Algorithms and Satisfiability

7. Satisfiability, Part I: Principles and Basic Algorithms

How to Think About What is True or False

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Thanks to Jörg Hoffmann for slide sources

Agenda

So far...

1. Dynamic Programming
2. Greedy Algorithms
3. Computational geometry algorithms: sweeping techniques
4. External-memory algorithms and data structures
5. Parallel algorithms
6. Amortized analysis

$\rightarrow$Techniques to make efficient algorithms and analyze their performance

What if an efficient algorithm does not exist?

What to do when you can't find an efficient algorithm?[1]

"I can't find an efficient algorithm, I guess I'm just too dumb."

In this lecture:

In this lecture we will study algorithms for:

1. Satisfiability: Given a Boolean formula, is it satisfiable?
   $\rightarrow$**NP**-complete

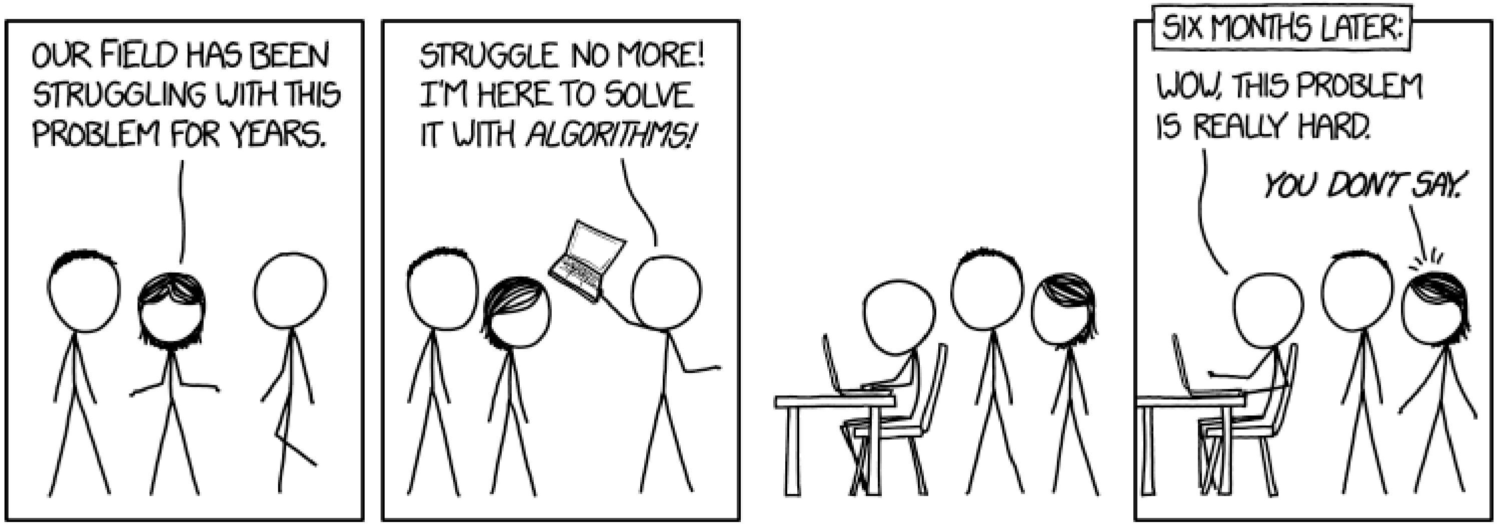
2. Classical Planning: Can we achieve our goal by applying a sequence of actions?
   $\rightarrow$**PSPACE**-complete

$\rightarrow$These problems cannot be solved in polynomial time (unless **P**=**NP**)!

But that is only worst case assymptotic complexity...

- Do the interesting real-world instances pertain to the worst case?
- Are interesting real-world instances small enough so that we can solve them?

You say I can't solve that? Hold my beer!



"We TOLD you it was hard." "Yeah, but now that I'VE tried, we KNOW it's hard." XKCD.com/1831

Two Questions

What algorithms can we use to solve these hard problems?
$\rightarrow$Explored in the Lectures and Exercises

How to solve these hard problems in practice using solvers?
$\rightarrow$Explored in the mini-projects

- Preparation: Install tool in your computer
- (shorter) Lecture
- Project: Use a solver to solve some problems
- Optional: submit your solution to receive feedback
- Exam relevant!: in 2021 no one submitted their solution and at least half of the students failed to answer the exam question!

The SAT Problem

SAT

Is a propositional logic formula ϕ satisfiable?

