Machine Intelligence

6. Reasoning Under Uncertainty, Part III: Inference in Bayesian networks

Putting the Machinery to Practical Use

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Thanks to Thomas D. Nielsen and Jörg Hoffmann for slide sources

Agenda

1. Introduction
2. Probabilistic Inference Tasks
3. Exact Inference: Naive Enumeration
4. Exact Inference: Variable Elimination
5. Naive Bayes Models
6. Approximate Inference
7. Conclusion

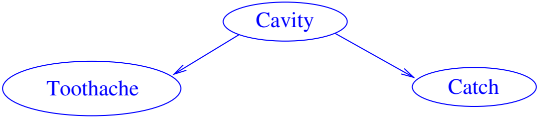
Our Agenda for This Topic

→ Our treatment of the topic "Probabilistic Reasoning" consists of Chapters 4-6.

- **Chapter 4**: All the basic machinery at use in Bayesian networks.
  → Sets up the framework and basic operations.
- **Chapter 5**: Bayesian networks: What they are and how to build them.
  → The most wide-spread and successful practical framework for probabilistic reasoning.
- **This Chapter**: Bayesian networks: how to use them.
  → How to use Bayesian Networks to answer our questions.

Reminder: Our Machinery

1. **Graph captures variable dependencies:** (Variables $X_1, \dots, X_n$)



$\rightarrow$ Given evidence $\mathbf{e}$, want to know $\mathbf{P}(X \mid e)$. Remaining vars: $\mathbf{Y}$.

2. **Normalization+Marginalization:**

$$\mathbf{P}(X \mid \mathbf{e}) = \alpha \mathbf{P}(X, \mathbf{e}) = \alpha \sum_{\mathbf{y} \in \mathbf{Y}} \mathbf{P}(X, \mathbf{e}, \mathbf{y})$$

$\rightarrow$ A sum over atomic events!

3. **Chain rule:** $X_1, \dots, X_n$ consistently with dependency graph.

$$\mathbf{P}(X_1, \dots, X_n) = \mathbf{P}(X_n \mid X_{n-1}, \dots, X_1) * \mathbf{P}(X_{n-1} \mid X_{n-2}, \dots, X_1) * \dots * \mathbf{P}(X_1)$$

4. **Exploit conditional independence:** Instead of $\mathbf{P}(X_i \mid X_{i-1}, \dots, X_1)$, we can use $\mathbf{P}(X_i \mid Parents(X_i))$.

$\rightarrow$ Bayesian networks!

Reminder: Recovering the Full Joint Probability Distribution

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Choose $X_1, \dots, X_n$ consistent with BN: $X_j \in Parents(X_i) \implies j < i$.

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$$\mathbf{P}(X_1, \dots, X_n) = \Pi_{i=1}^{n}\mathbf{P}(X_i \mid Parents(X_i))$$

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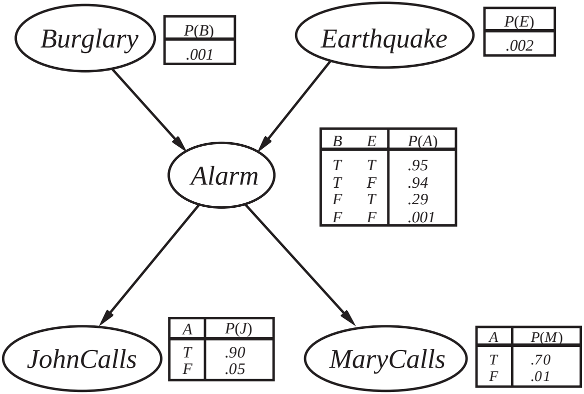
$\rightarrow$ Same for atomic events $P(x_1, \dots, x_n)$.

Reminder: Recovering a Probability for John, Mary, and the Alarm



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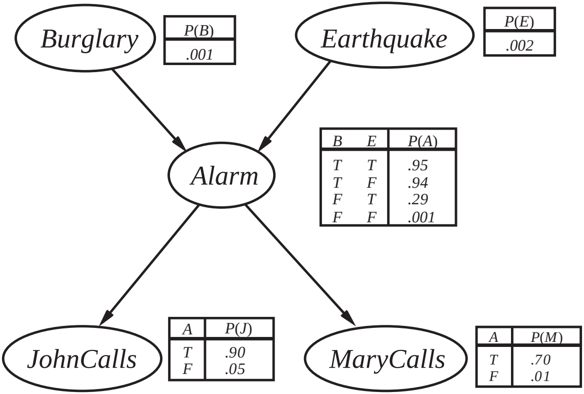

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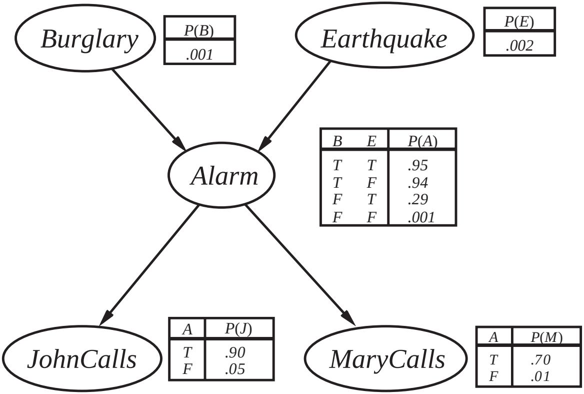

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P(j, m, a, \neg b, \neg e) &= P(j \mid a) \cdot P(m \mid a) \cdot P(a \mid \neg b, \neg e) \cdot P(\neg b) \cdot P(\neg e) \\
&= 0.9 * 0.7 * 0.001 * 0.999 * 0.998 \\
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$$

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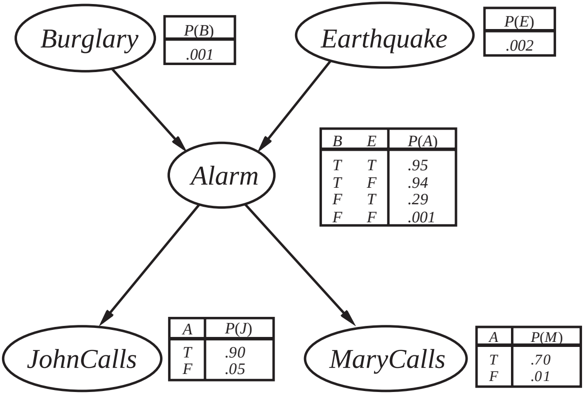

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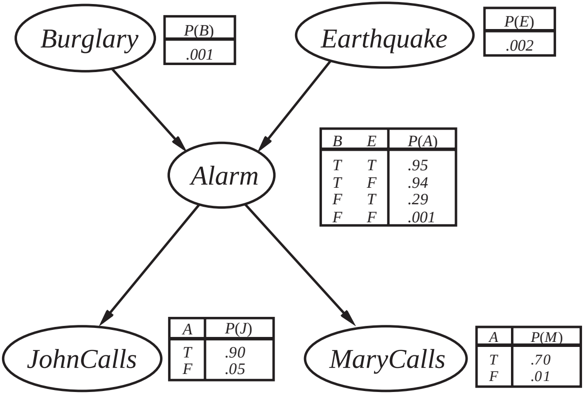

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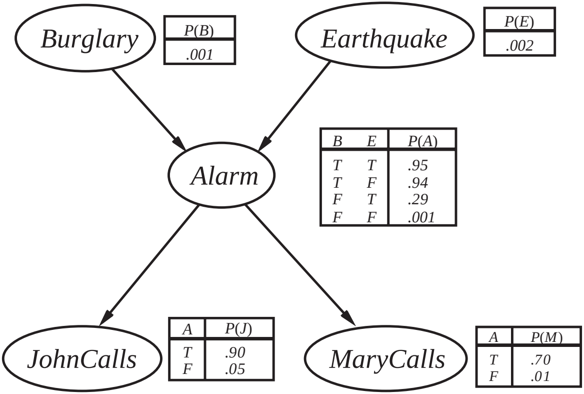

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- **Approximate Inference via Sampling:** What to do when computing the exact answer is too expensive?
  →We can approximate the solution via sampling methods!

