Supplemental Material

Generalized Lotka-Voltera

The generalized Lotka-Volterra model is

$$\frac{dN_i}{dt} = N_i \left(r_i + \sum_k a_{ik} N_k \right) \equiv f_i(N) \tag{1}$$

$$= r_i N_i + N_i \sum_k a_{ik} N_k \tag{2}$$

$$= r_i N_i + a_{ii} N_i^2 + N_i \sum_{k \neq i} a_{ik} N_k \tag{3}$$

Assume there is an equilibrium N^* where $N_i^* > 0$. For all i that equilibrium must satisfy

$$f_i(N^*) = 0 \tag{4}$$

$$r_i N_i^* + a_{ii}(N_i^*)^2 + N_i^* \sum_{k \neq i} a_{ik} N_k^* = 0 \tag{5}$$

$$r_i + a_{ii} N_i^* + \sum_{k \neq i} a_{ik} N_k^* = 0 \tag{6}$$

So

$$-a_{ii} N_i^* = r_i + \sum_{k \neq i} a_{ik} N_k^* \tag{7}$$

The diagonal of the Jacobian is

$$\frac{\partial f_i}{\partial N_i} = r_i + 2a_{ii} N_i + \sum_{k \neq i} a_{ik} N_k \tag{8}$$

We evaluate the Jacobian at the equilrium N_i^*, using the relationship given in equation 7

$$\left. \frac{\partial f_i}{\partial N_i} \right|_{N^*} = r_i + 2a_{ii} N_i^* + \sum_{k \neq i} a_{ik} N_k^* \tag{9}$$

$$= 2a_{ii} N_i^* - a_{ii} N_i^* \tag{10}$$

$$= a_{ii} N_i^* \tag{11}$$

Generalized Lotka-Volterra in a Metacommunity

Let N_i represent the population size for a particular taxa-location combination within the metacommunity. Let $t(i)$ return the taxa associated with i and let $l(i)$ return the location associated with i.

The generalized Lotka-Volterra model is

$$\frac{dN_i}{dt} = N_i \left(r_i + \sum_k a_{ik} N_k \right) - mN_i + \sum_k m_{ik} N_k \equiv f_i(N) \tag{12}$$

$$= r_i N_i + a_{ii} N_i^2 + N_i \sum_{k \neq i} a_{ik} N_k - mN_i + \sum_k m_{ik} N_k \tag{13}$$

where $a_{ii} = s_i$ represents the strength of intraspecific competition.

Species interactions only occur for different taxa in the same location. Thus

$$a_{ij} = \begin{cases} X_{ij}, & \text{if } l(i) = l(j) \text{ and } t(i) \neq t(j) \\ 0, & \text{otherwise} \end{cases} \tag{14}$$

where X_{ij} is a random variable whose distribution is parameterized as described below.

Individuals emigrate from a location at per-capita rate m. We assume that individual who emigrate select another location to immigrate to at random, with each of the $M - 1$ other locations are given equal weight. Thus the rate at which individuals emigrating from j arrive at i is given by

$$m_{ij} = \begin{cases} \dfrac{m}{M-1}, & \text{if } l(i) \neq l(j) \text{ and } t(i) = t(j) \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

where the 0s occur because migration can only occur when i and j refer to the same taxa (in different locations).

Jacobian

For the diagonal,

$$\frac{\partial f_i}{\partial N_i} = r_i + 2a_{ii} N_i + \sum_{k \neq i} a_{ik} N_k - m \tag{16}$$

For the off-diagonal,

$$\frac{\partial f_i}{\partial N_j} = a_{ij} N_i + m_{ij} \tag{17}$$

Evaluating the Jacobian at the equilibrium N^*. Assume there is an equilibrium N^* that satisfies $N_i^* > 0$ for all i.

$$N_i^* \left(r_i + \sum_k a_{ik} N_k^* \right) - m N_i^* + \sum_k m_{ik} N_k = 0 \tag{18}$$

Dividing by N_i^*

$$r_i + \sum_k a_{ik} N_k^* - m + \frac{1}{N_i^*} \sum_k m_{ik} N_k^* = 0 \tag{19}$$

Breaking apart the first sum to pull out intraspecific competition,

$$r_i + a_{ii} N_i^* + \sum_{k \neq i} a_{ik} N_k^* - m + \frac{1}{N_i^*} \sum_k m_{ik} N_k^* = 0 \tag{20}$$

So

$$r_i + \sum_{k \neq i} a_{ik} N_k^* - m = -a_{ii} N_i^* - \frac{1}{N_i^*} \sum_k m_{ik} N_k^* \tag{21}$$

So

$$\left. \frac{\partial f_i}{\partial N_i} \right|_{N^*} = r_i + 2 a_{ii} N_i^* + \sum_{k \neq i} a_{ik} N_k^* - m \tag{22}$$

$$= 2 a_{ii} N_i^* - a_{ii} N_i^* - \frac{1}{N_i^*} \sum_k m_{ik} N_k^* \tag{23}$$

$$= a_{ii} N_i^* - \frac{1}{N_i^*} \sum_k m_{ik} N_k^* \tag{24}$$

and

$$\left. \frac{\partial f_i}{\partial N_j} \right|_{N^*} = a_{ij} N_i^* + m_{ij} \tag{25}$$