Algorithms and Satisfiability

9. Binary Decision Diagrams

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Agenda

1. Introduction
2. Binary Decision Diagrams
3. Queries
4. Operations
5. Conclusions

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How To Represent Logical Formulas?

Option 1: Logical Formula (e.g. in CNF)

$(\neg x_1 \vee x_2) \wedge x_3$

Advantage: Compact representation

Disadvantage: SAT (and other operations) is hard

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Option 2: Truth table

x_1	x_2	x_3	value
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→So, does there exist some compact representation under which SAT is easy (in the size of the representation)?

Boolean Functions

Definition

A Boolean function $f : \{0,1\}^k \to \{0,1\}$ maps k Boolean values to true or false.

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(Reduced Ordered) Binary Decision Diagrams

Reduced Ordered Binary Decision Diagrams (or BDDs for short): representation of logical functions that follows the following key ideas:

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1. Decision Diagram: Use Directed Acyclic Graph
2. Reduced: Apply Reduction Rules
3. Ordered: Fixed Variable Ordering

"BDDs are one of the only really fundamental data structures that came out in the last twenty-five years" — Donald Knuth

Representing Logical Functions as a Decision Tree

Turn a truth table for the Boolean function into a decision tree.

x_1	x_2	x_3	value
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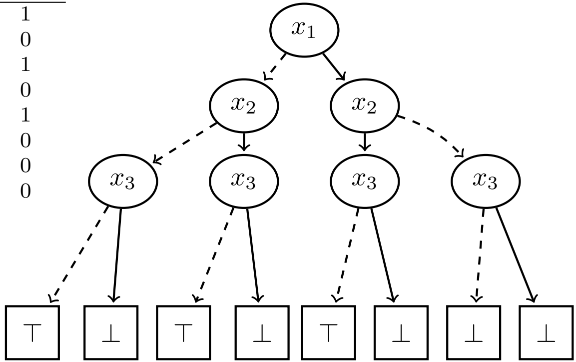

Notation: Nodes are labelled with the variable they checked (x_i), we use solid edge for $x_i = 1$ and dashed edge for $x_i = 0$.

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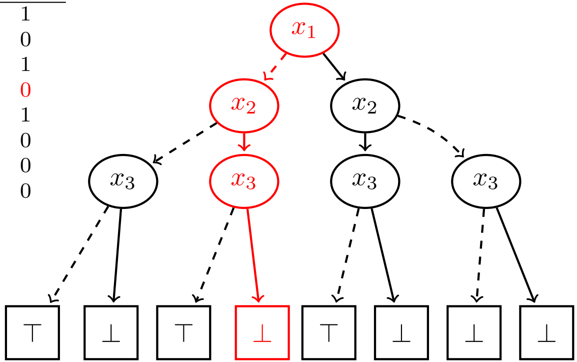

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Shannon Expansion

For any Boolean formula f and variable x, it can be written as:

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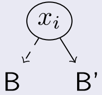
→Inner nodes correspond to a function represented via the Shannon expansion!

Notation

Node $A = (x_i, B, B')$ represents $f_A = (\overline{x_i} \wedge f_B) \vee (x_i \wedge f_{B'})$

- Variable $v(A) = x_i$ (nodes are labelled with the variable they check)
- Low child $l(A) = B$ (represented by a dashed edge)
- High child $r(A) = B'$ (represented by a solid edge)



Representing Logical Functions as a Decision Diagram

Diagram: Directed Acyclic Graph (DAG)
$\rightarrow$We can share a node in different parts of the diagram

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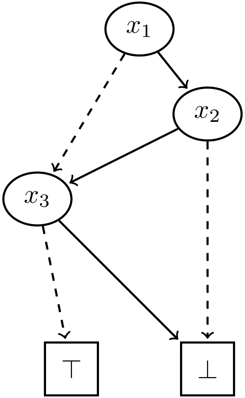

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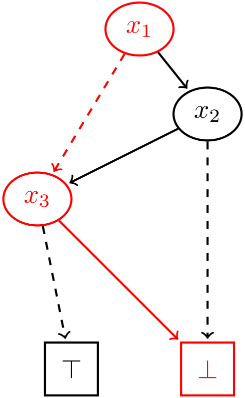

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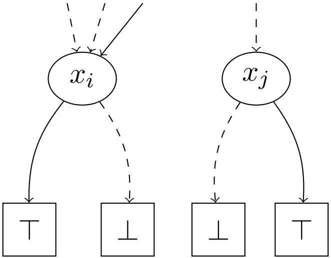
Reduction Rule #1: Eliminate Equivalent Leaves

There are only two unique leaves: $\perp$ and $\top$



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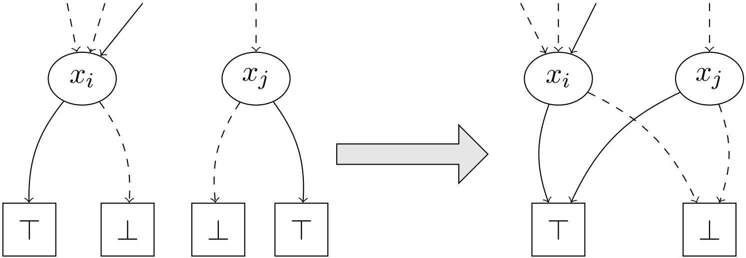
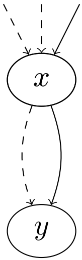


Reduction Rule #2: Eliminate Redundant Nodes

l(n) = r(n), then eliminate n and re-direct all its references to l(n)



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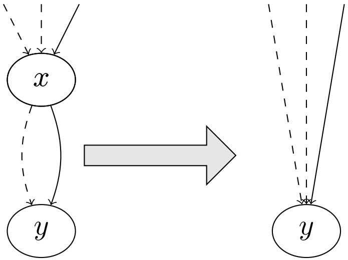

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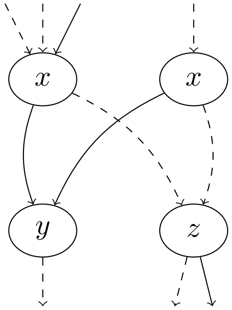
Two inner nodes n, n' are equivalent if and only if $v(n) = v(n')$, $l(n) = l(n')$, and $r(n) = r(n')$



