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Notes on Mixed Area Measures

Preliminary Basics

Let $\mathbb{K}^n$ denote the set of compact, convex sets in $\mathbb{R}^n$. Recall the *support function* h_K of a set $K \in \mathbb{K}^n$, defined as

$$h_K(x) = \sup_{p\in K} p\cdot x, \quad x \in \mathbb{R}^n.$$

It is easy to check that every h_K is a convex 1-homogeneous Lipschitz function $\mathbb{R}^n$. In particular, if h_K^* is the Legendre transform of h_K, then $h_K^*(x) \in \{0,\infty\}$ for all $x \in \mathbb{R}^n$. Moreover, $h_{\alpha K+L} = \alpha h_K + h_L$ for every $\alpha > 0$ and $K, L \in \mathbb{K}^n$.

Lemma 1. *If $K \in \mathbb{K}^n$, then $K = \{p \in \mathbb{R}^n : h_K^*(p) = 0\}$. In particular, if $h_K(x) \le h_L(x)$ for all $x \in \mathbb{R}^n$, then $K \subset L$.*

Proof. If $p \in K$, then $x \cdot p \le h_K(x)$ for all $x \in \mathbb{R}^n$. Therefore, $h_K^*(x) \le 0$, which forces $h_K^*(x) = 0$. Hence $K \subset \{p : h_K^*(p) = 0\}$. To obtain a contradiction, suppose there is $p \in \{p : h_K^*(p) = 0\} \setminus K$. By the hyperplane separation theorem, there exists $\alpha \in \mathbb{R}$ and $y \in \mathbb{R}^n$ such that $x \cdot y \le \alpha$ for all $x \in K$ and $p \cdot y > \alpha$. Therefore $h_K(y) \le \alpha < p \cdot y$, which contradicts the fact $h_K^*(y) = 0$.

If $h_K \le h_L$, then $h_L^* \le h_K^*$, so that $K = \{p : h_K^*(p) = 0\} \subset \{p : h_L^*(P) = 0\} = P$. □

Moreover, every continuous, convex homogeneous function on $\mathbb{R}^n$ is a support function.

Lemma 2. *If $u : \mathbb{R}^n \to \mathbb{R}$ is a continuous, homogeneous, and convex, then $u = h_K$ for some compact, convex K.*

Proof. One verifies by homogeneity that $u^*(p) \in \{0,\infty\}$ for all $p \in \mathbb{R}^n$. Therefore, let $K = \{p : u^*(p) = 0\}$. Then by the Fenchel duality theorem one has $u = h_K$. □

Primary Result

The main purpose of this section will be to prove Theorem 3.

Theorem 3. *Let $K_1,\dots,K_{n-1} \in \mathbb{K}^n$. Then there exists a unique positive Borel measure $S_{K_1,\dots,K_{n-1}}$ on $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ such that*

$$V(K_1,\dots,K_{n-1},K) = \int_{\mathbb{S}^{n-1}} h_K \, dS_{K_1,\dots,K_{n-1}}$$

for all $K \in \mathbb{K}^n$.

Definition 4. If $u \in C(\mathbb{S}^{n-1})$, then the *homogeneous extension* $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ is the function defined by

$$\tilde{u}(x) := \begin{cases} |x|u(x/|x|), & x \ne 0 \\ 0 & x = 0. \end{cases}$$

Definition 5. Let $C^2(\mathbb{S}^{n-1})$ denote the class of functions in $C(\mathbb{S}^{n-1})$ whose homogeneous extensions are in $C^2(\mathbb{R}^n \setminus \{0\})$.