- Propositional Logic: Satisfiability can refer to many different logics. In this course, we focus on one of the simplest!
- Satisfiable: A formula is satisfiable if it is possible to find an interpretation (assignment) that makes the formula true.

$\rightarrow$Does there exist an assignment that makes the formula true?
Examples:

- $(x \vee y) \wedge (\neg x \vee \neg y)$, Yes! $x = 0, y = 1$
- $(x \vee y) \wedge (\neg x) \wedge (\neg y)$, No!

Our Agenda for This Topic

→ Our treatment of the topic "SAT Solving" consists of Chapters 7 and 8.

- **This Chapter:** Basic definitions and concepts; resolution; DPLL.

  → Sets up the framework. Resolution is the quintessential reasoning procedure underlying most successful solvers.

- **Chapter 8:** Clause learning; practical problem structure.

  → State-of-the-art algorithms for satisfiability in propositional logic, and an important observation about how they behave.

- **Mini-project:** SAT modulo theories (SMT)

  → Extension beyond propositional formulas!

Our Agenda for This Chapter

- **Basics:** What's the SAT problem about?
  $\rightarrow$ Introduces what our problem is about.
- **Applications:** What is all this useful for?
  $\rightarrow$ Brief description of some of the applications of SAT.
- **Resolution:** How does resolution work? What are its properties?
  $\rightarrow$ Formally introduces the most basic reasoning method.
- **The Davis-Putnam (Logemann-Loveland) Procedure:** How to systematically test satisfiability?
  $\rightarrow$ The quintessential SAT solving procedure, DPLL.

Syntax of Propositional Logic

$\rightarrow$ Atoms Σ in propositional logic = Boolean variables.

Definition (Syntax). *Let Σ be a set of atomic propositions. Then:*

1. $\bot$ *and* $\top$ *are Σ-formulas.* ("False", "True")
2. *Each $P \in \Sigma$ is a Σ-formula.* ("Atom")
3. *If φ is a Σ-formula, then so is $\neg\varphi$.* ("Negation")

If φ and ψ are Σ-formulas, then so are:

4. $\varphi \wedge \psi$ ("Conjunction")
5. $\varphi \vee \psi$ ("Disjunction")
6. $\varphi \rightarrow \psi$ ("Implication")
7. $\varphi \leftrightarrow \psi$ ("Equivalence")

Notation: Atoms and negated atoms are called literals. Operator precedence: $\neg > \dots$ (we'll be using brackets except for negation).

Semantics of Propositional Logic

Definition (Interpretation). *Let Σ be a set of atomic propositions. An interpretation of Σ, also called a truth assignment, is a function $I : \Sigma \mapsto \{1, 0\}$. We set:*

$$
\begin{array}{lll}
I \models \top & & \\
I \not\models \bot & & \\
I \models P & \textit{iff} & P^I = 1 \\
I \models \neg\varphi & \textit{iff} & I \not\models \varphi \\
I \models \varphi \wedge \psi & \textit{iff} & I \models \varphi \text{ and } I \models \psi \\
I \models \varphi \vee \psi & \textit{iff} & I \models \varphi \text{ or } I \models \psi \\
I \models \varphi \rightarrow \psi & \textit{iff} & \textit{if } I \models \varphi, \textit{ then } I \models \psi \\
I \models \varphi \leftrightarrow \psi & \textit{iff} & I \models \varphi \textit{ if and only if } I \models \psi
\end{array}
$$

If $I \models \varphi$, we say that I satisfies φ, or that I is a model of φ. The set of all models of φ is denoted by $M(\varphi)$.

Semantics of Propositional Logic: Examples

Example

Formula: $\varphi = [(P \vee Q) \leftrightarrow (R \vee S)] \wedge [\neg(P \wedge Q) \wedge (R \wedge \neg S)]$

$\rightarrow$ For I with $I(P) = 1, I(Q) = 1, I(R) = 0, I(S) = 0$, do we have $I \models \varphi$? No: $(P \vee Q)$ is true but $(R \vee S)$ is false, so the left-hand side of the conjunction is false and the overall formula is false.

Example

Formula: $\varphi = \text{InSatisfiabilityClass} \rightarrow \text{HavingAGreatTime}$

$\rightarrow$ For I with $I(\text{InSatisfiabilityClass}) = 0$, $I(\text{HavingAGreatTime}) = 1$, do we have $I \models \varphi$? Yes: $\varphi = \psi_1 \rightarrow \psi_2$ is true iff either ψ_1 is false, or ψ_2 is true (i.e., $\psi_1 \rightarrow \psi_2$ has the same models as $\neg\psi_1 \vee \psi_2$).

