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PREFACE

This book is decicated to CUHK students studying ESSC and related subjects. The major aim of this book is to provide the mathematical foundation that many courses require. Worked examples are given to help students understand the methods used. Problems are attached at the end of each chapter to further assist students in getting familiar with the concepts. I am very grateful that Prof. Man-nin Chan and Dr. Andie Au-yeung for giving me the opportunity to write this book. Without their help, this book would never be created.

Benjamin Loi

Course List

Course Code	Course Name
Semester 1	
ESSC2020	Climate System Dynamics
ESSC2800	Introduction to Environmental Engineering
ESSC3120	Physics of the Earth
ESSC3200	Atmospheric Dynamics
ESSC3320	Hydrogeology
ESSC3800	Global Environmental Change
Semester 2	
ESSC2010	Solid Earth Dynamics
ESSC3010	Continuum Mechanics
ESSC3100	Structural Geology
ESSC3220	Atmospheric Chemistry
ESSC3300	Ocean and Climate
ESSC3600	Ecosystems and Climate

(4000-level courses are not included in this list)

CHAPTER 1

BASIC ALGEBRA

CONTENTS

1.1 INTRODUCTION

Algebra is one of the most fundamental mathematical tool used in Science. We would introduce some rules about algebraic operations, then look at some problems related to basic algebra.

Addition, Subtraction, Multiplication and Division Addition, Subtraction, Multiplication and Division are the most basic operations in the world of mathematics. Subtraction can be viewed as addition with a minus sign, division can be viewed as multiplication by interchanging the numerator and denominator.

Here are some important rules about these four operations:

$$\begin{aligned} a+b &= b+a && \text{Commutative Property of Addition} \\ -a &= (-1)a \\ (a+b)+c &= a+(b+c) && \text{Associative Property of Addition} \end{aligned}$$

$$\begin{aligned} ab &= ba && \text{Commutative Property of Multiplication} \\ a(b+c) &= ab+ac && \text{Distributive Property of Multiplication} \\ a(b-c) &= ab-ac \\ (a+b)c &= ac+bc \\ (a-b)c &= ac-bc \\ (ab)c &= a(bc) && \text{Associative Property of Multiplication} \end{aligned}$$

Equation Left-hand side and right-hand side of an equation are equal. If we do an operation on both side, they are still equal. Hence we can use this to our advantage and simplify equation by doing appropriate operations.

Example 1.1.1 Solve the following equation.

$$\begin{aligned} \frac{3x+8}{2} &= \frac{4x-5}{3} \\ 3x+8 &= 2\left(\frac{4x-5}{3}\right) \\ 3(3x+8) &= 2(4x-5) \\ 9x+24 &= 8x-10 \\ 9x-8x &= -10-24 \\ x &= -34 \end{aligned}$$

Example 1.1.2 Expand $(x+2)(2x+5)$.

$$\begin{aligned}(x+2)(2x+5) &= x(2x+5)+2(2x+5)\\ &= (2x^2+5x)+(4x+10)\\ &= 2x^2+9x+10\end{aligned}$$

Example 1.1.3 Make y as the subject for

$$\begin{aligned}x(y+3) &= \frac{y}{x}+4\\ x^2(y+3) &= y+4x\\ x^2y+3x^2 &= y+4x\\ x^2y-y &= 4x-3x^2\\ y(x^2-1) &= 4x-3x^2\\ y &= \frac{4x-3x^2}{x^2-1}\end{aligned}$$

Exponential and Logarithmic Functions Exponential function gives the value of base to the power of the index. While logarithmic function is the inverse operation of exponential function and retrieve the index with a given base. Example of such pairs are

$$\begin{aligned}10^2 = 100 &\leftrightarrow \log_{10}(100) = 2\\ \sqrt{9} = 9^{\frac{1}{2}} = 3 &\leftrightarrow \log_9(3) = \frac{1}{2}\\ e^3 \approx 20.086 &\leftrightarrow \log_e(e^3) = \ln(e^3) = 3\end{aligned}$$

where $e \approx 2.718$.

Here are some rules about exponentiation and logarithm:

$$\begin{aligned}a^{-b} &= \frac{1}{a^b}\\ a^{b+c} &= a^b a^c\\ a^{b-c} &= \frac{a^b}{a^c}\\ (a^b)^c &= a^{(bc)}\\ (ab)^c &= a^c b^c\end{aligned}$$

and

$$\begin{aligned}
\log(ab) &= \log a + \log b \\
\log\left(\frac{a}{b}\right) &= \log a - \log b \\
\log(a^b) &= b\log a \\
\log_b a &= \frac{\log_c a}{\log_c b}
\end{aligned}$$

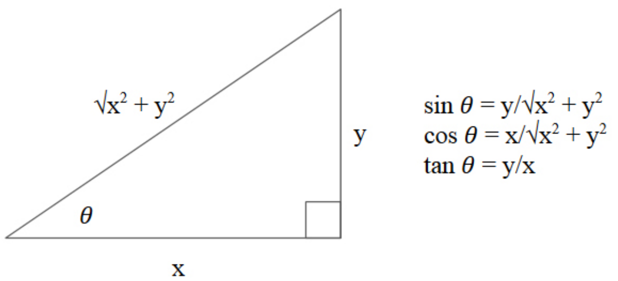
It is worth noting that exponentiation with a positive base always results in positive value, also we have $a^0 = 1$ and $\log_a 1 = 0$.

Example 1.1.4 Simplify the following expressions.

$$\begin{aligned}
\ln(xy^2e^{-3}) &= \ln x + \ln y^2 + \ln e^{-3} \\
&= \ln x + 2\ln y - 3\ln e \\
&= \ln x + 2\ln y - 3
\end{aligned}$$

$$\begin{aligned}
\frac{(e^4)^b}{(e^5)^{\ln c}} &= \frac{e^{4b}}{e^{5\ln c}} \\
&= \frac{e^{4b}}{e^{\ln(c^5)}} \\
&= \frac{e^{4b}}{c^5}
\end{aligned}$$

Trigonometric Functions The simplest trigonometric functions are sine, cosine and tangent. Related trigonometric functions are cosecant, secant and cotangent.



Trigonometric Ratios shown by a right-angled triangle.

Below are some trigonometric identities.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\csc\theta = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{1}{\tan\theta} = \tan(\frac{\pi}{2} - \theta)$$

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \\ \tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\end{aligned}$$

For a more detailed table of trigonometric identities that include sum-to-product and product-to-sum identities, the readers are referred to this link.

Alternative: Draw a triangle like the diagram in the previous page and set $y = 4$ and $x = 3$.

Example 1.1.5 Given that $\tan\theta = \frac{4}{3}$, without computing θ, find the value of $\sin\theta$.

$$\begin{aligned}\cot\theta &= \frac{1}{\tan\theta} = \frac{3}{4} \\ \csc^2\theta &= 1 + \cot^2\theta \\ \csc\theta &= \sqrt{1 + \frac{3}{4}^2} = \sqrt{\frac{25}{16}} = \frac{5}{4} \\ \sin\theta &= \frac{1}{\csc\theta} = \frac{4}{5}\end{aligned}$$

Factor Theorem A polynomial $p(x)$ has $(x-a)$ as a factor if and only if $p(a)$ equals to zero.

Example 1.1.6 Factorize $p(x) = 2x^2 + x - 15$.

Notice that $p(-3) = 0$, then $p(x)$ must have $(x-3)$ as a factor by factor theorem. Therefore, $p(x) = (x-3)(ax-b)$ for some a, b. Expanding $(x-3)(ax-b)$ gives

Strategy: Test $p(a) = 0$ for any a that is a factor of the constant term. In this case, -3 is a factor of -15.

$$ax^2 - (3a+b)x + 3b$$

By comparing the coefficients we have $a = 2, b = -5$. Thus,

$$p(x) = (x-3)(2x+5)$$

The equation $p(x) = (x-3)(2x+5) = 0$ has the solution of $x = 3$ or $-\frac{5}{2}$ from finding the values of x which cause the factor to become zero.

Quadratic Equation A quadratic equation is in the form of $ax^2 + bx + c = 0$. The roots are given by the following expression:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1.1.7 Solve $x^2 - 3x + 2 = 0$.

From the above expression, we have

Alternative: Factorize the left hand side and find the roots like Example 1.1.6.

$$\begin{aligned} x &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(2)}}{2(1)} \\ &= \frac{3}{2} \pm \frac{1}{2} = 1 \text{ or } 2 \end{aligned}$$

Complex Number The idea of complex number comes from the solution of quadratic equation when the value inside the square root is negative. We denote it with the symbol $\imath$ which is essentially $\sqrt{-1}$. As a result, $\imath^2 = -1$. For any positive number a, $\sqrt{-a^2} = \sqrt{a^2}\sqrt{-1} = a\imath$. Evaluation of complex numbers is similar to its real number counterpart.

Example 1.1.8 Evaluate the following expression.
(a) $(3+4\imath)(2-3\imath)$, and (b) $\frac{1+\imath}{1-\imath}$.

$$\begin{aligned}(3+4\imath)(2-3\imath) &= 3(2-3\imath)+4\imath(2-3\imath) \\ &= (6-9\imath)+(8\imath-12\imath^2) \\ &= (6-9\imath)+(8\imath+12) \\ &= 18-\imath\end{aligned}$$

$$\begin{aligned}\frac{1+\imath}{1-\imath} &= \frac{(1+\imath)(1+\imath)}{(1-\imath)(1+\imath)} \\ &= \frac{1+\imath+\imath-\imath^2}{1+\imath-\imath-\imath^2} \\ &= \frac{2\imath}{2} \\ &= \imath\end{aligned}$$

Example 1.1.9 Solve $x^2+2x+3=0$.

$$\begin{aligned}x &= \frac{-(2)\pm\sqrt{(2)^2-4(1)(3)}}{2(1)} \\ &= -1\pm\frac{\sqrt{-8}}{2} \\ &= -1\pm\frac{\sqrt{8}\imath}{2} \\ &= -1\pm\frac{2\sqrt{2}\imath}{2} \\ &= -1\pm\sqrt{2}\imath\end{aligned}$$

Unit Conversion When we tackle physical problems, often we need to take care of the units. Addition and Subtraction can only be carried out when involved quantities have the same units. Sometimes in the multiplication between fractions, it is desired to convert the units such that some can be cancelled out.

Example 1.1.10 Express 42.195 km in m.

Notice that $1\,\text{km} = 1000\,\text{m}$. Therefore,

$$42.195\,\text{km} = (42.195\,\text{km})(\frac{1000\,\text{m}}{1\,\text{km}}) = 42\,195\,\text{m}$$

Example 1.1.11 Find the conversion factor between $\mathrm{m\,s^{-1}}$ and km/h.

$$1\,\mathrm{m\,s^{-1}} = (1\,\mathrm{m\,s^{-1}})(\frac{1\,\mathrm{km}}{1000\,\mathrm{m}})(\frac{3600\,\mathrm{s}}{1\,\mathrm{h}}) = 3.6\,\mathrm{km/h}$$

Proportionality A variable y is proportional to another variable x if $y = kx$ where k is a constant. This is denotes as $y \propto x$. x here can be substituted by other expressions, like $y \propto x^2$, or $y \propto \ln x$. When the expression at the right hand side increases, the variable at the left hand side increases to the same extent.

Example 1.1.12 (a) If $y \propto x^3$, and x doubles, find the increase in y. (b) If $y \propto \frac{1}{x}$ and x halves, find the change in y.

(a) $y \propto x^3$ implies that $\frac{y}{x^3} = k$ for a fixed k. Hence we have

$$\begin{aligned}
\frac{y_n}{x_n^3} &= \frac{y_0}{x_0^3} = k \\
\frac{y_n}{y_0} &= \frac{x_n^3}{x_0^3} \\
&= \left(\frac{x_n}{x_0}\right)^3 \\
&= 2^3 = 8
\end{aligned}$$

Hence the new value of y is 8 times of the original.

(b) Similar to (a), we have

$$\begin{aligned}
x_n y_n &= x_0 y_0 \\
\frac{y_n}{y_0} &= \frac{x_0}{x_n} \\
&= \frac{1}{\left(\frac{x_n}{x_0}\right)} \\
&= \frac{1}{\left(\frac{1}{2}\right)} \\
&= 2
\end{aligned}$$

Thus the new value of y is 2 times of the original.

1.2 Basic Algebra in ESSC2020

1.2.1 Energy Balance

Example 1.2.1 The average surface temperature of the Earth is about 15 °C. Calculate the average outgoing radiative flux at Earth's surface.

Radiation from a surface follows Stefen-Boltzmann Law, which states that

$$E = \sigma T^4$$

where $\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$ and T is the temperature in Kelvin. This form is valid when the object considered is a perfect blackbody.

Common mistake: Temperature not changed to Kelvin when applying Stefen-Boltzmann Law.

Therefore the average outgoing radiative flux is

$$(5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}})(288.15\,\mathrm{K})^4 = 390.9\,\mathrm{W\,m^{-2}}$$

Example 1.2.2 Estimate the amount of energy released per second by the Sun, given that the surface temperature and radius of the Sun are about 5780 K and 7×10^5 km respectively.

By Stefen-Boltzmann Law, radiative flux at the Sun's surface is

$$(5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}})(5780\,\mathrm{K})^4 = 6.3284 \times 10^7\,\mathrm{W\,m^{-2}}$$

Common mistake: Radius of Sun not converted to meter in calculation.

Rate of energy release is calculated as radiative flux times surface area of the Sun, which is

$$(6.3284 \times 10^7\,\mathrm{W\,m^{-2}})(4\pi(7 \times 10^8\,\mathrm{m})^2) = 3.897 \times 10^{26}\,\mathrm{W}$$

This value is also called the luminosity of the Sun.

Example 1.2.3 Estimate the solar radiative flux at the proximity of the Earth, given that the distance between the Sun and Earth is 1.5×10^8 km.

The energy released by the Sun spreads out radially and uniformly. By conservation of energy, the solar flux density is its luminosity divided by the surface area of the imaginary sphere having a radius of Earth-Sun distance:

$$\frac{3.897 \times 10^{26}\,\mathrm{W}}{4\pi(1.5 \times 10^{11}\,\mathrm{m})^2} = 1378\,\mathrm{W\,m^{-2}}$$

The actual amount of solar flux at the Earth is roughly equal to what we have calculated in the above example. It is called the Solar constant, denoted as S_0, and has a value of about $1370\,\mathrm{W\,m^{-2}}$.

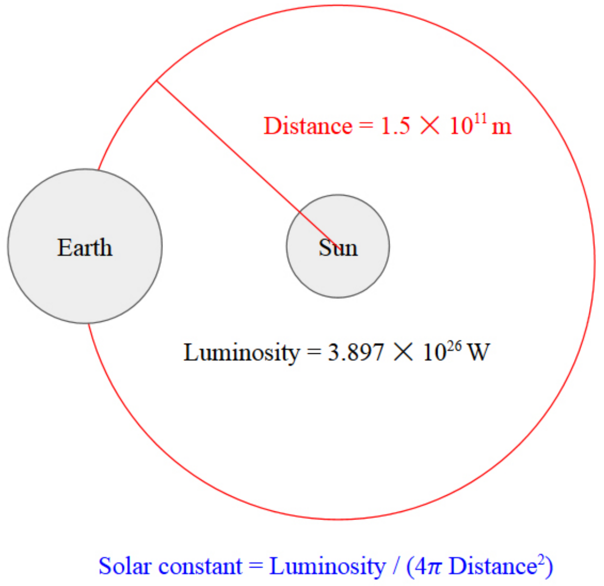
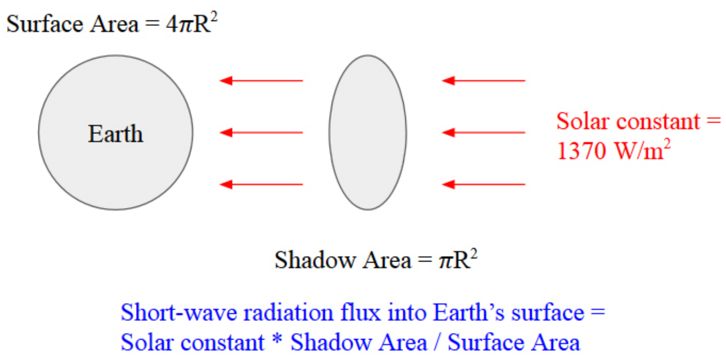


Figure showing the relation between Earth-Sun distance and Solar constant.

Since the cross-sectional area of the Earth is only πR^2_{Earth} but the total area of the Earth's surface is $4\pi R^2_{\text{Earth}}$, the average incoming solar flux into the Earth's surface is

$$\frac{S_0 \pi R^2_{\text{Earth}}}{4\pi R^2_{\text{Earth}}} = \frac{S_0}{4} = 342.5\,\mathrm{W\,m^{-2}}$$



Incoming solar flux, Earth's Shadow area and Surface area.

Example 1.2.4 Deep ocean has an albedo of about 0.06. Estimate the total amount of shortwave radiation absorbed by ocean water per unit area in one year.

The expression of radiation absorbed by a surface is

$$\frac{S_0}{4}(1-\alpha)$$

where α is the albedo, the fraction of radiation reflected away.

With $S_0 = 1370\,\mathrm{W/m^2}$ and $\alpha = 0.06$, the amount of radiation absorbed in one year is

$$\begin{aligned}\frac{1370\,\mathrm{W\,m^{-2}}}{4}(1-0.06)(1\,\mathrm{yr}) &= (342.5\,\mathrm{J\,s^{-1}\,m^{-2}})(0.94)(86400\times 365.25\ \mathrm{s})\\ &= 1.016\times 10^{10}\,\mathrm{J\,m^{-2}}\end{aligned}$$

Example 1.2.5 Find the would-be average temperature of Earth's surface if the greenhouse effect is not included.

At Earth's surface, the amount of radiation absorbed must balance the amount of radiation emitted on average such that the surface does not heat up or cool down in the long term. Thus we have

$$\begin{aligned}E_{abs} &= E_{emit}\\ \frac{S_0}{4}(1-\alpha_p) &= \sigma T_e^4\end{aligned}$$

where $\alpha_p \approx 0.3$ is the planetary albedo, the albedo for the Earth's surface as a whole. While T_e is the emission temperature, the temperature at which the total amount of radiation released by the Earth is equal to that would be emitted according to Stefen-Boltzmann Law.

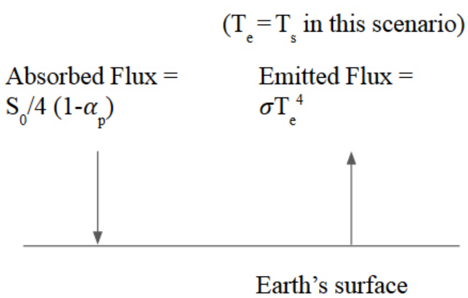


Diagram of fluxes if there is no Greenhouse Effect.

Further rearrangement gives

$$
\begin{aligned}
T_e^4 &= \frac{S_0}{4\sigma}(1-\alpha_p) \\
T_e &= \left[\frac{S_0}{4\sigma}(1-\alpha_p)\right]^{\frac{1}{4}} \\
&= \left[\left(\frac{1370\,\mathrm{W\,m^{-2}}}{4\sigma}\right)(1-0.3)\right]^{\frac{1}{4}} \\
&= \left[\left(\frac{342.5\,\mathrm{W\,m^{-2}}}{5.67\times10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}}\right)(0.7)\right]^{\frac{1}{4}} \\
&= 255\,\mathrm{K}
\end{aligned}
$$

In this case, the Earth's surface temperature is the same as the emission temperature, i.e. $T_s = 255\,\mathrm{K}$.

Example 1.2.6 Find the would-be average temperature of Earth's surface if the greenhouse effect is modelled by assuming the atmosphere is a homogeneous, perfectly absorbing medium for long-wave radiation.

We consider the energy balance for the atmosphere. Long-wave radiation from the Earth's surface is absorbed by the atmosphere and given by σT_s^4. Meanwhile radiation leaves the atmosphere by two pathways, one back to the Earth's surface and another to the space, since the atmosphere has two surfaces, outward and inward. Both of them are σT_a^4 where T_a is the atmosphere's temperature. Therefore, we have

$$
\begin{aligned}
E_{abs} &= E_{emit} \\
\sigma T_s^4 &= \sigma T_a^4 + \sigma T_a^4 = 2\sigma T_a^4
\end{aligned}
$$

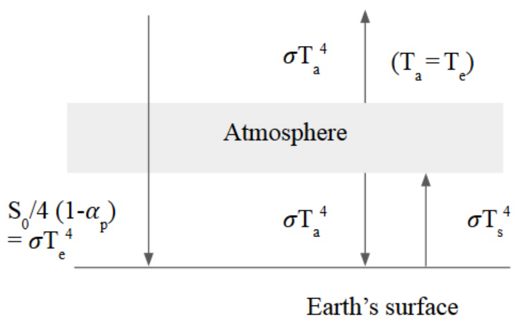


Diagram showing fluxes for simple Greenhouse Effect.

Notice that $T_a = T_e = 255\,\mathrm{K}$, the reason behind is that the only source of terrestrial radiation emitted to the space comes from the atmosphere. Then we have

$$\begin{aligned}T_s^4 &= 2T_e^4\\ T_s &= 2^{\frac{1}{4}}T_e\\ &= 2^{\frac{1}{4}}(255\,\mathrm{K})\\ &= 303\,\mathrm{K}\end{aligned}$$

Example 1.2.7 Estimate the average temperature of Earth's surface under greenhouse effect in which the atmosphere absorbs only a fraction of long-wave radiation from the ground.

In this case, we have to add a factor of ε in the expression of the amount of long-wave radiation absorbed by the atmosphere, which then becomes

$$\varepsilon\sigma T_\mathrm{s}^4$$

where ε is called absorptivity and quantifies the fraction of long-wave radiation captured by greenhouse gases in the atmosphere.

We also want to modify Stefen-Boltzmann Law to better reflect the fact that the atmosphere is not a perfect blackbody. Now the law has a new form of

$$E = \varepsilon\sigma T_\mathrm{a}^4$$

where ε here denotes emissivity which is equal to the absorptivity in above context due to Kirchhoff's Law.

Now consider the energy balance for the atmosphere again, similar to Example 1.2.6, we have

$$\begin{aligned}\varepsilon\sigma T_\mathrm{s}^4 &= 2\varepsilon\sigma T_\mathrm{a}^4\\ T_\mathrm{s} &= 2^{\frac{1}{4}}T_\mathrm{a}\end{aligned}$$

We should be careful that at this time $T_a \neq T_e$. Thus, we need to consider the energy balance for the Earth's surface as well. Absorption comes from two sources: solar radiation and long-wave radiation emitted from the atmosphere. Meanwhile the surface itself emits radiation according to Stefen-Boltzmann Law as well.

Therefore,

$$E_{\text{abs}} = E_{\text{emit}}$$
$$\frac{S_0}{4}(1-\alpha_p) + \varepsilon\sigma T_{\text{a}}^4 = \sigma T_{\text{s}}^4$$

Notice that $\frac{S_0}{4}(1-\alpha_p) = \sigma T_{\text{e}}^4$ just like example 1.2.5 if we consider the energy balance for the Earth as a whole. Now we have

$$\sigma T_{\text{e}}^4 + \varepsilon\sigma T_{\text{a}}^4 = \sigma T_{\text{s}}^4$$
$$T_{\text{e}}^4 + \varepsilon T_{\text{a}}^4 = T_{\text{s}}^4$$

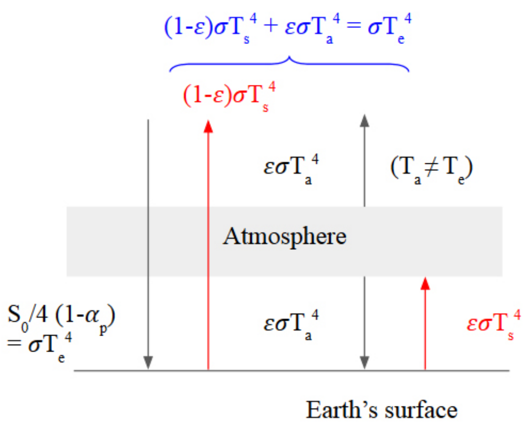


Diagram showing fluxes for leaky Greenhouse Effect.

Substituting $T_{\text{s}} = 2^{\frac{1}{4}} T_{\text{a}}$ as we have concluded above, we have

$$T_{\text{e}}^4 + \varepsilon T_{\text{a}}^4 = 2T_{\text{a}}^4$$
$$T_{\text{e}}^4 = (2-\varepsilon)T_{\text{a}}^4$$
$$T_{\text{a}}^4 = \left(\frac{1}{2-\varepsilon}\right) T_{\text{e}}^4$$
$$T_{\text{a}} = \left(\frac{1}{2-\varepsilon}\right)^{\frac{1}{4}} T_{\text{e}}$$

Emissivity ε of Earth's atmosphere has a typical value of 0.77. Substitution gives $T_a = 242\,\text{K}$, which indicates that the atmosphere's temperature is lower than the emission temperature. Again using the relation $T_s = 2^{\frac{1}{4}} T_a$, we arrive at

$$
\begin{aligned}
T_{\mathrm{a}} &= \left(\frac{1}{2-\varepsilon}\right)^{\frac{1}{4}} T_{\mathrm{e}} \\
T_s = 2^{\frac{1}{4}} T_{\mathrm{a}} &= \left(\frac{2}{2-\varepsilon}\right)^{\frac{1}{4}} T_{\mathrm{e}} \\
&= \left(\frac{2}{2-0.77}\right)^{\frac{1}{4}} (255\,\mathrm{K}) \\
&= 288\,\mathrm{K}
\end{aligned}
$$

corresponding to an estimated surface temperature about 15 °C, which is realistic when compared to observations.

Example 1.2.8 Estimate the value of emissivity of Mars' atmosphere provided that emission temperature and surface temperature of Mars are 211 K and 230 K.

From the expression we have just derived in Example 1.2.7, we have

$$
\begin{aligned}
T_s &= \left(\frac{2}{2-\varepsilon}\right)^{\frac{1}{4}} T_{\mathrm{e}} \\
\left(\frac{T_s}{T_e}\right)^4 &= \frac{2}{2-\varepsilon} \\
2-\varepsilon &= 2\left(\frac{T_e}{T_s}\right)^4 \\
\varepsilon &= 2\left[1-\left(\frac{T_e}{T_s}\right)^4\right]
\end{aligned}
$$

Substituting the corresponding values, we have

$$
\begin{aligned}
\varepsilon &= 2\left[1-\left(\frac{211\,\mathrm{K}}{230\,\mathrm{K}}\right)^4\right] \\
&= 0.583
\end{aligned}
$$

Example 1.2.9 Using the multi-layer model, in which the atmosphere is considered to consisted of multiple absorbing sub-layers, then by considering the energy balance at each sub-layer, starting from the top to bottom, we can conclude that

$$T_s = T_e(N+1)^{\frac{1}{4}}$$

where N is the amount of effective absorbing layers.

Using the expression $\frac{S_0}{4}(1-\alpha_p) = \sigma T_e^4$ in Example 1.2.5, it can be written as

$$T_s = \left[\frac{S_0}{4\sigma}(1-\alpha_p)(N+1)\right]^{\frac{1}{4}}$$

Now estimate the number of effective absorbing layers in Earth's atmosphere.

Rearrange the equation above and substituting $T_a = 255\,\mathrm{K}$, $T_s = 288\,\mathrm{K}$ found in previous examples, we have

$$\begin{aligned} N &= \left(\frac{T_s}{T_e}\right)^4 - 1 \\ &= \left(\frac{288\,\mathrm{K}}{255\,\mathrm{K}}\right)^4 - 1 \\ &= 0.627 \end{aligned}$$

Example 1.2.10 Venus has a very thick atmosphere. Given that the effective absorbing layers of Venus' atmosphere is around 125 and its emission temperature is $-45\,^\circ\mathrm{C}$. Estimate the surface temperature of Venus.

Common mistake: Temperature not in Kelvin when using the formula.

$$\begin{aligned} T_s &= T_e(N+1)^{\frac{1}{4}} \\ &= (228.15\,\mathrm{K})(125+1)^{\frac{1}{4}} \\ &= 764\,\mathrm{K} \end{aligned}$$

Not completed. To be written later.

1.3 Basic Algebra in ESSC2010

1.3.1 Half-life Calculation

Example 1.3.1 Half-life of the radioactive Potassium-40 isotope is 1.3 billion years. A rock sample has a parent-to-daughter ratio of 1:9. Estimate its age.

Relation between half-life τ and percentage of isotope remained is

$$\frac{N}{N_0} = \left(\frac{1}{2}\right)^{t/\tau}$$

Common mistake: Directly using the parent-to-daughter ratio as the fraction of remained isotope.

A parent-to-daughter ratio of 1:9 implies $\frac{1}{9+1} = \frac{1}{10}$ of Potassium-40 remains. Therefore,

$$\begin{aligned}
\frac{N}{N_0} = \left(\frac{1}{2}\right)^{t/1.3\,\text{Gya}} &= \frac{1}{10} \\
\ln\left(\frac{1}{2}\right)^{t/1.3\,\text{Gya}} &= \ln\frac{1}{10} \\
\frac{t}{1.3\,\text{Gya}}\ln\frac{1}{2} &= \ln\frac{1}{10}
\end{aligned}$$

$$\begin{aligned}
\therefore t &= (1.3\,\text{Gya})\frac{\ln\frac{1}{10}}{\ln\frac{1}{2}} \\
&= (1.3\,\text{Gya})\frac{-\ln 10}{-\ln 2} \\
&= (1.3\,\text{Gya})\frac{\ln 10}{\ln 2} \\
&= 4.32\,\text{Gya}
\end{aligned}$$

Example 1.3.2 Carbon-14 has a half-life of 5700 years. Find the fraction of carbon-14 remained if a tree is 20000 years old.

Common mistake: Mixing up the numerator (age) and denominator (half-life).

Substituting the age and half-life into the expression in the last example, we have

$$\begin{aligned}
\frac{N}{N_0} &= \left(\frac{1}{2}\right)^{20000\,\text{yr}/5700\,\text{yr}} \\
&= \left(\frac{1}{2}\right)^{3.509} \\
&= 0.0879
\end{aligned}$$

Example 1.3.3 A sedimentary rock sample with an estimated age of 200 Mya contains the radioactive Uranium-235 and its decay product Lead-207. The ratio of U^{235} and Pb^{207} inside the sample is measured to be 4.65:1. Estimate the half-life of Uranium-235.

A parent-to-daughter ratio of 4.65:1 implies $\frac{4.65}{4.65+1} \approx 0.823$ of Uranium-235 remains. Using the relation in Example 1.3.1, we have

$$\begin{aligned}
\frac{N}{N_0} = \left(\frac{1}{2}\right)^{200\,\mathrm{Mya}/\tau} &\approx 0.823 \\
\ln\left(\frac{1}{2}\right)^{200\,\mathrm{Mya}/\tau} &\approx \ln 0.823 \\
\frac{200\,\mathrm{Mya}}{\tau}\ln\frac{1}{2} &\approx \ln 0.823
\end{aligned}$$

$$\begin{aligned}
\therefore \tau &\approx (200\,\mathrm{Mya})\frac{\ln 0.5}{\ln 0.823} \\
&\approx 712\,\mathrm{Mya}
\end{aligned}$$

Also, $\frac{\ln 0.823}{\ln 0.5} = \frac{200\,\mathrm{Mya}}{712\,\mathrm{Mya}} = 0.281$ half-life has passed.

1.3.2 Earthquake Magnitude and Energy

Example 1.3.4 Given an earthquake has an average slip of 20 m, a total faulted area of $300\,\mathrm{km} \times 10\,\mathrm{km}$. Estimate its seismic moment and calculate the moment magnitude. You are given that shear modulus of the rock in question is 35 GPa.

The expression of seismic moment is

$$M_o = \mu DA$$

where μ, D, A are shear modulus, average slip and faulted area.

Substituting the values into the expression, we have

$$\begin{aligned}
M_o &= (35\,\mathrm{GPa})(20\,\mathrm{m})(300\,\mathrm{km} \times 10\,\mathrm{km}) \\
&= (35 \times 10^9\,\mathrm{Pa})(20\,\mathrm{m})(300 \times 10^3\,\mathrm{m} \times 10 \times 10^3\,\mathrm{m}) \\
&= 2.1 \times 10^{21}\,\mathrm{N\,m} \\
&= (2.1 \times 10^{21}\,\mathrm{N\,m})\left(\frac{1 \times 10^5\,\mathrm{dyne}}{1\,\mathrm{N}}\right)\left(\frac{1 \times 10^2\,\mathrm{cm}}{1\,\mathrm{m}}\right) \\
&= 2.1 \times 10^{28}\,\mathrm{dyne\,cm}
\end{aligned}$$

Moment magnitude and seismic moment are related by

$$M_w = 0.667 \log_{10} M_o - 10.733$$

where M_o has to be in dyne cm.

Common mistake: Seismic moment not in dyne cm when applying the formula.

Substituting the answer above, we have

$$\begin{aligned} M_w &= 0.667 \log_{10}(2.1 \times 10^{28}\,\text{dyne cm}) - 10.733 \\ &= 8.2 \end{aligned}$$

Example 1.3.5 Estimate the amount of seismic energy released by the 2011 Tohoku Earthquake which has a moment magnitude of 9.0.

The relation between seismic energy and moment magnitude is

$$\begin{aligned} \log_{10} E &= 1.5 M_w + 11.8 \\ E &= 10^{(1.5M_w + 11.8)} \end{aligned}$$

where E is in dyne cm like M_o in the calculation of moment magnitude.

