Petri Nets: Modelling, Properties and Analysis

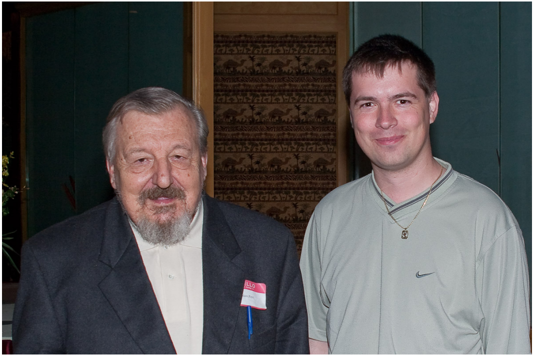
Jiri Srba

Department of Computer Science
Aalborg University, Denmark

Lecture 1

Petri Nets

- Graphical *and* mathematical modeling formalism.
- High level modeling of distributed/concurrent systems.
- Suggested by Carl Adam Petri in 1962, became one of the most popular models.
- Its own dedicated conference, yearly competition of tools ...



P/T Nets Syntax

Petri Net

A Petri net is a tupple $N = (P, T, F, W)$ where

- P is a finite set of places,
- T is a finite set of transitions such that $T \cap P = \emptyset$,
- $F \subseteq (P \times T) \cup (T \times P)$ is the set of arcs (flow relation),
- $W : F \to \mathbb{N}$ is the weight function.

A marking M on a net N is a function $M : P \to \mathbb{N}^0$.

A marked Petri net is a pair (N, M_0) where N is a Petri net and M_0 its initial marking.

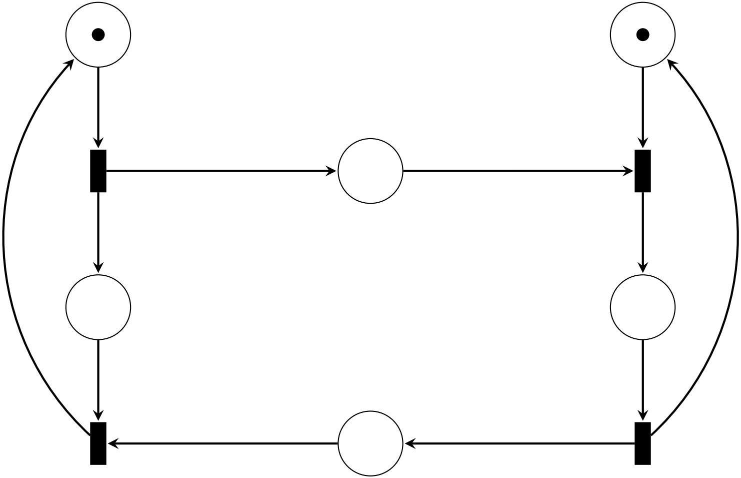
We extend the function W to the domain $(P \times T) \cup (T \times P)$ such that $W((p, t)) = 0$ if $(p, t) \notin F$ and $W((t, p)) = 0$ if $(t, p) \notin F$.

P/T Nets Semantics

A given Petri net $N = (P, T, F, W)$, together with a labelling function $\ell : T \to \mathcal{A}ct$ where $\mathcal{A}ct$ is a finite nonempty set of actions, determines a labelled transition system where

- states are all marking on N,
- $\mathcal{A}ct$ is the set of actions, and
- $M \xrightarrow{a} M'$ if there is a transition $t \in T$ with $\ell(t) = a$ and
  - $M(p) \geq W((p,t))$ for every $p \in P$ (transition t is enabled in M), and
  - $M'(p) = M(p) - W((p,t)) + W((t,p))$ for every $p \in P$ (transition t is fired).

