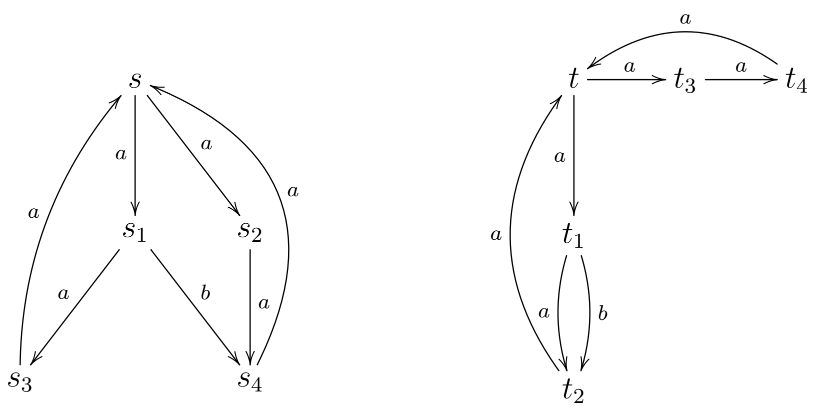
Exercise 1*

Consider the following labelled transition system.



Show that $s \sim t$ by finding a strong bisimulation R containing the pair (s, t).

Solution of Exercise 1

If we can show that $R = \{(s,t), (s_1,t_1), (s_3,t_2), (s_4,t_2), (s_2,t_3), (s_4,t_4)\}$ is a strong bisimulation, then $s \sim t$. Indeed R is a strong bisimulation since:

- Consider $(s,t) \in R$. Transitions from s:
  - If $s \xrightarrow{a} s_1$, match by doing $t \xrightarrow{a} t_1$, and $(s_1,t_1) \in R$.
  - If $s \xrightarrow{a} s_2$, match by doing $t \xrightarrow{a} t_3$, and $(s_2,t_3) \in R$.
  - These are all transitions from s.

  Transitions from t:
  - If $t \xrightarrow{a} t_1$, match by doing $s \xrightarrow{a} s_1$, and $(s_1,t_1) \in R$.
  - If $t \xrightarrow{a} t_3$, match by doing $s \xrightarrow{a} s_2$, and $(s_2,t_3) \in R$.
  - These are all transitions from t.
- Consider $(s_1,t_1) \in R$. Transitions from s_1:
  - If $s_1 \xrightarrow{a} s_3$, match by doing $t_1 \xrightarrow{a} t_2$ and $(s_3,t_2) \in R$.
  - If $s_1 \xrightarrow{b} s_4$, match by doing $t_1 \xrightarrow{b} t_2$ and $(s_4,t_2) \in R$.

  Transitions from t_1:
  - If $t_1 \xrightarrow{a} t_2$, match by doing $s_1 \xrightarrow{a} s_3$ and $(s_3,t_2) \in R$.
  - If $t_1 \xrightarrow{b} t_2$, match by doing $s_1 \xrightarrow{b} s_4$ and $(s_4,t_2) \in R$.

- Consider $(s_3, t_2) \in R$. Transitions from s_3:
  - If $s_3 \xrightarrow{a} s$, match by doing $t_2 \xrightarrow{a} t$ and $(s, t) \in R$.

  Transitions from t_2:
  - If $t_2 \xrightarrow{a} t$, match by doing $s_3 \xrightarrow{a} s$ and $(s, t) \in R$.
- Consider $(s_4, t_2) \in R$. Transitions from s_4:
  - If $s_4 \xrightarrow{a} s$, match by doing $t_2 \xrightarrow{a} t$ and $(s, t) \in R$.

  Transitions from t_2:
  - If $t_2 \xrightarrow{a} t$, match by doing $s_4 \xrightarrow{a} s$ and $(s, t) \in R$.
- Consider $(s_2, t_3) \in R$. Transitions from s_2:
  - If $s_2 \xrightarrow{a} s_4$, match by doing $t_3 \xrightarrow{a} t_4$ and $(s_4, t_4) \in R$.

  Transitions from t_3:
  - If $t_3 \xrightarrow{a} t_4$, match by doing $s_2 \xrightarrow{a} s_4$ and $(s_4, t_4) \in R$.
- Consider $(s_4, t_4) \in R$. Transitions from s_4:
  - If $s_4 \xrightarrow{a} s$, match by $t_4 \xrightarrow{a} t$ and $(s, t) \in R$.

  Transitions from t_4:
  - If $t_4 \xrightarrow{a} t$, match by $s_4 \xrightarrow{a} s$ and $(s, t) \in R$.

