Syntax and Semantics:
Exercise Session 2

Exercise 1.
Let $L_1 = \{aa, bb, bbb\}$, $L_2 = \{abba, aab, bb\}$
Specify the following languages:

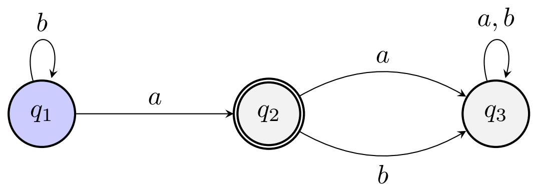
1. $L_1 \circ L_2$
2. $L_1 \cup L_2$
3. $L_1 \cap L_2$
4. $L_1 \setminus L_2$
5. Provide a few strings of L_2^*

Solution 1.

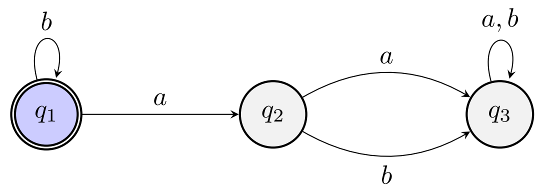
1. $L_1 \circ L_2 = \{aaabba, aaaab, aabb, bbabba, bbaab, bbbb, bbbabba, bbbaab, bbbbb\}$
2. $L_1 \cup L_2 = \{aa, bb, bbb, abba, aab, bb\}$
3. $L_1 \cap L_2 = \{bb\}$
4. $L_1 \setminus L_2 = \{aa, bbb\}$
5. Provide a few strings of L_2^* : $\varepsilon, abba, aab, bb, abbaaab, aabbb, aabaab, bbb, \ldots$

Exercise 2.
Given are the following three automata (where the initial state is denoted by dark blue).

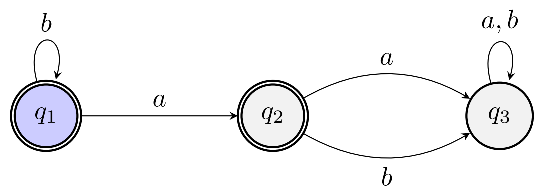
M_1



M_2



M_3



(a) Describe the sequence of states of M_1 for the following inputs:

(1) abbbab

(2) ababaab

(3) aaaaa

(4) ε

(b) Which of the previous sequences are final in M_1, M_2 and M_3?

(c) Describe the languages accepted by each of the three machines.

Solution 2.

(a) Describe the sequence of states of M_1 for the following inputs:

(1) abbbab: $q_1 \xrightarrow{a} q_2 \xrightarrow{b} q_3 \xrightarrow{b} q_3 \xrightarrow{b} q_3 \xrightarrow{a} q_3 \xrightarrow{b} q_3$

(2) ababaab: $q_1 \xrightarrow{a} q_2 \xrightarrow{b} q_3 \xrightarrow{a} q_3 \xrightarrow{b} q_3 \xrightarrow{a} q_3 \xrightarrow{a} q_3 \xrightarrow{b} q_3$

(3) aaaaa: $q_1 \xrightarrow{a} q_2 \xrightarrow{a} q_3 \xrightarrow{a} q_3 \xrightarrow{a} q_3 \xrightarrow{a} q_3$

(4) ε: q_1

(b) Which of the previous sequences are final in M_1, M_2 and M_3?

(1) M_1: None is final

(2) M_2: 4 is final

(3) M_3 4 is final

(c) Describe the languages accepted by each of the three matchines.

(1) $L(M_1) = \{b^n a \mid n \geq 0\}$

(2) $L(M_2) = \{b^n \mid n \geq 0\}$

(3) $L(M_3) = L(M_1) \cup L(M_2) = \{b^n a^m \mid n \geq 0, m \in \{0,1\}\}$

Exercise 3.
Give the state diagram for the following automaton and describe its language.
$M_4 = (Q, \Sigma, \delta, q_o, F)$ where:

$Q = \{s, q_1, q_2, r_1, r_2\}$

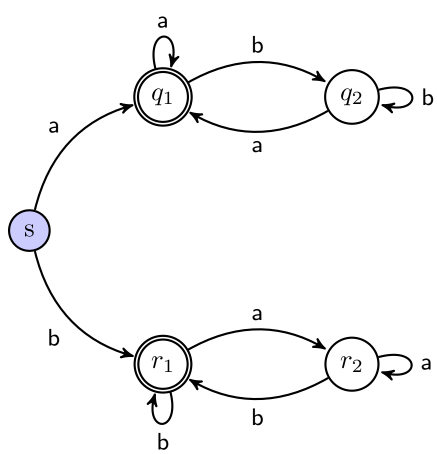
$\Sigma = \{a, b\}$

$q_0 = s$

$F = \{q_1, r_1\}$

δ	a	b
s	q_1	r_1
q_1	q_1	q_2
q_2	q_1	q_2
r_1	r_2	r_1
r_2	r_2	r_1

Solution 3.



