Intro to Ergodic Markov Processes

Math 758: Ergodic Theory

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Spring 2025

Probability Background

Certain standard constructions in measure theory are given special names in probability theory:

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4. The Lebesgue integral of a random variable is called the *expected value*:

$$\mathbb{E}[X] := \int_{\Omega} X \, d\mu.$$

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X : \Omega \to \mathbb{R}^n$ be a random variable.

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$$\mu := X_* \mathbb{P}.$$
2. We say that the *distribution* of X is μ, denoted by $X \sim \mu$.
3. Given any Borel probability measure μ on $\mathbb{R}^n$, we can find a probability measure space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable on Ω such that $X \sim \mu$.

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2. Always exists: either a Radon-Nikodym derivative or the orthogonal projection of X onto the space of $\mathcal{G}$-measurable random variables.
3. The conditional probability on $\mathcal{G}$ of an event $A \in \mathcal{F}$ is the (random) measure
$$\mathbb{P}[A \,|\, \mathcal{G}] := \mathbb{E}[\mathbf{1}_A \,|\, \mathcal{G}].$$

Markov Chains

Definition

A *Markov Process* in $\mathbb{R}^n$ with probability measures $\{\mathbb{P}_x : x \in \mathbb{R}^n\}$ on a measure space $(\Omega, \mathcal{F})$ is a family of random variables $\{X_t : t \geq 0\}$ satisfying

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1. $\mathbb{P}_x[X_0 = x] = 1$
2. (Markov Property) For every $x \in \mathbb{R}^n$ and $t, s \geq 0$ and Borel measurable A,

$$\mathbb{P}_x[X_{t+s} \in A \,|\, \mathcal{F}_s] = \mathbb{P}_{X_s}[X_t \in A]$$

$\mathcal{F}_t$ denotes the smallest sub-σ-algebra of $\mathcal{F}$ where X_s is measurable for each $0 \leq s \leq t$.

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1. Each $\{p_t(x, \cdot)\}$ is a Borel probability measure on $\mathbb{R}^n$ satisfying the *Chapman-Kolmogorov* property:

$$p_{t+s}(x, A) = \int_{\mathbb{R}^n} p_t(y, A) p_s(x, dy).$$

Examples: Brownian Motion

1. The transition kernels defined by

$$p_t(x, A) := \frac{1}{(2\pi t)^{n/2}} \int_A \exp\left(-\frac{1}{2t}|y-x|^2\right) \, dy$$

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2. Brownian motion is a continuous Markov process with many applications. For example, in 1827 Robert Brown discovered the phenomena when observing the irregular movement of pollen grains suspended in water.

Example:

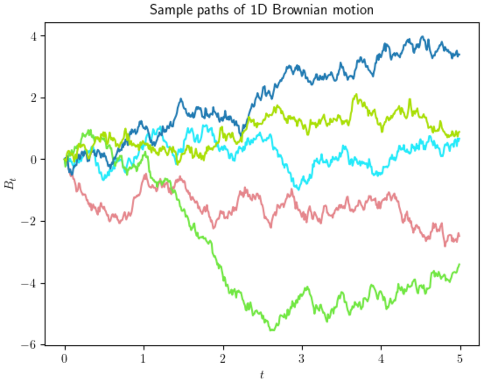

Figure: Five samples paths of a 1d Brownian Motion started at 0

Do Markov Processes Exist?

Theorem

Let $\{p_t(x, \cdot) : x \in \mathbb{R}^n, t \geq 0\}$ be a family of probability measures on $\mathbb{R}^n$ with the Chapman-Kolmogorov property

$$p_{t+s}(x, A) = \int p_t(y, A) p_s(x, \, dy).$$

Then there exists a probability space $(\Omega, \mathcal{F})$, a family of probability measure $\{\mathbb{P}_x : x \in \mathbb{R}^n\}$ and random variables $(X_t)_{t \geq 0}$ such that $(X_t)_{t \geq 0}$ is a Markov process with transition kernels $\{p_t(x, \cdot) : x \in \mathbb{R}^n, t \geq 0\}$.

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Sketch of Proof.

1. Let $\Omega = (\mathbb{R}^n)^{[0,\infty)}$ with the product topology and $\mathcal{F}$ be the associated Borel σ-algebra.

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2. Define a premeasure $\mathbb{P}_x$ on cylinders by

$$\mathbb{P}_x[(\omega_t)_{t\geq 0} : \omega_{t_1} \in A_1, \ldots, \omega_{t_n} \in A_n] = \int_{A_1} \cdots \int_{A_n} p_{t_n - t_{n-1}}(x_{n-1}, dx_n) \cdots p_{t_2 - t_1}(x_1, \, dx_2) \, p_{t_1}(x, dx_1)$$

for $0 \leq t_1 < \cdots < t_n$.

