Computability and Complexity

Lecture 11

NP-completeness
Cook-Levin Theorem (SAT is NP-complete)

given by Jiri Srba

What Is the Hardest Problem in NP?

Question:

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This is indeed possible and *SAT* is one example of such a problem. In fact there are many other problems in NP that are difficult for the class NP (*CLIQUE*, *HAMPATH*, *SUBSET-SUM*, *3SAT*, ...). We call them NP-complete problems.

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Consequence:

- If just one NP-complete problem can be solved in P then all other problems in NP will be solvable in P (hence P=NP).
- If P $\neq$ NP then none of the NP-complete problems is solvable in deterministic polynomial time.

NP-Completeness

Definition (NP-Completeness)

A language B is NP-complete iff

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If B is NP-complete and $B \in \mathrm{P}$,
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Theorem

If B is NP-complete, $B \leq_P C$, and $C \in \mathrm{NP}$,
then C is NP-complete.

Proof: Because $\leq_P$ is transitive (see the tutorial).

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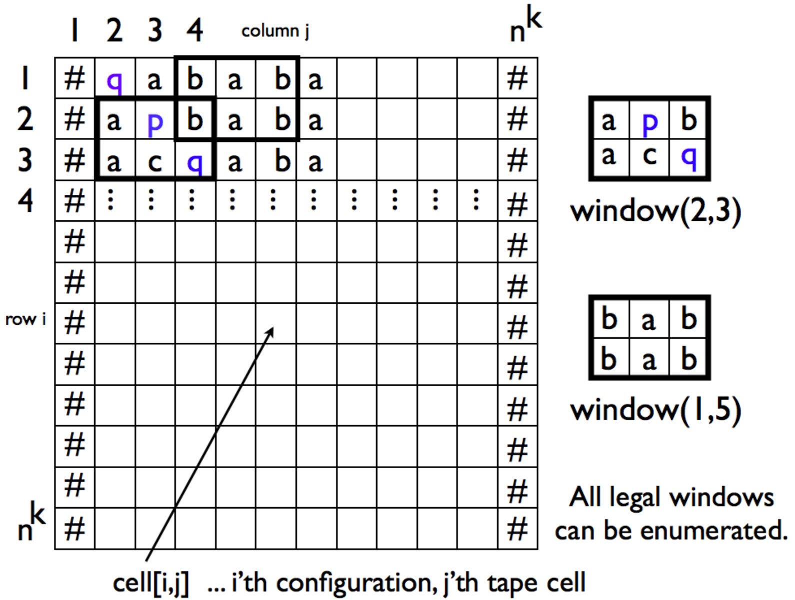
Our aim: for any input string w construct in polynomial time a Boolean formula ϕ such that

M accepts w if and only if ϕ is satisfiable.

Table of Configurations of M Run on *ababa*



Boolean Formula Describing Table of Configurations

Assume a table of configurations when M is run on $w = w_1 \dots w_n$. We will construct a formula ϕ such that M accepts w iff $\phi \in SAT$.

$$\phi \stackrel{\text{def}}{=} \phi_{cell} \wedge \phi_{start} \wedge \phi_{accept} \wedge \phi_{move}$$

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The set of variables contains:

$$x_{i,j,s}$$

where $1 \leq i, j \leq n^k$ and $s \in C$ is a tape symbol or a control state.
Note: There are only polynomially many variables w.r.t. to n.

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Intuition:

Variable $x_{i,j,s}$ is true if and only if cell$[i,j]$ contains the symbol s.

Definition of ϕ_{cell}

Every cell$[i, j]$ contains exactly one symbol s.

$$\phi_{cell} \stackrel{\text{def}}{=} \bigwedge_{1 \leq i,j \leq n^k} \left[\left(\bigvee_{s \in C} x_{i,j,s} \right) \wedge \bigwedge_{s,t \in C, s \neq t} \left(\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}} \right) \right]$$

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Note: ϕ_{cell} is of polynomial size w.r.t. to n.

Definition of ϕ_{start}

The first row contains the initial configuration $q_0 w_1 \ldots w_n$.

$$\phi_{start} \stackrel{\text{def}}{=} x_{1,1,\#} \;\wedge\; x_{1,2,q_0} \;\wedge\; x_{1,3,w_1} \;\wedge\; x_{1,4,w_2} \;\wedge\; \ldots\; x_{1,n+2,w_n} \;\wedge$$
$$x_{1,n+3,\sqcup} \wedge \ldots x_{1,n^k-1,\sqcup} \wedge x_{1,n^k,\#}$$

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There is an accepting configuration in the table.

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Definition of ϕ_{move}

Every window in the table is legal.

Let LW denote the set of all legal windows.

$$\phi_{move} \stackrel{\text{def}}{=} \bigwedge_{1 \leq i < n^k, 1 < j < n^k} \text{legal-window}(i, j)$$

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$$\bigvee_{\frac{a_1, a_2, a_3}{a_4, a_5, a_6} \in LW} x_{i,j-1,a_1} \wedge x_{i,j,a_2} \wedge x_{i,j+1,a_3} \wedge x_{i+1,j-1,a_4} \wedge x_{i+1,j,a_5} \wedge x_{i+1,j+1,a_6}$$

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Cook-Levin Theorem — Summary

- Let $A \in \mathrm{NP}$ be decided by poly-time nondeterministic TM M.
- For every $w \in \Sigma^*$ we constructed in polynomial time a formula $\phi \stackrel{\text{def}}{=} \phi_{cell} \wedge \phi_{start} \wedge \phi_{accept} \wedge \phi_{move}$.
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Corollary

The language *3SAT* is NP-complete.

Proof: A small modification of the proof for SAT. $\square$

Exam Questions

- Definition of NP-completeness.
- Theorems about NP-completeness.
- *SAT* and *3SAT* are NP-complete.