$L(M_4) = \{xwx \mid x \in \Sigma, w \in \Sigma^*\} \cup \{a\} \cup \{b\}$

Exercise 4.
For each of the following languages, give the state diagram of an automaton recognizing it.

(i) $L_1 = \{w \in \{1, 22\}^* \mid 11 \text{ is a prefix of } w\}$

(ii) $L_2 = \emptyset$ with alphabet $\Sigma = \{0, 1, 2\}$

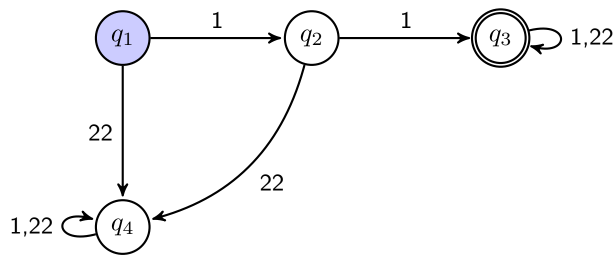
(iii) $L_3 = \{\varepsilon\}$ with alphabet $\Sigma = \{0, 1, 2\}$

(iv) $L_4 = \{w \in \{\text{go}, \text{stop}\}^* \mid w = \varepsilon \text{ or ends with stop}\}$

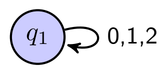
(v) $L_5 = \{w \in \{0, 1\}^* \mid w \text{ has } 001 \text{ as a prefix or } 11 \text{ as a suffix}\}$

Solution 4.

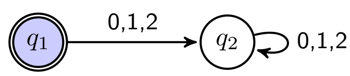
(i) $L_1 = \{w \in \{1, 22\}^* \mid 11 \text{ is a prefix of } w\}$



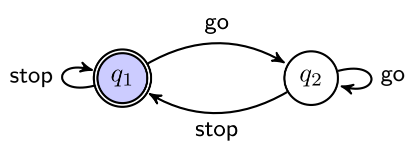
(ii) $L_2 = \emptyset$ with alphabet $\Sigma = \{0, 1, 2\}$



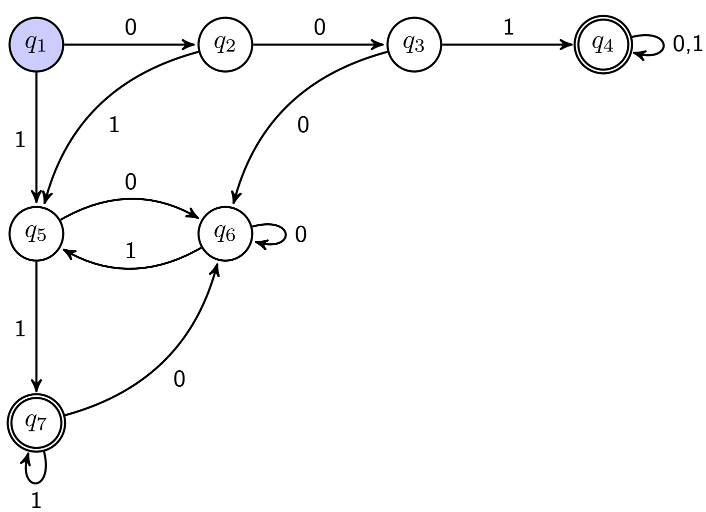
(iii) $L_3 = \{\varepsilon\}$ with alphabet $\Sigma = \{0, 1, 2\}$



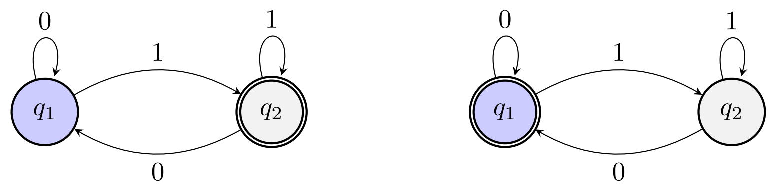
(iv) $L_4 = \{w \in \{\text{go}, \text{stop}\}^* \mid w = \varepsilon \text{ or ends with stop}\}$



(v) $L_5 = \{w \in \{0, 1\}^* \mid w \text{ has 001 as a prefix or 11 as a suffix}\}$



Exercise 5.
Consider the automata M_1 and M_2 drawn below.

