WS3

restart
with(*LinearAlgebra*) : *with*(*MTM*) :

Exercise 1

Exercise 1.1

Since $x \in R^4$ and $b \in R^{10}$ for the equation $Ax = b$ to be defined A has to be 10x4 matrix.

Exercise 1.2 should probably rethink this one

Since A only has four columns the column space of A can only describe a four dimensional space even though each vector may be in 10D. Therefore the output of the A will be some 4D subspace of 10D, but the error f may not be in this subspace. Therefore $\tilde{b}$ may not be in this subspace as well.

The least squares problem is when we find the vector b in the subspace that is the closest to $\tilde{b}$. That is we want to find $\min_x\left(\left|\tilde{b} - Ax\right|\right)$

> If A is $m \times n$ and $\mathbf{b}$ is in $\mathbb{R}^m$, a **least-squares solution** of $A\mathbf{x} = \mathbf{b}$ is an $\hat{\mathbf{x}}$ in $\mathbb{R}^n$ such that
> $$\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\mathbf{x}\|$$
> for all $\mathbf{x}$ in $\mathbb{R}^n$.

Exercise 1.3

We will have that $x \in R^{N^2}$
For b it is a little more tricky.
From the the 0 and 90 degree angles we get N values, for each.
From the 45 and 135 degree angles we get 2N-1 values for each.
Thus $b \in R^{6N-2}$
Again A will be an b by x matrix.

Exercise 1.4

> If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

This means that we have to find when x > b
$N^2 > 6N - 2$
$N^2 - 6N + 2 > 0$

$d = (-6)^2 - 4 \cdot 1 \cdot 2 = 36 - 8 = 28$

$N = \frac{6 + \sqrt{28}}{2} = N = 3 + \sqrt{7} \xrightarrow{\text{at 5 digits}} N = 5.6458$

$N = \frac{6 - \sqrt{28}}{2} = N = 3 - \sqrt{7} \xrightarrow{\text{at 5 digits}} N = 0.3542$

Thus when N is 6 or greater the column vectors of A must be linearly dependent.

Exercise 1.5 should probably be done better

To find the least squares solution we

The set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ coincides with the nonempty set of solutions of the normal equations $A^TA\mathbf{x} = A^T\mathbf{b}$.

Given an $m \times n$ matrix A with linearly independent columns, let $A = QR$ be a QR factorization of A as in Theorem 12. Then, for each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution, given by

$$\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b} \tag{6}$$

PROOF Let $\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$. Then

$$A\hat{\mathbf{x}} = QR\hat{\mathbf{x}} = QRR^{-1}Q^T\mathbf{b} = QQ^T\mathbf{b}$$

By Theorem 12, the columns of Q form an orthonormal basis for Col A. Hence, by Theorem 10, $QQ^T\mathbf{b}$ is the orthogonal projection $\hat{\mathbf{b}}$ of $\mathbf{b}$ onto Col A. Then $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$, which shows that $\hat{\mathbf{x}}$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$. The uniqueness of $\hat{\mathbf{x}}$ follows from Theorem 14. ■

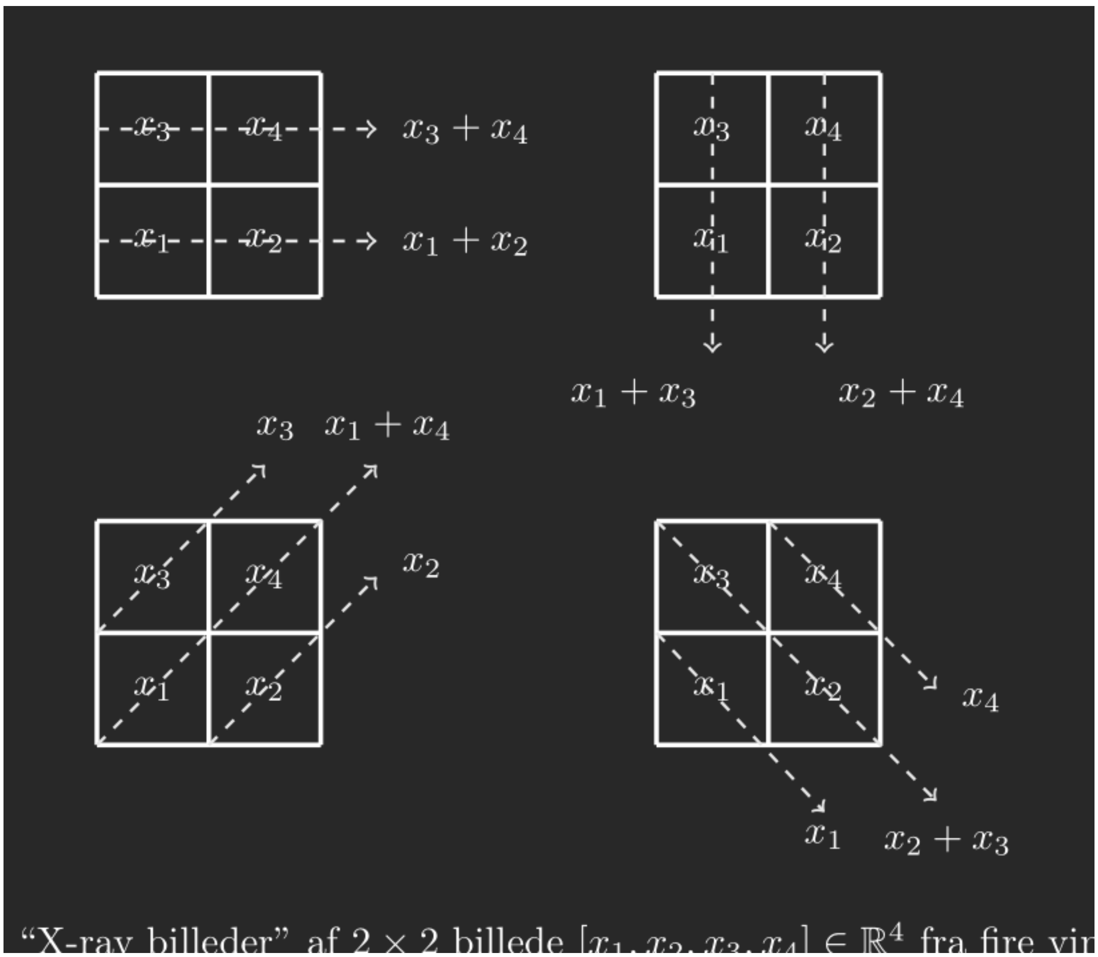
Exercise 1.7 might need to look at this again solution is not so good