Exercise 2*

Consider the CCS processes P and Q defined by:

$$\begin{aligned} P &\stackrel{\text{def}}{=} a.P_1 \\ P_1 &\stackrel{\text{def}}{=} b.P + c.P \end{aligned}$$

and

$$\begin{aligned} Q &\stackrel{\text{def}}{=} a.Q_1 \\ Q_1 &\stackrel{\text{def}}{=} b.Q_2 + c.Q \\ Q_2 &\stackrel{\text{def}}{=} a.Q_3 \\ Q_3 &\stackrel{\text{def}}{=} b.Q + c.Q_2 \quad . \end{aligned}$$

Show that $P \sim Q$ holds by finding an appropriate strong bisimulation.

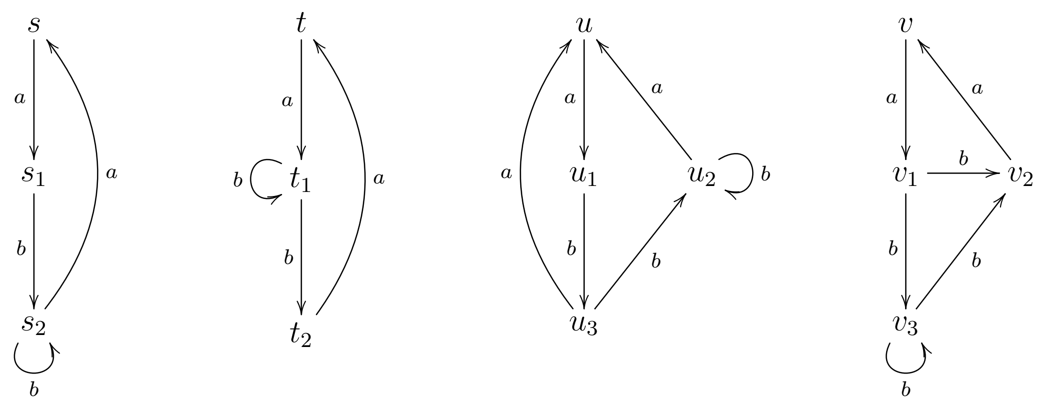
Solution of Exercise 2

Let $R = \{(P,Q),(P_1,Q_1),(P,Q_2),(P_1,Q_3)\}$. We only outline the proof; it follows along the lines as the proof in Exercise 1. You should complete the details.

- From $(P,Q) \in R$ either P or Q can do an a transition.
  - In either case the response is to match by making an a transition from the remaining state, so we end up in $(P_1,Q_1) \in R$.
- From $(P_1,Q_1) \in R$ we end up in either $(P,Q) \in R$ or $(P,Q_2) \in R$.
- From $(P,Q_2) \in R$ we can only end up in $(P_1,Q_3) \in R$.
- From $(P_1,Q_3) \in R$ we end up in either $(P,Q) \in R$ or $(P,Q_2) \in R$.

Exercise 3*

Consider the following labelled transition system.



Decide whether $s \overset{?}{\sim} t$, $s \overset{?}{\sim} u$, and $s \overset{?}{\sim} v$. Support your claims by giving a universal winning strategy either for the attacker (in the negative case) or the defender (in the positive case). In the positive case you can also define a strong bisimulation relating the pair in question.

Solution of Exercise 3

In this exercise you are asked to train yourself in the use of the game characterization for strong bisimulation. We therefore give universal winning strategy for the attacker or the defender in order to prove strong nonbisimilarity or bisimilarity. Let A denote the attacker and D the defender.

- Claim: $s \not\sim t$. The universal winning strategy for A is as follows.
  - In configuration (s, t), A chooses s and makes the move $s \xrightarrow{a} s_1$.
    - D's only possible response is to choose t and make the move $t \xrightarrow{a} t_1$. The current configuration is now (s_1, t_1)
  - In configuration (s_1, t_1), A chooses s_1 and makes the move $s_1 \xrightarrow{b} s_2$.

  Now the winning strategy depends on D's next move and is as follows. D can only choose the state t_1, but has two possible moves. Suppose D chooses $t_1 \xrightarrow{b} t_1$. Then the current configuration becomes (s_2, t_1). Now A choose s_2 and makes the move $s_2 \xrightarrow{a} s$. Then D looses since there are no a-transitions from t_1. If D uses the other possible move, namely $t_1 \xrightarrow{b} t_2$, the current configuration becomes (s_2, t_2). But then A chooses s_2 and makes the move $s_2 \xrightarrow{b} s_2$. Again D looses since there are no b-transitions from t_2.

