Exercise Session 12

Exercise 1.
(CLRS 25.1–4) Show that matrix multiplication defined by EXTEND-SHORTEST-PATH is associative. *Hint:* Let us write $A \odot B$ for EXTEND-SHORTEST-PATH(A, B). You have to prove that for arbitrary $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{p\times q}$ we have that $(A \odot B) \odot C = A \odot (B \odot C)$.

Exercise 2.
(CLRS 25.1–9) Modify FASTER-ALL-PAIRS-SHORTEST-PATHS so that it can determine whether the graph contains a negative-weight cycle. Justify the correctness of your solution.

Exercise 3.
Let $G = (V, E)$ be a weighted directed graph represented using the weight matrix $W = (w_{ij})$ where

$$w_{ij} = \begin{cases} 0 & \text{if } i = j \\ w(v_i, v_j) & \text{if } i \neq j \text{ and } (v_i, v_j) \in E \\ \infty & \text{if } i \neq j \text{ and } (v_i, v_j) \notin E \end{cases}$$

How would we delete an arbitrary vertex v from this graph, without changing the shortest-path distance between any other pair of vertices? Describe an algorithm that constructs a weighted directed graph $G' = (V \setminus \{v\}, E')$ such that shortest-path distance between any two vertices in G' is equal to the shortest-path distance between the same two vertices in G in $O(|V|^2)$ time.

★ **Exercise 4.**
Assume that $G = (V, E)$ is a directed acyclic graph represented using adjacency-lists.

(a) Describe an algorithm that computes the transitive closure of G, i.e. $G^* = (V, E^*)$, and analyse its running time.

(b) Can you generalise your solution to directed graphs that may contain cycles?

Exercise 5.
Rick has given Morty a detailed map of the Clackspire Labyrinth, which consists of a directed graph $G = (V, E)$ with non-negative edge weights W (indicating distance from one location in the map to the other), along with a list of dangerous locations $D \subset V$ that Morty has to avoid.

(a) Morty has to determine for each pair of locations $i, j \in V \setminus D$ the length of the shortest walk $i \rightsquigarrow j$ that does not have intermediate vertices in D. Describe the algorithm that Morty can use to solve the problem.

(b) Assume now that Morty also needs to pass by some location in the set $S \subseteq (V \setminus D)$. How does Morty compute the length of the shortest walk from $i \rightsquigarrow j$ that avoids dangerous locations D while ensuring to pass by some location in S?