Tutorial 1

Exercise 1 (compulsory)

Answer the following questions (and justify your answers):

1. Can a Turing machine ever write the blank symbol $\sqcup$ on its tape?
2. Can the tape alphabet Γ be equal to the input alphabet Σ?
3. Can the head of a Turing machine ever stay on the same cell for two subsequent steps of a computation?
4. Can the state set of a Turing machine consist of only a single state?

Solution:
Here are the correct answers:

1. A Turing machine can write a $\sqcup$, since $\sqcup \in \Gamma$ and the transition function has type $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$.
2. The tape alphabet Γ can never be equal to the input alphabet Σ, since $\sqcup \in \Gamma$, whereas $\sqcup \notin \Sigma$.
3. The head of a Turing machine can stay on the same cell for two consecutive steps of a computation if the head is at the leftmost tape cell and the machine tries to move left.
4. The state set of a Turing machine will always contain at least two states, since q_{accept} and q_{reject} are different.

Exercise 2 (compulsory)

Some of the following definitions are correct and some wrong (and some are even pure nonsense). Mark the correct definitions and for the incorrect ones underline the part of the definition which is wrong and explain why.

1. A language L is recognizable if the language halts in the state q_{accept} whenever $x \in L$.
2. A language L is recognizable if there exists a Turing machine M such that M, given input x, halts in the state q_{accept} if and only if $x \in L$.
3. A language L is recognizable if it halts in an accepting state q_{accept} whenever $x \in L$.
4. A language L is recognizable if there exists a Turing machine M which has a state q_{accept} and is a member of L.
5. A language L is recognizable if there exists a Turing machine M such that for any given input x the machine M run on x halts in the state q_{accept} if $x \in L$, and it either loops or halts in q_{reject} if $x \notin L$.
6. A language L is recognizable if every Turing machine M when run on a string x halts in the state q_{accept} if and only if $x \in L$.

Solution:
Here are the correct answer:

1. A language L is recognizable if the <u>language halts</u> in the state q_{accept} whenever $x \in L$.

   WRONG: *Language is not an input for a TM and we cannot run a TM on a language and ask if it halts or not; we can only ask whether a TM halts on a given string x.*

2. A language L is recognizable if there exists a Turing machine M such that for any given input x the machine M run on x halts in the state q_{accept} if and only if $x \in L$.

   CORRECT.

3. A language L is recognizable if it halts in an accepting state q_{accept} whenever $x \in L$.

   WRONG: *It is not clear what "it halts" refers to, probably to the language L and the same as in point 1. applies.*

4. A language L is recognizable if there exists a Turing machine M which has a state q_{accept} and is a member of L.

   WRONG: *A Turing machine cannot be a member of L, unless we talk about a string encoding of a TM, but also in this case it does not correctly define the term recognizable language. And by the way, every TM has the state q_{accept} by definition.*

5. A language L is recognizable if there exists a Turing machine M such that for any given input x the machine M run on x halts in the state q_{accept} if $x \in L$, and it either loops or halts in q_{reject} if $x \notin L$.

   CORRECT.

6. A language L is recognizable if every Turing machine M when run on a string x halts in the state q_{accept} if and only if $x \in L$.

   WRONG: *Not every Turing machine, but there should exist at least one such a TM.*

Exercise 3 (compulsory)

Complete the definition below and be formally precise.

A language $L \subseteq \Sigma^*$ is decidable iff ...

Solution:
There are several correct answers, for example:

- A language $L \subseteq \Sigma^*$ is decidable iff there exists a TM M such that M run on input $x \in \Sigma^*$ halts in q_{accept} if $x \in L$ and it halts in q_{reject} if $x \notin L$.
- A language $L \subseteq \Sigma^*$ is decidable iff there is a decider M such that M recognizes the language L.

Exercise 4 (compulsory)

Answer the following questions and give precise arguments.

1. Suppose that a Turing machine has its head at a symbol s and is in a state p which is different from q_{accept} and q_{reject}. How many distinct states may the machine be in after a transition?
2. Is it always the case that if a language is decidable then it is also recognizable?

Solution:
Here are the correct answers:

1. A Turing machine will always be in exactly one state after a transition step. This is due to the fact that the machine is deterministic, i.e., δ is a *function* and hence to every element of $Q \times \Gamma$ it assigns exactly one element of $Q \times \Gamma \times \{L, R\}$.
2. Yes, every decidable language is also recognizable. This follows directly from the definitions. Decidable languages are recognized by deciders (TMs that never loop) and they are just ordinary Turing machines.