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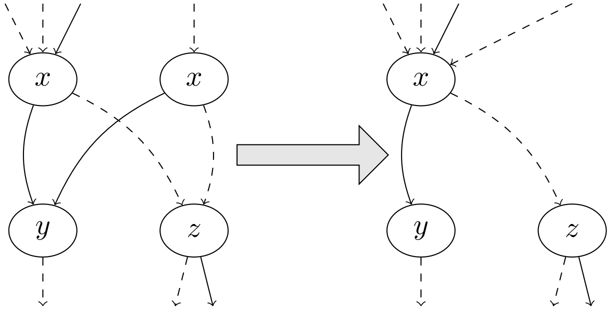

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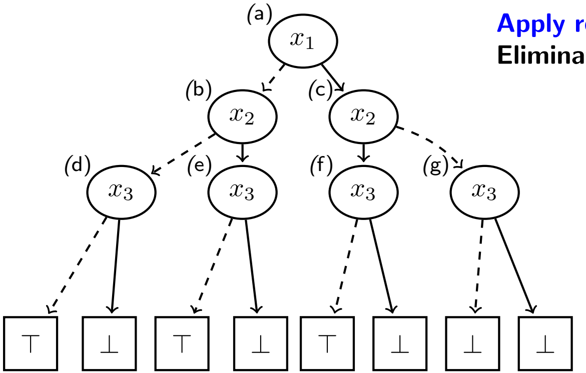

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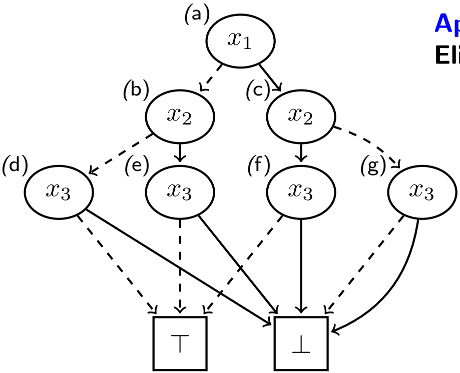

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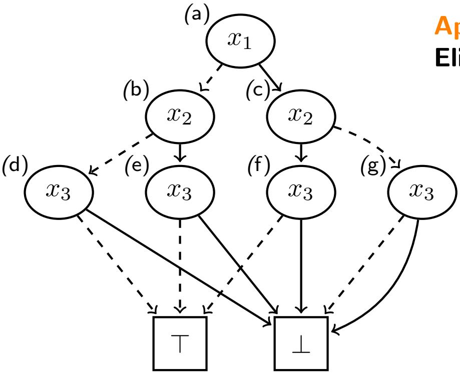

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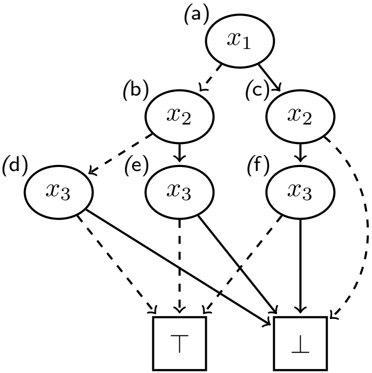

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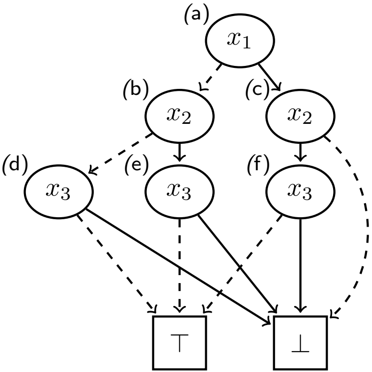

Apply reduction rule 3:
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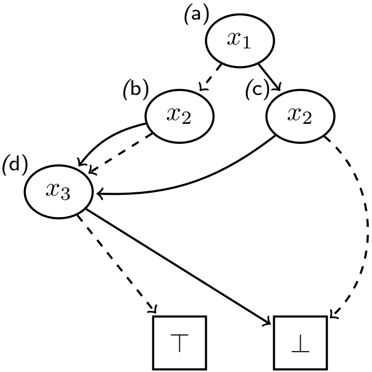

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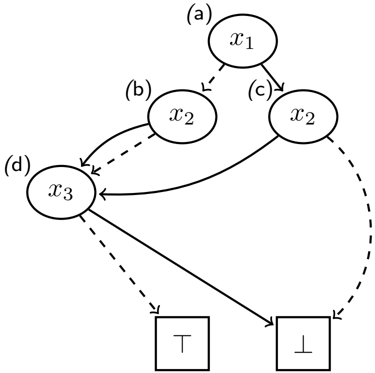

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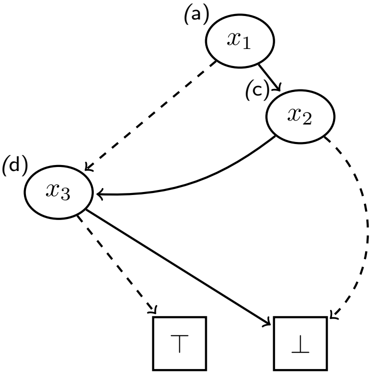

Apply reduction rule 2:
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The Reduce Operation: Pseudocode

reduce(A):

if *A is terminal* **then**

    // Eliminate equivalent leaves

    return UniqueTerminal$(A.value)$

$n_l = \textit{reduce}(l(A))$

$n_r = \textit{reduce}(r(A))$

if $n_l == n_r$ **then**

    // Eliminate redundant nodes

    result $= n_l$

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    result $=$ NodeUnique $(v(A), n_l, n_r)$

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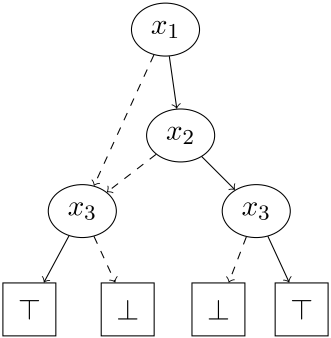
Overall time complexity: $\mathcal{O}(|A| \log(|A|))$

Questionnaire



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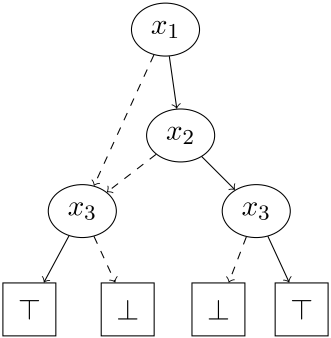

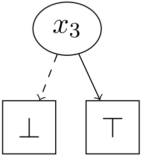

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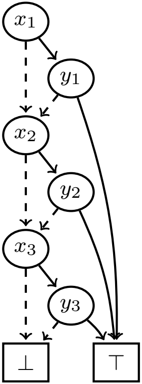
Proof sketch. By induction (base case is trivial for terminal nodes). Assume ROBDDs for functions $f(p_i, \ldots, p_n)$ are canonical, then show that ROBDDs for $f(p_{i-1}, \ldots, p_n)$ are canonical as well.

