Notes on Gutierrez and Gardner

Monge-Ampére Measures and the Brunn-Minkowski Theory

0.1. Constructing the Monge-Ampére Measure

Unless otherwise stated, we will denote by Ω a generic open subset of $\mathbb{R}^n$.

DEFINITION 1. Let $u \in C(\Omega)$. If $x \in \Omega$, we define the *sub-gradient* $\partial u(x)$ by

$$\partial u(x) = \left\{ p \in \mathbb{R}^n : u(t) \geq p \cdot (t-x) + u(x) \ \forall t \in \Omega \right\}.$$

Similarly, if $E \subset \Omega$ is any set, then

$$\partial u(E) = \bigcup_{x \in E} \partial u(x).$$

Usually u is at least a continuous function, and in many of the interesting examples, u is also convex. It can be shown that when u is convex, $\partial u(x) \neq \emptyset$ for all $x \in \Omega$.

THEOREM 2. *Suppose that $u \in C(\Omega)$ and that $K \subset \Omega$ is compact. Then $\partial u(K)$ is compact.*

PROOF. First, we show that $\partial u(K)$ is closed. Pick a sequence $\{p_k\}$ in $\partial u(K)$ which converges to some $p \in \mathbb{R}^n$. To each p_k, there exists $x_k \in K$ such that $u(t) \geq p_k \cdot (t - x_k) + u(x_k)$ for all $t \in \Omega$. Fix a convergence subsequence $\{x_{k_n}\}$ of the $\{x_k\}$, and let $x \in K$ denote the limit of this subsequence. Then the continuity of u guarantees that

$$u(t) \geq \lim_{n \to \infty} p_{k_n} \cdot (t - x_{k_n}) + u(x_{k_n}) = p \cdot (t-x) + u(x)$$

for all $t \in \Omega$, which means that $p \in \partial u(x) \subset \partial u(K)$.

Next, we show that $\partial u(K)$ is bounded. Fix a para-compact open set U with $K \subset U \subset \overline{U} \subset \Omega$. Then the compactness of K and $\overline{U}$ provides a $\delta > 0$ sufficiently small that $x + \delta v \in \overline{U}$ whenever $|v| = 1$ and $x \in K$. If $p \in \partial u(K)$ and $p \neq 0$, then fix an $x \in K$ with $p \in \partial u(x)$, and put $v = p/|p|$. Then we have

$$u(x + \delta v) \geq p \cdot (\delta v) + u(x) = \delta |p| + u(x).$$

Since u is bounded on $\overline{U}$ and on K, there is m and M such that $u(x) \geq m$ for all $x \in K$ and $u(x) \leq M$ for all $x \in \overline{U}$, and it follows

$$|p| \leq \frac{M - m}{\delta}.$$

This proves that $\partial u(K)$ is bounded, and we conclude $\partial u(K)$ is compact by the Heine-Borel theorem. □

COROLLARY 3. *If u is a convex function on Ω, then u is locally Lipschitz. That is, for every compact $K \subset \Omega$, there exists a constant C such that $|u(x) - u(y)| \leq C|x - y|$ for all $x, y \in K$.*

PROOF. Since $\partial u(K)$ is compact, there is $C > 0$ such that $|p| \leq C$ for all $p \in \partial u(K)$. Then for any $x, y \in K$, if $p_x \in \partial u(x)$ and $p_y \in \partial u(y)$,

$$u(x) - u(y) \geq p_y \cdot (x - y) \geq -M|x - y|, \quad u(y) - u(x) \geq p_x \cdot (y - x) \geq -M|x - y|$$

by an application of the Cauchy-Schwarz inequality. This is equivalent to the assertion of the Corollary. □

A useful tool which appears often in physics is called the *Legendre transform*

DEFINITION 4 (LEGENDRE TRANSFORM). The Legendre transform of a function $u : \Omega \to \mathbb{R}$ is the function $u^* : \mathbb{R}^n \to \mathbb{R}$ defined by

$$u^*(p) = \sup_{x \in \Omega} (x \cdot p - u(x)).$$

The function u^* acts as a sort of inverse function to the map $t \mapsto \partial u(t)$, a statement which we will make more precise in the next theorem.

LEMMA 5. *If $p \in \partial u(x)$ for some $x \in \Omega$, then $u^*(p) = x \cdot p - u(x)$, and $x \in \partial u^*(p)$.*

PROOF. By definition of the sub-gradient, we have $t \cdot p - u(t) \leq x \cdot p - u(x)$ for all $t \in \Omega$, and equality is achieved if $t = x$, which proves that $u^*(p) = x \cdot p - u(x)$. To see that $x \in \partial u^*(p)$, simply observe that for any $t \in \mathbb{R}^n$,

$$u^*(t) \geq x \cdot t - u(x) = x \cdot (t - p) + x \cdot p - u(x) = x \cdot (t - p) + u^*(p).$$

□

Given the Legendre transform of a function, it is sometimes possible to recover the original function.

THEOREM 6 (FENCHEL'S DUALITY THEOREM). *Suppose that $u \in C(\Omega)$ is a convex function. Then $u^{**} = u$.*

PROOF. By definition of u^*, it holds $p \cdot x \leq u(x) + u^*(p)$ for all $x \in \mathbb{R}^n$. Therefore, for a fixed x, $u^{**}(x) \leq u(x)$. To obtain the reverse inequality, let ℓ_x be a tangent line to u at x. It is easy to check that $\ell_x^{**} = \ell_x$, and since $\ell_x \leq u$ it holds $\ell_x^* \geq u^*$ and therefore $\ell_x(x) = \ell_x^{**}(x) \leq u^{**}(x)$. Since $\ell_x(x) = u(x)$, we are done. □

The first demonstration of the utility of the Legendre transform is the following technical lemma which we will use many times.

LEMMA 7. *If Ω is open and $u \in C(\Omega)$, then the set of points in $\mathbb{R}^n$ which belong to multiple subgradients of u is a Lebesgue null set. Formally, the set S defined by*

$$S = \left\{p \in \mathbb{R}^n : \exists x, y \in \Omega \text{ such that } p \in \partial u(x) \cap \partial u(y)\right\}$$

has $m(S) = 0$. In particular, if A and B are any disjoint subsets of Ω, then $m(\partial u(A) \cap \partial u(B)) = 0$.

PROOF. Fix a sequence $\{\Omega_k\}$ of para-compact open sets with $\Omega_k \subset \overline{\Omega}_k \subset \Omega$ and $\Omega = \bigcup_1^\infty \Omega_k$, and let

$$S_k = \left\{p \in \mathbb{R}^n : \exists x, y \in \Omega_k \text{ such that } p \in \partial u(x) \cap \partial u(y)\right\}.$$

Since $S \subset \bigcup_1^\infty S_k$, it is enough to show that $m(S_k) = 0$ for each k. Towards that end,fix k, and let u^* is the Legendre transform of $u|_{\Omega_k}$. Since each Ω_k is para-compact and u is continuous, we have $u|_{\Omega_k}$ is bounded hence $|u^*| < \infty$. Let $E = \{p : u^* \text{ is not differentiable at } p\}$. We claim that $S_k \subset E$. Since u^* is easily seen to be convex, we then obtain $m(S_k) = m(E) = 0$, as desired.

Towards this end, if $p \in S_k$, then there exist distinct $x, y \in \Omega_k$ with $p \in \partial u(x) \cap \partial u(y)$. By Lemma 5, it follows that $x, y \in \partial u^*(p)$. But, if u^* were differentiable at p, then $\partial u^*(p)$ would consist of a single element, therefore u^* is not differentiable at p.

Finally, since $\partial u(A) \cap \partial u(B) \subset S$ for any two disjoint A and B, we have $m(\partial u(A) \cap \partial u(B)) = 0$, completing the proof. □

We now bear the first fruit of this labor.

THEOREM 8. *If Ω is open and $u \in C(\Omega)$, then the family of sets*

$$\mathscr{S} := \{E \subset \Omega : E \text{ is Lebesgue measurable and } \partial u(E) \text{ is Lebesgue measurable}\}$$

contains the Borel σ-algebra on Ω. Moreover, the set function Mu defined by

$$Mu(E) := m(\partial u(E))$$

is a Borel measure on Ω.

Remark. The measure Mu is called the **Monge-Ampère** measure of u.

PROOF. We need to show that $\mathscr{S}$ is a σ-algebra on Ω. Since Lemma 2 shows that $\mathscr{S}$ contains the compacts, this will prove that $\mathscr{S}$ contains the Borel σ-algebra. Towards that end, it comes straight from the definition that for $\{E_k\} \subset \mathscr{S}$, then $\partial u\left(\bigcup_1^\infty E_k\right) = \bigcup_1^\infty \partial u(E_k)$, and as the latter set is Lebesgue measurable this proves $\mathscr{S}$ is closed under countable unions. Since Ω is σ-compact, we deduce $\Omega \in \mathscr{S}$, and trivially $\emptyset \in \mathscr{S}$.

If $E \in \mathscr{S}$, then since

$$\partial u(\Omega \setminus E) = (\partial u(\Omega) \setminus \partial u(E)) \cup (\partial u(\Omega \setminus E) \cap \partial u(E)),$$

and $m(\partial u(\Omega \setminus E) \cap \partial u(E)) = 0$ by Lemma 7, it follows $\partial u(\Omega \setminus E)$ differs from the Lebesgue measurable set $\partial u(\Omega) \setminus \partial u(E)$ by a set of measure 0, hence is Lebesgue measurable. This proves that $\Omega \setminus E \in S$, finishing the first part of the theorem.

To show that Mu is a measure, we trivially have $Mu(E) \geq Mu(\emptyset) = 0$ for every $E \in \mathscr{S}$. Also, Mu is finitely additive, since if $A \in \mathscr{S}$ and $B \in \mathscr{S}$ are disjoint, then $\partial u(A \cup B) = \partial u(A) \cup \partial u(B)$, so by inclusion-exclusion,

$$Mu(A \cup B) = Mu(A) + Mu(B) - m(\partial u(A) \cap \partial u(B)) = Mu(A) + Mu(B),$$

since $m(\partial u(A) \cap \partial u(B)) = 0$ by Lemma 7. Moreover, if $\{E_k\}$ is an ascending collection of sets in $\mathscr{S}$, then $\{\partial u(E_k)\}$ is an ascending collection of Lebesgue measurable sets, therefore

$$\lim_{n\to\infty} Mu(E_n) = \lim_{n\to\infty} m(\partial u(E_k)) = m\left(\bigcup_{k=1}^{\infty} \partial u(E_k)\right) = m\left(\partial u\left(\bigcup_{k=1}^{\infty} E_k\right)\right) = Mu\left(\bigcup_{k=1}^{\infty} E_k\right),$$

which means Mu is continuous from above. This is enough to show that Mu is countably additive, which completes the proof. □

EXAMPLE 9. Suppose that Ω is convex and $u \in C^2(\Omega)$ is a convex function. We claim that $dMu = \det H_u\, dx$, where H_u is the Hessian matrix of u.

To establish this fact, we rely on *Sard's Theorem*, which we will state without proof.

THEOREM 10 (SARD'S THEOREM). *If $\Omega \subset \mathbb{R}^n$ is open and $g : \Omega \to \mathbb{R}^n$ is continuously differentiable, then $m(\{f(x) : \det f'(x) = 0\}) = 0$.*

Continuing with this example, one has that Du is injective on $A := \{x : \det D^2 u(x) > 0\}$ (why?). Next, since u is has a continuous first derivative, it holds $Mu(E) = m(Du(E))$. Then we have

$$Mu(E) = Mu(E \cap A) + Mu(E \setminus A) = M(E \cap A),$$

since $Mu(E \setminus A) = m(\{Du(x) : \det D^2 u(x) = 0\}) = 0$, since the convexity of u implies that $\det D^2 u \geq 0$. Since Du is injective on $E \cap A$, the change of variables formula shows

$$Mu(E \cap A) = \int_{Du(E\cap A)} dx = \int_{E\cap A} \det D^2 u(x)\, dx = \int_E \det D^2 u(x)\, dx,$$

finishing the proof.

