# *Algorithms and Satisfiability*

## *Lecture 2*
## *Greedy Algorithms*

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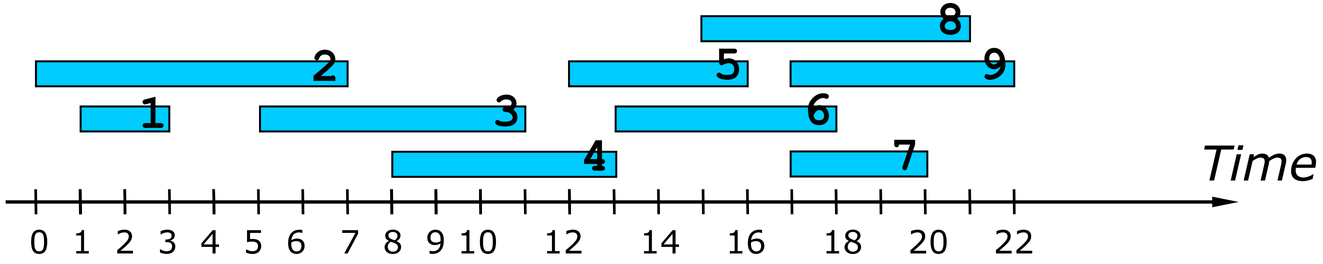
# Greedy algorithms

- Goals of the lecture:
  - *to understand the **principles** of the greedy algorithm design technique;*
  - *to understand the **example greedy algorithms** for activity selection and Huffman coding, to be able to **prove** that these algorithms find optimal solutions;*
  - *to be able to **apply** the greedy algorithm design technique.*

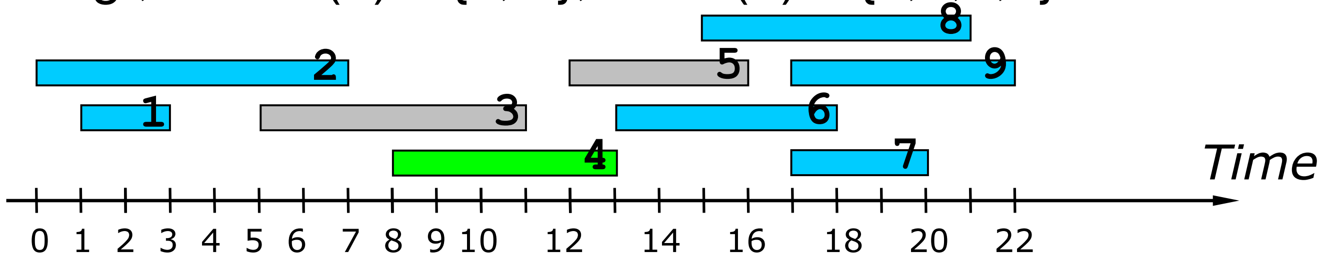
# Activity-Selection Problem

- Input:
  - A set of $n$ activities, each with start and end times: $A[i].s$ and $A[i].f$. The activity lasts during the period $[A[i].s, A[i].f)$
- Output:
  - The ***largest*** subset of mutually *compatible* activities
    - Activities are compatible if their intervals do not intersect



# “Straight-forward” solution

- Let’s just pick (schedule) one activity $A[k]$: ***k-nary choice***
  - This generates two set’s of activities compatible with it: *Before*($k$), *After*($k$)
    - E.g., *Before*(4) = {1, 2}; *After*(4) = {6,7,8,9}




  - Solution:

$$MaxN(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ \max\limits_{a \in A} \{ MaxN(Before(a)) + MaxN(After(a)) + 1 \} & \text{if } A \neq \emptyset. \end{cases}$$

