United Nations Educational, Scientific and Cultural Organization

ICTP-AP
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS ASIA-PACIFIC

GWData-Bootcamp

Mainly, for the Bayesian inference and computation methods in the gravitational-wave (GW) research

Bayesian inference for GW science

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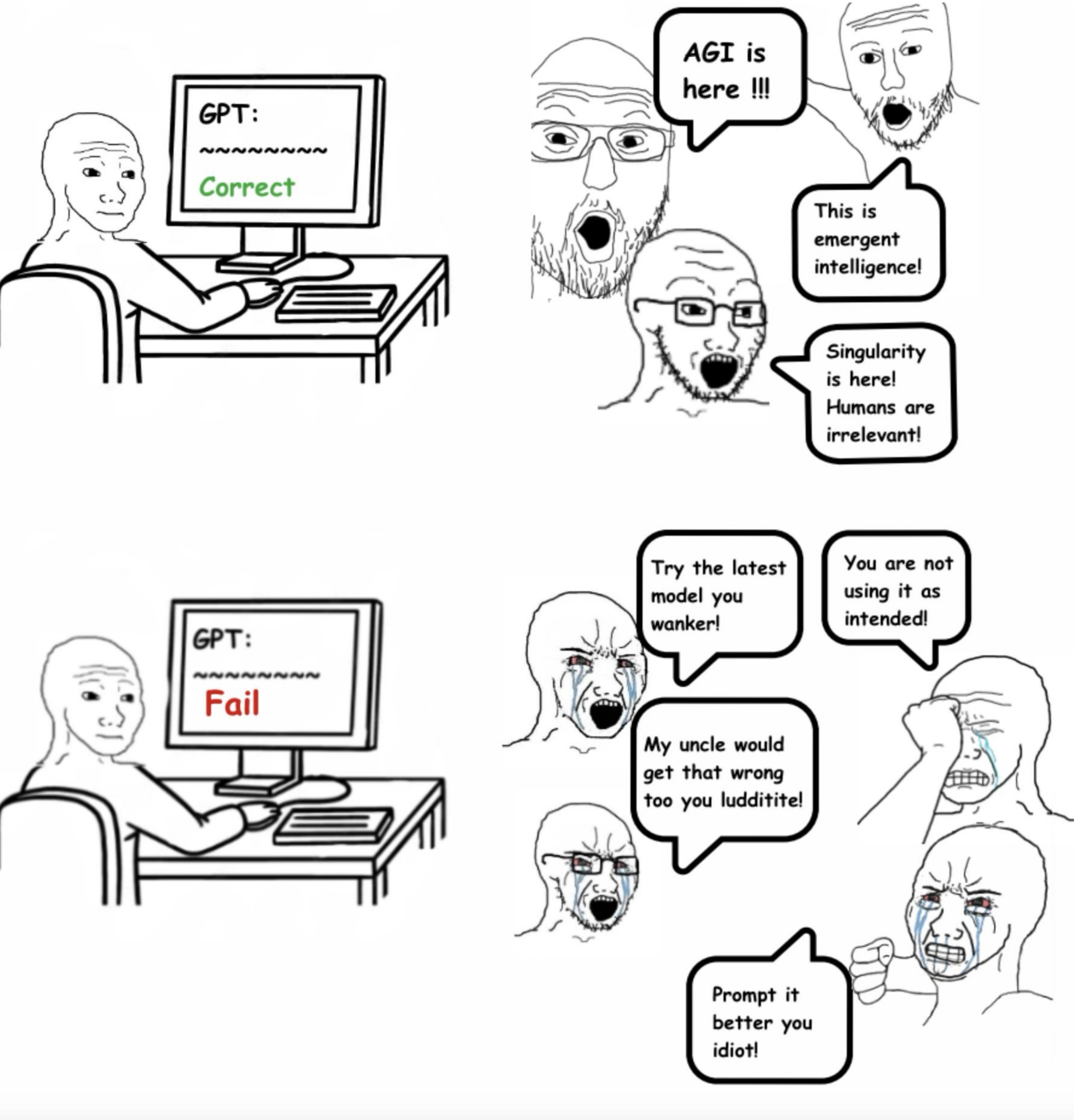
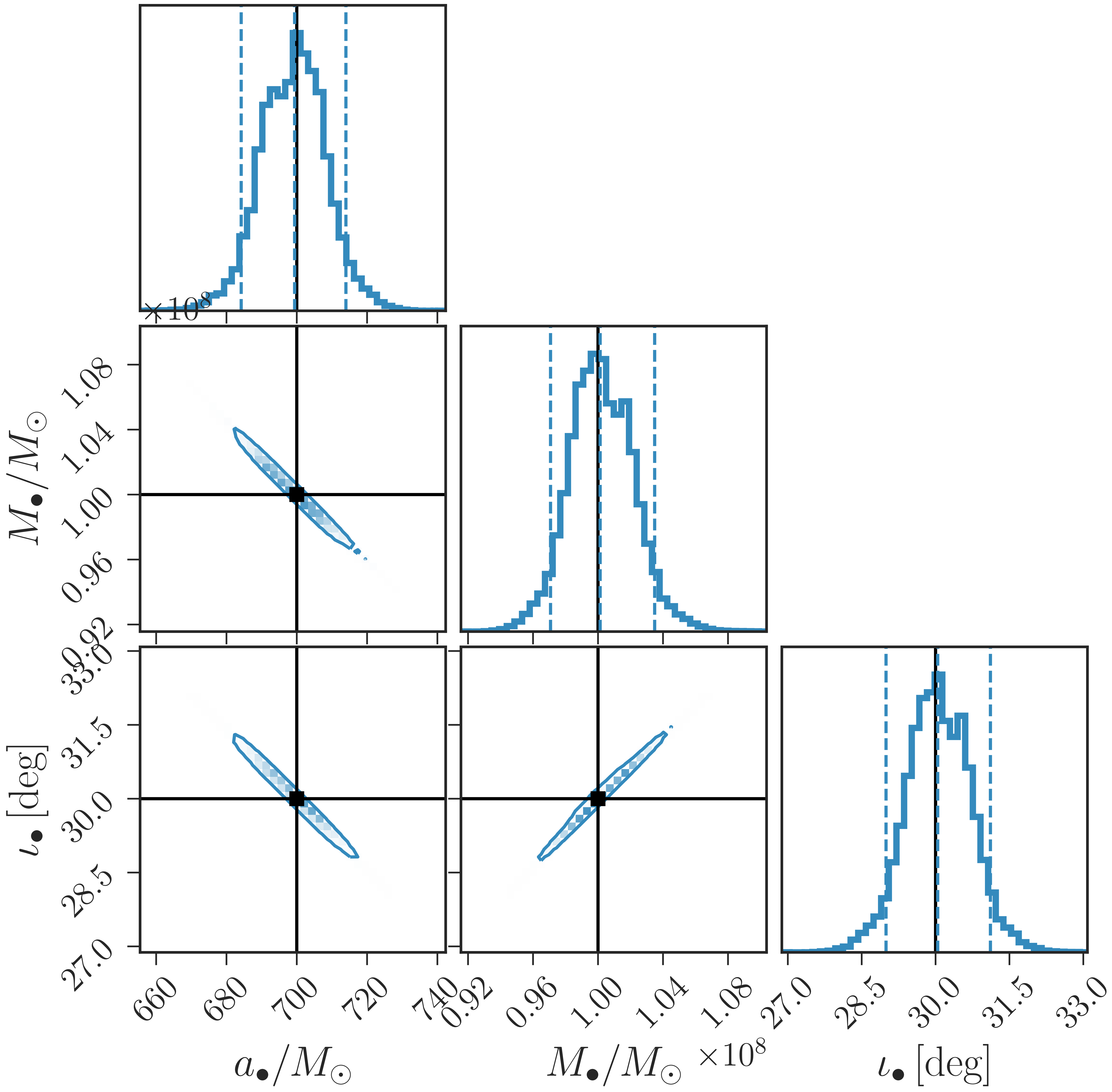
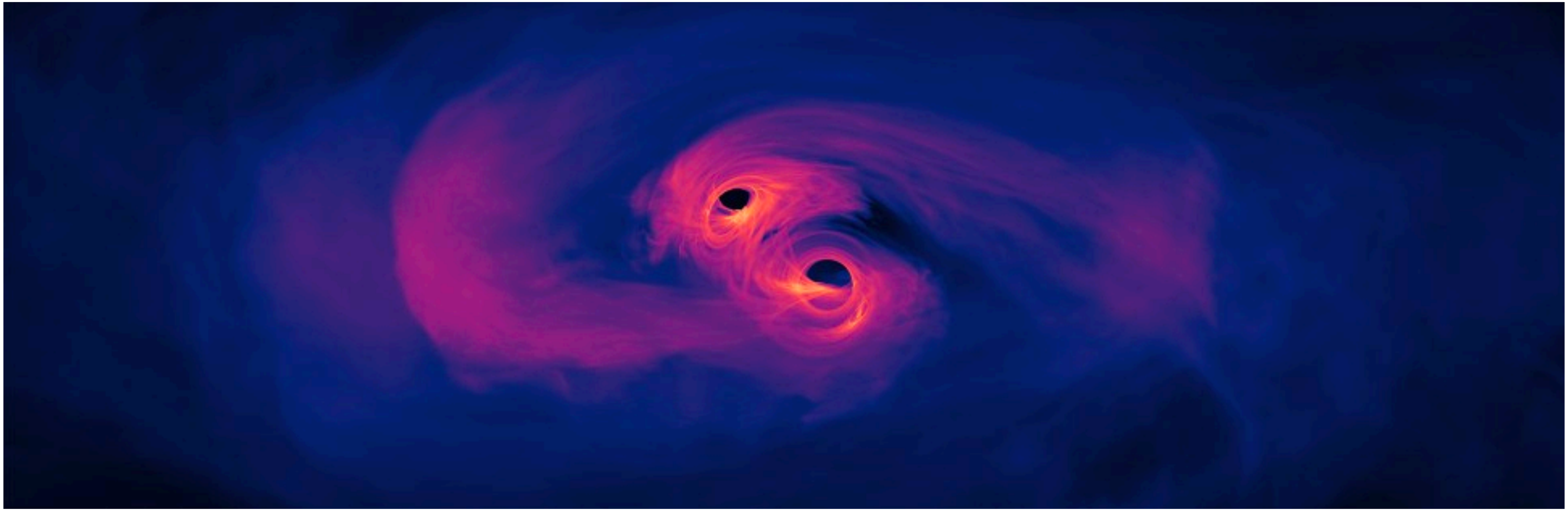
Dec. 16 & 17, 2023 (Beijing, Online)

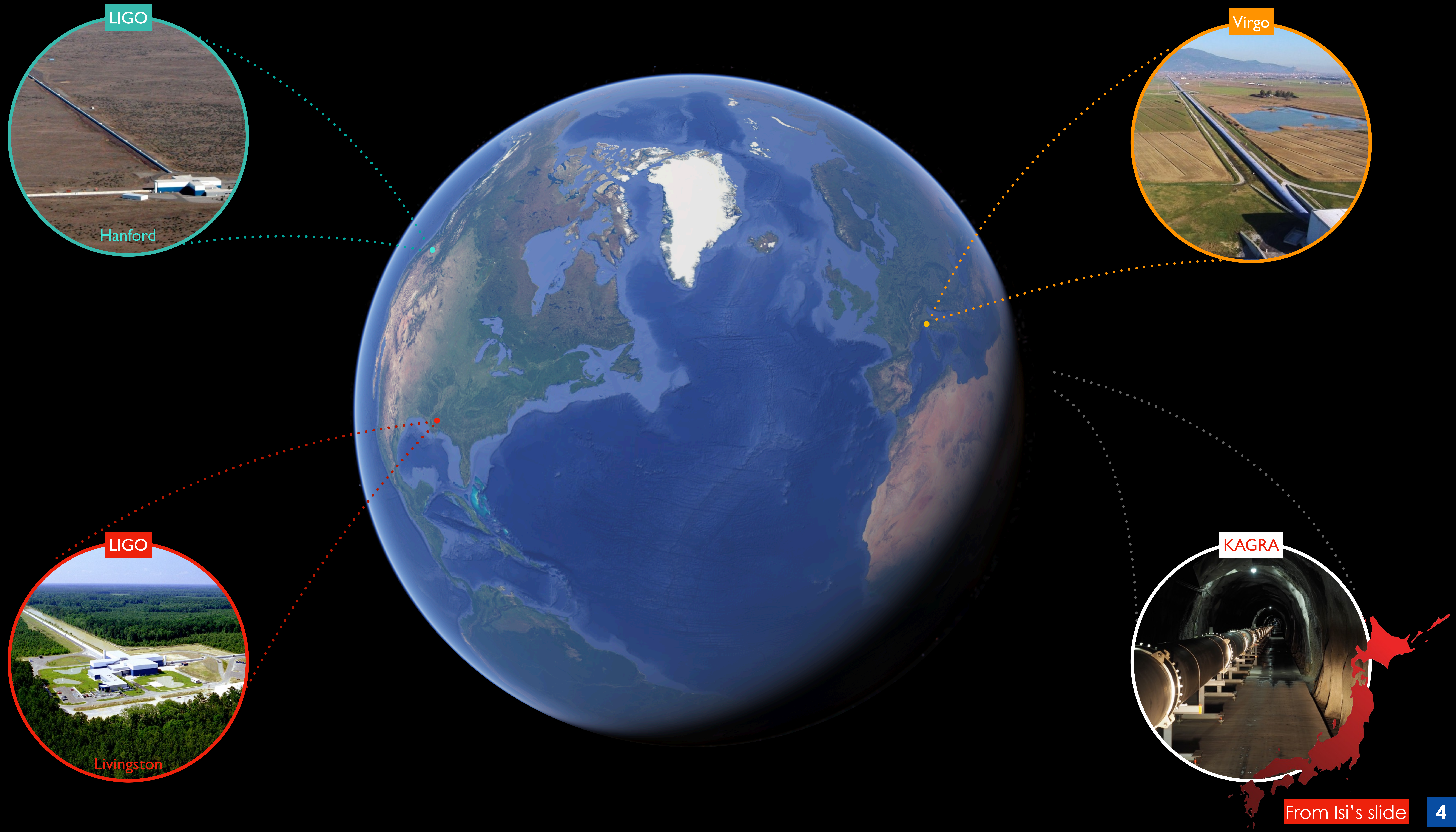

GPT:
~~~~~~~~
Correct
AGI is here !!!
This is emergent intelligence!
Singularity is here! Humans are irrelevant!
GPT:
~~~~~~~~
Fail
Try the latest model you wanker!
You are not using it as intended!
My uncle would get that wrong too you ludditite!
Prompt it better you idiot!


$M_\bullet/M_\odot$
$\iota_\bullet$ [deg]
$a_\bullet/M_\odot$
$M_\bullet/M_\odot$ $\times 10^8$
$\iota_\bullet$ [deg]

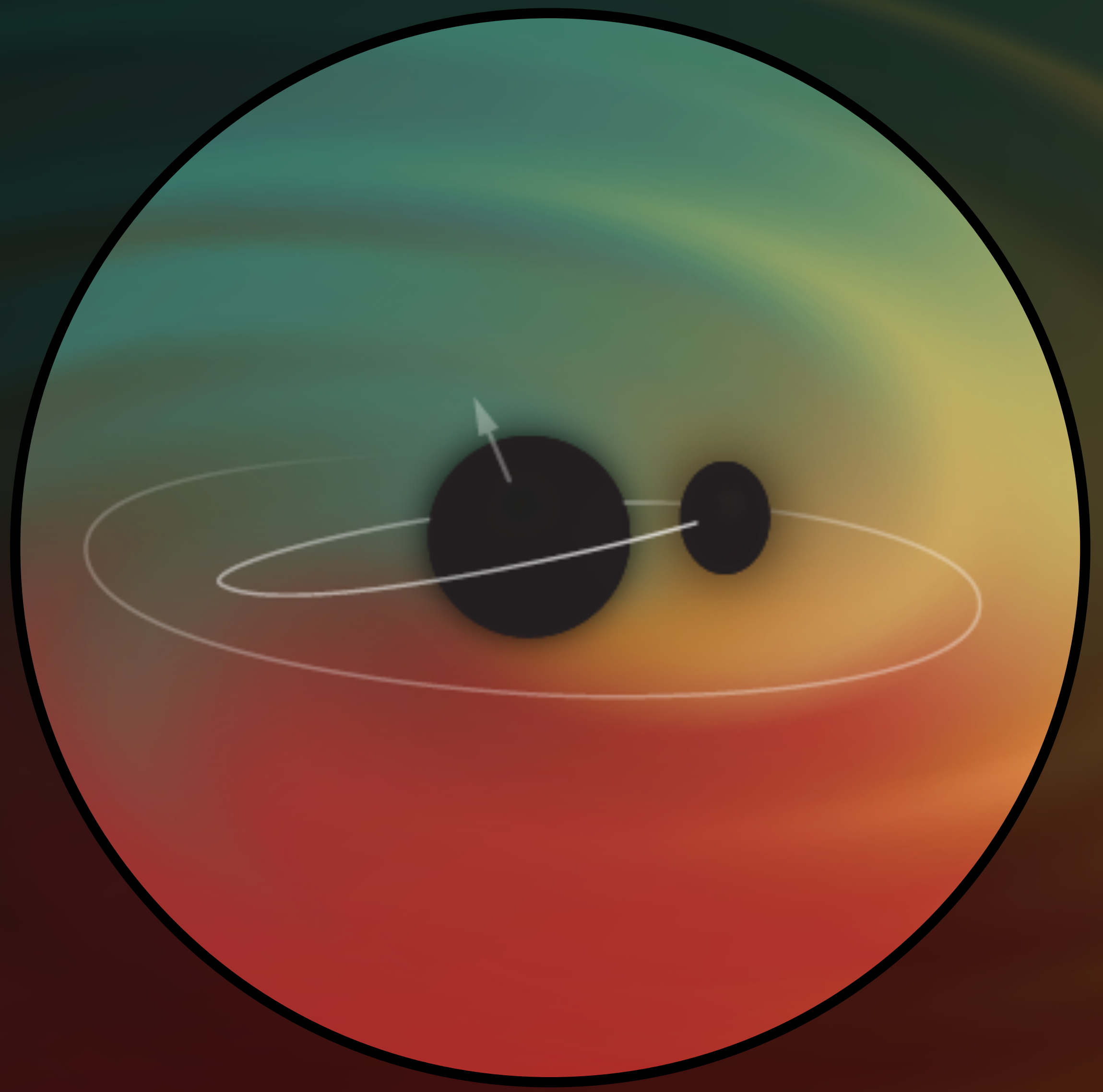


Simple background for GW

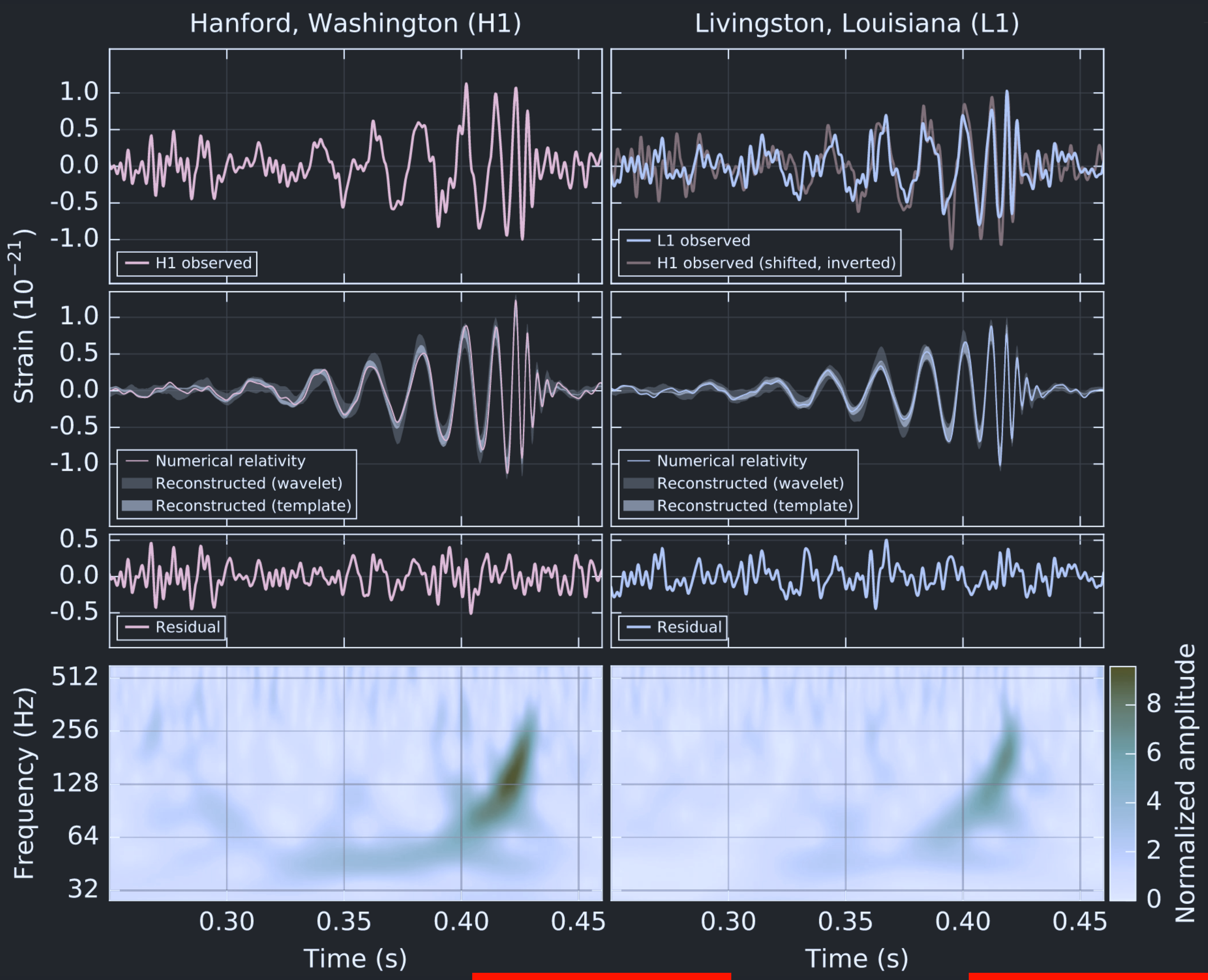
- Detectors
- Matched-filtering technique
- Parameter estimations


LIGO
Hanford
Virgo
LIGO
Livingston
KAGRA
From Isi's slide

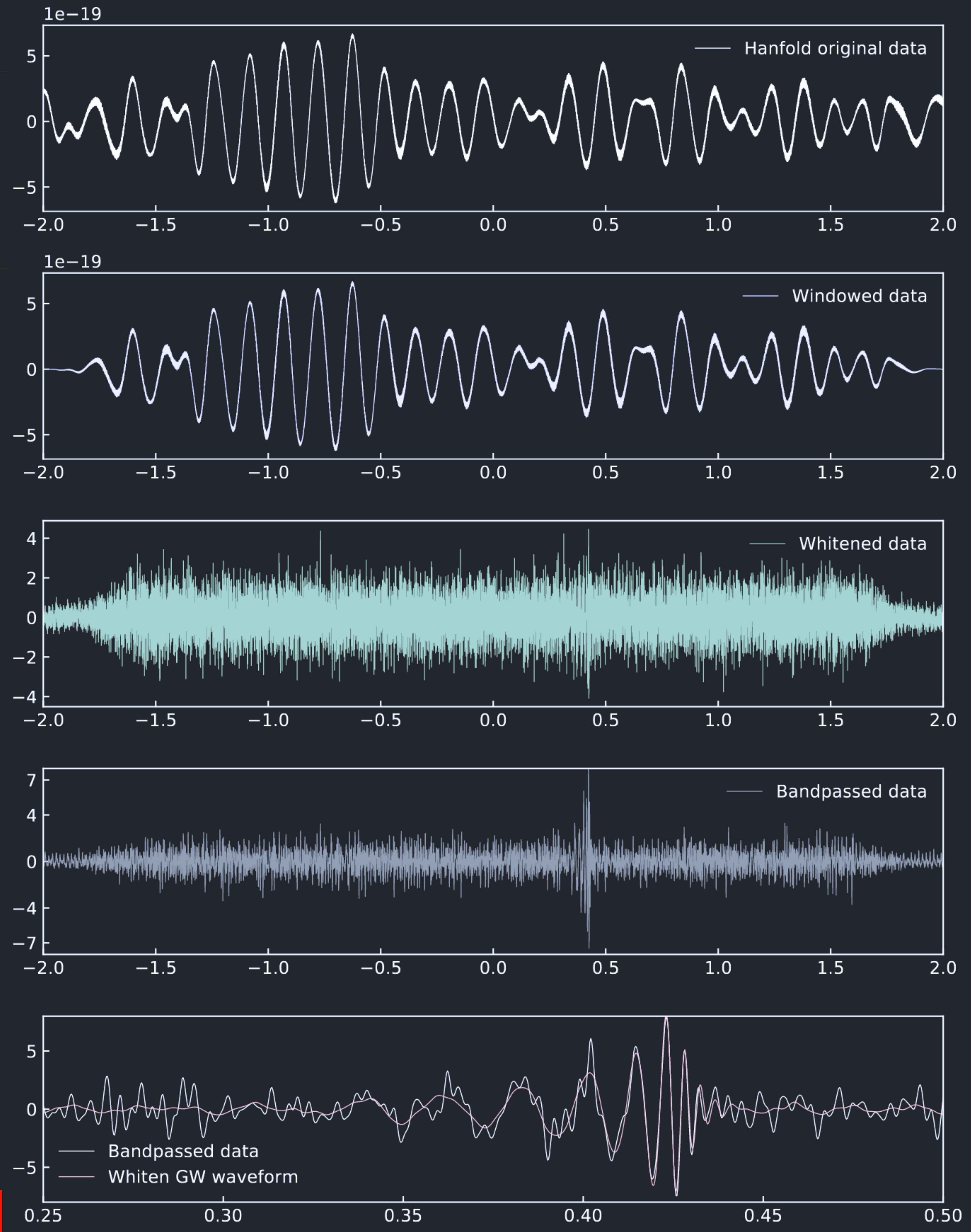
Typical strains from astrophysical sources $\lesssim 10^{-21}$,
displacement sensitivities δL of less than $\sim 10^{-18}$ m.
The radius of a proton is $\sim 8.5 \times 10^{-16}$ m.



