Quantum Cellular Automata

The Quantum Game of Life

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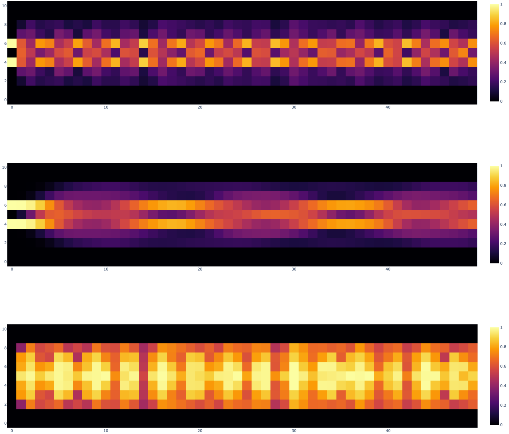

Classical Cellular Automata

- Grid of cells in one or more dimensions evolving over time steps
- State in time step $t + 1$ depends only on state in time step t

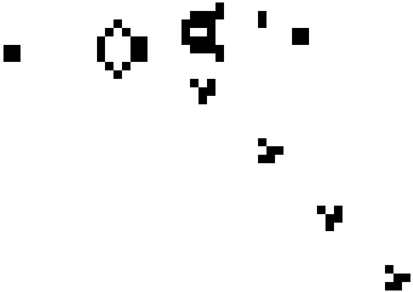

Figure: Gosper's glider gun from Conway's Game of Life

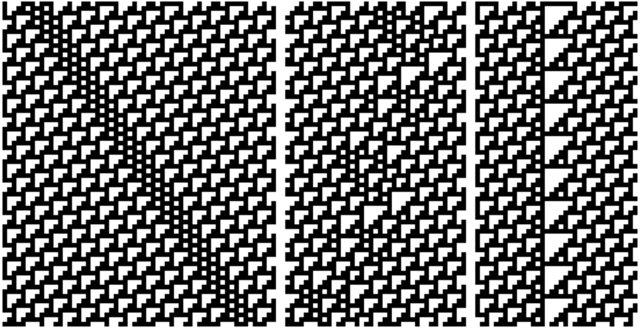

Figure: A glider in the rule-110 elementary cellular automaton

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$$0 := \textit{dead} \qquad\qquad 1 := \textit{alive}$$

$$|0\rangle := 0 \qquad\qquad |1\rangle := 1$$

$$000101000 \Longrightarrow |\psi\rangle = |0\rangle^{\otimes 3} \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle \otimes |0\rangle^{\otimes 3}$$

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To compute the probability that cell j is alive or dead, we measure with the corresponding observable

$$\hat{\hat{n}}_j = |0\rangle_j \langle 0| \qquad \hat{n}_j = |1\rangle_j \langle 1|$$

$$P(dead)_j = \langle\psi| \hat{\hat{n}}_j |\psi\rangle \qquad P(alive)_j = \langle\psi| \hat{n}_j |\psi\rangle$$

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How do we get $\hat{U}(k)$?

Hamiltonian

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$$\hat{\hat{n}}_j = |0\rangle_j \langle 0| \qquad\qquad \hat{n}_j = |1\rangle_j \langle 1|$$

$$\begin{aligned}
\hat{N}_i^{(2)} &= \hat{\hat{n}}_{i-2}\hat{\hat{n}}_{i-1}\hat{n}_{i+1}\hat{n}_{i+2} + \hat{\hat{n}}_{i-2}\hat{n}_{i-1}\hat{\hat{n}}_{i+1}\hat{n}_{i+2} \\
&+ \hat{\hat{n}}_{i-2}\hat{n}_{i-1}\hat{n}_{i+1}\hat{\hat{n}}_{i+2} + \hat{n}_{i-2}\hat{\hat{n}}_{i-1}\hat{\hat{n}}_{i+1}\hat{n}_{i+2} \\
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- As visible from the summation index i, constant boundary conditions are used

