Introduction to Linear Algebra

with Earth Science Applications

Draft Version

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Preface

This is a Linear Algebra textbook specifically designed for students who are majoring in any Earth Science related subjects like Geophysics and Atmospheric Sciences. With these target readers in mind, we set out to provide an adequate treatment about Linear Algebra concepts that enables them to tackle relevant Earth Science problems. In each chapter, we focus on a selected Linear Algebra topic, motivated by Earth Science examples and supplemented with Python programming tutorials. At the end of each chapter, a number of exercises are given for elucidating the concepts and working on Earth Science projects.

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Chapter 1

Introduction to Matrices and Linear Systems

Although the Earth System is well-known to be filled with non-linear processes, we still benefit from learning how to work with linear systems, by which many Earth Science problems can be approximated. This actually works well in a number of cases. For instance, in Atmosphere Sciences, we often consider what is called a *perturbation equation*, which assumes that deviations from the mean state are small enough to neglect quadratic terms. The most fundamental usage of Linear Algebra in Applied Sciences is to formulate, analyze and solve *linear systems of equations*. Some examples in Earth Sciences are mapping the depth of overlying soil layers underground, as well as chemical balances in various subsystems of the Earth. *Matrices* are one of the most central ingredients in Linear Algebra that can be used to described such systems, and we are going to address the basic aspects related to them in the first chapter.

1.1 Definition and Operations of Matrices

1.1.1 Basic Structure of Matrices

Matrices are rectangular arrays of numbers, the entries of which can be real or complex. For now we will work with the simpler case of real matrices first. A matrix having m rows and n columns is called an $m \times n$ matrix. The class of matrices with the same number of rows and columns, i.e. $m = n$, are known as ***square matrices***. Below shows some examples of matrices.

$$\begin{bmatrix} 1.17 & 2.01 & -2.15 & 5 \\ 1.44 & 3.61 & 2.88 & -3 \end{bmatrix}$$

A 2×4 real matrix.

$$\begin{bmatrix} \sqrt{2} - \frac{4}{\sqrt{5}}\imath \\ 0 \\ 1.27 \\ \sqrt{3}\imath \end{bmatrix}$$

A 4×1 complex matrix.

$$\begin{bmatrix} 3 & \sqrt{2} & 9 \\ 0 & -4\pi & \frac{1}{6} \\ 5.11 & 2 & -1 \end{bmatrix}$$

A 3×3 real, square matrix.

Given any matrix A, its entry at row i and column j will be denoted as A_{ij}. For example,

$$A = \begin{array}{l} \quad\ \text{Col 1} \\ \begin{bmatrix} 2 & 1 & 7 & \frac{8}{9} \\ 5 & -\frac{1}{3} & 5 & 0 \\ -3 & 4.38 & 6 & -1.66 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Row 2} \\ \end{array} \qquad A_{21} = 5$$

Short Exercise: Find A_{13}, A_{22}, A_{34} and A_{42} .[1]

1.1.2 Matrix Operations

Addition and Subtraction

Addition and subtraction between two matrices A and B are carried out *element-wise*, which means that if $C = A \pm B$, then $C_{ij} = A_{ij} \pm B_{ij}$. This implies that

[1] $A_{13} = 7$, $A_{22} = -\frac{1}{3}$, $A_{34} = -1.66$, A_{42} does not exist.

the two matrix operands must be of the same shape, and addition/subtraction is not defined for two matrices with different shapes. For instance, if we have

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 8.5 \\ 1 & -7 \end{bmatrix}$$

Then

$$\begin{aligned} A + B &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 8.5 \\ 1 & -7 \end{bmatrix} \\ &= \begin{bmatrix} 1+1 & 2+1 \\ 3+0 & 4+8.5 \\ 5+1 & 6+(-7) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 \\ 3 & 12.5 \\ 6 & -1 \end{bmatrix} \end{aligned}$$

Short Exercise: Find $A - B$.[2]

Scalar Multiplication

Multiplying a matrix by a number (*scalar*) constitutes a ***scalar multiplication***, in which all entries are multiplied by that scalar. It is illustrated by the example below.

$$\begin{aligned} A &= \begin{bmatrix} 2 & -5.3 & 6 \\ -1 & 4.1 & -3 \end{bmatrix} \\ 3A &= 3\begin{bmatrix} 2 & -5.3 & 6 \\ -1 & 4.1 & -3 \end{bmatrix} \end{aligned}$$

[2] $A - B = \begin{bmatrix} 0 & 1 \\ 3 & -4.5 \\ 4 & 13 \end{bmatrix}$

$$
\begin{aligned}
&= \begin{bmatrix} 3(2) & 3(-5.3) & 3(6) \\ 3(-1) & 3(4.1) & 3(-3) \end{bmatrix} \\
&= \begin{bmatrix} 6 & -15.9 & 18 \\ -3 & 12.3 & -9 \end{bmatrix}
\end{aligned}
$$

Short Exercise: Find $\frac{1}{4}A$.[3]

Matrix Multiplication/Matrix Product

Meanwhile, multiplication between two matrices, commonly referred to as ***matrix multiplication/matrix product***, is not element-wise. It can be only carried out if the number of columns of the first matrix A equals to the number of rows of the second matrix B, let's say r. In other words, they need to be of the shapes $m \times r$ and $r \times n$ respectively. The resulting matrix AB will then have the shape $m \times n$, which means that the number of rows/columns of the output matrix follows the first/second input matrix respectively. The following two examples explain this requirement.

$$
A = \begin{bmatrix} 1 & 2.1 & 2 \\ 1 & 3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 5 \\ \sqrt{7} \end{bmatrix}
$$

Since the shapes of A and B are 2×3 and 3×1 so that the number of columns in A and the number of rows in B are both 3, the matrix product AB is possible. The resulting matrix will be of size 2×1. On the other hand, BA is not defined if we reverse the order of the matrix product. Meanwhile, for

$$
C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 6 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 3.44 & 1.07 \\ 0 & 5.96 \\ -4.3 & 2.75 \end{bmatrix}
$$

as the number of columns in C is 4, which is not equal to the number of rows in D (3), the matrix product CD is undefined in this case. (However, DC is just

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[3] $\frac{1}{4}A = \begin{bmatrix} \frac{1}{2} & -1.325 & \frac{3}{2} \\ -\frac{1}{4} & 1.025 & -\frac{3}{4} \end{bmatrix}$

valid, and what will be its shape?[4]) Now we are ready to see how the entries in matrix product is exactly computed.

Definition 1.1.1 (Matrix Product/Matrix Multiplcation)**.** Given an $m \times r$ matrix A and another $r \times n$ matrix B, we denote the matrix product between A and B as AB that will have the shape of $m \times n$. To calculate any entry in AB at row i and column j, we select row i from the first matrix A and column j from the second matrix B. Subsequently, take the products within each of the r pairs of numbers from that row and column. Their sum will then be the required value of the element, i.e.

$$\begin{aligned}(AB)_{ij} &= A_{i1}B_{1j} + A_{i2}B_{2j} + A_{i3}B_{3j} + \ldots + A_{ir}B_{rj} \\ &= \sum_{k=1}^{r} A_{ik}B_{kj}\end{aligned}$$

again, r is the number of columns/rows in the first/second matrix.

Example 1.1.1. Calculate the matrix product $C = AB$, where

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Solution. The output will be a 2×2 matrix. Using the definition above, we have

$$\begin{aligned}C_{11} = (AB)_{11} &= A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} \\ &= (1)(1) + (3)(2) + (5)(3) = 22 \\ C_{12} = (AB)_{12} &= A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ &= (1)(4) + (3)(5) + (5)(6) = 49\end{aligned}$$

Hence the entries along the first row of C will be 22 and 49. The remaining entries at the second row can be found in a similar way, and the readers are

[4] DC will be a 3×4 matrix.

encouraged to do this themselves. You should be able to get

$$C = \begin{bmatrix} 22 & 49 \\ 28 & 64 \end{bmatrix}$$

□

Matrix product has some important properties, listed as follows.

Properties 1.1.2. If A, B, C are some matrices having compatible shapes (*conformable*) so that the matrix multiplication operations below are valid, then

$$\underbrace{A \cdots A}_{k \text{ times}} = A^k \qquad k\text{-th power of a (square) matrix}$$

$$(AB)C = A(BC) = ABC \qquad \text{Associative Property}$$

$$(A \pm B)C = AC \pm BC \qquad \text{Distributive Property}$$

$$A(B \pm C) = AB \pm AC \qquad \text{Distributive Property}$$

Another important observation is that, in general $AB \neq BA$ even if the matrix products AB and BA are both well-defined, so they are not *commutative*. However, there are some exceptions to this.[5]

Example 1.1.2. Calculate $-2A + 3B$, where

$$A = \begin{bmatrix} 1 & 6 & 9 \\ 4 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 8 & 6 \\ -5 & 0 & 3 \end{bmatrix}$$

Solution.

$$\begin{aligned} -2A + 3B &= -2\begin{bmatrix} 1 & 6 & 9 \\ 4 & 4 & 6 \end{bmatrix} + 3\begin{bmatrix} 4 & 8 & 6 \\ -5 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -2 & -12 & -18 \\ -8 & -8 & -12 \end{bmatrix} + \begin{bmatrix} 12 & 24 & 18 \\ -15 & 0 & 9 \end{bmatrix} \end{aligned}$$

[5] A trivial exception is that $A = B$.

$$= \begin{bmatrix} 10 & 12 & 0 \\ -23 & -8 & -3 \end{bmatrix}$$

□

Example 1.1.3. Compute $(A + 3B)(2A - B)$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 0 \\ 4 & -1 \end{bmatrix}$$

Solution. Using the distributive property in Properties 1.1.2, the expression can be expanded to

$$\begin{aligned} (A + 3B)(2A - B) &= A(2A - B) + (3B)(2A - B) \\ &= A(2A) + A(-B) + (3B)(2A) + (3B)(-B) \\ &= 2A^2 - AB + 6BA - 3B^2 \end{aligned}$$

Bear in mind that $AB \neq BA$. We calculate each of the terms, which gives

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} (1)(1) + (2)(3) & (1)(2) + (2)(5) \\ (3)(1) + (5)(3) & (3)(2) + (5)(5) \end{bmatrix} \\ &= \begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 4 & -1 \end{bmatrix} \\ &= \begin{bmatrix} (1)(-2) + (2)(4) & (1)(0) + (2)(-1) \\ (3)(-2) + (5)(4) & (3)(0) + (5)(-1) \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 \\ 14 & -5 \end{bmatrix} \end{aligned}$$

Similarly, it is not difficult to obtain

$$BA = \begin{bmatrix} -2 & -4 \\ 1 & 3 \end{bmatrix} \qquad B^2 = \begin{bmatrix} 4 & 0 \\ -12 & 1 \end{bmatrix}$$

Hence the final answer will be

$$\begin{aligned}
&2A^2 - AB + 6BA - 3B^2 \\
&= 2\begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix} - \begin{bmatrix} 6 & -2 \\ 14 & -5 \end{bmatrix} + 6\begin{bmatrix} -2 & -4 \\ 1 & 3 \end{bmatrix} - 3\begin{bmatrix} 4 & 0 \\ -12 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 14 & 24 \\ 36 & 62 \end{bmatrix} - \begin{bmatrix} 6 & -2 \\ 14 & -5 \end{bmatrix} + \begin{bmatrix} -12 & -24 \\ 6 & 18 \end{bmatrix} - \begin{bmatrix} 12 & 0 \\ -36 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -16 & 2 \\ 64 & 82 \end{bmatrix}
\end{aligned}$$

□

Alternatively, one can evaluate $C = A+3B$ and $D = 2A-B$ first, and subsequently calculate the matrix dot product CD. (This is actually easier and more efficient.) The readers should try this as an exercise.

Matrix Equation Manipulation

For any matrix equation, one can do addition, subtraction and multiplication on both sides of the equation. However, one important note is that multiplying a matrix to an equation requires that the same matrix to be inserted to the left (or right) on both sides, respecting the order. So, for a matrix equation like (assuming the shapes of matrices are compatible),

$$AB - C = DE + F \tag{1.1}$$

if we want to multiply the equation by some matrix G, then two possibilities are

$$G(AB - C) = G(DE + F)$$

$$(AB - C)G = (DE + F)G$$

but we have, in general

$$\begin{aligned} G(AB - C) &\neq (DE + F)G \\ (AB - C)G &\neq G(DE + F) \end{aligned}$$

Doing successive matrix multiplications follows the same principle, step by step. Using the same example of Equation (1.1), given another matrix H, we note some valid outcomes.

$$\begin{aligned} HG(AB - C) &= HG(DE + F) \\ (AB - C)GH &= (DE + F)GH \\ GH(AB - C) &= GH(DE + F) \\ H(AB - C)G &= H(DE + F)G \\ G(AB - C)H &= G(DE + F)H \end{aligned}$$

However, be careful that cancellation at both sides may not be correct. If $AB = AC$, then we cannot conclude that $B = C$ with certain. Nevertheless, in the next chapter we will see one of the scenarios where cancellation actually works.

1.2 Definition of Linear Systems of Equations

The prime application of matrices is to deal with ***linear systems (of equations)*** as mentioned in the introduction. To understand what a linear system is, we first have to know the definition of a ***linear equation*** (in multiple variables, let's say $x_1, x_2, \ldots$, or $x, y, \ldots$). In a linear equation, for any additive term, there is at most one variable (unknown), with a power of one, times some constant coefficient, like x, $-\sqrt{5}x$, $-y$, $2.33y$. This means that there are no cross-product terms such as $1.68xy$, variables with a power that is not one, like x^3, or non-linear functions, including $\sin x$, e^y. For n variables, a linear equation has the following form.

Definition 1.2.1 (Linear Equation)**.** A linear equation is an equation in the form of

$$\sum_{j=1}^{n} a_j x_j = a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = h$$

where $x_1, x_2, \ldots, x_n$ are the unknowns, while $a_1, a_2, \ldots, a_n$ and h are some constants. If $h = 0$, then it is known as a ***homogeneous linear equation***.

Short Exercise: Determine whether the equations below are (a) linear, and if they are linear, then (b) homogeneous or not. The unknowns are x, y, z. [6]

1. $3x + 4.7y = 2\sqrt{2}$
2. $\cos x + \ln y = 0$
3. $7\pi x - z = 2$
4. $x^2 + 3.8y^{-3/2} = 1$
5. $1.05x + 3.17y + 6.44z = 0$
6. $xyz = 8$

A system of linear equations are then simply a family of m linear equations in a set of some unknowns, $m \geq 1$.

Definition 1.2.2 (Linear System of Equations)**.** A linear system of size $m \times n$, i.e. m linear equations in n unknowns $(x_1, x_2, \ldots, x_n)$, has the form of

$$\begin{cases}
\sum_{j=1}^{n} a_j^{(1)} x_j = a_1^{(1)}x_1 + a_2^{(1)}x_2 + a_3^{(1)}x_3 + \cdots + a_n^{(1)}x_n & = h^{(1)} \\
\sum_{j=1}^{n} a_j^{(2)} x_j = a_1^{(2)}x_1 + a_2^{(2)}x_2 + a_3^{(2)}x_3 + \cdots + a_n^{(2)}x_n & = h^{(2)} \\
\vdots & \\
\sum_{j=1}^{n} a_j^{(m)} x_j = a_1^{(m)}x_1 + a_2^{(m)}x_2 + a_3^{(m)}x_3 + \cdots + a_n^{(m)}x_n & = h^{(m)}
\end{cases}$$

If $h^{(1)}, h^{(2)}, \ldots, h^{(m)}$ on R.H.S. are all zeros, i.e. all the equations are homoge-

[6]Linear/Inhomogeneous, Non-linear, Linear/Inhomogeneous, Non-linear, Linear/Homogeneous, Non-linear.

neous, then the system is called a ***homogeneous linear system (of equations)***.

It is not hard to see that for any homogeneous linear system, it always has the trivial solution of $x_j = 0$ for $j = 1, 2 \ldots, n$, or expressed as $\vec{x} = \mathbf{0}$. However, such trivial solution may not be the only solution to the system, as we shall see in Chapter 3. Below shows some examples of linear systems.

$$\begin{cases} 3.3x + 4y &= 5 \\ 7x + 9.7y &= 13.1 \end{cases}$$

A 2×2 linear system with two equations, two unknowns.

$$\begin{cases} x + 2y - 4z &= 3 \\ x - y + 3z &= -4 \end{cases} \tag{1.2}$$

A 2×3 linear system with two equations, three unknowns.

$$\begin{cases} x + 2.2y + 3z &= 0 \\ 2x + 3z &= 0 \\ 4x - 5.6y &= 0 \end{cases} \tag{1.3}$$

A 3×3 homogeneous linear system (homogeneous as the constants on R.H.S. are all zeros), notice that the coefficients of y and z in the second/third equations are zeros as well and do not appear explicitly.

The above formulation of a linear system closely resembles a tabular structure. Therefore, we are motivated to represent such systems with the language of matrices, which have an appearance of tabular arrays. Indeed, it is possible to rewrite an $m \times n$ linear system as $A\vec{x} = \vec{h}$, where A is an $m \times n$ matrix with entries copied from the coefficients in front of the variables, arranged like in Definition 1.2.2. In this book sometimes we will call it a *coefficient matrix*. Meanwhile, $\vec{x}$ is a *column vector* (an $n \times 1$ matrix) holding the n unknowns, and $\vec{h}$ is another column vector (an $m \times 1$ matrix) that contains the m constants on R.H.S. of the linear system.

Properties 1.2.3. For a linear system defined as in Definition 1.2.2, it can be rewritten as $A\vec{x} = \vec{h}$, where $A_{ij} = a_j^{(i)}$, $\vec{x} = x_j$, and $\vec{h} = h^{(i)}$.

Using the second example (Equation (1.2)) above as an illustration, we can easily verify that

$$\begin{cases} x + 2y - 4z &= 3 \\ x - y + 3z &= -4 \end{cases}$$

can be expressed as (you should check it by expanding the matrix product)

$$\begin{bmatrix} 1 & 2 & -4 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

An even simpler representation is the ***augmented matrix*** which omits the unknowns and concatenates A and $\vec{h}$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ 1 & -1 & 3 & -4 \end{array}\right]$$

Short Exercise: Write down the augmented matrix for the linear system in Equation (1.3).[7]

1.3 Elementary Row Operations

When we construct a matrix, it is natural to think about how to manipulate its structure. ***Elementary row operations*** provide such possibility in three ways, outlined in the following definition.

---

[7] $\left[\begin{array}{ccc|c} 1 & 2.2 & 3 & 0 \\ 2 & 0 & 3 & 0 \\ 4 & -5.6 & 0 & 0 \end{array}\right]$

Definition 1.3.1 (Elementary Row Operations)**.** Denote the p-th row of a matrix as R_p. The three types of elementary row operations are

1. Multiplying a row R_p by any non-zero constant $c \neq 0$;
2. Adding another row R_q times any non-zero constant $c \neq 0$, to a row R_p, such that the new p-th row becomes $R_p + cR_q$;
3. Swapping a row R_p with another row R_q.

To facilitate computation, we denote these three kinds of operations using the following notations.

1. $cR_p \to R_p$,
2. $R_p + cR_q \to R_p$,
3. $R_p \leftrightarrow R_q$

For example, the matrix A

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 11 \end{bmatrix}$$

can be transformed to a new matrix A'

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 5 \end{bmatrix}$$

if we apply the elementary row operation of subtracting $2R_1$ from R_2 (i.e. $R_2 - 2R_1 \to R_2$).

Short Exercise: Find out the resulting matrix A'' if we multiply the first row of A' by 3 and then subtract the second row from the first row.[8]

Attentive readers may have noticed that these three types of elementary row operations resemble we have been always doing to the equations when solving a

[8] $\begin{bmatrix} 0 & 3 & 4 \\ 3 & 3 & 5 \end{bmatrix}$

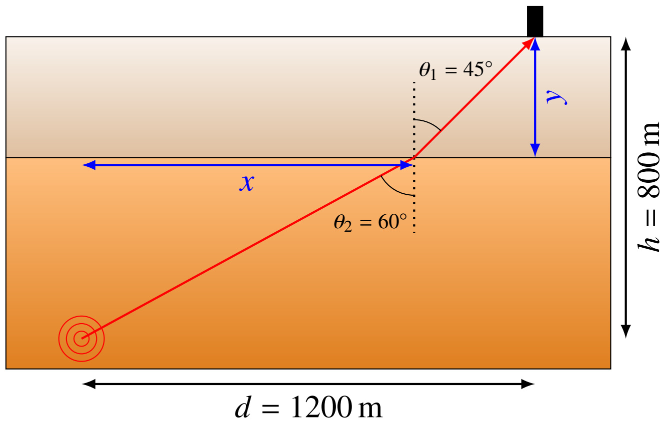

Figure 1.1: The underground schematic for the seismic ray in Example 1.4.1.

linear system as taught in high school. We re-introduce them as elementary row operations here first as they allow a systematic treatment of linear systems and matrices in later chapters.

1.4 Earth Science Applications

Example 1.4.1. Seismic wave follows *Snell's Law* like a light ray when it comes to refraction. Assuming the ground can be modelled as a two-layer system (see Figure 1.1), and we know a particular train of seismic wave generated from an underground source that reaches the ground receiver travels at an angle of $\theta_1 = 45°/\theta_2 = 60°$ to the vertical at the top/bottom layer. (θ_1 can be found by analyzing the seismic waveform, and then θ_2 can be estimated by Snell's Law given we know about the densities of the respective layers.) Given that the horizontal and vertical distance between the seismic source and the surface receiver are $d = 120$ m and $h = 80$ m, construct a linear system for this situation in two unknowns: the depth of the top layer y and the horizontal displacement x (in meters) where the wave reaches at the interface relative to the source.

Solution. We can deduce two equations from the given information. Consider the upper portion of the seismic ray, from basic trigonometry, we know that

$$\begin{aligned}\frac{d-x}{y} &= \tan\theta_1\\ d-x &= (\tan\theta_1)y\\ x+(\tan\theta_1)y &= d\end{aligned}$$

Similarly, for the lower portion of the seismic ray, we have

$$\begin{aligned}\frac{x}{h-y} &= \tan\theta_2\\ x &= (\tan\theta_2)h-(\tan\theta_2)y\\ x+(\tan\theta_2)y &= (\tan\theta_2)h\end{aligned}$$

The corresponding linear system is

$$\begin{cases} x+(\tan\theta_1)y & = d\\ x+(\tan\theta_2)y & = (\tan\theta_2)h\end{cases}$$

where x and y are the unknowns to be solved. d, h, θ_1 and θ_2 (and hence $\tan\theta_1$ and $\tan\theta_2$) are constants. Expressing the system in matrix form, we have

$$\begin{bmatrix}1 & \tan\theta_1\\ 1 & \tan\theta_2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}d\\ (\tan\theta_2)h\end{bmatrix}$$

Substituting the provided values for the constants ($\tan\theta_1 = \tan(45°) = 1$, $\tan\theta_2 = \tan(60°) = \sqrt{3}$), we have

$$\begin{bmatrix}1 & 1\\ 1 & \sqrt{3}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}120\\ 80\sqrt{3}\end{bmatrix}$$

□

Example 1.4.2. The radiation transfer across the atmosphere of any planet (including the Earth) in the Solar system can be compared to a *multi-layer model* with fully absorbing layers (note that it is a very simplistic approach). Assume there are N such layers and the total rate of incident Solar radiation reaching the surface is E_{in}. Each of the layers also emits radiation to the other two layers directly above/below it. The rate of radiative emission for the j-th layer that has a temperature T_j is $E_j = \sigma T_j^4$ according to the *Stefan–Boltzmann Law*, with $\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$. The overall scenario can be seen in Figure 1.2. Formulate a linear system that represents the energy equilibrium (incoming radiation = outgoing radiation) of all layers and the surface, with E_j being the unknowns, over $j = 1, 2, \ldots, N, N+1$.

Solution. Considering the energy equilibrium for the first (topmost) layer, we have

$$-2E_1 + E_2 = 0$$

Going down to the second layer, it is

$$E_1 - 2E_2 + E_3 = 0$$

In general, for the j-th layer in the middle, where j runs from 2 to N, we can similarly obtain

$$E_{j-1} - 2E_j + E_{j+1} = 0$$

Finally, for the surface (the $N+1$-th layer), we have

$$\begin{aligned} E_N - E_{N+1} + E_{in} &= 0 \\ E_N - E_{N+1} &= -E_{in} \end{aligned}$$

Summarizing all the $N+1$ equations, they can be expressed in matrix form as

$$\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & & 0 & 0 & 0 \\ 0 & 1 & -2 & & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & & -2 & 1 & 0 \\ 0 & 0 & 0 & & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_{N-1} \\ E_N \\ E_{N+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -E_{in} \end{bmatrix}$$

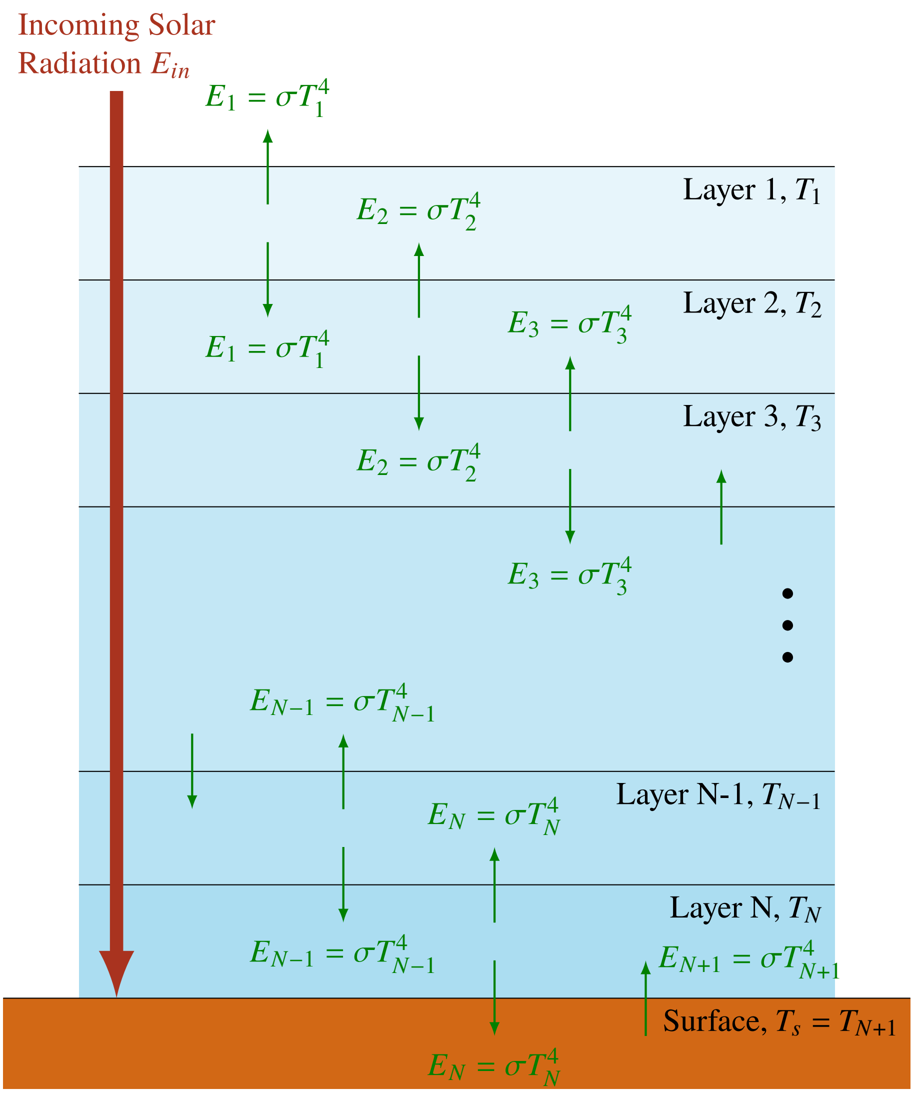

Figure 1.2: The atmospheric profile with multiple (N) absorbing layers in Example 1.4.2. The surface is treated as an extra N+1-th layer.

Particularly, for $N = 4$, it is

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -E_{in} \end{bmatrix}$$

□

Example 1.4.3. The seawater in oceans contains a variety of dissolved salts in the form of ions. Most of them are sodium (Na^+), magnesium (Mg^{2+}), chlorine (Cl^-) and sulphate (SO_4^{2-}). Consider a sample of seawater and assume the concentration of other ions are negligible. Their are two major constraints over the individual concentrations of each type of ions ($n = [Na^+]$, $m = [Mg^{2+}]$, $c = [Cl^-]$, $s = [SO_4^{2-}]$). First, the overall charge of the seawater has to be neutral. Second, their concentrations should add up to the measured salinity (the total mass concentration of salts, inferred by electrical conductivity). It is given that the salinity of the seawater sample is 34 psu (1 psu = 1 g kg^{-1} which is the unit preferred in oceanography). Write down the corresponding linear system that is consisted of two equations for this situation.

Example 1.4.4. There are four weather stations in proximity. Each of them measures the local air temperature T_i, where $i = 1, 2, 3, 4$. Assume that the spatial pattern of temperature over the region approximately follows a linear gradient such that both $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial y}$ can be treated as constants. Assign the location of the first station to be the origin $(0, 0)$, and the relative locations of the second/third/-fourth station are $(10, 20)$, $(25, 15)$, $(-10, 5)$ (in km). The measured temperature of the four stations at some time are 27.1 °C, 27.3 °C, 27.4 °C, 26.9 °C. Set up a linear system for finding $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial y}$.

(WIP)

We will talk about how to solve the linear systems in these four examples in Section 3.3.

1.5 Python Programming

We will use the package `numpy` and `scipy` throughout the book to solve linear algebra problems via *Python* programming. First, we can define a 2D `numpy` array that works as a matrix.

```
import numpy as np
myMatrix1 = np.array([[1, 4], [5, 3]])
print(myMatrix1)
```

which gives

```
[[1 4]
 [5 3]]
```

representing the matrix

$$\begin{bmatrix} 1 & 4 \\ 5 & 3 \end{bmatrix}$$

We can similarly define another matrix:

```
myMatrix2 = np.array([[1, 3], [5, 6]])
```

Addition, subtraction, and scalar multiplication are straight-forward.

```
myMatrix3 = 3*myMatrix1 - 4*myMatrix2
print(myMatrix3)
```

The above code produces

```
[[ -1   0]
 [ -5 -15]]
```

and you can verify the answer by hand. Meanwhile, matrix product is done by the function `np.matmul()`.

```
myMatrix4 = np.matmul(myMatrix1, myMatrix2) # or equivalently
    myMatrix1 @ myMatrix2
print(myMatrix4)
```

gives

```
[[21 27]
 [20 33]]
```

To select a specfic entry, use indexing by square brackets. The first index/second index represents row/column. Beware that each index starts at zero in *Python*. So putting the number 1 in the first/second index actually means the second row/column. So

```
print(myMatrix4[1,0])
```

refers to the entry at row 2, column 1 of `myMatrix4` which is 20. Also, we can select the i-th row (or the j-th column) by `<Matrix>[i-1, :]` (`<Matrix>[:, j-1]`), where the colon `:` implies selecting along the entire row (column). For example,

```
print(myMatrix3[0,:])
print(myMatrix4[:,1])
```

gives `[-1  0]` and `[27 33]` respectively. Now let's see how to perform elementary row operations. It will be easier and less error-prone if we copy the array before performing the operations.

```
myMatrix5 = np.copy(myMatrix4)
myMatrix5[0,:] = myMatrix5[0,:]/3
print(myMatrix5)
```

The lines above, when executed, divide the second row of `myMatrix5` (which is a copy of `myMatrix4`) by 3, and give

```
[[ 7  9]
 [20 33]]
```

Meanwhile, the subsequent lines below

```
myMatrix5[1,:] = myMatrix5[1,:] - 2*myMatrix5[0,:]
print(myMatrix5)
```

proceed to subtract 2 times the first row from the second row, and produce

```
[[ 7  9]
 [ 6 15]]
```

Row interchange is a bit more tricky.

```
myMatrix6 = np.copy(myMatrix4)
myMatrix6[[0, 1],:] = myMatrix6[[1, 0],:]
```

This swaps the first and second row. (You can swap columns in a similar way.) Printing out the new matrix by `print(myMatrix6)` shows

```
[[20 33]
 [21 27]]
```

An important pitfall is that, since our inputs to `np.array` are all integers, the previous arrays will automatically have a data type of `int` (integer). This may produce unexpected errors when the calculation leads to decimals/fractions. If it is the case, then we can avoid such bugs by declaring the array with the keyword `dtype=float` to use *floating point numbers*, like

```
myMatrix1 = np.array([[1, 4], [5, 3]], dtype=float)
```

when printed out via `print(myMatrix1)` it gives

```
[[1. 4.]
 [5. 3.]]
```

Notice the newly appeared decimal points after the original integers. Alternatively, we can add decimal points to the integer entries during the array initialization, as

```
myMatrix1 = np.array([[1., 4.], [5., 3.]])
```

1.6 Exercises

Exercise 1.1 Let

$$A = \begin{bmatrix} 1 & 2 \\ 5 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 3 \\ -2 & 7 \end{bmatrix}$$

Find:

(a) $A + B$,

(b) $2A - \frac{3}{2}B$,

(c) AB,

(d) BA.

Exercise 1.2 Let

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -1 \\ 4 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 & -2 \\ -2 & 1 & 3 \end{bmatrix}$$

Find:

(a) AB,

(b) BA.

Exercise 1.3 Let

$$A = \begin{bmatrix} 4 & 6 \\ 3 & 3 \end{bmatrix}$$
$$B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$
$$C = \begin{bmatrix} 3 & 9 & 1 \\ 4 & 3 & -1 \end{bmatrix}$$

Find:

(a) $(A + B)C$,

(b) $AC + BC$,

(c) $(AB)C$,

(d) $A(BC)$.

Exercise 1.4 Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 7 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 & -2 \\ 4 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

Find:

(a) $(A+B)(2A-B)$,

(b) $(\frac{3}{2}A-B)(-A+\frac{1}{2}B)$.

Exercise 1.5 Let

$$A=\begin{bmatrix}1&0&3\\2&1&6\\5&2&0\end{bmatrix} \qquad B=\begin{bmatrix}2&3&5\\1&3&8\\4&0&7\end{bmatrix}$$

Find:

(a) A^2,

(b) B^2,

(c) AB,

(d) BA.

Exercise 1.6 Rewrite the following system of linear equations in matrix form.

$$\begin{cases}3y-4z & =6\\5x-y+2z & =13\\6x+z & =8\end{cases}$$

Exercise 1.7 For the following matrix,

$$\begin{bmatrix}2&3&5&7\\1&2&4&8\\1&3&6&10\end{bmatrix}$$

Find the results if the following elementary row operations are applied on it:

(a) Multiplying the third row by a factor of 2, and then subtracting the third row by the second row,

(b) Adding the first row by 3 times the third row, and then interchanging the first and second row, and finally subtract the third row by 2 times the first row.

Exercise 1.8 The *dry adiabatic lapse rate*, which is the rate of decrease in air temperature when an unsaturated air parcel rises, is about $\Gamma_{dry} = 9.8\,°\text{C}\,\text{km}^{-1}$. When the temperature of the air parcel falls below the *dew point*, the air saturates and condensation occurs. Typically, dew point temperature of an air parcel will decrease at a rate of roughly $\Gamma_{dew} = 2\,°\text{C}\,\text{km}^{-1}$. Now, an air parcel with an initial air temperature/dew point temperature of $T_{a,ini} = 25.4\,°\text{C}$ / $T_{dew,ini} = 17.8\,°\text{C}$ at the ground starts to rise. Let z_{cd} and T_{cd} be the height above the ground (in km) and temperature (in °C) of the air parcel when condensation occurs. Construct a linear system with z_{cd} and T_{cd} as the unknowns to represent this situation.

Exercise 1.9 In some ancient Chinese Mathematics texts, the problem of *Chickens and Rabbits in the Same Cage* was posed. *"Now there are some chickens and rabbits placed in the same cage, with a total number of* 35 *heads and* 94 *legs. How many chickens and rabbits are there respectively?"* Given the fact that a chicken (rabbit) has two (four) legs (and obviously only one head), write down the corresponding linear system in terms of the numbers of chickens x and rabbits y.

Chapter 2

Inverses and Determinants

In this chapter, we are going to discuss two important concepts about matrices, which are their *inverses* and *determinants*. They will appear from time to time in the remaining parts of this book. To derive them, we need to introduce some prerequisite ideas first, including the *identity matrix*, *transpose*, and the methods of *Gaussian Elimination* and *cofactor expansion*.

2.1 Identity Matrices and Transpose

2.1.1 Identity Matrices

One important class of matrices is the ***identity matrices***. They are $n \times n$ square matrices, where n can be any positive integer, with entries along the *main diagonal* (where index of row = column) being 1 and other off-diagonal elements being 0. Usually, they are denoted by I_n, or simply I.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad\qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity matrices of size 2×2 and 3×3 with the main diagonal 1s highlighted.

Definition 2.1.1 (Identity Matrix)**.** An identity matrix of the square shape $n \times n$ I_n is defined as $[I_n]_{ij} = 1$, for $i = j$, and $[I_n]_{ij} = 0$, for $i \neq j$, where $1 \leq i, j \leq n$.

Short Exercise: Explicitly write down I_5.[1]

One important property of identity matrices is

Properties 2.1.2. Matrix product between any matrix A with an identity matrix I always returns A whenever the matrix product is defined. If A is of the shape $m \times n$, then $AI_n = I_m A = A$. If A is now a square matrix such that $m = n$ (and $I_m = I_n = I$), then we have $AI = IA = A$.

In other words, the identity I can be regarded to be the "1" in the world of matrices. This is one of the cases that $AB = BA$ commutes (if both of them are square and either one of them is the identity matrix). Using the matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

as an example, the readers can try to computed AI_3 and I_2A to see if the results are A itself.[2]

---

[1] $I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

[2]

$$\begin{aligned} AI_3 &= \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (a)(1) + (b)(0) + (c)(0) & (a)(0) + (b)(1) + (c)(0) & (a)(0) + (b)(0) + (c)(1) \\ (d)(1) + (e)(0) + (f)(0) & (d)(0) + (e)(1) + (f)(0) & (d)(0) + (e)(0) + (f)(1) \end{bmatrix} \\ &= \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = A \end{aligned}$$

The calculation of $I_2A = A$ is similar.

2.1.2 Transpose

Transpose of a matrix, denoted by adding the superscript ^{T}, is formed by interchanging its rows and columns, that is, flipping the elements about the main diagonal.

Definition 2.1.3 (Transpose)**.** The transpose of an $m \times n$ matrix A, denoted as A^T, is formed according to the relation $[A^T]_{pq} = A_{qp}$, $1 \leq p \leq n$, $1 \leq q \leq m$, i.e. swapping the row and column indices. Now A^T is an $n \times m$ matrix.

Two examples are given below to show the effect of applying transpose on matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -4 & 3 \\ 2 & -2 & 0 \\ -3 & 1 & 4 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 2 & -3 \\ -4 & -2 & 1 \\ 3 & 0 & 4 \end{bmatrix}$$

Particularly, in the second example, we have outlined the main diagonal of B (as well as B^T) and how the elements flip about it when transpose is carried out. Some useful properties about transpose are listed as follows.

Properties 2.1.4. For two matrices A and B, we have

1. $(cA)^T = cA^T$, where c is any constant;
2. $(A^T)^T = A$, i.e. transposing twice returns the original matrix (which is obvious);
3. $(A \pm B)^T = A^T \pm B^T$, if A and B have the same shape;
4. $(AB)^T = B^T A^T$, if A and B are conformable;

5. $A_{kk} = A^T_{kk}$ for any k that A_{kk} is defined, i.e. the main diagonal is unaffected by transpose.

