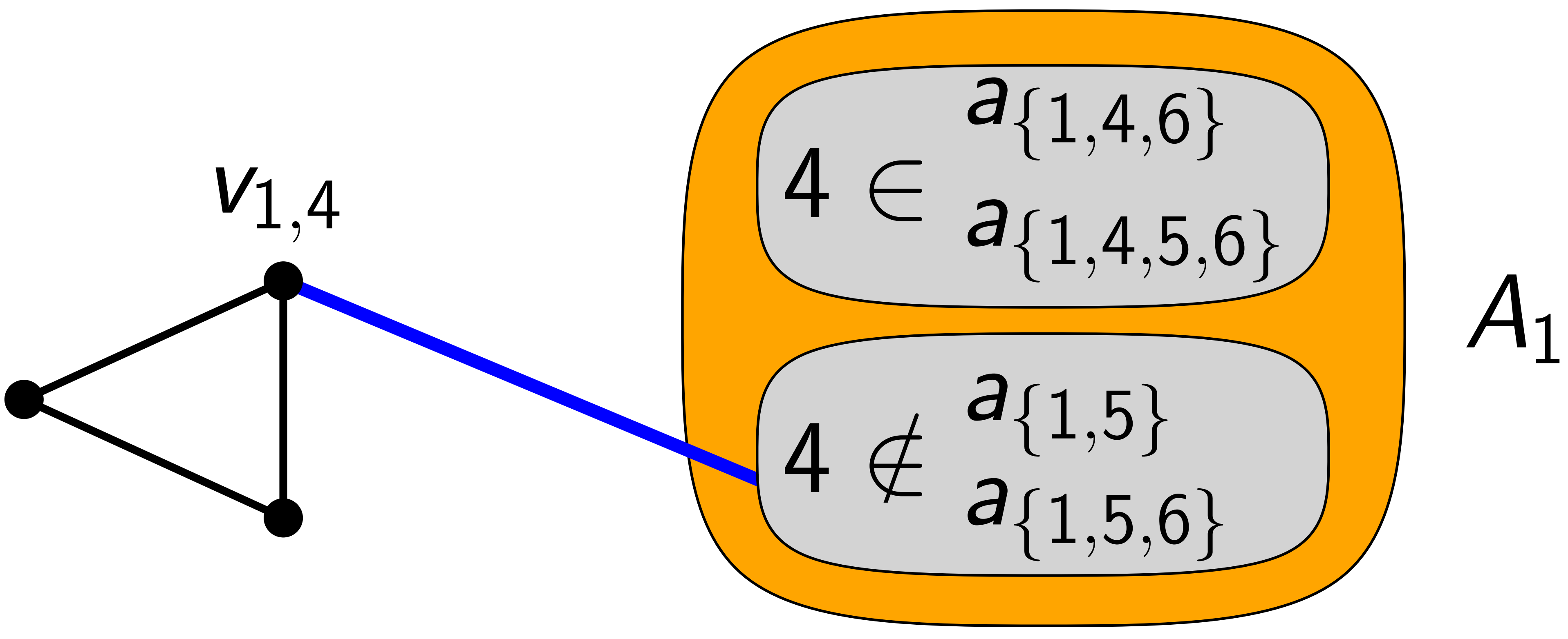

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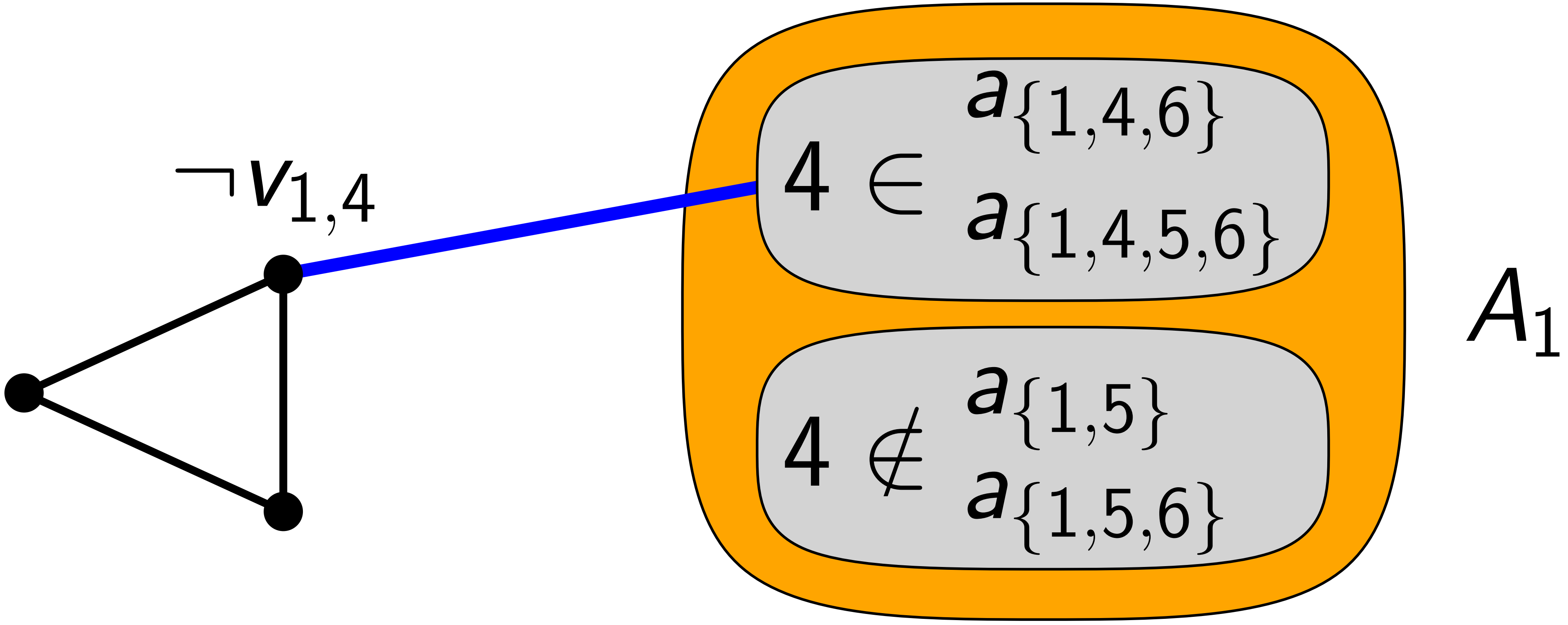


