Algorithms and Satisfiability

8. Satisfiability, Part II: SAT Solvers, DPLL, and Clause Learning

How to *Efficiently* Think About What is True or False

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Thanks to Jörg Hoffmann for slide sources

Agenda

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1. Introduction
2. DPLL $=$ (A Restricted Form of) Resolution
3. *Why* Did Unit Propagation Yield a Conflict?
4. Clause Learning
5. Phase Transitions: Where the *Really* Hard Problems Are
6. Conclusion

Reminder: Our Agenda for This Topic

$\rightarrow$ Our treatment of the topic "Satisfiability" consists of Chapters 7 and 8.

- **Chapter 7:** Basic definitions and concepts; resolution; DPLL.

  $\rightarrow$ Sets up the framework. Resolution is the quintessential reasoning procedure underlying most successful solvers.

- **This Chapter:** Clause learning; practical problem structure.

  $\rightarrow$ State-of-the-art algorithms for reasoning about propositional logic, and an important observation about how they behave.

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Reminder: To decide whether KB $\models \varphi$, decide satisfiability of $\theta :=$ KB $\cup \{\neg\varphi\}$: θ is unsatisfiable iff KB $\models \varphi$.

$\rightarrow$ Deduction can be performed using SAT solvers.

Reminder: General Problem Solving using Logic

(some new problem)

model problem in logic $\mapsto$ use off-the-shelf reasoning tool

(its solution)

- "Any problem that can be formulated as reasoning about logic."
- *Very* successful using propositional logic and modern solvers for SAT! (Propositional satisfiability testing, $\rightarrow$ **This Chapter**.)

Reminder: Conventions

Terminology and Notation

- A literal l is an atom or the negation thereof (e.g., $P, \neg Q$); the negation of a literal is denoted $\bar{l}$ (e.g., $\overline{\neg Q} = Q$).
- A clause C is a disjunction of literals. We identify C with the set of its literals (e.g., $P \vee \neg Q$ becomes $\{P, \neg Q\}$).
- We identify a CNF formula ψ with the set Δ of its clauses (e.g., $(P \vee \neg Q) \wedge R$ becomes $\{\{P, \neg Q\}, \{R\}\}$).
- The empty clause is denoted $\Box$.

$\rightarrow$ For the remainder of this chapter, we assume that the input is a set Δ of clauses.

Our Agenda for This Chapter

- **DPLL = (A Restricted Form of) Resolution:** How does this relate to what we did in the last chapter?

  → Mathematical understanding of DPLL.

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- **Phase Transitions: Where the Really Hard Problems Are:** Are *all* formulas "hard" to solve?

  → The answer is "no". And in some cases we can figure out exactly when they are/aren't hard to solve.

But – What About Local Search for SAT?

There's a wealth of research on local search for SAT, e.g.:

GSAT Algorithm

INPUT: a set of clauses Δ, $\textsc{Max-Flips}$, and $\textsc{Max-Tries}$
OUTPUT: a satisfying truth assignment of Δ, if found

for $i := 1$ to $\textsc{Max-Tries}$
  $I :=$ a randomly-generated truth assignment
  for $j := 1$ to $\textsc{Max-Flips}$
    if I satisfies Δ **then return** I
    $X :=$ a proposition reversing whose truth assignment gives
        the largest increase in the number of satisfied clauses
    $I := I$ with the truth assignment of X reversed
  end for
end for
return "no satisfying assignment found"

$\rightarrow$ Local search is not as successful in SAT applications, though it can be a good complement to the techniques presented here. Not covered here.

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DPLL = (A Restricted Form of) Resolution

Notation: Define the number of decisions of a DPLL run as the total number of times a truth value was set by either unit propagation or the splitting rule.

Theorem. *If DPLL returns "unsatisfiable" on Δ, then $\Delta \vdash \Box$ with a resolution derivation whose length is at most the number of decisions.*

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Proof Sketch. Consider first DPLL without the unit propagation rule.

Consider any leaf node N, for proposition X, both of whose truth values directly result in a clause C that has become empty.

Then for $X = 0$ the respective clause C must contain X; and for $X = 1$ the respective clause C must contain $\neg X$. Thus we can resolve these two clauses to a clause $C(N)$ that does not contain X.

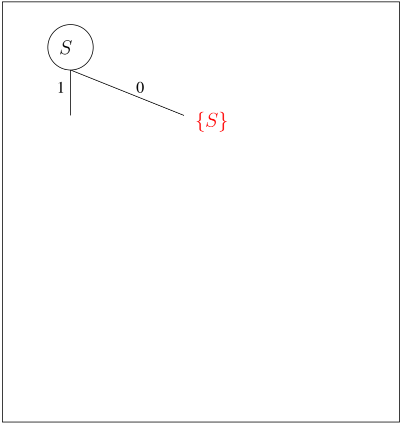
$C(N)$ can contain only the negations of the decision literals $l_1, \dots, l_k$ above N. Remove N from the tree, then iterate the argument. Once the tree is empty, we have derived the empty clause.

Unit propagation can be simulated via applications of the splitting rule, choosing a proposition that is constrained by a unit clause: One of the two truth values then immediately yields an empty clause.

DPLL = (A Restricted Form of) Resolution

Example: $\Delta = \{\{\neg Q, \neg P\}, \{P, \neg Q, \neg R, \neg S\}, \{Q, \neg S\}, \{R, \neg S\}, \{S\}\}$

DPLL: (Without UP; leaves annotated with clauses that became empty)



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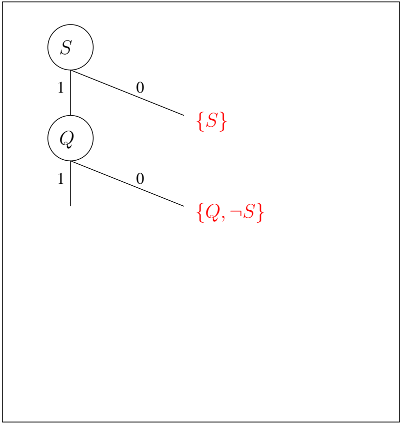

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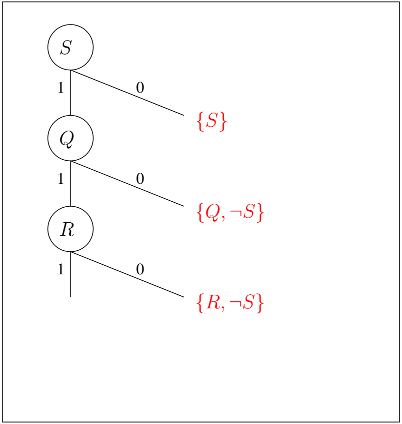

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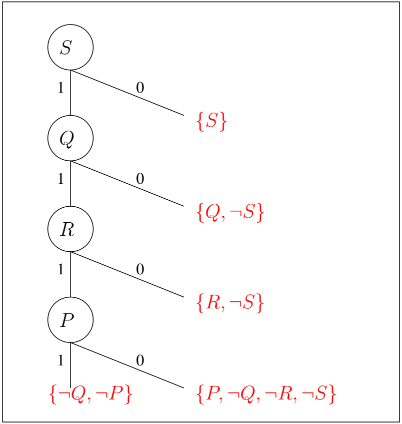