GW150914



Abbott et al. '2016



From Junjie's thesis

GW150914
1e−19
Hanfold origi
Window
Whiten
Hanford, Washington (H1)
Livingston, Louisiana (L1)
Need more informations about waveform
Strain (10^{-21})
H1 observed
L1 observed
H1 observed (shifted, inverted)
Numerical relativity
Reconstructed (wavelet)
Reconstructed (template)
Abbott et al. '2016
From Junjie's thesis

Gravitational-Wave Transient Catalog 3 (GWTC-3)

Compact binary coalescences from the second part of the third observing run (O3b)

Observing period: **1st Nov 2019 15:00 UTC to 27th Mar 2020 17:00 UTC**

35 O3b CBC candidates identified with inferred probability of astrophysical origin, **p_{astro} > 0.5**

125.5 days with 2 or more detectors observing

GW candidates identified by **4 search pipelines**

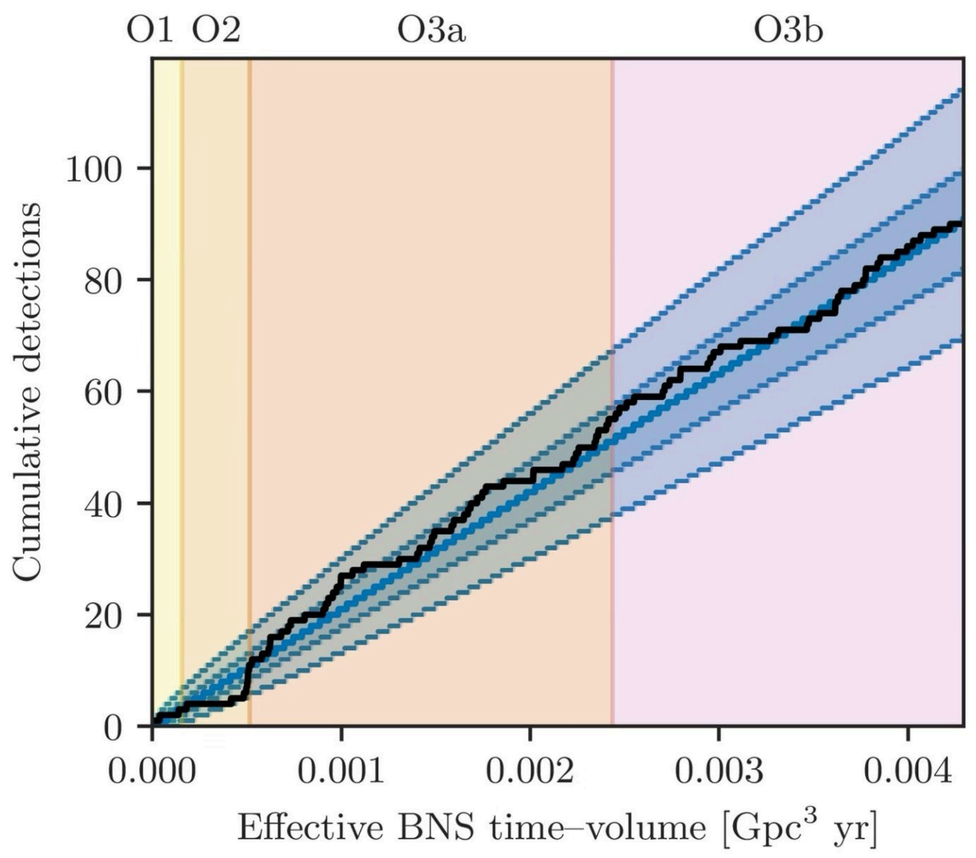
2 O3b exceptional events previously published

Total of **90 events in GWTC-3** with p_{astro} > 0.5

18 of our 35 candidates were previously reported during O3b as **low-latency public alerts**.
GWTC-3 also includes **17 O3b candidates** that are being reported for the first time.

Events of Note

GW200220_061928	**Most massive** binary system in O3b with total mass = 148 $M_\odot$
GW191219_163120	**NSBH merger** between a 1.17 $M_\odot$ NS and 31.1 $M_\odot$ BH, with the most extreme mass ratio (q = 0.038) measured to date
GW200115_042309	**NSBH merger** between a 1.44 $M_\odot$ NS and 5.9 $M_\odot$ BH
GW200210_092254	**NSBH or BBH merger:** less massive object has a mass of 2.83 $M_\odot$
GW191109_010717	**BBH merger** which is very likely to have negative spin
GW191129_134029	**Least massive definite BBH merger** in O3b, with total mass = 17.5 $M_\odot$



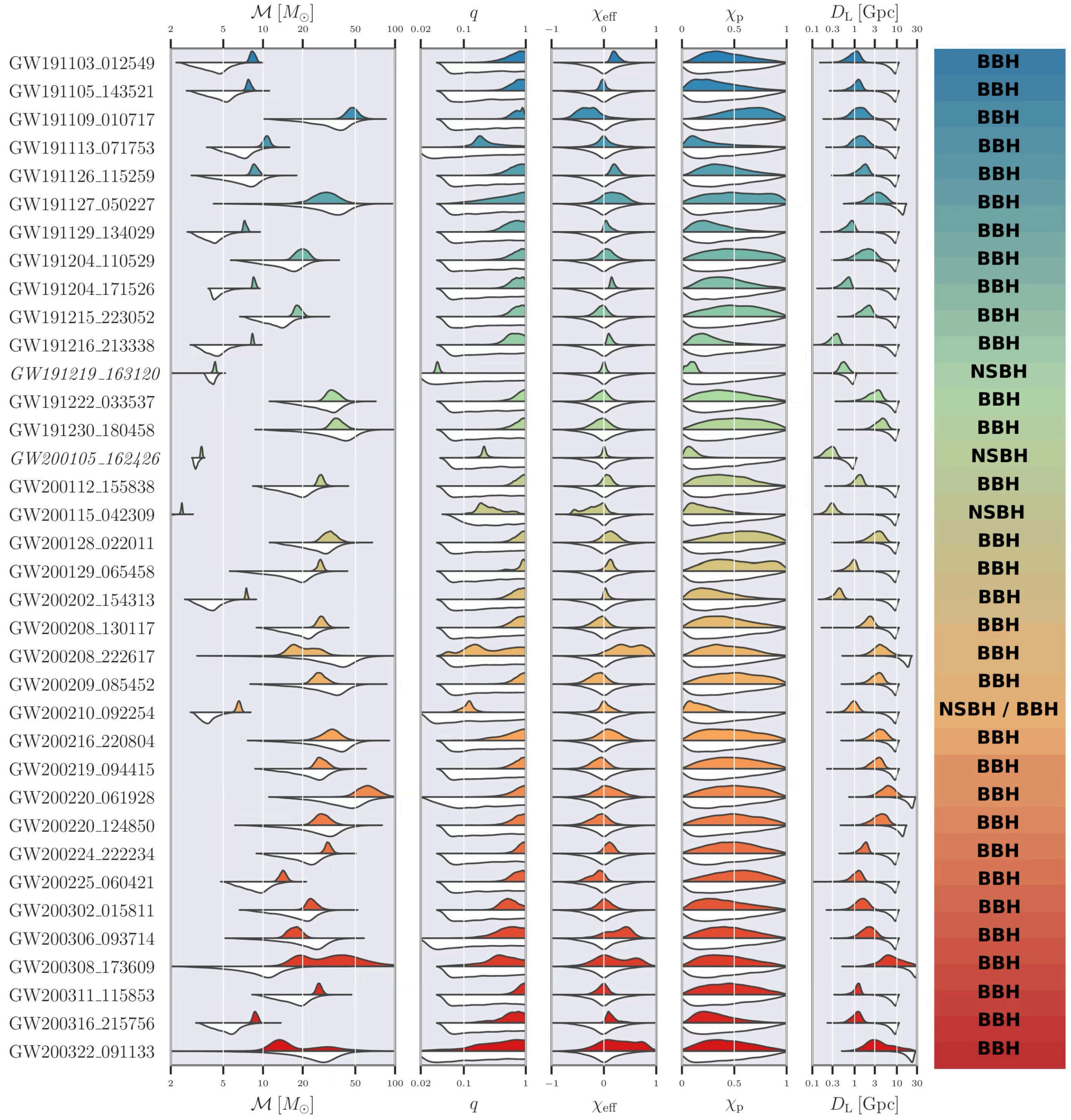
Growth in the number of LVK catalog candidates across observing runs. This figure shows the number of compact binary coalescence candidates with a probability of astrophysical origin p_{astro} > 0.5 versus the detector network's effective surveyed time-volume for binary neutron star (BNS) coalescences, which should be approximately proportional to the number of detections.

The colored bands indicate the different observing runs: O1, O2, O3a and O3b. The cumulative number of probable candidates is indicated by the solid black line.

Data for all GWTC-3 events are available from the **Gravitational-Wave Open Science Center: https://www.gw-openscience.org/**

Gravitational-Wave Transient Catalog 3 (GWTC-3)

Compact binary coalescences from the second part of the third observing run (O3b)

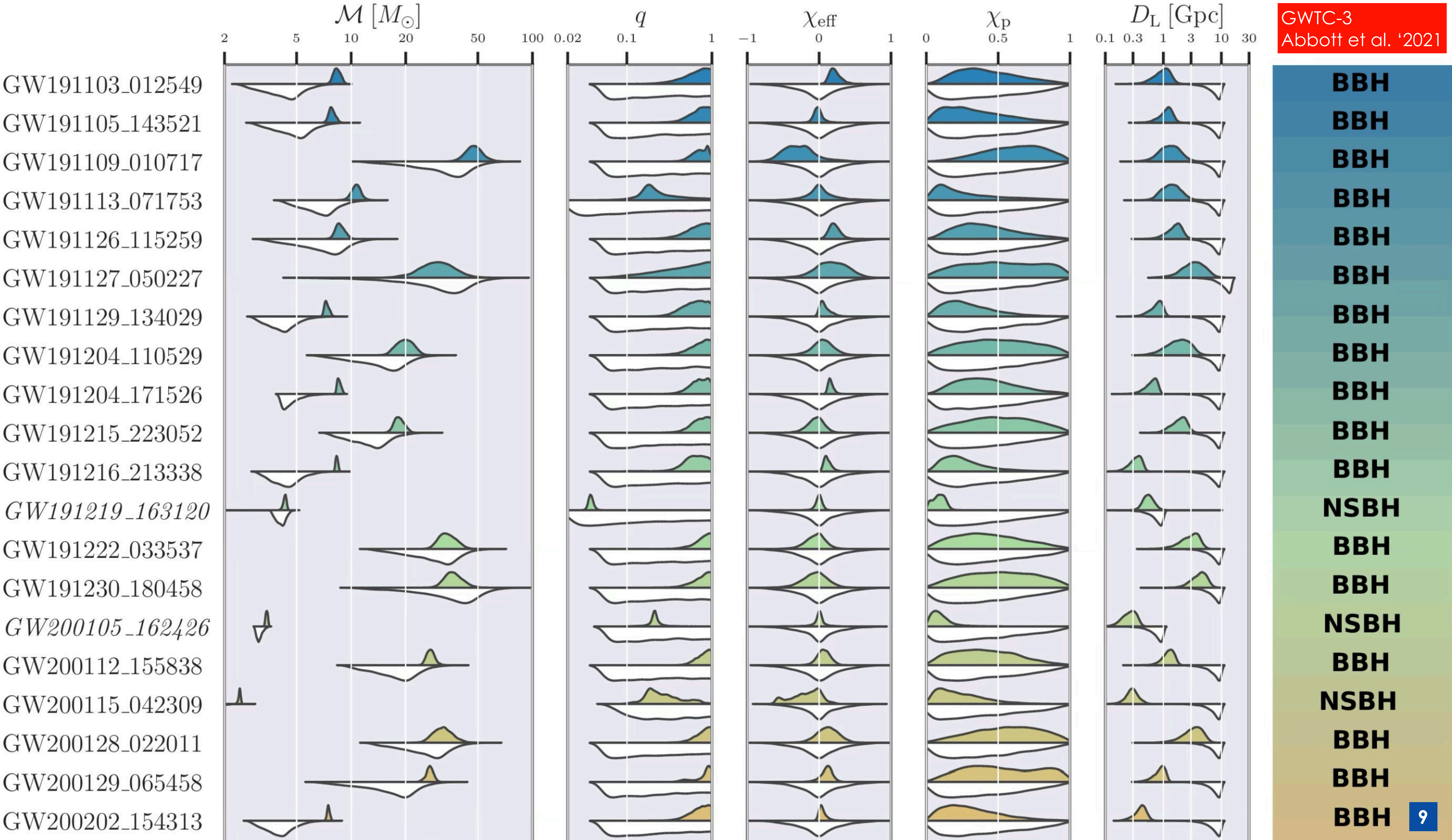


Properties of the events reported in the O3b catalog are listed above: chirp mass $\mathcal{M}$, in solar masses, mass ratio q, effective inspiral spin $\chi_{\rm eff}$, effective precession spin $\chi_{\rm p}$, and distance $D_{\rm L}$ in Gigaparsecs.

Also listed for each event is the most likely source classification. Events labelled BBH are those that we are confident are binary black hole coalescences. Events labelled NSBH are those that are possible neutron star and black hole coalescences. (We consider compact objects that are likely to have masses less than 3 times the mass of our sun to be possible neutron star candidates).

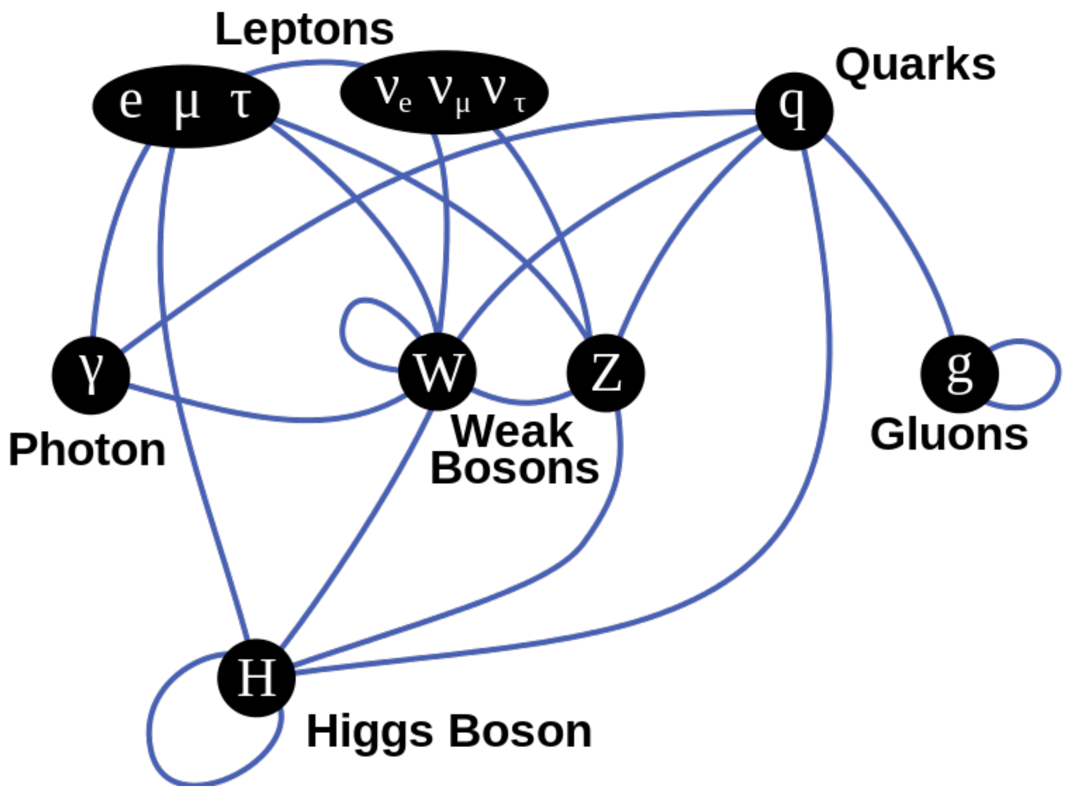
Credits: Martin Hendry, Hannah Middleton

GWTC-3
Abbott et al. '2021



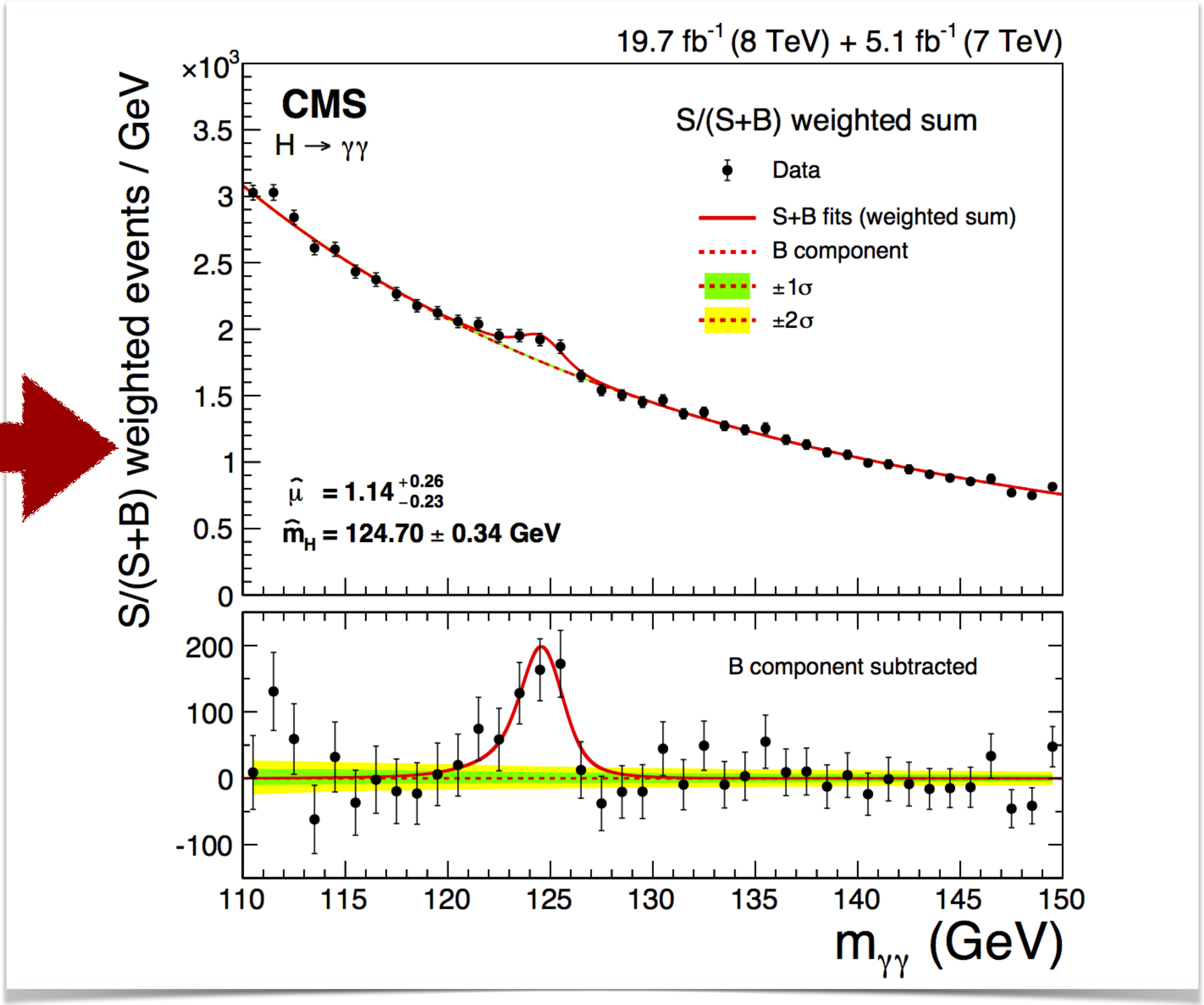
$\mathcal{M}\ [M_\odot]$
2 5 10 20 50 100
q
0.02 0.1 1
$\chi_{\rm eff}$
-1 0 1
$\chi_{\rm p}$
0 0.5 1
$D_{\rm L}$ [Gpc]
0.1 0.3 1 3 10 30
GWTC-3
Abbott et al. '2021
GW191103_012549 BBH
GW191105_143521 BBH
GW191109_010717 BBH
GW191113_071753 BBH
GW191126_115259 BBH
GW191127_050227 BBH
GW191129_134029 BBH
GW191204_110529 BBH
GW191204_171526 BBH
GW191215_223052 BBH
GW191216_213338 BBH
GW191219_163120 NSBH
GW191222_033537 BBH
GW191230_180458 BBH
GW200105_162426 NSBH
GW200112_155838 BBH
GW200115_042309 NSBH
GW200128_022011 BBH
GW200129_065458 BBH
GW200202_154313 BBH

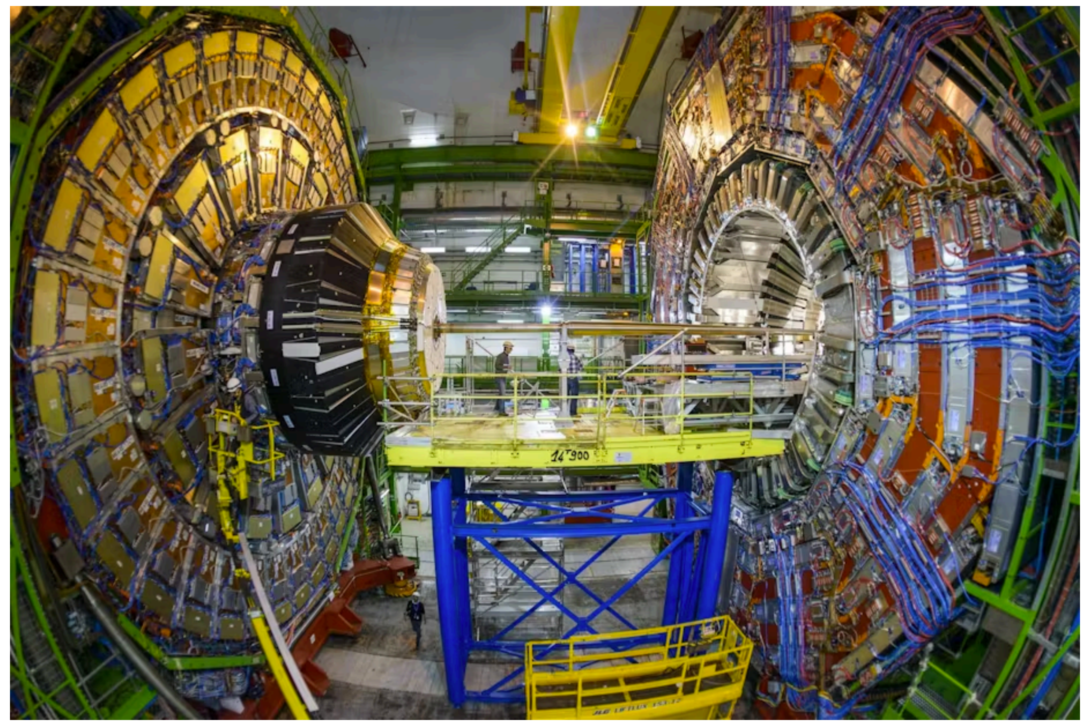
Example: the discovery of Higgs boson



Theory (model, hypothesis, …)

Stat.





Experiment (decay rate, …)

Credit: CERN

Intention

- Experiment: observing events [e.g., proton-proton (pp) collisions], measuring a set of characteristics:
    - Particle momenta, …
    - Energy of jets, …
- Compa
  - Par

We need a clear definition of PROBABILITY

    - Estimate the free parameters of the theory: $\mu = E(X)$
    - Quantify the uncertainty in the estimates: σ
  - Hypothesis test [5σ]:
    - Assess how well a given theory stands in agreement with the observed data:
      - p-value, Bayesian factor (BF)

Events & Probability

- Events
  - Repeatable (with independence)
    - e.g. coin-flip experiments
    - …
  - Disposable
    - e.g. How possible will it rain tomorrow?
    - …

Objective probability

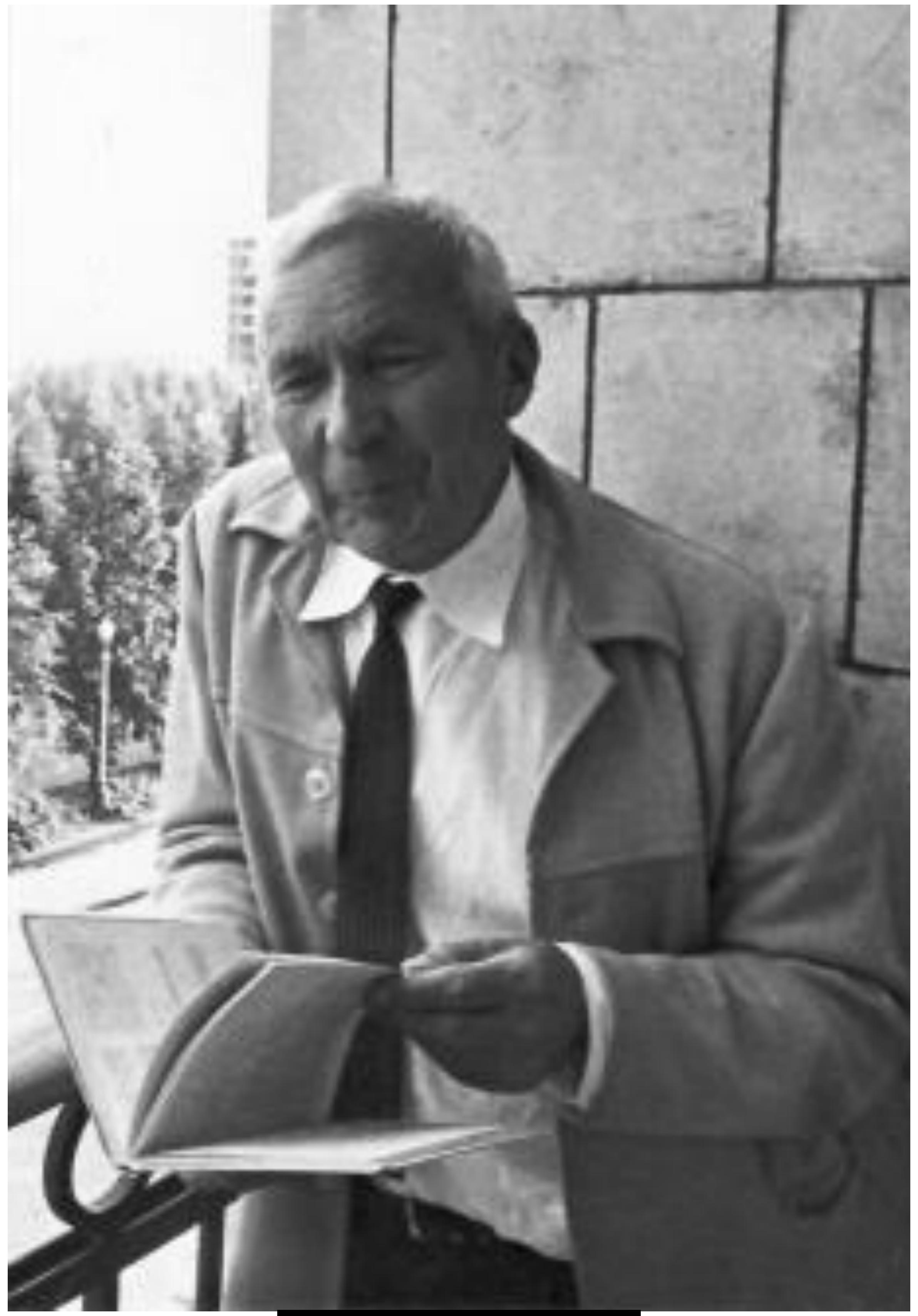
Both is the True Definition of Probability?