Common mistake: Interpreting the number evaluated from the formula as in Joule.

With $M_w = 9.0$, the total seismic energy released is

$$\begin{aligned} E &= 10^{1.5(9.0)+11.8} \\ &= 10^{25.3} \\ &= 2.00 \times 10^{25}\,\text{dyne cm} \\ &= (2.00 \times 10^{25}\,\text{dyne cm}) \left(\frac{1\,\text{N}}{1 \times 10^5\,\text{dyne}} \right) \left(\frac{1\,\text{m}}{1 \times 10^2\,\text{cm}} \right) \\ &= 2.00 \times 10^{18}\,\text{N m} = 2.00 \times 10^{18}\,\text{J} \end{aligned}$$

For reference, energy released by the Hiroshima atomic bomb is about 63×10^{12} J. Hence the seismic energy released by the 2011 Tohoku Earthquake is comparable to that of $2.00 \times 10^{18}\,\text{J}/63 \times 10^{12}\,\text{J} \approx 31750$ Hiroshima atomic bombs.

Example 1.3.6 In terms of energy, find how many times a $M_w = 8.0$ earthquake is larger than a $M_w = 6.5$ earthquake.

The required energy ratio is found by

$$\begin{aligned}\frac{E_1}{E_2} &= \frac{10^{(1.5M_{w1}+11.8)}}{10^{(1.5M_{w2}+11.8)}} \\ &= \frac{10^{(1.5(8.0)+11.8)}}{10^{(1.5(6.5)+11.8)}} \\ &= 10^{(1.5(8.0)+11.8)-(1.5(6.5)+11.8)} \\ &= 10^{2.25} \\ &= 178\end{aligned}$$

1.3.3 Tsunami Wave

Example 1.3.7 Calculate the change in the wave speed of tsunami as it moves from a water depth of H_0 to $H_n = \frac{1}{10}H_0$.

Wave speed of tsunami is simply

$$v = \sqrt{gH}$$

Hence

$$v \propto H^{\frac{1}{2}}$$

Therefore,

$$\begin{aligned}\frac{v_n}{v_0} &= \frac{H_n^{\frac{1}{2}}}{H_0^{\frac{1}{2}}} \\ &= \left(\frac{\frac{1}{10}H_0}{H_0}\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{10}\right)^{\frac{1}{2}} \\ &= 0.316\end{aligned}$$

Alternative: Directly compare the wave speed by substituting H_0 and H_n into the expression.

So we conclude that the wave speed decreases to 31.6% of the initial as it moves to shallower area.

Example 1.3.8 A tsunami wave is initially with an amplitude of 0.8 m and at a water depth of 3000 m. Calculate the new wave amplitude if it moves to a water depth of 20 m.

We use the relation between amplitude and water depth, which is

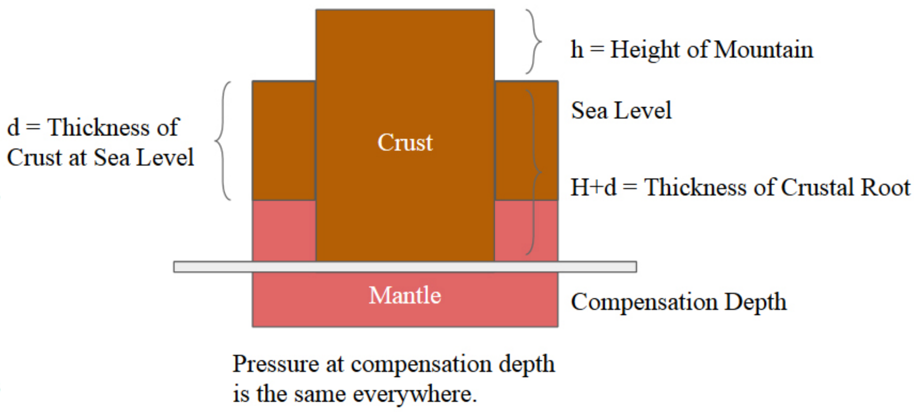
$$A \propto H^{-\frac{1}{4}}$$

This implies

$$\begin{aligned}
\frac{A_n}{A_0} &= \frac{H_n^{-\frac{1}{4}}}{H_0^{-\frac{1}{4}}} \\
A_n &= A_0 \left(\frac{H_n}{H_0}\right)^{-\frac{1}{4}} \\
&= (0.8\,\mathrm{m}) \left(\frac{20\,\mathrm{m}}{3000\,\mathrm{m}}\right)^{-\frac{1}{4}} \\
&= 2.80\,\mathrm{m}
\end{aligned}$$

1.3.4 Isostasy

Example 1.3.9 Given that crust and mantle have a density of $2.7\,\mathrm{g\,cm^{-3}}$ and $3.3\,\mathrm{g\,cm^{-3}}$ respectively. If a mountain is 4 km high, finds the total thickness of the crust there. It is given that average depth of the crust at sea level height is about 30 km.



Demonstration of Pratt's Isostasy.

By Archimedes' Principle, we have

$$\begin{aligned}
\rho_c g(h+d+H) &= \rho_c g d + \rho_m g H \\
\rho_c h + \rho_c H &= \rho_m H \\
H(\rho_m - \rho_c) &= \rho_c h \\
H &= \frac{\rho_c h}{\rho_m - \rho_c}
\end{aligned}$$

Substituting $h = 4\,\text{km}$, $\rho_c = 2.7\,\text{g}\,\text{cm}^{-3}$, and $\rho_m = 3.3\,\text{g}\,\text{cm}^{-3}$, we have

$$\begin{aligned}
H &= \frac{(2.7\,\text{g}\,\text{cm}^{-3})(4\,\text{km})}{(3.3\,\text{g}\,\text{cm}^{-3}) - (2.7\,\text{g}\,\text{cm}^{-3})} \\
&= (4.5)(4\,\text{km}) \\
&= 18\,\text{km}
\end{aligned}$$

From the diagram, we obtain the total crustal thickness as $4 + 30 + 18 = 52\,\text{km}$.

Example 1.3.10 If the maximum crustal thickness can be achieved is 70 km, roughly estimate the height of the highest mountain that is possible to be created.

Using the equation in the last example, we have

$$H = \frac{\rho_c h}{\rho_m - \rho_c}$$

Hence the total crustal thickness in terms of the mountain height h is

$$h + d + H = h + 30\,\text{km} + \frac{\rho_c h}{\rho_m - \rho_c}$$

h_{max} is then found by

$$\begin{aligned}
h_{max} + \frac{\rho_c}{\rho_m - \rho_c} h_{max} + 30\,\text{km} &= 70\,\text{km} \\
h_{max}\left[1 + \frac{(2.7\,\text{g}\,\text{cm}^{-3})}{(3.3\,\text{g}\,\text{cm}^{-3}) - (2.7\,\text{g}\,\text{cm}^{-3})}\right] &= 40\,\text{km} \\
h_{max}(1 + 4.5) &= 40\,\text{km} \\
h_{max} &= 7.27\,\text{km}
\end{aligned}$$

which is an underestimation when compared to the height of the world's highest mountain, the Himalayas, which is about 8.85 km.

1.4 PROBLEMS

Question 1.2.1 Find the value of Solar constant for Mars, given that the distance of Mars from the Sun is 2.28×10^8 km.

Question 1.3.1 If the current rate of subduction $0.09\,\mathrm{m^2\,s^{-1}}$ is applicable in the past, find the thickness of sediments that have been subducted in the last 3 Gyr if the mass of subducted sediments is equal to one-half the present mass of the continents?
Assume the density of the continents q_c is $2700\,\mathrm{kg\,m^{-3}}$, the density of sediments q_s is $2400\,\mathrm{kg\,m^{-3}}$, the continental area A_c is $1.9 \times 10^8\,\mathrm{km^2}$ and the mean continental thickness h_c is 35 km.

Question 1.3.2 If the area of the oceanic crust is $3.2 \times 10^8\,\mathrm{km^2}$ and new seafloor is now being created at the rate of $2.8\,\mathrm{km^2/yr}$, what is the mean age of the oceanic crust?
Assume that the rate of seafloor creation has been constant in the past.

Question 1.3.3 The frequency of aftershocks decreases roughly with the reciprocal of time after the main shock. This empirical relations was first descrbed by Fusakichi Omori in 1894 and is known as the Omori's law. It is expressed as

$$n(t) = \frac{k}{(c+t)^p}$$

where k is the constant defining the overall rate, c is the constant defining the initial decay and p is a third constant that modifies the decay rates and typically falls in the range 0.7-1.5.

Typically, the constant c ranges 10-60 seconds, which the decay rates changes during the time after the mainshock from seconds to minutes.

Considering an aftershock sequence that can be fit with these values: $k = 20$ eqs, $c = 30$ s and $p = 1$. What would be rate of earthquakes per day at 1,2,10,40 days after the mainshock respectively?

Not completed. To be written later.

CHAPTER 2

BASIC CALCULUS

CONTENTS

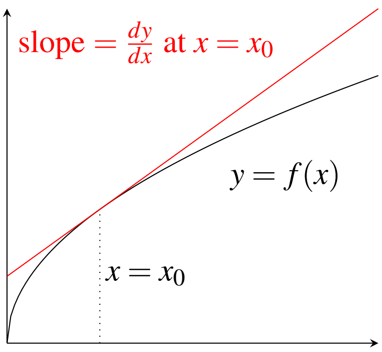
2.1 Introduction

2.1.1 Differentiation

Derivative and Differentiation Differentiation is a process of finding derivative, which is the rate of change of value of a function with respect to a variable, or geometrically, the slope of its tangent. A precise definition of derivative for a function $y = f(x)$ is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$$

in which both $\frac{d}{dx}$ and $'(x)$ denote differentiation against the variable x. However, we would not go into much detail about this, instead we would take a practical approach. Derivatives of common functions, as well as some rules of differentiation would be introduced.



A function (black line) with its tangent (red line) at a given point shown. The derivative of the function at that point is the slope of the highlighted tangent.

Derivatives of k and $kf(x)$ Derivative of a constant function $y = k$ is simply zero, since a constant function has a slope of zero everywhere. For any function multiplied by a constant, like $y = ku = kf(x)$, its derivative would be $\frac{dy}{dx} = \frac{d(ku)}{dx} = k\frac{du}{dx} = kf'(x)$. Here, k only serves as a proportionality constant and can be pulled out from the derivative.

Memo: Pulling down the power then decreasing the power by one.

Derivative of x^n Derivative of functions in the form of x^n, where $n \neq 0$ is a constant, is given by $\frac{dx^n}{dx} = nx^{n-1}$.

Distributive Law Derivative of a function composed by adding up multiple functions follows the distributive law, meaning that taking derivative on the entire function is equivalent to doing so on each term separately. Mathematically, derivative of $y = u + v = f(x) + g(x)$ is

$$\frac{dy}{dx} = \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} = f'(x) + g'(x)$$

Example 2.1.1 Find the derivative of $2x^2 + \frac{9}{x} + 5$.

$$\begin{aligned}\frac{d}{dx}(2x^2 + \frac{9}{x} + 5) &= \frac{d}{dx}(2x^2) + \frac{d}{dx}(9x^{-1}) + \frac{d}{dx}(5) \\ &= 2\frac{d}{dx}(x^2) + 9\frac{d}{dx}(x^{-1}) + \frac{d}{dx}(5) \\ &= 2(2x) + 9(-x^{-2}) + (0) \\ &= 4x - \frac{9}{x^2}\end{aligned}$$

Derivatives of Trigonometric Functions Derivatives of common trigonometric functions like $\sin x$ and $\cos x$ are given in the following table.

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x = \frac{1}{\cos x}$	$\sec x \tan x$
$\cot x = \frac{1}{\tan x}$	$-\csc^2 x$
$\csc x = \frac{1}{\sin x}$	$-\csc x \cot x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$

Derivatives of e^x and $\ln x$ Derivatives of exponential function e^x and logarithmic function $\ln x$, are e^x and $\frac{1}{x}$ respectively. Notice the derivative of e^x returns itself.

Product Rule If a function is composed by the product of two functions, i.e. $y = uv = f(x)g(x)$, then its derivative is found by the product rule, which is

$$\frac{dy}{dx} = \frac{d(uv)}{dx} = \frac{du}{dx}v + u\frac{dv}{dx} = f'(x)g(x) + f(x)g'(x)$$

For a product composed by more than two functions, we can apply product rule recursively.

Example 2.1.2 Find the derivative of $y = uv$, in which $u = e^x$ and $v = \cos x$.

$$\begin{aligned}\frac{d(e^x \cos x)}{dx} &= \frac{de^x}{dx}\cos x + e^x \frac{d\cos x}{dx} && \text{Product Rule}\\ &= e^x \cos x - e^x \sin x\end{aligned}$$

Subsequently, find the derivative of $y = uvw$, in which $u = x^2$, $v = e^x$, and $w = \cos x$.

$$\begin{aligned}\frac{d(x^2 e^x \cos x)}{dx} &= \frac{dx^2}{dx}(e^x \cos x) + x^2 \frac{d(e^x \cos x)}{dx} && \text{Product Rule}\\ &= 2xe^x \cos x + x^2 \frac{d(e^x \cos x)}{dx}\\ &= 2xe^x \cos x + x^2(e^x \cos x - e^x \sin x)\end{aligned}$$

Quotient Rule Similarly we have the quotient rule for function in form of $y = \frac{u}{v} = \frac{f(x)}{g(x)}$, which is

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Example 2.1.3 Find the derivative of $y = \frac{u}{v}$, for $u = \ln x$ and $v = x$.

$$\begin{aligned}\frac{d}{dx}\left(\frac{\ln x}{x}\right) &= \frac{\frac{d\ln x}{dx}x - \ln x\frac{dx}{dx}}{x^2} && \text{Quotient Rule}\\ &= \frac{\frac{1}{x}x - \ln x}{x^2}\\ &= \frac{1 - \ln x}{x^2}\end{aligned}$$

Chain Rule An important problem arises when we have to find the derivative of composite functions, like $(3x+2)^5$ or $\sin(\ln(x+1))$. A composite function is a function that enclose another function. The simplest composite function would be in the form of $y = f(z) = f(g(x))$ where $z = g(x)$ is a function of x and serves as the input to the function f. In the previous example $y = (3x+2)^5$, $y = f(z)$ would be z^5 and $z = g(x)$ would be $3x+2$.

Now to evaluate the the derivative of composite functions we apply Chain Rule, the name of which comes from the way that the resulted terms are chained one by one. For $y = f(z)$ and $z = g(x)$, Chain Rule gives

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = f'(z)g'(x)$$

Similarly, for $y = f(z)$, $z = g(u)$, and $u = h(x)$, we have

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{du}\frac{du}{dx} = f'(z)g'(u)h'(x)$$

In case of more variables, Chain Rule follows the same manner as above.

Example 2.1.4 Find the derivative of $y = (2x+1)^3$ by Chain Rule.

To use Chain Rule, we identify $y = f(z) = z^3$ and $z = g(x) = 2x+1$, then we have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dz}\frac{dz}{dx} && \text{Chain Rule} \\
&= \frac{dz^3}{dz}\frac{d(2x+1)}{dx} \\
&= (3z^2)\frac{d(2x+1)}{dx} \\
&= (3(2x+1)^2)(2) \\
&= 6(2x+1)^2
\end{aligned}$$

We could have directly written

$$\begin{aligned}
\frac{dy}{dz} &= \frac{d(2x+1)^3}{d(2x+1)} \\
&= 3(2x+1)^2
\end{aligned}$$

by treating $(2x+1)$ as a whole.

To verify the answer, we expand $(2x+1)^3$ and differentiate term by term.

$$\begin{aligned}
\frac{df}{dx} &= \frac{d}{dx}(2x+1)^3 \\
&= \frac{d}{dx}(8x^3+12x^2+6x+1) \\
&= \frac{d}{dx}(8x^3) + \frac{d}{dx}(12x^2) + \frac{d}{dx}(6x) + \frac{d}{dx}(1) \\
&= 8\frac{d}{dx}(x^3) + 12\frac{d}{dx}(x^2) + 6\frac{d}{dx}(x) + \frac{d}{dx}(1) \\
&= 8(3x^2) + 12(2x) + 6(1) + (0) \\
&= 24x^2 + 24x + 6 \\
&= 6(2x+1)^2
\end{aligned}$$

Example 2.1.5 Calculate the derivative of $e^{2x+\cos x}$.

Writing $y = f(z) = e^z$ and $z = g(x) = 2x + \cos x$, we have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dz}\frac{dz}{dx} && \text{Chain Rule} \\
&= \frac{d(e^z)}{dz}\frac{d(2x+\cos x)}{dx} \\
&= (e^z)(2-\sin x) \\
&= (2-\sin x)e^{2x+\cos x}
\end{aligned}$$

Example 2.1.6 Calculate the derivative of $\sin(\ln(x^2+1))$.

We let $y = f(z) = \sin z$, $z = g(u) = \ln u$, $u = h(x) = x^2+1$, then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dz}\frac{dv}{du}\frac{du}{dx} && \text{Chain Rule} \\
&= \frac{d(\sin(\ln(x^2+1))}{(\ln(x^2+1))}\frac{d(\ln(x^2+1))}{d(x^2+1)}\frac{d(x^2+1)}{dx}
\end{aligned}$$

Alternative: Write $\frac{d\sin z}{dz}\frac{d\ln u}{du}\frac{d(x^2+1)}{dx}$ as in Example 2.1.4.

Differentiating each by each, we have

$$\begin{aligned}
\frac{dy}{dx} &= (\cos(\ln(x^2+1))\left(\frac{1}{x^2+1}\right)(2x) \\
&= \frac{2x}{x^2+1}(\cos(\ln(x^2+1))
\end{aligned}$$

Common mistake: Treat x in the power as a constant and get $x(x^{x-1})$.

Example 2.1.7 Evaluate $\frac{d}{dx}x^x$.

$$\begin{aligned}
\frac{d}{dx}x^x &= \frac{d}{dx}e^{\ln(x^x)} \\
&= \frac{d}{dx}e^{x\ln x} \\
&= \left(\frac{d}{d(x\ln x)}e^{x\ln x}\right)\left(\frac{d}{dx}(x\ln x)\right) && \text{Chain Rule} \\
&= e^{x\ln x}\left(x\frac{d}{dx}\ln x + \ln x\frac{d}{dx}x\right) && \text{Product Rule} \\
&= e^{\ln(x^x)}\left(x\frac{1}{x} + \ln x\right) \\
&= x^x(1+\ln x)
\end{aligned}$$

Higher-order Derivatives We can define higher-order derivatives, which represent differentiation for multiple times. We use notations like $\frac{d^n}{dx^n}$ and $^{(n)}(x)$ to denote n-th order derivative with respect to x. Specifically, second-order derivative for the function $y = f(x)$ is written as

$$\frac{d^2y}{dx^2} = f''(x)$$

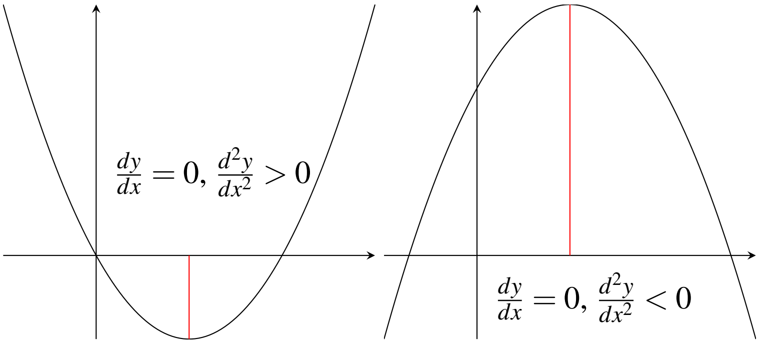
Example 2.1.8 Find $\frac{d^2}{dx^2}(e^{x^2})$.

$$\begin{aligned} \frac{d}{dx}(e^{x^2}) &= \frac{d(e^{x^2})}{d(x^2)}\frac{d(x^2)}{dx} && \text{Chain Rule} \\ &= (e^{x^2})(2x) \end{aligned}$$

$$\begin{aligned} \frac{d^2}{dx^2}(e^{x^2}) &= \frac{d}{dx}(\frac{d}{dx}(e^{x^2})) \\ &= \frac{d}{dx}((e^{x^2})(2x)) \\ &= (\frac{d}{dx}(e^{x^2}))(2x) + (e^{x^2})(\frac{d}{dx}(2x)) && \text{Product Rule} \\ &= ((e^{x^2})(2x))(2x) + (e^{x^2})(2) \\ &= (4x^2+2)e^{x^2} \end{aligned}$$

Critical Point and Stationary Value $y = f(x)$ reaches its local maximum or minimum only if $\frac{dy}{dx} = f'(x)$ is zero at that point. Such point is called the critical point and the value of the function there is called the stationary value. However, not all critical points are local maximum or minimum.

If indeed they are, we can distinguish them by the second-order derivative test. If $\frac{d^2y}{dx^2} = y''(x) > 0$ then the function concaves upwards, and the critical point is a local minimum, and if $\frac{d^2y}{dx^2} = y''(x) < 0$ then the function concaves downward, and it is a local maximum.



Left: Local minimum of a function. Right: Local maximum of a function.

If the result of the second-order derivative test is inconclusive, then we can observe the change of $\frac{dy}{dx} = f'(x)$ which is the slope near the critical point, if the sign changes from positive to negative, it is a local maximum, and the opposite implies a local minimum.

Example 2.1.9 Find the local maximum or minimum of $y = x^3 - 2x^2 + x - 6$, if there are any.

We set $\frac{dy}{dx} = 0$ to find the critical point, which leads to

$$\begin{aligned} \frac{d}{dx}(x^3) - \frac{d}{dx}(2x^2) + \frac{d}{dx}(x) - \frac{d}{dx}6 = 0 \\ 3x^2 - 4x + 1 = 0 \\ x = 1 \text{ or } \frac{1}{3} \end{aligned}$$

Alternative: Look at how $\frac{dy}{dx}$ changes sign near $x = 1$ and $x = \frac{1}{3}$.

We compute the second-order derivative for the test, which is

$$\frac{d^2y}{dx^2} = 6x - 4$$

At $x = 1$, $\frac{d^2y}{dx^2} = 6(1) - 4 = 2 > 0$, hence it is a local minimum.
At $x = \frac{1}{3}$, $\frac{d^2y}{dx^2} = 6(\frac{1}{3}) - 4 = -2 < 0$, hence it is a local maximum.

Taylor's Series Taylor's Series of a function is an infinite series which approximates the function near a particular point. For a function $y = f(x)$, at $x = a$, its Taylor's Series is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

where $f^{(n)}(a)$ denotes the n-th order derivative of $f(x)$ at $x = a$.

If $a = 0$, then it reduces to a Maclaurin Series, which has the form of

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

Example 2.1.10 Find the Taylor's Series of $\sin x$ at $x = 0$.

The n-th order derivatives of $f(x) = \sin x$ is

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \\ &\ldots \end{aligned}$$

and the pattern repeats for every 4 times of differentiation. Substituting $x = 0$, then we have

$$\begin{aligned} f(0) &= \sin 0 = 0 \\ f'(0) &= \cos 0 = 1 \\ f''(x) &= -\sin 0 = 0 \\ f'''(x) &= -\cos 0 = -1 \\ f^{(4)}(x) &= \sin 0 = 0 \\ &\ldots \end{aligned}$$

Hence the Taylor's Series of $\sin x$ at $x = 0$ is

$$\begin{aligned} \sin x &= 0 + x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{-1}{7!}x^7 + \ldots \\ &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots \end{aligned}$$

List of Common Taylor's Series We list out the commonly used Taylor's Series for some functions in the list below.

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\
\tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \ldots \\
(1+x)^n &= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots
\end{aligned}$$

However, the Taylor's Series does not always converge, that is, approaches the should-be value, if x is outside a certain range, called the radius of convergence. The readers are encouraged to check the radii of convergence of the Taylor's Series given above.

Example 2.1.11 Prove Euler's Formula, that is

$$e^{\imath\theta} = \cos\theta + \imath\sin\theta$$

Express both sides in Taylor's Series, we have

$$\begin{aligned}
\text{L.H.S.} &= 1 + \imath\theta + \frac{(\imath\theta)^2}{2!} + \frac{(\imath\theta)^3}{3!} + \frac{(\imath\theta)^4}{4!} + \ldots \\
&= 1 + \imath\theta - \frac{\theta^2}{2!} - \frac{\imath\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots
\end{aligned}$$

$$\begin{aligned}
\text{R.H.S} &= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots) + \imath(\theta - \frac{\theta^3}{3!} + \ldots) \\
&= 1 + \imath\theta - \frac{\theta^2}{2!} - \frac{\imath\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots = \text{L.H.S.}
\end{aligned}$$

Partial Derivatives Partial derivatives is the derivative with respect to a certain variable when the function being differentiated consists of multiple variables. To proceed, we treat all other variables as constants and carry out differentiation as usual. We use notations such as

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$$

to represent partial differentiation against x and y. Rules like Product Rule, Quotient Rule, and Chain Rule assumes similar forms but with partial derivative signs.

Example 2.1.12 Find the partial derivatives of $\sin(x^2y)$ with respect to x and y.

$$\begin{aligned}\frac{\partial}{\partial x}(\sin(x^2y)) &= \frac{\partial(\sin(x^2y))}{\partial(x^2y)}\frac{\partial(x^2y)}{\partial x} && \text{Chain Rule}\\ &= (\cos(x^2y))(2xy)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial y}(\sin(x^2y)) &= \frac{\partial(\sin(x^2y))}{\partial(x^2y)}\frac{\partial(x^2y)}{\partial y} && \text{Chain Rule}\\ &= (\cos(x^2y))(x^2)\end{aligned}$$

Clairaut's Theorem Clairaut's Theorem states that for mixed partial derivatives, the order of differentiation does not matter, if the relevent derivatives are continuous. Specfically, for second-order mixed partial derivatives, we have

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

or using subscripts to denote partial derivatives, we have

$$z_{xy} = z_{yx}$$

where $z(x,y)$ is a function of x and y.

Example 2.1.13 Verify Clairaut's Theorem for second-order partial derivatives on $z = x^2y^2 + xy^3$.

$$\begin{aligned}z_{xy} &= (x^2y^2 + xy^3)_{xy}\\ &= (2xy^2 + y^3)_y\\ &= 4xy + 3y^2\end{aligned}$$

$$\begin{aligned}z_{yx} &= (x^2y^2 + xy^3)_{yx}\\ &= (2x^2y + 3xy^2)_x\\ &= 4xy + 3y^2 = z_{xy}\end{aligned}$$

Critical Point and Stationary Value - Revisited For a function $z = f(x,y)$ with two variables, if z_x and z_y both are zero at a point then it is a critical point. To know if it is a local minimum or maximum, we apply an extended version of the second-order derivative test. If $z_{xx} > 0$ and $z_{xx}z_{yy} - z_{xy}^2 > 0$, it is a local minimum. If $z_{xx} < 0$ and $z_{xx}z_{yy} - z_{xy}^2 > 0$, it is a local maximum. Otherwise if $z_{xx}z_{yy} - z_{xy}^2 < 0$, then it is called a saddle point.

Example 2.1.14 Find the critical point of $z(x,y) = x^2 + y^2$ and determine its nature.

The first-order partial derivative is easily seen to be $z_x = 2x$ and $z_y = 2y$. Only at $(0,0)$, they will both become zero. Hence $(0,0)$ is the desired critical point. Next we apply the second-order derivative test by computing

$$\begin{aligned} z_{xx}z_{yy} - z_{xy}^2 &= (2)(2) - (0)^2 \\ &= 4 > 0 \end{aligned}$$

for any point on the surface. Therefore, it is a local minimum.

Chain Rule for Multiple Variables If $u = f(x,y,z)$ is a function of the intermediate variables $x(t)$, $y(t)$, and $z(t)$, which are functions of another variable t, then Chain Rule takes the form of

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

It follows the same manner for two intermediate variables.

Example 2.1.15 Find $\frac{dz}{dt}$ if $z = x^2 + e^{-y^2}$ with $x = \sin t$ and $y = \cos t$.

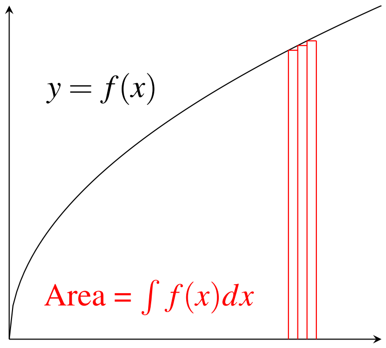
$$
\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} && \text{Chain Rule for Multiple Variables} \\
&= \frac{\partial(x^2+e^{-y^2})}{\partial x}\frac{dx}{dt} + \frac{\partial(x^2+e^{-y^2})}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial(x^2)}{dx}\frac{dx}{dt} + \frac{\partial(e^{-y^2})}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial(x^2)}{dx}\frac{d(\sin t)}{dt} + \frac{\partial(e^{-y^2})}{\partial(-y^2)}\frac{\partial(-y^2)}{\partial y}\frac{d(\cos t)}{dt} && \text{Chain Rule} \\
&= 2x\frac{d(\sin t)}{dt} + (e^{-y^2})(-2y)\frac{d(\cos t)}{dt} \\
&= 2x(\cos t) - 2ye^{-y^2}(-\sin t) \\
&= 2\sin t\cos t + 2\sin t\cos t e^{(-\cos t)^2}
\end{aligned}
$$

2.1.2 INTEGRATION

Integral and Integration Integration is the inverse operation of differentiation. An integral of $y = f(x)$ has the form of

$$\int f(x)dx$$

where $\int dx$ denotes integration with respect to x, with dx as an infinitely small segment along the x direction. This form is also called indefinite integral. Geometrically, it is the signed area under the curve $y = f(x)$ expressed in terms of x. When we calculate such integral, the end result has to include an integration constant since the area does not have an interval specified.



A function (black line) and its area integral shown. Individual bar has a width of dx and an area of $f(x)dx$. Their sum is the value of the integral $\int f(x)dx$.

Integral of $kf(x)$ Integral in form of $kf(x)$ where k is a constant is simply $\int kf(x)dx = k\int f(x)dx$. k can be pulled outside the integral sign, similar to what we have for differentiation.

Memo: Raising the power by 1 and dividing by the new power.

Integral of x^n Integral of functions in the form of x^n, where $n \neq -1$ is a constant, is given by $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$, where C denotes the integration constant.

Distributive Law Integral, just like derivative, follows the distributive law, which means that we can apply the integral separately on each additive term. If $f(x) = g(x) + h(x)$, then we have

$$\int f(x)dx = \int (g(x) + h(x))dx = \int g(x)dx + \int h(x)dx$$

Common mistake: Forgetting to add the integration constant.

Example 2.1.16 Find $\int (6x^3 + \frac{8}{x^2} + 1)dx$.

$$\begin{aligned}\int (6x^3+\frac{8}{x^2}+1)dx &= \int 6x^3dx+\int 8x^{-2}dx+\int dx\\ &= 6\int x^3dx+8\int x^{-2}dx+\int dx\\ &= 6(\frac{1}{4}x^4)+8(-x^{-1})+x+C\\ &= \frac{3}{2}x^4-\frac{8}{x}+x+C\end{aligned}$$

Integrals of e^x and $\frac{1}{x}$ Integration of e^x and $\frac{1}{x}$ gives e^x and $\ln|x|$. It is basically the reverse of differentiating e^x and $\ln x$ described in earlier section, but with an absolute sign in ln.

Integrals of Trigonometric Functions Below is a table summarizing integrals involving common trigonometric functions, which is closely related with the afore-mentioned table of derivatives of trigonometric functions.

$f(x)$	$\int f(x)dx\ (+C)$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$
$\sec x\tan x$	$\sec x$
$\csc^2 x$	$-\cot x$
$\csc x\cot x$	$-\csc x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$-\frac{1}{\sqrt{1-x^2}}$	$\cos^{-1} x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$

Example 2.1.17 Evaluate $\int(2\sec x\tan x+\frac{3}{x})dx$.

$$\begin{aligned}\int (2\sec x\tan x+\frac{3}{x})dx &= \int 2\sec x\tan x dx+\int \frac{3}{x}dx\\ &= 2\int \sec x\tan x dx+3\int \frac{1}{x}dx\\ &= 2\sec x+3\ln x+C\end{aligned}$$

Integration by Substitution Integration by substitution is an important integration technique which is essentially the reverse of differentiation by Chain Rule. It involves changing the differential dx into du, where u is a function of x, then integrating with respect to u. The formulation is

$$\int f(g(x))\frac{du}{dx}dx = \int f(u)du$$

where we choose $u = g(x)$ and $du = \frac{du}{dx}dx$. Now we would look at some examples to understand how it works.

Example 2.1.18 Integrate $x(x^2+3)^4$.

To use integration by substitution, we choose $u = x^2+3$ and $du = \frac{du}{dx}dx = 2xdx$, then we have

Alternative: Expand $x(x^2+3)^4$ and integrate term by term.

$$\begin{aligned}
\int x(x^2+3)^4 &= \int \frac{1}{2}(x^2+3)^4(2xdx) \\
&= \int \frac{1}{2}u^4 du && \text{Integration by Substitution} \\
&= \frac{1}{10}u^5 + C \\
&= \frac{1}{10}(x^2+3)^5 + C
\end{aligned}$$

Example 2.1.19 Evaluate $\int \tan(2x+7)dx$.