Terminology

Satisfiability

A formula φ is:

- satisfiable if there exists I that satisfies φ.
- unsatisfiable if φ is not satisfiable.
- falsifiable if there exists I that doesn't satisfy φ.
- valid if $I \models \varphi$ holds for all I. We also call φ a tautology.

Equivalence

Formulas φ and ψ are equivalent, $\varphi \equiv \psi$, if $M(\varphi) = M(\psi)$.

Entailment

Formula φ entails ψ ($\varphi \models \psi$), if $M(\psi) \subseteq M(\varphi)$.

General Problem Solving using SAT

(some new problem)

model problem in logic ↦ use off-the-shelf SAT solver

(its solution)

- "Any problem that can be formulated as SAT."
- *Very* successful using propositional logic and modern solvers for SAT! (Propositional satisfiability testing, **Chapter 8**.)

Applications

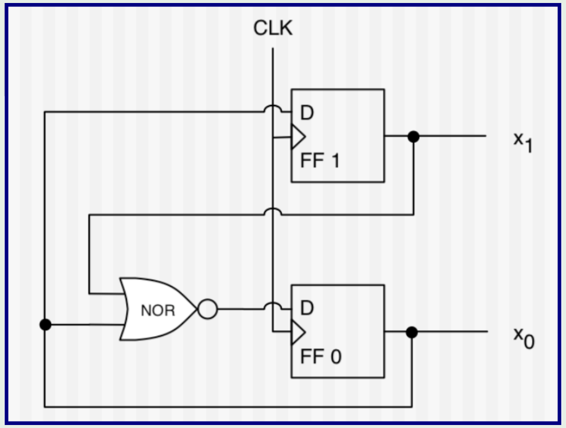
Lots of interesting problems can be formulated as SAT!

- Lots of NP problems
- Scheduling
- Hardware Verification
- Logical Deduction
- Planning **(Chapter 12)**

And we can even extend this by considering SAT modulo theories **(mini project)**

Example Application: Hardware Verification

Example



- Counter, repeatedly from $c = 0$ to $c = 2$.
- 2 bits x_1 and x_0; $c = 2 * x_1 + x_0$.
- ("FF" Flip-Flop, "D" Data IN, "CLK" Clock)
- **To Verify:** If $c < 3$ in current clock cycle, then $c < 3$ in next clock cycle.

Step 1: Encode into propositional logic.

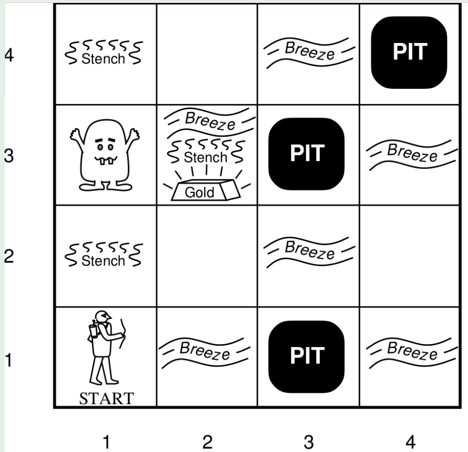
- Propositions: x_1, x_0; and x_1', x_0' (value in next cycle).
- Transition relation: $x_1' \leftrightarrow x_0$; $x_0' \leftrightarrow \neg(x_1 \vee x_0)$.
- Initial state: $\neg(x_1 \wedge x_0)$. Error property: $x_1' \wedge x_0'$.

Step 2: Transform to CNF, encode as set Δ of clauses.
$\rightarrow \{\{\neg x_1', x_0\}, \{x_1', \neg x_0\}, \{x_0', x_1, x_0\}, \{\neg x_o', \neg x_1\}, \{\neg x_0', \neg x_0\}, \{\neg x_1, \neg x_0\}, \{x_1'\}, \{x_0'\}\}$

Step 3: Call a SAT solver (up next).

Example Application: Logical Deduction (Wumpus)

Example



- The player cannot see the entire board, only if there is Stench or Breeze on the current cell.
- **To Verify:** After visiting [2,1] and [1,2], are we sure cell [2,2] is free?.

Step 1: Encode into propositional logic.