Lemma 6. *The set $C^2(\mathbb{S}^{n-1})$ is uniformly dense in $C(\mathbb{S}^{n-1})$.*

In the sequel, we will use $\|\cdot\|_\infty$ to denote the uniform norm on the sphere. This is, if $u \in C(\mathbb{S}^{n-1})$, then

$$\|u\|_\infty := \sup_{x\in\mathbb{S}^{n-1}} |u(x)|.$$

PROOF. The proof is by the Stone-Weierstrass theorem. Indeed, it is easy to check that $C^2(\mathbb{S}^{n-1})$ is closed under finite linear combinations and contains the constant functions. If $u, v \in C^2(\mathbb{S}^{n-1})$, then

$$\widetilde{uv}(x) = \frac{\tilde{u}(x)\,\tilde{v}(x)}{|x|},$$

which is twice differentiable except at 0. If $x \neq y$, then the function $f \in C^2(\mathbb{S}^{n-1})$ defined by $f(\theta) = |x\cdot\theta|$ satisfies $f(x) \neq f(y)$ by the Cauchy-Schwarz inequality, so $C^2(\mathbb{S}^{n-1})$ separates points. The hypotheses of the Stone-Weierstrass theorem are therefore satisfied. □

The real reason we care about the class $C^2(\mathbb{S}^{n-1})$ is because of the following Theorem.

THEOREM 7. *Every positive homogeneous element of $C^2(\mathbb{R}^n\setminus\{0\})$ is the difference of two support functions. That is, if $u \in C^2(\mathbb{R}^n\setminus\{0\})$, then there are compact, convex sets K and L such that $u = h_K - h_L$.*

We will prove this lemma with the aid of a simple linear algebra lemma.

LEMMA 8. *Let M be a subspace of $\mathbb{R}^n$, and suppose that $A : \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric linear map satisfying $A|_{M^\perp} = 0$. If A is positive semi-definite on M, then A is positive semi-definite on $\mathbb{R}^n$.*

PROOF. If $x \in \mathbb{R}^n$, then write $x = \alpha + \beta$, where $\alpha \in M$ and $\beta \in M^\perp$. Then by symmetry and the fact $A|_{M^\perp} = 0$,

$$x^\top Ax = (\alpha+\beta)^\top A(\alpha+\beta) = \alpha^\top A\alpha + \beta^\top A\alpha = \alpha^\top A\alpha + \alpha^\top A\beta = \alpha^\top A\alpha \geq 0,$$

which finishes the proof. □

We now prove Theorem 7.

PROOF. By 1-homogeneity of u, we have $D^2u(\alpha x) = \frac{1}{\alpha}D^2u(x)$ for $x \in \mathbb{R}^n\setminus\{0\}$ and $\alpha > 0$. That is, D^2u is -1-homogeneous. Next, we observe that $D^2u(x)x = 0$. Since by 1-homogeneity we have $\nabla u(\alpha x) = \nabla u(x)$, by definition of the Hessian,

$$\begin{aligned} 0 = \lim_{|h|\to 0}\frac{|\nabla u(x+h) - \nabla u(x) - D^2u(x)h|}{|h|} &= \lim_{t\to 0}\frac{|\nabla u((1+t)x) - \nabla u(x) - D^2u(x)tx|}{t|x|} \\ &= \frac{|D^2u(x)x|}{|x|}. \end{aligned}$$

Therefore $D^2u(x)x = 0$ so that $D^2u(x)|_{\operatorname{span}(x)} = 0$. Next, we claim that if $|x| = 1$ and $h \in x^\perp$, then $D^2h_B(x)h = h$. To see this, let $h = (h_1,\ldots,h_n)$, and $x = (x_1,\ldots,x_n)$. Then the inner product of the j-th row of $D^2h_B(x)$ and u is given by

$$\begin{aligned} -\sum_{i\neq j}h_ix_ix_j + h_j\sum_{i\neq j}x_i^2 &= -x_jh\cdot x + h_jx_j^2 + h_j\sum_{i\neq j}x_i^2 \\ &= h_j\sum_{i=1}^n x_i^2 \\ &= h_j. \end{aligned}$$

