Machine Intelligence

4. Reasoning under Uncertainty, Part I: Basics

(Our Machinery for) Thinking About What is Likely to be True

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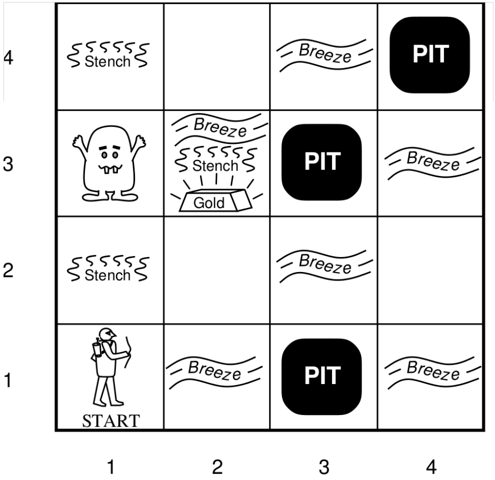
Thanks to Thomas D. Nielsen and Jörg Hoffmann for slide sources

Agenda

The Wumpus World



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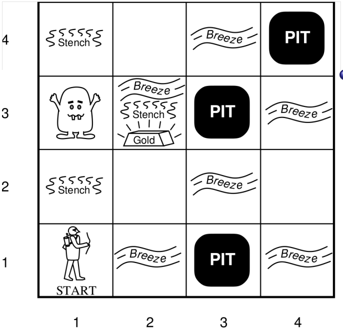

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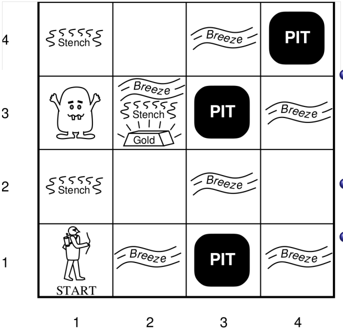

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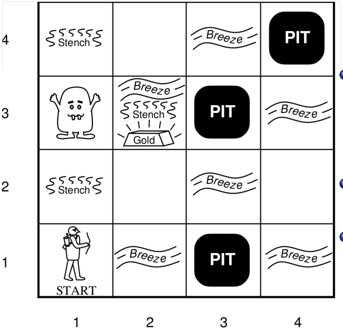

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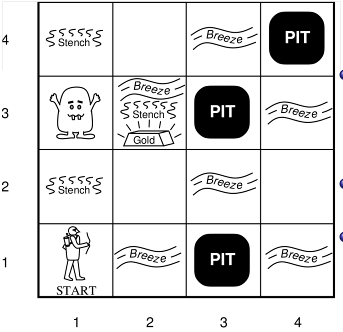

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- Cell adjacent (i.e. north, south, west, east) to Wumpus: $Stench$ (else: $None$).
- Cell adjacent to Pit: $Breeze$ (else: $None$).
- Cell that contains gold: $Glitter$ (else: $None$).
- You walk into a wall: $Bump$ (else: $None$).
- Wumpus shot by arrow: $Scream$ (else: $None$).

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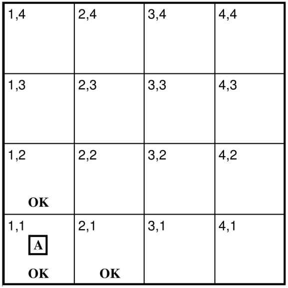

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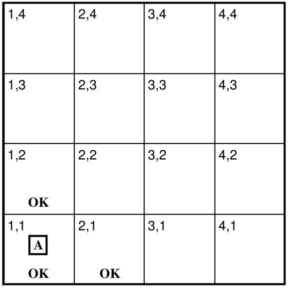

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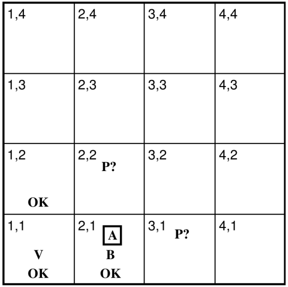

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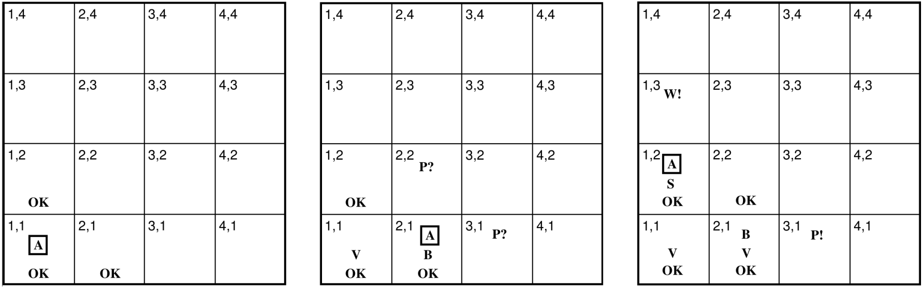

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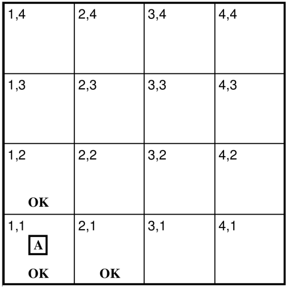

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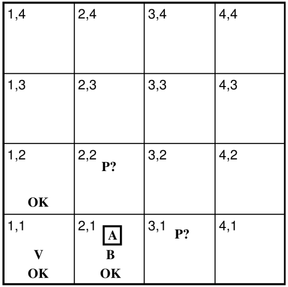