First, rewriting the integral as

$$\int \tan(2x+7)dx = \int \frac{\sin(2x+7)}{\cos(2x+7)}dx$$

We choose $u = \cos(2x+7)$, then by Chain Rule $du = \frac{du}{dx}dx = -2\sin(2x+7)dx$, subsequently

$$\begin{aligned}
\int \tan(2x+7)dx &= \int \frac{-1}{2\cos(2x+7)}(-2\sin(2x+7)dx) \\
&= -\int \frac{1}{2u}du && \text{Integration by Substitution} \\
&= -\frac{1}{2}\ln|u| + C \\
&= -\frac{1}{2}\ln|\cos(2x+7)| + C
\end{aligned}$$

Trigonometric Substitution Trigonometric substitution is a type of integration by substitution manipulating trigonometric functions. It is particularly useful when dealing with square root.

Example 2.1.20 Integrate $\frac{1}{x\sqrt{x^2+1}}$.

To use trigonometric substitution, observe that $\tan^2\theta + 1 = \sec^2\theta$. Letting $x = \tan\theta$ and $dx = \frac{dx}{d\theta}d\theta = \sec^2\theta d\theta$, we have

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2+1}}dx &= \int \frac{1}{\tan\theta\sqrt{\tan^2\theta+1}}(\sec^2\theta d\theta) \quad \text{Integration by substitution} \\
&= \int \frac{\sec^2\theta}{\tan\theta\sec\theta}d\theta \\
&= \int \csc\theta d\theta
\end{aligned}$$

Further observe that if we let $u = \csc\theta + \cot\theta$, then

$$\begin{aligned}
\frac{du}{d\theta} &= -\csc\theta\cot\theta - \csc^2\theta \\
&= -\csc\theta(\csc\theta + \cot\theta)
\end{aligned}$$

now we have

$$\begin{aligned}
\int \csc\theta d\theta &= \int \csc\theta \frac{\csc\theta + \cot\theta}{\csc\theta + \cot\theta}d\theta \\
&= \int -\frac{(-\csc\theta(\csc\theta + \cot\theta)d\theta)}{\csc\theta + \cot\theta} \\
&= \int -\frac{du}{u} \qquad \text{Integration by Substitution} \\
&= -\ln|u| + C \\
&= -\ln|\csc\theta + \cot\theta| + C
\end{aligned}$$

With $x = \tan\theta$, then $\cot\theta = \frac{1}{x}$ and $\csc\theta = \sqrt{1+\cot^2\theta} = \sqrt{1+\frac{1}{x^2}} = \frac{1}{x}\sqrt{x^2+1}$,

$$\begin{aligned}
\int \csc\theta d\theta &= -\ln|\csc\theta + \cot\theta| + C \\
&= -\ln|\frac{1}{x}\sqrt{x^2+1} + \frac{1}{x}| + C \\
&= -\ln|\frac{1}{x}(\sqrt{x^2+1}+1)| + C \\
&= -\ln|\frac{1}{x}| - \ln|\sqrt{x^2+1}+1| + C \\
&= \ln|x| - \ln|\sqrt{x^2+1}+1| + C
\end{aligned}$$

Partial Fraction Partial fraction is a technique that decomposes a fraction into parts that are easier to be integrated each by each. We would see how the method works in the following example.

Example 2.1.21 Integrate $\frac{2x+7}{(x-1)(x+4)}$.

We assume that the expression can be written in the form of partial fractions which is

$$\frac{3x+7}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}$$

where A, B are some constants. Factoring the right hand side gives

$$\frac{A}{x-1} + \frac{B}{x+4} = \frac{A(x+4)+B(x-1)}{(x-1)(x+4)} = \frac{(A+B)x+(4A-B)}{(x-1)(x+4)}$$

By comparing coefficients, we have

$$\begin{aligned}
A+B &= 3 \\
4A-B &= 7
\end{aligned}$$

which has a solution of $A = 2$, $B = 1$. Hence

$$\begin{aligned}
\int \frac{3x+7}{(x-1)(x+4)}dx &= \int (\frac{2}{x-1} + \frac{1}{x+4})dx \\
&= 2\ln|x-1| + \ln|x+4| + C \quad \text{Integration by Substitution}
\end{aligned}$$

For fractions that are more complicated, the form of partial fractions would change accordingly.

Integration by Parts Integration by parts is another important integration method which is an integration counterpart of Product Rule for differentiation. Mathematically, it is expressed as

$$\int u\frac{dv}{dx}dx = \int udv = uv - \int vdu = uv - \int v\frac{du}{dx}dx$$

Example 2.1.22 Evaluate $\int xe^x dx$.

Let $u = x$ and $v = e^x$, also note that $dv = e^x dx$, we have

$$\begin{aligned}
\int xe^x dx &= \int udv && \text{Integration by Substitution} \\
&= uv - \int vdu && \text{Integration by Parts} \\
&= xe^x - \int e^x dx \\
&= xe^x - e^x + C
\end{aligned}$$

Example 2.1.23 Find $\int \ln x dx$.

We can choose $u = \ln x$ and $v = x$ then the integral is in the form of $\int udv$.

$$\begin{aligned}
\int \ln x dx &= x\ln x - \int xd(\ln x) && \text{Integration by Parts} \\
&= x\ln x - \int x\left(\frac{1}{x}\right)dx \\
&= x\ln x - \int dx \\
&= x\ln x - x + C
\end{aligned}$$

Example 2.1.24 Find $\int e^x \sin x dx$.

Alternative: Start with $\int \sin x d(e^x)$.

Common mistake: Forgetting the minus sign in $d(\cos x) = -\sin x dx$.

$$\begin{aligned}
\int e^x \sin x dx &= \int -e^x d(\cos x) && \text{Integration by Substitution} \\
&= (-e^x \cos x) - \int \cos x d(-e^x) && \text{Integration by Parts} \\
&= (-e^x \cos x) + \int \cos x e^x dx \\
&= -e^x \cos x + \int e^x d(\sin x) && \text{Integration by Substitution} \\
&= -e^x \cos x + (e^x \sin x) - \int \sin x d(e^x) && \text{Integration by Parts} \\
&= e^x(\sin x - \cos x) - \int e^x \sin x dx
\end{aligned}$$

Common mistake: Forgetting to include the integration constant at the last step.

Hence

$$\begin{aligned}
\int e^x \sin x dx &= e^x(\sin x - \cos x) - \int e^x \sin x dx \\
2\int e^x \sin x dx &= e^x(\sin x - \cos x) \\
\int e^x \sin x dx &= \frac{1}{2} e^x(\sin x - \cos x) + C
\end{aligned}$$

Fundamental Theorem of Calculus Fundamental Theorem of Calculus relates indefinite integral to definite integral which has an interval, i.e. a lower limit and an upper limit, and is written as

$$\int_a^b f(x)dx$$

which represents the signed area under the curve $y = f(x)$ inside the interval $x = [a, b]$. If $f(x)$ integrates to $F(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Integration variables like x in the definite integral above is called a dummy variable since eventually the results do not depend on them and thus they can be replaced by any other dummy variables. Also, we have properties like

$$\begin{aligned}
\int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx \\
\int_b^a f(x)dx &= -\int_a^b f(x)dx
\end{aligned}$$

Another form of Fundamental Theorem of Calculus is

$$\frac{d}{du}\int_a^u f(x)dx = f(u)$$

where u is a function of x. The geometrical meaning of this form is, the rate of change of the area under the curve $y = f(x)$ as the upper limit u changes equals to the value of $f(u)$ there. Definite integrals have no integration constants, since it is cancelled out as in $F(b) - F(a)$.

Example 2.1.25 Find $\int_1^{10} e^{3x}dx$.

$$\begin{aligned}
\int_1^{10} e^{3x}dx &= \frac{1}{3}\int_1^{10} e^{3x}d(3x) && \text{Integration by Substitution} \\
&= \frac{1}{3}[e^{3x}]_1^{10} \\
&= \frac{1}{3}(e^{3(10)} - e^{3(1)}) && \text{Fundamental Theorem of Calculus} \\
&= \frac{1}{3}(e^{30} - e^3)
\end{aligned}$$

Caution: The limits are for the variable x. If you want to write $u = 3x$, the limits have to be changed accordingly.

Example 2.1.26 Find $\frac{d}{dx}\int_a^{x^2}(x^3+1)dx$.

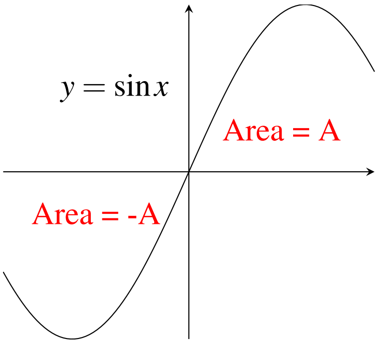
Let the upper limit be $u = x^2$, then applying Chain Rule, we have

$$\begin{aligned}
\frac{d}{dx}\int_a^{x^2}(x^3+1)dx &= \left[\frac{d}{du}\int_a^u (x^3+1)dx\right]\frac{du}{dx} && \text{Chain Rule} \\
&= (u^3+1)\frac{d(x^2)}{dx} && \text{Fundamental Theorem of Calculus} \\
&= (x^6+1)(2x)
\end{aligned}$$

Example 2.1.27 Find $\int_{-a}^{a} \sin x dx$.

Notice that $\sin(-x) = -\sin x$, that is, sine is an odd function. Then graphically, the signed area of sine inside $[-a, 0]$ and $[0, a]$ cancels out each other. Therefore, the integral evaluates to zero.

Alternative: Proceed as usual and compute the definite integral like Example 2.1.24.



A sine function which is shown to have equal area but different signs on both sides.

Similar logic can be applied to even functions. If $f(x)$ is even, we have

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

Multi-variable Integration To integrate a function in multiple independent variables, proceed similarly to the calculation of partial derivatives, treating all other variables as constants, except the one inside the differential. It can be viewed as the inverse of partial differentiation.

Caution: x here is treated as a constant here. Hence x can be pulled into the differential, $xdy = d(xy)$.

Common mistake: Integration constant not as a function of x.

Example 2.1.28 Find $\int f(x,y)dy$ where $f(x,y) = xe^{xy}$.

$$\begin{aligned} \int xe^{xy}dy &= \int e^{xy}d(xy) && \text{Integration by Substitution} \\ &= e^{xy} + C(x) \end{aligned}$$

where $C(x)$ is any function of x but not y. $C(x)$ would vanish and give us back the original expression if we are to apply the reverse operation, partial differentiation with respect to y.

2.1.3 MISCELLANEOUS

Operations on Differentials To evaluate differentials consisting of multiple variables like $d(xy)$ and $d(\frac{x}{y})$, we proceed as if calculating derivatives but without the differential in the denominator. Implicitly used is the Chain Rule for multiple variables.

Example 2.1.29 Evaluate $d(xy)$ and $d(\frac{x}{y})$.

$$d(xy) = xdy + ydx \qquad \text{Product Rule}$$
$$d(\frac{x}{y}) = \frac{ydx - xdy}{y^2} \qquad \text{Quotient Rule}$$

Limits and L'Hospital's Rule We have purposely omitted the treatment on limits, which is sometimes used to express where a quantity approaches when the independent variable tends to zero, infinity, or some singularities. Here we introduce the most important practical result of limits, which is the L'Hospital's Rule. It states that

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

as long as both the limits of $f(x)$ and $g(x)$ goes to zero, or infinity. It can be repeatedly applied if this condition holds.

Example 2.1.30 Find the limit of

$$\lim_{x\to 0} \frac{\sin x}{x}$$

Since $\lim_{x\to 0} \sin x$ and $\lim_{x\to 0} x$ both goes to zero, we can apply L'Hospital's Rule.

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} \qquad \text{L'Hospital's Rule}$$
$$= 1$$

Hyperbolic Functions Hyperbolic functions are composed of exponential functions which have properties similar to trigonometric functions. The two most commonly used hyperbolic functions are $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$. Their derivatives are

$$\begin{aligned}\frac{d(\sinh x)}{dx} &= \frac{d}{dx}(\frac{e^x - e^{-x}}{2}) \\ &= \frac{e^x + e^{-x}}{2} && \text{Chain Rule} \\ &= \cosh x\end{aligned}$$

and

$$\begin{aligned}\frac{d(\cosh x)}{dx} &= \frac{d}{dx}(\frac{e^x + e^{-x}}{2}) \\ &= \frac{e^x - e^{-x}}{2} && \text{Chain Rule} \\ &= \sinh x\end{aligned}$$

and conversely, the integrals of $\sinh x$ and $\cosh x$ are $\cosh x$ and $\sinh x$. It can also be observed that the second-order derivatives of $\sinh x$ and $\cosh x$ return themselves.

Some important hyperbolic identities are

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= 1 \\ \cosh^2 x + \sinh^2 x &= \cosh 2x \\ 2\sinh x \cosh x &= \sinh 2x\end{aligned}$$

Alternative: Write the expression in terms of exponential functions.

Example 2.1.31 Differentiate and integrate $\sinh^2 x \cosh x$.

$$\begin{aligned}\frac{d}{dx}(\sinh^2 x \cosh x) &= (\sinh^2 x)\frac{d}{dx}(\cosh x) + (\cosh x)\frac{d}{dx}(\sinh^2 x) && \text{Product Rule} \\ &= (\sinh^2 x)(\sinh x) + (\cosh x)(2\sinh x \cosh x) && \text{Chain Rule} \\ &= \sinh x(\sinh^2 x + 2\cosh^2 x)\end{aligned}$$

$$\begin{aligned}\int \sinh^2 x \cosh x dx &= \int \sinh^2 x d(\sinh x) && \text{Integration by Substitution} \\ &= \frac{1}{3}\sinh^3 x + C\end{aligned}$$

Displacement, Velocity, Acceleration Displacement or position of an object along an axis, denoted as $x(t)$, which is a function of time t, can be differentiated to obtain the velocity $v = \frac{dx}{dt}$. Differentiating once more gives the acceleration

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$.

Reversely, acceleration can be integrated to retrieve the velocity by $v = \int a(t)dt = \int \frac{dv}{dt}dt = \int dv$. Integrating the velocity similarly gives the displacement $s = \int v(t)dt$.

Example 2.1.32 An object's displacement is given by $x(t) = t^3 + 6t^2 + 12t + 4$ (in m). Find its velocity and acceleration at $t = 1$ (in s).

$$\begin{aligned} v &= \frac{ds}{dt} \\ &= \frac{d}{dt}(t^3 + 6t^2 + 12t + 4) \\ &= 3t^2 + 12t + 12 \end{aligned}$$

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= \frac{d}{dt}(3t^2 + 12t + 12) \\ &= 6t + 12 \end{aligned}$$

So at $t = 1\,\mathrm{s}$, the velocity and acceleration are $3(1)^2 + 12(1) + 12 = 27\,\mathrm{m\,s^{-1}}$ and $6(1) + 12 = 18\,\mathrm{m\,s^{-2}}$ respectively.

Example 2.1.33 If an object has a velocity of $v(t) = e^{-t}$. Find express the displacement in t if displacement at $t = 0$ is 0.

$$\begin{aligned} s &= \int v dt \\ &= \int e^{-t} dt \\ &= -\int e^{-t} d(-t) && \text{Integration by Substitution} \\ &= C - e^{-t} \end{aligned}$$

At $t = 0$, $s = 0$, so

$$\begin{aligned} C - e^{-0} &= 0 \\ C &= 1 \end{aligned}$$

Therefore, $s = 1 - e^{-t}$.

Example 2.1.34 Derive the commonly used formula for constant acceleration.

$$\begin{aligned} v &= \int_0^t a dt \\ &= u + at \\ s &= \int_0^t v dt \\ &= \int (u+at) dt \\ &= ut + \frac{1}{2}at^2 \end{aligned}$$

$$\begin{aligned} \int_u^v v dv &= \int_0^t \left(\frac{ds}{dt}\right)\left(\frac{dv}{dt}\right) dt \\ &= \int_0^s a ds \end{aligned}$$

Integrating both sides, we get

$$\begin{aligned} \left[\frac{1}{2}v^2\right]_u^v &= [as]_0^s \\ v^2 - u^2 &= 2as \end{aligned}$$

2.2 Basic Calculus in ESSC2020

2.3 Basic Calculus in ESSC3220

2.4 Problems

Question 2.1.1 Find the following derivatives.
(a) $\frac{d}{dx}(7x^4+6x^3+8x+11)$ (b) $\frac{d}{dx}(e^x+\frac{1}{x^2})$, (c) $\frac{d}{dx}(\tan x \ln x)$, and (d) $\frac{d}{dx}(\frac{x}{\sin x})$.

Question 2.1.2 Find the following derivatives.
(a) $\frac{d}{dx}(4x^2+5x+3)^3$, (b) $\frac{d}{dx}(\ln(\tan(x-1))$, (c) $\frac{d}{dx}(\cos^{-1}(x^2+1))$, (d) $\frac{d}{dx}e^{-x^2}$.

Question 2.1.3 Evaluate

$$\frac{d}{dx}(x^{\sin^2 x})$$

Question 2.1.4 Evaluate

$$\frac{d^2}{dx^2}(e^{\cos x})$$

Question 2.1.5 Find any local maxima or minima of
(a) $(x-3)(x^2+3x+1)$, (b) $\frac{x}{x^2+k^2}$, where k is a constant, (c) x^{e-x}, for $0<x<e$.

Question 2.1.6 Verify the Taylor's Series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

and

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

Question 2.1.7 Expand and obtain the first three terms of

$$(1+x)^{\frac{3}{2}}$$

Find the error of the approximation for $x = 1$.

Question 2.1.8 Find the partial derivatives with respect to x and y for the following expressions.
(a) $x^3+4xy^2+2y^3$, (b) $\frac{y}{\sqrt{x^2+y^2}}$, (c) $\cos(xe^y)$.

Question 2.1.9 Verify Clairaut's Theorem for second-order partial derivatives of

$$z = \frac{xe^y}{\ln x}$$

Question 2.1.10 Find any local maxima or minima for the following functions with two independent variables.
(a) $z = x^4+y^4$, (b) $z = xy^2$, (c) $z = \sin(\pi x)e^{-y^2}$.

Question 2.1.11 With $x = t^2$ and $y = e^t$, find $\frac{dz}{dt}$ if
(a) $z = x^2y$, and (b) $z = y\sin xy$.
Repeat for $x = \cos t$, $y = e^{-t^2}$.

Question 2.1.12 Calculate the following integrals.
(a) $\int(7x^2+\frac{4}{x^3})dx$, (b) $\int(\frac{1}{x}+\sec x\tan x)dx$.

Question 2.1.13 Evaluate the following integrals.
(a) $\int(x^2e^x)dx$, (b) $\int((x^3+3x+1)^2(x^2+1))dx$ (c) $\int(\sin^3 x)dx$, (d) $\int(\cos^4 x)dx$, (e) $\int(\sec^3 x)dx$, (f) $\int(x\ln x)dx$, (g) $\int(e^x\cos x)dx$.

Question 2.1.14 Compute the following integrals by trigonometric substitution.

$$\int\frac{dx}{\sqrt{x^2+a^2}}$$

and

$$\int\sqrt{x^2-a^2}dx$$

where a is some constant.

Question 2.1.15 Find

$$\int\frac{5x^2+2x-2}{(x-1)^2(2x+3)}dx$$

by assuming the partial fractions

$$\frac{5x^2+2x-2}{(x-1)^2(2x+3)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{2x+3}$$

Question 2.1.16 Find the following definite integrals.
(a) $\int_2^4(xe^{3x})dx$, (b) $\int_0^{2\pi}(\sin x\cos 2x)dx$.

Question 2.1.17 Evaluate the following expression.

$$\frac{d}{dx}\int_a^{\sin x}(4x^2+5)dx$$

Question 2.1.18 Evaluate the following integral.

$$\frac{d}{dx}\int_x^{x^3}(xe^{x^2+1})dx$$

Hints: Splitting the integral into

$$\frac{d}{dx}\int_a^{x^3}(xe^{x^2+1})dx-\frac{d}{dx}\int_a^{x}(xe^{x^2+1})dx$$

where a is any constant.

Question 2.1.19 Prove that

$$\int_{-a}^{a}(\frac{1}{1+e^x}-\frac{1}{2})dx=0$$

Question 2.1.20 Find the following integrals.
(a) $\int xye^{xy^2}dy$, (b) $\int \frac{x}{(x^2y+1)y}dx$.

Question 2.1.21 Integrate and differentiate

$$\tanh x=\frac{\sinh x}{\cosh x}$$

Question 2.1.22 Find the limit of

$$\lim_{x\to 0}\frac{1-e^{-x}\cos x}{e^{-x}\sin x}$$

Question 2.1.23 Evaluate

$$\int\frac{\sin x dx}{\sin x+\cos x}$$

by writing

$$\sin x=\frac{1}{2}(\sin x+\cos x)+\frac{1}{2}(\sin x-\cos x)$$

in the numerator.

Question 2.1.24 Evaluate

$$\int\frac{dx}{x\sqrt{x^2+a^2}}$$

by either trigonometric substitution, or making the substitution $u=\sqrt{x^2+a^2}$ and then apply the method of partial fraction.

Question 2.1.25 A ship spills oil which spreads on the sea surface uniformly in a radial manner. The rate of increase in the area covered by oil is $\frac{dA}{dt}=10\,\mathrm{m}^2$, find the moving speed of the oil edge, i.e. the rate of increase in radius $\frac{dr}{dt}$ by first expressing A in terms of r.

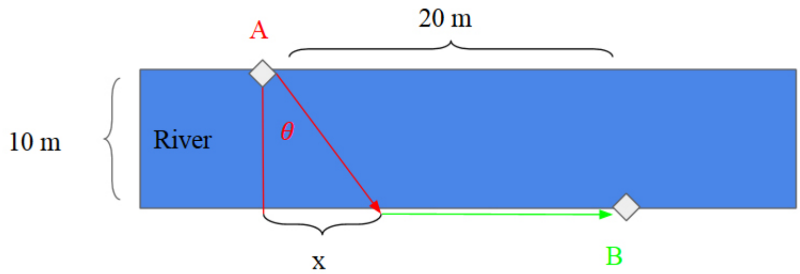
Question 2.1.26 A cone-shaped cup filled with water of a base radius of 4 cm and height of 10 cm is leaking out water at a rate of $\frac{dV}{dt} = -3\,\text{cm}^3$ through a hole at the bottom. Find the rate of change in the water level $\frac{dh}{dt}$ by first expressing V in terms of h.

Question 2.1.27 Population growth can be modelled by

$$\frac{dN}{dt} = kN(N_{\max} - N)$$

Find the value of N at the point where $\frac{d}{dt}(\frac{dN}{dt}) = \frac{d^2N}{dt^2} = 0$ which indicates the growth of population begins to decline.

Question 2.1.28 Point A and Point B marked on the following diagram is separated by a straight river with width = 10 m. Find the minimum time needed for a man to move from point A and point B if his swimming speed in the river is $2\,\text{m}\,\text{s}^{-1}$ and walking speed on land is $3\,\text{m}\,\text{s}^{-1}$.



Hints: Express the time t needed in terms of x indicated in the diagram then solve $\frac{dt}{dx} = 0$, where $0 < x < 20$. You should be able to get

$$t = \frac{\sqrt{10^2 + x^2}}{2} + \frac{20 - x}{3}$$

Also prove that the angle θ in this case is found by

$$\sin\theta = \frac{2}{3}$$

CHAPTER 3

ORDINARY DIFFERENTIAL EQUATION

CONTENTS

3.1 Introduction

Ordinary Differential Equation An ordinary differential equation (ODE) is a equation involving functions of only one independent variable and their derivatives. For examples,

$$\frac{dy}{dx} = xy$$
$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

Here x is the independent variable and $y = f(x)$ is the dependent variable. We are going to learn how to solve them, starting from the easiest one.

First-order ODE A first-order ODE. as its name suggests, is an ODE which involves derivative that is first-order only. This kind of ODE is often the easiest to solve. Nevertheless, there are still first-order ODEs that can't be solved analytically in elementary functions. The most common methods used are separation of variables and integrating factor.

Separation of Variables Separation of Variables requires rearranging the equation such that each side has terms involving only one variable and its corresponding differential, then integrating the both sides at the same time. To see how it works, take a look at the examples below.

Example 3.1.1 Solve $\frac{dy}{dx} = xy$.

We start with rearranging the equation to get

$$\frac{dy}{y} = xdx \qquad \text{Separation of Variables}$$

Common mistake: Forgetting to add the integration constant.

Integrating both sides gives

$$\int \frac{dy}{y} = \int xdx$$
$$\ln y = \frac{1}{2}x^2 + C$$
$$y = e^{(\frac{1}{2}x^2 + C)}$$
$$y = Ae^{\frac{1}{2}x^2}$$

where $A = e^C$.

Example 3.1.2 Solve $\frac{dy}{dx} = -\frac{x+3}{y}$.

We proceed as in Example 3.1.1, obtaining

$$\begin{aligned} \frac{dy}{dx} &= -\frac{x+3}{y} \\ ydy &= -(x+3)dx && \text{Separation of Variables} \\ \int ydy &= -\int (x+3)dx \\ \frac{1}{2}y^2 &= -\frac{1}{2}(x+3)^2 + C \\ (x+3)^2 + y^2 &= R^2 \end{aligned}$$

where $R^2 = 2C$. The solution represents a family of circles centered at $(-3, 0)$.

Integrating Factor For a first-order ODE in form of

$$\frac{dy}{dx} + P(x)y = Q(x)$$

separation of variables alone does not work so well. So the method of integrating factor is developed. By multiplying both sides by an integrating factor $e^{\int P(x)dx}$, we arrive at

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)y = e^{\int P(x)dx}Q(x)$$

Observe that the left hand side can be grouped such that it becomes

$$\frac{d}{dx}(e^{\int P(x)dx}y) = e^{\int P(x)dx}Q(x)$$

The readers can verify this by using Product Rule and Fundamental Theorem of Calculus.

Finally, we rearrange and integrate both sides to obtain

$$\begin{aligned} d(e^{\int P(x)dx}y) &= e^{\int P(x)dx}Q(x)dx \\ e^{\int P(x)dx}y &= \int e^{\int P(x)dx}Q(x)dx \end{aligned}$$

It looks rather complicated, so it is better to have some worked examples to demonstrate the principle.

Example 3.1.3 Solve $\frac{dy}{dx} + \frac{4}{x}y = 3$.

We identify $P(x) = \frac{4}{x}$, then we have the integrating factor as

$$\begin{aligned} e^{\int P(x)dx} &= e^{\int \frac{4}{x}dx} \\ &= e^{4\ln x} \\ &= x^4 \end{aligned}$$

Hence multiplying both sides by x^4, we have

$$\begin{aligned} x^4\frac{dy}{dx} + 4x^3y &= 3x^4 && \text{Integrating Factor} \\ \frac{d}{dx}(x^4y) &= 3x^4 \\ x^4y &= \int 3x^4dx \\ x^4y &= \frac{3}{5}x^5 + C \\ y &= \frac{3}{5}x + \frac{C}{x^4} \end{aligned}$$

Example 3.1.4 Solve $\frac{dy}{dx} + 2y = 1$.

Alternative: Use separation of variables.

This time, $P(x) = 2$, the integrating factor is

$$e^{\int_0^x 2dx} = e^{2x}$$

Now multiplying both sides by e^{2x}, then

$$\begin{aligned} e^{2x}\frac{dy}{dx} + 2e^{2x}y &= e^{2x} && \text{Integrating Factor} \\ \frac{d}{dx}(e^{2x}y) &= e^{2x} \\ e^{2x}y &= \int e^{2x}dx \\ e^{2x}y &= \frac{1}{2}e^{2x} + C \\ y &= \frac{1}{2} + Ce^{-2x} \end{aligned}$$

Exact Equations A first-order ODE is exact if it can be written as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

where $f(x,y)$ is a function of x and y. df is called the total differential here. In fact, it comes from Chain Rule for multiple variables discussed in the last chapter. Then we can solve it by integrating both sides to get

$$f(x,y) = C$$

In other words, a first-order ODE $Mdx + Ndy = 0$ is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Example 3.1.5 Solve $\frac{dy}{dx} = -\frac{2x+y}{x+2y}$.

Rearranging the equation gives

$$\begin{aligned}(x+2y)dy &= -(2x+y)dx \\ (2x+y)dx + (x+2y)dy &= 0\end{aligned}$$

The equation is exact since $\frac{\partial(2x+y)}{\partial y} = \frac{\partial(x+2y)}{\partial x} = 1$. $f(x,y)$ can be found by integrating $\int Mdx$ and $\int Ndy$, which gives

Common mistake: Integration constants have wrong dependence.

$$\begin{aligned}f &= \int (2x+y)dx = x^2 + xy + G(y) \\ f &= \int (x+2y)dy = xy + y^2 + H(x)\end{aligned}$$

Comparing the results we conclude that $f(x,y) = x^2 + y^2 + xy$. So we have

$$\begin{aligned}(2x+y)dx + (x+2y)dy &= 0 \\ df = d(x^2+y^2+xy) &= 0 \qquad \text{Exact Equation} \\ f(x,y) = x^2+y^2+xy &= C\end{aligned}$$

Implicit Differentiation Motivated by the above example, we want to verify its answer as it is not in a form where y is readily the subject. Instead the variables are related by an implicit equation. To do so we apply implicit differentiation, which is differentiation on both sides of such equations.

Example 3.1.6 By implicit differentiation, verify the answer in Example 3.1.5.

Alternative: Re-arrange the exact equation to obtain $\frac{dy}{dx} = -\frac{\partial f}{\partial x} / \frac{\partial f}{\partial y}$.

We take $\frac{d}{dx}$ on both sides and utilize Chain Rule, thus

$$\begin{aligned} \frac{d}{dx}(x^2+y^2+xy) &= \frac{d}{dx}C \\ 2x+2y\frac{dy}{dx}+x\frac{dy}{dx}+y &= 0 \\ \frac{dy}{dx} &= -\frac{2x+y}{x+2y} \end{aligned}$$

Example 3.1.7 Solve $ydx - xdy = y^3dy$.

Although the equation is not exact, observe that the left hand side resembles the result of Quotient Rule on $d\left(\frac{x}{y}\right)$, which is

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

Alternative: Identify $\frac{1}{y^2}$ as the integrating factor by the method described below.

Therefore, dividing both side by y^2, the equation becomes

$$\begin{aligned} \frac{ydx-xdy}{y^2} &= ydy \\ d\left(\frac{x}{y}\right) &= ydy \\ \frac{x}{y} &= \int ydy \\ \frac{x}{y} &= \frac{1}{2}y^2 + C \\ x &= \frac{1}{2}y^3 + Cy \end{aligned}$$

Integrating Factor - Revisited The method of integrating factor can be extended for non-exact equation that has the form $Mdx + Ndy$. We state two special cases of integrating factors without proof here.

If $g(x) = \frac{M_y - N_x}{N}$ is a function of x only, then

$$e^{\int g(x)dx}$$

is an integrating factor.

Similarly, if $h(y) = \frac{M_y - N_x}{-M}$ is a function of y only, then

$$e^{\int h(y)dy}$$

is an integrating factor.

We have to test whether the evaluated expression is a function depending on x or y only, as we have assumed.

Example 3.1.8 Solve $Mdx + Ndy = 0$ where $M = 3x^2y + y^3$, $N = 2xy^2$.

We would test whether $g = \frac{M_y - N_x}{N}$ is a function of x only and $h = \frac{M_y - N_x}{-M}$ is a function of y only.

$$\begin{aligned}\frac{M_y - N_x}{N} &= \frac{3x^2 + 3y^2 - 2y^2}{2xy^2} \\ &= \frac{3x^2 + y^2}{2xy^2}\end{aligned}$$

which is not a function of x only. On the other hand,

$$\begin{aligned}\frac{M_y - N_x}{-M} &= -\frac{3x^2 + 3y^2 - 2y^2}{3x^2y + y^3} \\ &= -\frac{3x^2 + y^2}{3x^2y + y^3} \\ &= -\frac{1}{y}\end{aligned}$$

which is a function of y only, subsequently

$$\begin{aligned}e^{\int h(y)dy} &= e^{\int -\frac{1}{y}dy} \\ &= e^{-\ln y} \\ &= \frac{1}{y}\end{aligned}$$

So we identify $\frac{1}{y}$ as the integrating factor. Now multiplying both sides of the equation by $\frac{1}{y}$, we get

$$(3x^2 + y^2)dx + 2xydy = 0$$

Now the equation is exact and we can proceed like Example 3.1.5. $f(x,y)$ here can be found from

$$f = \int (3x^2+y^2)dx = x^3+xy^2+G(y)$$
$$f = \int 2xydy = xy^2+H(x)$$

and so we conclude $f(x,y) = x^3+xy^2$, then we have

$$df = d(x^3+xy^2) = 0 \qquad \text{Exact Equation}$$
$$f(x,y) = x^3+xy^2 = C$$

Initial Conditions We have seen the solutions derived above consists of an integration constant C. To determine the value of C, we need a initial condition to tell us the value of the function at a given point.

Example 3.1.9 Solve $\frac{dy}{dx} = x+y$ provided that $y(0) = 3$.

$$\frac{dy}{dx} = y+x$$
$$\frac{dy}{dx} - y = x$$

The integrating factor is

$$e^{\int(-1)dx} = e^{-x}$$

By multiplying e^{-x} on both sides of the equation, we have

$$e^{-x}\frac{dy}{dx} - ye^{-x} = xe^{-x} \qquad \text{Integrating Factor}$$
$$\frac{d}{dx}(e^{-x}y) = xe^{-x}$$
$$e^{-x}y = \int xe^{-x}dx$$
$$e^{-x}y = -\int xd(e^{-x})$$
$$e^{-x}y = -xe^{-x} + \int e^{-x}dx$$
$$e^{-x}y = -xe^{-x} - e^{-x} + C$$
$$y = -(x+1) + Ce^{x}$$

Now substitute $x = 0$, $y = 3$, we have

$$\begin{aligned} -(0+1) + Ce^0 &= 3 \\ C &= 4 \end{aligned}$$

Hence the solution is

$$y = -(x+1) + 4e^x$$

Second-order ODE with Constant Coefficients We have spent some time talking about how to solve first-order ODE. And now it is the time to move to second-order ODE. We start with the most simplest form of such equation, which have constant coefficients and are in the form of

$$ay'' + by' + cy = 0$$

where a, b, c are constants. Here we use y' to denote $\frac{dy}{dx}$ and y'' to denote $\frac{d^2y}{dx^2}$.

