

# Algorithms and Satisfiability

## 8. Satisfiability, Part II: SAT Solvers, DPLL, and Clause Learning

How to *Efficiently* Think About What is True or False

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Thanks to Jörg Hoffmann for slide sources

# Agenda

- 1 Introduction
- 2 DPLL = (A Restricted Form of) Resolution
- 3 *Why* Did Unit Propagation Yield a Conflict?
- 4 Clause Learning
- 5 Phase Transitions: Where the *Really* Hard Problems Are
- 6 Conclusion

# Reminder: Our Agenda for This Topic

→ Our treatment of the topic “Satisfiability” consists of Chapters 7 and 8.

- **Chapter 7:** Basic definitions and concepts; resolution; DPLL.
  - Sets up the framework. Resolution is the quintessential reasoning procedure underlying most successful solvers.
- **This Chapter:** Clause learning; practical problem structure.
  - State-of-the-art algorithms for reasoning about propositional logic, and an important observation about how they behave.

# SAT

**The SAT Problem:** Given a propositional formula  $\varphi$ , decide whether or not  $\varphi$  is satisfiable.

- The first problem proved to be **NP**-complete!
- $\varphi$  is commonly assumed to be in CNF. This is without loss of generality, because any  $\varphi$  can in polynomial time be transformed into a satisfiability-equivalent CNF formula (cf. **Chapter 7**).
- Active research area, annual SAT conference, lots of tools etc. available: <http://www.satlive.org/>
- Tools addressing SAT are commonly referred to as **SAT solvers**.

**Reminder:** To decide whether  $\text{KB} \models \varphi$ , decide satisfiability of  $\theta := \text{KB} \cup \{\neg\varphi\}$ :  $\theta$  is unsatisfiable iff  $\text{KB} \models \varphi$ .

→ Deduction can be performed using SAT solvers.

# Reminder: General Problem Solving using Logic

(some new problem)



model problem in logic  $\mapsto$  use off-the-shelf reasoning tool



(its solution)

- “Any problem that can be formulated as reasoning about logic.”
- Very successful using propositional logic and modern solvers for SAT! (Propositional satisfiability testing,  $\rightarrow$  **This Chapter.**)

# Reminder: Conventions

## Terminology and Notation

- A literal  $l$  is an atom or the negation thereof (e.g.,  $P, \neg Q$ ); the negation of a literal is denoted  $\bar{l}$  (e.g.,  $\overline{\neg Q} = Q$ ).
- A clause  $C$  is a disjunction of literals. We identify  $C$  with the set of its literals (e.g.,  $P \vee \neg Q$  becomes  $\{P, \neg Q\}$ ).
- We identify a CNF formula  $\psi$  with the set  $\Delta$  of its clauses (e.g.,  $(P \vee \neg Q) \wedge R$  becomes  $\{\{P, \neg Q\}, \{R\}\}$ ).
- The empty clause is denoted  $\square$ .

→ For the remainder of this chapter, we assume that the input is a set  $\Delta$  of clauses.

# Our Agenda for This Chapter

- **DPLL = (A Restricted Form of) Resolution:** How does this relate to what we did in the last chapter?  
→ Mathematical understanding of DPLL.
- **Why Did Unit Propagation Yield a Conflict?** How can we analyze which mistakes were made in “dead” search branches?  
→ Knowledge is power, see next.
- **Clause Learning:** How can we learn from our mistakes?  
→ One of the key concepts, perhaps *the* key concept, underlying the success of SAT.
- **Phase Transitions: Where the Really Hard Problems Are:** Are *all* formulas “hard” to solve?  
→ The answer is “no”. And in some cases we can figure out exactly when they are/aren’t hard to solve.

# But – What About Local Search for SAT?

There's a wealth of research on local search for SAT, e.g.:

## GSAT Algorithm

**INPUT:** a set of clauses  $\Delta$ , MAX-FLIPS, and MAX-TRIES

**OUTPUT:** a satisfying truth assignment of  $\Delta$ , if found

**for**  $i := 1$  to MAX-TRIES

$I :=$  a randomly-generated truth assignment

**for**  $j := 1$  to MAX-FLIPS

**if**  $I$  satisfies  $\Delta$  **then return**  $I$

$X :=$  a proposition reversing whose truth assignment gives  
            the largest increase in the number of satisfied clauses

$I := I$  with the truth assignment of  $X$  reversed

**end for**

**end for**

**return** “no satisfying assignment found”

→ Local search is not as successful in SAT applications, though it can be a good complement to the techniques presented here. Not covered here.



# DPLL = (A Restricted Form of) Resolution

**Notation:** Define the **number of decisions** of a DPLL run as the total number of times a truth value was set by either unit propagation or the splitting rule.

**Theorem.** *If DPLL returns “unsatisfiable” on  $\Delta$ , then  $\Delta \vdash \square$  with a resolution derivation whose length is at most the number of decisions.*

**Proof Sketch.** Consider first DPLL without the unit propagation rule.

Consider any leaf node  $N$ , for proposition  $X$ , both of whose truth values directly result in a clause  $C$  that has become empty.

Then for  $X = 0$  the respective clause  $C$  must contain  $X$ ; and for  $X = 1$  the respective clause  $C$  must contain  $\neg X$ . Thus we can resolve these two clauses to a clause  $C(N)$  that does not contain  $X$ .

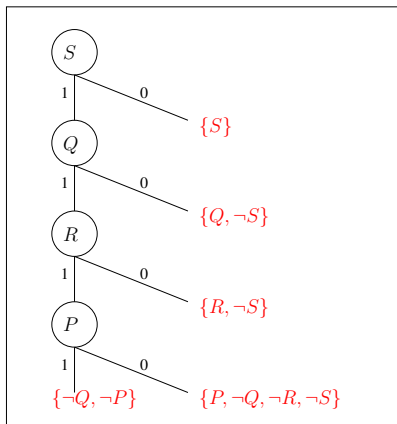
$C(N)$  can contain only the negations of the decision literals  $l_1, \dots, l_k$  above  $N$ . Remove  $N$  from the tree, then iterate the argument. Once the tree is empty, we have derived the empty clause.

Unit propagation can be simulated via applications of the splitting rule, choosing a proposition that is constrained by a unit clause: One of the two truth values then immediately yields an empty clause.