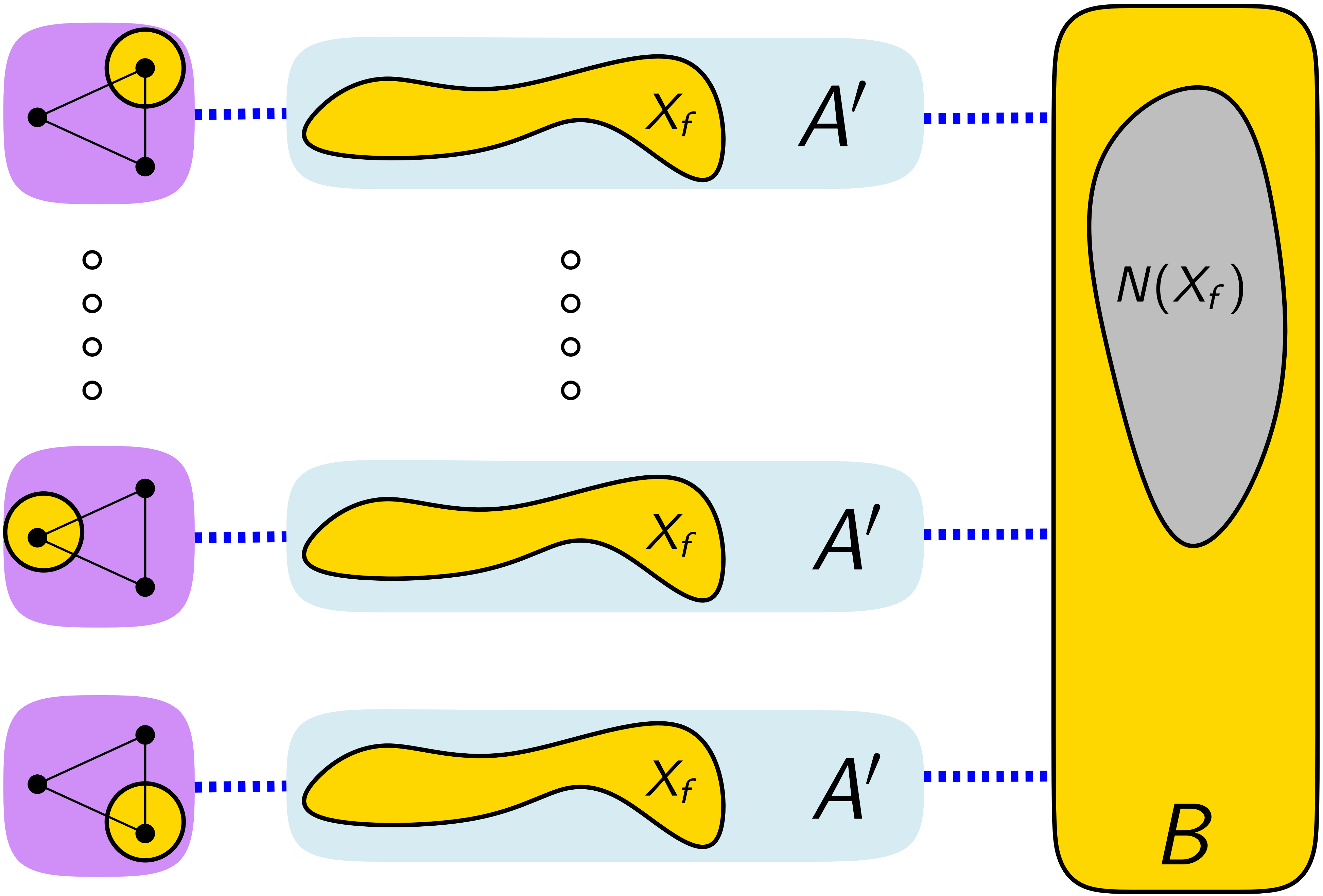
Clause Gadgets

Clause gadget $=$ a **triangle** and some edges with a copy of A'

Lemma

f satisfies C **iff** X_f can be completed with one vertex from the clause gadget

φ is satisfiable **iff** there exists an independent set of weight W



- The **linear rank-width** of this graph is at most $2k + 4$.
- This graph has $2^{O(k)}m$ vertices and can be constructed in $2^{O(k)}m$ time.

A $2^{o(\mathrm{rw}^2)}n^{O(1)}$ time algo. for **IS** $\Longrightarrow$ A $2^{o(k^2)}(n+m)^{O(1)}$ time algo. for **3-CNF-SAT** $\Longrightarrow$ **ETH** is false

Theorem [B., Korhonen and Nederlof, 2022+]
Under ETH, there are no $2^{o(\mathrm{rw}^2)}n^{O(1)}$ time algorithms for **Independent Set**