Example: Sender and Receiver



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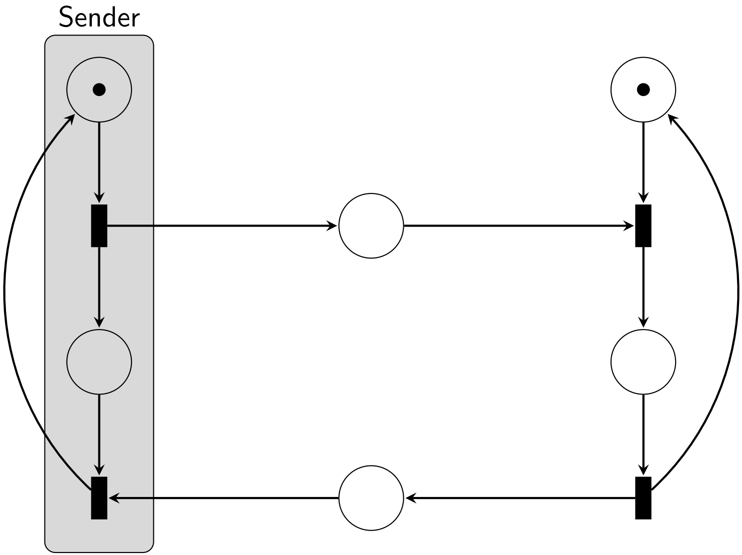

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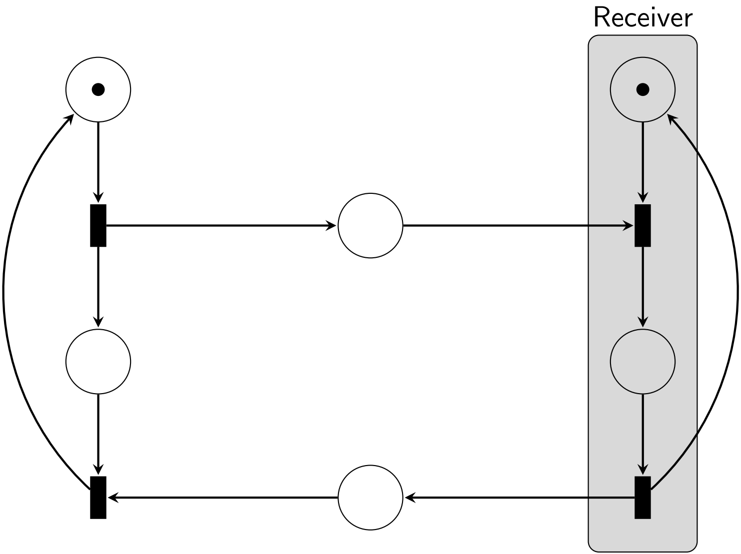

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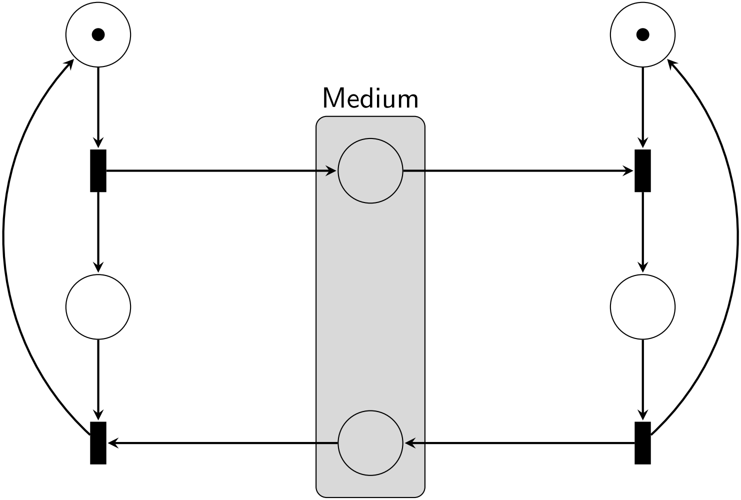

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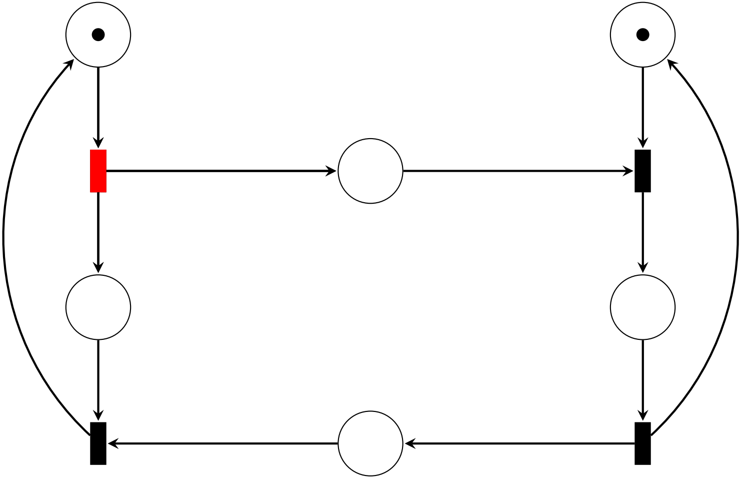