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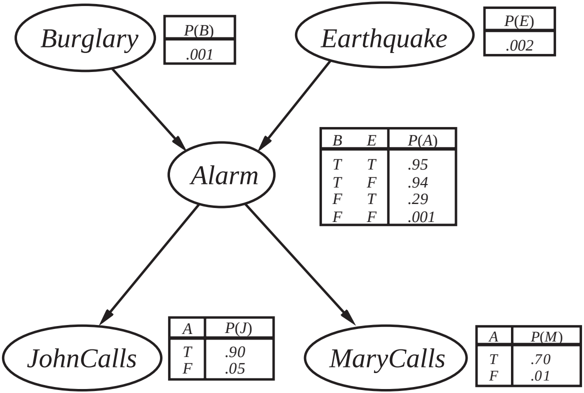
Inference for Mary and John

→ Observe evidence variables and draw conclusions on query variables.



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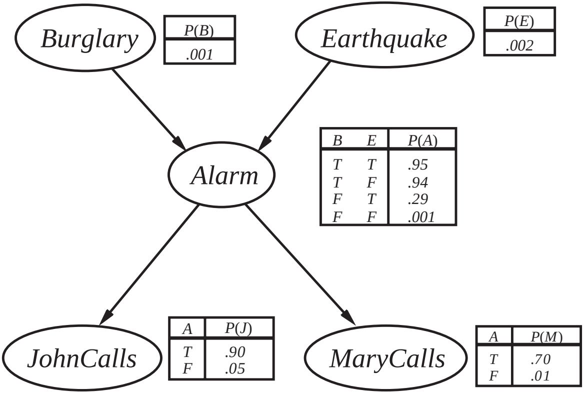

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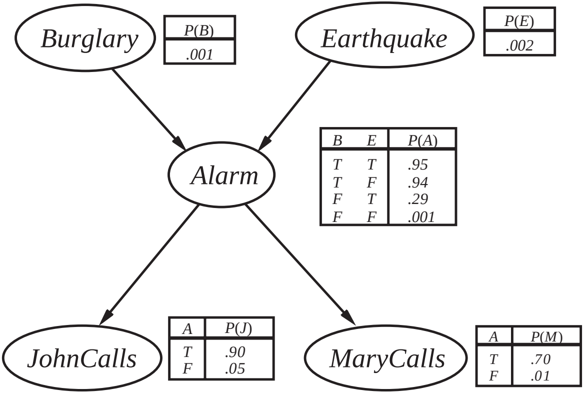

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Probabilistic Inference Tasks in Bayesian Networks

Definition (Probabilistic Inference Task). *Given random variables $X_1, \dots, X_n$, a probabilistic inference task consists of a set $\mathbf{X} \subseteq \{X_1, \dots, X_n\}$ of query variables, a set $\mathbf{E} \subseteq \{X_1, \dots, X_n\}$ of evidence variables, and an event $\mathbf{e}$ that assigns values to $\mathbf{E}$. We wish to compute the posterior probability distribution $\mathbf{P}(\mathbf{X} \mid \mathbf{e})$.*

Notes:

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Simplifying the Problem by Using Normalization

According to the definition of conditional probability:

$$P(X \mid \mathbf{E} = \mathbf{e}) = \frac{P(X, \mathbf{E} = \mathbf{e})}{P(\mathbf{E} = \mathbf{e})}$$

It is sufficient to compute for each $x \in D_X$ the value

$$P(X = x, \mathbf{E} = \mathbf{e}).$$

Together with

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Notation: As $\alpha = \frac{1}{\sum_{x \in D_X} P(X=x, \mathbf{E}=\mathbf{e})}$ is easily derived from $P(X \mid \mathbf{E} = \mathbf{e})$, we simply write $P(X \mid \mathbf{E} = \mathbf{e}) = \alpha P(X \mid \mathbf{E} = \mathbf{e})$ instead of $\frac{P(X, \mathbf{E}=\mathbf{e})}{P(\mathbf{E}=\mathbf{e})}$.

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Note:

- For each $\mathbf{y}$ the probability $P(X = x, \mathbf{E} = \mathbf{e}, \mathbf{Y} = \mathbf{y})$ can be computed from the network (in time linear in the number of random variables).
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Simplified notation:

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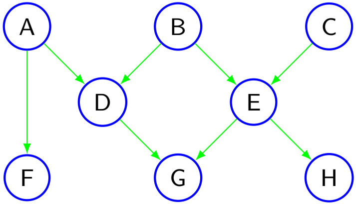
Naive Enumeration Example



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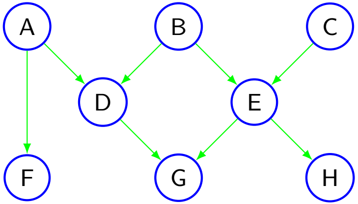

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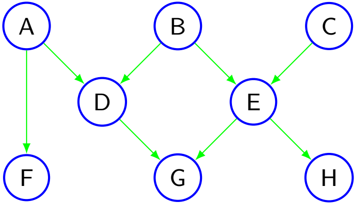

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Inserting evidence we get:

$$P(B \mid a, f, g, h) = \alpha \cdot \sum_{C,D,E} P(a, B, C, D, E, f, g, h)$$

and

$$\frac{1}{\alpha} = P(a, f, g, h) = \sum_{\mathrm{B}} P(B, a, f, g, h)$$

Inference by Enumeration: The Principle (A Reminder!)

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3. **Chain rule.** Order $X_1, \dots, X_n$ consistent with BN.

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$$\mathbf{P}(X \mid \mathbf{e}) = \alpha \mathbf{P}(X, \mathbf{e}); \text{ if } \mathbf{Y} \neq \emptyset \text{ then } \mathbf{P}(X \mid \mathbf{e}) = \alpha \sum_{\mathbf{y} \in \mathbf{Y}} \mathbf{P}(X, \mathbf{e}, \mathbf{y})$$

$\rightarrow$ Recover the summed-up probabilities $\mathbf{P}(X, \mathbf{e}, \mathbf{y})$ from BN!

3. **Chain rule.** Order $X_1, \dots, X_n$ consistent with BN.

$$\mathbf{P}(X_1, \dots, X_n) = \mathbf{P}(X_n \mid X_{n-1}, \dots, X_1)\mathbf{P}(X_{n-1} \mid X_{n-2}, \dots, X_1) \dots \mathbf{P}(X_1)$$

4. **Exploit conditional independence.** Instead of $\mathbf{P}(X_i \mid X_{i-1}, \dots, X_1)$, use $\mathbf{P}(X_i \mid Parents(X_i))$.

