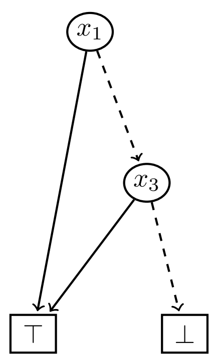
Exercise 1

Draw the Binary Decision Diagram associated with the following functions $f : \mathcal{B}^4 \to \mathcal{B}$ (having all 4 variables as input) under variable ordering x_1, x_2, x_3, x_4. For each BDD, indicate how many nodes and how many satisfying assignments represents.

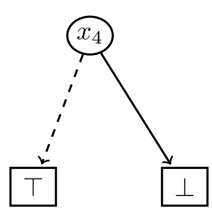
1. $x_1 \vee x_3$
2. $\neg x_4$
3. $x_4 \wedge \neg x_2 \wedge \neg x_1$
4. $(x_1 \vee x_3) \wedge (x_2 \vee x_4)$

Solution:

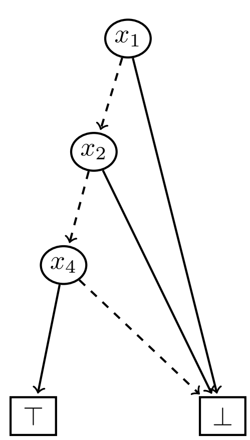
1. $x_1 \vee x_3$, 4 nodes, 12 assignments



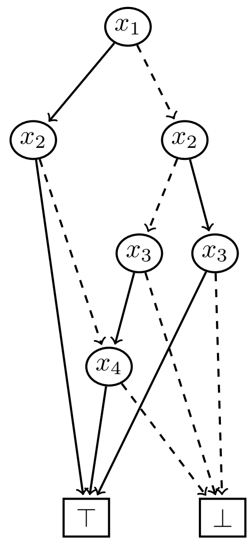
2. $\neg x_4$, 3 nodes, 8 assignments



3. $x_4 \wedge \neg x_2 \wedge \neg x_1$, 5 nodes, 2 assignments



4. $(x_1 \lor x_3) \land (x_2 \lor x_4)$, 8 nodes, 9 assignments



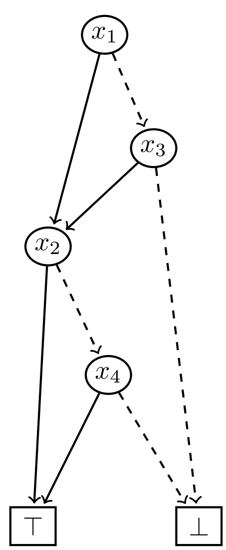
Exercise 2

For each of the BDDs in Exercise 1, does the variable ordering chosen minimize the number of nodes? Whenever a better variable ordering exist, provide it and draw the resulting BDD.

Solution:

Finding the best variable ordering is NP-hard in general. However in these small examples, the simple heuristic of trying to put related variables together is sufficient to find the best variable ordering.

1. $x_1 \lor x_3$: There is not a better variable ordering for this BDD
2. $\neg x_4$: There is not a better variable ordering for this BDD
3. $x_4 \land \neg x_2 \land \neg x_1$: There is not a better variable ordering for this BDD
4. $(x_1 \lor x_3) \land (x_2 \lor x_4)$: The best variable ordering is x_1, x_3, x_2, x_4. The resulting BDD is:



Exercise 3

Consider the following statements. Indicate whether they are true or false. In case they are false, indicate why.

Note: we denote by $|f|$ the number of nodes used to represent a Boolean function f.