# DPLL = (A Restricted Form of) Resolution

**Example:**  $\Delta = \{\{\neg Q, \neg P\}, \{P, \neg Q, \neg R, \neg S\}, \{Q, \neg S\}, \{R, \neg S\}, \{S\}\}$

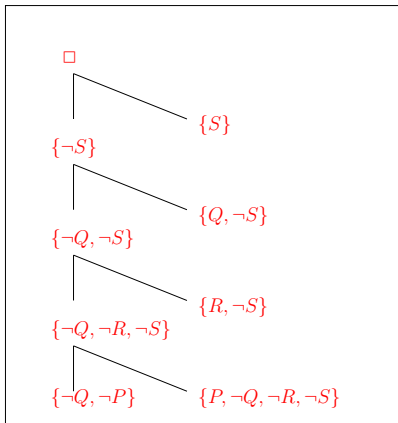
**DPLL:** (Without UP; leaves annotated with clauses that became empty)



# DPLL = (A Restricted Form of) Resolution

**Example:**  $\Delta = \{\{\neg Q, \neg P\}, \{P, \neg Q, \neg R, \neg S\}, \{Q, \neg S\}, \{R, \neg S\}, \{S\}\}$

**Resolution Proof from that DPLL Tree:**



# DPLL vs. Resolution: Discussion

**So What?** The theorem we just proved helps to *understand* DPLL:

→ DPLL is an effective practical method for conducting resolution proofs.

**In Fact:**

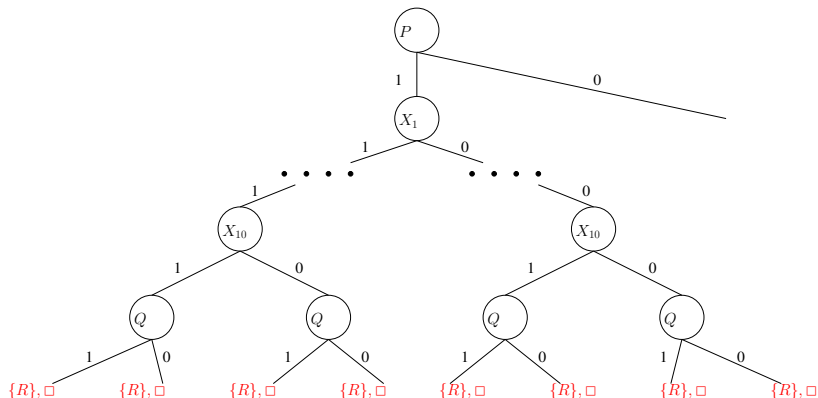
- DPLL = **tree resolution**.
- This is a fundamental weakness! There are inputs  $\Delta$  whose shortest tree-resolution proof is exponentially longer than their shortest (general) resolution proof.
  - In a tree resolution, each derived clause  $C$  is used only once (at its parent). **The same  $C$  is derived anew every time it is used!**

→ DPLL “makes the same mistakes over and over again”.

→ To the rescue: **clause learning**.

## Same Mistakes over Again: Example (Redundance1)

$$\Delta = \{\{\neg P, \neg Q, R\}, \{\neg P, \neg Q, \neg R\}, \{\neg P, Q, R\}, \{\neg P, Q, \neg R\}, \\ \{X_1, \dots, X_{100}\}, \{\neg X_1, \dots, \neg X_{100}\}\}$$



**Note:** Here, the problem could be avoided by splitting over different variables. **This is not so in general!** (See slide 36.)

# How To *Not* Make the Same Mistakes Over Again?

... it's not that difficult, really:

- A Figure out what went wrong.
- B Learn to not do that again in the future.

**And now for DPLL:**

A *Why Did Unit Propagation Yield a Conflict?*

→ This section. We will capture the “what went wrong” in terms of graphs over literals set during the search, and their dependencies.

B *What can we learn from that information?*

→ A new clause! Next section.

# Implication Graphs

## Notation/Terminology: Literals set along a branch of DPLL

- Value of  $P$  set by the splitting rule: **choice literal**,  $P$  for  $I(P) = 1$ , respectively  $\neg P$  for  $I(P) = 0$ .
- Value of  $P$  set by the UP rule: **implied literal**  $P$  respectively  $\neg P$ .
- Empty clause derived by UP: **conflict literal**  $\square$ .

**Definition (Implication Graph).** Let  $\Delta$  be a set of clauses, and consider any search branch  $\beta$  of DPLL on  $\Delta$ . The **implication graph**  $G^{\text{impl}}$  is a directed graph. Its vertices are the choice and implied literals along  $\beta$ , as well as a separate conflict vertex  $\square_C$  for every clause  $C$  that became empty.

Where  $\{l_1, \dots, l_k, l'\} \in \Delta$  **became unit** with implied literal  $l'$ ,  $G^{\text{impl}}$  includes the **arcs**  $\overline{l_1} \rightarrow l'$ ,  $\dots$ ,  $\overline{l_k} \rightarrow l'$ . Where  $C = \{l_1, \dots, l_k\} \in \Delta$  **became empty**,  $G^{\text{impl}}$  includes the **arcs**  $\overline{l_1} \rightarrow \square_C$ ,  $\dots$ ,  $\overline{l_k} \rightarrow \square_C$ .

- **How do we know that  $\overline{l_1}, \dots, \overline{l_k}$  are vertices in  $G^{\text{impl}}$ :** Because  $\{l_1, \dots, l_k, l'\}$  became unit respectively empty.
- **Vertices with indegree 0:** Choice literals, and unit clauses of  $\Delta$ .