Inference by Enumeration: The Principle (A Reminder!)

Given evidence $\mathbf{e}$, want to know $\mathbf{P}(X \mid \mathbf{e})$. Hidden variables: $\mathbf{Y}$.

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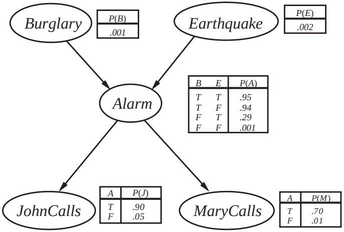
→ Given a Bayesian network BN, probabilistic inference tasks can be solved as sums of products of conditional probabilities from BN.

→ Sum over all value combinations of hidden variables.

Inference by Enumeration: John and Mary

- Want: $\mathbf{P}(Burglary \mid johncalls, marycalls)$.
  Hidden variables: $\mathbf{Y} = \{Earthquake, Alarm\}$.



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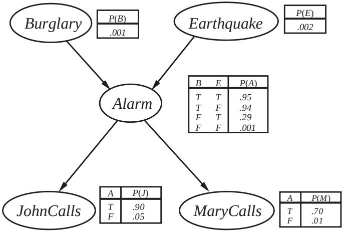

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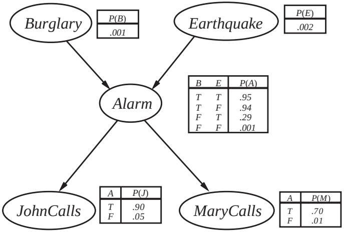

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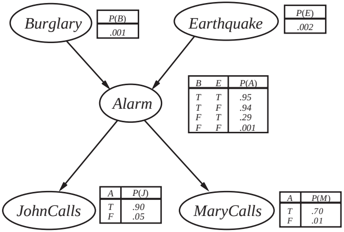

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- Move variables outwards (until we hit the first parent): $\mathbf{P}(B \mid j,m) =$
  $\alpha \mathbf{P}(B) \sum_{v_E} P(v_E) \sum_{v_A} \mathbf{P}(v_A \mid B, v_E) P(j \mid v_A) P(m \mid v_A)$

  $\rightarrow$ The probabilities of the outside-variables multiply the entire "rest of the sum" (compare slides 35 and 36).
- Continuation on next slide ...

Inference by Enumeration: John and Mary, ctd.

Chain rule and conditional independence, ctd.: $\mathbf{P}(B \mid j, m) =$

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$= \alpha \langle 0.00059224, 0.0014919 \rangle \approx \langle 0.284, 0.716 \rangle$

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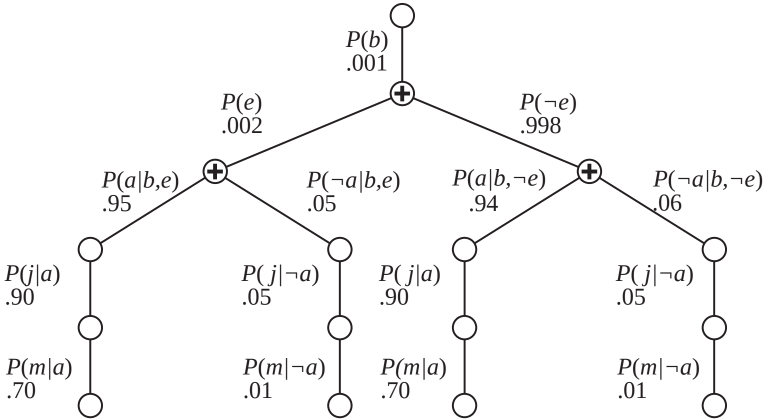
$\rightarrow$ This computation can be viewed as a "search tree", see next slide.

The Evaluation of $P(b \mid j, m)$, as a "Search Tree"



→ Inference by enumeration = a tree with "sum nodes" branching over values of hidden variables, and with non-branching "multiplication nodes".

Inference by Enumeration: Pseudo-Code

→ With bn.Vars being a variable ordering consistent with bn:

```
function ENUMERATION-ASK(X, e, bn) returns a distribution over X
  inputs: X, the query variable
          e, observed values for variables E
          bn, a Bayes net with variables {X} ∪ E ∪ Y   /* Y = hidden variables */

  Q(X) ← a distribution over X, initially empty
  for each value x_i of X do
     Q(x_i) ← ENUMERATE-ALL(bn.VARS, e_{x_i})
        where e_{x_i} is e extended with X = x_i
  return NORMALIZE(Q(X))
```

```
function ENUMERATE-ALL(vars, e) returns a real number
  if EMPTY?(vars) then return 1.0
  Y ← FIRST(vars)
  if Y has value y in e
     then return P(y | parents(Y)) × ENUMERATE-ALL(REST(vars), e)
     else return Σ_y P(y | parents(Y)) × ENUMERATE-ALL(REST(vars), e_y)
        where e_y is e extended with Y = y
```

Inference by Enumeration: Properties

Inference by Enumeration:

- Evaluates the tree in a depth-first manner.

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But: Variable Elimination.

- Improves on inference by enumeration through (A) avoiding repeated computation, and (B) avoiding irrelevant computation.
- In some special cases, variable elimination runs in polynomial time.

Agenda

1. Introduction
2. Probabilistic Inference Tasks
3. Exact Inference: Naive Enumeration
4. Exact Inference: Variable Elimination
5. Naive Bayes Models
6. Approximate Inference
7. Conclusion

Variable Elimination: Sketch of Ideas

(A) Avoiding repeated computation: Evaluate expressions from right to left, storing all intermediate results.

Variable Elimination: Sketch of Ideas

(A) Avoiding repeated computation: Evaluate expressions from right to left, storing all intermediate results. For query $P(B \mid j, m)$:

1. CPTs of BN yield factors (probability tables): $\mathbf{P}(B \mid j, m) =$
$$\alpha \underbrace{\mathbf{P}(B)}_{\mathbf{f}_1(B)} \sum_{v_E} \underbrace{P(v_E)}_{\mathbf{f}_2(E)} \sum_{v_A} \underbrace{\mathbf{P}(v_A \mid B, v_E)}_{\mathbf{f}_3(A,B,E)} \underbrace{P(j \mid v_A)}_{\mathbf{f}_4(A)} \underbrace{P(m \mid v_A)}_{\mathbf{f}_5(A)}$$

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2. Then the computation is performed in terms of factor product and summing out variables from factors: $\mathbf{P}(B \mid j, m) =$
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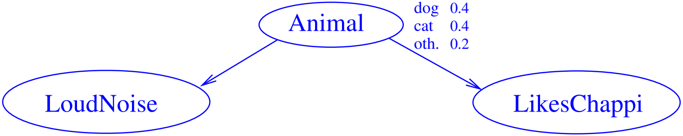
$\rightarrow$ The rightmost sum equals 1 and can be dropped.

Questionnaire

P(A)	
dog	0.4
cat	0.4
oth.	0.2



A	P(LN|A)
dog	0.7
cat	0.2
oth.	0.01

A	P(LC|A)
dog	0.8
cat	0.1
oth.	0.7

Question!