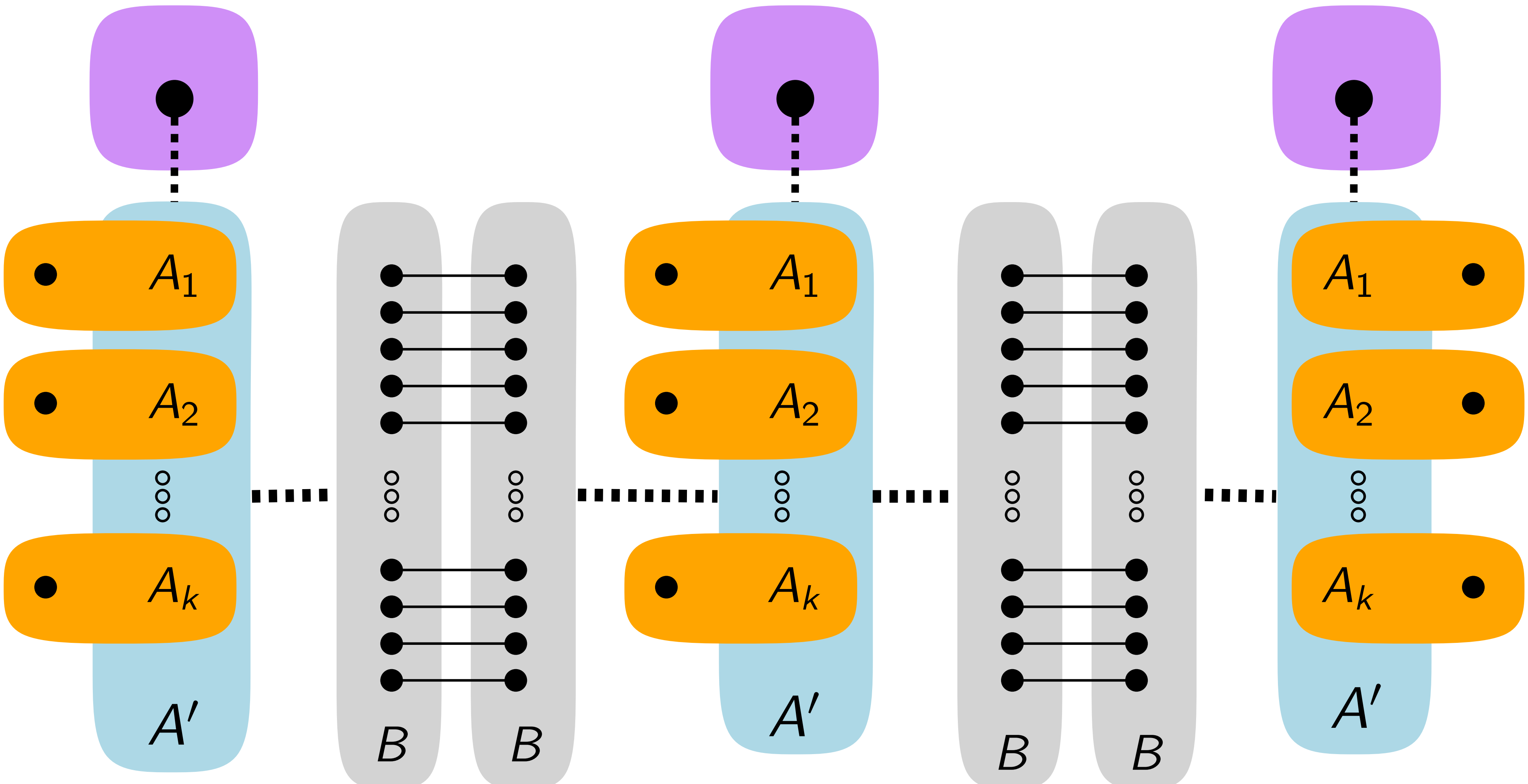
Other problems

Given a graph G, we can construct G' such that $\mathrm{rw}(G') \leqslant \mathrm{rw}(G) + 1$ and the following are equivalent:

- G has an **independent set** of size k
- G' has an **induced matching** of size k