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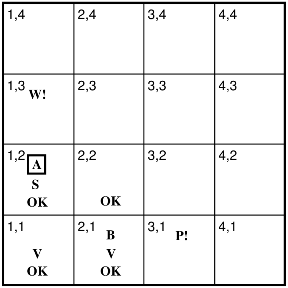

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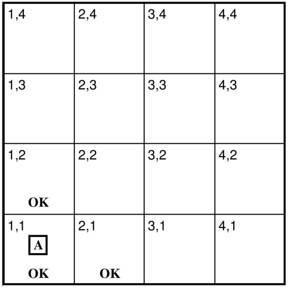

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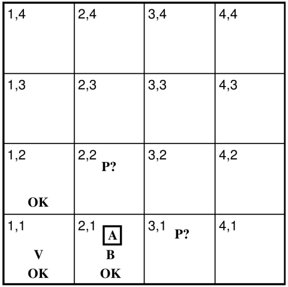

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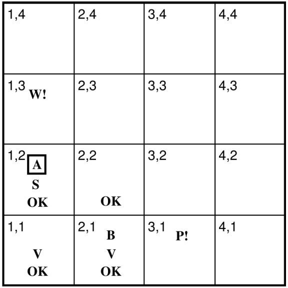

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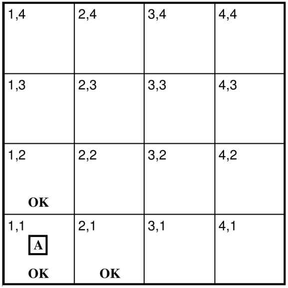

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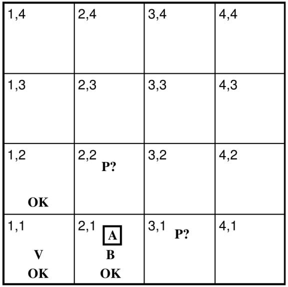

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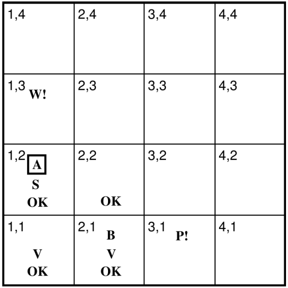

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Agents that Think Rationally

