Machine Intelligence

6. Reasoning Under Uncertainty, Part III: Inference in Bayesian networks

Putting the Machinery to Practical Use

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Our Agenda for This Topic

→ Our treatment of the topic "Probabilistic Reasoning" consists of Chapters 4-6.

- **Chapter 4**: All the basic machinery at use in Bayesian networks.

  → Sets up the framework and basic operations.

- **Chapter 5**: Bayesian networks: What they are and how to build them.

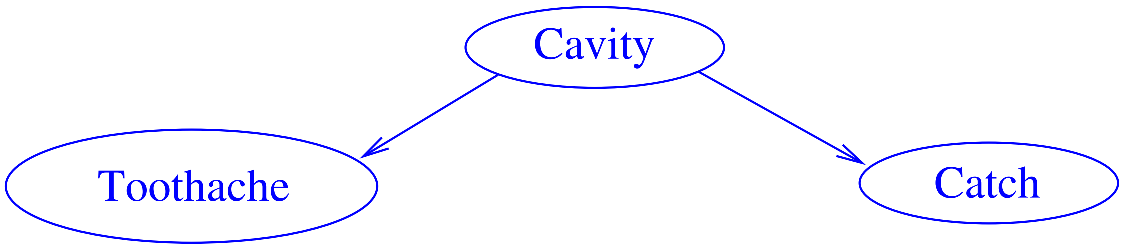
  → The most wide-spread and successful practical framework for probabilistic reasoning.

- **This Chapter**: Bayesian networks: how to use them.

  → How to use Bayesian Networks to answer our questions.

Reminder: Our Machinery

1. **Graph captures variable dependencies:** (Variables $X_1, \dots, X_n$)



$\rightarrow$ Given evidence $\mathbf{e}$, want to know $\mathbf{P}(X \mid e)$. Remaining vars: $\mathbf{Y}$.

2. **Normalization+Marginalization:**

$$\mathbf{P}(X \mid \mathbf{e}) = \alpha \mathbf{P}(X, \mathbf{e}) = \alpha \sum_{\mathbf{y} \in \mathbf{Y}} \mathbf{P}(X, \mathbf{e}, \mathbf{y})$$

$\rightarrow$ A sum over atomic events!

3. **Chain rule:** $X_1, \dots, X_n$ consistently with dependency graph.

$$\mathbf{P}(X_1, \dots, X_n) = \mathbf{P}(X_n \mid X_{n-1}, \dots, X_1) * \mathbf{P}(X_{n-1} \mid X_{n-2}, \dots, X_1) * \dots * \mathbf{P}(X_1)$$

4. **Exploit conditional independence:** Instead of $\mathbf{P}(X_i \mid X_{i-1}, \dots, X_1)$, we can use $\mathbf{P}(X_i \mid Parents(X_i))$.

$\rightarrow$ Bayesian networks!

Reminder: Recovering the Full Joint Probability Distribution

"A Bayesian network is a methodology for representing the full joint probability distribution."

$\rightarrow$ How to recover the full joint probability distribution $\mathbf{P}(X_1, \dots, X_n)$ from $BN = (\{X_1, \dots, X_n\}, E)$?

Chain rule: For any ordering $X_1, \dots, X_n$, we have:

$$\mathbf{P}(X_1, \dots, X_n) = \mathbf{P}(X_n \mid X_{n-1}, \dots, X_1)\mathbf{P}(X_{n-1} \mid X_{n-2}, \dots, X_1) \dots \mathbf{P}(X_1)$$

Choose $X_1, \dots, X_n$ consistent with BN: $X_j \in Parents(X_i) \implies j < i$.

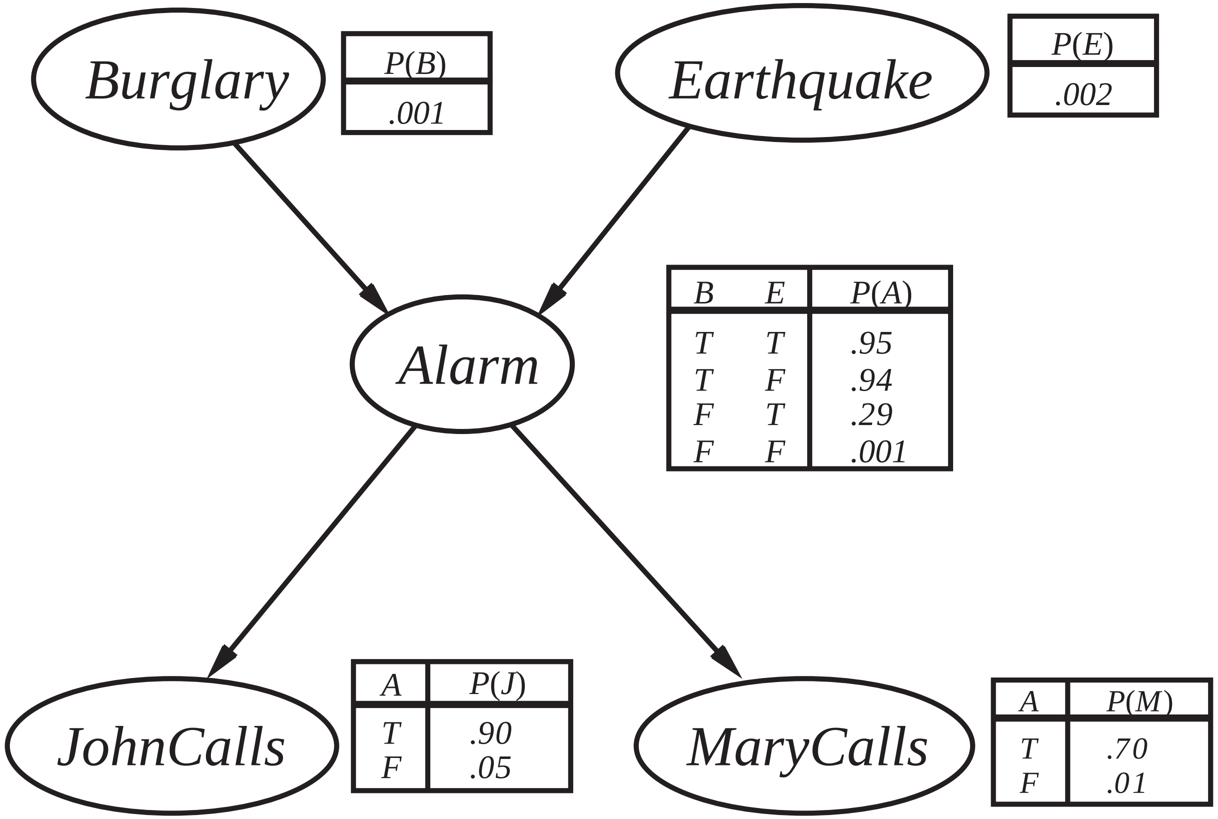
Exploit conditional independence: With BN assumption (A), instead of $\mathbf{P}(X_i \mid X_{i-1} \dots, X_1)$ we can use $\mathbf{P}(X_i \mid Parents(X_i))$:

$$\mathbf{P}(X_1, \dots, X_n) = \Pi_{i=1}^{n}\mathbf{P}(X_i \mid Parents(X_i))$$

The distributions $\mathbf{P}(X_i \mid Parents(X_i))$ are given by BN assumption (B).