# Implication Graphs: Example (Vanilla1) in Detail

$$\Delta = \{\{P, Q, \neg R\}, \{\neg P, \neg Q\}, \{R\}, \{P, \neg Q\}\}$$

UP ( $R \mapsto 1$ )

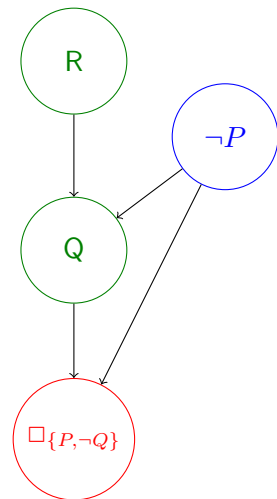
$$\{\{P, Q\}, \{\neg P, \neg Q\}, \{P, \neg Q\}\}$$

SP ( $P \mapsto 0$ )

$$\{\{Q\}, \{\neg Q\}\}$$

UP ( $Q \mapsto 1$ )

$$\{\square\}$$





# Implication Graphs: Example (Vanilla1) in Detail

$$\Delta = \{\{P, Q, \neg R\}, \{\neg P, \neg Q\}, \{R\}, \{P, \neg Q\}\}$$

1. UP Rule:  $R \mapsto 1$

Implied literal  $R$ .

$$\{\{P, Q\}, \{\neg P, \neg Q\}, \{P, \neg Q\}\}$$

2. Splitting Rule:

- 2a.  $P \mapsto 0$

Choice literal  $\neg P$ .

$$\{\{Q\}, \{\neg Q\}\}$$

- 3a. UP Rule:  $Q \mapsto 1$

Implied literal  $Q$ , arcs  $R \rightarrow Q$  and  $\neg P \rightarrow Q$ .

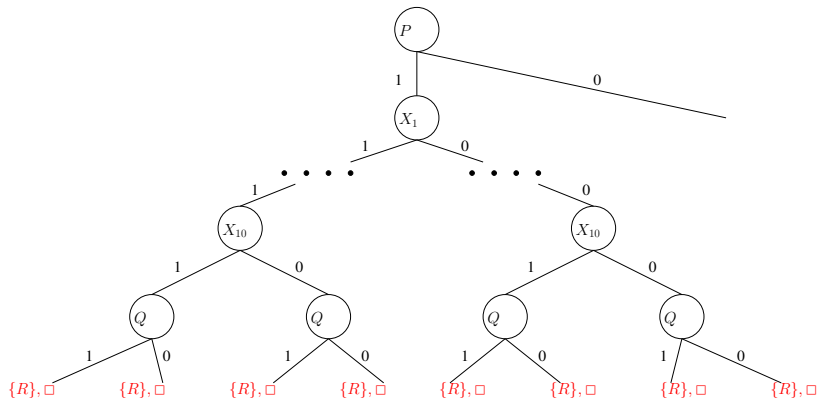
$$\{\square\}$$

Conflict literal  $\square$ , arcs  $\neg P \rightarrow \square_{\{P, \neg Q\}}$  and  $Q \rightarrow \square_{\{P, \neg Q\}}$ .

# Implication Graphs: Example (Redundance1)

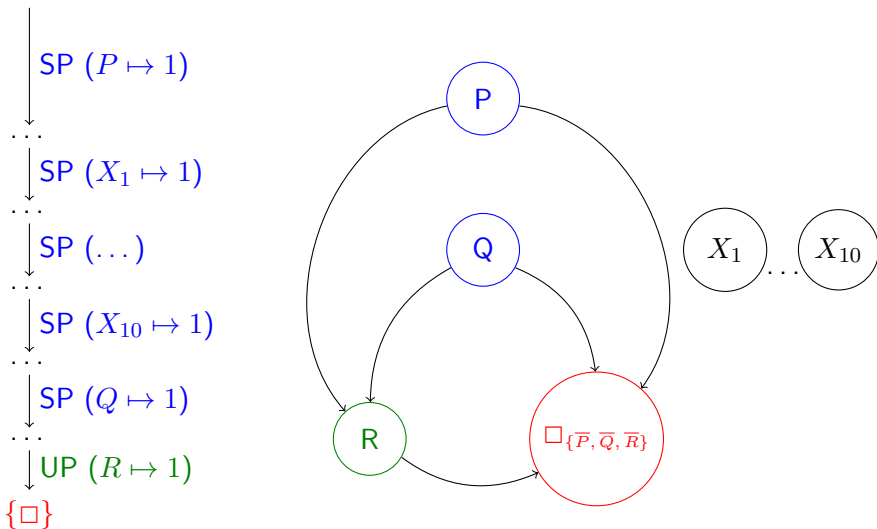
$$\Delta = \{\{\neg P, \neg Q, R\}, \{\neg P, \neg Q, \neg R\}, \{\neg P, Q, R\}, \{\neg P, Q, \neg R\}, \\ \{X_1, \dots, X_{100}\}, \{\neg X_1, \dots, \neg X_{100}\}\}$$

Choice:  $P, X_1, \dots, X_{10}, Q$ . Implied:  $R$ .



# Implication Graphs: Example (Redundance1)

$$\Delta = \{\{\neg P, \neg Q, R\}, \{\neg P, \neg Q, \neg R\}, \{\neg P, Q, R\}, \{\neg P, Q, \neg R\}, \{X_1, \dots, X_{100}\}, \{\neg X_1, \dots\}$$



# Implication Graphs: Example (Redundance2)

$$\Delta = \{ \{ \neg P, \neg Q, R \}, \{ \neg P, \neg Q, \neg R \}, \{ \neg P, Q, R \}, \{ \neg P, Q, \neg R \}, \\ \{ \neg Q, S \}, \{ \neg Q, \neg S \}, \{ X_1, \dots, X_{100} \}, \{ \neg X_1, \dots, \neg X_{100} \} \}$$

↓ SP ( $P \mapsto 1$ )

Choice:  $P, X_1, \dots, X_{10}, Q$ . Implied:  $R, S$ .

↓ SP ( $X_1 \mapsto 1$ )

↓ SP (...)

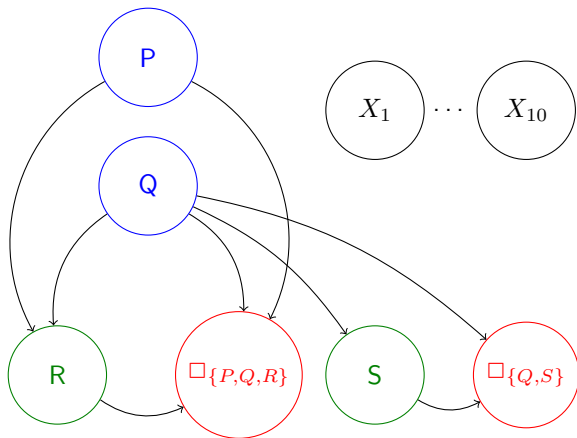
↓ SP ( $X_{10} \mapsto 1$ )

↓ SP ( $Q \mapsto 1$ )

↓ UP ( $R \mapsto 1$ )

↓ UP ( $S \mapsto 1$ )

{□}



# Questionnaire

**Recall:** Where  $\{l_1, \dots, l_k, l'\} \in \Delta$  became unit on search branch  $\beta$ , with implied literal  $l'$ ,  $G^{\text{impl}}$  includes the arcs  $\bar{l}_1 \rightarrow l', \dots, \bar{l}_k \rightarrow l'$ .