# Dynamic Programming Alg.

- The recurrence results in a dynamic programming algorithm
  - Sort activities on the end time (for simplicity assume also "sentinel" activities $A[0]$ and $A[n+1]$)
  - Let $S_{ij}$ – a set of activities after $A[i]$ and before $A[j]$ and compatible with $A[i]$ and $A[j]$.
  - Let's have a two-dimensional array, s.t., $c[i, j] = MaxN(S_{ij})$:

$$c[i,j] = \begin{cases} 0 & \text{if } S_{ij} = \emptyset, \\ \max_{a_k \in S_{ij}} \{c[i,k] + c[k,j] + 1\} & \text{if } S_{ij} \neq \emptyset. \end{cases}$$

  - $MaxN(A) = MaxN(S_{0,n+1}) = c[0, n+1]$

# Dynamic Programming Alg. II

- Does it really work correctly?
  - We have to prove the optimal sub-structure:
    - *If an optimal solution A to $S_{ij}$ includes A[k], then it also includes optimal solutions to $S_{ik}$ and $S_{kj}$*
    - To prove use “cut-and-paste” argument

- *What is the space used by this algorithm?*
- *What is the running time of this algorithm?*
  - Definitely $\Omega(n^2)$

# Activity Selection DP Alg. 2.0

- Alternative way of thinking about it – ***binary choice***:
  - Sort activities on the start time (have "sentinel" activity *A*[*n*+1] after all the other activities)
  - Let *next*(*i*) = min $\{k \mid k > i \wedge \neg overlaps(A[i], A[k])\}$
  - The subproblem is then to schedule all the activities starting with *i* and after.
  - *What is the recurrence?*

$$c[i]=\begin{cases} 0 & \text{if } i > n, \\ \max(1+c[next(i)],\ c[i+1]) & \text{otherwise}. \end{cases}$$

  - *MaxN*(*A*) = *c*[1]
  - *What is the running time and space used*?
    - Don't forget *next(i)*...

# Greedy choice

- What if we could choose “the best” activity (as of now) and be sure that it belongs to an optimal solution
  - We wouldn’t have to check out all these sub-problems and consider all currently possible choices!

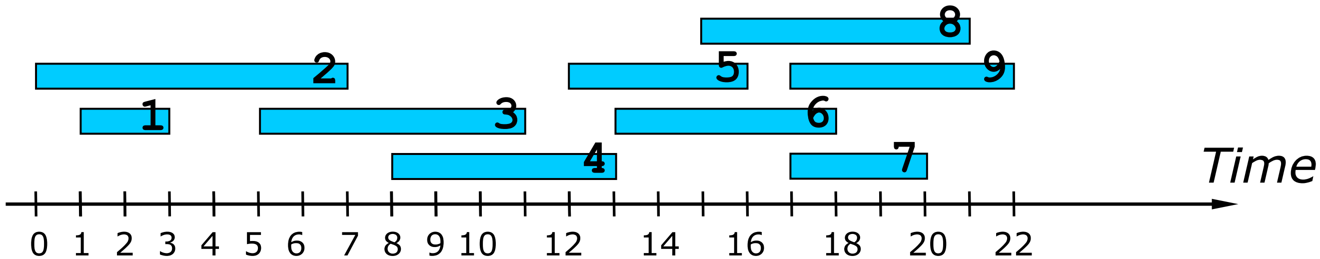
- Idea: Choose the activity that finishes first!
  - Intuition: leave as much time as possible for other activities
  - Then, solve *only one* sub-problem for the remaining compatible activities
  - (Have sentinel activity $A[0].f = 0$ – before all other)

```
MaxN(A[0..n], i)  //returns a set of activities
m ← i + 1
while m ≤ n and A[m].s < A[i].f do
   m ← m + 1
if m ≤ n then return {A[m]} ∪ MaxN(A, m)
              else return ∅
```

# Iterative algorithm

```
MaxNi(A[0..n])  //returns a set of activities
R ← {A[1]}
k ← 1              // A[k] is always the last added activity
for m = 2 to n
   if A[m].s ≥ A[k].f then  // first compatible A[m]
     R ← R ∪ {A[m]}
     k ← m
return R
```