Hamiltonian

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By solving the Schrödinger equation

$$\mathrm{i}\partial_t \left|\psi\right\rangle_t = \hat{H} \left|\psi\right\rangle_t$$

the time evolution operator $\hat{U}$ is given by

$$\hat{U}(t) = e^{-\mathrm{i}\hat{H}t} \tag{2}$$

Time Evolution Operator

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$$|\psi\rangle_t = \hat{U}(t)\,|\psi\rangle_0 \tag{3}$$

The time t needed for a state to flip under the action of operator $\hat{S}_i$ is $\frac{\pi}{2}$

For a fixed time step duration $t = \frac{\pi}{2}$, the state after k time steps is given by

$$|\psi\rangle_k = \left(\hat{U}\left(\frac{\pi}{2}\right) \right)^k |\psi\rangle_0 \tag{4}$$

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In particular, during 1 time step, the state of some $|\psi\rangle_k$ at time step k will evolve into

$$|\psi\rangle_{k+1} = \hat{U}\left(\frac{\pi}{2}\right) |\psi\rangle_k \tag{5}$$

Iterative Calculation

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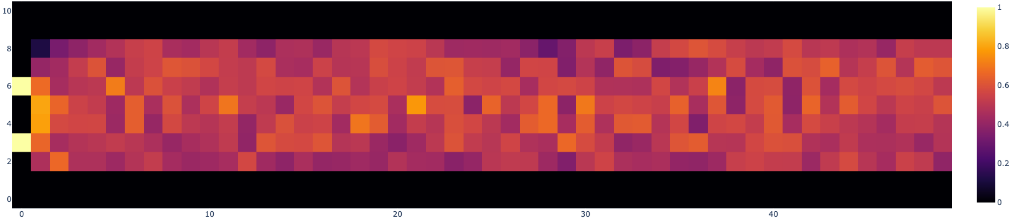

Figure: Time evolution of the initial state $|\psi\rangle_0 = |0\rangle^{\otimes 3} \otimes |1001\rangle \otimes |0\rangle^{\otimes 4}$ according to equation (5) shown as the distribution of alive cells

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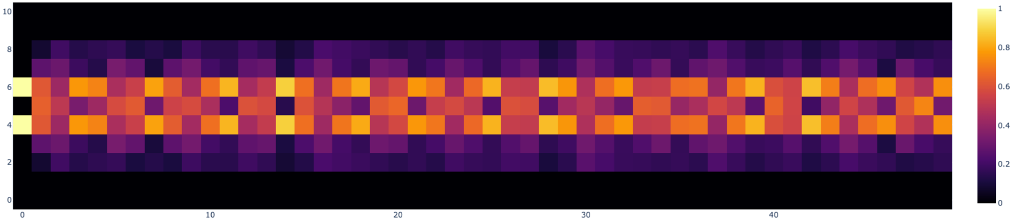

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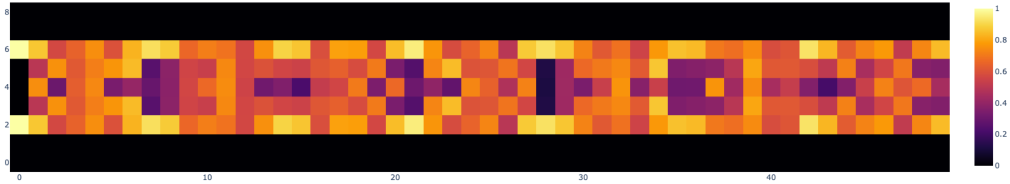

Figure: Time evolution of the initial state $|\psi\rangle_0 = |001000100\rangle$ according to equation (5) shown as the distribution of alive cells

Variable Time Steps

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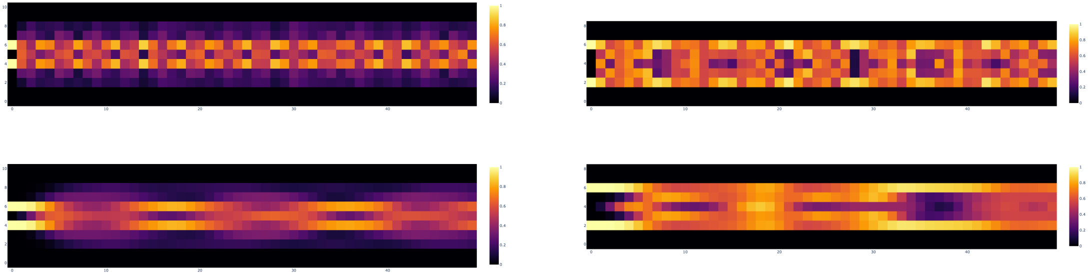