## Question!

**Can implication graphs have cycles?**

(A): Yes

(B): No

→ No, because the implication graph keeps track of chronological behavior along the current DPLL search branch  $\beta$ . Unit propagation cannot derive  $l'$  whose value was already set beforehand.

**In detail:** Assume there is a cycle. Consider the first time point along the search branch where the cycle occurs, because an arc  $(\bar{l}_i, l')$  is added to the implication graph. Then (a)  $l'$  is implied because  $\{l_1, \dots, l_k, l'\} \in \Delta$  just became unit; and (b) as  $(\bar{l}_i, l')$  closes a cycle,  $l'$  must already have been in the implication graph beforehand. (a) and (b) are in contradiction.

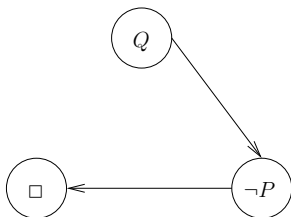
# Implication Graphs: A Remark

→ The implication graph is *not* uniquely determined by the choice literals.

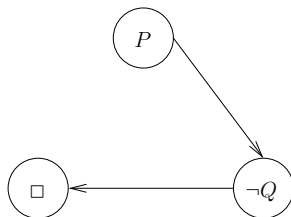
**Because:**

- The implication graph also depends on “ordering decisions” made during UP: Which unit clause is picked first.
- Example:  $\Delta = \{\{\neg P, \neg Q\}, \{Q\}, \{P\}\}$

Option 1:



Option 2:



# Conflict Graphs

→ A conflict graph captures “what went wrong” in a failed node.

**Definition (Conflict Graph).** Let  $\Delta$  be a set of clauses, and let  $G^{\text{impl}}$  be the implication graph for some search branch of DPLL on  $\Delta$ . A *conflict graph*  $G^{\text{confl}}$  is a **sub-graph** of  $G^{\text{impl}}$  induced by a subset of vertices such that:

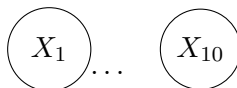
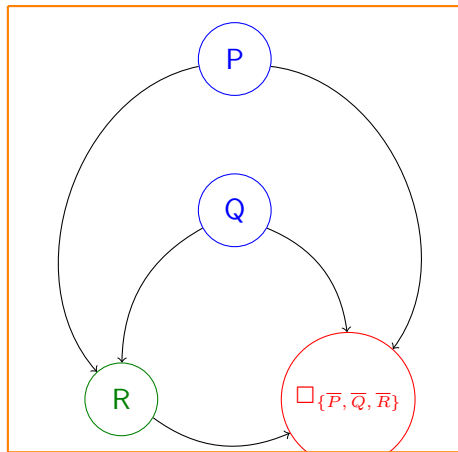
- ❶  $G^{\text{confl}}$  contains exactly one conflict vertex  $\square_C$ .
- ❷ If  $l'$  is a vertex in  $G^{\text{confl}}$ , then all parents of  $l'$ , i.e. vertices  $\bar{l}_i$  with a  $G^{\text{impl}}$  arc  $(\bar{l}_i, l')$ , are vertices in  $G^{\text{confl}}$  as well.
- ❸ All vertices in  $G^{\text{confl}}$  have a path to  $\square_C$ .

→ Conflict graph = Starting at a conflict vertex, backchain through the implication graph until reaching choice literals.

# Conflict Graphs: Example (Redundance1)

$$\Delta = \{\{\neg P, \neg Q, R\}, \{\neg P, \neg Q, \neg R\}, \{\neg P, Q, R\}, \{\neg P, Q, \neg R\}, \\ \{X_1, \dots, X_{100}\}, \{\neg X_1, \dots, \neg X_{100}\}\}$$

Choice:  $P, X_1, \dots, X_{10}, Q$ . Implied:  $R$ .

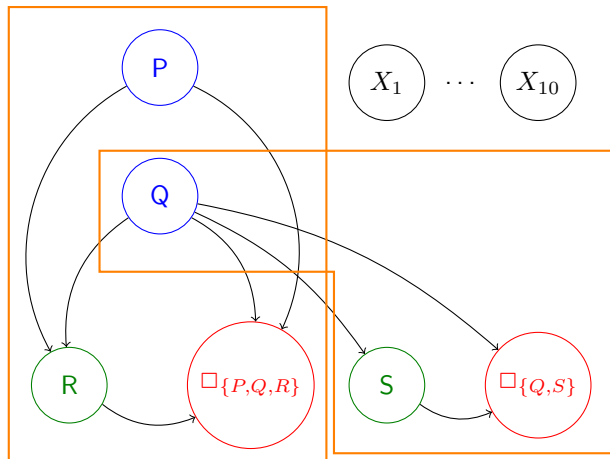




# Conflict Graphs: Example (Redundance2)

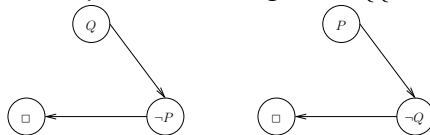
$\Delta = \{\{\neg P, \neg Q, R\}, \{\neg P, \neg Q, \neg R\}, \{\neg P, Q, R\}, \{\neg P, Q, \neg R\},$   
 $\{\neg Q, S\}, \{\neg Q, \neg S\}, \{X_1, \dots, X_{100}\}, \{\neg X_1, \dots, \neg X_{100}\}\}$

Choice:  $P, X_1, \dots, X_{10}, Q$ . Implied:  $R, S$ .