Subjective probability

Definition of "probability"

- Kolmogorov axioms (1933)
  - Consider a set $\mathcal{S}$ with subsets $A, B, \ldots$
    - For all $A \subset \mathcal{S}, \quad P(A) \geq 0$
    - $P(S) = 1$
    - If $A \cap B = \emptyset$, $P(A \cup B) = P(A) + P(B)$
  - Define the conditional probability of A given B:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$



Andrey Kolmogorov
1903-1987

Bayes' theorem

- From the definition of conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{and} \quad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

- Bayes' theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- Law of total probability: $P(B) = \sum_i P(B|A_i)P(A_i)$

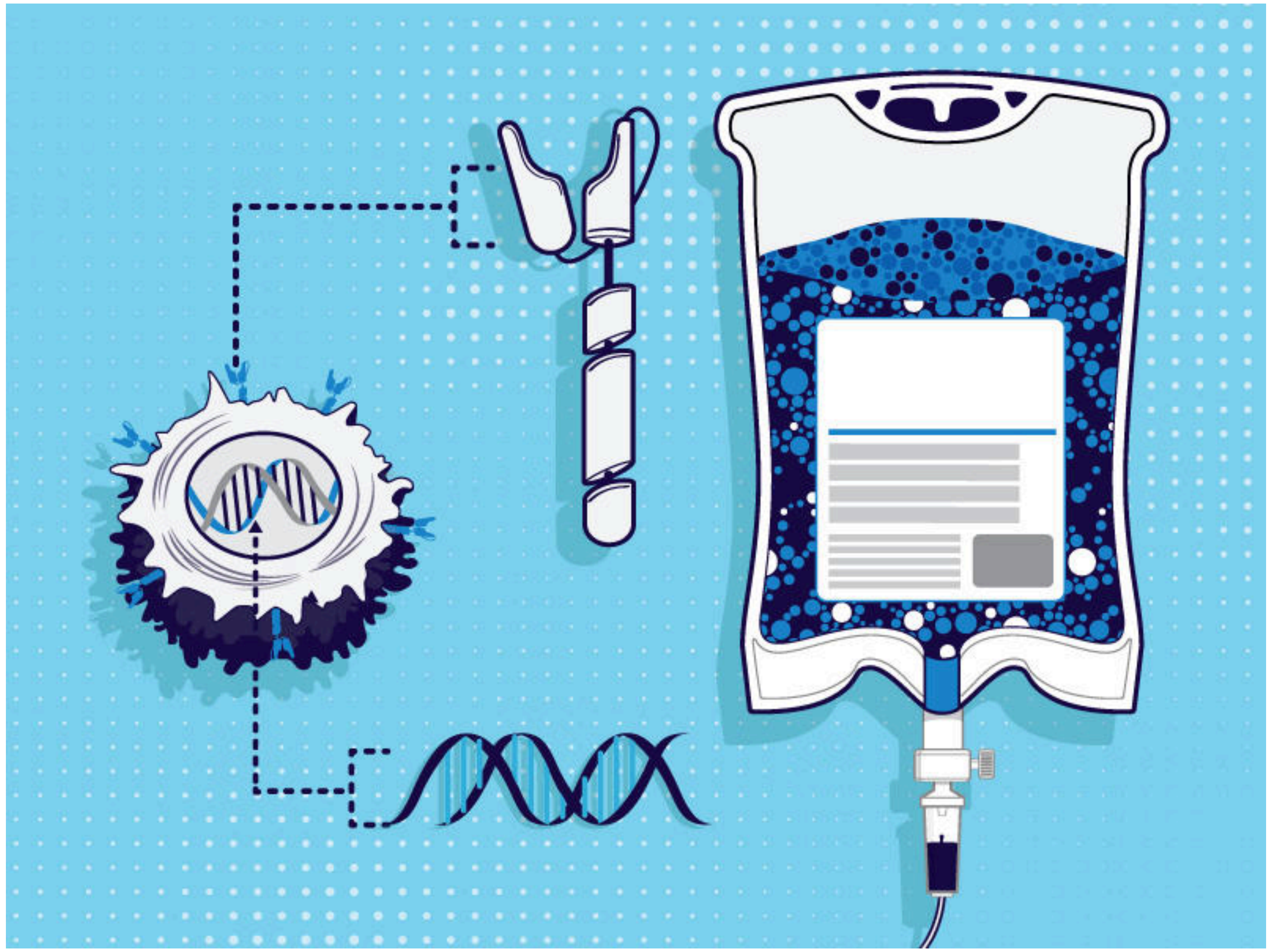
-

$$\Rightarrow P(A|B) = \frac{P(B|A)P(A)}{\sum_i P(B|A_i)P(A_i)}$$

Thomas Bayes
1702-1761

The simplest example: false positives

$$P(+\,|\,\text{Cancer}) = 0.9$$
$$P(-\,|\,\text{no Cancer}) = 0.9$$
$$P(\text{Cancer}) = 0.01$$



$$P(\text{Cancer}\,|\,+) = ?$$

GW detection

$$\mathbf{d}(t) = \begin{cases} \mathbf{n}(t) & \text{if signal not present} \\ \mathbf{n}(t) + \mathbf{h}(t;\boldsymbol{\Theta}) & \text{if signal } \mathbf{h}(t;\boldsymbol{\Theta}) \text{ present.} \end{cases}$$

$$P(\mathbf{h}\,|\,\mathbf{d}) = \frac{P(\mathbf{d}\,|\,\mathbf{h})\,\pi(\mathbf{h})}{P(\mathbf{d})} = \frac{P(\mathbf{d}\,|\,\mathbf{h})\,\pi(\mathbf{h})}{P(\mathbf{d}\,|\,\mathbf{h})\,\pi(\mathbf{h}) + P(\mathbf{d}\,|\,\mathbf{0})\,\pi(\mathbf{0})}$$

$$= \Lambda\left[\Lambda + \frac{\pi(\mathbf{0})}{\pi(\mathbf{h})}\right]^{-1} \quad \textbf{where } \Lambda = \frac{P(\mathbf{d}\,|\,\mathbf{h})}{P(\mathbf{d}\,|\,\mathbf{0})} = \frac{P(\mathbf{d}-\mathbf{h}\,|\,\mathbf{0})}{P(\mathbf{d}\,|\,\mathbf{0})}$$

Stationary, Gaussian white noise: $P(\mathbf{n}\,|\,\mathbf{0}) = \mathcal{N} e^{-\frac{1}{2}(\mathbf{n}|\mathbf{n})} = \mathcal{N} e^{-\frac{1}{2}\mathbf{n}^{\mathrm{T}}\cdot\mathbf{C}^{-1}\cdot\mathbf{n}}$

Finn '1992

Log-likelihood ratio: $\ln\,\Lambda = -\frac{1}{2}(\mathbf{d}-\mathbf{h}\,|\,\mathbf{d}-\mathbf{h}) + \frac{1}{2}(\mathbf{d}\,|\,\mathbf{d}) = (\mathbf{d}\,|\,\mathbf{h}) - \frac{1}{2}(\mathbf{h}\,|\,\mathbf{h})$

Rethink the interpretations

Both interpretations consistent with Kolmogorov axioms.

- Frequentist statistics:
  - Probability is interpreted as the frequency of the outcome of a repeatable experiment.

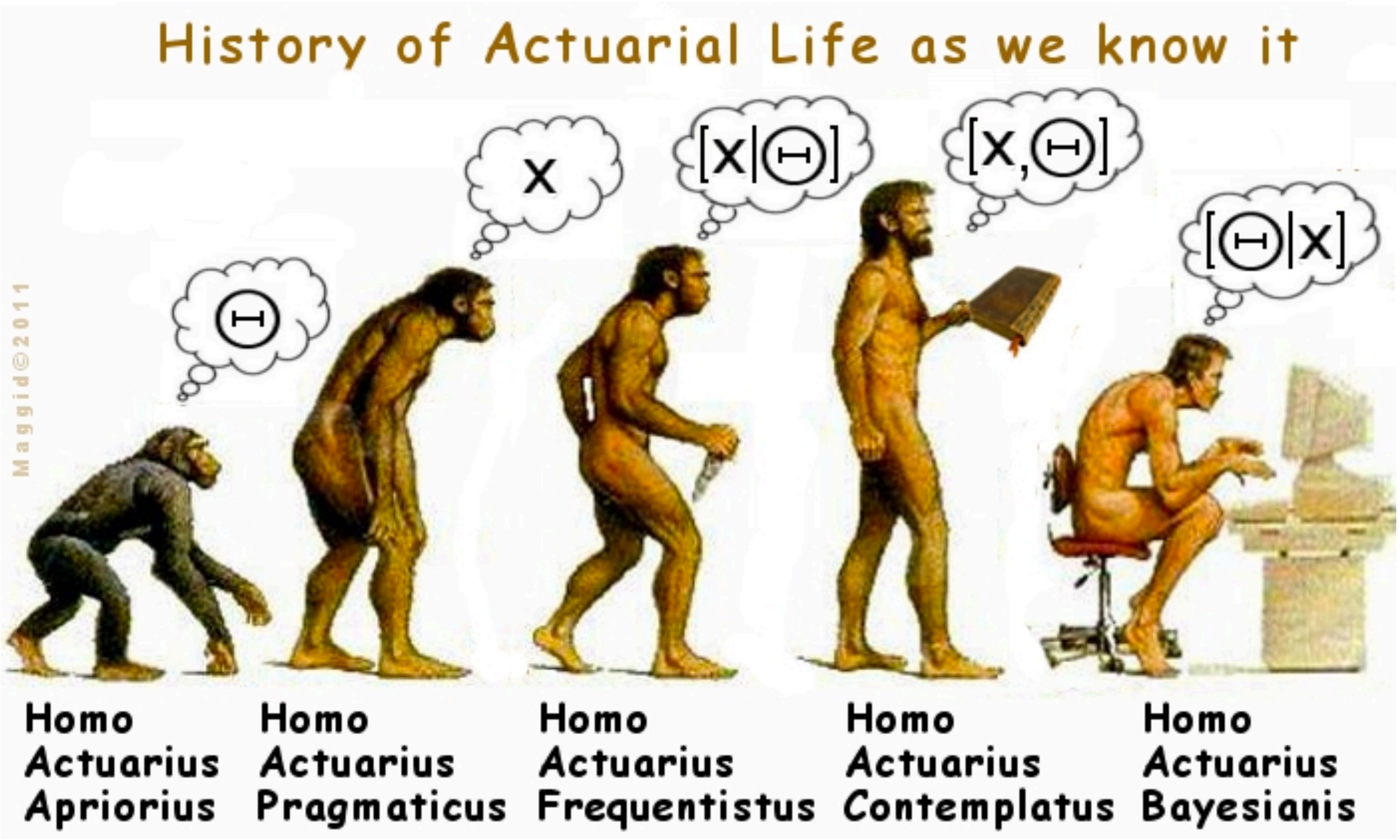
$$P(A) = \lim_{n \to \infty} \frac{\text{times outcomes of A}}{n}$$

  - assumes that there is an unknown but objectively fixed parameter (e.g. $\hat{\theta} = \text{argmax}_\theta \, p(x \mid \theta)$)

- Bayesian statistics:
  - Probability is more general and includes degree of belief (subjective probability)

$$P(A) = \text{degree of belief that A is true}$$

  - The prior degree of belief is updated by the data from the experiment.

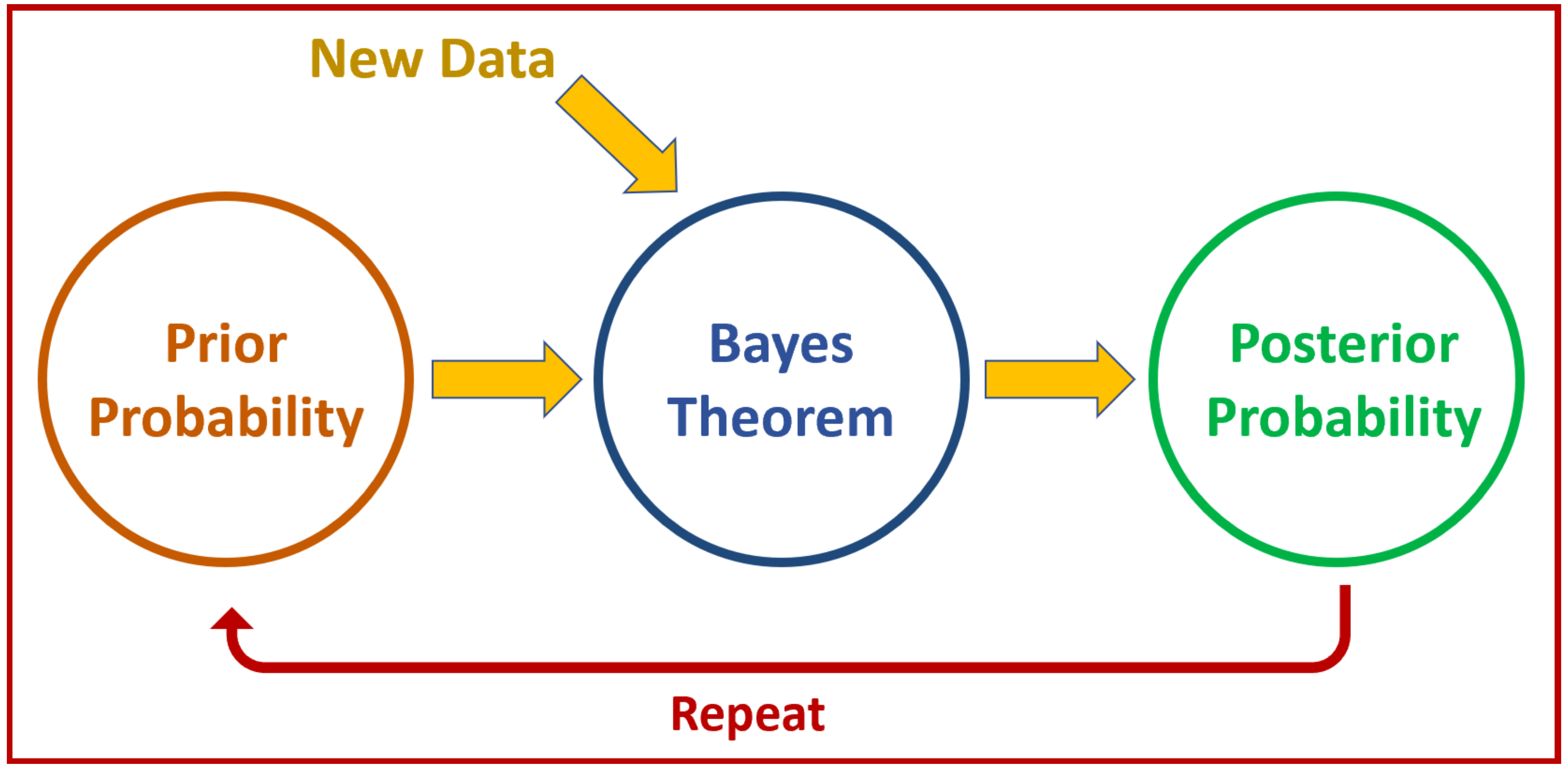

History of Actuarial Life as we know it
Maggid©2011
Θ
X
[X|Θ]
[X,Θ]
[Θ|X]
Homo Actuarius Apriorius
Homo Actuarius Pragmaticus
Homo Actuarius Frequentistus
Homo Actuarius Contemplatus
Homo Actuarius Bayesianis

Bayesian thinking

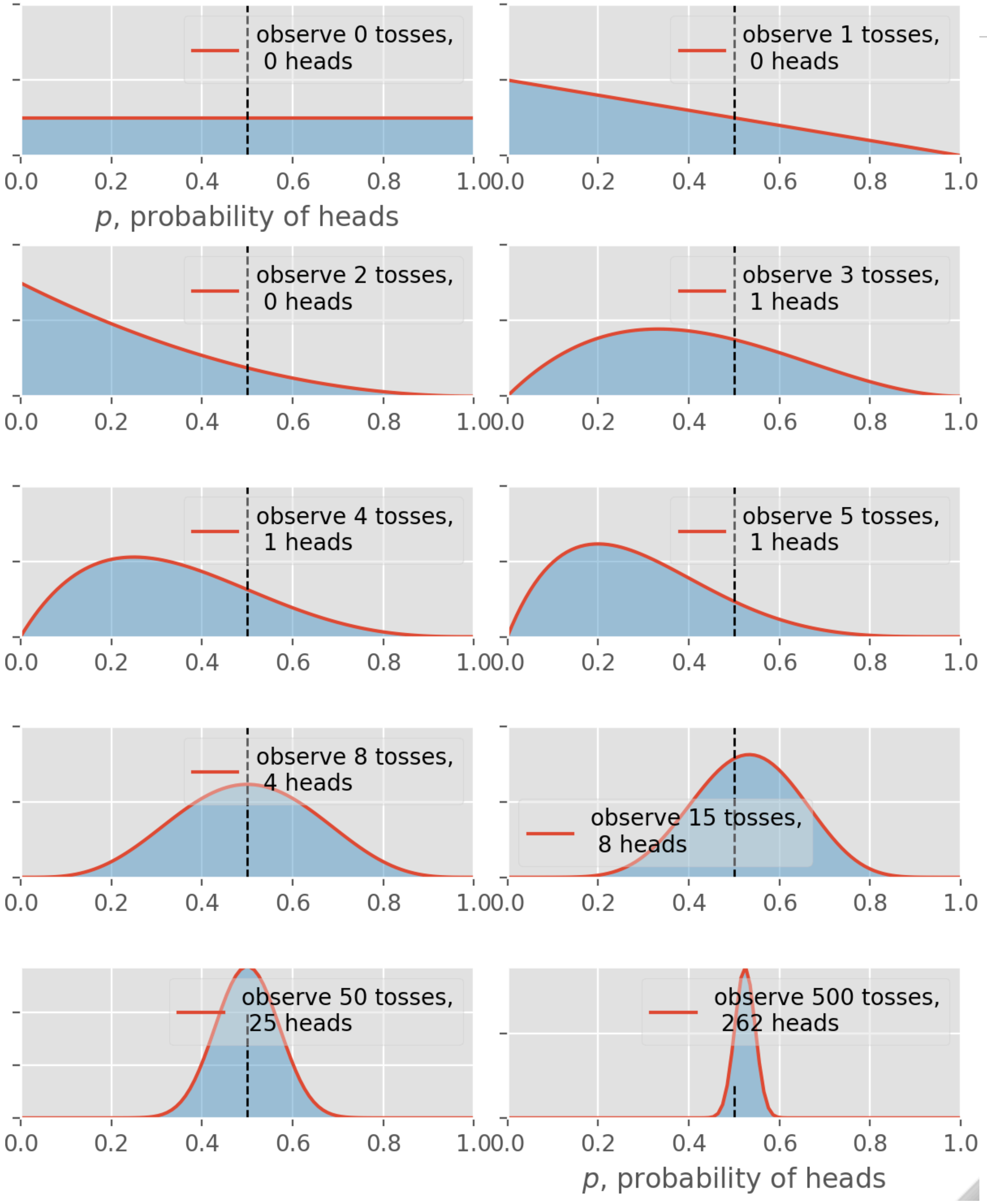
The prior degree of belief is updated by the data from the experiment

$$P(\Theta \mid X) = \frac{P(X \mid \Theta)P(\Theta)}{P(X)}$$

$$\propto P(X \mid \Theta)P(\Theta) \quad (\propto \text{ is proportional to })$$



Effect of the prior as N increases

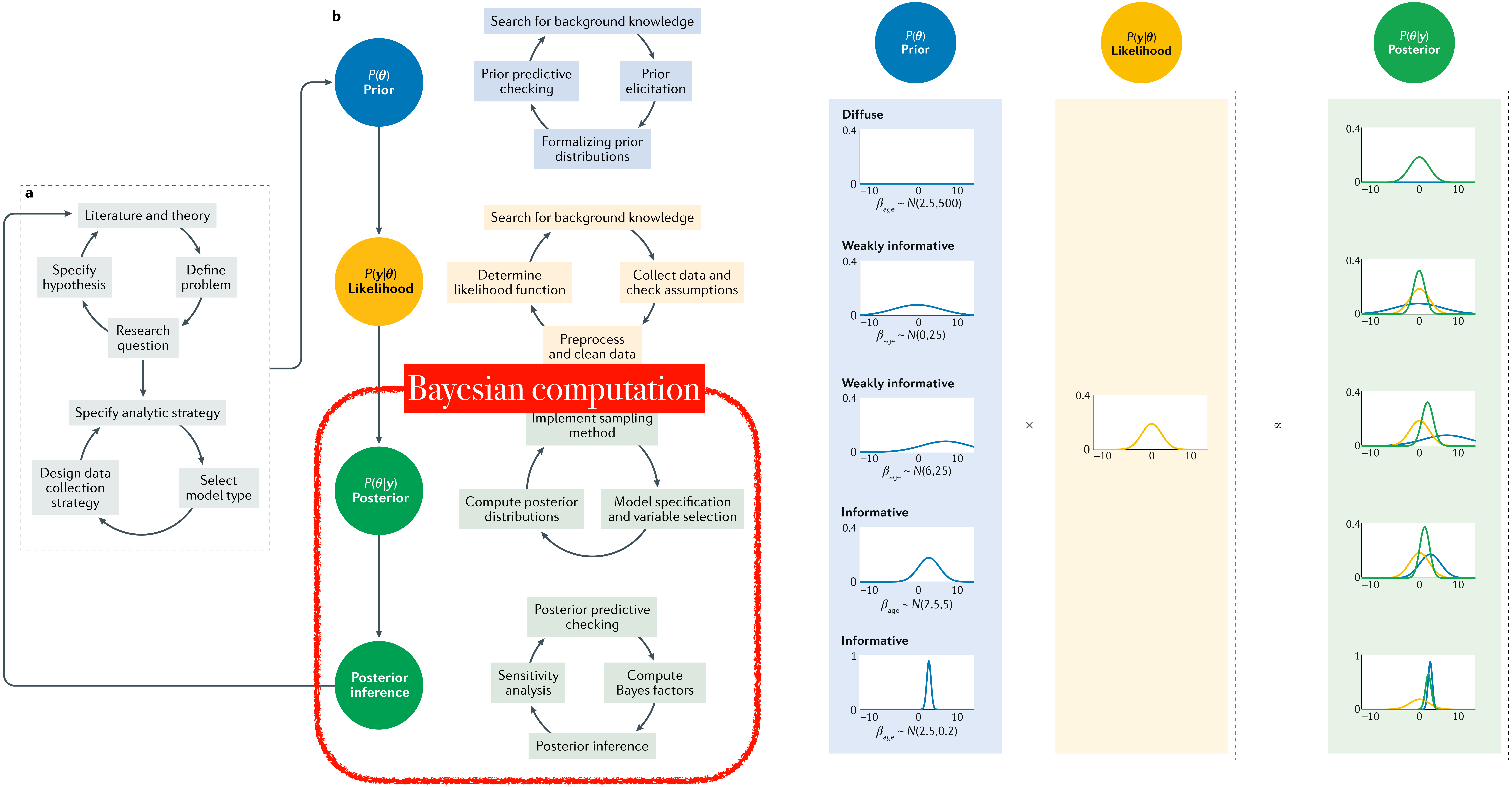


$$p(\theta \mid X) \propto \underbrace{p(X \mid \theta)}_{\text{likelihood}} \cdot \overbrace{p(\theta)}^{\text{prior}}$$

$$\ln(p(\theta \mid X)) = c + L(\theta; X) + \ln(p(\theta))$$

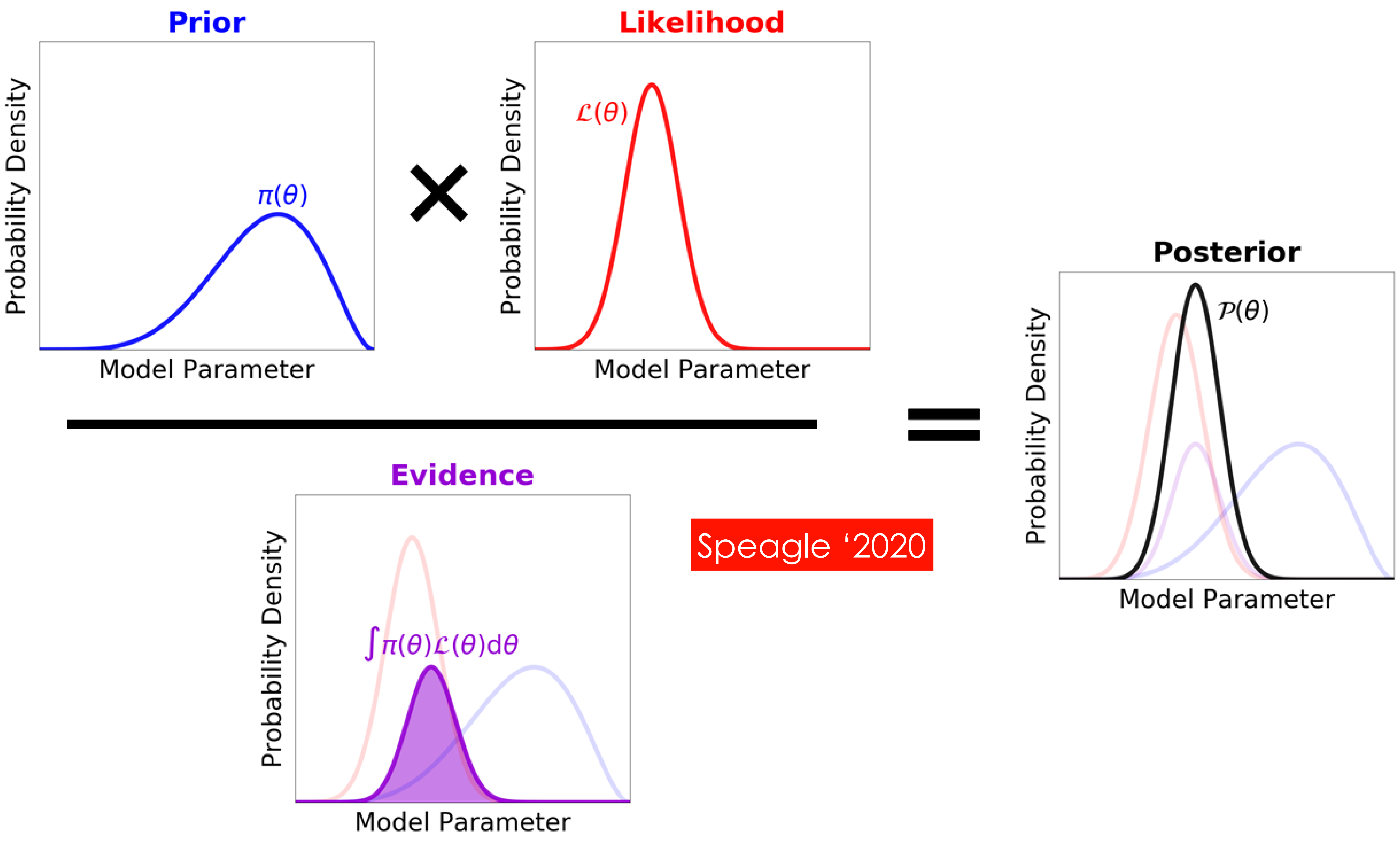
As the sample size increases, the chosen prior has less influence.

a.k.a. convergence of posterior distributions



van de Schoot et al. '2021

Bayesian inference



$$\frac{\pi(\boldsymbol{\Theta} \mid \mathcal{H})\, p(d \mid \mathcal{H}, \boldsymbol{\Theta})}{p(d \mid \mathcal{H})} = p(\boldsymbol{\Theta} \mid \mathcal{H}, d)$$

- Parameter estimation: Posterior
  - Find the posterior probability density $p(\boldsymbol{\Theta} \mid d, \mathcal{H})$
- Model selection: Evidence
  - compare different hypotheses though an odds ratio

$$O^{\mathcal{H}_1}_{\mathcal{H}_2} = \frac{p\left(\mathcal{H}_1 \mid d\right)}{p\left(\mathcal{H}_2 \mid d\right)} = \frac{p(d \mid \mathcal{H}_1)}{p(d \mid \mathcal{H}_2)} \frac{p(\mathcal{H}_1)}{p(\mathcal{H}_2)}$$

Parameter estimation in GW

Find the posterior probability density

$$p(\mathbf{\Theta} \mid \mathbf{d}, \mathscr{H}) = \frac{p(\mathbf{d} \mid \mathbf{\Theta}, \mathscr{H})\, p(\mathbf{\Theta} \mid \mathscr{H})}{p(\mathbf{d} \mid \mathscr{H})}$$

The priors $p(\mathbf{\Theta} \mid \mathscr{H})$ are based on what we know about the Universe

- Masses > 0 and usually $3M_\odot > M > 1M_\odot$ for the neutron star
- For binary black holes, we do not set the tidal deformability
- Luminosity distance $p(d_L) \propto d_L^{(D-1)}$, D is the dimension of space
- …….

