Basic Imperative Statements

Recall the abstract syntax of Bims

$S ::= \text{"}x = a\text{"} \mid \text{skip} \mid S_1 ; S_2 \mid \text{if } b \text{ then } S_1 \text{ else } S_2 \mid \text{while } b \text{ do } S$

$b ::= \text{"}a_1 = a_2\text{"} \mid \text{"}a_1 < a_2\text{"} \mid \neg b_1 \mid b_1 \wedge b_2 \mid (b_1)$

$a ::= n \mid x \mid a_1 + a_2 \mid a_1 * a_2 \mid a_1 - a_2 \mid (a_1)$

$n \in \mathbb{N}\mathrm{um}, \quad x \in \mathbb{V}\mathrm{ar}, \quad a \in \mathbb{A}\mathrm{exp}, \quad b \in \mathbb{B}\mathrm{exp} \quad S \in \mathbb{S}\mathrm{tm}$

Previously we proposed semantics for Aexp <u>without variables</u>

The semantics made use of semantic functions to interpret numerals

How can we interpret variables?

What changes in a system if we change the interpretation of a variable?

§1. States

Definition [State]: A state is a partial function

$$s : \mathbb{V}ar \rightharpoonup \mathbb{Z}$$

The set of all states is $\mathbb{S}tates = \mathbb{V}ar \rightharpoonup \mathbb{Z}$

It is usefull to represent a state by listing its content, e.g.,

$$s = [x \mapsto 2,\ y \mapsto 3,\ z \mapsto 5]$$

If s is a state, the updated state $s[x \mapsto v]$ is the state s' such that

$$s'(y) = \begin{cases} s(y) & , \ y \neq x \\ v & , \ y = x \end{cases}$$

If $s = [x \mapsto 2,\ y \mapsto 3,\ z \mapsto 5]$, then

$$s[y \mapsto 5] = [x \mapsto 2,\ y \mapsto 5,\ z \mapsto 5]$$
$$s[y \mapsto 0,\ z \mapsto 1] = [x \mapsto 2,\ y \mapsto 0,\ z \mapsto 1]$$
$$s[w \mapsto 3] = [x \mapsto 2,\ y \mapsto 3,\ z \mapsto 5,\ w \mapsto 3]$$

The concept of state allows us to provide a BS-semantics for the entire Aexp, including the variables.

The transition relations are of type

$$s \vdash a \longrightarrow_A v$$

The presence of the state is an indication of the fact that the semantics depends directly on states. In fact, each state has its own transition system.

This situation is generated by the fact that in different states the same variable has different interpretations.

$$a ::= n \mid x \mid a_1 + a_2 \mid a_1 * a_2 \mid a_1 - a_2 \mid (a_1)$$

BS-semantics for Aexp

[PLUS$_{BS}$]
$$\frac{s \vdash a_1 \longrightarrow_A v_1 \qquad s \vdash a_2 \longrightarrow_A v_2}{s \vdash a_1 + a_2 \longrightarrow_A v}\,, \quad v = v_1 + v_2$$

[MINUS$_{BS}$]
$$\frac{s \vdash a_1 \longrightarrow_A v_1 \qquad s \vdash a_2 \longrightarrow_A v_2}{s \vdash a_1 - a_2 \longrightarrow_A v}\,, \quad v = v_1 - v_2$$

[MULT$_{BS}$]
$$\frac{s \vdash a_1 \longrightarrow_A v_1 \qquad s \vdash a_2 \longrightarrow_A v_2}{s \vdash a_1 * a_2 \longrightarrow_A v}\,, \quad v = v_1 \cdot v_2$$

[PARENT$_{BS}$]
$$\frac{s \vdash a \longrightarrow_A v}{s \vdash (a) \longrightarrow_A v}$$

[NUM$_{BS}$]
$$s \vdash n \longrightarrow_A v \,, \quad v = \mathcal{N}[\![n]\!]$$

[VAR$_{BS}$]
$$s \vdash x \longrightarrow_A v \,, \quad s(x) = v$$

Notice the structure of the transition system:

for each state $s \in \mathbb{S}tates$, we have

$$\mathcal{T}_s = (\mathbb{A}_{exp} \cup \mathbb{Z}, \xrightarrow{s}_A, \mathbb{Z})$$

where instead of $a \xrightarrow{s}_A v$ we write $s \vdash a \longrightarrow_A v$.