Importance of Variable Ordering

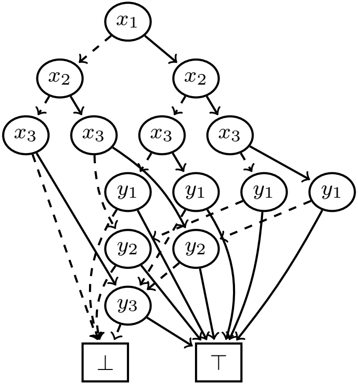
Choice of variable ordering can cause the representation of a function to be exponential or polynomial in the number of variables.

Example: $f(x_1, \dots, x_n, y_1, \dots, y_n) = (x_1 \wedge y_1) \vee \dots \vee (x_n \wedge y_n)$ $(n = 3)$.



Polynomial size:

$x_1, y_1, x_2, y_2, x_3, y_3$



Exponential size:

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- Static Variable Ordering:
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- Dynamic Variable Ordering:
  - Find the best ordering for a given BDD (or set of BDDs)
  - Optimization problem that is NP-complete
  - But we can just use some approximation

Agenda

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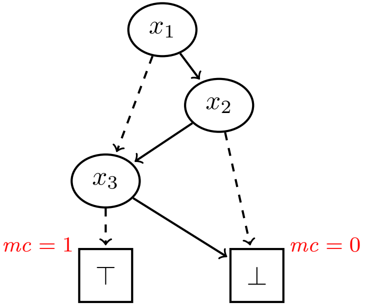

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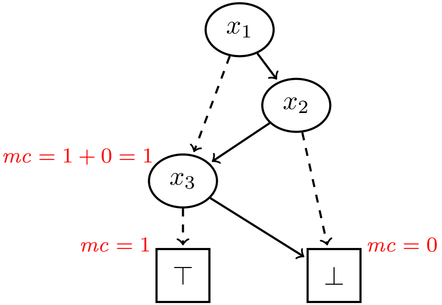

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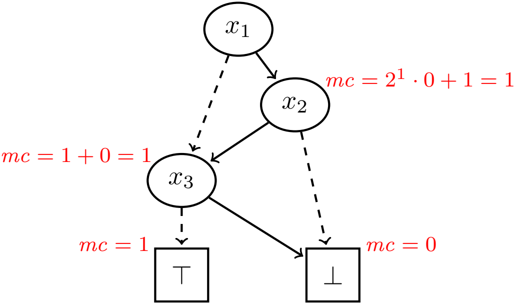

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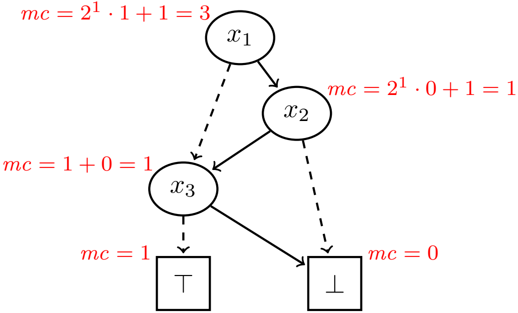

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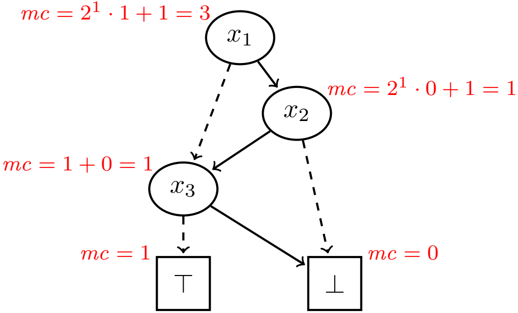

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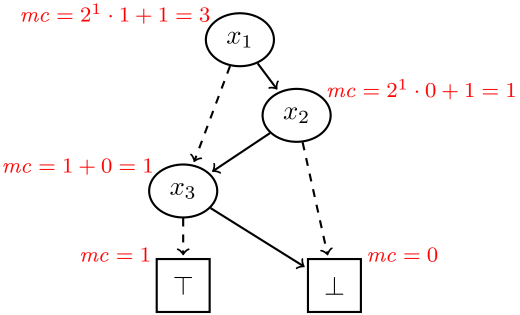
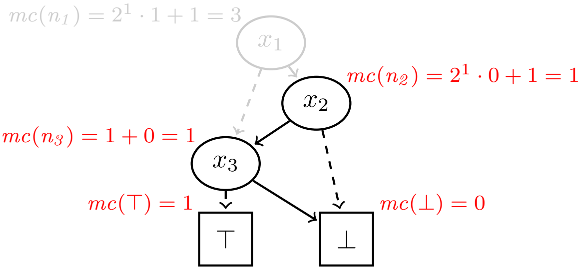


$MC(n_1) = 3$

Questionnaire



Question!

How many models does n_2 have over $x_1, \dots, x_3$? (MC (n_2)))

(A): 1 (B): 2

(C): 3 (D): 4

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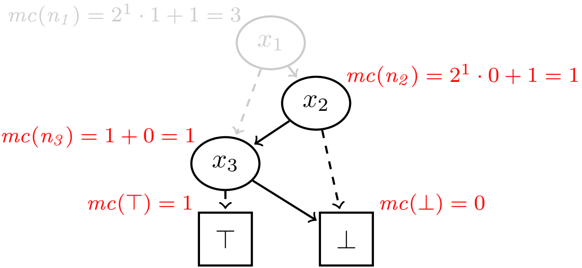

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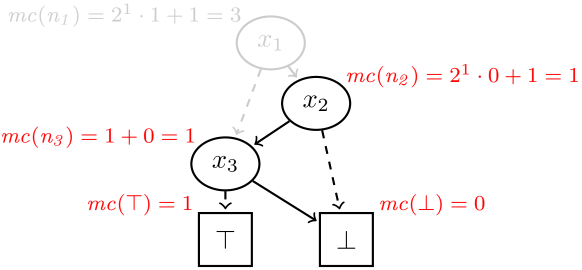