# Questionnaire

**Recall:** The implication graph depends on “ordering decisions” during UP: Which unit clause is picked first. E.g.  $\Delta = \{\{\neg P, \neg Q\}, \{Q\}, \{P\}\}$ .



## Question!

Does the **existence** of a conflict graph depend on these decisions?

(A): Yes

(B): No

→ Observe: A conflict graph exists iff  $\square$  is UP-derivable in the current simplified formula  $\Delta'$ . So the question is whether it can happen that, when propagating a unit clause  $\{l\}$  in  $\Delta'$ , on the resulting simplified  $\Delta''$  the UP calculus cannot derive the empty clause anymore.

The answer is no.  $\Delta''$  can be obtained in two steps: 1. Remove  $\bar{l}$  from every  $C \in \Delta'$  where  $\bar{l} \in C$  to obtain  $\Delta'_l$ . 2. Remove  $C \in \Delta'_l$  where  $l \in C$  to obtain  $\Delta''$ . 1. cannot hurt  $\square$ -derivability because every clause of  $\Delta'_l$  is a sub-clause of  $\Delta'$ , and smaller clauses can only be better. 2. cannot hurt  $\square$ -derivability because  $\bar{l}$  is not contained in  $\Delta'_l$  (so if  $l \in C$  then no derivative of  $C$  can ever become empty).

# Questionnaire, ctd.

## Question!

**How many conflict graphs do we get for the choice literal  $\neg R$ , when running UP on  $\{\{P, Q, R\}, \{\neg P, Q, R\}, \{S, R\}, \{\neg S, R\}\}$ ?**

(A): 0

(B): 1

(C): 2

(D): 3

→ (B) is correct: The only conflict we get is via  $\{S, R\}$ ,  $\{\neg S, R\}$ , and the choice literal  $\neg R$ .

## Question!

**And for the choice literals  $\neg Q$ ,  $\neg R$ ?**

→ (C) is correct: We get the above conflict, and another one via  $\{P, Q, R\}$ ,  $\{\neg P, Q, R\}$ , and the choice literals  $\neg Q$  and  $\neg R$ .

(Note: These choices can happen in DPLL on  $\Delta$ , if we choose  $\neg Q$  first.)

# Clause Learning

**Observe:** Conflict graphs encode *logical entailments*

$$\Delta \models \left( \bigwedge_{l \in \text{choiceLits}(G^{\text{confl}})} l \right) \rightarrow \perp$$

→ Given  $\Delta$ , setting all choice literals in a conflict graph results in failure.

**Observe:** We can re-write this!

$$\Delta \models \bigvee_{l \in \text{choiceLits}(G^{\text{confl}})} \bar{l}$$

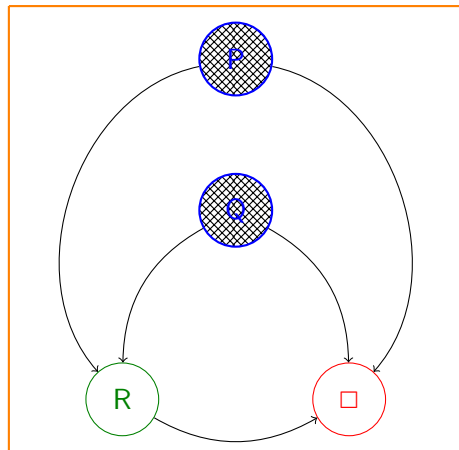
**Proposition (Clause Learning).** Let  $\Delta$  be a set of clauses, and let  $G^{\text{confl}}$  be a conflict graph at some time point during a run of DPLL on  $\Delta$ . Let  $\text{choiceLits}(G^{\text{confl}})$  be the choice literals in  $G^{\text{confl}}$ . Then  $\Delta \models \{\bar{l} \mid l \in \text{choiceLits}(G^{\text{confl}})\}$ .

→ The negation of the choice literals in a conflict graph is a valid clause.

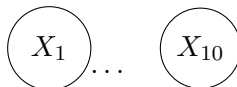
# Clause Learning: Example (Redundance1)

$$\Delta = \{\{\neg P, \neg Q, R\}, \{\neg P, \neg Q, \neg R\}, \{\neg P, Q, R\}, \{\neg P, Q, \neg R\}, \\ \{X_1, \dots, X_{100}\}, \{\neg X_1, \dots, \neg X_{100}\}\}$$

Choice:  $P, X_1, \dots, X_{10}, Q$ . Implied:  $R$ .



$\{\neg P, \neg Q\}$



# The Effect of Learned Clauses

(in Redundance1)

→ What happens after we learned a new clause  $C$ ?

1. **We add  $C$  into  $\Delta$ .** → Example:  $C = \{\neg P, \neg Q\}$ .

2. **We retract the last choice  $l'$ .**

→ Example: Retract the choice  $l' = Q$ .

**Observation:**  $C = \{\bar{l} \mid l \in \text{choiceLits}(G^{\text{confl}})\}$ . Before we learn the clause,  $G^{\text{confl}}$  must contain the most recent choice  $l'$ : otherwise, the conflict would have occurred earlier on. So  $C = \{\bar{l}_1, \dots, \bar{l}_k, \bar{l}'\}$  where  $l_1, \dots, l_k$  are earlier choices.

→ Example:  $l_1 = P$ ,  $C = \{\neg P, \neg Q\}$ ,  $l' = Q$ .

**Observation:** Given the earlier choices  $l_1, \dots, l_k$ , after we learned the new clause  $C = \{\bar{l}_1, \dots, \bar{l}_k, \bar{l}'\}$ ,  $\bar{l}'$  is now set by UP!

3. **We set the opposite choice  $\bar{l}'$  as an implied literal.**

→ Example: Set  $\neg Q$  as an implied literal.

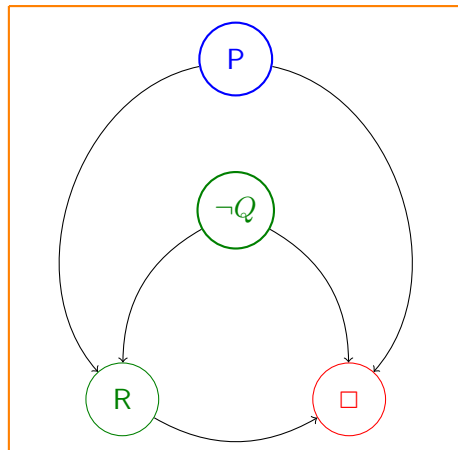
4. **We run UP and analyze conflicts.** Learned clause: earlier choices only!