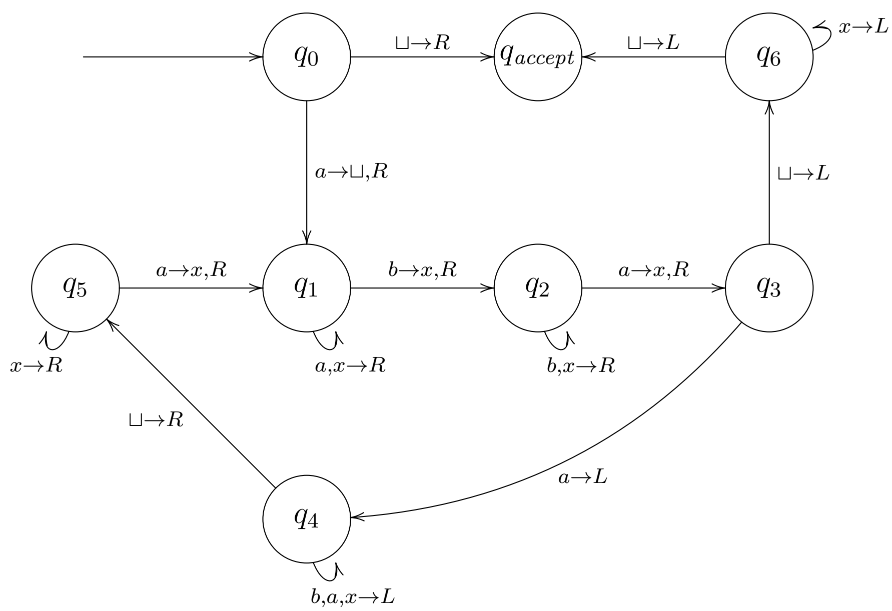
Exercise 5 (compulsory)

As a preparation for this exercise solve Exercise 3.2 part a) on page 187. Then you can proceed with the following tasks:

1. Draw a state diagram of a Turing machine M recognizing the language $\{a^n b^n a^n \mid n \geq 0\}$ over the alphabet $\Sigma = \{a, b\}$.
2. Consider the input string $w = aabbaa$. Write the whole sequence of configurations that M will enter when run on w.
3. Does M accept w?

Solution:
The correct solutions are as follows (note that your solution might still be correct even though your Turing machine looks differently). By writing e.g. $a, x \to R$ on an arrow, we understand that there are two arrows, one with $a \to R$ and one with $x \to R$.

1.



All missing transitions in the picture go implicitly to the state q_{reject}.

2. For the input string $aabbaa$ the machine M will pass through the following sequence of configurations:

$$
\begin{array}{rrr}
q_0 aabbaa \to & \sqcup q_1 abbaa \to & \sqcup a q_1 bbaa \to \\
\sqcup a x q_2 baa \to & \sqcup a x b q_2 aa \to & \sqcup a x b x q_3 a \to \\
\sqcup a x b q_4 x a \to & \sqcup a x q_4 b x a \to & \sqcup a q_4 x b x a \to \\
\sqcup q_4 a x b x a \to & q_4 \sqcup a x b x a \to & \sqcup q_5 a x b x a \to \\
\sqcup x q_1 x b x a \to & \sqcup x x q_1 b x a \to & \sqcup x x x q_2 x a \to \\
\sqcup x x x x q_2 a \to & \sqcup x x x x x q_3 \to & \sqcup x x x x q_6 x \to \\
\sqcup x x x q_6 x x \to & \sqcup x x q_6 x x x \to & \sqcup x q_6 x x x x \to \\
\sqcup q_6 x x x x x \to & q_6 \sqcup x x x x x \to & q_{accept} \sqcup x x x x x
\end{array}
$$

3. Yes.

Exercise 6 (optional, if you need more practice on TM basics)

If you don't feel comfortable with the computation of a TM and its design, try Exercise 3.2 on page 187 and Exercise 3.8 on page 188 (give the implementation-level descriptions but you can also draw the full state diagrams).

Exercise 7 (optional and mind-challenging)

Problem 3.9 on page 188 (in international edition), or Problem 3.22 on page 190 (in standard edition).