The preceding lemma establishes that working with $C^2(\Omega)$ functions is nicer than working with arbitrary convex functions. Fortunately, it turns out that all convex functions can be approximated by convex $C^2(\Omega)$ functions (see Exercise 17), so we would like to understand if these approximations also translate into approximations of the Monge-Ampére measure, in a certain precise sense. In particular, we see to show that compact convergence translates into vague convergence of the Monge-Ampére measures. This will be achieved by essentially demonstrating the Portmanteau theorem.

THEOREM 11. *Suppose that $\{u_k\}$ is a sequence of convex functions in Ω such that $u_k \to u \in C(\Omega)$ uniformly on compact sets. Then:*

(1) If $K \subset \Omega$ is compact, then

$$\limsup_{k\to\infty} \partial u_k(K) := \{p : p \in \partial u_k(K) \textit{ for infinitely many } k\} \subset \partial u(K),$$

and

$$\limsup_{k\to\infty} Mu_k(K) \leq Mu(K).$$

(2) *If U is an open set such that $U \subset \overline{U} \subset \Omega$, then for any compact $K \subset U$,*

$$\partial u(K) \subset \liminf_{k\to\infty} \partial u_k(U) := \{p : p \in \partial u_k(U) \text{ for all but finitely many } k\},$$

where the inclusion holds Lebesgue a.e. Moreover, if $U \subset \Omega$ is any open set,

$$Mu(U) \leq \liminf_{k\to\infty} Mu_k(U).$$

PROOF.

(1) If $p \in \limsup \partial u_k(K)$, then there is an infinite sequence of positive integers $\{k_n\}$ such that $p \in \partial u_{k_n}(K)$. Pick $x_n \in K$ such that $p \in \partial u_{k_n}(x_n)$. The compactness of K guarantees a subsequence $\{x_{n_j}\}$ that converges to some $x \in K$ as $j \to \infty$. Then for all $t \in \mathbb{R}^n$,

$$u(t) = \lim_{j\to\infty} u_{k_{n_j}}(t) \geq \lim_{j\to\infty} p\cdot(t - x_{n_j}) + u_{k_{n_j}}(x_{n_j}) = p\cdot(t-x) + u(x),$$

since $u_k \to u$ uniformly on K. This proves that $p \in \partial u(K)$. Now, since Mu_k and Mu are finite on the compacts, we can apply the Lebesgue-Fatou lemma to obtain

$$Mu(K) = \int \mathbb{1}_{\partial u(K)}\,dx \geq \int \limsup_{k\to\infty} \mathbb{1}_{\partial u_k(K)}\,dx \geq \limsup_{k\to\infty} \int \mathbb{1}_{\partial u_k(K)}\,dx = \limsup_{k\to\infty} Mu_k(K).$$

(2) Without loss of generality, assume $\overline{U}$ is compact. Define S as in Lemma 7 for the function u. We claim that $\partial u(U) \setminus S \subset \liminf \partial u_k(U)$. Towards this end, if $p \in \partial u(U) \setminus S$, fix a unique $x_0 \in K$ with $p \in \partial u(x_0)$. Then for all $x \in \Omega$ with $x \neq x_0$, we have $u(x) > p\cdot(x - x_0) + u(x_0)$. Otherwise, if there is equality for some $x_1 \neq x_0$, we have

$$u(t) \geq p(t-x_0) + u(x_0) = p(t-x_1) + u(x) + p(x_1 - x_0) - u(x_1) + u(x_0) = p(t - x_1) + u(x_1),$$

so that $p \in \partial u(x_1)$. If $\ell(x) := p\cdot(x-x_0) + u(x_0)$, this means $\delta := \min\{u(x) - \ell(x) : x \in \partial U\} > 0$. Next, since $u_k \to u$ uniformly on $\overline{U}$, there is K sufficiently large that $|u(x) - u_k(x)| < \delta/2$ for all $x \in \overline{U}$ and $k \geq K$. For all such k, let

$$\delta_k = \max_{x\in\overline{U}}(\ell(x) - u_k(x) + \delta/2).$$

By the compactness of $\overline{U}$, there is $x_k \in \overline{U}$ so that the equality is achieved. Then we have

$$u(x_0) + p(x_k - x_0) - u_k(x_k) + \delta/2 \geq u(x_0) + p(x - x_0) - u_k(x) + \delta/2$$

for all $x \in \overline{U}$, or equivalently.

$$u_k(x) \geq p(x - x_k) + u_k(x_k).$$

Since $\ell(x_0) - u_k(x_0) + \delta/2 = u(x_0) - u_k(x_0) + \delta/2 > 0$, we have $\delta_k > 0$, and since $\ell(x) - u_k(x) < \ell(x) - u(x) + \delta/2 < -\delta/2 < 0$ for $x \in \partial U$, we see that $x_k \notin \partial U$. This means $p \in \partial u_k(x_k) \subset \partial u_k(U)$ for all $k \geq K$, which means $p \in \liminf \partial u_k(U)$.

To show the "moreover," suppose that $U \subset \Omega$ is any open set. If $K \subset U$ is compact, then there is a bounded open set V with $K \subset V \subset \overline{V} \subset U$, and we have already shown that $\partial du(K) \subset \liminf \partial u_k(V)$. Fatou's lemma used in the same way as in part (1) shows that

$$Mu(K) \leq \liminf_{k\to\infty} Mu_k(V) \leq \liminf_{k\to\infty} Mu_k(U).$$

Therefore, if $\{K_m\}$ is a sequence of compact sets with $K_m \subset K_{m+1}$ for all m and $U = \bigcup_1^\infty K_m$, then taking $m \to \infty$ yields

$$Mu(U) = \lim_{m\to\infty} Mu(K_m) \leq \liminf_{k\to\infty} Mu_k(U),$$

as desired.

□

As an immediate consequence, we get

THEOREM 12. *Suppose that Ω is a convex open set and $\{u_k\}$ is a sequence of convex functions such that $u_k \to u \in C(\Omega)$ uniformly on compact sets. Then the sequence of Monge-Ampére measures Mu_k converges vaguely to Mu. That is, for all $f \in C_c(\Omega)$,*

$$\lim_{k\to\infty} \int f\, dMu_k = \int f\, dMu.$$

PROOF. This is a direct application of the portmanteau theorem using Lemma 11. □

EXERCISE 13. Prove that if μ is a finitely additive measure such that

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_k\right)$$

for every countable collection of sets $\{E_n\}$ such that $E_n \subset E_{n+1}$ for all $n \geq 1$, then μ is a measure. This fact was used without proof at the end of Theorem 8.

EXERCISE 14. Prove a special case of Sard's theorem when $u : \Omega \subset \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

EXERCISE 15. Suppose that μ is a Borel probability measure on a compact and convex set P. Define the integral $\int_P x\, d\mu(x)$ coordinate-wise. That is,

$$\int_P x\, d\mu(x) = \left(\int_P x_1\, d\mu(x), \ldots, \int_P x_n\, d\mu(x)\right)$$

if one writes $x = (x_1, \ldots, x_n)$. Then $\int_P x\, d\mu(x) \in P$. In words, the center of μ-mass of P is contained in P.

EXERCISE 16. Prove that if $\{\mu_n\}$ is a sequence of positive Borel measures on a metric space X converging vaguely (i.e., in the sense of Theorem 12) to some positive Borel measure μ, then for any open set U and compact set K,

$$\mu(U) \leq \liminf_{n\to\infty} \mu_n(U), \quad \mu(K) \geq \limsup_{n\to\infty} \mu_n(K).$$

Show that the converse holds if the $\{\mu_n\}$ and μ are finite on the compacts.

EXERCISE 17. Suppose that Ω is a convex open subset of $\mathbb{R}^n$, and $f : \Omega \to \mathbb{R}$ is convex. Show that there is a sequence $\{f_n\}$ of convex functions with derivatives of all orders in Ω such that $f_n \to f$ uniformly on compact subsets of Ω.

EXERCISE 18. Show that for any convex function $u \in C(\Omega)$, where Ω is open and convex in $\mathbb{R}$,

$$\int_\Omega u\phi''\, dx = \int_\Omega \phi\, dMu$$

for all $\phi \in C_c^\infty(\Omega)$. That is, $Mu = D^2u$, where the equality and derivatives are understood in the sense of distributions.

EXERCISE 19. Suppose that $u, v \in C(\Omega)$ are convex functions. Show that $M(u+v) \geq Mu + Mv$. (*Hint*: If $u, v \in C^2(\Omega)$, use Example 9 and the inequality

$$\det(A+B) \geq \det A + \det B$$

whenever A and B are symmetric positive semi-definite. For the general case approximate u and v by convex $C^2(\Omega)$ functions.)

Before proceeding, we next recall the following basic property of convex functions.

LEMMA 20. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and $u, v \in C(\overline{\Omega})$. If $u = v$ on $\partial\Omega$ and $v \geq u$ on Ω, then $\partial v(\Omega) \subset \partial u(\Omega)$.*

PROOF. Fix $p \in \partial v$ and $x' \in \Omega$ with $p \in \partial v(x')$. Let $\ell(x) = p \cdot (x - x') + v(x')$, and put

$$\delta = \max_{x \in \overline{\Omega}} \{\ell(x) - u(x)\}.$$

Since $\ell(x') - u(x') = v(x') - u(x') \geq 0$, it follows $\delta \geq 0$. If $\delta = 0$, then $\ell(x) - u(x) \leq 0$ for $x \in \overline{\Omega}$. It then follows $\ell(x) \leq u(x) \leq v(x)$, which makes it clear that $p \in \partial u(x')$.

Otherwise, suppose $\delta > 0$. Since u and v are continuous on $\overline{\Omega}$ there is $x_0 \in \overline{\Omega}$ such that $\delta = \ell(x_0) - u(x_0)$. Since $u = v$ on $\partial\Omega$, we have $\ell(x) - u(x) \leq 0$ for $x \in \partial\Omega$, hence $x_0 \in \Omega$. Then we have

$$\ell(x_0) - u(x_0) \geq \ell(x) - u(x), \quad x \in \Omega,$$

and re-arranging terms gives

$$u(x) \geq p \cdot (x - x_0) + u(x_0), \quad x \in \Omega,$$

so $p \in \partial u(x_0) \subset \partial u(\Omega)$, finishing the proof. □

We next present the following theorem, which we state without proof.

THEOREM 21 (ALEKSANDROV'S MAXIMUM PRINCIPLE). *If $\Omega \subset \mathbb{R}^n$ is a bounded, open, and convex set with diamter Δ, and $u \in C(\overline{\Omega})$ is convex with $u = 0$ on $\partial\Omega$, then*

$$|u(x_0)|^n \leq C_n \Delta^{n-1} d(x_0, \partial\Omega) Mu(\Omega),$$

where $x_0 \in \Omega$ and C_n is an absolute constant depending only on n.

We next introduce a notion of a convex hull for convex functions.