We tackle this problem by using a test solution, which has the property that its derivative returns itself times a constant so that the equation becomes simpler as we will see below. The function satisfies such requirement is $y = e^{rx}$ where r is some constant to be found. Substitution gives

$$\begin{aligned} ar^2e^{rx} + bre^{rx} + ce^{rx} &= 0 \\ ar^2 + br + c &= 0 \end{aligned}$$

which becomes a quadratic equation in r. This is called the auxiliary equation. Finding the two roots of r allows us to conclude that $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are the solutions we need. Note that their linear combination $c_1y_1 + c_2y_2 = c_1e^{r_1x} + c_2e^{r_2x}$ satisfy the equation as well. This is the complete general solution where c_1 and c_2 are to be determined by initial conditions.

Example 3.1.10 Solve $y'' - 3y' + 2y = 0$, where $y(0) = 5$ and $y'(0) = 0$

In this case, $a = 1$, $b = -3$, $c = 2$. The auxiliary equation is

$$r^2 - 3r + 2 = 0$$

which has $r = 1$ or 2 as the solutions. Hence the general solution is

$$y = c_1y_1 + c_2y_2 = c_1e^x + c_2e^{2x}$$

From the initial conditions, we have

$$c_1e^0 + c_2e^{2(0)} = 5$$
$$c_1 + c_2 = 5$$

and

$$c_1e^0 + 2c_2e^{2(0)} = 0$$
$$c_1 + 2c_2 = 0$$

Solving them gives $c_1 = 10$ and $c_2 = -5$. Therefore the full solution is

$$y = 10e^x - 5e^{2x}$$

Example 3.1.11 Solve $y'' - 2y' + y = 0$.

We have the auxiliary equation as

$$r^2 - 2r + 1 = 0$$
$$r = 1$$

Since it is a double root, we only have $y_1 = e^x$ as a solution. We want to find another solution that is different. It can be proved that $y_2 = xy_1$ is another solution that satisfies such equations and the readers are encouraged to do so. Hence the general solution is

$$y = c_1e^x + c_2xe^x$$

Example 3.1.12 Solve $y'' - 2y' + 5y = 0$, where $y(0) = 1$ and $y'(0) = 0$.

The auxiliary equation is

$$r^2 - 2r + 5 = 0$$
$$r = 1 \pm 2\imath$$

which are complex. Hence we have $y_1 = e^{(1+2\imath)x}$ and $y_2 = e^{(1-2\imath)x}$ as the solutions. However, we want them to be real-valued functions. In general, if $y_1 = e^{(a+b\imath)x}$ and $y_2 = e^{(a-b\imath)x}$ then by using Euler's Formula, their linear combination

$$\begin{aligned}
\frac{y_1 + y_2}{2} &= \frac{e^{(a+b\imath)x} + e^{(a-b\imath)x}}{2} \\
&= e^{ax}\frac{e^{\imath bx} + e^{-\imath bx}}{2} \\
&= e^{ax}\frac{(\cos(bx) + \imath\sin(bx)) + (\cos(bx) - \imath\sin(bx))}{2} \\
&= e^{ax}\cos(bx)
\end{aligned}$$

and

$$\begin{aligned}\frac{y_1 - y_2}{2\imath} &= \frac{e^{(a+b\imath)x} - e^{(a-b\imath)x}}{2\imath} \\ &= e^{ax}\frac{e^{\imath bx} - e^{-\imath bx}}{2\imath} \\ &= e^{ax}\frac{(\cos(bx) + \imath\sin(bx)) - (\cos(bx) - \imath\sin(bx))}{2\imath} \\ &= e^{ax}\sin(bx)\end{aligned}$$

are real-valued solutions that satisfy the equation. Hence in this case, the general solution is

$$y = e^x(c_1\cos(2x) + c_2\sin(2x))$$

with

$$y' = e^x(c_1\cos(2x) + c_2\sin(2x)) + e^x(-2c_1\sin(2x) + 2c_2\cos(2x))$$

As $y(0) = 1$ and $y'(0) = 0$, we have $c_1 = 1$ and $c_2 = \frac{-1}{2}$. So the complete solution is

$$y = e^x(\cos(2x) - \frac{1}{2}\sin(2x))$$

Euler's Equation Euler's equation is a variant of second-order ODE, which has a form of

$$ax^2y'' + bxy' + cy = 0$$

It can be shown that by letting $z = \ln x$ and $y(x) = Y(z)$, it becomes

$$aY'' + (b-a)Y' + cY = 0$$

which then can be solved by the method described above.

Example 3.1.13 Solve $x^2y'' - 4xy' + 6y = 0$.

Making the transform $z = \ln x$ and $y(x) = Y(z)$, it becomes

$$\begin{aligned}Y'' + (-4-1)Y' + 6Y &= 0 \\ Y'' - 5Y' + 6Y &= 0\end{aligned}$$

Common mistake: Forgetting to subtract a from the coefficient of Y'.

The auxiliary equation is

$$
\begin{aligned}
R^2 - 5R + 6 &= 0 \\
R &= 2 \text{ or } 3
\end{aligned}
$$

Hence the general solution is

$$
\begin{aligned}
Y &= c_1 e^{2z} + c_2 e^{3z} \\
y &= c_1 e^{2\ln x} + c_2 e^{3\ln x} \\
&= c_1 x^2 + c_2 x^3
\end{aligned}
$$

Non-homogeneous second-order ODE Up to this point all the second-order ODEs we tackle are homogeneous, that is, $G(x) = 0$ as in

$$f(y'', y', y) = G(x)$$

where $f(y'', y', y)$ only involves y and its derivatives but not x. Examples of non-homogeneous second-order ODEs are

$$
\begin{aligned}
y'' + 2y' + 3y &= \sin x \\
y'' - 5y' + 4y &= x
\end{aligned}
$$

We can solve them by trial and error or Variation of parameters to get the particular solution y_p which appears due to $G(x)$, then add it up with the complementary solution y_c for the complementary homogeneous ODE where $G(x)$ is removed.

Example 3.1.14 Solve $y'' - 5y' + 4y = 16x$, where $y(0) = 7$ and $y'(0) = 9$.

We try a solution of $y_p = ax + b$, substitution gives

$$-5(a) + 4(ax + b) = 16x$$

Comparing the coefficients, we have $a = 4$ and $b = 5$. So the particular solution is

$$y_p = 4x + 5$$

The complementary homogeneous ODE is

$$y'' - 5y' + 4y = 0$$

and has the auxiliary equation as

$$
\begin{aligned}
r^2 - 5r + 4 &= 0 \\
r &= 1 \text{ or } 4
\end{aligned}
$$

the complementary solutions are $y_1 = e^x$ and $y_2 = e^{4x}$. Subsequenly, the complete solution is given by

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 + y_p \\ &= c_1 e^x + c_2 e^{4x} + (4x+5) \end{aligned}$$

Substituting $y(0) = 7$ and $y'(0) = 9$, we have

$$\begin{aligned} c_1 e^{(0)} + c_2 e^{4(0)} + (4(0)+5) &= 7 \\ c_1 + c_2 &= 2 \\ c_1 e^{(0)} + 4c_2 e^{4(0)} + 4 &= 9 \\ c_1 + 4c_2 &= 5 \end{aligned}$$

Solving them leads to $c_1 = 1$ and $c_2 = 1$, and thus the complete solution is

$$y = e^x + e^{4x} + (4x+5)$$

Variation of Parameters Variation of Parameters is a powerful method of solving non-homogeneous second-order ODE with constant coefficients in form of $ay'' + by' + cy = G(x)$. Based on the complementary solution $y_c = c_1 y_1 + c_2 y_2$, we solve the following system,

$$\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ v_1' y_1' + v_2' y_2' &= \frac{G(x)}{a} \end{aligned}$$

After obtaining v_1' and v_2', we integrate them to retrieve v_1 and v_2. Then the particular solution is

$$y_p = v_1 y_1 + v_2 y_2$$

where c_1, c_2 in y_c are replaced by variables v_1, v_2 to produce y_p.

Example 3.1.15 Solve $y'' - y' = e^{-x}$.

The auxiliary equation for the complementary equation is

$$\begin{aligned} r^2 - r &= 0 \\ r &= 0 \text{ or } 1 \end{aligned}$$

Hence the complementary solution is

$$y_1 = 1, y_2 = e^x$$

Alternative: Test $y_p = Ae^{-x}$.

Common mistake: Forgetting to divide $G(x)$ by a, although in this case $a = 1$.

Variation of parameters requires us to solve

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0 \\ v_1'y_1' + v_2'y_2' &= e^{-x} \end{aligned} \qquad \text{Variation of Parameters}$$

Substitution gives

$$\begin{aligned} v_1' + v_2'e^x &= 0 \\ v_2'e^x &= e^{-x} \end{aligned}$$

Solving directly, or by Cramer's Rule, we have $v_1' = -e^{-x}$ and $v_2' = e^{-2x}$. Integration gives

$$v_1 = e^{-x}, v_2 = -\frac{1}{2}e^{-2x}$$

Hence the full solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 + y_p \\ &= c_1y_1 + c_2y_2 + v_1y_1 + v_2y_2 \\ &= c_1 + c_2e^x + (e^{-x})(1) - (\frac{1}{2}e^{-2x})(e^x) \\ &= c_1 + c_2e^x + \frac{1}{2}e^{-x} \end{aligned}$$

Partial Differential Equation - Preview To solve a partial differential equation having partial derivatives with respect to a single particular variable only, we proceed as if it is an ordinary differential equation with an appropriate integration constant.

Example 3.1.16 Solve $\frac{\partial y}{\partial x} = xz$ where x and z are independent variables.

We apply separation of variables and the equation becomes

$$\partial y = xz\partial x \qquad \text{Separation of Variables}$$

We can treat the equation as

$$dy = xzdx$$

Common mistake: Forgetting to include other independent variables in the integration constant.

while regarding z as a constant. Integrating both sides gives

$$\begin{aligned} \int dy &= \int xzdx \\ y &= \frac{1}{2}x^2z + C(z) \end{aligned}$$

where $C(z)$ is any function in z only.

3.2 ODE in ESSC3200

3.2.1 Hydrostatic Balance

Example 3.2.1 Derive the expression for hydrostatic balance, i.e. the weight of an air parcel is balanced by the vertical pressure gradient.

The weight of an air parcel with size $\delta x \delta y \delta z$ is simply $\rho g \delta x \delta y \delta z$. The buoyancy force due to pressure difference on top face A and bottom face B in the vertical direction is

$$\begin{aligned} F_z &= (-p_A + p_B)\delta x \delta y \\ &= (-(p_B + \frac{\partial p}{\partial z}\delta z) + p_B))\delta x \delta y \\ &= -\frac{\partial p}{\partial z}\delta x \delta y \delta z \end{aligned}$$

where we have used Taylor's series to the first order to rewrite p_A. Hydrostatic balance is then

$$\begin{aligned} -\frac{\partial p}{\partial z}\delta x \delta y \delta z &= \rho g \delta x \delta y \delta z \\ \frac{\partial p}{\partial z} &= -\rho g \end{aligned}$$

This equation relates the pressure level to any height. Therefore, we can always define a geopotential height $z(p)$, which is the height at which the air pressure is equal to p. A related quantity is the geopotential which is

$$\int_0^{z(p)} g dz$$

This quantity is useful when we later discuss the isobaric coordinates.

Example 3.2.2 Derive the pressure profile for an isothermal atmosphere with $T = T_0$ a constant.

We need to use the equation of state to establish the relationship between temperature, pressure and density. For the atmosphere, the ideal gas law is accurate to be used, which takes the form of

$$pV = nR^*T$$

where R^* is the universal gas constant $8.314\,\mathrm{J\,mol^{-1}\,K^{-1}}$. However, for atmospheric science, we usually use the form

$$p = \rho R_d T$$

where the gas constant of dry air $R_d = \frac{R^*}{M_d} = 287\,\mathrm{J\,kg^{-1}\,K^{-1}}$ with the molar mass of dry air $M_d = 28.97\,\mathrm{g\,mol^{-1}}$. Notice the values are only applicable for the Earth. With this, we can rewrite the hydrostatic equation as

$$\begin{aligned}\frac{\partial p}{\partial z} &= -\frac{p}{R_d T} g \\ \frac{\partial p}{p} &= -\frac{g}{R_d T} \partial z\end{aligned}$$

We solve the equation by integrating both sides with appropriate boundary conditions, viz. at $z = 0$, $p = p_0$ the surface pressure, which gives

$$\begin{aligned}\int_{p_0}^{p} \frac{dp}{p} &= -\int_0^z \frac{g}{R_d T_0} dz \\ \ln(\frac{p}{p_0}) &= -\frac{g}{R_d T_0} z \\ \frac{p}{p_0} &= e^{-\frac{g}{R_d T_0} z} \\ p &= p_0 e^{-\frac{g}{R_d T_0} z}\end{aligned}$$

where the quantity $\frac{RT}{g}$ is called the scale height and represents the vertical scale of the atmosphere. At altitude equals to one scale height the pressure drops to $\frac{1}{e} = 36.8\%$ of the surface pressure.

Example 3.2.3 Derive the pressure profile for an atmosphere having a constant lapse rate γ such that $T = T_0 - \gamma z$.

The hydrostatic equation is

$$\begin{aligned}\frac{\partial p}{\partial z} &= -\frac{p}{R_d (T_0 - \gamma z)} g \\ \frac{\partial p}{p} &= -\frac{g}{R_d (T_0 - \gamma z)} \partial z\end{aligned}$$

Integrating in a manner similar to Example 3.2.2, we have

$$\begin{aligned}
\int_{p_0}^{p} \frac{dp}{p} &= -\int_{0}^{z} \frac{g}{R_d(T_0 - \gamma z)} dz \\
\ln(\frac{p}{p_0}) &= \frac{g}{R_d\gamma}[\ln(T_0 - \gamma z)]_0^z \\
\ln(\frac{p}{p_0}) &= \frac{g}{R_d\gamma}\ln(\frac{T_0 - \gamma z}{T_0}) \\
\ln(\frac{p}{p_0}) &= \ln((\frac{T_0 - \gamma z}{T_0})^{\frac{g}{R_d\gamma}}) \\
\frac{p}{p_0} &= (\frac{T_0 - \gamma z}{T_0})^{\frac{g}{R_d\gamma}} \\
p &= p_0(\frac{T_0 - \gamma z}{T_0})^{\frac{g}{R_d\gamma}}
\end{aligned}$$

We can rearrange it to express the geopotential height z in terms of the pressure p as can be seen in the next example.

In the subsequent sections, we would use R to denote R_d.

Example 3.2.4 Calculate the geopotential height at 500 hPa if the sea level pressure is 1010 hPa and the layer in-between has a mean temperature of $0\,^{\circ}$C.

Using the results of Example 3.2.2, we have

$$z = \frac{R\langle T\rangle}{g}\ln\frac{p_0}{p}$$

which is called the hypsometric equation. $\langle T\rangle$ denotes the mean temperature in the layer. Substituting $T = 273\,$K, $p = 500$ hPa and $p_0 = 1010$ hPa, the value of z is 5536 m.

Alternative: If the reference height is not at sea level, we have $z - z_0 = \frac{R\langle T\rangle}{g}\ln\frac{p_0}{p}$

3.2.2 Thermodynamics of Dry Atmosphere

Example 3.2.5 Derive the expression of the dry adiabatic lapse rate.

To do so, we need to utilize the first law of thermodynamics for unit mass, which is

$$c_v dT = dQ - p d\alpha$$

where c_v is the specific heat capacity at constant volume and has a value of $718\,\mathrm{J\,kg^{-1}\,K^{-1}}$ for dry air while $\alpha = \frac{V}{m} = \frac{1}{\rho}$ is the specific volume.

Notice that from the ideal gas law

$$d(p\alpha) = d(RT)$$
$$pd\alpha + \alpha dp = RdT$$

To make use of it, we transform the first law of thermodynamics into another form as

$$c_v dT = dQ - pd\alpha$$
$$c_v dT + RdT = dQ - pd\alpha + (pd\alpha + \alpha dp)$$
$$c_p dT = dQ + \alpha dp$$

where $c_p = c_v + R$ is the specific heat capacity at constant pressure and has a value of $1005\,\mathrm{J\,kg^{-1}\,K^{-1}}$ for dry air.

For adiabatic motion, $dQ = 0$, so we have

$$c_p dT = \alpha dp$$

Invoking the hydrostatic balance, $\partial p = -\rho g \partial z$, we have

$$c_p dT = -\alpha\rho g dz$$
$$c_p dT = -g dz$$
$$\frac{dT}{dz} = -\frac{c_p}{g}$$

$\Gamma_d = \frac{c_p}{g} = 9.8\,\mathrm{K\,km^{-1}}$ is then the dry adiabatic lapse rate we want.

Example 3.2.6 If an air parcel is rising at a rate of $0.025\,\mathrm{m/s}$, find the magnitude of the external radiative heating required to keep its temperature unchanged.

We start with the first law of thermodynamics differentiated with respect to time, which is

$$c_p \frac{dT}{dt} = \frac{dQ}{dt} + \alpha \frac{dp}{dt}$$

Since the temperature is constant, $\frac{dT}{dt} = 0$. Noting that in atmospheric setting, $\frac{dp}{dt} \approx \frac{\partial p}{\partial z}\frac{dz}{dt}$. Together with the hydrostatic balance $\frac{\partial p}{\partial z} = -\rho g$, the equation is now

$$\frac{dQ}{dt} + \alpha(-\rho g)\frac{dz}{dt} = 0$$
$$\frac{dQ}{dt} = g\frac{dz}{dt}$$

So the required external heating is $(9.81\,\mathrm{m\,s^{-2}})(0.025\,\mathrm{m/s}) = 0.245\,\mathrm{W\,kg^{-1}}$.

Example 3.2.7 Derive the expression of potential temperature, which is the temperature a dry air parcel would have achieved if it moves adiabatically to a pressure level of 1000 hPa.

To establish the relation, we first make use of the first law of thermodynamics,

$$c_p dT = dQ + \alpha dp$$

Adiabatic processes implies $dQ = 0$, hence

$$\begin{aligned} c_p dT &= \alpha dp \\ c_p dT &= \frac{RT}{p} dp \\ \frac{dT}{T} &= \frac{R}{c_p}\frac{dp}{p} \end{aligned}$$

where we have used the equation of state. Integration with the boundary condition $T = \theta$ which is the potential temperature at $p = 1000$ hPa, gives

$$\begin{aligned} \int_{\theta}^{T} \frac{dT}{T} &= \int_{1000}^{p} \frac{R}{c_p}\frac{dp}{p} \\ \ln(\frac{T}{\theta}) &= \frac{R}{c_p}\ln(\frac{p}{1000}) \\ \ln(\frac{T}{\theta}) &= \ln((\frac{p}{1000})^{\frac{R}{c_p}}) \\ \frac{T}{\theta} &= (\frac{p}{1000})^{\kappa} \end{aligned}$$

where $\kappa = \frac{R}{c_p} = 0.286$. Potential temperature θ of an air parcel conserves if there is no heat exchange.

Example 3.2.8 Suppose an air parcel of temperature $-10\,^{\circ}$C is originally at a pressure level of 700 hPa is raised adiabatically to a pressure level of 500 hPa. Find its potential temperature and the new temperature at 500 hPa.

Using the expression of potential temperature derived above, we have

$$\theta = T(\frac{1000}{p})^{\kappa}$$

Substituting $T = 263.15\,$K, $p = 700$ hPa, we conclude that $\theta = 291.4\,$K. Using the value of θ just obtained and apply the expression again, at $p = 500$ hPa, $T = 239.0\,\text{K} = -34.15\,^{\circ}$C.

Common mistake: Not converting the temperature into Kelvin before calculation.

Example 3.2.9 Derive the relation about atmospheric stability and environmental potential temperature, which is

$$\frac{1}{\theta_e}\frac{\partial\theta_e}{\partial z} = \frac{1}{T_e}(\frac{\partial T_e}{\partial z} + \frac{g}{c_p}) = \frac{1}{T_e}(\frac{\partial T_e}{\partial z} + \Gamma_d) = \frac{1}{T_e}(-\Gamma_e + \Gamma_d)$$

Take natural logarithm on the expression of potential temperature, we have

$$\ln(\frac{\theta}{T}) = \frac{R}{c_p}\ln(\frac{1000}{p})$$
$$\ln\theta - \ln T = \frac{R}{c_p}(\ln 1000 - \ln p)$$

Differentiation on both sides with respect to z gives

$$\frac{\partial}{\partial z}\ln\theta - \frac{\partial}{\partial z}\ln T = -\frac{\partial}{\partial z}\frac{R}{c_p}\ln p$$
$$\frac{1}{\theta}\frac{\partial\theta}{\partial z} - \frac{1}{T}\frac{\partial T}{\partial z} = -\frac{R}{c_p}\frac{1}{p}\frac{\partial p}{\partial z}$$

Using hydrostatic balance and equation of state, we have

$$\begin{aligned}\frac{1}{\theta}\frac{\partial\theta}{\partial z} &= \frac{1}{T}\frac{\partial T}{\partial z} - \frac{R}{c_p}\frac{1}{\rho RT}(-\rho g)\\ &= \frac{1}{T}\frac{\partial T}{\partial z} + \frac{1}{T}\frac{g}{c_p}\\ &= \frac{1}{T}(\frac{\partial T}{\partial z} + \frac{g}{c_p}) = \frac{1}{T}(\frac{\partial T}{\partial z} + \Gamma_d) = \frac{1}{T}(-\Gamma + \Gamma_d)\end{aligned}$$

If $\frac{\partial\theta_e}{\partial z} > 0$, then environmental lapse rate is smaller than the dry adiabatic lapse rate $\Gamma_e = -\frac{\partial T_e}{\partial z} < \Gamma_d$, the atmosphere is stable. It is the opposite when $\frac{\partial\theta_e}{\partial z} < 0$ and the atmopsphere is unstable.

3.2.3 Buoyancy Wave

Example 3.2.10 Derive the expression for Brunt–Väisälä frequency of buoyancy wave.

By Archimedes' principle, buoyancy force on an air parcel is

$$\frac{d^2\delta z}{dt^2} = \frac{(\rho_e - \rho')Vg}{\rho' V} = \frac{\delta\rho}{\rho'}g$$

where the subscript e and superscript $'$ represents environment and parcel respectively. δz is the displacement of the parcel from the equilibrium level.

Taking natural logarithm on the expression of potential temperature and then differentiating the expression of potential temperature with p fixed gives

$$\begin{aligned}
\ln\theta - \ln T &= \frac{R}{c_p}(\ln 1000 - \ln p) \\
\ln\theta + \ln\frac{\rho R}{p} &= \frac{R}{c_p}(\ln 1000 - \ln p) \\
\frac{1}{\theta}\delta\theta + \frac{1}{\rho}\delta\rho &= 0
\end{aligned}$$

where $\delta\theta = \theta_e - \theta' = \theta_e$ since θ' following an air parcel conserves as the motion is adiabatic. The perturbations are small such that we have replaced ∂ with δ. Combining them together, we have

$$\begin{aligned}
\frac{d^2\delta z}{dt^2} + \frac{\delta\theta_e}{\theta_e}g &= 0 \\
\frac{d^2\delta z}{dt^2} + \frac{g}{\theta_e}(\frac{\partial\theta_e}{\partial z}\delta z) &= 0
\end{aligned}$$

This is a second-order ordinary differential equation. If the atmosphere is stable, i.e. $\frac{\partial\theta_e}{\partial z} > 0$, then the solution is in the form $\delta z = A\cos(Nt) + B\sin(Nt)$, where $N = \sqrt{\frac{g}{\theta_e}(\frac{\partial\theta_e}{\partial z})} = \sqrt{g\frac{\partial\ln\theta_e}{\partial z}}$ is called the Brunt–Väisälä frequency.

Example 3.2.11 At a certain height, the temperature is $10\,^\circ\mathrm{C}$ and the environmental lapse rate is $\Gamma = -\frac{\partial T}{\partial z} = 5\,^\circ\mathrm{C/km}$. Find the general expression for the displacement δz if at $t = 0$, the displacement δz is zero and $\frac{d\delta z}{dt}$ is $0.1\,\mathrm{m\,s^{-1}}$.

From the two examples above, the Brunt–Väisälä frequency is

$$N = \sqrt{\frac{g}{\theta_e}(\frac{\partial\theta_e}{\partial z})} = \sqrt{\frac{g}{T}(\frac{\partial T}{\partial z} + \frac{g}{c_p})}$$

Substituting $g = 9.81\,\mathrm{m\,s^{-2}}$, $T = 283.15\,\mathrm{K}$, $\frac{\partial T}{\partial z} = -0.005\,\mathrm{K\,m^{-1}}$, the value of N is $0.01284\,\mathrm{rad/s}$. Hence the solution has the form of

Common mistake: Wrong sign and non-SI unit for $\frac{\partial T}{\partial z}$.

$$\delta z = A\cos(0.01284t) + B\sin(0.01284t)$$

Applying the initial conditions at $t = 0$, we have

$$\delta z = A = 0$$
$$\frac{d\delta z}{dt} = 0.01284B = 0.1$$
$$B = 7.786\,\mathrm{m}$$

Example 3.2.12 Given the dispersion relation for propagating buoyancy wave is

$$\omega^2 = \frac{N^2k^2}{k^2+m^2}$$

where $k = \frac{2\pi}{\lambda_x}$ and $m = \frac{2\pi}{\lambda_z}$ are the zonal and vertical wavenumber, with a general solution for pertubations in the form of

$$A\cos(kx + mz - \omega t + \phi)$$

Using the value of N found in Example 3.2.11, find the horizontal phase speed if horizontal wavelength and vertical wavelength are 4 km and 100 km respectively.

Common mistake: Not converting the wavelength from km into m.

From the dispersion relation, we have

$$\begin{aligned}\omega &= \pm\sqrt{\frac{N^2k^2}{k^2+m^2}} \\ &= \pm\sqrt{\frac{(0.01284)^2(2\pi/4000)^2}{(2\pi/4000)^2+(2\pi/100000)^2}} \\ &= \pm 0.1283\,\mathrm{rad/s}\end{aligned}$$

Notice how the value of ω is close to N as the vertical wavelength is much longer than the horizontal one. Without the loss of generality, assume that the wave is propagating eastwards, such that ω and k are taken to be positive. Then the phase speed is

$$\begin{aligned}c_x &= \frac{\omega}{k} \\ &= \frac{\omega\lambda_x}{2\pi} \\ &= \frac{(0.1283)(4000)}{2\pi} = 8.17\,\mathrm{m\,s^{-1}}\end{aligned}$$

3.3 ODE in ESSC3300

3.4 ODE in ESSC3220

3.5 Problems

Question 3.1.1 Solve the following differential equations.
(a) $\frac{dy}{dx} = \frac{x^2}{y}$, (b) $\frac{dy}{dx} = \frac{\sqrt{y^2-1}}{x-1}$, (c) $\frac{dy}{dx} = \sec^2 x \tan y$.

Question 3.1.2 Solve the following differential equations.
(a) $\frac{dy}{dx} + x^3 y = 8$, (b) $\frac{dy}{dx} + \frac{5}{x} y = 7$, (c) $\frac{dy}{dx} + (\tan x) y = 5$.

Question 3.1.3 Solve the following differential equations.
(a) $\frac{dy}{dx} = -\frac{2x+3y}{3x-4y}$, (b) $(\cos x \cos y)dx - (\sin x \sin y)dy = 0$,
(c) $(\frac{e^{y^2}}{x})dx + (2ye^{y^2} \ln x)dy = 0$.

Question 3.1.4 Solve the following differential equations.
(a) $(x+2y)dx + xdy = 0$, (b) $xydx + (x^2+2y^2)dy = 0$,
(c) $\cos x dx + (\sin x + y + 1)dy = 0$.

Question 3.1.5 Solve

$$\frac{dy}{dx} + \frac{2}{x} y = 1$$

where $y(1) = 1$.

Question 3.1.6 Solve

$$(4x^2 + 2y^2)dx + xydy = 0$$

where $y(1) = 2$.

Question 3.1.7 Find the general solutions of the second-order ordinary differential equations below.
(a) $y'' + 3y' + 2y = 0$, (b) $y'' - 2y' + y = 0$, (c) $y'' + y + 1 = 0$.

Question 3.1.8 Solve

$$y'' + 3y' - 4y = 0$$

where $y(0) = 3$, $y'(0) = -2$.

Question 3.1.9 Solve

$$y'' + 4y' + 8y = 0$$

where $y(0) = -1$, $y'(0) = 4$.

Question 3.1.10 Find an solution for

$$x^2 y'' + 2xy' - 2y = 0$$

Question 3.1.11 Solve

$$y'' + 7y' + 10y = 5x^2 + 7x + 6$$

where $y(0) = \frac{1}{2}$, $y'(0) = \frac{3}{2}$.

Question 3.1.12 Solve

$$y'' + 3y' + 2y = 6e^x$$

where $y(0) = 2$, $y'(0) = -1$.

Question 3.1.13 Solve

$$y'' + y = \sin x$$

where $y(0) = 1$, $y'(0) = \frac{1}{2}$. The method of Variation of Parameters may be applied. However, if we attempt this question by trial and error, notice that $\sin x$ is already an solution of the complementary equation and vanishes upon substitution. Therefore, a better guess would be $x \sin x$ and $x \cos x$ instead.

Question 3.1.14 In this chapter, we have seen about implicit differentiation, for x and y related implicitly by a function $f(x, y) = c$, we have

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} / \frac{\partial f}{\partial y}$$

Similarly for three variables, $g(x, y, z) = c$, we have, for example

$$\frac{\partial z}{\partial x} = -\frac{\partial g}{\partial x} / \frac{\partial g}{\partial z}$$

where we keep y as a constant when computing $\frac{\partial z}{\partial x}$ which is the response of z to the change in x while moving along the curve $g(x, y, z) = c$. Prove that in this case

$$\frac{\partial x}{\partial y} \frac{\partial y}{\partial z} \frac{\partial z}{\partial x} = -1$$

Question 3.1.15 Do Problem 13 and 16 in the section Miscellaneous Problems for Chapter 1 of the textbook Differential Equations with Applications and Historical Notes (3rd Edition).

Question 3.1.16 A free falling object subjected to air resistance can be modelled to have an acceleration of

$$\frac{dv}{dt} = g - cv$$

where c is a constant representing the extent of air resistance and has a unit of s^{-1}. Express the velocity v as well as the downward displacement $s = \int v dt$ in terms of the time elapsed t since released from rest. If the object reaches 90% of the terminal speed at $t = 100\,\mathrm{s}$, find the value of c. How about if the acceleration is in the form of

$$\frac{dv}{dt} = g - bv^2$$

instead where b has a unit of m^{-1}?

Question 3.1.17 A chemical tracer P decays into another tracer Q, which in turn reacts to produce another product R. Both reactions are first-order, and hence have rate laws in form of

$$\begin{aligned}
\frac{d[P]}{dt} &= -k_1[P] \\
\frac{d[Q]}{dt} &= k_1[P] - k_2[Q] \\
\frac{d[R]}{dt} &= k_2[Q]
\end{aligned}$$

Initially, $[P] = 1$ and $[Q] = [R] = 0$. Express the concentrations of all tracers in terms of t. Find the time t_Q where $[Q]$ reaches maximum in terms of k_1, k_2. Briefly describe what happens to t_Q if k_1 is much greater or smaller than k_2.

Question 3.1.18 Two armies A and B fight on a battlefield. The evolution of the combat can be described by a system of differential equations using Lanchester's Law. Assume their attacking power can be described linearly by some constants k_A, k_B, then we have the dealt damages as

$$\begin{aligned}
\frac{d[A]}{dt} &= -k_B[B] \\
\frac{d[B]}{dt} &= -k_A[A]
\end{aligned}$$

If army A has 30000 men, army B has 20000 men, $k_A = 1000\,\mathrm{h}^{-1}$, $k_B = 2500\,\mathrm{h}^{-1}$, determine which army would be eliminated first. At what time does this occur?

CHAPTER 4

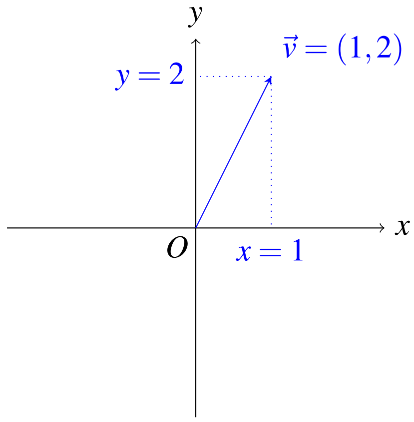
VECTOR AND VECTOR CALCULUS

CONTENTS

4.1 Introduction

4.1.1 Vector Geometry

Vectors Vectors are mathematical objects that have magnitude and direction, resembling an arrow. A vector is represented by a tuple of numbers, like $(1,2,3)$ and $(4,6,3,3)$. Each of those numbers are the components of the vector and their amount determines the number of dimension. For example, the vector $(4,6,3,3)$ is 4-dimensional. A vector is denoted by an arrow symbol or a bold letter, like $\vec{v}$ and $\mathbf{v}$.