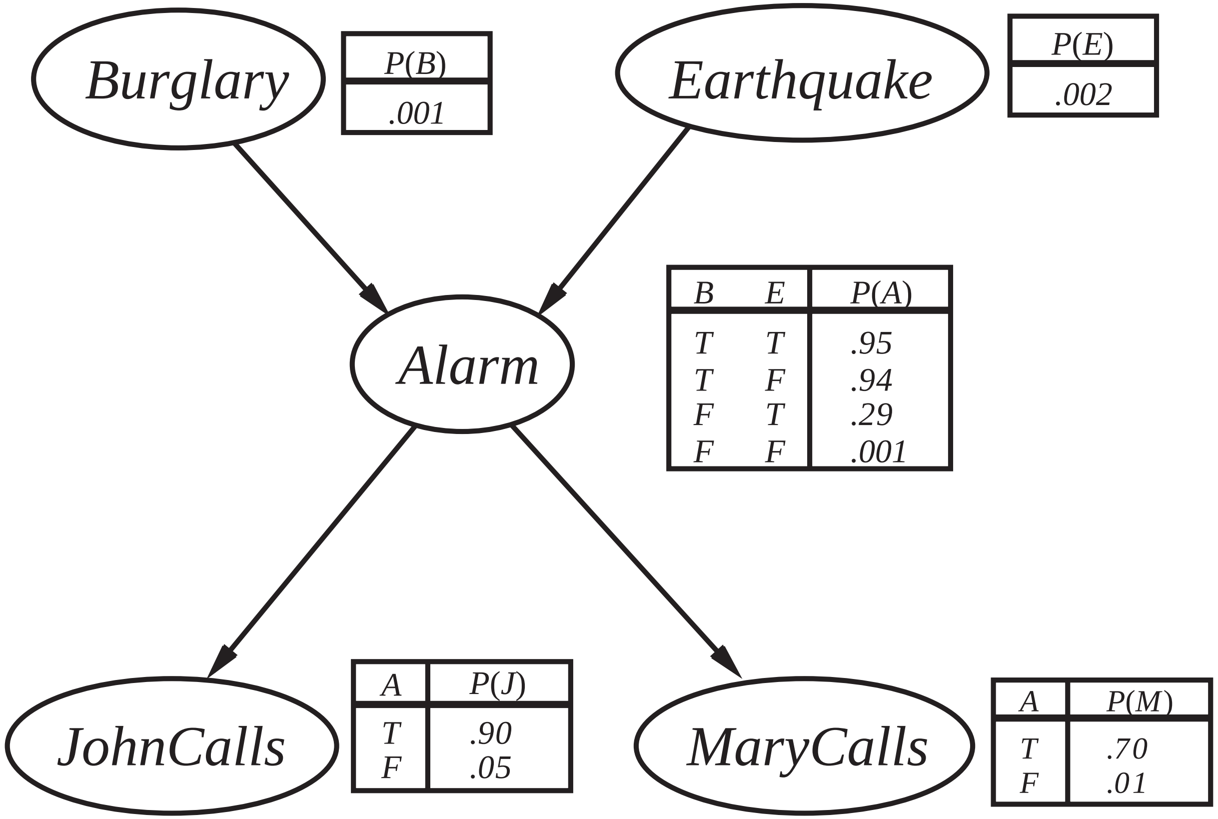
$\rightarrow$ Same for atomic events $P(x_1, \dots, x_n)$.

Reminder: Recovering a Probability for John, Mary, and the Alarm



$$P(j, m, a, \neg b, \neg e) =$$
$$=$$
$$=$$

Recovering a Probability for John, Mary, and the Alarm



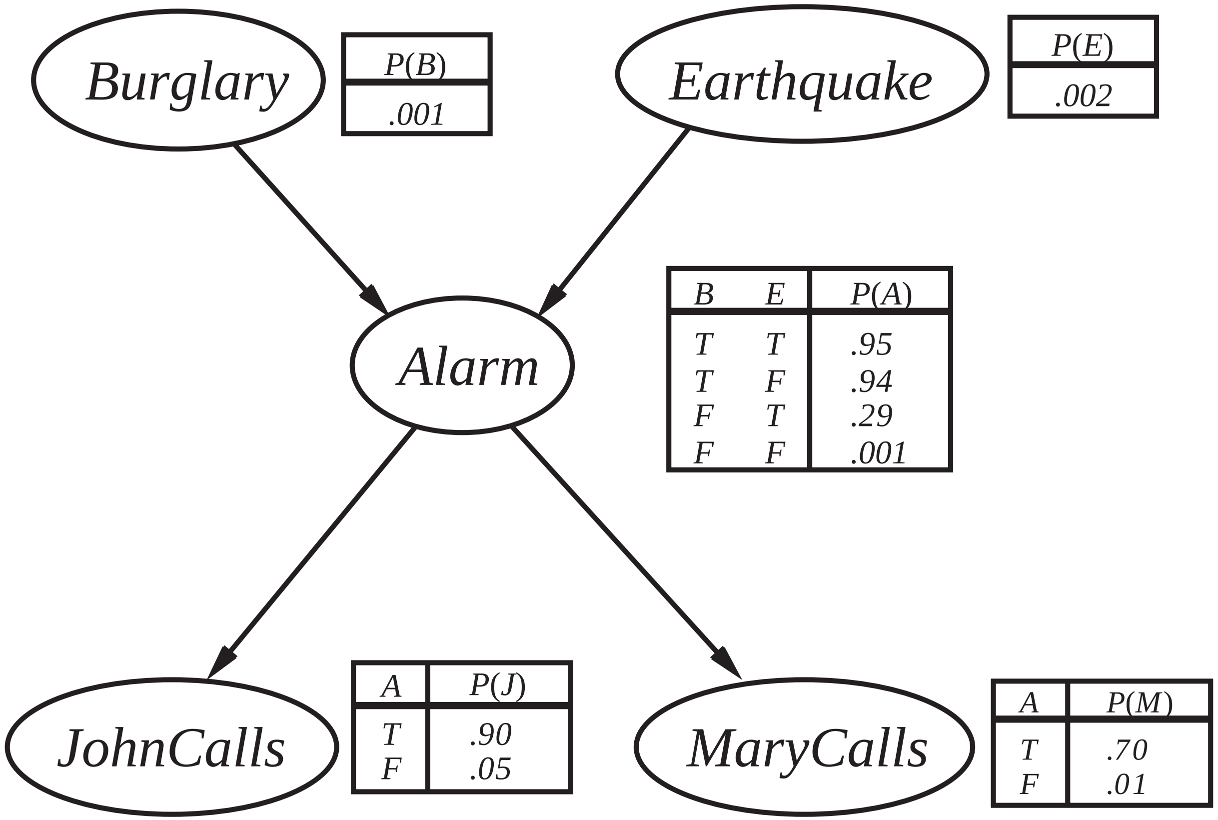
$$
\begin{aligned}
P(j, m, a, \neg b, \neg e) &= \\
&= \\
&=
\end{aligned}
$$

Our Agenda for This Chapter

- **Probabilistic Inference Tasks**: What problems do we want to solve?
  →Before doing algorithms that answer, we need to understand what the questions are about!
- **Exact Inference: Naive Enumeration**: A basic method to solve the probabilistic inference tasks
  →A first basic method we can use (but does not scale very well)
- **Exact Inference: Variable Elimination**: A more clever way of using the structure of the Bayesian Network to our advantage!
  →How to obtain the exact answer to our questions !
- **Naive Bayes Models**: A very simple Bayesian model that can always be computed quickly
  →A very simple case !
- **Approximate Inference via Sampling:** What to do when computing the exact answer is too expensive?
  →We can approximate the solution via sampling methods!

Inference for Mary and John

→ Observe evidence variables and draw conclusions on query variables.



What is $\mathbf{P}(Burglary \mid johncalls)$?

What is $\mathbf{P}(Burglary \mid johncalls, marycalls)$?

Probabilistic Inference Tasks in Bayesian Networks

Definition (Probabilistic Inference Task). *Given random variables $X_1, \dots, X_n$, a probabilistic inference task consists of a set $\mathbf{X} \subseteq \{X_1, \dots, X_n\}$ of query variables, a set $\mathbf{E} \subseteq \{X_1, \dots, X_n\}$ of evidence variables, and an event $\mathbf{e}$ that assigns values to $\mathbf{E}$. We wish to compute the posterior probability distribution $\mathbf{P}(\mathbf{X} \mid \mathbf{e})$.*

Notes:

- $\mathbf{Y} := \{X_1, \dots, X_n\} \setminus (\mathbf{X} \cup \mathbf{E})$ are the hidden variables.
- We assume that a BN for $X_1, \dots, X_n$ is given.
- In the remainder, for simplicity, $\mathbf{X} = \{X\}$ is a singleton.

Example: In $\mathbf{P}(Burglary \mid johncalls, marycalls)$, $X = Burglary$, $\mathbf{e} = johncalls, marycalls$, and $\mathbf{Y} = \{Alarm, EarthQuake\}$.

Simplifying the Problem by Using Normalization

According to the definition of conditional probability:

$$P(X \mid \mathbf{E} = \mathbf{e}) = \frac{P(X, \mathbf{E} = \mathbf{e})}{P(\mathbf{E} = \mathbf{e})}$$

It is sufficient to compute for each $x \in D_X$ the value

$$P(X = x, \mathbf{E} = \mathbf{e}).$$

Together with

$$P(\mathbf{E} = \mathbf{e}) = \sum_{x \in D_X} P(X = x, \mathbf{E} = \mathbf{e})$$

this gives the desired posterior distribution.

$\rightarrow$ To predict the desired conditional probability $P(X \mid \mathbf{E} = \mathbf{e})$, it suffices for us to compute the probability of (non-atomic) events: $P(X \mid \mathbf{E} = \mathbf{e})$!

Notation: As $\alpha = \frac{1}{\sum_{x \in D_X} P(X=x,\mathbf{E}=\mathbf{e})}$ is easily derived from $P(X \mid \mathbf{E} = \mathbf{e})$, we simply write $P(X \mid \mathbf{E} = \mathbf{e}) = \alpha P(X \mid \mathbf{E} = \mathbf{e})$ instead of $\frac{P(X,\mathbf{E}=\mathbf{e})}{P(\mathbf{E}=\mathbf{e})}$.

