Tutorial 7 (extra)

Exercise 1 (highly recommended)

Consider the following claim that some of you used during the second test.

Claim: If a language is co-recognizable then it cannot be decidable.

Does this claim hold or not? Give arguments for your answer.

Solution:
The claim is of course wrong. If a language is co-recognizable then it can still be decidable (but not all co-recognizable languages are of course decidable). In fact, every decidable language is by definition also recognizable and co-recognizable.

You might have gotten the impression that if a problem is co-recognizable then it is "sort of more difficult than being recognizable", but this is not the right intuition. In fact both recognizable and co-recognizable problems are essentially on the same "undecidability level", they just differ in the point whether we have a recognizer that accepts the positive or the negative instances of the problem.

Exercise 2 (for further practice on the simplest reductions)

1. Prove that $HALT_{TM} \leq_m A_{TM}$. (Note that your task is to find a mapping reduction in the opposite direction than the one provided in the book in Example 5.24 on page 236).

2. Prove that the language

$$EPSILON_{TM} \stackrel{\text{def}}{=} \{\langle M \rangle \mid M \text{ is a TM and } \epsilon \in L(M)\ \}$$

   is undecidable. Is $EPSILON_{TM}$ recognizable? Is $EPSILON_{TM}$ co-recognizable?

3. Prove that E_{TM} is undecidable. First, define the problem and then provide either the standard reduction or mapping reduction from a suitable undecidable problem.

Solution:

1. We will prove that $HALT_{TM} \leq_m A_{TM}$ by constructing a computable function f which on input $\langle M, w \rangle$ returns $\langle M', w \rangle$ such that

$$\langle M, w \rangle \in HALT_{TM} \text{ if and only if } \langle M', w \rangle \in A_{TM}.$$

   The idea is that we modify the machine M into a new machine M' such that M' will accept w if and only if M halted on w. The following TM M_f computes the function f.

   M_f = "On input $\langle M, w \rangle$:
   1. Construct the following machine M':
      M' = "On input x:
      1. Run M on x.
      2. If M accepted, then $\underline{M' \text{ accepts}}$.
      3. If M rejected, then $\underline{M' \text{ accepts}}$."
   2. Output $\langle M', w \rangle$."

   Clearly M_f computes the function f with the required property. Hence $HALT_{TM}$ is mapping reducible to A_{TM}.

2. Complete analogy with Exercise 3 from Exercise Set 5 (just replace 0010 with ϵ).

3. Lecture 5, slide 7.

Exercise 3 (for even further practice)

Prove that the problem

$$INFINITE_{TM} \stackrel{\text{def}}{=} \{\langle M \rangle \mid \ M \text{ is a TM and } L(M) \text{ contains infinitely many strings } \}$$

is undecidable.

Exercise 4 (only for ”feinschmeckers”)

The credit for this exercise goes to Morten Dahl and Morten Kühnrich.

Assume that $L \subseteq \{0, 1\}^*$ is an undecidable language. Prove that $L' \stackrel{\text{def}}{=} L \cup F$ remains undecidable for any finite language $F \subseteq \{0, 1\}^*$. Is this the case also if we allow F to be an infinite language?

Hint: Prove the undecidability claim by reduction from L to L'. The details of the proof are rather delicate and in some sense a part of the proof is non-constructive.