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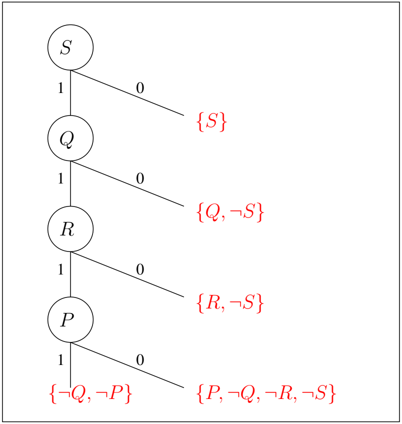

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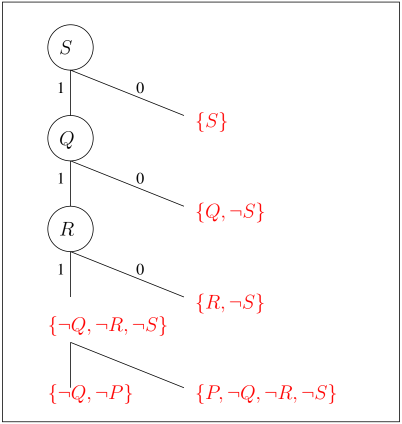

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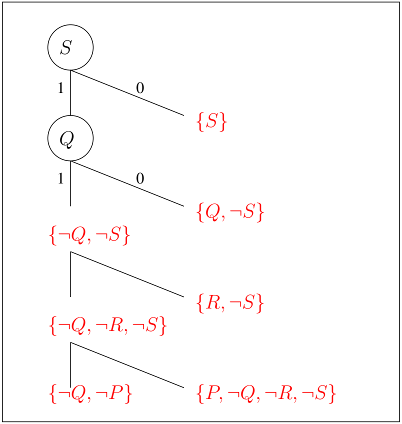

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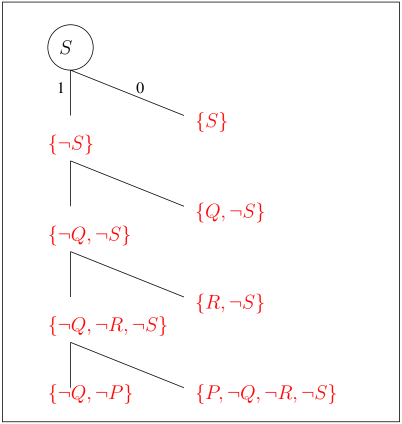

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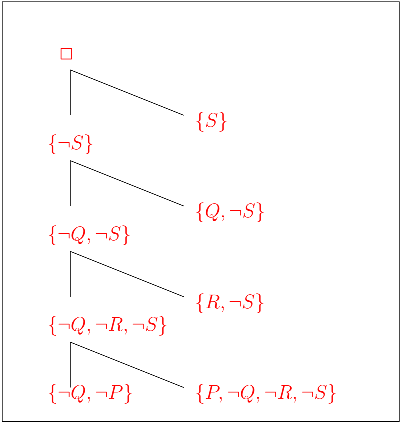

DPLL vs. Resolution: Discussion

So What? The theorem we just proved helps to *understand* DPLL:

→ DPLL is an effective practical method for conducting resolution proofs.

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→ To the rescue: clause learning.

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Same Mistakes over Again: Example (Redundance1)

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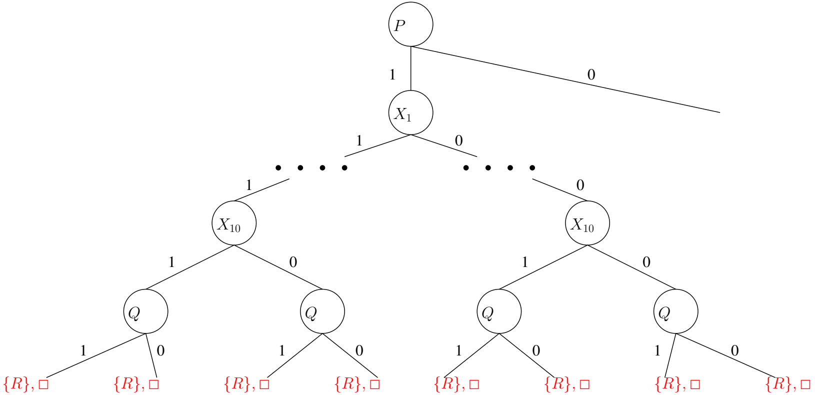

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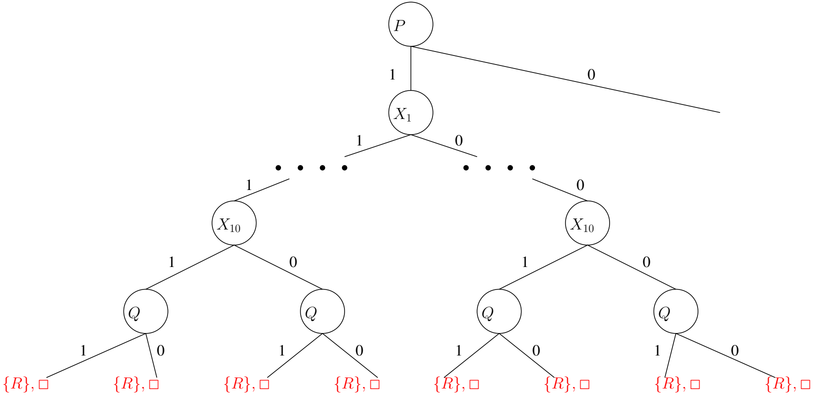

Note: Here, the problem could be avoided by splitting over different variables. This is not so in general! (See slide 36.)

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(A) Figure out what went wrong.

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And now for DPLL:

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$\rightarrow$ This section. We will capture the "what went wrong" in terms of graphs over literals set during the search, and their dependencies.

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(B) What can we learn from that information?

$\rightarrow$ A new clause! Next section.

Implication Graphs

Notation/Terminology: Literals set along a branch of DPLL

- Value of P set by the splitting rule: choice literal, P for $I(P) = 1$, respectively $\neg P$ for $I(P) = 0$.
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Where $\{l_1, \dots, l_k, l'\} \in \Delta$ became unit with implied literal l', G^{impl} includes the arcs $\overline{l_1} \rightarrow l'$, $\dots$, $\overline{l_k} \rightarrow l'$. Where $C = \{l_1, \dots, l_k\} \in \Delta$ became empty, G^{impl} includes the arcs $\overline{l_1} \rightarrow \Box_C$, $\dots$, $\overline{l_k} \rightarrow \Box_C$.

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- Vertices with indegree 0: Choice literals, and unit clauses of Δ.

