

# Matrixregning, Afsnit 2.1, 2.4

01. februar 2021

Sandsynlighedsteori og lineær algebra (SLIAL)

Forår 2021



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Del I

Matrix addition, multiplikation med  
skalar

# Summen af matricer

Hvis  $A$  og  $B$  begge er  $m \times n$ -matricer, defineres deres sum elementvist

## Eksempel

$$2 \begin{bmatrix} 0 & 5 & 10 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 0+1 & 5+2 & 10+3 \\ 2+4 & 4+5 & 6+6 \end{bmatrix} = \begin{bmatrix} 1 & 7 & 13 \\ 6 & 9 & 12 \end{bmatrix}$$

Ganges en matrix med en skalar, ganges skalaren ind på hvert element i matricen

## Eksempel

$$2 \begin{bmatrix} 0 & 5 & 10 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 \cdot 0 & 2 \cdot 5 & 2 \cdot 10 \\ 2 \cdot 2 & 2 \cdot 4 & 2 \cdot 6 \end{bmatrix} = \begin{bmatrix} 0 & 10 & 20 \\ 4 & 8 & 12 \end{bmatrix}$$

$$(-1) \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \end{bmatrix}$$

# Summen af matricer

## Egenskaber

Hvis  $A$ ,  $B$  og  $C$  er matricer af samme dimensioner, og  $r$  og  $s$  er skalarer, gælder

►  $A + B = B + A$

►  $(A + B) + C = A + (B + C)$

►  $A + O = A$ , hvor  $O$  er nulmatricen

►  $r(A + B) = rA + rB$

►  $(r + s)A = rA + sA$

►  $r(sA) = (rs)A$

►  $A - B := A + (-1)B$

$$a_{ij} + b_{ij} = b_{ij} + a_{ij}$$

$$(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij})$$

$$a_{ij} + 0 = a_{ij}$$

$$\underline{r(a_{ij} + b_{ij})} = \underline{ra_{ij}} + \underline{rb_{ij}}$$

$$(A - B)_{ij} = A_{ij} - B_{ij}$$



# Summen af matricer

## Egenskaber

Billig operation:  $C = A + B$  “koster”  $m \times n$  additioner, hvor  $m \times n$  er størrelse af  $A, B$

```
1  for (i=0; i<m; i++) {  
2      for (j=0; j<n; j++) {  
3          C[i][j] = A[i][j] + B[i][j];  
4      }  
5  }
```

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Del II

## Matrix multiplikation

# Matrixprodukt



$$\begin{array}{ccc}
 \begin{array}{c} \text{[ } \quad \longleftrightarrow \quad \text{]} \\ A \\ m \times n \end{array} & \cdot & \begin{array}{c} \text{[ } \quad \updownarrow \quad \text{]} \\ B \\ n \times p \end{array} \\
 = & C & \\
 m \times p & & 
 \end{array}$$

# Matrixprodukt



$$\begin{aligned}
 C_{ij} &= A_{i1}B_{1j} + \cdots + A_{in}B_{nj} \\
 &= \sum_{k=1}^n A_{ik} B_{kj} \\
 &\quad \left( \begin{array}{l} n \text{ mult.} \\ (n-1) \text{ addit.} \end{array} \right) \cdot m \cdot p \\
 &\quad \sim n \cdot m \cdot p
 \end{aligned}$$



# Matrixprodukt

## Eksempler

### Eksempel

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -3 \\ 0 & 1 & 1 \end{bmatrix} =$$

Handwritten annotations: A red box around the first row of the first matrix, a green box around the first column of the second matrix, and a red box around the scalar 3. A blue arrow points from the scalar 3 to the first column of the second matrix.

$$\begin{bmatrix} 1 \cdot 2 + 2 \cdot 1 + 3 \cdot 0 & 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 & 1 \cdot 0 + 2 \cdot (-3) + 3 \cdot 1 \\ 4 \cdot 2 + 5 \cdot 1 + 6 \cdot 0 & 4 \cdot 1 + 5 \cdot 1 + 6 \cdot 1 & 4 \cdot 0 + 5 \cdot (-3) + 6 \cdot 1 \end{bmatrix} = \begin{bmatrix} 4 & 6 & -1 \\ 13 & 15 & -9 \end{bmatrix}$$

Handwritten annotations: A red '2x3' above the first row of the result matrix. A red arrow points from the first row of the result matrix to the calculation  $1 \cdot 0 + 2 \cdot (-3) + 3 \cdot 1$ .

### Eksempel

$$\text{Udregn } \begin{bmatrix} 7 & 8 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} -2 & 8 \\ 6 & 5 \\ -5 & 8 \end{bmatrix}, \text{ hvis det er defineret}$$

Handwritten annotations: A red '2' above the second matrix, a blue '2' below the first matrix, and a blue arrow pointing from the blue '2' to the second matrix.

$$\begin{bmatrix} 7 & 8 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} -2 & 6 & 5 \\ 8 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 7 \cdot (-2) + 8 \cdot 8 & \dots & \dots \end{bmatrix}$$

Handwritten annotations: A blue '2' below the first matrix, a blue '2' below the second matrix, and a blue arrow pointing from the blue '2' to the second matrix.

