Syntax and Semantics:
Exercise Session 9

Exercise 1.

Evaluate the following expressions and describe their derivation trees –use the big-step semantics of $\mathbb{A}$exp.

(i) $(\bar{3} + \bar{12}) * (\bar{4} * (\bar{5} * \bar{8}))$

(ii) $(\bar{3} + (\bar{12} * \bar{4})) * (\bar{5} * \bar{8})$

(iii) $(\bar{3} + (\bar{12})) * ((\bar{4}) * (\bar{5} * \bar{8}))$

Solution 1.

(i) Here is a derivation tree for $(\bar{3} + \bar{12}) * (\bar{4} * (\bar{5} * \bar{8}))$

$$
{\scriptstyle[\text{MULT}_{BS}]}\dfrac{
{\scriptstyle[\text{PARENT}_{BS}]}\dfrac{
{\scriptstyle[\text{PLUS}_{BS}]}\dfrac{\bar{3} \to 3 \qquad \bar{12} \to 12}{\bar{3} + \bar{12} \to 15}
}{(\bar{3} + \bar{12}) \to 15}
\qquad
{\scriptstyle[\text{PARENT}_{BS}]}\dfrac{
{\scriptstyle[\text{MULT}_{BS}]}\dfrac{\bar{4} \to 4 \qquad
{\scriptstyle[\text{PARENT}_{BS}]}\dfrac{
{\scriptstyle[\text{MULT}_{BS}]}\dfrac{\bar{5} \to 5 \qquad \bar{8} \to 8}{\bar{5} * \bar{8} \to 40}
}{(\bar{5} * \bar{8}) \to 40}
}{\bar{4} * (\bar{5} * \bar{8}) \to 160}
}{(\bar{4} * (\bar{5} * \bar{8})) \to 160}
}{(\bar{3} + \bar{12}) * (\bar{4} * (\bar{5} * \bar{8})) \to 2400}
$$

(ii) Here is a derivation tree for $(\bar{3} + (\bar{12} * \bar{4})) * (\bar{5} * \bar{8})$

$$
{\scriptstyle[\text{MULT}_{BS}]}\dfrac{
{\scriptstyle[\text{PARENT}_{BS}]}\dfrac{
{\scriptstyle[\text{PLUS}_{BS}]}\dfrac{\bar{3} \to 3 \qquad
{\scriptstyle[\text{PARENT}_{BS}]}\dfrac{
{\scriptstyle[\text{MULT}_{BS}]}\dfrac{\bar{12} \to 12 \qquad \bar{4} \to 4}{\bar{12} * \bar{4} \to 48}
}{(\bar{12} * \bar{4}) \to 48}
}{\bar{3} + (\bar{12} * \bar{4}) \to 51}
}{(\bar{3} + (\bar{12} * \bar{4})) \to 51}
\qquad
\dfrac{\vdots}{(\bar{5} * \bar{8}) \to 40}
}{(\bar{3} + (\bar{12} * \bar{4})) * (\bar{5} * \bar{8}) \to 2040}
$$

where proof tree for the subexpression $(\bar{5} * \bar{8})$ is the same as in (i).

(iii) Here is a derivation tree for $(\bar{3} + (\bar{12})) * ((\bar{4}) * (\bar{5} * \bar{8}))$

$$
[\text{MULT}_{BS}]\dfrac{[\text{PARENT}_{BS}]\dfrac{[\text{PLUS}_{BS}]\dfrac{\bar{3} \to 3 \qquad [\text{PARENT}_{BS}]\dfrac{\bar{12} \to 12}{(\bar{12}) \to 12}}{\bar{3} + (\bar{12}) \to 15}}{(\bar{3} + (\bar{12})) \to 15} \qquad [\text{PARENT}_{BS}]\dfrac{[\text{MULT}_{BS}]\dfrac{[\text{PARENT}_{BS}]\dfrac{\bar{4} \to 4}{(\bar{4}) \to 4} \qquad \dfrac{\vdots}{(\bar{5} * \bar{8}) \to 40}}{(\bar{4}) * (\bar{5} * \bar{8}) \to 160}}{((\bar{4}) * (\bar{5} * \bar{8})) \to 160}}{(\bar{3} + (\bar{12})) * ((\bar{4}) * (\bar{5} * \bar{8})) \to 2400}
$$

Again, the proof tree for the subexpression $(\bar{5} * \bar{8})$ is the same as in (i).