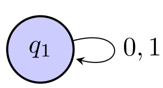


M_1 M_2

1. Construct an automaton that recognizes the language $L(M_1) \cap L(M_2)$
2. Prove that the regular languages are closed under intersection.
3. Construct an automaton for each of the following languages:
   (i) $\{0,1\}^* \setminus L(M_1)$
   (ii) $\{0,1\}^* \setminus L(M_2)$
   (iii) $\{0,1\}^* \setminus (L(M_1) \cap L(M_2))$
4. Prove that the set of regular languages is closed under complement.

Hint:
2. Similar construction with the one for union, only that $F = F_1 \times F_2$
4. Change the final states in not-final and reverse

Solution 5.
1. Construct an automaton that recognizes the language $L(M_1) \cap L(M_2)$



   Notice that $L(M_1) \cap L(M_2) = \emptyset$. Indeed $L(M_1) = \{w1 \mid w \in \{0,1\}^*\}$ and $L(M_2) = \{\varepsilon\} \cup \{w0 \mid w \in \{0,1\}^*\}$

2. Prove that the regular languages are closed under intersection.

   Consider two arbitrary automata $M_1 = (Q_1, \Sigma, q_0^1, \delta_1, F_1)$ and $M_2 = (Q_2, \Sigma, q_0^2, \delta_2, F_2)$. Here we assume without loss of generality that M_1 and M_2 share the same alphabet Σ. Let $M = (Q, \Sigma, q_0, \delta, F)$ be an automaton such that $Q = Q_1 \times Q_2, q_0 = (q_0^1, q_0^2), F = F_1 \times F_2$ and $\delta : Q \times \Sigma \to Q$ defined as $\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$

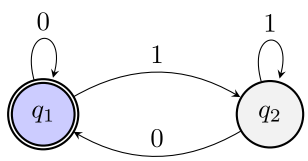
   Now we prove that $L(M) = L(M_1) \cap L(M_2)$.

($\subseteq$) Let $w \in L(M)$ and $(q_1, r_1)(q_2, r_2)...(q_n, r_n)$ be an accepting sequence of states for w in M. By construction $(q_n, r_n) \in F = F_1 \times F_2$, therefore $q_n \in F_1$ and $r_n \in F_2$. Moreover $q_1 = q_0^1$ and $r_1 = q_0^2$. By definition of δ, for all $j \in \{1, ..., n-1\}$, it holds that $\delta_1(q_j, a_{j+1}) = q_{j+1}$ and $\delta_2(r_j, a_{j+1}) = r_{j+1}$. Noting that the sequences $q_1, ..., q_n$ and $r_1, ..., r_n$ are accepting in M_1 and M_2, respectively, we obtain that $w \in L(M_1) \cap L(M_2)$.

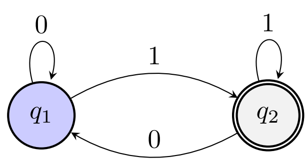
($\supseteq$) Consider $w \in L(M_1) \cap L(M_2)$. Then, $w \in L(M_1)$ and $w \in L(M_2)$ and there exist accepting sequences $q_1, ..., q_n$ and $r_1, ..., r_n$. By construction of M we have that the sequence $(q_1, r_1)(q_2, r_2)...(q_n, r_n)$ is accepting in M. Hence, $w \in L(M)$.

3. Construct an automaton for each of the following languages:

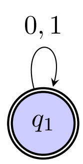
(i) $\{0, 1\}^* \setminus L(M_1)$



(ii) $\{0, 1\}^* \setminus L(M_2)$



(iii) $\{0, 1\}^* \setminus (L(M_1) \cap L(M_2)) = \{0, 1\}^*$
Note that $(L(M_1) \cap L(M_2)) = \emptyset$



4. Prove that the set of regular languages is closed under complement.

To prove that the set of regular languages is closed under complement we have to prove that for each regular language $L \subseteq \Sigma^*$ the language $\Sigma^* \setminus L$ is regular. To this end we will show that the language $\Sigma^* \setminus L$ admits an automaton $\overline{M}$ that recognizes it.

By hypothesis L is regular, therefore there exists an automaton $M = (Q, \Sigma, \delta, q_0, F)$ such that $L(M) = L$.

Let $\overline{M} = (Q, \Sigma, \delta, q_0, Q \setminus F)$. We will show that $w \in L$ iff $w \notin L(\overline{M})$.