$N := 2 : b := \langle 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \rangle :$

$A_T := \langle 0, 0, 1, 1 | 1, 1, 0, 0 | 1, 0, 1, 0 | 0, 1, 0, 1 | 0, 0, 1, 0 | 1, 0, 0, 1 | 0, 1, 0, 0 | 0, 0, 0, 1 | 0, 1, 1, 0 | 1, 0, 0, 0 \rangle :$

$A := transpose(A_T)$

$$A := \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{1.1.6.1}$$



First we find an orthogonal basis for A, using graham schmidt method

The Gram–Schmidt Process

Given a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal basis for W. In addition

$$\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\} \qquad \text{for } 1 \le k \le p$$

$p(y, z) := \frac{y \cdot z}{z \cdot z} \cdot z : c(y) := Column(A, y) :$

$v_1 := c(1) :$

$v_2 := c(2) - p(c(2), v_1) :$

$v_3 := c(3) - p(c(3), v_1) - p(c(3), v_2) :$

$v_4 := c(4) - p(c(4), v_1) - p(c(4), v_2) - p(c(4), v_3) :$

$Q := \left\langle \frac{v_1}{Norm(v_1, 2)} \middle| \frac{v_2}{Norm(v_2, 2)} \middle| \frac{v_3}{Norm(v_3, 2)} \middle| \frac{v_4}{Norm(v_4, 2)} \right\rangle :$

$R := transpose(Q) \cdot A :$

Given an $m \times n$ matrix A with linearly independent columns, let $A = QR$ be a QR factorization of A as in Theorem 12. Then, for each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution, given by

$$\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b} \tag{6}$$

PROOF Let $\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$. Then

$$A\hat{\mathbf{x}} = QR\hat{\mathbf{x}} = QRR^{-1}Q^T\mathbf{b} = QQ^T\mathbf{b}$$

By Theorem 12, the columns of Q form an orthonormal basis for Col A. Hence, by Theorem 10, $QQ^T\mathbf{b}$ is the orthogonal projection $\hat{\mathbf{b}}$ of $\mathbf{b}$ onto Col A. Then $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$, which shows that $\hat{\mathbf{x}}$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$. The uniqueness of $\hat{\mathbf{x}}$ follows from Theorem 14. ■

$$v := R^{-1} \cdot transpose(Q) \cdot b = \begin{bmatrix} \frac{55}{21} \\ \frac{62}{21} \\ \frac{34}{21} \\ \frac{41}{21} \end{bmatrix}$$

$M := \langle A_T \cdot A | A_T \cdot b \rangle :$

$$rref(M) = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{55}{21} \\ 0 & 1 & 0 & 0 & \frac{62}{21} \\ 0 & 0 & 1 & 0 & \frac{34}{21} \\ 0 & 0 & 0 & 1 & \frac{41}{21} \end{bmatrix}$$

Exercise 2

restart

with(*LinearAlgebra*) : *with*(*MTM*) :

Exercise 2.1

Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$
b. $(U\mathbf{x})\cdot(U\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$
c. $(U\mathbf{x})\cdot(U\mathbf{y}) = 0$ if and only if $\mathbf{x}\cdot\mathbf{y} = 0$

Since the length of the vector containing [a, b] has to be maintained by property a then the absolute value of d is given by $\sqrt{a^2+b^2}$.