DEFINITION 22 (CONVEX AND CONCAVE ENVELOPE). Suppose that $u \in C(\Omega)$, where Ω is a convex open set, and let

$$\mathscr{F}u = \{v : v(x) \leq u(x)\ \forall x \in \Omega,\ v \text{ convex in } \Omega\}$$

$$\mathscr{G}u = \{v : v(x) \geq u(x)\ \forall x \in \Omega,\ v \text{ concave in } \Omega\}.$$

Then the convex envelope of u is the function u_* defined as

$$u_* = \sup_{v \in \mathscr{F}u} v,$$

and the concave envelope of u is the function u^* defined as

$$u^* = \inf_{v \in \mathscr{G}u} v.$$

It is clear from the definitions that $u_* \leq u \leq u^*$, and that $\mathscr{F}(-u) = -\mathscr{G}u$ and hence $-u^* = (-u)_*$. We define the set of *contact points*

$$\mathscr{C}_* u := \{u_* = u\}, \quad \mathscr{C}^* u := \{u^* = u\}.$$

It turns out that these contact points characterize the sub-gradient of u.

LEMMA 23. *If Ω and u are as in Definition 22, then*

$$\partial u(\Omega) = \partial u_*(\mathscr{C}_* u).$$

PROOF. Evidently $x \in \mathscr{C}_* u$ implies that $\partial u_*(x) \subset \partial u(x)$, so that $\partial u_*(C_* u) \subset \partial u(\mathscr{C}_* u)$. Now, if $x_0 \notin C_* u$, then $u^*(x_0) < u(x_0)$, and since lines are convex functions it follows that no line can touch u at x_0. This means $\partial u(\Omega \setminus C_* u) = \emptyset$. Therefore $\partial u(\Omega) = \partial u(C_* u)$. By the same token, we have $\partial u(C_* u) \subset \partial u_*(C_* u)$, which finishes the proof. □

The following theorem provides a partial converse to Lemma 20

THEOREM 24. *Suppose that $u, v \in C(\overline{\Omega})$ are convex, and*
$$Mu(E) \leq Mv(E)$$
whenever $E \subset \Omega$ is a Borel set. Then
$$\min_{x\in\overline{\Omega}}\{u(x)-v(x)\} = \min_{x\in\partial\Omega}\{u(x)-v(x)\}.$$
In particular, if $u = v$ on $\partial\Omega$, then $u \geq v$ on Ω.

PROOF. Let $a = \min_{x\in\overline{\Omega}}\{u(x)-v(x)\}$ and $b = \min_{x\in\partial\Omega}\{u(x)-v(x)\}$. To obtain a contradiction, suppose that $a < b$. There exists $x_0 \in \Omega$ such that $a = u(x_0) - v(x_0)$. Let $\delta > 0$ be small enough that $\delta \cdot \mathrm{diam}\Omega^2 < \frac{1}{2}(b-a)$, and let
$$w(x) = v(x) + \delta|x-x_0|^2 + \frac{b+a}{2}.$$
Let $G = \{x \in \overline{\Omega} : u(x) < w(x)\}$. We aim to show that $Mu(G) > Mv(G)$, which is the desired to contradiction. Towards that end, first observe that $x_0 \in G$, so that G is nonempty and open. If there is $x \in G \cap \partial\Omega$, then $u(x) - v(x) \geq b$ and so
$$w(x) \leq u(x) - b + \delta|x-x_0|^2 + \frac{b-a}{2} \leq u(x) + \delta\mathrm{diam}\Omega^2 - \frac{b-a}{2} < u(x),$$
which means that $G \cap \partial\Omega = \emptyset$. In fact, we have the stronger claim that $w(x) < u(x)$ on $\partial\Omega$, so that $\partial G = \{x \in \Omega : u(x) = w(x)\}$. By Lemma 20 it follows $\partial w(G) \subset \partial u(G)$.

At this point we work in two cases. If $v \in C^2(\overline{\Omega})$, then by Example 9 we can write
$$dMw = dM(v + \delta|\cdot - x_0|^2) = \det\left[D^2 v + 2\delta I\right] dx,$$
where I is the $n \times n$ identity matrix. Now, for any two symmetric positive semi-definite matrices A and B, it holds
$$\det[A+B] \geq \det[A] + \det[B],$$
from which it follows
$$\det\left[D^2 v + 2\delta I\right] \geq \det D^2 v + (2\delta)^n.$$
Therefore,
$$Mu(G) \geq Mw(G) \geq \int_G \det D^2 v(x) + (2\delta)^n \, dx = Mv(G) + (2\delta)^n m(G) > Mv(G),$$
which is the desired contradiction.

For the second case, we suppose $v \notin C^2(\overline{\Omega})$. By mollifying v, we obtain a sequence $\{v_k\}$ of convex functions where $v_k \in C^2(\overline{\Omega})$ for each k and $v_k \to v$ uniformly on the compacts. For v_k, we have just shown that $M(v_k + \delta|\cdot - x_0|^2) \geq Mv_k + (2\delta)^n m$, so by Theorem 12,
$$\begin{aligned}\int_\Omega f\, dM(v + \delta|\cdot - x_0|^2) = \lim_{k\to\infty}\int_\Omega f\, dM(v_k + \delta|\cdot - x_0|^2) &\geq \lim_{n\to\infty}\int_\Omega f\, dMv_k + \int f(2\delta)^n\, dx \\ &= \int_\Omega f\, dMv + \int f(2\delta)^n\, dx.\end{aligned}$$
Since $f \in C_c(\Omega)$ is arbitrary, we deduce $M(v + \delta|\cdot - x_0|^2) \geq Mv + (2\delta)^n m$. Following the same argument as before, we then have
$$Mu(G) \geq Mw(G) \geq Mv(G) + (2\delta)^n m(G) > Mv(G),$$
finishing the proof. □

The focus of much of the next section is in the solving of a variant of *Dirichlet's Problem.* Formally, we wish to solve the generalized PDE

PROBLEM 25 (DIRICHLET'S PROBLEM). *Let Ω be an open set, μ a Borel measure on Ω, and $g \in C(\partial\Omega)$. Is there a convex function $u \in C(\overline{\Omega})$ such that $Mu = \mu$ in Ω and $u = g$ on $\partial\Omega$?*

The answer, as we will find, is essentially "yes." To motivate the solution to this problem, we observe that every convex function u is *subharmonic*. That is, for every $a \in \Omega$, there is $R > 0$ sufficiently small that $\overline{B}_R(a) \subset \Omega$ and

$$u(a) \le \int_{\mathbb{S}^{n-1}} u(a + R\xi)\, d\sigma(\xi),$$

where σ is the normalized surface measure of the sphere. Equivalently, the distributional Laplacian Δu is a positive linear functional on $C^2(\Omega)$, where $\Delta u = \mathrm{Trace} D^2 u$. Therefore, it should be no surprise that the essential techniques of solving this problem can be found within the study of harmonic and subharmonic functions. In particular we will use the famed "Perron method" to construct these solutions. Consequently, we will take a detour into harmonic function theory to build the necessary background.

0.1.1. Harmonic Functions. As mentioned, we will take a small detour into the theory of harmonic and subharmonic functions. If a familiarity with this subject matter is already acquired, then the reader can skip to Theorem 33.

We begin with the properties of harmonic functions, as most properties of sub-harmonic functions can be inferred from harmonic functions.

DEFINITION 26 (HARMONIC FUNCTION). Let $\Omega \subset \mathbb{R}^n$ be a nonempty open set, and suppose $u \in C^2(\Omega)$. We say that u is *harmonic* if $\Delta u \equiv 0$ on Ω, where

$$\Delta u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}.$$

It is easy to check that if u is harmonic, that translates and dilations of u are also harmonic. The following is an important example to keep in mind.

EXAMPLE 27. When $n = 2$, let $f(x) = \log|x|$, and when $n > 2$, let $f(x) = |x|^{2-n}$. Then f is harmonic on $\mathbb{R}^n \setminus \{0\}$. When $n = 2$, then it is easy to check that

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{2}{|x|}\left(1 - \frac{2x_1^2}{|x|^2}\right), \quad \frac{\partial^2 f}{\partial x_2^2} = \frac{2}{|x|}\left(1 - \frac{2x_2^2}{|x|^2}\right).$$

Summing then yields $\Delta f = 0$. Similarly, when $n > 2$, one has

$$\frac{\partial^2 f}{\partial x_j^2} = \left(1 - \frac{n}{2}\right)\frac{2}{|x|}\left(1 - \frac{n x_j^2}{|x|^2}\right),$$

and similarly summing gives $\Delta f = 0$.

THEOREM 28. *Suppose that $T \in SO(n)$ (where $SO(n)$ is the set of orthogonal linear maps on $\mathbb{R}^n$) and u is harmonic on an open $\Omega \subset \mathbb{R}^n$. Then $\Delta(u \circ T) = (\Delta u) \circ T$ on $T^{-1}(\Omega)$. In particular, $u \circ T$ is harmonic on $T^{-1}(\Omega)$.*

PROOF. The Fourier transforms of $\Delta(u \circ T)$ and $(\Delta u) \circ T$ are the same, when taken in the distribution sense. Therefore the two functions must be equal pointwise everywhere, so that $\Delta(u \circ T) = 0$. □

We next recall an identity of Green from advanced calculus, when Ω is a bounded open set with smooth boundary:

$$\int_{\Omega} (u\Delta v - v\Delta u)\, dx = \int_{\partial\Omega} (u D_n v - v D_n u)\, ds,$$

where u and v are C^2 functions on a neighborhood of $\overline{\Omega}$ and $n = n(x)$ is the outward unit normal vector to the tangent plane of $\overline{\Omega}$ at $x \in \partial\Omega$, and D_n is the directional derivative in the direction of n, and finally s is the surface measure of $\partial\Omega$. If u is harmonic and $v \equiv 1$ then one obtains the following identity:

$$\int_{\partial S} \nabla f \cdot n\, ds = 0.$$

If we specialize this identity, we obtain the mean-value property.

THEOREM 29. *If u is harmonic on Ω then for any $x \in \Omega$ and $r > 0$ small enough that $\overline{B}_r(x) \subset \Omega$, it holds*

$$u(a) = \int_{\mathbb{S}^{n-1}} u(a + r\xi)\, d\sigma(\xi).$$

PROOF. By shifting and scaling, without loss of generality $r = 1$ and $a = 0$. Fix $\epsilon \in (0,1)$, and let $\Omega = \{x : \epsilon < |x| < 1\}$. Let f be as in Example 27. If $n > 2$, then Green's identity shows

$$\int_{\partial\Omega} u D_n f - f D_n u\, ds = 0.$$

Writing $\partial\Omega = \mathbb{S}^{n-1} \cup (\epsilon\mathbb{S}^{n-1})$ and $\nabla f = (2-n)|x|^{-n}x$ and $n(x) = x/|x|$, on $\mathbb{S}^{n-1}$ we get $D_n f(x) = \nabla f(x) \cdot n(x) = (2-n)x \cdot x = 2-n$, and on $\epsilon\mathbb{S}^{n-1}$ we get $D_n f(x) = \nabla f(x) \cdot n(x) = (2-n)\epsilon^{1-n}$. So,

$$\int_{\partial\Omega} u D_n f - f D_n u\, ds = (2-n)\int_{\mathbb{S}^{n-1}} u\, ds - (2-n)\epsilon^{1-n}\int_{\epsilon\mathbb{S}^{n-1}} u\, ds,$$

since $\int_{\partial\Omega} D_n u\, ds = 0$ from the observation immediately preceding this theorem. So,

$$\int_{\mathbb{S}^{n-1}} u\, ds = \epsilon^{1-n}\int_{\epsilon\mathbb{S}^{n-1}} u\, ds,$$

so that

$$\int_{\mathbb{S}^{n-1}} u\, d\sigma = \int_{\mathbb{S}^{n-1}} u(\epsilon\xi)\, d\sigma(\xi).$$

Taking $\epsilon \to 0$ using the continuity of u then gives the desired claim. When $n = 2$, the same argument instead taking $f(x) = \log|x|$ proves the theorem. $\square$

EXERCISE 30. Prove that the following identity is valid for a harmonic $u \in C(\Omega)$:

$$u(a) = \frac{1}{m(B_r(a))}\int_{B_r(a)} u(x)\, dx$$

where $r > 0$ is small enough that $\overline{B}_r(x) \subset \Omega$. *Hint*: Convert the integral to polar coordinates.

