Computability and Complexity

Lecture 4

Reals are uncountable
Undecidability of the acceptance problem for TM
Complement of the acceptance problem for TM is unrecognizable

given by Jiri Srba

Bijection (Correspondence)

Definition

A function $f : A \to B$ is called bijection (correspondence) iff

- f is injective (one-to-one), i.e., $f(a) \neq f(b)$ whenever $a \neq b$, and
- f is surjective (onto), i.e., for every $b \in B$ there is $a \in A$ such that $f(a) = b$.

Examples:

- There is a bijection between natural numbers and even natural numbers.
- There is a bijection between natural and rational numbers.

Countable Sets

Definition

A set A is countable if it is either finite or there is a bijection between the set A and the set of natural numbers.

Facts:

- All even natural numbers are countable.
- The set of rational numbers is countable.

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Theorem

The set of real numbers is uncountable.

Proof: By contradiction via the diagonalization method.

There Exist Nonrecognizable Languages

Observation

There are only countably many Turing machines.

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Conclusion

There exist languages that are not recognizable.

Acceptance Problem for Turing Machines

Problem: "Given a TM M and a string w, does M accept w?

Language Formulation (Acceptance Problem for TM)

$$A_{TM} \stackrel{\text{def}}{=} \{\langle M, w\rangle \mid \ M \text{ is a TM}, w \text{ a string, and } M \text{ accepts } w\}$$

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Theorem (Turing, 1936)

The language A_{TM} is undecidable.

Proof of Undecidability of A_{TM}

By contradiction assume that there is a decider H for A_{TM}:

$$H(\langle M, w\rangle) = \begin{cases} \underline{\text{accept}} & \text{if } M \text{ accepts } w \\ \underline{\text{reject}} & \text{if } M \text{ does not accept } w \end{cases}$$

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From H we can build a machine D, which is also a decider:

$D =$ "On input $\langle M \rangle$

1. Run H on $\langle M, \langle M \rangle \rangle$.
2. If H accepted then D <u>rejects</u>. If H rejected then D <u>accepts</u>."

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So D cannot exist, so H cannot exist either (D was built from H).

This means that A_{TM} is undecidable. $\square$

The Language A_{TM} Is Recognizable

$$A_{TM} \stackrel{\text{def}}{=} \{\langle M, w\rangle \mid \ M \text{ is a TM, } w \text{ a string, and } M \text{ accepts } w\}$$

Theorem

The language A_{TM} is recognizable.

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Theorem

The language A_{TM} is recognizable.

Proof: Construct a recognizer U for A_{TM}.

$U =$ "On input $\langle M, w\rangle$:

1. Simulate M on w.
2. If M accepted then U <u>accepts</u>.
   If M rejected then U <u>rejects</u>."

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$A_{TM} \stackrel{\text{def}}{=} \{\langle M, w\rangle \mid$ M is a TM, w a string, and M accepts $w\}$

Theorem

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2. If M accepted then U accepts.
   If M rejected then U rejects."

Remarks:

- The machine U (also called the universal TM) is not a decider.
- It is a "programmable" TM, think of $\langle M, w\rangle$ as a program with input data.

Co-Recognizable Languages

Intuition for a recognizable language L (recognized by TM M):

- If $w \in L$ then M run on w will halt in accept state.
- If $w \notin L$ then M run on w will reject or loop.

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Definition (Co-Recognizable Language)

A language L is co-recognizable if $\overline{L}$ is recognizable.

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A language is decidable if and only if it is recognizable and co-recognizable.

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Proof:
"$\Rightarrow$": Every decidable language is recognizable and decidable languages are closed under complement.
"$\Leftarrow$": Assume that L is recognizable and co-recognizable. Hence there exists a recognizer M_1 for L and a recognizer M_2 for $\overline{L}$. We construct a decider M for L.
$M =$ "On input x:

1. Run M_1 on x and M_2 on x in parallel.
2. If M_1 accepted then M <u>accepts</u>.
   If M_2 accepted then M <u>rejects</u>."

Notice that M terminates on any input x as either $x \in L$ or $x \in \overline{L}$.

$\overline{A_{TM}}$ Is not Recognizable

Corollary

The language $\overline{A_{TM}}$ is co-recognizable but not recognizable.

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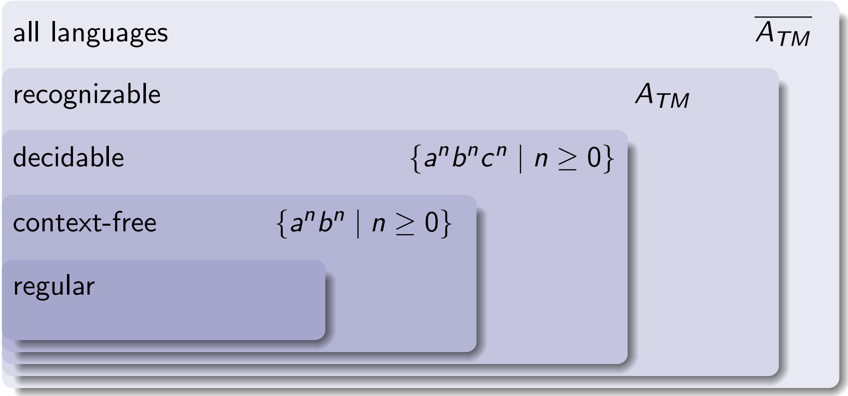
Proof: Co-recognizability follows directly from the definition. Unrecognazibility is proved by contradiction:

- Assume that $\overline{A_{TM}}$ is recognizable.
- Because we know that A_{TM} is recognizable, our theorem implies that $\overline{A_{TM}}$ and A_{TM} are both decidable.
- But we know that A_{TM} is not decidable. This is a contradiction, hence $\overline{A_{TM}}$ cannot be recognizable.

$\square$

Overview (Strict Hierarchy of Language Classes)



Exam Questions

- The language A_{TM} and its undecidability (including proof).
- Definition of co-recognizable languages.
- Proof that A_{TM} is recognizable and $\overline{A_{TM}}$ is co-recognizable.
- Theorem that language if decidable iff it is recognizable and co-recognizable.
- The language $\overline{A_{TM}}$ is not recognizable.