A 2D vector in the x-y plane.

Vector Addition, Subtraction, and Scalar Multiplication Vector addition and subtraction is element-wise. This means that we add/subtract the corresponding components each by each. It also implies that such operations are only valid for vectors having the same dimension. Multiplying a scalar to a vector means all components are multiplied by that scalar. Subtraction then can be viewed as addition with a (-1) factor.

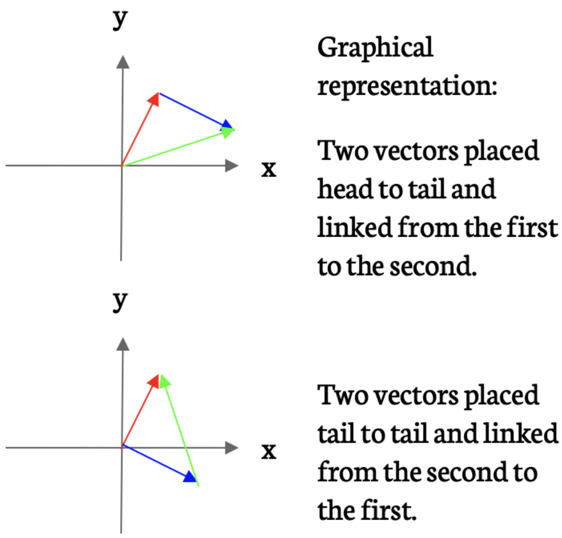
Here are some notable properties of vector addition and subtraction.

$$\vec{u}+\vec{v}=\vec{v}+\vec{u} \qquad \text{Commutative property of vector addition}$$
$$-\vec{u}=(-1)\vec{u}$$
$$(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w}) \qquad \text{Associative property of vector addition}$$

Example 4.1.1 Let $\vec{u}=(1,-1,2)$, $\vec{v}=(1,3,3)$. Find $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$.

$$\begin{aligned}\vec{u}+\vec{v}&=(1,-1,2)+(1,3,3)\\&=(1+1,-1+3,2+3)\\&=(2,2,5)\end{aligned}$$

$$
\begin{aligned}
\vec{u}-\vec{v} &= (1,-1,2)-(1,3,3) \\
&= (1-1,-1-3,2-3) \\
&= (0,-4,-1)
\end{aligned}
$$



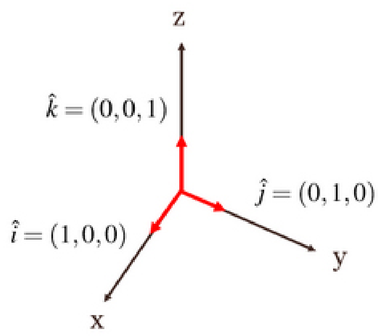
Vector Addition (red+blue) and Subtraction (red-blue) visualized. The results are the green vectors.

Example 4.1.2 Let $\vec{u}=(3,1,2)$, $\vec{v}=(1,2,0)$. Find $2\vec{u}-3\vec{v}$.

$$
\begin{aligned}
2\vec{u}-3\vec{v} &= 2(3,1,2)-3(1,2,0) \\
&= (6,2,4)-(3,6,0) \\
&= (6-3,2-6,4-0) \\
&= (3,-4,4)
\end{aligned}
$$

Standard Unit Vectors Vectors can be decomposed into standard unit vectors which are vectors of length 1 along the positive direction of the axes. In 2-dimensional and 3-dimensional space, we usually denote them as $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ corresponding to x-axis, y-axis, and z-axis respectively. For instance, $(3,2,9)$ can be written as

$$3(1,0,0)+2(0,1,0)+9(0,0,1)=3\hat{\imath}+2\hat{\jmath}+9\hat{k}$$



Standard Unit Vectors along the axes.

Length of a Vector, Direction Length, or called norm, of a vector, denoted by $||$, is found by generalizing Pythagora's Theorem. It is evaluated as the square root of the sum of squares of each components. Direction of a vector is simply its unit vector, which is the original vector divided by its length. Unit vector is usually written with the arrow symbol replaced by a hat $\hat{}$.

Example 4.1.3 Find the length and unit vector of $\vec{v} = (5,3,1,1)$.

$$\begin{aligned} |\vec{v}| &= \sqrt{5^2+3^2+1^2+1^2} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$

$$\begin{aligned} \hat{v} &= \frac{\vec{v}}{|\vec{v}|} \\ &= \frac{1}{6}(5,3,1,1) \\ &= (\frac{5}{6},\frac{3}{6},\frac{1}{6},\frac{1}{6}) \end{aligned}$$

Vector Products There are two types of vector product. One is dot product, which takes two vectors and produce a scalar. Another is cross product, which also takes two vectors but produce a new vector.

Dot Product Dot Product is the sum of products of the corresponding components between two vectors, denoted by $\vec{u} \cdot \vec{v}$. By definition, the length of a vector $\vec{v}$ can be written as $\sqrt{\vec{v} \cdot \vec{v}}$.

Example 4.1.4 Find the dot product between $\vec{u} = (1,2,3)$ and $\vec{v} = (4,5,6)$.

$$\begin{aligned}\vec{u}\cdot\vec{v} &= (1,2,3)\cdot(4,5,6)\\ &= (1)(4)+(2)(5)+(3)(6)\\ &= 32\end{aligned}$$

Property of Dot Product Some important properties of dot product are listed below.

$$\begin{aligned}\vec{u}\cdot\vec{v} &= \vec{v}\cdot\vec{u} && \text{Commutative Law of Dot Product}\\ \vec{u}\cdot(\vec{v}+\vec{w}) &= \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w} && \text{Distributive Law of Dot Product}\\ \vec{u}\cdot(\vec{v}-\vec{w}) &= \vec{u}\cdot\vec{v}-\vec{u}\cdot\vec{w}\\ (\vec{u}+\vec{v})\cdot\vec{w} &= \vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}\\ (\vec{u}-\vec{v})\cdot\vec{w} &= \vec{u}\cdot\vec{w}-\vec{v}\cdot\vec{w}\end{aligned}$$

Example 4.1.5 Verify $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$, for $\vec{u} = (1,2,1)$, $\vec{v} = (0,1,1)$, $\vec{w} = (1,1,3)$.

$$\begin{aligned}\vec{u}\cdot(\vec{v}+\vec{w}) &= (1,2,1)\cdot((0,1,1)+(1,1,3))\\ &= (1,2,1)\cdot(1,2,4)\\ &= (1)(1)+(2)(2)+(1)(4)\\ &= 9\end{aligned}$$

$$\begin{aligned}\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w} &= (1,2,1)\cdot(0,1,1)+(1,2,1)\cdot(1,1,3)\\ &= ((1)(0)+(2)(1)+(1)(1))+((1)(1)+(2)(1)+(1)(3))\\ &= 3+6\\ &= 9 = \vec{u}\cdot(\vec{v}+\vec{w})\end{aligned}$$

Example 4.1.6 Prove cosine law by vector notation for the triangle with side lengths of a, b, c.

$$\begin{aligned}c^2 = |\vec{c}|^2 &= |(\vec{a}-\vec{b})|^2\\ &= (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})\\ &= \vec{a}\cdot\vec{a}-\vec{b}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b} && \text{Distributive Law of Dot Product}\\ &= |\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b} && \text{Commutative Law of Dot Product}\\ &= |\vec{a}|^2+|\vec{b}|^2-2|\vec{a}||\vec{b}|\cos\theta\\ &= a^2+b^2-2ab\cos\theta\end{aligned}$$

Geometric Meaning of Dot Product Dot product is related to the angle between the two input vectors by

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$$

Then the dot product between a vector and itself is $\vec{v} \cdot \vec{v} = |\vec{v}|^2$. If the input vectors are perpendicular, then its dot product is zero. Moreover, by Cauchy-Schwarz Inequality, we have

$$-1 \leq \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \cos \theta \leq 1$$

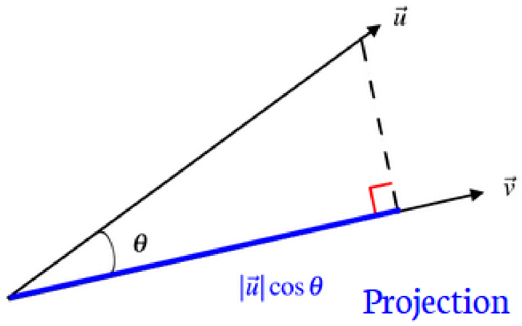
Hence θ in the cosine is always defined as a real number. Furthermore, the length of projection of $\vec{u}$ onto $\vec{v}$ is given by

$$|\text{proj}_v \vec{u}| = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$$

The projection vector is then

$$\text{proj}_v \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \hat{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$$

where we use the relation about unit vector $\hat{v} = \frac{\vec{v}}{|\vec{v}|}$.



Projection of $\vec{u}$ onto $\vec{v}$ indicated by the blue line.

Example 4.1.7 Prove $\vec{u} = (1,2,4)$ and $\vec{v} = (2,1,-1)$ are perpendicular to each other.

$$\begin{aligned}\vec{u} \cdot \vec{v} &= (1,2,4) \cdot (2,1,-1) \\ &= (1)(2) + (2)(1) + (4)(-1) \\ &= 0\end{aligned}$$

Since their dot product is zero, they are perpendicular to each other.

Example 4.1.8 Find the angle between $\vec{u} = (-1, -2, 3)$ and $\vec{v} = (4, 0, 4)$.

$$\begin{aligned}
\cos\theta &= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \\
&= \frac{(-1)(4) + (-2)(0) + (3)(4)}{(\sqrt{(-1)^2 + (-2)^2 + (3)^2})(\sqrt{(4)^2 + (0)^2 + (4)^2})} \\
&= \frac{8}{\sqrt{448}} \\
&= \frac{1}{\sqrt{7}}
\end{aligned}$$

Therefore, $\theta = 67.8°$.

Example 4.1.9 Find the projection vector of $\vec{u} = (1, 4, 6)$ onto $\vec{v} = (2, -3, 5)$.

Alternative: Find $\hat{v}$ first then compute $\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\hat{v}$.

$$\begin{aligned}
\text{proj}_v\vec{u} &= \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\vec{v} \\
&= \frac{(1, 4, 6) \cdot (2, -3, 5)}{(2, -3, 5) \cdot (2, -3, 5)}(2, -3, 5)
\end{aligned}$$

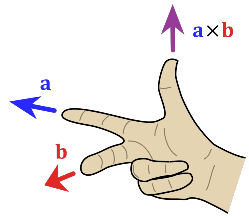
where we use the formula $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$, next

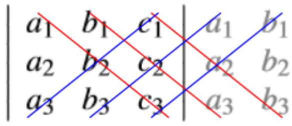
$$\begin{aligned}
\text{proj}_v\vec{u} &= \frac{(1)(2) + (4)(-3) + (6)(5)}{(2)(2) + (-3)(-3) + (5)(5)}(2, -3, 5) \\
&= \frac{20}{38}(2, -3, 5) \\
&= (\frac{20}{19}, \frac{-30}{19}, \frac{50}{19})
\end{aligned}$$

Cross Product Cross Product produces a 3D vector which is perpendicular to the two input 3D vectors, the direction of which is determined by the right-hand rule. For $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, the cross product is given by the determinant

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Moreover, we have $\hat{i}\times\hat{j}=\hat{k}$, $\hat{j}\times\hat{k}=\hat{i}$, $\hat{k}\times\hat{i}=\hat{j}$.





$$=(a_1b_2c_3+b_1c_2a_3+c_1a_2b_3)-(c_1b_2a_3+a_1c_2b_3+b_1a_2c_3)$$

Left: Demonstration of the Right-Hand Rule. (From Wikipedia) Right: Short-cut for Calculating 3×3 Determinant. Computation of 2×2 Determinant follows similar rule.

Example 4.1.10 Find the cross product of $\vec{u}=(2,1,1)$ and $\vec{v}=(-1,-1,2)$. Also verify that the resultant vector is perpendicular to $\vec{u}$ and $\vec{v}$ respectively.

$$\begin{aligned}\vec{u}\times\vec{v} &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & 1\\ -1 & -1 & 2\end{vmatrix}\\ &= ((1)(2)\hat{i}+(1)(-1)\hat{j}+(2)(-1)\hat{k})-((1)(-1)\hat{i}+(2)(2)\hat{j}+(1)(-1)\hat{k})\\ &= 3\hat{i}-5\hat{j}-\hat{k}=(3,-5,-1)\end{aligned}$$

$$\begin{aligned}(\vec{u}\times\vec{v})\cdot\vec{u} &= (3,-5,-1)\cdot(2,1,1)\\ &= (3)(2)+(-5)(1)+(-1)(1)\\ &= 0\end{aligned}$$

Therefore, the vector produced from cross product is perpendicular to $\vec{u}$. Similar goes for $\vec{v}$.

Property of Cross Product Some important properties of dot product are listed below.

$$\begin{aligned}\vec{u}\times\vec{v} &= -\vec{v}\times\vec{u} && \text{Anti-commutative Law of Dot Product}\\ \vec{u}\times(\vec{v}+\vec{w}) &= \vec{u}\times\vec{v}+\vec{u}\times\vec{w} && \text{Distributive Law of Dot Product}\\ \vec{u}\times(\vec{v}-\vec{w}) &= \vec{u}\times\vec{v}-\vec{u}\times\vec{w}\\ (\vec{u}+\vec{v})\times\vec{w} &= \vec{u}\times\vec{w}+\vec{v}\times\vec{w}\\ (\vec{u}-\vec{v})\times\vec{w} &= \vec{u}\times\vec{w}-\vec{v}\times\vec{w}\end{aligned}$$

It should be noted that it is generally not true that $(\vec{u}\times\vec{v})\times\vec{w}=\vec{u}\times(\vec{v}\times\vec{w})$.

Example 4.1.11 Verify $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v}+\vec{u}\times\vec{w}$, for $\vec{u}=(1,1,2)$, $\vec{v}=(1,0,1)$, $\vec{w}=(-1,1,0)$.

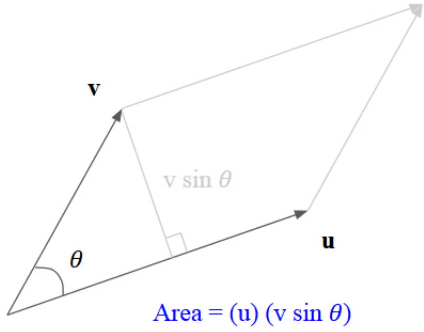
$$\begin{aligned}
\vec{u}\times(\vec{v}+\vec{w}) &= (1,1,2)\times((1,0,1)+(-1,1,0)) \\
&= (1,1,2)\times(0,1,1) \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} \\
&= -\hat{i}-\hat{j}+\hat{k} = (-1,-1,1)
\end{aligned}$$

$$\begin{aligned}
\vec{u}\times\vec{v}+\vec{u}\times\vec{w} &= (1,1,2)\times(1,0,1)+(1,1,2)\times(-1,1,0)) \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{vmatrix} \\
&= (\hat{i}+\hat{j}-\hat{k})+(-2\hat{i}-2\hat{j}+2\hat{k}) \\
&= -\hat{i}-\hat{j}+\hat{k} = (-1,-1,1) = \vec{u}\times(\vec{v}+\vec{w})
\end{aligned}$$

Geometric Meaning of Cross Product Similar to dot product, cross product has its own geometric interpretation. The relation between the magnitude of cross product and the angle between the two input vectors are given as

$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$$

which is also the area of parallelogram formed by $\vec{u}$ and $\vec{v}$. By extension, the area of triangle formed by $\vec{u}$ and $\vec{v}$ is $\frac{1}{2}|\vec{u}\times\vec{v}|$.



Cross Product and Area of Parallelogram.

Another observation is that if the two input vectors are parallel, the cross product returns a zero vector.

Example 4.1.12 Prove sine law by vector notation for the triangle with side lengths of a, b, c and angles α, β, γ.

Area of the triangle can be expressed by three different cross products as below

$$\frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}|\vec{b}\times\vec{c}| = \frac{1}{2}|\vec{c}\times\vec{a}|$$

Then, we have

$$|\vec{a}||\vec{b}|\sin\gamma = |\vec{b}||\vec{c}|\sin\alpha = |\vec{c}||\vec{a}|\sin\beta$$

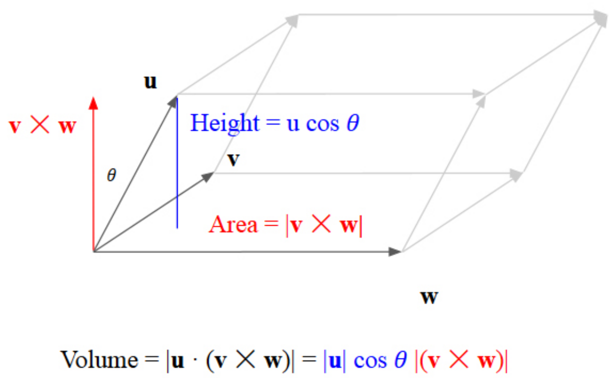
Dividing the equation by abc. we now have

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Triple Scalar Product The triple scalar product of $\vec{u}$, $\vec{v}$ and $\vec{w}$ is $\vec{u}\cdot(\vec{v}\times\vec{w})$. Its absolute value is the volume of the parallelepiped formed by the three vectors.

A quick way to obtain the triple scalar product is to compute the determinant

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



Triple Scalar Product and Volume of Parallelepiped.

Example 4.1.13 Find the volume of the parallelepiped formed by $\vec{u} = (1,0,1)$, $\vec{v} = (1,1,2)$, $\vec{w} = (3,2,5)$.

The volume is computed from the triple scalar product as

$$
\begin{aligned}
\vec{u}\cdot(\vec{v}\times\vec{w}) &= \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 3 & 2 & 5 \end{vmatrix} \\
&= (1)(1)(5)+(1)(1)(2)+(0)(2)(3) \\
&- (1)(1)(3)-(0)(1)(5)-(1)(2)(2) \\
&= 0
\end{aligned}
$$

The triple scalar product vanishes, which means that these three vectors are co-planar, i.e. lying on the same plane.

Line and Plane The equations of line and plane in 2D and 3D space are related to their normal vectors. If the normal vector of the line and plane are $a\hat{i}+b\hat{j}$ and $a\hat{i}+b\hat{j}+c\hat{k}$ respectively, then the equation must be in the form of $ax+by=k$ and $ax+by+cz=k$. The reverse is also true. It can be readily seen in the example below.

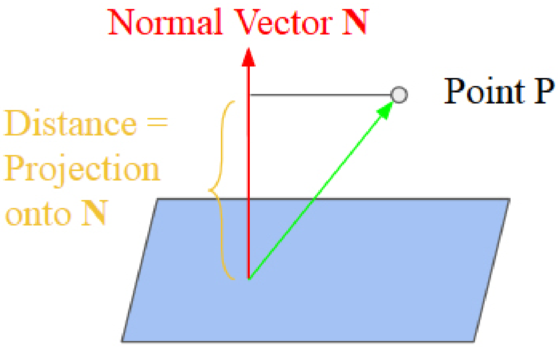
Example 4.1.14 Find the equation of the plane with a normal vector of $3\hat{i}+4\hat{j}+5\hat{k}$, given that the plane passes through the point $(1,2,6)$.

Any vector lying on the plane would be perpendicular to the normal vector. We can express such vector by $(x-1)\hat{i}+(y-2)\hat{j}+(z-6)\hat{k}$ where x,y,z lies on the plane. Then, we have

$$
\begin{aligned}
((x-1)\hat{i}+(y-2)\hat{j}+(z-6)\hat{k})\cdot(3\hat{i}+4\hat{j}+5\hat{k}) &= 0 \\
3(x-1)+4(y-2)+5(z-6) &= 0 \\
3x+4y+5z &= 41
\end{aligned}
$$

whose coefficients are seen to be determined by the normal vector.

Distance of Point to Plane Distance of a point to a plane is the projection of the vector which connects the plane to the point, onto the normal vector of the plane.



Distance of point P to the plane is indicated by yellow color (projection of green vector onto red vector). Meanwhile the black line is the distance of point P to the straight line along the $\hat{N}$ direction.

Example 4.1.15 Find the distance of the point $(2,3,5)$ to the plane $x+2y+3z=6$.

From the equation of the plane, it can be inferred that its normal vector is $\hat{\imath}+2\hat{\jmath}+3\hat{k}$. Now we can choose any point on the plane, like $(1,1,1)$. The required distance is then the projection of the vector $\vec{P}=(2-1)\hat{\imath}+(3-1)\hat{\jmath}+(5-1)\hat{k}=\hat{\imath}+2\hat{\jmath}+4\hat{k}$ onto the plane's normal vector $\vec{N}=\hat{\imath}+2\hat{\jmath}+3\hat{k}$.

$$\begin{aligned}
|\mathrm{proj}_N\vec{P}| &= \frac{\vec{P}\cdot\vec{N}}{|\vec{N}|} \\
&= \frac{(\hat{\imath}+2\hat{\jmath}+4\hat{k})\cdot(\hat{\imath}+2\hat{\jmath}+3\hat{k})}{|\hat{\imath}+2\hat{\jmath}+3\hat{k}|} \\
&= \frac{(1)(1)+(2)(2)+(4)(3)}{\sqrt{1^2+2^2+3^2}} \\
&= \frac{17}{\sqrt{14}}
\end{aligned}$$

4.1.2 Vector Calculus

Vector Parameterization Components of a vector can be some functions of a parameter such that as the parameter changes, the vector traces a curve in the space. This technique is called parameterization, which can be used to represent curves that are impossible to be described by a single equation.

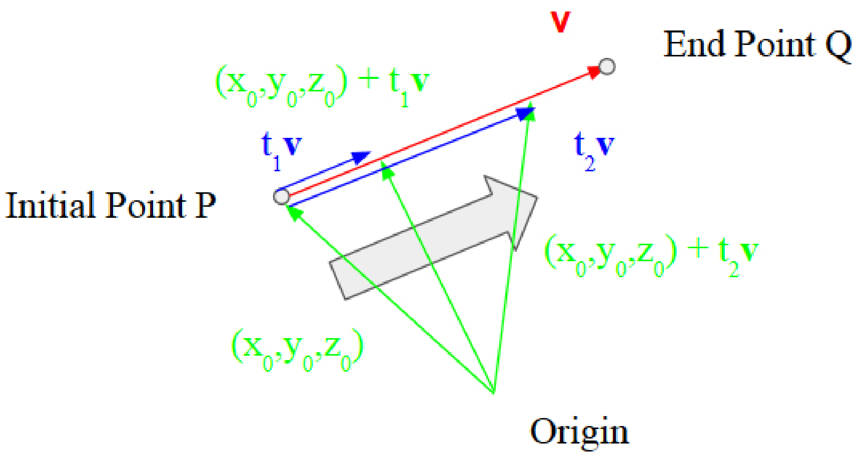
Example 4.1.16 Find a parameterization for the line segment connecting $(-1,1,5)$ to $(3,-1,3)$.

The vector connecting these two points is $\vec{v} = (3-(-1))\hat{i} + (-1-1)\hat{j} + (3-5)\hat{k}) = 4\hat{i} - 2\hat{j} - 2\hat{k}$. For any point (x,y,z) on the line segment, the vector between (x,y,z) and the given initial point $(x_0,y_0,z_0) = (-1,1,5)$ would be parallel to $\vec{v}$. Hence the line can be represented by

Alternative: Use $(3,-1,3)$ instead of $(-1,1,5)$ for derivation, with a different range of t.

$$\begin{aligned}(x,y,z)-(x_0,y_0,z_0) &= t\vec{v} \\ (x,y,z) &= (-1,1,5)+t(4,-2,-2) \\ &= (-1+4t, 1-2t, 5-2t)\end{aligned}$$

where $0 < t < 1$. If it is an infinitely long straight line passing through these two points rather than a segment, we have $-\infty < t < \infty$.



parameterization of a line segment, as t increases, the blue vector $t\vec{v}$ extends along the $\hat{v}$ direction and the green vector traces a straight line as indicated by the big grey arrow.

It is worthy to note that similar to the fact that a curve can be parameterized by one parameter, a surface can be parameterized by two parameters.

Example 4.1.17 If the line segment in Example 4.1.16 is extended to an infinitely long straight line, find the distance of the point $(1,1,1)$ to the line.

Alternative: Consider a plane with a normal $(4,-2,-2)$ and find the corresponding plane equation so that $(1,1,1)$ lies on it. Subsequently, find the intersection between the line and the plane.

We consider the vector linking any point on the straight line, let's say $(-1,1,5)$, to the point $(1,1,1)$ in question. The projection of this vector $\vec{u} = (1,1,1) - (-1,1,5) = (2,0,-4)$, onto any vector parallel to the direction of the line, like $\vec{v} = (4,-2,-2)$ we have found above, is

$$\begin{aligned}
\text{proj}_v \vec{u} &= \frac{u \cdot v}{|\vec{v}|^2}\vec{v} \\
&= \frac{(2,0,-4)\cdot(4,-2,-2)}{4^2+(-2)^2+(-2)^2}(4,-2,-2) \\
&= \frac{16}{24}(4,-2,-2) \\
&= (\frac{8}{3}, -\frac{4}{3}, -\frac{4}{3})
\end{aligned}$$

The displacement of the point from the line is then represented by $(2,0,-4) - (\frac{8}{3}, -\frac{4}{3}, -\frac{4}{3}) = (-\frac{2}{3}, \frac{4}{3}, -\frac{8}{3})$. Hence the distance is $\sqrt{(-\frac{2}{3})^2 + (\frac{4}{3})^2 + (-\frac{8}{3})^2} = \frac{\sqrt{84}}{3}$.

Example 4.1.18 Find a parameterization scheme for the ellipse $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$.

Alternative: Let $x = 2\sin t$ and $y = 3\cos t$ such that the ellipse is still traced but at a different starting point and in reverse direction.

Alternative: Replace t by kt where k is a constant, changing the speed of moving along the ellipse.

To do this we utilize the commonly used trigonometric identity,

$$\sin^2\theta + \cos^2\theta = 1$$

It is apparent that if we let $x = 2\cos t$, $y = 3\sin t$, then the equation automatically satisfy the identity. Hence we have $(x,y) = (2\cos t, 3\sin t)$. This would trace the ellipse in an anti-clockwise direction. The readers can convince themselves by graphing the vector with increasing value of t.

Example 4.1.19 Use a single equation to relate the parameterized variables $x = 2t+3$ and $y = 4t^2+1$.

$$\begin{aligned}
x &= 2t+3 \\
t &= \frac{x-3}{2}
\end{aligned}$$

Substituting this into $y = 4t^2 + 1$, we have

$$
\begin{aligned}
y &= 4\left(\frac{x-3}{2}\right)^2 + 1 \\
&= (x-3)^2 + 1 \\
&= x^2 - 6x + 10
\end{aligned}
$$

Displacement, Velocity, Acceleration Vectors Any vector representing the position, or the displacement of an object, expressed in functions of time t, such as $\vec{s}(t) = (x, y, z) = (t, 2t, t^2)$, can be differentiated component by component to obtain the velocity vector. This is very similar to what we have seen in the last part of Chapter 2 but generalized to vectors. Acceleration vector can be produced from the velocity vector by the same manner.

On the other hand, integrating the components each by each turns an acceleration vector to a velocity vector, and a velocity vector to a displacement vector.

Example 4.1.20 Given that the velocity of an object is $\vec{v} = 3t\hat{i} + 4t^2\hat{j} + 5\hat{k}$. Find its acceleration and displacement.

Displacement is found by

Common mistake: Forgetting to include the integration constants.

$$
\begin{aligned}
\vec{s} &= \int \vec{v} dt \\
&= \int 3t dt \hat{i} + \int 4t^2 dt \hat{j} + \int 5 dt \hat{k} \\
&= (\frac{3}{2}t^2 + C_x)\hat{i} + (\frac{4}{3}t^3 + C_y)\hat{j} + (5t + C_z)\hat{k}
\end{aligned}
$$

While acceleration is found by

$$
\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} \\
&= \frac{d(3t)}{dt}\hat{i} + \frac{d(4t^2)}{dt} dt \hat{j} + \frac{d(5)}{dt}\hat{k} \\
&= 3\hat{i} + 8t\hat{j}
\end{aligned}
$$

Arc Length of a Curve The arc length of a parameterized curve, or in other words, the distance travelled by the position vector along the path, from $t = a$ to

$t = b$, by Pythagora's Theorem, is

$$\int \sqrt{dx^2 + dy^2 + dz^2} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Notice that

$$\vec{v} = \frac{d\vec{s}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

we can rewrite the integral as

$$\int_a^b |\vec{v}| dt$$

Example 4.1.21 Find the arc length of the curve $x = 1$, $y = \sqrt{t}$, $z = t$, from $t = 0$ to $t = 4$.

$$\begin{aligned}\vec{v} &= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \\ &= \frac{d(1)}{dt}\hat{i} + \frac{d(\sqrt{t})}{dt}\hat{j} + \frac{d(t)}{dt}\hat{k} \\ &= \frac{1}{2\sqrt{t}}\hat{j} + \hat{k}\end{aligned}$$

Thus, the arc length is

$$\begin{aligned}\int_0^4 |\vec{v}| dt &= \int_0^4 \sqrt{(\frac{1}{2\sqrt{t}})^2 + (1)^2 dt} \\ &= \int_0^4 \sqrt{\frac{1}{4t} + 1 dt} \\ &= \int_0^4 \frac{\sqrt{1+4t}}{2\sqrt{t}} dt\end{aligned}$$

Let $u = \sqrt{t}$, then $du = \frac{1}{2\sqrt{t}} dt$, it becomes

$$\int_0^2 \sqrt{1+4u^2} du$$

Now let $u = \frac{1}{2}\tan\theta$, $du = \frac{1}{2}\sec^2\theta d\theta$, it becomes

$$\int_{u=0}^{u=2} \sqrt{1+\tan^2\theta}(\frac{1}{2}\sec^2\theta d\theta) = \frac{1}{2}\int_{u=0}^{u=2} \sec^3\theta d\theta$$

where the integral is calculated as

$$\begin{aligned}
\int \sec^3\theta d\theta &= \int \sec\theta d(\tan\theta) \\
&= [\sec\theta\tan\theta] - \int \tan\theta d(\sec\theta) \\
&= [\sec\theta\tan\theta] - \int \sec\theta\tan^2\theta d\theta \\
&= [\sec\theta\tan\theta] - \int \sec\theta(\sec^2\theta - 1)d\theta \\
&= [\sec\theta\tan\theta] - \int \sec^3\theta d\theta + \int \sec\theta d\theta
\end{aligned}$$

So we have

$$\begin{aligned}
2\int \sec^3\theta d\theta &= [\sec\theta\tan\theta] + \int \sec\theta d\theta \\
\int \sec^3\theta d\theta &= \frac{1}{2}[\sec\theta\tan\theta] + \frac{1}{2}\int \sec\theta d\theta \\
&= \frac{1}{2}[\sec\theta\tan\theta] + \frac{1}{2}\int \sec\theta\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}d\theta \\
&= \frac{1}{2}[\sec\theta\tan\theta] + \frac{1}{2}\int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}d\theta \\
&= \frac{1}{2}[\sec\theta\tan\theta] + \frac{1}{2}\int \frac{1}{\sec\theta+\tan\theta}d(\sec\theta+\tan\theta) \\
&= \frac{1}{2}[\sec\theta\tan\theta] + \frac{1}{2}[\ln|\sec\theta+\tan\theta|]
\end{aligned}$$

Changing back the variable to u, it becomes

$$\int \sec^3\theta d\theta = \frac{1}{2}[\sqrt{1+4u^2}(2u)] + \frac{1}{2}[\ln|\sqrt{1+4u^2}+2u|]$$

Therefore, the required arc length is

$$\begin{aligned}
\int_0^2 \sqrt{1+4u^2}du &= \frac{1}{2}\left(\frac{1}{2}[2u\sqrt{1+4u^2}]_0^2 + \frac{1}{2}[\ln|2u+\sqrt{1+4u^2}|]_0^2\right) \\
&= \sqrt{17} + \frac{1}{4}\ln(4+\sqrt{17})
\end{aligned}$$

Del operator Del operator is a crucial element of vector calculus. It can be regarded as a vector consisted of partial derivative operators. In 3D it is $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. Later on, we would see how it interacts with functions and vectors to produce gradient, divergence, and curl.

Gradient and Directional Derivative Gradient of a function $f(x,y,z)$ is denoted as ∇f, which is a vector $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ pointing towards the direction of greatest increase in f, hence comes its name. Directional Derivative is the rate of change of the function f in the direction $\hat{s}$, and is calculated as $\nabla f \cdot \hat{s}$, which, upon expansion, resembles Chain Rule for multiple variables.

Another property is that for a curve $f(x,y) = c$ or surface $f(x,y,z) = c$, its gradient vector is always normal to the curve or surface.

Example 4.1.22 Find the gradient of the function $f(x,y) = 2x^2 + 3y\cos y$. Specifically, calculate the gradient at (π,π), as well as the directional derivative along the direction $\hat{s} = \frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}$.

