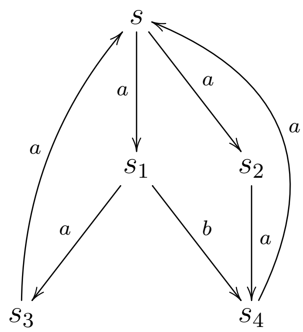
Exercise 1*

Consider the following labelled transition system.



1. Decide whether the state s satisfies the following formulae of Hennessy-Milner logic:

   (a) $s \stackrel{?}{\models} \langle a \rangle tt$

   (b) $s \stackrel{?}{\models} \langle b \rangle tt$

   (c) $s \stackrel{?}{\models} [a] ff$

   (d) $s \stackrel{?}{\models} [b] ff$

   (e) $s \stackrel{?}{\models} [a]\langle b \rangle tt$

   (f) $s \stackrel{?}{\models} \langle a \rangle\langle b \rangle tt$

   (g) $s \stackrel{?}{\models} [a]\langle a \rangle [a][b] ff$

   (h) $s \stackrel{?}{\models} \langle a \rangle \big(\langle a \rangle tt \wedge \langle b \rangle tt\big)$

   (i) $s \stackrel{?}{\models} [a] \big(\langle a \rangle tt \vee \langle b \rangle tt\big)$

   (j) $s \stackrel{?}{\models} \langle a \rangle \big([b][a] ff \wedge \langle b \rangle tt\big)$

   (k) $s \stackrel{?}{\models} \langle a \rangle \big([a](\langle a \rangle tt \wedge [b] ff) \wedge \langle b \rangle ff\big)$

2. Compute the following sets according to the denotational semantics for Hennessy-Milner logic.

   - $[\![[a][b] ff]\!] = ?$
   - $[\![\langle a \rangle \big(\langle a \rangle tt \wedge \langle b \rangle tt\big)]\!] = ?$
   - $[\![[a][a][b] ff]\!] = ?$
   - $[\![[a] \big(\langle a \rangle tt \vee \langle b \rangle tt\big)]\!] = ?$

Solution of Exercise 1

1. The state s satisfies the following formulae of Hennessy-Milner logic:

   (a) $s \models \langle a \rangle tt$

   (b) $s \not\models \langle b \rangle tt$

   (c) $s \not\models [a] ff$

   (d) $s \models [b] ff$

   (e) $s \not\models [a]\langle b \rangle tt$

   (f) $s \models \langle a \rangle \langle b \rangle tt$

   (g) $s \models [a]\langle a \rangle [a][b] ff$

   (h) $s \models \langle a \rangle \big(\langle a \rangle tt \wedge \langle b \rangle tt \big)$

   (i) $s \models [a]\big(\langle a \rangle tt \vee \langle b \rangle tt \big)$

   (j) $s \not\models \langle a \rangle \big([b][a] ff \wedge \langle b \rangle tt \big)$

   (k) $s \not\models \langle a \rangle \big([a](\langle a \rangle tt \wedge [b] ff) \wedge \langle b \rangle ff \big)$

2. The denotational semantics of the logical formulae is as follows:

$$
\begin{aligned}
[\![[a][b] ff]\!] &= [\cdot a \cdot][\![[b] ff]\!] \\
&= [\cdot a \cdot][\cdot b \cdot][\![ff]\!] \\
&= [\cdot a \cdot][\cdot b \cdot]\emptyset \\
&= [\cdot a \cdot]\{P \mid \forall P'. P \xrightarrow{b} P' \Rightarrow P' \in \emptyset\} \\
&= [\cdot a \cdot]\{s, s_3, s_2, s_4\} \\
&= \big\{P \mid \forall P'. P \xrightarrow{a} P' \Rightarrow P' \in \{s, s_3, s_2, s_4\}\big\} \\
&= \{s_1, s_2, s_3, s_4\}
\end{aligned}
$$

$$
\begin{aligned}
[\![\langle a \rangle \big(\langle a \rangle tt \wedge \langle b \rangle tt \big)]\!] &= \langle \cdot a \cdot \rangle [\![\langle a \rangle tt \wedge \langle b \rangle tt]\!] \\
&= \langle \cdot a \cdot \rangle \big([\![\langle a \rangle tt]\!] \cap [\![\langle b \rangle tt]\!] \big) \\
&= \langle \cdot a \cdot \rangle \big(\langle \cdot a \cdot \rangle Proc \cap \langle \cdot b \cdot \rangle Proc \big) \\
&= \langle \cdot a \cdot \rangle \big(\{s, s_1, s_2, s_3, s_4\} \cap \{s_1\} \big) \\
&= \langle \cdot a \cdot \rangle \{s_1\} \\
&= \{s\}
\end{aligned}
$$

$$
\begin{aligned}
[\![[a][a][b] ff]\!] &= [\cdot a \cdot][\cdot a \cdot][\cdot b \cdot]\emptyset \\
&= [\cdot a \cdot][\cdot a \cdot]\{s, s_2, s_3, s_4\} \\
&= [\cdot a \cdot]\{s_1, s_2, s_3, s_4\} \\
&= \{s, s_1, s_2\}
\end{aligned}
$$

$$
\begin{aligned}
\llbracket [a]\big(\langle a\rangle \mathit{tt} \vee \langle b\rangle \mathit{tt}\big) \rrbracket &= [\cdot a\cdot]\llbracket \langle a\rangle \mathit{tt} \vee \langle b\rangle \mathit{tt} \rrbracket \\
&= [\cdot a\cdot]\big(\langle\cdot a\cdot\rangle \mathit{Proc} \cup \langle\cdot b\cdot\rangle \mathit{Proc}\big) \\
&= [\cdot a\cdot]\{s, s_1, s_2, s_3, s_4\} \\
&= \{s, s_1, s_2, s_3, s_4\}
\end{aligned}
$$

