Syntax and Semantics:
Exercise Session 13

Exercise 1.
Consider the following grammar for arithmetic expressions $\mathbb{A}$:

$$n ::= 0 \mid s(n) \qquad\qquad a ::= n \mid a + a \mid a * a$$

Intuitively, $s(n)$ is the successor of n, e.g., $s(0)$ and $s(s(0))$ encodes 1 and 2, respectively.

1) Let the set of states be $\mathbb{A}$, the set of final states be $\{0, s(0), s(s(0)), s(s(s(0))), \ldots\}$ and consider the following *small step* transition rules for addition:

$$[r_1]\ s(n_1) + n_2 \Rightarrow n_1 + s(n_2) \qquad\qquad [r_2]\ 0 + a \Rightarrow a$$

$$[r_3]\ a_1 + a_2 \Rightarrow a_2 + a_1 \qquad\qquad [r_4]\ \frac{a_1 \Rightarrow a_1'}{a_1 + a_2 \Rightarrow a_1' + a_2}$$

Prove $s(s(s(0))) + s(s(0)) \Rightarrow^* s(s(s(s(s(0)))))$ using $[r_1], \ldots, [r_4]$. All intermediate computations have to be proven. (Intuitively, this corresponds to deducing 3+2 = 5.)

2) Complete the *small step* semantics from 1) by providing the rules for multiplication.

   Note: The complete semantics of $\mathbb{A}$ is intended to be the same as for arithmetic expressions in the language **Bims** from Hans Hüttel's book but with the modified representation of numbers as explained above.

3) Using the transition rules from 1) and 2), prove $s(s(0)) * s(s(0)) \Rightarrow^* s(s(s(s(0))))$. All intermediate computations have to be proven.

1) $s(s(s(0))) + s(s(0)) \overset{r_3}{\Longrightarrow} s(s(0)) + s(s(s(0)))$

$\overset{r_1}{\Longrightarrow} s(0) + s(s(s(s(0))))$

$\overset{r_1}{\Longrightarrow} 0 + s(s(s(s(s(0)))))$

$\overset{r_2}{\Longrightarrow} s(s(s(s(s(0)))))$

2)

$$[m_1] \quad s(n_1) * n_2 \Longrightarrow n_1 * n_2 + n_2$$

$$[m_2] \quad s(0) * a \Longrightarrow a$$

$$[m_3] \quad a_1 * a_2 \Longrightarrow a_2 * a_1$$

$$[m_4] \quad \frac{a_1 \Longrightarrow a_1'}{a_1 * a_2 \Longrightarrow a_1' * a_2}$$

3)

$$s(s(0)) * s(s(0)) \overset{m_1}{\Longrightarrow} s(0) * s(s(0)) + s(s(0))$$

$$m_4 \; \frac{s(0) * s(s(0)) \overset{m_2}{\Longrightarrow} s(s(0))}{s(0) * s(s(0)) + s(s(0)) \Longrightarrow s(s(0)) + s(s(0))}$$

$$\Downarrow m_2 + m_4$$

$$s(s(0)) + s(s(0))$$

$$r_1 \Downarrow$$

$$s(0) + s(s(s(0)))$$

$$r_1 \Downarrow$$

$$0 + s(s(s(s(0))))$$

$$r_2 \Downarrow$$

$$s(s(s(s(0))))$$

Exercise 2.

Consider the arithmetic and boolean expressions from Chapter 4 in Hans Hüttel's book that have been extended by the existential ($\exists$) and the universal ($\forall$) quantifier:

$$a ::= n \mid x \mid a_1 + a_2 \mid a_1 * a_2 \mid a_1 - a_2 \mid (a_1)$$
$$b ::= a_1 = a_2 \mid a_1 < a_2 \mid \neg b_1 \mid b_1 \wedge b_2 \mid b_1 \rightarrow b_2 \mid (b_1) \mid \exists x.(b_1) \mid \forall x.(b_1)$$

1. Consider the following *big step* transition rules for the existential quantifier:

$$[\exists_1]\ \frac{s[x \mapsto v] \vdash b_1 \rightarrow_b \mathbf{tt}}{s \vdash \exists x.(b_1) \rightarrow_b \mathbf{tt}}\ \text{FOR SOME } v \in \mathbb{Z}$$

$$[\exists_2]\ \frac{s[x \mapsto v] \vdash b_1 \rightarrow_b \mathbf{ff}}{s \vdash \exists x.(b_1) \rightarrow_b \mathbf{ff}}\ \text{FOR ALL } v \in \mathbb{Z}$$

   By combining the above rules with those from the book of Hans Hüttel, evaluate the expression $\exists x.(x * x = \underline{4})$ in the state $s = [x \mapsto 0]$ to its truth value.