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} \\ &= 4x\hat{\imath} + 3(\cos y - y\sin y)\hat{\jmath}\end{aligned}$$

At (π,π), $\nabla f = 4\pi\hat{\imath} - 3\hat{\jmath}$. Hence the direction of greatest increase in f is the unit vector $\frac{1}{\sqrt{(4\pi)^2+3^2}}(4\pi,3)$, and the directional derivative along $\hat{s} = \frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}$ is

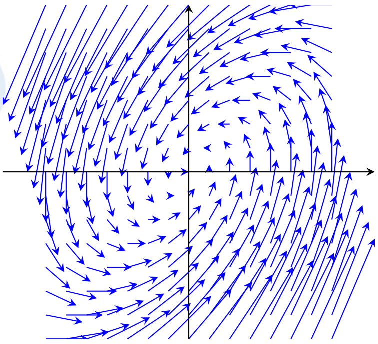
$$\begin{aligned}\nabla f \cdot \hat{s} &= (4\pi\hat{\imath} - 3\hat{\jmath}) \cdot (\frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}) \\ &= 2\sqrt{3}\pi - \frac{3}{2}\end{aligned}$$

Example 4.1.23 Find a normal vector for the ellipsoid $f(x,y,z) = x^2 + 2y^2 + z^2 = 4$ at $(1,1,1)$.

$$\nabla f = 2x\hat{\imath} + 4y\hat{\jmath} + 2z\hat{k}$$

$\nabla f = 2\hat{\imath} + 4\hat{\jmath} + 2\hat{k}$ at $(1,1,1)$, so $(2,4,2)$ is one of the possible normal vectors.

Vector Field, Divergence, Curl A vector field consisted of vectors, each assigned to a point across the space. Real-life examples are wind field and electric field. In three-dimensional setting, it is in the form of $\vec{F} = M(x,y,z)\hat{\imath} + N(x,y,z)\hat{\jmath} + P(x,y,z)\hat{k}$ where $M(x,y,z)$, $M(x,y,z)$, $M(x,y,z)$ are functions of x, y, z. Its property can be characterized by its divergence and curl.



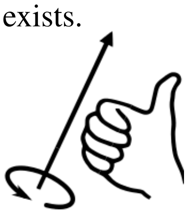
An example vector field $\vec{F} = -y\hat{i} + (x-y)\hat{j}$.

Divergence is $\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$. Positive divergence means the net flux is leaving and negative implies the net flux is entering.

Curl is $\nabla \times \vec{F}$, which expressed in determinant form, is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

It outputs a vector which indicated the local tendency of rotation according to the right-hand grip rule. For two-dimensional vector fields, only the $\hat{k}$ component exists.



Demonstration of Right-Hand Grip Rule. (From Wikipedia)

Example 4.1.24 Find the divergence and curl of the vector field $\vec{F} = M\hat{i} + N\hat{j}$, where $M(x,y) = x - 3y$, $N(x,y) = 3x + y$.

$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \\ &= \frac{\partial(x-3y)}{\partial x} + \frac{\partial(3x+y)}{\partial y} \\ &= 1 + 1 = 2 \end{aligned}$$

$$\begin{aligned}
\nabla\times\vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-3y & 3x+y & 0 \end{vmatrix} \\
&= (\frac{\partial(3x+y)}{\partial x} - \frac{\partial(x-3y)}{\partial y})\hat{k} \\
&= (3-(-3))\hat{k} = 6\hat{k}
\end{aligned}$$

Positive divergence and curl means that the flux is outgoing and there is an anti-clockwise rotation throughout the region.

Example 4.1.25 Find the divergence and curl of the vector field $\vec{F} = (\sin x)yz\hat{i} + xy^2z^2\hat{j} + (x+y)e^{-z^2}\hat{k}$.

$$\begin{aligned}
\nabla\cdot\vec{F} &= \frac{\partial((\sin x)yz)}{\partial x} + \frac{\partial(xy^2z^2)}{\partial y} + \frac{\partial(x+y)e^{-z^2}}{\partial z} \\
&= (\cos x)yz + 2xyz^2 - 2(x+y)ze^{-z^2}
\end{aligned}$$

$$\begin{aligned}
\nabla\times\vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\sin x)yz & xy^2z^2 & (x+y)e^{-z^2} \end{vmatrix} \\
&= (\frac{\partial((x+y)e^{-z^2})}{\partial y} - \frac{\partial(xy^2z^2)}{\partial z})\hat{i} + (\frac{\partial((\sin x)yz)}{\partial z} - \frac{\partial((x+y)e^{-z^2})}{\partial x})\hat{j} \\
&+ (\frac{\partial(xy^2z^2)}{\partial x} - \frac{\partial((\sin x)yz)}{\partial y})\hat{k} \\
&= (e^{-z^2} - 2xy^2z)\hat{i} + ((\sin x)y - e^{-z^2})\hat{j} + (y^2z^2 - (\sin x)z)\hat{k}
\end{aligned}$$

Laplacian Laplacian is the operator $\nabla^2 = \nabla\cdot\nabla$ which can be regarded as the second-order derivative for a multi-dimensional function. For a function $f(x,y,z)$, $\nabla^2 f = \nabla\cdot\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ is computed by first evaluating the gradient then taking the divergence of the gradient.

Example 4.1.26 Evaluate the laplacian for $f(x,y,z) = e^x yz$.

$$\begin{aligned}\nabla^2 f &= \nabla \cdot (\nabla f) \\ &= \nabla \cdot (e^x yz\hat{i} + e^x z\hat{j} + e^x y\hat{k}) \\ &= e^x yz\end{aligned}$$

Total Derivative and Local Derivative Recall Chain Rule for multiple variables, for a field quantity $u(x,y,z,t)$ associated to a physical object depending on time and space, we have

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} = \frac{\partial u}{\partial t} + \vec{v} \cdot \nabla u$$

where $\frac{du}{dt}$ is called the total derivative or material derivative, which is the rate of change in u of the moving element being traced. On the other hand, $\frac{\partial u}{\partial t}$ is the local derivative, which is the rate of change in u locally at the fixed position. Descriptions using the total derivative and local derivative are called Lagrangian and Eulerian respectively.

$\vec{v} \cdot \nabla u = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$ is the advection term which accounts for the change in u contributed by the movement of the element up or down the gradient.

Example 4.1.27 Given that a physical quantity $u(x,y,t)$ is increasing everywhere at a rate of 3 per second. And its gradient ∇u is $\hat{i} - 2\hat{j}$ per meter. If an element moves towards the south east at $4\,\mathrm{m\,s^{-1}}$, find the rate of change in u following the element.

This requires us to find the total derivative for the element. The velocity of the element is

$$\vec{v} = 4\hat{v} = 4(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}) = 2\sqrt{2}\hat{i} - 2\sqrt{2}\hat{j}$$

The local derivative is 3 per second everywhere, and hence the total derivative is computed as

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial t} + \vec{v} \cdot \nabla u \\ &= 3 + (2\sqrt{2}\hat{i} - 2\sqrt{2}\hat{j}) \cdot (\hat{i} - 2\hat{j}) \\ &= 3 + 6\sqrt{2}\end{aligned}$$

Example 4.1.28 A physical quantity $u(x,y,t)$ tracing a moving object is changing at a rate of 5 per second. The object moves to the west at a speed of $3\,\mathrm{m\,s^{-1}}$ and the gradient of the field ∇u is $-3\hat{i}+7\hat{j}$ per meter. Find the local rate of change in u at the instant.

Its material derivative is 5 per second. The local derivative is then

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{du}{dt} - \vec{v}\cdot\nabla u \\ &= 5-(-3\hat{i})\cdot(-3\hat{i}+7\hat{j}) \\ &= 5-9=-4\end{aligned}$$

4.1.3 LINE INTEGRAL AND MULTIPLE INTEGRAL

Line Integral A line integral is an integral integrated along a curve. Simple integral like $\int f(x)dx$ can be viewed as line integral along the x-axis. To evaluate such integral, we parameterize the path on which the integral is carried out, and integrate in terms of the parameter. In three-dimensional case, it turns the line integral into

$$\begin{aligned}\int f(x,y,z)ds = \int f(t)\frac{ds}{dt}dt &= \int f(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}dt \\ &= \int f(t)|\vec{v}|dt\end{aligned}$$

where ds is a small line segment along the path and equals to $\sqrt{dx^2+dy^2+dz^2}$

Example 4.1.29 Integrate $f(x,y) = x^2y$ along the straight line $y = 2x+3$ from $(0,3)$ to $(2,7)$.

We start with parameterizing the line, one of the possible choices that is $x = t$, $y = 2t+3$, with $0 < t < 2$. Then the integral is evaluated as

$$\begin{aligned}\int f(x,y)ds &= \int_0^2 x^2y\sqrt{dx^2+dy^2} \\ &= \int_0^2 t^2(2t+3)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt \\ &= \int_0^2 t^2(2t+3)\sqrt{1^2+2^2}dt \\ &= \sqrt{5}\int_0^2 2t^3+3t^2dt \\ &= \sqrt{5}[\frac{1}{2}t^4+t^3]_0^2 \\ &= 16\sqrt{5}\end{aligned}$$

Example 4.1.30 Integrate $f(x,y,z) = xy+z$ along the curve consisting of two sections, around the arc $x^2+y^2 = 1$, $z = 0$ from $(1,0,0)$ to $(0,1,0)$, and a vertical line from $(0,1,0)$ to $(0,1,1)$.

We parameterize the two sections, which are $x = \cos r$, $y = \sin r$ where $0 < r < \frac{\pi}{2}$, and $x = 0$, $y = 1$, $z = t$ where $0 < t < 1$. Then we calculate the two line integrals as

$$\begin{aligned}
I_1 &= \int_0^{\frac{\pi}{2}} (\cos r \sin r + 0)\sqrt{\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2} dr \\
&= \int_0^{\frac{\pi}{2}} \cos r \sin r \sqrt{(-\sin r)^2 + (\cos r)^2 + 0^2} dr \\
&= \int_0^{\frac{\pi}{2}} \sin r d(\sin r) \\
&= [\frac{1}{2}\sin^2 r]_0^{\frac{\pi}{2}} \\
&= \frac{1}{2}
\end{aligned}$$

Alternative: Integrate $\int z dz$ directly.

and

$$\begin{aligned}
I_2 &= \int_0^1 ((0)(1) + t)\sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2} dt \\
&= \int_0^1 t\sqrt{0^2 + 0^2 + 1^2} dt \\
&= \int_0^1 t dt \\
&= [\frac{1}{2}t^2]_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

Therefore, the required line integral is $I_1 + I_2 = \frac{1}{2} + \frac{1}{2} = 1$.

Line Integral - Revisited Another kind of line integral that is commonly seen is

$$\int \vec{F} \cdot d\vec{s} = \int \vec{F} \cdot (dx, dy, dz) = \int \vec{F} \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt$$

In physics, it is usually used in work done calculation where the vector field is a force field. Evaluation of such work done integral is similar to what we have above.

Example 4.1.31 For a helix parameterized as $x = \cos t$, $y = \sin t$, $z = \frac{t}{\pi}$, $0 < t < 4\pi$, find $\int \vec{F} \cdot d\vec{s}$, where $\vec{F} = y\hat{i} + x\hat{j} + e^z\hat{k}$.

$$\begin{aligned}\int \vec{F} \cdot d\vec{s} &= \int_0^4 (y, x, e^z) \cdot (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})dt \\ &= \int_0^4 (\sin t, \cos t, e^{\frac{t}{\pi}}) \cdot (-\sin t, \cos t, \frac{1}{\pi})dt \\ &= \int_0^4 (\cos^2 t - \sin^2 t + \frac{1}{\pi}e^{\frac{t}{\pi}})dt \\ &= \int_0^4 (\cos 2t + \frac{1}{\pi}e^{\frac{t}{\pi}})dt \\ &= \left[\frac{1}{2}\sin 2t + e^{\frac{t}{\pi}}\right]_0^{4\pi} \\ &= e^4 - 1\end{aligned}$$

Conservative Field A vector field $\vec{F}$ is conservative if it can be written as ∇f for some function $f(x,y,z)$. A conservative field has the property that, its work done integral from point a to point b can be re-written as

$$\begin{aligned}\int_a^b \vec{F} \cdot d\vec{s} &= \int_a^b \nabla f \cdot (dx, dy, dz) \\ &= \int_a^b (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \cdot (dx, dy, dz) \\ &= \int_a^b \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \int_a^b df = f(b) - f(a)\end{aligned}$$

where we have used Chain Rule for multiple variables. It means that the work done integral only depends on the value of $f(x,y,z)$ at a and b. Moreover, if the work done integral is carried out along a closed loop, then it simply evaluates to zero. A vector field is conservative if the curl is zero everywhere.

Example 4.1.32 Prove that $\vec{F} = y^2z^3\hat{i} + 2xyz^3\hat{j} + 3xy^2z^2\hat{k}$ is a conservative field. Find the work done integral from the point $(1,1,2)$ to $(1,2,4)$.

To prove that it is a conservative field, it is enough to show that

$$\begin{aligned}
\nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} \\
&= \left(\frac{\partial(3xy^2z^2)}{\partial y} - \frac{\partial(2xyz^3)}{\partial z} \right) \hat{i} + \left(\frac{\partial(y^2z^3)}{\partial z} - \frac{\partial(3xy^2z^2)}{\partial x} \right) \hat{j} \\
&+ \left(\frac{\partial(2xyz^3)}{\partial x} - \frac{\partial(y^2z^3)}{\partial y} \right) \hat{k} \\
&= (6xyz^2 - 6xyz^2)\hat{i} + (3y^2z^2 - 3y^2z^2)\hat{j} + (2yz^3 - 2yz^3)\hat{k} \\
&= \vec{0}
\end{aligned}$$

To find a function f such $\vec{F} = \nabla f$, we integrate the components as follows

$$\int y^2z^3 dx = xy^2z^3 + F(y,z)$$
$$\int 2xyz^3 dy = xy^2z^3 + G(x,z)$$
$$\int 3y^2z^2 dx = xy^2z^3 + H(x,y)$$

We choose $F = G = H = 0$ such that $f(x,y,z) = xy^2z^3$. And the required work done integral is

$$\begin{aligned}
f(1,2,4) - f(0,1,2) &= (1)(2)^2(4)^3 - (1)(1)^2(2)^3 \\
&= 248
\end{aligned}$$

Multiple Integral Multiple integral is a type of integral that have more than one integral signs, each integrated with respect to one variable. For two-dimensional space, it is in the form of

$$\iint f(x,y)dxdy = \iint f(x,y)dA$$

where $dxdy = dA$ is the area of a small rectangle with length dx and dy, which are small segments along the x and y direction. For three-dimensional space, it is in the form of

$$\iint f(x,y,z)dxdydz = \iint f(x,y,z)dV$$

where $dxdydz = dV$ is the volume of a small cuboid with length dx, dy and dz. To evaluate such integral, we have to identify the limits or boundaries, which can depend on other variables, and iterate from the innermost to the outermost integral.

Example 4.1.33 Integrate $\int_1^2 \int_0^1 \frac{x}{y} dxdy$ inside the rectangle $x = [0,1]$ and $y = [1,2]$.

Alternative: Integrate with respect to y first.

$$\int_1^2 \int_0^1 \frac{x}{y} dxdy = \int_1^2 \frac{1}{y} \left[\frac{1}{2}x^2\right]_0^1 dy$$
$$= \int_1^2 \frac{1}{2y} dy$$
$$= [\frac{1}{2} \ln y]_1^2 = \frac{\ln 2}{2}$$

Example 4.1.34 Derive the formula for area of a triangle with sides length a and b, with vertices at $(0,0)$, $(a,0)$, $(0,b)$.

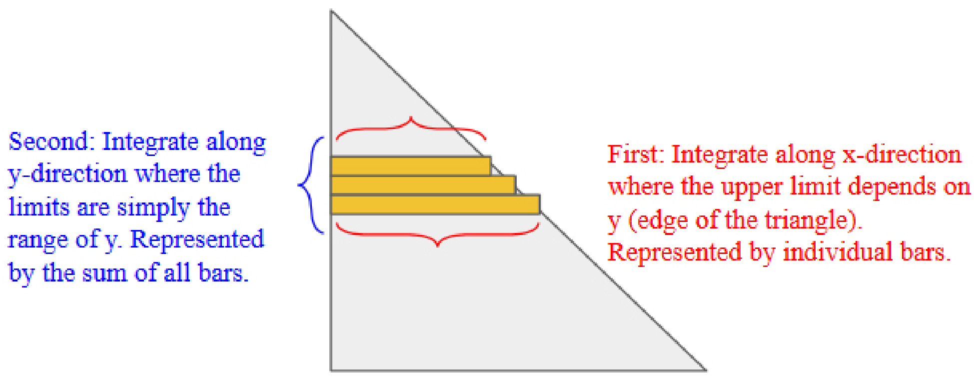
We evaluate the integral

Alternative: Evaluate $\int_0^a y(x)dx$ where $y(x) = b - \frac{b}{a}x$ is the altitude at x.

$$\iint f(x,y)dxdy = \iint f(x,y)dA$$

with appropriate limits and $f(x,y) = 1$ such that it represents the integrated area inside the triangle. To set up the limits, we need to find the equation of the line connecting $(a,0)$ to $(0,b)$, which is $x = a - \frac{a}{b}y$. If we choose to integrate with respect to x first, then the limits of x are determined by y, specifically from $x = 0$ to $x = a - \frac{a}{b}y$.

Alternative: Integrate with respect to y first, with the limits from $y = 0$ to $y = b - \frac{b}{a}x$. This is equivalent to the alternative suggested above.



Integration with respect to x then y in a triangular region.

Hence the area of the triangle is

$$\begin{aligned}\int_0^b \int_0^{a-\frac{a}{b}y} dxdy &= \int_0^b [x]_0^{a-\frac{a}{b}y} dy \\ &= \int_0^b (a - \frac{a}{b}y) dy \\ &= [ay - \frac{a}{2b}y^2]_0^b \\ &= \frac{1}{2}ab\end{aligned}$$

Example 4.1.35 Integrate $f(x,y) = x^2 + y^2$ inside the circle $x^2 + y^2 < 1$.

As suggested in the previous examples, we can choose the order of integration freely. As long as the integrand is continuous, the validity holds. This result is called Fubini's Theorem. We identify the limits depending on the order of integration, if we integrate along the x direction first, the limit of x should depend on y and are $\pm\sqrt{1-y^2}$, and after integrating along x direction, the limit of y would be simply ± 1. Hence the integral is

$$\begin{aligned}\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2) dxdy &= \int_{-1}^1 \left[\frac{1}{3}x^3 + xy^2\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy \\ &= \int_{-1}^1 \frac{2}{3}(1-y^2)\sqrt{1-y^2} + 2y^2\sqrt{1-y^2} dy\end{aligned}$$

Let $y = \sin\theta$, $dy = \cos\theta d\theta$, we have

$$\begin{aligned}&\int_{-1}^1 \left(\frac{2}{3}(1-y^2)\sqrt{1-y^2} + 2y^2\sqrt{1-y^2}\right) dy \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{2}{3}(1-\sin^2\theta)\sqrt{1-\sin^2\theta} + 2\sin^2\theta\sqrt{1-\sin^2\theta}\right) \cos\theta d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{2}{3}\cos^4\theta + 2(1-\cos^2\theta)\cos^2\theta\right) d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-\frac{4}{3}\cos^4\theta + 2\cos^2\theta\right) d\theta\end{aligned}$$

However,

$$
\begin{aligned}
\int \cos^4\theta &= \int \cos^3\theta d(\sin\theta) \\
&= \cos^3\theta\sin\theta - \int \sin\theta d(\cos^3\theta) \\
&= \cos^3\theta\sin\theta + \int 3\sin^2\theta\cos^2\theta d\theta \\
&= \cos^3\theta\sin\theta + \int 3(1-\cos^2\theta)\theta\cos^2\theta d\theta \\
&= \cos^3\theta\sin\theta + 3\int \cos^2 d\theta - \int 3\cos^4\theta d\theta
\end{aligned}
$$

Hence,

$$\int \cos^4\theta = \frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}\int \cos^2\theta$$

and

$$
\begin{aligned}
\int \cos^2\theta &= \int \frac{1}{2}(1+\cos 2\theta)d\theta \\
&= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-\frac{4}{3}\cos^4\theta + 2\cos^2\theta\right) d\theta \\
&= \left[-\frac{4}{3}(\frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta)) + 2(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \left[-\frac{1}{3}\cos^3\theta\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{\pi}{2}
\end{aligned}
$$

Example 4.1.36 Derive the volume of a pyramid with a square base of length a and height h.

Similar to what we do for calculating area, we want to find $\int\int\int dxdydz$ with suitable limits, which are $x = [-\frac{h-z}{h}\frac{a}{2}, \frac{h-z}{h}\frac{a}{2}]$, $y = [-\frac{h-z}{h}\frac{a}{2}, \frac{h-z}{h}\frac{a}{2}]$ and $z = [0,h]$, thus we have

$$\begin{aligned}
\int_0^h \int_{-\frac{h-z}{h}\frac{a}{2}}^{\frac{h-z}{h}\frac{a}{2}} \int_{-\frac{h-z}{h}\frac{a}{2}}^{\frac{h-z}{h}\frac{a}{2}} dxdydz &= \int_0^h \int_{-\frac{h-z}{h}\frac{a}{2}}^{\frac{h-z}{h}\frac{a}{2}} [x]_{-\frac{h-z}{h}\frac{a}{2}}^{\frac{h-z}{h}\frac{a}{2}} dydz \\
&= \int_0^h \int_{-\frac{h-z}{h}\frac{a}{2}}^{\frac{h-z}{h}\frac{a}{2}} a\frac{h-z}{h} dydz \\
&= \int_0^h a\frac{h-z}{h} [y]_{-\frac{h-z}{h}\frac{a}{2}}^{\frac{h-z}{h}\frac{a}{2}} dz \\
&= \int_0^h a^2 \frac{(h-z)^2}{h^2} dz \\
&= \int_0^h a^2 \frac{(h^2-2zh+z^2)}{h^2} dz \\
&= [a^2(z - \frac{z^2}{h} + \frac{z^3}{3h^2})]_0^h \\
&= \frac{1}{3}a^2h
\end{aligned}$$

Example 4.1.37 Find $\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2-y^2}} xyzdxdydz$ inside the part of a dome located in the upper first quadrant as indicated by the limits.

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2-y^2}} xyzdxdydz &= \int_0^1 \int_0^{\sqrt{1-z^2}} yz[\frac{1}{2}x^2]_0^{\sqrt{1-z^2-y^2}} dydz \\
&= \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{1}{2}yz(1-z^2-y^2)dydz \\
&= \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{1}{2}(yz - yz^3 - y^3z)dydz \\
&= \int_0^1 [\frac{1}{4}y^2z - \frac{1}{4}y^2z^3 - \frac{1}{8}y^4z]_0^{\sqrt{1-z^2}} dz \\
&= \int_0^1 (\frac{1}{8}z - \frac{1}{4}z^3 + \frac{1}{8}z^5)dz \\
&= [\frac{1}{16}z^2 - \frac{1}{16}z^4 + \frac{1}{48}z^6]_0^1 \\
&= \frac{1}{48}
\end{aligned}$$

Coordinate Transformation Sometimes we encounter multiple integrals that is easier to be represented in another coordinate system. This raises the question if we can do the integral in that coordinate system. Indeed, it is possible to do so, with the help of the Jacobian determinant. If we can express x and y in terms of new coordinates u and v, then the area differentials are related by

$$dxdy = |J|dudv$$

where

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

For three dimensional coordinates it follows the same essence.

Example 4.1.38 Integrate $f(x,y) = x + y$ in the square bound by $y - x = 0$, $y - x = 2$, $y + x = 0$, $y + x = 2$.

The equations for the square suggests a transformation of $u = y + x$, $v = y - x$, which implies that $x = \frac{u-v}{2}$ and $y = \frac{u+v}{2}$. The new limits are then $u = [0,2]$ and $v = [0,2]$. The Jacobian is

Alternative: Let $u = y - x$, $v = y + x$. Remember to take the absolute value of the Jacobian.

$$\begin{aligned} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= \frac{\partial(\frac{u-v}{2})}{\partial u}\frac{\partial(\frac{u+v}{2})}{\partial v} - \frac{\partial(\frac{u-v}{2})}{\partial v}\frac{\partial(\frac{u+v}{2})}{\partial u} \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2} \end{aligned}$$

Subsequently the integral becomes

$$\begin{aligned} \iint f(x,y)dxdy &= \int_0^2 \int_0^2 f(u,t)|J|dudv \\ &= \int_0^2 \int_0^2 (\frac{u-v}{2} + \frac{u+v}{2})|\frac{1}{2}|dudv \\ &= \int_0^2 \int_0^2 \frac{1}{2}ududv \\ &= \int_0^2 [\frac{1}{4}u^2]_0^2 dv \\ &= \int_0^2 dv \\ &= 2 \end{aligned}$$

Example 4.1.39 Redo the integral in Example 4.1.35 by using polar coordinates.

Polar coordinates transformation is

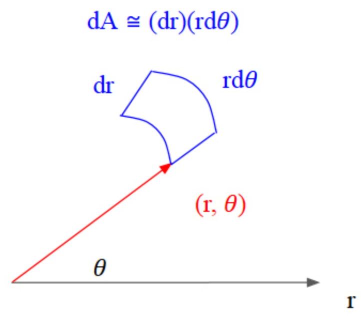
$$\begin{aligned} x(r,\theta) &= r\cos\theta \\ y(r,\theta) &= r\sin\theta \\ r^2 &= x^2+y^2 \end{aligned}$$

Hence the Jacobian determinant is

$$\begin{aligned} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} &= \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} \\ &= (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta) \\ &= r(\cos^2\theta + \sin^2\theta) = r \end{aligned}$$

Thus the area differentials are related by

$$dA = dxdy = rdrd\theta$$



Graphical Interpretation of the Area Differential in Polar Coordinates.

The limits for the circular region are $r=[0,1]$ and $\theta=[0,2\pi]$. The integral is then calculated as

$$\begin{aligned} \int_0^{2\pi}\int_0^1 r^2(rdrd\theta) &= \int_0^{2\pi}\int_0^1 r^3 rdrd\theta \\ &= \int_0^{2\pi} [\frac{1}{4}r^4]_0^1 d\theta \\ &= \frac{1}{4}\int_0^{2\pi} d\theta \\ &= \frac{1}{4}[\theta]_0^{2\pi} \\ &= \frac{1}{4}(2\pi) = \frac{\pi}{2} \end{aligned}$$

Example 4.1.40 Integrate $f(\rho) = \frac{1}{\rho}$ inside a sphere $0 < \rho < 1$ where ρ is the radial distance, with azimuth angle $0 < \theta < 2\pi$ and zenith angle $0 < \phi < \pi$. Spherical coordinates transformation is

$$\begin{aligned} x(\rho,\theta,\phi) &= \rho \sin\phi \cos\theta \\ y(\rho,\theta,\phi) &= \rho \sin\phi \sin\theta \\ z(\rho,\theta,\phi) &= \rho \cos\phi \end{aligned}$$

The Jacobian determinant is therefore

$$\begin{aligned} \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} &= \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} \\ &= -\rho^2\sin^3\phi(\cos^2\theta+\sin^2\theta) - \rho^2\sin\phi\cos^2\phi(\sin^2\theta+\cos^2\theta) \\ &= -\rho^2\sin\phi(\sin^2\phi+\cos^2\phi) \\ &= -\rho^2\sin\phi \end{aligned}$$

The volume differentials are related by

$$dV = dxdydz = |-\rho^2\sin\phi|d\rho d\theta d\phi = \rho^2\sin\phi d\rho d\theta d\phi$$

As a result,

$$\begin{aligned} \iiint \frac{1}{\rho} dV &= \int_0^\pi \int_0^{2\pi} \int_0^1 \frac{1}{\rho}\rho^2 \sin\phi d\rho d\theta d\phi \\ &= \int_0^\pi \int_0^{2\pi} \int_0^1 \rho\sin\phi d\rho d\theta d\phi \\ &= \int_0^\pi \int_0^{2\pi} [\frac{1}{2}\rho^2]_0^1 \sin\phi d\theta d\phi \\ &= \int_0^\pi \int_0^{2\pi} \frac{1}{2}\sin\phi d\theta d\phi \\ &= \int_0^\pi \frac{1}{2}[\theta]_0^{2\pi}\sin\phi d\phi \\ &= \int_0^\pi \frac{1}{2} 2\pi \sin\phi d\phi \\ &= \pi[-\cos\phi]_0^\pi = 2\pi \end{aligned}$$

Cylindrical coordinates are similar to polar coordinates but with an extra z dimension, and an area differential of $rdrd\theta dz$.

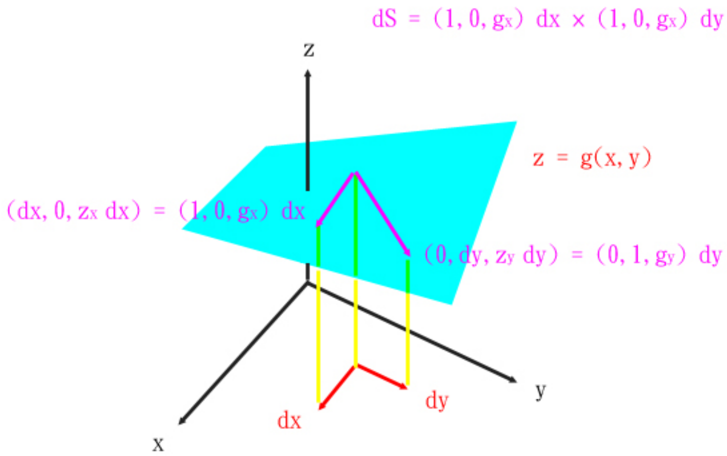
Surface Integral Similar to line integral, we have surface integral which integrates a function on a surface. Two-dimensional surface in a three-dimensional space can be explicitly defined as $z = g(x,y)$, however there are other cases using other coordinates, or an implicit equation. Surface integral of a function $f(x,y,z)$ is written as

$$\iint_R f(x,y,z)dS$$

where dS is called a surface element representing an infinitely small area on the integration surface R. We would see how to obtain dS in the next example.

Example 4.1.41 Integrate $f(x,y,z) = xyz$, on the surface $z = x+y$ with $x = [1,2]$ and $y = [1,2]$.

We want to express dS, given that x increments by dx and y increments by dy. If x increases by dx, then the surface vector moves by $dx\hat{i} + \frac{\partial z}{\partial x}dx\hat{k} = (1,0,z_x)dx$ and similar for y we have $dy\hat{j} + \frac{\partial z}{\partial y}dy\hat{k} = (0,1,z_y)dy$.



Surface Element in terms of dx and dy on the x-y plane where the surface is projected onto.

Thus the area of the surface element is the magnitude of their cross product

$$\begin{aligned}(1,0,z_x)dx \times (0,1,z_y)dy &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} dxdy \\ &= (-z_x\hat{i} - z_y\hat{j} + \hat{k})dxdy\end{aligned}$$

which is $\sqrt{(-z_x)^2+(-z_y)^2+1^2}dxdy = \sqrt{z_x^2+z_y^2+1}dxdy = \sqrt{1^2+1^2+1}dxdy = \sqrt{3}dxdy$.

Alternatively, using the property of gradient,

$$dS = \frac{|\nabla g|}{|\nabla g \cdot \hat{N}|}dA$$

where $g(x,y,z) = c$ defines the surface and $\hat{N}$ is the unit normal vector of the projection plane. In this question, $g(x,y,z) = x+y-z = 0$ and $\hat{N} = \hat{k}$ if we choose x-y plane as the projection plane, and thus

$$\begin{aligned}
dS &= \frac{|\nabla g|}{|\nabla g \cdot \hat{k}|}dxdy \\
&= \frac{|(1,1,-1)|}{|(1,1,-1)\cdot(0,0,1)|}dxdy \\
&= \sqrt{3}dxdy
\end{aligned}$$

On the surface, $f(x,y,z) = xyz = xy(x+y)$. The integral is then

$$\begin{aligned}
\iint_R f(x,y,z)dS &= \int_1^2\int_1^2 xy(x+y)\sqrt{3}dxdy \\
&= \int_1^2\int_1^2 \sqrt{3}(x^2y+xy^2)dxdy \\
&= \int_1^2 \sqrt{3}[\frac{1}{3}x^3y+\frac{1}{2}x^2y^2]_1^2 dy \\
&= \int_1^2 \sqrt{3}(\frac{7}{3}y+\frac{3}{2}y^2)dy \\
&= \sqrt{3}[\frac{7}{6}y^2+\frac{1}{2}y^3]_2^1 \\
&= \sqrt{3}[\frac{7}{6}y^2+\frac{1}{2}y^3]_2^1 \\
&= 7\sqrt{3}
\end{aligned}$$

Example 4.1.42 Integrate $f(x,y,z) = x^2+yz$ on the parameterized diamond-shaped surface $x = u+v$, $y = u-v$, $z = v-u$, where $0 < u < 1$ and $0 < v < 1$.

Similar to the idea in previous example, we want to find how the surface vector moves when u and v changes by du and dv. They are $(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u})du$ and

$(\frac{\partial x}{\partial v}, \frac{\partial x}{\partial v}, \frac{\partial x}{\partial v})dv$ respectively. The magnitude of their cross product forms the surface element as

$$dS = |(x_u, y_u, z_u) \times (x_v, y_v, z_v)|dudv$$

In this case,

$$\begin{aligned}(x_u, y_u, z_u) \times (x_v, y_v, z_v) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \\ &= -2\hat{j} - 2\hat{k}\end{aligned}$$

Thus the surface element is

$$dS = \sqrt{(-2)^2 + (-2)^2} dudv = 2\sqrt{2}dudv$$

And the integral is

$$\begin{aligned}&\int_0^1 \int_0^1 ((u+v)^2 + (u-v)(v-u))(2\sqrt{2}dudv) \\ &= \int_0^1 \int_0^1 8\sqrt{2}uvdudv \\ &= \int_0^1 8\sqrt{2}v[\frac{1}{2}u^2]_0^1 dv \\ &= \int_0^1 4\sqrt{2}vdv \\ &= 4\sqrt{2}[\frac{1}{2}v^2]_0^1 \\ &= 2\sqrt{2}\end{aligned}$$

Surface Integral - Revisited Just like line integral, there exists another kind of surface integral called the flux integral, which measures the net flux across a given surface. For a vector field $\vec{F}$ and an oriented surface with a unit normal vector $\hat{n}$ and surface element dS, the flux integral is

$$\iint_R \vec{F} \cdot \hat{n} dS = \iint_R \vec{F} \cdot d\vec{S}$$

where $\vec{F} \cdot \hat{n}$ gives the flux across the boundary per unit area. In two-dimensional space, the flux integral retains the same form with dS replaced by ds refers to a small line segment along the curve.