Marginalization: Inference as Summation

Want to compute: $P(X = x, \mathbf{E} = \mathbf{e})$

Problem: We do not know the value of hidden variables Y

We sum over all possible values of the hidden variables!

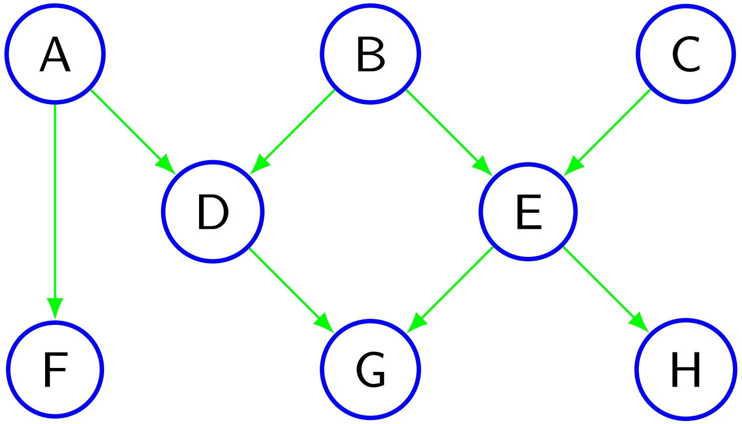
Let X be the variable of interest, $\mathbf{E}$ the evidence variables, and $\mathbf{Y} = Y_1, \dots, Y_l$ the remaining variables in the network not belonging to $X \cup \mathbf{E}$. Then

$$P(X = x, \mathbf{E} = \mathbf{e}) =$$

Simplified notation:

$$P(x, \mathbf{e}) =$$

Naive Enumeration Example



Find $P(B|a,f,g,h) = \frac{P(B,a,f,g,h)}{P(a,f,g,h)}$

We can if we have access to $P(A,B,C,D,E,F,G,H)$

$\rightarrow$ Reminder: We can recover the full joint probability distribution from our BN!

Inserting evidence we get:

$$P(B \mid a,f,g,h) = \alpha \cdot \sum_{C,D,E} P(a,B,C,D,E,f,g,h)$$

and

$$\frac{1}{\alpha} = P(a,f,g,h) = \sum_{\mathrm{B}} P(B,a,f,g,h)$$

Inference by Enumeration: The Principle (A Reminder!)

Given evidence $\mathbf{e}$, want to know $\mathbf{P}(X \mid \mathbf{e})$. Hidden variables: $\mathbf{Y}$.

1. **Bayesian network** BN **captures variable dependencies.**

2. **Normalization+Marginalization.**

$$\mathbf{P}(X \mid \mathbf{e}) = \alpha \mathbf{P}(X, \mathbf{e}); \text{ if } \mathbf{Y} \neq \emptyset \text{ then } \mathbf{P}(X \mid \mathbf{e}) = \alpha \textstyle\sum_{\mathbf{y} \in \mathbf{Y}} \mathbf{P}(X, \mathbf{e}, \mathbf{y})$$

$\rightarrow$ Recover the summed-up probabilities $\mathbf{P}(X, \mathbf{e}, \mathbf{y})$ from BN!

3. **Chain rule.** Order $X_1, \dots, X_n$ consistent with BN.

$$\mathbf{P}(X_1, \dots, X_n) = \mathbf{P}(X_n \mid X_{n-1}, \dots, X_1) \mathbf{P}(X_{n-1} \mid X_{n-2}, \dots, X_1) \dots \mathbf{P}(X_1)$$

4. **Exploit conditional independence.** Instead of $\mathbf{P}(X_i \mid X_{i-1}, \dots, X_1)$, use $\mathbf{P}(X_i \mid Parents(X_i))$.

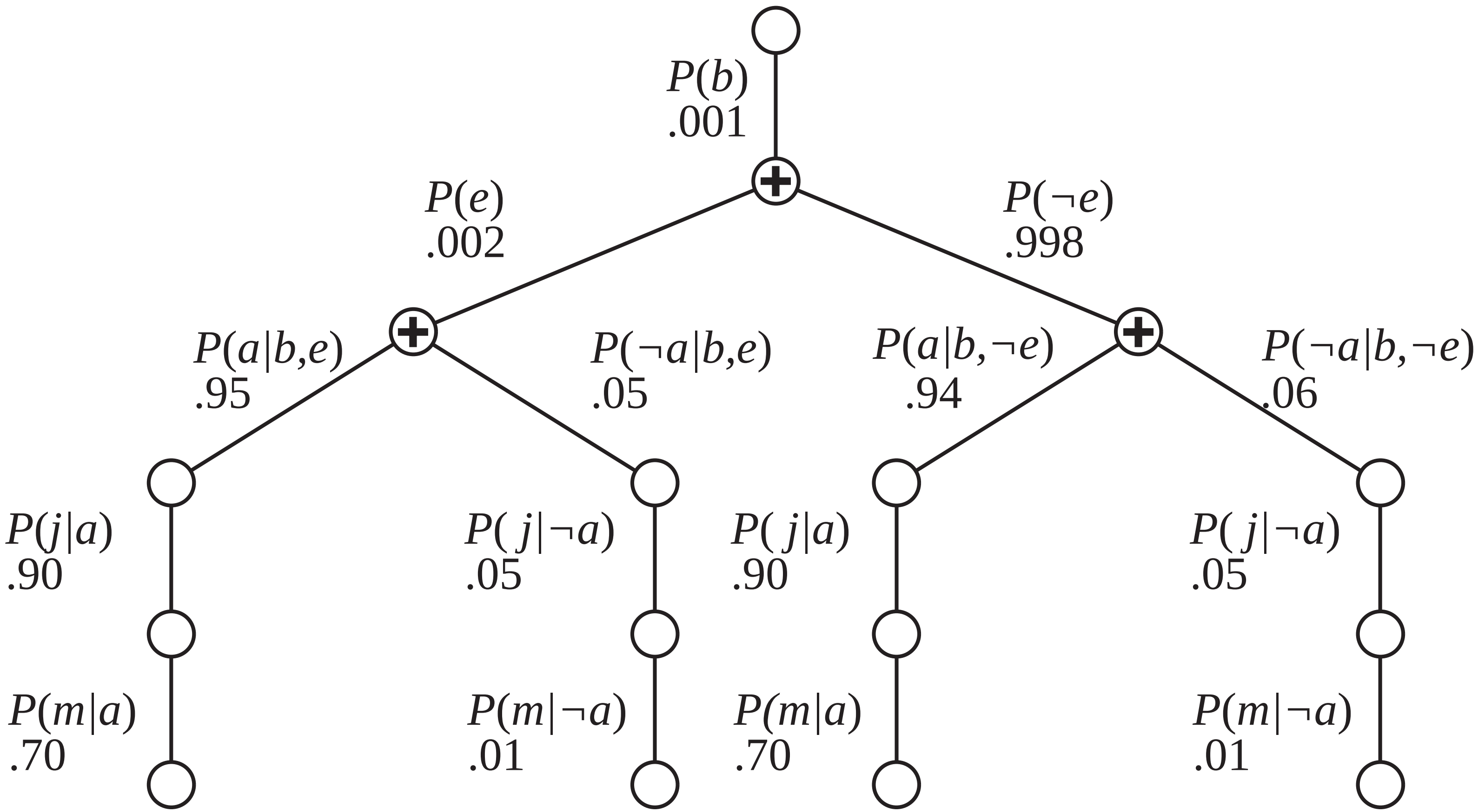
$\rightarrow$ Given a Bayesian network BN, probabilistic inference tasks can be solved as sums of products of conditional probabilities from BN.

$\rightarrow$ Sum over all value combinations of hidden variables.

Inference by Enumeration: John and Mary

Inference by Enumeration: John and Mary, ctd.

The Evaluation of $P(b \mid j, m)$, as a "Search Tree"



$\rightarrow$ Inference by enumeration = a tree with "sum nodes" branching over values of hidden variables, and with non-branching "multiplication nodes".