<u>Problem:</u> Give a BS-semantics for $\mathbb{B}exp$ for the complete case when $\mathbb{A}exp$ contains variables.

§2. Big-Step Semantics of Statements

What is the role of the statements in the abstract syntax of Bims?

$$S ::= \text{"}x = a\text{"} \mid \text{skip} \mid S_1 ; S_2 \mid \text{if } b \text{ then } S_1 \text{ else } S_2 \mid \text{while } b \text{ do } S$$

$$n \in \mathbb{N}\mathrm{um}, \quad x \in \mathbb{V}\mathrm{ar}, \quad a \in \mathbb{A}\mathrm{exp}, \quad b \in \mathbb{B}\mathrm{exp} \quad S \in \mathbb{S}\mathrm{tm}$$

The role of the statements is to change the state of the system. For this reason, the transitions for statements have the form

$$\langle S, s \rangle \longrightarrow s' \ , \text{ for } S \in \mathbb{S}\mathrm{tm}, \ s, s' \in \mathbb{S}\mathrm{tates}$$

"if we execute S in s, we get the final state s'".

The transition system is $\mathcal{T} = (T, \longrightarrow, F)$ where

$$T = (\mathbb{S}\mathrm{tm} \times \mathbb{S}\mathrm{tates}) \cup \mathbb{S}\mathrm{tates}$$

$$F = \mathbb{S}\mathrm{tates}$$

$\longrightarrow$ is defined by the following rules

$$S ::= \text{"}x := a\text{"} \mid \text{skip} \mid S_1 ; S_2 \mid \text{if } b \text{ then } S_1 \text{ else } S_2 \mid \text{while } b \text{ do } S$$

BS-Semantics for Stm

$[\mathrm{ASSGN}_{BS}]$ $\quad \langle x := a, s\rangle \longrightarrow s[x \mapsto v]\ ,\quad s \vdash a \longrightarrow_A v$

$[\mathrm{Skip}_{BS}]$ $\quad \langle \text{skip}, s\rangle \longrightarrow s$

$[\mathrm{COMP}_{BS}]$ $$\frac{\langle S_1, s\rangle \longrightarrow s'' \qquad \langle S_2, s''\rangle \longrightarrow s'}{\langle S_1 ; S_2, s\rangle \longrightarrow s'}$$

$[\text{if-}\mathrm{T}_{BS}]$ $$\frac{\langle S_1, s\rangle \longrightarrow s'}{\langle \text{if } b \text{ then } S_1 \text{ else } S_2, s\rangle \longrightarrow s'}\ ,\quad s \vdash b \longrightarrow_B \mathrm{T}$$

$[\text{if-}\bot_{BS}]$ $$\frac{\langle S_2, s\rangle \longrightarrow s'}{\langle \text{if } b \text{ then } S_1 \text{ else } S_2, s\rangle \longrightarrow s'}\ ,\quad s \vdash b \longrightarrow_B \bot$$

$[\text{while-}\mathrm{T}_{BS}]$ $$\frac{\langle S, s\rangle \longrightarrow s'' \qquad \langle \text{while } b \text{ do } S, s''\rangle \longrightarrow s'}{\langle \text{while } b \text{ do } S, s\rangle \longrightarrow s'}\ ,\quad s \vdash b \longrightarrow_B \mathrm{T}$$

$[\text{while-}\bot_{BS}]$ $\quad \langle \text{while } b \text{ do } S, s\rangle \longrightarrow s\ ,\quad s \vdash b \longrightarrow_B \bot$

Question: Are the rules of the BS-semantics for statements compositional?
I.e., are the premises making use of syntactic entities that are imediat constituents of the syntactic construct found in the conclusion?

Example: $\langle x := \underline{7} * \underline{5}\ ,\ [x \mapsto 2, y \mapsto 3] \rangle$

$$\left.\begin{array}{ll} [x \mapsto 2, y \mapsto 3] \vdash \underline{7} \rightarrow_A 7 & [NUM_{BS}] \\ [x \mapsto 2, y \mapsto 3] \vdash \underline{5} \rightarrow_A 5 & [NUM_{BS}] \end{array}\right| \overset{[MULT_{BS}]}{\Longrightarrow}$$

$$\Rightarrow [x \mapsto 2, y \mapsto 3] \vdash \underline{7} * \underline{5} \longrightarrow 35 \quad \overset{[ASSGN_{BS}]}{\Longrightarrow}$$

$$\Rightarrow \langle x := \underline{7} * \underline{5}\ ,\ [x \mapsto 2, y \mapsto 3] \rangle \longrightarrow [x \mapsto 35, y \mapsto 3]$$