Figure: Time evolution of the initial states $|\psi\rangle_0^{(left)} = |0\rangle^{\otimes 4} \otimes |101\rangle \otimes |0\rangle^{\otimes 4}$ and $|\psi\rangle_0^{(right)} = |001000100\rangle$ according to equation (6) with $t^{(top)} = \frac{\pi}{2}$ and $t^{(bottom)} = \frac{\pi}{10}$ shown as the distribution of alive cells

Entropy of entanglement

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$$|\Psi_{AB}\rangle = |\psi_A\rangle |\psi_B\rangle \tag{7}$$

$$\rho_A = Tr_B(|\Psi_{AB}\rangle \langle\Psi_{AB}|) \tag{8}$$

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If A represents one cell and B represents the rest of the system, $S(\rho_A)$ is called the **single site entropy**.

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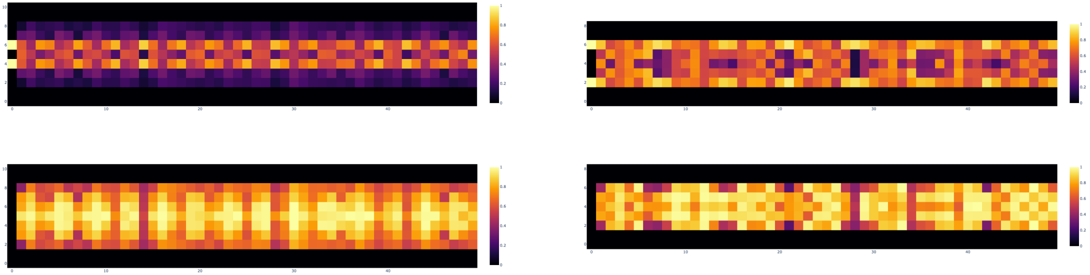

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Changing the rules

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- $\hat{N}_i$ defines how many neighbors are considered and how many of them need to be alive to flip a cell
- The bounds of the summation index i depend on $\hat{N}_i$ and on the the boundary conditions used

Rule 150

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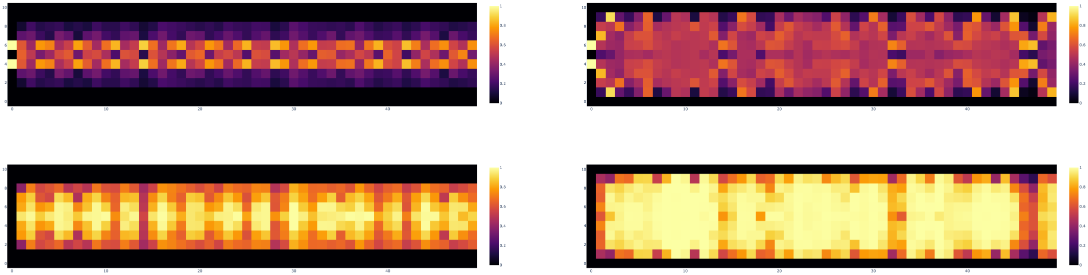

Figure: Time evolution of the initial state $|\psi\rangle_0 = |0\rangle^{\otimes 4} \otimes |101\rangle \otimes |0\rangle^{\otimes 4}$ according to rule F_{12} (left) and rule 150 (right) shown as the distribution of alive cells (top) and single site entropy (bottom)