Inference by Enumeration: Pseudo-Code

→ With $bn.\textsc{Vars}$ being a variable ordering consistent with bn:

function ENUMERATION-ASK(X, **e**, bn) **returns** a distribution over X
  inputs: X, the query variable
    e, observed values for variables **E**
    bn, a Bayes net with variables $\{X\} \cup \mathbf{E} \cup \mathbf{Y}$ /* **Y** = *hidden variables* */

  $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
  for each value x_i of X **do**
    $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL(bn.VARS, $\mathbf{e}_{x_i}$)
      where $\mathbf{e}_{x_i}$ is **e** extended with $X = x_i$
  return NORMALIZE($\mathbf{Q}(X)$)

---

function ENUMERATE-ALL($vars$, **e**) **returns** a real number
  if EMPTY?($vars$) **then return** 1.0
  $Y \leftarrow$ FIRST($vars$)
  if Y has value y in **e**
    then return $P(y \mid parents(Y)) \times$ ENUMERATE-ALL(REST($vars$), **e**)
    else return $\sum_y P(y \mid parents(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}_y$)
      where $\mathbf{e}_y$ is **e** extended with $Y = y$

Inference by Enumeration: Properties

Inference by Enumeration:

- Evaluates the tree in a depth-first manner.
- Space Complexity: Linear in the number of variables.
- Time Complexity:

Bad News: Not in general.

- Probabilistic inference is **#P**-hard.
- **#P** is harder than **NP** (i.e., **NP** $\subseteq$ **#P**).

But: Variable Elimination.

- Improves on inference by enumeration through (A) avoiding repeated computation, and (B) avoiding irrelevant computation.
- In some special cases, variable elimination runs in polynomial time.

Variable Elimination: Sketch of Ideas

(A) Avoiding repeated computation: Evaluate expressions from right to left, storing all intermediate results. For query $P(B \mid j, m)$:

1. CPTs of BN yield factors (probability tables): $\mathbf{P}(B \mid j, m) =$
$$\alpha \underbrace{\mathbf{P}(B)}_{\mathbf{f}_1(B)} \sum_{v_E} \underbrace{P(v_E)}_{\mathbf{f}_2(E)} \sum_{v_A} \underbrace{\mathbf{P}(v_A \mid B, v_E)}_{\mathbf{f}_3(A,B,E)} \underbrace{P(j \mid v_A)}_{\mathbf{f}_4(A)} \underbrace{P(m \mid v_A)}_{\mathbf{f}_5(A)}$$

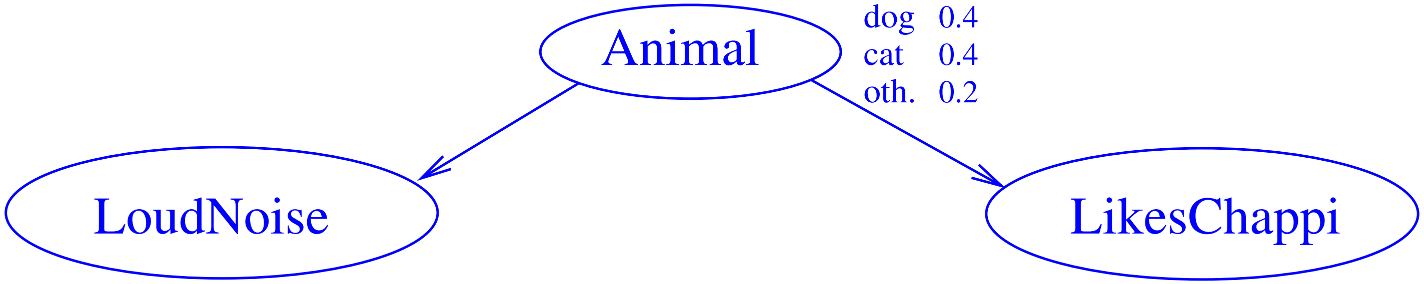
2. Then the computation is performed in terms of factor product and summing out variables from factors: $\mathbf{P}(B \mid j, m) =$
$$\alpha\, \mathbf{f}_1(B) \times \sum_{v_E} \mathbf{f}_2(E) \times \sum_{v_A} \mathbf{f}_3(A, B, E) \times \mathbf{f}_4(A) \times \mathbf{f}_5(A)$$

(B) Avoiding irrelevant computation: Repeatedly remove hidden variables that are leaf nodes. For query $P(JohnCalls \mid burglary)$:

$$\mathbf{P}(J \mid b) = \alpha P(b) \sum_{v_E} P(v_E) \sum_{v_A} P(v_A \mid b, v_E) \mathbf{P}(J \mid v_A) \sum_{v_M} P(v_M \mid v_A)$$

$\rightarrow$ The rightmost sum equals 1 and can be dropped.

Questionnaire



	P(A)
dog	0.4
cat	0.4
oth.	0.2

A	P(LN|A)
dog	0.7
cat	0.2
oth.	0.01

A	P(LC|A)
dog	0.8
cat	0.1
oth.	0.7

Question!

Say BN is the Bayesian network above. How can we compute $P(dog \mid loudnoise)$?

$P(d \mid ln) =$

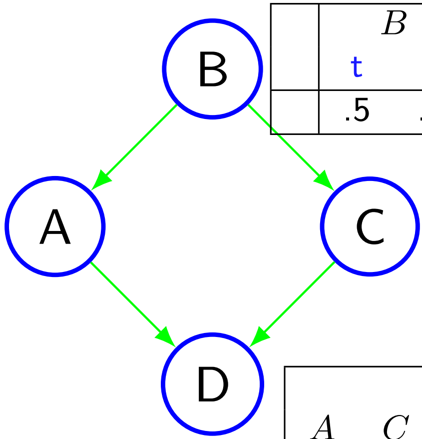
(A): $\alpha \sum_{v_{LC}} P(v_{LC} \mid d) P(ln \mid d)$

(B): $\alpha \sum_{v_{LC}} P(v_{LC} \mid d) P(ln \mid d) P(d)$

(C): $\alpha P(ln \mid d) P(d) \sum_{v_{LC}} P(v_{LC} \mid d)$

(D): $\alpha P(ln \mid d) P(d)$

Example



	B: t	B: f
	.5	.5

B	A: t	A: f
t	.7	.3
f	.1	.9

B	C: t	C: f
t	.7	.3
f	.2	.8

A	C	D: t	D: f
t	t	.9	.1
t	f	.7	.3
f	t	.8	.2
f	f	.4	.6

Naive Enumeration:

$$P(A, D = f) =$$

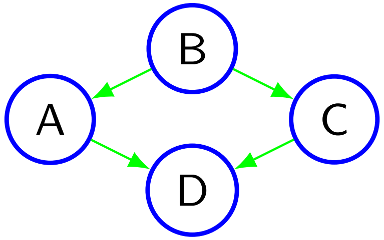
$$\sum_{b \in \{t,f\}} \sum_{c \in \{t,f\}} P(B = b, A, C = c, D = f) =$$

$$\sum_{b \in \{t,f\}} \sum_{c \in \{t,f\}} P(B = b)P(A \mid B = b)P(C = c \mid B = b)P(D = f \mid A, C = c)$$

Observation 1: $P(B = b)P(A \mid B = b)$ does not depend on C, so

$$= \sum_{b \in \{t,f\}} P(B = b)P(A \mid B = b) \sum_{c \in \{t,f\}} P(C = c \mid B = b)P(D = f \mid A, C = c)$$

Example continued



	B	
	t	f
	.5	.5

	A	
B	t	f
t	.7	.3
f	.1	.9

	C	
B	t	f
t	.7	.3
f	.2	.8

		D	
A	C	t	f
t	t	.9	.1
t	f	.7	.3
f	t	.8	.2
f	f	.4	.6

Observation 2: Let's precompute first factor F_1

$$\sum_b P(B=b)P(A \mid B=b) \underbrace{\sum_c P(C=c \mid B=b)P(D=f \mid A, C=c)}_{F_1} =$$

$$\sum_b P(B=b)P(A \mid B=b)F_1(B=b, A) = F_2(A)$$

Is F_1 a single numerical value?

Variable Elimination

The procedure operates on **factors**: functions of subsets of variables

Definition (Factor). A factor $F(V_1, \dots V_k)$ is a function mapping each combination of values of the variables $V_1, \dots, V_k$ to a number.

$\rightarrow$ Factors are not necessarily CPTs as numbers do not need to sum up to 1.

Required operations on factors:

- Restriction (setting selected variables to specific values)
- Multiplication (join tables and multiply the probabilities)
- Marginalization (summing out selected variables)

Restriction (of variable D to value $D = f$):

A	C	$F(A,C,D)$ D = t	$F(A,C,D)$ D = f
t	t	.9	.1
t	f	.7	.3
f	t	.8	.2
f	f	.4	.6

$\mapsto$

A	C	$F_2(A,C)$
t	t	.1
t	f	.3
f	t	.2
f	f	.6

Variable Elimination

The procedure operates on **factors**: functions of subsets of variables

Definition (Factor). A factor $F(V_1, \dots V_k)$ is a function mapping each combination of values of the variables $V_1, \dots, V_k$ to a number.