Implication Graphs: Example (Vanilla1) in Detail

$\Delta = \{\{P, Q, \neg R\}, \{\neg P, \neg Q\}, \{R\}, \{P, \neg Q\}\}$

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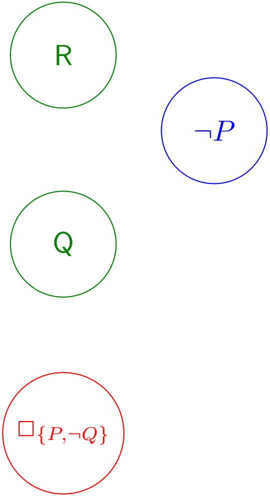

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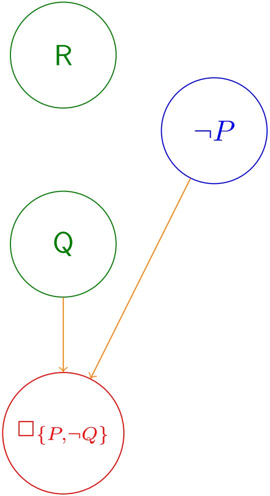

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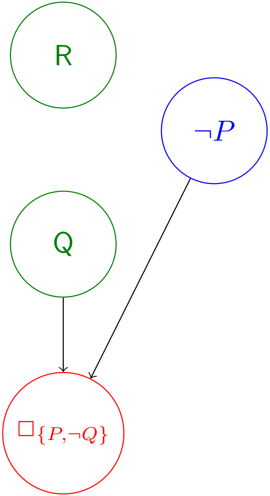

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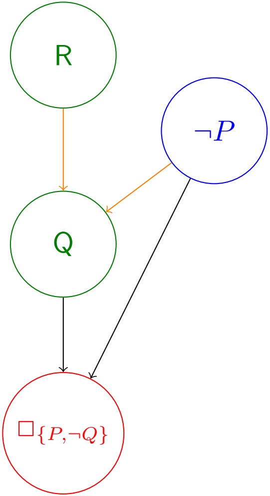

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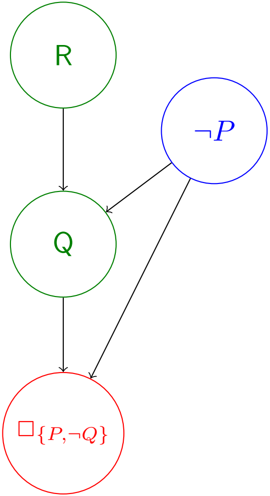

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1. UP Rule: $R \mapsto 1$
   Implied literal R.
   $\{\{P, Q\}, \{\neg P, \neg Q\}, \{P, \neg Q\}\}$
2. Splitting Rule:

2a. $P \mapsto 0$
   Choice literal $\neg P$.
   $\{\{Q\}, \{\neg Q\}\}$

3a. UP Rule: $Q \mapsto 1$
   Implied literal Q, arcs $R \to Q$ and $\neg P \to Q$.
   $\{\Box\}$
   Conflict literal $\Box$, arcs $\neg P \to \Box_{\{P,\neg Q\}}$ and $Q \to \Box_{\{P,\neg Q\}}$.

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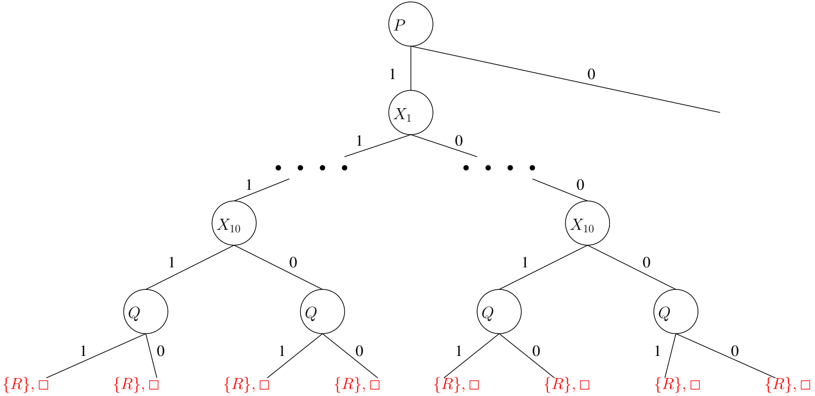
Choice: $P, X_1, \dots, X_{10}, Q$. Implied: R.



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$$\Delta = \{\{\neg P, \neg Q, R\}, \{\neg P, \neg Q, \neg R\}, \{\neg P, Q, R\}, \{\neg P, Q, \neg R\}, \{X_1, \dots, X_{100}\}, \{\neg X_1, \dots$$

$$\downarrow \text{SP } (P \mapsto 1)$$

$$\dots$$

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$$\downarrow \text{SP } (\dots)$$

$$\dots$$

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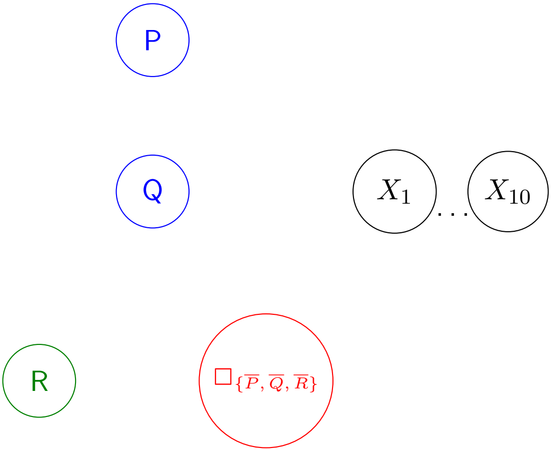

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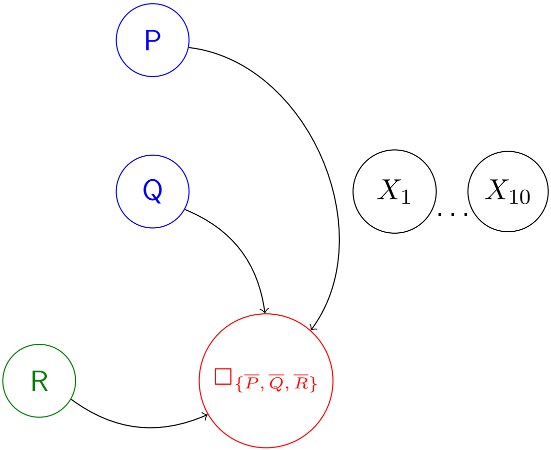

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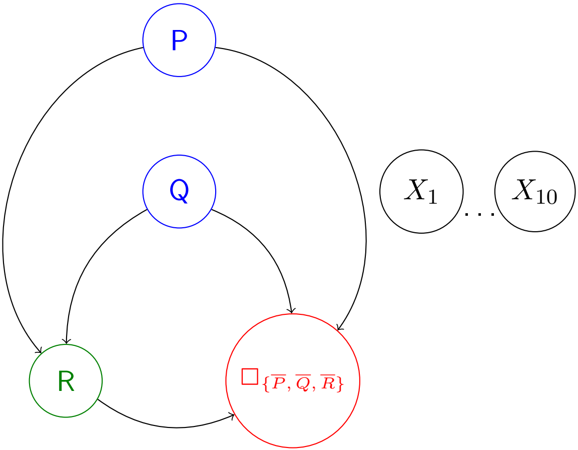

Implication Graphs: Example (Redundance2)

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Choice: $P, X_1, \dots, X_{10}, Q$. Implied: R, S.