Example 4.1.43 For a vertical plane $x = 1$, $0 < y < 3$, $0 < z < 3$ which is oriented towards the positive x-direction, find the flux across the plane if $\vec{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$.

The oriented surface has a unit normal vector of $(1,0,0)$ everywhere. Project the surface onto the y-z plane, it is easy to see $dS = dydz$, thus we have

$$\begin{aligned}
\iint_R \vec{F} \cdot \hat{n} dS &= \int_0^3 \int_0^3 (1,0,0) \cdot (yz,xz,xy) dydz \\
&= \int_0^3 \int_0^1 yzdydz \\
&= \int_0^3 [\frac{1}{2}y^2]_0^3 dz \\
&= \frac{9}{2} \int_0^3 dz \\
&= \frac{9}{2} [\frac{1}{2}z^2]_0^3 \\
&= \frac{81}{4}
\end{aligned}$$

Example 4.1.44 For a surface $z = x^2 + y^2$, within the circular region $x^2 + y^2 < 1$, find the flux under the vector field $\vec{F} = x\sqrt{x^2+y^2}\hat{i} + y\sqrt{x^2+y^2}\hat{j}$.

We follow the similar procedure of Example 4.1.41 and find $\hat{n}dS$ as

Alternative: Use the cross-product method in Example 4.1.41 to find the normal vector, which also gives you $\hat{n}dS$.

$$\hat{n}dS = \frac{\nabla g}{|\nabla g \cdot \hat{N}|} dA$$

the only difference from the formula in Example 4.1.41 is that $|\nabla g|$ becomes ∇g, with $g(x,y,z) = x^2 + y^2 - z = 0$. The formula in Example 4.1.41 finds dS but here we want $\hat{n}dS$. Using x-y plane as the projection plane, we have

$$\begin{aligned}
\hat{n}dS &= \frac{\nabla g}{|\nabla g \cdot \hat{k}|} dxdy \\
&= \frac{(2x,2y,-1)}{|(2x,2y,-1) \cdot (0,0,1)|} dxdy \\
&= (2x,2y,-1) dxdy
\end{aligned}$$

However, we want the upward direction being the positive direction, so we use $\hat{n}dS = (-2x, -2y, 1)dxdy$ instead. The integral is

$$\begin{aligned}\iint_R \vec{F} \cdot \hat{n}dS &= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x\sqrt{x^2+y^2}, y\sqrt{x^2+y^2}, 0) \cdot (-2x, -2y, 1)dxdy \\ &= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} -2(x^2+y^2)\sqrt{x^2+y^2}dxdy\end{aligned}$$

Using polar coordinates, it now becomes

$$\begin{aligned}\int_0^{2\pi} \int_0^1 (-2r^3)rdrd\theta &= \int_0^{2\pi} \int_0^1 -2r^4 drd\theta \\ &= \int_0^{2\pi} [-\frac{2}{5}r^5]_0^1 d\theta \\ &= \int_0^{2\pi} -\frac{2}{5}d\theta \\ &= -\frac{2}{5}[\theta]_0^{2\pi} \\ &= -\frac{4\pi}{5}\end{aligned}$$

Green's Theorem For line integral or flux integral along a closed loop in two dimensional space, Green's Theorem states that they can be re-written using curl or divergence which allows easier computation, under the condition that the vector field has continuous first partial derivatives. For a vector field $\vec{F} = M\hat{i} + N\hat{j}$, the curl form of Green's Theorem for work done integral is

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{s} = \oint_C Mdx + Ndy &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy \\ &= \iint_R (\nabla \times \vec{F}) \cdot \hat{k}dxdy\end{aligned}$$

with the work done integral carried out in anti-clockwise direction. Meanwhile the divergence form of Green's Theorem for flux integral is

$$\begin{aligned}\oint_C \vec{F} \cdot \hat{n}ds = \oint_C Mdy - Ndx &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy \\ &= \iint_R \nabla \cdot \vec{F}dxdy\end{aligned}$$

where $\hat{n}$ is the outward unit normal vector and the flux integral represents the outward flux. Here $\oint_C$ means integration along the closed loop C and R is the region enclosed by C.

We shall see that these two forms of Green's Theorem are special cases for Stokes' Theorem and Divergence Theorem which are to be discussed later.

Example 4.1.45 Find the work done integral along the closed loop $x^2+y^2=4$ in anti-clockwise direction under $\vec{F}=M\hat{\imath}+N\hat{\jmath}$, where $M=y^2$ and $N=x^2$.

We apply the curl form of Green's Theorem and obtain

$$\begin{aligned}\oint_C \vec{F}\cdot d\vec{s} &= \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy && \text{Green's Theorem}\\ &= \int_{-2}^{2}\int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}}(2x-2y)dxdy\end{aligned}$$

Alternative: Evaluate the line integral directly by using the parameterization $x = 2\cos t$, $y = 2\sin t$, $0<t<2\pi$.

Using polar coordinates, the integral becomes

$$\int_0^{2\pi}\int_0^2 2(r\cos\theta - r\sin\theta)rdrd\theta$$

If we choose to integrate with respect to θ first, it can be readily seen that the integral is zero.

Example 4.1.46 Find the inward flux across the square $x=[0,1]$, $y=[0,1]$ with $\vec{F}=M\hat{\imath}+N\hat{\jmath}$, where $M=1-x$ and $N=1-y$.

We apply the divergence form of Green's Theorem and have the outward flux as

Alternative: Evaluate flux across each edge and add them up.

$$\begin{aligned}\oint_C \vec{F}\cdot\hat{n}ds &= \iint_R \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)dxdy && \text{Green's Theorem}\\ &= \int_0^1\int_0^1 (-1)+(-1)dxdy\\ &= -2\int_0^1\int_0^1 dxdy\\ &= -2\end{aligned}$$

Since the question requires the inward flux, we add a negative sign and the answer is 2.

Common mistake: Forgetting to take into account of the orientation.

Stokes' Theorem Stokes' Theorem is an extension of the curl form of Green's Theorem, which says that for any work done integral along the closed loop C in three-dimensional space, we have

$$\oint_C \vec{F} \cdot d\vec{s} = \iint_R (\nabla \times \vec{F}) \cdot \hat{n} dS$$

where R can be any surface having C as the boundary and $\hat{n}$ as its unit normal vector with its direction determined by using the right-hand grip rule on the integration loop C. We would see how it works in the next example.

In addition, we see that if the curl is zero, then by Stokes' Theorem the work done integral along any closed loop would be zero, which constitutes a conservative field as we have discussed earlier.

Example 4.1.47 Integrate $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ as a work done integral along the circle $x^2 + y^2 = 1$, $z = 0$, in anti-clockwise direction.

The curl is

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} \\ &= \hat{i} + \hat{j} + \hat{k} \end{aligned}$$

Alternative: Simply use the circular region $x^2 + y^2 < 1$, $z = 0$. This reduces Stokes' Theorem into Green's Theorem.

For demonstration purpose, we choose the half-spherical surface $g(x,y,z) = x^2 + y^2 + z^2 = 1$, $z > 0$ to be used when applying Stokes' Theorem, and project the surface onto the x-y plane to do the integration, where the procedure can be referenced from Example 4.1.40, and therefore we have

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{s} &= \iint_R (\nabla \times \vec{F}) \cdot \hat{n} dS && \text{Stokes' Theorem} \\ &= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (1,1,1) \cdot \frac{\nabla g}{|\nabla g \cdot \hat{k}|} dxdy \\ &= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (1,1,1) \cdot \frac{(2x,2y,2z)}{2z} dxdy \\ &= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (\frac{x}{z} + \frac{y}{z} + 1) dxdy \\ &= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (\frac{x+y}{\sqrt{1-x^2-y^2}} + 1) dxdy \end{aligned}$$

We use the fact that both $\frac{x}{\sqrt{1-x^2-y^2}}$ and $\frac{y}{\sqrt{1-x^2-y^2}}$ are odd functions and the integration region is symmetrical to eliminate them. Hence the integral is simply the area of the circular region

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dxdy = \iint_R dA = \pi$$

Example 4.1.48 Given that the curl of $\vec{F} = (z-y)\hat{\imath} + (x-z)\hat{\jmath} + (y-x)\hat{k}$ is $\nabla \times \vec{F} = (2,2,2)$. Find the upward flux of any arbitrary surface that has a square boundary $-1 < x < 1$, $-1 < y < 1$, $z = 0$ under the vector field $\nabla \times F = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$.

We should be careful that the vector field in question is $\nabla \times \vec{F}$ not $\vec{F}$. Nevertheless, we can use Stokes' Theorem to relate the upward flux under $\nabla \times \vec{F}$ and line integral of $\vec{F}$, which is

$$\iint_R (\nabla \times \vec{F}) \cdot \hat{n} dS = \oint_C \vec{F} \cdot d\vec{s} \qquad \text{Stokes' Theorem}$$

where left hand side is exactly what we want, but cannot compute directly due to the arbitrary nature of R, so we proceed to calculate the right hand side, which gives us

Alternative: By extension, use Stokes' Theorem once more time to relate the two integrals to any flux integral with a fixed integration surface chosen freely.

$$\oint_C \vec{F} \cdot d\vec{s} = \oint_C (z-y, x-z, y-z) \cdot d\vec{s}$$

Breaking the boundary into four edges, calculating the corresponding line integral each by each along anti-clockwise direction, and adding them up gives the answer. For the right edge where $x = 1$, $z = 0$, the line integral is

$$\int_{-1}^{1} (-y, 1, y) \cdot (0, dy, 0) = \int_{-1}^{1} dy = [y]_{-1}^{1} = 2$$

The evaluation for other edges follow similar procedures, and the final answer is $2+2+2+2 = 8$.

Divergence Theorem Divergence Theorem is a generalzed version of the divergence form of Green's Theorem, which states that the outward flux integral across a closed surface R is related to the divergence by

$$\oiint_R \vec{F} \cdot \hat{n} dS = \iiint_D \nabla \cdot \vec{F} dV$$

where D is the volume enclosed by the surface. If the divergence is zero everywhere, then the flux across any closed surface would also be zero.

Example 4.1.49 Find the outward flux across the cylinder $x^2+y^2=4$, $0<z<3$ under the vector field $\vec{F}=2x\hat{i}+5y\hat{j}+8z\hat{k}$. Also, find the flux across the curved surface of the cylinder.

The divergence is

$$\nabla\cdot\vec{F}=\frac{\partial(2x)}{\partial x}+\frac{\partial(5y)}{\partial y}+\frac{\partial(8z)}{\partial z}=2+5+8=15$$

Hence by Divergence Theorem the required outward flux is

$$\begin{aligned}\oiint_R \vec{F}\cdot\hat{n}dS &= \iiint_D \nabla\cdot\vec{F}dV && \text{Divergence Theorem}\\ &= 15\iiint_D dV\end{aligned}$$

where $\iiint_D dV=\pi(2)^2(3)=12\pi$ is simply the volume of the cylinder and thus the outward flux across the cylinder is 180π.

Alternative: Compute the flux across the curved surface by using cylindrical coordinates.

The flux across the curved surface can be found by subtracting the flux across the cylinder by the outward fluxes on the top and bottom. Outward flux on the bottom $z=0$ is

$$\iint_{\text{bottom}}(2x\hat{i}+5y\hat{j}+8(0)\hat{k})\cdot(0,0,-1)dA=0$$

Meanwhile the outward flux on the top $z=3$ is

$$\begin{aligned}\iint_{\text{top}}(2x\hat{i}+5y\hat{j}+8(3)\hat{k})\cdot(0,0,1)dA &= \iint_{\text{top}}24\pi dA\\ &= 24(\pi(2)^2)=96\pi\end{aligned}$$

Thus the flux across the curved surface is $180\pi-96\pi=84\pi$.

Example 4.1.50 Find

$$\oiint_R(x^2+2y^2+3z^2)dS$$

on the spherical surface $x^2+y^2+z^2=1$ of radius 1.

Notice that the unit outward normal vector of the spherical surface is simply (x,y,z). Then we are able to rewrite the integral as

$$\oiint_R\vec{F}\cdot\hat{n}dS=\oiint_R(x\hat{i}+2y\hat{j}+3z\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})dS$$

then by Divergence Theorem, we have

$$\begin{aligned}
\oiint_R (x\hat{i}+2y\hat{j}+3z\hat{k})\cdot\hat{n}dS &= \iiint_D \nabla\cdot(x\hat{i}+2y\hat{j}+3z\hat{k})dV \quad \text{Divergence Theorem} \\
&= 6\iiint_D dV \\
&= 6(\frac{4}{3}\pi(1)^3) = 8\pi
\end{aligned}$$

Alternative: Evaluate the surface integral directly by breaking the surface into upper half and lower half.

It is noted that both Stokes' Theorem and Divergence Theorem can only be applied if the first partial derivatives of the vector field are continuous as for Green's Theorem.

4.1.4 MISCELLANEOUS

Helmholtz's Theorem Helmholtz's Theorem is a crucial result in vector calculus which states that, a smooth vector field can be decomposed into

$$\vec{F} = \nabla\phi + \nabla\times\vec{A}$$

if the divergence and curl of $\vec{F}$ tends to zero when approaching infinity distance. A physical interpretation of the two terms on the right is the divergent part and rotating part of $\vec{F}$ respectively. It can be shown that the divergent part is non-rotating, and the rotating part is non-divergent, i.e. $\nabla\times\nabla\phi = \vec{0}$ and $\nabla\cdot(\nabla\times\vec{A}) = 0$, or in the other words, curl of gradient and divergence of curl are both zero.

By extension, if $\vec{F}$ itself is non-divergent, then only the rotating part remains and it can be written as $\nabla\times\vec{A}$. Furthermore, if $\vec{F}$ is two-dimensional, for example, a horizontal flow $(u, v, 0)$, then it can be written as $\nabla\times(0, 0, \psi)$, the curl of a vector aligned in the vertical direction. Expanding the expression, we have

$$u = -\frac{\partial\psi}{\partial y}$$
$$v = \frac{\partial\psi}{\partial x}$$

where a negative sign is added by custom. ψ in this case is often referred to as the stream function.

Example 4.1.51 Verify that if we rewrite a horizontal flow using stream function, then it is non-divergent by itself.

To this end, we only need to show that the divergence is zero. Using stream function, we have

$$\begin{aligned}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= -\psi_{yx} + \psi_{xy} \\ &= 0\end{aligned}$$

where we have used the Clairaut's Theorem, allowing us to switch the order of partial derivatives.

4.2 Vector / Vector Calculus in ESSC3200

4.2.1 Equations of Motion

Example 4.2.1 Express Pressure Gradient Force using vector notation.

Consider an air parcel with size $\delta x \delta y \delta z$, along the x-direction the net force per unit mass due to pressure difference on the left face A and right face B is

$$\begin{aligned} F_x &= \frac{1}{m}(p_A - p_B)\,\delta y \delta z \\ &= \frac{1}{m}\left(p_A - (p_A + \frac{\partial p}{\partial x}\delta x)\right)\delta y \delta z \end{aligned}$$

where we expand p_B using Taylor's series to the first order, subsequently

$$\begin{aligned} F_x &= -\frac{1}{m}\frac{\partial p}{\partial x}\delta x \delta y \delta z \\ &= -\frac{1}{\rho \delta x \delta y \delta z}\frac{\partial p}{\partial x}\delta x \delta y \delta z \\ &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \end{aligned}$$

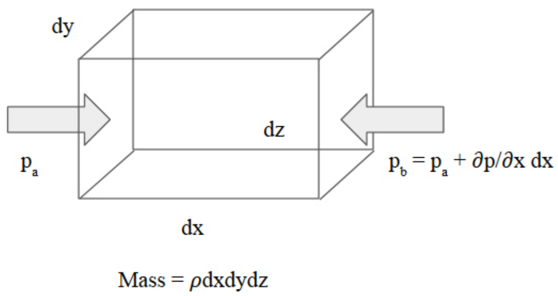


Illustration of Pressure Gradient Force.

Similarly,

$$\begin{aligned} F_y &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\ F_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}$$

Hence

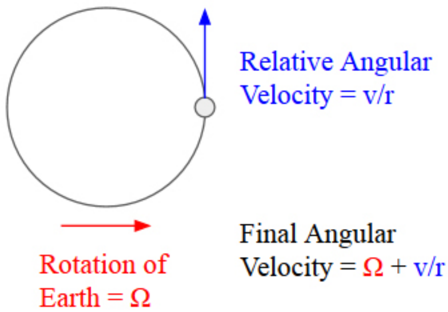
$$\begin{aligned} F &= F_x\hat{i} + F_y\hat{j} + F_z\hat{k} \\ &= -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}\right) \\ &= -\frac{1}{\rho}\nabla p \end{aligned}$$

Example 4.2.2 Prove that the expression of Coriolis Force is $-2\vec{\Omega} \times \vec{v}$.

The radial component of Coriolis force is found by considering the centrifugal force for a tangential motion with a velocity U relative to the rotating frame, which is

$$F_{\text{cen}} = R\omega_{\text{abs}}^2 = R(\Omega + \frac{U}{R})^2 = \Omega^2 R + 2\Omega U + \frac{U^2}{R}$$

$\omega_{\text{abs}} = \Omega + \omega = \Omega + \frac{U}{R}$ is the angular velocity observed in the absolute frame. The terms $2\Omega U$ is the desired radial component of Coriolis Force. The last term is the curvature term.



Angular velocities of an air parcel in the rotating frame and inertial frame.

For an air parcel which is at rest in the rotating frame and has an absolute velocity $U_{\text{abs}} = R\Omega$ initially, Conservation of Angular Momentum requires that under a small radial displacement dR and thus a change in the relative tangential velocity dU, RU_{abs} remains the same, and we have

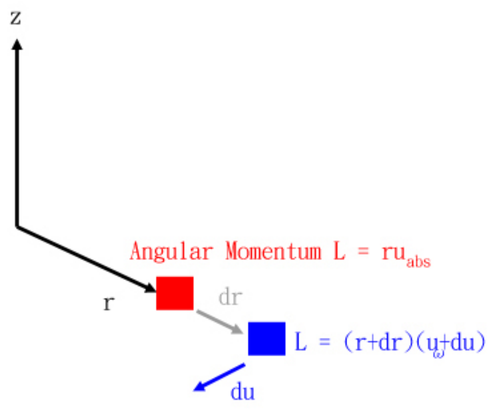
$$\begin{aligned} RU_{\text{abs}} &= (R + dR)(\Omega(R + dR) + dU) \\ \Omega R^2 &= \Omega R^2 + 2\Omega R dR + \Omega dR^2 + RdU + dRdU \end{aligned}$$

where the new U_{abs} at the right hand side comes from adding the would-be velocity at $R + dR$ caused by rotation of the frame to dU.

Neglecting second-order terms and take the time derivative, we have the tangential component of Coriolis Force as

$$2\Omega R dR + R dU = 0$$
$$dU = -2\Omega dR$$
$$\frac{dU}{dt} = -2\Omega \frac{dR}{dt} = -2\Omega V$$

where V is the radial velocity.



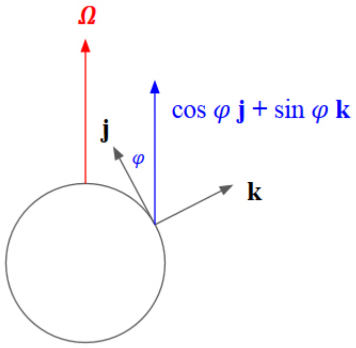
Angular Momentum before and after the displacement.

Here we use the right-handed cylindrical coordinate systems with unit vectors $\hat{R}, \hat{\theta}, \hat{z}$, where $\hat{R}$ and $\hat{\theta}$ represents the radial and azimuthal direction and $\hat{z}$ is along the axis of rotation. The radial velocity and tangential velocity are then expressed in vectors $V\hat{R}$ and $U\hat{\theta}$. Notice the notations for velocities are different from those usually defined in a local coordinate system. Earth's angular velocity $\vec{\Omega}$ is simply $\Omega\hat{z}$. Now expand $-2\vec{\Omega} \times \vec{v}$, we have

$$\begin{aligned} -2\vec{\Omega} \times \vec{v} &= -2\Omega\hat{z} \times (V\hat{R} + U\hat{\theta} + W\hat{z}) \\ &= -2\begin{vmatrix} \hat{R} & \hat{\theta} & \hat{z} \\ 0 & 0 & \Omega \\ V & U & W \end{vmatrix} \\ &= 2\Omega U\hat{R} - 2\Omega V\hat{\theta} \end{aligned}$$

which is consistent with the above findings. Despite using a cylindrical coordinate system to prove, the formula holds for any other right-handed system.

Example 4.2.3 Find the expression of Coriolis Force under the local coordinate system $\hat{i}$, $\hat{j}$, $\hat{k}$, which represent zonal, meridional, and zenith direction respectively.



$\hat{\Omega}$ *expressed in terms of* $\hat{j}$ *and* $\hat{k}$.

In the local coordinate system, $\vec{\Omega} = \Omega\cos\varphi\hat{j} + \Omega\sin\varphi\hat{k}$ where φ denotes the latitude. With the velocity given by $\vec{v} = u\hat{i} + v\hat{j} + w\hat{k}$, we have the Coriolis Force as

$$\begin{aligned} -2\vec{\Omega}\times\vec{v} &= -2\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \Omega\cos\varphi & \Omega\sin\varphi \\ u & v & w \end{vmatrix} \\ &= (2\Omega\sin\varphi v - 2\Omega\cos\varphi w)\hat{i} - 2\Omega\sin\varphi u\hat{j} + 2\Omega\cos\varphi u\hat{k} \end{aligned}$$

If we consider motions on the horizontal plane only, then the expression of Coriolis Force is reduced to

$$\begin{aligned} F_{\text{Cor}} &= 2\Omega\sin\varphi v\hat{i} - 2\Omega\sin\varphi u\hat{j} \\ &= fv\hat{i} - fu\hat{j} \\ &= -f\hat{k}\times\vec{v}_H \end{aligned}$$

where $f = 2\Omega\sin\varphi$ is the Coriolis parameter, $\vec{v}_H$ is the horizontal velocity.

To conclude the three examples above, the equation of horizontal motion is

$$\frac{d\vec{v}_H}{dt} = -\frac{1}{\rho}\nabla_H p - f\hat{k}\times\vec{v}_H + \vec{F}_f$$

where $\vec{F}_f$ denotes friction, $\nabla_H = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0)$. It is noteworthy that while they are called forces, their expression actually refers to acceleration, i.e. force per unit mass. We also convenient leave out the curvature terms.

Example 4.2.4 For a 1500 kg car moving eastwards at a speed of $20\,\text{m}\,\text{s}^{-1}$ at $35\,°\text{N}$, find the Coriolis Force acting on it in terms of local coordinates.

Coriolis Force acting on the car per unit mass is, using a cylindrical coordinate system as in Example 4.2.2,

$$\begin{aligned}-2\vec{\Omega}\times\vec{v} &= -2\Omega\hat{z}\times u\hat{\theta} \\ &= 2\Omega u\hat{R} \\ &= 2(7.292\times 10^{-5}\,\mathrm{s}^{-1})(20\,\mathrm{m\,s}^{-1})\hat{R} \\ &= (0.002917\,\mathrm{m\,s}^{-2})\hat{R}\end{aligned}$$

Converting to the local coordinate system, we have

$$\hat{R} = -\sin\varphi\hat{j}+\cos\varphi\hat{k}$$

Hence the required force is

$$\begin{aligned}\vec{F} &= (1500\,\mathrm{kg})(0.002917\,\mathrm{m\,s}^{-2})(-\sin(35^\circ)\hat{j}+\cos(35^\circ)\hat{k}) \\ &= (-2.51\,\mathrm{N})\hat{j}+(3.58\,\mathrm{N})\hat{k}\end{aligned}$$

Example 4.2.5 For a pressure field $p(x,y) = p_0 + p'\sin x\cos y$, find the flow velocity at $(1,1)$ if the flow is under geostrophic balance, i.e. the pressure gradient force is balanced by Coriolis Force.

Geostrophic balance implies that the equation of horizontal motion takes the form

$$\frac{d\vec{v}_H}{dt} = -\frac{1}{\rho}\nabla_H p - f\hat{k}\times\vec{v_H} = \vec{0}$$

Taking the cross product $\hat{k}\times$ on the equation of motion, we have

Alternative: Write out the equation of motion in x and y directions.

$$\begin{aligned}-f\hat{k}\times(\hat{k}\times\vec{v}_H) &= \frac{1}{\rho}\hat{k}\times\nabla_H p \\ f\vec{v}_H &= \frac{1}{\rho}\hat{k}\times\nabla_H p \\ \vec{v}_H &= \frac{1}{f\rho}\hat{k}\times\nabla_H p\end{aligned}$$

where we have used the vector identity $\hat{k}\times(\hat{k}\times\vec{v}) = -\vec{v}$ if $\vec{v}$ has no k-component.

In the question,

$$\begin{aligned}\nabla_H p &= \frac{\partial p}{\partial x}\hat{i}+\frac{\partial p}{\partial y}\hat{j} \\ &= p'\cos x\cos y\hat{i} - p'\sin x\sin y\hat{j}\end{aligned}$$

Hence we have

$$
\begin{aligned}
\vec{v}_H &= \frac{1}{f\rho}\hat{k} \times (p' \cos x \cos y \hat{i} - p' \sin x \sin y \hat{j}) \\
&= \frac{1}{f\rho}(p' \sin x \sin y \hat{i} + p' \cos x \cos y \hat{j})
\end{aligned}
$$

and therefore the flow velocity at $(1,1)$ is simply $(0.708\frac{p'}{f\rho}, 0.292\frac{p'}{f\rho})$.

4.2.2 Continuity Equation

Example 4.2.6 Express the continuity equation, i.e. the conservation of mass, in vector notation.

Consider a volume element fixed in space with the size $\delta x \delta y \delta z$. The mass contained inside the volume is $\rho \delta x \delta y \delta z$. The change in the mass caused by advection in x-direction is given by the difference in flux on the left face A and right face B as follows

$$
\begin{aligned}
\left(\frac{\partial m}{\partial t}\right)_x &= ((\rho u)_A - (\rho u)_B)\delta y \delta z \\
&= ((\rho u)_A - ((\rho u)_A + \frac{\partial(\rho u)}{\partial x}\delta x))\delta y \delta z \\
&= -\frac{\partial(\rho u)}{\partial x}\delta x \delta y \delta z
\end{aligned}
$$

where we expand $(\rho u)_B$ using Taylor's series to the first order. It is noted that local derivative instead of material derivative is used for the expression of change in mass is because the volume element considered is fixed in space.

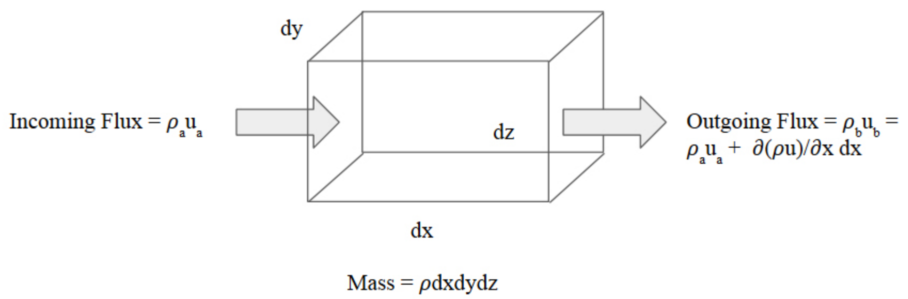


Illustration of fluxes across the volume boundary.

Similarly,

$$\left(\frac{\partial m}{\partial t}\right)_y = -\frac{\partial(\rho v)}{\partial y}\delta x\delta y\delta z$$
$$\left(\frac{\partial m}{\partial t}\right)_z = -\frac{\partial(\rho w)}{\partial z}\delta x\delta y\delta z$$

Then, we conclude the total change in mass is given by

$$\frac{\partial m}{\partial t} = -(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z})\delta x\delta y\delta z$$

Since mass is just $\rho\delta x\delta y\delta z$ and the size $\delta x\delta y\delta z$ is fixed, it simplifies to

$$\frac{\partial \rho}{\partial t}\delta x\delta y\delta z = -(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z})\delta x\delta y\delta z$$
$$\frac{\partial \rho}{\partial t} = -(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z})$$
$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\vec{v})$$

which means that local change in density is caused by the divergence of the flux $\rho\vec{v}$.

Under the special case of constant density, the continuity equation is reduced to

$$\nabla\cdot\vec{v} = 0$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Example 4.2.7 Given a homogeneous fluid, if $u = u_0 \cos x \sin z$, $v = v_0 e^{-y^2}$, find the expression for the vertical velocity w.

In a homogeneous fluid, the density is constant, hence from the continuity equation we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$
$$\frac{\partial w}{\partial z} = -(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$
$$\frac{\partial w}{\partial z} = u_0 \sin x \sin z + 2v_0 y e^{-y^2}$$

Integration on both sides gives

$$\begin{aligned} w &= \int (u_0 \sin x \sin z + 2v_0 y e^{-y^2}) dz \\ w &= -u_0 \sin x \cos z + 2v_0 y z e^{-y^2} + w_0(x,y) \end{aligned}$$

where $w_0(x,y)$ is some function that depends on x, y but does not depend on z.

4.2.3 Advection

Example 4.2.8 At a weather station, the temperature falls at $0.02\,^\circ\mathrm{C/km}$ towards the north-west direction. The wind is blowing from the west to the east with a speed of $5\,\mathrm{m\,s^{-1}}$. Given that the air is being heated by radiation at a rate of $0.5\,^\circ\mathrm{C/hr}$, find the local temperature change at the station.

The relation between local derivative, material derivative and advection of temperature is

$$\frac{\partial T}{\partial t} = \frac{dT}{dt} - \vec{v} \cdot \nabla T$$

In SI units, $\vec{v} = (5,0)$, $\nabla T = (2\cos(45^\circ) \times 10^{-5}, -2\sin(45^\circ) \times 10^{-5}) = (1.414 \times 10^{-5}, -1.414 \times 10^{-5})$, and for the air $\frac{dT}{dt} = \frac{0.5}{3600} = 1.389 \times 10^{-4}$. Then the required answer is

$$\begin{aligned} \frac{\partial T}{\partial t} &= 1.389 \times 10^{-4} - (5,0) \cdot (1.414 \times 10^{-5}, -1.414 \times 10^{-5}) \\ &= 6.819 \times 10^{-5}\,^\circ\mathrm{C/s} \end{aligned}$$

4.2.4 Circulation

Example 4.2.9 Derive Kelvin's Circulation Theorem.

Notice the integration is done in an anti-clockwise direction. Starting with Chain Rule, we have

$$\begin{aligned} \frac{dC}{dt} = \frac{d}{dt} \oint_C \vec{v} \cdot d\vec{s} &= \oint_C \frac{dv}{dt} \cdot d\vec{s} + \oint_C \vec{v} \cdot d(\frac{d\vec{s}}{dt}) \\ &= \oint_C \frac{dv}{dt} \cdot d\vec{s} \end{aligned}$$

The second term vanishes because

$$\begin{aligned}\oint_C \vec{v}\cdot d(\frac{d\vec{s}}{dt}) &= \oint_C \vec{v}\cdot d(\vec{v})\\ &= \oint_C \frac{1}{2}d(\vec{v}\cdot\vec{v}) = 0\end{aligned}$$

since the integral over a scalar around an closed loop is zero. Subsequently, using the equation of motion for an inviscid fluid, we have

$$\frac{dC}{dt} = \oint_C (-\frac{1}{\rho}\nabla p - \vec{g})\cdot d\vec{s} = \oint_C -\frac{1}{\rho}\nabla p\cdot d\vec{s} = \oint_C -\frac{1}{\rho}dp$$

as the gravity $\vec{g}$ is a conservative force. If ρ is a function of p only, which means that the atmosphere is barotropic, then

$$\frac{dC}{dt} = 0$$

since it can be written as a closed integral of a scalar function solely in p.

Example 4.2.10 By Kelvin's Circulation Theorem, derive the change in circulation caused by land-sea temperature contrast which leads to land-sea breeze.

Kelvin's Circulation Theorem is

$$\frac{d}{dt}\oint_C \vec{v}\cdot d\vec{s} = \frac{dC}{dt} = -\oint_C \frac{1}{\rho}dp = -\oint_C \frac{RT}{p}dp$$

in which we apply the equation of state.

