Exercise Session 04

Exercise 1.
Consider the following recurrence $T(n) = T(2n/3) + \Theta(1)$. Prove that $T(n) = O(\lg n)$.

Solution 1.
The recurrence is of the form $T(n) = aT(n/b) + f(n)$ where $a = 1$, $b = 3/2$, and $f(n) = c$ for some constant $c > 0$. We can solve the recurrence using the master method. Note that $n^{\log_b a} = 1$, and $f(n) = c = \Theta(n^{\log_b a}) = \Theta(1)$. Therefore, according to the second case of the Master theorem we have that $T(n) = \Theta(\lg n)$. Then, by Theorem 3.1 in CLRS we can conclude that $T(n) = O(\lg n)$.

Exercise 2.
Consider the following recurrence:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(n-1) + \Theta(n) & \text{if } n > 1 \end{cases}$$

Prove that $T(n) = O(n^2)$ using the substitution method.

Solution 2.
To prove that $T(n) = O(n^2)$, it suffices to show that there exists $n_0 \in \mathbb{N}$ and $c > 0$ such that for all $n \geq n_0$, $T(n) \leq cn^2$. In what follows we prove the statements for $n_0 = 1$.

Base Case ($n = 1$). By definition of T and Θ-notation, there exists a constant d such that $T(n) \leq d$. Thus, for $c \geq d$ we have $d \leq c = cn^2$.

Inductive Step ($n > 1$). Then we have that

$$\begin{aligned} T(n) &= T(n-1) + \Theta(n) && \text{(def. } T\text{)} \\ &\leq T(n-1) + en && \text{(for some constant } e > 0\text{)} \\ &\leq c(n-1)^2 + en && \text{(inductive hypothesis)} \\ &\leq c(n-1)^2 + cn && \text{(assuming } c \geq e\text{)} \\ &= cn^2 - 2cn + c + cn && ((n-1)^2 = n^2 - 2n + 1) \\ &\leq cn^2 - 2cn + cn + cn && (n > 1) \\ &\leq cn^2\,. \end{aligned}$$

Note that for both the base case and the inductive step to hold we need to have $c \geq d$ and $c \geq e$. This can be easily achieved by choosing $c = \max\{e, d\}$.

Exercise 3.
Consider the following recurrence:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil - 1) + \Theta(n) & \text{if } n > 1 \end{cases}$$

Use the substitution method to prove that $T(n) = O(n \lg n)$.
Hint: be careful when you choose the base case because $n = 0$ and $n = 1$ may not work

Solution 3.
To prove that $T(n) = O(n \lg n)$, it suffices to show that there exists $n_0 \in \mathbb{N}$ and $c > 0$ such that for all $n \geq n_0$, $T(n) \leq cn \lg n$. We will prove this by induction on n and we will choose $n_0 = 2$. This choice of n_0 allows us to "skip" the case $T(1)$ which is problematic because $T(1) = d_0$ for some $d_0 > 0$ but $cn \lg n = 0$.

Before we start our proof by induction we need to understand which are the base cases. Clearly $n = 2$ is a base case, but it is not the only one. Indeed by definition of T, if we take $n = 3$ we have that $T(3) = T(1) + T(1) + \Theta(3)$, this means that the case $n = 3$ cannot be described by a sub-cases where $n \geq 2$. Analogously, also $n = 4$ needs to be treated as a base case because $T(4) = T(2) + T(1) + \Theta(4)$. Since for any $n \geq 5$ we have that $n/2 > 2$ we are now sure that the value of $T(n)$ for $n \geq 5$ can be described in terms of $T(2)$, $T(3)$, and $T(4)$.

Base Case We consider separately the 3 cases for $n = 2, 3, 4$. By definition of T and Θ-notation, there exist constants $d_0, d_1 > 0$ such that $T(0) \leq d_0$ and $T(1) \leq d_1$.