Say BN is the Bayesian network above. How can we compute $P(dog \mid loudnoise)$?

$P(d \mid ln) =$

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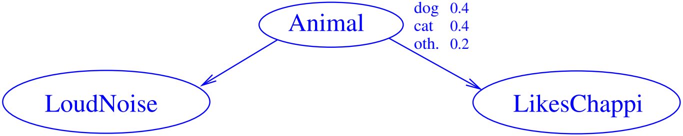

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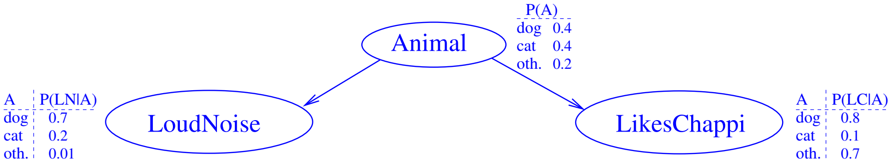

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(B) Yes.

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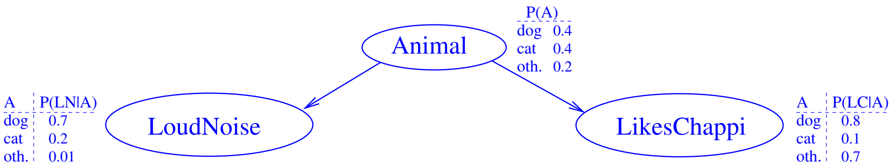

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$P(d \mid ln) =$

(A): $\alpha \sum_{v_{LC}} P(v_{LC} \mid d) P(ln \mid d)$

(B): $\alpha \sum_{v_{LC}} P(v_{LC} \mid d) P(ln \mid d) P(d)$

(C): $\alpha P(ln \mid d) P(d) \sum_{v_{LC}} P(v_{LC} \mid d)$

(D): $\alpha P(ln \mid d) P(d)$

(A) No: need to multiply by $P(d)$.

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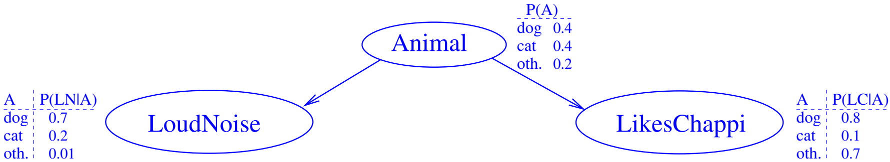
Questionnaire



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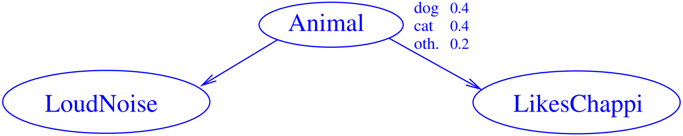
Questionnaire



	P(A)
dog	0.4
cat	0.4
oth.	0.2

A	P(LN|A)
dog	0.7
cat	0.2
oth.	0.01

A	P(LC|A)
dog	0.8
cat	0.1
oth.	0.7

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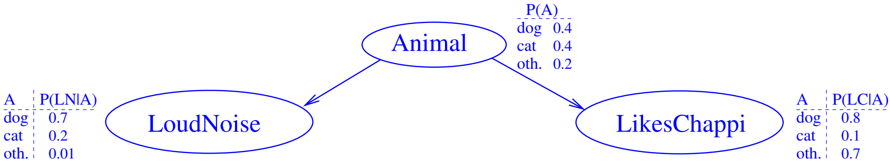
$\rightarrow$ So what is $P(dog \mid loudnoise)$? We compute $\alpha\langle P(ln \mid d)P(d), P(ln \mid c)P(c), P(ln \mid o)P(o)\rangle = \alpha\langle 0.7 * 0.4, 0.2 * 0.4, 0.01 * 0.2\rangle \approx \langle 0.77, 0.22, 0.01\rangle$. Hence $P(dog \mid loudnoise) =$

Questionnaire



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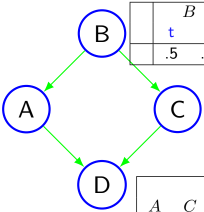
Example



	B: t	B: f
	.5	.5

B	A: t	A: f
t	.7	.3
f	.1	.9

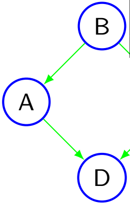
B	C: t	C: f
t	.7	.3
f	.2	.8

A	C	D: t	D: f
t	t	.9	.1
t	f	.7	.3
f	t	.8	.2
f	f	.4	.6

Naive Enumeration:

$$P(A, D = f) =$$

Example



	B	
	t	f
	.5	.5

B	A t	A f
t	.7	.3
f	.1	.9

B	C t	C f
t	.7	.3
f	.2	.8

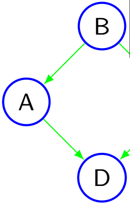
A	C	D t	D f
t	t	.9	.1
t	f	.7	.3
f	t	.8	.2
f	f	.4	.6

Naive Enumeration:

$$P(A, D = f) =$$

$$\sum_{b \in \{t,f\}} \sum_{c \in \{t,f\}} P(B = b, A, C = c, D = f) =$$

Example



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	t	f
	.5	.5

B	A	
	t	f
t	.7	.3
f	.1	.9

B	C	
	t	f
t	.7	.3
f	.2	.8

A	C	D	
		t	f
t	t	.9	.1
t	f	.7	.3
f	t	.8	.2
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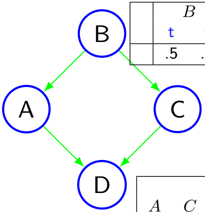
Example



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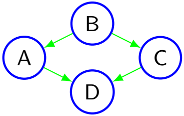
$$\sum_{b \in \{t,f\}} \sum_{c \in \{t,f\}} P(B = b, A, C = c, D = f) =$$

$$\sum_{b \in \{t,f\}} \sum_{c \in \{t,f\}} P(B = b)P(A \mid B = b)P(C = c \mid B = b)P(D = f \mid A, C = c)$$

Observation 1: $P(B = b)P(A \mid B = b)$ does not depend on C, so

$$= \sum_{b \in \{t,f\}} P(B = b)P(A \mid B = b) \sum_{c \in \{t,f\}} P(C = c \mid B = b)P(D = f \mid A, C = c)$$

Example continued



	B = t	B = f
	.5	.5

B	A = t	A = f
t	.7	.3
f	.1	.9

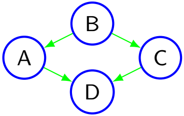
B	C = t	C = f
t	.7	.3
f	.2	.8

A	C	D = t	D = f
t	t	.9	.1
t	f	.7	.3
f	t	.8	.2
f	f	.4	.6

Observation 2: Let's precompute first factor F_1

$$\sum_b P(B=b)P(A \mid B=b) \underbrace{\sum_c P(C=c \mid B=b)P(D=f \mid A, C=c)}_{F_1} =$$

Example continued



	B	
	t	f
	.5	.5

	A	
B	t	f
t	.7	.3
f	.1	.9

	C	
B	t	f
t	.7	.3
f	.2	.8

		D	
A	C	t	f
t	t	.9	.1
t	f	.7	.3
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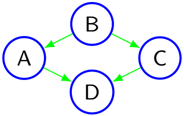
$$\sum_b P(B=b)P(A \mid B=b)F_1(B=b, A) = F_2(A)$$

Is F_1 a single numerical value?