# Matrixprodukt

## Eksempler



### Eksempel

Hvad er element  $(\underline{2}, \underline{3})$  i produktet  $\begin{bmatrix} -1 & 14 & 3 \\ 11 & -2 & 4 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 6 & 13 \\ 77 & 9 \end{bmatrix}$

*(Handwritten annotations: a green underline under the 2 in (2,3), a red underline under the 3 in (2,3), a green underline under the second row of the first matrix, and a blue bracket and wavy line under the third column of the second matrix.)*

$$\text{Svar: } 11 \cdot 1 + (-2) \cdot 1 + 4 \cdot 1 \\ = 11 - 2 + 4 = 13$$

# Identitetsmatricen

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$\underline{I}A = A = A\underline{I}$$

Pas på: hvis  $A$  har størrelse  $m \times n$ , så

- ▶ “den venstre”  $I$  har størrelse  $m \times m$
- ▶ “den højre”  $I$  har størrelse  $n \times n$

$$A \quad m \times n$$

$$\underline{\underline{A}} \quad n \{ \underline{\underline{I}} \}$$

$$\underline{\underline{I}} \quad m \{ A \}$$

# Egenskaber

Matrixproduktet har følgende egenskaber:  
(givet, at matricerne har passende dimensioner)

- ▶  $A(BC) = (AB)C$
- ▶  $A(B + C) = \underline{AB} + \underline{AC}$
- ▶  $\cdot \underline{(B + C)A} = \underline{BA} + \underline{CA}$
- ▶  $r(AB) = (rA)B = A(rB)$ , hvor  $r$  er en vilkårlig skalar
- ▶  $IA = A = AI$ , hvor  $I$  er identitetsmatricen

## PAS PÅ!



Hvorfor nævner vi både  $A(B + C)$  og  $(B + C)A$  i egenskaberne?

Matrixproduktet opfylder generelt *ikke*  $AB = BA$

## Eksempel

$$\begin{aligned}
 \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} &= \begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 & 1 \cdot 0 + 1 \cdot 1 \\ 0 \cdot 1 + 2 \cdot 1 & 0 \cdot 0 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \\
 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} &= \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 1 + 0 \cdot 2 \\ 1 \cdot 1 + 1 \cdot 0 & 1 \cdot 1 + 1 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}
 \end{aligned}$$

# PAS (stadig) PÅ!

Der er andre eksempler, hvor matrixproduktet opfører sig anderledes, end man måske kunne håbe

- ▶  $AB = AC$  medfører ikke nødvendigvis  $B = C$
- ▶  $AB = O$  medfører ikke nødvendigvis  $A = O$  eller  $B = O$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = B$$

$$A \cdot B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \cdot 0 + 1 \cdot 0 & 0 \cdot 1 + 1 \cdot 0 \\ 0 \cdot 0 + 0 \cdot 0 & 0 \cdot 1 + 0 \cdot 0 \end{bmatrix}$$

$$C = O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AB = O = AC$$

$$B \neq C$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

# Matrixprodukt

## Egenskaber

Dyr operation:  $C = AB$  "koster"  $m \times n \times p$  additioner og multiplikationer, hvor  $m \times n$  er størrelse af  $A$ , og  $n \times p$  er størrelse af  $B$ .

$C=0$

```

1  for (i=0; i<m; i++) {
2    for (j=0; j<p; j++) {
3      for (k=0; k<n; k++) {
4        C[i][j] += A[i][k] B[k][j];
5      }
6    }
7  }
```

---

Kan gøres billigere: se Strassen algoritme!

# Matrix power



$$A^k = \underbrace{AA \dots A}_{k \text{ ganger}}$$

$$A^0 = I$$

$$A^1 = A$$

$$A^2 = \underbrace{A \cdot A}_2$$

;

$$\underbrace{A}_n \cdot \{A$$

$$A \sim n \times n$$



Del III

Transponeret matrix

# Transponeret matrix

Matricer har en ekstra operation: *transponering*

Hvis  $A$  er en  $m \times n$ -matrix, er  $A^T$   $n \times m$ -matricen med  $(A^T)_{ij} = A_{ji}$

Eksempel

$$\begin{array}{c} 2 \\ \left[ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array} \right]^T = \begin{array}{c} 4 \\ \left[ \begin{array}{cc} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{array} \right] \end{array} \end{array}$$

Handwritten annotations: A green circle highlights the element 2 in the first row, second column of the original matrix. A blue circle highlights the element 5 in the second row, first column. Blue arrows show the mapping of these elements to their positions in the transposed matrix: from (1,2) to (2,1) and from (2,1) to (1,2). The dimensions 2 and 4 are written next to the matrices, and the number 2 is written below the transposed matrix.