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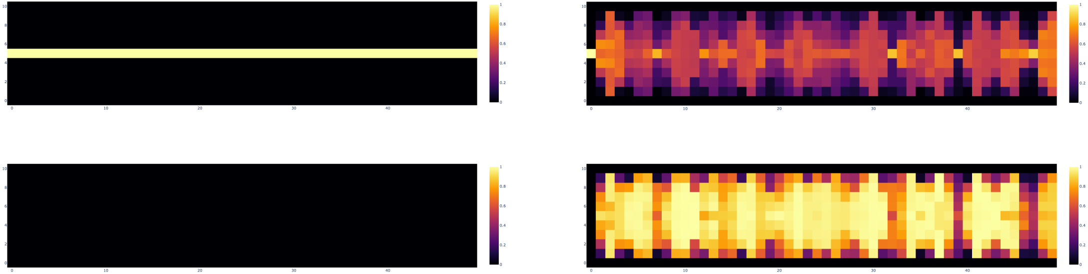

Figure: Time evolution of the initial state $|\psi\rangle_0 = |0\rangle^{\otimes 5} \otimes |1\rangle \otimes |0\rangle^{\otimes 5}$ according to rule F_{12} (left) and rule 150 (right) shown as the distribution of alive cells (top) and single site entropy (bottom)

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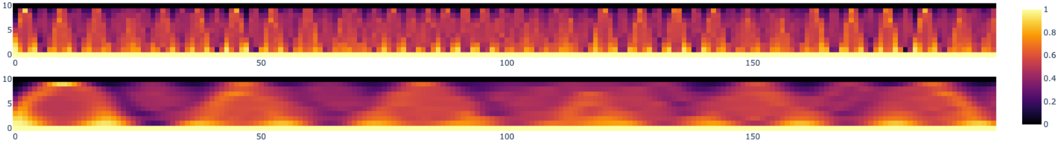

Figure: Time evolution of the initial state $|\psi\rangle_0 = \bigotimes_{k=0}^{N-1}\left[R_x\left(\pi\frac{k}{N-1}\right)|1\rangle\right]$, with $R_x(\theta)$ the x-rotation gate and $N = 11$, according to rule 150 with $t^{(top)} = \frac{\pi}{2}$ and $t^{(bottom)} = \frac{\pi}{10}$ shown as the distribution of alive cells

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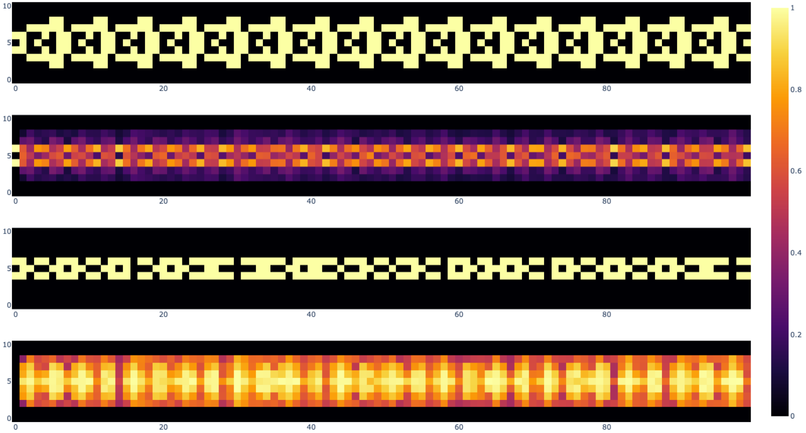
Classical

Quantum

Rounded

single site
entropy



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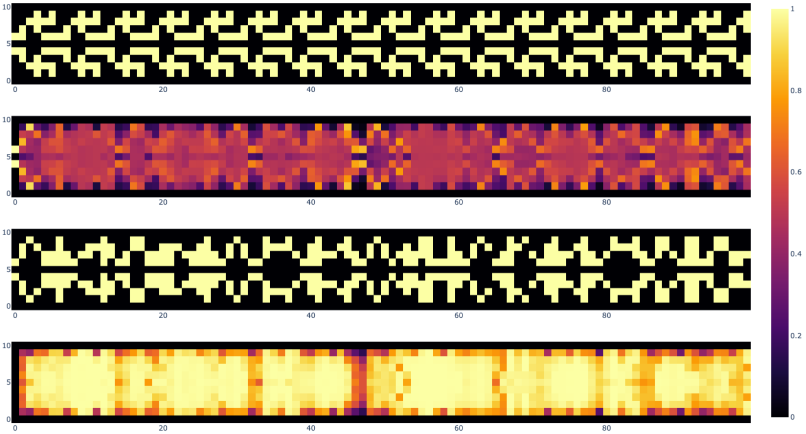