Exercise 2.

Suggest a new small-step semantics for $\mathbb{A}$exp, which is deterministic. (Hint: Use rules to ensure that the evaluation is done from left to right.)

Solution 2.

Consider the small-step semantics for $\mathbb{A}$exp seen in class. It can be turned to a deterministic semantics by replacing the rules [PLUS-R$_{ss}$], [MULT-R$_{ss}$] and, [SUB-R$_{ss}$] with the following.

$$
[\text{PLUS-R}_{ss}]\frac{a_2 \Rightarrow a_2'}{v + a_2 \Rightarrow v + a_2'} \qquad [\text{MULT-R}_{ss}]\frac{a_2 \Rightarrow a_2'}{v * a_2 \Rightarrow v * a_2'}
$$

$$
[\text{SUB-R}_{ss}]\frac{a_2 \Rightarrow a_2'}{v - a_2 \Rightarrow v - a_2'}
$$

Intuitively, these rules can be applied only when the left hand side of the expression is fully evaluated (i.e., it's a normal form). This is known as the leftmost selection strategy.

Exercise 3.

Give a big-step and a small-step semantics for $\mathbb{B}$exp for the case

$$
b ::= a_1 = a_2 \mid a_1 < a_2 \mid \neg b_1 \mid b_1 \wedge b_2
$$

assuming that we already have a big-step and a small-step semantics for $\mathbb{A}$exp defined by $\to_A$ and $\Rightarrow_A$ respectively.

Solution 3.

The big-step operational semantics for $\mathbb{B}$exp can be defined as

$$[\mathrm{EQ}_\top]\,\frac{a_1 \to_A v \quad a_2 \to_A v}{a_1 = a_2 \to_B \top} \qquad [\mathrm{EQ}_\bot]\,\frac{a_1 \to_A v_1 \quad a_2 \to_A v_2}{a_1 = a_2 \to_B \bot} \text{ if } v_1 \neq v_2$$

$$[\mathrm{LT}_\top]\,\frac{a_1 \to_A v_1 \quad a_2 \to_A v_2}{a_1 < a_2 \to_B \top} \text{ if } v_1 < v_2 \quad [\mathrm{LT}_\bot]\,\frac{a_1 \to_A v_1 \quad a_2 \to_A v_2}{a_1 < a_2 \to_B \bot} \text{ if } v_1 \geq v_2$$

$$[\mathrm{NEG}_\top]\,\frac{b \to_B \bot}{\neg b \to_B \top} \qquad [\mathrm{NEG}_\bot]\,\frac{b \to_B \top}{\neg b \to_B \bot}$$

$$[\mathrm{AND}_\top]\,\frac{b_1 \to_B \top \quad b_2 \to_B \top}{b_1 \wedge b_2 \to_B \top}$$

$$[\mathrm{AND1}_\bot]\,\frac{b_1 \to_B \bot}{b_1 \wedge b_2 \to_B \bot} \qquad [\mathrm{AND2}_\bot]\,\frac{b_2 \to_B \bot}{b_1 \wedge b_2 \to_B \bot}$$

Notice that the above semantics is nondeterministic. Indeed if both b_1 and b_2 evaluate to $\bot$ the expression $b_1 \wedge b_2$ can be applied both to $\mathrm{AND1}_\bot$ and $\mathrm{AND2}_\bot$. However, in both cases $b_1 \wedge b_2$ is evaluated to $\bot$. An alternative semantics for $\mathbb{B}$exp which is deterministic can be obtained from the above by replacing rule $\mathrm{AND2}_\bot$ with

$$[\mathrm{AND2}_\bot]\,\frac{b_1 \to_B \top \quad b_2 \to_B \bot}{b_1 \wedge b_2 \to_B \bot}\,.$$