This proves $D^2h_B(x)h = h$. Next, let $R \geq 0$ be chosen such that $-R \leq \theta_1^\top D^2u(\theta_2)\theta_1$ for all $\theta_1,\theta_2 \in \mathbb{S}^{n-1}$. By continuity, R can be made finite. Then, for a fixed $x \in \mathbb{S}^{n-1}$ and all $h \in x^\perp$, we have

$$h^\top(D^2u(x) + RD^2h_B(x))h = h^\top D^2u(x)h + Rh^\top h = |h|^2\left(\left(\frac{h}{|h|}\right)^\top D^2u(x)\left(\frac{h}{|h|}\right) + R\right) \geq 0.$$

Since $D^2u(x) + RD^2u(x)$ is symmetric and and $(D^2u(x) + RD^2u(x))|_{\operatorname{span}(x)} = 0$, by Lemma 8, this shows that $D^2u(x) + RD^2h_B(x)$ is a positive semi-definite matrix on $\mathbb{R}^n$. Since D^2u and RD^2h_B are -1 homogeneous, it follows $D^2u(x) + RD^2h_B(x)$ is positive semi-definite for all $x \in \mathbb{R}^n\setminus\{0\}$. Therefore, $u + Rh_B$ is convex and positive homogeneous, so there exists a compact convex set K such that $u + Rh_B = h_K$ by Lemma 2. This proves $u = h_K - h_{RB}$, which was to be shown. □

We can now prove Theorem 3.

PROOF. Define a linear operator T as follows: if $u \in C^2(\mathbb{S}^{n-1})$, then $u = h_K - h_L$ for some compact, convex sets K and L by Theorem 7. Therefore, define

$$Tu := V(K_1, \dots, K_{n-1}, K) - V(K_1, \dots, K_{n-1}, L).$$

To see that T is well-defined, if $h_K - h_L = h_{K'} - h_{L'}$ for $K', L' \in \mathcal{K}^n$, then $K + L' = L + K'$, so

$$V(K_1, \dots, K_{n-1}, K + L') = V(K_1, \dots, K_{n-1}, K' + L),$$

and by using the multi-linearity of mixed volumes we obtain

$$\begin{aligned} T(h_K - h_L) &= V(K_1, \dots, K_{n-1}, K) - V(K_1, \dots, K_{n-1}, L) \\ &= V(K_1, \dots, K_{n-1}, K') - V(K_1, \dots, K_{n-1}, L') = T(h_{K'} - h_{L'}). \end{aligned}$$

To see that T is linear, if $v = h_M - h_N \in C^2(\mathbb{S}^{n-1})$ and $\alpha \geq 0$, we have $\alpha u + v = h_{\alpha K + M} - h_{\alpha L + N}$

$$\begin{aligned} T(\alpha u + v) &= V(K_1, \dots, K_{n-1}, \alpha K + M) - V(K_1, \dots, K_{n-1}, \alpha L + N) \\ &= \alpha(V(K_1, \dots, K_{n-1}, K) - V(K_1, \dots, K_{n-1}, L)) + V(K_1, \dots, K_{n-1}, M) - V(K_1, \dots, K_{n-1}, N) \\ &= \alpha Tu + Tv \end{aligned}$$

where again we used the multi-linearity of mixed volumes. If $\alpha \leq 0$, then $\alpha u + v = h_{-\alpha L + M} - h_{-\alpha K + N}$, hence

$$\begin{aligned} T(\alpha u + v) &= V(K_1, \dots, K_{n-1}, -\alpha L + M) - V(K_1, \dots, K_{n-1}, -\alpha K + N) \\ &= \alpha(V(K_1, \dots, K_{n-1}, K) - V(K_1, \dots, K_{n-1}, L)) + V(K_1, \dots, K_{n-1}, M) - V(K_1, \dots, K_{n-1}, N) \\ &= \alpha Tu + Tv, \end{aligned}$$

which establishes linearity of T.