- Propositions: $Stench[i,j], Breeze[i,j], Wumpus[i,j], Pit[i,j]$;
- Knowledge Base: $KB = \bigwedge_{i,j} Wumpus[i,j] \implies Stench[i,j+1] \wedge Stench[1,2] \wedge \dots$
- Question: Q $= \neg(Wumpus[2,2] \wedge Pit[2,2])$.

Step 2: Transform $KB \wedge \neg(Q)$ to CNF, encode as set Δ of clauses.

Step 3: Call a SAT solver (up next). If unsatisfiable, then we can conclude Q.

The Truth Table Method

Want: Determine whether φ is satisfiable, valid, etc.

Method: Build the truth table, enumerating all interpretations of Σ.

Example

Is $\varphi = ((P \vee H) \wedge \neg H) \rightarrow P$ valid?

P	H	$P \vee H$	$(P \vee H) \wedge \neg H$	$(P \vee H) \wedge \neg H \rightarrow P$
0	0	0	0	1
0	1	1	0	1
1	0	1	1	1
1	1	1	0	1

$\rightarrow$ Yes. φ is true for all possible combinations of truth values.

$\rightarrow$ Is this a good method for answering these questions? No! For N propositions, the truth table has 2^N rows. [Satisfiability (validity) testing is **NP**-hard (**co-NP**-hard), but that pertains to *worst-case* behavior.]

Normal Forms

The two quintessential normal forms: (there are others as well)

- A formula is in conjunctive normal form (CNF) if it consists of a conjunction of disjunctions of literals:

$$\bigwedge_{i=1}^{n}\left(\bigvee_{j=1}^{m_i} l_{i,j}\right)$$

- A formula is in disjunctive normal form (DNF) if it consists of a disjunction of conjunctions of literals:

$$\bigvee_{i=1}^{n}\left(\bigwedge_{j=1}^{m_i} l_{i,j}\right)$$

$\rightarrow$ Given a propositional formula φ, we can in polynomial time construct a CNF/DNF formula ψ that is satisfiable if and only if φ is. (Proof omitted)

Transformation to Normal Form

CNF Transformation (DNF Transformation: Analogously)

Exploit the equivalences:

1. $(\varphi \leftrightarrow \psi) \equiv [(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)]$ (Eliminate "$\leftrightarrow$")
2. $(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi)$ (Eliminate "$\rightarrow$")
3. $\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi)$ and $\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$ (Move "$\neg$" inwards)
4. $[(\varphi_1 \wedge \varphi_2) \vee (\psi_1 \wedge \psi_2)] \equiv [(\varphi_1 \vee \psi_1) \wedge (\varphi_2 \vee \psi_1) \wedge (\varphi_1 \vee \psi_2) \wedge (\varphi_2 \vee \psi_2)]$ (Distribute "$\vee$" over "$\wedge$")

Example: $((P \vee H) \wedge \neg H) \rightarrow P$

(eliminate $\rightarrow$):	$\neg((P \vee H) \wedge \neg H) \vee P$	
(move $\neg$ inwards):	$(\neg(P \vee H) \vee H) \vee P$	$((\neg P \wedge \neg H) \vee H) \vee P$
(distr. "$\vee$" over "$\wedge$"):	$(((\neg P \vee H) \wedge (\neg H \vee H))) \vee P$	
(distr. "$\vee$" over "$\wedge$"):	$(((\neg P \vee H \vee P) \wedge (\neg H \vee H \vee P)))$	

$\rightarrow$ Note: The formula may grow exponentially! ("Distribute" step)

$\rightarrow$ However, satisfiability-preserving CNF transformation is polynomial!

Questionnaire

Question!

A CNF formula is . . .

(A): Valid iff at least one disjunction is valid.

(B): Valid iff every disjunction is valid.

(C): Satisfiable if at least one disjunction is satisfiable.

(D): Satisfiable if every disjunction is satisfiable.

→ (A): No, other parts of the global conjunction may be false under any one given interpretation.

→ (B): Yes: The CNF is a conjunction of valid formulas, so is valid itself. (Compare the CNF transformation of the example formula on slide 26).

→ (C): No since we need *all* disjuncts to be satisfied together.

→ (D): No since we need all disjuncts to be satisfied together *by the same interpretation*.