The small-step semantics for $\mathbb{B}$exp can be defined as

$$[\text{EQ-L}]\ \frac{a_1 \Rightarrow_A a_1'}{a_1 = a_2 \Rightarrow_B a_1' = a_2} \qquad [\text{EQ-R}]\ \frac{a_2 \Rightarrow_A a_2'}{v_1 = a_2 \Rightarrow_B v_1 = a_2'}$$

$$[\text{EQ}_\top]\ \frac{}{v = v \Rightarrow_B \top} \qquad [\text{EQ}_\bot]\ \frac{}{v_1 = v_2 \Rightarrow_B \bot}\ \text{if } v_1 \neq v_2$$

$$[\text{LT-L}]\ \frac{a_1 \Rightarrow_A a_1'}{a_1 < a_2 \Rightarrow_B a_1' < a_2} \qquad [\text{LT-R}]\ \frac{a_2 \Rightarrow_A a_2'}{v_1 < a_2 \Rightarrow_B v_1 < a_2'}$$

$$[\text{LT}_\top]\ \frac{}{v_1 < v_2 \Rightarrow_B \top}\ \text{if } v_1 < v_2 \quad [\text{LT}_\bot]\ \frac{}{v_1 < v_2 \Rightarrow_B \bot}\ \text{if } v_1 \geq v_2$$

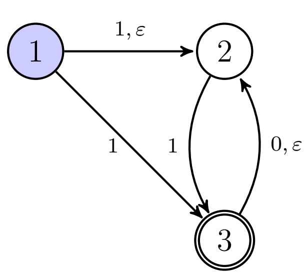
$$[\text{NEG}]\ \frac{b \Rightarrow_B b'}{\neg b \Rightarrow_B \neg b'} \qquad [\text{NEG}_\bot]\ \frac{}{\neg \top \Rightarrow_B \bot} \qquad [\text{NEG}_\top]\ \frac{}{\neg \bot \Rightarrow_B \top}$$

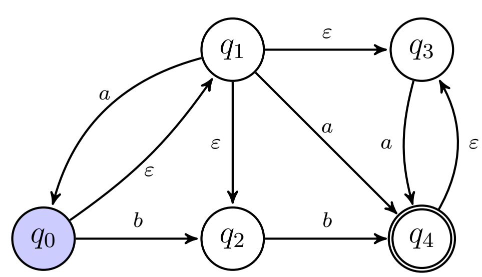
$$[\text{AND-L}]\ \frac{b_1 \Rightarrow_B b_1'}{b_1 \wedge b_2 \Rightarrow_B b_1' \wedge b_2} \qquad [\text{AND-R}]\ \frac{b_2 \Rightarrow_B b_2'}{\top \wedge b_2 \Rightarrow_B \top \wedge b_2'}$$

$$[\text{AND1}_\bot]\ \frac{}{\bot \wedge b_2 \Rightarrow_B \bot} \quad [\text{AND2}_\bot]\ \frac{}{\top \wedge \bot \Rightarrow_B \bot} \quad [\text{AND}_\top]\ \frac{}{\top \wedge \top \Rightarrow_B \top}$$

It is worth noticing that the above semantics is deterministic, since each binary operator evaluates its right hand side only after having completed the evaluation of the left hand side. This is known as leftmost selection strategy.

Exercise 4.
Convert the following non-deterministic automata to deterministic ones.





Solution 4.
We obtain the following DFAs.