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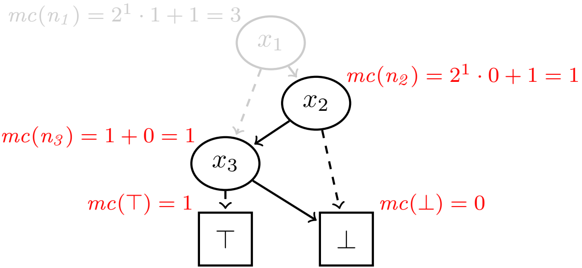

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Similarly, $MC(n_3) = 2^2 \cdot 1 = 4$, $MC(\top) = 2^3 \cdot 1 = 8$

$\rightarrow$We must specify over how many variables we perform model counting

Agenda

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- Reduce (f)
  $\rightarrow$Reduces the canonical form of a function f represented as a non-reduced BDD

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Keys:

- Recursively iterate over the BDD
- Dynamic programming: Use cache to store intermediate results. Never re-do the same work twice!

But. . . how to obtain a BDD in the first place?

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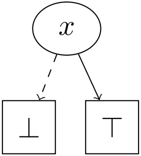
You can easily obtain the BDDs that represent simple functions:



And then, use operations on them to construct more complex functions!

$$(\neg x_1 \lor x_2) \land x_3$$

The Apply Operation

Given compatible OBDDs B_f and B_g that represent formulas f and g, apply($\Box, B_f, B_g$) computes an OBDD representing $f \Box g$.

- Compatible: operands must use the same variable ordering
- $\Box$ is some binary operation on Boolean formulas (e.g. $\vee, \wedge, \oplus$)
- Unary operations can be handled too.
  for example, negation: $\neg x = x \oplus 1$

Using the Shannon expansion, $f \Box g$ can be expanded as:

$$f \Box g \equiv (\neg x \wedge (f[x=0] \Box g[x=0])) \vee (x \wedge (f[x=1] \Box g[x=1]))$$

The Apply Operation: Pseudocode

apply$(\Box, A, B)$:

if *A is terminal and B is terminal* **then**
  return UniqueTerminal$(A.value \Box B.value)$

if $v(A) == v(B)$ **then**
  $v = v(A)$
  $n_l = apply(\Box, l(A), l(B))$
  $n_r = apply(\Box, r(A), r(B))$
else if *B is terminal or* $v(B) > v(A)$ **then**
  $v = v(A)$
  $n_l = apply(\Box, l(A), B)$
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else
  $v = v(B)$
  $n_l = apply(\Box, A, l(B))$
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result $=$ NodeUnique (v, n_l, n_r)

return result

The Apply Operation: Pseudocode

apply($\Box, A, B$):

if *A is terminal and B is terminal* **then**
    return UniqueTerminal($A.value \Box B.value$)
if *(A,B) in Cache* **then**
    return Cache[(A, B)]
if $v(A) == v(B)$ **then**
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    $n_r = apply(\Box, A, r(B))$
result = NodeUnique (v, n_l, n_r)
Cache[(A,B)] = result
return result

The Apply Operation: Cases

1. apply $(\square, \boxed{\text{u}}, \boxed{\text{v}}) = \boxed{u \square v}$

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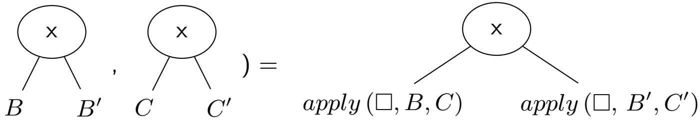

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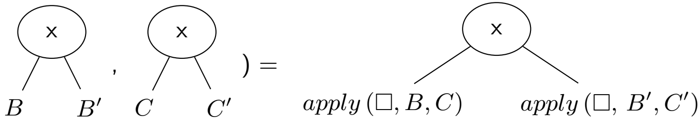

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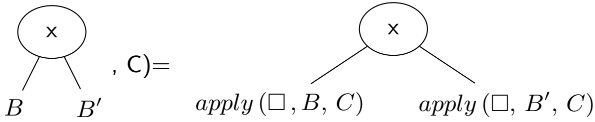

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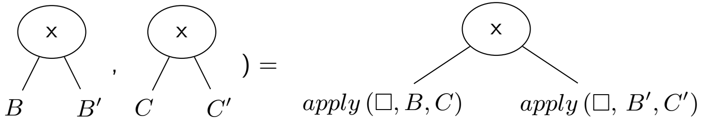
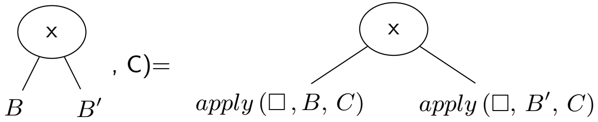
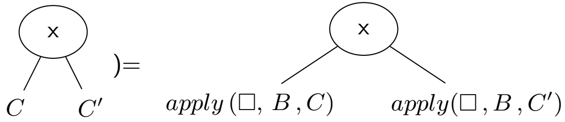
2. apply $(\square,$ 

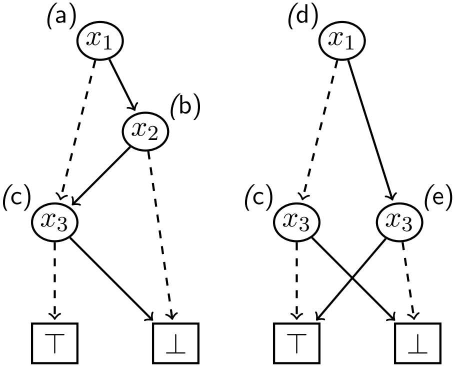
3. apply $(\square,$ 

   when C is terminal node, or non-terminal with $v(C) > x$

4. apply $(\square,\ B,$ 

   when B is terminal node, or non-terminal with $v(B) > x$

The Apply Operation: Example (Conjunction)



