Exercise 1

Perform a trace of DPLL until you find the first conflict. Start by using the splitting rule and assign the value T to the proposition S and then assign T to Q.

$\Delta_b = \{\{\neg P, Q\}, \{\neg Q, E\}, \{\neg E, S\}, \{\neg Q, F\}, \{\neg E, \neg F\}, \{P, F\}, \{\neg E, \neg S, G\}, \{\neg P, \neg Q, \neg F\}\}$

Whenever you encounter a conflict, draw the corresponding implication graph as well as its conflict graph, and mention which clause can be learned with the clause learning method. Highlight vertices of choice and implied literals with different colors. What is the learned clause?

Note: You do not need to continue with the DPLL with clause learning procedure after learning the clause.

Solution:

1. Splitting rule: $S \mapsto T$:
   $\{\{\neg P, Q\}, \{\neg Q, E\}, \{\neg Q, F\}, \{\neg E, \neg F\}, \{P, F\}, \{\neg E, G\}, \{\neg P, \neg Q, \neg F\}\}$
2. Splitting rule: $Q \mapsto T$:
   $\{\{E\}, \{F\}, \{\neg E, \neg F\}, \{P, F\}, \{\neg E, G\}, \{\neg P, \neg F\}\}$
3. UP: $E \mapsto T$:
   $\{\{F\}, \{\neg F\}, \{P, F\}, \{G\}, \{\neg P, \neg F\}\}$
4. UP: $F \mapsto T$:
   $\{\Box, \{G\}, \{\neg P\}\}$
5. UP: $G \mapsto T$:
   $\{\Box, \{\neg P\}\}$
6. UP: $P \mapsto F$:
   $\{\Box\}$

The implication and conflict graphs are shown in Figure 1. The clause learned is $\{\neg Q\}$.

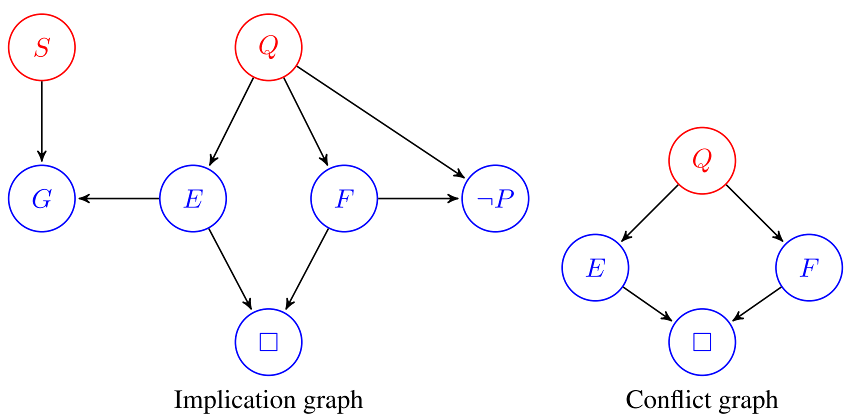


Figure 1: Implication and conflict graphs for Exercise 1.

Exercise 2

Perform DPLL with clause learning as explained in the slides of Chapter 8 on the following set of clauses (give the full DPLL trace). Start by using the splitting rule and assign the value T to the proposition S and then assign T to Q. Whenever you encounter a conflict, draw the corresponding implication graph as well as its conflict graph, and mention which clause can be learned with the clause learning method. Highlight vertices of choice and implied literals with different colors. Then use this information and continue with the DPLL procedure, backtracking the last choice as specified in slide 33 of Chapter 8. Do this until the clause set is proven to be satisfiable or unsatisfiable.

$\Delta_a = \{\{R, P, \neg Q\}, \{\neg P, \neg Q\}, \{\neg R, Q\}, \{\neg R, \neg Q\}, \{R, \neg P, Q\}, \{\neg S, Q, R\}, \{S, \neg P\}\}$

Solution:

DPLL Trace:

$\Delta_a = \{\{R, P, \neg Q\}, \{\neg P, \neg Q\}, \{\neg R, Q\}, \{\neg R, \neg Q\}, \{R, \neg P, Q\}, \{\neg S, Q, R\}, \{S, \neg P\}\}$

1. Splitting Rule (S)

1a. $S \mapsto T$: $\{\{R, P, \neg Q\}, \{\neg P, \neg Q\}, \{\neg R, Q\}, \{\neg R, \neg Q\}, \{R, \neg P, Q\}, \{Q, R\}\}$

2a. Splitting Rule (Q)

2aa. $Q \mapsto T$: $\{\{R, P\}, \{\neg P\}, \{\neg R\}\}$

3aa. UP: $R \mapsto F$: $\{\{P\}, \{\neg P\}\}$

4aa. UP: $P \mapsto T$: $\{\Box\}$

$\rightarrow$ Learning clause $\{\neg Q\}$, see Figure 2.

- (a) Add $\{\neg Q\}$ to Δ
- (b) Going back to last splitting rule: $Q \mapsto T$
- (c) Set $Q \mapsto F$ as an implied literal
- (d) Continue: $\{\{R, P, \neg Q\}, \{\neg P, \neg Q\}, \{\neg R, Q\}, \{\neg R, \neg Q\}, \{R, \neg P, Q\}, \{Q, R\}, \{\neg Q\}\}$

2ab. UP $Q \mapsto F$: $\{\{\neg R\}, \{R, \neg P\}, \{R\}\}$

3ab. UP $R \mapsto F$: $\{\{\neg P\}, \Box\}$

4ab. UP $P \mapsto F$: $\{\Box\}$

$\rightarrow$ Learning clause $\{\neg S\}$, see Figure 2.