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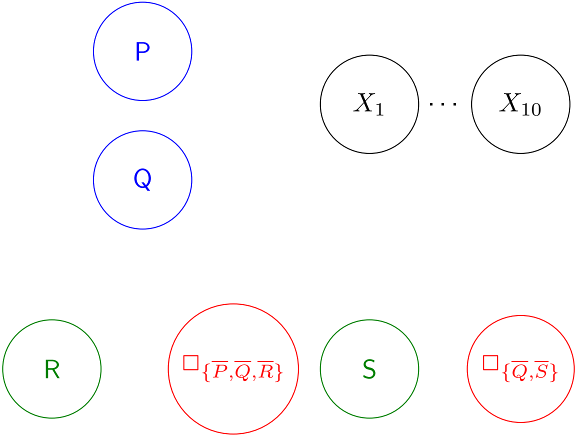

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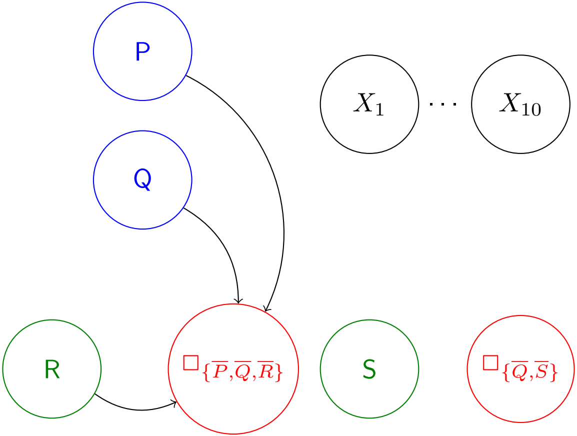

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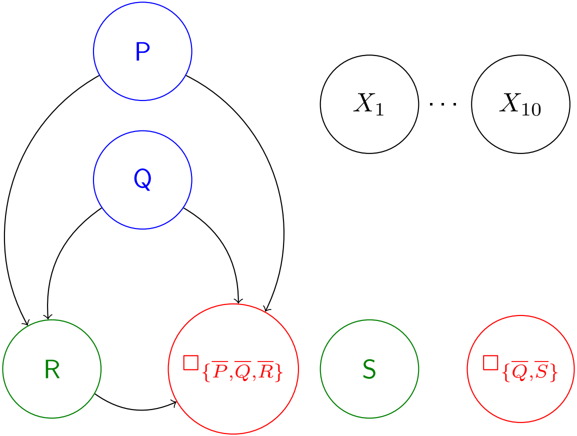

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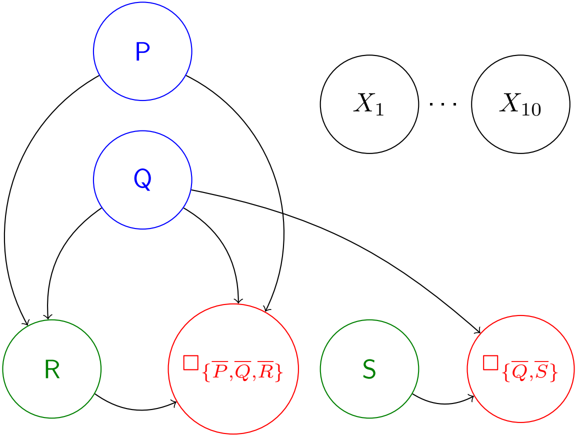

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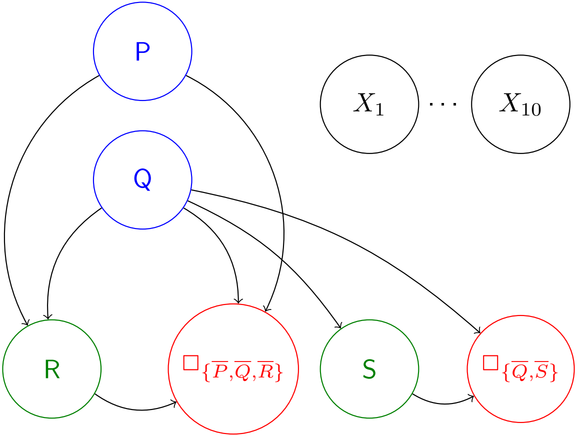

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Recall: Where $\{l_1, \dots, l_k, l'\} \in \Delta$ became unit on search branch β, with implied literal l', G^{impl} includes the arcs $\overline{l_1} \rightarrow l'$, ..., $\overline{l_k} \rightarrow l'$.

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→ The implication graph is *not* uniquely determined by the choice literals.

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- Example: $\Delta = \{\{\neg P, \neg Q\}, \{Q\}, \{P\}\}$

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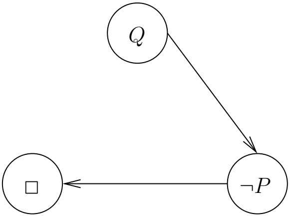
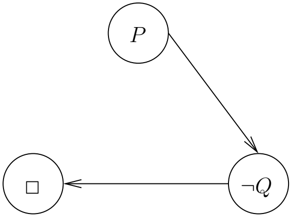


Option 2:



Conflict Graphs

→ A conflict graph captures "what went wrong" in a failed node.

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Definition (Conflict Graph). *Let Δ be a set of clauses, and let G^{impl} be the implication graph for some search branch of DPLL on Δ. A conflict graph G^{confl} is a* **sub-graph** *of G^{impl} induced by a subset of vertices such that:*

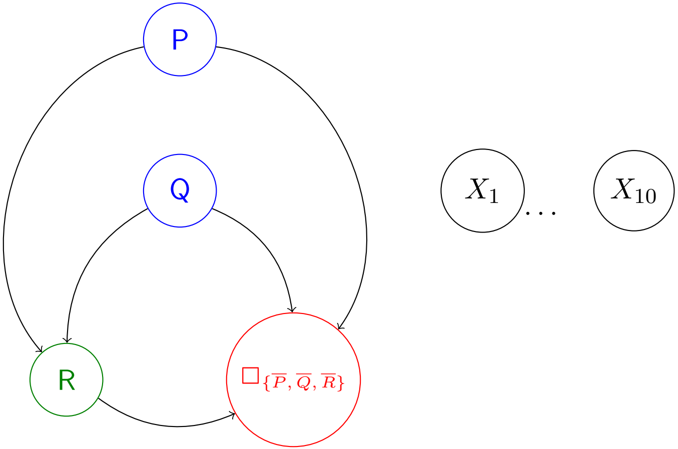
1. *G^{confl} contains exactly one conflict vertex $\Box_C$.*
2. *If l' is a vertex in G^{confl}, then all parents of l', i.e. vertices $\overline{l_i}$ with a G^{impl} arc $(\overline{l_i}, l')$, are vertices in G^{confl} as well.*
3. *All vertices in G^{confl} have a path to $\Box_C$.*

→ Conflict graph = Starting at a conflict vertex, backchain through the implication graph until reaching choice literals.

Conflict Graphs: Example (Redundance1)

$$\Delta = \{\{\neg P, \neg Q, R\}, \{\neg P, \neg Q, \neg R\}, \{\neg P, Q, R\}, \{\neg P, Q, \neg R\}, \{X_1, \dots, X_{100}\}, \{\neg X_1, \dots, \neg X_{100}\}\}$$

Choice: $P, X_1, \dots, X_{10}, Q$. Implied: R.