THEOREM 31 (MAXIMUM PRINCIPLE FOR HARMONIC FUNCTIONS). *If Ω is a connected open set, and u is harmonic on Ω, the only way u attains a maximum or minimum on Ω is if u is constant.*

PROOF. Without loss of generality, suppose that u attains a maximum in Ω. Let $E = \{x \in \Omega : u(x) = \sup_\Omega u\}$. By hypothesis, $E \neq \emptyset$. To obtain a contradiction, suppose that there exists $a \in E$ and a ball $B_r(a)$ contained in Ω such that $B_r(a) \cap E^c \neq \emptyset$. Then we have

$$\frac{1}{m(B_r(a))}\int_{B_r(a)} u(x)\, dx = \frac{1}{m(B_r(a))}\int_{B_r(a)\cap E^c} u(x)\, dx + u(a)\frac{m(B_r(a)\cap E)}{m(B_r(a))}.$$

By the continuity of u and the fact $B_r(a) \cap E^c$ is a nonempty open set (so has nonzero measure), we have $\frac{1}{m(B_r(a))}\int_{B_r(a)\cap E^c} u(x)\, dx < \frac{m(B_r(a)\cap E^c)}{m(B_r(a))}u(a)$, therefore

$$\frac{1}{m(B_r(a))}\int_{B_r(a)} u(x)\, dx < \frac{m(B_r(a)\cap E^c)}{m(B_r(a))}u(a) + u(a)\frac{m(B_r(a)\cap E)}{m(B_r(a))} = u(a).$$

But this is a contradiction, since $u(a) = \frac{1}{m(B_r(a))}\int_{B_r(a)} u(x)\, dx$ by Exercise 30. Therefore, for every $a \in E$, there is a ball $B_r(a) \subset E$. This means E is open. But, since u is continuous, E is also closed. Since Ω is connected and E is nonempty, it follows $E = \Omega$, which is the stated claim. $\square$

COROLLARY 32. *If Ω is a connected open set and $u \in C(\overline{\Omega})$ is harmonic on Ω, then u attains its maximum and minimum values on $\partial\Omega$.*

THEOREM 33 (DIRICHLET'S PROBLEM.). *If $\Omega \subset \mathbb{R}^n$ is bounded and strictly convex, and $g \in C(\partial\Omega)$, then there is a unique convex $u \in C(\overline{\Omega})$ such that $Mu = 0$ on Ω and $u = g$ on $\partial\Omega$.*

PROOF. We will explicitly construct a solution and then prove that it works. Let

$$\mathscr{F} = \{a : a \text{ is an affine functional and } a \le g \text{ on } \partial\Omega\}$$

and put

$$u(x) = \sup_{a\in\mathscr{F}} a(x).$$

Since u is a supremum of convex functions, u is convex, continuous on Ω, and moreover $u \le g$ on $\partial\Omega$.

The first thing we show is that in fact $u = g$ on $\partial\Omega$. Fix $\epsilon > 0$, and let $\delta > 0$ be so that $|g(x) - g(y)| < \epsilon$ whenever $x, y \in \partial\Omega$ and $|x - y| < \delta$. Fix $\xi \in \partial\Omega$, and let P be a supporting hyperplane to $\overline{\Omega}$ at ξ. That is, $\overline{\Omega} \subset \{x : P(x) \ge 0\}$, and $\xi \in \{x : P(x) = 0\}$. By the strict convexity of Ω, there is $\eta > 0$ small enough that $S := \{x \in \overline{\Omega} : P(x) \le \eta\} \subset B_\delta(\xi)$. Put

$$M = \min_{x\in\partial\Omega\cap S^c} g(x),$$

and consider the affine function $a(x) = g(\xi)-\epsilon-AP(x)$, where $A \ge 0$ is chosen so large that $A \ge \eta^{-1}(g(\xi)-\epsilon-M)$. It follows $a(\xi) = g(\xi)-\epsilon$. If $x \in \partial\Omega\cap S$, then $|g(\xi)-g(x)| \le \epsilon$ and so $g(x) \ge g(\xi)-\epsilon = g(\xi)-\epsilon-AP(x)+AP(x) \ge g(\xi) - \epsilon - AP(x) = a(x)$ since $AP(x) \ge 0$. If $x \in \partial\Omega \cap S^c$, then $P(x) > \eta$ and our choice of A yields

$$\begin{aligned} g(x) \ge M &= a(x) + \epsilon - g(\xi) + AP(x) + M \\ &\ge a(x) + M - g(\xi) + \epsilon + A\eta \\ &\ge a(x). \end{aligned}$$

This proves that $a(x) \le g(x)$ on $\partial\Omega$, so $a \in \mathscr{F}$ and we get $u(\xi) \ge a(\xi) = g(\xi) - \epsilon$. Since ϵ was arbitrary, we deduce $u(\xi) \ge g(\xi)$, so that $u = g$ on $\partial\Omega$.

Next, we need to show that u is continuous on $\overline{\Omega}$. Clearly u is continuous in Ω. To get continuity on $\partial\Omega$, suppose that $\xi \in \partial\Omega$, and $\{x_n\} \subset \overline{\Omega}$ converges to ξ. We show $u(x_n) \to g(\xi)$. Letting a be the affine functional as before, we have $u(x_n) \ge a(x_n)$ and therefore

$$\liminf u(x_n) \ge \liminf a(x_n) = g(\xi) - \epsilon$$

by continuity of a. Since ϵ is arbitrary, we get $\liminf u(x_n) \ge g(\xi)$. Fix a function h, harmonic in Ω, such that $h \in C(\overline{\Omega})$ and $h|_{\partial\Omega} = g$. By the maximum principle, if $a \in \mathscr{F}$ then $a \le h$ in Ω, so $u \le h$. This means $\limsup u(x_n) \le \limsup h(x_n) = g(\xi)$ by continuity of h, and this finishes the proof of continuity.

All that remains to do is show that $Mu(\Omega) = 0$. To do this, we show

$$\partial u(\Omega) \subset \{p \in \mathbb{R}^n : \exists x, y \in \Omega,\ x \ne y,\ p \in \partial u(x) \cap \partial u(y)\}.$$

Indeed, suppose that $p \in \partial u(x_0)$ for some $x_0 \in \Omega$. Let $a(x) = p\cdot(x-x_0)+u(x_0)$, so that $u \ge a$. Observe that there is $\xi \in \partial\Omega$ such that $g(\xi) = a(\xi)$. Otherwise, for some $\epsilon > 0$, we have $g(x) \ge a(x) + \epsilon$ on $\partial\Omega$ for some $\epsilon > 0$, hence $u(x) \ge a(x)+\epsilon$ on Ω and thus $u(x_0) \ge a(x_0)+\epsilon = u(x_0)+\epsilon > u(x_0)$, a contradiction. Therefore $u(x_0) = a(x_0)$ and $u(\xi) = a(\xi)$. Let $z = tx_0 + (1-t)\xi$ for some $0 < t < 1$. Since Ω is strictly convex, $z \in \Omega$. By convexity, $u(z) \le tu(x_0)+(1-t)u(\xi) = ta(x_0)+(1-t)a(\xi) = a(z)$. But $a(z) \le u(z)$, so a is also a supporting hyperplane to z, hence $p \in \partial u(z)$, as desired.

Uniqueness of u comes from Theorem 24. □

For the next two theorems, for a strictly convex open set Ω, Borel measure μ on Ω, and $g \in C(\partial\Omega)$, we define

$$\mathscr{F}(\mu, g) = \{v \in C(\overline{\Omega}) : v \text{ convex},\ Mv \ge \mu \text{ in } \Omega,\ v = g \text{ on } \partial\Omega\}.$$

Suppose that $\mathscr{F}(\mu, g) \ne \emptyset$, and so fix $v \in \mathscr{F}(\mu, g)$. By Theorem 33, there exists a unique convex $W \in C(\overline{\Omega})$ solving $MW = 0$ and $W = g$ on $\partial\Omega$. Then we have $MW \le \mu \le Mv$ for $v \in \mathscr{F}(\mu, g)$, therefore $v \le W$ in Ω by Theorem 24. This means the functions in $\mathscr{F}(\mu, g)$ are uniformly bounded from above.

THEOREM 34. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open strictly convex domain, and μ be a Borel measure in Ω. Suppose that $g \in C(\partial\Omega)$ and $\{u_j\}$ is a sequence of convex functions such that*

(1) $u_j = g$ on $\partial\Omega$,
(2) $Mu_j \to \mu$ vaguely in Ω,

(3) there is a fixed constant $M < \infty$ such that $Mu_j \le M$ for all $j \ge 1$.

Then there is a subsequence $\{u_{j_k}\}$ and a convex $u \in C(\overline{\Omega})$ such that $u_{j_k} \to u$ uniformly on compacts, $\mu = Mu$, and $u = g$ in $\partial\Omega$. By Theorem 33, we know that

PROOF. We have $u_j \in \mathscr{F}(u_j, g)$ for every $j \ge 1$, and therefore the u_j are uniformly bounded from above. We need to show that $\{u_j\}$ are also uniformly bounded from below. Let a be the affine functional □

0.2. The Brunn-Minkowski Theory

In this section we explore applications of Monge-Ampére measures. We require a few additional tools.

DEFINITION 35 (SUPPORT OF A CONVEX SET). Let $P \subset \mathbb{R}^n$ be convex set. The support of P is the function $h_P : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ defined by

$$h_P(x) = \sup_{p \in P} p \cdot x.$$

LEMMA 36. *For any convex set P, the support function h_P is convex and lower semi-continuous. Moreover, h_P is everywhere continuous if and only if P is bounded.*

PROOF. Since the family of functions $x \mapsto p \cdot x$ is convex for every $p \in \mathbb{R}^n$, and the supremum of convex functions is again convex, it is clear that h_P is convex. For lower continuity, we observe that for any $a \in \mathbb{R}$,

$$\{x : h_P(x) \le a\} = \bigcap_{p \in \mathbb{R}^n} \{x : p \cdot x \le a\}.$$

Since each set in the right-hand intersection is closed, the left-hand set is closed and lower semi-continuity follows.

On the other hand, suppose that h_P is everywhere continuous. Then h_P is uniformly bounded on the unit sphere $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. Let $\Lambda < \infty$ be the sup of $|h_P|$ on $\mathbb{S}^{n-1}$. If $p \in P$, then it holds

$$|p| = (p/|p|, p) \le h_P(p/|p|) \le \Lambda,$$

which proves that P is bounded. On the other hand, if P is bounded, say by some $M < \infty$, then for all $q \in P$ and $x \in \mathbb{R}^n$, it holds $q \cdot x \le M|x| < \infty$ by the Cauchy-Schwarz inequality, so that $h_P(x) \le M|x| < \infty$. By the same token, if $y \in \mathbb{R}^n$, it holds $|q \cdot (x - y)| \le M|x - y|$, so $q \cdot x \le q \cdot y + M|x - y| \le h_P(y) + M|x - y|$ and $q \cdot y \le q \cdot x + M|x - y| \le h_P(x) + M|x - y|$. Taking the sup over all $q \in P$ gives

$$h_P(x) \le h_P(y) + M|x - y|, \quad h_P(y) \le h_P(x) + M|x - y|,$$

which proves that h_P is Lipschitz continuous on $\mathbb{R}^n$. □

LEMMA 37. *Let P and Q be convex sets and $t \ge 0$.*

(1) $h_{tP+Q} = th_P + h_Q$
(2) $h_P = h_{\overline{P}}$
(3) $|h_P^*| \in \{0, \infty\}$, and $\overline{P} = \{x : h_P^*(x) \le 0\}$.
(4) $h_P \le h_Q$ if and only if $\overline{P} \subset \overline{Q}$.