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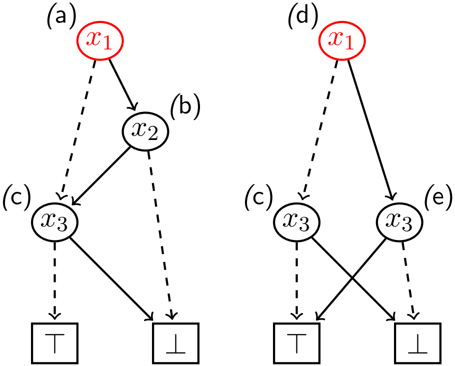

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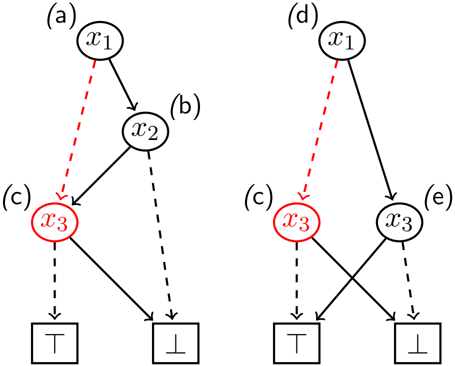

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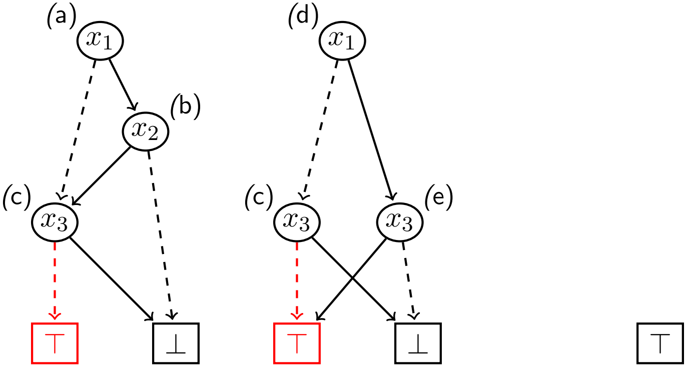

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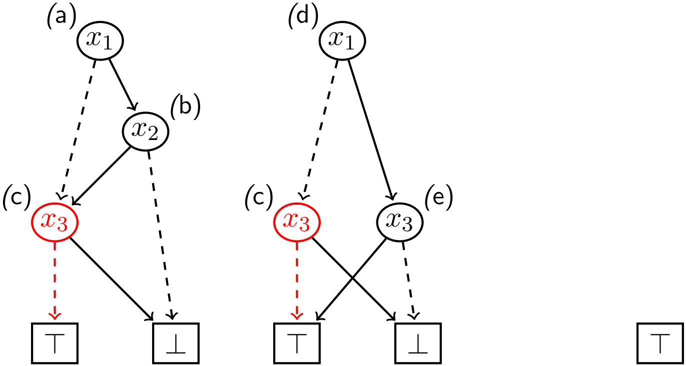

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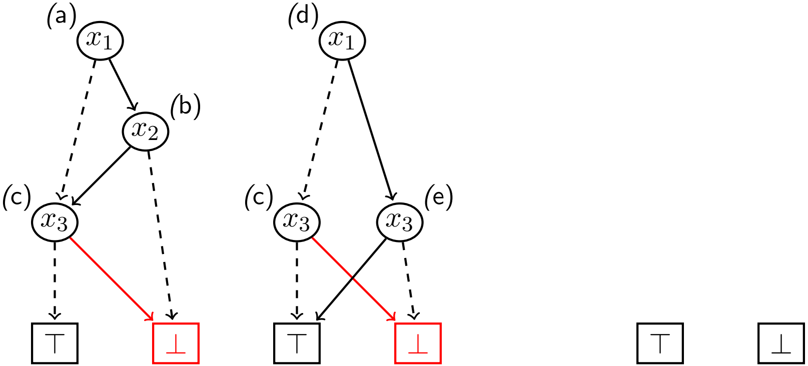

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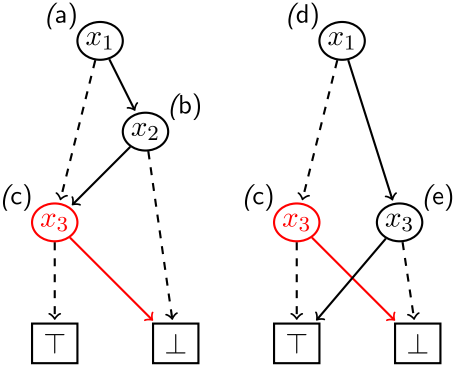

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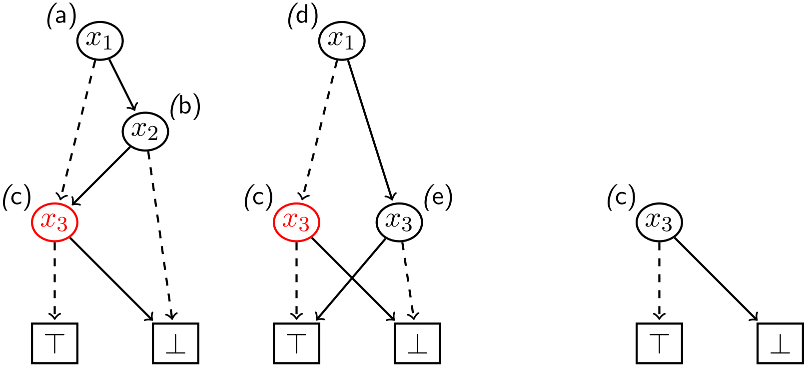

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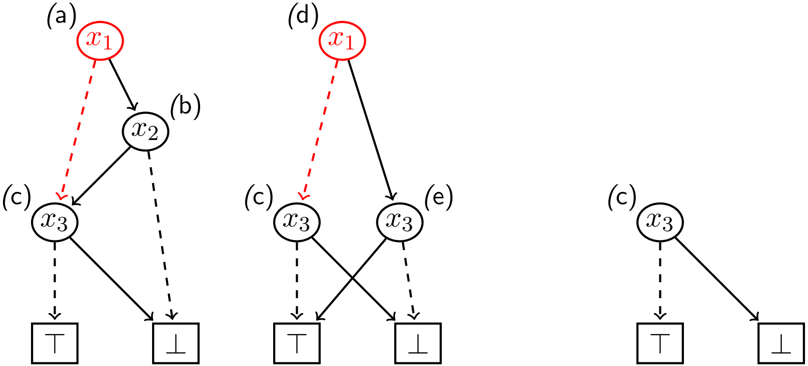