$(n = 2)$

$$
\begin{aligned}
T(2) &= T(1) + T(0) + e && \text{(for some constant } e > 0\text{)} \\
&\leq d_0 + d_1 + e \\
&\leq 2c \lg 2 && \text{(for } c \geq (d_0 + d_1 + e)/2\text{)} \\
&= cn \lg n && (n = 2)
\end{aligned}
$$

$(n = 3)$

$$
\begin{aligned}
T(3) &= T(1) + T(1) + e' && \text{(for some constant } e' > 0\text{)} \\
&\leq 2d_1 + e' \\
&\leq 3c \lg 3 && \text{(for } c \geq (2d_1 + e')/3 \lg 3\text{)} \\
&= cn \lg n && (n = 3)
\end{aligned}
$$

$(n = 4)$

$$
\begin{aligned}
T(4) &= T(2) + T(1) + e'' && \text{(for some constant } e'' > 0\text{)} \\
&\leq 2c + d_1 + e'' && \text{(for } c \geq (d_0 + d_1 + e)/2\text{, as proven before)} \\
&\leq 4c \lg 4 && \text{(for } c \geq (d_1 + e'')/6\text{)} \\
&= cn \lg n && (n = 4)
\end{aligned}
$$

Inductive Step ($n > 3$). Then we have that

$$
\begin{aligned}
T(n) &= T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil - 1) + \Theta(n) && \text{(def. } T\text{)} \\
&\leq T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil - 1) + c'n && \text{(for some constant } c' > 0\text{)} \\
&\leq c\lfloor n/2 \rfloor \lg \lfloor n/2 \rfloor + c(\lceil n/2 \rceil - 1) \lg(\lceil n/2 \rceil - 1) + c'n && \text{(inductive hypothesis)} \\
&\leq cn \lg(n/2) + c'n && (\lfloor n/2 \rfloor \leq n/2 \text{ and } (\lceil n/2 \rceil - 1) \leq n/2) \\
&= cn \lg(n) - cn + c'n \\
&\leq cn \lg(n) && \text{(assuming } c \geq c'\text{)}
\end{aligned}
$$

Let's now recall all the conditions we imposed on c

$$
\begin{aligned}
c &\geq (d_0 + d_1 + e)/2 \\
c &\geq (2d_1 + e')/3 \lg 3 \\
c &\geq (d_1 + e'')/6 \\
c &\geq c'
\end{aligned}
$$

By taking $c = \max\{(d_0 + d_1 + e)/2, (2d_1 + e')/3 \lg 3, (d_1 + e'')/6, c'\}$ the above proof works.

Exercise 4.
The factorial of n, is usually recursively defined as

$$n! = \begin{cases} 1 & \text{if } n = 0, \\ n \cdot (n-1)! & \text{if } n > 0 \end{cases}$$

(a) Prove that $n! = \Omega(2^n)$.

(b) Prove that $n! = O(n^n)$.

(c) Prove that $\lg n! = O(n \lg n)$.

Solution 4.

(a) To prove that $n! = \Omega(2^n)$ it suffices to show that for all $n \geq 1$, $n! \geq c2^n$ for some suitable constant $c > 0$ (notice that this corresponds to chose $n_0 = 1$ in the definition of Ω-notation).

Base Case ($n = 1$). $n! = 1! = 1 \geq 2 = 2^n$. Thus, for $n = 1$, $n! \geq c2^n$ holds when $c \geq 1/2$.

Inductive Step ($n > 1$). We have that

$$\begin{aligned} n! &= n \cdot (n-1)! && \text{(def. factorial)} \\ &\geq n \cdot c2^{n-1} && \text{(inductive hypothesis)} \\ &\geq 2 \cdot c2^{n-1} && (n \geq 2) \\ &= c2^n \,. \end{aligned}$$

Thus, for $c \geq 1/2$ we have that $n! \geq c2^n$ for all $n \geq 1$ from which we conclude $n! = \Omega(2^n)$.

(b) To prove that $n! = O(n^n)$ it suffices to show that for all $n \geq 1$, $n! \leq cn^n$ for some suitable constant $c > 0$.