PROOF.

(Proof of 1). When $t \ge 0$, the equality $h_{tP} = th_P$ is true by linearity of the inner product. Therefore, it is enough to show that $h_{P+Q} = h_P + h_Q$ whenever P and Q are convex. For a fixed x, suppose both $h_P(x)$ and $h_Q(x)$ are finite. Then, if $\alpha > 0$, there exist $x_P \in P$ and $x_Q \in Q$ with $h_P(x) - \alpha < x_P \cdot p$ and $h_Q(x) - \alpha < x_Q \cdot x$. Adding, we get $h_P(x) + h_Q(x) - 2\alpha < (x_P + x_Q) \cdot x \le h_{P+Q}(x)$. Since $\alpha > 0$ was arbitrary, it follows $h_P + h_Q \le h_{P+Q}$. The proofs of the $h_P = \infty$ or $h_Q = \infty$ cases are virtually the same, *mutatis mutandis.*

On the other hand, $h_{P+Q} \le h_P + h_Q$ follows immediately from

$$h_{P+Q}(x) = \sup_{p+q \in P+Q} (p+q)\cdot x = \sup_{p+q\in P+Q} [p\cdot x + q \cdot x] \le h_P(x) + h_Q(x).$$

This proves 1.

(Proof of 2). For a fixed x, let $v_x : P \to \mathbb{R}$ be defined by $v_x(p) = p\cdot x$. Then v_x is uniformly continuous, from which it follows from elementary analysis that $h_P(x) = \sup_{p\in P} v_x(p) = \sup_{p\in \overline{P}} v_x(p) = h_{\overline{P}}(x)$.

(Proof of 3). Observe that

$$E := \{x : h_P^*(x) \le 0\} = \{x : x\cdot u \le h_P(u)\ \forall u \in \mathbb{R}^n\}.$$

From the definition of h_P and by part (2), we see that $\overline{P} \subset E$. To obtain a contradiction, suppose there is $x \in E \setminus \overline{P}$. By the hyperplane separation theorem, there is $\alpha \in R$ and $y \in \mathbb{R}^n$ such that

$$x \cdot y > \alpha, \quad p \cdot y \le \alpha$$

for all $p \in \overline{P}$. Thus $h_P(y) < x\cdot y$, which is a contradiction since $x \in E$.

(Proof of 4). Since $h_P = h_{\overline{P}}$ and $h_Q = h_{\overline{Q}}$, clearly $\overline{P} \subset \overline{Q}$ implies $h_P \le h_Q$. For the converse, observe that $h_P \le h_Q$ means $h_P^* \ge h_Q^*$, so

$$\overline{P} = \{x : h_P^*(x) \le 0\} \subset \{x : h_Q^*(x) \le 0\} = \overline{Q}$$

by part 3.

□

Lemma 38. *Suppose that $h : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a positive homogeneous and lower semi-continuous convex function. Then there exists a unique closed convex set P such that $h = h_P$.*

Proof. Since h is positive homogeneous, the proof of Lemma 37.3 shows that the Legendre transform h^* takes values in 0 or $\pm\infty$. Let $P = \{x : h^*(x) \le 0\}$. Clearly P is closed and convex. Moreover,

$$h(x) = h^{**}(x) = \sup_{y\in\mathbb{R}^n} (y\cdot x - h^*(y)) = \sup_{y\in P} y \cdot x = h_P(x).$$

To see that P is unique, if U is some other closed set with $h_U = h$, then Lemma 37.4 shows that $U = P$. □

The support function allows us to define a nice topology on the space of compact, convex sets in $\mathbb{R}^n$.

Definition 39 (Hausdorff Distance). Let $\mathcal{H}^n$ be the set of all compact, convex sets in $\mathbb{R}^n$. We equip $\mathcal{H}^n$ with the *Hausdorff metric* d_H defined as

$$d_H(K,F) := \inf\{\delta > 0 : K \subset F + B_\delta,\ F \subset K + B_\delta\},$$

where B_δ is the closed ball of radius δ about 0.

The next lemma just checks that the Hausdorff distance is well-defined.

Lemma 40. *d_H as defined in Definition 39 is an invariant metric on $\mathcal{H}^n$.*

Proof. Clearly $d_H \ge 0$ and d_H is symmetric and invariant. We just need show that $d_H(K,F) = 0$ if and only if $K = F$ and the triangle inequality. For the first assertion, trivially $K = F$ means $d_H(K,F) = 0$. To show the converse, fix $x \in K$. For any $\delta > 0$, there exists $y \in F$ such that $x = y + v$ for some $v \in B_\delta$. It follows $|x - y| \le \delta$, and since δ was arbitrary, x is a limit point of F and therefore $x \in F$, hence $K \subset F$ The same argument with the roles of K and F reversed show that $F \subset K$, which means $K = F$.

For the triangle inequality, let K, F, N be compact convex sets. Fix $\delta_1 > d_H(K,F)$ and $\delta_2 > d_H(F,N)$. Then since $B_{\delta_1} + B_{\delta_2} = B_{\delta_1+\delta_2}$,

$$N + B_{\delta_1+\delta_2} = (N + B_{\delta_2}) + B_{\delta_1} \supset F + B_{\delta_1} \supset K$$

and

$$K + B_{\delta_1+\delta_2} = (K + B_{\delta_1}) + B_{\delta_2} \supset F + \delta_2 \supset N.$$

This proves that $d_H(K,N) \le \delta_1 + \delta_2$, and since $\delta_1 > d_H(K,F)$ and $\delta_2 > d_H(F,N)$ were arbitrary, it follows $d_H(K,N) \le d_H(K,F) + d_H(F,N)$. □

THEOREM 41. *Let $\|\cdot\|_u$ denote the uniform norm on the sphere $\mathbb{S}^{n-1}$. Then for any $K, F \in \mathscr{H}^n$,*

$$d_H(K,F) = \|h_K - h_F\|_u.$$

PROOF. Fix $\delta > d_H(K,F)$. Then $K \subset F + B_\delta$ and $F \subset K + B_\delta$. Therefore, for $x \in \mathbb{S}^{n-1}$, we have

$$h_K(x) \le h_{F+B_\delta}(x) = h_F(x) + h_{B_\delta}(x) = h_F(x) + \delta$$

and

$$h_F(x) \le h_{K+B_\delta}(x) = h_K(x) + h_{B_\delta}(x) = h_K(x) + \delta.$$

This proves $|h_F(x) - h_K(x)| \le \delta$ for all $x \in \mathbb{S}^{n-1}$, hence

$$\|h_F - h_K\|_u \le \delta.$$

Taking $\delta \to d_H(K,F)$ yields $\|h_F - h_K\|_u \le d_H(K,F)$. On the other hand, if $\delta = \|h_F - h_K\|_u$, then for all $y \in \mathbb{S}^{n-1}$,

$$h_F(y) \le h_K(y) + \delta, \quad h_K(y) \le h_F(y) + \delta.$$

If $x \in \mathbb{R}^n$ and $x \neq 0$, then putting $y = x/|x|$ gives

$$h_F(x) \le h_K(x) + \delta|x|, \quad h_K(x) \le h_F(x) + \delta|x|$$

by the positive homogeneity of h_F and h_K. Therefore $h_F \le h_{K+B_\delta}$ and $h_K \le h_{F+B_\delta}$, which means $F \subset K + B_\delta$ and $K \subset F + B_\delta$. Thus $\delta \ge d_H(K,F)$, finishing the proof. □

We are now ready to prove an extremely important property about $\mathscr{H}^n$, which is called the Blaschke Selection Theorem.

THEOREM 42 (BLASCHKE SELECTION THEOREM). *For any $n \ge 1$, the space $\mathscr{H}^n$ has the Heine-Borel property. In particular, $\mathscr{H}^n$ is complete.*

PROOF. Suppose that $\{F_k\}$ is a bounded sequence of elements in $\mathscr{H}^n$. Then it holds $C = \sup_{i,j} d_H(F_i, F_j) + d_H(F_1, \{0\}) < \infty$, so by Theorem 41, we have

$$\|h_{F_k}\|_u \le \sup_{i,j} \|h_{F_i} - h_{F_j}\|_u + \|h_{F_1}\|_u = C < \infty.$$

This means that the functions $\{h_{F_k}\}$ are uniformly bounded on $\mathbb{S}^{n-1}$. Moreover, for all $p \in F_k$, we have

$$\|p\| = |(p/|p|, p)| \le C.$$

So, for any $x, y \in \mathbb{S}^{n-1}$ and $p \in F_k$, it holds

$$|p \cdot (x - y)| \le C|x - y|,$$

and therefore $p \cdot x \le h_{F_k}(y) + C|x-y|$ and $p \cdot y \le h_{F_k}(x) + C|x-y|$. Taking the sup over all such p gives $h_{F_k}(x) \le h_{F_k}(y) + C|x-y|$ and $h_{F_k}(y) \le h_{F_k}(x) + C|x-y|$, which means $|h_{F_k}(x) - h_{F_k}(y)| \le C|x-y|$. This proves that the $\{h_{F_k}\}$ are equicontinuous. The Arzelà-Ascoli theorem guarantees a subsequence $\{h_{F_{k_j}}\}$ which converges uniformly on $\mathbb{S}^{n-1}$. Let h be the limit of this subsequence. Then since h_{F_k} is convex and positive homogeneous for all k, it holds h has the same properties, and h is also continuous. Therefore, there is a unique closed and convex set F such that $h = h_F$. Since $\|h_F\|_u \le C$, it is also the case F is bounded, which means $F \in \mathscr{H}^n$. Since

$$d_H(F_{k_j}, F) = \|h_{F_{k_j}} - h_F\|_u,$$

it follows that $F_{k_j} \to F$ in $\mathscr{H}^n$, which shows that every bounded sequence in $\mathscr{H}^n$ has a convergent subsequence. This proves the Heine-Borel property, which also implies the completeness of $\mathscr{H}^n$. □

EXERCISE 43. Suppose that $\{E_n\} \subset \mathscr{H}^n$ is a sequence of decreasing sets. That is, $E_{n+1} \subset E_n$ for each n. Prove that there exists $E \in \mathscr{H}^n$ such that $E_n \to E$ as $n \to \infty$, and that $E \subset E_n$ for all $n \ge 1$. Show that the same result holds if the inclusions are reversed. That is, if $E_{n+1} \supset E_n$ then $E_n \to E$ as $n \to \infty$ and $E \supset E_n$ for all n.

EXERCISE 44. Prove that $\mathscr{H}^n$ is a separable metric space, without relying on Theorem 42.

EXERCISE 45. Show that $\mathscr{H}^n$ is complete without using Theorem 42.

EXERCISE 46. Show that

$$d_H(U,V) = \max\left\{\sup_{x\in U} d(x,V), \sup_{x\in V} d(x,U)\right\},$$

where for $P \in \mathscr{H}^n$, $d(x,P) = \inf_{y\in P}|x-y|$.