Short Exercise: Show that $(ABC)^T = C^T B^T A^T$ if the matrices have compatible shapes for the matrix multiplcation.[3]

2.1.3 Symmetric Matrices

A ***symmetric matrix*** has its elements mirrored about the main diagonal. Taking transpose of such a matrix will leave it unchanged. Implicitly, it is required to be a square matrix.

Definition 2.1.5 (Symmetric Matrix)**.** If an $n \times n$ square matrix A and its transpose A^T are equal, i.e. $A_{pq} = [A^T]_{pq} = A_{qp}$ for all $1 \leq p, q \leq n$, or simply $A = A^T$, then A, and also A^T, are symmetric.

As an example,

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 0 & 4 \\ -2 & 4 & 3 \end{bmatrix}$$

is a 3×3 symmetric matrix.

Short Exercise: Show that $Y = XX^T$ and $Z = X^T X$ are symmetric for any matrix X.[4]

In contrast, we also have ***skew-symmetric matrices*** such that $A^T = -A$. This automatically requires elements along the main diagonal to be all zeros.

$$\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

A 3×3 skew-symmetric matrix.

---

[3]By (4), $(ABC)^T = ((AB)(C))^T = C^T(AB)^T = C^T B^T A^T$

[4]By Properties 2.1.4, $Y^T = (XX^T)^T = (X^T)^T(X)^T = XX^T = Y$, similar goes for $Z = X^T X$.

2.2 Inverses

2.2.1 Definition and Properties of Inverses

Inverse of a square matrix, denoted by appending the superscript $^{-1}$, is another square matrix such that the matrix product between these two matrices (in either order) yields an identity matrix.

Definition 2.2.1 (Inverse)**.** An $n \times n$ square matrix B is said to be the inverse of another $n \times n$ square matrix A if $AB = BA = I_n$. This inverse matrix is denoted as $B = A^{-1}$, and the relation becomes $AA^{-1} = A^{-1}A = I$. The opposite direction also holds, i.e. A is the inverse of A^{-1}. Hence, we say that A and A^{-1} are the inverse of each other.

If there exists an inverse A^{-1} for the square matrix A, then both A and A^{-1} are called ***invertible***. Otherwise, A is said to be ***singular***. This is another situation in which a matrix product $AB = BA$ (if $B = A^{-1}$) can commute.[5]

In the last chapter, we only define addition, subtraction, and multiplication for matrices, omitting division like it is an elephant in the room. The inverse serves as a remedy for this by acting as the reciprocal in the world of matrices. This allows us to "divide" on both sides of a matrix equation provided the relevant inverse exists. Remember, in the last chapter, we mentioned that cancellation may not work for something like $AB = AC$. But if A^{-1} exists, then by multiplying it to the left of both sides of the equation, we can effectively "divide by A"

$$AB = AC$$

[5] $AA^{-1} = I$ implies $A^{-1}A = I$ and vice versa. However, while looking innocent, showing this is actually not trivial. A heuristic way to "prove" it is to note that

$$\begin{aligned} AA^{-1} &= I \\ AA^{-1}A &= IA \\ A(A^{-1}A) &= A \end{aligned}$$

which implies that multiplying A by $A^{-1}A$ returns A itself, so it should be reasonable to assume $A^{-1}A = I$.

$$\begin{aligned}
A^{-1}AB &= A^{-1}AC \\
(A^{-1}A)B &= (A^{-1}A)C && \text{(Properties 1.1.2)} \\
IB &= IC && \text{(Definition 2.2.1)} \\
B &= C && \text{(Properties 2.1.2)}
\end{aligned}$$

so that cancellation holds in this situation. Take the matrix equation $AG = H$ as another example, if A has an inverse A^{-1}, then we may do a matrix "division" as follows:

$$\begin{aligned}
AG &= H \\
A^{-1}AG &= A^{-1}H \\
((A^{-1}A)G = IG =)\ G &= A^{-1}H \quad \text{(Properties 1.1.2 and 2.1.2, Definition 2.2.1)}
\end{aligned}$$

In addition, the inverse of a matrix, if exists, must be unique.

Properties 2.2.2 (Uniqueness of Inverse)**.** If A has an inverse A^{-1}, it is unique.

Proof. This property can be proved easily by first assuming that the invertible matrix A has two different inverses, B and C. Subsequently, by Definition 2.2.1, we have $BA = I$ (and also $AC = I$). Multiplying by C to the right on both sides gives

$$\begin{aligned}
BAC &= IC \\
B(AC) &= C && \text{(Properties 1.1.2 and 2.1.2)} \\
B(I) &= C && (AC = I \text{ from assumption}) \\
B &= C && \text{(Properties 2.1.2)}
\end{aligned}$$

So, B and C are actually the same matrix, implying that the inverse of A is unique. $\square$

Example 2.2.1. Let

$$A = \begin{bmatrix} 4 & 6 \\ 3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{5}{2} & -3 \\ -\frac{3}{2} & 2 \end{bmatrix}$$

Show that A and B are inverse to each other.

Solution.

$$\begin{aligned} AB &= \begin{bmatrix} 4 & 6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \frac{5}{2} & -3 \\ -\frac{3}{2} & 2 \end{bmatrix} \\ &= \begin{bmatrix} (4)(\frac{5}{2}) + (6)(-\frac{3}{2}) & (4)(-3) + (6)(2) \\ (3)(\frac{5}{2}) + (5)(-\frac{3}{2}) & (3)(-3) + (5)(2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \end{aligned}$$

We leave it to the readers for showing $BA = I$ too as an exercise. Hence, $AB = BA = I$, A and B are indeed the inverse of each other. □

The followings are some properties of inverses.

Properties 2.2.3. If a square matrix A is invertible and has an inverse A^{-1}, then

1. $(cA)^{-1} = \frac{1}{c}A^{-1}$, for any constant $c \neq 0$;
2. $(A^{-1})^{-1} = A$, i.e. the inverse of an inverse returns the original matrix;
3. $(A^n)^{-1} = (A^{-1})^n$, for any positive integer n;
4. $(AB)^{-1} = B^{-1}A^{-1}$, provided that B is invertible too (and they are square matrices of the same size);
5. $(A^T)^{-1} = (A^{-1})^T$.

However, $(A \pm B)^{-1}$ may not be equal to $A^{-1} \pm B^{-1}$, or even may be singular. We shall briefly prove (4) here.

Proof. It is given that A and B is invertible, and by Definition 2.2.1, we have $AA^{-1} = I$, as well as

$$BB^{-1} = I$$

Multiplying by A and A^{-1} to the left and right on both sides of above respectively yields

$$\begin{aligned}
ABB^{-1}A^{-1} &= AIA^{-1} \\
AB(B^{-1}A^{-1}) &= (AI)A^{-1} = AA^{-1} && \text{(Properties 1.1.2 and 2.1.2)} \\
&= I && \text{(Definition 2.2.1)}
\end{aligned}$$

This shows that multiplying AB by $B^{-1}A^{-1}$ produces an identity matrix, and therefore $(AB)^{-1} = B^{-1}A^{-1}$ is the unique inverse of AB by Definition 2.2.1 and Properties 2.2.2. □

Short Exercise: Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ if A, B and C are invertible and conformable.[6]

(4) of Properties 2.2.3 explicitly shows that the product AB is invertible if A and B are themselves invertible. The converse is actually true as well. Hence

Properties 2.2.4. For two square matrices A and B, AB is invertible if and only if A and B are invertible.

2.2.2 (Reduced) Row Echelon Form

Naturally, the next question is how to compute the inverse of any square matrix. For this, we have to understand a specific form of matrices called the ***(reduced) row echelon form*** first. A matrix is in reduced row echelon form (*rref*) when it satisfies the following requirements.

[6] By (4), $(ABC)^{-1} = ((AB)(C))^{-1} = C^{-1}(AB)^{-1} = C^{-1}B^{-1}A^{-1}$

Definition 2.2.5 ((Reduced) Row Echelon Form)**.** A matrix is in row echelon form if

1. The first non-zero number in every row is 1, which is known as the *"Leading 1"* (sometimes referred to as a *pivot*),
2. *"Leading 1"* of a lower row must appear farther to the right than that of any higher row,
3. Any row consisted of all zeros is placed at the bottom;
4. If additionally, any column containing a leading 1 (sometimes called a *pivotal column*) have zeros elsewhere in that column, then it is in *reduced* row echelon form.

It is apparent that all identity matrices are in (reduced) row echelon form. Examples of row echelon form (but not *reduced*), with the leading 1s highlighted are

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 4 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Meanwhile, examples of *reduced* row echelon form are

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The following matrices are *not* in row echelon form. (why?)[7]

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

[7] P violates (2) and Q does not satisfy (1) and (3) of Definition 2.2.5.

Short Exercise: Decide if the following matrices are in (reduced) row echelon form or not.[8]

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

We have studied about elementary row operations in the last chapter, which now can be used to transform matrices into their reduced row echelon form. The procedure is comprised of two major parts, the *forward phase*, converting the matrix to row echelon form first, and the *backward phase*, eventually transforming it into reduced row echelon form. The first phase is also named ***Gaussian Elimination***, and together they are called ***Gauss-Jordan Elimination***[9]. We demonstrate the entire procedure using an example.

Example 2.2.2. Carry out Gauss-Jordan Elimination on the following matrix to make it become reduced row echelon form.

$$A = \begin{bmatrix} 2 & 0 & 4 & 6 \\ 3 & 3 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Solution. At each step of the forward phase, the strategy is to look at on the leftmost column that has at least one non-zero entries (any column consisting of full zeros is ignored). Along that column, we either find an existing leading 1, or create a leading 1 via multiplying some row having a starting entry a that is as large as possible in magnitude, by the constant $1/a$. (The leading entry selected by this algorithm is commonly called the ***pivot***, and the process is called ***pivoting***.) The row holding the leading 1 is subsequently put at the top, by an interchanging of rows if needed. In this example, such rows will be highlighted

[8] Yes, Yes (reduced), No.

[9] Often we just write Gaussian Elimination in place of Gauss-Jordan Elimination.

in red.

$$\begin{bmatrix} 2 & 1 & 4 & 6 \\ 3 & 3 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & 2 & 3 \\ 3 & 3 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} \qquad \frac{1}{2}R_1 \to R_1$$

We have picked the first row R_1 for the leading 1 through multiplying it by a factor of $\frac{1}{2}$ here, but a leading 1 can be obtained from the other two rows as well. Subsequently, we make all the elements below the leading 1 along that *pivotal column* become zero, by adding the top row (which holds the leading 1), times $-a_i$ (where a_i is the corresponding leading entry of row i) to the other rows. Those zeros produced in this way will be highlighted in blue.

$$\begin{bmatrix} 1 & \frac{1}{2} & 2 & 3 \\ 3 & 3 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & 2 & 3 \\ 0 & \frac{3}{2} & -5 & -9 \\ 1 & 2 & 3 & 4 \end{bmatrix} \qquad R_2 - 3R_1 \to R_2$$

$$\to \begin{bmatrix} 1 & \frac{1}{2} & 2 & 3 \\ 0 & \frac{3}{2} & -5 & -9 \\ 0 & \frac{3}{2} & 1 & 1 \end{bmatrix} \qquad R_3 - R_1 \to R_3$$

The first iteration is finished. We now repeat the same process over the remaining submatrix made up of elements that are not yet highlighted in colour, from left to right recursively.

$$\begin{bmatrix} 1 & \frac{1}{2} & 2 & 3 \\ 0 & \frac{3}{2} & -5 & -9 \\ 0 & \frac{3}{2} & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & 2 & 3 \\ 0 & 1 & -\frac{10}{3} & -6 \\ 0 & \frac{3}{2} & 1 & 1 \end{bmatrix} \qquad \frac{2}{3}R_2 \to R_2$$

$$\to \begin{bmatrix} 1 & \frac{1}{2} & 2 & 3 \\ 0 & 1 & -\frac{10}{3} & -6 \\ 0 & 0 & 6 & 10 \end{bmatrix} \qquad R_3 - \frac{3}{2}R_2 \to R_3$$

$$\to \begin{bmatrix} 1 & \frac{1}{2} & 2 & 3 \\ 0 & 1 & -\frac{10}{3} & -6 \\ 0 & 0 & 1 & \frac{5}{3} \end{bmatrix} \qquad \frac{1}{6}R_3 \to R_3$$

Now, all entries below every leading 1 are zeros, and the forward phase is completed. We have obtained the row echelon form as an intermediate product.

The backward phase is done similarly but in a bottom-up fashion, from right to left. By adding appropriate multiples of lower rows to higher rows, we turn all the non-zero elements above the leading 1 along every pivotal column into zeros. Non-pivotal columns (the last column here) are ignored.

$$\begin{aligned}
\begin{bmatrix} 1 & \frac{1}{2} & 2 & 3 \\ 0 & 1 & -\frac{10}{3} & -6 \\ 0 & 0 & 1 & \frac{5}{3} \end{bmatrix} &\to \begin{bmatrix} 1 & \frac{1}{2} & 2 & 3 \\ 0 & 1 & 0 & -\frac{4}{9} \\ 0 & 0 & 1 & \frac{5}{3} \end{bmatrix} & R_2 + \frac{10}{3}R_3 \to R_2 \\
&\to \begin{bmatrix} 1 & \frac{1}{2} & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{4}{9} \\ 0 & 0 & 1 & \frac{5}{3} \end{bmatrix} & R_2 - 2R_3 \to R_1 \\
&\to \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{9} \\ 0 & 1 & 0 & -\frac{4}{9} \\ 0 & 0 & 1 & \frac{5}{3} \end{bmatrix} & R_1 - \frac{1}{2}R_2 \to R_1
\end{aligned}$$

The matrix is now in reduced row echelon form as required. The amount of leading 1s in the rref of the matrix is known as its *rank*, which equals to 3 here. □

Short Exercise: Repeat the example above but start by interchanging R_1 and R_3.[10]

From the short exercise above, we can see that even if we apply different elementary row operations (particularly for the creation of leading 1s) during Gauss-Jordan Elimination, we will acquire the same reduced echelon form in the end. In fact,

Theorem 2.2.6 (Uniqueness of Reduced Row Echelon Form)**.** Reduced row echelon form of a matrix is unique.

We shall omit the proof here. The following properties further reveal how elementary row operations are associated with reduced row echelon form.

[10]For checking, after the first iteration, it will be (WIP) and the end result will be the same.

Properties 2.2.7. If a matrix can be transformed into another matrix by elementary row operations, they are said to be ***row equivalent***.

Since for any pair of row equivalent matrices, either of them can be transformed into the other one by elementary row operations, and hence can be further transformed into the reduced row echelon form of the other matrix, by Theorem 2.2.6, the uniqueness of rref implies that

Properties 2.2.8. Row equivalent matrices have the same reduced row echelon form. Particularly, they are row equivalent to this rref. If two matrices have different reduced row echelon forms, then they are not row equivalent, and vice versa.

Let's go through one more simple example about Gauss-Jordan Elimination.

Example 2.2.3. Transform the following matrix into reduced row echelon form.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 6 & 4 & 1 \\ 2 & 3 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

Solution. One possible way to do the forward elimination is

$$\begin{bmatrix} 2 & 2 & 1 \\ 6 & 4 & 1 \\ 2 & 3 & 2 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 6 & 4 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 2 \\ 2 & 1 & 0 \end{bmatrix} \qquad R_1 \leftrightarrow R_2$$

$$\to \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{6} \\ 2 & 2 & 1 \\ 2 & 3 & 2 \\ 2 & 1 & 0 \end{bmatrix} \qquad \frac{1}{6}R_1 \to R_1$$

$$\to \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{6} \\ 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{5}{3} & \frac{5}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \qquad \begin{aligned} R_2 - 2R_1 &\to R_2 \\ R_3 - 2R_1 &\to R_3 \\ R_4 - 2R_1 &\to R_4 \end{aligned}$$

$$\rightarrow \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{6} \\ 0 & 1 & 1 \\ 0 & \frac{5}{3} & \frac{5}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \qquad \frac{3}{2}R_2 \to R_2$$

$$\rightarrow \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{6} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} R_3 - \frac{5}{3}R_2 &\to R_3 \\ R_4 + \frac{1}{3}R_2 &\to R_4 \end{aligned}$$

The backward elimination is straight-forward.

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{6} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_1 - \frac{2}{3}R_2 \to R_1$$

The rank of the matrix can be readily seen to be 2. □

2.2.3 Finding Inverses by Gaussian Elimination

With Gaussian Elimination, obtaining the inverse A^{-1} of any invertible matrix A is now possible. We start by writing out an identity matrix I of the same shape and concatenate this identity matrix to the right of A, leading to an augmented form of $[A|I]$. Then we carry out elementary row operations simultaneously on both sides of $[A|I]$ such that the matrix to the left, originally as A, is reduced to the identity matrix I by Gaussian Elimination. The identity matrix to the right will then be transformed into the desired inverse by the same set of elementary operations, such that the concatenated matrix will appear as $[I|A^{-1}]$,

Example 2.2.4. Find the inverse of

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

by Gaussian Elimination

Solution. Appending an 3×3 identity matrix to the right, we have

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2 - 2R_3 \to R_2$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right] \quad R_2 - R_3 \to R_2$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1 & -5 & 10 \\ 0 & 1 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right] \quad R_1 - 5R_3 \to R_1$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right] \quad R_1 - 4R_2 \to R_1$$

Hence the required inverse is

$$A^{-1} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & 3 \\ 0 & 1 & -2 \end{bmatrix}$$

□

Short Exercise: Verify the inverse of A^{-1} above is just A by the same method. [11]

The underlying reason why the above procedure can produce the inverse matrix is the equivalence between elementary row operations and multiplication by appropriate ***elementary matrices***.

[11] You should be able to retrieve the matrix A back. The first column of A^{-1} already holds a leading 1 and elements below which are zeros. A possible next step is to multiply R_2 by -1 and then subtract R_3 by R_2.

Properties 2.2.9 (Elementary Matrices)**.** Any elementary row operation on an $m \times n$ matrix can be represented by multiplying it to the left with a suitable *elementary matrix*. Such a matrix is essentially the one appeared after applying that particular elementary row operation on an identity matrix. For the three types of elementary row operations described in Definition 1.3.1:

1. $cR_p \to R_p$, $c \neq 0$,
2. $R_p + cR_q \to R_p$,
3. $R_p \leftrightarrow R_q$

their corresponding elementary matrices E are square ($m \times m$), and *invertible* (see the following remark) in which

1. $E_{kk} = 1$ for any k, except $E_{pp} = c$;
2. $E_{kk} = 1$ for all k, with $E_{pq} = c$;
3. $E_{kk} = 1$ for any k, except $E_{pp} = 0$ and $E_{qq} = 0$, with $E_{pq} = E_{qp} = 1$.

Entries not mentioned are all zeros.

Since it is quite abstract, it is useful to have some actual examples.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Multiplying } R_2 \text{ by a factor of 2: } 2R_2 \to R_2$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Adding 3 times } R_2 \text{ to } R_1\text{: } R_1 + 3R_2 \to R_1$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \text{Swapping } R_1 \text{ and } R_3\text{: } R_1 \leftrightarrow R_3$$

Any elementary row operation can be apparently undone by an inverse elementary row operation (addition vs subtraction, multiplication vs division ($c \neq 0$), swapping twice). Accordingly, any elementary matrix has another corresponding elementary matrix as its inverse, and the readers are invited to think about their forms in the exercise below.

Short Exercise: Write down the inverses of the three example elementary matrices above.[12]

For instance, consider a matrix

$$\begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

then the action of subtracting R_2 from R_3, $R_3 - R_2 \to R_3$. can be expressed as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 1 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 1 \\ -3 & -5 & 1 \end{bmatrix}$$

Short Exercise: Find out the 3×3 elementary matrix for subtracting 2 times the third row from the first row. What happens when we apply this elementary matrix to the left of the matrix above? [13]

Now we are ready to see why finding inverses by Gaussian Elimination works.

Theorem 2.2.10. If a matrix A can be converted to an identity matrix I as its reduced row echelon form by Gaussian Elimination, then it is invertible since the same steps can in turn be applied on I, producing its inverse A^{-1}.

Using the language of Properties 2.2.8, the matrix A has to be row equivalent to I for A^{-1} to exist. This also means if Gaussian Elimination fails to reduce

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12 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

13 $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 1 \\ -3 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 14 & 1 \\ 2 & 5 & 1 \\ -3 & -5 & 1 \end{bmatrix}$

A to I (i.e. the reduced row echelon form of A is some matrix other than the identity), then A^{-1} does not exist.

Proof. Assume A is invertible and hence $AA^{-1} = I$ (Definition 2.2.1). From Properties 2.2.9, When doing Gaussian Elimination over A, the i-th elementary row operation executed can be represented by an elementary matrix, denoted as E_i, for $i = 1, 2, \ldots, n$ where n is the total number of steps. If we multiply these E_i successively to the left on both sides of the equation $AA^{-1} = I$, we have

$$\begin{aligned}
E_n \cdots E_3 E_2 E_1 A A^{-1} &= E_n \cdots E_3 E_2 E_1 I \\
(E_n \cdots E_3 E_2 E_1 A) A^{-1} &= E_n \cdots E_3 E_2 E_1 I && \text{(Properties 1.1.2)} \\
(I) A^{-1} &= E_n \cdots E_3 E_2 E_1 I \\
A^{-1} &= E_n \cdots E_3 E_2 E_1 I && \text{(Properties 2.1.2)}
\end{aligned}$$

from the second line to the third line, we have $E_n \cdots E_3 E_2 E_1 A = I$ because the elementary row operations during Gaussian Elimination, represented by E_i, $i = 1, 2, \ldots, n$, reduce A to I as we demand in the assumption. With $A^{-1} = E_n \cdots E_3 E_2 E_1 I$, we immediately see that the same set of elementary matrices and hence elementary row operations can also transform I into A^{-1}, explicitly showing that A is invertible. □

As a corollary, because we have $E_n \cdots E_3 E_2 E_1 A = I$ from above, and all E_i are invertible by Properties 2.2.9, we can multiply their inverses $E_i' = E_i^{-1}$ (which are also elementary matrices), to the left on both sides successively, where i runs backwards from n to 1. This leads to

$$\begin{aligned}
E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_n^{-1} E_n \cdots E_3 E_2 E_1 A &= E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_n^{-1} I \\
A &= E_1' E_2' E_3' \cdots E_n'
\end{aligned}$$

as each of the pairs $E_n^{-1} E_n$, $E_{n-1}^{-1} E_{n-1}$, . . ., $E_2^{-1} E_2$, $E_1^{-1} E_1$ cancels out to produce I, and hence

Properties 2.2.11. All invertible matrices can be written as a product of some sequence of elementary matrices.

2.3 Determinants

2.3.1 Computing Determinants

The ***determinant*** of a *square* matrix A, denoted by $\det(A)$ or $|A|$, is a number associated with certain intrinsic properties of the matrix which can help us to find its inverse (Determinant of non-square matrices is undefined). Determinant of a 1×1 matrix is equal to the matrix's only entry. Determinants of 2×2 and 3×3 matrices can be calculated by a trick called ***Sarrus' Rule***.

Sarrus' Rule

Properties 2.3.1 (Sarrus' Rule)**.** Determinants of size 2×2 and 3×3 matrices can be found by the Sarrus' Rule. For a 2×2 matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

Its determinant is computed by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$|A| = a_{11}a_{22} - a_{21}a_{12}$$

which is the product of elements crossed by the red arrow, minus the blue one. Similarly, for a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Its determinant can be found by

$$\left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right.$$

$$\begin{aligned} |A| = &(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) \\ &- (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12}) \end{aligned}$$

Example 2.3.1. Find the determinant of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -5 & 0 & -3 \\ 4 & 3 & 1 \end{bmatrix}$$

Solution. By Sarrus's Rule (Properties 2.3.1), we have

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 2 & 4 \\ -5 & 0 & -3 \\ 4 & 3 & 1 \end{vmatrix} \\ &= ((1)(0)(1) + (2)(-3)(4) + (4)(-5)(3)) \\ &\quad - ((4)(0)(4) + (3)(-3)(1) + (1)(-5)(2)) \\ &= (0 - 24 - 60) - (0 - 9 - 10) \\ &= -65 \end{aligned}$$

□

Cofactor Expansion

Another commonly used method to calculate determinants is *Cofactor Expansion*, also known as *Laplace Expansion*. Before discussing cofactor expansion, it is necessary to know what *cofactors* are.

Definition 2.3.2 (Cofactor and Minor)**.** The ***cofactor*** C_{ij} at the (i, j) position of a matrix A is simply the determinant of the submatrix formed by deleting the i-th row and j-th column of A: M_{ij} (called the ***minor*** at (i, j)), times the factor of $(-1)^{i+j}$, that is, $C_{ij} = (-1)^{i+j} M_{ij}$.

The $(-1)^{i+j}$ factor can be visualized as a checkerboard pattern like

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

So, for a matrix like

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$

Its cofactor at $(2, 1)$ is

$$\begin{aligned} C_{21} &= (-1)^{(2+1)} \begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} && \text{(Definition 2.3.2)} \\ &= (-1)((3)(7) - (5)(5)) && \text{(Properties 2.3.1)} \\ &= 4 \end{aligned}$$

Short Exercise: Find C_{13} and C_{32} for the matrix above.[14]

With ***Cofactor (Laplace) Expansion***, the determinant of a matrix is computed as the sum of products between each entry and the corresponding cofactor along a picked row/column of it.

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[14] $C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = (1)((2)(5) - (3)(4)) = -2$, similarly $C_{32} = 4$.

Properties 2.3.3 (Cofactor/Laplace Expansion)**.** The determinant of a $n \times n$ square matrix A, $|A|$, can be found by selecting either a fixed row i, or column j, and adding up the products of every element-cofactor pair along that row/column. For the former case (selected the i-th row), the determinant is computed as

$$\begin{aligned} |A| &= A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in} \\ &= \sum_{k=1}^{n} A_{ik}C_{ik} \end{aligned}$$

For the latter case (fixed the j-th column), the determinant is similarly found by

$$\begin{aligned} |A| &= A_{1j}C_{1j} + A_{2j}C_{2j} + \cdots + A_{nj}C_{nj} \\ &= \sum_{k=1}^{n} A_{kj}C_{kj} \end{aligned}$$

where each of the cofactors C_{ij} is defined as in Definition 2.3.2. Important: regardless of which row or column is chosen, the result is always the same.

Example 2.3.2. Again, for the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$

Find its determinant via cofactor expansion.

Solution. According to Properties 2.3.3, if we choose the first row to be expanded, its determinant is

$$\begin{aligned} |A| &= A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} \\ &= (1)((-1)^{1+1}\begin{vmatrix} 4 & 6 \\ 5 & 7 \end{vmatrix}) + (3)((-1)^{1+2}\begin{vmatrix} 2 & 6 \\ 3 & 7 \end{vmatrix}) \\ &\quad + (5)((-1)^{1+3}\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}) & \text{(Definition 2.3.2)} \end{aligned}$$

$$= (1)(-2) + (3)(4) + (5)(-2) = 0 \qquad \text{(Properties 2.3.1)}$$

□

Short Exercise: Confirm the answer by carrying out cofactor expansion on another row or column.[15]

Example 2.3.3. Find the determinant of

$$A = \begin{bmatrix} 1 & 4 & 4 & 4 \\ 2 & 0 & 4 & 6 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 3 & 1 \end{bmatrix}$$

Solution. It is a 4×4 matrix and we have to apply cofactor expansion. We can choose row or column that contains some zero(s) to reduce the computation. Here we pick the second column and by Properties 2.3.3, we have

$$\begin{aligned} |A| &= (4)(-1)^{1+2} \begin{vmatrix} 2 & 4 & 6 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{vmatrix} + (0)(-1)^{2+2} \begin{vmatrix} 1 & 4 & 4 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{vmatrix} \\ &+ (1)(-1)^{3+2} \begin{vmatrix} 1 & 4 & 4 \\ 2 & 4 & 6 \\ 6 & 3 & 1 \end{vmatrix} + (2)(-1)^{4+2} \begin{vmatrix} 1 & 4 & 4 \\ 2 & 4 & 6 \\ 2 & 1 & 0 \end{vmatrix} \end{aligned}$$

By Sarrus' Rule (Properties 2.3.1), we can calculate each of the four 3×3 determinants (the detailed calculations are omitted, notice that we don't need to actually compute the second determinant) and obtain

$$|A| = (-4)(-6) + 0 + (-1)(50) + (2)(18) = 10$$

□

Finally, we can derive two simple results about determinants from the perspective of cofactor expansion.

[15] You should able to get $|A| = 0$, no matter which row/column is selected.

Properties 2.3.4. If a matrix have a row/column with full zeros, or two identical/proportional rows/columns, then it has a determinant of zero.

The first case is trivial (just do the expansion along the row/column with full zeros) and we will show the second case alongside the introduction of the properties of determinants in the upcoming subsection.

2.3.2 Properties of Determinants

There are some notable properties about determinants. First of all, it is very easy to see that determinants for any $n \times n$ identity matrix I_n is just 1. Second, there is a close relation between elementary row operations/elementary matrices and (their effects on) determinants, noted as follows.

Properties 2.3.5. The three types of elementary row operations in Definition 1.3.1, when applied on some square matrix A,

1. $cR_p \rightarrow R_p$, $c \neq 0$,
2. $R_p + cR_q \rightarrow R_p$,
3. $R_p \leftrightarrow R_q$,

change the determinant of A by a factor of c, 1 (unchanged), and -1 (switching the sign), respectively.

Properties 2.3.6. The three types of elementary matrices E in Properties 2.2.9 that correspond to the elementary row operations in Definition 1.3.1,

1. $E_{kk} = 1$ for any k, except $E_{pp} = c$ ($cR_p \rightarrow R_p$, $c \neq 0$),
2. $E_{kk} = 1$ for all k, with $E_{pq} = c$ ($R_p + cR_q \rightarrow R_p$),
3. $E_{kk} = 1$ for any k, except $E_{pp} = 0$ and $E_{qq} = 0$, with $E_{pq} = E_{qp} = 1$ ($R_p \leftrightarrow R_q$),

have a determinant of c, 1, and -1, respectively.

Since the determinants of elementary matrices, by Properties 2.3.6, coincide exactly with the factors by how the determinant of a square matrix A changes when the corresponding elementary row operations are applied on A (represented by multiplication to the left of A by these elementary matrices) as shown in Properties 2.3.5, we conclude that

Theorem 2.3.7. For any elementary matrix E and a square matrix A, we have

$$\det(EA) = \det(E)\det(A)$$

This theorem will be of use when we later prove other properties of determinant. However, before doing so, we will demonstrate how to utilize Properties 2.3.5 (or equivalently 2.3.6) to ease the calculation of determinants.

Example 2.3.4. Re-do Example 2.3.3 utilizing Properties 2.3.5.

Solution. We can factor out the 2 in second row and subtract 3 times the third row from the fourth row. By Properties 2.3.5, we have

$$\begin{aligned}
|A| = \begin{vmatrix} 1 & 4 & 4 & 4 \\ 2 & 0 & 4 & 6 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 3 & 1 \end{vmatrix} &= 2\begin{vmatrix} 1 & 4 & 4 & 4 \\ 1 & 0 & 2 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 3 & 1 \end{vmatrix} \\
&= 2\begin{vmatrix} 1 & 4 & 4 & 4 \\ 1 & 0 & 2 & 3 \\ 2 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix}
\end{aligned}$$

The determinant in the last line can be computed by doing cofactor expansion along the fourth row which now contains two zeros. With Properties 2.3.3 and 2.3.1, it is

$$\begin{vmatrix} 1 & 4 & 4 & 4 \\ 1 & 0 & 2 & 3 \\ 2 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix} = 0 + (-1)^{4+2}(-1)\begin{vmatrix} 1 & 4 & 4 \\ 1 & 2 & 3 \\ 2 & 1 & 0 \end{vmatrix} + 0 + (-1)^{4+4}(1)\begin{vmatrix} 1 & 4 & 4 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= 0 + (-1)(9) + 0 + (1)(14) = 5$$

and hence $|A| = 2(5) = 10$. □

With Theorem 2.3.7, we can unearth the relation between invertibility of a square matrix and its determinant.

Properties 2.3.8. An invertible matrix has a non-zero determinant. Otherwise, a singular matrix has a determinant of zero.

Proof. Let's denote the matrix in question as A. For the case in which A is invertible, by Properties 2.2.11 it can be written as the product of some elementary matrices $E_1, E_2, \ldots, E_{n-1}, E_n$, i.e.

$$A = E_1 E_2 \cdots E_{n-1} E_n$$

Taking the determinant of both sides, we have

$$\det(A) = \det(E_1 E_2 \cdots E_{n-1} E_n)$$

By repetitively using Theorem 2.3.7, we have

$$\begin{aligned}
\det(A) &= \det(E_1(E_2 \cdots E_{n-1} E_n)) \\
&= \det(E_1)\det(E_2 \cdots E_{n-1} E_n) \\
&= \det(E_1)\det(E_2)\det(\cdots E_{n-1} E_n) \\
&= \det(E_1)\det(E_2)\cdots\det(E_{n-1})\det(E_n)
\end{aligned}$$

Since by Properties 2.3.6, all elementary matrices have a non-zero determinant (particularly we have required $c \neq 0$ when multiplying a row), i.e. $\det(E_i) \neq 0$ for all i, we have $\det(A) \neq 0$. We will not go through the details for singular matrices, which are put in the footnote below only for reference.[16] □

Other properties of determinants include:

[16] By Theorem 2.2.10, singular matrices have reduced row echelon forms that are not the identity. Observe that all other square rrefs that are not the identity must have at least one row of full zeros, and by Properties 2.3.4 has a determinant of zero.

Properties 2.3.9. For any $n \times n$ square matrices A and B, we have

1. $\det(A^T) = \det(A)$,
2. $\det(kA) = k^n \det(A)$, for any constant k,
3. $\det(AB) = \det(A)\det(B)$, and
4. $\det(A^{-1}) = \dfrac{1}{\det(A)}$, if A is invertible.

By extension, $\det(A_1 A_2 \cdots A_n) = \det(A_1)\det(A_2)\cdots\det(A_n)$.

For instance, if

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 5 \\ 1 & 0 \end{bmatrix}$$

then

$$|A| = (2)(9) - (3)(5) = 3 \qquad |B| = (4)(0) - (5)(1) = -5$$

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 3 \\ 5 & 9 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (2)(4) + (3)(1) & (2)(5) + (3)(0) \\ (5)(4) + (9)(1) & (5)(5) + (9)(0) \end{bmatrix} \\ &= \begin{bmatrix} 11 & 10 \\ 29 & 25 \end{bmatrix} \\ |AB| &= (11)(25) - (10)(29) \\ &= -15 = (3)(-5) = |A||B| \end{aligned}$$

So we can see in this case, $\det(AB) = \det(A)\det(B)$ indeed. We put the formal proof for (3) of Properties 2.3.9 in the footnote for reference.[17]

[17] There are two cases to consider, A being invertible or singular. If A is singular, then by Properties 2.2.4, AB is also singular. And by Properties 2.3.8, both $\det(A)$ and $\det(AB)$

Short Exericse: Prove (4) of Properties 2.3.9.[18]

2.3.3 Finding Inverses by Adjugate

An alternative method to compute the inverse of a matrix is by using its ***adjugate***, which is the transpose of its cofactor matrix associated to it.

Definition 2.3.10 (Adjugate)**.** For a matrix A, its adjugate is defined as

$$[\text{adj}(A)]_{pq} = (C_{pq})^T = C_{qp}$$

where C_{pq} is the cofactor of A at (p, q), formulated as in Definition 2.3.2.

Properties 2.3.11. The inverse of a matrix A can be computed from its adjugate

---

will be zero, and the equality holds trivially. Otherwise, if A is invertible, then we can follow the idea in the proof of Properties 2.3.8, and let $A = E_1E_2 \cdots E_{n-1}E_n$ as a sequence of elementary matrices. By using Theorem 2.3.7 back and forth, we have

$$\begin{aligned}
\det(AB) &= \det(E_1E_2 \cdots E_{n-1}E_nB) \\
&= \det(E_1)\det(E_2)\cdots\det(E_{n-1})\det(E_n)\det(B) && \text{(Theorem 2.3.7)} \\
&= (\det(E_1)\det(E_2)\cdots\det(E_{n-1})\det(E_n))\det(B) \\
&= \det(E_1E_2\cdots E_{n-1}E_n)\det(B) && \text{(Theorem 2.3.7)} \\
&= \det(A)\det(B)
\end{aligned}$$

So the equality is true in both cases.

[18]Consider $A^{-1}A = I$, and take determinant on both sides. By (3), we have

$$\begin{aligned}
\det\left(A^{-1}A\right) &= \det(I) \\
\det\left(A^{-1}\right)\det(A) &= 1 && \text{(The identity always has a determinant of 1)} \\
\det\left(A^{-1}\right) &= \frac{1}{\det(A)}
\end{aligned}$$

by

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$$

From this formula, it is obvious that singular matrices, having a determinant of zero, does not has an inverse.

Example 2.3.5. For a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

It is not difficult to see that the determinant is $ad - bc$, and the adjugate matrix is

$$\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So the inverse, if $ad - bc \neq 0$, is

$$\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 2.3.6. Find the inverse of the following matrix by evaluating its adjugate.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 4 & 11 \end{bmatrix}$$

Solution. First of all, by Sarrus' Rule (Properties 2.3.1)

$$\begin{aligned} |A| &= ((1)(3)(11) + (2)(5)(1) + (3)(1)(4)) \\ &\quad - ((3)(3)(1) + (1)(5)(4) + (2)(1)(11)) \\ &= (33 + 10 + 12) - (9 + 20 + 22) \end{aligned}$$

$$= 4$$

The adjugate matrix is

$$\operatorname{adj}(A) = \begin{bmatrix} \begin{vmatrix} 3 & 5 \\ 4 & 11 \end{vmatrix} & -\begin{vmatrix} 1 & 5 \\ 1 & 11 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 4 & 11 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 11 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 13 & -6 & 1 \\ -10 & 8 & -2 \\ 1 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 13 & -10 & 1 \\ -6 & 8 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

(be careful of not forgetting the transpose!) Putting the pieces together according to the formula in Properties 2.3.11, we have

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \operatorname{adj}(A) \\ &= \frac{1}{4} \begin{bmatrix} 13 & -10 & 1 \\ -6 & 8 & -2 \\ 1 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{13}{4} & -\frac{5}{2} & \frac{1}{4} \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix} \end{aligned}$$

□

A summarizing point to be emphasized is that

Theorem 2.3.12 (Equivalence Statements)**.** For a square matrix A, the followings are equivalent:

(a) A is invertible, i.e. A^{-1} exists,

(b) $\det(A) \neq 0$,

(c) The reduced row echelon form of A is I.

which is just a rephrasing of Properties 2.3.8 and Theorem 2.2.10. Particularly, invertibility is equivalent to a non-zero determinant. We will see the expansion of these equivalence statements in later chapters.