Exercises:

(i) $\langle x := x * (\underline{2} + y)\ ,\ [x \mapsto 2, y \mapsto 3] \rangle \longrightarrow ?$

(ii) $\langle \text{if } x<y \text{ then } z := \underline{5} \text{ else } z := \underline{2}\ ,\ [x \mapsto 2, y \mapsto 3, z \mapsto 3] \rangle \longrightarrow ?$

Problem: Let $S = i := \underline{6};\ \text{while } i \neq \underline{0} \text{ do } (x := x + i;\ i := i - \underline{2})$
and $s = [x \mapsto 5]$

Is there a transition $\langle S, s \rangle \rightarrow s'$?

Sketch: With $S' \equiv (x \coloneqq x + i; i \coloneqq i - \underline{2})$, first prove that

$\langle S', [x \mapsto 5, i \mapsto 6] \rangle \rightarrow [x \mapsto 11, i \mapsto 4]$
$\langle S', [x \mapsto 11, i \mapsto 4] \rangle \rightarrow [x \mapsto 15, i \mapsto 2]$
$\langle S', [x \mapsto 15, i \mapsto 2] \rangle \rightarrow [x \mapsto 17, i \mapsto 0]$

- Then, prove $\langle \text{while } i \neq \underline{0} \text{ do } S', [x \mapsto 17, i \mapsto 0] \rangle \rightarrow [x \mapsto 17, i \mapsto 0]$
- Use the above to prove $\langle \text{while } i \neq \underline{0} \text{ do } S', [x \mapsto 15, i \mapsto 2] \rangle \rightarrow [x \mapsto 17, i \mapsto 0]$
- Use the above to prove $\langle \text{while } i \neq \underline{0} \text{ do } S', [x \mapsto 11, i \mapsto 4] \rangle \rightarrow [x \mapsto 17, i \mapsto 0]$
- Use the above to prove $\langle \text{while } i \neq \underline{0} \text{ do } S', [x \mapsto 5, i \mapsto 6] \rangle \rightarrow [x \mapsto 17, i \mapsto 0]$
- Use the above to prove $\langle S, [x \mapsto 5] \rangle \rightarrow [x \mapsto 17, i \mapsto 0]$

Definition [Termination]

Given a BS-semantics, the execution of a statement S terminates from start state s if there exists a state s' such that

$$\langle S, s \rangle \to s'$$

S loops forever on s if there is no state s' such that $\langle S, s \rangle \to s'$.

S always terminates if S terminate for all $s \in$ States.

S always loops forever if S loops forever for all states $s \in$ States.

Problem: Show that S = while 0=0 do skip always loops forever.

Hint: For any state $s \in$ States prove by induction on $n \geq 0$ that $\langle S, s \rangle \to s'$ has no derivation tree of height n.

(Sketch) Base case: There is no derivation tree of height zero because the rule [while-$\perp_{BS}$] is not applicable. Induction step: Since $\langle skip, s \rangle \to s''$ implies that $s = s''$ and $\langle S, s \rangle \to s'$ has no derivation tree of height n by induction hypothesis, rule [while-T$_{BS}$] is not applicable. Hence, $\langle S, s \rangle \to s'$ has no derivation tree of height $n + 1$.

§3. Small-Step Semantics of Statements

To design a SS-semantics of statements, we have to consider two types of transitions.

$$\langle S, s\rangle \Rightarrow s' \qquad\qquad \langle S, s\rangle \Rightarrow \langle S', s'\rangle$$

The transition system is $(\Gamma, \Rightarrow, F)$, where

- $\Gamma = (\mathbf{Stm} \times \mathbf{States}) \cup \mathbf{States}$
- $F = \mathbf{States}$
- $\Rightarrow$ is given by the following rules

$$S ::= \text{"}x = a\text{"} \mid \text{skip} \mid S_1 ; S_2 \mid \text{if } b \text{ then } S_1 \text{ else } S_2 \mid \text{while } b \text{ do } S$$

SS-semantics for Stm

$[\text{ASSGN}_{SS}]$ $\quad \langle x := a, s\rangle \Rightarrow s[x \mapsto v] \quad , \quad s \vdash a \rightarrow_A v$

