An Introduction to Quantum Computing
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The system:

- We have a 2-state quantum system with basis states $|0\rangle$ and $|1\rangle$, and joint state:

$$|\psi\rangle = \alpha\,|0\rangle \;+\; \beta\,|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1,\ \alpha, \beta \in \mathbb{C} \tag{1}$$

  where $|\alpha|^2$ is the probability for measuring state $|0\rangle$, and $|\beta|^2$ is the probability for measuring state $|1\rangle$.
- In vector form the states are $|0\rangle = [1, 0]^{\mathsf{T}}$ and $|1\rangle = [0, 1]^{\mathsf{T}}$.
- We define two states as:

$$|+\rangle \;\equiv\; \frac{1}{\sqrt{2}}\,|0\rangle \;+\; \frac{1}{\sqrt{2}}\,|1\rangle \tag{2}$$

$$|-\rangle \;\equiv\; \frac{1}{\sqrt{2}}\,|0\rangle \;-\; \frac{1}{\sqrt{2}}\,|1\rangle \tag{3}$$

Questions:

1. What are α and β in the system described in (1) above when $\alpha = \beta$?
2. What are $|+\rangle + |-\rangle$ and $|+\rangle - |-\rangle$ in (2)–(3) expressed by $|0\rangle$ and $|1\rangle$ in the general case of α and β?
3. And finally, what is then $|\psi\rangle$ expressed by $|0\rangle$ and $|1\rangle$?
4. How can the arbitrary state $|\psi\rangle$ in (1) be expressed by α, β, $|+\rangle$, and $|-\rangle$?

Solutions:

1. When $\alpha = \beta = \kappa$ in (1) we have:

$$|\alpha|^2 + |\beta|^2 = 1 \rightarrow 2\kappa^2 = 1 \rightarrow \kappa = \frac{1}{\sqrt{2}} = \alpha = \beta \qquad (4)$$

corresponding to probability $\frac{1}{2}$ for state $|0\rangle$, and probability $\frac{1}{2}$ for state $|1\rangle$.

Solutions:

2. We can express (2)–(3) in matrix-vector form as:

$$\sqrt{2} \cdot \begin{bmatrix} |+\rangle \\ |-\rangle \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} |0\rangle \\ |1\rangle \end{bmatrix} \tag{5}$$

Multiplying by the matrix inverse on both sides of the equation:

$$\frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} |+\rangle \\ |-\rangle \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} |0\rangle \\ |1\rangle \end{bmatrix} \tag{6}$$

Moving a bit around we reach:

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}} \tag{7}$$

$$|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}} \tag{8}$$

3. The joint state $|\psi\rangle$ is generally given by:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \tag{9}$$

Using (7)–(8) we get:

$$|\psi\rangle = \alpha\,\frac{|+\rangle + |-\rangle}{\sqrt{2}} + \beta\,\frac{|+\rangle - |-\rangle}{\sqrt{2}} \tag{10}$$

4. Separating the states $|+\rangle$ and $|-\rangle$ in (10) we get:

$$|\psi\rangle = \alpha\,\frac{|+\rangle + |-\rangle}{\sqrt{2}} + \beta\,\frac{|+\rangle - |-\rangle}{\sqrt{2}} \tag{11}$$

which may be rewritten as:

$$|\psi\rangle = \frac{\alpha + \beta}{\sqrt{2}}\,|+\rangle + \frac{\alpha - \beta}{\sqrt{2}}\,|-\rangle \tag{12}$$