- Let's run it:




- *What is the running time?*

# Greedy-choice property

- Does it find an optimal solution?:
  - We have to prove the *optimal sub-structure* property (we did that already)
  - We have to prove the *greedy-choice property*, i.e., that our locally optimal choice $a_1$ belongs to some globally optimal solution.
    - Let $A$ be an optimal solution and $x$ be an activity with smallest finishing time **in** $\boldsymbol{A}$.
    - $a_1.f \leq x.f$, as $a_1$ is the activity with the smallest finishing time overall.
    - Thus, we can replace $x$ with $a_1$ in $A$ to get a valid solution of the same size!
  - *Greedy exchange* proof.

# Greedy-choice property

- The challenge is to choose the right interpretation of "the best choice":
  - How about the activity that starts first?
    - Show a *counter-example*
  - The shortest activity?
  - The activity that overlaps the smallest number of the remaining activities?

# Greedy choice property

- How about the activity that starts first?

| a1 | a2 | a3 |
|---|---|---|
| 1 | 2 | 4 |
| 10 | 3 | 6 |

  - {a2, a3}, but not a1 that starts first.

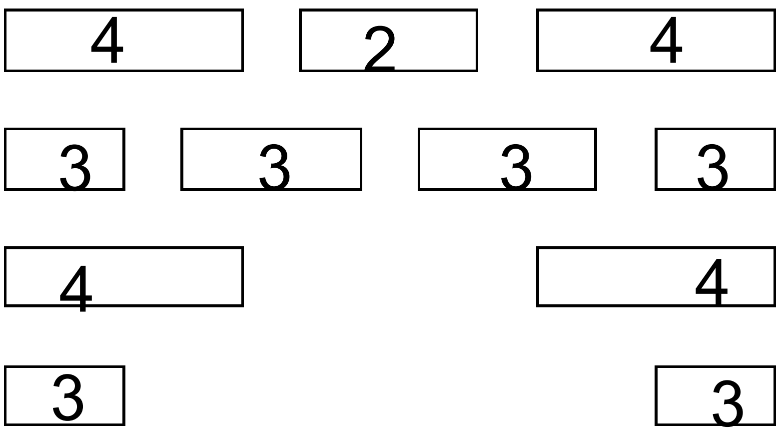
- The shortest activity?

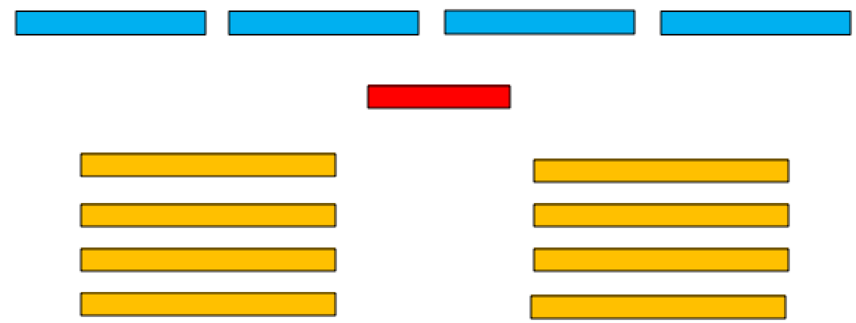
| a1 | a2 | a3 | a4 |
|---|---|---|---|
| 1 | 11 | 21 | 9 |
| 10 | 20 | 30 | 12 |

  - {a1, a2, a3}, but not a4 that is the shortest activity.

# Greedy choice property

- The activity that overlaps the smallest number of the remaining activities?



- The second row gives the maximum-size set of mutually compatible activities, but it does not include the activity with the smallest overlaps, i.e., the one with 2.