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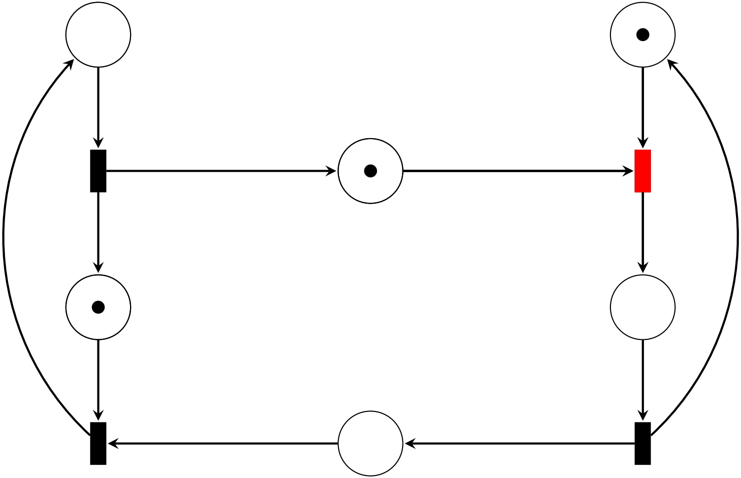

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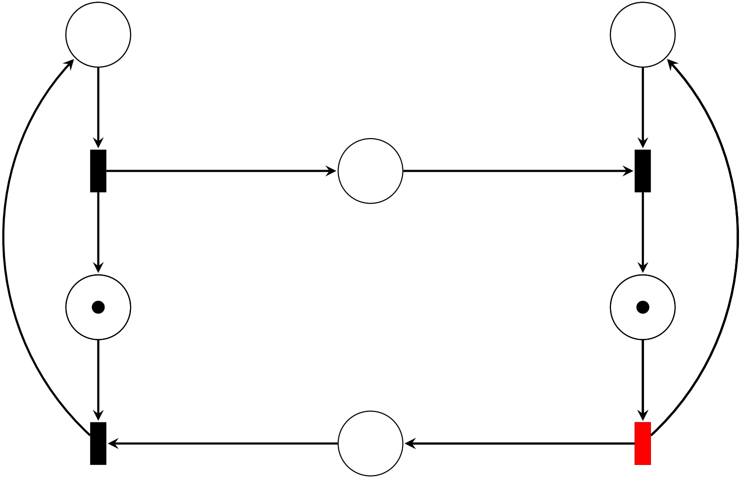

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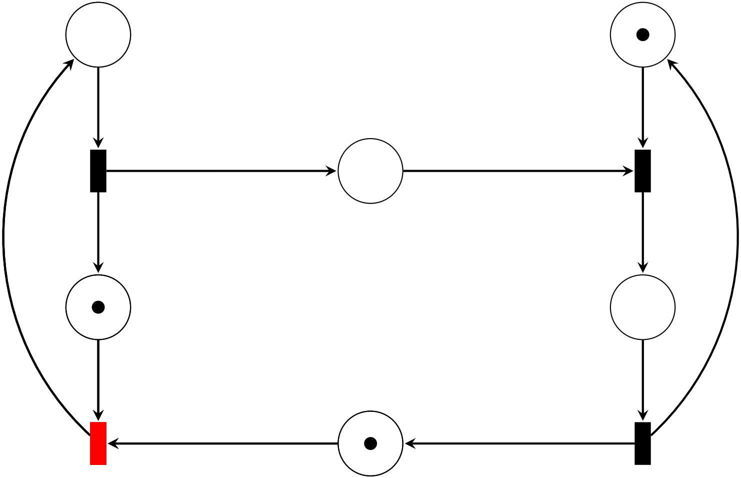