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Required operations on factors:

- Restriction (setting selected variables to specific values)
- Multiplication (join tables and multiply the probabilities)
- Marginalization (summing out selected variables)

Multiplication + Marginalization (of variable C):

B	C	$F_1(B,C)$
t	t	.7
t	f	.3
f	t	.2
f	f	.8

A	C	$F_2(A,C)$
t	t	.1
t	f	.3
f	t	.2
f	f	.6

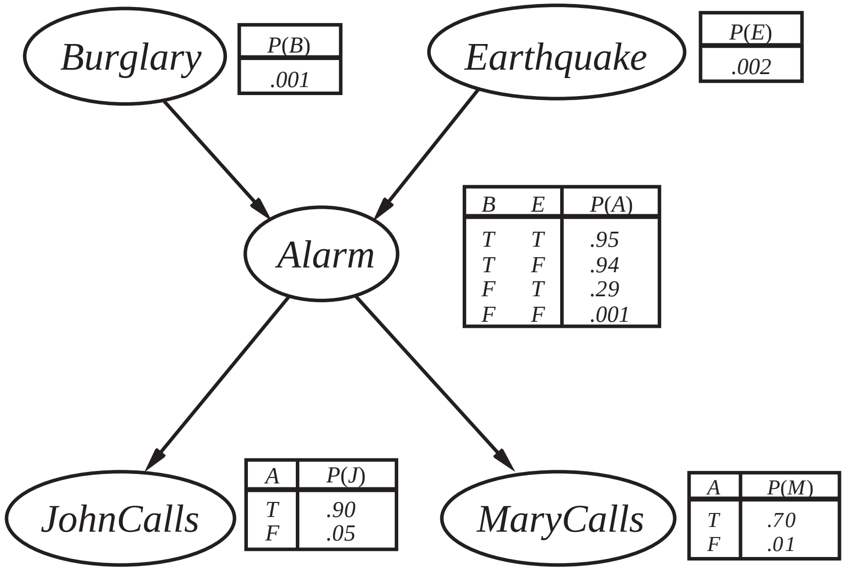
$\mapsto$

b	a	$F(B,A)$
t	t	$.7 \cdot .1 + .3 \cdot .3 = .16$
t	f	$.7 \cdot .2 + .3 \cdot .6 = .32$
f	t	$.2 \cdot .1 + .8 \cdot .3 = .26$
f	f	$.2 \cdot .2 + .8 \cdot .6 = .52$

Variable Elimination Algorithm

1. Factors = CPT in BN
2. For all variables in the evidence, restrict all factors with the observed value
3. Fix any order of the remaining variables, $X_1, \ldots, X_n$.
4. **for** $i := 1, \ldots, n$ **do**
   a. $\mathcal{F} = \{F \mid F \in \text{Factors}, F \text{ depends on } X_i\}$
   b. $T = \pi_{F \in \mathcal{F}}$ (Compute product of factors)
   c. $F_{new} = \text{Marginalize}(T, X_i)$
   d. Factors = Factors $\setminus \mathcal{F} \cup \{N\}$ (Replace all factors $\mathcal{F}$ by new factor N)

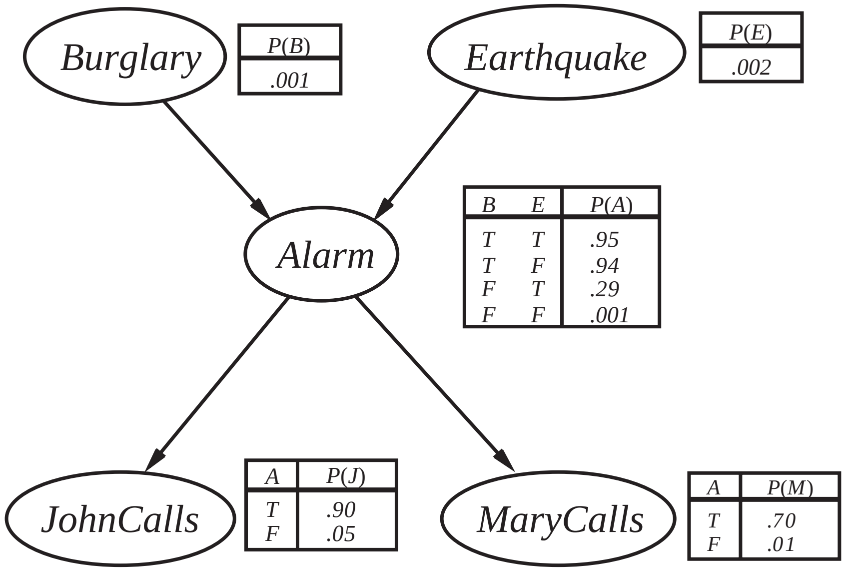
Variable Elimination Algorithm: Alarm Example



Bad ordering for computing $P(M, B = t)$: A, J, E

1. Factors = CPT in BN
2. For all variables in the evidence, restrict all factors with the observed value
3. Fix any order of the remaining variables, $X_1, \dots, X_n$.
4. **for** $i := 1, \dots, n$ **do**
   a. $\mathcal{F} = \{F \mid F \in$ Factors, F depends on $X_i\}$
   b. $T = \pi_{F \in \mathcal{F}}$ (Compute product of factors)
   c. $F_{new} =$ Marginalize(T, X_i)
   d. Factors = Factors $\setminus \mathcal{F} \cup \{N\}$ (Replace all factors $\mathcal{F}$ by new factor N)

Alarm Example



Bad ordering for computing $P(MC, B = t)$: A, J, E

$$\sum_{eq \in \{t,f\}} \sum_{jc \in \{t,f\}} \sum_{a \in \{t,f\}} P(B = t)P(EQ = eq)P(A = a \mid B = t, EQ = eq)P(JC = jc \mid A = a)P(MC \mid A = a) =$$

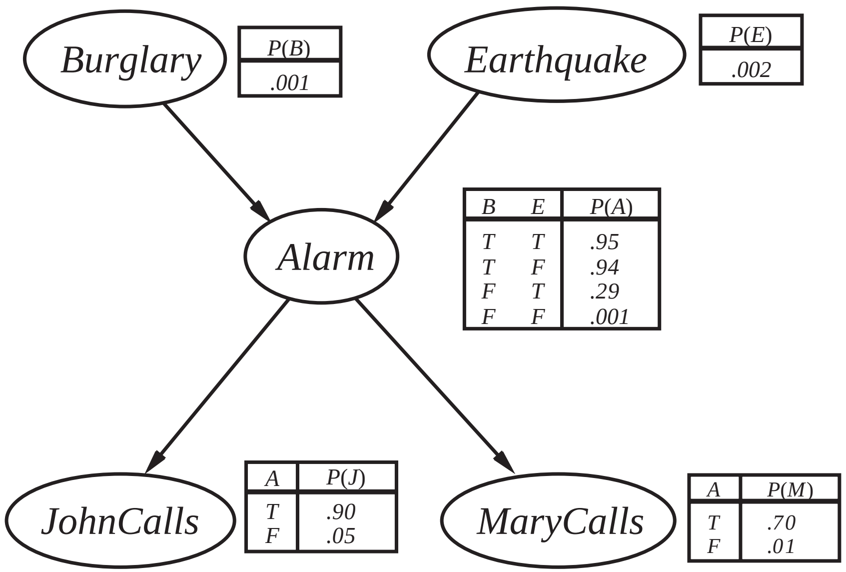
$$\sum_{eq \in \{t,f\}} \sum_{jc \in \{t,f\}} P(B = t)P(EQ = eq)F_1(eq, jc, MC) =$$

$$\sum_{eq \in \{t,f\}} P(B = t)P(EQ = eq)F_2(eq, MC) =$$

$$P(B = t)F_3(MC)$$

Largest factor (F_1) is function of 3 variables!

Alarm Example continued



Question!

What's the maximum number of columns of the largest table generated by variable elimination under ordering: J,E,A?