# Data Compression

- *Data compression* problem – strings *S* and *S'*:
  - $S \rightarrow S' \rightarrow S$, such that $|S'|<|S|$
- Text compression by coding with *variable-length* code:
  - Obvious idea – *assign short codes to frequent characters*: "**abracadabra**"

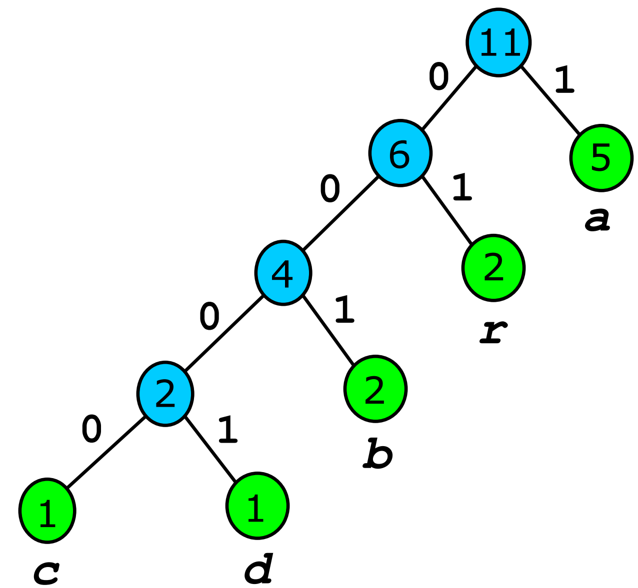
Frequency table:

| | **a** | **b** | **c** | **d** | **r** |
|---|---|---|---|---|---|
| Frequency | 5 | 2 | 1 | 1 | 2 |
| Fixed-length code | 000 | 001 | 010 | 011 | 100 |
| Variable-length code | 1 | 001 | 0000 | 0001 | 01 |