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   for $0 \leq t_1 < \cdots < t_n$.
3. Since $(\mathbb{R}^n)^{[0,\infty)}$ is locally compact Hausdorff, $\mathbb{P}_x$ extends to a probability measure on $\mathcal{F}$.

$\square$

Markov Existence Theorem

Sketch of Proof.

1. Let $X_t(\omega) = \omega_t$.
2. By construction,

$$\mathbb{P}_x[X_t \in A] = \mathbb{P}_x[(\omega_t)_{t \geq 0} : \omega_t \in A] = p_t(x, A).$$

3. The Chapman-Kolmogorov property implies

$$\mathbb{P}_x[X_{t+s} \in A \,|\, \mathcal{F}_s] = \mathbb{P}_{X_s}[X_t \in A]$$

for any $0 \leq s < t$.

$\square$

Stationary Measures

1. In light of the Markov existence theorem, we can always assume that $\Omega = (\mathbb{R}^n)^{[0,\infty)}$ and $X_t(\omega) = \omega_t$.
2. For a probability measure π on $\mathbb{R}^n$ define

$$\mathbb{P}_\pi[A] := \int_{\mathbb{R}^n} \mathbb{P}_x[A]\, d\pi(x).$$

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$$\mathbb{P}_\pi[A] := \int_{\mathbb{R}^n} \mathbb{P}_x[A]\, d\pi(x).$$

3. **Question:** Is there a Borel probability measure π on $\mathbb{R}^n$ (called a *stationary measure*) such that

$$\mathbb{P}_\pi[X_t \in A] = \mathbb{P}_\pi[X_0 \in A] = \pi(A)$$

for all $t \geq 0$?

Stationary Measures and Ergodic Flows

1. For each $s \geq 0$, let $\sigma_s : (\mathbb{R}^n)^{[0,\infty)} \to (\mathbb{R}^n)^{[0,\infty)}$ be the map $\sigma_s((\omega_t)_{t\geq 0}) = (\omega_{s+t})_{t\geq 0}$.

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3. The existence of a stationary measure π is equivalent to finding a measure where $\mathbb{P}_\pi$ is invariant under σ_t for every $t \geq 0$.
   - Just check $(\sigma_t)_\sharp \mathbb{P}_\pi = \mathbb{P}_\pi$ on cylinder sets using Chapman-Kolmogorov

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4. **Issue:** For a general Markov process, the state space is not compact.

Markov process with no invariant measure

1. Unfortunately, a stationary measure does not always exist.
2. Example: Brownian motion

Theorem

A Brownian motion B_t with transition kernels

$$p_t(x, A) := \frac{1}{(2\pi t)^{n/2}} \int_A \exp\left(-\frac{1}{2t}|y-x|^2\right) dy$$

does not have a stationary measure.

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2. If $\phi \in C_c^\infty(\mathbb{R}^n)$ is a test function,

$$\mathbb{E}_x[\phi(B_t)] = \phi * G_t(x)$$

where $G_t(x) = \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{1}{2t}|x|^2\right)$ is the Gaussian kernel.

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3. G_t satisfies the heat equation $\partial_t G_t = \triangle G_t$. Thus $\partial_t(\phi * G_t) = \triangle(\phi * G_t)$.

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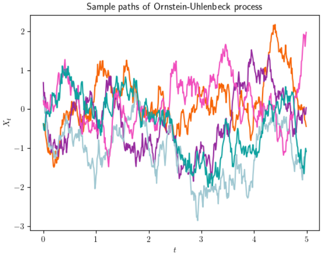
5. **No probability measure is a constant multiple of the Lebesgue measure!**

Sometimes there are stationary measures

- The Ornstein-Uhlenbeck (OU) process:

$$X_t = X_0 + \sqrt{2}B_t - \int_0^t X_s \, ds$$

where B_t is a Brownian motion.



- The OU process has stationary distribution

$$\pi(A) = \frac{1}{(2\pi)^{n/2}} \int_A \exp\left(-\frac{|x|^2}{2}\right) \, dx.$$

Tight Probability Measures

Definition

Let X be a locally compact Hausdorff space. A family of probability measures $\mathcal{P} \subset \mathcal{M}(X)$ is *tight* if for any $\epsilon > 0$ there is a compact $K \subset X$ where

$$\mu(K) \geq 1 - \epsilon$$

for all $\mu \in \mathcal{P}$.