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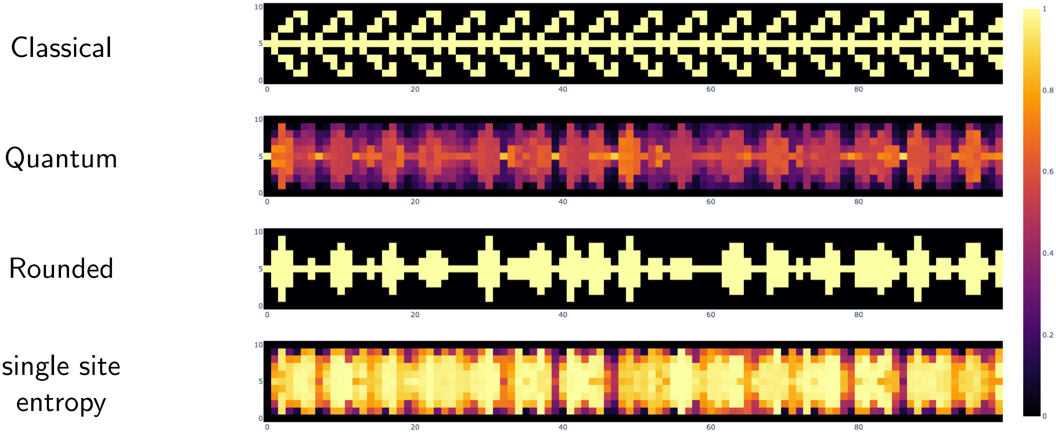

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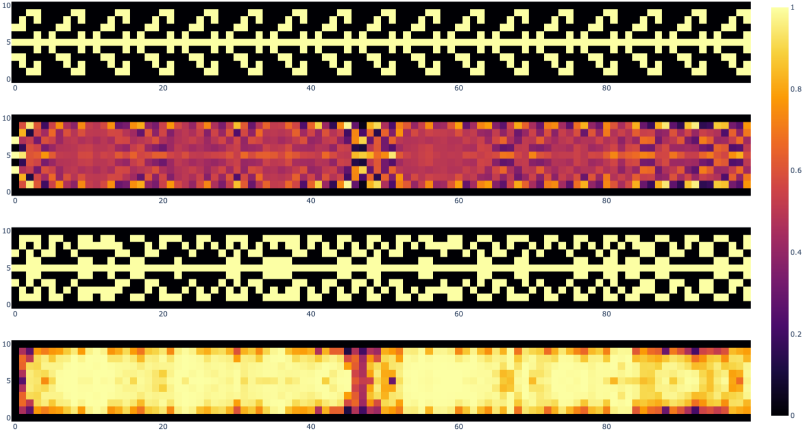

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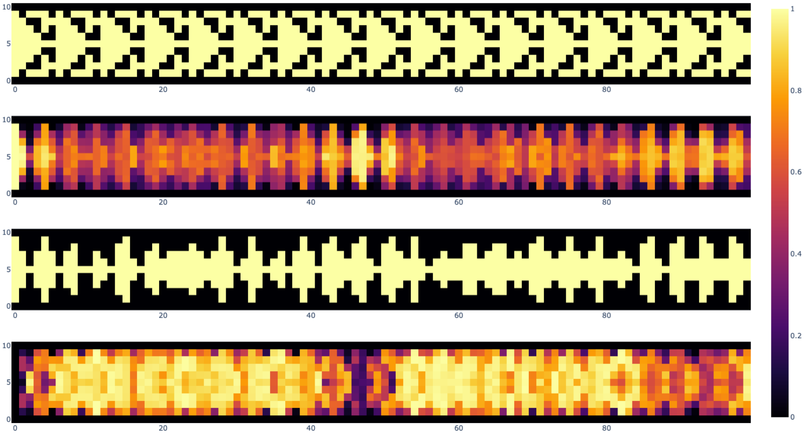