Base Case ($n = 1$). $n! = 1! = 1 = n^n$. Thus, for $n = 1$, $n! \leq cn^n$ holds when $c \geq 1$.

Inductive Step ($n > 1$). We have that

$$\begin{aligned} n! &= n \cdot (n-1)! && \text{(def. factorial)} \\ &\leq n \cdot c(n-1)^{n-1} && \text{(inductive hypothesis)} \\ &\leq n \cdot cn^{n-1} \\ &= cn^n \end{aligned}$$

Thus, for $c \geq 1$ we have that $n! \leq cn^n$ for all $n \geq 1$ from which we conclude $n! = O(n^n)$.

(c) To prove that $\lg n! = O(n \lg n)$ it suffices to show that for all $n \geq 1$, $\lg n! \leq n \lg n$ for some suitable constant $c > 0$.

Base Case ($n = 1$). $\lg n! = \lg 1 = 0 = n \lg n$. Thus, for $n = 1$, $\lg n! \leq cn \lg n$ holds for any $c > 0$.

Inductive Step ($n > 1$). We have that

$$\begin{aligned} \lg n! &= \lg(n \cdot (n-1)!) && \text{(def. factorial)} \\ &= \lg n + \lg((n-1)!) \\ &\leq \lg n + c(n-1)\lg(n-1) && \text{(inductive hypothesis)} \\ &\leq \lg n + c(n-1)\lg n && \text{(lg is monotone)} \\ &\leq c \lg n + c(n-1)\lg n && \text{(assuming } c \geq 1\text{)} \\ &= cn \lg n \end{aligned}$$

Thus, for $c \geq 1$ we have that $\lg n! \leq cn \lg n$ for all $n \geq 1$ from which we conclude $\lg n! = O(n \lg n)$.

Alternatively, we can prove the same result by simply recalling that $n! \geq n^n$ for all $n \leq 1$ (as proven before, by fixing $c = 1$). Therefore, we can apply the logarithm on both sides (recall that the logarithm is a monotone function) obtaining $\lg n! \leq \lg n^n = n \lg n$.

★ Exercise 5.

Consider the recurrence $T(n) = T(9n/10) + T(n/10) + cn$ where c is a constant such that $c > 0$. Prove that $T(n) = O(n \lg n)$.

Solution 5.

To prove $T(n) = O(n \lg n)$ we need to show that $T(n) \leq dn \lg n$ for a suitable constant $d > 0$.

$$
\begin{aligned}
T(n) &= T(9n/10) + T(n/10) + cn && \text{(def. } T\text{)} \\
&\leq d(9n/10)\lg(9n/10) + d(n/10)\lg(n/10) + cn && \text{(inductive hypothesis)} \\
&= d(9n/10)\lg(9/10) + d(9n/10)\lg n + d(n/10)\lg(1/10) + d(n/10)\lg n + cn \\
&= dn \lg n + dn\big((9/10)\lg(9/10) + (1/10)\lg(1/10)\big) + cn \\
&\leq dn \lg n
\end{aligned}
$$

The last inequality is true if

$$
dn\big((9/10)\lg(9/10) + (1/10)\lg(1/10)\big) + cn \leq 0\,. \tag{1}
$$

The inequality (1) is equivalent to $d\big((9/10)\lg(9/10) + (1/10)\lg(1/10)\big) \leq -c$. Since $\lg(9/10) < 0$ and $\lg(1/10) < 0$ we have that $\big((9/10)\lg(9/10) + (1/10)\lg(1/10)\big) < 0$, so that when we multiply both sides of the inequality by this factor we have to reverse the inequality:

$$
d \geq \frac{-c}{(9/10)\lg(9/10) + (1/10)\lg(1/10)}\,. \tag{2}
$$

Note that the right hand side of the above inequality is positive since $c > 0$, therefore it suffice to pick any value of d satisfying (2).
Note that the above proof can be easily generalised proving that $T(n) = T(\alpha n) + T((1-\alpha)n) + cn$ for any value of α such that $0 < \alpha < 1$ (*cf.* CLRS 4.4–9). We suggest the curious of you to try it out.