Syntax and Semantics: Exercise Session 1

1. Let us assume that we are given the sets $A = \{1, 2, 3\}$ and $B = \{2, 3\}$.

   (a) Does $A \subseteq B$ hold true?
   Solution: No, since $1 \notin B$.

   (b) Does $B \subseteq A$ hold true?
   Solution: Yes, since $2, 3 \in A$.

   (c) In what relation stands B with respect to A?
   Solution: B is a subset of A. Alternatively, A is a superset of B.

   (d) Find the set $A \cup B$.
   Solution: A.

   (e) How does one call $A \cup B$?
   Solution: The union of A and B.

   (f) Find the set $A \cap B$.
   Solution: B.

   (g) How does one call $A \cap B$?
   Solution: The intersection of A and B.

   (h) Find the set $A \times B$.
   Solution: $\{(a, b) \mid a \in A, b \in B\} = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$.

   (i) How does one call $A \times B$?
   Solution: The product set of A and B.

   (j) Find $\mathcal{P}(B)$, the power set of B.
   Solution: The powerset of B is given by all subsets of B, i.e., $\mathcal{P}(B) = \{A \mid A \subseteq B\}$. Hence, the answer is $\{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$. Note that the elements of the powerset are sets.

   (k) If E is some finite set, what is the cardinality of $\mathcal{P}(E)$? That is, how many elements does $\mathcal{P}(E)$ have?
   Solution: Since there are exactly $2^{|E|}$ binary functions from E to $\{0, 1\}$, where $|\cdot|$ is the cardinality of a set, it holds that $|\mathcal{P}(E)| = 2^{|E|}$.

2. Formalize the following informal statements by using predicate logic.

(a) There exists a real number x such that $x + x$ is greater than 8.
Solution: $\exists x. x + x > 8$

(b) Every real number x is the double of some real number y.
Solution: $\forall x. \exists y. x = 2y$

(c) All natural numbers are positive.
Solution: $\forall x \in \mathbb{N}. x > 0$

3. Consider the sets $C = \{1, 2, 17, 484\}$ and $D = \{x, y, z\}$ and consider the set of pairs $R = \{(1, x), (2, y), (17, y), (17, z)\}$.

(a) Does R encode a function (from C to D)?
Solution: No, since 17 is mapped to both y and z.

(b) Does R encode a relation (among C and D)?
Solution: Yes.

(c) Does $R \cup \{484\}$ encode a relation (among C and D)?
Solution: No, since $R \cup \{484\} \not\subseteq C \times D$.

(d) Let $X \setminus Y$ denote the elements of X not present in Y. With this, does $R \setminus \{(17, z)\}$ encode a function (from C to D)? If so, is the function total or partial?
Solution: Yes, it encodes a function. The function is partial because 484 is not mapped to any value in D.

4. Prove by induction on k that

$$1 + \sum_{i=1}^{k} (2i + 1) = (k + 1)^2$$

for all $k \geq 1$.

Solution: Base case $k = 1$: One readily verifies that $1 + (2 + 1) = (1 + 1)^2$. Induction step $k \to k + 1$: By observing that

$$1 + \sum_{i=1}^{k+1} (2i + 1) = 1 + \sum_{i=1}^{k} (2i + 1) + (2(k + 1) + 1),$$

an application of the induction hypothesis on $1 + \sum_{i=1}^{k} (2i + 1)$ yields

$$1 + \sum_{i=1}^{k+1} (2i + 1) = (k + 1)^2 + (2(k + 1) + 1).$$

This, in turn, yields the claim because:

$$\begin{aligned}1 + \sum_{i=1}^{k+1}(2i+1) &= (k+1)^2 + (2(k+1)+1) \\ &= k^2 + 2k + 1 + 2k + 2 + 1 \\ &= k^2 + 4k + 4 \\ &= (k+2)^2 \\ &= ((k+1)+1)^2\end{aligned}$$

5. Using the formula from 1 (k), compute first the cardinality of $\mathcal{P}(\mathcal{P}(B))$, where $B = \{2,3\}$ as in Exercise 1. Afterwards, compute $\mathcal{P}(\mathcal{P}(B))$ itself and check whether the cardinality of your answer matches the cardinality obtained in the first part of the exercise.

   Solution: It holds that $|\mathcal{P}(B)| = 2^{|B|} = 4$ and $|\mathcal{P}(\mathcal{P}(B))| = 2^{|\mathcal{P}(B)|} = 2^4 = 16$. For the second part of the exercise, we note that we are asked to provide all subsets of $\mathcal{P}(B) = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}$. Consequently, $\mathcal{P}(\mathcal{P}(B))$ has the following elements:

   - Subsets of $\mathcal{P}(B)$ with no elements:
     $$\emptyset$$
   - Subsets of $\mathcal{P}(B)$ with exactly one element:
     $$\{\emptyset\}, \{\{2\}\}, \{\{3\}\}, \{\{2,3\}\}$$
     (Note that $\emptyset$ and $\{\emptyset\}$ are two different sets!)
   - Subsets of $\mathcal{P}(B)$ with exactly two elements:
     $$\{\emptyset, \{2\}\}, \{\emptyset, \{3\}\}, \{\emptyset, \{2,3\}\}, \{\{2\}, \{3\}\}, \{\{2\}, \{2,3\}\}, \{\{3\}, \{2,3\}\}$$
   - Subsets of $\mathcal{P}(B)$ with exactly three elements:
     $$\big\{\{2\}, \{3\}, \{2,3\}\big\}, \big\{\emptyset, \{3\}, \{2,3\}\big\}, \big\{\emptyset, \{2\}, \{2,3\}\big\}, \big\{\emptyset, \{2\}, \{3\}\big\}$$
   - Subsets of $\mathcal{P}(B)$ with exactly four elements:
     $$\{\emptyset, \{2\}, \{3\}, \{2,3\}\}$$