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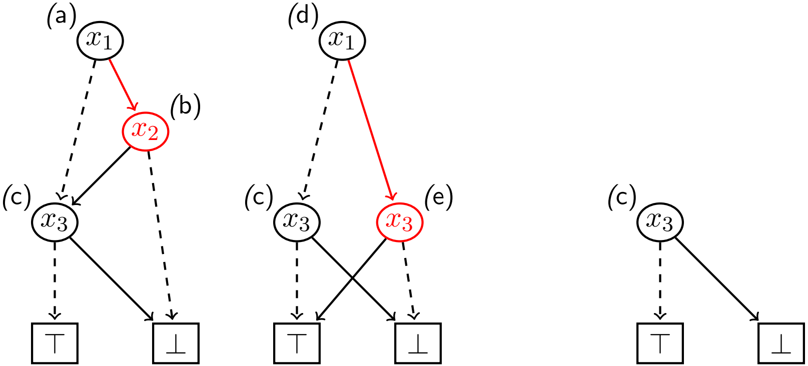

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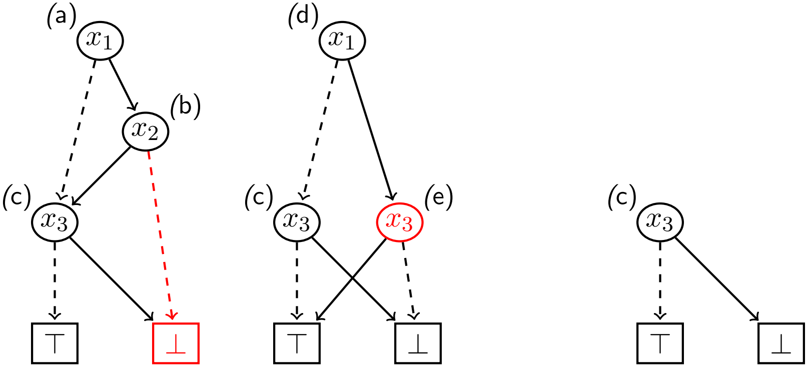

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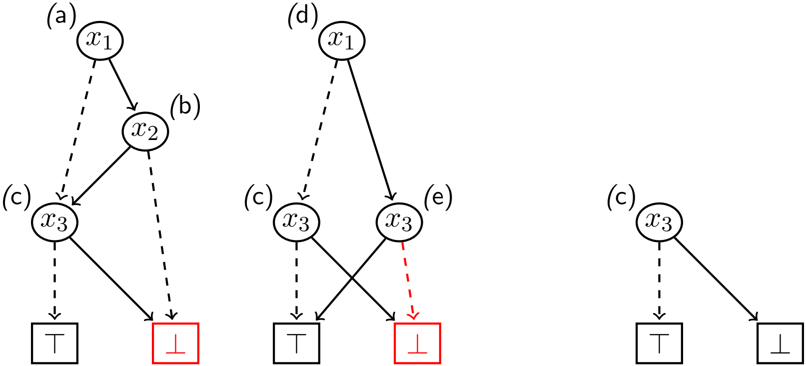

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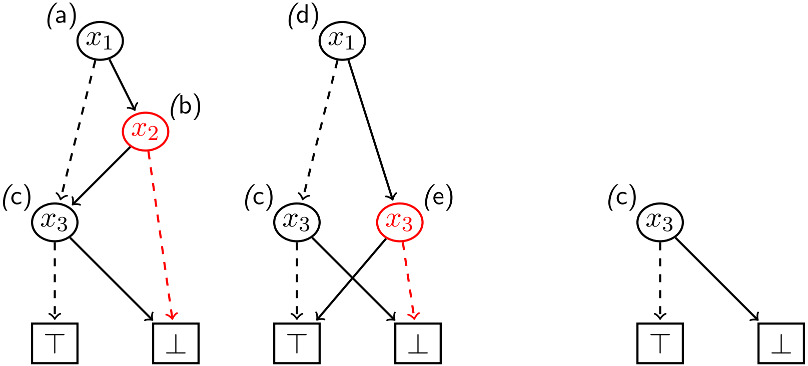

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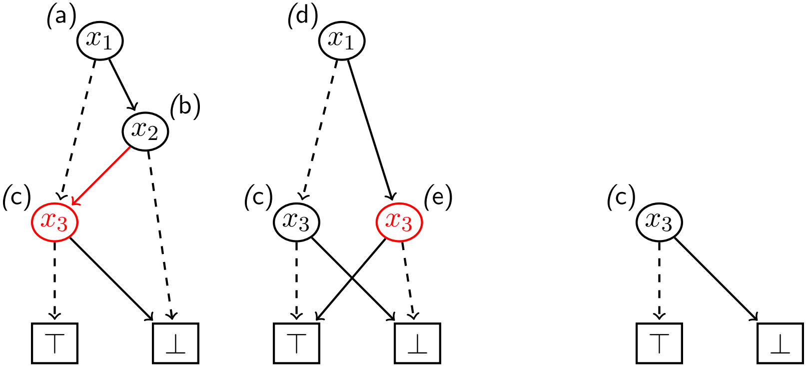

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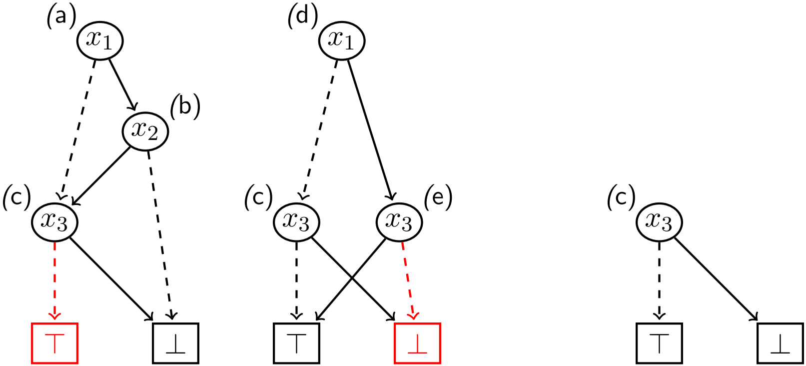

