Machine Intelligence

4. Background for Reasoning Under Uncertainty-probabilities

A Reminder of some basic probability concepts

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Summary

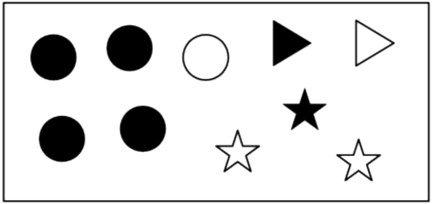
- The following slides recap some basic concepts in probability that you likely are already familiar with.
- During the lecture we will go quickly over these concepts, so please, go over these slides beforehand so that you don't get lost during the lecture.

Probability theory is built on the foundation of variables and worlds.

Worlds described by the variables:

- Filled: {true, false}
- Shape: {circle, triangle, star}

as well as position.



Probability measures

Ω: set of all possible worlds (for a given, fixed set of variables). A **probability measure over** Ω, is a function P, that assigns **probability values**

$$P(\Omega') \in [0, 1]$$

to subsets $\Omega' \subseteq \Omega$, such that

Axiom 1: $P(\Omega) = 1$.

Axiom 2: if $\Omega_1 \cap \Omega_2 = \emptyset$, then $P(\Omega_1 \cup \Omega_2) = P(\Omega_1) + P(\Omega_2)$.

Simplification for finite Ω

If all variables have a finite domain, then

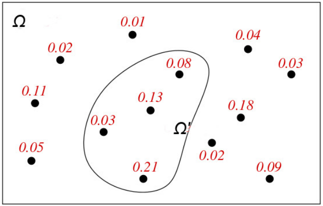
- Ω is finite, and
- a probability distribution is defined by assigning a probability value

$$P(\omega)$$

  to each individual possible world $\omega \in \Omega$.

For any $\Omega' \subseteq \Omega$ then

$$P(\Omega') = \sum_{\omega \in \Omega'} P(\omega)$$



$$P(\Omega') = 0.08 + 0.13 + 0.03 + 0.21 = 0.45$$

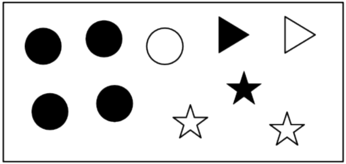
From now on, we will only consider variables with finite domains.

Probabilities of Propositions

A probability distribution over possible worlds defines probabilities for propositions α:

$$P(\alpha) = \sum_{\omega \in \Omega : \alpha \text{ is true in } \omega} P(\omega)$$

Example



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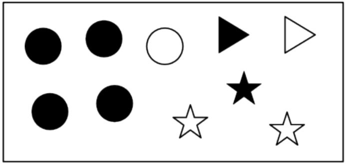

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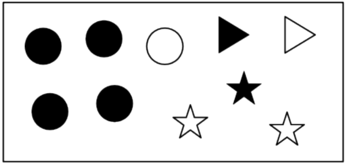

Assume probability for each world is 0.1:

- $P(\mathit{Shape} = \mathit{circle}) = 0.5$
- $P(\mathit{Filled} = \mathit{false}) = 0.4$
- $P(\mathit{Shape} = c \wedge \mathit{Filled} = f) =$

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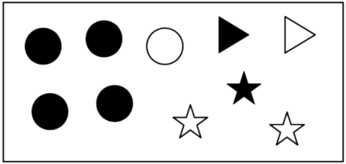

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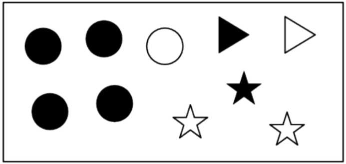

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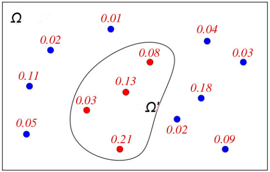
Another example



$$\begin{aligned} P(\mathit{Color} = \mathit{red}) &= 0.08 + 0.13 + 0.03 + 0.21 \\ &= 0.45 \end{aligned}$$

Axiom

If $\mathcal{A}$ and $\mathcal{B}$ are disjoint, then $P(\mathcal{A} \cup \mathcal{B}) = P(\mathcal{A}) + P(\mathcal{B})$.

Example

Consider a deck with 52 cards. If $\mathcal{A} = \{2, 3, 4, 5\}$ and $\mathcal{B} = \{7, 8\}$, then

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More generally

If $\mathcal{C}$ and $\mathcal{D}$ are not disjoint, then $P(\mathcal{C} \cup \mathcal{D}) = P(\mathcal{C}) + P(\mathcal{D}) - P(\mathcal{C} \cap \mathcal{D})$.

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If $\mathcal{C} = \{2, 3, 4, 5\}$ and $\mathcal{D} = \{\spadesuit\}$, then

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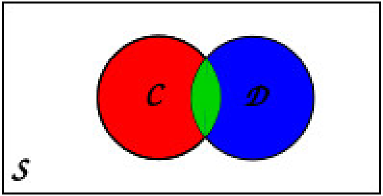

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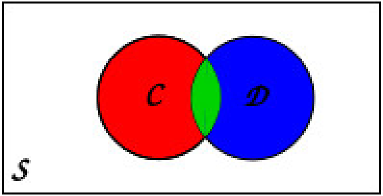
Example

If $\mathcal{C} = \{2, 3, 4, 5\}$ and $\mathcal{D} = \{\spadesuit\}$, then

$$P(\mathcal{C} \cup \mathcal{D}) = \frac{4}{13} + \frac{1}{4} - \frac{4}{52} = \frac{25}{52}.$$



Definition

The **conditional probability** of proposition p given e is $P(p \mid e) = \frac{P(p \wedge e)}{P(e)}$

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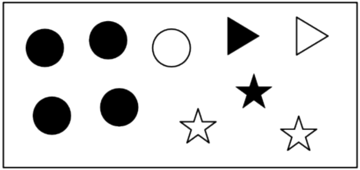

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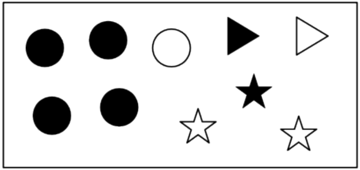

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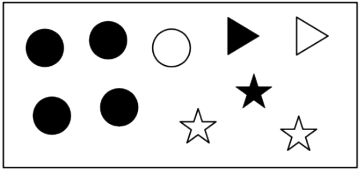

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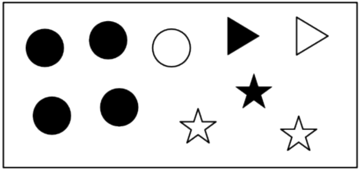

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$$P(\mathit{S{=}circle} \mid \mathit{Fill} = f \wedge \mathit{S{=}star}) = \frac{P(\mathit{S{=}star} \wedge \mathit{S{=}circle} \wedge \mathit{Fill} = f)}{P(\mathit{Fill} = f \wedge \mathit{S{=}star})} = \frac{0}{0.2} = 0$$

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Example: Two dice rolls are independent!

- $P(\mathit{Dice1} = 6 \wedge \mathit{Dice2} = 6) = 1/36$.