Think Before You Act!

```
function KB-AGENT(percept) returns an action
  persistent: KB, a knowledge base
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  TELL(KB, MAKE-PERCEPT-SENTENCE(percept, t))
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→ "Thinking" = Reasoning about knowledge represented using logic.

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  → The basic insight about how to invert the "direction" of conditional probabilities.
- **Conditional Independence:** How to capture and exploit complex relations between random variables?
  → Explains the difficulties arising when using Bayes' rule on multiple evidences. Conditional independence is used to ameliorate these difficulties.

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Propositional Logic: Syntax

Atomic Propositions

Boolean variables are now seen as **atomic propositions**. Convention: start with lowercase letter.

Constraints	Logic
$A =$ *true*	a
$A =$ *false*	$\neg a$

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Propositions (Formulas)

Using **logical connectives** more complex propositions are constructed:

$\neg p$	**not** p
$(p \wedge q)$	p **and** q
$(p \vee q)$	p **or** q
$(p \rightarrow q)$	p **implies** q

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$$\textit{rains} \rightarrow (\textit{umbrella} \vee \textit{home})$$

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An **interpretation** π for a set of atomic propositions $a_1, a_2, \ldots, a_n$ is an assignment of a truth value to each proposition (= possible world when atomic propositions seen as boolean variables):

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false	*true*	*false*
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Knowledge as Propositional Formulas

Satisfiability

A formula φ is:

- satisfiable if there exists I that satisfies φ.
- unsatisfiable if φ is not satisfiable.
- falsifiable if there exists I that doesn't satisfy φ.
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$\rightarrow$ Knowledge Base = set of formulas, interpreted as a conjunction.

Definition (Model). A model of a a knowledge base KB is an interpretation I in which all the formulas in the knowledge base are true: $I \models \varphi$ for all $\varphi \in$ KB.

$\rightarrow$a model is a possible world that satisfies the constraint.

We denote by $M(\varphi)$ the set of all models of φ (i.e., the set of possible worlds where the formula is true).

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deriving of a conclusion by reasoning

Remember (slide 4)? Does our knowledge of the cave entail a definite Wumpus position?

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Logical consequence (entailment)

Definition (Entailment). *Let Σ be a set of atomic propositions. We say that a set of formulas KB entails a formula φ, written $KB \models \varphi$, if φ is true in all models of KB, i.e., $M(\bigwedge_{\psi \in KB}) \subseteq M(\varphi)$. In this case, we also say that φ follows from KB.*

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A formula φ is a **logical consequence** of a knowledge base *KB*, if every model of *KB* is a model of φ. Written:

$$KB \models g$$

(whenever *KB* is true, then φ also is true).

Example: $KB = \{man \rightarrow mortal, man\}$. Then

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Simple Example

$$KB = \begin{cases} p \leftarrow q. \\ q. \\ r \leftarrow s. \end{cases}$$

	p	q	r	s
I_1	*true*	*true*	*true*	*true*
I_2	*false*	*false*	*false*	*false*
I_3	*true*	*true*	*false*	*false*
I_4	*true*	*true*	*true*	*false*
I_5	*true*	*true*	*false*	*true*

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Model?

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I_1	*true*	*true*	*true*	*true*	is a model of KB
I_2	*false*	*false*	*false*	*false*	not a model of KB
I_3	*true*	*true*	*false*	*false*	is a model of KB
I_4	*true*	*true*	*true*	*false*	is a model of KB
I_5	*true*	*true*	*false*	*true*	not a model of KB

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I_2	*false*	*false*	*false*	*false*	not a model of KB
I_3	*true*	*true*	*false*	*false*	is a model of KB
I_4	*true*	*true*	*true*	*false*	is a model of KB
I_5	*true*	*true*	*false*	*true*	not a model of KB

...

Which of p, q, r, q logically follow from KB?

Simple Example

$$KB = \begin{cases} p \leftarrow q. \\ q. \\ r \leftarrow s. \end{cases}$$

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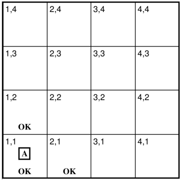
$$KB \models p,\ KB \models q,\ KB \not\models r,\ KB \not\models s$$

Proof by Contradiction



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Suppose that there exists an interpretation I in $M(KB)$ where the Wumpus is not at cell $(2,2)$. Can we conclude the cell $(2,2)$ is free?

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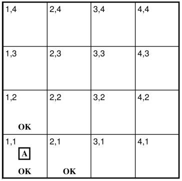

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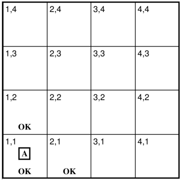

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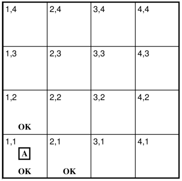

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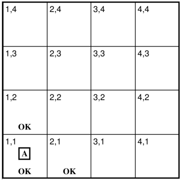

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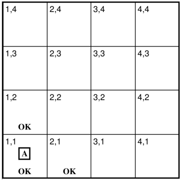