Parameter estimation in GW

Find the posterior probability density

$$p(\mathbf{\Theta} \mid \mathbf{d}, \mathscr{H}) = \frac{p(\mathbf{d} \mid \mathbf{\Theta}, \mathscr{H})\, p(\mathbf{\Theta} \mid \mathscr{H})}{p(\mathbf{d} \mid \mathscr{H})}$$

Computing likelihood $p(\mathbf{d} \mid \mathbf{\Theta}, \mathscr{H}) = \mathscr{N} e^{-\frac{1}{2}(\mathbf{d}-\mathbf{h}|\mathbf{d}-\mathbf{h})}$ is the key in the GW

The evidence $p(\mathbf{d} \mid \mathscr{H})$ can be ignored here (I will explain it later)

- Independent of the parameters, just a number

Model selection in GW

$$O^{\mathscr{H}_1}_{\mathscr{H}_2} = \frac{p\left(\mathscr{H}_1 \mid d\right)}{p\left(\mathscr{H}_2 \mid d\right)} = \frac{p(d \mid \mathscr{H}_1)}{p(d \mid \mathscr{H}_2)} \frac{p(\mathscr{H}_1)}{p(\mathscr{H}_2)} \propto \frac{Z_1}{Z_2}$$

The hypotheses $\mathscr{H}_1, \mathscr{H}_2$ correspond to different waveform models

- Binary neutron stars versus binary black holes
- Waveform with aligned spin versus precessing binary
- Waveform predicted by general relativity versus alternative theories of gravity
- Natural GW signal versus artificial signal
- Add more parameters in the GW waveform
- …….

Model comparison & Occam's penalty

$$O^{\mathcal{H}_1}_{\mathcal{H}_2} = \frac{p\left(\mathcal{H}_1 \mid d\right)}{p\left(\mathcal{H}_2 \mid d\right)} = \frac{p(d \mid \mathcal{H}_1)}{p(d \mid \mathcal{H}_2)} \frac{p(\mathcal{H}_1)}{p(\mathcal{H}_2)} \sim \mathcal{B}$$

$$p\left(d \mid \boldsymbol{\theta}_1, \mathcal{H}_1\right) = p\left(d \mid \boldsymbol{\theta}_1, \boldsymbol{\lambda} = 0, \mathcal{H}_2\right)$$

$$p(\boldsymbol{\lambda} \mid d, \mathcal{H}_2) = \frac{p(\lambda \mid \mathcal{H}_2) p(d \mid \lambda, \mathcal{H}_2)}{p(d \mid \mathcal{H}_2)}$$

$$= \frac{p(\lambda \mid \mathcal{H}_2) p(d \mid \lambda, \mathcal{H}_2)}{\int p(\lambda', \mathcal{H}_2) p(d \mid \lambda', \mathcal{H}_2) d\lambda'}$$

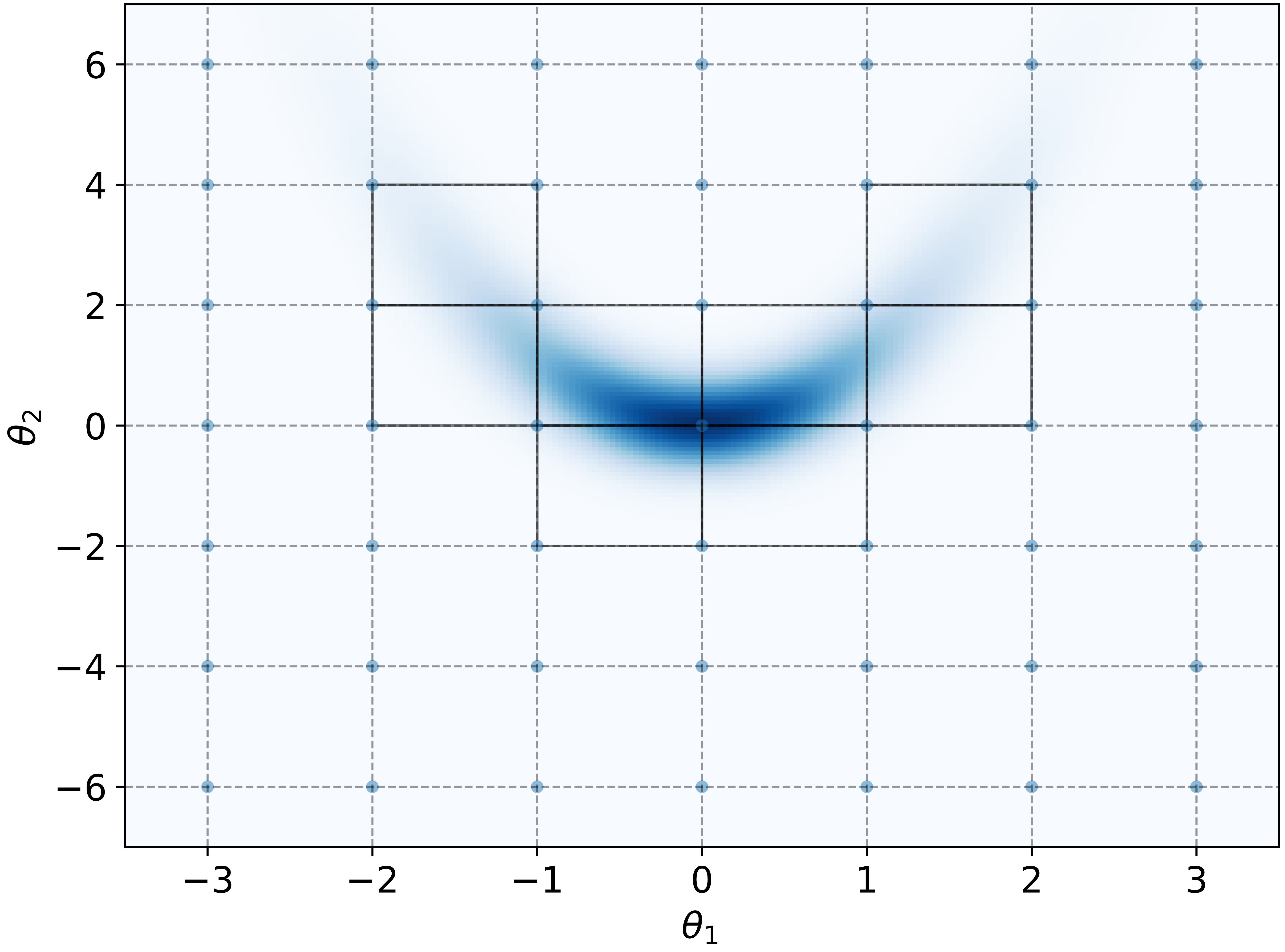
$$p(d \mid \lambda) = c e^{-(\lambda-\mu)^2/\left(2\sigma^2\right)}$$

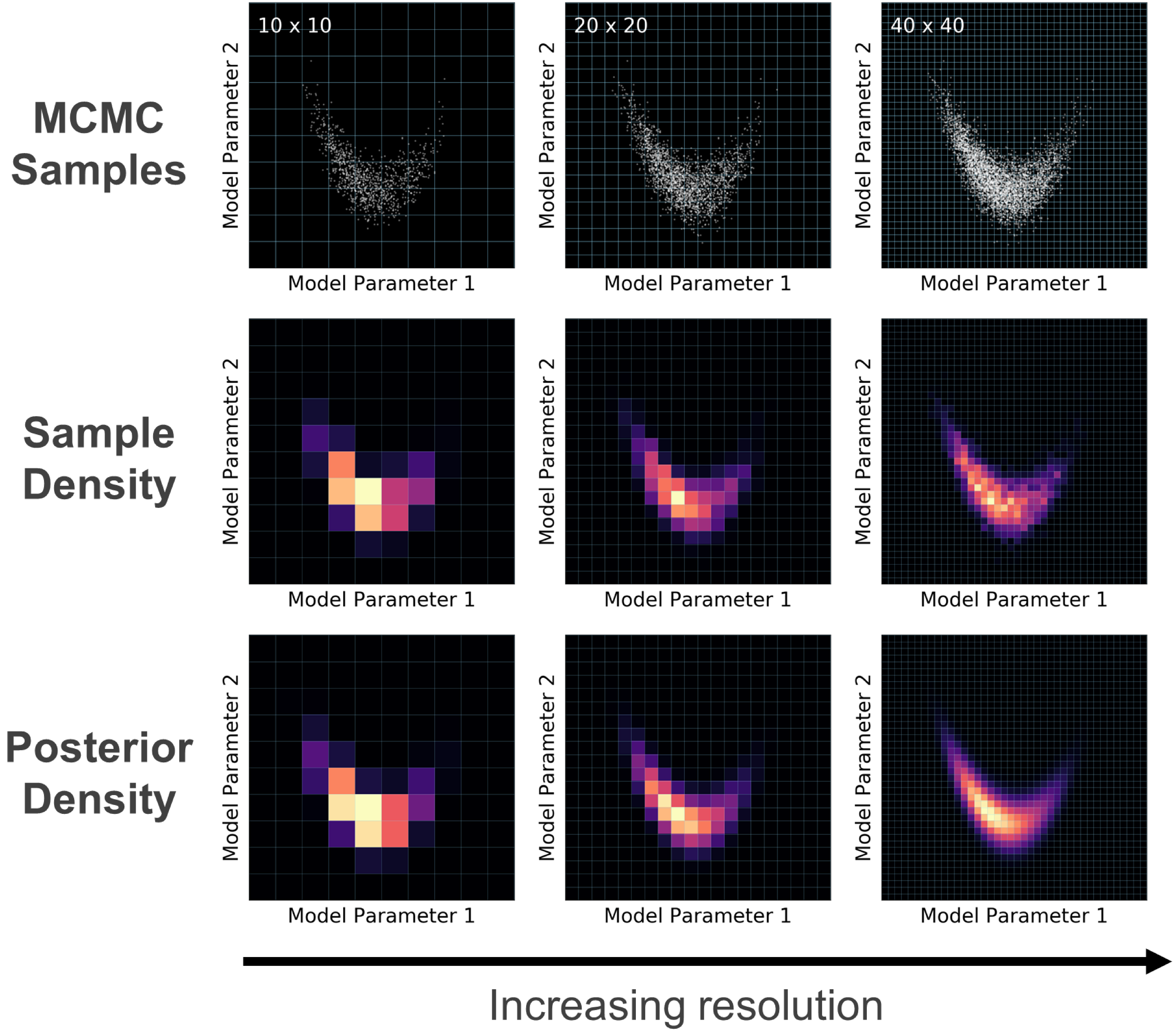
$$\mathcal{B} = \frac{p\left(\lambda = 0 \mid d, \mathcal{H}_2\right)}{p\left(\lambda = 0 \mid \mathcal{H}_2\right)} = \frac{1}{\sqrt{2\pi}} e^{-\mu^2/(2\sigma^2)} \frac{\Delta\lambda}{\sigma}.$$

Savage-Dickey density ratio

Zimmerman et al. '2019

Bayesian computation method





$$20 \times 20 \times \ldots \times 20 = 20^{\mathrm{D}}$$

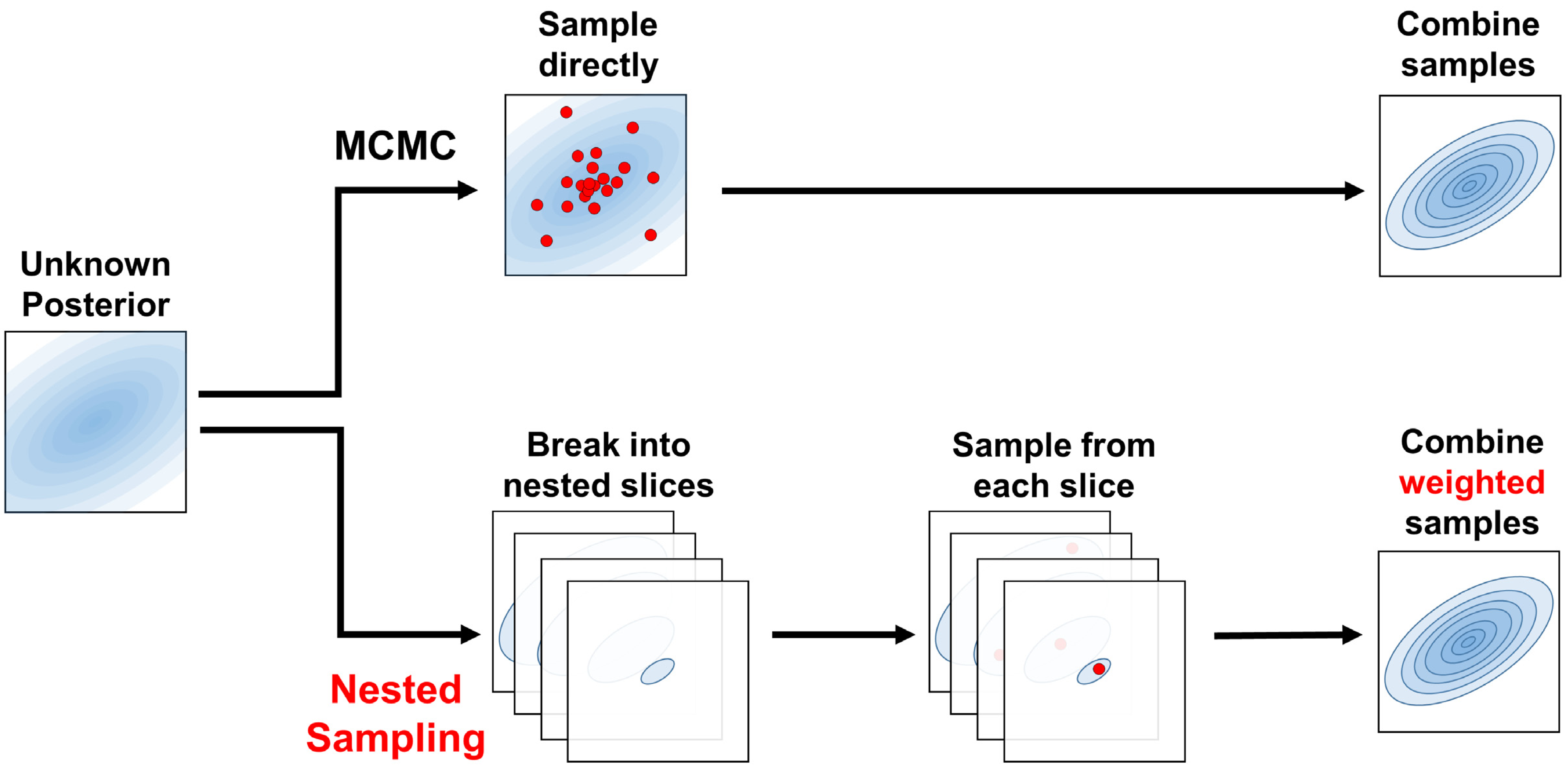
When $D = 15$

$$20^{\mathrm{D}} = 3.2768 \times 10^{19}$$

$$1\ \mathrm{ms} \rightarrow 10^{8}\ \mathrm{yrs}$$

Speagle '2020

Bayesian computation method



Speagle '2019

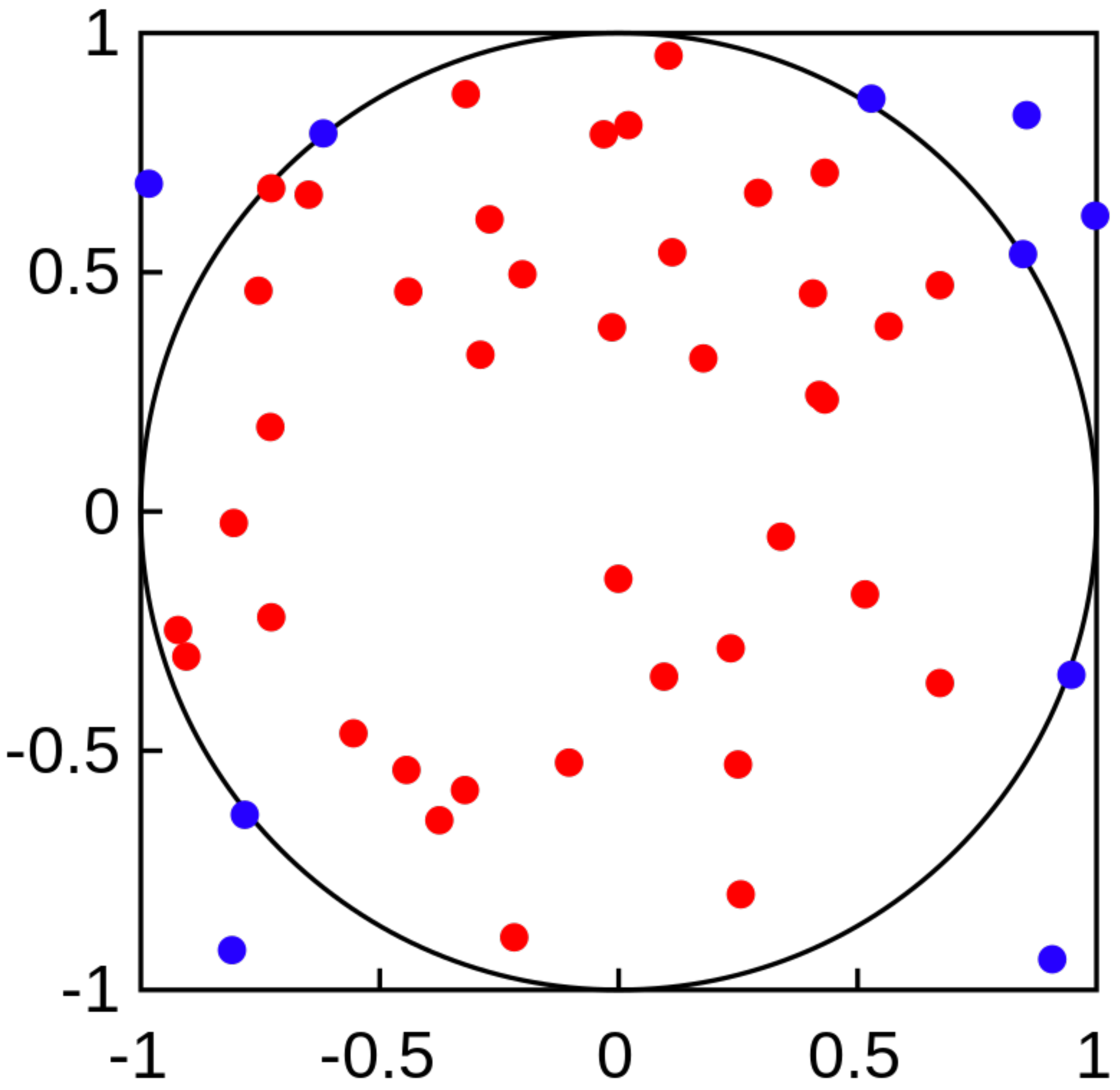
The Best of the 20th Century: Editors Name Top 10 Algorithms