Clausal Form

$\rightarrow$ For the remainder of this chapter, we assume that the input is a set Δ of clauses: (The same will be assumed in **Chapter 8**)

Terminology and Notation

- A literal l is an atom or the negation thereof (e.g., $P, \neg Q$); the negation of a literal is denoted $\bar{l}$ (e.g., $\overline{\neg Q} = Q$).
- A clause C is a disjunction of literals. We identify C with the set of its literals (e.g., $P \vee \neg Q$ becomes $\{P, \neg Q\}$).
- We identify a CNF formula ψ with the set Δ of its clauses (e.g., $(P \vee \neg Q) \wedge R$ becomes $\{\{P, \neg Q\}, \{R\}\}$).
- The empty clause is denoted $\Box$.

$\rightarrow$ An interpretation I satisfies a clause C iff there exists $l \in C$ such that $I \models l$. I satisfies Δ iff, for all $C \in \Delta$, we have $I \models C$.

Satisfiability in the Clausal Form: Rim Cases

It's normally simple . . .

- E.g., I with $I(P) = 0, I(Q) = 0, I(R) = 0$ does not satisfy $\Delta = \{\{P, \neg Q\}, \{R\}\}$.

. . . but can be confusing in the "rim cases":

- Does there exist I so that $I \models \Box$? No, there exists no literal $l \in \Box$ that we can satisfy.
- With $\Delta = \{\Box\}$, does there exist I so that $I \models \Delta$? No, because we can't satisfy $\Box$.
- With $\Delta = \{\}$, does there exist I so that $I \models \Delta$? Yes, because I satisfies all clauses $C \in \Delta$ (trivial as there is no clause in Δ).

Deduction

Basic Concepts in Deduction

- **Inference rule**: Rule prescribing how we can infer new formulas.

  $\rightarrow$ For example, if the KB is $\{\dots, (\varphi \rightarrow \psi), \dots, \varphi, \dots\}$ then ψ can be deduced using the inference rule $\frac{\varphi, \varphi \rightarrow \psi}{\psi}$.
- **Calculus**: Set $\mathcal{R}$ of inference rules.
- **Derivation**: φ can be derived from KB using $\mathcal{R}$, KB $\vdash_{\mathcal{R}} \varphi$, if starting from KB there is a sequence of applications of rules from $\mathcal{R}$, ending in φ.
- **Soundness**: $\mathcal{R}$ is sound if all derivable formulas do follow logically: if KB $\vdash_{\mathcal{R}} \varphi$, then KB $\models \varphi$.
- **Completeness**: $\mathcal{R}$ is complete if all formulas that follow logically are derivable: if KB $\models \varphi$, then KB $\vdash_{\mathcal{R}} \varphi$.

$\rightarrow$ If $\mathcal{R}$ is sound and complete, then to check whether KB $\models \varphi$, we can check whether KB $\vdash_{\mathcal{R}} \varphi$.

The Resolution Rule

Definition (Resolution Rule). *Resolution uses the following inference rule (with exclusive union $\dot\cup$ meaning that the two sets are disjoint):*

$$\frac{C_1\dot\cup\{l\}, C_2\dot\cup\{\bar{l}\}}{C_1 \cup C_2}$$

If Δ contains parent clauses of the form $C_1\dot\cup\{l\}$ and $C_2\dot\cup\{\bar{l}\}$, the rule allows to add the resolvent clause $C_1 \cup C_2$. l and $\bar{l}$ are called the resolution literals.

Example: $\{P, \neg R\}$ resolves with $\{R, Q\}$ to $\{P, Q\}$.

Lemma. *The resolvent follows from the parent clauses.*

Proof. If $I \models C_1\dot\cup\{l\}$ and $I \models C_2\dot\cup\{\bar{l}\}$, then I must make at least one literal in $C_1 \cup C_2$ true.