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In this course, we don't cover algorithms for testing satisfiability (this is part of the Algorithms and Satisfiability course on DAT6). But the principles are similar to what we covered in **Chapter 3**.

Agenda

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- Anyway, this still doesn't allow to compare the plausibility of different causes.

→ Logic does not allow to weigh different alternatives, and it does not allow to express incomplete knowledge ("cavity does not always come with a toothache, nor vice versa").

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- If we receive new knowledge (e.g., if we know $gum\_disease$) is true, the probability of $cavity$ changes!

$\rightarrow$ Probabilities represent and measure the uncertainty that stems from lack of knowledge.

Agenda

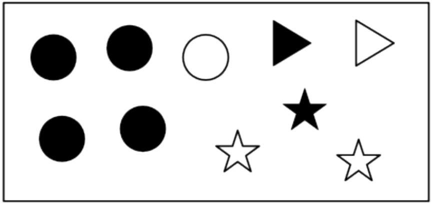
Probability Measures

Probability theory is built on the foundation of variables and worlds.

Worlds described by the variables:

- Filled: {true, false}
- Shape: {circle, triangle, star}

as well as position.



Probability measures

Ω: set of all possible worlds (for a given, fixed set of variables). A **probability measure over** Ω, is a function P, that assigns **probability values**

$$P(\Omega') \in [0, 1]$$

to subsets $\Omega' \subseteq \Omega$, such that

Axiom 1: $P(\Omega) = 1$.

Axiom 2: if $\Omega_1 \cap \Omega_2 = \emptyset$, then $P(\Omega_1 \cup \Omega_2) = P(\Omega_1) + P(\Omega_2)$.

Simplification for finite Ω

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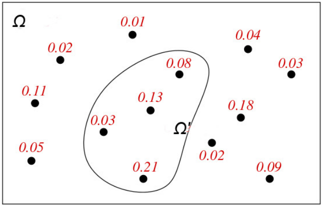

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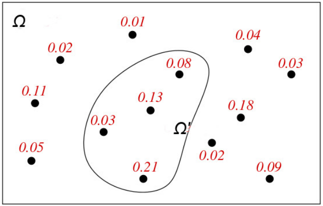

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Note: In general, random variables can have arbitrary domains. Here, we consider finite-domain random variables only, and Boolean random variables most of the time.

Random Variables and Distributions

Definition (Random Variables). Variables defining possible worlds on which probabilities are defined are called random variables.

Distributions

For a random variable A, and $a \in D_A$ we have the probability

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The **probability distribution of** A (written $P(A)$) is the function on D_A that maps each a to its probability $P(A = a)$.

Example:

$$\mathbf{P}(Headache) = \langle F \mapsto 0.1, T \mapsto 0.9 \rangle$$

$$\mathbf{P}(Weather) = \langle sunny \mapsto 0.7, rain \mapsto 0.2, cloudy \mapsto 0.08, snow \mapsto 0.02 \rangle$$

Joint Probability Distributions

Extension to several random variables:

$$P(A_1, \dots, A_k)$$

is the joint distribution of $A_1, \dots, A_k$. The joint distribution maps tuples $(a_1, \dots, a_k)$ with $a_i \in D_{A_i}$ to the probability

$$P(A_1 = a_1, \dots, A_k = a_k)$$

Example: $\mathbf{P}(Headache, Weather) =$

	$Headache = true$	$Headache = false$
$Weather = sunny$	$P(W = sunny \wedge headache)$	$P(W = sunny \wedge \neg headache)$
$Weather = rain$		
$Weather = cloudy$		
$Weather = snow$		

Terminology:

- Given random variables $\{X_1, \dots, X_n\}$, an atomic event (world) is an assignment of values to all variables.
- Given random variables $\{X_1, \dots, X_n\}$, the full joint probability distribution, denoted $\mathbf{P}(X_1, \dots, X_n)$, lists the probabilities of all atomic events.

$\rightarrow$ All worlds are disjoint (their pairwise conjunctions all are $\perp$); the sum of all fields is 1 (corresponds to their disjunction $\top$).

Probabilities of propositions

A probability distribution over possible worlds defines probabilities for formulas φ:

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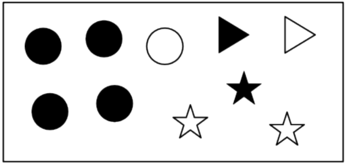

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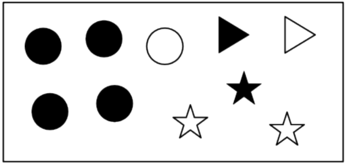

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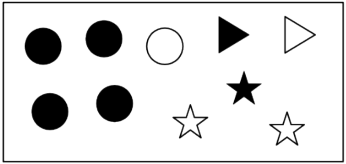

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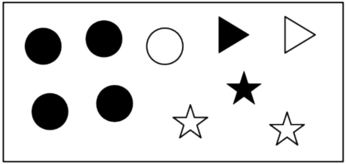

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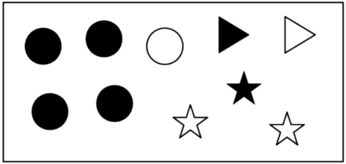

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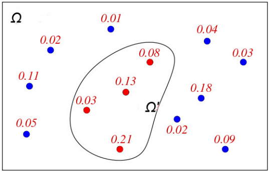

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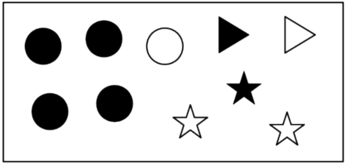
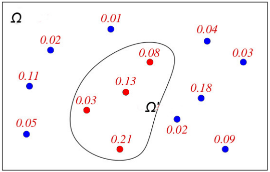


Assume probability for each world is 0.1:

- $P(\mathit{Shape} = \mathit{circle}) = 0.5$
- $P(\mathit{Filled} = \mathit{false}) = 0.4$
- $P(\mathit{Shape} = c \wedge \mathit{Filled} = f) = 0.1$

Another example



$$P(\mathit{Color} = \mathit{red}) = 0.08 + 0.13 + 0.03 + 0.21 = 0.45$$

Basic probability axioms

Axiom

If $\mathcal{A}$ and $\mathcal{B}$ are disjoint, then $P(\mathcal{A} \cup \mathcal{B}) = P(\mathcal{A}) + P(\mathcal{B})$.

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Consider a deck with 52 cards. If $\mathcal{A} = \{2, 3, 4, 5\}$ and $\mathcal{B} = \{7, 8\}$, then

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If $\mathcal{C}$ and $\mathcal{D}$ are not disjoint, then $P(\mathcal{C} \cup \mathcal{D}) = P(\mathcal{C}) + P(\mathcal{D}) - P(\mathcal{C} \cap \mathcal{D})$.