2.4 Python Programming

To create an identity matrix of size n, we use `np.identity(n)`. For example,

```
import numpy as np
I4 = np.identity(4)
print(I4)
```

returns

```
[[1. 0. 0. 0.]
 [0. 1. 0. 0.]
 [0. 0. 1. 0.]
 [0. 0. 0. 1.]]
```

Applying transpose on a matrix is simple where we just add `.T` after the array variable, like

```
myMatrix1 = np.array([[1., 0., 3.],
                      [1., 4., 1.],
                      [-1., 2., 4.]])
print(myMatrix1)
print(myMatrix1.T)
```

yields

```
[[ 1.  0.  3.]
 [ 1.  4.  1.]
 [-1.  2.  4.]]
[[ 1.  1. -1.]
 [ 0.  4.  2.]
 [ 3.  1.  4.]]
```

Finding the inverse of a matrix requires the `scipy.linalg` library and call the `inv` function.

```python
from scipy import linalg
myMatrix2 = linalg.inv(myMatrix1)
print(myMatrix2)
print(myMatrix1@myMatrix2) # Check: should give the identity
```

gives the expected results of

```
[[ 0.4375    0.1875   -0.375   ]
 [-0.15625  0.21875  0.0625 ]
 [ 0.1875  -0.0625   0.125   ]]
[[1. 0. 0.]
 [0. 1. 0.]
 [0. 0. 1.]]
```

Meanwhile, we can use the `det` function to calculate the determinant of a matrix as follows. First,

```python
print(linalg.det(myMatrix1))
```

gives the expected output of `32.0`. As another example,

```python
myMatrix3 = np.array([[3., 1., 3., 2.],
                      [0., -1., -3., 1.],
                      [1., -1., -2., 0.],
                      [2., 0., 1., 0.]])
print(linalg.det(myMatrix3))
```

produces an extremely small value of `1.1102230246251562e-16`. In fact, the matrix

$$\begin{bmatrix} 3 & 1 & 3 & 2 \\ 0 & -1 & -3 & 1 \\ 1 & -1 & -2 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix}$$

has a determinant of exactly zero. It is an artifact of numerical error when using floating point numbers. If we keep going ahead and computes its inverse by `linalg.inv(myMatrix3)`, we will obtain an absurd output of

```
[[ 1.200959e+15 -2.401919e+15  3.602879e+15 -3.602879e+15]
 [ 6.004799e+15 -1.200959e+16  1.801439e+16 -1.801439e+16]
 [-2.401919e+15  4.803839e+15 -7.205759e+15  7.205759e+15]
 [-1.200959e+15  2.401919e+15 -3.602879e+15  3.602879e+15]]
```

that have entries of extremely large magnitude. This phenomenon is due to the extremely small "determinant", through Properties 2.3.11, magnifies the adjugate by being in the denominator. (The actual computation does not use Properties 2.3.11 directly but this is a heuristic perspective to view the problem.) To prevent this, we can add a `if` condition to look for singularity, defining a function like

```
def safe_inv(matrix):
    if np.abs(linalg.det(matrix)) < np.finfo(float).eps:
        print("Warning: The matrix is highly singular!")
        return(np.nan)
    else:
        return(linalg.inv(matrix))
```

where `np.finfo(float).eps` gives the so-called *machine epsilon* ϵ (the order of relative round-off error) of `float` and we want the absolute value of the determinant be larger than that. Subsequently, calling `safe_inv(myMatrix3)` will print a warning. Finally, we note that we can use `sympy` to acquire the reduced row echelon form of a matrix. Let's use the matrix in Example 2.2.3 for demonstration.

```
import sympy

myMatrix4 = np.array([[2., 2., 1.],
                      [6., 4., 1.],
                      [2., 3., 2.],
                      [2., 1., 0.]])
myMatrix4_sympy = sympy.Matrix(myMatrix4) # Convert the numpy
    array to a sympy matrix
print(myMatrix4_sympy.rref())
```

then returns two objects

```
(Matrix([
[1, 0, -0.5],
[0, 1,  1.0],
[0, 0,    0],
[0, 0,    0]]), (0, 1))
```

The first one is the reduced row echelon form we want, and the second is a tuple which keeps the column indices of the pivots. `sympy` also does *zero testing* such that

```
myMatrix3_sympy = sympy.Matrix(myMatrix3)
print(myMatrix3_sympy**(-1))
```

raises properly the error of

```
NonInvertibleMatrixError("Matrix det == 0; not invertible.")
   sympy.matrices.common.NonInvertibleMatrixError: Matrix det
   == 0; not invertible.
```

2.5 Exercises

Exercise 2.1 Find the determinant of the matrix below by inspection.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 10 & 11 & 12 \\ 0 & 0 & 0 & 13 & 14 \\ 0 & 0 & 0 & 0 & 15 \end{bmatrix}$$

By the same logic, derive a general formula for the determinant of any upper(lower)-triangular matrix.[19]

Exercise 2.2 Let

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 6 \\ 0 & 1 \end{bmatrix}$$

Verify:

(a) $(AB)^T = B^T A^T$,

(b) $(AB)^{-1} = B^{-1}A^{-1}$, and

(c) $\det(AB) = \det(A)\det(B)$.

[19] An upper(lower)-triangular matrix is a matrix who elements below (above) the main diagonal are all zeros.

for this particular case.

Exercise 2.3 If

$$A = \begin{bmatrix} 3 & 2 & 9 \\ 1 & 2 & 3 \\ 4 & 0 & 4 \end{bmatrix}$$

Find its inverse by

(a) Gaussian Elimination, and

(b) Determinant and adjugate.

Exercise 2.4 Let

$$A = \begin{bmatrix} 0 & 2 & 5 \\ 0 & 4 & 9 \\ 1 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 4 & 6 \\ 3 & 5 & 8 \end{bmatrix}$$

Verify:

(a) $(AB)^T = B^T A^T$,

(b) $(AB)^{-1} = B^{-1}A^{-1}$, and

(c) $\det(AB) = \det(A)\det(B)$.

for this particular case.

Exercise 2.5 Show that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix}$$

is singular.

Exercise 2.6 Given

$$A = \begin{bmatrix} 1 & 9 & 1 & 4 \\ 0 & 6 & 2 & 8 \\ 1 & 9 & 3 & 9 \\ 0 & 9 & 0 & 1 \end{bmatrix}$$

Find its determinant, inverse, and determinant of the inverse.

Exercise 2.7 For the following matrix,

$$A = \begin{bmatrix} p & 1 & 2 \\ 0 & 2 & p \\ 4 & -2 & 0 \end{bmatrix}$$

Find the values of p such that A is invertible.

Exercise 2.8 Show that for any square matrix A, $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric. Hence show with an explicit formula that any square matrix A can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

Exercise 2.9 Prove that if A is an invertible $n \times n$ matrix, $|A| \neq 0$, then we have

$$\det(\operatorname{adj}(A)) = (\det(A))^{n-1}$$

using Properties 2.3.9 and 2.3.11.

Chapter 3

Solutions for Linear Systems

The last chapter has introduced the necessary machinery for solving linear systems of equations and now we are going to see how to apply them under suitable circumstances. Remember, in the first chapter, we have formulated some problems about linear systems of equations appearing in Earth Science, and they will be solved accordingly.

3.1 Number of Solutions for Linear Systems

Before tackling any linear system, we may like to know there are how many solutions. In fact, there are only three possibilities.

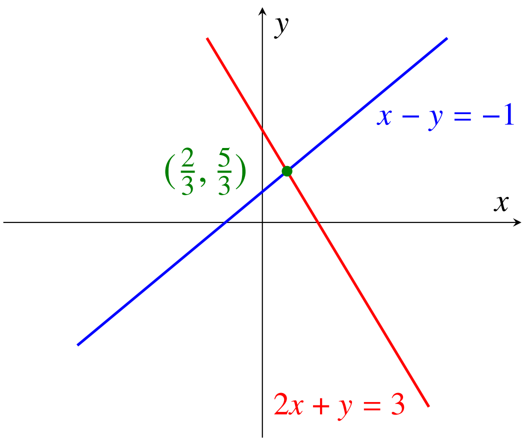
Theorem 3.1.1 (Number of Solutions for a Linear System)**.** For a system of linear equations $A\vec{x} = \vec{h}$ (recall Definition 1.2.2 and Properties 1.2.3), it has either:

1. No solution,
2. An unique solution, or
3. Infinitely many solutions.

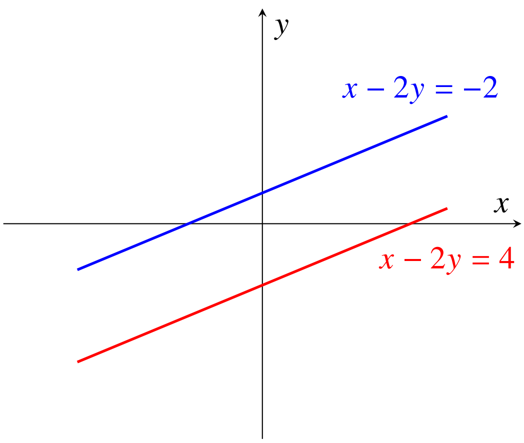
for the unknowns $\vec{x}$.

This can be illustrated by considering a linear system with two equations and two unknowns, with each equation representing a line. There are three types of scenarios.

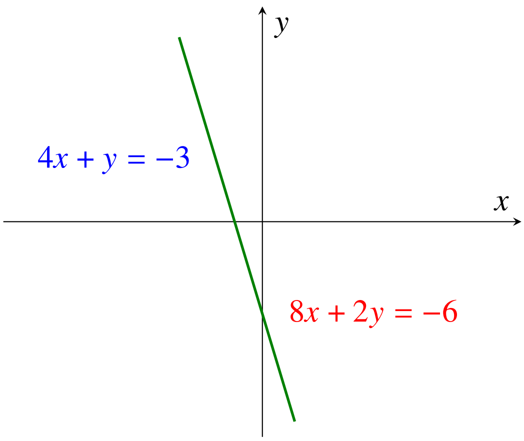
$$\begin{cases} a_1x + b_1y &= h_1 \\ a_2x + b_2y &= h_2 \end{cases}$$



One solution: Two non-parallel lines (red/blue) intersecting at one point (green).

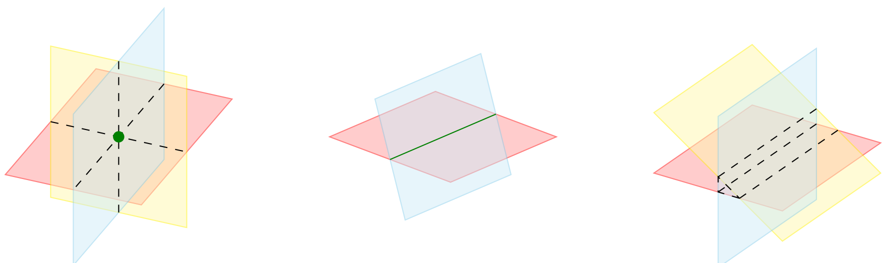


No solution: Two parallel lines never touch each other.



Infinitely many solutions: Two parallel lines overlap each other.

It goes similarly for any linear system of three unknowns in which equations represent planes instead, and the intersection of two non-parallel planes will be a line. We show three possible scenarios below. The readers can try to imagine and drawing out other possibilities.



One solution (left): Three planes (red/yellow/blue) intersecting at one point (green). Infinitely many solutions (middle): Two planes intersecting along a straight line. No solution (right): Three planes intersecting pair-wise along three non-intersecting parallel lines.

In fact, this theorem about the existence of solutions is true for any number of variables and equations.[1] If there is any solution, then the system is called

[1]Some readers may think if there can be finitely many solutions only. Unfortunately, it is

consistent. Otherwise, if no solution exists, then it is known as ***inconsistent***.

Clearly, the next task is about how to find out which case the linear system belongs to. The following theorem reveals the relationship between the number of solutions for a *square* linear system and the determinant of its coefficient matrix.

Theorem 3.1.2. For a square linear system $A\vec{x} = \vec{h}$, if the coefficient matrix A is invertible, i.e. $\det(A) \neq 0$, then there is always only one unique solution. However, if A is singular, $\det(A) = 0$, then it has either no solution, or infinitely many solutions.

As a consequence, if the homogeneous linear system $A\vec{x} = \mathbf{0}$ has as singular coefficient matrix with $\det(A) = 0$, since it always has a trivial solution of $\vec{x} = \mathbf{0}$, the above theorem implies that the homogeneous system must have infinitely many solutions (since it will not have no solution). We defer the arguments for Theorem 3.1.2, as well as the discussion about non-square systems, until we start actually solving linear systems in the next subsection.

Short Exercise: By inspection, determine the number of solutions for the following linear systems.[2]

$$\begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 4 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & 3 \\ 1 & 5 & 2 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3.2 Solving Linear Systems

Now it is the time to get down to solving linear systems (preferably written in matrix form), and we have two methods to choose:

impossible. Assume there are two distinct solutions $\vec{x}_1, \vec{x}_2$ to the system $A\vec{x} = \vec{h}$, then it is easy to show by construction all $\vec{x}_t = t\vec{x}_1 + (1-t)\vec{x}_2$ for any t will be valid solutions which are infinitely many.

[2]These two homogeneous linear system has a determinant of -1 and 0, and hence by Theorem 3.1.2 the first system has a unique solution and the second one has infinitely many solutions.

1. By Gaussian Elimination, for linear system in any shape, or
2. By Inverse, which is apparently only applicable for square, invertible coefficient matrices.

3.2.1 Solving Linear Systems by Gaussian Elimination

Like in Section 2.2.3, applying Gaussian Elimination on the augmented matrix (introduced at the end of Section 1.2) of a linear system can yield the solution to the right. The principle involving elementary row operations is the same as that in Properties 2.2.9 and Theorem 2.2.10, but with $A\vec{x} = \vec{h}$ instead of $AA^{-1} = I$. Let A_{rref} be the reduced row echelon form of A, and $E_1, E_2, \ldots, E_n$ be the elementary matrices used in the Gaussian Elimination process to arrive at the rref. For any solution $\vec{x}$ to the system $A\vec{x} = \vec{h}$, we multiply the elementary matrices one by one to the left on both sides of the equation, leading to

$$\begin{aligned}
(E_n \cdots E_2E_1)A\vec{x} &= (E_n \cdots E_2E_1)\vec{h} \\
(E_n \cdots E_2E_1A)\vec{x} = A_{\text{rref}}\vec{x} &= (E_n \cdots E_2E_1)\vec{h}
\end{aligned}$$

hence $\vec{x}$ will be the solution to $A_{\text{rref}}\vec{x} = \tilde{h}$ at the same time where $\tilde{h} = E_n \cdots E_2E_1\vec{h}$. Therefore, the solutions of $A\vec{x} = \vec{h}$ and $A_{\text{rref}}\vec{x} = \tilde{h}$ coincide, which can be inferred more directly from the latter system. In addition, the coefficient matrix A can be non-square, but we will look at the easier case of a square coefficient matrix first.

Square Systems

Example 3.2.1. Solve the following linear system by Gaussian Elimination.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 4 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 8 \end{bmatrix}$$

Solution. We rewrite the system in augmented form and then apply Gaussian Elimination, with the aim to reduce the coefficient matrix on the left to the identity matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 1 & 1 & 4 & 10 \\ 2 & 0 & 3 & 8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{r} R_2 - R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{r} R_2 - 2R_3 \to R_2 \\ R_1 - R_3 \to R_1 \end{array}$$

which translates to

$$\begin{cases} x = 1 \\ y = 1 \\ z = 2 \end{cases} \qquad \text{or} \qquad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Note that we have successfully converted the coefficient matrix to the identity along the way, which by Theorem 2.3.12 (c) to (a) implies that the coefficient matrix is invertible. This explains the first part of Theorem 3.1.2 as every unknown is now associated to a single leading 1 in the corresponding column of the identity matrix acquired from the reduction process and a unique solution can be derived. □

Example 3.2.2. Solve the linear system

$$\begin{bmatrix} 3 & 7 & 2 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$$

if possible.

Solution. Again, we apply Gaussian Elimination on the augmented matrix to obtain

$$\left[\begin{array}{ccc|c} 3 & 7 & 2 & 8 \\ 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 3 & 7 & 2 & 8 \\ 0 & 2 & 1 & 2 \end{array}\right] \qquad R_1 \leftrightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 4 & 2 & 2 \\ 0 & 2 & 1 & 2 \end{array}\right] \qquad R_2 - 3R_1 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 1 & 2 \end{array}\right] \qquad \frac{1}{4}R_2 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad R_3 - 2R_2 \rightarrow R_3$$

The last row corresponds to $0 = 1$ which is contradictory. As a consequence, the system is inconsistent, i.e. no solution exists. □

Example 3.2.3. Find all solutions for the following linear system.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Solution. Gaussian Elimination leads to

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 5 & 3 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \qquad R_2 - 2R_1 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{c} R_3 - R_2 \rightarrow R_3 \\ R_1 - 2R_2 \rightarrow R_1 \end{array}$$

Now, the last row corresponds to $0 = 0$, which is vacuous and implies that one equation is spurious. This also means we can assign an unknown as a ***free variable (parameter)*** for expressing other variables. We will choose unknown(s) that is/are not linked to any pivot in the reduced coefficient matrix. As the variables x and y already correspond to the two pivots in the first/second column, we can let $z = t$ where t represents a free parameter. Then the first/second row gives $x = 1 + t$, $y = -t$ respectively, and

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

with $-\infty < t < \infty$. The first column vector appearing alone

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is the so-called ***particular solution***, which can be any vector $\vec{x} = \vec{x}_p$ that satisfies the inhomogeneous part of the system $A\vec{x} = \vec{h}$. Meanwhile, the second column vector multiplied by the free parameter t

$$t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

is known as the ***complementary solution***, the family of all vectors $\vec{x} = \vec{x}_c$ that satisfy the homogeneous part $A\vec{x} = \mathbf{0}$. Combined together, they form the ***general solution*** $\vec{x}_g = \vec{x}_p + \vec{x}_c$ as the complete set of solutions to the linear system. □

Short Exercise: Try plugging in any number t to the general solution and verify the consistency.[3]

[3] Let's say $t = 1$ and $\tilde{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, then clearly $A\tilde{x} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. $\tilde{x}$ can become a new particular solution by noting that the original solution form can be rewritten as

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + (t-1) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + t' \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \tilde{x} + t' \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

where we "extract" $\tilde{x}$ from shifting the free parameter via $t' = t - 1$, and it represents the same general solution as the original expression.

The complementary solution encompasses all possible solutions to the homogeneous part $A\vec{x} = \mathbf{0}$ of the linear system $A\vec{x} = \vec{h}$. For broader situations, it can contain more than one pairs of free parameter and column vector (or none, if the homogeneous part only permits the trivial solution of all zeros), and the complementary solution becomes a *linear combination* of multiple *linearly independent* column vectors who satisfy $A\vec{x} = \mathbf{0}$ on their own. (We will clarify about these concepts in Chapter 6.) The amount of free variables is decided by the number of columns in the coefficient matrix (unknowns), minus the number of pivots (constraints) in its reduced row echelon form. This quantity is called *nullity* and in the last example it equals to 1. In case of multiple free variables, we assign the corresponding number of free parameters to non-pivotal unknowns and apply the same procedure as in the example above to acquire a set of complementary solution. Any column vector that constitutes the complementary solution (followed by a free parameter) can be scaled by any non-zero factor as we desire.[4]

Meanwhile, the particular solution can be set to any valid solution to the linear system (the choice does not affect the structure of its complementary part, see the footnote to the last short exercise). If the linear system is itself homogeneous, then the zero vector $\mathbf{0}$ can always be chosen as a particular solution which does not appear explicitly.

We have seen in the previous two examples that if the reduced row echelon form of the square coefficient matrix has some row of full zeros, then

[4]Using the last example as a demonstration,

$$\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{t}{2}\begin{bmatrix}2\\-2\\2\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} + s\begin{bmatrix}2\\-2\\2\end{bmatrix}$$

where we use $s = \frac{t}{2}$ as a new free parameter. Notice that the old column vector $\begin{bmatrix}1\\-1\\1\end{bmatrix}$ in the original expression of complementary solution, and the newly generated column vector $\begin{bmatrix}2\\-2\\2\end{bmatrix}$ that is double in length, both satisfy the homogeneous part $A\vec{x} = 0$.

it either leads to no solution (if inconsistent) or infinitely many solutions (if consistent). Since such a matrix at the same time has a determinant of zero (by Properties 2.3.4) and is singular, this establishes the second part of Theorem 3.1.2.

For non-square coefficient matrices, two cases occur.

1. There are more equations (rows) than unknowns (columns). The system is ***overdetermined***. The rref of coefficient matrix then must have at least one row of full zeros. If any one of them is inconsistent, then contradiction will arise just like in Example 3.2.2 and there will be no solution. However, if all zero rows are consistent (i.e. $0 = 0$), then there still can be a unique solution or infintely many of them.

2. There are fewer equations (rows) than unknowns (columns). The system is said to be ***underdetermined***. There must be unknown(s) that is/are non-pivot(s) in the reduced row echelon form of the coefficient matrix. Hence free variables, and infinitely many solutions ensue if there is no inconsistent row (if there is at least one row of $0 = k$ where k is a non-zero constant, then there is no solution). The calculation is similar to that in Example 3.2.3.

Let's see some examples for non-square linear systems.

Overdetermined Systems

Example 3.2.4. Find the solution to the following overdetermined system, if any.

$$\begin{bmatrix} 1 & 4 & 0 \\ 2 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 3 \\ 5 \end{bmatrix}$$

Solution.

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 4 \\ 2 & 2 & 3 & 8 \\ 1 & 1 & 2 & 3 \\ 0 & 3 & 1 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 0 & 4 \\ 0 & -6 & 3 & 0 \\ 0 & -3 & 2 & -1 \\ 0 & 3 & 1 & 5 \end{array}\right] \qquad \begin{array}{r} R_2 - 2R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 4 & 0 & 4 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -3 & 2 & -1 \\ 0 & 3 & 1 & 5 \end{array}\right] \qquad -\frac{1}{6}R_2 \to R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 4 & 0 & 4 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & \frac{5}{2} & 5 \end{array}\right] \qquad \begin{array}{r} R_3 + 3R_2 \to R_3 \\ R_4 - 3R_2 \to R_4 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 4 & 0 & 4 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & \frac{5}{2} & 5 \end{array}\right] \qquad 2R_3 \to R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & 4 & 0 & 4 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 10 \end{array}\right] \qquad R_4 - \frac{5}{2}R_3 \to R_4$$

The last row is inconsistent and hence the overdetermined system has no solution. □

Example 3.2.5. Show that there are infinitely many solution to the following overdetermined system.

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \end{bmatrix}$$

Solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 1 & 2 & 5 & 3 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -3 & -1 \\ 0 & -1 & -3 & -1 \end{array}\right] \qquad \begin{array}{r} R_2 - R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \\ R_4 - R_1 \to R_4 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{r} R_3 + R_2 \to R_3 \\ R_4 + R_2 \to R_4 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_1 - R_2 \to R_1$$

The two rows of full zeros indicate that two out of the four equations are redundant and there are effectively two constraints only, over the three variables. We can let the non-pivotal unknown $z = t$ be a free variable like in Example 3.2.3, and obtain $x = 1 + t$, $y = 1 - 3t$ from the first two rows. Thus the general solution is

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 + t \\ 1 - 3t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

where $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is a particular solution and the nullity is 1. □

Underdetermined Systems

Example 3.2.6. Solve the following underdetermined system.

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Solution.

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \\ 1 & 0 & 3 & 2 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & -1 & 1 & 1 & -1 \end{array}\right] \quad \begin{array}{l} R_2 - R_1 \longrightarrow R_2 \\ R_3 - R_1 \longrightarrow R_3 \end{array}$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 + R_2 \longrightarrow R_3$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 - R_2 \longrightarrow R_1$$

The last zero row is consistent. The first two columns are pivotal and we can let the remaining two unknowns that are not associated to any leading 1, $u = s$ and $v = t$ be free variables. From the first two equations, we retrieve $x = -1 - 3s - 2t$ and $y = 1 + s + t$, and therefore the general solution is

$$\vec{x} = \begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} = \begin{bmatrix} -1 - 3s - 2t \\ 1 + s + t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

with $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ as a particular solution and the nullity being 2. □

3.2.2 Solving Linear Systems by Inverse

For a square linear system $A\vec{x} = \vec{h}$, if A has a non-zero determinant and is invertible, then we can apply its inverse to recover the solution. Remember that multiplying a matrix by its inverse returns an identity matrix, hence it is possible to multiply the inverse A^{-1} (or heuristically, "dividing" by A) to the left on both sides of the equation $A\vec{x} = \vec{h}$ to cancel out the A on L.H.S., which reads

$$\begin{aligned} A^{-1}A\vec{x} = (A^{-1}A)\vec{x} &= A^{-1}\vec{h} \\ \vec{x} = I\vec{x} &= A^{-1}\vec{h} \quad \text{(Definition 2.2.1 and Properties 2.1.2)} \end{aligned} \tag{3.1}$$

This solution is unique, guaranteed by Theorem 3.1.2.

Example 3.2.7. Solve the linear system $A\vec{x} = \vec{h}$ by the inverse method, where

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix} \qquad \vec{h} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

Solution. It can be checked that the inverse of the coefficient matrix is

$$A^{-1} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

The readers are encouraged to derive the inverse by themselves. Subsequently, we have the solution to the linear system as

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\vec{h} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

□

Doing Gaussian Elimination to find the inverse and then compute the solution by $\vec{x} = A^{-1}\vec{h}$ (Equation (3.1)) is, at first sight somehow the same as using Gaussian Elimination directly to solve the linear system as sketched in Section 3.2.1. However, in computer, calculation of inverse can be unstable (see Section 2.4) and there are some other practical reasons not to take the first approach, which will be discussed in Section 3.4.

Besides, Theorem 2.3.12 can be extended as below by incorporating Theorem 3.1.2:

Theorem 3.2.1 (Equivalence Statement, ver. 2)**.** For a square matrix A, the followings are equivalent:

(a) A is invertible, i.e. A^{-1} exists,

(b) $\det(A) \neq 0$,

(c) The reduced row echelon form of A is I,

(d) The linear system $A\vec{x} = \vec{h}$ has a unique solution for any $\vec{h}$, particularly $A\vec{x} = \mathbf{0}$ has only the trivial solution $\vec{x} = \mathbf{0}$.

Cramer's Rule

3.3 Earth Science Applications

Now we are going to revisit and find the solutions to the two linear system problems in Section 1.4.

Example 3.3.1. Solve for the horizontal displacement x and depth of top layer y in the seismic ray problem of Example 1.4.1.

Solution. The linear system is

$$\begin{bmatrix} 1 & 1 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 120 \\ 80\sqrt{3} \end{bmatrix}$$

Since it is just a 2×2 coefficient matrix, we can directly use the expression in Example 2.3.5 to find its inverse, which is

$$\frac{1}{\sqrt{3}-1} \begin{bmatrix} \sqrt{3} & -1 \\ -1 & 1 \end{bmatrix} = \frac{1+\sqrt{3}}{2} \begin{bmatrix} \sqrt{3} & -1 \\ -1 & 1 \end{bmatrix}$$

and solve the system by multiplying the inverse following the method demonstrated in Section 3.2.2, leading to

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1+\sqrt{3}}{2} \begin{bmatrix} \sqrt{3} & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 120 \\ 80\sqrt{3} \end{bmatrix} = \begin{bmatrix} 60 + 20\sqrt{3} \\ 60 - 20\sqrt{3} \end{bmatrix}$$

Therefore the required horizontal displacement and depth of top layer are about 94.6 m and 25.4 m respectively. □

Example 3.3.2. Find the radiative loss E_j and hence temperature T_j in each layer of the multi-layer model in Example 1.4.2. In particular, what is the temperature at the surface ($j = N + 1$)?

Solution. The linear system is

$$\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & & 0 & 0 & 0 \\ 0 & 1 & -2 & & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & & -2 & 1 & 0 \\ 0 & 0 & 0 & & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_{N-1} \\ E_N \\ E_{N+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -E_{in} \end{bmatrix}$$

where N is any positive integer. Since N can be arbitrarily large, we may wish to avoid the direct computation of a massive inverse. Instead, we resort to a tactful way of row reduction to reveal the pattern of R_j. Rather than starting the reduction at the top as usual, we build up at the bottom, subtracting the lower row from the row directly above it and then moving up a row, repeated until we reach the top.

$$
\left[\begin{array}{ccccccc|c}
-2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & & 0 & 0 & 0 & 0 \\
\vdots & & & \ddots & & & \vdots & \vdots \\
0 & 0 & 0 & & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & -1 & -E_{in}
\end{array}\right]
$$

$$
\rightarrow \left[\begin{array}{ccccccc|c}
-2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & & 0 & 0 & 0 & 0 \\
\vdots & & & \ddots & & & \vdots & \vdots \\
0 & 0 & 0 & & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & & 1 & -1 & 0 & -E_{in} \\
0 & 0 & 0 & \cdots & 0 & 1 & -1 & -E_{in}
\end{array}\right] \qquad R_N + R_{N+1} \rightarrow R_N
$$

$$
\rightarrow \left[\begin{array}{ccccccc|c}
-2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & & 0 & 0 & 0 & 0 \\
\vdots & & & \ddots & & & \vdots & \vdots \\
0 & 0 & 0 & & -1 & 0 & 0 & -E_{in} \\
0 & 0 & 0 & & 1 & -1 & 0 & -E_{in} \\
0 & 0 & 0 & \cdots & 0 & 1 & -1 & -E_{in}
\end{array}\right] \qquad R_{N-1} + R_N \rightarrow R_{N-1}
$$

$$
\rightarrow \vdots \qquad \text{(Keep going up)}
$$

$$\rightarrow \left[\begin{array}{ccccccc|c} -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & & 0 & 0 & 0 & -E_{in} \\ \vdots & & & \ddots & & & \vdots & \vdots \\ 0 & 0 & 0 & & -1 & 0 & 0 & -E_{in} \\ 0 & 0 & 0 & & 1 & -1 & 0 & -E_{in} \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 & -R_{in} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccccc|c} -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & & 0 & 0 & 0 & -E_{in} \\ 0 & 1 & -1 & & 0 & 0 & 0 & -E_{in} \\ \vdots & & & \ddots & & & \vdots & \vdots \\ 0 & 0 & 0 & & -1 & 0 & 0 & -E_{in} \\ 0 & 0 & 0 & & 1 & -1 & 0 & -E_{in} \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 & -E_{in} \end{array}\right] \qquad R_2 + R_3 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccccccc|c} -1 & 0 & 0 & \cdots & 0 & 0 & 0 & -E_{in} \\ 1 & -1 & 0 & & 0 & 0 & 0 & -E_{in} \\ 0 & 1 & -1 & & 0 & 0 & 0 & -E_{in} \\ \vdots & & & \ddots & & & \vdots & \vdots \\ 0 & 0 & 0 & & -1 & 0 & 0 & -E_{in} \\ 0 & 0 & 0 & & 1 & -1 & 0 & -E_{in} \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 & -E_{in} \end{array}\right] \qquad R_1 + R_2 \rightarrow R_1$$

From the first row, we readily obtain $E_1 = E_{in}$. The second row yields the equation

$$\begin{aligned} E_1 - E_2 &= -E_{in} \\ E_2 &= E_1 + E_{in} = E_{in} + E_{in} = 2E_{in} \end{aligned}$$

Similarly, the subsequent rows are all in the form of $E_j = E_{j-1} + E_{in}$, and inductively we have $E_j = jE_{in}$. $E_1 = E_{in}$ is the emission of radiation from the Earth as a whole as viewed from the space, and the corresponding *emission temperature* is $T_e = T_1 = \sqrt[4]{E_1/\sigma} = \sqrt[4]{E_{in}/\sigma}$ by Stefan–Boltzmann Law. The surface releases terrestrial radiation at the rate of $E_{N+1} = (N+1)E_{in}$ and has a temperature of $T_{N+1} = \sqrt[4]{E_{N+1}/\sigma} = \sqrt[4]{(N+1)E_{in}/\sigma} = (N+1)^{1/4}\sqrt[4]{E_{in}/\sigma} =$

$(N+1)^{1/4}T_e$, i.e. the surface temperature is $(N+1)^{1/4}$ times the emission temperature. Our earth has an emission temperature of 255 K and a surface temperature of 288 K on average (notice that we have to use Kelvin instead of degree Celsius!), which leads to an effective number of absorbing layers $N = (288/255)^4 - 1 = 0.627$. □

3.4 Python Programming

For solving square linear systems in the form of $A\vec{x} = \vec{h}$, we can again import the `scipy.linalg` library and call the `solve` function with the coefficient matrix A as the first argument and $\vec{h}$ placed in the second one.

```
import numpy as np
from scipy import linalg

A = np.array([[1., 0., 1.],
              [2., 2., 3.],
              [1., 2., 0.]])
h = np.array([0., -1., 1.])
x = linalg.solve(A,h)
```

This corresponds to the linear system

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

which has a solution of

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

`print(x)` then gives the correct output of `[ 1. -0. -1.]`. However, if A is a singular matrix like the one shown in Section 2.4

```
A = np.array([[3., 1., 3., 2.],
              [0., -1., -3., 1.],
              [1., -1., -2., 0.],
              [2., 0., 1., 0.]]) # "myMatrix3" in the last
                 chapter
h = np.array([0., 1., 1., -1.])
x = linalg.solve(A,h)
print(x)
```

raises a warning and an unreasonable output of

```
LinAlgWarning: Ill-conditioned matrix
(rcond=3.42661e-18): result may not be accurate.
  x = linalg.solve(A,h)
[ 4.803839e+15  2.401919e+16 -9.607679e+15 -4.803839e+15]
```

Again, we can use the `sympy` package for the rescue as follows.

```
import sympy

A_sympy = sympy.Matrix(A)
h_sympy = sympy.Matrix(h)
A_sympy.solve(h_sympy)
```

which raises the same "not invertible" error as in Section 2.4. We note that, unfortunately, there is no simple way to deal with over/under-determined systems using either `scipy` or `sympy`. Moreover, there are two questions that may come to the curious readers when reading the programming sections of these two chapters. First, which of `scipy` and `sympy` should we choose over another? Second, why we don't compute the inverse of A and solve the system by something along the line of `x = linalg.inv(A) @ h`? For the first question, we note that `scipy` is numerical while `sympy` is symbolic, which means that if we are dealing with real data we may find `scipy` adequate and more efficient, while if we are focusing on the theoretical part of Mathematics we can obtain a more analytical solution with `sympy`. To the second question, we refer the readers to this excellent Stack Overflow post (31256252).

3.5 Exercises

Exercise 3.1 Solve the following linear system.

$$\begin{cases} 5x + y + 3z & = 6 \\ 2x - y + z & = \frac{7}{2} \\ 3x + 2y - 4z & = -\frac{13}{2} \end{cases}$$

Exercise 3.2 Solve $A\vec{x} = \vec{h}_k$, where

$$A = \begin{bmatrix} 6 & 7 & 7 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\vec{h}_1 = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \qquad \vec{h}_2 = \begin{bmatrix} 19/4 \\ 1 \\ 5/4 \end{bmatrix}$$

Exercise 3.3 Derive the solution to the following linear system.

$$\begin{cases} 3x + 4z & = 2 \\ x + y + 2z & = -1 \\ x - 2y & = 0 \end{cases}$$

Exercise 3.4 Solve the following linear system.

$$\begin{cases} m + n - p - 3q & = 2 \\ m - q & = 5 \\ 3m + 2n - 2p - 7q & = 9 \end{cases}$$

How about if the R.H.S. of the third equation is equal to 3 instead?

Exercise 3.5 For the following linear system,

$$\begin{bmatrix} 1 & 0 & \alpha \\ 0 & \alpha & 0 \\ \alpha & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ \alpha \end{bmatrix}$$

Find the values of α so that the system has no solution, or infinitely many solutions.

Exercise 3.6 *Ohm's law* relates voltage drop of a current due to resistance by $V = IR$. In addition, *Kirchhoff's Second Law* states that: The voltage gain balances the voltage drop around any closed loop (net voltage change must be zero). The clockwise convention is adopted, i.e. around a loop, a battery with its positive terminal facing the clockwise direction is considered a voltage gain, and clockwise current passing through a resistor is deemed as a voltage drop. Together with the knowledge that current at a junction must conserve (*Kirchhoff's First Law*), find I_1, I_2, I_3 (assumed flowing in the direction as indicated) for the circuit in Figure 3.1.

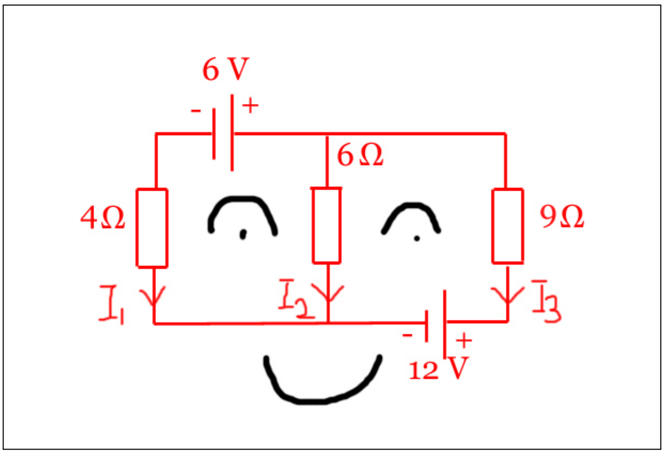


Figure 3.1: The circuit for Exercise 3.6

You will obtain two equations by considering any two loops with Kirchhoff's Second Law, and one from Kirchhoff's First Law. So, there are three equations, for the three unknown currents.

Exercise 3.7 The *shallow water equations* (see Figure 3.2) describe the evolution of gravity wave under some approximations such as *hydrostatic balance* and a sufficiently shallow fluid depth, and has the form of

$$
\begin{cases}
\dfrac{\partial \eta}{\partial t} + H(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}) & = 0 \\
\dfrac{\partial u}{\partial t} & = -g\dfrac{\partial \eta}{\partial x} \\
\dfrac{\partial v}{\partial t} & = -g\dfrac{\partial \eta}{\partial y}
\end{cases}
$$

when the Coriolis effect is ignored. By assuming a travelling wave solution

$$
\begin{aligned}
u &= \tilde{U}\cos(kx + ly - \omega t) \\
v &= \tilde{V}\cos(kx + ly - \omega t) \\
\eta &= \tilde{\eta}\cos(kx + ly - \omega t)
\end{aligned}
$$

where $\tilde{U}$, $\tilde{V}$, $\tilde{\eta}$ are some constants to be determined, show that the equations become

$$
\begin{cases}
\omega\tilde{\eta} - kH\tilde{U} - lH\tilde{V} & = 0 \\
\omega\tilde{U} - kg\tilde{\eta} & = 0 \\
\omega\tilde{V} - lg\tilde{\eta} & = 0
\end{cases}
$$

By requiring that $\tilde{U}$, $\tilde{V}$, $\tilde{\eta}$ have a non-trivial solution so that they are not all zeros, derive the dispersion relation of gravity wave, which is

$$
\begin{aligned}
\omega^2 &= gH(k^2 + l^2) \\
\omega &= c\kappa
\end{aligned}
$$

where $c = \sqrt{gH}$ is the wave speed, and $\kappa = \sqrt{k^2 + l^2}$ is the total wavenumber.

Exercise 3.8 Solve for the condensation height and temperature z_{cd} and T_{cd} in Exercise 1.8.

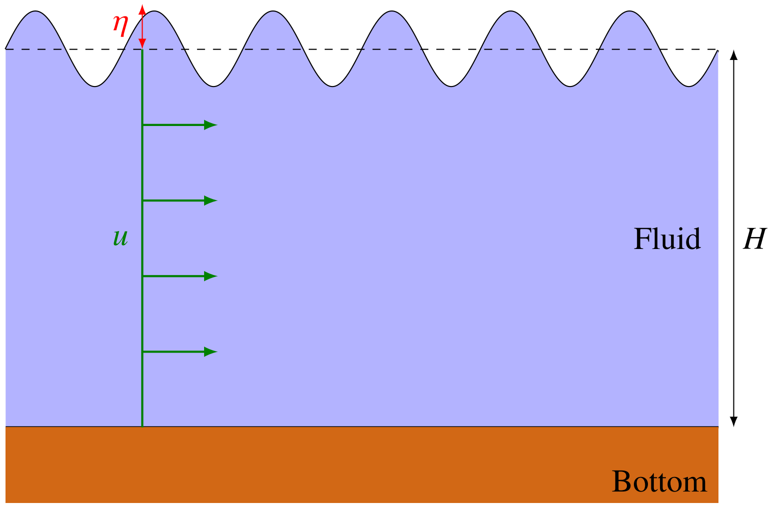


Figure 3.2: The x-z cross-section of shallow water system in Exercise 3.7. η is the height of free surface, H is the mean depth of the fluid, and u is the fluid velocity along x-axis.