→ Example:  $C = \{\neg P\}$ , see next slide.

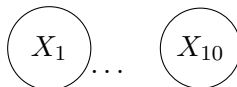
# The Effect of Learned Clauses: Example (Redundance1)

$$\Delta = \{\{\neg P, \neg Q, R\}, \{\neg P, \neg Q, \neg R\}, \{\neg P, Q, R\}, \{\neg P, Q, \neg R\}, \\ \{X_1, \dots, X_{10}\}, \{\neg X_1, \dots, \neg X_{10}\}, \{\neg P, \neg Q\}\}$$

Choice:  $P, X_1, \dots, X_{10}$ . Implied:  $\neg Q, R$ .

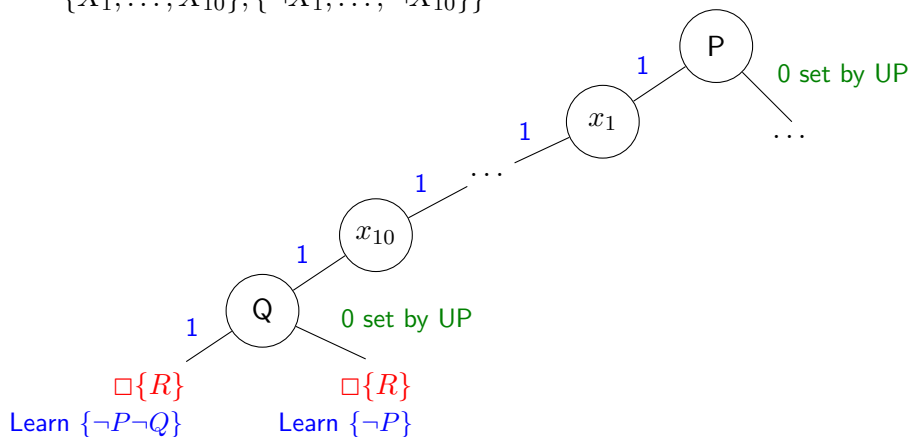


$\{\neg P\}$



# NOT Same Mistakes over Again: Example (Redundance1)

$$\Delta = \{\{\neg P, \neg Q, R\}, \{\neg P, \neg Q, \neg R\}, \{\neg P, Q, R\}, \{\neg P, Q, \neg R\}, \\ \{X_1, \dots, X_{10}\}, \{\neg X_1, \dots, \neg X_{10}\}\}$$



**Note:** Here, the problem could be avoided by splitting over different variables. **This is not so in general!** (see next slide)



# Clause Learning vs. Resolution

## Remember (slide 13):

- ① **DPLL = tree resolution**: Each derived clause  $C$  (not in  $\Delta$ ) is derived anew every time it is used.
- ② There exist  $\Delta$  whose **shortest tree-resolution proof is exponentially longer than their shortest (general) resolution proof**.

## This is no longer the case with clause learning!

- ① **We add each learned clause  $C$  to  $\Delta$ , can use it as often as we like.**
- ② **Clause learning renders DPLL equivalent to full resolution** [Beame et al. (2004); Pipatsrisawat and Darwiche (2009)]. (In how far exactly this is the case was an open question for ca. 10 years, so it's not as easy as I made it look here ...)

→ In particular: Selecting different variables/values to split on can *provably* not bring DPLL up to the power of DPLL+Clause Learning. (cf. slide 15, and previous slide)

# “DPLL + Clause Learning”?

**Disclaimer:** We have only seen *how to learn a clause from a conflict*. We will *not* cover how the overall DPLL algorithm changes, given this learning. Slides 33 – 35 are merely meant to give a *rough intuition* on “backjumping”.

**Just for the record:** (not exam or exercises relevant)

- One *could* run “DPLL + Clause Learning” by always backtracking to the maximal-level choice variable contained in the learned clause.
- But the actual algorithm is called **Conflict-Directed Clause Learning (CDCL)**, and differs from DPLL more radically:

$L := 0; I := \emptyset$

**repeat**

    execute UP

**if** a conflict was reached **then** //  $C = \{\overline{l_1}, \dots, \overline{l_k}, \overline{l'}\}$

**if**  $L = 0$  **then return** UNSAT

$L := \max_{i=1}^k \text{level}(l_i)$ ; erase  $I$  below  $L$

        add  $C$  into  $\Delta$ ; add  $\overline{l'}$  to  $I$  at level  $L$

**else**

**if**  $I$  is a total interpretation **then return**  $I$

        choose a new decision literal  $l$ ; add  $l$  to  $I$  at level  $L$

$L := L + 1$

# Remarks

## WHICH clause(s) to learn?

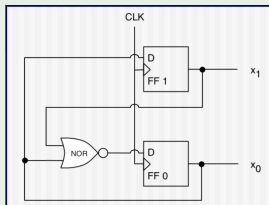
- While we only select  $choiceLits(G^{\text{confl}})$ , much more can be done.
- For any **cut** through  $G^{\text{confl}}$ , with  $choiceLits(G^{\text{confl}})$  on the “left-hand” side of the cut and the conflict literals on the right-hand side, the literals on the left border of the cut yield a learnable clause.
- Must take care to *not learn too many clauses* ...

## Origins of clause learning:

- Clause learning originates from **explanation-based (no-good) learning** developed in the CSP community.
- The distinguishing feature here is that the “no-good” is a clause:  
→ **The exact same type of constraint as the rest of  $\Delta$ .**

# Questionnaire

## Example



- Counter, repeatedly from  $c = 0$  to  $c = 2$ .
- **To Verify:** If  $c < 3$  in current clock cycle, then  $c < 3$  in next clock cycle.
- $\Delta = \{\{\neg x'_1, x_0\}, \{x'_1, \neg x_0\}, \{x'_0, x_1, x_0\}, \{\neg x'_0, \neg x_1\}, \{\neg x'_0, \neg x_0\}, \{\neg x_1, \neg x_0\}, \{x'_1\}, \{x'_0\}\}$

## Question!

Which clause do we learn after running UP on  $\Delta$ ?