(A): 1 (B): 2

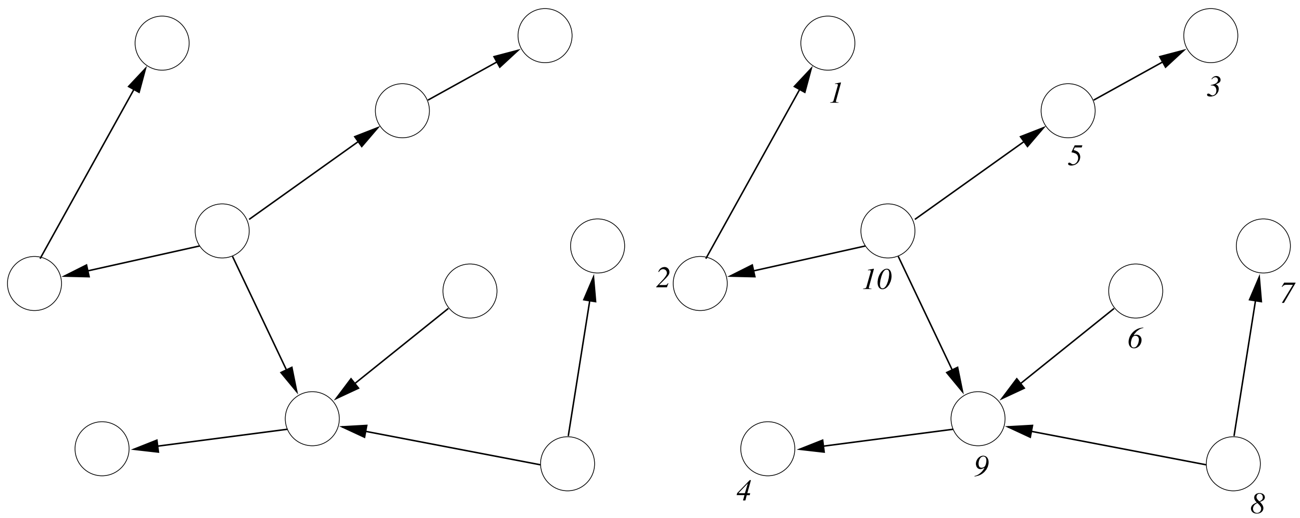
(C): 3 (D): 4

Alarm Example continued

Variable Elimination Runtime

An important easy case:

- A graph is called singly connected, or a polytree, if there is at most one undirected path between any two nodes in the graph.



→ Is our BN for Mary & John a polytree?

For singly connected network: any elimination order that "peels" variables from outside will only create factors of only one variable.

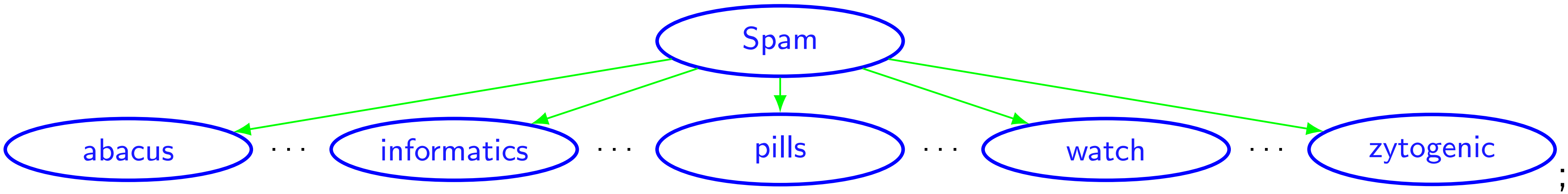
The complexity of inference is therefore linear in the total size of the network (= combined size of all conditional probability tables).

Naive Bayes Model

Example: Spam filter

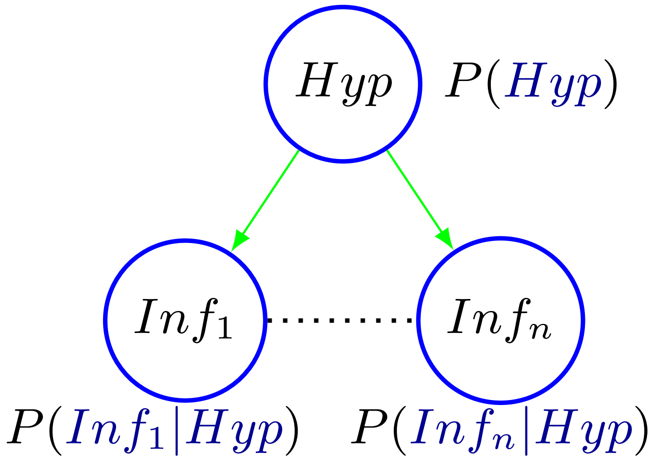
- A single query variable: Spam
- Many observable features (e.g. words appearing in the body of the message): abacus,. . . ,informatics, pills, . . . , watch,. . . , zytogenic

Network Structure:



- Inference with large number of variables possible
- Essentially how Thunderbird spam filter works

Naïve Bayes models



We want the posterior probability of the hypothesis variable Hyp given the observations $\{\text{Inf}_1 = e_1, \ldots, \text{Inf}_n = e_n\}$:

$$P(\text{Hyp}|\text{Inf}_1 = e_1, \ldots, \text{Inf}_n = e_n) = \frac{P(\text{Inf}_1 = e_1, \ldots, \text{Inf}_n = e_n|\text{Hyp})P(\text{Hyp})}{P(\text{Inf}_1 = e_1, \ldots, \text{Inf}_n = e_n)}$$
$$= \alpha \cdot P(\text{Inf}_1 = e_1|\text{Hyp}) \cdot \ldots \cdot P(\text{Inf}_n = e_n|\text{Hyp})P(\text{Hyp})$$

Note: The model assumes that the information variables are independent given the hypothesis variable.

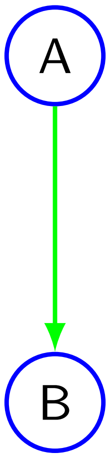
Sampling Inference

Using a BN as a Sample Generator

Observation: can use Bayesian network as random generator that produces states $\mathbf{X} = \mathbf{x}$ according to distribution P defined by the network.

$\rightarrow$ We just sample a random value for each variable always picking a value for parents(X) before picking a value for X

Example:



	A: t	A: f
	.2	.8

A	B: t	B: f
t	.7	.3
f	.4	.6

- Generate random numbers r_A, r_B uniformly from [0,1].
- Set $A = t$ if $r_A \leq .2$ and $A = f$ else.
- Depending on the value of A and r_B set B to t or f.

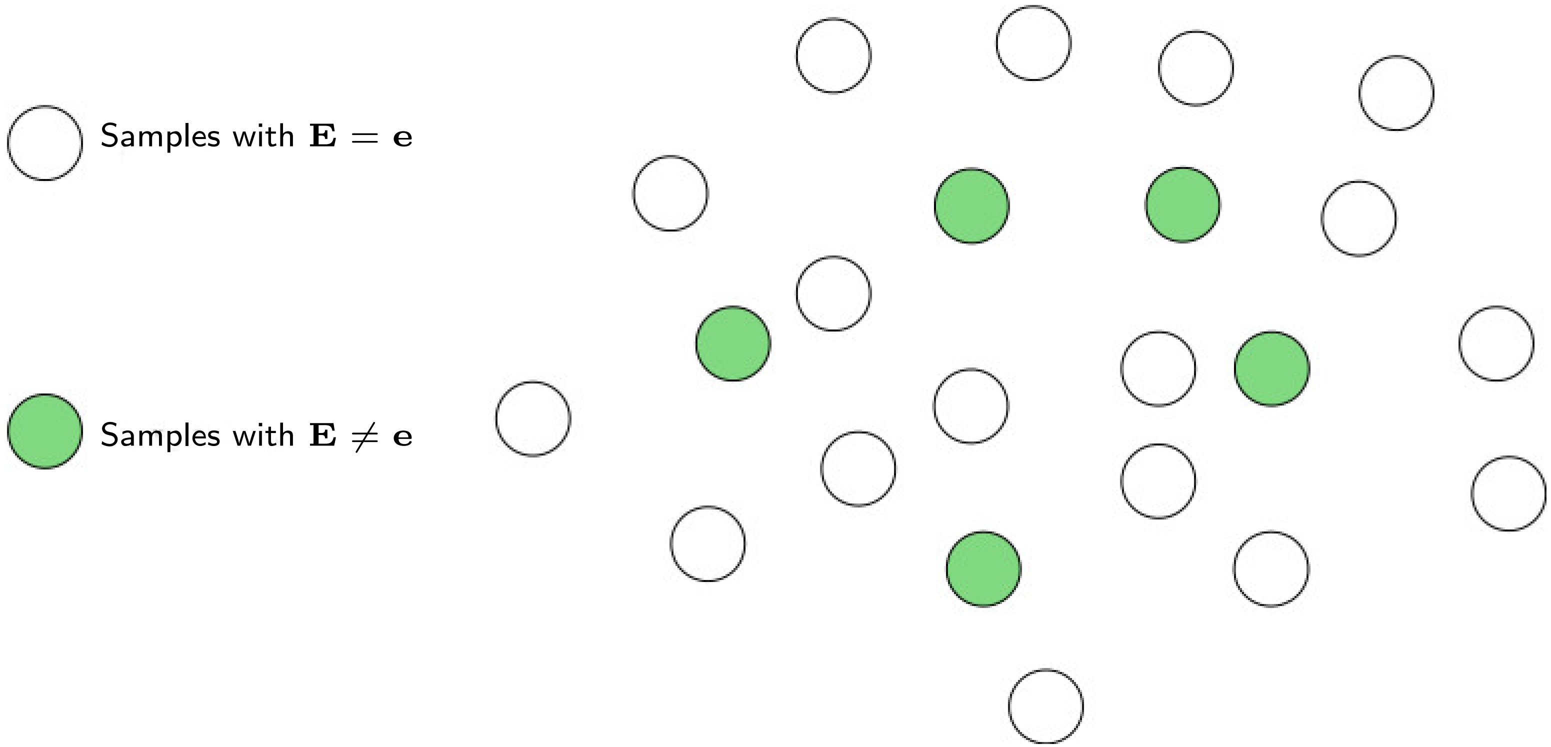
Random generation of one state: linear in size of network.