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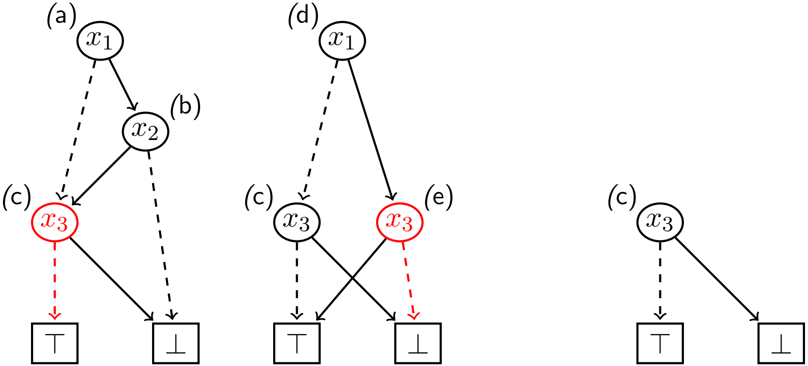

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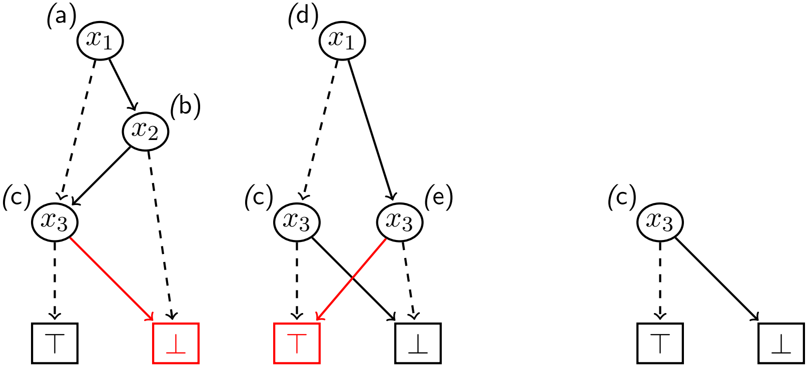

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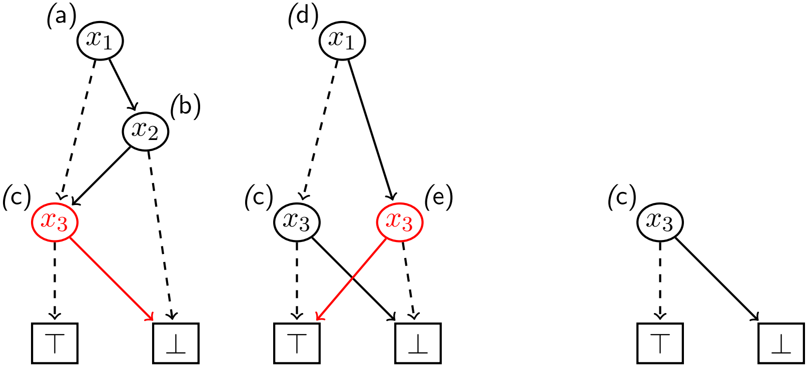

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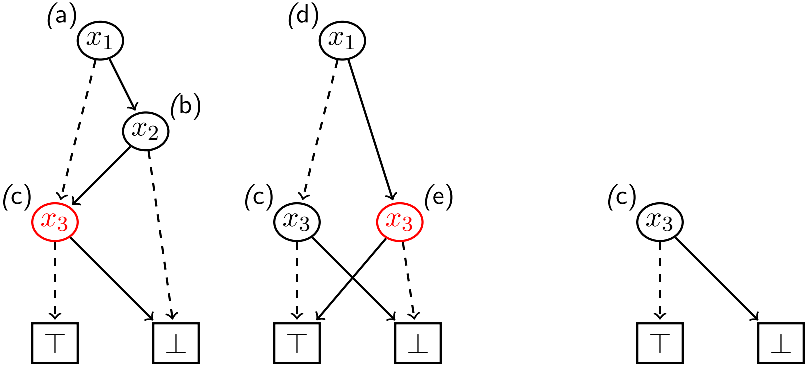

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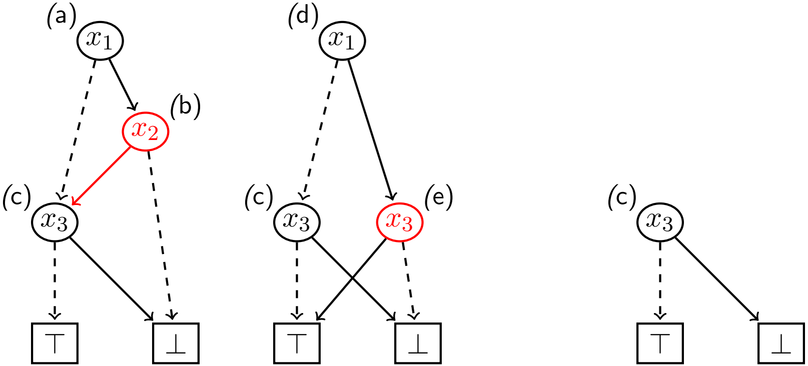

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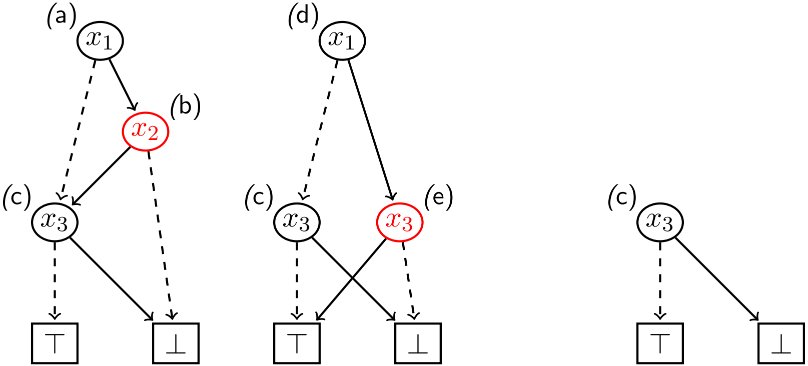

