Exercise 1

a) What are all the possible values that can be stored in the variable x after the execution of the following parallel program when assignment is assumed to be an atomic instruction?

$$\texttt{x:=10;}\ \big(\texttt{(x:=x*2; x:=x-11; x:=x+2)} \parallel \texttt{x:=x-5}\big)$$

b) Does the following parallel program satisfy the given pre and post condition?

$$\texttt{[x=0] (x++} \parallel \texttt{x++) [x=2]}$$

Solution of Exercise 1

a) The possible values stored in the variable x are 1 or 6. Note that there are *four* different executions.

b) The correctness depends on the level of granuality we consider. If the increment of x is considered as an atomic operation, then the answer is yes. On the other hand, the assignment `x++` can be compiled to assembly language as follows (`R` is a register and addr_x is the address where the value of x is stored)

```
LOAD R, addr_x
INC R
STORE R, addr_x
```

and then thanks to the interleaving of the instructions a possible value stored in x is also 1.

Exercise 2

Let R be a binary relation on a set A. Let us define the binary relation

$$E \stackrel{\text{def}}{=} \{(x, x) \mid x \in A\}.$$

It is trivially true that $R \cup E$ is a reflexive relation.

- Argue that $R \cup E$ is a reflexive closure of R.

Solution of Exercise 2

Since $R \cup E$ is indeed a reflexive relation and contains R, it suffices to show that $R \cup E$ is minimal. Suppose R'' is a binary relation such that (i) $R \subseteq R''$ and (ii) R'' is reflexive. We will show that $R \cup E \subseteq R''$. Let $(x, y) \in R \cup E$. Then at least one of the following must hold. Either $(x, y) \in R$ or $(x, y) \in E$. If $(x, y) \in R$, then $(x, y) \in R''$ because of (i). If $(x, y) \in E$, then $x = y$ so $(x, y) \in R''$ because of (ii). Hence $R \cup E \subseteq R''$.

Exercise 3

Let R be a binary relation on a set A. Let us define the binary relation

$$R^{-1} \stackrel{\text{def}}{=} \{(y, x) \mid (x, y) \in R\}.$$

- Argue that $R \cup R^{-1}$ is a symmetric relation.
- Argue that $R \cup R^{-1}$ is a symmetric closure of R.

Solution of Exercise 3

- Let $(x, y) \in R \cup R^{-1}$. If $(x, y) \in R$ then by the definition of R^{-1}, $(y, x) \in R^{-1}$, which establishes that $(y, x) \in R \cup R^{-1}$. If $(x, y) \in R^{-1}$ then by the definition of R^{-1}, $(y, x) \in R$, which establishes that $(y, x) \in R \cup R^{-1}$.
- Arguing that $R \cup R^{-1}$ is a symmetric closure of R is done similarly to Exercise 2. Let R'' be a binary relation that contains R and is symmetric. Then it is easy to see that $R \cup R^{-1} \subseteq R''$ by arguing as in Exercise 2.

Exercise 4

Consider a binary relation $succ \subseteq \mathbb{N} \times \mathbb{N}$ defined by

$$succ = \{(n, n + 1) \mid n \in \mathbb{N}\} .$$

Answer the following questions:

- What relation is the transitive closure of $succ$?
- What relation is the reflexive and transitive closure of $succ$?
- What relation is the symmetric, reflexive and transitive closure of $succ$?

Solution of Exercise 4

Consider a binary relation $succ \subseteq \mathbb{N} \times \mathbb{N}$ defined by

$$succ = \{(n, n + 1) \mid n \in \mathbb{N}\} .$$

Answer the following questions:

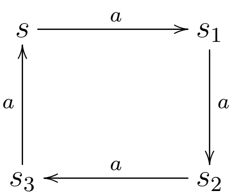
- What relation is the transitive closure of $succ$?
  It is the standard strict ordering relation $<$ on natural numbers.

