Supplemental Material

Generalized Lotka-Voltera

The generalized Lotka-Volterra model is

$$\frac{dN_i}{dt} = N_i \left(r_i + \sum_k a_{ik} N_k \right) \equiv f_i(N) \tag{1}$$

$$= r_i N_i + N_i \sum_k a_{ik} N_k \tag{2}$$

$$= r_i N_i + a_{ii} N_i^2 + N_i \sum_{k \neq i} a_{ik} N_k \tag{3}$$

Assume there is an equilibrium N^* where $N_i^* > 0$. For all i that equilibrium must satisfy

$$f_i(N^*) = 0 \tag{4}$$

$$r_i N_i^* + a_{ii} (N_i^*)^2 + N_i^* \sum_{k \neq i} a_{ik} N_k^* = 0 \tag{5}$$

$$r_i + a_{ii} N_i^* + \sum_{k \neq i} a_{ik} N_k^* = 0 \tag{6}$$

So

$$-a_{ii} N_i^* = r_i + \sum_{k \neq i} a_{ik} N_k^* \tag{7}$$

The diagonal of the Jacobian is

$$\frac{\partial f_i}{\partial N_i} = r_i + 2 a_{ii} N_i + \sum_{k \neq i} a_{ik} N_k \tag{8}$$

We evaluate the Jacobian at the equilrium N_i^*, using the relationship given in equation 7

$$\left. \frac{\partial f_i}{\partial N_i} \right|_{N^*} = r_i + 2 a_{ii} N_i^* + \sum_{k \neq i} a_{ik} N_k^* \tag{9}$$

$$= 2 a_{ii} N_i^* - a_{ii} N_i^* \tag{10}$$

$$= a_{ii} N_i^* \tag{11}$$

For the off-diagonal

$$\left. \frac{\partial f_i}{\partial N_j} \right|_{N^*} = a_{ij} N_i^* \tag{12}$$

Generalized Lotka-Volterra in a Metacommunity

Let N_i represent the population size for a particular taxa-location combination within the metacommunity. Let $t(i)$ return the taxa associated with i and let $l(i)$ return the location associated with i.

The generalized Lotka-Volterra model is

$$\frac{dN_i}{dt} = N_i\left(r_i + \sum_k a_{ik}N_k\right) - mN_i + \sum_k m_{ik}N_k \equiv f_i(N) \tag{13}$$

$$= r_iN_i + a_{ii}N_i^2 + N_i\sum_{k\neq i} a_{ik}N_k - mN_i + \sum_k m_{ik}N_k \tag{14}$$

where $a_{ii} = s_i$ represents the strength of intraspecific competition.

Species interactions only occur for different taxa in the same location. Thus

$$a_{ij} = \begin{cases} X_{ij}, & \text{if } l(i) = l(j) \text{ and } t(i) \neq t(j) \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

where X_{ij} is a random variable whose distribution is parameterized as described below.

Individuals emigrate from a location at per-capita rate m. We assume that individual who emigrate select another location to immigrate to at random, with each of the $M-1$ other locations are given equal weight. Thus the rate at which individuals emigrating from j arrive at i is given by

$$m_{ij} = \begin{cases} \dfrac{m}{M-1}, & \text{if } l(i) \neq l(j) \text{ and } t(i) = t(j) \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

where the 0s occur because migration can only occur when i and j refer to the same taxa (in different locations).

Jacobian

For the diagonal,

$$\frac{\partial f_i}{\partial N_i} = r_i + 2a_{ii}N_i + \sum_{k\neq i} a_{ik}N_k - m \tag{17}$$

For the off-diagonal,

$$\frac{\partial f_i}{\partial N_j} = a_{ij}N_i + m_{ij} \tag{18}$$

Evaluating the Jacobian at the equilibrium N^*. Assume there is an equilibrium N^* that satisfies $N_i^* > 0$ for all i.

$$N_i^* \left(r_i + \sum_k a_{ik} N_k^* \right) - m N_i^* + \sum_k m_{ik} N_k = 0 \tag{19}$$

Dividing by N_i^*

$$r_i + \sum_k a_{ik} N_k^* - m + \frac{1}{N_i^*} \sum_k m_{ik} N_k^* = 0 \tag{20}$$

Breaking apart the first sum to pull out intraspecific competition,

$$r_i + a_{ii} N_i^* + \sum_{k \neq i} a_{ik} N_k^* - m + \frac{1}{N_i^*} \sum_k m_{ik} N_k^* = 0 \tag{21}$$

So

$$r_i + \sum_{k \neq i} a_{ik} N_k^* - m = -a_{ii} N_i^* - \frac{1}{N_i^*} \sum_k m_{ik} N_k^* \tag{22}$$

So

$$\left. \frac{\partial f_i}{\partial N_i} \right|_{N^*} = r_i + 2 a_{ii} N_i^* + \sum_{k \neq i} a_{ik} N_k^* - m \tag{23}$$

$$= 2 a_{ii} N_i^* - a_{ii} N_i^* - \frac{1}{N_i^*} \sum_k m_{ik} N_k^* \tag{24}$$

$$= a_{ii} N_i^* - \frac{1}{N_i^*} \sum_k m_{ik} N_k^* \tag{25}$$

and

$$\left. \frac{\partial f_i}{\partial N_j} \right|_{N^*} = a_{ij} N_i^* + m_{ij} \tag{26}$$

Dealing with migration

We can write the Jacobian evaluated at N^* as the sum of two terms where there first term is DA where D is the diagonal matrix with entries N^* The second term represents the impact of migration. Along the diagonal this term is $-\frac{1}{N_i^*} \sum_k m_{ik} N_k^*$, and on the off-diagonal it is m_{ij}.

$$J = DA + M \tag{27}$$

where M is the matrix that holds the migration terms.

Finding a feasible equilibrium when there is migration

The generalized Lotka-Volterra model is

$$\frac{dN_i}{dt} = N_i \left(r_i + \sum_k a_{ik} N_k \right) - mN_i + \sum_k m_{ik} N_k \equiv f_i(N) \tag{28}$$

$$= r_i N_i + a_{ii} N_i^2 + N_i \sum_{k \neq i} a_{ik} N_k - mN_i + \sum_k m_{ik} N_k \tag{29}$$

where $a_{ii} = s_i$ represents the strength of intraspecific competition.

At a feasibile equilibrium,

$$f_i(N^*) = 0$$

$$\frac{1}{N_i} f_i(N^*) = 0$$

$$r_i + a_{ii} N_i^* + \sum_{k \neq i} a_{ik} N_k^* - m + \frac{1}{N_i^*} \sum_k m_{ik} N_k^* = 0 \tag{30}$$

Which we could write as

$$r + AN^* + stuff = 0 \tag{31}$$

And if $stuff$ were 0 we could solve for N^* as

$$N^* = -A^{-1} \cdot r \tag{32}$$

Let

$$g_i(N^*) = -m + \frac{1}{N_i^*} \sum_k m_{ik} N_k^* \tag{33}$$

So we are trying to solve

$$r + AN^* + g(N^*) = 0 \tag{34}$$

So

$$AN^* = -g(N^*) - r \tag{35}$$

and

$$N^* = -A^{-1} \left(g(N^*) + r \right) \tag{36}$$

What if we used the approximation $N^* = -A^{-1} \cdot r$ when evaluating g? Then,

$$N^* = -A^{-1} \left(g(-A^{-1} \cdot r) + r \right) \tag{37}$$

This may at least help with coding.