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If $\mathcal{C} = \{2, 3, 4, 5\}$ and $\mathcal{D} = \{\spadesuit\}$, then

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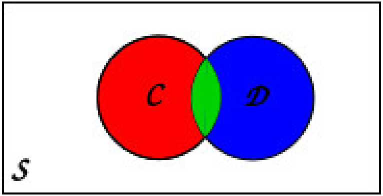

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If $\mathcal{C} = \{2, 3, 4, 5\}$ and $\mathcal{D} = \{\spadesuit\}$, then

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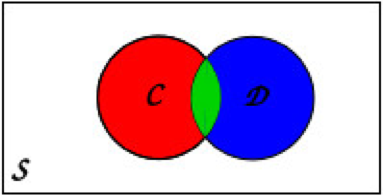

Agenda

Updating Your Beliefs

→ Do probabilities change as we gather new knowledge?

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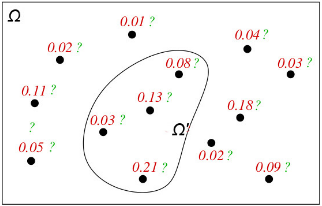

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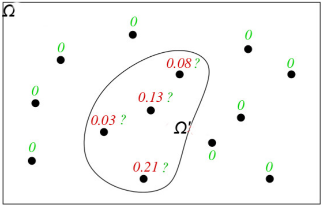

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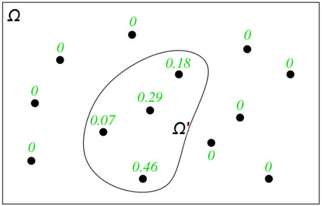

- worlds that are not consistent with evidence have probability 0
- the probabilities of worlds consistent with evidence are proportional to their probability before observation, and they must sum to 1

Conditional Probabilities

Definition. *Given propositions p and e where $P(e) \neq 0$, the conditional probability, or posterior probability, of p given e, written $P(p \mid e)$, is defined as:*

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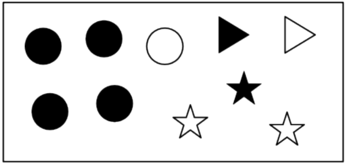

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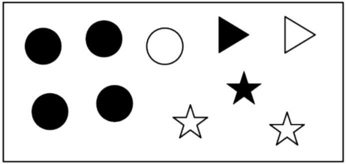

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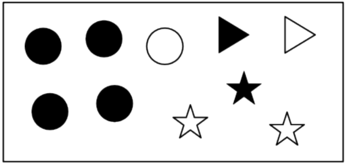

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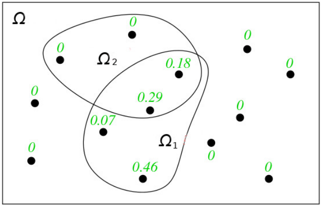

- e and p are represented by possible worlds Ω_1 and Ω_2
- division by $P(\Omega_1)$ already in green numbers

$$P(\Omega_2 \mid \Omega_1) = 0.18 + 0.29$$

Conditional Probability Distributions

Definition. *Given random variables X and Y, the conditional probability distribution of X given Y, written $\mathbf{P}(X \mid Y)$, is the table of all conditional probabilities of values of X given values of Y.*

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Example: $\mathbf{P}(Weather \mid Headache) =$

	$Headache = true$	$Headache = false$
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$\rightarrow$ "The probability of sunshine given that I have a headache?" If you're susceptible to headaches depending on weather conditions, this makes sense. Otherwise, the two variables are independent (we'll get to this later).

Agenda

The Product Rule

Proposition (Product Rule). *Given propositions* A *and* B, $P(a \wedge b) = P(a \mid b)P(b)$.
(Direct from definition.)

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Notation: $\mathbf{P}(X, Y) = \mathbf{P}(X \mid Y)\mathbf{P}(Y)$ is a system of equations:

$$\begin{aligned}
P(W = sunny \wedge headache) &= P(W = sunny \mid headache)P(headache) \\
P(W = rain \wedge headache) &= P(W = rain \mid headache)P(headache) \\
\dots &= \dots \\
P(W = snow \wedge \neg headache) &= P(W = snow \mid \neg headache)P(\neg headache)
\end{aligned}$$

$\rightarrow$ Similar for unconditional distributions, $\mathbf{P}(X, Y) = \mathbf{P}(X)\mathbf{P}(Y)$.

The Chain Rule

Proposition (Chain Rule). *Given random variables $X_1, \dots, X_n$, we have* $\mathbf{P}(X_1, \dots, X_n) = \mathbf{P}(X_n \mid X_{n-1}, \dots, X_1) * \mathbf{P}(X_{n-1} \mid X_{n-2}, \dots, X_1) * \dots * \mathbf{P}(X_2 \mid X_1) * \mathbf{P}(X_1)$.

Example: $P(\neg brush \wedge cavity \wedge toothache)$
$= P(toothache \mid cavity, \neg brush)P(cavity, \neg brush)$
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Note: This works *for any ordering* of the variables.

$\rightarrow$ We can recover the probability of atomic events from sequenced conditional probabilities for any ordering of the variables.

$\rightarrow$ First of the four basic techniques in Bayesian networks.

Marginalization

→ Extracting a sub-distribution from a larger joint distribution:

Proposition (Marginalization). *Given sets* $\mathbf{X}$ *and* $\mathbf{Y}$ *of random variables, we have:*

$$\mathbf{P}(\mathbf{X}) = \sum_{\mathbf{y} \in \mathbf{Y}} \mathbf{P}(\mathbf{X}, \mathbf{y})$$

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Example: (Note: Equation system!)

$$\mathbf{P}(Cavity) = \sum_{y \in Toothache} \mathbf{P}(Cavity, y)$$

$$P(cavity) = P(cavity, toothache) + P(cavity, \neg toothache)$$

$$P(\neg cavity) = P(\neg cavity, toothache) + P(\neg cavity, \neg toothache)$$

Questionnaire

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Say $P(dog) = 0.4$**,** $\neg dog \leftrightarrow cat$**, and** $P(likeslasagna \mid cat) = 0.5$**. Then** $P(likeslasagna \wedge cat) =$

(A): 0.2

(B): 0.5

(C): 0.475

(D): 0.3

Question!

Can we compute the value of $P(likeslasagna)$**, given the above informations?**

(A): Yes.

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Can we compute the value of $P(likeslasagna)$**, given the above informations?**

(A): Yes. (B): No.

$\rightarrow$ No. We don't know the probability that *dogs* like lasagna, i.e., $P(likeslasagna \mid dog)$.

Normalization: Idea

Problem: We know $P(cavity \wedge toothache)$ but don't know $P(toothache)$:

$$P(cavity \mid toothache) = \frac{P(cavity \wedge toothache)}{P(toothache)} = \frac{0.12}{P(toothache)}$$