Prokhorov's Theorem

Theorem (Prokhorov)

Let X be a locally compact Hausdorff space, and suppose that $\mathcal{P} \subset \mathcal{M}(X)$ is a weak-$*$ closed and tight family of probability measures. Then $\mathcal{P}$ is weak-$*$ compact.

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2. Tightness of $\mathcal{P}$ implies that μ is a probability measure.
3. Shows that $\mathcal{P}$ is weak-$*$ sequentially compact.

$\square$

Markov-Kakutani Fixed Point Theorem

Theorem (Markov-Kakutani Fixed Point Theorem)

Let X be a locally convex topological vector space, and suppose that $K \subset X$ is a nonempty compact convex subset of X. If $\mathcal{H}$ is a family of continuous commuting affine operators mapping K to itself, then there exists $x_0 \in K$ such that $Tx_0 = x_0$ for all $T \in \mathcal{H}$.

Proof.

For the proof, see Conway 1990 Theorem 10.1. □

A stationary measure existence theorem

Theorem

Let $(X_t)_{t\geq 0}$ be a Markov process with transition kernels $p_t(x,\cdot)$ such that $x \mapsto p_t(x,\cdot)$ is weak-$*$ continuous. Suppose further that there exists $x \in \mathbb{R}^n$ such that the family $\{p_t(x,\cdot) : t \geq 0\}$ is tight. Then there is a stationary measure π for the process $(X_t)_{t\geq 0}$.

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If there is a function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x) \to \infty$ as $|x| \to \infty$, and

$$\sup_{t\geq 0} \mathbb{E}_x[V(X_t)] < \infty$$

for some x, then the conditions of the theorem are satisfied.

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2. Let $P_t^* \mu = \int p_t(y,\cdot)\, d\mu(y)$.
3. Claim: $P_t^* : K \to K$.

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5. Since P_t^* is affine and weak-$*$ continuous, $P_t^* : K \to K$.

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4. By Markov-Kakutani fixed point theorem, $\exists \pi \in K$ such that

$$P_t^* \pi = \pi$$

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4. By Markov-Kakutani fixed point theorem, $\exists \pi \in K$ such that

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   for all $t \geq 0$.
5. π **is a stationary measure,** since

$$\mathbb{P}_\pi[X_t \in A] = P_t^* \pi(A) = \pi(A)$$

   for all Borel $A \subset \mathbb{R}^n$.

Ergodic Stationary Measures

- In general, deducing the existence of a measure π so that $\mathbb{P}_\pi$ is ergodic for the flow σ_t is done on a case-by-case basis.
- Not surprising: Cannot even be done for Markov subshifts since not every stochastic matrix is irreducible!

Perron-Frobenius for Markov Operators

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2. The process X is *Feller* if $P_t : C_b(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$ and $P_t f \to f$ uniformly as $t \to 0$.

Perron-Frobenius for Markov Operators

Theorem (Perron-Frobenius for Markov Processes)

Suppose that X is a Feller process. Suppose that there is a stationary measure π such that for any $x \in \mathbb{R}^n$, it holds $p_t(x, \cdot) \to \pi$ in the weak-$*$ sense as $t \to \infty$. Then π is the unique stationary distribution and $\mathbb{P}_\pi$ is mixing, i.e.,

$$\lim_{t \to \infty} \mathbb{P}_\pi[\sigma_t^{-1} A \cap B] = \mathbb{P}_\pi[A]\mathbb{P}_\pi[B]$$

for all events A and B.

Perron-Frobenius Analogue

Proof.

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1. Suppose μ is another stationary measure.
2. For any $f \in C_0(\mathbb{R}^n)$, it holds

$$\int f \, d\mu = \lim_{t \to \infty} \iint f(y) \, p_t(x, \, dy) \, d\mu(x) = \int f \, d\pi$$

by the assumptions of the theorem and the fact μ is stationary.

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Uniqueness of π:

1. Suppose μ is another stationary measure.
2. For any $f \in C_0(\mathbb{R}^n)$, it holds

$$\int f\, d\mu = \lim_{t\to\infty} \iint f(y)\, p_t(x,\, dy)\, d\mu(x) = \int f\, d\pi$$

   by the assumptions of the theorem and the fact μ is stationary.
3. Thus $\mu = \pi$.

□

Perron-Frobenius Analogue

Proof.