Markov Chain Monte Carlo

Prometheus in high-dimensional integral

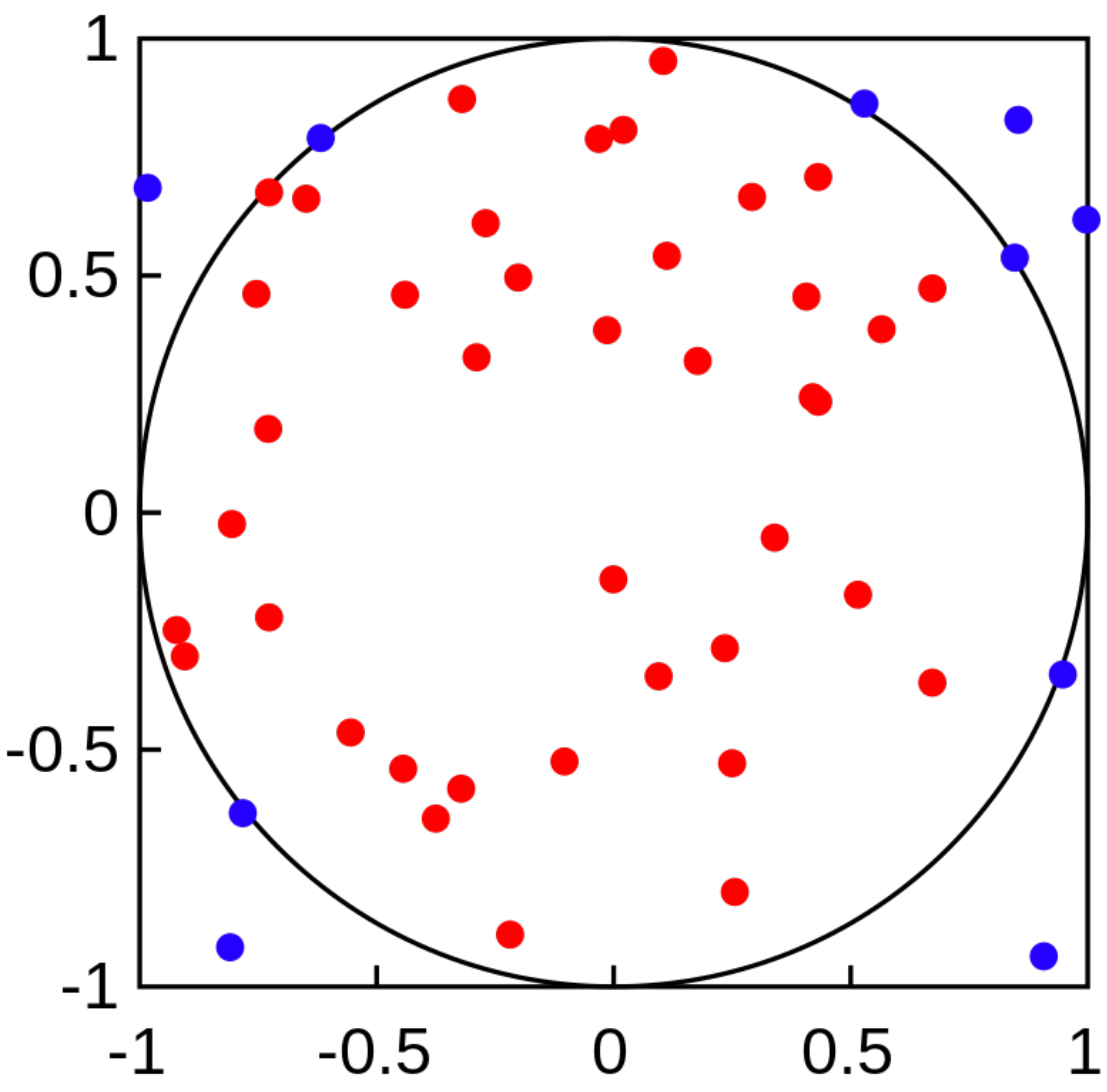
Save Bayesian statistics

Monte Carlo methods



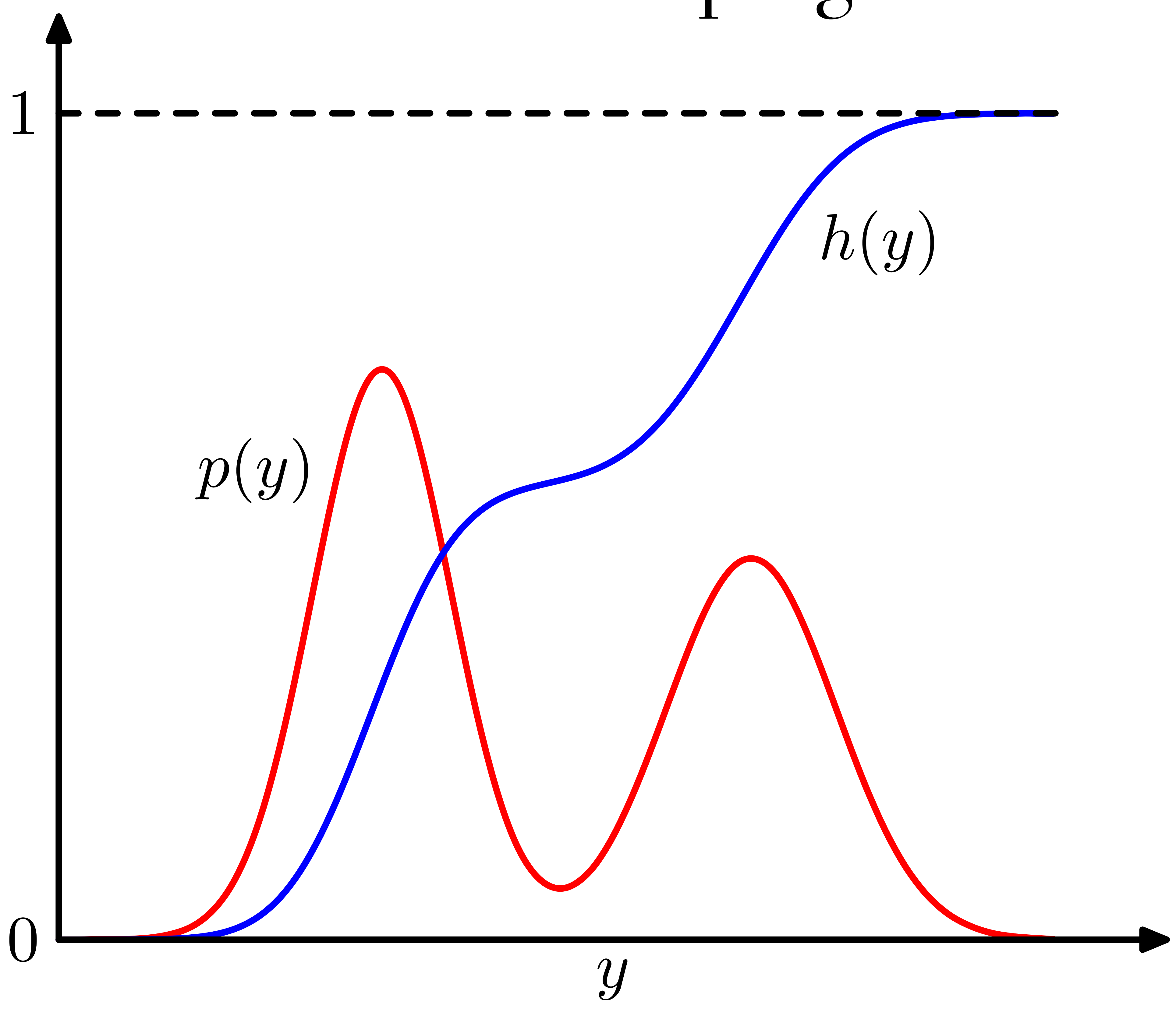
Monte Carlo integration

Monte Carlo methods: Basic sampling



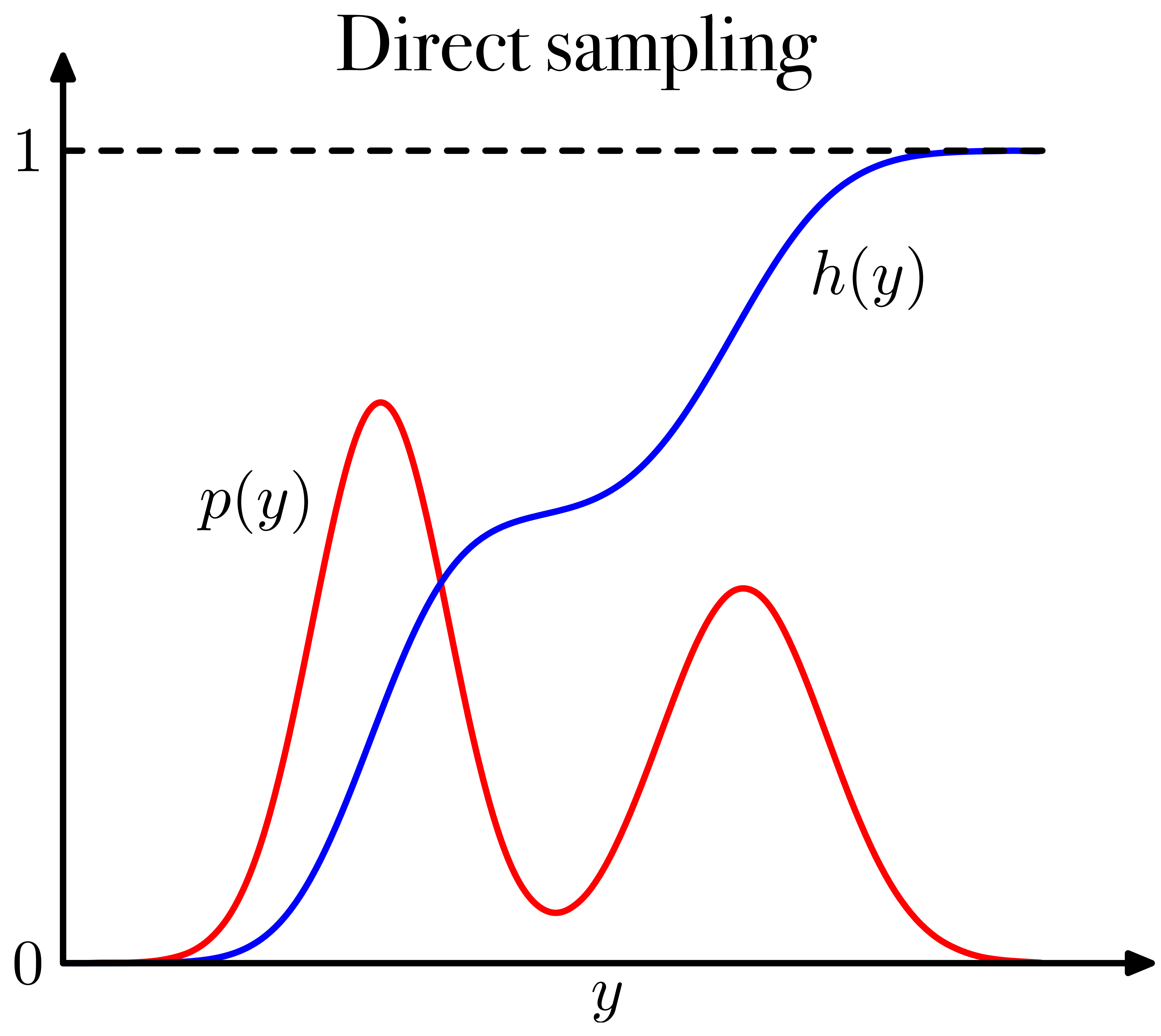
Monte Carlo integration

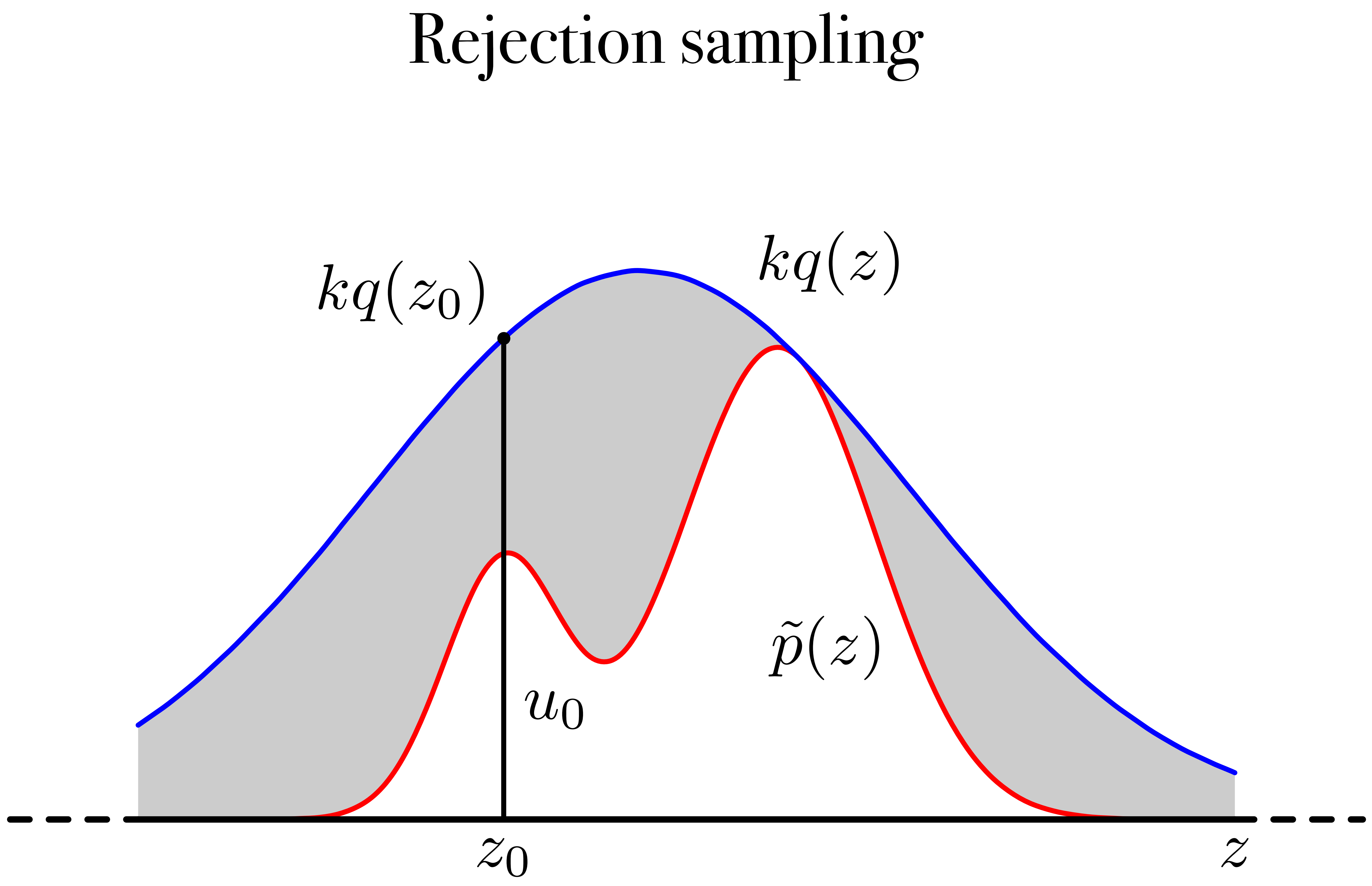
Direct sampling



Bishop '2006

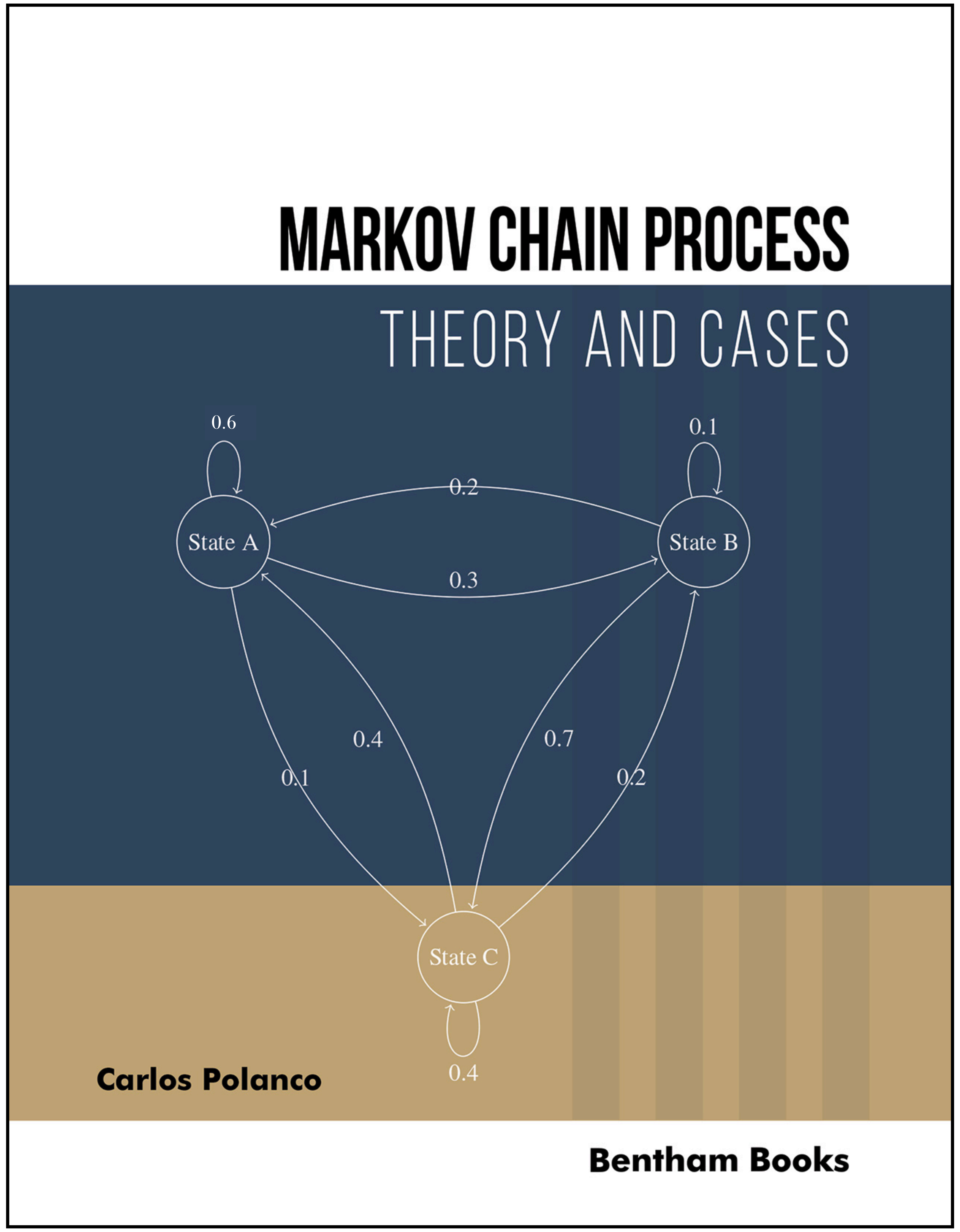
Monte Carlo methods: Basic sampling





Bishop '2006

Markov Chain

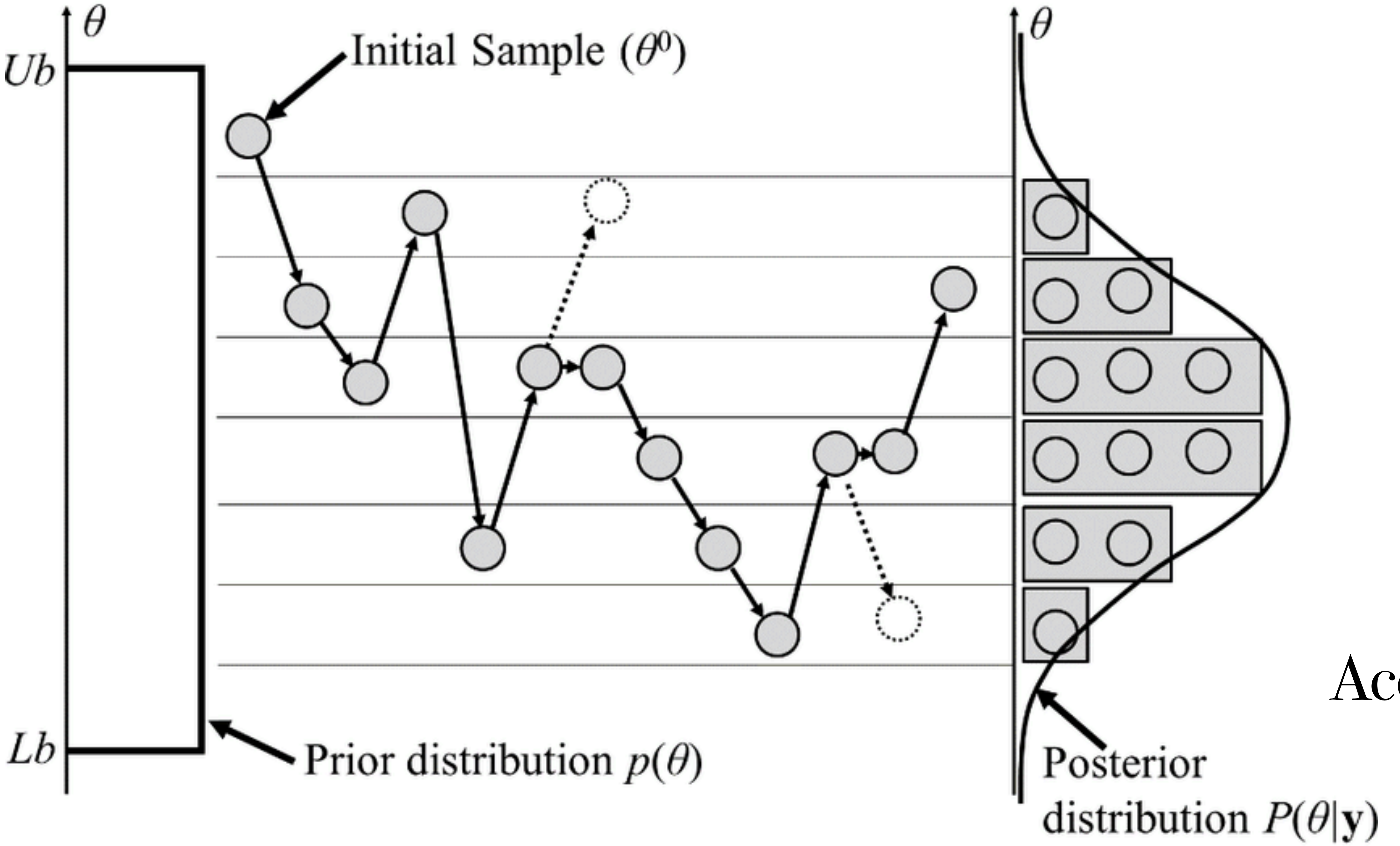


Three states: A (lord), B (soldier), C (farmer)

$$P = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.1 & 0.7 \\ 0.4 & 0.2 & 0.4 \end{bmatrix}$$

$$P(X^{t+1} \mid X^1, \ldots, X^t) = P(X^{t+1} \mid X^t)\,.$$

Markov Chain Monte Carlo (MCMC)



Detailed Balance Condition

$$\pi(i)P(i,j) = \pi(j)P(j,i)$$

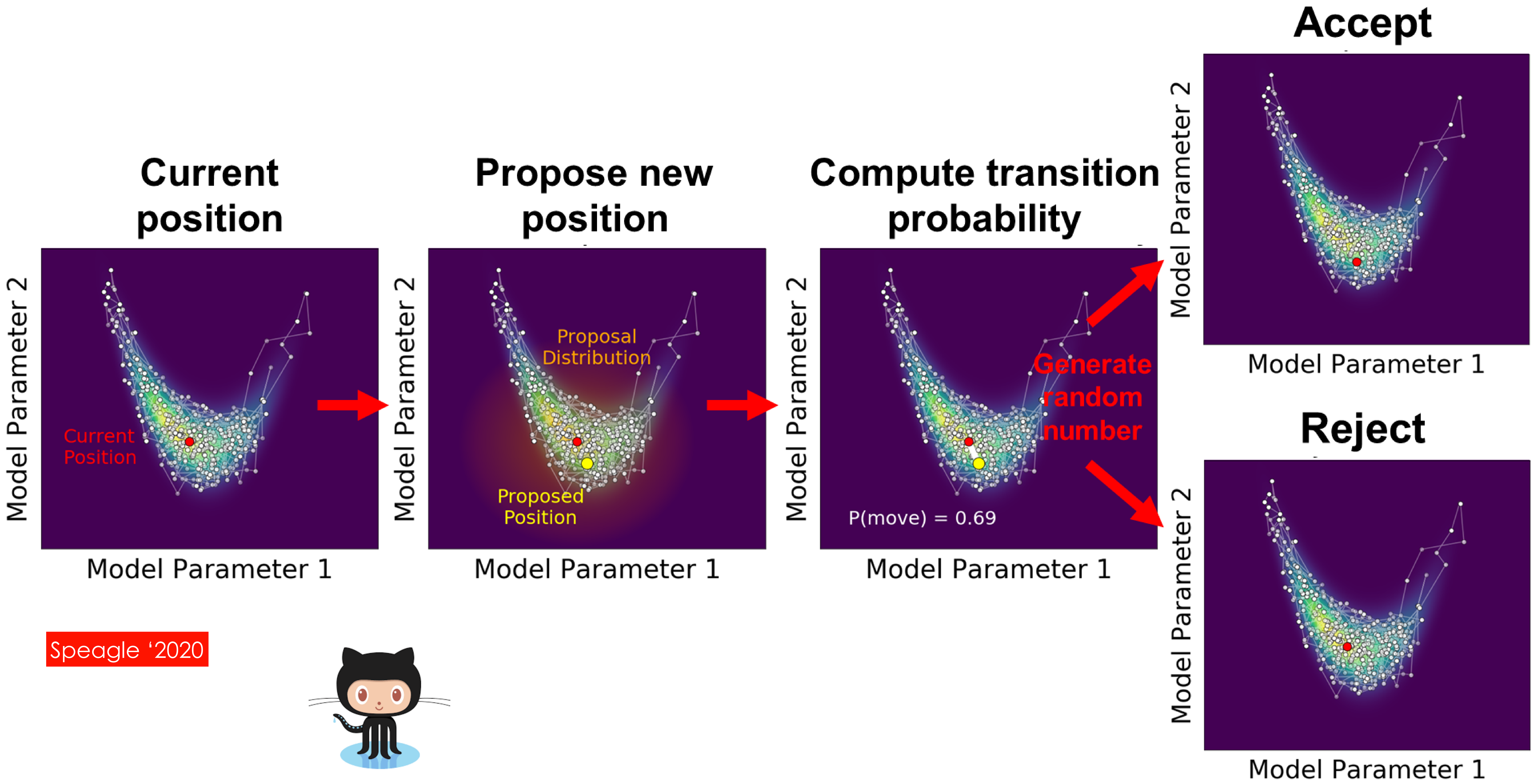
$$\pi(i)Q(i,j)\alpha(i,j) = \pi(j)Q(j,i)\alpha(j,i)$$

$$P(i,j) = Q(i,j)\alpha(i,j)$$

Metropolis-Hastings sampling

Accept rate $\alpha(i,j) = \min\left\{\frac{\pi(j)Q(j,i)}{\pi(i)Q(i,j)}, 1\right\}$

Metropolis-Hastings algorithm: detailed balance



Speagle '2020

Limitations of MCMC

For how many iterations should we run Markov chain Monte Carlo?

Charles C. Margossian

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Andrew Gelman

Department of Statistics and Political Science, Columbia University, New York, NY

"All human wisdom is contained in these two words: wait and hope."

— Alexandre Dumas, *The Count of Monte Cristo*

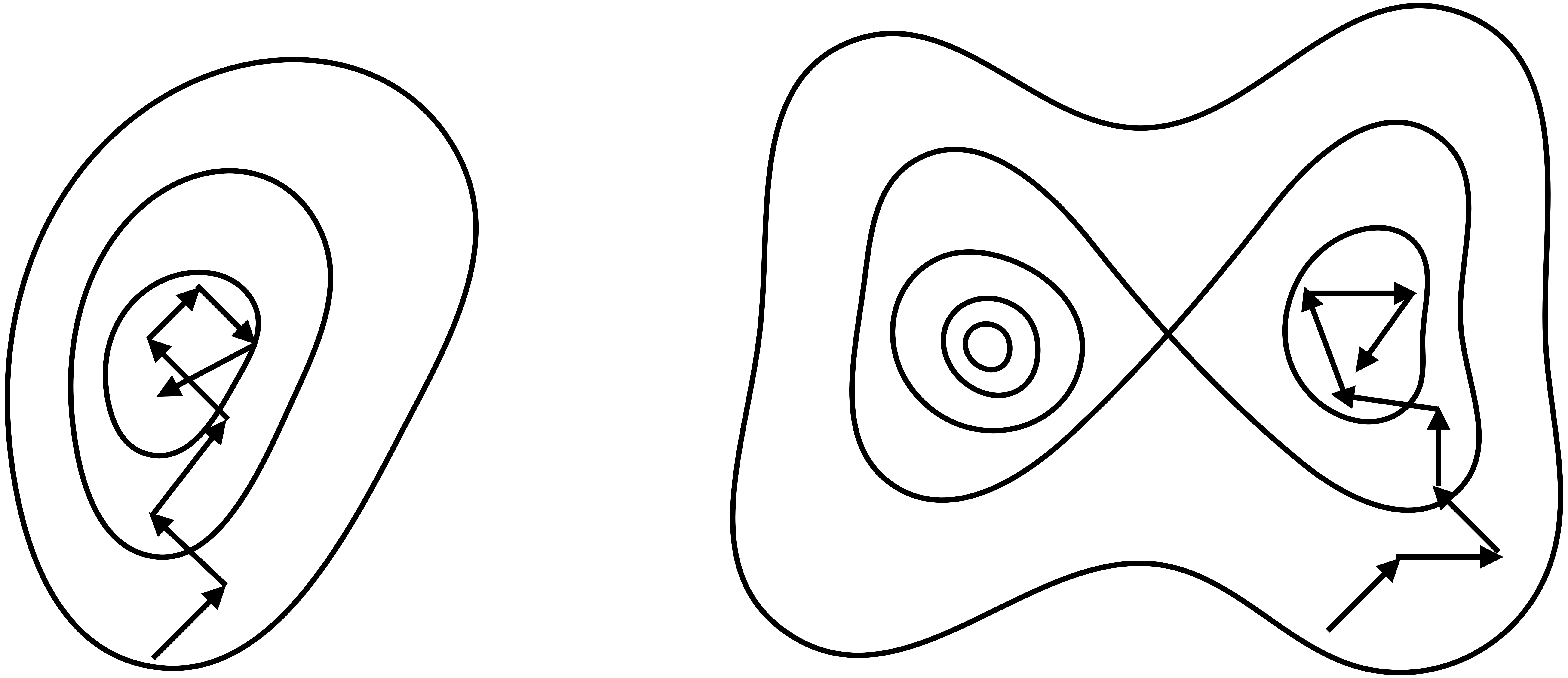
Burn-in & Thin-in

Auto-correlation statistics

Gelman-Rubin statistics

Zeus package: convergence
Emcee package: autocorrelation
Margossian & Gelman '2023
Gelman & Rubin '1992

Multimodal: trapped in local maximum



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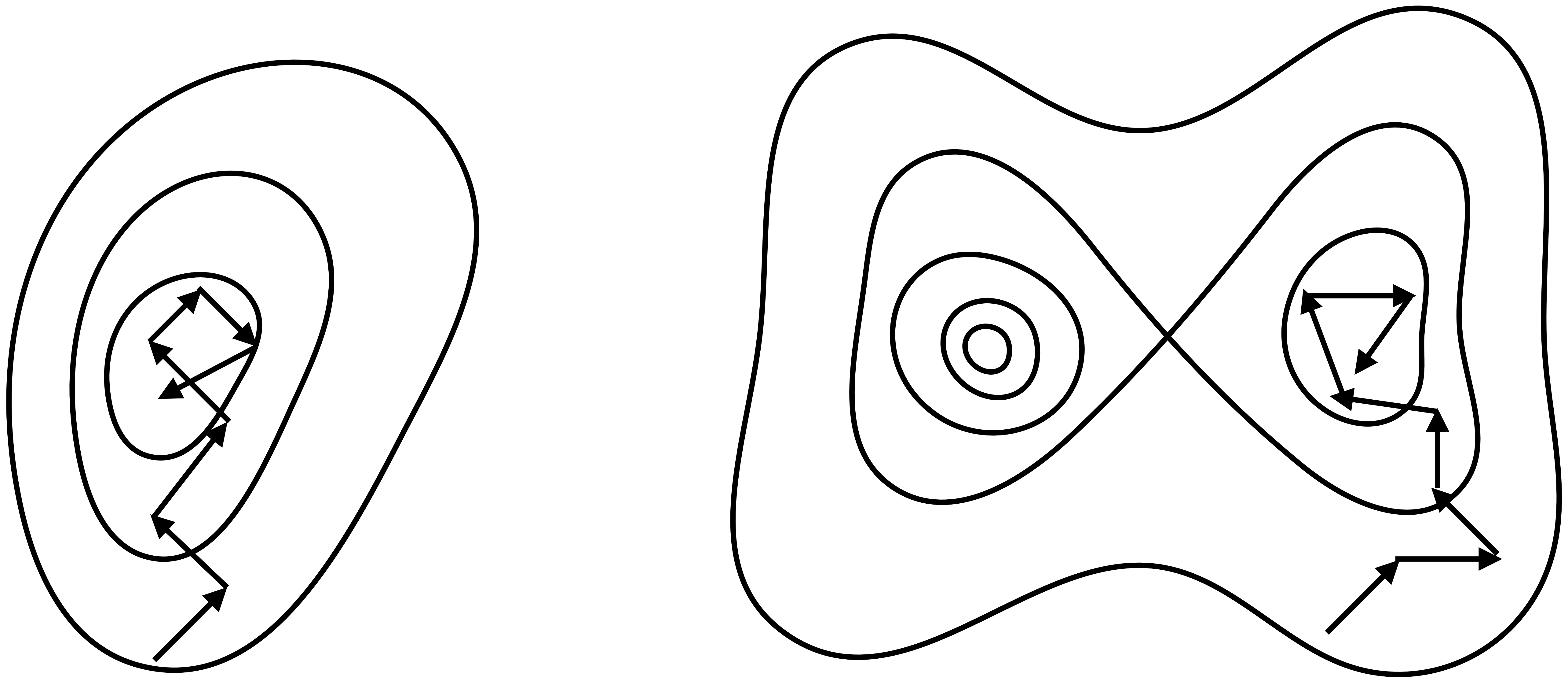


