Syntax and Semantics: Exercise Session 13

Exercise 1.
Consider the following grammar for arithmetic expressions $\mathbb{A}$:

$$n ::= 0 \mid s(n) \qquad\qquad a ::= n \mid a + a \mid a * a$$

Intuitively, $s(n)$ is the successor of n, e.g., $s(0)$ and $s(s(0))$ encodes 1 and 2, respectively.

1) Let the set of states be $\mathbb{A}$, the set of final states be $\{0, s(0), s(s(0)), s(s(s(0))), \dots\}$ and consider the following *small step* transition rules for addition:

$$[r_1]\ s(n_1) + n_2 \Rightarrow n_1 + s(n_2) \qquad [r_2]\ 0 + a \Rightarrow a$$

$$[r_3]\ a_1 + a_2 \Rightarrow a_2 + a_1 \qquad [r_4]\ \frac{a_1 \Rightarrow a_1'}{a_1 + a_2 \Rightarrow a_1' + a_2}$$

   Prove $s(s(s(0))) + s(s(0)) \Rightarrow^* s(s(s(s(s(0)))))$ using $[r_1], \dots, [r_4]$. All intermediate computations have to be proven. (Intuitively, this corresponds to deducing 3+2 = 5.)

2) Complete the *small step* semantics from 1) by providing the rules for multiplication.

   Note: The complete semantics of $\mathbb{A}$ is intended to be the same as for arithmetic expressions in the language **Bims** from Hans Hüttel's book but with the modified representation of numbers as explained above.

3) Using the transition rules from 1) and 2), prove $s(s(0)) * s(s(0)) \Rightarrow^* s(s(s(s(0))))$. All intermediate computations have to be proven.

Exercise 2.
Consider the arithmetic and boolean expressions from Chapter 4 in Hans Hüttel's book that have been extended by the existential ($\exists$) and the universal ($\forall$) quantifier:

$$
\begin{aligned}
a &::= n \mid x \mid a_1 + a_2 \mid a_1 * a_2 \mid a_1 - a_2 \mid (a_1) \\
b &::= a_1 = a_2 \mid a_1 < a_2 \mid \neg b_1 \mid b_1 \wedge b_2 \mid b_1 \rightarrow b_2 \mid (b_1) \mid \exists x.(b_1) \mid \forall x.(b_1)
\end{aligned}
$$

1. Consider the following *big step* transition rules for the existential quantifier:

$$
\begin{array}{ll}
[\exists_1] & \dfrac{s[x \mapsto v] \vdash b_1 \rightarrow_b \mathbf{tt}}{s \vdash \exists x.(b_1) \rightarrow_b \mathbf{tt}} \text{ FOR SOME } v \in \mathbb{Z} \\
[\exists_2] & \dfrac{s[x \mapsto v] \vdash b_1 \rightarrow_b \mathbf{ff}}{s \vdash \exists x.(b_1) \rightarrow_b \mathbf{ff}} \text{ FOR ALL } v \in \mathbb{Z}
\end{array}
$$

   By combining the above rules with those from the book of Hans Hüttel, evaluate the expression $\exists x.(x * x = \underline{4})$ in the state $s = [x \mapsto 0]$ to its truth value.

2. Provide the *big step* transition rules for the universal quantifier.

Exercise 3.

Consider the language $L'' = \{w \in \Sigma^* \mid w = a^n \text{ with } n \geq 1\}$ over the singleton alphabet $\Sigma = \{a\}$. With this, consider the big step transitions $\rightarrow \subseteq L'' \times \mathbb{N}$ given by

$$[r_1]\ \frac{}{a \rightarrow 1} \qquad\qquad [r_2]\ \frac{w \rightarrow k}{aw \rightarrow k'}\ k' = k \cdot |aw|,$$

where $|aw|$ denotes the length of the word aw.

1. Using the big step semantics, prove $aa \rightarrow 2$ and $aaa \rightarrow 6$.
2. Using the big step semantics and induction, prove that $a^n \rightarrow v$ with $v = 1 \cdot 2 \cdot \ldots \cdot n$ for all $n \geq 1$. Note that a^n is the word consisting of n symbols a.

Exercise 4.
Consider the language $L'' = \{\underline{0}, \underline{1}\}^+$ of binary strings. With this, we consider the big step semantics $\to \subseteq L'' \times \mathbb{N}$ given by

$$[r_0]\ \frac{}{\underline{0} \to 0}, \qquad [r_1]\ \frac{}{\underline{1} \to 1}, \qquad [r_2]\ \frac{w \to k'}{\underline{0}w \to k}\ k = k', \qquad [r_3]\ \frac{w \to k'}{\underline{1}w \to k}\ k = 2^{|w|} + k'$$

As usual, $|w|$ denotes the length of w, while σw stands for a concatenation of $\sigma \in \{\underline{0}, \underline{1}\}$ and $w \in L''$. Essentially, $\to$ assigns each binary string its decimal value (i.e., the value represented by the string with 2 as the base of the numeral system).

1) Using the big step semantics, prove $\underline{10} \to 2$ and $\underline{110} \to 6$.

2) Consider the language of ternary strings $L''' = \{\underline{0}, \underline{1}, \underline{2}\}^+$. Provide the big step semantics $\to_3 \subseteq L''' \times \mathbb{N}$ that maps each ternary string to its decimal value (i.e., the value represented by the string with 3 as the base of the numeral system).