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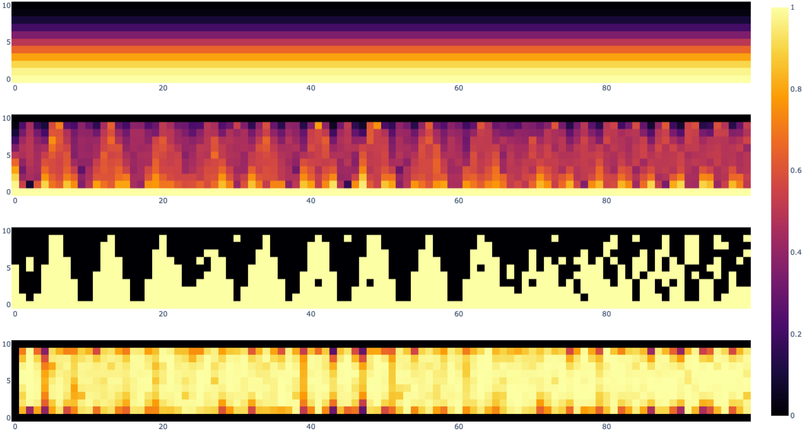

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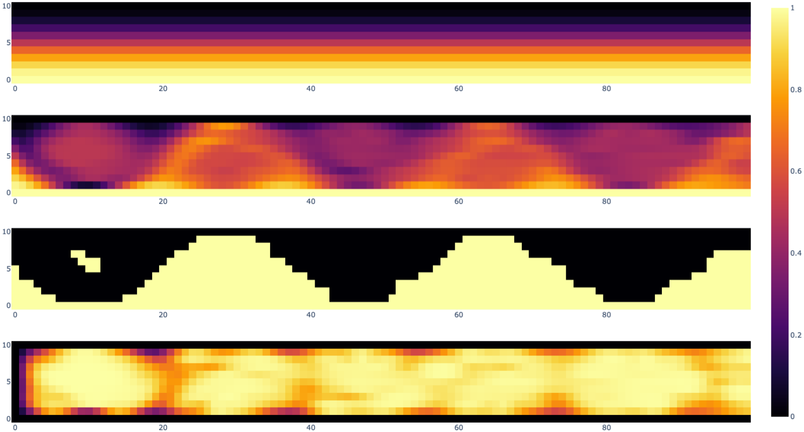

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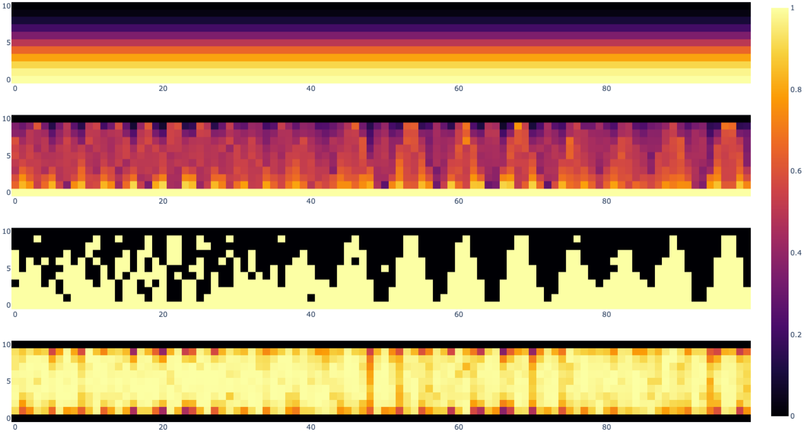