$[\text{Skip}_{SS}]$ $\quad \langle \text{skip}, s\rangle \Rightarrow s$

$[\text{COMP1}_{SS}]$ $\quad \dfrac{\langle S_1, s\rangle \Rightarrow \langle S_1', s'\rangle}{\langle S_1 ; S_2, s\rangle \Rightarrow \langle S_1' ; S_2, s'\rangle}$

$[\text{COMP2}_{SS}]$ $\quad \dfrac{\langle S_1, s\rangle \Rightarrow s'}{\langle S_1 ; S_2, s\rangle \Rightarrow \langle S_2, s'\rangle}$

$[\text{If-T}_{SS}]$ $\quad \langle \text{if } b \text{ then } S_1 \text{ else } S_2, s\rangle \Rightarrow \langle S_1, s\rangle \quad , \quad s \vdash b \rightarrow_B T$

$[\text{If-}\bot_{SS}]$ $\quad \langle \text{if } b \text{ then } S_1 \text{ else } S_2, s\rangle \Rightarrow \langle S_2, s\rangle \quad , \quad s \vdash b \rightarrow_B \bot$

$[\text{while}_{SS}]$ $\quad \langle \text{while } b \text{ do } S, s\rangle \Rightarrow \langle \text{if } b \text{ then } (S ; \text{while } b \text{ do } S) \text{ else skip}, s\rangle$

Definition [Termination]:

Given an SS-semantics, the execution of a statement <u>S terminates from the start state s</u>, if there exists a state s' such that

$$\langle S, s \rangle \Rightarrow^* s'$$

Otherwise, we say that <u>S loops forever in s</u>, i.e., there exists an infinite transition sequence

$$\langle S, s \rangle \Rightarrow \langle S_1, s_1 \rangle \Rightarrow \ldots \langle S_k, s_k \rangle \Rightarrow \ldots$$

<u>S always loops forever</u> if it loops forever for every starting state.

<u>S always terminates</u> if it terminates for every start state.

Exercise Find all the transitions (if there are any) in the transition sequence starting from $\langle S, s\rangle$, where

$$S = \text{if } \; x > \underline{3} \text{ then } (x := \underline{2} + x \,;\, y := 4) \text{ else skip}$$

and $s = [x \mapsto 4]$

Construct the derivation tree for each transition.

Exercise: We have seen that "while 0=0 do skip" always loops forever in the BS-semantics.

Prove that in the SS-semantics, for any state s we have

$$\langle \text{while } 0=0 \text{ do skip}, s\rangle \Rightarrow^3 \langle \text{while } 0=0 \text{ do skip}\rangle.$$

Sketch:

$\langle$while $0 = 0$ do skip, $s\rangle$

$\Rightarrow \langle$if $0 = 0$ then (skip; while $0 = 0$ do skip) else skip , $s\rangle$

$\Rightarrow \langle$skip; while $0 = 0$ do skip, $s\rangle$

$\Rightarrow \langle$while $0 = 0$ do skip, $s\rangle$

§4. Equivalence of the BS and SS semantics for $\mathbb{B}$ims.

Theorem 1: Let $S \in \mathbb{S}tm$ and $s \in \mathbb{S}tates$.

If $\langle S, s\rangle \rightarrow s'$, then $\langle S, s\rangle \Rightarrow^* s'$.

Proof: • We must show that the implication is true for all the transitions of type $\langle S, s\rangle \rightarrow s'$ in the BS-semantics.

• Each BS-transition respects one BS-transition rule

• Hence, it is sufficient to prove the implication stated by the theorem for each transition rule.

Exercise: Study the inductive proof in Hüttel's book (pag. 56–57).

Lemma 1: Let $S_1, S_2 \in \mathbf{Stm}$ and $s \in \mathbf{States}$.
If $\langle S_1, s\rangle \Rightarrow^k s'$, then $\langle S_1; S_2, s\rangle \Rightarrow^k \langle S_2, s'\rangle$.

Proof: We prove by induction on k that

if $\langle S_1, s\rangle \Rightarrow^k s'$, then $\langle S_1; S_2, s\rangle \Rightarrow^k \langle S_2, s'\rangle$.

- the case $k=0$ is void and the case $k=1$ is an instance of $[\text{COMP2}_{ss}]$.
- the case $k+1$: suppose that $\langle S_1, s\rangle \Rightarrow^{k+1} s'$.