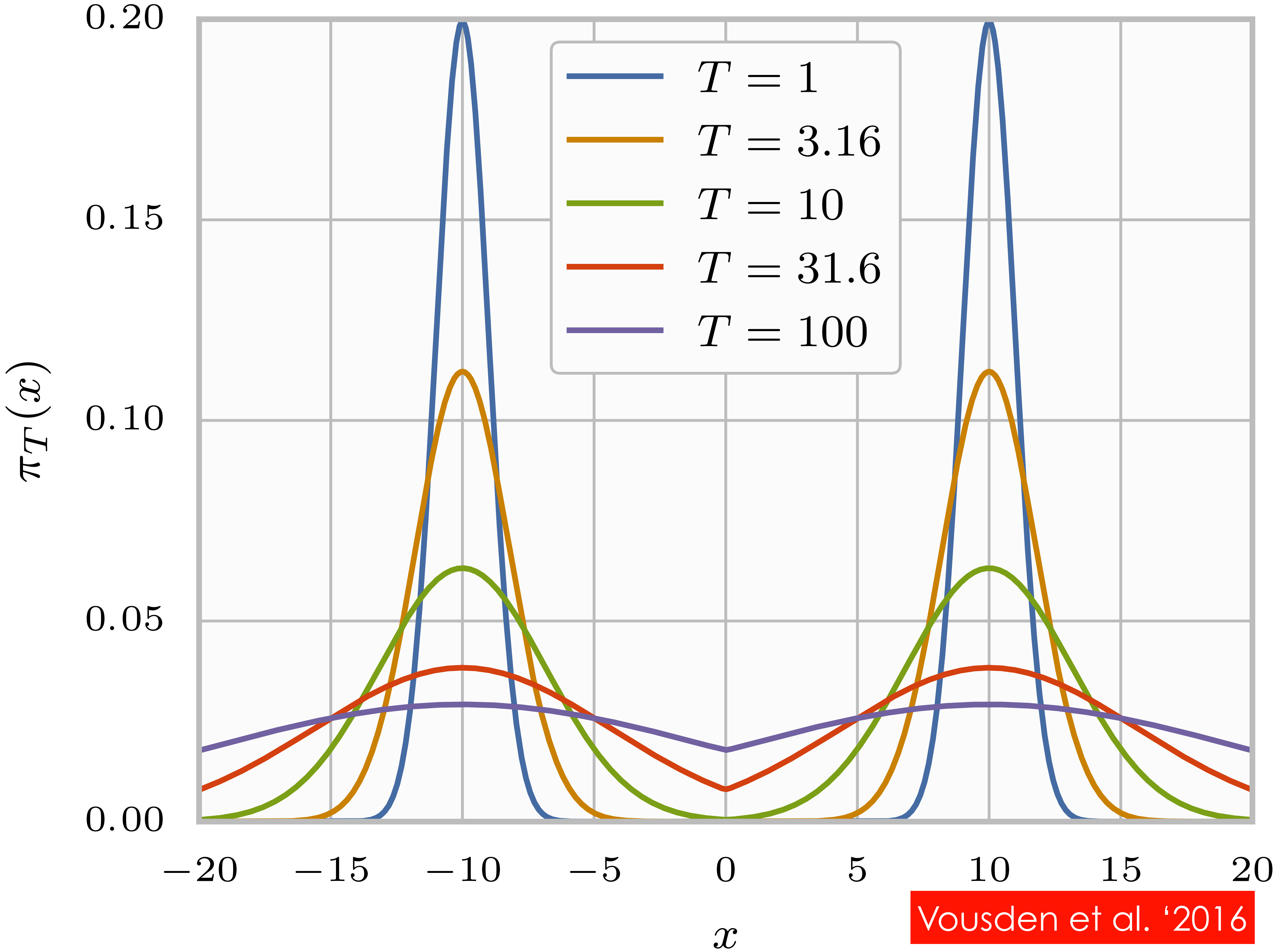
with Stepping-stone sampling algorithm

→ Evidence

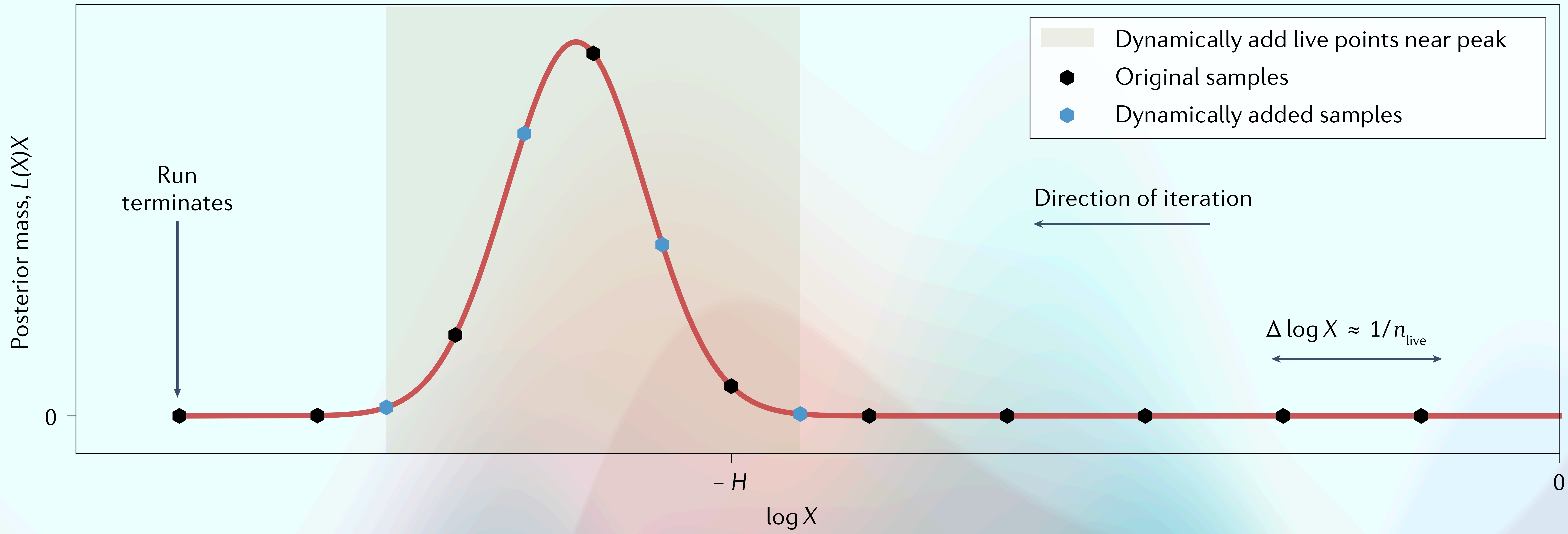
Maturana-Russel et al. '2019

PTMCMC (parallel-tempering):

$$\pi_T(\theta) \propto L(\theta)^{1/T} p(\theta)$$



Vousden et al. '2016



Approximate the evidence by integrating the prior in nested "shells" of iso-likelihood.

Nested sampling method

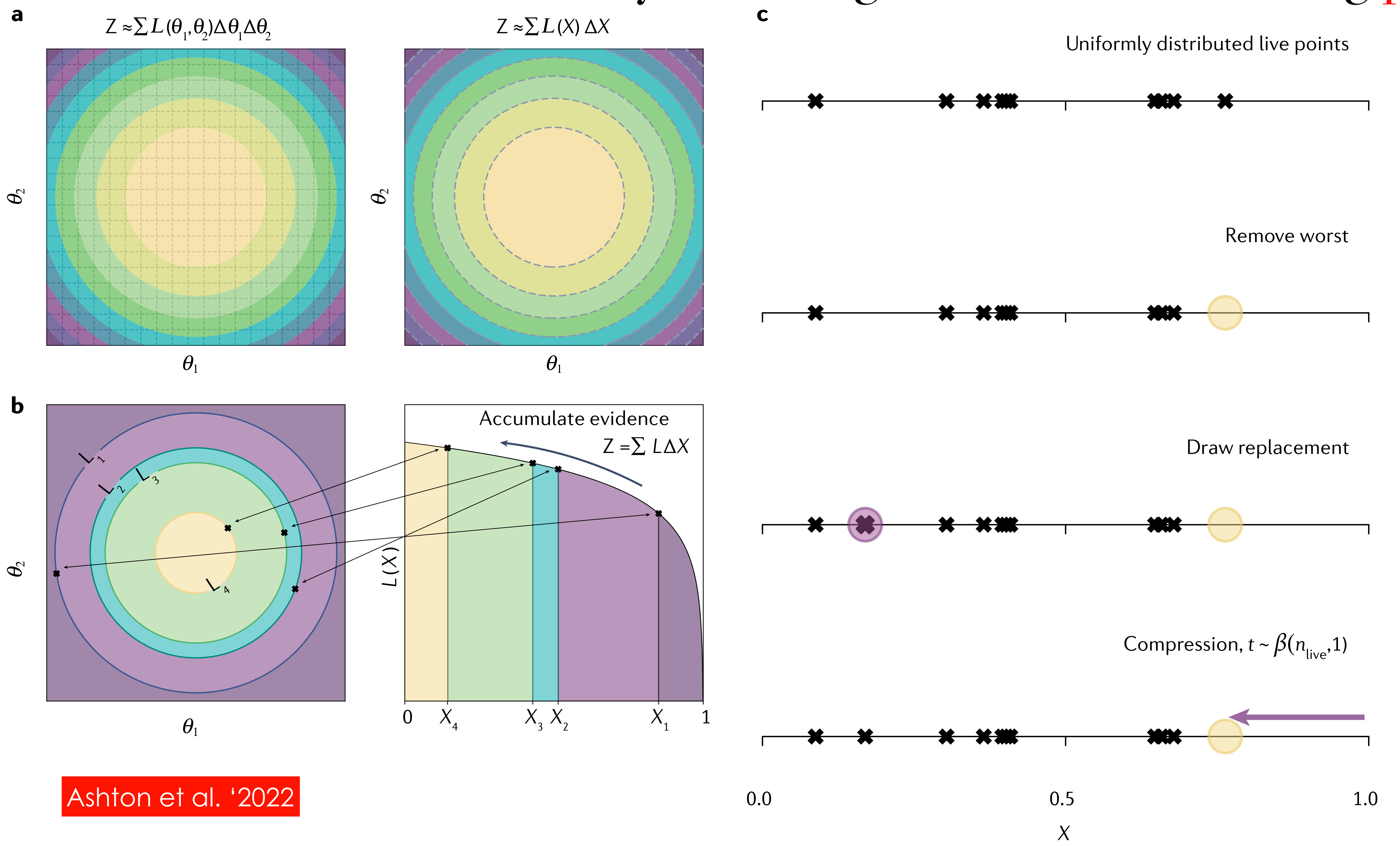
Well-defined stopping criteria

Calculate the evidence

Solve multimodal problem

Nested sampling

Goal of efficiently evaluating evidence and returning posterior estimate



$$\mathcal{Z} = \int \tilde{\mathcal{P}}(\boldsymbol{\Theta})\,\mathrm{d}\boldsymbol{\Theta}$$

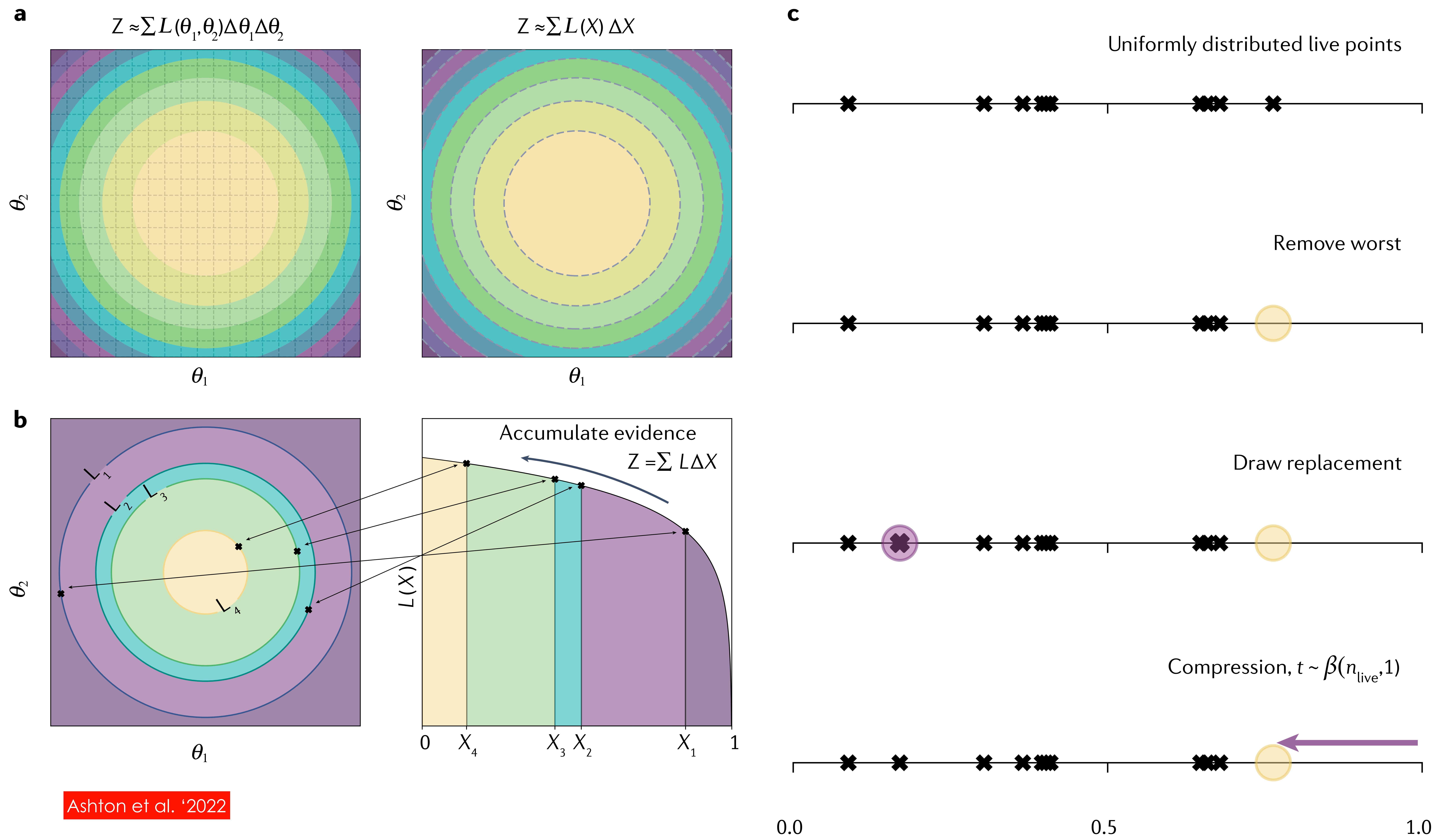
$$= \int_{\Omega_{\boldsymbol{\Theta}}} \mathcal{L}(\boldsymbol{\Theta})\pi(\boldsymbol{\Theta})\,\mathrm{d}^N\boldsymbol{\Theta}$$

$$= \int L(X)\,\mathrm{d}X$$

$$\approx \sum_{i=1}^{n} L\left(X_i\right)\Delta X_i$$

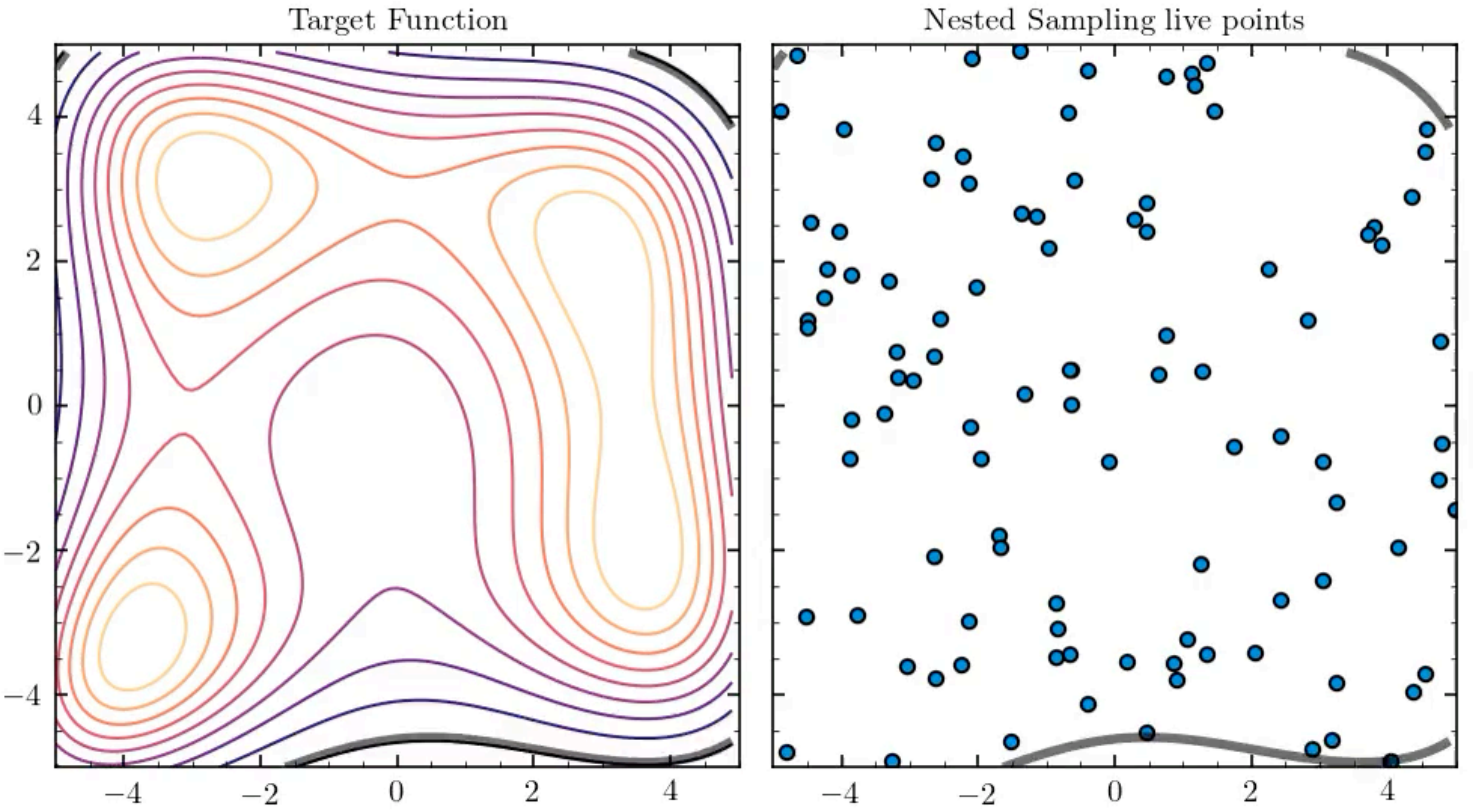
$$X(\lambda) = \int_{\{\boldsymbol{\Theta}:\mathcal{L}(\boldsymbol{\Theta})>\lambda\}} \pi(\boldsymbol{\Theta})\,\mathrm{d}\boldsymbol{\Theta}$$

