Petri Nets: Modelling, Properties and Analysis

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Lecture 2

Decidability of Behavioural Properties

Reachability	decidable*, TOWER-hard
Coverability	EXPSPACE-complete
Boundedness	EXPSPACE-complete
Liveness	decidable, TOWER-hard
Deadlock	decidable*, TOWER-hard
Reversibility	decidable, TOWER-hard
Home Marking	decidable, TOWER-hard
Bisimulation	undecidable

* — Ackermannian decision algorithm

Petri nets are not a Turing-powerful model.

Polynomial Time Reductions Among Selected Problems

Overview of Polynomial Time Reductions

boundedness $\leq$ reachability $\equiv$ deadlock $\leq$ liveness

Theorem

Reachability is polynomial time reducible to deadlock.

Theorem

Deadlock is polynomial time reducible to non-liveness.

More reductions in [Dufourd, Finkel: Polynomial-Time Many-One Reductions for Petri Nets].

Petri Nets with Inhibitor Arcs

Inhibitor Arcs

We extend a Petri net $N = (P, T, F, W)$ with a new set of inhibitor arcs $I \subseteq P \times T$.

Now $M \xrightarrow{a} M'$ if there is a transition $t \in T$ with $\ell(t) = a$ and

- $M(p) \geq W((p,t))$ for every $(p,t) \in F$,
- $M(p) = 0$ for every $p \in P$ such that $(p,t) \in I$, and
- $M'(p) = M(p) - W((p,t)) + W((t,p))$.

Theorem

Reachability, coverability, boundedness, liveness, presence of deadlocks, reversibility and home marking problems are undecidable for Petri nets with inhibitor arcs.

By reduction from reachability/boundedness problems of two-counter Minsky machines.

Two-Counter Minsky Machine

Definition: Minsky Machine with Nonnegative Counters c_1 and c_2

$$
\begin{aligned}
1 &: Ins_1 \\
2 &: Ins_2 \\
3 &: Ins_3 \\
&\vdots \\
e &: HALT
\end{aligned}
$$

where each instruction Ins_j is of one of the forms:

Inc j: $c_i := c_i + 1;$ goto k

Test j: if $c_i > 0$ then $(c_i := c_i - 1;$ goto $k)$ else goto ℓ

$$i \in \{1, 2\},\ 1 \le k, \ell \le e$$

Two-Counter Minsky Machine

A configuration of a Minsky machine is of the form (j, n_1, n_2). A computational step is defined in the natural way. Observe that the computation is deterministic.

W.l.o.g. we can assume that whenever the Minsky machine reaches the halting label e then both counters are empty. (Why?)

Undecidability of Halting [Minsky'67]

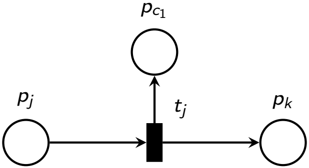
The question whether a given Minsky machine starting with both counters empty eventually halts with both counters empty, i.e. whether from $(1, 0, 0)$ we can reach $(e, 0, 0)$, is undecidable.

Undecidability of Boundedness [Kuzmin, Chalyy'11]

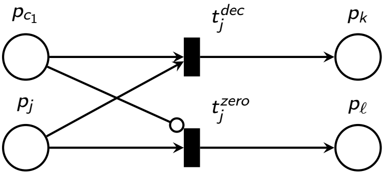
The question whether for a given Minsky machine starting with both counters empty there is a constant K such that any configuration (j, n_1, n_2) reachable from $(1, 0, 0)$ satisfies $n_1 + n_2 \leq K$ is undecidable.

Reduction: Faithful Simulation of Minsky Machine

j: $c_1 := c_1 + 1;$ goto k



j: if $c_1 > 0$ then $(c_1 := c_1 - 1;$ goto $k)$ else goto ℓ



Initial marking has one token in p_1 and all other places are empty.