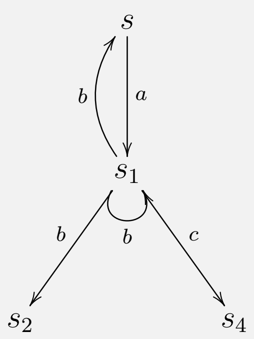
Exercise 2

Find (one) labelled transition system with an initial state s such that it satisfies (at the same time) the following properties:

- $s \models \langle a\rangle(\langle b\rangle\langle c\rangle \mathit{tt} \wedge \langle c\rangle \mathit{tt})$
- $s \models \langle a\rangle\langle b\rangle([a]\mathit{ff} \wedge [b]\mathit{ff} \wedge [c]\mathit{ff})$
- $s \models [a]\langle b\rangle([c]\mathit{ff} \wedge \langle a\rangle \mathit{tt})$

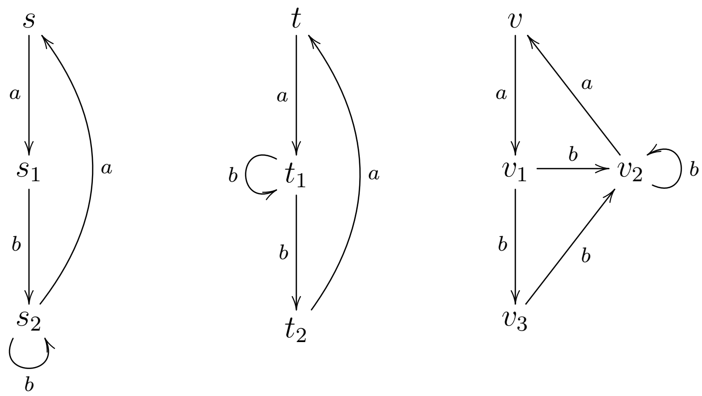
Solution of Exercise 2

One possible solution is as follows.



Exercise 3*

Consider the following labelled transition system.



It is true that $s \not\sim t$, $s \not\sim v$ and $t \not\sim v$. Find a distinguishing formula of Hennessy-Milner logic for the pairs

- s and t
- s and v
- t and v.

Solution of Exercise 3

Distinguishing HML-formulae are as follows.

- Let $F_1 = \langle a\rangle[b]\langle b\rangle tt$. Then $s \models F_1$, but $t \not\models F_1$.
- Let $F_2 = \langle a\rangle[b]\langle a\rangle tt$. Then $s \models F_2$ but $v \not\models F_3$.
- Let $F_3 = \langle a\rangle\langle b\rangle\big(\langle a\rangle tt \wedge \langle b\rangle tt\big)$. Then $t \not\models F_3$ but $v \models F_3$.

Exercise 4*

For each of the following CCS expressions decide whether they are strongly bisimilar and if no, find a distinguishing formula in Hennessy-Milner logic.

1. $b.a.Nil + b.Nil$ and $b.(a.Nil + b.Nil)$
2. $a.(b.c.Nil + b.d.Nil)$ and $a.b.c.Nil + a.b.d.Nil$
3. $a.Nil \,|\, b.Nil$ and $a.b.Nil + b.a.Nil$
4. $(a.Nil \,|\, b.Nil) + c.a.Nil$ and $a.Nil \,|\, (b.Nil + c.Nil)$

Home exercise: verify your claims in CAAL and check whether you found the same distinguishing formula as the tool.

Solution of Exercise 4

For each pair we provide a strong bisimulation containing it if they are strongly bisimilar, otherwise we show a distinguishing formula in Hennessy-Milner logic.

1. $b.a.Nil + b.Nil$ and $b.(a.Nil + b.Nil)$ are not bisimilar.

   $F_1 = [b]\langle b\rangle tt$ is a distinguishing formula:

   $$b.a.Nil + b.Nil \not\models F_1 \qquad \text{but} \qquad b.(a.Nil + b.Nil) \models F_1 \,.$$

2. $a.(b.c.Nil + b.d.Nil)$ and $a.b.c.Nil + a.b.d.Nil$ are not bisimilar.

   $F_2 = [a]\big(\langle b\rangle\langle c\rangle tt \wedge \langle b\rangle\langle d\rangle tt\big)$ is a distinguishing formula:

   $$a.(b.c.Nil + b.d.Nil) \models F_2 \qquad \text{but} \qquad a.b.c.Nil + a.b.d.Nil \not\models F_2 \,.$$

3. $a.Nil \mid b.Nil$ and $a.b.Nil + b.a.Nil$ are bisimilar.

   Let $A = a.Nil$ and $B = b.Nil$, then the following is a strong bisimulation containing the pair $(A \mid B, a.B + b.A)$ (note that this is exactly the pair $\big(a.Nil \mid b.Nil,\ a.b.Nil + b.a.Nil\big)$).

   $$R = \{(A \mid B, a.B + b.A), (Nil \mid B, B), (A \mid Nil, A), (Nil, Nil)\} \,.$$

4. $(a.Nil \mid b.Nil) + c.a.Nil$ and $a.Nil \mid (b.Nil + c.Nil)$ are not bisimilar.

   $F_3 = [a]\langle c\rangle tt$ is a distinguishing formula:

   $$(a.Nil \mid b.Nil) + c.a.Nil \not\models F_3 \qquad \text{but} \qquad a.Nil \mid (b.Nil + c.Nil) \models F_3 \,.$$

Exercise 5 (optional)

Prove that for every Hennessy-Milner formula F and every state $p \in Proc$:

$$p \models F \quad \text{if and only if} \quad p \in [\![F]\!] \,.$$

Hint: use induction on the structure of the formula F.