Exercise 3.9 Solve the *Chickens and Rabbits in the Same Cage* problem in Exercise 1.9. If we now introduce a new type of mystical creature who has one head and three legs, and throw them in another cage along with some chickens and rabbits, find all possible numbers of the three species if the cage now has 48 heads and 122 legs.

Chapter 4

Introduction to Vectors

After three chapters of discussion about matrices, it is time to talk about another closely related object type in linear algebra, namely, vectors. While *vectors* and *vector spaces* have strictly mathematical definitions which make them abstract, we will take a more physical point of view with the special case of (finite-dimensional) geometric vectors first.

4.1 Definition and Operations of Geometric Vectors

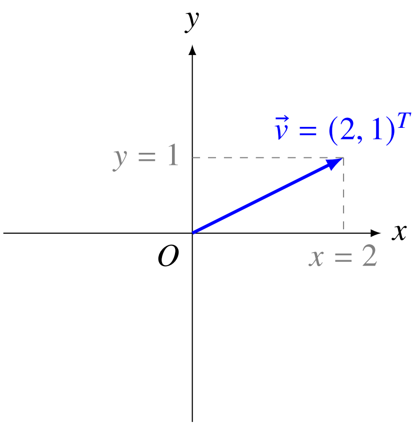
4.1.1 Basic Structure of Vectors in the Real n-space $\mathbb{R}^n$

A ***(geometric) vector*** is a physical quantity represented by an ordered tuple of *components* (numbers), e.g. $(1, 8, 7, 4)$, $(1 - \imath, 1 + 3\imath, 2)$. It has a *magnitude (length)* and *direction*, resembling an arrow. Some real-life examples are: two-dimensional flow velocity (u, v), relative position of an airplane to a ground radar (x, y, z).

Definition 4.1.1 (n-dimensional Geometric Vector)**.** A n-dimensional geometric vector consists of n ordered elements called ***components*** and are denoted by either an arrow or boldface, like $\vec{v}$ or $\mathbf{v}$. It is usually written out in two forms, as a *column vector* or an *ordered n-tuple*:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = (v_1, v_2, v_3, \ldots, v_n)^T$$

A n-dimensional vector can be treated as an $n \times 1$ (***column vector***) as suggested above, or a $1 \times n$ matrix (***row vector***) depending on the situation. The form of a column vector is taken more often than the row vector one and so the column form is assumed throughout the book if it is not further specified. That is why the superscript ^{T} is added for the n-tuple form to reflect that it is in fact a column vector despite written horizontally.



A 2D vector drawn in an x-y plane.

Movement 移動速度和方向	1-min Average Strength 一分鐘平均強度		Distance/Bearing from HK 與香港的距離和方位角
WNW 西北偏西 (288°) 18 km/h	70 kt (130 km/h)	TY (Cat. 1) 一級颱風	SSE 東南偏南 116 km
WNW 西北偏西 (289°) 20 km/h	70 kt (130 km/h)	TY (Cat. 1) 一級颱風	WSW 西南偏西 178 km

Forecast for *Typhoon Higos* (taken from Hong Kong Weather Watch). Its horiztonal movement is a two-dimensional vector, even though the speed and direction are given instead of the velocities in x and y-direction (they can be converted to each other).

Implicit in the definition of n-dimensional vectors is the n-dimensional *space* they are residing in. Assume the components of those vectors are all real, then the set of all such vectors constitutes the ***real n-space*** $\mathbb{R}^n$.

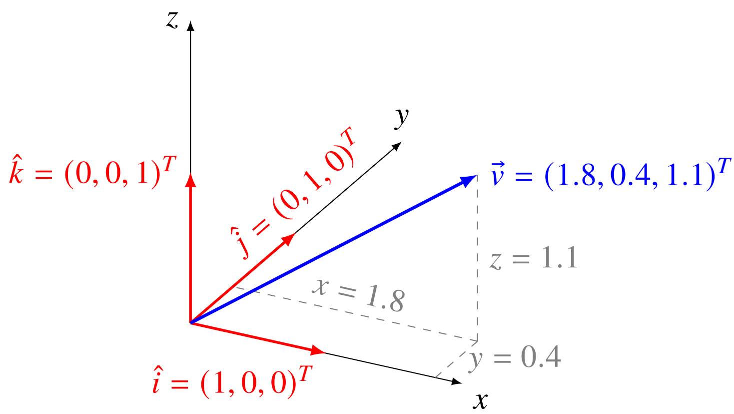
Definition 4.1.2 (The Real n-space $\mathbb{R}^n$)**.** The real n-space $\mathbb{R}^n$ is defined as the set of all possible n-tuples $\vec{v} = (v_1, v_2, v_3, \ldots, v_n)^T$ as defined in Definition 4.1.1, where v_i can take any *real* value, for $i = 1, 2, 3, \ldots, n$. Such objects in $\mathbb{R}^n$ are known as n-dimensional *real* vectors.

While we have not clearly defined what a vector space is, we note that $\mathbb{R}^n$ fulfills the requirements of a vector space in a mathematical sense. A more detailed discussion of this aspect will be deferred to Chapter 6. Meanwhile, the complex counterpart will be explored in Chapter 8.

An n-dimensional real geometric vectors as described in Definition 4.1.1 and 4.1.2 can be written as the sum of n ***standard unit vectors*** that have a magnitude of 1 and are oriented in the positive direction along the p-th coordinates axes. They are denoted by $\hat{e}_p$, where p can be from 1 to n. The coordinate axes are perpendicular (or more generally, *orthogonal*, introduced later in this chapter) to each other and this coordinate system is known as the ***Cartesian (coordinate) system***. Particularly, in the three-dimensional real space $\mathbb{R}^3$, $\hat{e}_1 = \hat{\imath} = (1, 0, 0)^T$, $\hat{e}_2 = \hat{\jmath} = (0, 1, 0)^T$, $\hat{e}_3 = \hat{k} = (0, 0, 1)^T$ correspond to "an arrow" of length 1 pointing in the positive direction of the x, y, z axes respectively.

Definition 4.1.3 (Standard Unit Vector)**.** A standard unit vector $\hat{e}_p$ in the real n-space $\mathbb{R}^n$ (Definition 4.1.2) has n components, consisted of 1 at the p-th entry and 0 elsewhere. Mathematically, for $1 \leq q \leq n$, $[\hat{e}_p]_q = 1$ when $q = p$ and $[\hat{e}_p]_q = 0$ when $q \neq p$.

Below is an example of a geometric vector in the three-dimensional xyz space ($\mathbb{R}^3$).



$$\vec{v} = \begin{bmatrix} 1.8 \\ 0.4 \\ 1.1 \end{bmatrix} = 1.8 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0.4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1.1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1.8\hat{\imath} + 0.4\hat{\jmath} + 1.1\hat{k}$$
$$= (1.8, 0.4, 1.1)^T$$

where we have written $\vec{v}$ in two forms, as an n-tuple and a sum of the three standard unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$.

4.1.2 Fundamental Vector Operations

Addition and Subtraction

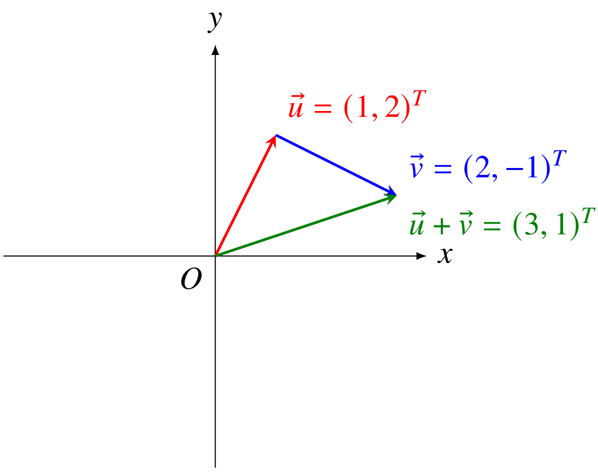
Same as their matrix counterpart, addition and subtraction between vectors are element-wise, and hence only valid for vectors of the same dimension. For $\vec{w} = \vec{u} \pm \vec{v}$, we have $w_i = u_i \pm v_i$. If

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

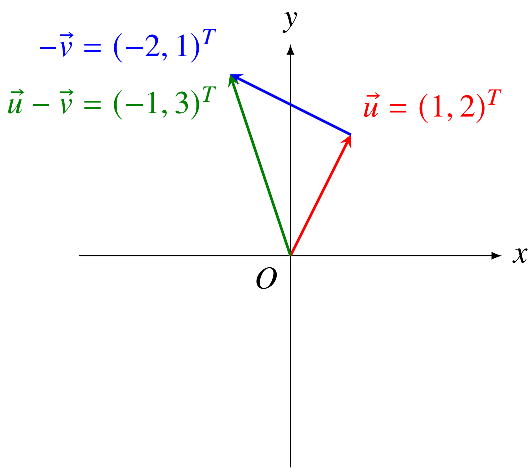
then

$$\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\vec{u} - \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$



Addition: The tail of the blue vector is placed to the head of the red vector, and the resultant green vector is from the origin to the head of blue vector.



Subtraction: Similar to addition but with the blue vector oriented in the opposite direction.

Scalar Multiplication

Multiplying a scalar (be it a real or complex number) to a vector means that all components are multiplied by that scalar.

$$2\begin{bmatrix}2\\0\\1\\9\end{bmatrix}=\begin{bmatrix}4\\0\\2\\18\end{bmatrix}$$

Looking back at vector subtraction, it can be viewed as addition with a factor of -1 in the front.

$$\begin{bmatrix}7\\5\\9\end{bmatrix}-\begin{bmatrix}3\\6\\9\end{bmatrix}=\begin{bmatrix}7\\5\\9\end{bmatrix}+(-1)\begin{bmatrix}3\\6\\9\end{bmatrix}=\begin{bmatrix}7\\5\\9\end{bmatrix}+\begin{bmatrix}-3\\-6\\-9\end{bmatrix}=\begin{bmatrix}4\\-1\\0\end{bmatrix}$$

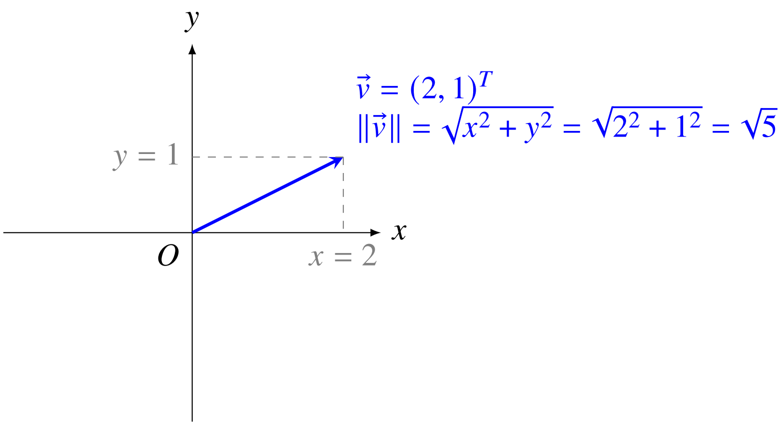
Length and Unit Vector

Length (magnitude), or more formally ***Euclidean norm***, of a vector $\vec{v}$ is based on a generalized version of ***Pythagoras' Theorem***, and is evaluated as the square root of the sum of squares of components.

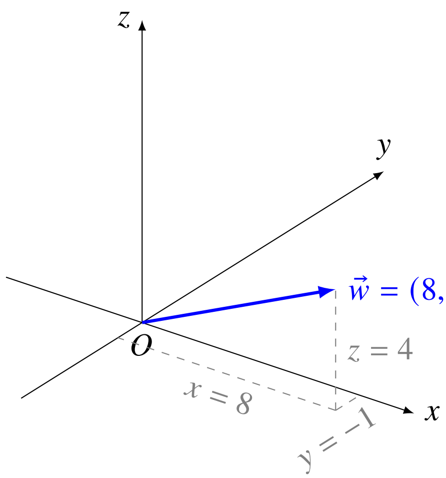
Definition 4.1.4 (Vector Length)**.** Length, or magnitude of a n-dimensional *real* vector $\vec{v}$, denoted by $||\vec{v}||$, is given by

$$\begin{aligned}||\vec{v}|| &= \sqrt{v_1^2+v_2^2+v_3^2+\cdots+v_n^2}\\ &= \sqrt{\sum_{k=1}^{n} v_k^2}\end{aligned}$$

For instance, the length of a two-dimensional vector follows the usual Pythagoras' Theorem as below.



Here is another example which is three-dimensional.



$$\vec{w} = \begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix} \qquad\qquad \|\vec{w}\| = \sqrt{8^2 + (-1)^2 + 4^2} = 9$$

We can create a ***unit vector*** that has a length of 1 from any vector $\vec{v}$ and orients in the same direction as $\vec{v}$. It is simply created by dividing (normalizing) $\vec{v}$ by its distance $\|\vec{v}\|$.

Definition 4.1.5 (Unit Vector)**.** The unit vector corresponding to a non-zero vector $\vec{v}$ is denoted as $\hat{v}$ and is given by

$$\hat{v} = \frac{1}{\|\vec{v}\|}\vec{v}$$

where the length $\|\vec{v}\|$ is defined as in Definition 4.1.4.

Note that despite vectors can carry physical units, unit vectors are all physically *dimensionless* when formulated in this way.

Short Exercise: Find the unit vector for $\vec{w} = (8, -1, 4)^T$ in the previous example, and verify that it has a length of 1.[1]

4.2 Special Vector Operations

Now we are going to introduce two special types of vector operations: *dot product*, and *cross product*.

4.2.1 Dot Product

(Real) Dot product (or ***scalar product***) is defined for two (real) vectors that have the same number of dimension. It is the sum of products of paired components between the two vectors. In other words, it can be regarded to be the matrix product between a row vector ($1 \times m$ matrix) and a column vector ($m \times 1$ matrix).

Definition 4.2.1 (Dot Product (Real))**.** The dot product between two n-dimensional *real* vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are denoted as either $\vec{u} \cdot \vec{v}$, or by

[1] $\|\vec{w}\| = 9$, $\hat{w} = \frac{\vec{w}}{\|\vec{w}\|} = \frac{1}{9}(8, -1, 4)^T = (\frac{8}{9}, -\frac{1}{9}, \frac{4}{9})^T$, $\|\hat{w}\| = \sqrt{(\frac{8}{9})^2 + (-\frac{1}{9})^2 + (\frac{4}{9})^2} = 1$.

matrix notation $\mathbf{u}^T\mathbf{v}$. They are defined as

$$\begin{aligned}\vec{u}\cdot\vec{v} = \mathbf{u}^T\mathbf{v} &= u_1v_1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n \\ &= \sum_{k=1}^{n} u_kv_k\end{aligned}$$

which is a scalar quantity.

Conversely, it can be said that entries of a matrix product are vector dot products between the corresponding rows and columns. It is emphasized that we are restricting ourselves to real entries since complex vectors introduce a bit of extra complications. Then, for two *real* matrices expressed in the form of combined row/column vectors that are $\mathbb{R}^n$,

$$A = \begin{bmatrix} —\vec{u}^{(1)T}— \\ \hline —\vec{u}^{(2)T}— \\ \hline \vdots \\ \hline —\vec{u}^{(p)T}— \end{bmatrix} = \left[\begin{array}{c|c|c|c} | & | & & | \\ \vec{u}^{(1)} & \vec{u}^{(2)} & \cdots & \vec{u}^{(p)} \\ | & | & & | \end{array}\right]^T \quad B = \left[\begin{array}{c|c|c|c} | & | & & | \\ \vec{v}^{(1)} & \vec{v}^{(2)} & \cdots & \vec{v}^{(q)} \\ | & | & & | \end{array}\right]$$

$$= \begin{bmatrix} \vec{u}_1^{(1)} & \vec{u}_2^{(1)} & \cdots & \vec{u}_n^{(1)} \\ \vec{u}_1^{(2)} & \vec{u}_2^{(2)} & \cdots & \vec{u}_n^{(2)} \\ \vdots & \vdots & & \vdots \\ \vec{u}_1^{(p)} & \vec{u}_2^{(p)} & \cdots & \vec{u}_n^{(p)} \end{bmatrix} \qquad = \begin{bmatrix} \vec{v}_1^{(1)} & \vec{v}_1^{(2)} & \cdots & \vec{v}_1^{(q)} \\ \vec{v}_2^{(1)} & \vec{v}_2^{(2)} & & \vec{v}_2^{(q)} \\ \vdots & & \ddots & \vdots \\ \vec{v}_n^{(1)} & \vec{v}_n^{(2)} & \cdots & \vec{v}_n^{(q)} \end{bmatrix}$$

(notice those transposes in the expression of A) their matrix product AB can be written as

$$AB = \begin{bmatrix} \vec{u}^{(1)}\cdot\vec{v}^{(1)} & \vec{u}^{(1)}\cdot\vec{v}^{(2)} & \cdots & \vec{u}^{(1)}\cdot\vec{v}^{(q)} \\ \vec{u}^{(2)}\cdot\vec{v}^{(1)} & \vec{u}^{(2)}\cdot\vec{v}^{(2)} & \cdots & \vec{u}^{(2)}\cdot\vec{v}^{(q)} \\ \vdots & \vdots & & \vdots \\ \vec{u}^{(p)}\cdot\vec{v}^{(1)} & \vec{u}^{(p)}\cdot\vec{v}^{(2)} & \cdots & \vec{u}^{(p)}\cdot\vec{v}^{(q)} \end{bmatrix}$$

We can use dot product to express the length of a real vector.

Properties 4.2.2. The length of a real vector, as defined in Definition 4.1.4, can be written using its dot product between itself as

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} \qquad \text{or} \qquad \|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$$

Notice that $\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2 \geq 0$. This quantity is always strictly greater than zero ($\vec{v} \cdot \vec{v} > 0$) unless $\vec{v} = \mathbf{0}$ is the zero vector (then $\vec{v} \cdot \vec{v} = 0$), which makes sense physically given that it represents length.

Example 4.2.1. If $\vec{u} = (1, 2, 3, 4, 5)^T$ and $\vec{v} = (-1, 0, 2, 1, -2)^T$, find the dot product $\vec{u} \cdot \vec{v} = \mathbf{u}^T \mathbf{v}$.

Solution.

$$\vec{u} \cdot \vec{v} = (1)(-1) + (2)(0) + (3)(2) + (4)(1) + (5)(-2) = -1$$

Alternatively,

$$\mathbf{u}^T \mathbf{v} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ -2 \end{bmatrix} = -1$$

□

Here are some properties of dot product.

Properties 4.2.3. For three n-dimensional real vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, the following properties hold.

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \vec{v} \cdot \vec{u} && \text{Symmetry Property} \\ \vec{u} \cdot (\vec{v} \pm \vec{w}) &= \vec{u} \cdot \vec{v} \pm \vec{u} \cdot \vec{w} && \text{Distributive Property} \\ (\vec{u} \pm \vec{v}) \cdot \vec{w} &= \vec{u} \cdot \vec{w} \pm \vec{v} \cdot \vec{w} && \text{Distributive Property} \end{aligned}$$

$$(a\vec{u}) \cdot (b\vec{v}) = ab(\vec{u} \cdot \vec{v}) \qquad \text{where } a, b \text{ are some constants}$$

Additionally, if A is an $n \times n$ square matrix, then

$$\begin{aligned} \vec{u} \cdot (A\vec{v}) &= \mathbf{u}^T (A\mathbf{v}) = (A^T\mathbf{u})^T \mathbf{v} = (A^T\vec{u}) \cdot \vec{v} \\ (A\vec{u}) \cdot \vec{v} &= (A\mathbf{u})^T \mathbf{v} = \mathbf{u}^T (A^T\mathbf{v}) = \vec{u} \cdot (A^T\vec{v}) \end{aligned}$$

where we have used Definition 4.2.1 and Properties 2.1.4.

Example 4.2.2. For $\vec{u} = (1, 3, 1)^T$ and $\vec{v} = (2, -1, 1)^T$, find $\|(\vec{u} + \vec{v})\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$.

Solution. By Properties 4.2.3, we can rewrite the expression as

$$\begin{aligned} (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) &= \vec{u} \cdot (\vec{u} + \vec{v}) + \vec{v} \cdot (\vec{u} + \vec{v}) \\ &= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \\ &= \vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} \end{aligned}$$

Subsequently,

$$\begin{aligned} &\vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} \\ &= (1, 3, 1)^T \cdot (1, 3, 1)^T + 2((1, 3, 1)^T \cdot (2, -1, 1)^T) + (2, -1, 1)^T \cdot (2, -1, 1)^T \\ &= (1^2 + 3^2 + 1^2) + 2((1)(2) + (3)(-1) + (1)(1)) + (2^2 + (-1)^2 + 1^2) \\ &= 11 + 2(0) + 6 \\ &= 17 \end{aligned}$$

Alternatively, one can calculate $\vec{w} = \vec{u} + \vec{v} = (1, 3, 1)^T + (2, -1, 1)^T = (3, 2, 2)^T$ and find $\vec{w} \cdot \vec{w} = \|\vec{w}\|^2$ instead. (which is easier and faster) $\square$

Example 4.2.3. Given $\vec{u}$ and $\vec{v}$ as defined in the example above, if

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & -1 \end{bmatrix}$$

verify that $\vec{u} \cdot (A\vec{v}) = (A^T\vec{u}) \cdot \vec{v}$.

Solution.

$$\begin{aligned}
A\vec{v} &= \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} (1)(2) + (2)(-1) + (1)(1) \\ (2)(2) + (0)(-1) + (3)(1) \\ (1)(2) + (1)(-1) + (-1)(1) \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} \\
\vec{u} \cdot (A\vec{v}) &= (1, 3, 1)^T \cdot (1, 7, 0)^T \\
&= (1)(1) + (3)(7) + (1)(0) \\
&= 22
\end{aligned}$$

On the other hand,

$$\begin{aligned}
A^T\vec{u} &= \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} (1)(1) + (2)(3) + (1)(1) \\ (2)(1) + (0)(3) + (1)(1) \\ (1)(1) + (3)(3) + (-1)(1) \end{bmatrix} \\
&= \begin{bmatrix} 8 \\ 3 \\ 9 \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
(A^T\vec{u}) \cdot \vec{v} &= (8, 3, 9)^T \cdot (2, -1, 1)^T \\
&= (8)(2) + (3)(-1) + (9)(1) \\
&= 22
\end{aligned}
$$

□

Geometric Meaning of Dot Product

The geometric meaning of dot product is embedded in the relation below.

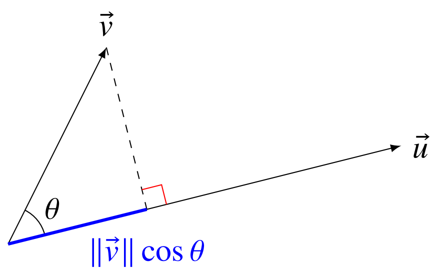
Properties 4.2.4. For two real vectors $\vec{u}$ and $\vec{v}$ that are of the same dimension, we have

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$. Furthermore, if $\hat{u}$ and $\hat{v}$ are unit vectors (Definition 4.1.5) such that $\|\vec{u}\| = \|\vec{v}\| = 1$, it reduces to

$$\hat{u} \cdot \hat{v} = \cos\theta$$

This means that the dot product between two vectors $\vec{u}$ and $\vec{v}$ is geometrically the signed product between $\vec{u}$ and the parallel component (projection) of $\vec{v}$ onto $\vec{u}$ (or vice versa), which is illustrated in the figure below. While an angle has a clear physical meaning only in a two/three-dimensional space, such relation generalizes the idea of an angle to higher dimensions.



Example 4.2.4. Find the angle between $\vec{u}$ and $\vec{v}$ in Example 4.2.1.

Solution. From Example 4.2.1, we have $\vec{u} \cdot \vec{v} = -1$, and

$$\begin{aligned} \|\vec{u}\| &= \sqrt{1^2 + 2^2 + 3^2 + 4^2 + 5^2} = \sqrt{55} \\ \|\vec{v}\| &= \sqrt{(-1)^2 + 0^2 + 2^2 + 1^2 + (-2)^2} = \sqrt{10} \end{aligned}$$

By Properties 4.2.4, we have

$$\begin{aligned} \cos\theta &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} \\ &= \frac{-1}{(\sqrt{55})(\sqrt{10})} \\ &\approx -0.0426 \\ \theta &\approx 1.613\,\text{rad} = 92.44^\circ \end{aligned}$$

□

By Properties 4.2.4, if the absolute value of the dot product $|\vec{u} \cdot \vec{v}|$ is equal to $\|\vec{u}\|\|\vec{v}\|$, where $\vec{u}$ and $\vec{v}$ are non-zero vectors, then it implies that $\cos\theta = \pm 1$, θ is either 0 or π, and hence the two vectors are parallel. In constrast,

Properties 4.2.5. If the dot product between two real vectors $\vec{u}$ and $\vec{v}$ is zero ($\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} = 0$), then by Properties 4.2.4, $\cos\theta = 0$ and the angle θ between $\vec{u}$ and $\vec{v}$ is $\frac{\pi}{2}$. In this case, $\vec{u}$ and $\vec{v}$ are said to be perpendicular, or *orthogonal* to each other.

From this, the concept of "***orthogonal***" becomes an extension of "perpendicular" in higher dimensions. It is easy to see that the standard unit vectors of $\mathbb{R}^n$ are orthogonal. Note that *the zero vector is regarded to be orthogonal to any vector*, so even if $\vec{u}$ or $\vec{v}$ is a zero vector, this properties still hold.

Some may notice that as $-1 \leq \cos\theta \leq 1$, if $|\vec{u} \cdot \vec{v}| > \|\vec{u}\|\|\vec{v}\|$, then θ in Properties 4.2.4 will be ill-defined. However, the ***Cauchy–Schwarz Inequality*** ensures this will not happen.

Theorem 4.2.6 (Cauchy–Schwarz Inequality)**.** Given two *real* n-dimensional vectors $\vec{u}$ and $\vec{v}$ ($\vec{u}, \vec{v} \in \mathbb{R}^n$), the following inequality holds.

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$$

$$|u_1v_1 + u_2v_2 + \cdots + u_nv_n| \leq \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Proof. Consider $\vec{w} = \vec{u} + t\vec{v}$, where t is any scalar, then $\|\vec{w}\|^2 = \vec{w} \cdot \vec{w} \geq 0$ by Properties 4.2.2. Also, $\vec{w} \cdot \vec{w}$ can be written as a quadratic polynomial in t:

$$\vec{w} \cdot \vec{w} = (\vec{u} + t\vec{v}) \cdot (\vec{u} + t\vec{v}) = \|\vec{u}\|^2 + 2t(\vec{u} \cdot \vec{v}) + t^2\|\vec{v}\|^2$$

Since this quantity is always greater than or equal to zero, i.e. the quadratic polynomial has no root or a repeated root, it means that the discriminant must be negative or zero. So,

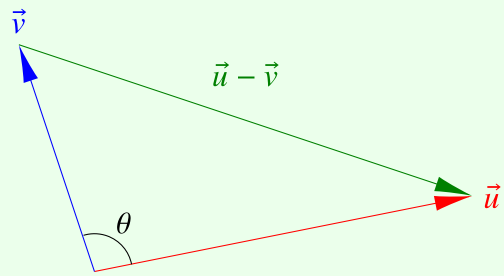
$$\begin{aligned}
\Delta = b^2 - 4ac &\leq 0 \\
(2(\vec{u} \cdot \vec{v}))^2 - 4\|\vec{u}\|^2\|\vec{v}\|^2 &\leq 0 \\
(\vec{u} \cdot \vec{v})^2 - \|\vec{u}\|^2\|\vec{v}\|^2 &\leq 0 \\
(\vec{u} \cdot \vec{v})^2 &\leq \|\vec{u}\|^2\|\vec{v}\|^2 \\
|\vec{u} \cdot \vec{v}| &\leq \|\vec{u}\|\|\vec{v}\|
\end{aligned}$$

□

Short Exercise: Think about under what circumstances the Cauchy–Schwarz Inequality turns into an equality (i.e. $|\vec{u} \cdot \vec{v}| = \|\vec{u}\|\|\vec{v}\|$).[2]

Example 4.2.5. Prove the ***Cosine Law*** by considering the triangle below

[2]When $\vec{u}$ and $\vec{v}$ are parallel, i.e. $\vec{u} = k\vec{v}$ for some scalar k, or $\vec{v} = \mathbf{0}$.



and expanding the dot product $\|(\vec{u} - \vec{v})\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})$.

Solution. Let denote the lengths $\|\vec{u}\|$, $\|\vec{v}\|$, $\|(\vec{u} - \vec{v})\|$ be a, b, c, then

$$\begin{aligned}
c^2 = \|(\vec{u} - \vec{v})\|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) && \text{(Properties 4.2.2)}\\
&= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} && \text{(Properties 4.2.3)}\\
&= \|\vec{u}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 && \text{(Properties 4.2.2 and 4.2.3)}\\
&= \|\vec{u}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta + \|\vec{v}\|^2 && \text{(Properties 4.2.4)}\\
&= a^2 - 2ab\cos\theta + b^2
\end{aligned}$$

□

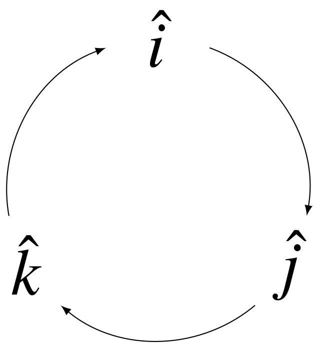
4.2.2 Cross Product

Another important type of vector product is the ***cross product*** (or sometimes just ***vector product***), which produces a three-dimensional real vector from two other three-dimensional real vectors. *The output vector will be orthogonal to the two input vectors*, and the direction is determined by the ***right hand rule***. Motivated by these requirements, we have the following basic definitions of cross product between the three standard unit vectors in $\mathbb{R}^3$.

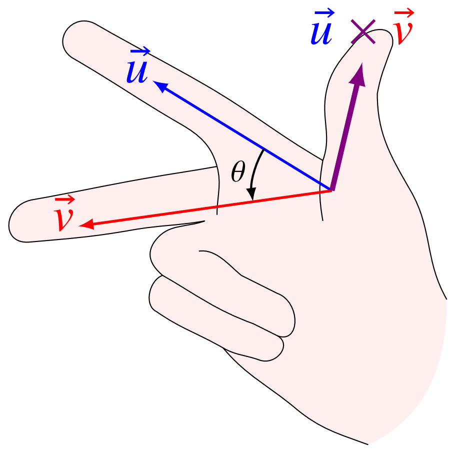
Definition 4.2.7. The computation of cross products (denoted by $\times$) involving any two of the standard unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ in $\mathbb{R}^3$ obeys the following rules.

1. $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{\imath} = -\hat{k}$,

2. $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{j} = -\hat{i}$,
3. $\hat{k} \times \hat{i} = \hat{j}$, $\hat{i} \times \hat{k} = -\hat{j}$, and
4. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \mathbf{0}$



A cyclic diagram for memorizing Definition 4.2.7. A clockwise / anti-clockwise permutation produces a positive / negative unit vector of the third.



Demonstration of the right hand rule.

The properties of cross product are noted below. One major difference setting cross product apart from the dot product is its anti-symmetric property.

Properties 4.2.8. For two $\mathbb{R}^3$ vectors $\vec{u}$ and $\vec{v}$, we have

$$\begin{aligned} \vec{u} \times \vec{v} &= -\vec{v} \times \vec{u} && \text{Anti-symmetry Property} \\ \vec{u} \times (\vec{v} \pm \vec{w}) &= \vec{u} \times \vec{v} \pm \vec{u} \times \vec{w} && \text{Distributive Property} \\ (\vec{u} \pm \vec{v}) \times \vec{w} &= \vec{u} \times \vec{w} \pm \vec{v} \times \vec{w} && \text{Distributive Property} \\ (a\vec{u}) \times (b\vec{v}) &= ab(\vec{u} \times \vec{v}) && \text{where } a, b \text{ are some constants} \end{aligned}$$

The calculation of cross product then follows from these rules, leading to the determinant shorthand below.

Properties 4.2.9. For $\vec{u} = (u_1, u_2, u_3)^T, \vec{v} = (v_1, v_2, v_3)^T \in \mathbb{R}^3$, their cross product $\vec{u} \times \vec{v}$ can be written in the form of a determinant as

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Proof. Starting from Definition 4.2.7 and Properties 4.2.8, we have

$$\begin{aligned}
\vec{u} \times \vec{v} &= (u_1\hat{i} + u_2\hat{j} + u_3\hat{k}) \times (v_1\hat{i} + v_2\hat{j} + v_3\hat{k}) \\
&= u_1v_1(\hat{i} \times \hat{i}) + u_1v_2(\hat{i} \times \hat{j}) + u_1v_3(\hat{i} \times \hat{k}) \\
&\quad + u_2v_1(\hat{j} \times \hat{i}) + u_2v_2(\hat{j} \times \hat{j}) + u_2v_3(\hat{j} \times \hat{k}) \\
&\quad + u_3v_1(\hat{k} \times \hat{i}) + u_3v_2(\hat{k} \times \hat{j}) + u_3v_3(\hat{k} \times \hat{k}) && \text{(Properties 4.2.8)} \\
&= u_1v_1(\mathbf{0}) + u_1v_2(\hat{k}) - u_1v_3(\hat{j}) \\
&\quad - u_2v_1(\hat{k}) + u_2v_2(\mathbf{0}) + u_2v_3(\hat{i}) \\
&\quad + u_3v_1(\hat{j}) - u_3v_2(\hat{i}) + u_3v_3(\mathbf{0}) && \text{(Definition 4.2.7)} \\
&= (u_2v_3 - u_3v_2)\hat{i} + (u_3v_1 - u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}
\end{aligned}$$

Meanwhile, cofactor expansion (Properties 2.3.3) along the first row of the given determinant form

$$\begin{aligned}
\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} &= \hat{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\
&= (u_2v_3 - u_3v_2)\hat{i} + (u_3v_1 - u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}
\end{aligned}$$

yields the identical result. □

Example 4.2.6. Given two $\mathbb{R}^3$ vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

Find $\vec{u} \times \vec{v}$.

Solution.

$$\begin{aligned}
\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 2 \\ 3 & -1 & 1 \end{vmatrix} \\
&= \hat{\imath} \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} && \text{(Cofactor expansion along the first row)} \\
&= 2\hat{\imath} + 5\hat{\jmath} - \hat{k} = (2, 5, -1)^T
\end{aligned}$$

□

Short Exercise: Check if $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$ by finding the corresponding dot products.[3]
Short Exercise: Following the short exercise above, show in general, $\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\vec{u} \times \vec{v}) = 0$.[4]

Geometric Meaning of Cross Product

Similar to vector dot product, vector cross product has a geometric interpretation.

[3] $\vec{u} \cdot (\vec{u} \times \vec{v}) = (1, 0, 2)^T \cdot (2, 5, -1)^T = (1)(2) + (0)(5) + (2)(-1) = 0$, $\vec{v} \cdot (\vec{u} \times \vec{v}) = (3, -1, 1)^T \cdot (2, 5, -1)^T = (3)(2) + (-1)(5) + (1)(-1) = 0$. The zero dot product in both cases shows they are orthogonal via Properties 4.2.5.

[4] From the derivation of Properties 4.2.9, $\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\hat{\imath} + (u_3v_1 - u_1v_3)\hat{\jmath} + (u_1v_2 - u_2v_1)\hat{k}$, and $\vec{u} \cdot (\vec{u} \times \vec{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0$ where all terms cancel out, and it is similar for $\vec{v}$.

Properties 4.2.10. Given two vectors $\vec{u}$ and $\vec{v}$ which are both of $\mathbb{R}^3$, the magnitude (length) of $\vec{u} \times \vec{v}$ is related to the angle between $\vec{u}$ and $\vec{v}$ as

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$$

From this, we immediately know that if $\vec{u}$ and $\vec{v} = k\vec{u}$, where k is some constant, are two parallel vectors, their cross product will be a zero vector as $\theta = 0$ (or π) and $\sin\theta = 0$. This is equivalent to the statement of $\vec{u} \times \vec{u} = \mathbf{0}$[5] (notice that it is not 0 but $\mathbf{0}$ since it always outputs a vector!). (You can also arrive at this conclusion with Properties 4.2.8.[6])

Example 4.2.7. If $\vec{u} = (1, 2, 3)^T$, and $\vec{v} = (-1, 1, 2)^T$, find $(\vec{u} + 2\vec{v}) \times (\vec{u} - \vec{v})$.

Solution. Observe that

$$\begin{aligned}
(\vec{u} + 2\vec{v}) \times (\vec{u} - \vec{v}) &= \vec{u} \times (\vec{u} - \vec{v}) + 2\vec{v} \times (\vec{u} - \vec{v}) \\
&= \vec{u} \times \vec{u} - \vec{u} \times \vec{v} + 2\vec{v} \times \vec{u} - 2\vec{v} \times \vec{v} \\
&= \mathbf{0} - \vec{u} \times \vec{v} - 2\vec{u} \times \vec{v} - 2(\mathbf{0}) \\
&= -3\vec{u} \times \vec{v}
\end{aligned}$$

where the fact that $\vec{u} \times \vec{u} = \mathbf{0}$, $\vec{v} \times \vec{v} = \mathbf{0}$ and Properties 4.2.8 are used. Now, with Properties 4.2.9, we have

$$-3\vec{u} \times \vec{v} = -3 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{vmatrix}$$

[5]By Properties 4.2.9,

$$\vec{u} \times \vec{u} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

and the determinant vanishes by Properties 2.3.4 due to the identical second/third row.

[6]The anti-symmetric property requires $\vec{u} \times \vec{u} = -\vec{u} \times \vec{u}$ and hence $2(\vec{u} \times \vec{u}) = \mathbf{0}$.

$$
\begin{aligned}
&= -3\left(\hat{\imath}\begin{vmatrix}2 & 3\\1 & 2\end{vmatrix} - \hat{\jmath}\begin{vmatrix}1 & 3\\-1 & 2\end{vmatrix} + \hat{k}\begin{vmatrix}1 & 2\\-1 & 1\end{vmatrix}\right) \quad &&\text{(Cofactor expansion along the first row)}\\
&= -3(\hat{\imath} - 5\hat{\jmath} + 3\hat{k})\\
&= -3\hat{\imath} + 15\hat{\jmath} - 9\hat{k} = (-3, 15, -9)^T
\end{aligned}
$$

The readers can try the alternative of computing $\vec{u} + 2\vec{v}$ and $\vec{u} - \vec{v}$ first and then their cross product. □

Finally, cancellation of dot product or cross product at both sides of an equation is generally not correct, and here is a table summarizing the inputs and outputs of dot/cross product for clarification.