- *How much do we save in this case?*

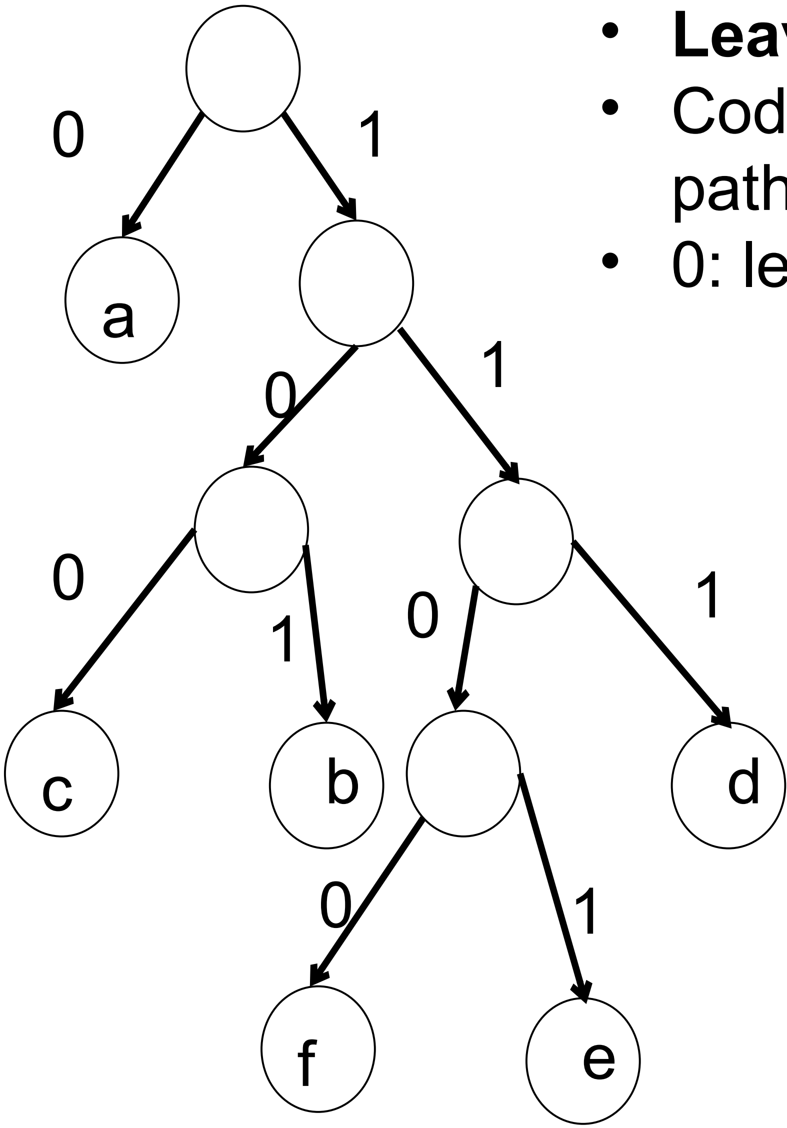
# Prefix code

- Optimal code for the given frequencies:
  - Achieves the minimal length of the coded text
- *Prefix code*: no codeword is a prefix of another
  - It can be shown that optimal coding can be done with prefix code




- We can store all codewords in a *binary trie* – very easy to decode
  - Coded characters in leaves
  - Each node contains the sum of the frequencies of all descendants

# Decoding using a binary trie




- **Leaves** represent characters.
- Codeword for a character is the simple path from the root to that character.
- 0: left 1:right

Decode: **001011101**

- Go to Socrative and write in your answer.

# Optimal Code/Trie

- The *cost* of the coding trie *T*:

$$B(T)=\sum_{c\in C} f(c)\,d_T(c)$$

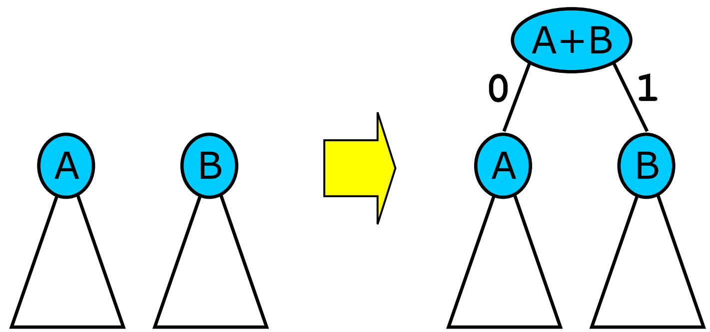
  - *C* – the alphabet,
  - $f(c)$ – frequency of character *c*,
  - $d_T(c)$ – depth of *c* in the trie (length of code in bits)

- Optimal trie – the one that minimizes $B(T)$

- Observation – optimal trie is always full:
  - Every non-leaf node has two children. Why?

# Huffman Algorithm - Idea

- Huffman algorithm, builds the code trie bottom up. Consider a forest of trees:
  - Initially – one separate node for each character.
  - In each step – join two trees into a larger tree




  - Repeat this until one tree (trie) remains.
  - Which trees to join? Greedy choice – the trees with the **smallest** frequencies!

# Huffman Algorithm

```
Huffman(C)
Q.build(C) // Builds a min-priority queue on frequencies
for i ← 1 to n-1 do
   Allocate new node z
   x ← Q.extractMin()
   y ← Q.extractMin()
   z.setLeft(x)  // corresponding to bit 0
   z.setRight(y) // corresponding to bit 1
   z.setF(x.f() + y.f())
   Q.insert(z)
return Q.extractMin() // Return the root of the trie
```