Naturally, the support function allows us to define the Monge-Ampére measure of a convex set rather than a convex function.

DEFINITION 47 (MONGE-AMPÉRE MEASURE FOR CONVEX SETS). The Monge-Ampére measure for a set $P \in \mathscr{H}^n$, denoted by MP, is defined as $MP := Mh_P$, where h_P is the support of P.

Note that MP is well defined for any bounded convex set by Lemma 36 and Theorem 8. Note that if $P \in \mathscr{H}^n$, the Legendre transform h_P^* of h_P only assumes the values 0 or $+\infty$, hence one can write $P = \{x : h_P^*(x) = 0\} = \{x : h_P^*(x) < \infty\}$. From this it is clear that $\partial h_P(\mathbb{R}^n) = P$, so that $MP(\mathbb{R}^n) = m(P)$. That is, MP is always a finite Borel measure (hence inner and outer regular) when $P \in \mathscr{H}^n$. Naturally, we want a few approximation lemmas.

LEMMA 48. *Suppose that $\{P_j\} \subset \mathscr{H}^n$ converges in $\mathscr{H}^n$ to some $P \in \mathscr{H}^n$. Then $MP_j \to MP$ vaguely as $j \to \infty$. That is, for every $f \in C_c(\mathbb{R}^n)$,*

$$\lim_{j\to\infty}\int_{\mathbb{R}^n} f\, dMP_j = \int_{\mathbb{R}^n} f\, dMP.$$

PROOF. After a conversion into polar coordinates, it follows from Theorem 41 that $P_j \to P$ in $\mathscr{H}^n$ implies $h_{P_j} \to h_P$ uniformly on compact sets. The proof is finished by applying Theorem 12. □

Working with the supports directly is sometimes unnecessary, given their general lack of smoothness. Therefore, for $P \in \mathscr{H}^n$, we introduce the family $\mathscr{C}_P$ of convex functions v such that $v - h_P$ is uniformly bounded in $\mathbb{R}^n$. Clearly $h_P \in \mathscr{C}_P$. We will later see that $\mathscr{C}_P$ always consists of many functions (including smooth functions!) which can be manufactured in great abundance.

LEMMA 49. *Let $P \in \mathscr{H}^n$ and $\epsilon > 0$ be given. Then there exists a smooth convex function $u \in C^\infty(\mathbb{R}^n)$ such that $\sup_{\mathbb{R}^n}|u - h_P| \le \epsilon$. In particular, $u \in \mathscr{C}_P$.*

PROOF. We mollify the support function h_P, which is Lipschitz continuous thanks to Lemma 37. Following standard mollification procedure, we take a positive smooth function ρ supported in the unit ball about the origin with $\int \rho = 1$, and consider the convolution $h_P * \rho_\delta$. Here, $\rho_\delta(x) := \delta^{-n}\rho(\delta^{-1}x)$ for $\delta > 0$. Let $C_1 = \mathrm{Lip}(h_P)$. Then for $x \in \mathbb{R}^n$, it holds

$$\begin{aligned}
|h_P * \rho_\delta(x) - h_P(x)| \le \int_{\mathbb{R}^n} \rho_\delta(y)|h_P(x-y) - h_P(x)|\, dy &\le C_1 \int |y|\rho_\delta(y)\, dy \\
&= C_1 \int_B |\delta y|\rho(y)\, dy \\
&\le \delta C_1 \int_B \rho(y)\, dy \\
&= C_1\delta,
\end{aligned}$$

where B is the unit ball about the origin and we used the fact $\operatorname{supp}\rho \subset B$. It is easy to check that $h_P * \rho_\delta$ is smooth and convex for $\delta > 0$. Choosing δ small enough that $C_1\delta \le \epsilon$ and taking the supremum over all $x \in \mathbb{R}^n$ finishes the proof. □

LEMMA 50. *Suppose that $u \in \mathscr{C}_P$. Then $Mu(\mathbb{R}^n) = m(P)$.*

PROOF. We first assume that P has nonempty interior. Since translating P changes its support (hence elements of $\mathscr{C}_P$) by a linear function, we may assume without loss of generality assume $0 \in P^\circ$. First, we have $Mh_P(\mathbb{R}^n) = m(P)$. Fix $\epsilon > 0$. For $k \geq 1$, there exists $C_k > 0$ such that $kB \subset \{x : h_P(x) \leq C_k\}$, where B is the ball about the origin, by the positive homogeneity of h_P. Moreover, since 0 is in the interior of P, we have that $\{x : h_P(x) \leq C_k\}$ is bounded, and is clearly closed, hence is compact.

Next, for $\epsilon > 0$, by definition of u there is $C_k' > 0$ and C_k'' sufficiently large that

$$\{x : h_P(x) \leq C_k\} \subset \{x : (1+\epsilon)h_P - C_k' \leq v(x)\} \subset \{x : h_P(x) \leq C_k''\}$$

Let $G = \{x : (1+\epsilon)h_P(x) - C_k' \leq v(x)\}$. Then G is compact, and $(1+\epsilon)h_P - C_k' = v$ on ∂G. Therefore, by Lemma 20

$$Mu(kB) \leq Mu(G^\circ) \leq M[(1+\epsilon h_P - C_k')](G^\circ) = (1+\epsilon)Mh_P(G^\circ) \leq (1+\epsilon)Mh_P(\mathbb{R}^n).$$

Taking $k \to \infty$ gives, by continuity of measures,

$$Mu(\mathbb{R}^n) \subset (1+\epsilon)Mh_P(\mathbb{R}^n).$$

Since $\epsilon > 0$ was arbtirary, we have $Mu(\mathbb{R}^n) \leq Mh_P(\mathbb{R}^n) = m(P)$. Now, switching the roles of u and h_P in constructing the set G above gives

$$m(P) = Mh_P(\mathbb{R}^n) \leq Mu(\mathbb{R}^n),$$

which finishes the proof when P has empty interior.

If P has empty interior, then for $p \in \partial u(\mathbb{R}^n)$, it holds $x \cdot p - u(x) \leq v^*(p) < \infty$ for all $x \in \mathbb{R}^n$, so for some $M > 0$ large, we have $x \cdot p - h_P(x) \leq x \cdot p - u(x) + M \leq v^*(P) + M$. Therefore, $h_P^*(p) < \infty$, which means $h_P^*(p) \leq 0$ by positive homogeneity, and therefore $p \in P$. This gives

$$\partial w(\mathbb{R}^n) \subset P,$$

so $0 \leq Mu(\mathbb{R}^n) \leq m(P) = 0$, which finishes the proof. □

EXERCISE 51. Show that $0 \in P^\circ$ is both a necessary and sufficient condition for $\{x : h_P(x) \leq C\}$ to be compact for every $C > 0$.

EXERCISE 52. Show that a function $f : \mathbb{R}^n \to \mathbb{R}$ is the support of a polyhedron (i.e., the convex hull of finitely many points) if and only if there are $x_1, \ldots, x_m$ such that $f(x) = \max\{x_1 \cdot x, \ldots, x_m \cdot x\}$ for all $x \in \mathbb{R}^n$.

EXERCISE 53. A compact convex set $P \subset \mathbb{R}^n$ is called a *polygon* if it is the convex hull of finitely many points). Show that the set of polygons is dense in $\mathscr{H}^n$ with its metric.

EXERCISE 54. Let K be a closed subset of $\mathbb{R}^n$. Show that K is convex if and only if $\frac{1}{2}K + \frac{1}{2}K \subset K$.

EXERCISE 55. We call a set $K \in \mathscr{H}^n$ a convex body if $m(K) > 0$, where m is the Lebesgue measure. Show that this is equivalent to saying K has a nonempty interior. (Hint: if K has nonempty interior $m(K) > 0$ is trivial. For the converse, use the fact $m(\frac{1}{2}K) > 0$ and the fact $\frac{1}{2}K + \frac{1}{2}K \subset K$.)

EXERCISE 56. If A and B are symmetric positive semi-definite, then $\det[A+B] \geq \det[A] + \det[B]$. *Hint*: Use the fact that there is a unique symmetric positive semi-definite matrix X such that $A = X^\top X$. Show that

$$\det[I + A] \geq 1 + \det[A].$$

THEOREM 57. *Let $K_1, \ldots, K_m$ be compact, convex sets in $\mathbb{R}^n$. Then the function $f : [0,\infty)^m \to [0,\infty)$ defined by*

$$f(t_1, \ldots, t_m) = m(t_1 K_1 + \cdots + t_m K_m)$$

is a homogeneous polynomial of degree n, and the coefficients of this polynomial are all non-negative.

We prove this theorem with the aid of the following lemma.

LEMMA 58. *Suppose that $A_1,\dots,A_m$ are $n\times n$ positive semi-definite matrices. Then the function $f : [0,\infty)^m \to [0,\infty)$ defined by $f(t_1,\dots,t_m) = \det[t_1A_1+\cdots+t_mA_m]$ is a homogeneous polynomial of degree n whose coefficients are all non-negative.*

PROOF. Fi □

DEFINITION 59 (MIXED VOLUME). Let $K_1,\dots,K_m$ be convex bodies. By Theorem 57, for $t_1,\dots,t_m \geq 0$, one may write

$$m(t_1K_1+\cdots+t_mK_m) = \sum_{|\alpha|=n} c_\alpha t^\alpha,$$

where $t = (t_1,\dots,t_m)$ and the sum ranges over all multi-indices $\alpha = (\alpha_1,\dots,\alpha_m)$ with $|\alpha| = n$. Then the coefficient c_α, which shall henceforth be denoted $V(K_1[\alpha_1],\dots,K_{})$, is called the α-th mixed volume. When $\alpha = (1,\dots,1)$

We next endeavor to show the following is true.

THEOREM 60. *For a fixed $m \geq 1$ and multindex $\alpha = (\alpha_1,\dots,\alpha_m)$ with $|\alpha| = n$, the mixed volume functional $MV_\alpha : (\mathscr{H}^n)^m \to [0,\infty)$ is continuous.*

Towards that end, we first demonstrate that the volume of a convex set is continuous.

THEOREM 61. *The mapping $f : \mathscr{H}^n \to [0,\infty)$ defined by $f(A) = m(A)$ is continuous.*

PROOF. Suppose that $\{A_j\} \subset \mathscr{H}^n$ converges to some $A \in \mathscr{H}^n$. Then the sequence of support functions $\{h_{A_j}\}$ converges uniformly to h_A on compact sets. Therefore, we deduce by Lemma 11 that

$$\liminf_{j\to\infty} MA_j(\mathbb{R}^n) \geq MA(\mathbb{R}^n).$$

However, we also know that $\partial h_{A_j}(\mathbb{R}^n) = \partial h_{A_j}(0)$ for every j, and the same is true of h_A (why?), therefore we have

$$\limsup_{j\to\infty} MA_j(\mathbb{R}^n) = \limsup_{j\to\infty} MA_j(\{0\}) \leq MA(\{0\}) = MA(\mathbb{R}^n).$$

However, $MA_j(\mathbb{R}^n) = m(A_j)$ and $MA(\mathbb{R}^n) = m(A)$, and we deduce

$$\lim_{j\to\infty} m(A_j) = m(A),$$

finishing the proof. □

Given the nice analytical properties of support functions, it is beneficial to have a way of computing mixed volumes by way of support functions. Before we prove the main result, we need a few estimation lemmas.