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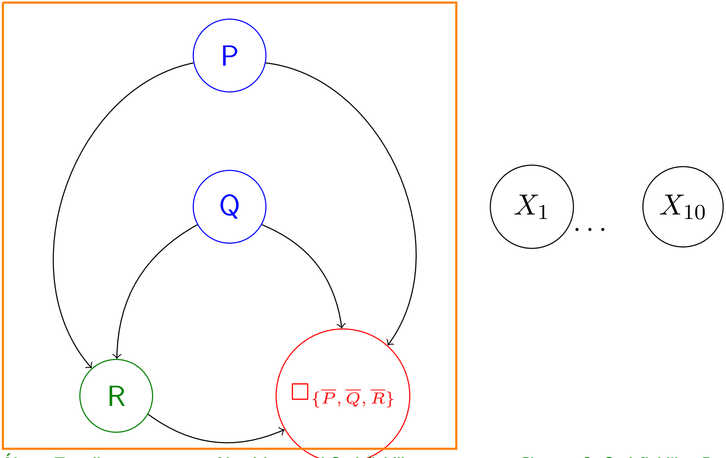

Conflict Graphs: Example (Redundance2)

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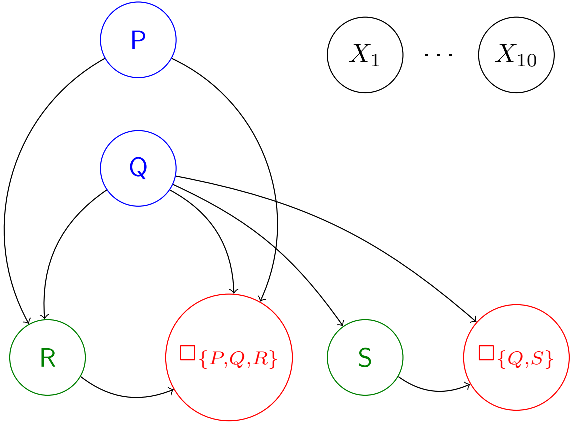
Choice: $P, X_1, \dots, X_{10}, Q$. Implied: R, S.



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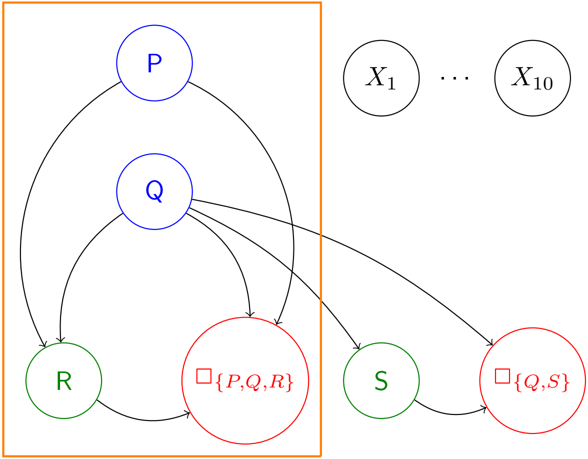

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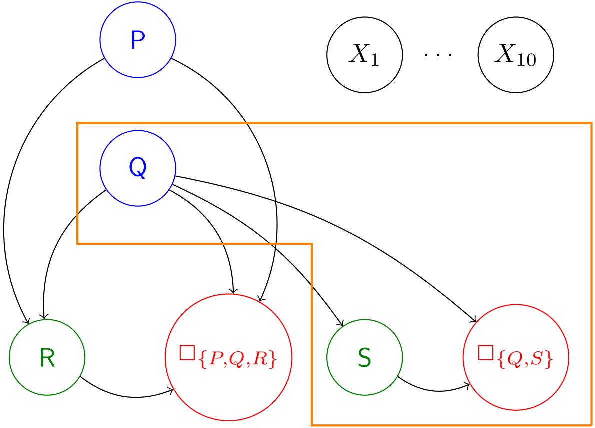
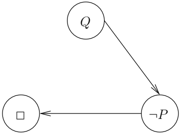
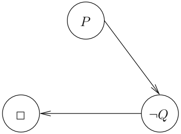


Questionnaire

Recall: The implication graph depends on "ordering decisions" during UP: Which unit clause is picked first. E.g. $\Delta = \{\{\neg P, \neg Q\}, \{Q\}, \{P\}\}$.





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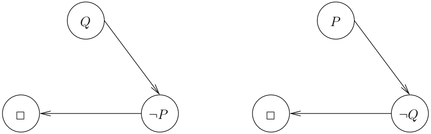

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Does the existence of a conflict graph depend on these decisions?

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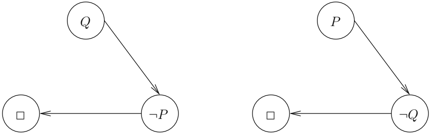

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$\rightarrow$ Observe: A conflict graph exists iff $\Box$ is UP-derivable in the current simplified formula Δ'. So the question is whether it can happen that, when propagating a unit clause $\{l\}$ in Δ', on the resulting simplified Δ'' the UP calculus cannot derive the empty clause anymore.

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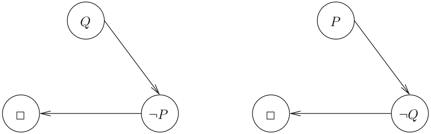

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The answer is no. Δ'' can be obtained in two steps: 1. Remove $\bar{l}$ from every $C \in \Delta'$ where $\bar{l} \in C$ to obtain Δ'_l. 2. Remove $C \in \Delta'_l$ where $l \in C$ to obtain Δ''. 1. cannot hurt $\Box$-derivability because every clause of Δ'_l is a sub-clause of Δ', and smaller clauses can only be better. 2. cannot hurt $\Box$-derivability because $\bar{l}$ is not contained in Δ'_l (so if $l \in C$ then no derivative of C can ever become empty).

Questionnaire, ctd.

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How many conflict graphs do we get for the choice literal $\neg R$, when running UP on $\{\{P, Q, R\}, \{\neg P, Q, R\}, \{S, R\}, \{\neg S, R\}\}$?

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(B): 1

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$\rightarrow$ (B) is correct: The only conflict we get is via $\{S, R\}$, $\{\neg S, R\}$, and the choice literal $\neg R$.

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And for the choice literals $\neg Q, \neg R$?

$\rightarrow$ (C) is correct: We get the above conflict, and another one via $\{P, Q, R\}$, $\{\neg P, Q, R\}$, and the choice literals $\neg Q$ and $\neg R$.

(Note: These choices can happen in DPLL on Δ, if we choose $\neg Q$ first.)

Agenda

Clause Learning

Observe: Conflict graphs encode *logical entailments*

$$\Delta \models (\bigwedge_{l \in choiceLits(G^{\mathsf{confl}})} l) \to \bot$$

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Proposition (Clause Learning). *Let Δ be a set of clauses, and let G^{confl} be a conflict graph at some time point during a run of DPLL on Δ. Let $choiceLits(G^{\text{confl}})$ be the choice literals in G^{confl}. Then $\Delta \models \{\bar{l} \mid l \in choiceLits(G^{\text{confl}})\}$.*

$\rightarrow$ The negation of the choice literals in a conflict graph is a valid clause.

Clause Learning: Example (Redundance1)

$$\Delta = \{\{\neg P, \neg Q, R\}, \{\neg P, \neg Q, \neg R\}, \{\neg P, Q, R\}, \{\neg P, Q, \neg R\}, \{X_1, \dots, X_{100}\}, \{\neg X_1, \dots, \neg X_{100}\}\}$$

Choice: $P, X_1, \dots, X_{10}, Q$. Implied: R.