(A): ☐

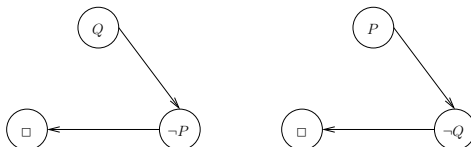
(B): None

→ We learn the clause  $\square$ . There are no choice literals, so the learned clause is empty.

→ In case there are no choice literals, the contradiction follows “without assumptions”, so we learn immediately that the input formula is unsatisfiable. This special case happens only if the input formula can be proved unsatisfiable using unit propagation (which is never the case in practice).

# Questionnaire, ctd.

**Recall:** The implication graph depends on “ordering decisions” during UP: Which unit clause is picked first. E.g.  $\Delta = \{\{\neg P, \neg Q\}, \{Q\}, \{P\}\}$ .



## Question!

**May the learned clause also change?**

(A): Yes

(B): No

→ Yes. Depending on which conflict UP ended up deriving, the conflict graph may differ, and thus the learned clause may differ.

(Note: In the example above, the learned clause in both cases is  $\square$  because there aren't any choice variables.)

# Questionnaire, ctd.

## Question!

**Which clauses can we learn after choosing  $\neg R$  and running UP on  $\{\{P, Q, R\}, \{\neg P, Q, R\}, \{S, R\}, \{\neg S, R\}\}$ ?**

(A):  $\{\neg S, R\}$ .

(B):  $\{Q, R\}$ .

(C):  $\{R\}$ .

(D):  $\square$ .

→ (A), (B): No: Neither  $S$  (in (A)) nor  $\neg Q$  (in (B)) is a choice literal.

→ (C): Yes, via the conflict from  $\{S, R\}, \{\neg S, R\}$ , choice literal  $\neg R$ .

→ (D): No: While UP does derive a conflict, that conflict depends on the choice literal  $\neg R$ .

# Questionnaire, ctd.

## Question!

**Which clauses can we learn after choosing  $\neg Q, \neg R$  and running UP on  $\{\{P, Q, R\}, \{\neg P, Q, R\}, \{S, R\}, \{\neg S, R\}\}$ ?**

(A):  $\{\neg S, R\}$ .

(B):  $\{Q, R\}$ .

(C):  $\{R\}$ .

(D):  $\square$ .

→ (A): No.  $S$  is not a choice literal.

→ (B): Yes, via the conflict  $\{P, Q, R\}, \{\neg P, Q, R\}$ , with choice literals  $\neg Q, \neg R$ .

→ (C): Yes, via the conflict from  $\{S, R\}, \{\neg S, R\}$ , choice literal  $\neg R$ .

→ (D): No (same as on previous slide).

(Note: These choices can happen in DPLL on  $\Delta$ , if we choose  $\neg Q$  first.)

# Damn, Where Are the Hard Problems?

## Err, what?

- SAT is **NP**-hard. Worst case for DPLL is  $2^n$ , with  $n$  propositions.
- Imagine I gave you as homework to make a formula family  $\{\phi\}$  where DPLL runtime necessarily is in the order of  $2^n$ .
  - I promise you're not gonna find this easy ... (although it is of course possible: e.g., the "Pigeon Hole Problem").
- People noticed by the early 90s that, in practice, the DPLL worst case does not tend to happen.

→ Modern SAT solvers successfully tackle practical instances where  $n > 1000000$ .



# Damn, Where Are the Hard Problems? Ctd.

**So, what's the problem?** Science is about *understanding the world*.

→ Are “hard cases” just pathological outliers? Can we say something about the *typical case*?

**Difficulty 1:** What is the “typical case” in applications? E.g., what is the “average” Hardware Verification instance?

→ Consider precisely defined random distributions instead.

**Difficulty 2:** Search trees get very complex, and are difficult to analyze mathematically, even in trivial examples. Never mind examples of practical relevance ...

→ The most successful works are empirical.

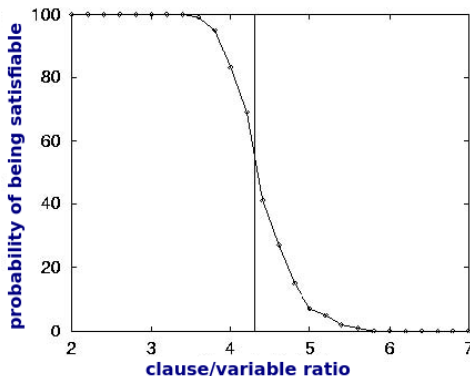
(Interesting theory is mainly concerned with *hand-crafted* formulas, like the Pigeon Hole Problem.)

# Phase Transitions in SAT [Mitchell *et al.* (1992)]

**Fixed clause length model:** Fix **clause length**  $k$ ;  $n$  **variables**. Generate  $m$  **clauses**, by uniformly choosing  $k$  variables  $P$  for each clause  $C$ , and for each variable  $P$  deciding uniformly whether to add  $P$  or  $\neg P$  into  $C$ .

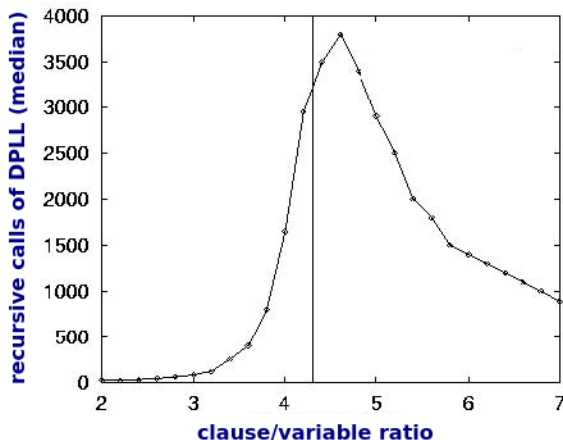
**Order parameter:** **Clause/variable ratio**  $\frac{m}{n}$ .

**Phase transition:** (Fixing  $k = 3$ ,  $n = 50$ )



# Does DPLL Care?

Oh yes, it does! Extreme runtime peak at the phase transition!