Example continued



	B	
	t	f
	.5	.5

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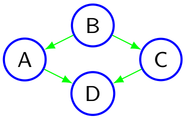
Is F_1 a single numerical value? No!, a table $F_1(A, B)$ because it takes different values for each value of variables A and B.

Example continued



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f	.2	.8

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t	t	.9	.1
t	f	.7	.3
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f	f	.4	.6

$\mapsto$

b	a	$F_1(B, A)$
t	t	$.7 \cdot .1 + .3 \cdot .3 = .16$
t	f	$.7 \cdot .2 + .3 \cdot .6 = .32$
f	t	$.2 \cdot .1 + .8 \cdot .3 = .26$
f	f	$.2 \cdot .2 + .8 \cdot .6 = .52$

and

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	.5	.5

B	A = t	A = f
t	.7	.3
f	.1	.9

b	a	$F_1(B, A)$
t	t	.16
⋮	⋮	⋮

$\mapsto$

a	$F_2(A)$
t	...
f	...

Variable Elimination

The procedure operates on **factors**: functions of subsets of variables

Definition (Factor). A factor $F(V_1, \ldots V_k)$ is a function mapping each combination of values of the variables $V_1, \ldots, V_k$ to a number.

→ Factors are not necessarily CPTs as numbers do not need to sum up to 1.

Required operations on factors:

- Restriction (setting selected variables to specific values)
- Multiplication (join tables and multiply the probabilities)
- Marginalization (summing out selected variables)

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Restriction (of variable D to value $D = f$):

		$F(A,C,D)$	
		D	
A	C	t	f
t	t	.9	.1
t	f	.7	.3
f	t	.8	.2
f	f	.4	.6

$\mapsto$

A	C	$F_2(A,C)$
t	t	.1
t	f	.3
f	t	.2
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A	C	$F_2(A,C)$
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Required operations on factors:

- Restriction (setting selected variables to specific values)
- Multiplication (join tables and multiply the probabilities)
- Marginalization (summing out selected variables)

Multiplication + Marginalization (of variable C):

B	C	$F_1(B,C)$
t	t	.7
t	f	.3
f	t	.2
f	f	.8

A	C	$F_2(A,C)$
t	t	.1
t	f	.3
f	t	.2
f	f	.6

$\mapsto$

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Variable Elimination Algorithm

1. Factors = CPT in BN
2. For all variables in the evidence, restrict all factors with the observed value
3. Fix any order of the remaining variables, $X_1, \dots, X_n$.
4. **for** $i := 1, \dots, n$ **do**
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   b. $T = \pi_{F \in \mathcal{F}}$ (Compute product of factors)
   c. $F_{new} =$ Marginalize(T, X_i)
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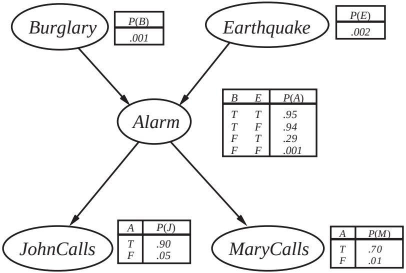
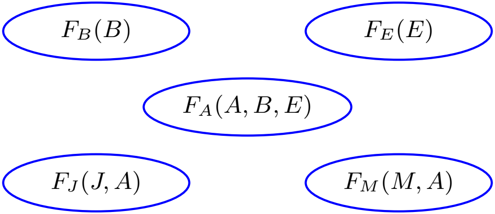
Variable Elimination Algorithm: Alarm Example



Bad ordering for computing $P(M, B = t)$: A, J, E



1. Factors = CPT in BN

Variable Elimination Algorithm: Alarm Example

Bad ordering for computing $P(M, B = t)$: A, J, E

$F_B(B)$ $F_E(E)$

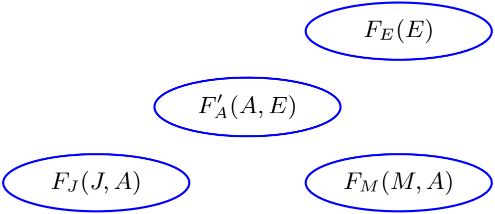
$F_A(A, B, E)$

$F_J(J, A)$ $F_M(M, A)$

1. Factors = CPT in BN
2. For all variables in the evidence, restrict all factors with the observed value

Variable Elimination Algorithm: Alarm Example

Bad ordering for computing $P(M, B = t)$: A, J, E



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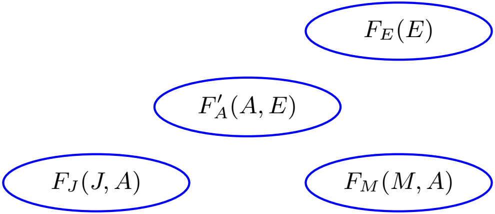

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Variable Elimination Algorithm: Alarm Example

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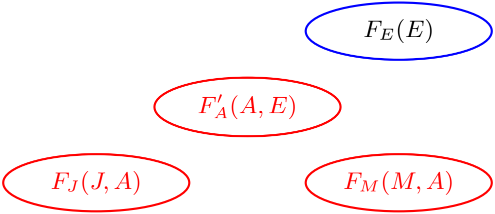

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Variable Elimination Algorithm: Alarm Example

Bad ordering for computing $P(M, B = t)$: A, J, E

$F_E(E)$

$T(A, J, M, E)$

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2. For all variables in the evidence, restrict all factors with the observed value
3. Fix any order of the remaining variables, $X_1, \dots, X_n$.
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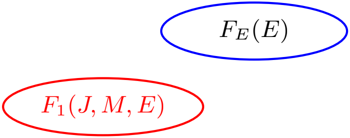
   c. $F_{new} = \text{Marginalize}(T, X_i)$
   d. Factors $=$ Factors $\setminus \mathcal{F} \cup \{N\}$ (Replace all factors $\mathcal{F}$ by new factor N)

Variable Elimination Algorithm: Alarm Example

Bad ordering for computing $P(M, B = t)$: A, J, E



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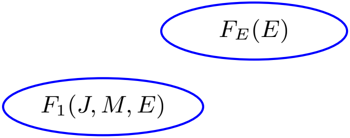
   d. Factors = Factors $\setminus \mathcal{F} \cup \{N\}$ (Replace all factors $\mathcal{F}$ by new factor N)

Variable Elimination Algorithm: Alarm Example

Bad ordering for computing $P(M, B = t)$: A, J, E



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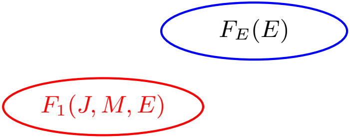
Variable Elimination Algorithm: Alarm Example

Bad ordering for computing $P(M, B = t)$: A, J, E



1. Factors = CPT in BN
2. For all variables in the evidence, restrict all factors with the observed value
3. Fix any order of the remaining variables, $X_1, \ldots, X_n$.
4. **for** $i := 1, \ldots, n$ **do**
   a. $\mathcal{F} = \{F \mid F \in \text{Factors}, F \text{ depends on } X_i\}$
   b. $T = \pi_{F \in \mathcal{F}}$ (Compute product of factors)
   c. $F_{new} = \text{Marginalize}(T, X_i)$
   d. Factors $=$ Factors $\setminus \mathcal{F} \cup \{N\}$ (Replace all factors $\mathcal{F}$ by new factor N)