Sampling Inference

Approximate Inference from Samples

To compute an approximation of $P(\mathbf{E} = \mathbf{e})$ ($\mathbf{E}$ a subset of the variables in the Bayesian network):

- generate a (large) number of random states
- count the frequency of states in which $\mathbf{E} = \mathbf{e}$.



Accuracy

Hoeffding Bound

- p: true probability $P(\mathbf{E} = \mathbf{e})$
- s: estimate for p from sample of size n
- ϵ: an error bound > 0.

Then

$$P(|s - p| > \epsilon) \leq 2e^{-2n\epsilon^2}$$

Required Sample Size

To obtain an estimate that with probability at most δ has an error greater than ϵ, it is sufficient to take

$$n = -ln(\delta/2)/(2\epsilon^2) \text{ samples.}$$

Example

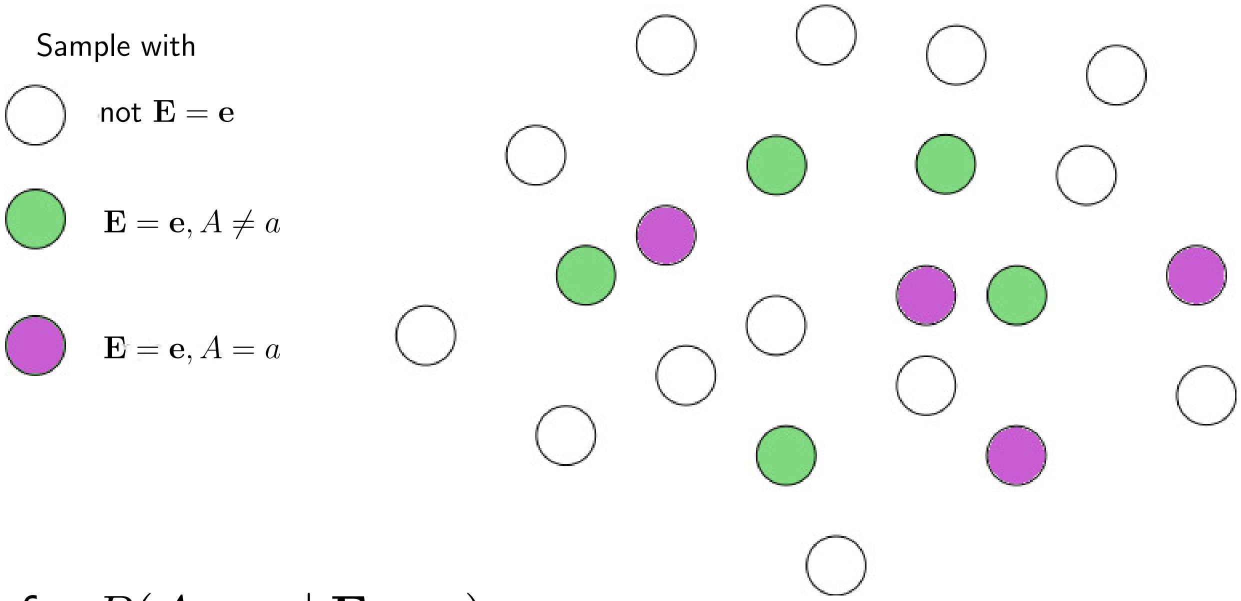
To get an error ϵ of less than 0.1 in 95% of the cases ($\delta = 0.05$), we need:

$$n > -ln(0.05/2)/(2 \cdot 0.1^2) \approx 184 \text{ samples}$$

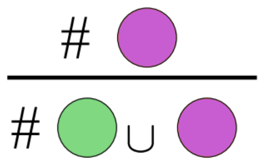
How many samples do we need if the error should be less than 0.01? 18444 samples

Rejection Sampling

The simplest approach: **Rejection Sampling**



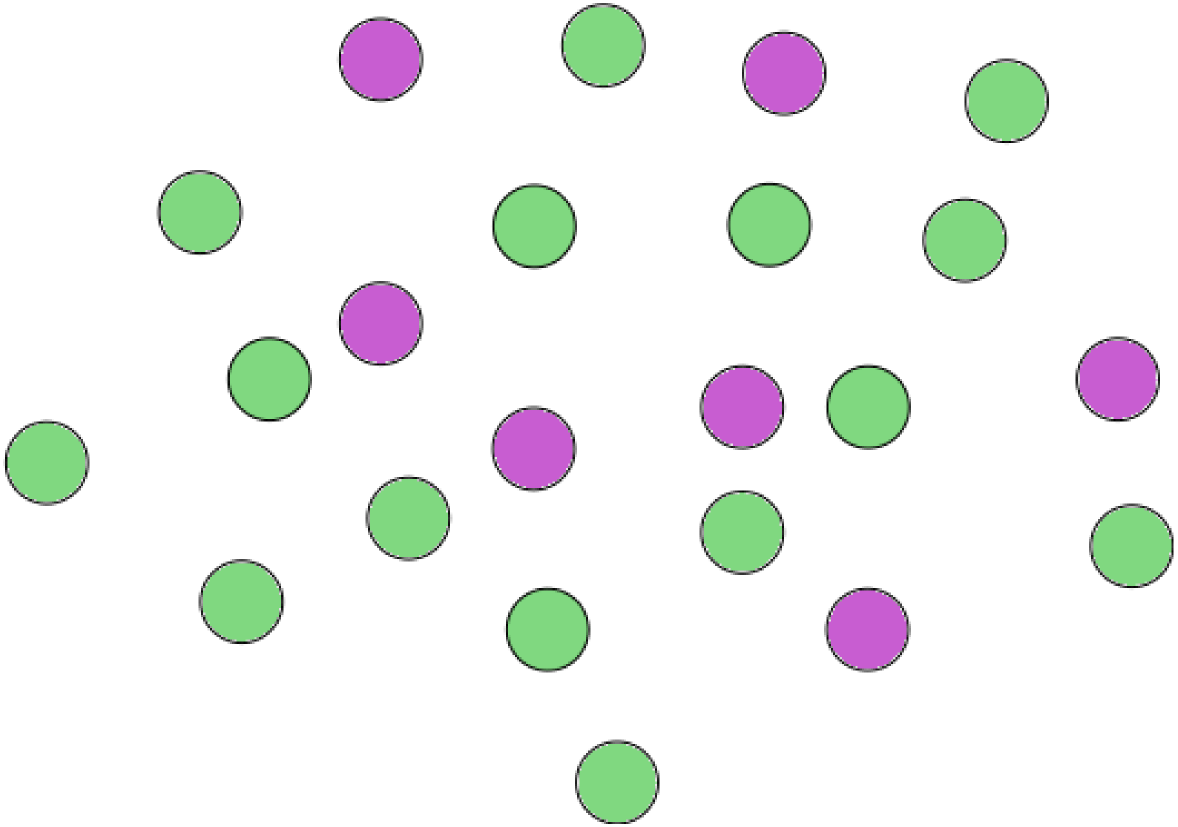
Approximation for $P(A = a \mid \mathbf{E} = \mathbf{e})$:



Sampling from the conditional distribution

Problem with rejection sampling: samples with $\mathbf{E} \neq \mathbf{e}$ are useless!

Ideally: would draw samples directly from the conditional distribution $P(\mathbf{A} \mid \mathbf{E} = \mathbf{e})$.



A Wrong Sampling Method

First idea (not to be followed)

1. Fix evidence variables to their observed states.
2. Sample from non-evidence variables.
3. Count frequency as before

Problem: This gives a sampling distribution

$$\prod_{X \in \mathbf{X} \setminus \mathbf{E}} P(X \mid \mathrm{pa}(X) \setminus \mathbf{E}, \mathrm{pa}(X) \cap \mathbf{E})$$

somewhere between $P(\mathbf{X})$ and $P(\mathbf{X} \mid \mathbf{e})$.