- G' has an **induced forest** of size $2k$

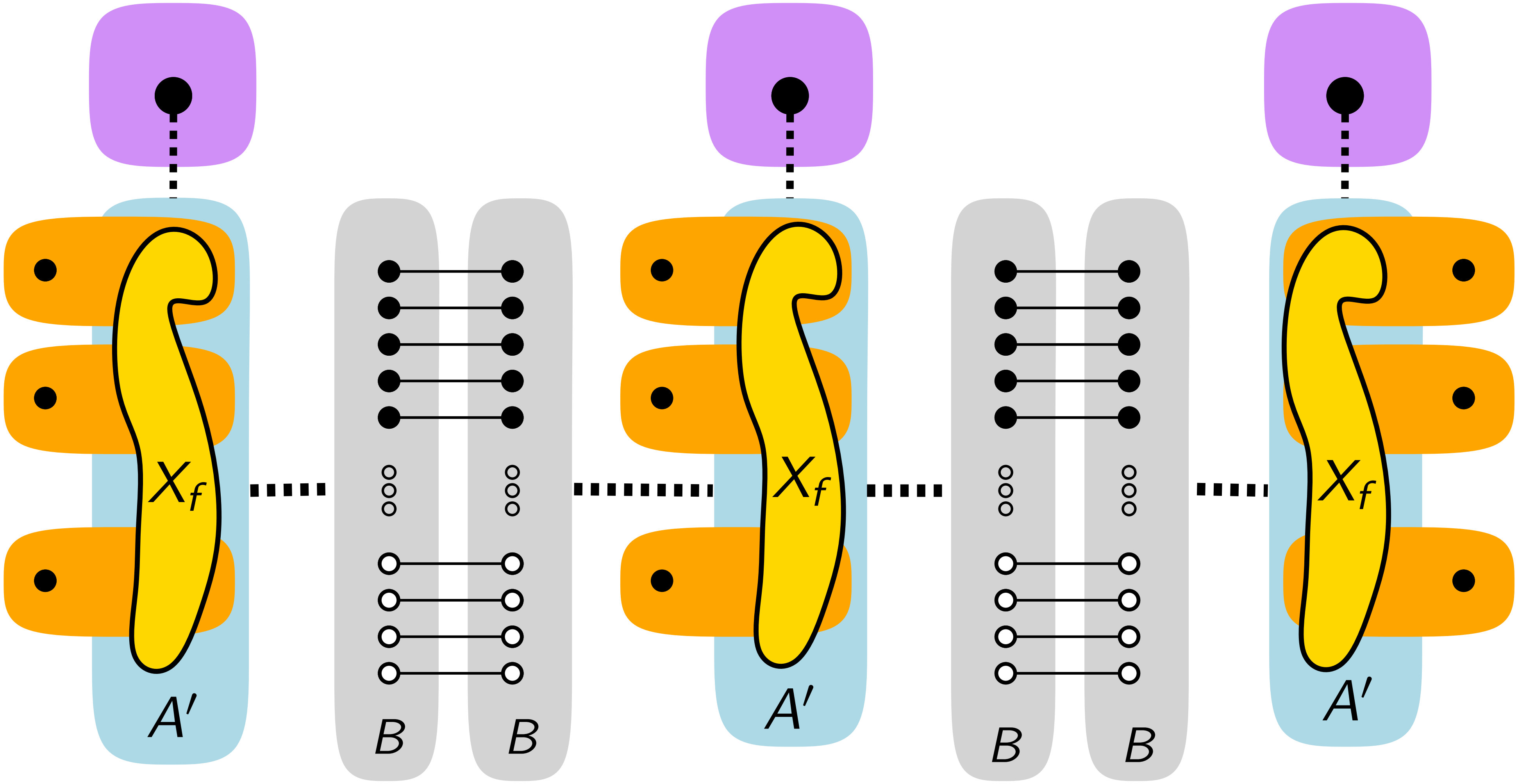
Theorem [B., Korhonen and Nederlof, 2022+]
Under ETH, there are no $2^{o(\mathrm{rw}^2)} n^{O(1)}$ time algorithms for **Max. Induced Matching** and **FVS**.