We next assert that T is positive. That is, if $u \geq 0$, then $Tu \geq 0$. Indeed, if $u = h_K - h_L \geq 0$, then $L \subset K$, so $V(K_1, \dots, K_{n-1}, K) \geq V(K_1, \dots, K_{n-1}, L)$ by the monotonicity of mixed volumes. It follows that $Tu \geq 0$. We claim that this implies T is a bounded linear operator on $C^2(\mathbb{S}^{n-1})$. Indeed, if $u \in C^2(\mathbb{S}^{n-1})$, then $\|u\|_\infty - u \in C^2(\mathbb{S}^{n-1})$ and $\|u\|_\infty - u \geq 0$, from which it follows $T(\|u\|_\infty - u) \geq 0$, or equivalently

$$T(u) \leq T(\|u\|_\infty) = V(K_1, \dots, K_{n-1}, B)\|u\|_\infty.$$

Repeating this argument with $\|u\|_\infty + u$ shows $-T(u) \leq V(K_1, \dots, K_{n-1}, B)\|u\|_\infty$, and we deduce

$$|Tu| \leq V(K_1, \dots, K_{n-1}, B)\|u\|_\infty.$$

Since $C^2(\mathbb{S}^{n-1})$ is uniformly dense in $C(\mathbb{S}^{n-1})$ by Theorem 7, it follows that T extends uniquely to a continuous positive linear operator on $C(\mathbb{S}^{n-1})$. By the Riesz-Kakutani theorem, there exists a unique positive Borel measure $S_{K_1,\dots,K_{n-1}}$ on $\mathbb{S}^{n-1}$ such that

$$Tf = \int_{\mathbb{S}^{n-1}} f \, dS_{K_1,\dots,K_{n-1}}$$

for all $f \in C(\mathbb{S}^{n-1})$. Taking $f = h_K$ for any $K \in \mathcal{K}^n$ yields

$$V(K_1, \dots, K_{n-1}, K) = \int_{\mathbb{S}^{n-1}} h_K \, dS_{K_1,\dots,K_{n-1}},$$

as desired. □

COROLLARY 9. *For every $K, L, K_1, \dots, K_{n-1} \in \mathcal{K}^n$,*

$$|V(K_1, \dots, K_{n-1}, K) - V(K_1, \dots, K_{n-1}, L)| \leq V(K_1, \dots, K_{n-1}, B) d_H(K, L).$$

If $K_1 = \cdots = K_{n-1} = K$, then we shall simply denote $S_K := S_{K_1,\dots,K_{n-1}}$ as the *surface measure* of K. We now prove a couple of results related to surface measures.

THEOREM 10. *Mixed area measures obey the following properties:*

(1) *(Symmetry)* $S_{K_1,\dots,K_{n-1}} = S_{K_{\sigma(1)},\dots,K_{\sigma(n-1)}}$ *for all permutations* σ.
(2) *(Linearity)* $S_{\alpha K+\alpha L,\dots,K_{n-1}} = \alpha S_{K,\dots,K_{n-1}} + S_{L,\dots,K_{n-1}}$ *for* $K, L \in \mathcal{K}^n$ *and* $\alpha \geq 0$.
(3) *(Translation Invariance)* $S_{K_1+x_1,\dots,K_{n-1}+x_n} = S_{K_1,\dots,K_{n-1}}$ *for all* $x_1,\dots,x_{n-1} \in \mathbb{R}^n$.
(4) *(Rotational Invaraince)* $S_{UK_1,\dots,UK_{n-1}} = S_{K_1,\dots,K_{n-1}}$ *for all rotations* $U \in O(n)$.
(5) *(Weak Continuity) If* $\{K_j^m\}_{m\geq 1} \subset \mathcal{K}^n$ *converges in the Hausdorff metric to* $K_j \in \mathcal{K}^n$ *for each* $1 \leq j \leq n-1$, *then*

$$\lim_{m\to\infty}\int f\,dS_{K_1^m,\dots,K_{n-1}^m} = \int f\,dS_{K_1,\dots,K_{n-1}}$$

for every $f \in C(\mathbb{S}^{n-1})$.

