

Supplemental Material

Generalized Lotka-Volterra

The generalized Lotka-Volterra model is

$$\frac{dN_i}{dt} = N_i \left(r_i + \sum_k a_{ik} N_k \right) \equiv f_i(N) \quad (1)$$

$$= r_i N_i + N_i \sum_k a_{ik} N_k \quad (2)$$

$$= r_i N_i + a_{ii} N_i^2 + N_i \sum_{k \neq i} a_{ik} N_k \quad (3)$$

Assume there is an equilibrium N^* where $N_i^* > 0$. For all i that equilibrium must satisfy

$$f_i(N^*) = 0 \quad (4)$$

$$r_i N_i^* + a_{ii} (N_i^*)^2 + N_i^* \sum_{k \neq i} a_{ik} N_k^* = 0 \quad (5)$$

$$r_i + a_{ii} N_i^* + \sum_{k \neq i} a_{ik} N_k^* = 0 \quad (6)$$

So

$$-a_{ii} N_i^* = r_i + \sum_{k \neq i} a_{ik} N_k^* \quad (7)$$

The diagonal of the Jacobian is

$$\frac{\partial f_i}{\partial N_i} = r_i + 2a_{ii} N_i + \sum_{k \neq i} a_{ik} N_k \quad (8)$$

We evaluate the Jacobian at the equilibrium N_i^* , using the relationship given in equation 7

$$\left. \frac{\partial f_i}{\partial N_i} \right|_{N^*} = r_i + 2a_{ii} N_i^* + \sum_{k \neq i} a_{ik} N_k^* \quad (9)$$

$$= 2a_{ii} N_i^* - a_{ii} N_i^* \quad (10)$$

$$= a_{ii} N_i^* \quad (11)$$

For the off-diagonal

$$\left. \frac{\partial f_i}{\partial N_j} \right|_{N^*} = a_{ij} N_i^* \quad (12)$$

Generalized Lotka-Volterra in a Metacommunity

Let N_i represent the population size for a particular taxa-location combination within the metacommunity. Let $t(i)$ return the taxa associated with i and let $l(i)$ return the location associated with i .

The generalized Lotka-Volterra model is

$$\frac{dN_i}{dt} = N_i \left(r_i + \sum_k a_{ik} N_k \right) - m N_i + \sum_k m_{ik} N_k \equiv f_i(N) \quad (13)$$

$$= r_i N_i + a_{ii} N_i^2 + N_i \sum_{k \neq i} a_{ik} N_k - m N_i + \sum_k m_{ik} N_k \quad (14)$$

where $a_{ii} = s_i$ represents the strength of intraspecific competition.

Species interactions only occur for different taxa in the same location. Thus

$$a_{ij} = \begin{cases} X_{ij}, & \text{if } l(i) = l(j) \text{ and } t(i) \neq t(j) \\ 0, & \text{otherwise} \end{cases} \quad (15)$$

where X_{ij} is a random variable whose distribution is parameterized as described below.

Individuals emigrate from a location at per-capita rate m . We assume that individual who emigrate select another location to immigrate to at random, with each of the $M - 1$ other locations are given equal weight. Thus the rate at which individuals emigrating from j arrive at i is given by

$$m_{ij} = \begin{cases} \frac{m}{M - 1}, & \text{if } l(i) \neq l(j) \text{ and } t(i) = t(j) \\ 0, & \text{otherwise} \end{cases} \quad (16)$$

where the 0s occur because migration can only occur when i and j refer to the same taxa (in different locations).

Jacobian

For the diagonal,

$$\frac{\partial f_i}{\partial N_i} = r_i + 2a_{ii} N_i + \sum_{k \neq i} a_{ik} N_k - m \quad (17)$$

For the off-diagonal,

$$\frac{\partial f_i}{\partial N_j} = a_{ij} N_i + m_{ij} \quad (18)$$

Evaluating the Jacobian at the equilibrium N^* . Assume there is an equilibrium N^* that satisfies $N_i^* > 0$ for all i .

$$N_i^* \left(r_i + \sum_k a_{ik} N_k^* \right) - m N_i^* + \sum_k m_{ik} N_k^* = 0 \quad (19)$$

Dividing by N_i^*

$$r_i + \sum_k a_{ik} N_k^* - m + \frac{1}{N_i^*} \sum_k m_{ik} N_k^* = 0 \quad (20)$$

Breaking apart the first sum to pull out intraspecific competition,

$$r_i + a_{ii} N_i^* + \sum_{k \neq i} a_{ik} N_k^* - m + \frac{1}{N_i^*} \sum_k m_{ik} N_k^* = 0 \quad (21)$$

So

$$r_i + \sum_{k \neq i} a_{ik} N_k^* - m = -a_{ii} N_i^* - \frac{1}{N_i^*} \sum_k m_{ik} N_k^* \quad (22)$$

So

$$\left. \frac{\partial f_i}{\partial N_i} \right|_{N^*} = r_i + 2a_{ii} N_i^* + \sum_{k \neq i} a_{ik} N_k^* - m \quad (23)$$

$$= 2a_{ii} N_i^* - a_{ii} N_i^* - \frac{1}{N_i^*} \sum_k m_{ik} N_k^* \quad (24)$$

$$= a_{ii} N_i^* - \frac{1}{N_i^*} \sum_k m_{ik} N_k^* \quad (25)$$

and

$$\left. \frac{\partial f_i}{\partial N_j} \right|_{N^*} = a_{ij} N_i^* + m_{ij} \quad (26)$$

Dealing with migration

We can write the first term of the Jacobian evaluated at N^* as the sum of two terms where the first term is DA where D is the diagonal matrix with entries N^* . The second term represents the impact of migration. Along the diagonal this term is $-\frac{1}{N_i^*} \sum_k m_{ik} N_k^*$, and on the off-diagonal it is m_{ij} .

$$J = DA + M \quad (27)$$

where M is the matrix that holds the migration terms.

Finding a feasible equilibrium when there is migration

The generalized Lotka-Volterra model is

$$\frac{dN_i}{dt} = N_i \left(r_i + \sum_k a_{ik} N_k \right) - m N_i + \sum_k m_{ik} N_k \equiv f_i(N) \quad (28)$$

$$= r_i N_i + a_{ii} N_i^2 + N_i \sum_{k \neq i} a_{ik} N_k - m N_i + \sum_k m_{ik} N_k \quad (29)$$

where $a_{ii} = s_i$ represents the strength of intraspecific competition.

At a feasible equilibrium,

$$f_i(N^*) = 0$$

$$\frac{1}{N_i} f_i(N^*) = 0$$

$$r_i + a_{ii} N_i^* + \sum_{k \neq i} a_{ik} N_k^* - m + \frac{1}{N_i^*} \sum_k m_{ik} N_k^* = 0 \quad (30)$$

Which we could write as

$$r + AN^* + stuff = 0 \quad (31)$$

And if *stuff* were 0 we could solve for N^* as

$$N^* = -A^{-1} \cdot r \quad (32)$$

Let

$$g_i(N^*) = -m + \frac{1}{N_i^*} \sum_k m_{ik} N_k^* \quad (33)$$

So we are trying to solve

$$r + AN^* + g(N^*) = 0 \quad (34)$$

So

$$AN^* = -g(N^*) - r \quad (35)$$

and

$$N^* = -A^{-1} (g(N^*) + r) \quad (36)$$

What if we used the approximation $N^* = -A^{-1} \cdot r$ when evaluating g ? Then,

$$N^* = -A^{-1} (g(-A^{-1} \cdot r) + r) \quad (37)$$

This may at least help with coding.