Weighted Dominating Set



Theorem [B., Korhonen and Nederlof, 2022+]
Under ETH, there are no $2^{o(\mathrm{rw}^2)} n^{O(1)}$ time algorithms for **Weighted** Dominating Set.

Weighted Dominating Set



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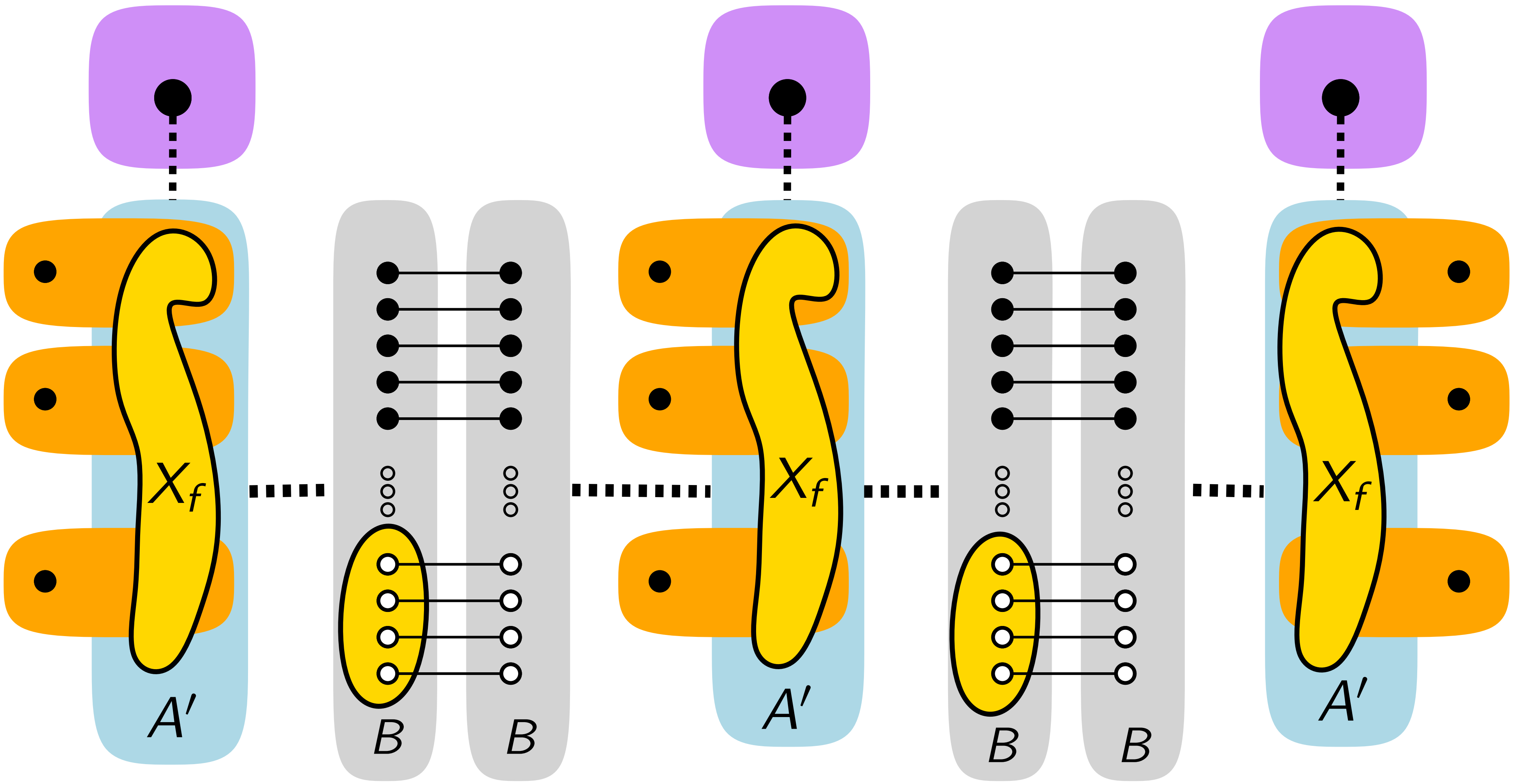
Weighted Dominating Set