(iff := if and only if)

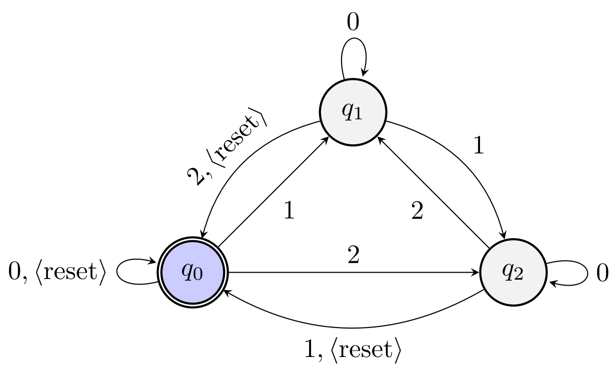
($\Rightarrow$) Assume $w \in L$ and $q_0, ..., q_n$(for $n = |w|$) be the accepting sequence of w in M. By construction $q_0, ..., q_n$ is also a sequence in $\overline{M}$ for the input word w. But since by hypothesis $q_n \in F$ we have that $q_n \notin Q \setminus F$. Hence $\overline{M}$ does not accept w. Thus $w \notin L(\overline{M})$

($\Leftarrow$) We proceed by contraposition: Let $w \notin L$, we have to show that $w \in L(\overline{M})$. If $w \notin L$, then $w \notin L(M)$. This means that M is in a state in $Q \setminus F$ after reading w. This, in turn, implies that $\overline{M}$ accepts w, i.e., $w \in L(\overline{M})$.

Note: $([w \notin L(\overline{M}) \implies w \in L] \equiv [w \notin L \implies w \in L(\overline{M})])$ Because $([A \implies B] \equiv [\neg B \implies \neg A])$. This last equivalence is contraposition.

Exercise 6.

Consider the following automaton over the alphabet $\Sigma = \{0, 1, 2, \langle\text{reset}\rangle\}$.



(i) Give examples of accepted and nonaccepted words (at least five for each).

(ii) Prove that the language $L(M)$ can be characterized as follows: Suppose that M keeps a running count of the sum of the numerical input symbols it reads and it a) subtracts 3 when the count exceeds 2 and b) resets the count to 0 if it reads $\langle\text{reset}\rangle$. Then, $L(M) = \{w \mid count(w) = 0(\text{mod } 3)\}$.

(iii) Generalize the automaton such that $L(M) = \{w \mid count(w) = 0(\text{mod } 4)\}$

Solution 6.

(i) Give examples of accepted and nonaccepted words (at least five for each).

Accepted: ε, 0, $\langle\text{reset}\rangle$, 2022, 12, 21, ...

Non-accepted: 1, 22, 11, 2$\langle\text{reset}\rangle$1, 221, ...

(ii) Prove that the language $L(M)$ can be characterized as follows: Suppose that M keeps a running count of the sum of the numerical input symbols it reads and it a) subtracts 3 when the count exceeds 2 and b) resets the count to 0 if it reads $\langle\text{reset}\rangle$. Then, $L(M) = \{w \mid count(w) = 0(\text{mod } 3)\}$.

Consider an input $w \in \{0, 1, 2, \langle\text{reset}\rangle\}^*$. We will prove that if the sequence of states of M for the input w ends in q_i, then $counts(w) = i \bmod 3$ for $i \in \{0, 1, 2\}$.

We proceed by induction on the length of w.

Base case $|w| = 0$

Then the sequence of M for w is q_0. It's immediate to check that $count(w) = count(\varepsilon) = 0$ therefore $count(w) = 0 \mod 3$.

Inductive step $|w| = n > 0$

Let $w = w'a$ for $w' \in \{0, 1, 2, \langle\text{reset}\rangle\}^*$ and $a \in \{0, 1, 2, \langle\text{reset}\rangle\}$.

Let $s_0, ..., s_{n-1}$ the sequence of states of M for the input w'. We consider 3 possible cases.

1) $[s_{n-1} = q_0]$ By inductive hypothesis we have that $count(w') = 0 \mod 3$. Now we proceed by cases on a
   (i) $[a \in \{0, \langle\text{reset}\rangle\}]$ Then the sequence of states on M for w is $s_0, ..., s_{n-1}, q_0$, and the thesis follows.
   (ii) $[a = 1]$ Then the sequence of states on M for w is $s_0, ..., s_{n-1}, q_1$, and the thesis follows.
   (iii) $[a = 2]$ Then the sequence of states on M for w is $s_0, ..., s_{n-1}, q_2$, and the thesis follows.

The cases for when $s_{n-1} = q_1$ and $s_{n-1} = q_2$ are similar to the previous one.

Since all accepting sequences in M end in q_0, we can conclude that $L(M) = \{w \mid count(w) = 0 (\text{mod } 3)\}$.

(iii) Generalize the automaton such that $L(M) = \{w \mid count(w) = 0 (\text{mod } 4)\}$