Then, $\langle S_1, s\rangle \Rightarrow \langle S_1^+, s^+\rangle \Rightarrow^k s'$

Using the inductive hypothesis, $\langle S_1^+; S_2, s^+\rangle \Rightarrow^k \langle S_2, s'\rangle$

$[\text{COMP1}_{ss}]$ implies $\langle S_1; S_2, s\rangle \Rightarrow \langle S_1^+; S_2, s^+\rangle$ $\Big\} \Rightarrow$

$\Rightarrow \langle S_1; S_2, s\rangle \Rightarrow^{k+1} \langle S_2, s'\rangle$

Corollary 1: Let $S_1, S_2 \in \mathbf{Stm}$ and $s \in \mathbf{States}$.
If $\langle S_1, s\rangle \Rightarrow^* s'$, then $\langle S_1; S_2, s\rangle \Rightarrow^* \langle S_2, s'\rangle$.

Lemma 2: For all $S_1, S_2 \in \mathbf{Stm}$, $s, s' \in \mathbf{States}$, if $\langle S_1;S_2, s\rangle \Rightarrow^k s''$, then there exists $s' \in \mathbf{States}$ and $k', k'' \in \mathbb{N}$ with $k'+k''=k$, such that

$$\langle S_1, s\rangle \Rightarrow^{k'} s' \quad \text{and} \quad \langle S_2, s'\rangle \Rightarrow^{k''} s''$$

Proof: Induction on k: • The cases $k=0$ and $k=1$: are both vacously true.

• The case $k+1$: Assume that $\langle S_1;S_2, s\rangle \Rightarrow^{k+1} s''$

- there exists a configuration γ s.t. $\langle S_1;S_2, s\rangle \Rightarrow \gamma \Rightarrow^k s''$
- γ was obtained using either [COMP1$_{ss}$] or [COMP2$_{ss}$]
- hence, either $\gamma = \langle S_1';S_2, s'''\rangle$ or $\gamma = \langle S_2, s'''\rangle$
- if $\gamma = \langle S_1';S_2, s'''\rangle$

$$[\text{COMP1}_{ss}]\ \frac{\langle S_1, s\rangle \Rightarrow \langle S_1', s'''\rangle}{\langle S_1;S_2, s\rangle \Rightarrow \langle S_1';S_2, s'''\rangle}$$

and $\langle S_1';S_2, s'''\rangle \Rightarrow^k s'$

- from the inductive hypothesis, $\exists k_{11}, k_{12}$ with $k = k_{11}+k_{12}$ and $\exists s' \in \mathbf{States}$

s.t. $\langle S_1', s'''\rangle \Rightarrow^{k_{11}} s'$ and $\langle S_2, s'\rangle \Rightarrow^{k_{21}} s''$ $\Rightarrow \begin{cases} k' = k_{11}+1 \\ k'' = k_{21} \end{cases}$

but $\langle S_1, s\rangle \Rightarrow \langle S_1', s'''\rangle$ $\Rightarrow$ $\langle S_1, s\rangle \Rightarrow^{k_{11}+1} s'$

Lemma 2: For all $S_1, S_2 \in \mathbf{Stm}$, $s, s' \in \mathbf{States}$, if $\langle S_1; S_2, s\rangle \Rightarrow^k s''$, then there exists $s' \in \mathbf{States}$ and $k', k'' \in \mathbb{N}$ with $k' + k'' = k$, such that

$$\langle S_1, s\rangle \Rightarrow^{k'} s' \quad \text{and} \quad \langle S_2, s'\rangle \Rightarrow^{k''} s''$$

Proof: Induction on k: • The cases $k=0$ and $k=1$: are both vacously true.

• The case $k+1$: Assume that $\langle S_1; S_2, s\rangle \Rightarrow^{k+1} s''$

- there exists a configuration γ s.t. $\langle S_1; S_2, s\rangle \Rightarrow \gamma \Rightarrow^k s''$
- γ was obtained using either $[\text{COMP1}_{ss}]$ or $[\text{COMP2}_{ss}]$
- hence, either $\gamma = \langle S_1'; S_2, s'''\rangle$ or $\gamma = \langle S_2, s'''\rangle$
- if $\gamma = \langle S_2, s'''\rangle$

$$[\text{COMP2}_{ss}] \quad \frac{\langle S_1, s\rangle \Rightarrow s'''}{\langle S_1; S_2, s\rangle \Rightarrow \langle S_2, s'''\rangle}$$

and $\langle S_2, s'''\rangle \Rightarrow^k s''$

- hence, if we take $s' = s'''$, $k' = 1$ and $k'' = k$, we have the proof.