Syntax and Semantics: Exercise Session 6

Exercise 1.
Give context-free grammars that generate the following languages with the alphabet $\Sigma = \{0, 1\}$.

$$L_1 = \{w \in \Sigma^* \mid w \text{ contains at least three occurrences of } 1\}$$
$$L_2 = \{w \in \Sigma^* \mid w \text{ starts and ends with the same symbol}\}$$
$$L_3 = \{w \in \Sigma^* \mid \text{the length of } w \text{ is odd and its middle symbol is } 0\}$$
$$L_4 = \emptyset$$
$$L_5 = \{\varepsilon\}$$

Solution 1.

1. L_1 is recognized by a CFG with the following rules
$$S \to A\,1\,A\,1\,A\,1\,A \qquad A \to \varepsilon \mid \Sigma\,A \qquad \Sigma \to 0 \mid 1$$
2. L_2 is recognized by a CFG with the following rules
$$S \to 0 \mid 1 \mid 1\,A\,1 \mid 0\,A\,0 \qquad A \to \varepsilon \mid \Sigma\,A \qquad \Sigma \to 0 \mid 1$$
3. L_3 is recognized by a CFG with the following rules
$$S \to 0 \mid \Sigma\,S\,\Sigma \qquad \Sigma \to 0 \mid 1$$
4. Remember that words recognized by a CFG have to be formed only by terminals. Therefore, L_4 can be recognized by a CFG with the following rule $S \to S$.

5. L_5 is recognized by a CFG with the following rule $S \to \varepsilon$

Exercise 2.
Consider the following CFG G with start variable A.

$$A \to X\,A\,X \mid S \qquad S \to a\,T\,b \mid b\,T\,a \qquad T \to X\,T\,X \mid X \mid \varepsilon \qquad X \to a \mid b$$

1. Describe G formally by giving all its components.
2. Give five strings in $\mathcal{L}(G)$.
3. Give five strings not in $\mathcal{L}(G)$.
4. Which of the following derivations is allowed in G?

$$\begin{array}{lll} T \Rightarrow aba & T \Rightarrow^* aba & XXX \Rightarrow^* aba \\ T \Rightarrow^* XX & T \Rightarrow^* XXX & S \Rightarrow^* \varepsilon \end{array}$$

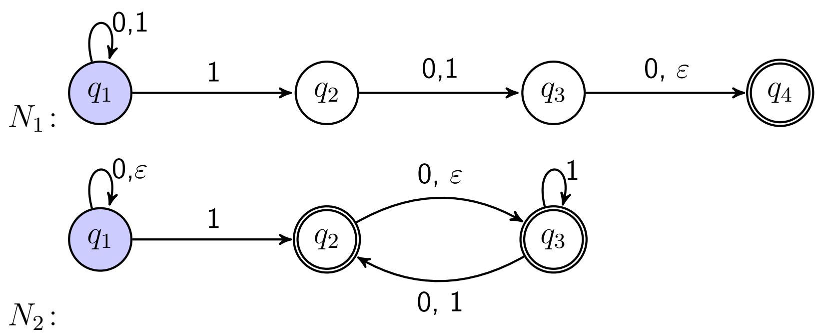
Solution 2.

1. $V = \{A, X, S, T\}$; $\Sigma = \{a, b\}$; start variable A; rules as given.
2. $\{ab, aaabb, abaa, ba, bbab\} \subseteq \mathcal{L}(G)$.
3. Let $\Sigma = \{a, b\}$. One can note that

   $$\mathcal{L}(G) = \{\Sigma^n x \Sigma^* y \Sigma^n \mid x \neq y,\, n \in \mathbb{N}\} \subseteq \Sigma^*$$

   therefore, $\{\varepsilon, a, b, aa, bb\} \cap \mathcal{L}(G) = \emptyset$.
4. (a) $T \not\Rightarrow aba$
   (b) $T \Rightarrow X\,T\,X \Rightarrow a\,T\,X \Rightarrow a\,X\,X \Rightarrow a\,b\,X \Rightarrow a\,b\,a$.
   (c) $X\,X\,X \Rightarrow a\,X\,X \Rightarrow a\,b\,X \Rightarrow a\,b\,a$
   (d) $T \Rightarrow X\,T\,X \Rightarrow X\,X$
   (e) $T \Rightarrow X\,T\,X \Rightarrow X\,X\,X$
   (f) $S \not\Rightarrow^* \varepsilon$

Exercise 3.
Provide a CFG equivalent to the following NFAs



Solution 3.

- N_1 recognizes the language $(0 \cup 1)^* 1 (0 \cup 1)(0 \cup \varepsilon)$; a CFG for it can be

$$S \to 0\,S \mid 1\,S \mid 1\,A \qquad A \to 0\,B \mid 1\,B \qquad B \to 0 \mid \varepsilon$$

- N_2 recognizes the language $0^* 1 (0 \cup 1)^*$; a CFG for it can be

$$S \to Z1A \qquad Z \to \varepsilon \mid 0\,Z \qquad A \to \varepsilon \mid \Sigma\,A \qquad \Sigma \to 0 \mid 1$$

Exercise 4.
Provide CFGs equivalent to each of the following regular expressions:

$$0^* 1 0^* \qquad 1 \cup 0^* \emptyset^* \qquad (01^+)^+$$

Solution 4.

- A CFG for $0^* 1 0^*$ can be

$$S \to Z\,1\,Z \qquad Z \to \varepsilon \mid 0\,Z$$

- A CFG for $1 \cup 0^* \emptyset^* = 1 \cup 0^*$ can be

$$S \to 1 \mid Z \qquad Z \to \varepsilon \mid 0\,Z$$

- A CFG for $(01^+)^+$ can be

$$S \to A \mid A\,S \qquad A \to 0\,B \qquad B \to 1 \mid 1\,B$$