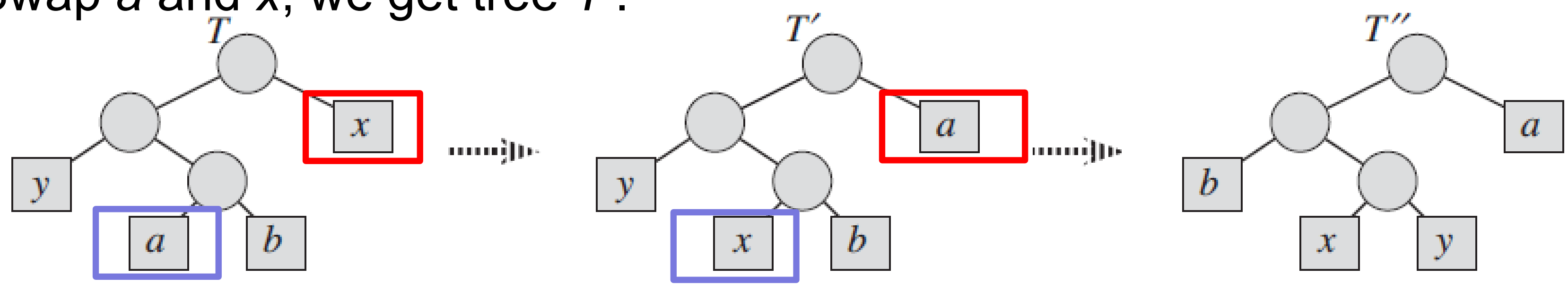
- *What is its running time?*
- Run the algorithm on: "**`oho ho, ole`**"

# Correctness of Huffman

- Greedy choice property:
  - Let *x, y* – two characters with lowest frequencies. Then there exists an optimal prefix code where codewords for *x* and *y* have the same length and differ only in the last bit
  - Let's prove it:
    - Transform an optimal trie *T* into one (*T''*), where *x* and *y* are max-depth siblings. Compare the costs.

# Greedy choice property

- Let *x* and *y* be the two characters with lowest frequencies.
- Let's assume that we have an optimal code tree *T*, where leaves a and b are two siblings of the maximum depth.
- Swap *a* and *x*, we get tree *T'*.




$$
\begin{aligned}
B(T) - B(T') &= \sum_{c \in C} c.freq \cdot d_T(c) - \sum_{c \in C} c.freq \cdot d_{T'}(c) \\
&= x.freq \cdot d_T(x) + a.freq \cdot d_T(a) - x.freq \cdot \boxed{d_{T'}(x)} - a.freq \cdot \boxed{d_{T'}(a)} \\
&= x.freq \cdot d_T(x) + a.freq \cdot d_T(a) - x.freq \cdot \boxed{d_T(a)} - a.freq \cdot \boxed{d_T(x)} \\
&= \boxed{(a.freq - x.freq)}\boxed{(d_T(a) - d_T(x))} \\
&\geq 0 ,
\end{aligned}
$$

**$B(T) \geq B(T')$**

Since *x* and *y* are the two characters with lowest frequencies, we have **$x.freq \leq a.freq$**

In tree *T, a* and *b* are two siblings of maximum depth. Thus, we have **$d_T(a) \geq d_T(x)$**

# Correctness of Huffman

- Optimal sub-structure property:
  - Let *x, y* – characters with minimum frequency
  - $C' = C - \{x,y\} \cup \{z\}$, such that $f(z) = f(x) + f(y)$
  - Let *T'* be an optimal code trie for *C'*
  - Replace leaf *z* in *T'* with internal node with two children *x* and *y*
  - The result tree *T* is an optimal code trie for *C*
- Proof a little bit more involved than a simple "cut-and-paste" argument

# Elements of Greedy Algorithms

- Greedy algorithms are used for optimization problems
  - A number of choices have to be made to arrive at an optimal solution.
  - At each step, make the “locally best” choice, without considering all possible choices and solutions to sub-problems induced by these choices (compare to dynamic programming).
  - After the choice, only one sub-problem remains (smaller than the original).