Theorem [B., Korhonen and Nederlof, 2022+]
Under ETH, there are no $2^{o(\mathrm{rw}^2)} n^{O(1)}$ time algorithms for **Weighted** Dominating Set.

Boolean-width [Bui-Xuan, Telle, and Vatshelle, 2011]
Defined from $\text{boolw}(A) := \log_2 |\{N(X) \cap \overline{A} \mid X \subseteq A\}|$.

Equivalent to clique-width and rank-width!

Theorem [Bui-Xuan, Telle, and Vatshelle, 2011]
For all graph G, we have $\log_2 \text{rw} \leq \text{boolw} \leq O(\text{rw}^2)$.

Theorem [Bui-Xuan, Telle and Vatshelle, 2011]
For the Universal k-rank cut, we have

$$\log_2 |\{N(X) \cap B \mid X \subseteq A\}| = \Omega(k^2)$$

Boolean-width [Bui-Xuan, Telle, and Vatshelle, 2011]
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For all graph G, we have $\log_2 \text{rw} \leq \text{boolw} \leq O(\text{rw}^2)$.

Theorem [B., Korhonen and Nederlof, 2022+]
There are graphs with rank-width k and boolean-width $\Omega(k^2)$ for arbitrary large k.

Conclusion

Theorem [B., Korhonen and Nederlof, 2022+]
Under ETH, there are no $2^{o(\mathrm{rw}^2)} n^{O(1)}$ time algorithms for

- Independent Set
- **Weighted** Dominating Set
- Maximum Induced Matching
- Feedback Vertex Set

First non-trivial ETH lower bounds for **rank-width**

Using $|\{N(X) \cap \overline{A} \mid X \subseteq A\}|$ leads to optimal algorithms for for several **problems** in terms of many **width parameters**: tree-width, clique-width, rank-width and mim-width!

Results on Independent Set

Best known:	Upper bound	ETH lower bound
tree-width clique-width	$2^{O(k)} n^{O(1)}$ [Folklore]	$2^{o(k)} n^{O(1)}$ [Folklore]
rank-width	$2^{O(k^2)} n^{O(1)}$ [Bui-Xuan et al., 2012]	$2^{o(k^2)} n^{O(1)}$ [Us, 2023]
mim-width	$n^{O(k)}$ [Bui-Xuan et al., 2013]	$n^{o(k/\log k)}$ [Bakkane and Jaffke, 2022+]

Results on Independent Set

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mim-width	$n^{O(k)}$ [Bui-Xuan et al., 2013]	$n^{o(k)}$ [Me, 2023+]

Open questions

What about:

- **Unweighted** Dominating Set? ($2^{O(\mathrm{rw}^2)} n^{O(1)}$)
- q-**Coloring**? ($2^{O(q\mathrm{rw}^2)} n^{O(1)}$)
- Chromatic Number? ($n^{2^{O(\mathrm{rw}^2)}}$)

Thank you!