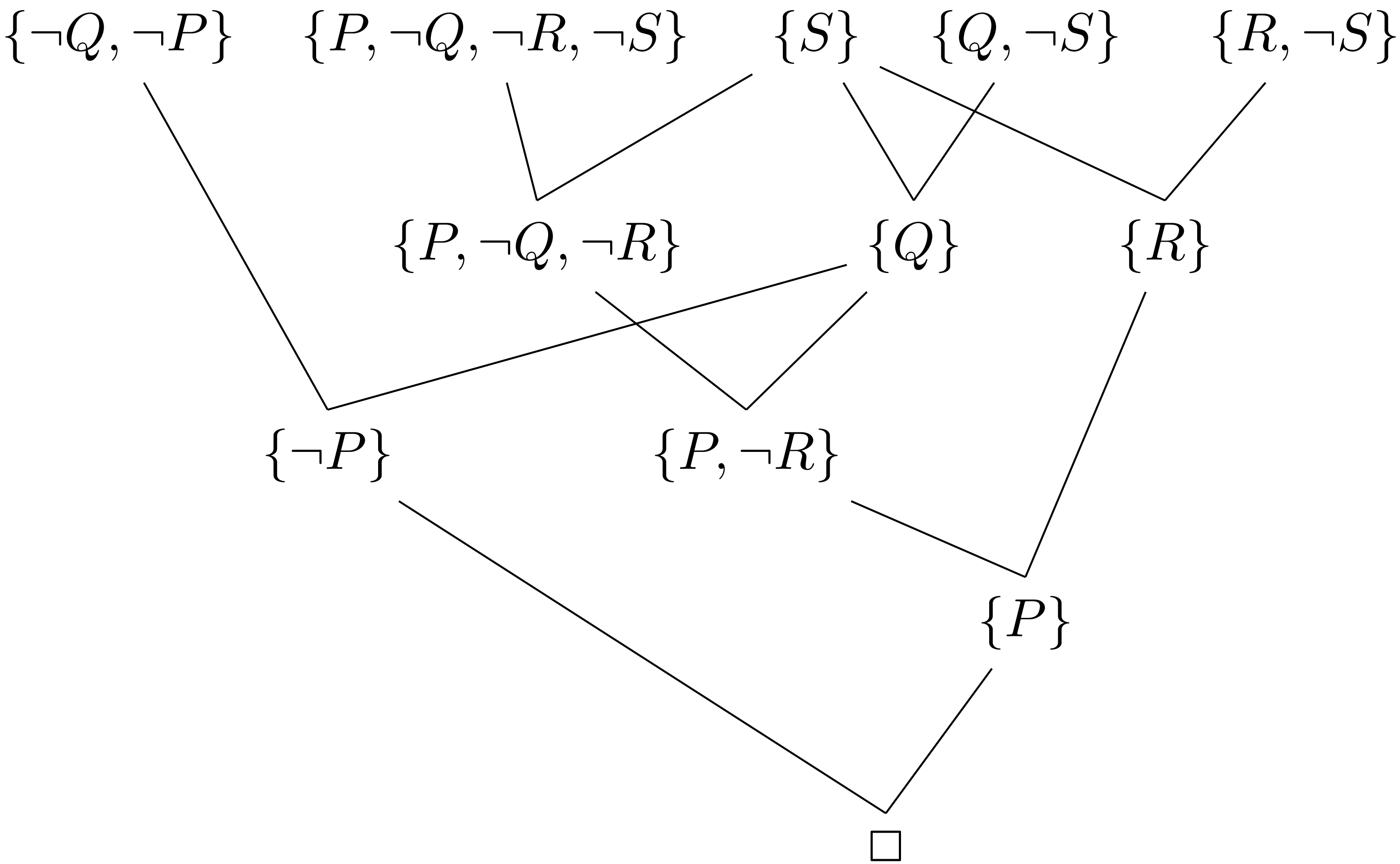
Theorem (Soundness). *If $\Delta \vdash D$, then $\Delta \models D$.* (Direct from Lemma.)

$\rightarrow$ What about the other direction? Is the resolvent *equivalent* to its parents?
No, because to satisfy the resolvent it is enough to satisfy one of C_1, C_2. E.g.: Setting $I(P) = 0$ and $I(Q) = 1$, we satisfy $\{P, Q\}$ but do not satisfy $\{P, \neg R\}$ when setting the resolution literal to $I(R) = 1$.

Using Resolution: A Simple Example

$\Delta = \{\{\neg Q, \neg P\}, \{P, \neg Q, \neg R, \neg S\}, \{Q, \neg S\}, \{S\}, \{R, \neg S\}\}$

Derive $\Box$ by applying the resolution rule.



Using Resolution: A Frequent Mistake

Question: Given clauses $C_1\dot{\cup}\{P,Q\}$ and $C_2\dot{\cup}\{\neg P,\neg Q\}$, can we resolve them to $C_1 \cup C_2$?

Answer: NO!

Observation 1: Consider $\Delta = \{\{P,Q\},\{\neg P,\neg Q\}\}$, and assume we were able to resolve as above. Then we could derive the empty clause. However, Δ is satisfiable (e.g. $P := T, Q := F$), so this deduction would be unsound.

Observation 2: The proof of the lemma on slide 32 is not valid for the hypothetical resolution of $C_1\dot{\cup}\{P,Q\}$ and $C_2\dot{\cup}\{\neg P,\neg Q\}$ to $C_1 \cup C_2$. This is due to Observation 1: An interpretation can set, e.g., $P := T, Q := F$, satisfying *both* $\{P,Q\}$ *and* $\{\neg P,\neg Q\}$ *together*, avoiding the need to satisfy either of C_1 or C_2.