DEFINITION 62 (HOMOGENEOUS EXTENSION). For a function $u \in C(\mathbb{S}^{n-1})$, the homogeneous extension of u is the function $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\tilde{u}(x) = \begin{cases} |x|u(x/|x|) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

THEOREM 63. *If $K_1,\dots,K_n \in \mathscr{H}^n$ and $Q_1,\dots,Q_n \in \mathscr{H}^n$ are such that $K_j \subseteq Q_j$ for each j, then*

$$V(K_1,\dots,K_n) \leq V(Q_1,\dots,Q_n).$$

PROOF. Since $K_j \subset Q_j$, it holds $h_{K_j} \leq h_{Q_j}$. Let $c > 0$. If $v_j = \max\{h_{K_j}, h_{Q_j} - c\}$, then $v_j \in \mathscr{C}_{Q_1}$. Indeed,

$$h_{Q_j} - c \leq v \leq h_{Q_j},$$

which is what we claimed. Therefore,

$$M[h_{Q_1},\dots,h_{Q_n}] = M[v_1,\dots,v_n],$$

and in particular,

$$\begin{aligned} V(Q_1,\dots,Q_n) &= M[v_1,\dots,v_n](\mathbb{R}^n) \\ &\geq M[v_1,\dots,v_n]\left(\bigcap_{j=1}^n \{h_{K_j} > h_{Q_j} - c\}\right) \\ &= M[h_{K_1},\dots,h_{K_n}]\left(\bigcap_{j=1}^n \{h_{K_j} > h_{Q_j} - c\}\right). \end{aligned}$$

As $c \to \infty$, we have $\bigcap_{j=1}^n \{h_{K_j} > h_{Q_j} - c\} \nearrow \mathbb{R}^n$, so

$$V(Q_1,\dots,Q_n) \geq M[h_{K_1},\dots,h_{K_n}](\mathbb{R}^n) = V(K_1,\dots,K_n),$$

as desired. □

DEFINITION 64. We denote by $C^2(\mathbb{S}^{n-1})$ the set of function $u \in C(\mathbb{S}^{n-1})$ whose homogeneous extensions are in $C^2(\mathbb{R}^n \setminus \{0\})$.

THEOREM 65. *The space $C^2(\mathbb{S}^{n-1})$ is uniformly dense in $C(\mathbb{S}^{n-1})$.*

PROOF. The proof is essentially by the Stone-Weierstrass Theorem. □

THEOREM 66. *Every positive homogeneous element of $C^2(\mathbb{R}^n \setminus \{0\})$ is the difference of two support functions. That is, if $u \in C^2(\mathbb{R}^n \setminus \{0\})$, then there are compact, convex sets K and L such that $u = h_K - h_L$.*

We will prove this lemma with the aid of a simple linear algebra lemma.

LEMMA 67. *Let M be a subspace of $\mathbb{R}^n$, and suppose that $A : \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric linear map satisfying $A|_{M^\perp} = 0$. If A is positive semi-definite on M, then A is positive semi-definite on $\mathbb{R}^n$.*

PROOF. If $x \in \mathbb{R}^n$, then write $x = \alpha + \beta$, where $\alpha \in M$ and $\beta \in M^\perp$. Then by symmetry and the fact $A|_{M^\perp} = 0$,

$$x^\top Ax = (\alpha+\beta)^\top A(\alpha+\beta) = \alpha^\top A\alpha + \beta^\top A\alpha = \alpha^\top A\alpha + \alpha^\top A\beta = \alpha^\top A\alpha \geq 0,$$

which finishes the proof. □

We now prove Theorem 66.

PROOF. By 1-homogeneity of u, we have $D^2u(\alpha x) = \frac{1}{\alpha} D^2u(x)$ for $x \in \mathbb{R}^n \setminus \{0\}$ and $\alpha > 0$. That is, D^2u is -1-homogeneous. Next, we observe that $D^2u(x)x = 0$. Since by 1-homogeneity we have $\nabla u(\alpha x) = \nabla u(x)$, by definition of the Hessian,

$$\begin{aligned} 0 = \lim_{|h|\to 0} \frac{|\nabla u(x+h) - \nabla u(x) - D^2u(x)h|}{|h|} &= \lim_{t\to 0} \frac{|\nabla u((1+t)x) - \nabla u(x) - D^2u(x)tx|}{t|x|} \\ &= \frac{|D^2u(x)x|}{|x|}. \end{aligned}$$

Therefore $D^2u(x)x = 0$ so that $D^2u(x)|_{\mathrm{span}(x)} = 0$. Next, we claim that if $|x| = 1$ and $h \in x^\perp$, then $D^2h_B(x)h = h$. To see this, let $h = (h_1,\dots,h_n)$, and $x = (x_1,\dots,x_n)$. Then the inner product of the j-th row of $D^2h_B(x)$ and u is given by

$$\begin{aligned} -\sum_{i\neq j} h_i x_i x_j + h_j \sum_{i\neq j} x_i^2 &= -x_j h\cdot x + h_j x_j^2 + h_j \sum_{i\neq j} x_i^2 \\ &= h_j \sum_{i=1}^n x_i^2 \\ &= h_j. \end{aligned}$$

This proves $D^2 h_B(x)h = h$. Next, let $R \geq 0$ be chosen such that $-R \leq \theta_1^\top D^2 u(\theta_2)\theta_1$ for all $\theta_1, \theta_2 \in \mathbb{S}^{n-1}$. By continuity, R can be made finite. Then, for a fixed $x \in \mathbb{S}^{n-1}$ and all $h \in x^\perp$, we have

$$h^\top (D^2 u(x) + RD^2 h_B(x))h = h^\top D^2 u(x)h + Rh^\top h = |h|^2 \left(\left(\frac{h}{|h|}\right)^\top D^2 u(x) \left(\frac{h}{|h|}\right) + R \right) \geq 0.$$

Since $D^2 u(x) + RD^2 u(x)$ is symmetric and and $(D^2 u(x) + RD^2 u(x))|_{\mathrm{span}(x)} = 0$, by Lemma 67, this shows that $D^2 u(x) + RD^2 h_B(x)$ is a positive semi-definite matrix on $\mathbb{R}^n$. Since $D^2 u$ and $RD^2 h_B$ are -1 homogeneous, it follows $D^2 u(x) + RD^2 h_B(x)$ is positive semi-definite for all $x \in \mathbb{R}^n \setminus \{0\}$. Therefore, $u + Rh_B$ is convex and positive homogeneous, so there exists a compact convex set K such that $u + Rh_B = h_K$ by Lemma 38. This proves $u = h_K - h_{RB}$, which was to be shown. □

Corollary 68. *The set of functions $\{h_K - h_L : K, L \in \mathcal{K}^n\}$ is uniformly dense in $C(\mathbb{S}^{n-1})$.*

Theorem 69. *Suppose $K_1, \ldots, K_n$ are compact, convex sets. Then there exists a unique finite Borel measure $S_{K_1,\ldots,K_n}$ on $\mathbb{S}^{n-1}$ such that*

$$V(K_1, \ldots, K_n, K) = \int_{\mathbb{S}^{n-1}} h_K \, dS_{K_1,\ldots,K_n}$$

for all $K \in \mathcal{K}^n$.

Proof. Define a linear operator T as follows: if $u \in C^2(\mathbb{S}^{n-1})$, then $u = h_K - h_L$ for some compact, convex sets K and L by Theorem 66. Therefore, define

$$Tu := V(K_1, \ldots, K_{n-1}, K) - V(K_1, \ldots, K_{n-1}, L).$$

To see that T is well-defined, if $h_K - h_L = h_{K'} - h_{L'}$ for $K', L' \in \mathcal{K}^n$, then $K + L' = L + K'$, so

$$V(K_1, \ldots, K_{n-1}, K + L') = V(K_1, \ldots, K_{n-1}, K' + L),$$

and by using the multi-linearity of mixed volumes we obtain

$$\begin{aligned} T(h_K - h_L) &= V(K_1, \ldots, K_{n-1}, K) - V(K_1, \ldots, K_{n-1}, L) \\ &= V(K_1, \ldots, K_{n-1}, K') - V(K_1, \ldots, K_{n-1}, L') = T(h_{K'} - h_{L'}). \end{aligned}$$

To see that T is linear, if $v = h_M - h_N \in C^2(\mathbb{S}^{n-1})$ and $\alpha \geq 0$, we have $\alpha u + v = h_{\alpha K + M} - h_{\alpha L + N}$

$$\begin{aligned} T(\alpha u + v) &= V(K_1, \ldots, K_{n-1}, \alpha K + M) - V(K_1, \ldots, K_{n-1}, \alpha L + N) \\ &= \alpha(V(K_1, \ldots, K_{n-1}, K) - V(K_1, \ldots, K_{n-1}, L)) + V(K_1, \ldots, K_{n-1}, M) - V(K_1, \ldots, K_{n-1}, N) \\ &= \alpha Tu + Tv \end{aligned}$$

where again we used the multi-linearity of mixed volumes. If $\alpha \leq 0$, then $\alpha u + v = h_{-\alpha L + M} - h_{-\alpha K + N}$, hence

$$\begin{aligned} T(\alpha u + v) &= V(K_1, \ldots, K_{n-1}, -\alpha L + M) - V(K_1, \ldots, K_{n-1}, -\alpha K + N) \\ &= \alpha(V(K_1, \ldots, K_{n-1}, K) - V(K_1, \ldots, K_{n-1}, L)) + V(K_1, \ldots, K_{n-1}, M) - V(K_1, \ldots, K_{n-1}, N) \\ &= \alpha Tu + Tv, \end{aligned}$$

which establishes linearity of T.

We next assert that T is positive. That is, if $u \geq 0$, then $Tu \geq 0$. Indeed, if $u = h_K - h_L \geq 0$, then $L \subset K$, so $V(K_1, \ldots, K_{n-1}, K) \geq V(K_1, \ldots, K_{n-1}, L)$ by the monotonicity of mixed volumes. It follows that $Tu \geq 0$. We claim that this implies T is a bounded linear operator on $C^2(\mathbb{S}^{n-1})$. Indeed, if $u \in C^2(\mathbb{S}^{n-1})$, then $\|u\|_\infty - u \in C^2(\mathbb{S}^{n-1})$ and $\|u\|_\infty - u \geq 0$, from which it follows $T(\|u\|_\infty - u) \geq 0$, or equivalently

$$T(u) \leq T(\|u\|_\infty) = V(K_1, \ldots, K_{n-1}, B)\|u\|_\infty.$$

Repeating this argument with $\|u\|_\infty + u$ shows $-T(u) \leq V(K_1, \ldots, K_{n-1}, B)\|u\|_\infty$, and we deduce

$$|Tu| \leq V(K_1, \ldots, K_{n-1}, B)\|u\|_\infty.$$

Since $C^2(\mathbb{S}^{n-1})$ is uniformly dense in $C(\mathbb{S}^{n-1})$ by Theorem 66, it follows that T extends uniquely to a continuous positive linear operator on $C(\mathbb{S}^{n-1})$. By the Riesz-Kakutani theorem, there exists a unique positive Borel measure $S_{K_1,\dots,K_{n-1}}$ on $\mathbb{S}^{n-1}$ such that

$$Tf = \int_{\mathbb{S}^{n-1}} f\, dS_{K_1,\dots,K_{n-1}}$$

for all $f \in C(\mathbb{S}^{n-1})$. Taking $f = h_K$ for any $K \in \mathscr{K}^n$ yields

$$V(K_1,\dots,K_{n-1},K) = \int_{\mathbb{S}^{n-1}} h_K\, dS_{K_1,\dots,K_{n-1}},$$

as desired. □

EXERCISE 70. Show that the surface measure of the ball satisfies the following relationship with the Lebesgue measure:

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_{\mathbb{S}^{n-1}} f(r\theta) r^{n-1}\, dS_B(\theta)\, dr.$$

EXERCISE 71. For any $K \in \mathscr{K}^n$, show that

$$\int_{\mathbb{S}^{n-1}} u\, dS_K(u) = 0,$$

where the integral is taken coordinate-wise.