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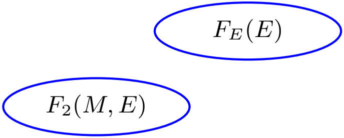

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Variable Elimination Algorithm: Alarm Example

Bad ordering for computing $P(M, B = t)$: A, J, E

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Variable Elimination Algorithm: Alarm Example

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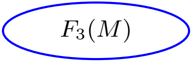

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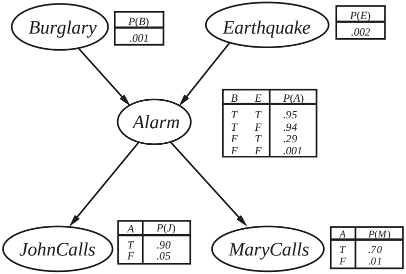

Bad ordering for computing $P(MC, B = t)$: A, J, E

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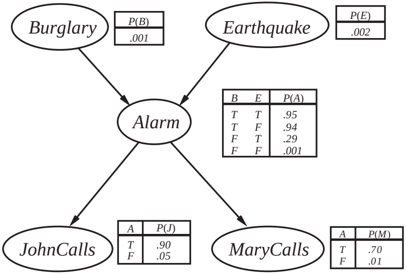

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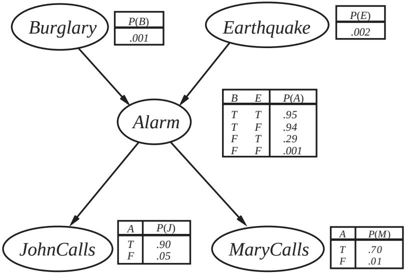

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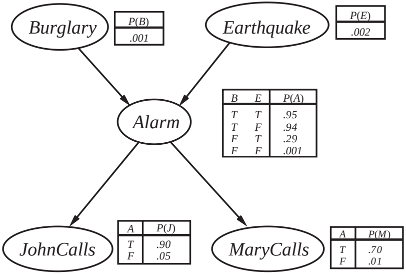

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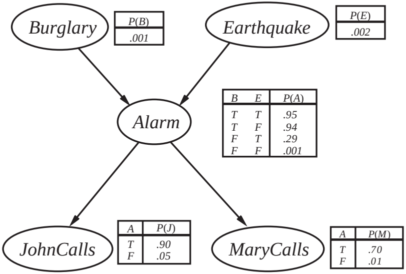

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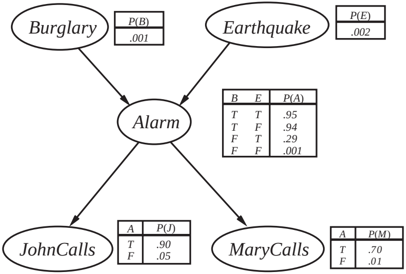

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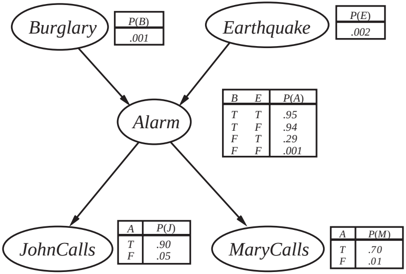

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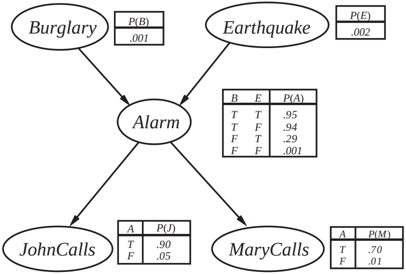

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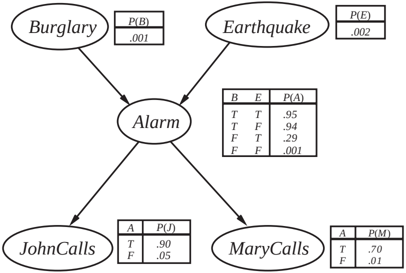
Largest factor (F_1) is function of 3 variables!

Alarm Example continued



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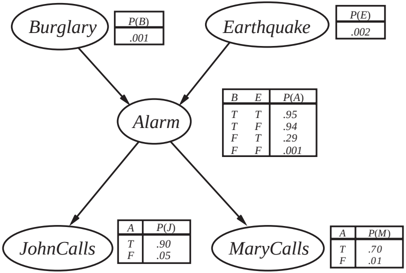

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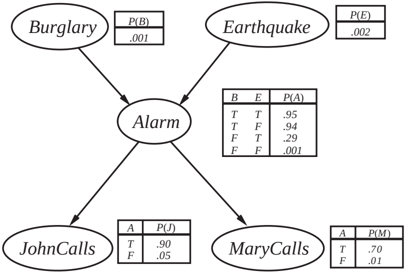

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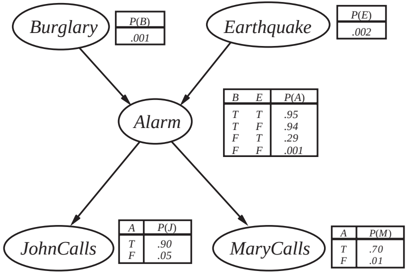

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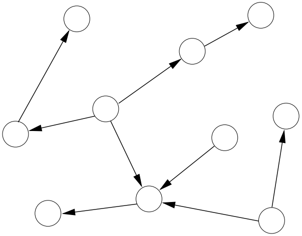
$$P(B = t)F_3(MC)$$

Largest factor ($P(A \mid B = t, EQ)$) is function of 2 variables!

Variable Elimination Runtime

An important easy case:

- A graph is called singly connected, or a polytree, if there is at most one undirected path between any two nodes in the graph.



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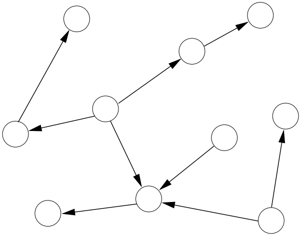

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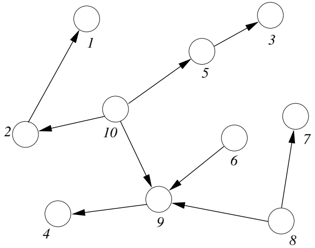

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For singly connected network: any elimination order that “peels” variables from outside will only create factors of only one variable.

The complexity of inference is therefore linear in the total size of the network (= combined size of all conditional probability tables).