Exercise 2.2

$c := \dfrac{a}{\mathrm{sqrt}(a^2+b^2)} : s := \dfrac{b}{\mathrm{sqrt}(a^2+b^2)} : G := \langle c, -s | s, c \rangle :$
G

$$\begin{bmatrix} \dfrac{a}{\sqrt{a^2+b^2}} & \dfrac{b}{\sqrt{a^2+b^2}} \\ -\dfrac{b}{\sqrt{a^2+b^2}} & \dfrac{a}{\sqrt{a^2+b^2}} \end{bmatrix} \quad (1.2.2.1)$$

$$\frac{a}{\mathrm{sqrt}(a^2+b^2)} \cdot \frac{b}{\mathrm{sqrt}(a^2+b^2)} - \frac{b}{\mathrm{sqrt}(a^2+b^2)} \cdot \frac{a}{\mathrm{sqrt}(a^2+b^2)} = 0$$

$$G.\langle a, b\rangle = \begin{bmatrix} \dfrac{a^2}{\sqrt{a^2+b^2}} + \dfrac{b^2}{\sqrt{a^2+b^2}} \\ 0 \end{bmatrix}$$

$$a \cdot \left\langle \frac{a}{\mathrm{sqrt}(a^2+b^2)}, -\frac{b}{\mathrm{sqrt}(a^2+b^2)} \right\rangle + b \cdot \left\langle \frac{b}{\mathrm{sqrt}(a^2+b^2)}, \frac{a}{\mathrm{sqrt}(a^2+b^2)} \right\rangle$$

$$\left\langle \frac{a^2}{\mathrm{sqrt}(a^2+b^2)} + \frac{b^2}{\mathrm{sqrt}(a^2+b^2)}, -\frac{ab}{\mathrm{sqrt}(a^2+b^2)} + \frac{ab}{\mathrm{sqrt}(a^2+b^2)} \right\rangle$$

$$\left\langle \frac{a^2}{\mathrm{sqrt}(a^2+b^2)} + \frac{b^2}{\mathrm{sqrt}(a^2+b^2)}, -0 \right\rangle$$

$\dfrac{a^2+b^2}{\mathrm{sqrt}(a^2+b^2)}$ multiply top and bottom by bottom

$$\frac{a^2+b^2 \cdot \mathrm{sqrt}(a^2+b^2)}{a^2+b^2}$$

$\mathrm{sqrt}(a^2+b^2)$

Exercise 2.3

So we should have to show that the dot product between the columns is still zero. Luckily most of the coloumns do not change.

We see that the dot product between a changed column and a modifed one will still be zero because every non zero scalar will be multiplied be zero.
This just leaves the dot product between the modified columns, which will be like in exercise 2.2 except that there can be zeroes multiplied by zeroes, which add nothing thus this will also be 0.

Except for the rows that have been modifed scalar will just be it self. In the modified rows we get.

$$x_1 \cdot 0 + \dots + x_{i-1} \cdot c + \dots + x_{i+1} \cdot s + x_n \cdot 0$$

$$x_i \cdot c + x_j \cdot s = \frac{a\,x_i}{\sqrt{a^2+b^2}} + \frac{b\,x_j}{\sqrt{a^2+b^2}}$$

$$\frac{a \cdot a}{\sqrt{a^2+b^2}} + \frac{b \cdot b}{\sqrt{a^2+b^2}}$$

From exercise 2.2 this is d

$$x_i \cdot (-s) + x_j \cdot c = \frac{a\,x_j}{\sqrt{a^2+b^2}} - \frac{x_i\,b}{\sqrt{a^2+b^2}}$$

$$\frac{a \cdot b}{\sqrt{a^2+b^2}} - \frac{a \cdot b}{\sqrt{a^2+b^2}} = 0$$

Exercise 2.4

If a matrix is orthogonal it follows that $A^T = A^{-1} \Rightarrow A^T A = I$
Thus we have that if both M and N are orthegonal, then the following can be written.

$$(M \cdot N)^T \cdot (M \cdot N) = N^T \cdot M^T \cdot M \cdot N = N^T \cdot N = I$$

$$(M \cdot N) \cdot (M \cdot N)^T = M \cdot N \cdot N^T \cdot M^T = M \cdot M^T = I$$

Exercise 2.5

a) Q is always orthogonal because we only multiply it with some version of G, which we already know is also orthogonal.

b) $A = QR = QGG^T R = QIR = QR$

c) In the second for loop we increment over j and the term with j will always end up being zero. So for i = 1 we will make scalar 2 to m zero in R, i = 2 we will make scalor 3 to m zero in R and so on until we reach i = n.