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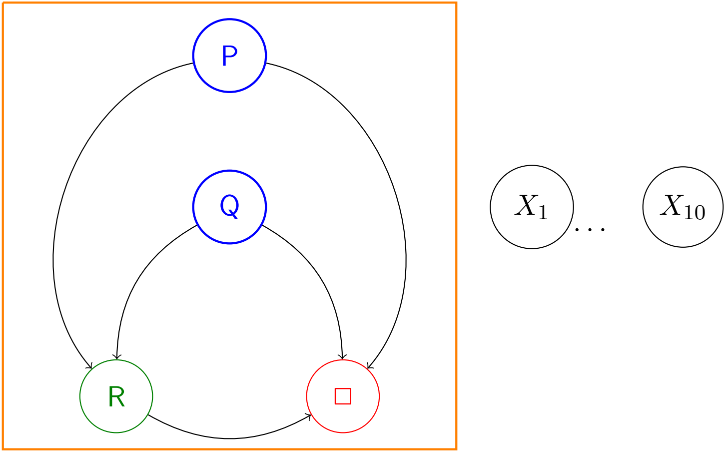

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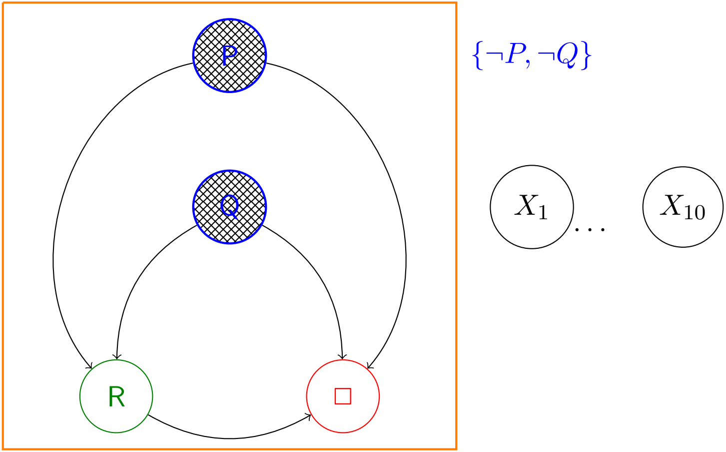

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   → Example: Set $\neg Q$ as an implied literal.
4. **We run UP and analyze conflicts.** Learned clause: earlier choices only!

   → Example: $C = \{\neg P\}$, see next slide.

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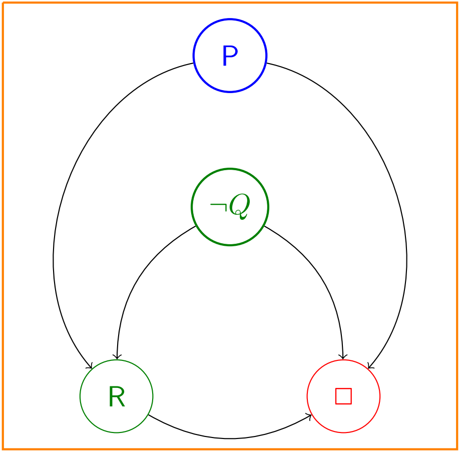

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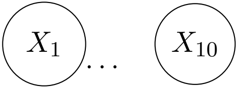

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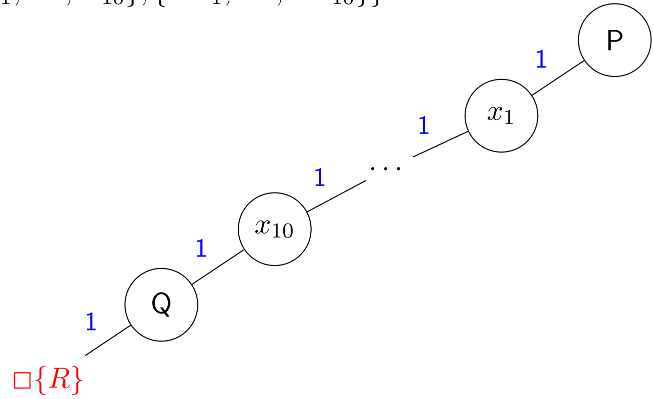

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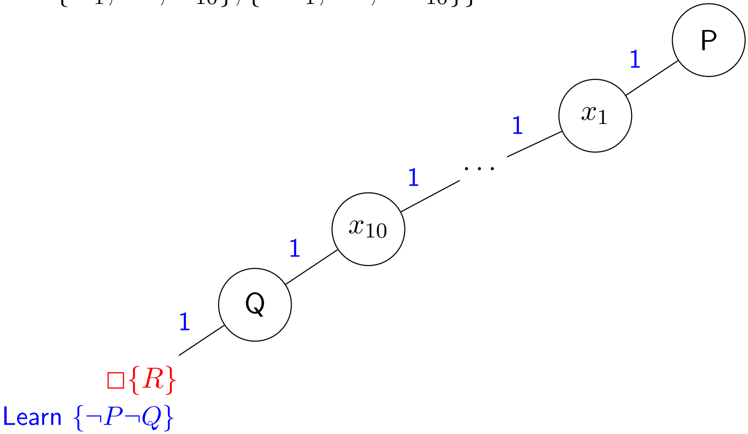

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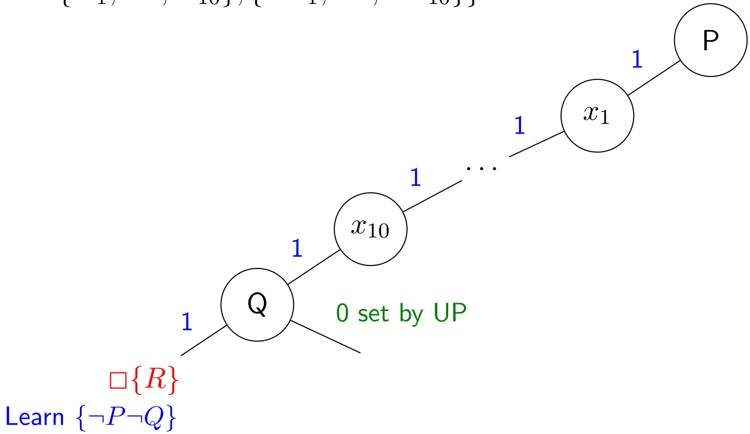

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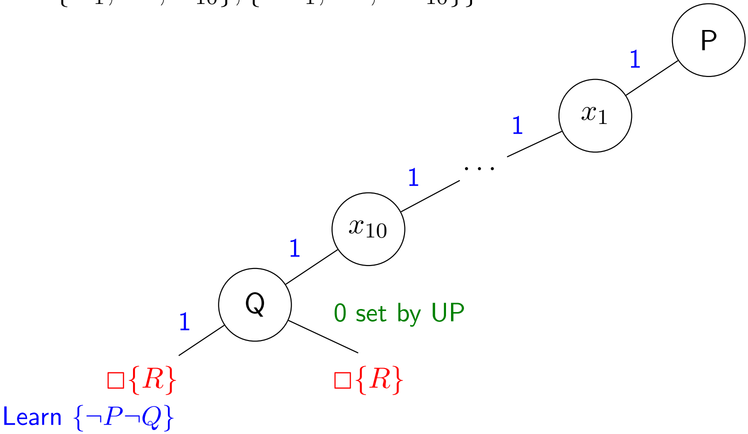

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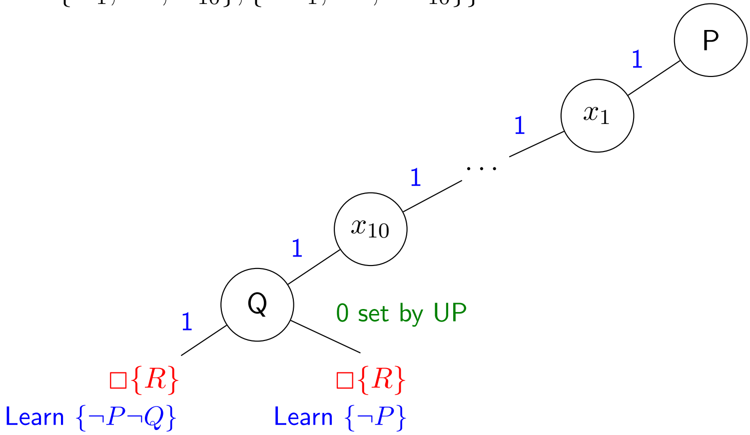