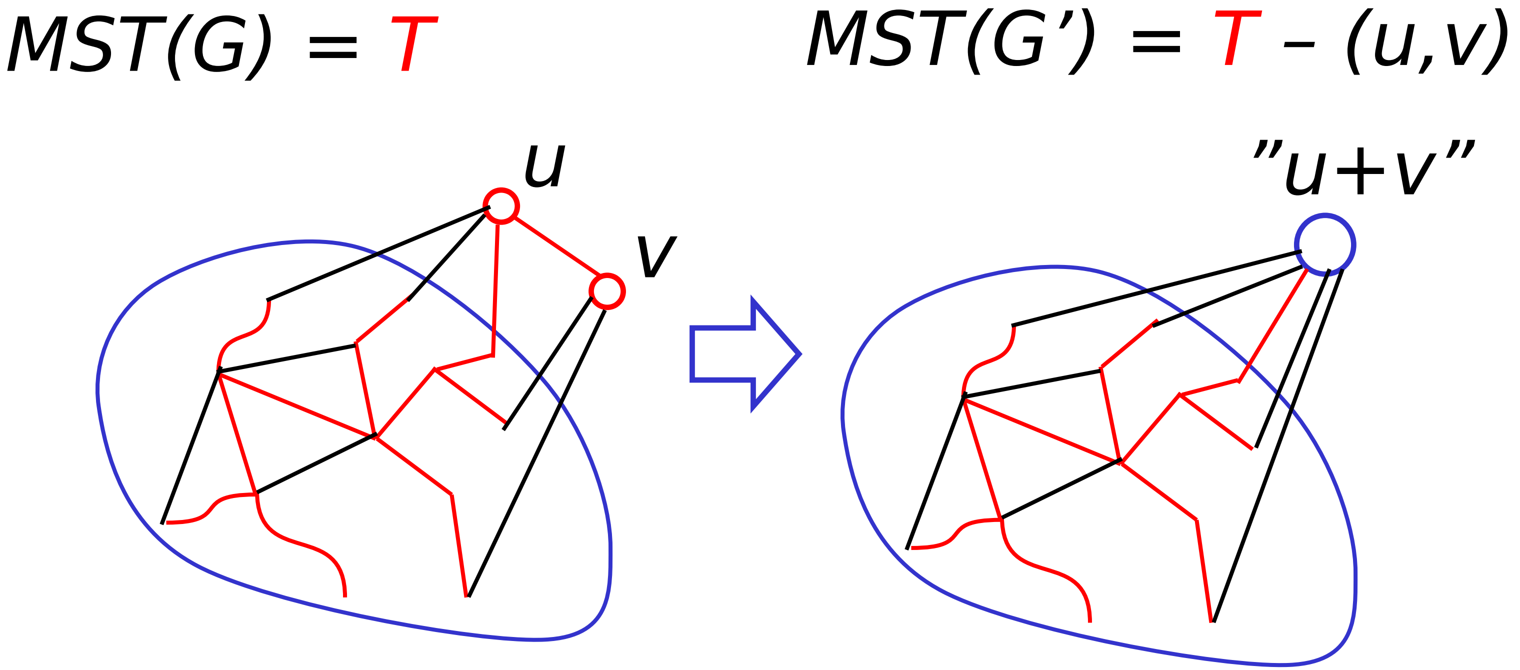
- Greedy algorithms usually sort or use priority queues.

# Elements of Greedy Algorithms

- First, one has to prove the *optimal sub-structure* property
  - the simple “cut-and-paste” argument may work
  - Not always possible (sign it is a hard problem):
    - Longest (vs. shortest) simple unweighted path
    - Maximum clique in a graph (vs. activity selection)

- The main challenge is to decide the interpretation of “the best” so that it leads to a global optimal solution, i.e., you can prove the *greedy choice property*
  - The proof is usually constructive: takes a hypothetical optimal solution without the specific greedy choice and transforms into one that has this greedy choice.
  - Or you find counter-examples demonstrating that your greedy choice does not lead to a global optimal solution.

# Prim's alg. for MST




- Optimal substructure: ***If*** (*u*,*v*) is in an MST *T* ***then*** *T* - (*u*,*v*) is an MST of G'
- "Cut and paste" argument
  - If *G'* would have a cheaper ST *T'*, then we would get a cheaper ST of *G*: *T'* + (*u*, *v*)
- Greedy choice: choose an edge incident on the "supervertex" with a minimum weight