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$$P(\neg cavity \mid toothache) = \frac{P(\neg cavity \wedge toothache)}{P(toothache)} = \frac{0.08}{P(toothache)}$$

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Step 3: Fixing $toothache$ to be true, view $P(cavity \wedge toothache)$ vs. $P(\neg cavity \wedge toothache)$ as the relative weights of $P(cavity)$ vs. $P(\neg cavity)$ within $toothache$. Then normalize their summed-up weight to 1:

$$1 = \alpha(0.12 + 0.08) \Rightarrow \alpha = 1/(0.12 + 0.08) = 1/0.2 = 5$$

$\rightarrow$ α is the normalization constant scaling the sum of relative weights to 1.

Normalization: Formal

Definition. *Given a vector* $\langle w_1, \dots, w_k \rangle$ *of numbers in* $[0,1]$ *where* $\sum_{i=1}^{k} w_i \leq 1$*, the normalization constant* α *is* $\alpha\langle w_1, \dots, w_k \rangle := 1/\sum_{i=1}^{k} w_i$.

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Proposition (Normalization). *Given a random variable* X *and an event* $\mathbf{e}$*, we have* $\mathbf{P}(X \mid \mathbf{e}) = \alpha\mathbf{P}(X, \mathbf{e})$.

Proof.

Normalization: Formal

Definition. *Given a vector* $\langle w_1, \dots, w_k \rangle$ *of numbers in* $[0, 1]$ *where* $\sum_{i=1}^{k} w_i \leq 1$*, the normalization constant* α *is* $\alpha\langle w_1, \dots, w_k \rangle := 1/\sum_{i=1}^{k} w_i$.

Example: $\alpha\langle 0.12, 0.08 \rangle = 5\langle 0.12, 0.08 \rangle = \langle 0.6, 0.4 \rangle$.

Proposition (Normalization). *Given a random variable* X *and an event* $\mathbf{e}$*, we have* $\mathbf{P}(X \mid \mathbf{e}) = \alpha\mathbf{P}(X, \mathbf{e})$.

Proof. For each value x of X, $P(X = x \mid \mathbf{e}) = P(X = x \wedge \mathbf{e})/P(\mathbf{e})$. So all we need to prove is that $\alpha = 1/P(\mathbf{e})$.

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Normalization+Marginalization: Given "query variable" X, "observed event" $\mathbf{e}$, and "hidden variables" set $\mathbf{Y}$: $\mathbf{P}(X \mid \mathbf{e}) = \alpha\mathbf{P}(X, \mathbf{e}) = \alpha \sum_{\mathbf{y} \in \mathbf{Y}} \mathbf{P}(X, \mathbf{e}, \mathbf{y})$.

$\rightarrow$ Second of the four basic techniques in Bayesian networks.

Questionnaire

Question!

Say we know $P(likeschappi \wedge dog) = 0.32$ **and** $P(\neg likeschappi \wedge dog) = 0.08$**. Can we compute** $P(likeschappi \mid dog)$**?**

(A): Yes. (B): No.

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(A): Yes. (B): No.

$\rightarrow$ Yes, because we can compute $P(dog) = P(likeschappi \wedge dog) + P(\neg likeschappi \wedge dog)$, and thus $P(likeschappi \mid dog) = \frac{P(likeschappi \wedge dog)}{P(dog)}$.

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Inserting $LikesChappi$ for X and dog for $\mathbf{e}$, we get $\mathbf{P}(LikesChappi \mid dog) = \alpha \mathbf{P}(LikesChappi, dog) = \alpha \langle P(likeschappi \wedge dog), P(\neg likeschappi \wedge dog) \rangle = \alpha \langle 0.32, 0.08 \rangle$.

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→ So what is $P(likeschappi \mid dog)$?

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→ So what is $P(likeschappi \mid dog)$? 0.8, because $\alpha = 1/P(dog) = 1/(0.32 + 0.08) = 2.5$.

Agenda

Bayes' Rule

Proposition (Bayes' Rule). *Given propositions A and B where $P(a) \neq 0$ and $P(b) \neq 0$, we have:*

$$P(a \mid b) = \frac{P(b \mid a)P(a)}{P(b)}$$

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Notation: (System of equations)

$$\mathbf{P}(X \mid Y) = \frac{\mathbf{P}(Y \mid X)\mathbf{P}(X)}{\mathbf{P}(Y)}$$

Applying Bayes' Rule

Example: Say we know that $P(toothache \mid cavity) = 0.6$, $P(cavity) = 0.2$, and $P(toothache) = 0.2$.

$\rightarrow$ We can compute $P(cavity \mid toothache)$:

Applying Bayes' Rule

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- Also, causal dependencies are often easier to assess.

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$\rightarrow$ Bayes' rule allows to perform diagnosis (observing a symptom, what is the cause?) based on prior probabilities and causal dependencies.

Questionnaire

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Say $P(dog) = 0.4$**,** $P(likeschappi \mid dog) = 0.8$**, and** $P(likeschappi) = 0.5$**. What is** $P(dog \mid likeschappi)$**?**

(A): 0.8

(B): 0.64

(C): 0.9

(D): 0.32

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$\rightarrow$ Is $P(likeschappi \mid dog)$ causal or diagnostic? Causal; liking or not liking dog food may be caused by being or not being a dog.

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→ By Bayes' rule, $P(dog \mid likeschappi) = \frac{P(likeschappi|dog)P(dog)}{P(likeschappi)} = \frac{0.8*0.4}{0.5} = 0.64$ so (B).

→ Is $P(likeschappi \mid dog)$ causal or diagnostic? Causal; liking or not liking dog food may be caused by being or not being a dog.

→ Is $P(dog \mid likeschappi)$ causal or diagnostic? Diagnostic; liking Chappi does not cause anybody to be a dog.

Bayes' rule for variables

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Consider the variables

- Temp : $\mathrm{sp}(\text{Temp}) = \{l, m, h\}$
- Sensor : $\mathrm{sp}(\text{Sensor}) = \{l, m, h\}$

$$P(\text{Temp}) = (0.1, 0.6, 0.3)$$

$P(\text{Sensor}|\text{Temp}) =$

		Temp		
		l	m	h
Sensor	l	0.8	0.1	0.05