Likelihood Weighting

We would like to sample from

$$P(\mathbf{X}, \mathbf{e}) = \underbrace{\prod_{X \in \mathbf{X} \setminus \mathbf{E}} P(X \mid \mathrm{pa}(X) \setminus \mathbf{E}, \mathrm{pa}(X) \cap \mathbf{E})}_{\text{Part 1}} \cdot \underbrace{\prod_{E \in \mathbf{E}} P(E = e \mid \mathrm{pa}(E) \setminus \mathbf{E}, \mathrm{pa}(E) \cap \mathbf{E})}_{\text{Part 2}}$$

So instead weigh each generated sample with a weight corresponding to Part 2.

Estimate $P(X = x \mid \mathbf{e})$ as

$$\hat{P}(X = x \mid \mathbf{e}) = \frac{\sum_{sample:X=x} w(sample)}{\sum_{sample} w(sample)},$$

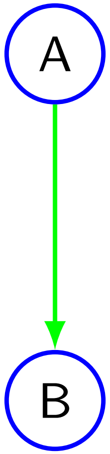
where

$$w(sample) = \prod_{E \in \mathbf{E}} P(E = e \mid \mathrm{pa}(E) = \pi) \qquad \text{(Part 2)}$$

and π is the values of $\mathrm{pa}(E)$ under $sample$ and $\mathbf{e}$.

Likelihood Sampling: Example

Sample state where B=t



	A: t	A: f
	.2	.8

A	B: t	B: f
t	.7	.3
f	.4	.6

- Initialize weight $W = 1$
- Generate random number r_A uniformly from [0,1].
- Set $A = t$ if $r_1 \leq .2$ and $A = f$ else.
- Set B to f. Update weight depending on the value of A to: $P(B = f \mid A = a)$

So:

- 20% of the time we sample $\langle A = t, B = f\rangle$ with weight
- and 80% of the time we sample $\langle A = f, B = f\rangle$ with weight

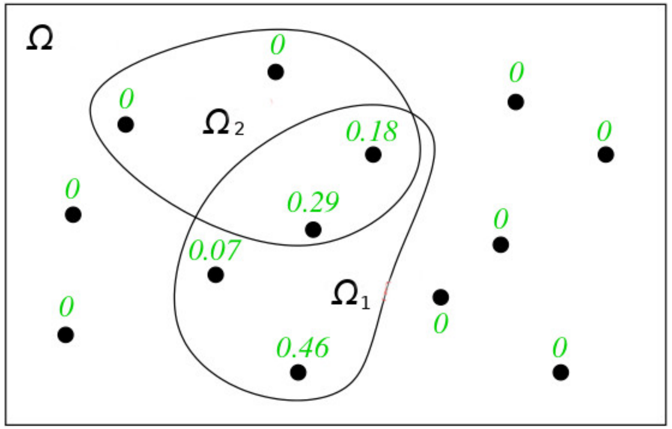
Importance Sampling I

Importance sampling

Likelihood weighting is an instance of importance sampling, where

- samples are weighted and can come from (almost) any proposal distribution.

- $\mathbf{S}$: the set of all variables defining possible worlds (includes the variables A and $\mathbf{E}$).
- Possible worlds then are tuples $\mathbf{s}$ of values



$$P(A = a \mid \mathbf{E} = \mathbf{e}) = \sum_{\mathbf{s} \in \Omega} P(A = a \mid \mathbf{S} = \mathbf{s}) P(\mathbf{S} = \mathbf{s} \mid \mathbf{E} = \mathbf{e})$$

Observe that:

- $P(\mathbf{S} = \mathbf{s} \mid \mathbf{E} = \mathbf{e})$ are the "green numbers"
- $P(A = a \mid \mathbf{S} = \mathbf{s})$ is 0 or 1, depending on whether $A = a$ in $\mathbf{s}$.

Importance Sampling II

If $\mathbf{s}_1, \dots, \mathbf{s}_n$ are sampled according to $P(\mathbf{S} = \mathbf{s} \mid \mathbf{E} = \mathbf{e})$, then

$$P(A = a \mid \mathbf{E} = \mathbf{e}) = \sum_{\mathbf{s} \in \Omega} P(A = a \mid \mathbf{S} = \mathbf{s}) P(\mathbf{S} = \mathbf{s} \mid \mathbf{E} = \mathbf{e}) \approx \frac{1}{n} \sum_{i=1}^{n} P(A = a \mid \mathbf{S} = \mathbf{s}_i)$$

Let Q be any probability distribution according to which we can sample possible worlds $\mathbf{s}_i$ (called a **proposal distribution**). Then:

$$\sum_{\mathbf{s} \in \Omega} P(A = a \mid \mathbf{S} = \mathbf{s}) P(\mathbf{S} = \mathbf{s} \mid \mathbf{E} = \mathbf{e}) = \sum_{\mathbf{s} \in \Omega} P(A = a \mid \mathbf{S} = \mathbf{s}) \frac{P(\mathbf{S} = \mathbf{s} \mid \mathbf{E} = \mathbf{e})}{Q(\mathbf{S} = \mathbf{s})} Q(\mathbf{S} = \mathbf{s})$$

If $\mathbf{s}_1, \dots, \mathbf{s}_n$ are sampled according to Q, then the right side is approximated by

$$\frac{1}{n} \sum_{i=1}^{n} P(A = a \mid \mathbf{S} = \mathbf{s}_i) \frac{P(\mathbf{S} = \mathbf{s}_i \mid \mathbf{E} = \mathbf{e})}{Q(\mathbf{S} = \mathbf{s}_i)}$$

which then is also an approximation of $P(A = a \mid \mathbf{E} = \mathbf{e})$.

Importance Sampling III

Importance Sampling

- Generate random samples $\mathbf{s}_i$ according to some proposal distribution Q.
- Estimate $P(A = a \mid \mathbf{E} = \mathbf{e})$ by $\frac{1}{n}\sum_{i=1}^{n} P(A = a \mid \mathbf{S} = \mathbf{s}_i)\frac{P(\mathbf{S}=\mathbf{s}_i|\mathbf{E}=\mathbf{e})}{Q(\mathbf{S}=\mathbf{s}_i)}$

Observations and Issues

- $P(A = a \mid \mathbf{S} = \mathbf{s}_i)$ is still only 0-1-valued
- $P(\mathbf{S} = \mathbf{s}_i \mid \mathbf{E} = \mathbf{e})$ is usually easy to compute, because $\mathbf{s}_i$ contains a value for all variables.
- $P(\mathbf{S} = \mathbf{s}_i \mid \mathbf{E} = \mathbf{e}) = 0$ if $\mathbf{s}_i$ does not satisfy $\mathbf{E} = \mathbf{e}$, i.e. samples that do not comply with the evidence don't count.
- The best approximation is obtained when Q is close to (identical to) $P(\mathbf{S} \mid \mathbf{E} = \mathbf{e})$.

Summary

- Bayesian networks (BN) are a wide-spread tool to model uncertainty, and to reason about it. A BN represents conditional independence relations between random variables. It consists of a graph encoding the variable dependencies, and of conditional probability tables (CPTs).
- Probabilistic inference requires to compute the probability distribution of a set of query variables, given a set of evidence variables whose values we know. The remaining variables are hidden.
- Inference by enumeration takes a BN as input, then applies Normalization+Marginalization, the Chain rule, and exploits conditional independence. This can be viewed as a tree search that branches over all values of the hidden variables.
- Variable elimination avoids unnecessary computation. It runs in polynomial time for poly-tree BNs. In general, exact probabilistic inference is **#P**-hard. Approximate probabilistic inference methods exist.
- When exact inference is infeasible, approximate inference can be used to obtaain estimates faster.
- Approximate inference can be done via sampling from the distribution the BN represents. We can apply a weighting function to correct our estimations when we are interested in a diffrent distribution.

Topics We Didn't Cover Here

- More advance inference by sampling: A whole zoo of methods for doing this exists.
- Clustering: Pre-combining subsets of variables to reduce the runtime of inference.
- Compilation to SAT: More precisely, to "weighted model counting" in CNF formulas. Model counting extends DPLL with the ability to determine the number of satisfying interpretations. Weighted model counting allows to define a mass for each such interpretation (= the probability of an atomic event).
- Dynamic BN: BN with one slice of variables at each "time step", encoding probabilistic behavior over time.
- Relational BN: BN with predicates and object variables.
- First-order BN: Relational BN with quantification, i.e., probabilistic logic. E.g., the BLOG language developed by Stuart Russel and co-workers.

Reading

- *Chapter 8: Reasoning with Uncertainty* from the book "Artificial Intelligence:Foundations of Computational Agents" (2nd edition). In particular:
  - Section 8.4 "Probabilistic Inference"
  - Section 8.4.1 "Variable Elimination for Belief Networks"
  - Section 8.6 "Stochastic Simulation"
  - Section 8.6.5 "Importance Sampling"

For a further reading on the topic you can also read:

- *Chapter 14: Probabilistic Reasoning* from the book "Artificial Intelligence:A Modern Approach (4th edition)