Undecidable Problems for Petri Nets with Inhibitor Arcs

From the halting problem:

- **Reachability:** can we place a token to p_e while all other places are empty?
- **Coverability:** can we mark the place p_e?
- **Liveness:** once a token is placed to p_e, make sure the net is live (careful about inhibitor arcs).
- **Deadlock:** the net has a deadlock if and only if p_e can be marked.
- **Reversibility:** add a transition from p_e to p_1 and set the initial marking with just one token in place p_e.
- **Home marking problem:** add a new place that is always incremented by any instruction and once p_e is marked, empty that place and return to the initial marking.

From the boundedness problem:

- **Boundedness:** the net is bounded if and only if the Minsky machine is bounded.

Bounded Petri Nets with Inhibitor Arcs

Fact

All the before mentioned problems are decidable for bounded Petri nets with inhibitor arcs.

Proof: Bounded nets have only finitely many reachable markings—the LTS fragment reachable from the initial marking is finite.

There can be exponentially many reachable markings w.r.t. to the size of the net.

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Competition

Draw a bounded Petri net with 5 places, 5 transitions, weights only 1 and one token in the initial marking with as large number of reachable markings as possible. Use TAPAAL and the query EF false (disable state-equations and reductions) to find out how many markings are reachable in your net.

Coverability Tree for Analysis of Petri Nets

Let ω be a new symbol such that for any $n \in \mathbb{N}^0$ we have $\omega > n$, $\omega \geq \omega$ and $\omega + n = \omega - n = \omega$. Given a marked Petri net (N, M_0), we construct its coverability tree as follows:

- Set M_0 as root, tag it as new.
- While there is some new marking M do:
  - If M appears on a path from root to M, tag it as old and continue with another new marking.
  - If M has no enabled transitions, tag it as deadlock and continue with another new marking.
  - For each $t \in T$ and M' such that $M \xrightarrow{t} M'$ do:
    - If there is $M'' \neq M'$ on a path from root to M' such that $M'' \leq M'$ then set $M'(p) = \omega$ for every p such that $M''(p) < M'(p)$.
    - Add M' as a child of M with edge label t. Tag M' as new.

Termination of Coverability Tree Construction

Assume a coverability tree for a marked net (N, M_0).

Theorem [Termination]

The coverability tree is finite (but can be of exponential size).

Proof:

- By König's Lemma: if the tree is infinite then it must contain an infinite branch (due to the fact that every node has only finitely many children).
- By Dickson's Lemma: every infinite sequence of k-tupples over $\mathbb{N}^0$ (markings in our case) contains an infinite nondecreasing subsequence w.r.t. coordinate-wise ordering (our $\leq$ ordering on markings).

Correctness of Coverability Tree

Assume a coverability tree for a marked net (N, M_0).

Theorem [Soundness]

Let M be a marking in the coverability tree and let M' be a marking obtained from M by replacing ω symbols with some concrete nonnegative integers. Then from M_0 one can reach a marking M'' such that $M' \leq M''$.

Proof: follows from monotonicity (if $M_1 \xrightarrow{t} M_2$ and $M_1' \geq M_1$ then also $M_1' \xrightarrow{t} M_2'$ such that $M_2' \geq M_2$).

Theorem [Completeness]

Let M be a marking reachable from M_0. Then there is a marking M' in the coverability tree such that $M \leq M'$.

Proof: induction on the length of the computation from M_0 to M.

Coverability Graph

Coverability Graph

The set of nodes of a reachability graph contains all nodes of the coverability tree and transitions correspond to standard transition firings (extended with ω).

If the net is bounded, we call it the reachability graph.

Selected Properties of Coverability Tree/Graph

Assume a coverability tree/graph for a marked net (N, M_0).

- The net is bounded if and only if its coverability tree does not contain any node with the symbol ω.
- The net is 1-bounded (safe) if and only if it is only 0 and 1 that appear in its coverability tree.
- A transition is L1-live if and only if it appears in the coverability tree.
- A marking M is coverable if and only if the coverability tree contains a node M' such that $M \leq M'$.
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Exercise: find two nets with isomorphic coverability trees but different sets of reachable markings.