Ashton et al. ‘2022



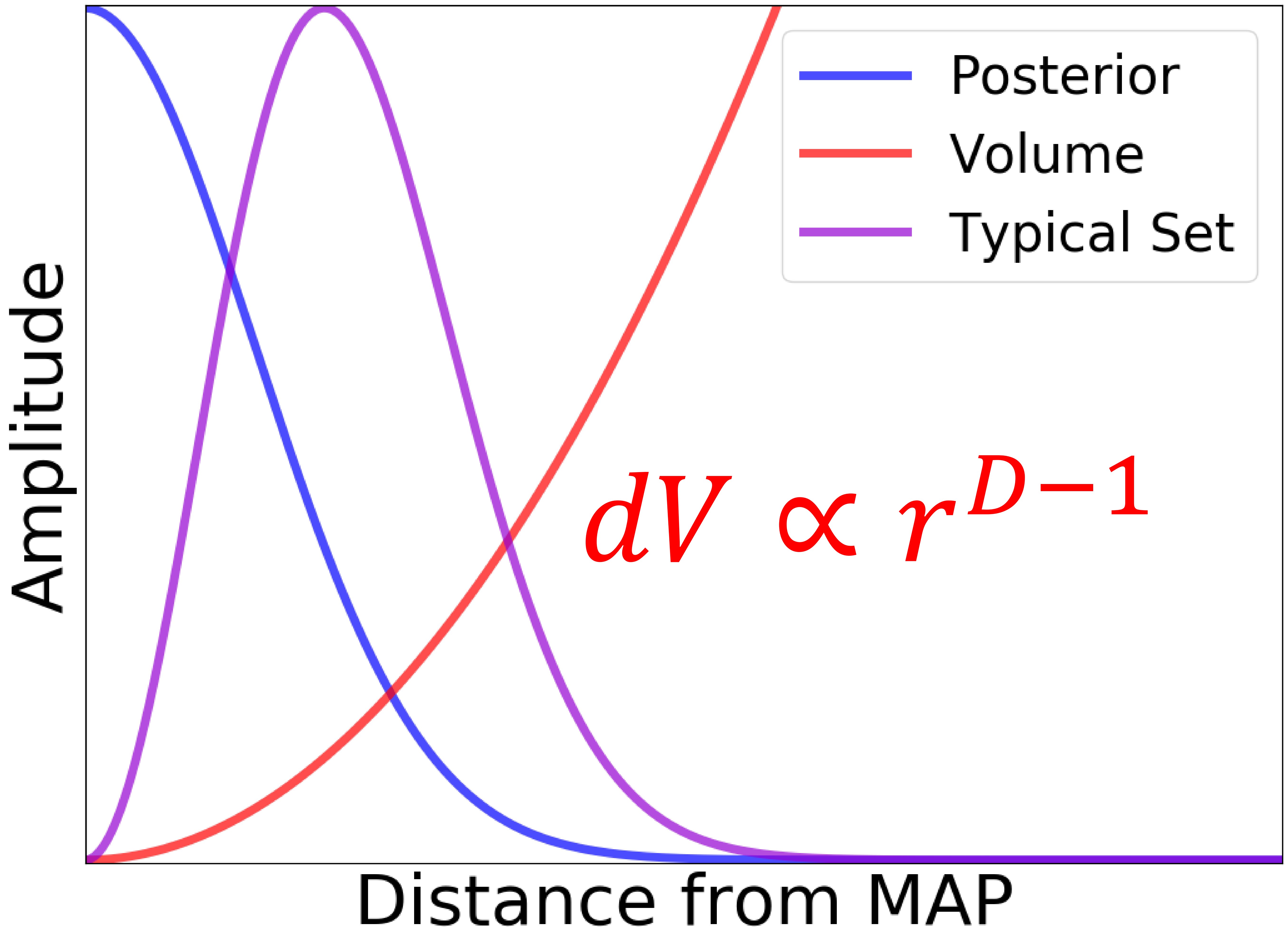
Ashton et al. ‘2022

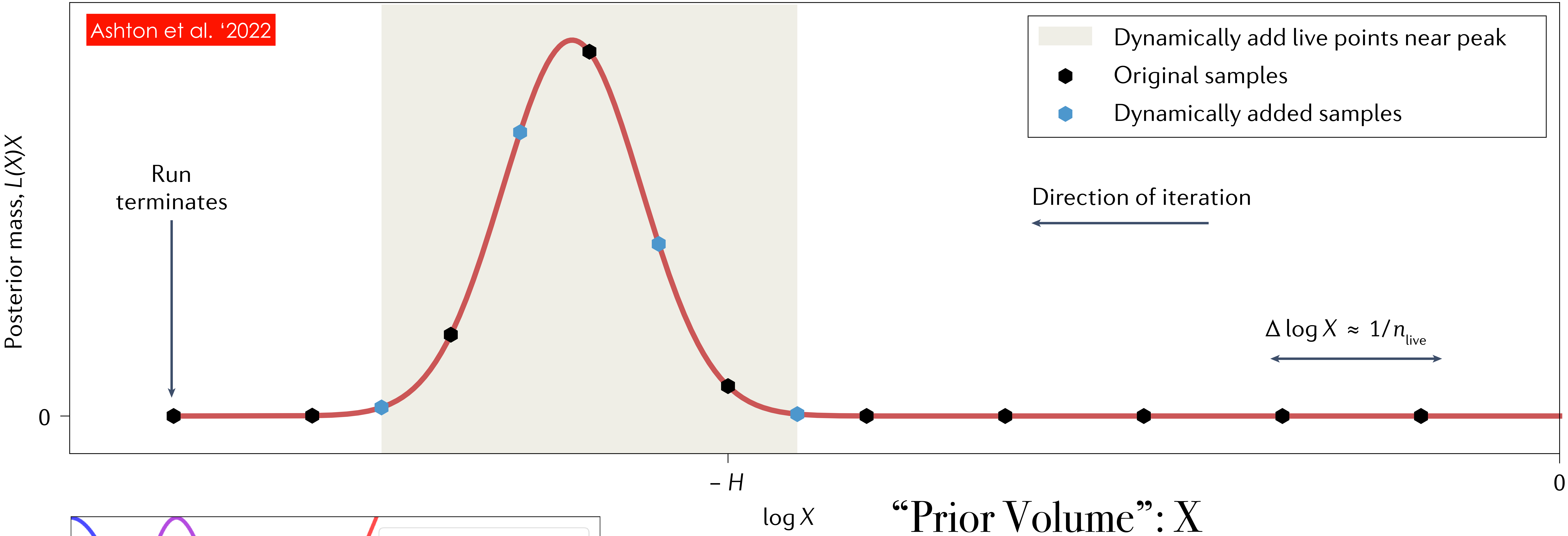
Nested sampling



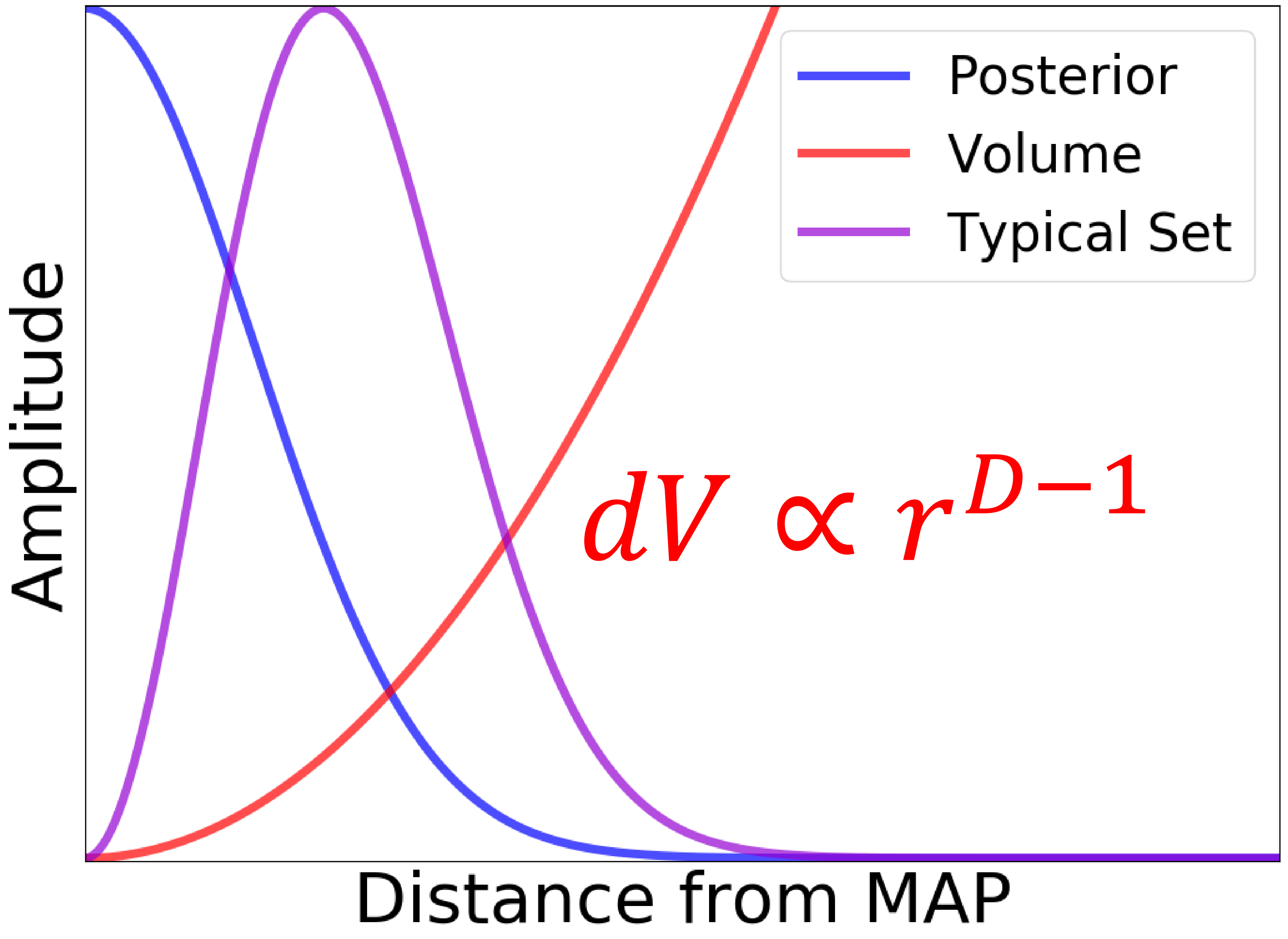
From https://polychord.io/

Posterior mass, $L(X)X$



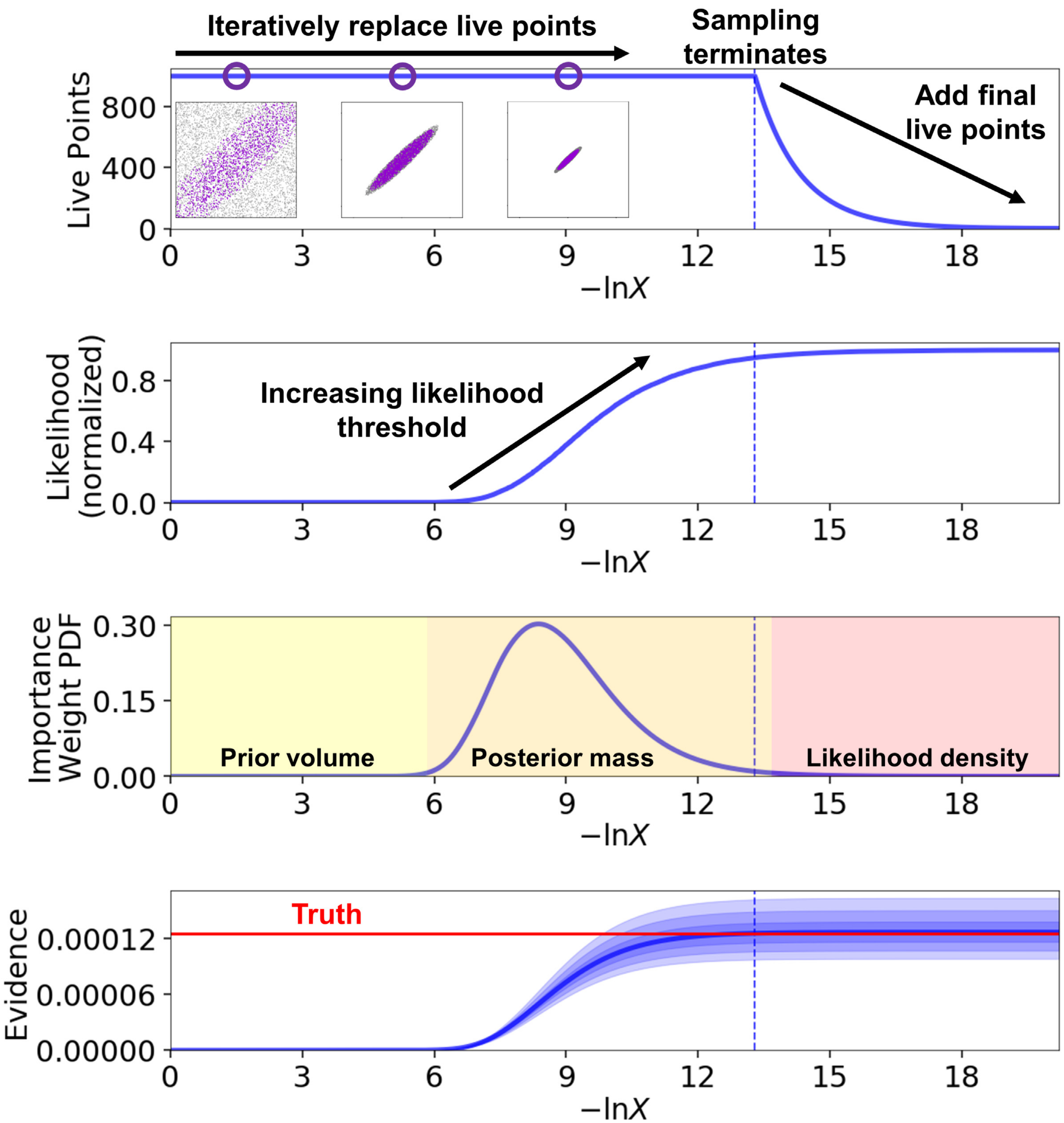


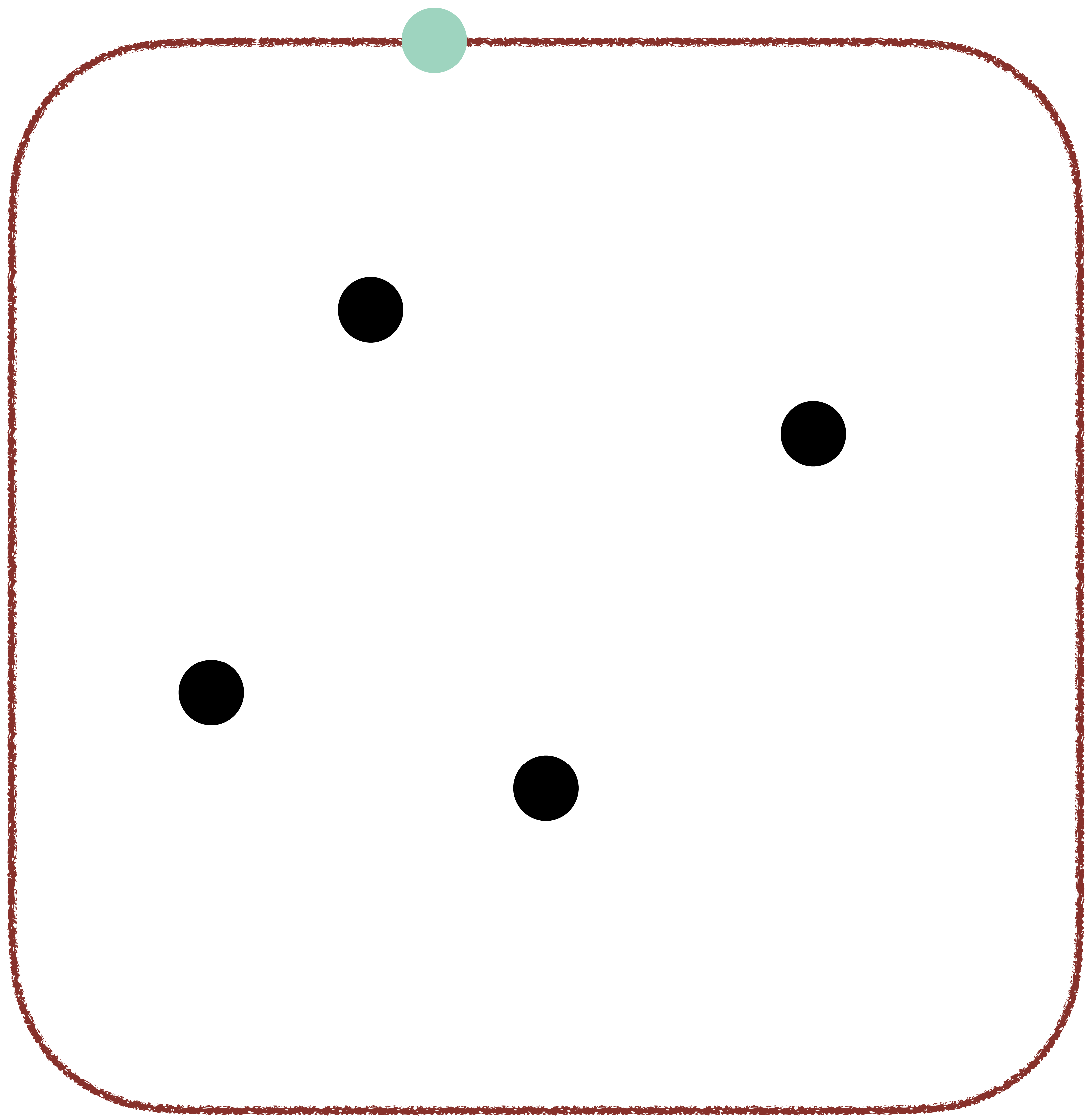
"Prior Volume": X

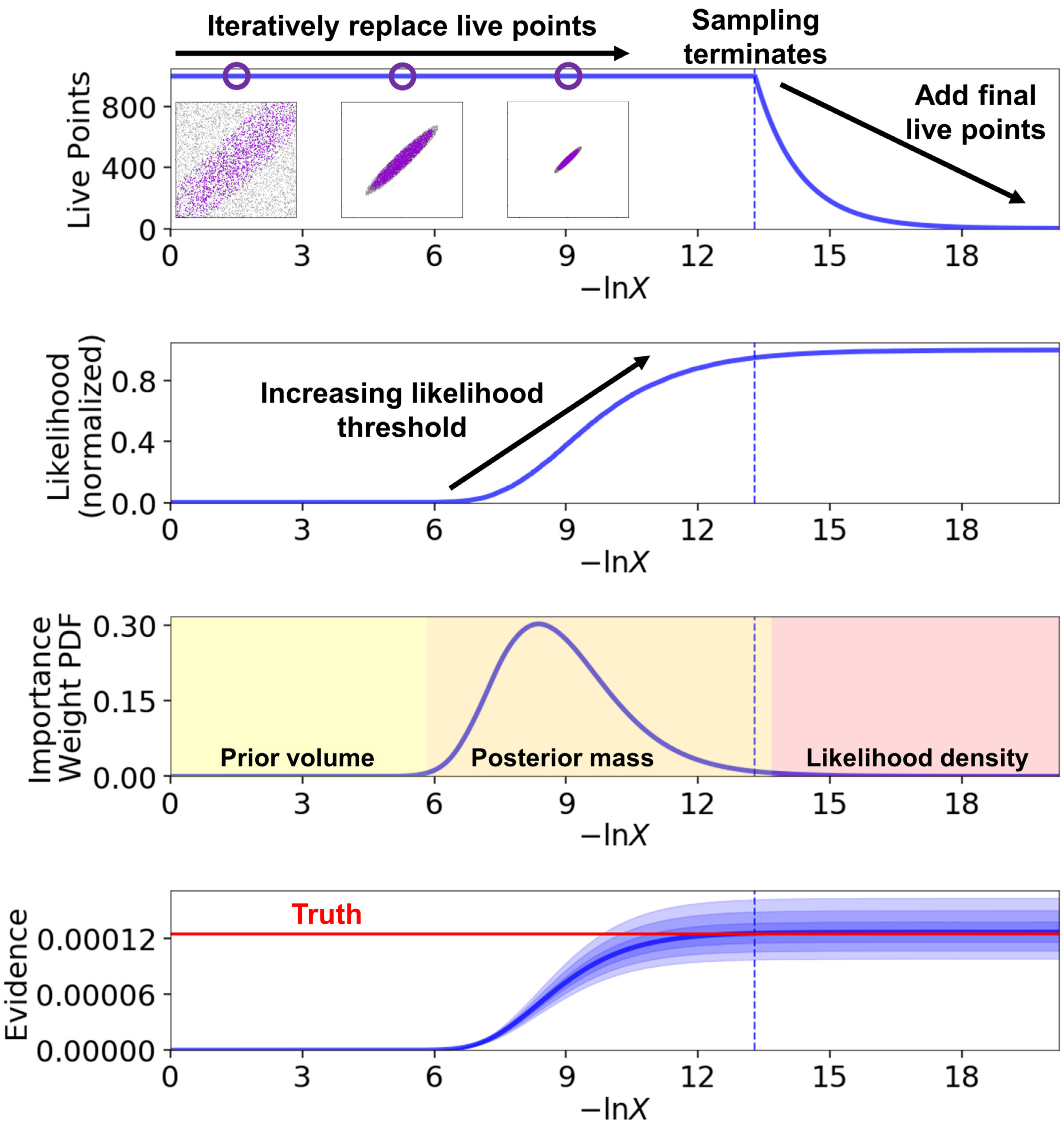


K-L divergence $$H = \int P(\Theta) \log\left(\frac{P(\Theta)}{\pi(\Theta)}\right) \mathrm{d}\Theta$$

$$\Delta \log Z \approx \sqrt{\frac{H}{n_{\mathrm{live}}}}$$


Iteratively replace live points
Sampling terminates
Add final live points
Live Points
800
400
0
0
3
6
9
12
15
18
−lnX
Likelihood (normalized)
0.8
0.4
0.0
Increasing likelihood threshold
Importance Weight PDF
0.30
0.15
0.00
Prior volume
Posterior mass
Likelihood density
Evidence
0.00012
0.00006
0.00000
Truth





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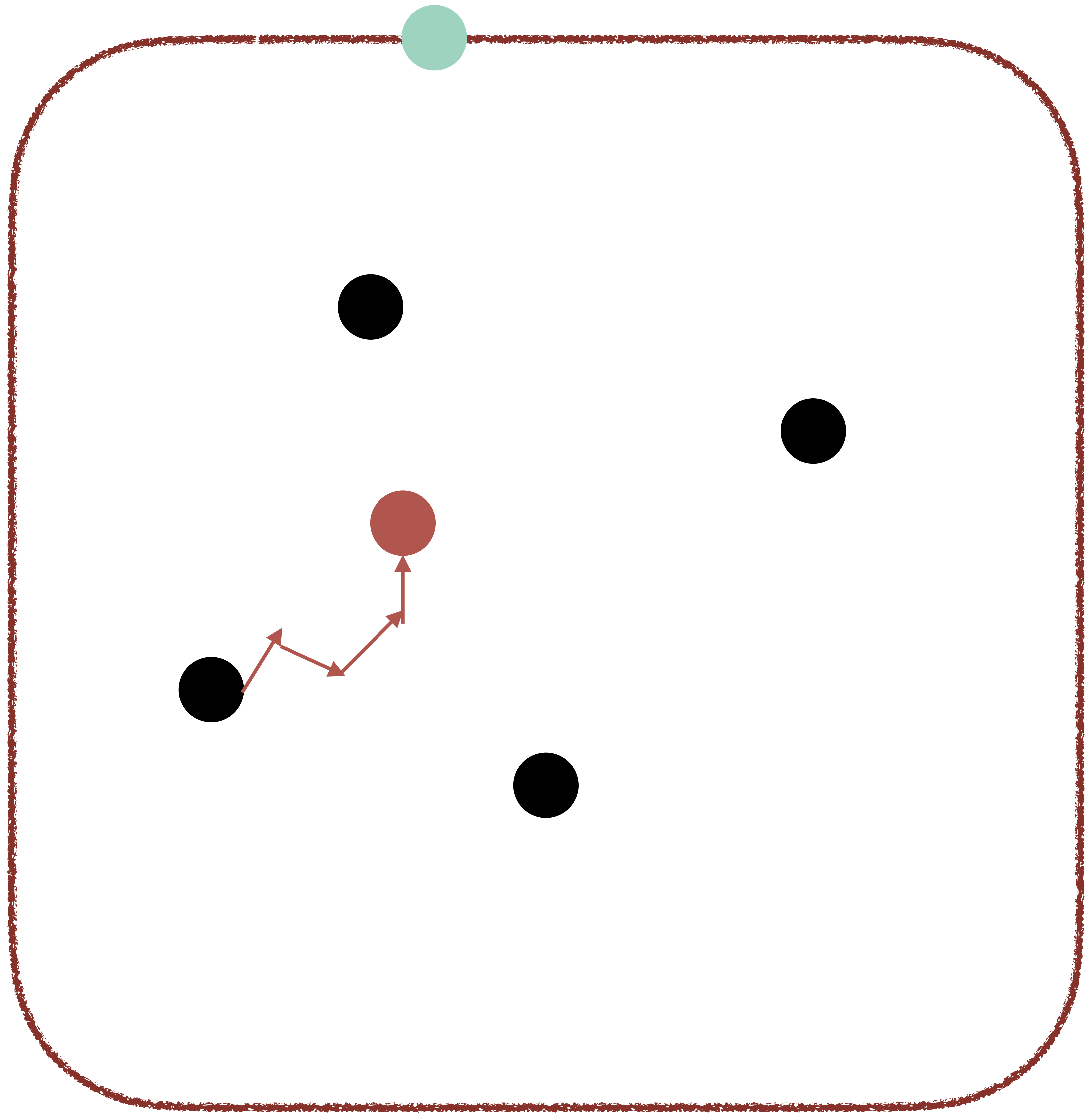

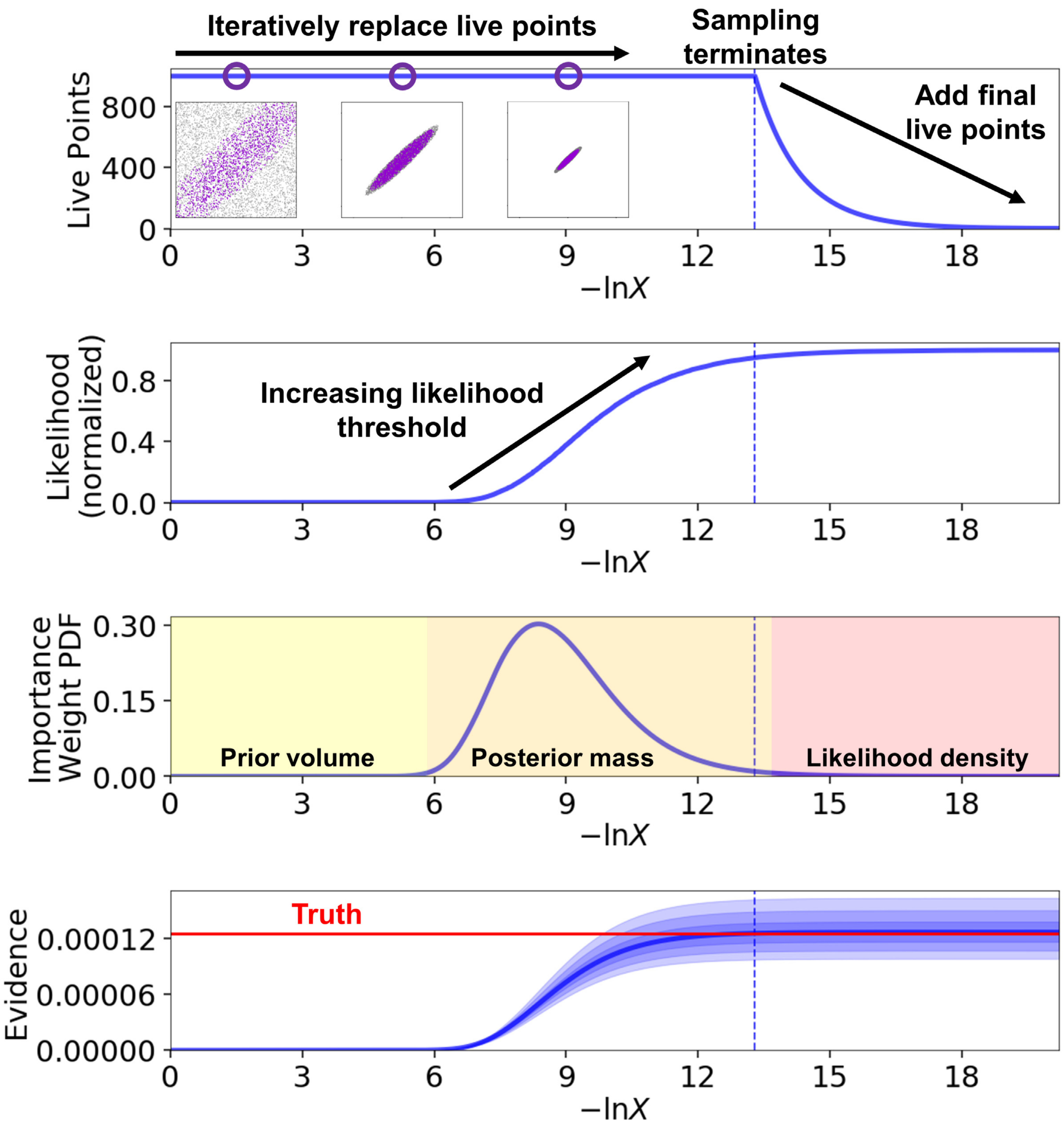
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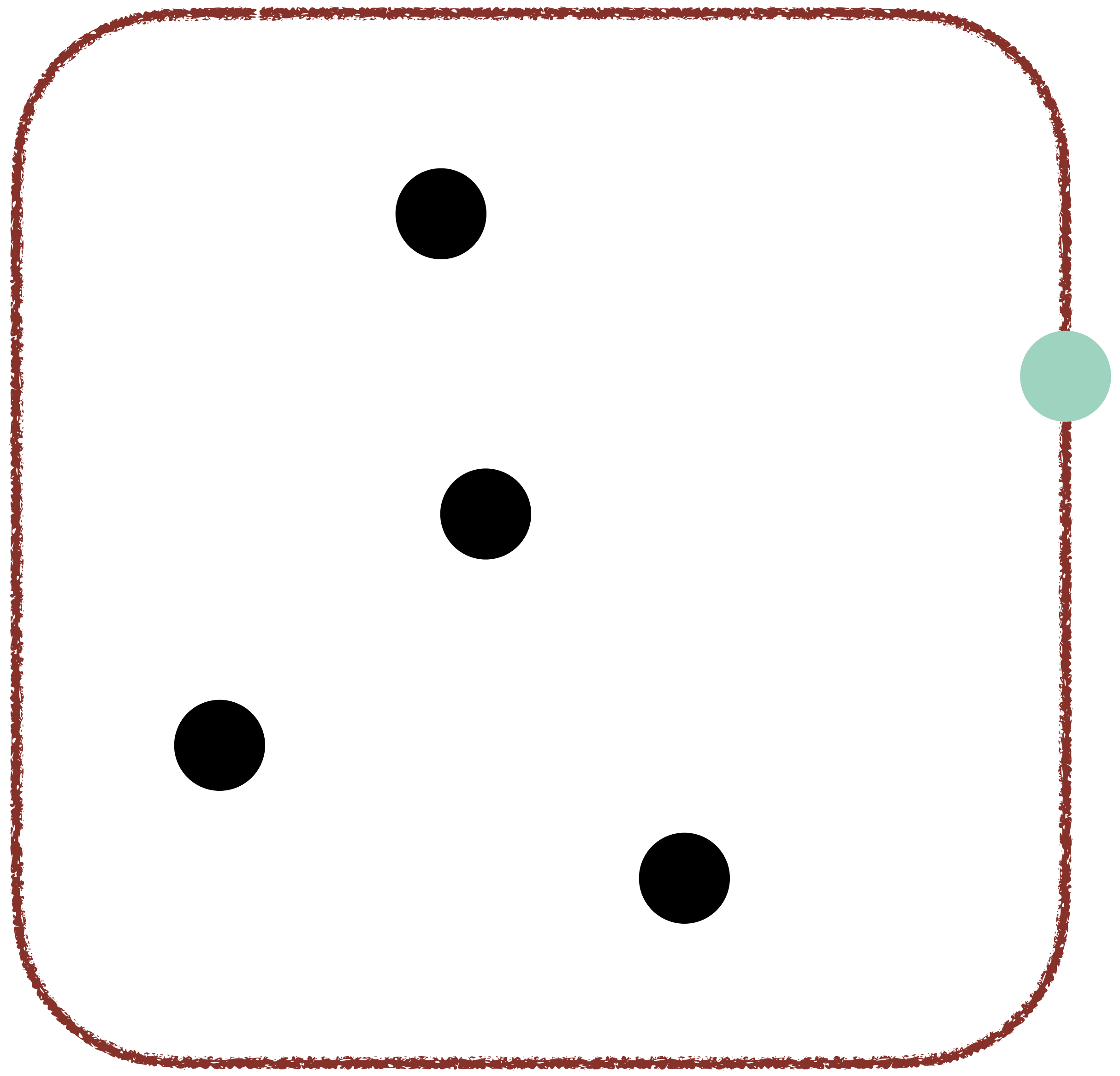


Bilby

A user-friendly Bayesian inference library. Fulfilling all your Bayesian dreams.