	Input	Output
Dot Product, or Scalar Product ($\cdot$)	Two real vectors of the same dimension ($\mathbb{R}^n$), the order does not matter (symmetric)	A scalar
Cross Product, or Vector Product ($\times$)	Two three-dimensional real vectors ($\mathbb{R}^3$), the order is important (anti-symmetric)	Another three-dimensional vector

4.3 Earth Science Applications

Example 4.3.1. The *Coriolis Effect* is a phenomenon describing the deflection of motion due to rotation of the Earth. It introduces an apparent force known as the *Coriolis Force* which is given by $\overrightarrow{F_{\text{cor}}} = -2\vec{\Omega} \times \vec{v}$ where $\Omega = \left\|\vec{\Omega}\right\| = 7.292 \times 10^{-5}\,\text{rad}\,\text{s}^{-1}$ represents the angular speed of Earth's rotation, and $\vec{\Omega}$ is oriented in the direction of the North Pole. Define the local frame of reference (see Figure 4.1) with the x-direction being the zonal direction, y-direction being the meridional direction, and z-direction being the zenith direction (normal to the Earth's surface), then we have $\vec{v} = (u, v, w) = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$ as the flow velocity in this local Cartesian coordinate system with unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ along the x, y, z axes. It can be seen that $\vec{\Omega} = (\Omega\cos\varphi)\hat{j} + (\Omega\sin\varphi)\hat{k}$ where φ is the latitude. Now by expanding $\overrightarrow{F_{\text{cor}}} = -2\vec{\Omega}\times\vec{v}$ show that the components of Coriolis Force along the local x, y, z directions are

$$\begin{aligned} F_{\text{cor},x} &= 2\Omega(v\sin\varphi - w\cos\varphi) \\ F_{\text{cor},y} &= -2\Omega u\sin\varphi \\ F_{\text{cor},z} &= 2\Omega u\cos\varphi \end{aligned}$$

The *Coriolis Parameter* f is usually used to denote the factor $2\Omega\sin\varphi$.

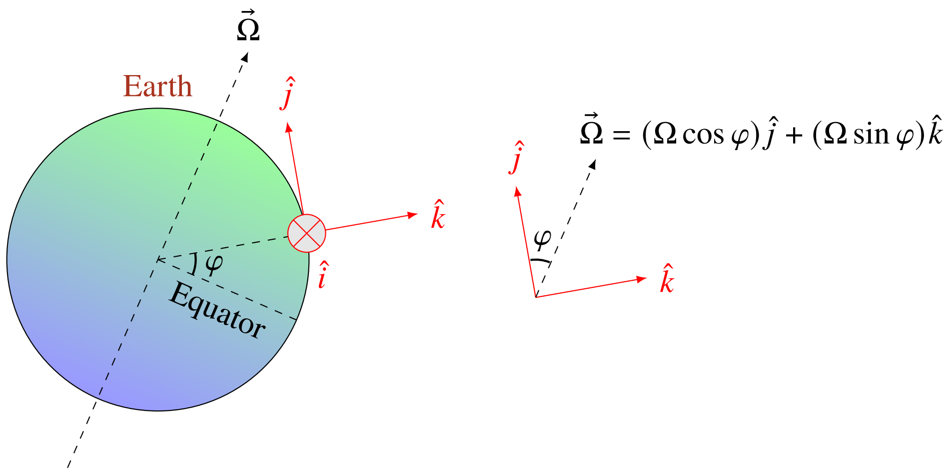


Figure 4.1: An illustration of the coordinate frame in Example 4.3.1.

Solution. Using Properties 4.2.9 to expand $\overrightarrow{F_{\text{cor}}}$ gives

$$\begin{aligned} -2\overrightarrow{\Omega}\times\vec{v} &= -2((\Omega\cos\varphi)\hat{j} + (\Omega\sin\varphi)\hat{k}) \times (u\hat{i} + v\hat{j} + w\hat{k}) \\ &= -2\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \Omega\cos\varphi & \Omega\sin\varphi \\ u & v & w \end{vmatrix} \\ &= -2[(w\Omega\cos\varphi - v\Omega\sin\varphi)\hat{i} + (u\Omega\sin\varphi)\hat{j} - (u\Omega\cos\varphi)\hat{k}] \end{aligned}$$

$$= [2\Omega(v \sin\varphi - w\cos\varphi)]\hat{\imath} + (-2\Omega u \sin\varphi)\hat{\jmath} + (2\Omega u \cos\varphi)\hat{k}$$

The $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ components correspond to $F_{\text{cor},x}$, $F_{\text{cor},y}$, $F_{\text{cor},z}$ respectively. Assume w is negligible, then $F_{\text{cor},x} = fv$ and $F_{\text{cor},y} = -fu$. □

4.4 Python Programming

We can use one-dimensional `numpy` arrays as vectors.

```
import numpy as np

myVec1 = np.array([-1., 2., 4.])
myVec2 = np.array([2., 1., 3.])
```

Addition, subtraction, and scalar multiplication works just like for matrices.

```
myVec3 = -myVec1 + 2*myVec2
print(myVec3)
```

gives the expected output of `[5. 0. 2.]`. We can select a component of any vector by indexing. Again, remember that indices in *Python* start from zero. `print(myVec3[1])` then returns `0.0`. The magnitude of a vector can be checked with `np.linalg.norm`. For example,

```
print(np.linalg.norm(myVec1))
```

produces `4.58257569495584` ($\sqrt{(-1)^2 + 2^2 + 4^2} = \sqrt{21}$). Dot product is computed via `np.dot` as follows.

```
myDot = np.dot(myVec1, myVec2)
print(myDot)
```

which outputs `12.0` (as $(-1)(2) + (2)(1) + (4)(3) = 12$). Similarly, cross product is found by `np.cross`.

```
myCross = np.cross(myVec1, myVec2)
print(myCross)
```

then gives

```
[ 2. 11. -5.]
```

and we can check if the cross product is orthogonal to the two input vectors.

```
# All lines below return zero.
print(np.dot(myVec1, myCross))
print(np.dot(myVec2, myCross))
print(np.dot(myVec3, myCross))
```

Dot product is defined for any two vectors with the same dimension, but cross product is only defined for three-dimensional vectors (or in some other sense two-dimensional), so

```
myVec4 = np.array([1., 3., 2., 0.])
myVec5 = np.array([2., 1., 0., -1.])
print(np.dot(myVec4, myVec5))
```

yields a valid output of 5.0, but

```
print(np.cross(myVec4, myVec5))
```

raises the error of

```
ValueError: incompatible dimensions for cross product
(dimension must be 2 or 3)
```

Finally, we note that following this Stack Overflow post (2827393), we can compute the unit vector of any given vector and angle between two vectors (based from the second observation in Properties 4.2.4, $\theta = \cos^{-1}(\hat{u} \cdot \hat{v})$).

```
def unit_vector(vector):
    """ Returns the unit vector of the vector.  """
    return vector / np.linalg.norm(vector)

def angle_between(v1, v2):
    """ Returns the angle in radians between vectors 'v1' and
        'v2'. """
    v1_u = unit_vector(v1)
    v2_u = unit_vector(v2)
    return np.arccos(np.clip(np.dot(v1_u, v2_u), -1.0, 1.0))
```

The `np.clip` is to avoid numerical round-off error that causes the dot product of the two normalized input vectors to just fall outside (e.g. `1.0000000000000002`) the valid range $[-1, 1]$ of $\cos^{-1}$. The naive way of (here the lists will be cast to one-dimensional arrays automatically during calculation.)

```
np.arccos(np.dot([1., 0, 0], [2., 0, 0]))
```

leads to the warning of

```
RuntimeWarning: invalid value encountered in arccos
nan
```

but

```
angle_between([1., 0, 0], [2., 0, 0])
```

gives `0.0` properly. Trying this on `myVec4` and `myVec5` with

```
print(unit_vector(myVec4))
print(angle_between(myVec4, myVec5))
```

produces a unit vector of `[0.267 0.802 0.535 0.    ]`, and an angle of `0.993757` (in radians).

4.5 Exercises

Exercise 4.1 For $\vec{u} = (1, 3, 3, 7)^T$ and $\vec{v} = (1, 2, 2, 5)^T$, find

(a) $\vec{u} + \vec{v}$,

(b) $\frac{3}{2}\vec{u} - \frac{1}{2}\vec{v}$,

(c) $\vec{u} \cdot \vec{v}$,

(d) $\vec{v} \cdot \vec{u}$,

(e) $(\vec{u} - 2\vec{v}) \cdot (2\vec{u} + \vec{v})$.

Exercise 4.2 For $\vec{u} = (7, 4, 1)^T$, $\vec{v} = (8, 1, 1)^T$, and

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Verify that

(a) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$,

(b) $\vec{u} \cdot (\vec{Av}) = (A^T \vec{u}) \cdot \vec{v}$,

(c) Compute $(3\vec{u} - 4\vec{v}) \cdot (\vec{u} \times \vec{v})$, is the answer what you expect?

Exercise 4.3 For $\vec{u} = (1, -3, 9)^T$ and $\vec{v} = (1, -2, 4)^T$, find

(a) Their unit vectors $\hat{u}$ and $\hat{v}$,

(b) The angle between them, by calculating their dot product,

(c) The cross product $\vec{u} \times \vec{v}$, and

(d) Show that the vector obtained from the cross product above is orthogonal (perpendicular) to $\vec{u}$ and $\vec{v}$, by calculating the corresponding dot products.

Exercise 4.4 The following table contains incomplete data about the movement of several typhoons at some moments. Complete the table by filling in the blanks. The first one has been done as an example.

Typhoon Name	Time	Speed	Direction	Vector Form
Nuri	2008/08/22, 08:00	$13\,\mathrm{km\,h^{-1}}$	315°	$(-9.192, 9.192)$
Vicente	2012/07/24, 02:00	$18\,\mathrm{km\,h^{-1}}$	299°	
Linfa	2015/07/09, 23:00			$(-13.595, -6.339)$
Mangkhut	2018/09/16, 22:00		288°	$(\quad, 7.725)$

Exercise 4.5 Prove the Triangular Inequality.

$$||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$$

Exercise 4.6 Prove the Parallelogram Law. (See Figure 4.2)

$$2||\vec{u}||^2 + 2||\vec{v}||^2 = ||\vec{u} + \vec{v}||^2 + ||\vec{u} - \vec{v}||^2$$

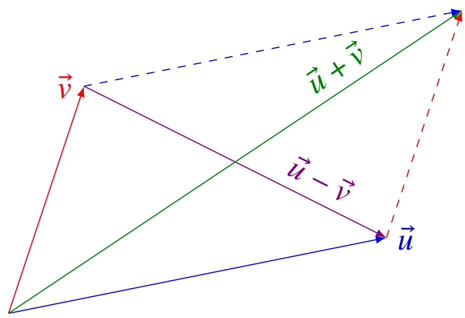


Figure 4.2: The parallelogram constructed by vectors for Exercise 4.6.

Exercise 4.7 Show that Coriolis Force derived in Example 4.3.1 does zero work and hence is consistent with the fact that it is an apparent force and never produces/consumes energy by itself.

Chapter 5

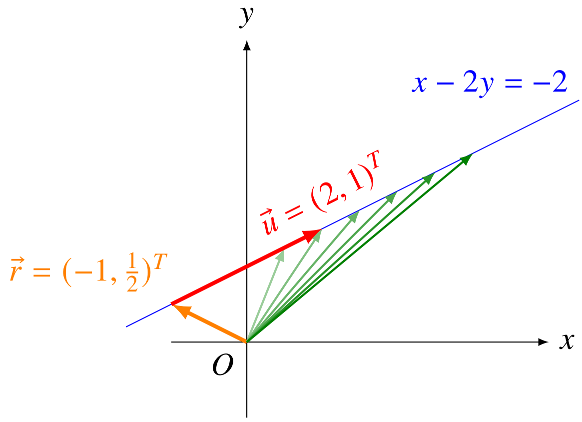
More on Vector Geometry

Vectors provides valuable assistance when it comes to describing geometric objects. In this chapter we are going to exploit the knowledge learnt in the previous chapters to solve geometry problems and inspect more deeply the intimate relationship between vectors, dot/cross products, and geometry.

5.1 Lines and Planes

(Straight) lines and *planes* are geometric shapes of importance in two/three-dimensional real spaces ($\mathbb{R}^2$ and $\mathbb{R}^3$) and due to their simplicity they will be frequently seen. They can be expressed either in terms of (a) an equation, and (b) vectors. We will start from the easier case of a line.

Since a straight line is a one-dimensional object, the vector form of such a line can be expressed by a fixed vector that points to its initial position, plus another vector oriented along the line's direction, times an arbitrary parameter which controls its extension or contraction, so that it traces out the line when the parameter changed continuously.



The graph of $x - 2y = -2$ can take the vector form of $\overrightarrow{OP} = \vec{r} + t\vec{u} = (-1, \frac{1}{2})^T + t(2, 1)^T$. The orange/red arrow represents the initial position/direction, and the locus of green arrow is controlled by t like a slider. The cases for $t = 0.75, 1, 1.25, 1.5, 1.75, 2$ are shown.

Short Exercise: Choose any value of t and substitute that value into the expression of $\overrightarrow{OP}$ above to see if the x and y-components satisfy the starting equation. Also, try to increase/decrease the value of t to observe how the vector traces out the desired straight line.[1]

5.1.1 Translating Equation Form to Vector Form

The general equation form of a line on an x-y plane is $ax + by = h$, resembling a linear system of one equation with two unknowns. From Section 3.2.1, it can be observed that it has infinitely many solutions and possesses a free variable. Let $y = t$, then rearranging the equation we have $x = (h - bt)/a$ where t is any scalar. Denote the origin as O and any point on the line as P, then

$$\overrightarrow{OP} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{h}{a} - \frac{b}{a}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{h}{a} \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{b}{a} \\ 1 \end{bmatrix}$$

[1]Let's say $t = -0.25$, $\overrightarrow{OP} = (-1, 0.5)^T + (-0.25)(2, 1)^T = (-1.5, 0.25)^T$, $x - 2y = (-1.5) - 2(0.25) = -2$.

This is one possible vector form (*parameterization*) of the line. Its idea can be borrowed from Example 3.2.3, with $(\frac{h}{a}, 0)^T$ being the initial position/particular solution, and $(-\frac{b}{a}, 1)^T$ as the direction of that line, multiplied by a free parameter (complementary solution) to complete the general solution. For example, if we have $3x - 2y = 5$, then by the same method, we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{3} + \frac{2}{3}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Bear in mind that the direction vector (representing complementary solution) can be scaled freely. In addition, any initial position vector (particular solution) can be chosen as long as it links to a point on the line and satisfies the equation. (You may refer to the discussion about particular/complementary solution in Section 3.2.1.) Hence there is no unique vector form for a line. For instance,

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + t_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

is equivalent to

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

for the line equation $2x - y = -1$.

Short Exercise: Check the equivalence of the two vector forms above by choosing a value for t_1 and finding the corresponding t_2 so that the vector points to the same position.[2]

[2]For example, if $t_1 = 1$, we have $(1,3)^T + (1)(2,4)^T = (3,7)^T$ as a point on the line, and for the another vector form $(-1,-1)^T + t_2(1,2)^T = (3,7)^T$ to coincide we will have $t_2 = 4$. In this case, it can be shown that the general relation between the two forms is determined by $t_2 = 2t_1 + 2$, as

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + t_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \left(\begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) + 2t_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + (2t_1 + 2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Short Exercise: What is the vector form of the equation $ax + by = h$ for the degenerate case $a = 0$?[3]

5.1.2 Recovering Equation Form from Vector Form

On the other hand, inferring line equation from the vector form is not straightforward at first sight. Since the vector form of a line always contains an arbitrary parameter, which is absent in the equation form, the motivation is to remove the parameter through some manipulation.

Remember that from Properties 4.2.5 the dot product between orthogonal (perpendicular) vectors returns zero. This means that by carrying out dot product with the ***normal vector*** of the line which is orthogonal to the direction vector, on both sides of the vector form will eliminate the parameter and recover the line equation. For example, given that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

We know that $(4, -1)^T$ is a normal vector orthogonal to the direction vector (see the next short exercise). So, by taking dot product with $(4, -1)^T$ on both sides, we have

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix} &= \left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix} \\
4x - y &= ((1)(4) + (3)(-1)) + ((1)(4) + (4)(-1))t = 1 + 0t = 1 \\
\Rightarrow 4x - y &= 1
\end{aligned}$$

Notice that the coefficients of the equation are the same as the components of the normal vector.

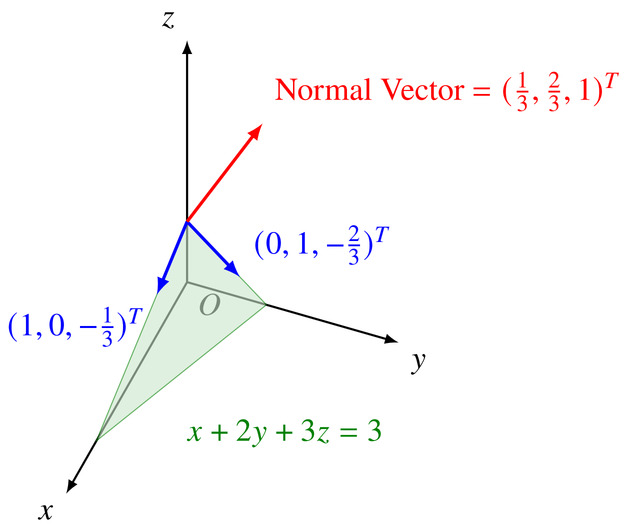
[3] The equation is reduced to $y = \frac{h}{b}$ and we select $x = t$ as the free variable instead.
$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ \frac{h}{b} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{h}{b} \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Short Exercise: Verify that (a, b) is always orthogonal to $(b, -a)$, and vice versa.[4]

5.1.3 Generalizing to Higher Dimensions

Similar concepts can be applied on the equation and vector form for planes. General form of equation of a plane in three-dimensional space is $ax+by+cz = h$, which is a linear system of one equation with three unknowns. From the analysis in Section 3.2.1 we know there are two free variables and two (non-parallel) direction vectors for such a plane. By assigning any two (non-pivotal) unknowns as the free variables, we then obtain the vector form of the plane.

Recall from Section 4.2.2, the cross product of any two non-parallel vectors on the plane will give a third vector normal to the plane. Subsequently, we can take the dot product with this newly obtained normal vector to convert the vector form back to a plane equation just like what we have done for lines in the last subsection. Again, the coefficients of the plane equation match the components of the normal vector, differed at most by a multiplicative factor.



[4] $(a, b)^T \cdot (b, -a)^T = (a)(b) + (b)(-a) = 0$

The plane represented by the equation $x + 2y + 3z = 3$. Notice that the normal vector can be found via computing $(1, 0, -\frac{1}{3})^T \times (0, 1, -\frac{2}{3})^T = (\frac{1}{3}, \frac{2}{3}, 1)^T$. The normal vector is magnified for the purpose of illustration.

Example 5.1.1. Transform the plane equation $2x + 3y + z = 4$ to vector form and convert the acquired vector form back to the starting equation to check consistency.

Solution. For the first part, we can let $y = s$, $z = t$, then from the plane equation we have $x = \frac{1}{2}(4 - 3s - t)$ and hence

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(4 - 3s - t) \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

where $-\infty < s < \infty$, $-\infty < t < \infty$ are the two free parameters. To recover the original equation, we can find the normal vector by doing cross product on the two direction vectors obtained above. By Properties 4.2.9, we can acquire a normal vector of

$$\begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{vmatrix} = \hat{i} + \frac{3}{2}\hat{j} + \frac{1}{2}\hat{k}$$

The next step is to take the dot product on both sides of the vector equation with the normal vector just retrieved.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} + s \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\begin{aligned} x + \frac{3}{2}y + \frac{1}{2}z &= 2 + s(0) + t(0) = 2 \\ \to 2x + 3y + z &= 4 \end{aligned}$$

□

The correspondence between the coefficients of a linear equation and components of its normal vector is not a coincidence. In fact, even for higher dimensional cases, where there is no intuitive geometric interpretation, it is still true.

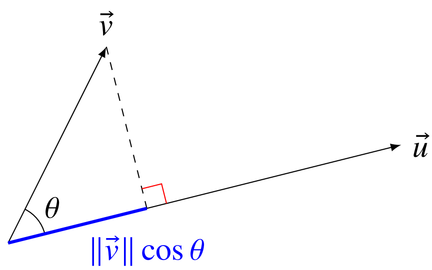
Properties 5.1.1. An equation of the form $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = h$ in $\mathbb{R}^n$ has a normal vector of $(a_1, a_2, a_3, \ldots, a_n)^T$.

The procedures carried in the last example can be similarly applied to higher dimensional situations where the equation now represents a ***hyperplane***[5].

5.2 Further Geometric Applications of Dot Product

5.2.1 Projection

We have mentioned in Properties 4.2.4 that dot product between two vectors is related to the projection of one vector onto another. By rearranging the formula of Properties 4.2.4, we can derive the length of projection as follows.



Properties 5.2.1. For two real vectors $\vec{u}$ and $\vec{v}$ having the same dimension, the

[5]A hyperplane in the n-dimensional real space $\mathbb{R}^n$ can be think of as an "$(n-1)$-dimensional flat surface".

(signed) scalar projection of $\vec{v}$ onto $\vec{u}$ is computed according to

$$\text{proj}_u v = \|\vec{v}\| \cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}$$

If we want to give directionality to the projection, then we can supply its unit vector $\hat{u}$ to make it a ***vector projection***:

$$\begin{aligned} \overrightarrow{\text{proj}}_u v &= (\text{proj}_u v)\hat{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}\hat{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2}\vec{u} \\ &= (\text{proj}_u v)\frac{\vec{u}}{\|\vec{u}\|} \end{aligned}$$

where we have used Definition 4.1.5 to write out the unit vector $\hat{u}$ as $\frac{\vec{u}}{\|\vec{u}\|}$.

Example 5.2.1. Find the projection of $\vec{v} = -2\hat{\imath} + 3\hat{\jmath} - \hat{k}$ onto $\vec{u} = 4\hat{\imath} + \hat{\jmath} - 3\hat{k}$.

Solution. According to Properties 5.2.1, The signed scalar projection of $\vec{v}$ into $\vec{u}$ is

$$\begin{aligned} \text{proj}_u v &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|} \\ &= \frac{(-2)(4) + (3)(1) + (-1)(-3)}{\sqrt{(4)^2 + (1)^2 + (-3)^2}} \\ &= -\frac{2}{\sqrt{26}} = -\frac{\sqrt{26}}{13} \end{aligned}$$

and the vector projection is

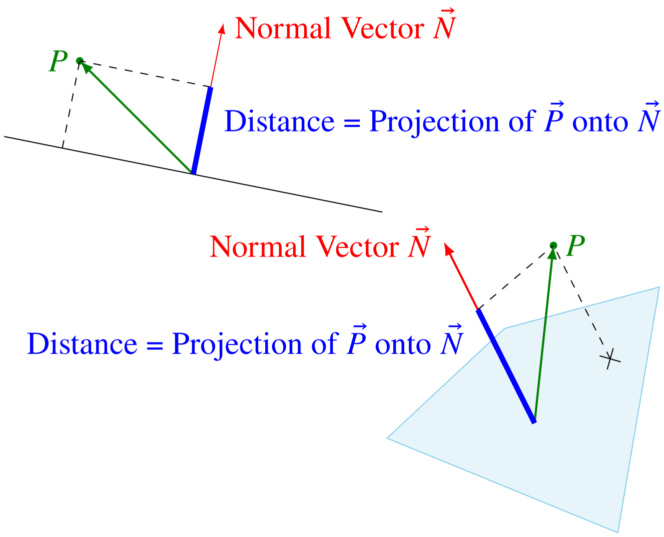
$$\begin{aligned} \overrightarrow{\text{proj}}_u v &= (\text{proj}_u v)\frac{\vec{u}}{\|\vec{u}\|} \\ &= (-\frac{\sqrt{26}}{13})\left(\frac{4\hat{\imath} + \hat{\jmath} - 3\hat{k}}{\sqrt{26}}\right) \end{aligned}$$

$$= -\frac{1}{13}(4\hat{i} + \hat{j} - 3\hat{k}) = (-\frac{4}{13}, -\frac{1}{13}, \frac{3}{13})^T$$

□

5.2.2 Distance

Distance of a point to a line/plane (in $\mathbb{R}^2/\mathbb{R}^3$ respectively) can be found by projecting any vector starting somewhere from the line/plane to the point, onto the normal vector of that line/plane, as illustrated in the figures below.



Example 5.2.2. Find the distance from the plane $x - 2y + 3z = 6$ to the point $(3, 3, 6)^T$.

Solution. From the equation of the plane, and by Properties 5.1.1, it can be inferred that the normal vector of the plane is $\hat{i} - 2\hat{j} + 3\hat{k}$. We can select any point on the plane as we wish, let's say $(4, 2, 2)^T$, and the vector from such a point to the point $(3, 3, 6)^T$ in question is simply their difference $(3, 3, 6)^T - (4, 2, 2)^T =$

$-\hat{\imath}+\hat{\jmath}+4\hat{k}$. Subsequently, the distance is found from the length of the projection of this vector $-\hat{\imath}+\hat{\jmath}+4\hat{k}$ onto the normal vector of the plane $\hat{\imath}-2\hat{\jmath}+3\hat{k}$. By Properties 5.2.1, it is

$$\frac{(-\hat{\imath}+\hat{\jmath}+4\hat{k})\cdot(\hat{\imath}-2\hat{\jmath}+3\hat{k})}{\|\hat{\imath}-2\hat{\jmath}+3\hat{k}\|} = \frac{(-1)(1)+(1)(-2)+(4)(3)}{\sqrt{(1)^2+(-2)^2+(3)^2}} = \frac{9}{\sqrt{14}}$$

□

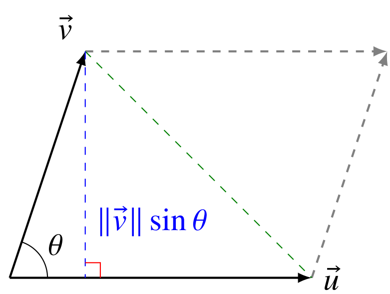
Sometimes the calculation may lead to a negative value for the projection and we may want to take the absolute value. The case of finding the distance of a point to a line of $\mathbb{R}^3$ is considered in Exercise 5.3.

5.3 Further Geometric Applications of Cross Product

Unless specified, all vectors in this section is assumed to be of $\mathbb{R}^3$.

5.3.1 Area

The area of the parallelogram formed by two vectors $\vec{u}$, $\vec{v}$ are simply the absolute value of their cross product.



Properties 5.3.1. Directly from Properties 4.2.10, the area of the parallelogram formed by two vectors $\vec{u}$ and $\vec{v}$ is

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$$

Similarly, the area of triangle made by $\vec{u}$ and $\vec{v}$ is half of the above:

$$\frac{1}{2}\|\vec{u} \times \vec{v}\| = \frac{1}{2}\|\vec{u}\|\|\vec{v}\| \sin\theta$$

Example 5.3.1. Find the area of the parallelogram formed by $\vec{u} = (-1, -2, 4)^T$ and $\vec{v} = (3, 0, 1)^T$.

Solution. By Properties 4.2.9, the cross product between the two given vectors is

$$\begin{aligned}\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & -2 & 4 \\ 3 & 0 & 1 \end{vmatrix} \\ &= -2\hat{\imath} + 13\hat{\jmath} + 6\hat{k}\end{aligned}$$

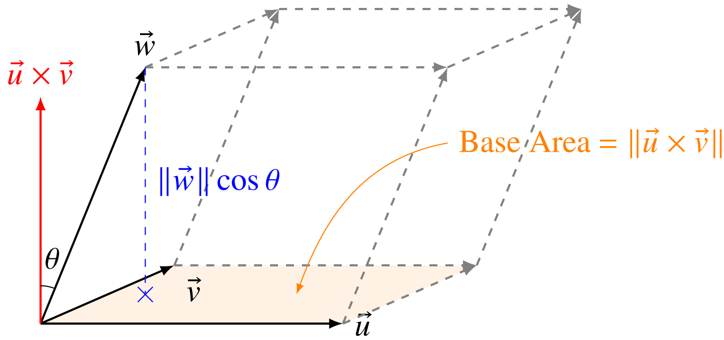
Therefore, as suggested by Properties 5.3.1, the required area is

$$\begin{aligned}\|\vec{u} \times \vec{v}\| &= \sqrt{(-2)^2 + (13)^2 + (6)^2} \\ &= \sqrt{209}\end{aligned}$$

□

5.3.2 Volume

Meanwhile, volume of parallelepiped (see the figure below) formed by three vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ is given by the absolute value of the so-called ***scalar triple product*** as follows.



Properties 5.3.2 (Scalar Triple Product)**.** The volume of parallelepiped formed by three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is calculated as

$$\|\vec{u} \times \vec{v}\| \|\vec{w}\| \cos\theta = |(\vec{u} \times \vec{v}) \cdot \vec{w}| = \operatorname{abs}\left(\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \right)$$

where

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

is the scalar triple product of $\vec{u}$, $\vec{v}$, and $\vec{w}$. Also, by applying Properties 2.3.5, the determinant form of scalar triple product indicates that

$$\begin{aligned} (\vec{u} \times \vec{v}) \cdot \vec{w} &= (\vec{v} \times \vec{w}) \cdot \vec{u} = (\vec{w} \times \vec{u}) \cdot \vec{v} \\ &= -(\vec{v} \times \vec{u}) \cdot \vec{w} = -(\vec{w} \times \vec{v}) \cdot \vec{u} = -(\vec{u} \times \vec{w}) \cdot \vec{v} \end{aligned}$$

Proof. We will prove the determinant formula shown above for $(\vec{u} \times \vec{v}) \cdot \vec{w}$ briefly. By Properties 4.2.9, we have

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\hat{i} + (u_3v_1 - u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$$

and then according to Definition 4.2.1

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)^T \cdot (w_1, w_2, w_3)^T$$

$$= (u_2v_3 - u_3v_2)(w_1) + (u_3v_1 - u_1v_3)(w_2) + (u_1v_2 - u_2v_1)(w_3)$$

which is equal to

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = w_1(u_2v_3 - u_3v_2) - w_2(u_1v_3 - u_3v_1) + w_3(u_1v_2 - u_2v_1)$$

where we do cofactor expansion (Properties 2.3.3) along the third row of the determinant. □

If the volume of parallelepiped evaluated from the scalar triple product is zero, it implies that the three vectors involved are ***co-planar***, i.e. lying on the same plane.

Properties 5.3.3. Given three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, if their scalar triple product $(\vec{u} \times \vec{v}) \cdot \vec{w} = 0$ equals to zero, then $\vec{u}$, $\vec{v}$, and $\vec{w}$ are co-planar and lie on the same plane, and vice versa.

Note that if $\vec{w} = \alpha\vec{u} + \beta\vec{v}$, where α and β are some scalars, then $\vec{u}$, $\vec{v}$, $\vec{w}$ are co-planar, and $(\vec{u}\times\vec{v})\cdot\vec{w} = (\vec{u}\times\vec{v})\cdot(\alpha\vec{u}+\beta\vec{v}) = \alpha((\vec{u}\times\vec{v})\cdot\vec{u})+\beta((\vec{u}\times\vec{v})\cdot\vec{v}) = \alpha(0) + \beta(0) = 0$ as both $\vec{u} \cdot (\vec{u} \times \vec{v})$ and $\vec{v} \cdot (\vec{u} \times \vec{v})$ equal to zero.

Example 5.3.2. Find the volume of the parallelepiped formed by $\vec{u} = (1, -2, 2)^T$, $\vec{v} = (-1, -1, 1)^T$ and $\vec{w} = (2, 1, 0)^T$.

Solution. By Properties 5.3.2, the triple scalar product of the three given vectors is

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} 1 & -2 & 2 \\ -1 & -1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = -3$$

and the volume is $|-3| = 3$. □

Generalization to other dimensions

Given that the volume of parallelepiped formed by three vectors is equal to the absolute value of the corresponding matrix determinant as derived above, it is natural to ask if similar results hold for other numbers of dimension. In fact, Properties 5.3.2 can be generalized to include length, area and the so-called *n-volume* (Volume equivalent of n vectors in the n-dimensional space).

Properties 5.3.4. For n vectors of $\mathbb{R}^n$, their n-volume is the absolute value of the determinant of matrix formed by these column (or row) vectors. When $n = 1, 2, 3$, the n-volume corresponds to the usual notions of length, area and volume.

We can check the legitimacy of the last sentence in Properties 5.3.4 by noticing it is consistent with Properties 5.3.1 about area of two vectors on the x-y plane. Given $\vec{u} = (u_1, u_2)^T$ and $\vec{v} = (v_1, v_2)^T$, by Properties 5.3.4 the area of the parallelogram formed by them is

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1v_2 - v_1u_2$$

Alternatively, we can treat $\vec{u}$ and $\vec{v}$ as two three-dimensional vectors $(u_1, u_2, 0)^T$ and $(v_1, v_2, 0)^T$ such that they have a zero z-component and remain lying on the x-y plane. Then according to the previous Properties 5.3.1, the area is computed by $\|\vec{u} \times \vec{v}\|$, where by Properties 4.2.9,

$$\begin{aligned}
\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix} \\
&= \hat{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \quad \text{(Cofactor Expansion along the third row)} \\
&= (u_1v_2 - v_1u_2)\hat{k} = (0, 0, u_1v_2 - v_1u_2)^T
\end{aligned}$$

Hence $\|\vec{u} \times \vec{v}\| = \sqrt{(0)^2 + (0)^2 + (u_1v_2 - v_1u_2)^2} = u_1v_2 - v_1u_2$, which coincides with the expression we just derived from Properties 5.3.4.

Remarks

The solution of a linear system can be considered as a point/line/plane/hyperplane too, depending on the number of free variables and thus direction vectors in the complementary part (0/1/2 or more). We may also like to call it a *solution space*. However, while such shapes surely occupy space geometrically, we have been shying away from defining what really means by a *vector space* mathematically, which will be the main point of discussion in the next chapter.

5.4 Useful Vector Identities

In this section, we will prove some key vector identities that may be of utilities to some readers.

Properties 5.4.1 (Vector Triple Product)**.** The ***vector triple product*** of three vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ is defined as

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

Proof. By Properties 4.2.9, the L.H.S. can be expanded into

$$\begin{aligned}
&\vec{u} \times (\vec{v} \times \vec{w}) \\
&= (u_1\hat{i} + u_2\hat{j} + u_3\hat{k}) \\
&\quad \times [(v_2w_3 - v_3w_2)\hat{i} + (v_3w_1 - v_1w_3)\hat{j} + (v_1w_2 - v_2w_1)\hat{k}] \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_2w_3 - v_3w_2 & v_3w_1 - v_1w_3 & v_1w_2 - v_2w_1 \end{vmatrix}
\end{aligned}$$

The $\hat{i}$ component along the x-direction is

$$\begin{aligned}
&u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3) \\
&= u_2w_2v_1 + u_3w_3v_1 - u_2v_2w_1 - u_3v_3w_1
\end{aligned}$$

$$
\begin{aligned}
&= u_1w_1v_1 + u_2w_2v_1 + u_3w_3v_1 - u_1v_1w_1 - u_2v_2w_1 - u_3v_3w_1 \\
&= (u_1w_1 + u_2w_2 + u_3w_3)v_1 - (u_1v_1 + u_2v_2 + u_3v_3)w_1 \\
&= (\vec{u} \cdot \vec{w})v_1 - (\vec{u} \cdot \vec{v})w_1
\end{aligned}
$$

which is equal to the $\hat{i}$ component on the R.H.S. and similar results can be shown for the $\hat{j}$, $\hat{k}$ components and the equality establishes. □

Properties 5.4.2 (Jacobi Identity)**.**

$$\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = \mathbf{0}$$

Proof. By Properties 5.4.1, we have

$$
\begin{aligned}
&\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) \\
= &[(\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}] \\
&+ [(\vec{v} \cdot \vec{u})\vec{w} - (\vec{v} \cdot \vec{w})\vec{u}] \\
&+ [(\vec{w} \cdot \vec{v})\vec{u} - (\vec{w} \cdot \vec{u})\vec{v}] \\
= &[(\vec{u} \cdot \vec{w})\vec{v} - (\vec{w} \cdot \vec{u})\vec{v}] \\
&+ [(\vec{v} \cdot \vec{u})\vec{w} - (\vec{u} \cdot \vec{v})\vec{w}] \\
&+ [(\vec{w} \cdot \vec{v})\vec{u} - (\vec{v} \cdot \vec{w})\vec{u}] \\
= &0\vec{v} + 0\vec{w} + 0\vec{u} = \mathbf{0}
\end{aligned}
$$

□

Properties 5.4.3 (Lagrange's Identity)**.**

$$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$$

Proof. Manipulating the geometric formulae of dot/cross product, we have

$$
\begin{aligned}
\|\vec{u} \times \vec{v}\|^2 &= \|\vec{u}\|^2\|\vec{v}\|^2 \sin^2\theta && \text{(Properties 4.2.10)} \\
&= \|\vec{u}\|^2\|\vec{v}\|^2(1 - \cos^2\theta)
\end{aligned}
$$

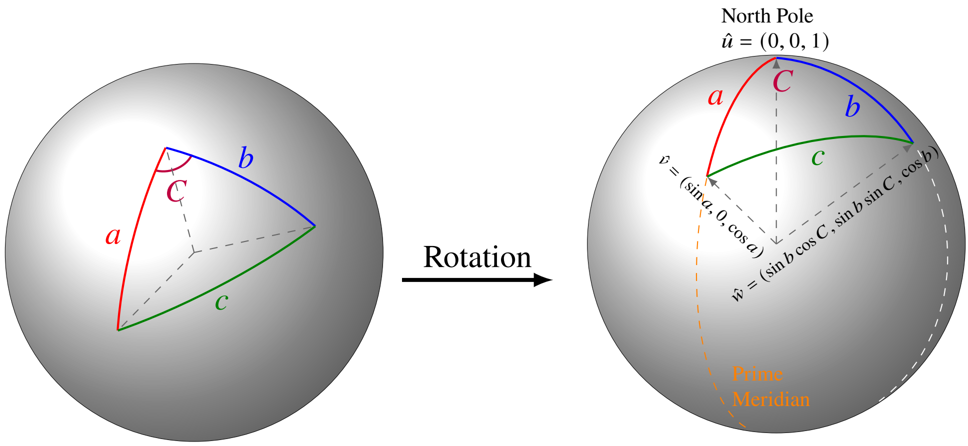


Figure 5.1: The spherical triangle on a unit sphere as described in Properties 5.4.4.

$$
\begin{aligned}
&= \|\vec{u}\|^2 \|\vec{v}\|^2 - \|\vec{u}\|^2 \|\vec{v}\|^2 \cos^2\theta \\
&= \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2 \qquad \text{(Properties 4.2.4)}
\end{aligned}
$$

□

The last identity is the ***Cosine Law for Spherical Trigonometry***.

Properties 5.4.4 (Cosine Law for Spherical Trigonometry)**.**

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

where a, b, c are the (subtended angle of) three arcs (in radians) of a spherical triangle on a unit sphere and C is the angle between the two arcs a and b, as shown in Figure 5.1.

Proof. For the given spherical triangle, we can always rotate the coordinate system (see Figure 5.1) while keeping its shape intact, such that the corner C is positioned exactly at the north pole ($\hat{u} = (0, 0, 1)^T$) and one of the two arcs starting from corner C (let's say a) lies along the Prime Meridian (angle from the

x-axis is $0°$, i.e. $y = 0$). The vector $\hat{v}$ at the end of arc a will then have a direction of $(\sin a, 0, \cos a)^T$. The vector $\hat{w}$ to the remaining corner at the intersection of arcs b and c will similarly have a z-component of $\cos b$, and its projection on x-y plane will be $\sin b$ and the x/y-component will then be $\sin b \cos C$ and $\sin b \sin C$, i.e. $\hat{w} = (\sin b \cos C, \sin b \sin C, \cos b)^T$. Now consider the dot product $\hat{v} \cdot \hat{w}$. The geometric meaning of dot product (Properties 4.2.4) implies that it is the angle between $\hat{v}$ and $\hat{w}$, that is, $\hat{v} \cdot \hat{w} = \cos c$. On the other hand,

$$\begin{aligned}
\hat{v} \cdot \hat{w} &= (\sin a, 0, \cos a)^T \cdot (\sin b \cos C, \sin b \sin C, \cos b)^T \\
&= (\sin a)(\sin b \cos C) + (0)(\sin b \sin C) + (\cos a)(\cos b) \\
&= \cos a \cos b + \sin a \sin b \cos C
\end{aligned}$$

Therefore, equaling the two expressions of $\hat{v} \cdot \hat{w}$ gives the desired formula of $\cos c = \cos a \cos b + \sin a \sin b \cos C$. □

5.5 Earth Science Applications

Example 5.5.1. Derive the *Haversine Formula* for finding the great-circle distance between any two points on a sphere with their latitudes/longitudes provided. Hence find the distance between New York (40.73 °N, 73.94 °W) and Warsaw (52.24 °N, 21.02 °E).