  Remark: there is another winning strategy for the attacker which is easier to describe; try to find it.

- Claim: $s \sim u$: The universal winning strategy for D is as follows.
  - Starting in (s, u), A has two possible moves. Either (a) $s \xrightarrow{a} s_1$ or (b) $u \xrightarrow{a} u_1$.
    - If A chooses (a), then D takes the move $u \xrightarrow{a} u_1$, and the current configuration becomes (s_1, u_1).
    - If A chooses (b), then D takes the move $s \xrightarrow{a} s_1$, and the current configuration again becomes (s_1, u_1).
  - In configuration (s_1, u_1), A can choose either (a) $s_1 \xrightarrow{b} s_2$, or (b) $u_1 \xrightarrow{b} u_3$.
    - If A chooses (a), then D takes the move $u_1 \xrightarrow{b} u_3$, and the current configuration becomes (s_2, u_3).
    - If A chooses (b), then D takes the move $s_1 \xrightarrow{b} s_2$, and the current configuration again becomes (s_2, u_3).
  - In configuration (s_2, u_3), A can choose either (a) $s_2 \xrightarrow{b} s_2$ or (b) $s_2 \xrightarrow{a} s$ or (c) $u_3 \xrightarrow{a} u$ or (d) $u_3 \xrightarrow{b} u_2$.
    - If A chooses (a), then D takes the move $u_3 \xrightarrow{b} u_2$ and the current configuration becomes (s_2, u_2).
    - If A chooses (b), then D takes the move $u_3 \xrightarrow{a} u$ and the current configuration becomes (s, u) which is exactly the start configuration.
    - If A chooses (c), then D takes the move $s_2 \xrightarrow{a} s$ and the current configuration becomes (s, u) which is the start configuration.
    - If A chooses (d), then D takes the move $s_2 \xrightarrow{b} s_2$ and the current configuration becomes (s_2, u_2) as when the attacker played (a). Hence from now we only need to consider games form the state (s_2, u_2).

Now we can argue that D has a winning strategy. From (s_2, u_2), D's response to any move from A will be to take the same transition. This means that the next configuration is either (s_2, u_2) or (s, u). The game will be infinite, and hence D is the winner.

- Claim: $s \not\sim v$: The universal winning strategy for A is as follows.
  - In configuration (s, v), A makes the move $s \xrightarrow{a} s_1$.
    * Now D must make the move $v \xrightarrow{a} v_1$ and the current configuration becomes (s_1, v_1).
  - In configuration (s_1, v_1), A chooses $v_1 \xrightarrow{b} v_2$.
    * D must make the move $s_1 \xrightarrow{b} s_2$. The current configuration is (s_2, v_2).

  Now A wins since from (s_2, v_2) as he can choose to make the move $s_2 \xrightarrow{b} s_2$. Since there are no b-transitions from v_2, D looses.

Exercise 4

Prove that for any CCS processes P and Q the following laws hold:

- $P \mid Nil \sim P$
- $P + Nil \sim P$
- $P \mid Q \sim Q \mid P$

Hint: define appropriate binary relations on processes and prove that they are strong bisimulations.

Solution of Exercise 4

The general idea in this exercise is that in order to prove that $P \sim Q$ you define some binary relation R such that $(P, Q) \in R$, and then proceed to prove that R is indeed a strong bisimulation.

- Define $R = \{(P|Nil, P) \quad | \quad P \text{ is a CCS process}\}$. We show that R is a strong bisimulation.
  - Suppose for some $\alpha \in Act$ that $P|Nil \xrightarrow{\alpha} P'|Nil$. We now have to find some process $\tilde{P}$ such that $P \xrightarrow{\alpha} \tilde{P}$ and $(P'|Nil, \tilde{P}) \in R$. Now use the transition relation. The only rule that could have been used is the COM1-rule.

$$\frac{P \xrightarrow{\alpha} P'}{P|Nil \xrightarrow{\alpha} P'|Nil}.$$

Now set $\tilde{P} = P'$. Then we are finished since we now know that $P \xrightarrow{\alpha} P'$ and by the definition of R, $(P'|Nil, \tilde{P}) = (P'|Nil, P') \in R$.

– Symmetrically we must prove that when $P \xrightarrow{\alpha} P'$, then some $\tilde{P}$ exists so that $P|Nil \xrightarrow{\alpha} \tilde{P}$ and $(\tilde{P}, P') \in R$. But this is easy. By using the COM1-rule we have

$$\frac{P \xrightarrow{\alpha} P'}{P|Nil \xrightarrow{\alpha} P'|Nil}.$$

So we simply let $\tilde{P} = P'|Nil$. And again by definition of R, we have that $(\tilde{P}, P') = (P'|Nil, P') \in R$. This proves that R, is a bisimulation. And since $(P|Nil, P) \in R$, this means that $P|Nil \sim P$.