Mixing:

1. Enough to check that for any bounded continuous function $f : (\mathbb{R}^n)^m \to \mathbb{R}$

$$\lim_{s\to\infty} \mathbb{E}_\pi[f(X_{t_1}, \ldots, X_{t_m}) f(X_{t_1+s}, \ldots, X_{t_m+s})] = \mathbb{E}_\pi[f(X_{t_1}, \ldots, X_{t_m})]^2$$

$\square$

Perron-Frobenius Analogue

Proof.

1. By Markov property,

$$\mathbb{E}_\pi[f(X_{t_1},\ldots,X_{t_m})f(X_{t_1+s},\ldots,X_{t_m+s})\,|\,\mathcal{F}_s]$$
$$= f(X_{t_1},\ldots,X_{t_m})\mathbb{E}_{X_s}[f(X_{t_1},\ldots,X_{t_m})]$$

for $s > t_m$.

Perron-Frobenius Analogue

Proof.

1. By Markov property,

$$\begin{aligned}&\mathbb{E}_\pi[f(X_{t_1},\dots,X_{t_m})f(X_{t_1+s},\dots,X_{t_m+s})\,|\,\mathcal{F}_s]\\&=f(X_{t_1},\dots,X_{t_m})\mathbb{E}_{X_s}[f(X_{t_1},\dots,X_{t_m})]\end{aligned}$$

   for $s > t_m$.

2. Let

$$g(x)=\mathbb{E}_x[f(X_{t_1},\dots,X_{t_m})].$$

Perron-Frobenius Analogue

Proof.

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   for $s > t_m$.

2. Let

$$g(x) = \mathbb{E}_x[f(X_{t_1},\ldots,X_{t_m})].$$

3. g is continuous, so

$$\lim_{s\to\infty}\int g(x_{m+1})p_{s-t_m}(x_m,\, dx_{m+1}) = \int g\, d\pi.$$

Perron-Frobenius Analogue

Proof.

1. By DCT,

$$\lim_{s\to\infty} \iint f(x_1,\dots,x_m) \int g(x_{m+1}) p_{s-t_m}(x_m, dx_{m+1})$$
$$d(X_{t_1},\dots,X_{t_m})_* \mathbb{P}_x \, d\pi(x)$$
$$= \int g \, d\pi \iint f(x_1,\dots,x_n) d(X_{t_1},\dots,X_{t_m})_* \mathbb{P}_x \, d\pi(x)$$
$$= \mathbb{E}_\pi[f(X_{t_1},\dots,X_{t_m})]^2$$

$\square$

Proof.

But:

$$\iint f(x_1, \ldots, x_m) \int g(x_{m+1}) p_{s-t_m}(x_m, dx_{m+1})$$
$$d(X_{t_1}, \ldots, X_{t_m})_* \mathbb{P}_x \, d\pi(x)$$
$$= \mathbb{E}_\pi[f(X_{t_1}, \ldots, X_{t_m}) f(X_{t_1+s}, \ldots, X_{t_m+s})].$$

$\square$

Calculating Stationary Measures

1. Let X be a Feller process and

$$P_t f(x) = \mathbb{E}_x[f(X_t)].$$

2. (Infinitesimal Generator) Let A be the unbounded operator

$$Af := \lim_{h \to 0} \frac{P_h f - f}{h}$$

functions $f \in C_b(\mathbb{R}^n)$ where the limit exists and is uniform.

Calculating Stationary Measures

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2. Example: generator of OU process is the **Ornstein-Uhlenbeck Operator**:

$$A\phi(x) := -x \cdot \nabla\phi(x) + \triangle\phi(x), \quad \phi \in C_b^2(\mathbb{R}^n).$$

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3. Assume that $d\pi = \phi \, dx$ for some $\phi \geq 0$. Then:

$$A^*\phi(x) := \text{div}(x\phi(x)) + \triangle\phi(x) = 0.$$

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4. **The Gaussian** $d\pi = \frac{1}{(2\pi)^{n/2}} \exp(-|x|^2/2)\, dx$ **is unique stationary measure for OU process.**

Recap

1. For continuous time Markov processes in $\mathbb{R}^n$, stationary measures give rise to shift invariant measures on $(\mathbb{R}^n)^{[0,\infty)}$.
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Recap

1. For continuous time Markov processes in $\mathbb{R}^n$, stationary measures give rise to shift invariant measures on $(\mathbb{R}^n)^{[0,\infty)}$.
2. Stationary measures may or may not exist
3. When they exist, one usually has to compute the infinitesimal generator and solve a PDE to find the stationary distribution
4. More advanced techniques are required to determine properties such as mixing

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