Solution. Denote the latitudes/longitudes of the two locations by $\varphi_{1,2}$ and $\lambda_{1,2}$. Staring from the Cosine Law for Spherical Trigonometry (Properties 5.4.4) with corner C still fixed at north pole but arc a not necessarily along the Prime Meridian, we have $C = \lambda_2 - \lambda_1$, $a = \frac{\pi}{2} - \varphi_1$, $b = \frac{\pi}{2} - \varphi_2$, and

$$\begin{aligned}
\cos c &= \cos a \cos b + \sin a \sin b \cos C \\
\cos c &= \cos\left(\frac{\pi}{2} - \varphi_1\right) \cos\left(\frac{\pi}{2} - \varphi_2\right) + \sin\left(\frac{\pi}{2} - \varphi_1\right) \sin\left(\frac{\pi}{2} - \varphi_2\right) \cos(\lambda_2 - \lambda_1) \\
\cos c &= \sin \varphi_1 \sin \varphi_2 + \cos \varphi_1 \cos \varphi_2 \cos(\lambda_2 - \lambda_1)
\end{aligned}$$

The *haversine* of an angle θ is $\mathrm{hav}(\theta) = \sin^2(\frac{\theta}{2}) = \frac{1}{2}(1 - \cos\theta)$ and therefore $\cos\theta = 1 - 2\,\mathrm{hav}(\theta)$. Subsequently,

$$\begin{aligned}
\cos c &= \sin\varphi_1 \sin\varphi_2 + \cos\varphi_1 \cos\varphi_2 (1 - 2\,\mathrm{hav}(\lambda_2 - \lambda_1)) \\
\cos c &= \sin\varphi_1 \sin\varphi_2 + \cos\varphi_1 \cos\varphi_2 - 2\cos\varphi_1 \cos\varphi_2\,\mathrm{hav}(\lambda_2 - \lambda_1) \\
\cos c &= \cos(\varphi_2 - \varphi_1) - 2\cos\varphi_1 \cos\varphi_2\,\mathrm{hav}(\lambda_2 - \lambda_1) \\
(1 - 2\,\mathrm{hav}(c)) &= (1 - 2\,\mathrm{hav}(\varphi_2 - \varphi_1)) - 2\cos\varphi_1 \cos\varphi_2\,\mathrm{hav}(\lambda_2 - \lambda_1) \\
\mathrm{hav}(c) &= \mathrm{hav}(\varphi_2 - \varphi_1) + \cos\varphi_1 \cos\varphi_2\,\mathrm{hav}(\lambda_2 - \lambda_1)
\end{aligned}$$

where we have used the trigonometric identity $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$ in the middle. The Haversine Formula is now established and we can use it to calculate the angle c subtended by the arc between two locations and hence their distance by $d = rc$ where r is the radius (of the Earth, 6370 km). For New York (40.73 °N, 73.94 °W) and Warsaw (52.24 °N, 21.02 °E), $\lambda_1 = -73.94°$, $\lambda_2 = 21.02°$, $\varphi_1 = 40.73°$, $\varphi_2 = 52.24°$, and

$$\begin{aligned}
\mathrm{hav}(c) &= \mathrm{hav}(52.24° - 40.73°) \\
&\qquad + \cos(40.73°)\cos(52.24°)\,\mathrm{hav}(21.02° - (-73.94°)) \\
&= \mathrm{hav}(11.51°) + \cos(40.73°)\cos(52.24°)\,\mathrm{hav}(94.96°) \\
&= \sin^2(\frac{11.51°}{2}) + \cos(40.73°)\cos(52.24°)\sin^2(\frac{94.96°}{2}) \\
\sin^2(\frac{c}{2}) &\approx 0.26214 \\
c &\approx 61.6° = 1.075\,\mathrm{rad}
\end{aligned}$$

and therefore the required distance is $d = rc = (6370\,\mathrm{km})(1.075\,\mathrm{rad}) \approx 6848\,\mathrm{km}$. The value computed by the Haversine Formula will be slightly off from the true value since the Earth is not a perfect sphere but rather an oblate one. □

Example 5.5.2. The Earth's magnetic field can be approximated by a magnetic dipole, so that the magnetic field lines on the Earth's surface are oriented from the geomagnetic North Pole to geomagnetic South Pole (like longitudinal lines but for the geomagnetic dipole). In 2020, the geomagnetic North Pole is at

80.7 °N, 72.7 °W. Find the magnetic declination (angle from the geographic North to geomagnetic North) at Tokyo (35.65 °N, 139.84 °E) according to this *geomagnetic dipole model*.

Solution. To find the magnetic declination we need to calculate the three arcs of the spherical triangle with its three corners at the geographic/geomagnetic North Pole and Tokyo. The arc distance between geographic/geomagnetic North Pole d is simply $90° - 80.7° = 9.3°$. Similarly, the arc from the geographic North Pole to Tokyo is $a = 90° - 35.65° = 54.35°$. We can use the Haversine Formula derived in the last example to obtain the arc from the geomagnetic North Pole to Tokyo, which yields

$$\begin{aligned}
\operatorname{hav}(t) &= \operatorname{hav}(80.7° - 35.65°) \\
&\quad + \cos(35.65°)\cos(80.7°)\operatorname{hav}((-72.7°) - 139.84°) \\
&= \operatorname{hav}(45.05°) + \cos(35.65°)\cos(80.7°)\operatorname{hav}(-212.54°) \\
&\approx 0.26777 \\
c &\approx 62.3°
\end{aligned}$$

Denote the declination angle by D. By Properties 5.4.4, we have

$$\begin{aligned}
\cos d &= \cos a \cos t + \sin a \sin t \cos D \\
\cos(9.3°) &= \cos(54.35°)\cos(62.3°) + \sin(54.35°)\sin(62.3°)\cos D \\
\cos D &\approx 0.9951 \\
D &\approx \pm 5.7°
\end{aligned}$$

To determine the sign, we note that concluded from the longitudes of Tokyo and geomagnetic North, the geomagnetic North is located to the east of Tokyo, and hence $D = 5.7$ °E. However, we note that the actual declination is 7.8 °W which has an opposite sign and is far from our answer (you can extract the value from https://www.ngdc.noaa.gov/geomag/calculators/magcalc.shtml). The reason is that the geomagnetic dipole is only a rough first-order approximation, while in reality the Earth's magnetic field has a much more complex structure. □

5.6 Python Programming

Projection as in Properties 5.2.1 can be calculated by `numpy` functions and let's wrap them up in our self-defined function as below.

```
def scalar_projection(u, v):
    """ Calculates the scalar projection of v onto u. """
    return np.dot(u,v) / np.linalg.norm(u)
```

This computes the scalar projection of $\vec{v}$ onto $\vec{u}$. Testing with Example 5.2.1 shows

```
u = np.array([4., 1., -3.])
v = np.array([-2., 3., -1.])
print(scalar_projection(u, v))
```

a consistent output of `-0.39223`. Incorporating the unit vector function (`unit_vector()`) defined in the last chapter's programming section, we obtain the vector projection.

```
def vector_projection(u, v):
    """ Calculates the vector projection of v onto u. """
    return scalar_projection(u, v) * unit_vector(u)

print(vector_projection(u, v))
```

This results in `[-0.3077 -0.0769 0.2308]` which matches the example's answer. Area of parallelogram formed by two vectors is the magnitude of their cross product and the corresponding function is typed below.

```
def area_parallelogram(u, v):
    """ Calculate the area of parallelogram formed by two
        vectors u and v. """
    return np.linalg.norm(np.cross(u,v))
```

`print(area_parallelogram(u, v))` then gives `18.974`. Meanwhile, the function to compute volume of parallelepiped made up of three vectors can be defined such that it uses the determinant formula in Properties 5.3.2.

```
def volume_parallelepiped(u, v, w):
    """ Calculate the volume of parallelepiped formed by two
        vectors u, v, w. """
    return np.abs(np.linalg.det(np.c_[u,v,w]))
```

```
w = np.array([1., 2., -3.])
print(volume_parallelepiped(u, v, w))
```

(`np.c_[]` is a short hand of combining arrays column by column) produces `14.00000...04` due to numerical round-off error (the true answer would be just 14). Finally, let's conclude this section by defining the Haversine Formula in Example 5.5.1.

```
def Haversine_dist(latlon1, latlon2):
    """ Haversine Formula for computing the great-circle
        distance between two places on the Earth.
        Input: (lat1, lon1), (lat2, lon2) in degrees.
        Output: Great-circle distance in km.
    """
    R_Earth = 6370 # Earth's Radius
    lat1, lon1 = latlon1[0], latlon1[1]
    lat2, lon2 = latlon2[0], latlon2[1]
    # Converting degree to radian
    lat1_rad, lon1_rad, lat2_rad, lon2_rad = np.deg2rad(lat1),
         np.deg2rad(lon1), np.deg2rad(lat2), np.deg2rad(lon2)
    # Haversine's Formula
    hav_c = np.sin((lat2_rad-lat1_rad)/2)**2 + np.cos(lat1_rad
        )*np.cos(lat2_rad)*np.sin((lon2_rad-lon1_rad)/2)**2
    arc_c = 2*np.arcsin(np.sqrt(hav_c)) # Inverting to get the
         great-circle arc angle
    return(R_Earth*arc_c) # Arc angle to arc length
```

Using the latitudes and longitudes of New York and Warsaw in Example 5.5.1 for testing, `Haversine_dist((40.73, -73.94), (52.24, 21.02))` outputs `6847.76`.

5.7 Exercises

Exercise 5.1 Parameterize the following equations into vector form.

(a) $6x + 8y = 9$,

(b) $x + 9y + 9z = 7$,

(c) $y = 3, -\infty < x < \infty$, and

(d) $2x + z = 9, -\infty < y < \infty$.

Exercise 5.2 Eliminate the parameters and find the direct equation.

(a)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} + s \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 8 \\ 0 \\ 5 \end{bmatrix}$$

where $-\infty < s, t < \infty$.

Exercise 5.3 Find the distance of the point $(3, 2, 9)^T$ to the plane $x+2y+5z = 10$, as well as the distance of the point $(3, 2, 9)^T$ to the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

where $-\infty < t < \infty$.

Exercise 5.4 Prove that the shortest distance between two lines, $\vec{u} = \vec{a} + s\hat{l}$ and $\vec{v} = \vec{b} + t\hat{m}$, where $-\infty < s, t < \infty$, $\vec{a}, \vec{b}$ are some arbitrary vectors and $\hat{l}, \hat{m}$ are some fixed, non-parellel unit vectors representing direction of the two lines, is

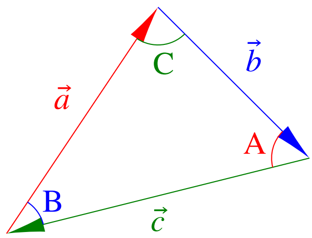
$$\text{Dist}(u, v) = \frac{(\hat{a} - \hat{b}) \cdot (\hat{l} \times \hat{m})}{\left\|\hat{l} \times \hat{m}\right\|}$$

Hints: Geometrically, the distance between these two lines is the projection of any vector from one line to another onto the vector normal to the plane made by $\hat{l}$ and $\hat{m}$.

$$\frac{(\vec{v}-\vec{u})\cdot(\hat{l}\times\hat{m})}{\left\|\hat{l}\times\hat{m}\right\|}$$

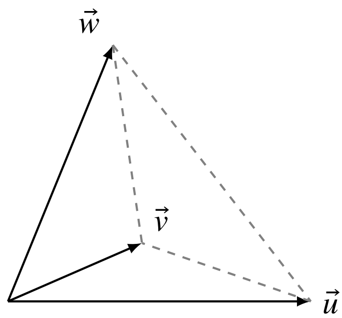
Draw a diagram to convince yourself it is true. What does it imply if $\vec{a}\cdot(\hat{l}\times\hat{m}) = \vec{b}\cdot(\hat{l}\times\hat{m})$?

Exercise 5.5 Prove Sine Law with vector notation by considering the triangle below



and equating three expressions of its area $\frac{1}{2}\left\|\vec{a}\times\vec{b}\right\| = \frac{1}{2}\left\|\vec{b}\times\vec{c}\right\| = \frac{1}{2}\|\vec{c}\times\vec{a}\|$. Properties 5.3.1 will be useful.

Exercise 5.6 By extending Properties 5.3.2, derive a vector formula for the volume of a tetrahedron (pyramid).



Exercise 5.7 For $\vec{u} = (1,2,3)^T$, $\vec{v} = (2,1,5)^T$, $\vec{w} = (1,4,0)^T$, find

(a) Area of the parallelogram formed by $\vec{u}$ and $\vec{v}$,

(b) Volume of the parallelepiped formed by $\vec{u}$, $\vec{v}$ and $\vec{w}$,

(c) Redo the above for $\vec{w} = (1,5,4)^T$, what does the result tell you?

Exercise 5.8 Find the geometric interpretation of solutions of the following systems of linear equations.

(a)

$$\begin{cases} x + 2y + 2z & = 3 \\ 3x - y + 3z & = 2 \\ x - 2y - z & = -1 \end{cases}$$

(b)

$$\begin{cases} 2x - y - z & = 3 \\ x + y + 2z & = -1 \\ x + 4y + 7z & = -6 \end{cases}$$

Chapter 6

Vector Spaces and Coordinate Bases

The previous chapters have provided a basic understanding of matrices and vectors separately. What bridge these two quantities together are the concepts of *vector (sub)spaces*, *linear combination*, *span*, *linear independence*. With all of these, we will revisit the process of Gaussian Elimination from the view of *column-row factorization*. Then, we will learn how to find *coordinate bases* for vector spaces and also investigate about the so-called *four fundamental subspaces* induced by a matrix and see how they are interconnected.

6.1 Making of the Real n-space $\mathbb{R}^n$

6.1.1 $\mathbb{R}^n$ as a Vector Space

We have briefly mentioned in Definition 4.1.2 that the real n-space $\mathbb{R}^n$ is mathematically a vector space, but without stating the actual requirements. In fact, to be qualified as a ***vector space***, a set has to satisfy the ten axioms below. We will limit ourselves to ***real vector spaces*** for now.

Definition 6.1.1 (Axioms of a Real Vector Space)**.** A real vector space is a non-empty set $\mathcal{V}$ with the zero vector $\mathbf{0}$, such that for all elements (vectors) $\vec{u}, \vec{v}, \vec{w} \in \mathcal{V}$ in the set, and real numbers (as the *scalars*) $a, b \in \mathbb{R}$ (for a complex vector space replace $\mathbb{R}$ by $\mathbb{C}$ here), we have

1. $\vec{u} + \vec{v} \in \mathcal{V}$ (Closure under Vector Addition: Addition between two vectors is defined and the resulting vector is still in the vector space.)
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Commutative Property of Addition)
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (Associative Property of Addition)
4. $\vec{u} + \mathbf{0} = \mathbf{0} + \vec{u} = \vec{u}$ (Zero Vector as the Additive Identity)
5. For any $\vec{u}$, there exists $\vec{w}$ such that $\vec{u} + \vec{w} = \mathbf{0}$. This $\vec{w}$ is denoted as $-\vec{u}$. (Existence of Additive Inverse)
6. $a\vec{u} \in \mathcal{V}$ (Closure under Scalar Multiplication: Multiplying a vector by any scalar (real number for a real vector space) is defined and the resulting vector is still in the vector space.)
7. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ (Distributive Property of Scalar Multiplication)
8. $(a + b)\vec{u} = a\vec{u} + b\vec{u}$ (Distributive Property of Scalar Multiplication)
9. $a(b\vec{u}) = (ab)\vec{u}$ (Associative Property of Scalar Multiplication)
10. $1\vec{u} = \vec{u}$ (The real number 1 as the Multiplicative Identity)

The real n-space $\mathbb{R}^n$ satisfies all the axioms above and is finite-dimensional, particularly n-dimensional (the notion of dimension here should be intuitive, but we will go through it more precisely later), with addition and scalar multiplication being the usual ones as defined in Section 4.1.2, and the zero vector is simply $\mathbf{0} = (0, 0, 0, \ldots, 0)^T$ with n zeros. We will not do it here but interested readers can try to justify all of them. To build the definition of a vector space from these axioms allows the generalization and application of its utilities to more abstract structures. However, for most usages, we will focus on $\mathbb{R}^n$[1], and the vector space axioms are provided above mainly for reference. We defer the treatment of complex vector spaces to Chapter 8.

[1]We actually have a very good reason to do so, as we will see in the next chapter: any n-dimensional real vector space is *isomorphic* to and can be treated like $\mathbb{R}^n$.

6.1.2 Subspaces of $\mathbb{R}^n$

It will be very boring if we consider only the entire $\mathbb{R}^n$ as a vector space. In last chapter, we show that geometrically there can be lower-dimensional shapes like lines/planes/hyperplanes residing in $\mathbb{R}^n$. This raises the question if we can similarly find ***subspaces*** embedded in $\mathbb{R}^n$ that is a subset of $\mathbb{R}^n$ which still fulfills the vector space axioms such that it is a vector space in its own right. Nevertheless, to determine if a subset of vector space is a subspace, we don't need to check all the ten axioms but rather just two of them.

Theorem 6.1.2 (Criteria for a Subspace)**.** If $\mathcal{W}$ is a non-empty subset of a (real) vector space $\mathcal{V}$ (i.e. $\mathcal{W} \subseteq \mathcal{V}$), then $\mathcal{W}$ is called a (real) subspace of $\mathcal{V}$ if the following criteria are satisfied:

1. For any $\vec{u}, \vec{v} \in \mathcal{W}$, $\vec{u} + \vec{v} \in \mathcal{W}$ (Closed under Addition)

2. For any scalar a ($\in \mathbb{R}$) and $\vec{u} \in \mathcal{W}$, $a\vec{u} \in \mathcal{W}$ (Closed under Scalar Multiplication), particularly when $a = 0$, $0\vec{u} = \mathbf{0} \in \mathcal{W}$ so that a subspace always contains the zero vector of $\mathcal{V}$.

These are the same conditions of (1) and (6) in Definition 6.1.1. Or equivalently, for any $\vec{u}, \vec{v} \in \mathcal{W}$ and two scalars a and b, $a\vec{u} + b\vec{v} \in \mathcal{W}$.

Example 6.1.1. Consider the following subsets of $\mathbb{R}^2$ and decide if they are subspaces of $\mathbb{R}^2$ by verifying the two criteria listed in Theorem 6.1.2.

(a) The line $x - 2y = 0$,

(b) The y-axis,

(c) The positive y-axis,

(d) The line $2x + y = 1$,

(e) The parabola $y = x^2$,

(f) The point $(-1, 1)^T$,

(g) The first quadrant $x > 0$, $y > 0$,

(h) The origin $\mathbf{0} = (0, 0)^T$,

(i) $\mathbb{R}^2$ itself.

Solution. (a) The vector form of the line is $\mathcal{W} = \{(x, y)^T = t(2, 1)^T \mid -\infty < t < \infty\}$. To check the first condition, let's say $\vec{u} = t_1(2, 1)^T \in \mathcal{W}$ and $\vec{v} = t_2(2, 1)^T \in \mathcal{W}$ are vectors in $\mathcal{W}$ for some t_1 and t_2, then $\vec{u} + \vec{v} = t_1(2, 1)^T + t_2(2, 1)^T = (t_1 + t_2)(2, 1)^T = s(2, 1)^T \in \mathcal{W}$ also lies on the straight line and is a vector in $\mathcal{W}$ where $s = t_1 + t_2$, so $\mathcal{W}$ is closed under addition. To check the second condition, this time we simply let $\vec{u} = t(2, 1)^T \in \mathcal{W}$. Subsequently, $a\vec{u} = at(2, 1)^T = r(2, 1)^T \in \mathcal{W}$, for any scalar a and $r = at$, so it is closed under scalar multiplication. Hence the line $x - 2y = 0$ is a subspace of $\mathbb{R}^2$.

(b) Same arguments as above but with $\mathcal{W} = \{(x, y)^T = t(0, 1)^T \mid -\infty < t < \infty\}$, so the y-axis is also a subspace of $\mathbb{R}^2$.

(c) For any point on the positive y-axis, multiplying it by a negative number places it on the negative y-axis instead, so it is not closed under scalar multiplication and thus not a subspace of $\mathbb{R}^2$.

(d) Denote the collection of points on the line as $\mathcal{W}$. Pick $\vec{u} = (1, -1)^T \in \mathcal{W}$ and $\vec{v} = (0, 1)^T \in \mathcal{W}$, then $\vec{u} + \vec{v} = (1, 0)^T \notin \mathcal{W}$ as $2(1) + (0) = 2 \neq 1$, so it is not closed under addition and fails to be a subspace of $\mathbb{R}^2$.

(e) Denote the collection of points on the parabola as $\mathcal{W}$. Pick $\vec{u} = (1, 1)^T \in \mathcal{W}$ and $\vec{v} = (2, 4)^T \in \mathcal{W}$, then $\vec{u} + \vec{v} = (3, 5)^T \notin \mathcal{W}$ is apparently not on the parabola, so it is not closed under addition and can't be a subspace of $\mathbb{R}^2$.

(f) It is easy to see that it fails to be closed under either addition or scalar multiplication (for example, take $a(-1, 1)^T$ with $a \neq 1$) and is not a subspace of $\mathbb{R}^2$.

(g) Denote the collection of points on the first quadrant as $\mathcal{W}$. Pick $\vec{u} = (1,1)^T \in \mathcal{W}$, then $(-1)\vec{u} = -(1,1)^T = (-1,-1)^T \notin \mathcal{W}$ is outside the first quadrant. Therefore, it is not closed under scalar multiplication and hence not a subspace of $\mathbb{R}^2$.

(h) It trivially satisfies the two criteria ($\mathbf{0}$ is the only element in the set, $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $a\mathbf{0} = \mathbf{0}$ for any scalar a) and is a subspace of $\mathbb{R}^2$.

(i) $\mathbb{R}^2$ is a vector space to begin with and technically a subset of itself (it contains itself, although not a proper one) so by definition it is a subspace of $\mathbb{R}^2$.

□

Generalizing the above discussion, we can easily infer that for $\mathbb{R}^2$, only the origin (the zero subspace), an infinitely long straight line that passes through the origin, or $\mathbb{R}^2$ itself can be its subspaces (see the schematic in Figure 6.1). We often use the phrase ***proper subspaces*** to exclude the accommodating vector space itself ($\mathbb{R}^2$ in this case). For any $\mathbb{R}^n$, the ***zero subspace*** $\{\mathbf{0}\}$ and ***improper subspace*** $\mathbb{R}^n$ are always two subspaces of it.

Short Exercise: Determine if the following subsets of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.[2]

(a) The origin $\mathbf{0} = (0,0,0)^T$,

(b) The point $(1,2,3)^T$,

(c) The line $(x,y,z)^T = t(-1,1,2)^T$ for any scalar t,

(d) The line $(x,y,z)^T = (1,-1,3) + t(1,2,-1)^T$ for any scalar t,

(e) The plane $x + 2y - 3z = 0$,

(f) The plane $x + y + 4z = 5$,

(g) $\mathbb{R}^3$ itself,

(h) The sphere $x^2 + y^2 + z^2 = 1$,

[2]Yes, No, Yes, No, Yes, No, Yes, No, No. In fact, all possible subspaces of $\mathbb{R}^3$ are $\{\mathbf{0}\}$, any infinitely long line/extending plane through the origin and $\mathbb{R}^3$ itself.

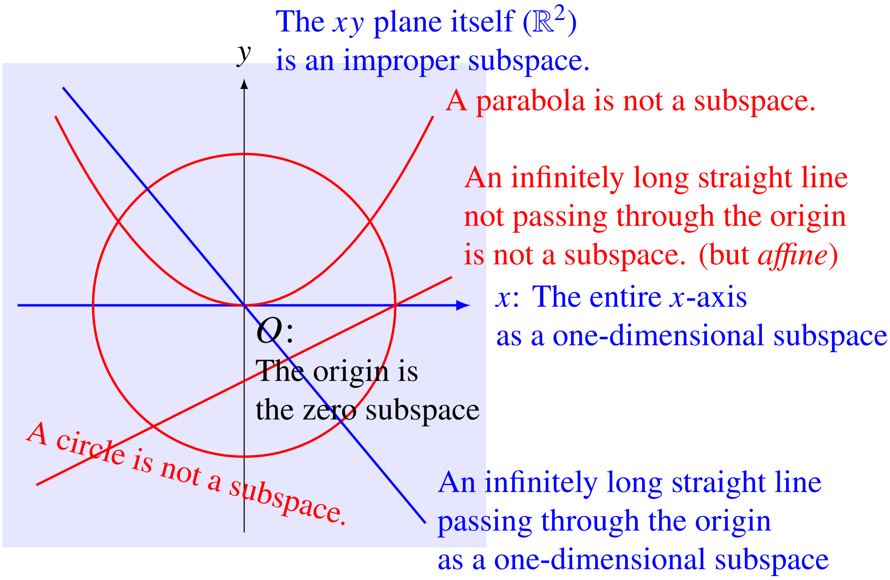


Figure 6.1: Some examples (blue) and non-examples (red) of subspaces in $\mathbb{R}^2$.

(i) The cone $x^2 + y^2 = z^2$.

Further generalization motivated by the short exercise above leads to an intuitive result that, for $\mathbb{R}^n$, all its possible subspaces are geometrically "flat shapes" that pass through the origin and extend infinitely. On the other hand, any "curved shape" will not qualify as a subspace. From now on, we assume all vector (sub)spaces mentioned are finite-dimensional unless otherwise specified.

6.1.3 Span by Linear Combinations of Vectors

The last section sees subspaces from a top-down perspective as some subsets of a larger vector space. Here, we are going to take another look at them with a bottom-up perspective, about how to generate a subspace of $\mathbb{R}^n$ from some vectors. To do so, we need to first understand what is a ***linear combination*** of vectors.

Definition 6.1.3 (Linear Combination of Vectors)**.** A linear combination of vectors $\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}, \ldots, \vec{v}^{(q)} \in \mathcal{V}$ where $\mathcal{V}$ is some vector space has the form of

$$\sum_{j=1}^{q} c_j \vec{v}^{(j)} = c_1\vec{v}^{(1)} + c_2\vec{v}^{(2)} + c_3\vec{v}^{(3)} + \cdots + c_q\vec{v}^{(q)}$$

where the coefficients c_j are some scalars (real numbers for a real vector space) and the amount of vectors q is *finite*.

As a small example, if there are two vectors $\vec{u} = (1,2)^T$ and $\vec{v} = (3,4)^T \in \mathbb{R}^2$, then $\vec{h} = (5,6)^T \in \mathbb{R}^2$ can be written as a linear combination of $\vec{u}$ and $\vec{v}$ because $\vec{h} = (5,6)^T = -(1,2)^T + 2(3,4)^T = -\vec{u} + 2\vec{v}$.

Short Exercise: If $\vec{h} = (1,4)^T$ instead, express it as a linear combination of $\vec{u}$ and $\vec{v}$.[3]

Attentive readers may realize that the short exercise above can be considered as a task to find out the solution (if any) for the system

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Extending this, to decide whether a vector $\vec{h} \in \mathbb{R}^n$ can be written as the linear combination of other vectors $\vec{v}^{(j)} \in \mathbb{R}^n$, $j = 1, 2, \ldots, q$, is equivalent to determining whether the linear system $A\vec{x} = \vec{h}$ has a solution, where A equals to (writing out $\vec{v}^{(j)}$ in a matrix column by column)

$$A = \left[\begin{array}{c|c|c|c} | & | & & | \\ \vec{v}^{(1)} & \vec{v}^{(2)} & \cdots & \vec{v}^{(q)} \\ | & | & & | \end{array}\right]$$

Here, the matrix product $A\vec{x}$ is a compact way to represent a linear combination of the column vectors that compose A.

[3] $(1,4)^T = 4(1,2)^T - (3,4)^T$.

Properties 6.1.4. A linear combination $c_1\vec{v}^{(1)} + c_2\vec{v}^{(2)} + c_3\vec{v}^{(3)} + \cdots + c_q\vec{v}^{(q)}$ made up of some vectors $\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}, \ldots, \vec{v}^{(q)} \in \mathbb{R}^n$ as in Definition 6.1.3, can be expressed by the matrix product $A\vec{x}$, where

$$A = \left[\begin{array}{c|c|c|c} | & | & & | \\ \vec{v}^{(1)} & \vec{v}^{(2)} & \cdots & \vec{v}^{(q)} \\ | & | & & | \end{array}\right] \qquad \vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_q \end{bmatrix}$$

From now on, we will just simply write $A = [\vec{v}^{(1)}|\vec{v}^{(2)}|\cdots|\vec{v}^{(q)}]$ and similarly for other matrices when applicable to save space.

From this perspective, the first/second/last column of a matrix A can be extracted by

$$A\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad A\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad A\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

and it goes similarly for any other column. Below is a small example to demonstrate the equivalence between matrix-vector products and linear combinations.

$$\begin{bmatrix} 5 & 1 & -1 & 2 \\ 2 & 3 & 0 & 7 \\ 4 & -2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 & -1 & 2 \\ 2 & 3 & 0 & 7 \\ 4 & -2 & 3 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -1 & 2 \\ 2 & 3 & 0 & 7 \\ 4 & -2 & 3 & 1 \end{bmatrix}\left(\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$= (-1)\begin{bmatrix} 5 \\ 2 \\ -4 \end{bmatrix} + (2)\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + (3)\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 4 \\ 1 \end{bmatrix}$$

Example 6.1.2. Show that $\vec{h} = (2, 4, 3)^T$ cannot be written as a linear combination of $\vec{v}^{(1)} = (-1, 0, 1)^T$ and $\vec{v}^{(2)} = (1, 1, 0)^T$.

Solution. Following the above discussion, the objective is equivalent to showing that the linear system

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

has no solution. We can apply the method of Gaussian Elimination as demonstrated in Section 3.2.1, which leads to

$$\left[\begin{array}{cc|c} -1 & 1 & 2 \\ 0 & 1 & 4 \\ 1 & 0 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \\ -1 & 1 & 2 \end{array}\right] \qquad R_1 \leftrightarrow R_3$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{array}\right] \qquad R_3 + R_1 - R_2 \to R_3$$

The last row is inconsistent and hence there is no solution to the linear system and $\vec{h}$ cannot be expressed by a linear combination of $\vec{v}^{(1)}$ and $\vec{v}^{(2)}$. □

With the idea of linear combination, we can define the ***span*** generated by a *finite* set of vectors.

Definition 6.1.5 (Span)**.** The span of q vectors in a set $\mathcal{S} = \{\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}, \dots, \vec{v}^{(q)}\}$ where all of them are from the same vector space $\mathcal{V}$, i.e. $\vec{v}^{(j)} \in \mathcal{V}$ for $j = 1, 2, \dots, q$, is consisted of all their possible linear combinations as given in Definition 6.1.3, and is denoted by

$$\text{span}(\mathcal{S}) = \{\sum_{j=1}^{q} c_j \vec{v}^{(j)} \mid \text{for any scalar } c_j \text{ with } \vec{v}^{(j)} \in \mathcal{S}\}$$

Again we will limit ourselves to the cases where the coefficients c_j are real and q has to be finite. If the $\vec{v}^{(j)}$ are from the real n-space, i.e. $\vec{v}^{(j)} \in \mathbb{R}^n$, then as suggested by Properties 6.1.4, the span can be thought in the form of

$$\text{span}(\mathcal{S}) = \{A\vec{x} \mid \text{for any } \vec{x} \in \mathbb{R}_q\}$$

with $A = [\vec{v}^{(1)}|\vec{v}^{(2)}|\cdots|\vec{v}^{(q)}]$ is an $n \times q$ matrix and $\vec{x} = (c_1, c_2, \dots, c_q)^T$ being the coefficient vector.

For example, the span of $\mathcal{S}_1 = \{(-1, 1)^T\}$ is simply $t(-1, 1)^T$ where $-\infty < t < \infty$, or the line $y = -x$. The span of $\mathcal{S}_2 = \{(1, 0, 2)^T, (0, 1, -1)^T\}$ (notice that the two vectors are not a constant multiple of each other and thus non-parallel) is $s(1, 0, 2)^T + t(0, 1, -1)^T$ where $-\infty < s, t < \infty$, or represented by the plane $2x - y - z = 0$ (see Section 5.1.3). Adding more vectors in the spanning set does not always imply the span will be larger. For example, the span of $\mathcal{S}_3 = \{(1, 0)^T, (0, 1)^T\}$ and $\mathcal{S}_4 = \{(1, 0)^T, (0, 1)^T, (1, 1)^T, (1, -1)^T\}$ are both apparently $\mathbb{R}^2$. This issue will be addressed in later subsections.

Example 6.1.3. Show that any vector in $\mathbb{R}^2$ can be written as infinitely many different linear combinations of the four vectors in the set $\mathcal{S}_4$ mentioned above.

Solution. This is to decide if the linear system

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

has infinitely many solutions for any pair of (x, y). The augmented form

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & x \\ 0 & 1 & 1 & -1 & y \end{array}\right]$$

is already in reduced row echelon form. There is a corresponding pivot for both x and y in the first two columns, and no zero row is present, which means that there would not be any inconsistency and we can always construct a family of solutions by setting the non-pivotal unknowns to be free variables, let's say $c_3 = s$ and $c_4 = t$. Then we have $c_1 = x - s - t$, $c_2 = y - s + t$ from the rows. As a result, any linear combination in the form of

$$(x - s - t)(1, 0)^T + (y - s + t)(0, 1)^T + s(1, 1)^T + t(1, -1)^T$$

will produce the vector $(x, y)^T$ with any value of s and t as desired, and there are infinitely many of them. This example shows that a vector (in this case any arbitrary vector of $\mathbb{R}^2$) can possibly be written as more than one linear combinations of the constituent vectors in the spanning set (here $\mathcal{S}_4$). □

An essential property of spans is that they are subspaces and vice versa. This fact integrates the top-down (it is a subset of a larger vector space) and bottom-up (it is formed by linear combinations of vectors) view of subspaces.

Properties 6.1.6. The span of a subset of some vectors in $\mathcal{V}$ is a subspace of $\mathcal{V}$. A subspace of $\mathcal{V}$ is always some span (not necessarily unique) of some vectors in $\mathcal{V}$.

We leave the proof for showing the span $\longrightarrow$ subspace direction in the footnote[4] and that for the subspace $\longrightarrow$ span direction is out of the scope. Subsequently, we

[4]We check if the two criteria in Theorem 6.1.2 hold for a span. Let the span be the one defined in Definition 6.1.5, then any vector in the span can be written as $\sum_{j=1}^{q} c_j \vec{v}^{(j)}$ for some constants c_j. Let $\vec{u} = \sum_{j=1}^{q} \alpha_j \vec{v}^{(j)} \in \text{span}(\mathcal{S})$ and $\vec{v} = \sum_{j=1}^{q} \beta_j \vec{v}^{(j)} \in \text{span}(\mathcal{S})$ are both in the span for some sets of constants α_j and β_j, then their sum $\vec{u} + \vec{v} = \sum_{j=1}^{q} \alpha_j \vec{v}^{(j)} + \sum_{j=1}^{q} \beta_j \vec{v}^{(j)} = \sum_{j=1}^{q} (\alpha_j + \beta_j) \vec{v}^{(j)} = \sum_{j=1}^{q} \gamma_j \vec{v}^{(j)} \in \text{span}(\mathcal{S})$ where $\gamma_j = \alpha_j + \beta_j$ is also in the span and hence it closed under addition. Similarly, writing $a\vec{w} = a(\sum_{j=1}^{q} \beta_j \vec{v}^{(j)}) = \sum_{j=1}^{q} (a\beta_j) \vec{v}^{(j)}$ shows that $a\vec{w} \in \text{span}(\mathcal{S})$ and the span is closed under scalar multiplication and we are done.

say $\mathcal{W} = \text{span}(\mathcal{S})$ is a subspace of $\mathcal{V}$ ($\mathcal{W} \subseteq \mathcal{V}$) *generated* by the set $\mathcal{S}$ and $\mathcal{S}$ is known as a ***spanning/generating set*** for $\mathcal{W}$. This duality between subspace and span is consistent when we look at them from a geometric point of view: as mentioned at the end of Section 6.1.2 before, subspaces can be thought of as "flat shapes", or put differently, "linear objects" of infinite extent; Meanwhile a span is precisely consisted of all possible linear combinations of vectors. These spanning vectors represent straight directions that extend infinitely long and also produce a "linear shape". (also see Figure 6.2) Applying Properties 6.1.6 on Definition 6.1.5, we can say that the span generated by the column vectors $\vec{v}^{(j)}$ in $A = [\vec{v}^{(1)}|\vec{v}^{(2)}| \cdots |\vec{v}^{(q)}]$ forms a subspace better known as the ***column space*** of A.

Definition 6.1.7 (Column Space)**.** The column space of an $p \times q$ matrix A is the span generated by the q column vectors that make up A as suggested in Definition 6.1.5.

Finally, a result related to Properties 6.1.6 is noted below.

Properties 6.1.8. Any subspace of $\mathcal{V}$ that contains a subset $\mathcal{S}$ of some vectors in $\mathcal{V}$ also contains $\text{span}(\mathcal{S})$.

6.1.4 Linear Independence, CR Factorization

Another key concept in this chapter is the problem of linear independence, which has profound implications in Linear Algebra. Given a set of vectors, if every one of them can not be expressed as the linear combination of other members, or speaking loosely, each of them is not "dependent" on other vectors, then such a set of vectors is said to be ***linearly independent***. Otherwise, if at least one of them can be expressed as some linear combination of other vectors, then the set is known as ***linearly dependent***.

To check linear independence of q vectors, one may indeed directly show that for every vector $\vec{v}^{(j)}$ in the set, $j = 1, 2, 3, \ldots, q$, it cannot be written as the linear combination of other vectors $\vec{v}^{(k)}$ in the set, $k \neq j$. A slightly easier way is looking at the linear combination of just the first $j-1$ vectors (from $\vec{v}^{(1)}$ up to $\vec{v}^{(j-1)}$) for $\vec{v}^{(j)}$. However, it is very tedious if the amount of vectors is large. Fortunately, we have a theorem which significantly simplifies our work.

Theorem 6.1.9. For a set of vectors $\mathcal{S} = \{\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}, \ldots, \vec{v}^{(q)}\}$ where $\vec{v}^{(j)} \in \mathcal{V}$, $j = 1, 2, \ldots, q$, they are linearly independent if and only if, the equation $c_1\vec{v}^{(1)} + c_2\vec{v}^{(2)} + c_3\vec{v}^{(3)} + \cdots + c_q\vec{v}^{(q)} = \mathbf{0}$ has the trivial solution where all the coefficients are zeros ($c_j = \mathbf{0}$) as its unique solution. Using the language in Properties 6.1.4 when $\vec{v}^{(j)} \in \mathbb{R}^n$, it means that the homogeneous linear system $A\vec{x} = \mathbf{0}$ where $A = [\vec{v}^{(1)}|\vec{v}^{(2)}|\vec{v}^{(3)}|\cdots|\vec{v}^{(q)}]$ only has the trivial solution $\vec{x} = \mathbf{0}$.