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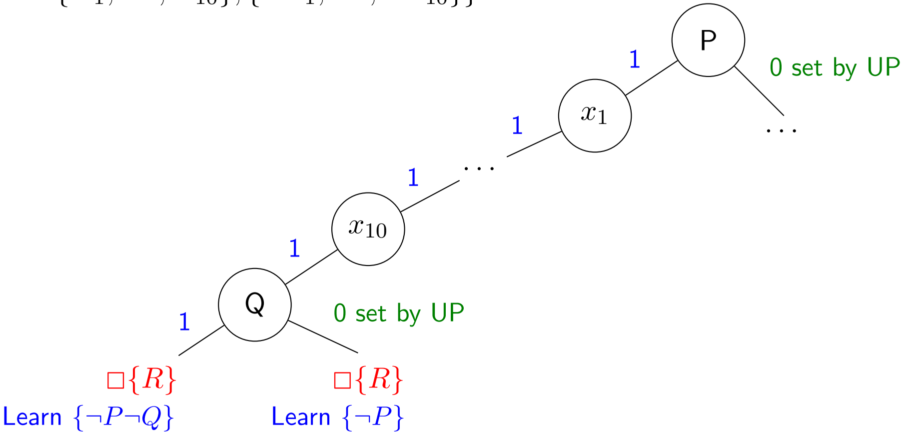

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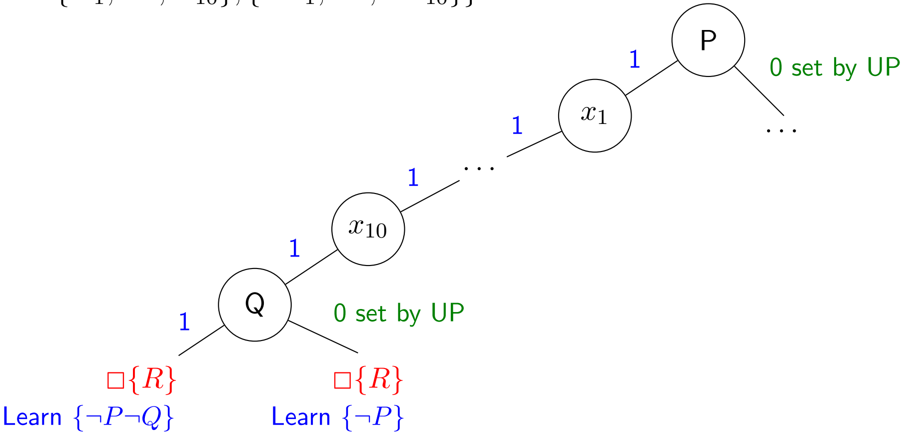

Note: Here, the problem could be avoided by splitting over different variables. This is not so in general! (see next slide)

Clause Learning vs. Resolution

Remember (slide 13):

1. DPLL $=$ tree resolution: Each derived clause C (not in Δ) is derived anew every time it is used.
2. There exist Δ whose shortest tree-resolution proof is exponentially longer than their shortest (general) resolution proof.

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$\rightarrow$ In particular: Selecting different variables/values to split on can *provably* not bring DPLL up to the power of DPLL+Clause Learning. (cf. slide 15, and previous slide)

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Disclaimer: We have only seen *how to learn a clause from a conflict*. We will *not* cover how the overall DPLL algorithm changes, given this learning. Slides 33 – 35 are merely meant to give a *rough intuition* on "backjumping".

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- One *could* run "DPLL + Clause Learning" by always backtracking to the maximal-level choice variable contained in the learned clause.
- But the actual algorithm is called Conflict-Directed Clause Learning (CDCL), and differs from DPLL more radically:

$L := 0$; $I := \emptyset$
repeat
  execute UP
  if a conflict was reached **then** $// \ C = \{\overline{l_1}, \dots, \overline{l_k}, \overline{l'}\}$
    if $L = 0$ **then return** UNSAT
    $L := \max_{i=1}^{k}$ *level*(l_i); erase I below L
    add C into Δ; add $\overline{l'}$ to I at *level* L
  else
    if I is a total interpretation **then return** I
    choose a new decision literal l; add l to I at *level* L
    $L := L + 1$

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WHICH clause(s) to learn?

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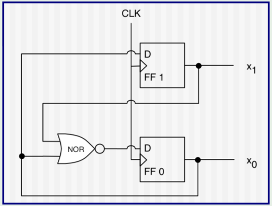
Origins of clause learning:

- Clause learning originates from explanation-based (no-good) learning developed in the CSP community.
- The distinguishing feature here is that the "no-good" is a clause:
  $\rightarrow$ The exact same type of constraint as the rest of Δ.

Questionnaire

Example



- Counter, repeatedly from $c = 0$ to $c = 2$.
- **To Verify:** If $c < 3$ in current clock cycle, then $c < 3$ in next clock cycle.
- $\Delta = \{\{\neg x'_1, x_0\}, \{x'_1, \neg x_0\}, \{x'_0, x_1, x_0\}, \{\neg x'_o, \neg x_1\}, \{\neg x'_0, \neg x_0\}, \{\neg x_1, \neg x_0\}, \{x'_1\}, \{x'_0\}\}$

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Questionnaire

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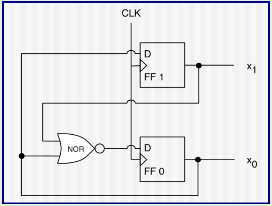

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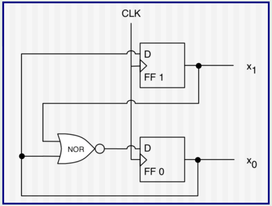

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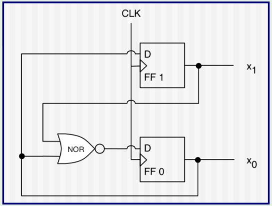

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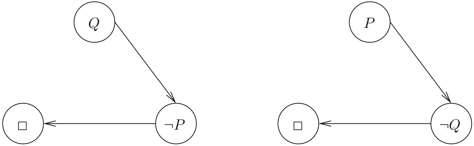
→ In case there are no choice literals, the contradiction follows "without assumptions", so we learn immediately that the input formula is unsatisfiable. This special case happens only if the input formula can be proved unsatisfiable using unit propagation (which is never the case in practice).

Questionnaire, ctd.

Recall: The implication graph depends on "ordering decisions" during UP: Which unit clause is picked first. E.g. $\Delta = \{\{\neg P, \neg Q\}, \{Q\}, \{P\}\}$.



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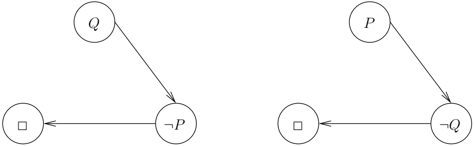

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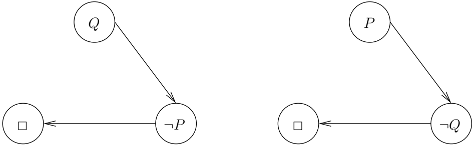

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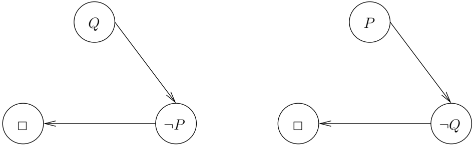

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(Note: In the example above, the learned clause in both cases is □ because there aren't any choice variables.)