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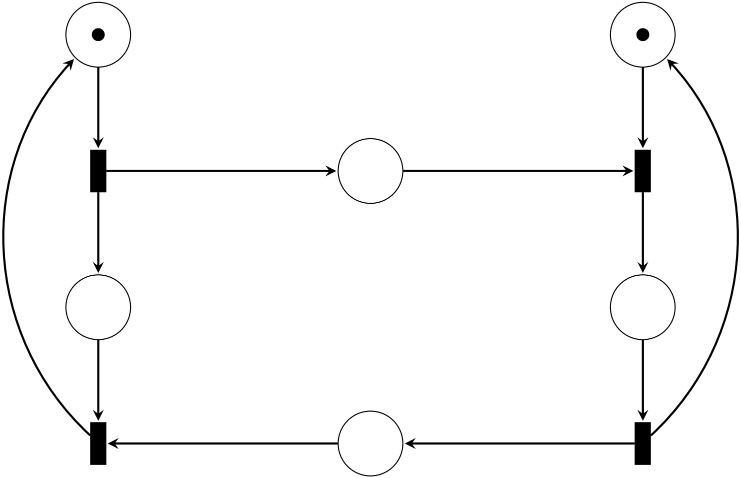

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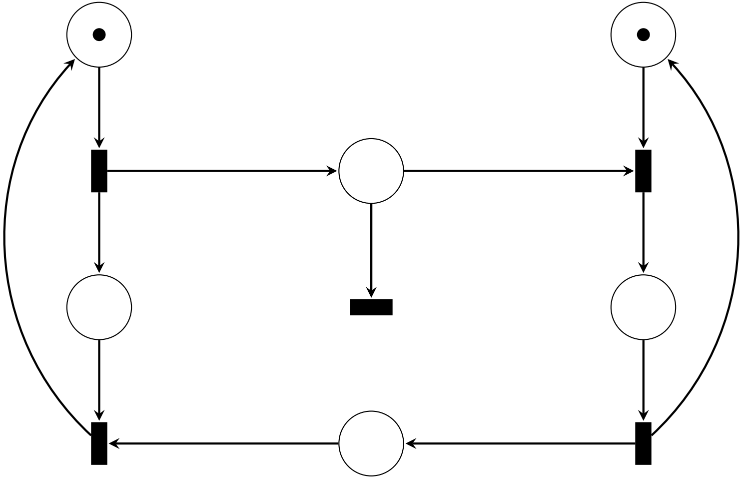

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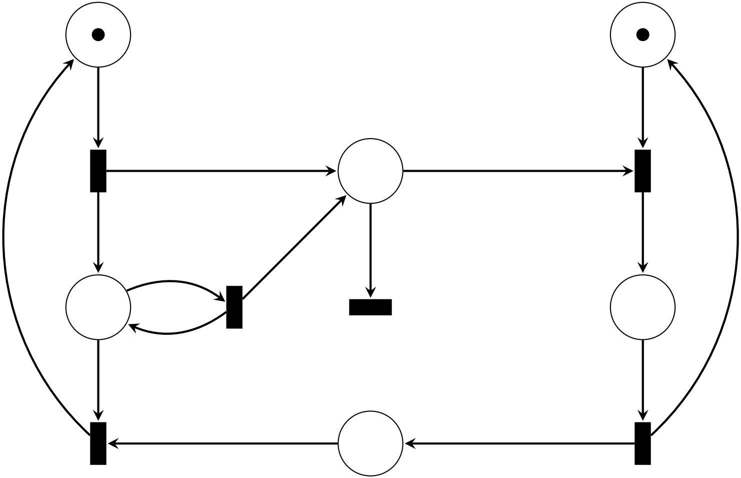

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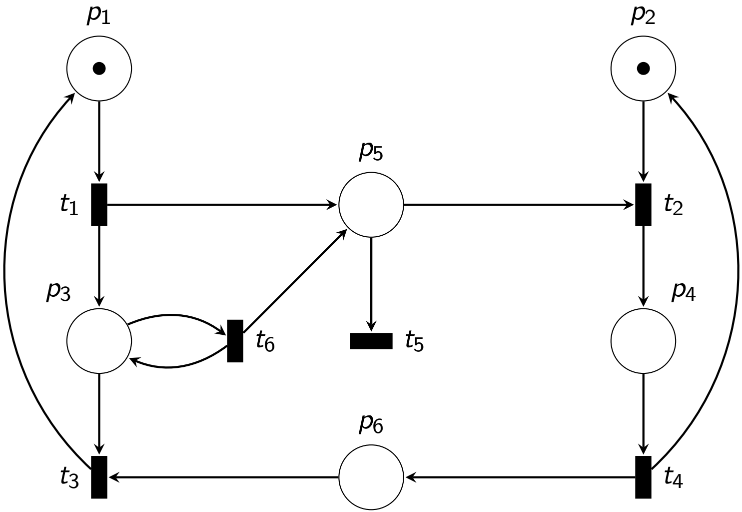

Exercise: draw a fragment of the LTS generated by the net.