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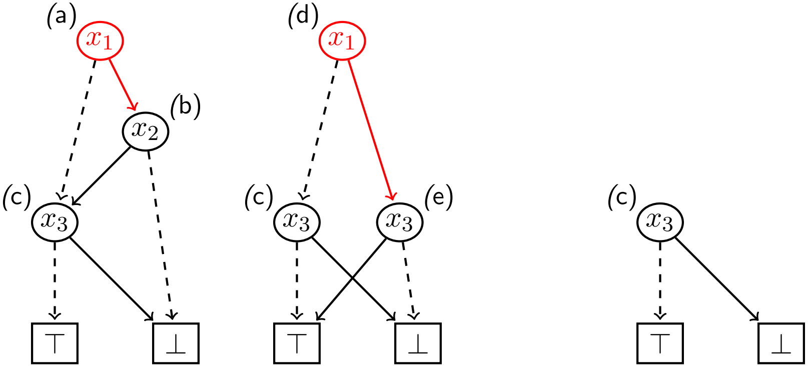

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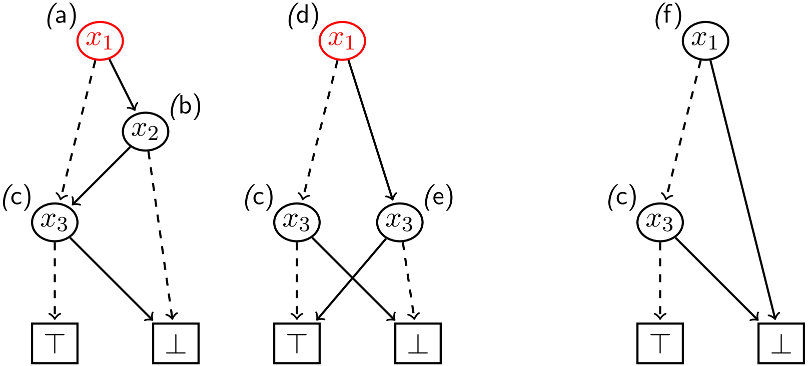

1. $(a, d) \mapsto (x_1,\ c,\ \bot) \mapsto f$
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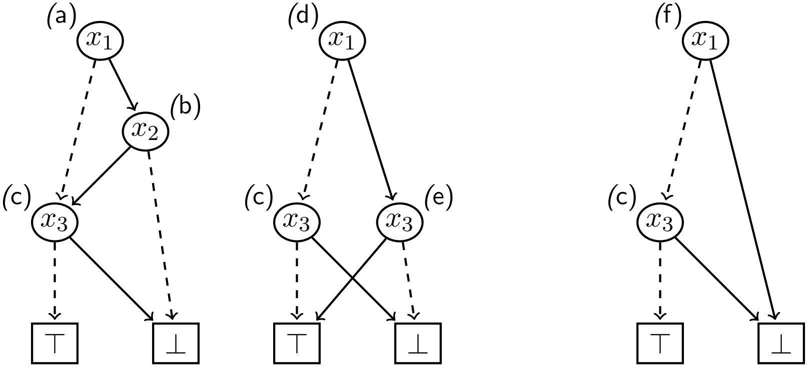

1. $(a, d) \mapsto (x_1,\ c,\ \perp) \mapsto f$
2. $(c, c) \mapsto (x_3,\ \top,\ \perp) \mapsto c$
3. $(\top, \top) \mapsto \top$
4. $(\perp, \perp) \mapsto \perp$
5. $(b, e) \mapsto (x_2,\ \perp,\ \perp) \mapsto \perp$
6. $(\perp, e) \mapsto \perp$
7. $(\perp, \perp) \mapsto \perp$
8. $(c, e) \mapsto (x_3,\ \perp,\ \perp) \mapsto \perp$
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So, is the BDD representing each CNF formula of polynomial size?

Answer: NO! If the CNF formula has N clauses, each represented with a BDD with K nodes then in the worst case we have K^N nodes.

Complexity of Operations (extracted from Darwiche and Marquis (2002)

	Queries				Transformations				
L	CO	VA	EQ	CT	$\bigwedge_n$	$\wedge$	$\bigvee_n$	$\vee$	$\neg C$
NNF	$\circ$	$\circ$	$\circ$	$\circ$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
OBDD$<$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\bullet$	$\checkmark$	$\bullet$	$\checkmark$	$\checkmark$
DNF	$\checkmark$	$\circ$	$\circ$	$\circ$	$\bullet$	$\checkmark$	$\checkmark$	$\checkmark$	$\bullet$
CNF	$\circ$	$\checkmark$	$\circ$	$\circ$	$\checkmark$	$\checkmark$	$\bullet$	$\checkmark$	$\bullet$

$\checkmark$: Polynomial Time, $\bullet$ ($\circ$) Exponential time (unless P=NP)

Agenda

Summary

Reduced Ordered Binary Decision Diagrams (BDDs for short)

- Directed acyclic graph with two leave nodes $\perp$, $\top$
- Each node decomposes the function using the Shannon expansion
- Reduced
  any node with two identical children is removed
  equivalent nodes are merged
- Ordered
  Splitting variables always follow the same order along all paths
  $x_1 < x_2 < x_3 < \ldots < x_n$
- Properties:
  - Canonical representation
  - Efficient queries
  - Efficient operations

Some Caveats

We made some simplifying assumptions, and didn't mention a lot of optimizations. For example:

- Typically, negation is encoded on the edges
  - The same node is used to represent a function and its negation
  - Negation can be performed in constant time
  - One needs to handle this during the apply operation
- In our version of apply we always reach the terminals. However, we can return early in some cases by encoding some additional if conditions
  - $f \vee f = f$
  - $f \vee \bot = f$
  - $f \vee \top = \top$
  - $f \wedge f = f$
  - ...

Some Extensions

- What if instead of two terminals ($\perp, \top$), I have many (e.g. Numbers) $\rightarrow$Algebraic Decision Diagrams (ADD)

  (you can also use multiple BDDs or auxiliary variables in the BDD)

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- Can I represent more complex arithmetic expressions?
  $\rightarrow$Edge-valued Multi-valued Decision Diagrams (EVMDDs)

Reading

- The art of Computer Programming. Knuth. Section 7.1.4. Binary Decision Diagrams
- Graph-based algorithms for Boolean function manipulation [Bryant (1986)].
  →Original paper exploring BDDs
- A Knowledge Compilation Map [Darwiche and Marquis (2002)].
  →A very good summary of how BDDs relate to other alternative representations

References I

Randal E. Bryant. Graph-based algorithms for boolean function manipulation. *IEEE Transactions on Computers*, 35(8):677–691, 1986.

Adnan Darwiche and Pierre Marquis. A knowledge compilation map. *J. Artif. Intell. Res.*, 17:229–264, 2002.