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Which clauses can we learn after choosing $\neg R$ and running UP on $\{\{P, Q, R\}, \{\neg P, Q, R\}, \{S, R\}, \{\neg S, R\}\}$?

(A): $\{\neg S, R\}$.

(B): $\{Q, R\}$.

(C): $\{R\}$.

(D): $\Box$.

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(D): $\Box$.

$\rightarrow$ (A), (B): No: Neither S (in (A)) nor $\neg Q$ (in (B)) is a choice literal.

$\rightarrow$ (C): Yes, via the conflict from $\{S, R\}$, $\{\neg S, R\}$, choice literal $\neg R$.

$\rightarrow$ (D): No: While UP does derive a conflict, that conflict depends on the choice literal $\neg R$.

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Questionnaire, ctd.

Question!

Which clauses can we learn after choosing $\neg Q, \neg R$ and running UP on $\{\{P, Q, R\}, \{\neg P, Q, R\}, \{S, R\}, \{\neg S, R\}\}$?

(A): $\{\neg S, R\}$.

(B): $\{Q, R\}$.

(C): $\{R\}$.

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→ (A): No. S is not a choice literal.

→ (B): Yes, via the conflict $\{P, Q, R\}$, $\{\neg P, Q, R\}$, with choice literals $\neg Q, \neg R$.

→ (C): Yes, via the conflict from $\{S, R\}$, $\{\neg S, R\}$, choice literal $\neg R$.

→ (D): No (same as on previous slide).

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(Note: These choices can happen in DPLL on Δ, if we choose $\neg Q$ first.)

Agenda

Damn, Where Are the Hard Problems?

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Err, what?

- SAT is **NP**-hard. Worst case for DPLL is 2^n, with n propositions.
- Imagine I gave you as homework to make a formula family $\{\phi\}$ where DPLL runtime necessarily is in the order of 2^n.

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- People noticed by the early 90s that, in practice, the DPLL worst case does not tend to happen.

$\rightarrow$ Modern SAT solvers successfully tackle practical instances where $n > 1000000$.

Damn, Where Are the Hard Problems? Ctd.

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→ The most successful works are empirical.

(Interesting theory is mainly concerned with *hand-crafted* formulas, like the Pigeon Hole Problem.)

Phase Transitions in SAT [Mitchell *et al.* (1992)]

Fixed clause length model: Fix clause length k; n variables. Generate m clauses, by uniformly choosing k variables P for each clause C, and for each variable P deciding uniformly whether to add P or $\neg P$ into C.

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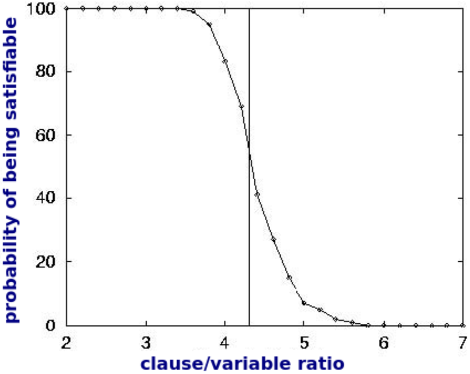
Phase transition: (Fixing $k = 3$, $n = 50$)



Does DPLL Care?

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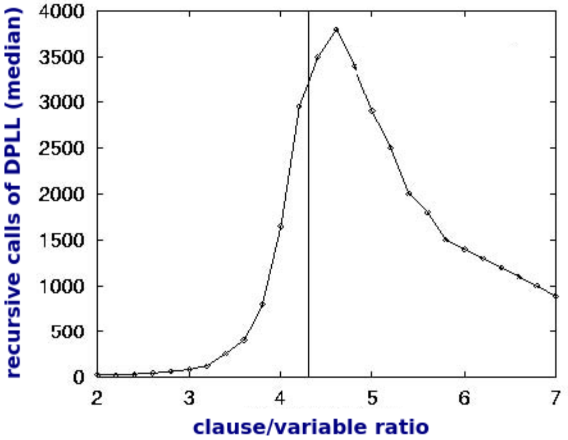
Oh yes, it does! Extreme runtime peak at the phase transition!



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- Over-Constrained: Satisfiability likelihood close to 0. Most DPLL search paths short, conflict reached after few applications of splitting rule. ("Broad but shallow")
- Critically Constrained: At the phase transition, many *almost-successful* DPLL search paths. ("Close, but no cigar")

The Phase Transition Conjecture

Conjecture: [Cheeseman *et al.* (1991)]

Phase Transition Conjecture

"All **NP**-complete problems have at least one order parameter, and the hard to solve problems are around a critical value of this order parameter. This critical value (a phase transition) separates one region from another, such as over-constrained and under-constrained regions of the problem space."

$\rightarrow$ [Cheeseman *et al.* (1991)] confirmed this for Graph Coloring and Hamiltonian Circuits. Later work confirmed it for SAT (see previous slides), and for numerous other **NP**-complete problems.

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- Benchmark design: Choose instances from phase transition region.

  → Commonly used in competitions etc. (In SAT, random phase transition formulas are the most difficult for DPLL-style searches.)
- Predicting solver performance: Yes, but very limited because:

→ All this works only for the particular considered *distributions of instances*! Not meaningful for any other instances.

Questionnaire, ctd.

Question!

Say I encode a Wumpus problem into Δ that turns out to have clause/variable ratio 10. Which is true?

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Extreme example: Say we generate clauses including only positive literals ...

Practical example: Many Verification problems have huge numbers of clauses but are still satisfiable.

$\rightarrow$ For example, consider the straightforward encoding for "Exactly one of n variables $x_1, \dots, x_n$ is true." We get the clause $\{x_1, \dots, x_n\}$ ("at least one is true") and, for every $1 \leq i \neq j \leq n$, the clause $\{\neg x_i, \neg x_j\}$ ("at most one is true"). The clause/variable ratio is $\frac{n^2-n+2}{2n}$, but the formula is satisfiable.

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$\rightarrow$ (D): Depends on *how* I generate the CNF. If I use Mitchell *et al.* (1992)'s methods then yes. If I use a different method, then no.

Agenda

Summary

- Implication graphs capture how UP derives conflicts. Their analysis enables us to do clause learning. DPLL with clause learning is called CDCL. It corresponds to full resolution, not “making the same mistakes over again”.
- CDCL is state of the art in applications, routinely solving formulas with millions of propositions.
- In particular random formula distributions, typical problem hardness is characterized by phase transitions.

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What about local search?

- Better on random instances.
- No “dramatic” progress in last decade.
- Parameters are difficult to adjust.

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- Resolution special cases: There's a universe in between unit resolution and full resolution: trade-off inference vs. search.
- Proof complexity: Can one resolution special case X simulate another one Y polynomially? Or is there an exponential separation (example families where X is exponentially less effective than Y)?

Reading

- The Art of Computer Programming by Donald E. Knuth, Vol 4. Section 7.2.2.2
- Handbook of Satisfiability, Hans van Maaren, Armin Biere, Toby Walsh.

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