- (a) Add $\{\neg S\}$ to Δ
- (b) Going back to last splitting rule: $S \mapsto T$
- (c) Set $S \mapsto F$ as an implied literal
- (d) Continue: $\{\{R, P, \neg Q\}, \{\neg P, \neg Q\}, \{\neg R, Q\}, \{\neg R, \neg Q\}, \{R, \neg P, Q\}, \{\neg S, Q, R\}, \{S, \neg P\}, \{\neg Q\}, \{\neg S\}\}$

1b. UP $S \mapsto F$: $\{\{R, P, \neg Q\}, \{\neg P, \neg Q\}, \{\neg R, Q\}, \{\neg R, \neg Q\}, \{R, \neg P, Q\}, \{\neg P\}, \{\neg Q\}\}$

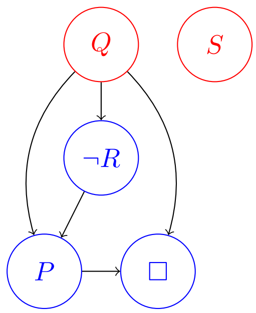
2b. UP $P \mapsto F$: $\{\{R, \neg Q\}, \{\neg R, Q\}, \{\neg R, \neg Q\}, \{\neg Q\}\}$

3b. UP $Q \mapsto F$: $\{\{\neg R\}\}$

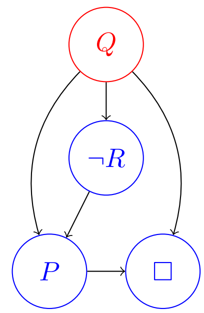
4b. UP $R \mapsto F$: $\{\{\}\}$

$\Rightarrow$ Δ_a is satisfiable

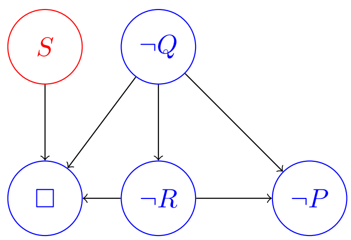
From the conflict graph in 4aa we learn the clause $\{\neg Q\}$. Having this, we perform a backtracking in the DPLL trace until we reach the splitting rule where we assigned T to Q. As we know that Q must have the value F, we continue the DPLL procedure accordingly until we reach another conflict. From the conflict graph in 4ab we learn the clause $\{\neg S\}$ and backtrack to the root of the tree. Then continue the DPLL procedure until finding the clause set $\{\}$ at 4b. We conclude that the clause set is satisfiable with the assignment $S \mapsto F$, $Q \mapsto F$, $R \mapsto F$ and $P \mapsto F$.



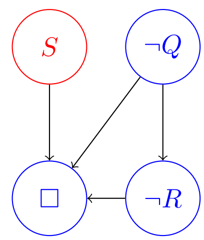
(a) Implication graph in 4aa.



(b) Conflict graph in 4aa.



(c) Implication graph in 4ab.



(d) Conflict graph in 4ab.

Figure 2: Implication and conflict graphs for Exercise 19(a). Choice literals are indicated by red circles, implied literals by blue circles.

Exercise 3

Consider the following statements. For each statement say whether it is correct or not and briefly explain why in a couple of sentences.

1. Every implication graph results in exactly one conflict graph.
2. If a branch of DPLL ends in the empty clause, we can always learn a new clause.
3. If a branch of DPLL ends in the empty clause without having applied the splitting rule, we cannot learn a new clause.
4. If a branch of DPLL ends in the empty clause, can we learn more than one clause such that none of them is a subset of the other? If yes, provide an example where this happens. If no, explain why not.

Solution:

1. False, there may be more than one conflict graph, if we have multiple conflicts.
2. True, we can learn a new clause by constructing the corresponding implication and conflict graphs.
3. False, in that case we will learn the empty clause, so the formula has been proven to be unsatisfiable.
4. Yes, whenever we have multiple conflict graphs, we can end up with more than one clause so that none of them subsumes the other (e.g. {A, B} and {A, C}).

   For example, consider the following set of clauses:

   $\Delta = \{\{A, B, D\}, \{A, C, E\}, \{A, \neg E\}, \{A, \neg D\}\}$

   Choices: $B \mapsto 0$, $C \mapsto 0$, $A \mapsto 0$, Implied: $D \mapsto 0$, $E \mapsto 0$

   In that case, the implication graph contains two conflicts: $\square_{\{A,B,D\}}$ and $\square_{\{A,C,E\}}$. This results in two different conflict graphs, and we learn the two clauses {A, B} and {A, C}.

Exercise 4

Considering the procedure to generate random k-SAT formulas by with n variables and m clauses by Mitchell et al. (1992), indicate for each statement below whether it is correct or not and briefly explain why in one or two sentences.

1. Increasing the number of clauses makes the problem more constrained.
2. Problems more constrained are harder to solve.
3. Increasing the number of clauses makes the problem harder to solve.
4. Problems more constrained are easier to solve.
5. Increasing the number of clauses makes the problem easier to solve.

Solution:

1. True, adding more clauses can only make the problem harder to satisfy (all clauses must be satisfied). As clauses are randomly selected, independently of the previous ones, increasing the number of clauses will make the problem more constrained on average.
2. False; If the problem is underconstrained, then constraining it makes it harder (until reaching the phase transition). But over-constrained problems will be easier to solve the more constrained they are.
3. False; by 1. we know that it will be more constrained, but as argued in 2. this does not make the problem harder to solve.
4. False; If the problem is over-constrained, then constraining it makes it easier. But under-constrained problems will be harder to solve the more constrained they are (until reaching the phase transition)
5. False; by 1. we know that it will be more constrained, but as argued in 4. this does not make the problem easier to solve.