2. Provide the *big step* transition rules for the universal quantifier.

2) $$[\forall_1]\ \frac{s[x \mapsto v] \vdash b_1 \rightarrow_b tt}{s \vdash \forall x.(b_1) \rightarrow_b tt}\ \text{for all } v \in \mathbb{Z}$$

$$[\forall_2]\ \frac{s[x \mapsto v] \vdash b_1 \rightarrow_b ff}{s \vdash \forall x.(b_1) \rightarrow_b ff}\ \text{for some } v \in \mathbb{Z}$$

1)

$$\exists_1\ \dfrac{Eq_{BSS}\ \dfrac{Mult_{BSS}\ \dfrac{Var_{BSS}\ \dfrac{}{s[x \mapsto 2] \vdash x \rightarrow_a 2}}{s[x \mapsto 2] \vdash x * x \rightarrow_a 4} \qquad Num_{BSS}\ \dfrac{}{s[x \mapsto 2] \vdash \underline{4} \rightarrow_a 4}}{s[x \mapsto 0][x \mapsto 2] \vdash x * x = \underline{4} \rightarrow_b tt}}{s[x \mapsto 0] \vdash \exists x.(x * x = \underline{4}) \rightarrow tt}$$

(Note: $s[x \mapsto 0][x \mapsto 2] = s[x \mapsto 2]$)

Exercise ~~4~~ 3.
Consider the language $L'' = \{w \in \Sigma^* \mid w = a^n \text{ with } n \geq 1\}$ over the singleton alphabet $\Sigma = \{a\}$. With this, consider the big step transitions $\rightarrow \subseteq L'' \times \mathbb{N}$ given by

$$[r_1]\ \frac{}{a \rightarrow 1} \qquad\qquad [r_2]\ \frac{w \rightarrow k}{aw \rightarrow k'}\ k' = k \cdot |aw|,$$

where $|aw|$ denotes the length of the word aw.

1. Using the big step semantics, prove $aa \rightarrow 2$ and $aaa \rightarrow 6$.

2. Using the big step semantics and induction, prove that $a^n \rightarrow v$ with $v = 1 \cdot 2 \cdot \ldots \cdot n$ for all $n \geq 1$. Note that a^n is the word consisting of n symbols a.

1)
$$r_2\ \cfrac{r_2\ \cfrac{r_1\ \cfrac{}{a \rightarrow 1}}{aa \rightarrow 2}\ 2 = 1 \cdot |aa|}{aaa \rightarrow 6}\ 6 = 2 \cdot |aaa|$$

2) $\underline{n = 1:}$ $a^1 \rightarrow 1$ because $r_1\ \frac{}{a \rightarrow 1}$

$\underline{n \rightarrow n+1:}$ By I.H., we have $a^n \rightarrow n!$.
With this, we obtain

$$r_1\ \frac{a^n \rightarrow n!}{aa^n \rightarrow k'}\ k' = n! \cdot |aa^n| = n!\,(n+1) = (n+1)!\,,$$

thus implying $a^{n+1} \rightarrow (n+1)!$

Exercise 4.

Consider the language $L'' = \{\underline{0}, \underline{1}\}^+$ of binary strings. With this, we consider the big step semantics $\rightarrow \subseteq L'' \times \mathbb{N}$ given by

$$[r_0]\ \frac{}{\underline{0} \rightarrow 0}, \qquad [r_1]\ \frac{}{\underline{1} \rightarrow 1}, \qquad [r_2]\ \frac{w \rightarrow k'}{\underline{0}w \rightarrow k}\ k = k', \qquad [r_3]\ \frac{w \rightarrow k'}{\underline{1}w \rightarrow k}\ k = 2^{|w|} + k'$$

As usual, $|w|$ denotes the length of w, while σw stands for a concatenation of $\sigma \in \{\underline{0}, \underline{1}\}$ and $w \in L''$. Essentially, $\rightarrow$ assigns each binary string its decimal value (i.e., the value represented by the string with 2 as the base of the numeral system).

1) Using the big step semantics, prove $\underline{10} \rightarrow 2$ and $\underline{110} \rightarrow 6$.

2) Consider the language of ternary strings $L''' = \{\underline{0}, \underline{1}, \underline{2}\}^+$. Provide the big step semantics $\rightarrow_3 \subseteq L''' \times \mathbb{N}$ that maps each ternary string to its decimal value (i.e., the value represented by the string with 3 as the base of the numeral system).

   [Hint: You may wish to double check your semantics by showing $\underline{12} \rightarrow_3 5$ and $\underline{212} \rightarrow_3 23$. This is only a suggestion and does not contribute to the score.]

1)

$$r_3\ \dfrac{r_0\ \dfrac{}{\underline{0} \rightarrow 0}}{\underline{10} \rightarrow 2}\ 2 = 2^{|\underline{0}|} + 0$$

$$r_3\ \dfrac{\dfrac{\dots}{\underline{10} \rightarrow 2}}{\underline{110} \rightarrow 6}\ 6 = 2^{|\underline{10}|} + 2$$

2)

$$t_0\ \frac{}{\underline{0} \rightarrow 0} \qquad t_1\ \frac{}{\underline{1} \rightarrow 1} \qquad t_2\ \frac{}{\underline{2} \rightarrow 2}$$

$$t_3\ \frac{w \rightarrow k'}{\underline{0w} \rightarrow k}\ k = k' \qquad t_4\ \frac{w \rightarrow k'}{\underline{1w} \rightarrow k}\ k = 3^{|w|} + k'$$

$$t_5\ \frac{w \rightarrow k'}{\underline{2w} \rightarrow k}\ k = 2 \cdot 3^{|w|} + k'$$