Agenda

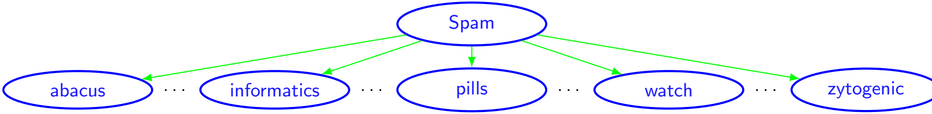
1. Introduction
2. Probabilistic Inference Tasks
3. Exact Inference: Naive Enumeration
4. Exact Inference: Variable Elimination
5. Naive Bayes Models
6. Approximate Inference
7. Conclusion

Naive Bayes Model

Example: Spam filter

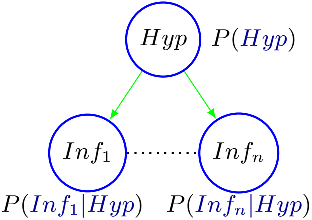
- A single query variable: Spam
- Many observable features (e.g. words appearing in the body of the message): abacus,. . . ,informatics, pills, . . . , watch,. . . , zytogenic

Network Structure:



- Inference with large number of variables possible
- Essentially how Thunderbird spam filter works

Naïve Bayes models

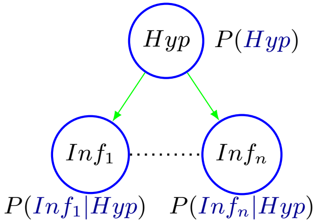


We want the posterior probability of the hypothesis variable Hyp given the observations $\{\mathsf{Inf}_1 = e_1, \ldots, \mathsf{Inf}_n = e_n\}$:

$$P(\mathsf{Hyp}|\mathsf{Inf}_1 = e_1, \ldots, \mathsf{Inf}_n = e_n) = \frac{P(\mathsf{Inf}_1 = e_1, \ldots, \mathsf{Inf}_n = e_n|\mathsf{Hyp})P(\mathsf{Hyp})}{P(\mathsf{Inf}_1 = e_1, \ldots, \mathsf{Inf}_n = e_n)}$$

Note: The model assumes that the information variables are independent given the hypothesis variable.

Naïve Bayes models



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$$= \alpha \cdot P(\text{Inf}_1 = e_1|\text{Hyp}) \cdot \ldots \cdot P(\text{Inf}_n = e_n|\text{Hyp})P(\text{Hyp})$$

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Sampling Inference

Using a BN as a Sample Generator

Observation: can use Bayesian network as random generator that produces states $\mathbf{X} = \mathbf{x}$ according to distribution P defined by the network.

$\rightarrow$ We just sample a random value for each variable always picking a value for parents(X) before picking a value for X

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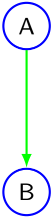
Example:



A=t	A=f
.2	.8

A	B=t	B=f
t	.7	.3
f	.4	.6

- Generate random numbers r_A, r_B uniformly from [0,1].
- Set $A = t$ if $r_A \leq .2$ and $A = f$ else.
- Depending on the value of A and r_B set B to t or f.

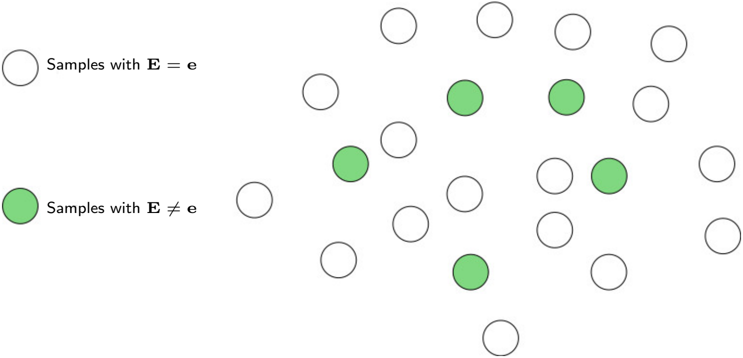
Random generation of one state: linear in size of network.

Sampling Inference

Approximate Inference from Samples

To compute an approximation of $P(\mathbf{E}=\mathbf{e})$ ($\mathbf{E}$ a subset of the variables in the Bayesian network):

- generate a (large) number of random states
- count the frequency of states in which $\mathbf{E}=\mathbf{e}$.



Accuracy

Hoeffding Bound

- p: true probability $P(\mathbf{E} = \mathbf{e})$
- s: estimate for p from sample of size n
- ϵ: an error bound > 0.

Then

$$P(|s - p| > \epsilon) \leq 2e^{-2n\epsilon^2}$$

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To obtain an estimate that with probability at most δ has an error greater than ϵ, it is sufficient to take

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To get an error ϵ of less than 0.1 in 95% of the cases ($\delta = 0.05$), we need:

$$n > -ln(0.05/2)/(2 \cdot 0.1^2) \approx 184 \text{ samples}$$

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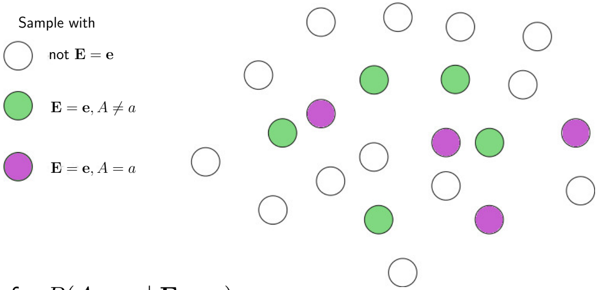
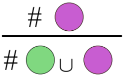
How many samples do we need if the error should be less than 0.01? 18444 samples

Rejection Sampling

The simplest approach: **Rejection Sampling**



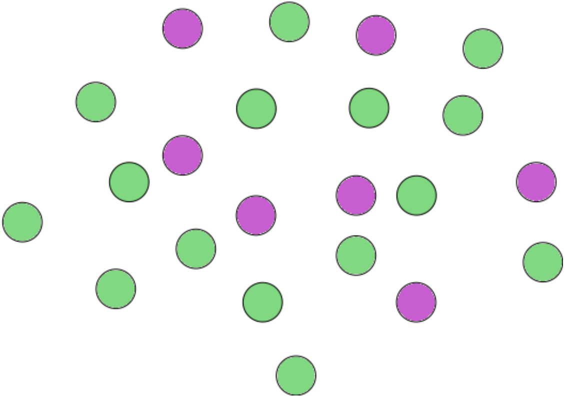
Approximation for $P(A = a \mid \mathbf{E} = \mathbf{e})$:



Sampling from the conditional distribution

Problem with rejection sampling: samples with $\mathbf{E} \neq \mathbf{e}$ are useless!

Ideally: would draw samples directly from the conditional distribution $P(\mathbf{A} \mid \mathbf{E} = \mathbf{e})$.



A Wrong Sampling Method

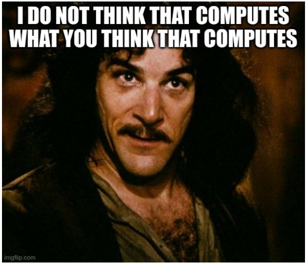
First idea (not to be followed)

1. Fix evidence variables to their observed states.
2. Sample from non-evidence variables.
3. Count frequency as before

A Wrong Sampling Method

First idea (not to be followed)

1. Fix evidence variables to their observed states.
2. Sample from non-evidence variables.
3. Count frequency as before



Problem: This gives a sampling distribution

$$\prod_{X \in \mathbf{X} \setminus \mathbf{E}} P(X \mid \mathrm{pa}(X) \setminus \mathbf{E}, \mathrm{pa}(X) \cap \mathbf{E})$$

somewhere between $P(\mathbf{X})$ and $P(\mathbf{X} \mid \mathbf{e})$.

