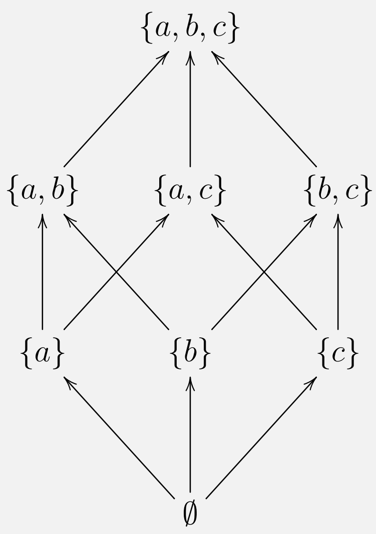
Exercise 1*

Draw a graphical representation of the complete lattice $(2^{\{a,b,c\}}, \subseteq)$ and compute supremum and infimum of the following sets:

1. $\sqcap\{\{a\},\{b\}\} = ?$
2. $\sqcup\{\{a\},\{b\}\} = ?$
3. $\sqcap\{\{a\},\{a,b\},\{a,c\}\} = ?$
4. $\sqcup\{\{a\},\{a,b\},\{a,c\}\} = ?$
5. $\sqcap\{\{a\},\{b\},\{c\}\} = ?$
6. $\sqcup\{\{a\},\{b\},\{c\}\} = ?$
7. $\sqcap\{\{a\},\{a,b\},\{b\},\emptyset\} = ?$
8. $\sqcup\{\{a\},\{a,b\},\{b\},\emptyset\} = ?$

Solution of Exercise 1

The complete lattice:



1. $\sqcap\{\{a\},\{b\}\} = \emptyset$
2. $\sqcup\{\{a\},\{b\}\} = \{a,b\}$
3. $\sqcap\{\{a\},\{a,b\},\{a,c\}\} = \{a\}$
4. $\sqcup\{\{a\},\{a,b\},\{a,c\}\} = \{a,b,c\}$

5. $\sqcap\{\{a\},\{b\},\{c\}\} = \emptyset$
6. $\sqcup\{\{a\},\{b\},\{c\}\} = \{a,b,c\}$
7. $\sqcap\{\{a\},\{a,b\},\{b\},\emptyset\} = \emptyset$
8. $\sqcup\{\{a\},\{a,b\},\{b\},\emptyset\} = \{a,b\}$

Exercise 2

Prove that for any partially ordered set $(D, \sqsubseteq)$ and any $X \subseteq D$, if supremum of X ($\sqcup X$) and infimum of X ($\sqcap X$) exist then they are uniquely defined. (Hint: use the definition of supremum and infimum and antisymmetry of $\sqsubseteq$.)

Solution of Exercise 2

We prove the claim for the supremum (least upper bound) of X. The arguments for the infimum are symmetric. Let $d_1, d_2 \in D$ be two supremums of a given set X. This means that $X \sqsubseteq d_1$ and $X \sqsubseteq d_2$ as both d_1 and d_2 are upper bounds of X. Now because d_1 is the least upper bound and d_2 is an upper bound we get $d_1 \sqsubseteq d_2$. Similarly, d_2 is the least upper bound and d_1 is an upper bound so $d_2 \sqsubseteq d_1$. However, from antisymmetry and $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_1$ we get that $d_1 = d_2$.

Exercise 3

Let $(D, \sqsubseteq)$ be a complete lattice. What are $\sqcup\emptyset$ and $\sqcap\emptyset$ equal to?

Solution of Exercise 3

- $\sqcup\emptyset = \bot = \sqcap D$.
- $\sqcap\emptyset = \top = \sqcup D$.

Exercise 4*

Consider the complete lattice $(2^{\{a,b,c\}}, \subseteq)$. Define a function $f : 2^{\{a,b,c\}} \to 2^{\{a,b,c\}}$ such that f is monotonic.

- Compute the greatest fixed point by using directly the Tarski's fixed point theorem.

- Compute the least fixed point by using the Tarski's fixed point theorem for finite lattices (i.e. by starting from $\perp$ and by applying repeatedly the function f until the fixed point is reached).

Solution of Exercise 4

For example we define f as follows (note that there are many possibilites):

S	$f(S)$
$\emptyset$	$\{a\}$
$\{a\}$	$\{a\}$
$\{b\}$	$\{a\}$
$\{c\}$	$\{a\}$
$\{a,b,c\}$	$\{a,b\}$
$\{a,b\}$	$\{a,b\}$
$\{a,c\}$	$\{a,b\}$
$\{b,c\}$	$\{a,b\}$

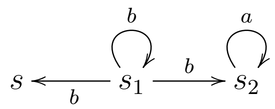
The function f is monotonic which we can verify by a case inspection.

- According to Tarski's fixed point theorem, the greatest fixed point $gfp(f)$ is given by $gfp(f) = \sqcup A$, where
$$A = \left\{x \in 2^{\{a,b,c\}} \mid x \sqsubseteq f(x)\right\}.$$
In our case, by the definition of f, we get $A = \left\{\emptyset, \{a\}, \{a,b\}\right\}$. The supremum of A in $2^{\{a,b,c\}}$ is $\{a,b\}$. So, by Tarski's fixed point theorem, the greatest fixed point of f is $\{a,b\}$.

- First note that $\perp = \sqcap 2^{\{a,b,c\}} = \emptyset$. We now repeatedly apply f until it stabilizes
$$f(\emptyset) = \{a\}$$
$$f(f(\emptyset)) = f(\{a\}) = \{a\}$$
and hence the least fixed point of f is $\{a\}$.

Exercise 5

Consider the following labelled transition system.



Compute for which sets of states $[\![X]\!] \subseteq \{s, s_1, s_2\}$ the following formulae are true.

- $X = \langle a \rangle tt \vee [b]X$
- $X = \langle a \rangle tt \vee ([b]X \wedge \langle b \rangle tt)$

Solution of Exercise 5

- $X = \langle a \rangle tt \vee [b]X$
  - The equation holds for the following sets of states: $\{s_2, s\}, \{s_2, s_1, s\}$.
- $X = \langle a \rangle tt \vee ([b]X \wedge \langle b \rangle tt)$
  - The equation holds only for the set $\{s_2\}$.