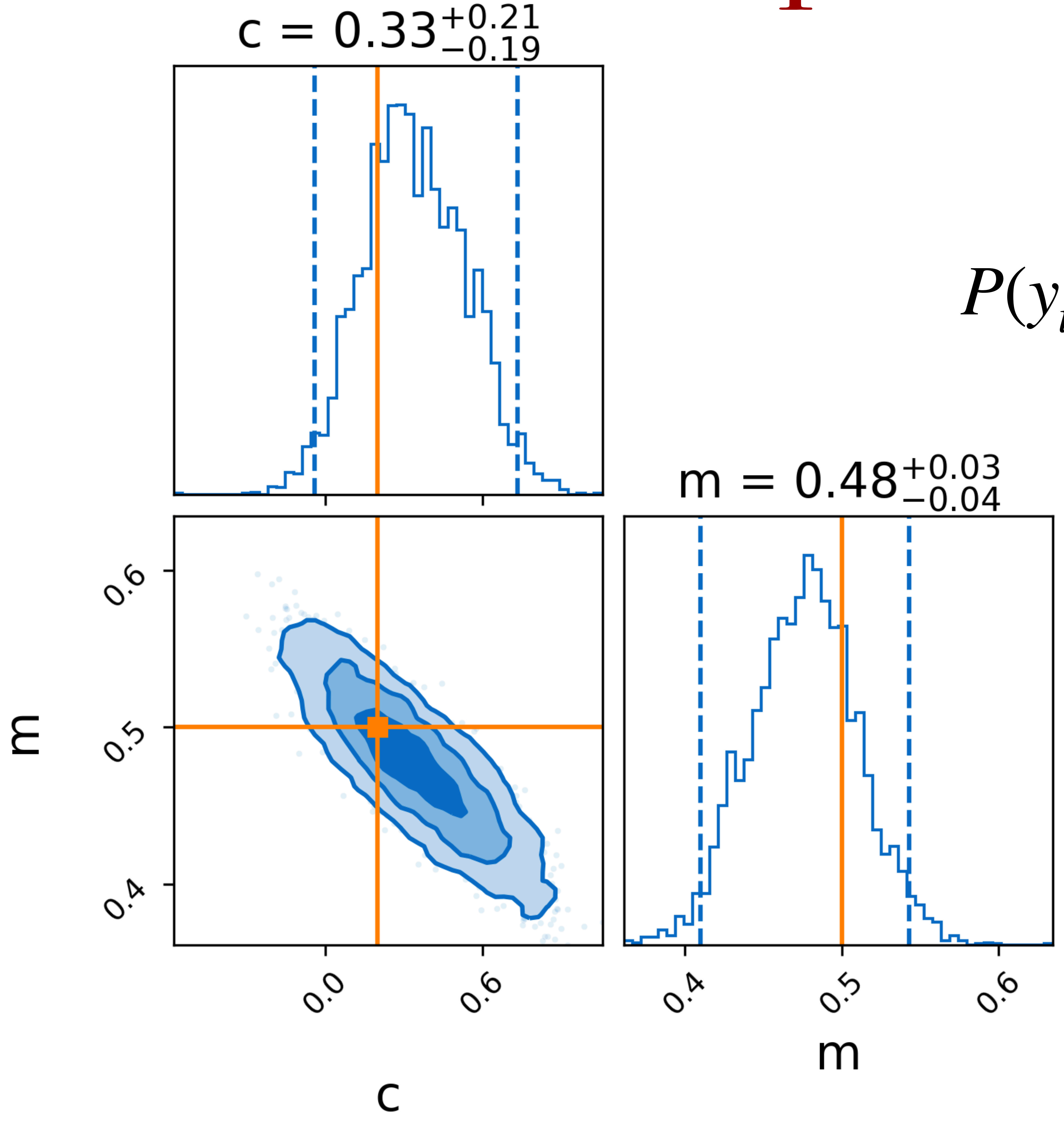
Nested sampling family

- dynesty
- UltraNest
- nestle
- pymultinest
- cpnest
- nessai
- pypolychord

MCMC family

- emcee
- ptemcee
- bilby-mcmc
- zeus
- ptmcmcsampler
- PyMC3

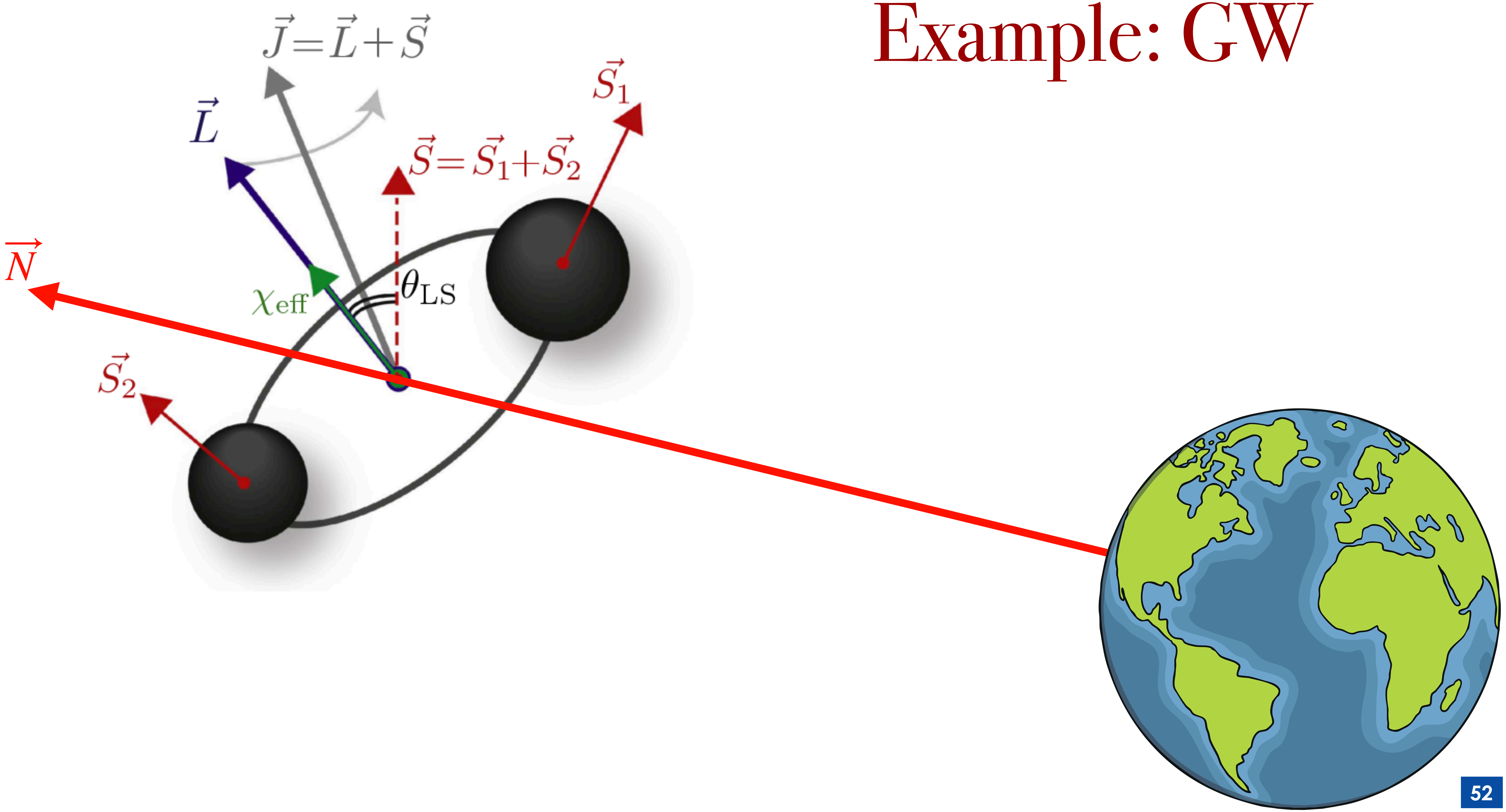
Example: linear regression



$$P(y_i, t_i \,|\, m, c, \mathscr{H}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(y_i - (t_i m + c))^2}{2\sigma^2} \right)$$

https://lscsoft.docs.ligo.org/bilby/basics-of-parameter-estimation.html

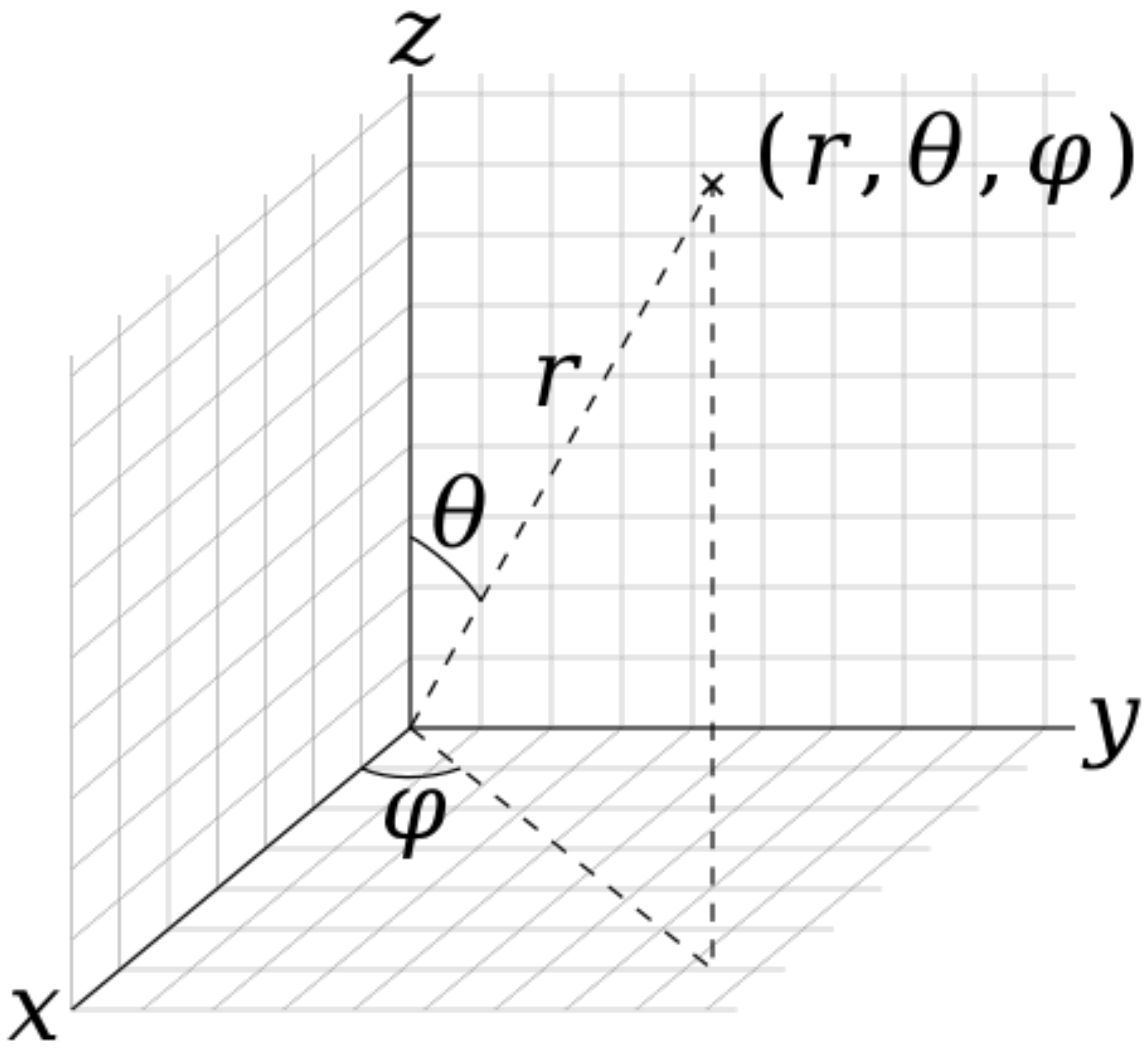
Example: GW



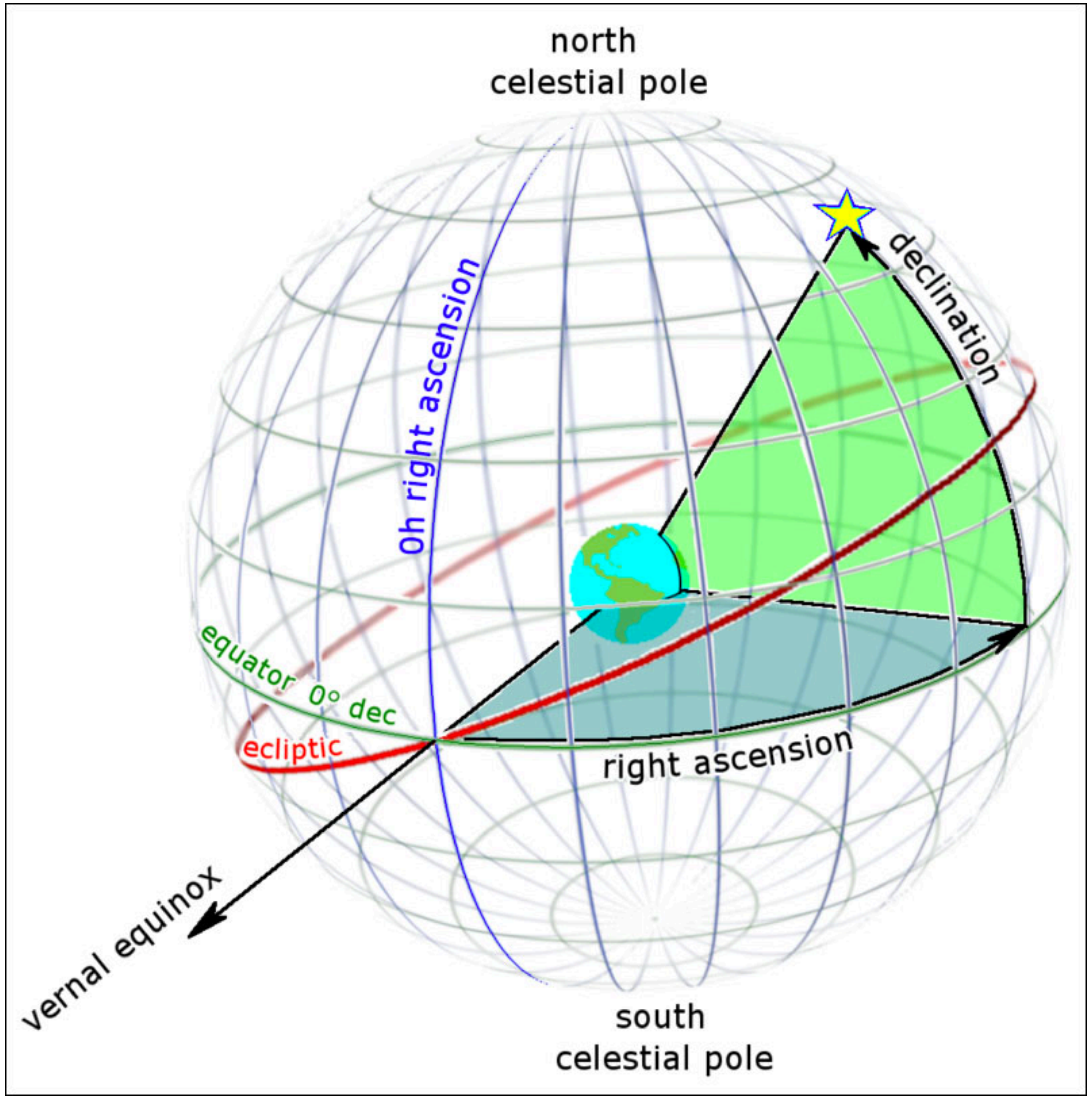
$\vec{J} = \vec{L} + \vec{S}$
$\vec{L}$
$\vec{S_1}$
$\vec{S} = \vec{S_1} + \vec{S_2}$
$\overrightarrow{N}$
$\chi_{\rm eff}$
$\theta_{\rm LS}$
$\vec{S_2}$

Priors for spherical coordinates

$$\mathrm{d}V = \mathrm{d}r \cdot r\mathrm{d}\theta \cdot r\sin\theta\mathrm{d}\varphi$$



Priors for RA, Dec, distance



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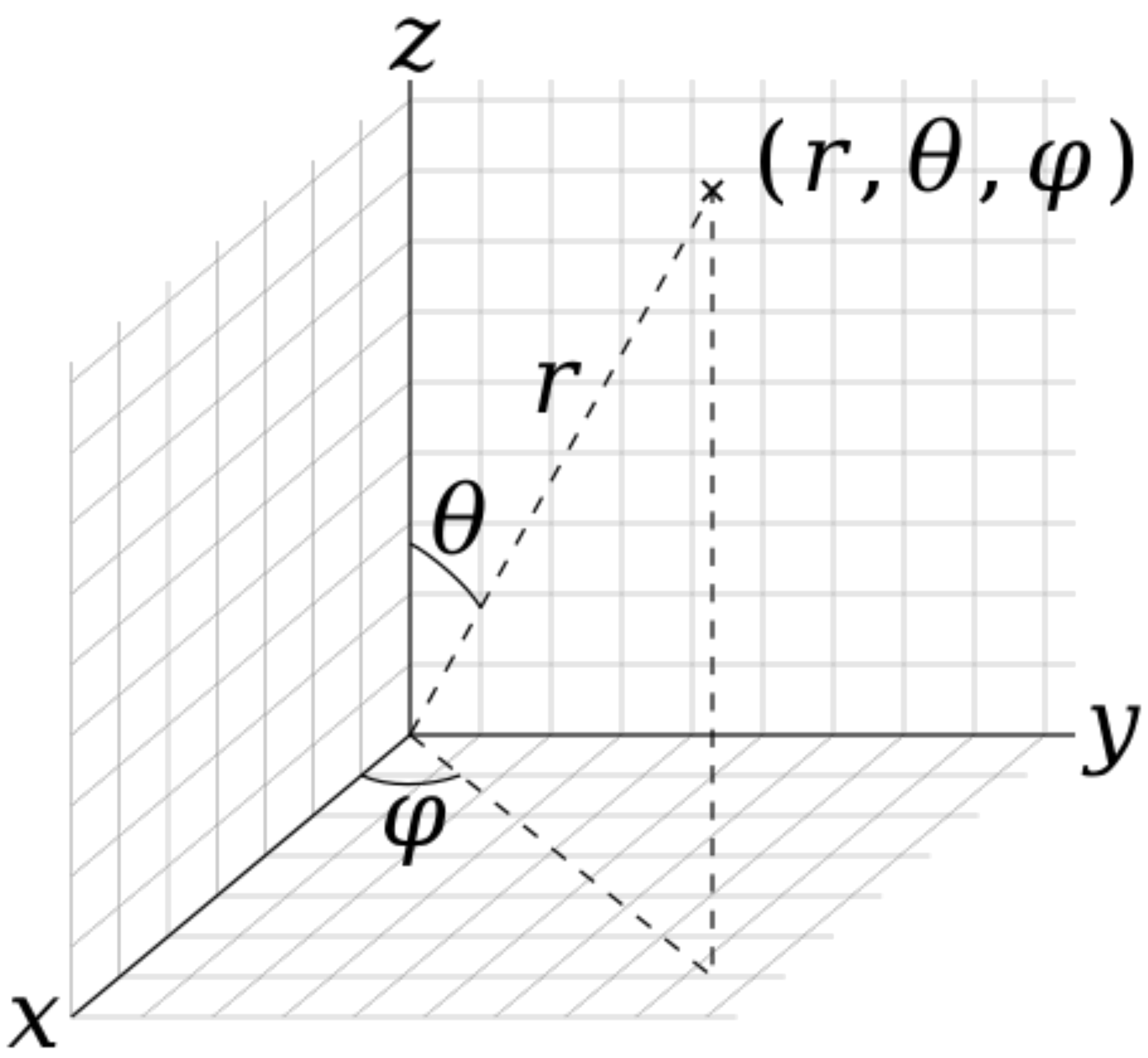
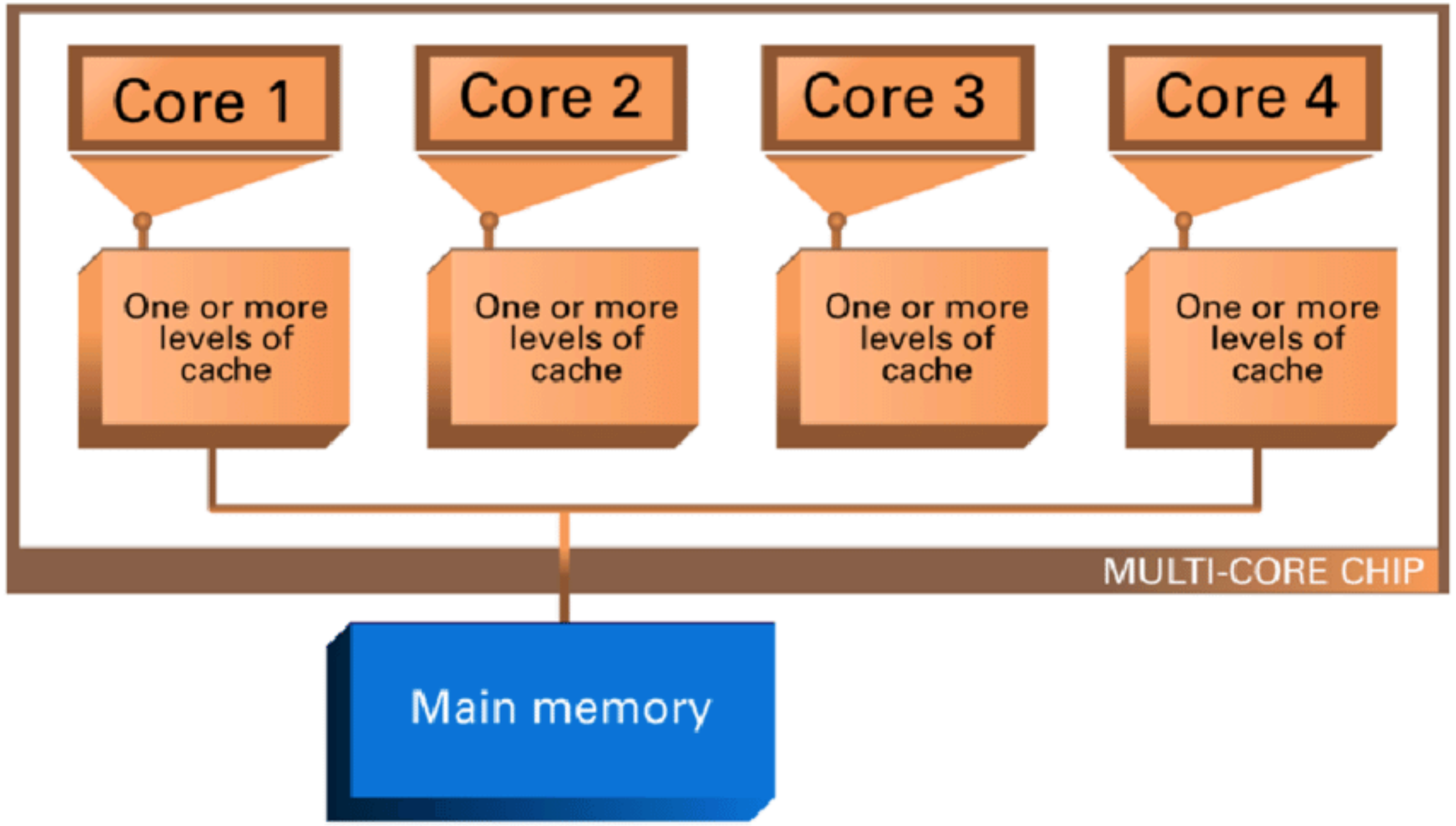


Parallel Bilby = OpenMPI + Bilby for GW

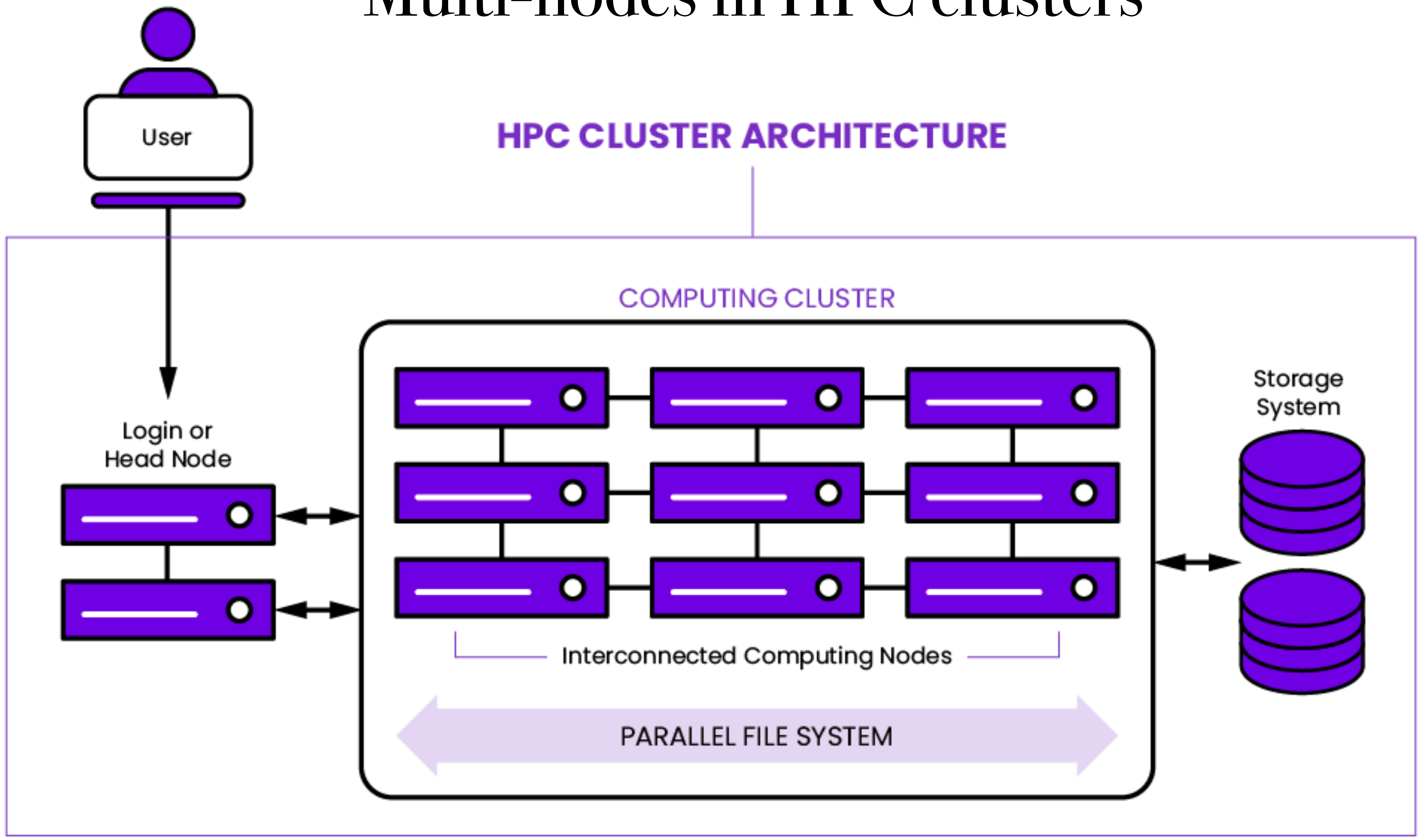
OpenMP/multiprocessing in Python

Shared-memory multi-processor



MPI: Message Passing Interface

Multi-nodes in HPC clusters




THE AMAZING
Thomas Bayes
P(A|B) = P(B|A) P(A) / P(B)
P(A)
P(B)