	m	0.15	0.8	0.1
	h	0.05	0.1	0.85

Question!

Assume we observe $S =$ low, what is our updated belief in the temperature?

(A): $P(T = low \mid S = low) = 0.1$

(B): $P(T = low \mid S = low) = 0.52$

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$$P(T|\text{low}) = \frac{P(\text{low}|T)P(T)}{\sum_T P(\text{low}|T)P(T)} =$$

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$P(\text{low}|T)P(T) =$

	l	m	h
$S = \text{low}$	0.08	0.06	0.015

$\rightarrow \frac{P(S = \text{low})}{0.155}$

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	Temp=l	Temp=m	Temp=h
$S = $ low	0.08 / 0.155	0.06 / 0.155	0.015 / 0.155

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	l	m	h
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$\rightarrow$

$P(S = \text{low})$
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	l	m	h
$S = $ low	0.08	0.06	0.015

$\rightarrow \frac{P(S = \text{low})}{0.155}$

Agenda

Working with the Full Joint Probability Distribution

Example:

	$toothache$	$\neg toothache$
$cavity$	0.12	0.08
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→ How to compute $P(cavity \mid toothache)$? $\frac{P(cavity \wedge toothache)}{P(toothache)}$

→ All relevant probabilities can be computed using the full joint probability distribution, by expressing propositions as disjunctions of atomic events.

Working with the Full Joint Probability Distribution??

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→ Bayesian networks. (First, we do the simple case.)

Independence

Definition. *Events a and b are independent if $P(a \wedge b) = P(a)P(b)$.*

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Examples:

- $P(\mathit{Dice1} = 6 \wedge \mathit{Dice2} = 6) = 1/36$.
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Definition. *Random variables X and Y are independent if $\mathbf{P}(X, Y) = \mathbf{P}(X)\mathbf{P}(Y)$.* (System of equations!)

Example: Football statistics

Results for Bayern Munich and SC Freiburg in seasons 2001/02 and 2003/04. (Not counting the matches Munich vs. Freiburg):

$$D_{Munich)} = D_{Freiburg} = \{Win,\ Draw,\ Loss\}$$

2001/02

Munich: `LWDWWWWWWWWLDLDLDLWLDWWWDWDDWWWW`

Freiburg: `WLLDDWLDWDWLLLDDLWDDLLDLLLLLLWLW`

2003/04

Munich: `WDWWLDWWDWLWWDDWDWLWWWDDWWWLWWLL`

Freiburg: `LDDWDWLWLLLWWLWLWLLDWLDDWDLLLWLD`

Summary:

Munich	Freiburg W	D	L	
W	12	9	15	36
D	3	4	9	16
L	6	4	2	12
	21	17	26	

Independence of Outcomes

The joint distribution of *Munich* and *Freiburg*:

$P($*Munich*,*Freiburg*$)$:

Munich	*Freiburg* W	D	L	$P($*Munich*$)$
W	.1875	.1406	.2344	.5625
D	.0468	.0625	.1406	.25
L	.0937	.0625	.0312	.1875
$P($*Freiburg*$)$	.3281	.2656	.4062	

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$P(\textit{Munich}, \textit{Freiburg})$:

	Freiburg			
Munich	W	D	L	$P(\textit{Munich})$
W	.1875	.1406	.2344	.5625
	.571	.529	.577	
D	.0468	.0625	.1406	.25
	.143	.235	.346	
L	.0937	.0625	.0312	.1875
	.285	.235	.077	
$P(\textit{Freiburg})$	.3281	.2656	.4062	

Conditional distribution: $P(\textit{Munich} \mid \textit{Freiburg})$

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Conditional distribution: $P(\text{Munich} \mid \text{Freiburg})$

We have (almost):

$$P(\text{Munich} \mid \text{Freiburg}) = P(\text{Munich})$$

The variables *Munich* and *Freiburg* are **independent**.

Independent Variables

Definition of Independence

The variables $A_1, \dots, A_k$ and $B_1, \dots, B_m$ are **independent** if

$$P(A_1, \dots, A_k \mid B_1, \dots, B_m) = P(A_1, \dots, A_k)$$

This is equivalent to:

$$P(B_1, \dots, B_m \mid A_1, \dots, A_k) = P(B_1, \dots, B_m)$$

and also to:

$$P(A_1, \dots, A_k, B_1, \dots, B_m) = P(A_1, \dots, A_k) \cdot P(B_1, \dots, B_m)$$

Compact Specifications by Independence

Independence properties can greatly simplify the specification of a joint distribution:

	$F =$			
$M =$	W	D	L	$P(M)$
W				.5625
D		M and F are independent		.25
L				.1875
$P(F)$	.3281	.2656	.4062	

The probability for each possible world then is defined, e.g.

$$P(M = D, F = L) = 0.25 \cdot 0.4062 = 0.10155$$

$\rightarrow$ Independence can be exploited to represent the full joint probability distribution more compactly.

$\rightarrow$ Usually, random variables are independent only under particular conditions: conditional independence, see next section.

Agenda

Questionnaire

$$\begin{array}{ll} \mathit{Hair\ length:} & D_{\mathit{Hair\ length}} = \{\mathit{long}, \mathit{short}\} \\ \mathit{Height:} & D_{\mathit{Height}} = \{\mathit{tall}, \mathit{medium}\} \end{array}$$

Question!

Are Hair length and Height independent?

(A): Yes

(B): No

Questionnaire

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Question!

Are Hair length and Height independent?

(A): Yes (B): No

Joint Distribution:

	Sex			
	male		*female*	
	Hair length		*Hair length*	
Height	*long*	*short*	*long*	*short*
tall	0.06	0.24	0.07	0.03
medium	0.04	0.16	0.28	0.12

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medium	0.04	0.16	0.28	0.12

$P(\textit{Hair length}, \textit{Height})$ $P(\textit{Height})$, $P(\textit{Height} \mid \textit{Hair length})$:

	Hair length		
Height	*long*	*short*	
tall	0.13	0.27	0.4
	0.289	0.49	
medium	0.32	0.28	0.6
	0.711	0.51	

