Syntax and Semantics
Exercise Session 10

Exercise 1.
Is there a big-step transition for each of the following cases? If so, prove it.

(i) $\langle \texttt{x} := \texttt{2};\ (\texttt{skip};\ \texttt{y} := \texttt{3}),\ [x \mapsto 3, y \mapsto 5]\rangle \to\ ?$

(ii) $\langle \texttt{if x} < \texttt{y then z} := \texttt{5 else S},\ [x \mapsto 2, y \mapsto 3, z \mapsto 5]\rangle \to\ ?$
where $\texttt{S} = (\texttt{if x} + \texttt{1} < \texttt{y then z} := \texttt{2 else z} := \texttt{3})$

(iii) $\langle \texttt{skip};\ \texttt{x} := \texttt{3};\ \texttt{while}\,\texttt{x} \leq \texttt{5}\,\texttt{do}\,(\texttt{x} := \texttt{x} + \texttt{1};\ \texttt{y} := \texttt{2}),\ \sigma\rangle \to\ ?$ where $\sigma = [x \mapsto 2, y \mapsto 0]$.

Solution 1.
(i) Let $\sigma = [x \mapsto 3, y \mapsto 5]$.

$$\text{[COMP]}\ \dfrac{\langle \texttt{x} := \texttt{2},\ \sigma\rangle \to \sigma_1 \qquad \text{[COMP]}\ \dfrac{\langle \texttt{skip},\ \sigma_1\rangle \to \sigma_1 \qquad \langle \texttt{y} := \texttt{3},\ \sigma_1\rangle \to \sigma_2}{\langle \texttt{skip};\ \texttt{y} := \texttt{3},\ \sigma_1\rangle \to \sigma_2}}{\langle \texttt{x} := \texttt{2};\ (\texttt{skip};\ \texttt{y} := \texttt{3}),\ \sigma\rangle \to \sigma_2}$$

where $\sigma_1 = [x \mapsto 2, y \mapsto 5]$ and $\sigma_2 = [x \mapsto 2, y \mapsto 3]$. This proves that

$$\langle \texttt{x} := \texttt{2};\ (\texttt{skip};\ \texttt{y} := \texttt{3}),\ [x \mapsto 3, y \mapsto 5]\rangle \to [x \mapsto 2, y \mapsto 3]$$

(ii) Let $\sigma = [x \mapsto 2, y \mapsto 3, z \mapsto 5]$. We have the following derivation

$$\text{[IF-}\top\text{]}\ \dfrac{\langle \texttt{z} := \texttt{5},\ \sigma\rangle \to [x \mapsto 2, y \mapsto 3, z \mapsto 5]}{\langle \texttt{if x} < \texttt{y then z} := \texttt{5 else S},\ \sigma\rangle \to [x \mapsto 2, y \mapsto 3, z \mapsto 5]}$$

since the side condition for the rule [IF-$\top$] holds, as shown by the following derivation

$$\text{[LT-}\top\text{]}\ \dfrac{\sigma \vdash \texttt{x} \to_A 2 \qquad \sigma \vdash \texttt{y} \to_A 3}{\sigma \vdash \texttt{x} < \texttt{y} \to_B \top}$$

Notice that it was not necessary to evaluate $\texttt{S}$.

(iii) Let $\sigma = [x \mapsto 2, y \mapsto 0]$ and $\texttt{W} = \texttt{while}\,\texttt{x} \leq \texttt{5}\,\texttt{do}\,(\texttt{x} := \texttt{x} + \texttt{1};\ \texttt{y} := \texttt{2})$

$$\dfrac{\langle \texttt{skip},\ \sigma\rangle \to \sigma \qquad \dfrac{\langle \texttt{x} := \texttt{3},\ \sigma\rangle \to \sigma_1 \qquad \dfrac{\dfrac{\langle \texttt{x} := \texttt{x}+\texttt{1},\ \sigma_1\rangle \to \sigma_2 \quad \langle \texttt{y} := \texttt{2},\ \sigma_2\rangle \to \sigma_3}{\langle \texttt{x} := \texttt{x}+\texttt{1};\ \texttt{y} := \texttt{2},\ \sigma_1\rangle \to \sigma_3} \qquad \dfrac{\vdots}{\langle \texttt{W},\ \sigma_3\rangle \to \sigma_5}}{\langle \texttt{W},\ \sigma_1\rangle \to \sigma_5}}{\langle \texttt{x} := \texttt{3};\ \texttt{W},\ \sigma\rangle \to \sigma_5}}{\langle \texttt{skip};\ \texttt{x} := \texttt{3};\ \texttt{W},\ \sigma\rangle \to \sigma_5}$$

where the derivation tree for $\langle \mathtt{W},\ \sigma_3\rangle \to \sigma_5$ is given below