# Transponeret matrix

## Egenskaber

- ▶  $(A^T)^T = A$
- ▶  $(A + B)^T = A^T + B^T$
- ▶  $(rA)^T = r(A^T)$  for en vilkårlig skalar  $r$
- ▶  $(AB)^T = B^T A^T$

$$A \sim m \times \underline{n}$$

$$B \sim \underline{n} \times p$$

$$A^T \sim \underline{n} \times m$$

$$B^T \sim p \times \underline{n}$$

$$B^T A^T$$

## Del IV

# Vektorer, $\mathbb{R}^n$ , skalarprodukt

# Vektorer

Skal betragte vektorer som  $n \times \textcolor{blue}{1}$  matrixer:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

Men vi skal skrive  $x_i$  istedenfor  $x_{i,1}$

# Matrix-vektor produkt



- ▶  $A \sim m \times n$
- ▶  $x \sim n \times 1$
- ▶  $Ax \sim m \times 1 \in \mathbb{R}^m$
- ▶  $y \sim m \times 1 \in \mathbb{R}^m$
- ▶  $y^T A \sim \underline{1} \times n$

$$(A^T y)^T = y^T (A^T)^T \\ = y^T A$$

# Skalarprodukt

$$x, y \in \mathbb{R}^n$$

$$\triangleright x, y \sim n \times \underline{1}$$

Skalarprodukt:

$$x^T y = y^T x = x_1 y_1 + \cdots + x_n y_n$$

$$= \sum_{i=1}^n x_i y_i$$

$$\begin{aligned} x^T &\sim 1 \times n \\ y &\sim \underline{n} \times 1 \end{aligned}$$

$$x^T y \sim 1 \times 1$$

Del V

# Strassen algorithm for matrix multiplikation



# Divide-and-conquer

$$A, B, C \sim \underline{2n} \times \underline{2n}$$

$$A = \begin{pmatrix} \overset{\cdot}{A}_{11} & \overset{\cdot}{A}_{12} \\ \overset{\cdot}{A}_{21} & \overset{\cdot}{A}_{22} \end{pmatrix} \quad B = \begin{pmatrix} \overset{\cdot}{B}_{11} & \overset{\cdot}{B}_{12} \\ \overset{\cdot}{B}_{21} & \overset{\cdot}{B}_{22} \end{pmatrix}$$

$$C = \begin{pmatrix} \overset{\cdot}{C}_{11} & \overset{\cdot}{C}_{12} \\ \overset{\cdot}{C}_{21} & \overset{\cdot}{C}_{22} \end{pmatrix} = \begin{pmatrix} \overset{\cdot}{A}_{11}\overset{\cdot}{B}_{11} + \overset{\cdot}{A}_{12}\overset{\cdot}{B}_{21} & \overset{\cdot}{A}_{11}\overset{\cdot}{B}_{12} + \overset{\cdot}{A}_{12}\overset{\cdot}{B}_{22} \\ \overset{\cdot}{A}_{21}\overset{\cdot}{B}_{11} + \overset{\cdot}{A}_{22}\overset{\cdot}{B}_{21} & \overset{\cdot}{A}_{21}\overset{\cdot}{B}_{12} + \overset{\cdot}{A}_{22}\overset{\cdot}{B}_{22} \end{pmatrix}$$

hvor  $A_{ij}, B_{ij}, C_{ij} \sim n \times n$ .

No free lunch på denne måde:

- Beregning af  $AB$  koster  $\sim (2n)^3 = 8n^3$  operationer
- Beregning af  $A_{ij}B_{jk}$  koster  $\sim n^3$ , men vi har 8 små matricer at gange

# Divide-and-conquer

## Strassen algoritme

$$\begin{aligned}
 M_1 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\
 M_3 &= A_{11}(B_{12} - B_{22}) \\
 M_5 &= (A_{11} + A_{12})B_{22} \\
 M_7 &= (A_{12} - A_{22})(B_{21} + B_{22})
 \end{aligned}$$

$$\begin{aligned}
 M_2 &= (A_{21} + A_{22})B_{11} \\
 M_4 &= A_{22}(B_{21} - B_{11}) \\
 M_6 &= (A_{21} - A_{11})(B_{11} + B_{12})
 \end{aligned}$$

Da

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 - M_2 + M_3 + M_6 \end{pmatrix}$$

- Beregning af  $AB$  koster  $\sim (2n)^3 = 8n^3$  operationer
- Beregning af  $M_i$  koster  $\sim n^3$ , og vi har kun 7 små matricer at gange

# Divide-and-conquer

## Strassen algoritme

- Anvend Strassen's algoritme igen (rekursion) for at beregne  $M_i$ ,  $i = 1, \dots, 7$

Antager:  $A, B, C \sim 2^N \times 2^N$ , og la antallet af operationer i Strassen's algoritmen være  $f(N)$ . Husk at den naive algoritme koster  $(2^N)^3$ .

- $f(1) = 1$
- $f(N) = 7f(N-1) + \ell(2^{N-1} \cdot 2^{N-1})$ , hvor  $\ell$  er konstant (antallet av ekstra additioner/subtraktioner)
- $f(N) \approx 7^N = (2^N)^{\log_2 7} \approx (2^N)^{2.8074} < (2^N)^3$ , som i den naive algoritme