PROOF. Points (1)-(4) are easy to check and left to the reader. For point (5), by the continuity of mixed volumes, we have

$$\lim_{m\to\infty}\int u\,dS_{K_1^m,\dots,K_{n-1}^m} = \int u\,dS_{K_1,\dots,K_{n-1}}$$

for all $u \in C^2(\mathbb{S}^{n-1})$ by Lemma 7. Now, if $f \in C(\mathbb{S}^{n-1})$ and $\epsilon > 0$, by Theorem **??**, there is $u \in C^2(\mathbb{S}^{n-1})$ with $\|f-u\|_\infty < \epsilon$. Then for all $m \geq 1$, we have

$$-\epsilon V(K_1^m,\dots,K_{n-1}^m,B) + \int u\,dS_{K_1^m,\dots,K_{n-1}^m} \leq \int f\,dS_{K_1^m,\dots,K_{n-1}^m} \leq \int u\,dS_{K_1^m,\dots,K_{n-1}^m} + \epsilon V(K_1^m,\dots,K_{n-1}^m,B).$$

Taking $m \to \infty$ gives

$$-\epsilon V(K_1,\dots,K_{n-1},B) + \int u\,dS_{K_1,\dots,K_{n-1}} \leq \liminf_{m\to\infty}\int f\,dS_{K_1^m,\dots,K_{n-1}^m}$$

and

$$\limsup_{m\to\infty}\int f\,dS_{K_1^m,\dots,K_{n-1}^m} \leq \int u\,dS_{K_1,\dots,K_{n-1}} + \epsilon V(K_1,\dots,K_{n-1},B).$$

Since

$$-2\epsilon V(K_1,\dots,K_{n-1},B) + \int f\,dS_{K_1,\dots,K_{n-1}} \leq -\epsilon V(K_1,\dots,K_{n-1},B) + \int u\,dS_{K_1,\dots,K_{n-1}}$$

and

$$\int u\,dS_{K_1,\dots,K_{n-1}} + \epsilon V(K_1,\dots,K_{n-1},B) \leq 2\epsilon V(K_1,\dots,K_{n-1},B) + \int f\,dS_{K_1,\dots,K_{n-1}},$$

we may take $\epsilon \to 0$ to obtain

$$\int f\,dS_{K_1,\dots,K_{n-1}} \leq \liminf_{m\to\infty}\int f\,dS_{K_1^m,\dots,K_{n-1}^m} \leq \limsup_{m\to\infty}\int f\,dS_{K_1^m,\dots,K_{n-1}^m} \leq \int f\,dS_{K_1,\dots,K_{n-1}},$$

which is the stated claim. □

THEOREM 11. *Suppose that* $S_K = S_L$, *where* K *and* L *are compact, convex sets in* $\mathbb{R}^n$. *Then* $K = L + \alpha$, *where* $\alpha \in \mathbb{R}^n$ *is a constant.*

PROOF. One has

$$V(K,\dots,K,L) = \int h_L\,dS_K = \int h_L\,dS_L = m(L)$$

and

$$V(L,\dots,L,K) = \int h_K\,dS_L = \int h_K\,dS_K = m(K).$$

By the Brunn-Minkowski inequality, one has

$$m(L)^n = V(K,\dots,K,L)^n \geq m(K)^{n-1}m(L), \quad m(K)^n = V(L,\dots,L,K)^n \geq m(L)^{n-1}m(K).$$

Therefore, $m(L) = m(K)$, and more importantly, equality holds in the Brunn-Minkowski inequality. This means K and L are homothetic. Since $S_K = S_L$, it follows K and L are translates. □

When equality holds for the j-th order mixed volumes is a famous open problem in convex geometry.

COROLLARY 12. *For a fixed* $L \in \mathcal{K}^n$, *the map* $K \mapsto \int h_K\,dS_L$ *is minimized uniquely by translates of* L.