Proof. The "if" direction: We need to show that $c_j = \mathbf{0}$ being the only solution to $c_1\vec{v}^{(1)} + c_2\vec{v}^{(2)} + c_3\vec{v}^{(3)} + \cdots + c_q\vec{v}^{(q)} = \mathbf{0}$ implies that $\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}, \ldots, \vec{v}^{(q)}$ are linearly independent. We can prove the contrapositive where the opposite of the conclusion, $\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}, \ldots, \vec{v}^{(q)}$ are linearly dependent, implies the opposite of the hypothesis, i.e. there is non-trivial solution to the equation. This requires that at least one of these vectors, without the loss of generality let's say $\vec{v}^{(1)}$, can be written as the linear combination of other vectors in the form of

$$\vec{v}^{(1)} = a_2\vec{v}^{(2)} + a_3\vec{v}^{(3)} + \cdots + a_q\vec{v}^{(q)}$$

Rearranging gives

$$\vec{v}^{(1)} - a_2\vec{v}^{(2)} - a_3\vec{v}^{(3)} - \cdots - a_q\vec{v}^{(q)} = \mathbf{0}$$

which shows that the coefficients $c_1 = 1, c_2 = -a_2, c_3 = -a_3, \ldots, c_q = -a_q$ is another solution other than $c_j = \mathbf{0}$ to $c_1\vec{v}^{(1)} + c_2\vec{v}^{(2)} + c_3\vec{v}^{(3)} + \cdots + c_q\vec{v}^{(q)} = \mathbf{0}$ (concerning c_1 particularly).
The "only if" direction: We want to show the converse that linear independence of $\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}, \ldots, \vec{v}^{(q)}$ only permits $c_j = \mathbf{0}$ as the unique solution to $c_1\vec{v}^{(1)} + c_2\vec{v}^{(2)} + c_3\vec{v}^{(3)} + \cdots + c_q\vec{v}^{(q)} = \mathbf{0}$. To do so, we can again resort to its contrapositive, i.e. the existence of an alternative solution of $c_j = a_j$ which are not all zeros to the equation in question, means that the vectors $\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}, \ldots, \vec{v}^{(q)}$ are linearly dependent. Choose one of the a_j that is not zero and denote it by a_k, then

$$a_1\vec{v}^{(1)} + \cdots + a_{k-1}\vec{v}^{(k-1)} + a_k\vec{v}^{(k)} + a_{k+1}\vec{v}^{(k+1)} + \cdots + a_q\vec{v}^{(q)} = \mathbf{0}$$

$$\vec{v}^{(k)} = -\frac{a_1}{a_k}\vec{v}^{(1)} - \cdots - \frac{a_{k-1}}{a_k}\vec{v}^{(k-1)} - \frac{a_{k+1}}{a_k}\vec{v}^{(k+1)} - \cdots - \frac{a_q}{a_k}\vec{v}^{(q)}$$

where we have divided the equation by the non-zero a_k to avoid dividing by zero and rearranged it to show that $\vec{v}^{(k)}$ can be written in some linear combination of other vectors $\vec{v}^{(j)}$, $j \neq k$ as shown above, and thus vectors in $\mathcal{S}$ are linearly dependent. □

As a corollary, any set containing the zero vector $\mathbf{0}$ must be linearly dependent. (Why?)[5]

Example 6.1.4. Determine if $\vec{u} = (1, 2, 1)^T$, $\vec{v} = (3, 4, 2)^T$, $\vec{w} = (6, 8, 1)^T$ are linearly independent.

By Theorem 6.1.9, this is equivalent to decide if $A\vec{x} = \mathbf{0}$, where $A = [\vec{u}|\vec{v}|\vec{w}]$ has the trivial solution as the only solution. With the help of Theorem 3.1.2, we know that it is equivalent to check if $\det(A)$ is zero or not. Since

$$|A| = \begin{vmatrix} 1 & 3 & 6 \\ 2 & 4 & 8 \\ 1 & 2 & 1 \end{vmatrix} = 6 \neq 0$$

We conclude that $A\vec{x} = \mathbf{0}$ only has the trivial solution $\vec{x} = \mathbf{0}$ and these three vectors are linearly independent.

Short Exercise: Redo the above example with $\vec{u} = (1, 1, 3)^T$, $\vec{v} = (1, 3, 2)^T$, $\vec{w} = (2, 8, 3)^T$.[6]

Including our earlier discussion in Section 3.2.1, Theorem 6.1.9 gives some interesting results.

[5]For any such a set $\mathcal{S}_0 = \{\vec{u}_1, \vec{u}_2, \dots, \mathbf{0}\}$, the linear system $c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_0\mathbf{0} = \mathbf{0}$ has a family of infinitely many solution with $c_j = 0$ for $j \neq 0$ and any value of c_0, which by Theorem 6.1.9 they are linearly dependent.

[6]The determinant of $A = [\vec{u}|\vec{v}|\vec{w}]$ in the case is $|A| = 0$, and hence by the remark for Theorem 3.1.2 the linear system $A\vec{x} = \mathbf{0}$ has infinitely many solutions, and these three vectors are linearly dependent by Theorem 6.1.9.

1. If there are q vectors of $\mathbb{R}^p$ in a set and $p < q$, i.e. the amount of vectors is more than their dimension, then $A = [\vec{v}^{(1)}|\vec{v}^{(2)}|\vec{v}^{(3)}|\cdots|\vec{v}^{(q)}]$ is an $p \times q$ matrix which has more columns (q) than rows (p). In this case $A\vec{x} = \mathbf{0}$ must have at least one free variables and thus infinitely many solutions, hence the vectors must be linearly dependent.

2. Otherwise ($p \geq q$), we can solve $A\vec{x} = \mathbf{0}$ by Gaussian Elimination to see if it only has the trivial solution. If so (not), the vectors are linearly independent (dependent). Alternatively, if A is a square matrix, then we may check if its determinant is non-zero, just like what have been done in Example 6.1.4. Gaussian Elimination still works for any square matrix, and in case of linear independence (dependence), A will (not) be reduced to an identity matrix.

In many cases the number of vectors are indeed not equal to their dimension so the method of using determinant to check linear independence in the last example does not apply and we need to resort to Gaussian Elimination. In fact, Gaussian Elimination can disclose more information than just if a set of vectors is linearly (in)dependent as an entirety in both cases, but also how exactly these vectors are dependent on each other, soon to be explained. Before doing so, we note that the above observations lead to an extension of Theorem 3.2.1.

Theorem 6.1.10 (Equivalence Statement, ver. 3)**.** For an $n \times n$ real square matrix A, the followings are equivalent:

(a) A is invertible, i.e. A^{-1} exists,

(b) $\det(A) \neq 0$,

(c) The reduced row echelon form of A is I,

(d) The linear system $A\vec{x} = \vec{h}$ has a unique solution for any $\vec{h}$, particularly $A\vec{x} = \mathbf{0}$ has only the trivial solution $\vec{x} = \mathbf{0}$,

(e) The n column vectors $\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}, \dots, \vec{v}^{(n)}$ of $\mathbb{R}^n$ as in $A = [\vec{v}^{(1)}|\vec{v}^{(2)}|\vec{v}^{(3)}|\cdots|\vec{v}^{(n)}]$ are linearly independent.

We now revisit the procedure of Gaussian Elimination and show that it actually explicitly reveals the so-called ***dependence relations*** between vectors (how a vector can be written as some linear combination of other vectors) as a by-product when determining linear (in)dependence. Let's illustrate this by an example: Given

$$\begin{aligned}
\vec{v}^{(1)} &= (1, 2, 1)^T \\
\vec{v}^{(2)} &= (2, 4, 2)^T \\
\vec{v}^{(3)} &= (1, -1, -1)^T \\
\vec{v}^{(4)} &= (2, 1, 0)^T \\
\vec{v}^{(5)} &= (0, -3, -2)^T
\end{aligned}$$

Note that the vectors are related by these dependence relations: $\vec{v}^{(2)} = 2\vec{v}^{(1)}$, $\vec{v}^{(4)} = \vec{v}^{(1)} + \vec{v}^{(3)}$ and $\vec{v}^{(5)} = -\vec{v}^{(1)} + \vec{v}^{(3)}$, while $\vec{v}^{(1)}$ and $\vec{v}^{(3)}$ are themselves linearly independent. Construct

$$\begin{aligned}
A &= [\vec{v}^{(1)}|\vec{v}^{(2)}|\vec{v}^{(3)}|\vec{v}^{(4)}|\vec{v}^{(5)}] \\
&= \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 2 & 4 & -1 & 1 & -3 \\ 1 & 2 & -1 & 0 & -2 \end{bmatrix}
\end{aligned}$$

by concatenating the five vectors column by column. Now we carry out Gaussian Elimination as follows.

$$\begin{aligned}
\begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 2 & 4 & -1 & 1 & -3 \\ 1 & 2 & -1 & 0 & -2 \end{bmatrix} &\to \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & -3 & -3 & -3 \\ 0 & 0 & -2 & -2 & -2 \end{bmatrix} & \begin{aligned} R_2 - 2R_1 &\to R_2 \\ R_3 - R_1 &\to R_3 \end{aligned} \\
&\to \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \end{bmatrix} & -\frac{1}{3}R_2 \to R_2 \\
&\to \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & R_3 + 2R_2 \to R_3
\end{aligned}$$

$$\to \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_1 - R_2 \to R_1$$

The new columns in the above rref matrix $A_{\text{rref}} = [\vec{v}^{(1)\prime}|\vec{v}^{(2)\prime}|\vec{v}^{(3)\prime}|\vec{v}^{(4)\prime}|\vec{v}^{(5)\prime}]$ follow the exact same dependence relation: $\vec{v}^{(2)\prime} = 2\vec{v}^{(1)\prime}$, $\vec{v}^{(4)\prime} = \vec{v}^{(1)\prime} + \vec{v}^{(3)\prime}$ and $\vec{v}^{(5)\prime} = -\vec{v}^{(1)\prime} + \vec{v}^{(3)\prime}$. $\vec{v}^{(1)\prime}$ and $\vec{v}^{(3)\prime}$ are clearly still linearly independent of each other too. This demonstrates that dependence relations (and by extension linear independent vectors) are preserved under elementary row operations during Gaussian Elimination. (The detailed argument is put in the following footnote.[7])

We now introduce a helper theorem so that we may proceed.

Theorem 6.1.11 (Plus/Minus Theorem)**.** Let $\mathcal{S} = \{\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}, \dots, \vec{v}^{(q)}\}$ be a set of vectors in the vector space $\mathcal{V}$, i.e. $\vec{v}^{(j)} \in \mathcal{V}$, we have the following two results.

---

[7] We will show this for the case of row addition/subtraction only because the other two types of elementary row operations are easy to check. Without loss of generality, take a dependence relation in the form of $\vec{v}^{(r+1)} = c_1\vec{v}^{(1)} + c_2\vec{v}^{(2)} + \cdots + c_r\vec{v}^{(r)}$, i.e.

$$\begin{bmatrix} \vdots \\ \vec{v}_p^{(r+1)} \\ \vdots \\ \vec{v}_q^{(r+1)} \\ \vdots \end{bmatrix} = c_1 \begin{bmatrix} \vdots \\ \vec{v}_p^{(1)} \\ \vdots \\ \vec{v}_q^{(1)} \\ \vdots \end{bmatrix} + c_2 \begin{bmatrix} \vdots \\ \vec{v}_p^{(2)} \\ \vdots \\ \vec{v}_q^{(2)} \\ \vdots \end{bmatrix} + \cdots + c_r \begin{bmatrix} \vdots \\ \vec{v}_p^{(r)} \\ \vdots \\ \vec{v}_q^{(r)} \\ \vdots \end{bmatrix}$$

The elementary row operation of adding c_q times row R_q to row R_p produces a new matrix A' with column vectors

$$\vec{v}^{(j)\prime} = \begin{bmatrix} \vdots \\ \vec{v}_p^{(j)} + c_q\vec{v}_q^{(j)} \\ \vdots \\ \vec{v}_q^{(j)} \\ \vdots \end{bmatrix}$$

(a) If $\mathcal{S}$ is a linearly independent set and $\vec{v}$ is not in span$(\mathcal{S})$, then $\mathcal{S} \cup \{\vec{v}\}$ formed after inserting $\vec{v}$ into the set is still linearly independent,

(b) If $\vec{w}$ is a vector in $\mathcal{S}$ that can be expressed as a linear combination of other vectors in the set, then the new set $\mathcal{S} - \{\vec{w}\}$ formed after removing $\vec{w}$ from $\mathcal{S}$ has the same span, i.e.

$$\text{span}(\mathcal{S}) = \text{span}(\mathcal{S} - \{\vec{w}\})$$

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for all j. Therefore,

$$\vec{v}^{(r+1)'} = \begin{bmatrix} \vdots \\ \vec{v}_p^{(r+1)} + c_q \vec{v}_q^{(r+1)} \\ \vdots \\ \vec{v}_q^{(r+1)} \\ \vdots \end{bmatrix}$$
$$= \begin{bmatrix} \vdots \\ (c_1\vec{v}_p^{(1)} + c_2\vec{v}_p^{(2)} + \cdots + c_r\vec{v}_p^{(r)}) + c_q(c_1\vec{v}_q^{(1)} + c_2\vec{v}_q^{(2)} + \cdots + c_r\vec{v}_q^{(r)}) \\ \vdots \\ (c_1\vec{v}_q^{(1)} + c_2\vec{v}_q^{(2)} + \cdots + c_r\vec{v}_q^{(r)}) \\ \vdots \end{bmatrix}$$

by the original dependence relation, which is then equal to

$$\begin{bmatrix} \vdots \\ c_1(\vec{v}_p^{(1)} + c_q\vec{v}_q^{(1)}) \\ \vdots \\ c_1\vec{v}_q^{(1)} \\ \vdots \end{bmatrix} + \begin{bmatrix} \vdots \\ c_2(\vec{v}_p^{(2)} + c_q\vec{v}_q^{(2)}) \\ \vdots \\ c_2\vec{v}_q^{(2)} \\ \vdots \end{bmatrix} + \cdots + \begin{bmatrix} \vdots \\ c_r(\vec{v}_p^{(r)} + c_q\vec{v}_q^{(r)}) \\ \vdots \\ c_r\vec{v}_q^{(r)} \\ \vdots \end{bmatrix}$$
$$= c_1\vec{v}^{(1)'} + c_2\vec{v}^{(2)'} + \cdots + c_r\vec{v}^{(r)'}$$

This shows that the same dependence relation holds between the new column vectors $\vec{v}^{(j)'}$, $j = 1, 2, \ldots, r + 1$.

Proof. We include the proof for (a) as a footnote since (a) is less of a concern.[8] For (b), assign the vector $\vec{v}^{(k)}$ that is being removed where $1 \leq k \leq q$ as $\vec{w}$. We can write $\vec{w} = a_1\vec{v}^{(1)} + a_2\vec{v}^{(2)} + \cdots + a_{k-1}\vec{v}^{(k-1)} + a_{k+1}\vec{v}^{(k+1)} + \cdots + a_q\vec{v}^{(q)}$ using other vectors in $\mathcal{S}$ where a_j, $j \neq k$ are some constants. For any vector $\vec{v} = b_1\vec{v}^{(1)} + b_2\vec{v}^{(2)} + \cdots + b_{k-1}\vec{v}^{(k-1)} + b_k\vec{v}^{(k)} + b_{k+1}\vec{v}^{(k+1)} + \cdots + b_q\vec{v}^{(q)}$ in $\text{span}(\mathcal{S})$ with b_j being the coefficients, it can be rewritten as a linear combination of the remaining vectors:

$$\begin{aligned}
\vec{v} &= b_1\vec{v}^{(1)} + b_2\vec{v}^{(2)} + \cdots + b_{k-1}\vec{v}^{(k-1)} + b_k\vec{v}^{(k)} + b_{k+1}\vec{v}^{(k+1)} + \cdots + b_q\vec{v}^{(q)} \\
&= b_1\vec{v}^{(1)} + b_2\vec{v}^{(2)} + \cdots + b_{k-1}\vec{v}^{(k-1)} + b_{k+1}\vec{v}^{(k+1)} + \cdots + b_q\vec{v}^{(q)} + b_k\vec{v}^{(k)} \\
&= b_1\vec{v}^{(1)} + b_2\vec{v}^{(2)} + \cdots + b_{k-1}\vec{v}^{(k-1)} + b_{k+1}\vec{v}^{(k+1)} + \cdots + b_q\vec{v}^{(q)} + b_k\vec{v}^{(k)} \\
&\quad + b_k(a_1\vec{v}^{(1)} + a_2\vec{v}^{(2)} + \cdots + a_{k-1}\vec{v}^{(k-1)} + a_{k+1}\vec{v}^{(k+1)} + \cdots + a_q\vec{v}^{(q)}) \\
&= (b_1 + b_ka_1)\vec{v}^{(1)} + (b_2 + b_ka_2)\vec{v}^{(2)} + (b_{k-1} + b_ka_{k-1})\vec{v}^{(k-1)} \\
&\quad + (b_{k+1} + b_ka_{k+1})\vec{v}^{(k+1)} + \cdots + (b_q + b_ka_q)\vec{v}^{(q)} \\
&\in \text{span}(\mathcal{S} - \{\vec{u}_k\}) = \text{span}(\mathcal{S} - \{\vec{w}\})
\end{aligned}$$

Therefore for all $\vec{v} \in \text{span}(\mathcal{S})$, $\vec{v} \in \text{span}(\mathcal{S} - \{\vec{w}\})$ and hence $\text{span}(\mathcal{S}) \subseteq \text{span}(\mathcal{S} - \{\vec{w}\})$. It is trivial to show $\text{span}(\mathcal{S} - \{\vec{w}\}) \subseteq \text{span}(\mathcal{S})$, and thus $\text{span}(\mathcal{S}) = \text{span}(\mathcal{S} - \{\vec{w}\})$. This part of the theorem is very relevant to the span of sets $\mathcal{S}_3$ and $\mathcal{S}_4$ in the previous Example 6.1.3. □

[8] We will prove the contrapositive that if $\mathcal{S}$ is a linearly independent set then, $\mathcal{S} \cup \{\vec{v}\}$ is linearly dependent if and only $\vec{v}$ is in $\text{span}(\mathcal{S})$. The "if" direction is trivial by the definition of span and linear dependence. For the converse, if $\mathcal{S} \cup \{\vec{v}\}$ is linearly dependent, then there is non-trivial solution $c_j = d_j$ where d_j are not all zeros to the equation $c_1\vec{v}^{(1)} + c_2\vec{v}^{(2)} + \cdots + c_q\vec{v}^{(q)} + c_v\vec{v} = \mathbf{0}$ by Theorem 6.1.9. Since $\mathcal{S}$ is linearly independent, $d_v \neq 0$, for otherwise $d_v = 0$ and then at least one of the $c_j = d_j$ ($j \neq v$) will be non-zero and lead to a non-trivial solution to $c_1\vec{v}^{(1)} + c_2\vec{v}^{(2)} + \cdots + c_q\vec{v}^{(q)} = \mathbf{0}$ which contradicts the linear independence of $\mathcal{S}$, so we have $d_1\vec{u}_1 + d_2\vec{u}_2 + \cdots + d_q\vec{u}_q + d_v\vec{v} = \mathbf{0}$ and because $d_v \neq 0$ we can obtain

$$\vec{v} = -\frac{1}{d_v}(d_1\vec{v}^{(1)} + d_2\vec{v}^{(2)} + \cdots + d_q\vec{v}^{(q)})$$

showing that $\vec{v}$ is a linear combination of $\vec{v}^{(j)} \in \mathcal{S}$.

With these results, Gaussian Elimination enables us to carry out the ***Column-row (CR) Factorization*** over a matrix. First, note that by part (b) of Theorem 6.1.11 above, the column space (Definition 6.1.7) of a matrix A can be expressed as the span of a *minimal generating set* by removing linearly dependent column vectors in A (which does not change the span and still generates the same subspace) and only keeping the linearly independent ones. Meanwhile, the linear (in)dependence of the column vectors of A can be inferred by Gaussian Elimination as just demonstrated in the last example. After obtaining the minimal generating set, we can express any vector in the column space as a unique linear combination of these linearly independent vectors inside the set due to the following properties.

Properties 6.1.12. For a set of vectors $\mathcal{S} = \{\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}, \ldots, \vec{v}^{(q)}\}$, $\vec{v}^{(j)} \in \mathcal{V}$ for $j = 1, 2, \cdots, q$, which are linearly independent, any vector $\vec{v} \in \text{span}(\mathcal{S})$ in their span can be written as a unique linear combination of the vectors in $\mathcal{S}$. Otherwise, if the vectors in $\mathcal{S}$ are linearly dependent, there will be infinitely many such linear combinations to assemble $\vec{v}$.

Again, we will simply provide the proof in a footnote for reference.[9] Return to

[9] We will show the first part only. Since $\vec{v}$ already belongs to span$(\mathcal{S})$, it must be possible to express $\vec{v}$ as some linear combination(s) of vectors $\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}, \ldots, \vec{v}^{(q)}$ in $\mathcal{S}$ by Definition 6.1.5. Now it suffices to show that it is unique. Assume the contrary that there are two distinct linear combinations of vectors in $\mathcal{S}$ that represent $\vec{v}$, and hence we can express it by

$$\begin{aligned}\vec{v} &= d_1\vec{v}^{(1)} + d_2\vec{v}^{(2)} + d_3\vec{v}^{(3)} + \cdots + d_q\vec{v}^{(q)} \\ &= g_1\vec{v}^{(1)} + g_2\vec{v}^{(2)} + g_3\vec{v}^{(3)} + \cdots + g_q\vec{v}^{(q)}\end{aligned}$$

where d_j, g_j are two sets of coefficients are not exactly the same. Subtracting one expression by another leads to

$$\begin{aligned}\begin{matrix}(d_1\vec{v}^{(1)} + d_2\vec{v}^{(2)} + d_3\vec{v}^{(3)} + \cdots + d_q\vec{v}^{(q)}) \\ -\,(g_1\vec{v}^{(1)} + g_2\vec{v}^{(2)} + g_3\vec{v}^{(3)} + \cdots + g_q\vec{v}^{(q)})\end{matrix} &= \vec{v} - \vec{v} \\ (d_1 - g_1)\vec{v}^{(1)} + (d_2 - g_2)\vec{v}^{(2)} + (d_3 - g_3)\vec{v}^{(3)} + \cdots + (d_q - g_q)\vec{v}^{(q)} &= \mathbf{0}\end{aligned}$$

Since d_j, g_j are assumed to be not identical, it is a non-trivial solution to the equation $c_1\vec{v}^{(1)} + c_2\vec{v}^{(2)} + c_3\vec{v}^{(3)} + \cdots + c_q\vec{v}^{(q)} = \mathbf{0}$, where $c_j = d_j - g_j$ are not all zeros. This our example, where

$$\begin{aligned}
A &= [\vec{v}^{(1)}|\vec{v}^{(2)}|\vec{v}^{(3)}|\vec{v}^{(4)}|\vec{v}^{(5)}] \\
&= \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 2 & 4 & -1 & 1 & -3 \\ 1 & 2 & -1 & 0 & -2 \end{bmatrix}
\end{aligned}$$

We have found that the corresponding rref is

$$A_{\text{rref}} = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So that the first/third column vectors are linearly independent, and the second/-fourth/fifth column vectors that are linearly dependent on the first/third ones can be expressed as a linear combination of them. Now let

$$C = [\vec{v}^{(1)}|\vec{v}^{(3)}] = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & -1 \end{bmatrix}$$

using the two linear independent vectors. By Properties 6.1.12 and 6.1.4, each of the $\vec{v}^{(j)}$ can be expressed as a unique linear combination in the form of a matrix product between C and a column vector that contains the coefficients in front of the chosen linear independent vectors that make up the $\vec{v}^{(j)}$. The required column vector is exactly the corresponding column in the rref which retains the dependence relations, with row(s) of all zeros removed. For instance,

$$\begin{aligned}
\vec{v}^{(5)} &= -\vec{v}^{(1)\prime} + \vec{v}^{(3)\prime} \\
\begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} &= -1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} && \text{(Properties 6.1.4)}
\end{aligned}$$

contradicts our assumption and hence the linear combination of $\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}, \ldots, \vec{v}^{(q)}$ to generate $\vec{v}$ must be unique.

$$= [\vec{v}^{(1)}|\vec{v}^{(3)}]\begin{bmatrix}-1\\1\end{bmatrix} = C\begin{bmatrix}-1\\1\end{bmatrix}$$

Denote

$$R = \begin{bmatrix}1 & 2 & 0 & 1 & -1\\0 & 0 & 1 & 1 & 1\end{bmatrix}$$

which is consisted of the non-zero rows of A_{rref}, then similarly

$$\vec{v}^{(1)} = \begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}1 & 1\\2 & -1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = CR_1$$

$$\vec{v}^{(2)} = \begin{bmatrix}2\\4\\2\end{bmatrix} = \begin{bmatrix}1 & 1\\2 & -1\\1 & -1\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix} = CR_2$$

$$\vec{v}^{(3)} = \begin{bmatrix}1\\-1\\-1\end{bmatrix} = \begin{bmatrix}1 & 1\\2 & -1\\1 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = CR_3$$

$$\vec{v}^{(4)} = \begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}1 & 1\\2 & -1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = CR_4$$

where R_j is the j-th column of R. Therefore,

$$\begin{aligned}A = [\vec{v}^{(1)}|\vec{v}^{(2)}|\vec{v}^{(3)}|\vec{v}^{(4)}|\vec{v}^{(5)}] &= \begin{bmatrix}1 & 2 & 1 & 2 & 0\\2 & 4 & -1 & 1 & -3\\1 & 2 & -1 & 0 & -2\end{bmatrix}\\
&= [CR_1|CR_2|CR_3|CR_4|CR_5]\\
&= C([R_1|R_2|R_3|R_4|R_5]) = CR = \begin{bmatrix}1 & 1\\2 & -1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & 0 & 1 & -1\\0 & 0 & 1 & 1 & 1\end{bmatrix}\end{aligned}$$

from the second line to the third line we use the fact that the same matrix multiplied to the left in every column of another matrix can be factored out (why?)[10] and this is the desired CR Factorization of A. In general, for any matrix, its CR Factorization is derived as follows.

[10]For an $m \times r$ matrix A, and an $r \times n$ matrix B, we have, by Definition 1.1.1 (with a slightly

Properties 6.1.13 (CR Factorization)**.** The Column-row Factorization of any matrix A that has a rref of A_{rref}, is given by $A = CR$, where C contains the r linearly independent columns of A at which the r leading 1s of A_{rref} are located, and R is simply the first r rows of A_{rref}, with all the full-zero rows below removed.

The k-th row of R contains the coefficients of k-th column vector in C required to generate each column in the original A matrix.

Example 6.1.5. Show that $\vec{u} = (2, 1, -1, 1)^T$, $\vec{v} = (1, 2, 1, -1)^T$, $\vec{w} = (0, 1, 1, 2)^T$ are linearly independent and find the CR Factorization of $A = [\vec{u}|\vec{v}|\vec{w}]$. What if $\vec{w} = (1, -1, -2, 2)^T$ instead?

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different notation),

$$\begin{aligned}
A[B_1|B_2|\cdots|B_n] &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & & a_{2r} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \left[\begin{array}{c|c|c|c} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ \vdots & & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{array}\right] \\
&= \left[\begin{array}{c|c|c|c} \sum_{k=1}^r a_{1k}b_{k1} & \sum_{k=1}^r a_{1k}b_{k2} & \cdots & \sum_{k=1}^r a_{1k}b_{kn} \\ \sum_{k=1}^r a_{2k}b_{k1} & \sum_{k=1}^r a_{2k}b_{k2} & & \sum_{k=1}^r a_{2k}b_{kn} \\ \vdots & & \ddots & \vdots \\ \sum_{k=1}^r a_{mk}b_{k1} & \sum_{k=1}^r a_{mk}b_{k2} & \cdots & \sum_{k=1}^r a_{mk}b_{kn} \end{array}\right] \\
&= [AB_1|AB_2|\cdots|AB_n]
\end{aligned}$$

where B_j now denotes the j-th column of B:

$$B_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \end{bmatrix} \qquad \text{and hence } AB_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & & a_{2r} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^r a_{1k}b_{kj} \\ \sum_{k=1}^r a_{2k}b_{kj} \\ \vdots \\ \sum_{k=1}^r a_{mk}b_{kj} \end{bmatrix}$$

Solution. From Theorem 6.1.9, we need to show that the system $A\vec{x} = \mathbf{0}$ has only the trivial solution $\vec{x} = \mathbf{0}$, where

$$A = [\vec{u}|\vec{v}|\vec{w}] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

To do so we can apply Gaussian Elimination as below.

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] \qquad R_1 \leftrightarrow R_4$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right] \qquad \begin{array}{r} R_2 - R_1 \to R_2 \\ R_3 + R_1 \to R_3 \\ R_4 - 2R_1 \to R_4 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right] \qquad \frac{1}{3}R_2 \to R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \qquad R_4 - 3R_2 \to R_4$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \qquad \frac{1}{3}R_3 \to R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_4 + 3R_1 \to R_4$$

(the zero column to the right can be omitted) The forward phase leads to a redundant row and the presence of pivots in every column indicates that the trivial solution of $\vec{x} = 0$ is the only solution, hence the three vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly independent. The backward phase concerning the matrix A itself is instantaneous, yielding its rref:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

By Properties 6.1.13, the CR Factorization of A is then trivially

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = CR$$

with $C = A$ and $R = I_3$ as all columns in A are linearly independent. In general, if the n column vectors in an $m \times n$ matrix A are linearly independent[11], then its rref will be in the form of

$$A_{\text{rref}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \end{bmatrix}$$

where the top is an $n \times n$ identity matrix I_n, followed by $m - n$ rows of full zeros at the bottom. The CR Factorization of A will then be simply comprised of $C = A$ and $R = I_n$. For the second case where $\vec{w} = (1, -1, -2, 2)^T$, we can

[11] It is necessary that $m \geq n$.

repeat the same analysis by deriving the rref of the modified A matrix.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ -1 & 1 & -2 \\ 1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & 1 & 1 \end{bmatrix} \qquad R_1 \leftrightarrow R_4$$

$$\to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \\ 0 & 3 & -3 \end{bmatrix} \qquad \begin{matrix} R_2 - R_1 \to R_2 \\ R_3 + R_1 \to R_3 \\ R_4 - 2R_1 \to R_4 \end{matrix}$$

$$\to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & -3 \end{bmatrix} \qquad \frac{1}{3}R_2 \to R_2$$

$$\to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_4 - 3R_2 \to R_4$$

$$\to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_1 + R_2 \to R_1$$

The final rref reveals that $\vec{u}$ and $\vec{v}$ are two linearly independent vectors in the column space of A, in addition to the dependence relation of $\vec{w} = \vec{u} - \vec{v}$. Hence its new CR Factorization, by Properties 6.1.13, is

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ -1 & 1 & -2 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

□

6.2 Coordinate Bases for $\mathbb{R}^n$ and its Subspaces

6.2.1 Coordinate Bases for $\mathbb{R}^n$

Back in Definition 4.1.3, we have introduced the n standard unit vectors $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$ for the real n-space $\mathbb{R}^n$. Obviously the standard unit vectors are linearly independent and their span is exactly $\mathbb{R}^n$. We often refer to the coefficients x_j in front of $\hat{e}_j$ of a vector $\vec{v} = (x_1, x_2, \ldots, x_n)^T = x_1\hat{e}_1 + x_2\hat{e}_2 + \cdots + x_n\hat{e}_n$ in $\mathbb{R}^n$ as the *Cartesian* coordinates of $\vec{v}$. The coordinates x_j are unique, guaranteed by Properties 6.1.12. However, sometimes we may want to express an $\mathbb{R}^n$ vector in another ***coordinate basis (system)*** with axes different from the standard unit vectors (other than the ***standard basis***). Motivated by the properties of the Cartesian coordinate system above, in which the standard unit vectors are linearly independent and span $\mathbb{R}^n$ such that every vector in $\mathbb{R}^n$ can be expressed as a unique linear combination of them (Properties 6.1.12 again), we require all other coordinate bases for $\mathbb{R}^n$ to carry the same properties. The coefficients of this linear combination will then be the coordinates of that vector in this basis.

Definition 6.2.1 (Coordinate Basis for $\mathbb{R}^n$)**.** A coordinate basis for $\mathbb{R}^n$ should consist of n vectors in $\mathbb{R}^n$ which

(a) are linearly independent, and

(b) span (generate) $\mathbb{R}^n$.

Some may wonder why the definition above has explicitly stated that the number of vectors in a coordinate basis for $\mathbb{R}^n$ is exactly n, although many people would probably think it is reasonable and accept this without a doubt. For the sake of completeness, we will explain that this is a result coming naturally from the conditions of linear independence and spanning $\mathbb{R}^n$. We have previously shown that Theorem 6.1.9 implies that in $\mathbb{R}^n$ if there are more vectors q than the dimension n then they will be linearly dependent. So linear independence requires $q \leq n$. To span $\mathbb{R}^n$, it is apparent that $q \geq n$.[12] Hence the number of vectors q must be equal to n.

[12] To formally show this, express the span of q $\mathbb{R}^n$ vectors $c_1\vec{v}^{(1)} + c_2\vec{v}^{(2)} + c_3\vec{v}^{(3)} + \cdots + c_q\vec{v}^{(q)}$

The following theorem shows that we actually only need to check either one of the conditions in Definition 6.2.1.

Theorem 6.2.2. A set of n vectors of $\mathbb{R}^n$ is linearly independent if and only if they span $\mathbb{R}^n$.

Proof. Linear Independence $\to$ Spanning $\mathbb{R}^n$: Assume $\vec{v}^{(1)}, \vec{v}^{(2)}, \ldots, \vec{v}^{(n)}$ are linear independent with $A = [\vec{v}^{(1)}|\vec{v}^{(2)}|\cdots|\vec{v}^{(n)}]$ being a square matrix. The application of part (e) $\to$ (d) of Theorem 6.1.10 immediately shows that there is always a (unique) solution to $A\vec{x} = \vec{h}$ for any $\vec{h}$ of $\mathbb{R}^n$. Recall that $A\vec{x}$ represents the span of $\{\vec{v}^{(1)}, \vec{v}^{(2)}, \ldots, \vec{v}^{(n)}\}$ (Definition 6.1.5) so it implies that these vectors generates the entire $\mathbb{R}^n$.

Spanning $\mathbb{R}^n$ $\to$ Linear Independence: Assume that $\vec{v}^{(1)}, \vec{v}^{(2)}, \ldots, \vec{v}^{(n)}$ are linear dependent, then by (c) and (e) of Theorem 6.1.10 the reduced row echelon form of $A = [\vec{v}^{(1)}|\vec{v}^{(2)}|\cdots|\vec{v}^{(n)}]$ is not the identity matrix and contains at least one row of full zeros. Following a logic similar to Footnote 12, these vectors cannot span $\mathbb{R}^n$ and the contrapositive is proved. □

Example 6.2.1. Show that $\mathcal{B} = \{\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}\} = \{(1,2,1)^T_E, (-1,1,0)^T_E, (1,-1,2)^T_E\}$ forms a basis for $\mathbb{R}^3$ and express $\vec{v}$ in $\mathcal{B}$ (a.k.a. $[\vec{v}]_B$) where $[\vec{v}]_E = (2,1,2)^T_E$, the subscript E (B) emphasizes that the coordinates are relative to standard basis $\mathcal{E}$ (the alternative basis $\mathcal{B}$).

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by $A\vec{x}$ where $A = [\vec{v}^{(1)}|\vec{v}^{(2)}|\vec{v}^{(3)}|\cdots|\vec{v}^{(q)}]$ is an $n \times q$ matrix and $\vec{x} = (c_1, c_2, c_3, \ldots, c_q)^T$ consists of q coefficients as unknowns (Properties 6.1.4). If $q < n$, then $A\vec{x} = \vec{h}$ is an overdetermined system so that we can always find some row of full zeros in the rref of A to the left of the augmented matrix as we solve the system by Gaussian Elimination. Along the column vector to the right of the augmented matrix that undergoes the reduction process together, we can always set the number on such a row to some non-zero number (let's say, 1) if not already to make sure it is inconsistent. Invert the entire process of Gaussian Elimination over the augmented matrix to recover A from its reduced form. To the right of the augmented matrix will then appear $\vec{h}_{\text{inconst}}$. This system $A\vec{x} = \vec{h}_{\text{inconst}}$ is inconsistent by the design above (just do the same steps of Gaussian Elimination again and the inconsistent 1 to the right will reappear), which shows that the span does not include $\vec{h}_{\text{inconst}}$ and cannot cover the entire $\mathbb{R}^n$.

Solution. By Definition 6.2.1 and Theorem 6.2.2, the first part is equivalent to checking if the three $\mathbb{R}^3$ vectors in $\mathcal{B}$ are linearly independent. By Theorem 6.1.10, we can simply check if $\det(A)$ is non-zero where

$$A = \left[[\vec{v}^{(1)}]_E | [\vec{v}^{(2)}]_E | [\vec{v}^{(3)}]_E \right] = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

A simple calculation reveals that $\det(A) = 6 \neq 0$ so $\mathcal{B}$ is indeed a basis for $\mathbb{R}^3$. To express $(2, 1, 2)^T_E$ in $\mathcal{B}$ is to find $[\vec{v}]_B = ([v_1]_B, [v_2]_B, [v_3]_B)^T_B$ where $[v_j]_B$ is the j-th component (coefficient) of $\vec{v}$ in the $\mathcal{B}$ coordinate system such that the corresponding linear combination of $\vec{v}^{(j)}$ produces the required vector when we work in the $\mathcal{E}$ basis as well:

$$\begin{aligned} [v_1]_B(\vec{v}^{(1)}) + [v_2]_B(\vec{v}^{(2)}) + [v_3]_B(\vec{v}^{(3)}) &= \vec{v} \\ [v_1]_B(1, 2, 1)^T_E + [v_2]_B(-1, 1, 0)^T_E + [v_3]_B(1, -1, 2)^T_E &= (2, 1, 2)^T_E \end{aligned}$$

or put in matrix form,

$$\begin{aligned} [v_1]_B[\vec{v}^{(1)}]_E + [v_2]_B[\vec{v}^{(2)}]_E + [v_3]_B[\vec{v}^{(3)}]_E &= [\vec{v}]_E \\ A[\vec{v}]_B = \left[[\vec{v}^{(1)}]_E | [\vec{v}^{(2)}]_E | [\vec{v}^{(3)}]_E \right] \begin{bmatrix} [v_1]_B \\ [v_2]_B \\ [v_3]_B \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} [v_1]_B \\ [v_2]_B \\ [v_3]_B \end{bmatrix} &= \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

We can either use matrix inverse or Gaussian Elimination to solve for the $[v_j]_B$, yielding $[v_1]_B = 1$, $[v_2]_B = -\frac{1}{2}$, $[v_3]_B = \frac{1}{2}$, and hence $[\vec{v}]_B = (1, -\frac{1}{2}, \frac{1}{2})^T_B$. The matrix equation

$$A[\vec{v}]_B = \left[[\vec{v}^{(1)}]_E | [\vec{v}^{(2)}]_E | [\vec{v}^{(3)}]_E \right] \begin{bmatrix} [v_1]_B \\ [v_2]_B \\ [v_3]_B \end{bmatrix} = [\vec{v}]_E$$

shows that A transforms the coordinate system for a given vector from the $\mathcal{B}$ to $\mathcal{E}$, and we will write $A = P^E_B$ (thus $P^E_B[\vec{v}]_B = [\vec{v}]_E$) for clarity in the future. Notice that $P^E_B[\vec{v}^{(j)}]_B = P^E_B(e_j)_B$ returns the j-th basis vector in $\mathcal{B}$ (j-th column of P^E_B) expressed in the standard basis $\mathcal{E}$, namely $[\vec{v}^{(j)}]_E$, where $(e_j)_B = [\vec{v}^{(j)}]_B$

is the numeric coordinate representation (emphasized by the absence of hat symbol over e) of the j-th basis vector in the $\mathcal{B}$ coordinate system with the j-th component being 1 and other being 0. For instance[13],

$$P_B^E[\vec{v}^{(2)}]_B = P_B^E(e_2)_B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = [\vec{v}^{(2)}]_E$$

From now on, we simply omit the subscript E and write $(\vec{v})$ in place of $[\vec{v}]_E$ if not specified, to denote vectors in the standard basis as implicitly assumed before. □

6.2.2 Coordinate Bases for Subspaces of $\mathbb{R}^n$

Now that we are able to construct a coordinate basis for $\mathbb{R}^n$, it is natural to ask if we can also extend this and come up with some coordinate basis for any subspace of $\mathbb{R}^n$ (since a subspace is itself a vector space too), in the sense that any vector in the subspace can be uniquely expressed by the basis vectors (*linear independence*) and the basis *spans* the subspace exactly, just like any basis for $\mathbb{R}^n$. Actually, we have already done this for the column space of a matrix A back in the derivation of CR Factorization in Section 6.1.4 where we created a minimal generating set from the column vectors that compose A. For other subspaces of $\mathbb{R}^n$ the procedure is similar. If we are given a subspace as some span of some vectors, then to find a basis for it, we carry out CR Factorization as if these vectors are columns of the matrix and retain the linearly independent vectors. These linearly independent vectors still span the subspace by part (b) of Theorem 6.1.11, and any vector in the subspace can be written as a unique linear combination of them by Properties 6.1.12.