# Why Does DPLL Care?

## Intuitive explanation:

**Under-Constrained:** Satisfiability likelihood close to 1. Many solutions, first DPLL search path usually successful. (“Deep but narrow”)

**Over-Constrained:** Satisfiability likelihood close to 0. Most DPLL search paths short, conflict reached after few applications of splitting rule. (“Broad but shallow”)

**Critically Constrained:** At the phase transition, many *almost-successful* DPLL search paths. (“Close, but no cigar”)

# The Phase Transition Conjecture

**Conjecture:** [Cheeseman *et al.* (1991)]

## Phase Transition Conjecture

“All **NP**-complete problems have at least one **order parameter**, and the hard to solve problems are around a critical value of this order parameter. This critical value (a **phase transition**) separates one region from another, such as over-constrained and under-constrained regions of the problem space.”

→ [Cheeseman *et al.* (1991)] confirmed this for Graph Coloring and Hamiltonian Circuits. Later work confirmed it for SAT (see previous slides), and for numerous other **NP**-complete problems.

# Why Should We Care?

## Enlightenment:

- Phase transitions contribute to the fundamental understanding of the behavior of search, even if it's only in random distributions.
- There are interesting theoretical connections to phase transition phenomena in physics. (See [Gomes and Selman (2005)] for a short summary.)

## Ok, but what can we use these results for?

- **Benchmark design**: Choose instances from phase transition region.  
→ Commonly used in competitions etc. (In SAT, random phase transition formulas are the most difficult for DPLL-style searches.)
- **Predicting solver performance**: Yes, but very limited because:

→ All this works only for the particular considered *distributions of instances*! Not meaningful for any other instances.

# Questionnaire, ctd.

## Question!

Say I encode a Wumpus problem into  $\Delta$  that turns out to have clause/variable ratio 10. Which is true?

(A):  $\Delta$  is very likely to be unsatisfiable.

(B):  $\Delta$  is very likely to be satisfiable.

→ Neither is true. The clause/var ratio 4.3 has a meaning only for Mitchell *et al.* (1992)'s “fixed clause length model” for generating random CNF formulas.

→ For other CNF formulas, the clause/variable ratio is completely meaningless!

**Extreme example:** Say we generate clauses including only positive literals ...

**Practical example:** Many Verification problems have huge numbers of clauses but are still satisfiable.

→ For example, consider the straightforward encoding for “Exactly one of  $n$  variables  $x_1, \dots, x_n$  is true.” We get the clause  $\{x_1, \dots, x_n\}$  (“at least one is true”) and, for every  $1 \leq i \neq j \leq n$ , the clause  $\{\neg x_i, \neg x_j\}$  (“at most one is true”). The clause/variable ratio is  $\frac{n^2-n+2}{2n}$ , but the formula is satisfiable.

# Questionnaire, ctd.

## Question!

**Say I sit down tonight and write a random  $\Delta$  with clause/variable ratio 1.1. Which are true?**

(A): I'm bored.

(B):  $\Delta$  is satisfiable.

(C): All slides for next week are prepared already.

(D):  $\Delta$  is very likely to be satisfiable.

→ (A), (C): Definitely true ...

→ (B): Definitely not a certainty, for any way of generating random CNFs (unless we include only positive, or only negative, literals).

→ (D): Depends on *how* I generate the CNF. If I use Mitchell *et al.* (1992)'s methods then yes. If I use a different method, then no.



# Summary

- **Implication graphs** capture how UP derives conflicts. Their analysis enables us to do **clause learning**. DPLL with clause learning is called **CDCL**. It corresponds to full resolution, not “making the same mistakes over again”.
- CDCL is state of the art in applications, routinely solving formulas with millions of propositions.
- In particular random formula distributions, typical problem hardness is characterized by **phase transitions**.

# State of the Art in SAT

## SAT competitions:

- Since beginning of the 90s <http://www.satcompetition.org/>
- Distinguish **random** vs. **industrial** vs. **handcrafted** benchmarks.
- Largest industrial instances:  $> 1000000$  propositions.

## State of the art is CDCL:

- **Vastly superior on handcrafted and industrial benchmarks.**
- Key techniques: **Clause Learning!** Also: Efficient implementation (UP!), good branching heuristics, random restarts, portfolios.

## What about local search?

- **Better on random instances.**
- No “dramatic” progress in last decade.
- Parameters are difficult to adjust.

# Topics We Didn't Cover Here

- **Variable/value selection heuristics:** A whole zoo is out there.
- **Implementation techniques:** One of the most intensely researched subjects. Famous “watched literals” technique for UP had huge practical impact.
- **Local search:** In space of all truth value assignments. GSAT (slide 9) had huge impact at the time (1992), caused huge amount of follow-up work. Less intensely researched since clause learning hit the scene in the late 90s.
- **Portfolios:** How to combine several SAT solvers effectively?
- **Random restarts:** Tackling heavy-tailed runtime distributions.
- **Tractable SAT:** Polynomial-time sub-classes (most prominent: 2-SAT, Horn formulas).
- **MaxSAT:** Assign weight to each clause, maximize weight of satisfied clauses (= optimization version of SAT).
- **Resolution special cases:** There's a universe in between unit resolution and full resolution: trade-off inference vs. search.
- **Proof complexity:** Can one resolution special case X simulate another one Y polynomially? Or is there an exponential separation (example families where X is exponentially less effective than Y)?

# Reading

- The Art of Computer Programming by Donald E. Knuth, Vol 4. Section 7.2.2.2
- Handbook of Satisfiability, Hans van Maaren, Armin Biere, Toby Walsh.

# References I

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- Peter Cheeseman, Bob Kanefsky, and William M. Taylor. Where the really hard problems are. In J. Mylopoulos and R. Reiter, editors, *Proceedings of the 12th International Joint Conference on Artificial Intelligence (IJCAI'91)*, pages 331–337, Sydney, Australia, August 1991. Morgan Kaufmann.
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- Knot Pipatsrisawat and Adnan Darwiche. On the power of clause-learning SAT solvers with restarts. In Ian P. Gent, editor, *Proceedings of the 15th International Conference on Principles and Practice of Constraint Programming (CP'09)*, volume 5732 of *Lecture Notes in Computer Science*, pages 654–668. Springer, 2009.