$\leadsto$ No, *Hair length* and *Height* are not independent. Tall people have shorter hair! Does that sound right to you?

Conditional Independence

Definition. *Given sets of random variables* $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}$, *we say that* $\mathbf{Z}_1$ *and* $\mathbf{Z}_2$ *are conditionally independent given* $\mathbf{Z}$ *if:*

$$\mathbf{P}(\mathbf{Z}_1, \mathbf{Z}_2 \mid \mathbf{Z}) = \mathbf{P}(\mathbf{Z}_1 \mid \mathbf{Z})\mathbf{P}(\mathbf{Z}_2 \mid \mathbf{Z})$$

We alternatively say that $\mathbf{Z}_1$ *is conditionally independent of* $\mathbf{Z}_2$ *given* $\mathbf{Z}$.

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$P(\textit{Hair length}, \textit{Height} \mid \textit{Sex} = \textit{female})$, $P(\textit{Height} \mid \textit{Sex} = \textit{female})$, $P(\textit{Height} \mid \textit{Hair length}, \textit{Sex} = \textit{female})$:

	Hair length		
Height	*long*	*short*	
tall	0.14	0.06	0.2
	0.2	0.2	
medium	0.56	0.24	0.8
	0.8	0.8	

$\leadsto$ *Hair length* and *Height* are independent given *Sex*=*female*.

Also: *Hair length* and *Height* are independent given *Sex*=*male*.

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$\rightsquigarrow$ *Hair length* and *Height* are independent given *Sex*=*female*.

Also: *Hair length* and *Height* are independent given *Sex*=*male*.

$\rightsquigarrow$ *Hair length* and *Height* are independent given *Sex*.

Note: The definition is symmetric regarding the roles of $\mathbf{Z}_1$ and $\mathbf{Z}_2$: $hairlength$ is conditionally independent of $height$, *and vice versa*.

Bayes' Rule with Multiple Evidence

Example: Say we know from medicinical studies that $P(cavity) = 0.2$, $P(toothache \mid cavity) = 0.6$, $P(toothache \mid \neg cavity) = 0.1$, $P(catch \mid cavity) = 0.9$, and $P(catch \mid \neg cavity) = 0.2$. Now, in case we did observe the symptoms toothache and catch (the dentist's probe catches in the aching tooth), what would be the likelihood of having a cavity? What is $P(cavity \mid catch, toothache)$?

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$$P(cavity \mid catch, toothache) = \frac{P(catch, toothache \mid cavity)P(cavity)}{P(catch, toothache)}$$

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Question!

So, is everything fine? Do we just need some more medicinical studies?

(A): Yes. (B): No.

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$\rightarrow$ No! We would need $\mathbf{P}(toothache \wedge catch \mid Cavity)$, i.e., causal dependencies for all combinations of symptoms! ($\gg 2$, in general)

Bayes' Rule with Multiple Evidence, ctd.

Second attempt: First Normalization (slide 34), then Chain Rule (slide 30) using ordering $X_1 = Cavity, X_2 = Catch, X_3 = Toothache$:

$$\mathbf{P}(Cavity \mid catch, toothache) =$$
$$\alpha \mathbf{P}(Cavity, catch, toothache) =$$
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Close, but no Banana: Less red (i.e.unknown) probabilities, but still $\mathbf{P}(toothache \mid catch, Cavity)$.

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But: They are independent given the presence or absence of cavity!

$\rightarrow$ See next slide.

Conditional Independence, ctd.

Proposition. *If* $\mathbf{Z}_1$ *and* $\mathbf{Z}_2$ *are conditionally independent given* $\mathbf{Z}$*, then* $\mathbf{P}(\mathbf{Z}_1 \mid \mathbf{Z}_2, \mathbf{Z}) = \mathbf{P}(\mathbf{Z}_1 \mid \mathbf{Z})$.

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Conditional Independence, ctd.

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→ In the presence of conditional independence, we can drop variables from the right-hand side of conditional probabilities.

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$\rightarrow$ Third of the four basic techniques in Bayesian networks. Last missing technique: "Capture variable dependencies in a graph"; illustration see Conclusions, details see **Next Chapter**.

Agenda

Summary

- Reasoning can be attained by a combination of logic and probability.
- Deduction is about deriving conclusions that follow logically from our knowledge base.
- Uncertainty is unavoidable in many environments, namely whenever agents do not have perfect knowledge.
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- Conditional probabilities express the likelihood of an event given observed evidence.
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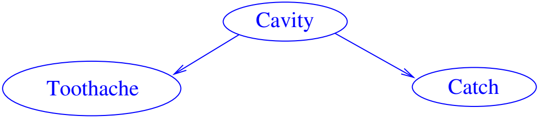
  → Bayesian networks (up next) do this, in a comprehensive manner (see next slides for some spoilers of where are we headed).

Exploiting Conditional Independence: Overview

1. **Graph captures variable dependencies:** (Variables $X_1, \dots, X_n$)



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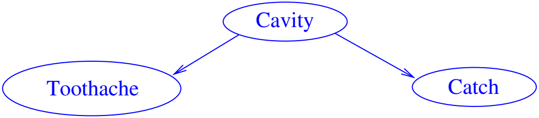

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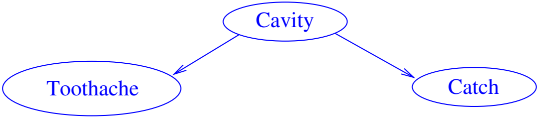

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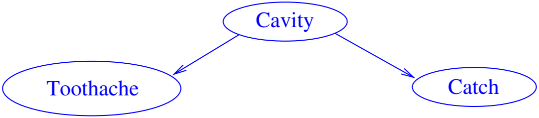

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→ Bayesian networks!

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$\alpha\langle 0.9 * 0.6 * 0.2, 0.2 * 0.1 * 0.8\rangle = \alpha\langle 0.108, 0.016\rangle$. So $\alpha \approx 8.06$ and
$\mathbf{P}(cavity \mid toothache \wedge catch) \approx 0.87$.