$$
\dfrac{\dfrac{\langle \mathtt{x := x + 1},\ \sigma_3\rangle \to \sigma_4 \quad \langle \mathtt{y := 2},\ \sigma_4\rangle \to \sigma_4}{\langle \mathtt{x := x + 1;\ y := 2},\ \sigma_3\rangle \to \sigma_4} \qquad \dfrac{\dfrac{\dfrac{\langle \mathtt{x := x + 1},\ \sigma_4\rangle \to \sigma_5 \quad \langle \mathtt{y := 2},\ \sigma_5\rangle \to \sigma_5}{\langle \mathtt{x := x + 1;\ y := 2},\ \sigma_4\rangle \to \sigma_5} \qquad \langle \mathtt{W},\ \sigma_5\rangle \to \sigma_5}{\langle \mathtt{W},\ \sigma_4\rangle \to \sigma_5}}{\langle \mathtt{W},\ \sigma_3\rangle \to \sigma_5}
$$

and the states are

$$
\begin{array}{lll}
\sigma_1 = [x \mapsto 3, y \mapsto 0] & \sigma_2 = [x \mapsto 4, y \mapsto 0] & \sigma_3 = [x \mapsto 4, y \mapsto 2] \\
\sigma_4 = [x \mapsto 5, y \mapsto 2] & \sigma_5 = [x \mapsto 6, y \mapsto 2] &
\end{array}
$$

The above derivation proves that

$$\langle \mathtt{skip;\ x := 3;\ while\, x} \leq \mathtt{5\, do\, (x := x + 1;\ y := 2)},\ [x \mapsto 2, y \mapsto 0]\rangle \to [x \mapsto 6, y \mapsto 2]$$

Exercise 2.
Let $\mathtt{S} = \mathtt{while}\ \neg(\mathtt{2} < (\mathtt{1} + \mathtt{1}))\ \mathtt{do}\ (\mathtt{if\ x < x\ then\ x := 2\ else\ skip})$.

(i) Prove that S loops forever in the small-step semantics.

(ii) Prove that there exists $k \geq 0$ s.t. in the SS-semantics for any state $s \in \mathbb{S}tates$, $\langle \mathtt{S},\ s\rangle \Rightarrow^k \langle \mathtt{S},\ s\rangle$.

Solution 2.
It suffices to show (ii), indeed it proves that the small-step semantics for S reduces $\langle \mathtt{S},\ s\rangle$ to itself after some number of steps, meaning that the same steps will be taken over and over again without leading to a normal form.

The following derivation steps show that in 4 transitions $\langle \mathtt{S},\ s\rangle$ reduces to itself, i.e., $\langle \mathtt{S},\ s\rangle \Rightarrow^4 \langle \mathtt{S},\ s\rangle$

$$
\begin{aligned}
\langle \mathtt{S},\ s\rangle &\Rightarrow \langle \mathtt{if}\ \neg(\mathtt{2} < (\mathtt{1} + \mathtt{1}))\ \mathtt{then}\ (\mathtt{if\ x < x\ then\ x := 2\ else\ skip;\ S})\ \mathtt{else\ skip},\ s\rangle \\
&\Rightarrow \langle \mathtt{if\ x < x\ then\ x := 2\ else\ skip;\ S},\ s\rangle \\
&\Rightarrow \langle \mathtt{skip;\ S},\ s\rangle \\
&\Rightarrow \langle \mathtt{S},\ s\rangle
\end{aligned}
$$

Exercise 3.
Find all the transitions (if there are any) in the transition sequence starting from $\langle \mathtt{S},\ s\rangle$ in the small-step semantics, for each of the following cases:

(i) $\mathtt{S} = \mathtt{if}\ (\neg(\mathtt{x} > \mathtt{3}) \vee \mathtt{y} > \mathtt{2})\ \mathtt{then}\ (\mathtt{x := x + 3;\ y := 2})\ \mathtt{else\ skip}$ and $s = [x \mapsto 0, y \mapsto 0]$.

(ii) $\mathtt{S} = \mathtt{while}\ \neg(\mathtt{x} < \mathtt{y})\ \mathtt{do}\ (\mathtt{x := x - 1;\ y := y + 1})$ and $s = [x \mapsto 3, y \mapsto 0]$.

Solution 3.

(i) $\mathtt{S} = \mathtt{if}\ (\neg(\mathtt{x} > \mathtt{3}) \vee \mathtt{y} > \mathtt{2})\ \mathtt{then}\ (\mathtt{x} := \mathtt{x} + \mathtt{3};\ \mathtt{y} := \mathtt{2})\ \mathtt{else\ skip}$ and $s = [x \mapsto 0]$. Then we have

$$\begin{aligned}
&\langle \mathtt{S},\ s\rangle = \langle \mathtt{if}\ (\neg(\mathtt{x} > \mathtt{3}) \vee \mathtt{y} > \mathtt{2})\ \mathtt{then}\ (\mathtt{x} := \mathtt{x} + \mathtt{3};\ \mathtt{y} := \mathtt{2})\ \mathtt{else\ skip},\ [x \mapsto 0]\rangle \\
&\Rightarrow \langle \mathtt{x} := \mathtt{x} + \mathtt{3};\ \mathtt{y} := \mathtt{2},\ [x \mapsto 0]\rangle && (\text{IF-}\top\text{, since } [x \mapsto 0] \vdash (\neg(\mathtt{x} > \mathtt{3}) \vee \mathtt{y} > \mathtt{2}) \to_B \top) \\
&\Rightarrow \langle \mathtt{y} := \mathtt{2},\ [x \mapsto 3]\rangle && (\text{COMP2 and ASSIGN}) \\
&\Rightarrow [x \mapsto 3, y \mapsto 2] && (\text{ASSIGN})
\end{aligned}$$