EXERCISE 72. An alternative metric one can introduce on $\mathscr{K}^n$ is the so-called "mean width" metric ρ_W, defined by

$$\rho_W(K,L) = 2V_1([K,L]) - V_1(K) - V_1(L).$$

Here, $V_1(A) := V(B[n-1], A)$ is the first intrinsic volume and $[K,L]$ is the smallest convex set containing K and L. Prove that ρ_W is indeed a metric and that the metric topology generated by ρ_W is the same as the one generated by the Hausdorff metric. Moreover, show that if $\{K_n\}$ and $\{L_n\}$ are sequences in $\mathscr{K}^n$ is such that $d_H(K_n, L_n) \to 0$, then $\rho_W(K_n, L_n) \to 0$, but the converse need not hold.

EXERCISE 73. Fix a compact convex set G with nonempty interior, and let $\mathscr{G}_j(E) := V(G[n-j], E[j])$ for $E \in \mathscr{K}^n$. Let $\mathscr{E}$ denote the set of $K \in \mathscr{K}^n$ such that $K \subset G$. Define a function $\rho : \mathscr{E}^n \times \mathscr{E}^n \to [0,\infty)$ by

$$\rho(K,L) = \sum_{j=1}^{n} 2\mathscr{G}_j([K,L]) - \mathscr{G}_j(K) - \mathscr{G}_j(L).$$

Here, $[K,L]$ is the smallest convex set containing K and L. Using techniques from complex geometry beyond the scope of this text, one can show that ρ satisfies a quasi-triangle inequality. That is, there exists a constant $D > 1$ depending on G and n such that $\rho(K,L) \le D(\rho(K,U) + \rho(U,L))$. The purpose of this exercise will be to show that ρ satisfies the other axioms of a metric. Namely, show that $\rho \ge 0$, that $\rho(K,L) = \rho(L,K)$ and $\rho(K,L) = 0$ if and only if $K = L$

EXERCISE 74 (CONTINUATION OF 73). We continue with the notation of Exercise 73. The quasi-metric ρ generates a topology on $\mathscr{E}$. It is a deep result that this topology is actually metrizable. Show that this topology is the same as the one generated on $\mathscr{E}$ by the Hausdorff distance d_H. In particular, show the even stronger result that there exists a constant $A > 0$, depending only on G and n, such that

$$\rho(K,L) \le A d_H(K,L), \quad \forall K, L \subset G.$$

(*Hint*: To show equivalence, it is enough to show that a sequence converging in ρ converges in d_H and vice-versa. Use mixed area measures to prove the estimate $\rho \le A d_H$. Use the Theorem 42 to show convergence in ρ implies convergence in d_H.)

0.3. Answers to Selected Exercises.

SOLUTION (16).

PROOF. Fix $F \subset U$ a compact set. By Urysohn's lemma, there is a compactly supported continuous function u supported in U such that $0 \le u \le 1$ and $u(x) = 1$ for $x \in F$. Then

$$\mu(F) \le \int u\, d\mu = \lim_{n\to\infty} \int u\, d\mu_n \le \liminf_{n\to\infty} \mu_n(U).$$

Since μ is an inner-regular Borel measure, taking the supremum over all such F gives

$$\mu(U) \le \liminf_{n\to\infty} \mu_n(U).$$

On the other hand, if K is compact, then pick an open set V such that $K \subset V$. Again taking u to be a compactly supported continuous function supported in V with $0 \le u \le 1$ and $u(x) = 1$ on K, we have

$$\limsup_{n\to\infty} \mu_n(K) \le \lim_{n\to\infty} \int u\, d\mu_n = \int u\, d\mu \le \mu(V).$$

Since μ is an outer-regular Borel measure, taking the infimum over all open sets $V \supset K$ gives

$$\limsup_{n\to\infty} \mu_n(K) \le \mu(K).$$

To demonstrate the converse, □

SOLUTION (18).

PROOF. When $u \in C^2(\Omega)$, an integration by parts (using the fact ϕ has compact support in Ω) shows that

$$\int_\Omega u\phi''\, dx = \int_\Omega \phi u''\, dx = \int_\Omega \phi\, dMu$$

by Example 9. If u is any convex function in Ω, then Exercise 17 guarantees a sequence of smooth convex functions $\{u_n\}$ such that $u_n \to u$ uniformly on compact subsets of Ω. Thus, for a fixed $\phi \in C_c^\infty(\Omega)$, uniform convergence of $\{u_n\}$ on $\operatorname{supp}\phi$ gives

$$\lim_{n\to\infty} \int_\Omega u_n\phi''\, dx = \int_\Omega u\phi''\, dx.$$

However, since u_n is smooth, we have just shown that

$$\int_\Omega u_n\phi''\, dx = \int_\Omega \phi\, dMu_n,$$

and by Theorem 12,

$$\lim_{n\to\infty} \int_\Omega \phi\, dMu_n = \int_\Omega \phi\, dMu,$$

finishing the proof. □

SOLUTION. We prove the theorem with the assistance of the following lemma.

LEMMA. *Let $\mathscr{F}$ be a bounded family of sets in $\mathbb{R}^n$ (i.e., there is a ball B such that $A \in \mathscr{F}$ implies $A \subset B$), and let G denote the smallest convex set containing all elements of $\mathscr{F}$. Then*

$$h_G = \sup_{A\in\mathscr{F}} h_A.$$

PROOF. Since $A \subset G$ for all $A \in \mathscr{F}$, we have

$$\sup_{A\in\mathscr{F}} h_A \le h_G.$$

For the converse, if $p \in G$ then there are positive constants $\lambda_1, \dots, \lambda_m$ with $\lambda_1 + \cdots + \lambda_m = 1$ and $x_1, \dots, x_m$ such that $x_k \in A_k$ for some $A_k \in \mathscr{F}$ such that $p = \lambda_1 x_1 + \cdots + \lambda_m x_m$. Then for $x \in \mathbb{R}^n$, it holds

$$x \cdot p = \lambda_1 (x \cdot x_1) + \cdots + \lambda_m (x \cdot x_m) \le \max(h_{A_1}(x), \dots, h_{A_m}(x)) \le \sup_{A\in\mathscr{F}} h_A(x).$$

Since $p \in G$ is arbitrary, $h_G(x) \le \sup_{A \in \mathscr{F}} h_A(x)$. $\square$

With the lemma proved, we can solve the rest of the problem.

PROOF. Suppose that $K = \operatorname{conv}(x_1, \dots, x_m)$ for finitely many $x_1, \dots, x_m \in \mathbb{R}^n$. Then by the lemma,

$$h_K(x) = \max\{x_1 \cdot x, \dots, x_m \cdot x\},$$

which proves one direction. For the converse, suppose that $f(x) = \max\{x_1 \cdot x, \dots, x_m \cdot x\}$. Since f is convex, continuous, and positively homogeneous, $f = h_K$ for some compact K. If $L = \operatorname{conv}(x_1, \dots, x_m)$, then we have $h_L = \max\{x_1 \cdot x, \dots, x_m \cdot x\}$, so $h_L = h_K$. This means $L = K$, so f is the support function of a polyhedron. $\square$

SOLUTION (53). There are multiple ways to approach this problem. We present a purely analytical one.

PROOF. Fix $\epsilon > 0$, and for each $p \in \partial K$, let $U_p = \{x \in \mathbb{S}^{n-1} : h_K(x) - p \cdot x < \epsilon\}$. Since for each $x \in \mathbb{S}^{n_1}$ there exists $p \in \partial K$ with $h_K(x) = p \cdot x$, it holds that $\{U_p\}$ is an open cover of $\mathbb{S}^{n-1}$. Therefore, choose a finite sub-cover $U_{p_1}, \dots, U_{p_k}$, and let $P = \operatorname{conv}(p_1, \dots, p_k) \subset K$. Then for $x \in \mathbb{R}^n$, $h_P(x) = \max\{p_1 \cdot x, \dots, p_k \cdot x\}$, and if $\theta \in \mathbb{S}^{n-1}$, there is p_j with $\theta \in U_{p_j}$ hence $0 \le h_K(\theta) - h_P(\theta) \le h_K(\theta) - \theta \cdot p_j < \epsilon$. It follows $d_H(K, P) < \epsilon$. $\square$

A careful examination of this proof shows that, by replacing the sets U_p by $\{x \in \mathbb{S}^{n-1} : h_K(x) - p \cdot x < \delta p \cdot x\}$ for a suitable $\delta > 0$, we can find a polyhedron P such that $P \subset K \subset (1+\epsilon)P$ and $d_H(K, P) < \epsilon$.

SOLUTION (55). We make use of the following classical result of measure theory, known as Steinhaus' Theorem:

LEMMA (STEINHAUS' THEOREM). *Let $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^n$ be Lebesgue measurable sets, and suppose $m(E) > 0$ and $m(F) > 0$. Then $E + F$ has a nonempty interior.*

PROOF. There are many proofs of this fact, we present in our opinion the cleanest solution. Define a function $f : \mathbb{R}^n \to [0, \infty)$ by $f(x) = m(E \cap (x - F))$. Let $U = \{x : f(x) > 0\}$. Since $f = \mathbb{1}_E * \mathbb{1}_F$, where $*$ is the convolution, it follows f is continuous, so that U is open. Moreover, $U \subset E + F$, since $f(x) > 0$ means $E \cap (x - F) \neq \emptyset$ and therefore $x \in E + F$. To see that $U \neq \emptyset$, observe that $\|f\|_1 = \|\mathbb{1}_E\|_1 \|\mathbb{1}_F\|_1 = m(E)m(F) > 0$, where $\|\cdot\|_1$ is the canonical norm on $L^1(\mathbb{R}^n)$. This proves f is not identically 0, hence $E + F$ contains the nonempty open set U. $\square$

The rest of the problem now follows in a straightforward fashion:

PROOF. If K has nonempty interior then K contains a nonempty open ball, therefore K has positive measure. For the converse, we have $m(K/2) = 2^{-n} m(K) > 0$, so by Steinhaus' Theorem $\frac{1}{2}K + \frac{1}{2}K$ has a nonempty interior. Since $\frac{1}{2}K + \frac{1}{2}K \subset K$ by convexity, the problem is finished. $\square$

SOLUTION. 46

PROOF. Let

$$\eta_1 = \sup_{x \in U} d(x, V), \quad \eta_2 = \sup_{x \in V} d(x, U), \quad \eta = \max\{\eta_1, \eta_2\}.$$

It then follows that $U \subset \{x : d(x, V) \le \eta_1\}$ and $V \subset \{x : d(x, U) \le \eta_2\}$. It is easy to see that $\{x : d(x, V) \le \eta_1\} = V + B_{\eta_1} \subset V + B_\eta$ and $\{x : d(x, U) \le \eta_2\} = U + B_{\eta_2} \subset U + B_\eta$, which means that $d_H(U, V) \le \eta$. For the converse, fix $\delta > d_H(U, V)$. Then $U \subset V + B_\delta$ and $V \subset U + B_\delta$, which means $U \subset \{x : d(x, V) \le \delta\}$ and $V \subset \{x : d(x, U) \le \delta\}$. Therefore $d(x, U) \le \delta$ for all $x \in V$ and $d(x, V) \le \delta$ for all $x \in U$, hence $\eta_1 \le \delta$ and $\eta_2 \le \delta$, so that $\eta \le \delta$. Since $\delta > d_H(U, V)$ was arbitrary, we get $\eta \le d_H(U, V)$, which finishes the proof. $\square$