- This time we show that $P + Nil \sim P$ by giving a universal winning strategy for the defender. Remember that the game is played on the LTS, so we will just denote the states of the LTS by the CCS-expression. If the attacker chooses $P + Nil$, then the only possible moves are those of P since Nil has no transitions. So if $P \xrightarrow{a} P'$, the attacker can make the move $P + Nil \xrightarrow{a} P'$. But then the defender can make the move $P \xrightarrow{a} P'$. The current configuration is now (P', P'). From now on the defenders strategy is do to the same as the attacker. Either the game is infinite, in which case the defender wins. Or the game is finite. But then the defender wins, since the attacker cannot make any move because both processes are stuck. Similarly if the attacker plays $P \xrightarrow{a} P'$. Then the defender moves $P + Nil \xrightarrow{a} P'$, and the configuration again becomes (P', P').

- We show now that $R = \{(P|Q, Q|P) \mid P, Q \text{ are CCS-expressions}\}$ is a strong bisimulation. We only give an outline of the proof, the method is the same as in the first bullet. Suppose $P|Q \xrightarrow{a} P'|Q'$.

  – If COM3-rule was applied, we can argue as follows:

$$\frac{P \xrightarrow{a} P' \quad Q \xrightarrow{\overline{a}} Q'}{P|Q \xrightarrow{\tau} P'|Q'}$$

  But then since $\overline{\overline{a}} = a$ we can use the same rule to derive:

$$\frac{Q \xrightarrow{\overline{a}} Q' \quad P \xrightarrow{a} P'}{Q|P \xrightarrow{\tau} Q'|P'}.$$

  And by the definition of R, we know that $(P'|Q', Q'|P') \in R$.

  – If COM1 or COM2 rule was used, we do the following analysis. Suppose the COM1-rule was the one used. Then we know that

$$\frac{P \xrightarrow{a} P'}{P|Q \xrightarrow{a} P'|Q}.$$

Again one can now apply the COM2-rule and derive

$$\frac{P \xrightarrow{a} P'}{Q|P \xrightarrow{a} Q|P'},$$

and $(P'|Q, Q|P') \in R$. In order to finish the proof we need to argue for the symmmetric case (i.e. when the rule COM2 was used from $P|Q$). The argument for this case is similar as before.

The case when $Q|P \xrightarrow{a} Q'|P'$ is completely symmetric.

Exercise 5

Argue that any two strongly bisimilar processes have the same sets of traces, i.e., that

$$s \sim t \text{ implies } \mathit{Traces}(s) = \mathit{Traces}(t).$$

Hint: you can find useful the game characterization of strong bisimilarity.

Solution of Exercise 5

Assume that $s \sim t$. We will show both trace inclusions as follows.

- $\mathit{Traces}(s) \subseteq \mathit{Traces}(t)$: Let $w = a_1 a_2 \dots a_n$ be a trace from $\mathit{Traces}(s)$. The attacker will play the sequence w in n-rounds of the strong bisimulation game, always from the left processes s. As $s \sim t$, the defender has to be able to answer to such an attack and hence he has to be able to do the same sequence w from the right process t. This means that $w \in \mathit{Traces}(t)$.
- $\mathit{Traces}(t) \subseteq \mathit{Traces}(s)$: The argument is completely symmetric, the attacker plays the whole sequence from the right process t and the defender has to be able to match it in the left process.

This implies that $\mathit{Traces}(s) = \mathit{Traces}(t)$.

Exercise 6

Is it true that any relation of strong bisimilarity must be reflexive, transitive and symmetric? If yes then prove it, if not then give counter examples, i.e.

- define an LTS and a binary relation on states which is not reflexive but it is a strong bisimulation

- define an LTS and a binary relation on states which is not symmetric but it is a strong bisimulation
- define an LTS and a binary relation on states which is not transitive but it is a strong bisimulation.

Solution of Exercise 6

The answer is no for all the cases and the relation R of strong bisimulation from Exercise 1 can serve as a counter example for reflexivity and symmetry.

Exercise 7 (optional)

Argue that $s \sim t$ iff the defender has a winning strategy in the strong bisimulation game starting from the pair (s, t).

Hint: show that from knowing defender's universal winning strategy you can find a strong bisimulation and that given a strong bisimulation you can define defender's universal winning strategy.