For an afternoon case where the land has a temperature of $T_l = 28\,^\circ\mathrm{C}$, the sea has a temperature of $T_s = 25\,^\circ\mathrm{C}$, both assumed to be roughly constant with height in the boundary layer, the change in circulation along a closed curve across the coast between $p_1 = 1000\,\mathrm{hPa}$ and $p_2 = 900\,\mathrm{hPa}$ is then

$$\begin{aligned}\frac{dC}{dt} &= -\oint_C \frac{RT}{p}dp\\ &= -\oint_C RT\,d(\ln p)\\ &= -\int_{p_1}^{p_2} RT_l d(\ln p) - \int_{p_2}^{p_1} RT_s d(\ln p)\\ &= \int_{p_2}^{p_1} R(T_l - T_s)d(\ln p)\end{aligned}$$

Substituting the values gives the answer as

$$R\Delta T\Delta(\ln p) = (287\,\mathrm{J\,kg^{-1}\,K^{-1}})(3\,\mathrm{K})(\ln\frac{1000}{900}) = 90.72\,\mathrm{m^2\,s^{-2}}$$

If the distance between the land and sea is $L = 100\,\mathrm{km} = 1\times10^5\,\mathrm{m}$, then the vertical distance between the two pressure levels is relatively negligible, and hence we can estimate on average the rate of change in the wind speed perpendicular to the coast by

$$\frac{dC}{dt} = \frac{d}{dt}\oint_C \vec{v}\cdot d\vec{s} = 2L\frac{dv}{dt}$$

and hence

$$\frac{dv}{dt} = \frac{90.72\,\mathrm{m^2\,s^{-2}}}{2\times10^5\,\mathrm{m}} = 4.54\times10^{-4}\,\mathrm{m\,s^{-2}}$$

Additionally, the change in circulation can be expressed in a more convenient form, even when the temperature is dependent on pressure, as

$$\begin{aligned}
-\int_{p_1}^{p_2}\frac{1}{\rho_l}dp - \int_{p_2}^{p_1}\frac{1}{\rho_s}dp &= -\int_{z(p_1)}^{z(p_2)}\frac{1}{\rho_l}\frac{\partial p_l}{\partial z_l}dz_l - \int_{z(p_2)}^{z(p_1)}\frac{1}{\rho_s}\frac{\partial p_s}{\partial z_s}dz_s \\
&= -\int_{z(p_1)}^{z(p_2)}\frac{1}{\rho_l}(-\rho_l g)dz_l - \int_{z(p_2)}^{z(p_1)}\frac{1}{\rho_s}(-\rho_s g)dz_s \\
&= \int_{z(p_1)}^{z(p_2)} g\,dz_l - \int_{z(p_1)}^{z(p_2)} g\,dz_s \\
&= g(\Delta z_l - \Delta z_s)
\end{aligned}$$

where z is the geopotential height.

Example 4.2.11 Derive Bjerknes' Circulation Theorem.

Here we use C to denote the relative circulation and C_a the absolute circulation, similarly $\vec{v}$ the relative velocity and $\vec{v}_a$ the absolute velocity. By Kelvin's Circulation Theorem, we have

$$\frac{dC_a}{dt} = \frac{d}{dt}\oint_C \vec{v}_a\cdot d\vec{s} = \frac{d}{dt}\oint_C \vec{v}\cdot d\vec{s} + \frac{d}{dt}\oint_C (\vec{\Omega}\times\vec{r})\cdot d\vec{s}$$

where the cross product between the Earth's angular velocity and the displacement vector from Earth's center $\vec{\Omega}\times\vec{r}$ represents the velocity contributed by the Earth's rotation. By Stoke's Theorem, its integral is then

$$\oint_C (\vec{\Omega}\times\vec{r})\cdot d\vec{s} = \iint_R (\nabla\times(\vec{\Omega}\times\vec{r}))\cdot\hat{n}\,dA$$

Notice that

$$\vec{\Omega} \times \vec{r} = \Omega r \cos\varphi \hat{\theta}$$

where φ is the latitude and $\hat{\theta}$ is a unit vector in the azimuthal direction under the spherical coordinates. We use the following formula without proof to compute its curl

$$\begin{aligned}
\nabla \times (\vec{\Omega} \times \vec{r}) &= \frac{1}{r^2 \cos\varphi} \begin{vmatrix} r\cos\varphi\hat{\theta} & r\hat{\varphi} & \hat{r} \\ \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial r} \\ (\Omega r\cos\varphi)(r\cos\varphi) & 0 & 0 \end{vmatrix} \\
&= 2\Omega\cos\varphi\hat{\varphi} + 2\Omega\sin\varphi\hat{r} = 2\vec{\Omega}
\end{aligned}$$

For a small area enclosed by the curve, we have

$$\begin{aligned}
\frac{dC_a}{dt} &= \frac{d}{dt}\oint_C \vec{v}\cdot d\vec{s} + \frac{d}{dt}\oint_C (\vec{\Omega}\times\vec{r})\cdot d\vec{s} \\
-\oint_C \frac{1}{\rho}dp &= \frac{dC}{dt} + \frac{d}{dt}\iint_R 2\vec{\Omega}\cdot\hat{n}dA \\
\frac{dC}{dt} &= -\oint_C \frac{1}{\rho}dp - \frac{d}{dt}((2\vec{\Omega}\cdot\hat{n})A)
\end{aligned}$$

where we apply Kelvin's Circulation Theorem on the absolute circulation.

Example 4.2.12 A cylindrical air column in a barotropic atmosphere at $35^\circ N$ has a radius of 100 km. If it is initially at rest and contracts such that its radius becomes 50 km, find the mean tangential velocity at the perimeter.

By integrating Bjerknes' Circulation Theorem, we have

$$\Delta C = -(2\vec{\Omega}\cdot\hat{n})\Delta A$$

the $-\oint_C \frac{1}{\rho}dp$ term does not appear due to the barotropic assumption. Substitution gives

$$\begin{aligned}
\Delta C &= -2(7.292\times10^{-5}\,\text{rad/s})(\sin(35^\circ))(\pi(50000\,\text{m})^2 - \pi(100000\,\text{m})^2) \\
&= 1970966\,\text{m}^2\,\text{s}^{-1}
\end{aligned}$$

and because

$$\Delta C = \Delta\oint \vec{v}\cdot d\vec{s} = 2\pi r v_{\parallel}$$

we have the final answer as

$$v_{\parallel} = \frac{\Delta C}{2\pi r} = \frac{1970966\,\text{m}^2\,\text{s}^{-1}}{2\pi(50000\,\text{m})} = 6.274\,\text{m}\,\text{s}^{-1}$$

4.2.5 VORTICITY

Example 4.2.13 Find the expression for vorticity, which is a local measure of rotation and defined as circulation per unit area when the region is infinitesimal, i.e.

$$\lim_{A\to 0} \frac{\oint_C \vec{v}\cdot d\vec{s}}{\iint_R dA} = \lim_{A\to 0} \frac{C}{A}$$

Similar to circulation, absolute and relative vorticity can be defined. Just like how we derive Bjerknes' Circulation Theorem, we have

$$\lim_{A\to 0} \frac{\oint_C \vec{v}_a\cdot d\vec{s}}{dA} = \lim_{A\to 0} \frac{\oint_C \vec{v}\cdot d\vec{s}}{dA} + \lim_{A\to 0} \frac{\oint_C (\vec{\Omega}\times\vec{r})\cdot d\vec{s}}{dA}$$
$$\lim_{A\to 0} \frac{\iint_R (\nabla\times\vec{v}_a)\cdot\hat{n}dA}{dA} = \lim_{A\to 0} \frac{\iint_R (\nabla\times\vec{v})\cdot\hat{n}dA}{dA} + \lim_{A\to 0} \frac{\iint_R 2\vec{\Omega}\cdot\hat{n}dA}{dA}$$

where the second term on the right again manifests the rotation of the Earth and we use Stokes' Theorem. Since A tends to zero, then we conclude for any arbitrary location

$$(\nabla\times\vec{v}_a)\cdot\hat{n} = (\nabla\times\vec{v})\cdot\hat{n} + 2\vec{\Omega}\cdot\hat{n}$$

in which the absolute vorticity $\nabla\times\vec{v}_a$ is related to the relative vorticity $\nabla\times\vec{v}$. For synoptic scale motion, we only need to consider the vertical component of vorticity, hence $\hat{n} = \hat{k}$, subsequently, we have

$$\eta = \zeta + 2\vec{\Omega}\cdot\hat{k} = \zeta + 2\Omega\sin\varphi = \zeta + f$$

where η and ζ represents the vertical component of absolute vorticity and relative vorticity, and the Coriolis parameter f is the planetary vorticity. Note that

$$\zeta = (\nabla\times\vec{v})\cdot\hat{k} = \hat{k}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Example 4.2.14 Derive the barotropic potential vorticity equation in the shallow water system.

By Kelvin's Circulation Theorem, and the fact that vorticity is just circulation per unit area, we have

$$\frac{d}{dt}C_a = \frac{d}{dt}(\eta A) = \frac{d}{dt}((\zeta+f)A) = 0$$

for a fluid column in a barotropic and shallow water environment, where hydrostatic balance holds, so we notice that the horizontal pressure gradient force

$$\begin{aligned}\frac{\partial p}{\partial z} &= -\rho g \\ p &= -\rho g z + p_s(x,y) \\ \frac{\partial p}{\partial x} &= \frac{\partial p_s(x,y)}{\partial x}\end{aligned}$$

is constant with height if the fluid has a homogeneous density. By the virtue of geostrophic balance, then the velocity of the fluid column and the vorticity are constant with height too, which indicates the fluid column will move as a whole. Therefore, Ah is also a constant following the flow, and

$$\frac{d}{dt}((\zeta+f)A) = \frac{d}{dt}(\frac{(\zeta+f)Ah}{h}) = \frac{d}{dt}(\frac{\zeta+f}{h}) = 0$$

In a shallow water system, we also have from the continuity equation for a homogeneous fluid

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Integrating with respect to height, we have

$$w(z) = -\int_0^z (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})dz = -(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})z$$

since the horizontal velocity does not depend on height. At $z = h$, the vertical velocity is just the change in height of the fluid column. Thus

$$\frac{dh}{dt} = -(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})h$$

This means that the change in height of the air column depends on the height itself as well as the convergence. Now plugging this equation to the barotropic potential vorticity equation, we have

$$\begin{aligned}\frac{d}{dt}(\frac{\zeta+f}{h}) &= -\frac{(\zeta+f)}{h^2}\frac{dh}{dt} + \frac{1}{h}\frac{d}{dt}(\zeta+f) = 0 \\ \frac{d}{dt}(\zeta+f) &= \frac{(\zeta+f)}{h}\frac{dh}{dt} \\ &= \frac{(\zeta+f)}{h}(-(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})h) \\ &= -(\zeta+f)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})\end{aligned}$$

Convergence increases the magnitude of vorticity while divergence reduces it.

Example 4.2.15 A barotropic air column initially with a zero relative vorticity at 30°N extends to a fixed tropopause with a height of 10 km. If it moves over a mountain of 2 km tall at 45°N and thus the column is squashed to 8 km thick, find the change in its relative vorticity by the barotropic potential vorticity equation.

First, from the barotropic potential vorticity equation, we know that $\frac{\zeta+f}{h}$ conserves. Hence

$$\frac{\zeta'+f'}{h'} = \frac{\zeta+f}{h}$$
$$\frac{\zeta'+2(7.292\times10^{-5}\,\mathrm{rad/s})\sin(45^\circ)}{8\,\mathrm{km}} = \frac{2(7.292\times10^{-5}\,\mathrm{rad/s})\sin(30^\circ)}{10\,\mathrm{km}}$$
$$\zeta' = -4.48\times10^{-5}\,\mathrm{s}^{-1}$$

4.2.6 Barotropic Rossby Wave

4.3 Vector / Vector Calculus in ESSC3300

4.4 Vector / Vector Calculus in ESSC3120

4.4.1 Gravity

Gravity Potential Formulation

By the law of universal gravitation, the gravitational attraction , the gravitational attraction $\boldsymbol{F}$ exerted by M on m

$$\boldsymbol{F} = -G\frac{mM}{r^2}\hat{r}$$

$\hat{r}$ is a unit vector in the direction of increase in coordinate r, which is directed away from the center of reference at the mass M. The negative sign in the equation indicates that the force $\boldsymbol{F}$ acts in the opposite direction, towards the attracting mass M. The constant G is the constant of universal gravitation.

The law of conservation of energy means that *the total energy of a closed system* is constant. Two forms of energy need to be considered here. First is the potential energy, which an object has by virtue of its position relative to the origin of force and the second is work done against the action of force during a change in position.

$$\textit{Total Energy} = K.E. + P.E.$$

Therefore, in general, if a constant force F moves through a small distance dr in the same direction as the force, the work done is $dW = Fdr$ and the change in potential energy dE_p is given

$$dE_p = -dW = -\boldsymbol{F}d\boldsymbol{r} = -(F_x dx + F_y dy + F_z dz)$$

The expression in brackets is called the *scalar product* of the vector $\boldsymbol{F}$ and d$\boldsymbol{r}$. $(F dr cos\theta)$

The gravitational potential is the potential energy of a unit mass in a field of gravitational attraction. Let the potential denoted as U_g

$$mdU_g = F dr = -m\boldsymbol{a_g} d\boldsymbol{r}$$
$$-\frac{dU_g}{dr}\hat{r} = \boldsymbol{\nabla} U_g = \boldsymbol{a_g}$$
$$\frac{dU_g}{dr} = G\frac{M}{r^2}$$

Therefore, the gravitational potential is given by

$$U_g = -G\frac{M}{r}$$

Assuming m_i to be the mass of particle at distance r_i from P, this the gravitational acceleration is

$$a_g = -G\frac{m_1}{r_1^2}\hat{r} - G\frac{m_2}{r_2^2}\hat{r} - G\frac{m_3}{r_3^2}\hat{r} - ...$$

Since vector sum is quite complicated, an alternate approach is to utilize the gravitational potential and compute acceleration by differentiation. Thus, the potential is given by

$$U_g = -G\frac{m_1}{r_1} - G\frac{m_2}{r_2} - G\frac{m_3}{r_3} - ...$$

Rather than represented as an assemblage of discrete particles, objects in real world are represented as continuous mass distribution, therefore we can subdivide the volume into discrete volume and if the density of each matter in the volume is known, the mass of the small element can be computed. By integrating over the volume of the body its gravitational potential can be calculated.

$$U_g = -G\int\int\int \frac{\rho(x,y,z)}{r(x,y,z)} dxdydz = \int \frac{G\rho(\boldsymbol{r'})d^3r'}{|\boldsymbol{r}-\boldsymbol{r'}|}$$

The integration gives the gravitational potential and acceleration at points inside and outside a hollow or homogeneous solid sphere. The values outside a sphere at distance r from its center are the same as if the entire mass E of the sphere were concentrated at its center.

Example 4.4.1 Consider a solid sphere with radius r, compute the gravitational potential of the sphere from $r = \infty$ to $r = 0$ (outside the sphere and inside the sphere).
TODO

Example 4.4.2 What is the gravity anomalies of a sphere at depth? Assume a sphere of radius R and density contrast$\Delta\rho$ with its center at depth z below the surface.

$$\text{With the mass contrast } \Delta M = \frac{4}{3}\pi R^3 \Delta\rho \text{ and } r^2 = x^2 + z^2$$
$$\Delta g_z = \Delta g sin\theta = G\frac{M}{r^2}\frac{z}{r}$$
$$\Delta g_z = \frac{4}{3}\pi G \Delta\rho R^3 \frac{z}{(z^2+x^2)^{(3/2)}}$$
$$= \frac{4}{3}\pi G \left(\frac{\Delta\rho R^3}{z^2}\right)\left[\frac{1}{1+\left(\frac{x}{z}\right)^2}\right]^{3/2}$$

Note: The answer shows plenty of parameters and variables and sometimes they are confusing. It is always essential to distinguish which are parameters (independent variables), dependent variables and variable. In the above case, x is the variable. We can understand the gravity abnormaly by it shape. So, what is the shape of the answer illustrated above? (Hint: try google search: y = 1/(1+x^2)^(3/2))

Example 4.4.3 What is the gravity anomalies of a infinite long cylinder along y axis at depth? Assume a sphere of radius R and density contrast$\Delta\rho$ with its center at depth z below the surface. TODO

Example 4.4.4 A thin borehole is drilled through the center of the Earth, and a ball is dropped into the borehole. Assume the Earth to be a homogeneous solid sphere. Show that the ball will oscillate back and forth from one side of the Earth to the other. How long does it take to traverse the Earth and reach the other side?

Using the derived gravitational potential in Example 4.4.1, we have

$$a_g = -\frac{dU_g}{dr} = -\frac{4\pi G\rho r}{3}$$
$$\frac{d^2r}{dt^2} = -\frac{4\pi G\rho r}{3} = -w_0^2 r$$

Recall the harmonic oscillation formulation and its solution, we have

$$r(t) = Asin(w_0 t) \text{ with } w_0 = \sqrt{\frac{4\pi G\rho}{3}}$$

Therefore, the time required to transverse the Earth and reach the other side requires half the period of the oscillation motion, which is

$$\text{Time required} = \frac{\pi}{w_0}$$

4.4.2 THERMODYNAMICS

Heat Conduction

Three dimensional heat flow equation is given by

$$\frac{\partial T}{\partial t} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \text{ with } \kappa = \frac{k}{\rho c_P}$$

The equation may be solved by any set of boundary conditions using the method of *separation of variable*.

Separation of variable refers to a method to solve partial differential equation under a special condition of the partial differential equation is homogeneous and linear. *Homogeneous* equation refers to the coefficients in the equation are constant (or independent to the variables).

Example 4.4.4 Penetration of external heat into the Earth - 1D half space heat conduction.

Considering the surface temperature of the Earth varies cyclically with angular frequency w, so that at time t the surface temperature is equal to $T_0 coswt$. Therefore, the study of temperature profile with depth at a certain time is a 1-D half space heat conduction with a time depend boundary condition.

The surface-temperatue variation $T_0 coswt$ can be expressed with the aid of complex numbers as the real part of $T_0 e^{iwt}$. Let z be the depth of a point below a surface. Using the *separation of variables*, the heat conduction equation can be separated and written as two ordinary equations with the same constant.

Let $T(x,t) = \theta(t)Z(z)$ and substitute into heat conduction equation,

$$\frac{\partial \theta(t)Z(z)}{\partial t} = \kappa \frac{\partial^2 \theta(t)Z(z)}{\partial z^2}$$
$$\frac{1}{\theta(t)}\frac{d\theta(t)}{dt} = \kappa \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = iw$$

Therefore, the time dependence of the temperature is given by

$$\theta(t) = \theta_0 e^{iwt}$$

The equation for the temperature spatial distribution is

$$\frac{d^2Z}{dz^2} = i\frac{w}{\kappa}Z$$

Let $-n^2 = iw/k$,

$$\frac{d^2Z}{dz^2} = -n^2 Z$$

which has the solutions

$$Z = Z_0 e^{inz} \text{ and } Z = Z_1 e^{-inz}$$

where

$$in = \sqrt{i\frac{w}{\kappa}} = \sqrt{\frac{w}{2\kappa}}(1+i)$$

The two possible solutions for depth variation:

$$Z = Z_1 e^{inz} = Z_1 e^{(\sqrt{w/2k})(1+i)z}$$
$$Z = Z_0 e^{-inz} = Z_0 e^{-(\sqrt{w/2k})(1+i)z}$$

The temperature must decrease with increasing depth z below the surface, so only the second solution is acceptable. Combining the solutions for θ and Z, we get

$$T(z,t) = Z_0 e^{-z\sqrt{w/2k}(1+i)}\theta_0 e^{iwt}$$
$$= Z_0\theta_0 e^{-\sqrt{w/2k}} e^{i(wt - z\sqrt{w/2\kappa})}$$

Considering the real part of the solution and define penetrating depth $d = \sqrt{2\kappa/w}$, the equation reduces to

$$T(z,t) = T_0 e^{-z/d} cos\left(wt - \frac{z}{d}\right)$$

The parameter d is called *decay depth* of the temperature, which the amplitude of temperature fluctuation at this depth is attenuated to 1/e of its value on the surface. The solution can also rewritten as

$$T(z,t) = T_0 e^{-z/d} cosw(t - t_d)$$

where the phase difference or delay time $t_d = z/wd$
represents the length of time by which the temperature at depth z lags behind the surface temperature.

4.4.3 Geomagnetism

4.5 Vector / Vector Calculus In ESSC3010

4.6 Problems

Question 4.1.1 For two vectors $\vec{u} = (1,2,4)$ and $\vec{v} = (5,2,0)$, find
(a) $\vec{u}+\vec{v}$, (b) $\vec{u}-\vec{v}$, (c) $\vec{u}\cdot\vec{v}$, (d) $\vec{u}\times\vec{v}$, (e) the angle between them, (f) the projection of $\vec{u}$ on $\vec{v}$.

Question 4.1.2 For two vectors $\vec{u} = (1,4,3)$ and $\vec{v} = (-3,5,2)$, find
(a) $\vec{u}+2\vec{v}$, (b) $2\vec{u}-\vec{v}$, (c) $\vec{v}\cdot\vec{u}$, (d) $\vec{v}\times\vec{u}$, (e) the angle between them, (f) the projection of $\vec{v}$ on $\vec{u}$.

Question 4.1.3 Prove that if $\vec{v} = k\vec{u}$ where k is a constant, then their cross product equals to zero, by writing out the determinant.

Question 4.1.4 Prove that
(a) $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = |\vec{u}|^2 - |\vec{v}|^2$,
(b) $(\vec{u}+\vec{v})\times(\vec{u}-\vec{v}) = -2\vec{u}\times\vec{v}$.

Question 4.1.5 Find the volume of the parallelepiped formed by the following vectors.
(a) (2,1,3), (1,1,4), (2,3,2), (b) (3,1,1), (2,2,1), (0,4,1).

Question 4.1.6 Find the equation of the line with a normal vector of $2\hat{\imath}+3\hat{\jmath}$ which passes through the point (1,1). Also, find the equation of the plane with a normal vector $\hat{\imath}-2\hat{\jmath}+\hat{k}$ which passes through the point (4,3,2).

Question 4.1.7 Find the distance of the point (1,6,3) to the plane $2x+2y+1=7$.

Question 4.1.8 Find the distance of the line $x=1+t$, $y=2-t$, $z=3-2t$ to the plane $4x+2y+z=11$.

Question 4.1.9 Verify that one of the possible parameterization schemes for the ellipse $\frac{x^2}{4^2}+\frac{y^2}{5^2}=1$ is $x=4\cos t$, $y=5\sin t$. Hence describe the shape of the curve parameterized by $x=4\cos t$, $y=5\sin t$, $z=\frac{2}{\pi}t$.

Question 4.1.10 A wheel of radius a is initially at rest. The bottom of the wheel is marked with a red dot. The wheel is then rolled to the right. Find a parameterization of the path traced by the red dot. This type of curve is called a cycloid.

Question 4.1.11 The velocity of an object is $\vec{v}=(\sin t)\hat{\imath}+e^{-t}\hat{\jmath}+t\hat{k}$. Find its acceleration and displacement.

Question 4.1.12 Find the arc length of the curve $x=e^{-t}\cos t, y=e^{-t}\sin t, z=t$, from $t=0$ to $t=3$.

Question 4.1.13 Find the gradient of the function $f(x,y)=2\sin xe^{-y^2}$. Calculate the gradient at $(-\pi,-1)$ and $(\pi,1)$. If a surface is defined as $z=f(x,y)$, find the normal vector at $(0,0,0)$.

Question 4.1.14 Find a normal vector for the surface $z=x^2+y$ at $(1,1,2)$.

Question 4.1.15 Find the divergence and curl of the following vector fields.
(a) $\vec{F}=(x-y)\hat{\imath}+(y-z)\hat{\jmath}+(z-x)\hat{k}$,
(b) $\vec{F}=e^{-x^2+y^2}(x+y)\hat{\imath}+e^{-x^2+y^2}(y-x)\hat{\jmath}$,
(c) $\vec{F}=x^2\sin ye^{-z}\hat{\imath}+x^2\cos ye^{-z}\hat{\jmath}+xyz\hat{k}$.

Question 4.1.16 Calculate the Laplacian of $f(x,y,z)=\frac{\sin z}{xy}$, which is $\nabla^2 f$.

Question 4.1.17 Given a physical quantity $u(x,y,t)$, if near a location, $\frac{\partial u}{\partial t}=-5$ per unit time, and its gradient is $\nabla u=-2\hat{\imath}+3\hat{\jmath}$. If an element moves towards the north-west at 1 unit length per unit time, find the rate of change in u tracing the element.

Question 4.1.18 Given a physical quantity $u(x,y,t)$ of an moving object is increasing at a rate of $\frac{du}{dt} = 3$ per second. It travels towards a bearing of 030° (positive y-axis as north and positive x-axis as east) at a speed of 2 unit length per unit time. Find the local rate of change in u at that moment.

Question 4.1.19 Evaluate

$$\int f(x,y,z)ds$$

along the straight line $x = t$, $y = 2t+1$, $z = 3t+4$, from $0 < t < 5$, where $f(x,y,z)$ is (a) xyz, and (b) $e^x(y+z)$.

Question 4.1.20 Evaluate

$$\int \frac{xy}{z} ds$$

along the curve $x = \cos t$, $y = \sin t$, $z = e^{-t}$, $0 < t < \pi$.

Question 4.1.21 Compute

$$\int \vec{F} \cdot d\vec{s}$$

along the curve $x = \cos t$, $y = \sin t$, $z = \ln t$, $1 < t < \pi$, for $\vec{F} = -y\hat{i} + x\hat{j} + z\hat{k}$.

Question 4.1.22 Compute

$$\int \vec{F} \cdot d\vec{s}$$

along the curve $x = t$, $y = t^2$, $z = e^{-t}$, $-1 < t < 1$, for $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$.

Question 4.1.23 Show that

$$\vec{F} = (ye^z + \cos x \cos y)\hat{i} + (xe^z - \sin x \sin y)\hat{j} + xye^z\hat{k}$$

is conservative. Hence find the value of work done integral

$$\int \vec{F} \cdot d\vec{s}$$

(a) starting from $(0,0,0)$ to (π,π,π), (b) along any closed loop.

Question 4.1.24 Integrate

$$\int_1^e \int_\pi^0 \frac{\sin x}{y} dxdy$$

inside the rectangle $x = [0, \pi]$ and $y = [1, e]$.

Question 4.1.25 Integrate $f(x,y) = x^2 y$ inside the circle $x^2 + y^2 < 1$.

Question 4.1.26 Integrate $f(x,y,z) = xy + z$ inside the upper first quadrant of a dome with radius of 1.

Question 4.1.27 Integrate

$$\iint xydxdy$$

in the region bounded by $xy = 1$, $xy = 4$, $\frac{y}{x} = 4$, $\frac{y}{x} = \frac{1}{4}$. First, show that if we substitute $u = \sqrt{xy}$ and $v = \sqrt{\frac{y}{x}}$, then $x = \frac{u}{v}$, and $y = uv$. Next, evaluate its Jacobian and apply coordinate transformation.

Question 4.1.28 Integrate $f(x,y) = \sqrt{x^2 + y^2}$ within the circular region $x^2 + y^2 < 1$ by using polar coordinates.

Question 4.1.29 Integrate $f(\rho, \phi) = \rho \sin \phi$ inside a sphere of radius 1 by using spherical coordinates. $0 < \rho < 1$ is the radial distance, $0 < \theta < 2\pi$ is the azimuth angle and $0 < \phi < \pi$ is the zenith angle.

Question 4.1.30 Integrate $f(x,y,z) = \sqrt{\frac{1}{4x^2+4y^2+1}}$, on the surface $z = x^2 + y^2$, inside the square region $-1 < x < 1$ and $-1 < y < 1$.

Question 4.1.31 For a plane $x + 2y + 4z = 12$ that is oriented towards the positive z-direction, find the flux across its upper first quadrant

$$\iint_R \vec{F} \cdot \hat{n} dS$$

under $\vec{F} = x^2\hat{i} + y^2\hat{j} + xy\hat{k}$. Since it is only the upper first quadrant, the limits of x and y should be set accordingly.

Question 4.1.32 For a surface $z = \sqrt{x^2+y^2}$ which has a shape like a inverted cone, within the circular region $x^2+y^2 < 1$, find the upward flux

$$\iint_R \vec{F} \cdot \hat{n} dS$$

across the surface if $\vec{F} = x\hat{i} + y\hat{j} + \frac{1}{z}e^z\hat{k}$.

Question 4.1.33 Find the work done integral

$$\oint_C \vec{F} \cdot d\vec{s}$$

along the closed triangle $x = 1$, $y = 1$, $x+y = 2$ in clockwise direction with $\vec{F} = xy^2\hat{\imath} - x^2y\hat{\jmath}$ by Green's Theorem or piecewise integration.

Question 4.1.34 Find the outward flux across the circle $x^2+y^2 = 1$

$$\oint_C \vec{F} \cdot \hat{n} dS$$

with with $\vec{F} = x^2\hat{\imath} + y^2\hat{\jmath}$ by Green's Theorem or direct integration.

Question 4.1.35 Compute the work done integral of $\vec{F} = x^2y\hat{i} + y^2z\hat{j} + z^2x\hat{k}$ along the circle $y^2+z^2 = 1$, $x = 0$, along anti-clockwise direction facing in the positive x-direction.

Question 4.1.36 Find the outward flux across the sphere $x^2+y^2+z^2 = 1$ under $\vec{F} = x\hat{i} + y\hat{j} + e^{-z^2}\hat{k}$.

Question 4.1.37 Show that if we write a horizontal flow using stream function, then the curl of the flow is

$$\nabla^2\psi = \nabla \cdot \nabla\psi = \psi_{xx} + \psi_{yy}$$

Question 4.1.38 For a velocity field $\vec{v} = -ye^{-t}\hat{\imath} + xe^{-t}\hat{\jmath}$, and a scalar field $s = ke^{-x^2}\cos y(1-e^{-t})$ where k is some constant, find $\frac{ds}{dt}$ of a element moving according to the given velocity field when it is present at $x = 2$, $y = -2$ with $t = 3$.

Question 4.1.39 Evaluate

$$I = \int_0^\infty e^{-x^2} dx$$

by noticing

$$\begin{aligned} I^2 &= \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy \\ &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy \end{aligned}$$

since x and y are dummy variables and they are independent of each other. Then, apply polar coordinate transformation. The integration region is effectively an semi-infinite circle in the first quadrant, with $0 < r < \infty$, $0 < \theta < \frac{\pi}{2}$. This is a famous result for the Gaussian function.

Question 4.1.40 Evaluate

$$\oint_C \frac{xdy - ydx}{mx^2 + ny^2}$$

along a closed circle $x^2 + y^2 = 1$, by noticing

$$d(y/x) = \frac{xdy - ydx}{x^2}$$

and determining the new limits.

Question 4.1.41 By Divergence Theorem, calculate

$$\oiint_R (x^4 + y^4 + z^4) dS$$

on the spherical surface $x^2 + y^2 + z^2 = 1$.

Question 4.1.42 Compute the work done along the upper arc of semi-circle $x^2 + y^2 = 4$, $y > 0$, under the vector field $\vec{F} = (e^{(x+1)^2} + (y+1))\hat{\imath} + (\ln(y+1) - (x+1))\hat{\jmath}$, by Green's / Stokes' Theorem along an appropriate closed curve.

Question 4.5.1 Show that the "half-width" w of the gravity anomaly over a sphere and the depth z to the center of the sphere are related by z = 0.652w. (Hint Example 4.4.2)

Example: low density salt dome ($\rho = 2150kmm^{-3}$) intruding higher-density carbonate rocks $\rho_o = 2500kgm^{-3}$) results a negative gravity anomaly.

CHAPTER 5

LINEAR ALGEBRA

CONTENTS

5.1 Introduction

Before the writing of this book, the author has organized a linear algebra workshop. The lecture material and exercises can be found (not available temporarily) here. In the following parts, we would cover the areas which were not mentioned in the workshop.

Cramer's Rule Cramer's Rule can be used to find a unique solution for a system of equations if it exists. Given a system of

$$\begin{aligned} a_1x+b_1y+c_1z &= d_1 \\ a_2x+b_2y+c_2z &= d_2 \\ a_3x+b_3y+c_3z &= d_3 \end{aligned}$$

where a_n, b_n, c_n, d_n can be constants or functions. Then the solution is given by

$$x=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, y=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, z=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

where the determinants in numerator is the determinant in denominator with the first, second and third column replaced by (d_1, d_2, d_3) respectively. For different amount of variables the procedure is similar.

Example 5.1.1 Solve the following system.

$$\begin{aligned} x+2y+3z &= 14 \\ x+y+z &= 6 \\ x-2y+z &= 0 \end{aligned}$$

Alternative: Use Gaussian Elimination or Inverse.

By Cramer's Rule, we have

$$x=\frac{\begin{vmatrix} 14 & 2 & 3 \\ 6 & 1 & 1 \\ 0 & -2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix}}=\frac{-6}{-6}=1$$

$$y = \frac{\begin{vmatrix} 1 & 14 & 3 \\ 1 & 6 & 1 \\ 1 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix}} = \frac{-12}{-6} = 2$$

$$z = \frac{\begin{vmatrix} 1 & 2 & 14 \\ 1 & 1 & 6 \\ 1 & -2 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix}} = \frac{-18}{-6} = 3$$

Example 5.1.2 Solve the following system.

$$(\cos t)x + (\sin t)y = 0$$
$$(\cot t)x + (\tan t)y = \cos t$$

From Cramer's Rule, we have

$$x = \frac{\begin{vmatrix} 0 & \sin t \\ \cos t & \tan t \end{vmatrix}}{\begin{vmatrix} \cos t & \sin t \\ \cot t & \tan t \end{vmatrix}} = \frac{-\sin t \cos t}{\sin t - \cos t}$$

$$y = \frac{\begin{vmatrix} \cos t & 0 \\ \cot t & \cos t \end{vmatrix}}{\begin{vmatrix} \cos t & \sin t \\ \cot t & \tan t \end{vmatrix}} = \frac{\cos^2 t}{\sin t - \cos t}$$

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