(ii) Let $\mathtt{S} = \mathtt{while}\ \neg(\mathtt{x} < \mathtt{y})\ \mathtt{do}\ (\mathtt{x} := \mathtt{x} - \mathtt{1};\ \mathtt{y} := \mathtt{y} + \mathtt{1})$ and and $s = [x \mapsto 3, y \mapsto 0]$. Then we have

$$\begin{aligned}
&\langle \mathtt{S},\ s\rangle = \langle \mathtt{while}\ \neg(\mathtt{x} < \mathtt{y})\ \mathtt{do}\ (\mathtt{x} := \mathtt{x} - \mathtt{1};\ \mathtt{y} := \mathtt{y} + \mathtt{1}),\ [x \mapsto 3, y \mapsto 0]\rangle \\
&\Rightarrow \langle \mathtt{if}\ \neg(\mathtt{x} < \mathtt{y})\ \mathtt{then}\ \mathtt{x} := \mathtt{x} - \mathtt{1};\ \mathtt{y} := \mathtt{y} + \mathtt{1};\ \mathtt{S}\ \mathtt{else\ skip},\ [x \mapsto 3, y \mapsto 0]\rangle && (\text{WHILE}) \\
&\Rightarrow \langle \mathtt{x} := \mathtt{x} - \mathtt{1};\ \mathtt{y} := \mathtt{y} + \mathtt{1};\ \mathtt{S},\ [x \mapsto 3, y \mapsto 0]\rangle && (\text{IF-}\top) \\
&\Rightarrow \langle \mathtt{y} := \mathtt{y} + \mathtt{1};\ \mathtt{S},\ [x \mapsto 2, y \mapsto 0]\rangle && (\text{COMP2 and ASSIGN}) \\
&\Rightarrow \langle \mathtt{while}\ \neg(\mathtt{x} < \mathtt{y})\ \mathtt{do}\ (\mathtt{x} := \mathtt{x} - \mathtt{1};\ \mathtt{y} := \mathtt{y} + \mathtt{1}),\ [x \mapsto 2, y \mapsto 1]\rangle && (\text{COMP2 and ASSIGN}) \\
&\Rightarrow \langle \mathtt{if}\ \neg(\mathtt{x} < \mathtt{y})\ \mathtt{then}\ \mathtt{x} := \mathtt{x} - \mathtt{1};\ \mathtt{y} := \mathtt{y} + \mathtt{1};\ \mathtt{S}\ \mathtt{else\ skip},\ [x \mapsto 2, y \mapsto 1]\rangle && (\text{WHILE}) \\
&\Rightarrow \langle \mathtt{x} := \mathtt{x} - \mathtt{1};\ \mathtt{y} := \mathtt{y} + \mathtt{1};\ \mathtt{S},\ [x \mapsto 2, y \mapsto 1]\rangle && (\text{IF-}\top) \\
&\Rightarrow \langle \mathtt{y} := \mathtt{y} + \mathtt{1};\ \mathtt{S},\ [x \mapsto 1, y \mapsto 1]\rangle && (\text{COMP2 and ASSIGN}) \\
&\Rightarrow \langle \mathtt{while}\ \neg(\mathtt{x} < \mathtt{y})\ \mathtt{do}\ (\mathtt{x} := \mathtt{x} - \mathtt{1};\ \mathtt{y} := \mathtt{y} + \mathtt{1}),\ [x \mapsto 1, y \mapsto 2]\rangle && (\text{COMP2 and ASSIGN}) \\
&\Rightarrow \langle \mathtt{if}\ \neg(\mathtt{x} < \mathtt{y})\ \mathtt{then}\ \mathtt{x} := \mathtt{x} - \mathtt{1};\ \mathtt{y} := \mathtt{y} + \mathtt{1};\ \mathtt{S}\ \mathtt{else\ skip},\ [x \mapsto 1, y \mapsto 2]\rangle && (\text{WHILE}) \\
&\Rightarrow \langle \mathtt{skip},\ [x \mapsto 1, y \mapsto 2]\rangle && (\text{IF-}\bot) \\
&\Rightarrow [x \mapsto 1, y \mapsto 2] && (\text{SKIP})
\end{aligned}$$

Exercise 4.

Prove that regular expressions are closed under intersection.

Solution 4.

Any regular expression can be encoded into a DFA and vice versa. DFAs are closed under intersection. Therefore regular expressions are closed under intersection as well.