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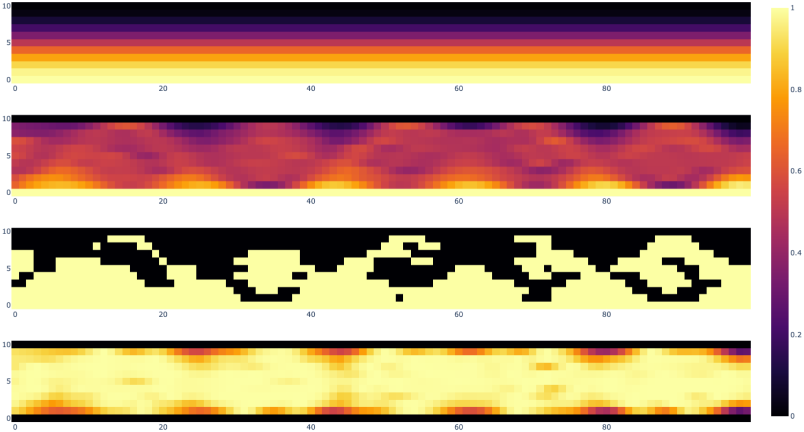

Desciption for plots in slide "The Quantum Game of Life"

- Classical: The time evolution corresponding to a classical cellular automaton
- Quantum: The time evolution of the distribution of alive cells
- Rounded: The rounded version of "Quantum"
- single site entropy: The time evolution of the single site entropy

Desciption for plots in slide "The Quantum Game of Life"

Plots in order:

- rule: f_{12}, $t = \frac{\pi}{2}$, $|\psi\rangle_0 = |0\rangle^{\otimes 4} \otimes |101\rangle \otimes |0\rangle^{\otimes 4}$, start time step $k = 0$
- rule: 150, $t = \frac{\pi}{2}$, $|\psi\rangle_0 = |0\rangle^{\otimes 4} \otimes |101\rangle \otimes |0\rangle^{\otimes 4}$, start time step $k = 0$
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- rule: 150, $t = \frac{\pi}{2}$, $|\psi\rangle_0 = |0\rangle^{\otimes 3} \otimes |10101\rangle \otimes |0\rangle^{\otimes 3}$, start time step $k = 0$
- rule: 150, $t = \frac{\pi}{2}$, $|\psi\rangle_0 = |0\rangle \otimes |1\rangle^{\otimes 9} \otimes |0\rangle$, start time step $k = 0$
- rule: 150, $t = \frac{\pi}{2}$, $|\psi\rangle_0 = \bigotimes_{k=0}^{N-1} \left[R_x \left(\pi \frac{k}{N-1} \right) |1\rangle \right]$, start time step $k = 0$
- rule: 150, $t = \frac{\pi}{10}$, $|\psi\rangle_0 = \bigotimes_{k=0}^{N-1} \left[R_x \left(\pi \frac{k}{N-1} \right) |1\rangle \right]$, start time step $k = 0$
- rule: 150, $t = \frac{\pi}{2}$, $|\psi\rangle_0 = \bigotimes_{k=0}^{N-1} \left[R_x \left(\pi \frac{k}{N-1} \right) |1\rangle \right]$, start time step $k = 70$
- rule: 150, $t = \frac{\pi}{10}$, $|\psi\rangle_0 = \bigotimes_{k=0}^{N-1} \left[R_x \left(\pi \frac{k}{N-1} \right) |1\rangle \right]$, start time step $k = 70$

Wolfram Code

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A naming system to define the rules of a cellular automaton

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Rule 150:
"A cell is flipped if the number of alive cells among its nearest neighbors is 1"

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1 1 1	1 1 0	1 0 1	1 0 0	0 1 1	0 1 0	0 0 1	0 0 0
1	0	0	1	0	1	1	0

Wolfram Code

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"A cell is flipped if the number of alive cells among its nearest neighbors is 1"

1 1 1	1 1 0	1 0 1	1 0 0	0 1 1	0 1 0	0 0 1	0 0 0
1	0	0	1	0	1	1	0

The corresponding wolfram code for this rule is $10010110b = \mathbf{150}$

Wolfram Code

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"A cell is flipped if the number of alive cells among its nearest neighbors is 1"

1 1 1	1 1 0	1 0 1	1 0 0	0 1 1	0 1 0	0 0 1	0 0 0
1	0	0	1	0	1	1	0

The corresponding wolfram code for this rule is $10010110b = \mathbf{150}$

The wolfram code for rule-F_{12} is 2266898040