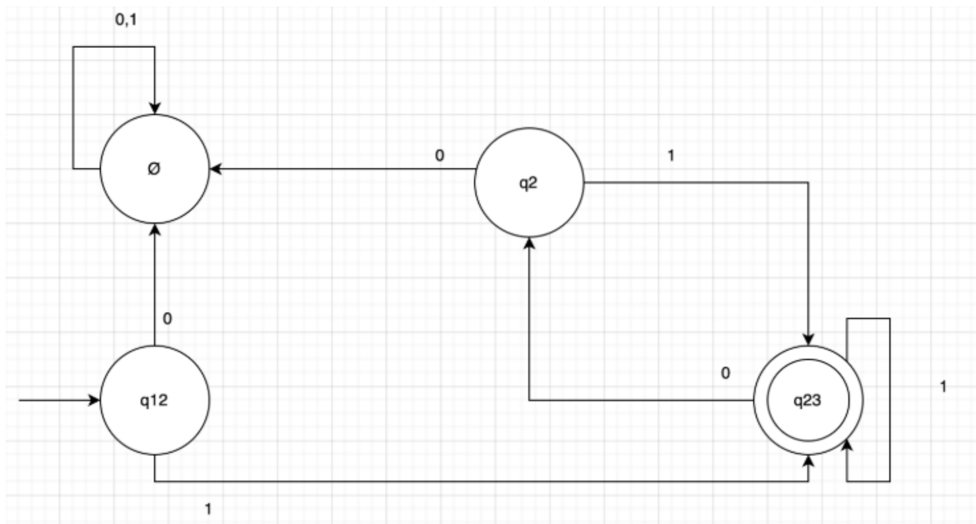


Figure 1: DFA conversion of the first NFA.

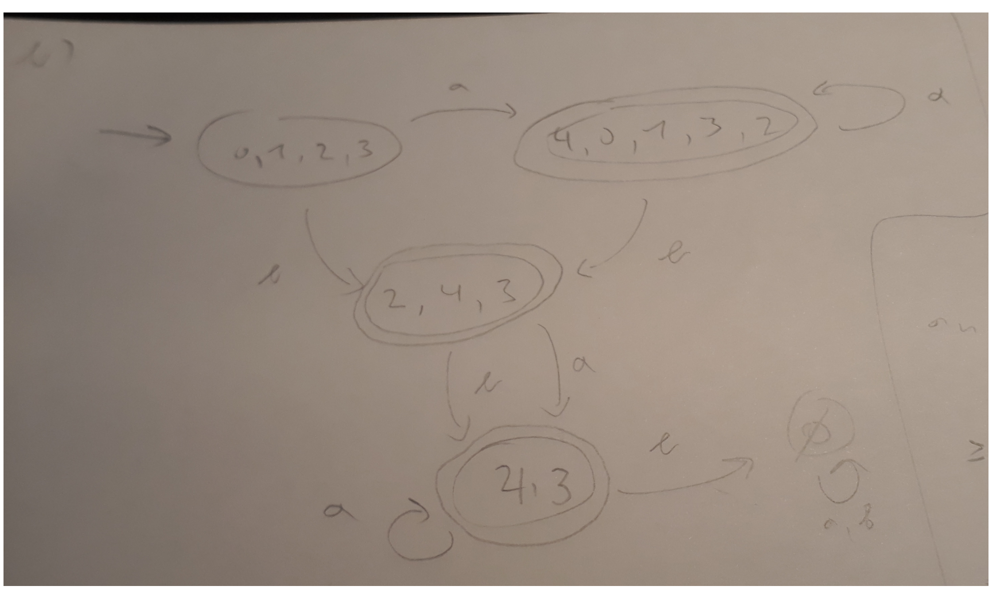
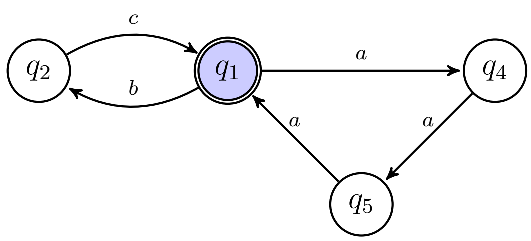


Figure 2: DFA conversion of the second NFA.

Exercise 5.
Consider the regular expression $(bc \cup aaa)^*$. Provide an NFA that recognizes the same language as the regular expression.

Solution 5.
An NFA that recognizes the regular language $(bc \cup aaa)^*$ is the following.



Exercise 6.
Consider the language $L = \{a^k b^{2k} \mid k \geq 0\}$.

(i) Prove that L is not regular.

(ii) Prove that L is context-free.

Solution 6.

(i) To prove that L is not regular, we make use of the Pumping Lemma for regular languages. Assume that L is regular and that $p \geq 0$ is its pumping length. Take the string $s = a^p b^{2p}$. Clearly, $|s| \geq p$ and $s \in L$. Consider an arbitrary split of s of the form xyz such that $|y| > 0$ and $|xy| \leq p$. By the given assumptions we have that xy only contains a's and y is not empty. Then $xy^0z \notin L$ since it has $2p$ b's (many b's as s) but a number of a's that is (strictly) less than p.

(ii) The following context-free grammar generates L

$$S \to \varepsilon \mid aSbb\,.$$