- What relation is the reflexive and transitive closure of $succ$?
  It is the standard less or equal relation $\leq$ on natural numbers.

- What relation is the symmetric, reflexive and transitive closure of $succ$?
  It is the relation $\mathbb{N} \times \mathbb{N}$.

Exercise 5*

Let us consider the following labelled transition system.



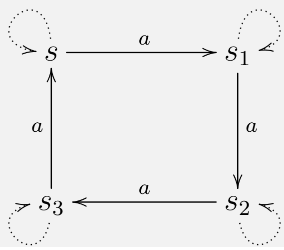
- Define the labelled transition system as a triple $(Proc, Act, \{\xrightarrow{a} \mid a \in Act\})$.
- What is the reflexive closure of the binary relation $\xrightarrow{a}$? (A drawing is fine.)
- What is the symmetric closure of the binary relation $\xrightarrow{a}$? (A drawing is fine.)
- What is the transitive closure of the binary relation $\xrightarrow{a}$? (A drawing is fine.)

Solution of Exercise 5

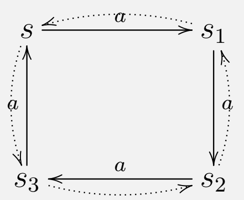
- – $Proc = \{s, s_1, s_2, s_3\}$
  – $Act = \{a\}$
  – $\xrightarrow{a} = \{(s, s_1), (s_1, s_2), (s_2, s_3), (s_3, s)\}$

  In the following three diagrams the dotted arrows indicate the pairs of states that must be added to the relation $\xrightarrow{a}$ in order to obtain the indicated closures.

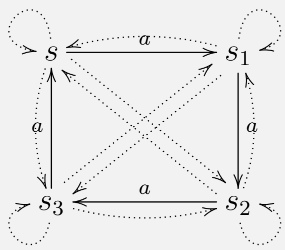
- Reflexive closure



- Symmetric closure



- Transitive closure



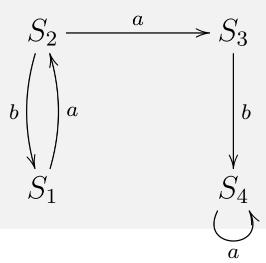
Exercise 6

Assume a given labelled transition system $T = (\mathit{Proc}, \mathit{Act}, \{\xrightarrow{a} \mid a \in \mathit{Act}\})$ such that the sets Proc and Act are finite.

- Does it imply that $\xrightarrow{a}$ is also a finite set? Why?
- Draw an example of an LTS with four states and two actions.
- How can your example be described by a sequential fragment of CCS (with Nil, action prefixing, nondeterminism and recursive definitions of names)?
- Show that in general any finite LTS T can be described by using only a sequential fragment of CCS.

Solution of Exercise 6

- Yes, there are at most $|\mathit{Proc}|^2$ elements in $\xrightarrow{a}$.
- An LTS with four states and two actions (you probably have a different one).



- CCS description of the LTS

$$
\begin{aligned}
S_1 &\stackrel{\text{def}}{=} a.S_2 \\
S_2 &\stackrel{\text{def}}{=} a.S_3 + b.S_1 \\
S_3 &\stackrel{\text{def}}{=} b.S_4 \\
S_4 &\stackrel{\text{def}}{=} a.S_4
\end{aligned}
$$

- Describing a finite LTS using CCS can be done as follows. For each $S \in \mathit{Proc}$ add a (possibly recursive) definition of a new process constant S as follows:
  - If $S \not\longrightarrow$ then add the defining equation $S \stackrel{\text{def}}{=} Nil$, otherwise
  - let $S \xrightarrow{a_1} S_1,\ S \xrightarrow{a_2} S_2,\ \ldots,\ S \xrightarrow{a_n} S_n$ be all the transitions going out of S. Then add the following defining equation $S \stackrel{\text{def}}{=} a_1.S_1 + a_2.S_2 + \cdots + a_n.S_n$.