Questionnaire

Question!

What are resolvents of $\{P, \neg Q, R\}$ and $\{\neg P, Q, R\}$?

(A): $\{Q, \neg Q, P, R\}$.

(B): $\{P, \neg P, R, S\}$.

(C): $\{R\}$.

(D): $\{Q, \neg Q, R\}$.

$\rightarrow$ (A): No. If we resolve on P then it disappears completely.

$\rightarrow$ (B): No. By resolving on Q we get this clause except S, and although the larger clause always is sound as well of course, we are not allowed to deduce it by the rule.

$\rightarrow$ (C): No. If we resolve on P then we get both Q and $\neg Q$ into the clause, similar if we resolve on Q.
$\rightarrow$ We can resolve on only ONE literal at a time, cf. slide 34.

$\rightarrow$ (D): Yes, this is what we get by resolving on P.

The DPLL Procedure

Call on input Δ and the empty partial interpretation I:

function DPLL(Δ, I) **returns** a partial interpretation I, or "unsatisfiable"
/* Unit Propagation (UP) Rule: */
  Δ' := a copy of Δ; $I' := I$
  while Δ' contains a unit clause $C = \{l\}$ **do**
    extend I' with the respective truth value for the proposition underlying l
    simplify Δ' /* remove false literals and true clauses */
/* Termination Test: */
  if $\Box \in \Delta'$ **then return** "unsatisfiable"
  if $\Delta' = \{\}$ **then return** I'
/* Splitting Rule: */
  select some proposition P for which I' is not defined
  $I'' := I'$ extended with one truth value for P; Δ'' := a copy of Δ'; simplify Δ''
  if $I''' :=$ DPLL$(\Delta'', I'') \neq$ "unsatisfiable" **then return** I'''
  $I'' := I'$ extended with the other truth value for P; $\Delta'' := \Delta'$; simplify Δ''
  return DPLL(Δ'', I'')

$\rightarrow$ In practice, of course one uses flags etc. instead of "copy".

DPLL: Example (Vanilla1)

$$\Delta = \{\{P, Q, \neg R\}, \{\neg P, \neg Q\}, \{R\}, \{P, \neg Q\}\}$$

1. UP Rule: $R \mapsto 1$
   $\{\{P, Q\}, \{\neg P, \neg Q\}, \{P, \neg Q\}\}$
2. Splitting Rule:

2a. $P \mapsto 0$
$\{\{Q\}, \{\neg Q\}\}$

3a. UP Rule: $Q \mapsto 1$
$\{\Box\}$

2b. $P \mapsto 1$
$\{\{\neg Q\}\}$

3b. UP Rule: $Q \mapsto 0$
$\{\}$

DPLL: Example (Vanilla2)

$$\Delta = \{\{\neg Q, \neg P\}, \{P, \neg Q, \neg R, \neg S\}, \{Q, \neg S\}, \{R, \neg S\}, \{S\}\}$$

1. UP Rule: $S \mapsto 1$
   $\{\{\neg Q, \neg P\}, \{P, \neg Q, \neg R\}, \{Q\}, \{R\}\}$
2. UP Rule: $Q \mapsto 1$
   $\{\{\neg P\}, \{P, \neg R\}, \{R\}\}$
3. UP Rule: $R \mapsto 1$
   $\{\{\neg P\}, \{P\}\}$
4. UP Rule: $P \mapsto 1$
   $\{\Box\}$

Properties of DPLL

Unsatisfiable case:

- What can we say if "unsatisfiable" is returned?

  $\rightarrow$ In this case, we know that Δ is unsatisfiable: Unit propagation is *sound*, in the sense that it does not reduce the set of solutions. ($=$ Soundness of calculus, cf. next two slides.)

Satisfiable case:

- What can we say when a partial interpretation I is returned?

  $\rightarrow$ Any extension of I to a complete interpretation satisfies Δ. (By construction, I suffices to satisfy all clauses.)

DPLL is an example of a successful algorithmic pattern: Search $+$ Inference

- DPLL $\approx$ Search $=$ Backtracking, with Inference() $=$ unit propagation.
- Unit propagation is sound: It does not reduce the set of solutions. (Also: $=$ Soundness of calculus, cf. next slide.)