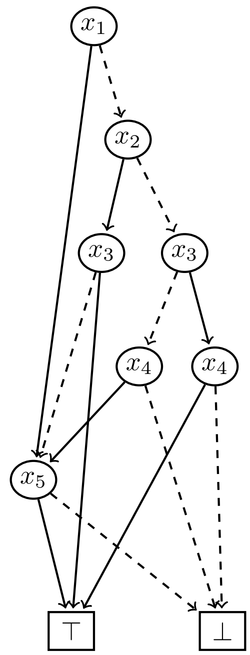
- For any Boolean function there is always a single, unique, BDD that represents it.
- Each path from the root of the BDD to a leaf node corresponds exactly with a total assignment of all variables.
- If a Boolean function f is represented as a BDD, then for each satisfiable assignment there is a corresponding path from the root node to $\top$.
- Consider 3 Boolean functions f,g, and h such that $h = f \vee g$. All of them are represented as BDDs under the same variable ordering. Then, $|f| + |g| \leq |h|$.
- Consider 3 Boolean functions f,g, and h such that $h = f \vee g$. All of them are represented as BDDs under the same variable ordering. Then, $\max(|f|, |g|) \leq |h|$.

Solution:

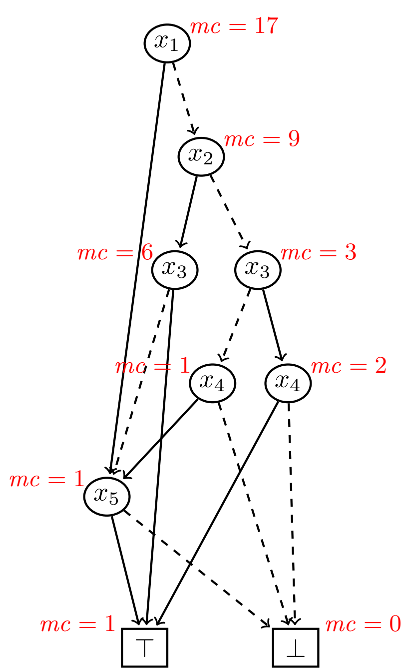
- False. The statement would be true for a fixed variable ordering, but there may be multiple representations of the same function as different BDDs by changing the variable ordering.
- False. Each path corresponds to a partial assignment.
- True.
- False, it could be the case that $|f|$ and $|g|$ are arbitrarily large while $|h|$ is a single node (e.g., representing the function $\top$).
- False, it could be the case that $|f|$ and $|g|$ are arbitrarily large while $|h|$ is a single node (e.g., representing the function $\top$).

Exercise 4

Consider the following BDD. What is the number of satisfying assignments of the Boolean function that it represents? To count it, run the model counting algorithm, labelling each node with the corresponding $mc(f)$ value.



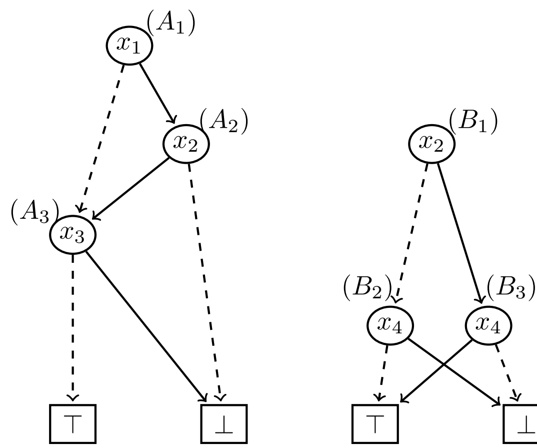
Solution:



The total number of assignments is $MC = 17$, as the root node of our BDD is on the first level (x_1).

Exercise 5

Compute the conjunction of these two BDDs, and draw the resulting BDD. Mark each node of the resulting BDD with identifiers (either re-using one of the existing identifiers or choosing a new one: $C_1, \ldots,$). For each recursive call to the *apply* procedure, specify what is the input (the identifiers of the two input nodes) and the output (the identifier of the output node).



Solution:

Input	$\mapsto$ Output
(A_1, B_1)	$\mapsto C_5$
(A_3, B_1)	$\mapsto C_3$
(A_3, B_2)	$\mapsto C_1$
$(\top, B_2)$	$\mapsto B_2$
$(\perp, B_2)$	$\mapsto \perp$
(A_3, B_3)	$\mapsto C_2$
$(\top, B_3)$	$\mapsto B_3$
$(\perp, B_3)$	$\mapsto \perp$
(A_2, B_1)	$\mapsto C_4$
(A_2, B_1)	$\mapsto C_4$
$(\perp, B_2)$	$\mapsto \perp$
(A_3, B_3)	$\mapsto C_2$