Likelihood Weighting

We would like to sample from

$$P(\mathbf{X}, \mathbf{e}) = \underbrace{\prod_{X \in \mathbf{X} \setminus \mathbf{E}} P(X \mid \mathrm{pa}(X) \setminus \mathbf{E}, \mathrm{pa}(X) \cap \mathbf{E})}_{\text{Part 1}} \cdot \underbrace{\prod_{E \in \mathbf{E}} P(E = e \mid \mathrm{pa}(E) \setminus \mathbf{E}, \mathrm{pa}(E) \cap \mathbf{E})}_{\text{Part 2}}$$

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So instead weigh each generated sample with a weight corresponding to Part 2.

Estimate $P(X = x \mid \mathbf{e})$ as

$$\hat{P}(X = x \mid \mathbf{e}) = \frac{\sum_{sample:X=x} w(sample)}{\sum_{sample} w(sample)},$$

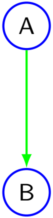
where

$$w(sample) = \prod_{E\in\mathbf{E}} P(E = e \mid \mathrm{pa}(E) = \pi) \qquad \text{(Part 2)}$$

and π is the values of $\mathrm{pa}(E)$ under $sample$ and $\mathbf{e}$.

Likelihood Sampling: Example

Sample state where B=t



A: t	A: f
.2	.8

A	B: t	B: f
t	.7	.3
f	.4	.6

- Initialize weight $W = 1$
- Generate random number r_A uniformly from [0,1].
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- Set B to f. Update weight depending on the value of A to: $P(B = f \mid A = a)$

So:

- 20% of the time we sample $\langle A = t, B = f \rangle$ with weight

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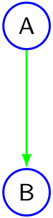

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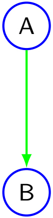

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Importance Sampling I

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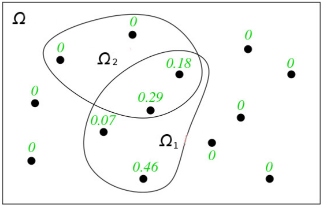
- $\mathbf{S}$: the set of all variables defining possible worlds (includes the variables A and $\mathbf{E}$).
- Possible worlds then are tuples $\mathbf{s}$ of values



$$P(A = a \mid \mathbf{E} = \mathbf{e}) = \sum_{\mathbf{s} \in \Omega} P(A = a \mid \mathbf{S} = \mathbf{s}) P(\mathbf{S} = \mathbf{s} \mid \mathbf{E} = \mathbf{e})$$

Observe that:

- $P(\mathbf{S} = \mathbf{s} \mid \mathbf{E} = \mathbf{e})$ are the "green numbers"
- $P(A = a \mid \mathbf{S} = \mathbf{s})$ is 0 or 1, depending on whether $A = a$ in $\mathbf{s}$.

Importance Sampling II

If $\mathbf{s}_1, \dots, \mathbf{s}_n$ are sampled according to $P(\mathbf{S} = \mathbf{s} \mid \mathbf{E} = \mathbf{e})$, then

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Let Q be any probability distribution according to which we can sample possible worlds $\mathbf{s}_i$ (called a **proposal distribution**). Then:

$$\sum_{\mathbf{s} \in \Omega} P(A = a \mid \mathbf{S} = \mathbf{s}) P(\mathbf{S} = \mathbf{s} \mid \mathbf{E} = \mathbf{e}) = \sum_{\mathbf{s} \in \Omega} P(A = a \mid \mathbf{S} = \mathbf{s}) \frac{P(\mathbf{S} = \mathbf{s} \mid \mathbf{E} = \mathbf{e})}{Q(\mathbf{S} = \mathbf{s})} Q(\mathbf{S} = \mathbf{s})$$

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If $\mathbf{s}_1, \ldots, \mathbf{s}_n$ are sampled according to Q, then the right side is approximated by

$$\frac{1}{n} \sum_{i=1}^{n} P(A = a \mid \mathbf{S} = \mathbf{s}_i) \frac{P(\mathbf{S} = \mathbf{s}_i \mid \mathbf{E} = \mathbf{e})}{Q(\mathbf{S} = \mathbf{s}_i)}$$

which then is also an approximation of $P(A = a \mid \mathbf{E} = \mathbf{e})$.

Importance Sampling III

Importance Sampling

- Generate random samples $\mathbf{s}_i$ according to some proposal distribution Q.
- Estimate $P(A = a \mid \mathbf{E} = \mathbf{e})$ by $\frac{1}{n}\sum_{i=1}^{n} P(A = a \mid \mathbf{S} = \mathbf{s}_i)\frac{P(\mathbf{S}=\mathbf{s}_i \mid \mathbf{E}=\mathbf{e})}{Q(\mathbf{S}=\mathbf{s}_i)}$

Observations and Issues

- $P(A = a \mid \mathbf{S} = \mathbf{s}_i)$ is still only 0-1-valued
- $P(\mathbf{S} = \mathbf{s}_i \mid \mathbf{E} = \mathbf{e})$ is usually easy to compute, because $\mathbf{s}_i$ contains a value for all variables.
- $P(\mathbf{S} = \mathbf{s}_i \mid \mathbf{E} = \mathbf{e}) = 0$ if $\mathbf{s}_i$ does not satisfy $\mathbf{E} = \mathbf{e}$, i.e. samples that do not comply with the evidence don't count.
- The best approximation is obtained when Q is close to (identical to) $P(\mathbf{S} \mid \mathbf{E} = \mathbf{e})$.

Agenda

1 Introduction

2 Probabilistic Inference Tasks

3 Exact Inference: Naive Enumeration

4 Exact Inference: Variable Elimination

5 Naive Bayes Models

6 Approximate Inference

7 Conclusion

Summary

- Bayesian networks (BN) are a wide-spread tool to model uncertainty, and to reason about it. A BN represents conditional independence relations between random variables. It consists of a graph encoding the variable dependencies, and of conditional probability tables (CPTs).
- Probabilistic inference requires to compute the probability distribution of a set of query variables, given a set of evidence variables whose values we know. The remaining variables are hidden.
- Inference by enumeration takes a BN as input, then applies Normalization+Marginalization, the Chain rule, and exploits conditional independence. This can be viewed as a tree search that branches over all values of the hidden variables.
- Variable elimination avoids unnecessary computation. It runs in polynomial time for poly-tree BNs. In general, exact probabilistic inference is **#P**-hard. Approximate probabilistic inference methods exist.
- When exact inference is infeasible, approximate inference can be used to obtaain estimates faster.

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- Dynamic BN: BN with one slice of variables at each "time step", encoding probabilistic behavior over time.
- Relational BN: BN with predicates and object variables.
- First-order BN: Relational BN with quantification, i.e., probabilistic logic. E.g., the BLOG language developed by Stuart Russel and co-workers.

Reading

- *Chapter 8: Reasoning with Uncertainty* from the book "Artificial Intelligence:Foundations of Computational Agents" (2nd edition). In particular:
  - Section 8.4 "Probabilistic Inference"
  - Section 8.4.1 "Variable Elimination for Belief Networks"
  - Section 8.6 "Stochastic Simulation"
  - Section 8.6.5 "Importance Sampling"

For a further reading on the topic you can also read:

- *Chapter 14: Probabilistic Reasoning* from the book "Artificial Intelligence:A Modern Approach (4th edition)