UP = Unit Resolution

The Unit Propagation (UP) Rule ...

while Δ' contains a unit clause $\{l\}$ **do**
  extend I' with the respective truth value for the proposition underlying l
  simplify Δ' /* remove false literals */

... corresponds to a calculus:

Definition (Unit Resolution). *Unit Resolution is the calculus consisting of the following inference rule:*

$$\frac{C\dot{\cup}\{\bar{l}\}, \{l\}}{C}$$

That is, if Δ contains parent clauses of the form $C\dot{\cup}\{\bar{l}\}$ and $\{l\}$, the rule allows to add the resolvent clause C.

$\rightarrow$ Unit propagation = Resolution restricted to the case where one of the parent clauses is unit.

UP/Unit Resolution: Soundness/Completeness

Soundness:

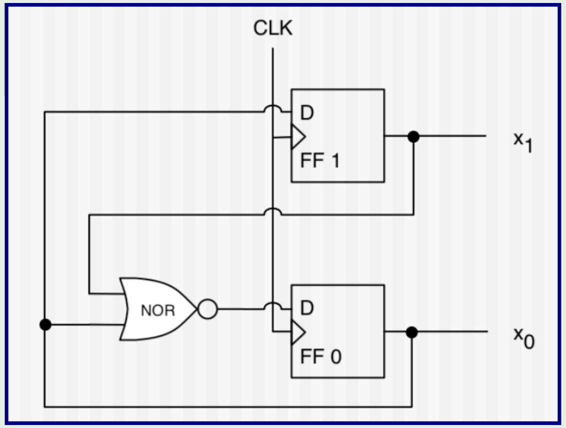
- Need to show: *If Δ' can be derived from Δ by UP, then $\Delta \models \Delta'$.*
- Yes, because any derivation made by unit resolution can also be made by (full) resolution, which we already know has this property.
- (Intuitively: if Δ' contains the unit clause $\{l\}$, then l must be made true so $C\dot{\cup}\{\bar{l}\}$ implies C.)

Completeness:

- Need to show: *If $\Delta \models \Delta'$, then Δ' can be derived from Δ by UP.*
- No. UP makes only limited inferences, as long as there are unit clauses. It does not guarantee to infer everything that can be inferred.
- Example: $\{\{P, Q\}, \{P, \neg Q\}, \{\neg P, Q\}, \{\neg P, \neg Q\}\}$ is unsatisfiable but UP cannot derive the empty clause $\Box$.

Questionnaire

Example



- Counter, repeatedly from $c = 0$ to $c = 2$.
- **To Verify:** If $c < 3$ in current clock cycle, then $c < 3$ in next clock cycle.
- $\Delta = \{\{\neg x_1', x_0\}, \{x_1', \neg x_0\}, \{x_0', x_1, x_0\}, \{\neg x_o', \neg x_1\}, \{\neg x_0', \neg x_0\}, \{\neg x_1, \neg x_0\}, \{x_1'\}, \{x_0'\}\}$

Question!

How many recursive calls to DPLL are made on Δ?

(A): 0 (B): 1

(C): 4 (D): 11

$\rightarrow$ The correct answer is (B): UP derives the empty clause (via $\{x_1'\}$, $\{\neg x_1', x_0\}$, $\{\neg x_0', \neg x_0\}$, $\{x_0'\}$) in the first recursive call, so exactly 1 search node is generated.

Summary

- SAT: Is a propositional logic formula ϕ satisfiable?
  - Hard problem in general (NP-hard)
  - Many applications
- Propositional logic formulas are built from atomic propositions, with the connectives "and, or, not".
- Every propositional formula can be brought into conjunctive normal form (CNF), which can be identified with a set of clauses.
- Resolution is a deduction procedure based on trying to derive the empty clause. It is refutation-complete, and can be used to prove $\mathrm{KB} \models \varphi$ by showing that $\mathrm{KB} \cup \{\neg\varphi\}$ is unsatisfiable.
- SAT solvers decide satisfiability of CNF formulas. This can be used for deduction, and is highly successful as a general problem solving technique (e.g., in Verification).
- DPLL = backtracking with inference performed by unit propagation (UP), which iteratively instantiates unit clauses and simplifies the formula.

Further Reading

The main material for the course are the post-handouts. If you are interested on more detailed overview of the topic, you can check these books:

- The Art of Computer Programming by Donald E. Knuth, Vol 4. Section 7.2.2.2
- Handbook of Satisfiability, Hans van Maaren, Armin Biere, Toby Walsh.