Exercise: Model a Car Crossing with Two Traffic Lights

- One traffic light in north/south direction,
- one traffic light in east/west direction,
- traffic light cycle: red, red and yellow, green, yellow, red, ...
- guarantee mutual exclusion of green.

Use TAPAAL

`http://www.tapaal.net`

and ignore the timing intervals (in Menu/View disable showing of $[0, \mathrm{inf})$ intervals).

Finite Capacity Semantics

Our Petri net semantics assumes an infinite (unbounded) capacity.

Let $K(p)$ be a maximum allowed number of tokens in the place p.

Definition [Finite Capacity Semantics]

Let M be a marking such that $M(p) \leq K(p)$ for all $p \in P$. In finite capacity semantics we have $M \xrightarrow{a} M'$ only if $M'(p) \leq K(p)$ for all $p \in P$.

Fact: Finite capacity semantics can be simulated by nets with the infinite capacity. (For each place p add a complementary place p'; be careful about weighted arcs!).

For the rest of the course, we consider the infinite capacity semantics.

Petri Nets as Language Generators

Let $N = (P, T, F, W)$ be a Petri net. Let M be a marking on N and let F be a set of accepting markings.

Definition [Language of a Petri Net]

The language of N generated from M by the set F is defined as

$$L(M) = \{w \in \mathcal{A}ct^* \mid M \xrightarrow{w} M' \text{ and } M' \in F\} .$$

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Exercise: Draw a net (with an initial marking and final markings) that generates the language $\{a^n b^m \mid n, m \geq 1 \text{ and } m \leq 3n\}$.

Behavioural Properties of Petri Nets

Let $\longrightarrow = \cup_{a \in \mathcal{A}ct} \xrightarrow{a}$.
We write $M' \geq M$ if $M'(p) \geq M(p)$ for all places $p \in P$.

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Reachability Problem

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Boundedness Problem

Given a marked Petri net (N, M_0), is there a number k such that whenever $M_0 \longrightarrow^* M$ for some M then $M(p) \leq k$ for all $p \in P$?
If yes, then N is called k-bounded.

Remark: 1-bounded nets are also called safe nets.

Behavioural Properties of Petri Nets

Liveness

Given a marked Petri net (N, M_0), whenever $M_0 \longrightarrow^* M$ then for each $t \in T$ there is some M' s.t. $M \longrightarrow^* M'$ and M' enables t ?

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Liveness degree of a transition t in a marked net (N, M_0):

- **L0-live**: if t is not enabled in any marking reachable from M_0.
- **L1-live**: if t is enabled in some marking reachable from M_0.
- **L2-live**: for any given number k, there is a firing sequence from M_0 where t is fired at least k times.
- **L3-live**: if t is fired infinitely often in some infinite firing sequence from M_0.
- **L4-live**: if t is L1-live in any marking reachable from M_0.

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Lk Liveness

A marked Petri net (N, M_0) is Lk live for $k \in \{0, 1, 2, 3, 4\}$ if every transition in N is Lk live. The net N is strictly Lk live if it is Lk live but not L$(k+1)$ live.

Behavioural Properties of Petri Nets

Deadlock Problem

Given a marked Petri net (N, M_0), is there a marking M such that $M_0 \longrightarrow^* M$ and M does not enable any transition at all?

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Home Marking Problem

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Bisimulation Problem

Given marked Petri nets (N, M_0) and (N', M_0'), are the markings M_0 and M_0' in labelled transition systems generated by the nets strongly bisimilar?

Decidability of Behavioural Properties

Reachability	decidable*, TOWER-hard
Coverability	EXPSPACE-complete
Boundedness	EXPSPACE-complete
Liveness	decidable, TOWER-hard
Deadlock	decidable*, TOWER-hard
Reversibility	decidable, TOWER-hard
Home Marking	decidable, TOWER-hard
Bisimulation	undecidable

* decision algorithm with Ackermann running time (2019)
** tower of exponents of elementary height in the input size (2019)
ELEMENTARY — tower of exponents of fixed height

Petri nets are not a Turing-powerful model.