[13] In contrast, the standard unit vector $\hat{e}_2$ with a hat is a true vector with $[\hat{e}_2]_E = (0, 1, 0)_E^T$. $P_B^E[\hat{e}_2]_B = [\hat{e}_2]_E$ and thus

$$[\hat{e}_2]_B = (P_B^E)^{-1}[\hat{e}_2]_E = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{5}{6} & \frac{1}{6} & \frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = (\frac{1}{3}, \frac{1}{6}, -\frac{1}{6})_B^T$$

Properties 6.2.3. For the subspace generated by a spanning set $\{\vec{v}^{(1)}, \vec{v}^{(2)}, \dots, \vec{v}^{(q)}\}$, its basis can be found by applying CR Factorization (Properties 6.1.13) over $A = \left[\vec{v}^{(1)}|\vec{v}^{(2)}|\dots|\vec{v}^{(q)}\right]$. Then a possible basis is $\{\vec{v}^{(j)}\}$ for all j such that the j-th column of rref of A contains a leading 1.

Example 6.2.2. Find a basis for the subspace generated by $\mathcal{G} = \{(1,0,2,1)^T, (1,-1,1,-2)^T, (0,0,1,0)^T, (-2,1,0,1)^T\}$. Hence express $(3,-1,4,0)^T$ in this basis.

Solution. By Properties 6.2.3, we need to find the CR Factorization for

$$A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & -1 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix}$$

We proceed with Gaussian Elimination.

$$\begin{aligned}
\begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & -1 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix} &\to \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 4 \\ 0 & -3 & 0 & 3 \end{bmatrix} & \begin{aligned} R_3 - 2R_1 &\to R_3 \\ R_4 - R_1 &\to R_4 \end{aligned} \\
&\to \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 4 \\ 0 & -3 & 0 & 3 \end{bmatrix} & -R_2 \to R_2 \\
&\to \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{aligned} R_3 + R_2 &\to R_3 \\ R_4 + 3R_2 &\to R_4 \end{aligned} \\
&\to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} & R_1 - R_2 \to R_1
\end{aligned}$$

So a possible basis for the subspace = span($\mathcal{G}$) is its first three generating vectors $\mathcal{B} = \{(1,0,2,1)^T, (1,-1,1,-2)^T, (0,0,1,0)^T\}$. To express $\vec{v} = (3,-1,4,0)^T$ in this basis, we need to find $[\vec{v}]_B$ just like in Example 6.2.1, which is derived by

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 2 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} [v_1]_B \\ [v_2]_B \\ [v_3]_B \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 4 \\ 0 \end{bmatrix}$$

We repeat the same steps of Gaussian Elimination over the augmented matrix where now the left portion consisted of the three linearly independent basis vectors only.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -1 & 0 & -1 \\ 2 & 1 & 1 & 4 \\ 1 & -2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & -2 \\ 0 & -3 & 0 & -3 \end{array}\right] \quad \begin{array}{r} R_3 - 2R_1 \to R_3 \\ R_4 - R_1 \to R_4 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & -3 & 0 & -3 \end{array}\right] \quad -R_2 \to R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{r} R_3 + R_2 \to R_3 \\ R_4 + 3R_2 \to R_4 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 - R_2 \to R_1$$

The last full-zero row is consistent and $[\vec{v}]_B = ([v_1]_B, [v_2]_B, [v_3]_B)_B^T = (2,1,-1)_B^T$. As a final note, since the fourth vector $(-2,1,0,1)^T$ in the generating set can be written as a non-zero linear combination of the first three vectors (with the coefficients of $-1,-1,3$), we can replace any one of the three vectors in the basis by $(-2,1,0,1)^T$. □

In general, any vector $\vec{v}^{(j)}$ in a basis can be replaced by another vector that is the linear combination of the basis vectors where the coefficient corresponding to $\vec{v}^{(j)}$ is particularly not zero.[14] Moreover, we can now properly define the "dimension" of any subspace of $\mathbb{R}^n$. It is simply the number of vectors in its basis. Some may wonder if it is possible for two bases of the same vector space to have different number of vectors so that the notion of its dimension will be problematic. In fact, all bases of a finite-dimensional vector (sub)space must possess the same amount of vectors, and we simply note this below.[15]

Properties 6.2.4. If $\mathcal{V}$ is a vector space with a finite basis, then all bases of $\mathcal{V}$ are finite and have the same number of vectors.

From the statement above, we see that if we can find any basis with exactly n vectors for a vector space $\mathcal{V}$ where n is finite, then n will be the unique integer such that every basis $\mathcal{V}$ is consisted of this number of vectors. n is then referred to as the ***dimension*** of $\mathcal{V}$, and we define $\dim(\mathcal{V}) = n$. $\mathcal{V}$ is then known as a ***finite-dimensional*** vector space. If a vector space is not finite-dimensional, i.e. a finite basis cannot be found, then it is called ***infinite-dimensional***. Moreover,

Properties 6.2.5. For any subspace $\mathcal{W}$ of a vector space $\mathcal{V}$, $\dim(\mathcal{W}) \leq \dim(\mathcal{V})$. If $\dim(\mathcal{W}) = \dim(\mathcal{V})$, $\mathcal{W} = \mathcal{V}$.

and

Theorem 6.2.6. If a vector space $\mathcal{V}$ is generated by a set $\mathcal{G}$ with a finite amount of vectors, then some subset of $\mathcal{G}$ is a basis for $\mathcal{V}$, and $\mathcal{V}$ has finite bases.

which is a broader restatement of Properties 6.2.3.

Finally, we expand Theorem 6.2.2 (Equivalent requirements of a basis) to any finite-dimensional vector (sub)space. The results are simply stated below.

[14]This preserves the span and linear independence. See Properties 6.3.4 later for the span (plus (a) of Properties 6.2.7 for linear independence).

[15]It, along with other results below, actually comes from a more general theorem called the *Steinitz Replacement Theorem*.

Properties 6.2.7. If $\mathcal{V}$ is a vector space with $\dim(\mathcal{V}) = n$, then

(a) Any generating set for $\mathcal{V}$ contains at least n vectors. If, furthermore, it is made of exactly n vectors, then it is also a basis for $\mathcal{V}$,

(b) Any linearly independent subset of $\mathcal{V}$ that has exactly n vectors is a basis for $\mathcal{V}$,

(c) Every linearly independent subset $\mathcal{G}_1$ of $\mathcal{V}$ with $m \leq n$ vectors can be extended to a basis for $\mathcal{V}$, i.e. there exists another subset $\mathcal{G}_2$ of $\mathcal{V}$ with $n - m$ vectors such that $\mathcal{B} = \mathcal{G}_1 \cup \mathcal{G}_2$ is a basis for $\mathcal{V}$.

A point worth mentioning is that part (c) of the properties above allows the possibility of completing a basis from its fragment, which will be used in many arguments from time to time.

6.2.3 Direct Sum Representation

Since we can create subspaces from multiple individual vectors, we may like to know if we can go one step further and make a larger vector space from smaller subspaces by composing them together. This then leads to the *direct sum* representation. Let's begin with the definition of ***sum of subspaces*** first.

Definition 6.2.8 (Subspace Sum)**.** Given two subspaces $\mathcal{W}_1, \mathcal{W}_2$, of a vector space $\mathcal{V}$, their subspace sum is

$$\mathcal{W}_1 + \mathcal{W}_2 = \{\vec{w}_1 + \vec{w}_2 \mid \vec{w}_1 \in \mathcal{W}_1, \vec{w}_2 \in \mathcal{W}_2\}$$

consisted of all possible vectors resulted from addition between any pair of vectors from $\mathcal{W}_1, \mathcal{W}_2 \subseteq \mathcal{V}$ respectively. Note that $(\mathcal{W}_1 + \mathcal{W}_2) \subseteq \mathcal{V}$ is a subspace of $\mathcal{V}$.

For example, if $\mathcal{W}_1 = \text{span}(\{(1, 0, 1)^T\})$ and $\mathcal{W}_2 = \text{span}(\{(1, 1, 0)^T, (0, 1, 1)^T\})$, then according to the definition of span in Definition 6.1.5 and that of subspace sum above, $\mathcal{W}_1 + \mathcal{W}_2 = \text{span}(\{(1, 0, 1)^T, (1, 1, 0)^T, (0, 1, 1)^T\})$, which is just the span of union of the generating vectors from the two smaller spans, and can be shown to be equal to $\mathbb{R}^3$ following the same idea used when doing Example 6.2.1. Extending this, we have

$$\mathcal{W}_1 + \mathcal{W}_2 + \cdots + \mathcal{W}_n = \{\vec{w}_1 + \vec{w}_2 + \cdots \vec{w}_n \mid \vec{w}_j \in \mathcal{W}_j, 1 \leq j \leq n\}$$

In the small example above, $\dim(\mathcal{W}_1)+\dim(\mathcal{W}_2) = 1+2 = 3 = \dim(\mathcal{W}_1+\mathcal{W}_2)$, as the spanning vectors collected from the two subspaces are linearly independent of each other, i.e. the basis vector in $\mathcal{W}_1$ cannot be expressed as the linear combination of those in $\mathcal{W}_2$ and vice versa. In this case, the dimensions of the two subspaces can be *directly* added together, and hence it constitutes a ***direct sum***, whose requirement is given below.

Definition 6.2.9 (Direct Sum)**.** A direct sum between two subspaces $\mathcal{W}_1, \mathcal{W}_2$ is their subspace sum $\mathcal{W}_1 + \mathcal{W}_2$ as defined in Definition 6.2.8 which additionally satisfies $\mathcal{W}_1 \cap \mathcal{W}_2 = \{\mathbf{0}\}$, and is denoted as $\mathcal{W}_1 \oplus \mathcal{W}_2$, and we have $\dim(\mathcal{W}_1 \oplus \mathcal{W}_2) = \dim(\mathcal{W}_1) + \dim(\mathcal{W}_2)$.

Here we show the condition of $\mathcal{W}_1 \cap \mathcal{W}_2 = \{\mathbf{0}\}$ is equivalent to the above condition that the basis vectors from $\mathcal{W}_1$ and $\mathcal{W}_2$ combined are linearly independent. Let $\vec{u}^{(1)}, \vec{u}^{(2)}, \dots, \vec{u}^{(p)}$ and $\vec{v}^{(1)}, \vec{v}^{(2)}, \dots, \vec{v}^{(q)}$ be the basis vectors for $\mathcal{W}_1$ and $\mathcal{W}_2$ respectively. If these basis vectors are linearly independent, then by Theorem 6.1.9, the equation

$$c_1\vec{u}^{(1)} + c_2\vec{u}^{(2)} + \cdots + c_p\vec{u}^{(p)} + c_{p+1}\vec{v}^{(1)} + \cdots + c_{p+q}\vec{v}^{(q)} = \mathbf{0}$$

only has $c_j = 0$ as the trivial solution, $1 \leq j \leq p+q$. Rearranging, we have

$$\begin{aligned} &c_1\vec{u}^{(1)} + c_2\vec{u}^{(2)} + \cdots + c_p\vec{u}^{(p)} \in \mathcal{W}_1 \\ = &-c_{p+1}\vec{v}^{(1)} - \cdots - c_{p+q}\vec{v}^{(q)} \in \mathcal{W}_2 \end{aligned}$$

But since $c_j = 0$ is the only solution to this, it shows that there is only the zero vector in both $\mathcal{W}_1$ and $\mathcal{W}_2$ at the same time. The converse essentially follows the same argument in reverse. We say that $\mathcal{W}_1$ and $\mathcal{W}_2$ are a ***complement*** to each other in $\mathcal{W}_1 \oplus \mathcal{W}_2$, as any non-zero vector not in one of them will be found in another.

Properties 6.2.10 (Complement)**.** If a vector space can be written as a direct sum of two smaller subspaces, i.e. $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2$, then $\mathcal{W}_1 = \mathcal{W}_2^C$ and $\mathcal{W}_2 = \mathcal{W}_1^C$ are said to be the complement (denoted by the superscript ^{C}) to each other in $\mathcal{V}$.

As a counter-example, consider Example 6.2.2, suppose $\mathcal{W}_1 = \text{span}(\mathcal{B}_1) = \text{span}(\{(1, 0, 2, 1)^T, (1, -1, 1, -2)^T\})$ and $\mathcal{W}_2 = \text{span}(\mathcal{B}_2) = \text{span}(\{(0, 0, 1, 0)^T, (-2, 1, 0, 1)^T\})$ be the subspaces spanned the first/last two vectors in $\mathcal{G}$ respectively. It is not hard to see that $\mathcal{B}_1$ and $\mathcal{B}_2$ are themselves linearly independent (and hence they are bases for $\mathcal{W}_1$ and $\mathcal{W}_2$ individually), and $\dim(\mathcal{W}_1) = \dim(\mathcal{W}_2) = 2$. Nevertheless, in that example, we already know that the four vectors, when put together, are not linearly independent: $(-2, 1, 0, 1)^T$ is equal to $-(1, 0, 2, 1)^T - (1, -1, 1, -2)^T + 3(0, 0, 1, 0)^T$, and they only generates a three-dimensional subspace. Hence $\dim(\mathcal{W}_1 + \mathcal{W}_2) = \dim(\text{span}(\mathcal{B}_1) + \text{span}(\mathcal{B}_2)) = \dim(\text{span}(\mathcal{G})) = 3 \neq 4 = 2 + 2 = \dim(\mathcal{W}_1) + \dim(\mathcal{W}_2)$, and therefore they cannot form a direct sum. Geometrically, these two subspaces are like two planes intersecting along a straight line.

The direct sum of multiple subspaces are then recursively defined as

$$\begin{aligned}
&\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \cdots \oplus \mathcal{W}_{n-1} \oplus \mathcal{W}_n \\
=& (\cdots((\mathcal{W}_1 \oplus \mathcal{W}_2) \oplus \mathcal{W}_3) \oplus \cdots \oplus \mathcal{W}_{n-1}) \oplus \mathcal{W}_n
\end{aligned}$$

where we add up the subspaces one by one. Below shows an example of this.

Example 6.2.3. Given $\mathcal{W}_1 = \text{span}\{(1, 0, 2, 1, 0)^T, (2, 1, 0, 0, -1)^T\}$, $\mathcal{W}_2 = \text{span}\{(0, 3, 1, 0, 0)^T, (0, 0, -1, -2, 1)^T\}$, $\mathcal{W}_3 = \text{span}\{(1, 1, -3, 0, -1)^T\}$, show that $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ is a valid direct sum which equals to $\mathbb{R}^5$.

Solution. First, let's derive $\mathcal{W}_1 \oplus \mathcal{W}_2$. It is obvious that the two generating vectors from each of $\mathcal{W}_1$ and $\mathcal{W}_2$ are linearly independent themselves as they are not constant multiples of another. Now following similar ideas in Example 6.2.2, we are going to show that every column in the matrix formed by combining basis vectors of both $\mathcal{W}_1$ and $\mathcal{W}_2$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 2 & 0 & 1 & -1 \\ 1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

is pivotal after Gaussian Elimination, as follows.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 2 & 0 & 1 & -1 \\ 1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -4 & 1 & -1 \\ 0 & -2 & 0 & -2 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad \begin{array}{r} R_3 - 2R_1 \to R_3 \\ R_4 - R_1 \to R_4 \end{array}$$

$$\to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 13 & -1 \\ 0 & 0 & 6 & -2 \\ 0 & 0 & 3 & 1 \end{bmatrix} \qquad \begin{array}{r} R_3 + 4R_2 \to R_3 \\ R_4 + 2R_2 \to R_4 \\ R_5 + R_2 \to R_5 \end{array}$$

$$\to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 6 & -2 \\ 0 & 0 & 13 & -1 \end{bmatrix} \qquad R_3 \leftrightarrow R_5$$

$$\to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 6 & -2 \\ 0 & 0 & 13 & -1 \end{bmatrix} \qquad \frac{1}{3}R_3 \to R_3$$

$$\to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & -\frac{16}{3} \end{bmatrix} \qquad \begin{array}{r} R_4 - 6R_3 \to R_4 \\ R_5 - 13R_3 \to R_5 \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{16}{3} \end{bmatrix} \qquad -\frac{1}{4}R_4 \to R_4$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_5 + \frac{16}{3}R_4 \to R_5$$

and we are done (the backward phase is optional for this). Therefore, the four column vectors are linearly independent when considered as a whole and the direct sum $\mathcal{W}_1 \oplus \mathcal{W}_2 = \text{span}(\{(1,0,2,1,0)^T, (2,1,0,0,-1)^T, (0,3,1,0,0)^T, (0,0,-1,-2,1)^T\})$ makes sense, with $\dim(\mathcal{W}_1 \oplus \mathcal{W}_2) = \dim(\mathcal{W}_1) + \dim(\mathcal{W}_2) = 2 + 2 = 4$, $\mathcal{W}_1 \oplus \mathcal{W}_2 \subset \mathbb{R}^5$. Now, we attempt to compose $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 = (\mathcal{W}_1 \oplus \mathcal{W}_2) \oplus \mathcal{W}_3$, which requires showing that the only generating vector $(1,1,-3,0,-1)^T$ in $\mathcal{W}_3$ is linearly independent from (the basis vectors of) $\mathcal{W}_1 \oplus \mathcal{W}_2$. One way to do this is to show that the augmented system formed by appending $(1,1,-3,0,-1)^T$ to the matrix at the start

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 \\ 2 & 0 & 1 & -1 & -3 \\ 1 & 0 & 0 & -2 & 0 \\ 0 & -1 & 0 & 1 & -1 \end{array}\right]$$

has no solution and thus $(1,1,-3,0,-1)^T$ cannot be written as their linear combination (see part (a) of Theorem 6.1.11). We can simply repeat the exactly same reduction steps performed above, which would lead to

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & -\frac{7}{3} \end{array}\right]$$

where the last row is inconsistent. Therefore $(1, 1, -3, 0, -1)^T$ is linearly independent from the preceding four vectors and $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ is a valid direct sum, and $\dim(\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3) = \dim(\mathcal{W}_1 \oplus \mathcal{W}_2) + \dim(\mathcal{W}_3) = 4 + 1 = 5$. By Properties 6.2.5, $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 = \mathbb{R}^5$. □

The importance of direct sum is that the coordinates of two vectors in respective bases from the two subspaces can be simply concatenated when we add up both the vectors and bases, and *this representation will be unique*. Going in the opposite direction, we can also split the coordinates of a direct sum back into the respective subspaces. Let's illustrate this with $\mathcal{W}_1$ and $\mathcal{W}_2$ in the above example. Using the given sets of generating vectors $\mathcal{B}_1 = \{(1, 0, 2, 1, 0)^T, (2, 1, 0, 0, -1)^T\}$ and $\mathcal{B}_2 = \{(0, 3, 1, 0, 0)^T, (0, 0, -1, -2, 1)^T\}$ as bases for $\mathcal{W}_1$ and $\mathcal{W}_2$, the coordinates $(1, 2)^T_{B_1}$ and $(1, -1)^T_{B_2}$ in the $\mathcal{B}_1$ and $\mathcal{B}_2$ system, represent the vectors

$$\begin{aligned} (1, 0, 2, 1, 0)^T + (2, 1, 0, 0, -1)^T &= (5, 2, 2, 1, -2)^T \\ \text{and } (0, 3, 1, 0, 0)^T - (0, 0, -1, -2, 1)^T &= (0, 3, 2, 2, -1)^T \end{aligned}$$

in $\mathbb{R}^5$ respectively. When they are summed, it yields $(5, 2, 2, 1, -2)^T + (0, 3, 2, 2, -1)^T = (5, 5, 4, 3, -3)^T$. The basis formed by combining $\mathcal{B}_1$ and $\mathcal{B}_2$ will be

$$\begin{aligned} \mathcal{B}_1 \cup \mathcal{B}_2 = \{&(1, 0, 2, 1, 0)^T, (2, 1, 0, 0, -1)^T, \\ &(0, 3, 1, 0, 0)^T, (0, 0, -1, -2, 1)^T\} \end{aligned}$$

and the merged coordinates $(1, 2, 1, -1)^T_{B_1+B_2}$ then correspond exactly to

$$\begin{aligned} &(1, 0, 2, 1, 0)^T + 2(2, 1, 0, 0, -1)^T + (0, 3, 1, 0, 0)^T - (0, 0, -1, -2, 1)^T \\ = &(5, 5, 4, 3, -3)^T \in \mathcal{W}_1 \oplus \mathcal{W}_2 \subset \mathbb{R}^5 \end{aligned}$$

The new coordinate representation $(1, 2, 1, -1)^T_{B_1+B_2}$ is unique as $\mathcal{B}_1 \cup \mathcal{B}_2$ has been shown to be linearly independent in Example 6.2.3 and Properties 6.1.12 applies over the direct sum $\mathcal{W}_1 \oplus \mathcal{W}_2$, and it can be partitioned cleanly as $(1, 2, 1, -1)^T_{B_1+B_2} = (1, 2)^T_{B_1} + (1, -1)^T_{B_2}$.

On the other hand, the uniqueness property will not hold if the subspace sum is not a direct sum. Let's use Example 6.2.2 again as a demonstration, where $\mathcal{B}_1 = \{(1,0,2,1)^T, (1,-1,1,-2)^T\}$ and $\mathcal{B}_2 = \{(0,0,1,0)^T, (-2,1,0,1)^T\}$ and we have already shown that they are not linearly independent when combined. Take

$$\begin{aligned}
(1,2)^T_{B_1} &= (1,0,2,1)^T + 2(1,-1,1,-2)^T = (3,-2,4,-3)^T \\
\text{and } (-1,1)^T_{B_2} &= -(0,0,1,0)^T + (-2,1,0,1)^T = (-2,1,-1,1)^T
\end{aligned}$$

Their concatenated sum will be

$$\begin{aligned}
&(1,2,-1,1)^T_{B_1+B_2} \\
&= (1,0,2,1)^T + 2(1,-1,1,-2)^T - (0,0,1,0)^T + (-2,1,0,1)^T \\
&= (1,-1,3,-2)^T = (3,-2,4,-3)^T + (-2,1,-1,1)^T
\end{aligned}$$

but

$$\begin{aligned}
(0,1,2,0)^T_{B_1+B_2} &= (1,-1,1,-2)^T + 2(0,0,1,0)^T \\
&= (1,-1,3,-2)^T = (1,2,-1,1)^T_{B_1+B_2}
\end{aligned}$$

is aptly an alternative representation.

Another aspect about direct sum is that all finite-dimensional, particularly n-dimensional vector spaces with some basis consisted of n vectors can be regarded to be a direct sum of the n one-dimensional subspaces generated by each of the basis vectors individually. We have provided a schematic (Figure 6.2) to better illustrate this.

6.3 The Four Fundamental Subspaces Induced by Matrices

6.3.1 Row Space, Column Space

In Definition 6.1.7, we have developed the notion of column space. For a $m \times n$ matrix $A = [\vec{v}^{(1)}|\vec{v}^{(2)}|\cdots|\vec{v}^{(n)}]$, its column space is the subspace generated by

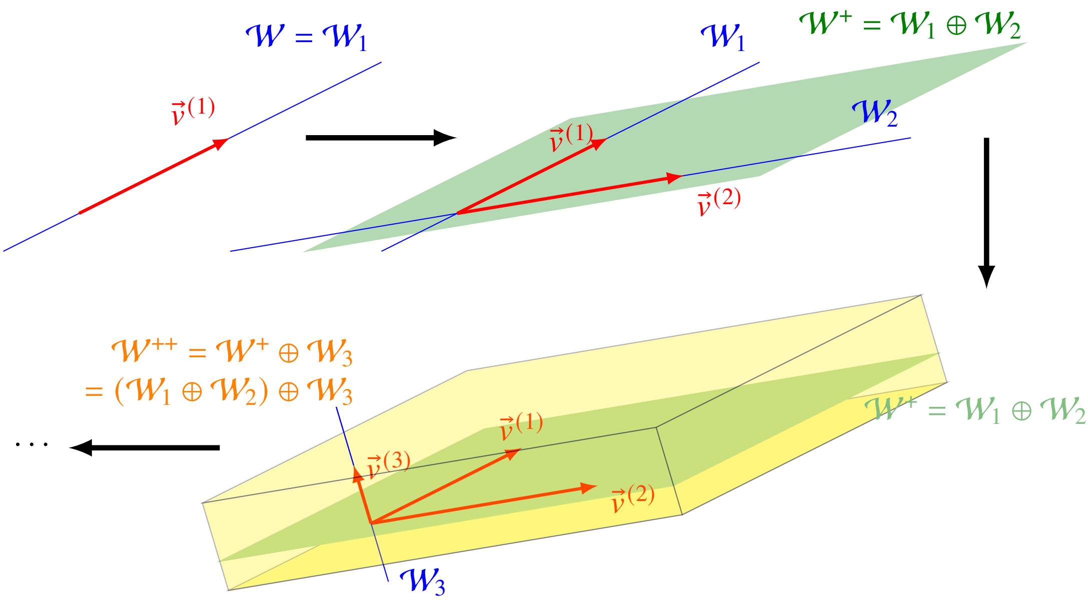


Figure 6.2: Iteratively adding one-dimensional subspaces to the direct sum. Note that the lines, plane and "cuboid" all extend infinitely. We can only visualize up to a three-dimensional direct sum but it goes on even for higher dimensions.

the n $\mathbb{R}^m$ vectors $\vec{v}^{(j)}$, $j = 1, 2, \ldots, n$. Similarly, we can also define the ***row space*** of a matrix. We formally define both of them as below.

Definition 6.3.1 (Column/Row Space)**.** For an $m \times n$ real matrix A, its column space $C(A)$ is the subspace spanned by its n column vectors, $\vec{v}^{(1)}, \vec{v}^{(2)}, \ldots, \vec{v}^{(n)} \in \mathbb{R}^m$ as in $A = [\vec{v}^{(1)}|\vec{v}^{(2)}|\cdots|\vec{v}^{(n)}]$; Meanwhile its row space $\mathcal{R}(A)$ is the subspace spanned by its m row vectors $\vec{w}^{(1)T}, \vec{w}^{(2)T}, \ldots, \vec{w}^{(m)T} \in \mathbb{R}^n$ as in

$$A = \begin{bmatrix} \vec{w}^{(1)T} \\ \hline \vec{w}^{(2)T} \\ \hline \vdots \\ \hline \vec{w}^{(m)T} \end{bmatrix}$$

Notice that the row (column) space of a matrix $\mathcal{R}(A) = C(A^T)$ ($C(A) = \mathcal{R}(A^T)$) is just the column (row) space of its transpose.

For instance, in Example 6.2.2, the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & -1 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix}$$

actually has a column space of $C(A) = \text{span}(\{(1, 0, 2, 1)^T, (1, -1, 1, -2)^T, (0, 0, 1, 0)^T, (-2, 1, 0, 1)^T\}) = \text{span}(\{(1, 0, 2, 1)^T, (1, -1, 1, -2)^T, (0, 0, 1, 0)^T\})$ of dimension 3 despite the vectors are in $\mathbb{R}^4$. In deriving this result we have produced the reduced row echelon form of A, which is

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

from which we can see the number of pivots, or *rank*, is also 3. In fact, just like the case above, the ***rank*** of a matrix always indicates the dimension of its column space. This is due to Properties 6.2.3 and 6.2.4, leading to the following equivalent definition.

Definition 6.3.2 (Rank)**.** The rank of a matrix A is the number of leading 1s in its reduced row echelon form, which is also the amount of linearly independent vectors in any basis of its column space, i.e. the dimension of the column space.

We can approach this from another angle, which involves restating previous results related to Gaussian Elimination. In Section 6.1.4, we have shown that elementary row operations preserve (the amount of) linear independent vectors, hence

Properties 6.3.3. Elementary row operations does not change the number of dimensions in the column space of a matrix.

The matrix A will then have the same number of dimensions in its column space throughout the Gaussian Elimination procedure, which coincides with the number of linearly independent vectors and thus pivots in the final reduced row echelon form, establishing the equivalence in Definition 6.3.2. However, notice that elementary row operations do change the actual column space. On the other hand, for row space, we have an even stronger result.

Properties 6.3.4. Elementary row operations does not change the row space of a matrix, and thus its dimension.

which is not hard to accept. Swapping rows, and multiplying a row by some constant obviously does not affect the span of rows in the matrix. Adding to/subtracting from a row R_p (also as a row vector $\vec{w}^{(p)T}$) by the constant multiple of another row R_q ($\vec{w}^{(q)T}$) also will not alter it. To see this, observe that the newly resulted row vector is just a linear combination of the two input rows, i.e. the new R_p becomes $\vec{w}^{(r)T} = \vec{w}^{(p)T} + c\vec{w}^{(q)T}q$ (and hence $\vec{w}^{(p)T} = \vec{w}^{(r)T} - c\vec{w}^{(q)T}$). Using part (b) of Theorem 6.1.11 twice, we have

$$\begin{aligned}
&= \text{span}(\{\ldots, \vec{w}^{(p)}, \ldots, \vec{w}^{(q)}, \ldots, \vec{w}^{(r)}\}) \\
\mathcal{R}(A) &= \text{span}(\{\ldots, \vec{w}^{(p)}, \ldots, \vec{w}^{(q)}, \ldots\}) \\
&= \text{span}(\{\ldots, \vec{w}^{(r)}, \ldots, \vec{w}^{(q)}, \ldots\}) = \mathcal{R}(A')
\end{aligned}$$

where A' denotes the matrix after the addition/subtraction elementary row operation. Our next key theorem relies on the observation that the dimensions of row and column space of a matrix in its reduced row echelon form are the same, or in other words,

Properties 6.3.5. A matrix in reduced row echelon form has the same amount of (linearly independent) vectors in the basis of its row and column space.

We will not read off the detailed arguments in the proof, but instead note that it is essentially an analysis of positions of the leading 1s and zeros in any reduced row echelon form. However, we will give an example to elucidate how it holds. Take a reduced row echelon form of

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is obvious that its column space is spanned by the basis $\{(1, 0, 0, 0)^T, (0, 1, 0, 0)^T, (0, 0, 1, 0)^T\}$, while a basis of its row space can be simply formed by the first three non-zero row vectors $\{(1, 1, 0, 0, 1), (0, 0, 1, 0, 0), (0, 0, 0, 1, 1)\}$. In this case, the dimension of row/column space of the reduced row echelon form is both 3. With these observations, we can derive the desired result, sometimes referred to as *"Column rank equals to row rank"*.

Properties 6.3.6. For any matrix, the dimension of its column space is equal to that of its row space, i.e.

$$\dim(\mathcal{C}(A)) = \dim(\mathcal{R}(A)) = \dim(\mathcal{C}(A^T))$$

Proof. Any matrix has a unique reduced row echelon form due to Theorem 2.2.6, whose row/column space has the same number of dimensions by Properties 6.3.5. According to Properties 6.3.3 and 6.3.4, the elementary row operations done to convert the matrix to its reduced row echelon form leave both the dimensions of row and column space conserved, and thus the column rank and row rank in the starting matrix are equal. $\square$

Example 6.3.1. Given a matrix

$$A = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 1 & 2 & 1 & -1 \\ 1 & 0 & -5 & 3 \end{bmatrix}$$

find a basis for its column/row space $\mathcal{C}(A)$ and $\mathcal{R}(A)$ and check if Properties 6.3.6 holds.

Solution. We first apply Gaussian Elimination to A, which leads to

$$\begin{aligned}
\begin{bmatrix} 1 & 1 & -2 & 1 \\ 1 & 2 & 1 & -1 \\ 1 & 0 & -5 & 3 \end{bmatrix} &\to \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & -1 & -3 & 2 \end{bmatrix} && \begin{matrix} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{matrix} \\
&\to \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} && R_3 - R_2 \to R_3 \\
&\to \begin{bmatrix} 1 & 0 & -5 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} && R_1 - R_2 \to R_1
\end{aligned}$$

The number of pivotal columns is 2, and from the reduced row echelon form we obtain the dependence relations where the first two column vectors $(1,1,1)^T$ and $(1,2,0)^T$ are linearly independent while the last two column vectors $(-2,1,-5)^T = -5(1,1,1)^T + 3(1,2,0)^T$ and $(1,-1,3)^T = 3(1,1,1)^T - 2(1,2,0)^T$ are linear combinations of the previous two. Hence $\mathcal{C}(A)$ has a basis of $\{(1,1,1)^T, (1,2,0)^T\}$ and $\dim(\mathcal{C}(A)) = 2$. On the other hand, to find the row space we consider A^T and repeat the elimination process again as follows. However, notice that according to the dependence relations for the column vectors in A above, we can immediately do the corresponding addition/subtraction operations for the rows in A^T, to reduce the third/fourth rows, obtaining

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ -2 & 1 & -5 \\ 1 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} &R_3 + 5R_1 - 3R_2 \to R_3, \\ &R_4 - 3R_1 + 2R_2 \to R_4 \end{aligned}$$

and the next step is straight-forward:

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} &\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & R_2 - R_1 \to R_2 \\ &\to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & R_1 - R_2 \to R_1 \end{aligned}$$

which reveals that the first two columns (representing the first two row vectors in A) are linearly independent and the third column (the last row vector in A) is redundant ($(1, 0, -5, 3)^T = 2(1, 1, -2, 1)^T - (1, 2, 1, -1)^T$). Therefore $\mathcal{R}(A)$ has a basis of $\{(1, 1, -2, 1)^T, (1, 2, 1, -1)^T\}$, and $\dim(\mathcal{R}(A)) = 2 = \dim(\mathcal{C}(A))$, and Properties 6.3.6 is true in this case. □

Finally, in view of Definitions 6.1.5 and 6.3.1, the analysis about solving linear systems in Section 3.2 can be summarized as

Properties 6.3.7. A linear system $A\vec{x} = \vec{h}$ is consistent if and only if $\vec{h}$ is in the column space of A.

6.3.2 Null Space, Rank-Nullity Theorem

As we have briefly mentioned in the end of last chapter, the solution of a linear system $A\vec{x} = \vec{h}$, where A is an $m \times n$ matrix and $\vec{x} \in \mathbb{R}^n$, can be viewed as some sort of a solution space. In Section 3.2.1 we know that it is made up of the particular and complementary solution, where the latter corresponding to the family of $\vec{x} = \vec{x}_0$ (= $\vec{x}_c$ using the notation in that section) that satisfies the homogeneous part $A\vec{x} = \mathbf{0}$. The set $\vec{x}_0 \in \mathbb{R}^n$ can be shown to form a subspace of $\mathbb{R}^n$[16], and this subspace is then called the ***null space*** of A.

Definition 6.3.8 (Null Space)**.** For an $m \times n$ real matrix A, its null space $\mathcal{N}(A)$ is the subspace consisted of all solution vectors $\vec{x} = \vec{x}_0 \in \mathbb{R}^n$ to the matrix equation $A\vec{x} = \mathbf{0}$. The dimension of null space is called ***nullity***.

This definition of nullity as the dimension of null space is consistent with that in Section 3.2.1 where nullity is initially given by the number of columns in the matrix minus the amount of leading 1s in its rref (rank), or equivalently the number of non-pivotal columns. To see this, observe that any solution $\vec{x}_0$ to $A\vec{x} = \mathbf{0}$ is also the solution to $A_{\text{rref}}\vec{x} = \mathbf{0}$ and vice versa, via elementary matrices. Hence the null space and nullity of A will be the same as that of A_{rref}. Previously we have assigned free variables to the non-pivotal columns (let's say there is k of them) of A_{rref} and derive $\vec{x}_0$ where they are generated by k pairs of free variables and column vectors $(\vec{x}_0^{(1)}, \vec{x}_0^{(2)}, \ldots, \vec{x}_0^{(k)})$. It is clear that such a procedure will ensure these k vectors are linearly independent as each of them has a component of 1 in the position corresponding to that particular free variable in the rref and 0s in other positions corresponding to other free variables (see Example 3.2.6 for instance). We claim that they also span the entire null space of A_{rref}.[17] Hence by the definition given in Section 6.2.2 they form a basis for the null

[16] To show this we check the two conditions in Theorem 6.1.2. Let $\vec{x}_0^{(1)}$ and $\vec{x}_0^{(2)}$ be two vectors in the null space $\vec{x}_0$. Then we have: 1. $A(\vec{x}_0^{(1)} + \vec{x}_0^{(2)}) = A\vec{x}_0^{(1)} + A\vec{x}_0^{(2)} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, so $\vec{x}_0^{(1)} + \vec{x}_0^{(2)} \in \vec{x}_0$, and 2. $A(a\vec{x}_0^{(1)}) = a(A\vec{x}_0^{(1)}) = a\mathbf{0} = \mathbf{0}$, hence $a\vec{x}_0^{(1)} \in \vec{x}_0$.

[17] Assume the contrary so that the span of $\{\vec{x}_0^{(1)}, \vec{x}_0^{(2)}, \ldots, \vec{x}_0^{(k)}\}$ does not cover the whole null space, then the dimension of null space has to be greater than k by Properties 6.2.5. Without loss of generality, let the "correct" dimension of null space to be $k+1$. Then by (c) of Properties 6.2.7, there exists $\vec{x}_0^{(k+1)}$ such that $\{\vec{x}_0^{(1)}, \vec{x}_0^{(2)}, \ldots, \vec{x}_0^{(k)}, \vec{x}_0^{(k+1)}\}$ are basis of the null space. This $\vec{x}_0^{(k+1)}$ can be made to take the value of 0 at all k positions where the free variables reside by subtracting it by appropriate multiples of $\vec{x}_0^{(j)}$ without altering the null space (c.f. Footnote 14), and by doing so non-zero components of $\vec{x}_0^{(k+1)}$ only appear in positions corresponding to leading 1s in A_{rref}. This causes a contradiction since

space of A_{rref} as well as A and by Properties 6.2.4 the dimension of null space of A is also k.

Using Definitions 6.3.1, 6.3.2, and 6.3.8 to rephrase, the preceding discussion means that the rank of a matrix plus its nullity equals to its number of columns, which leads to the so-called ***Rank-nullity Theorem***.

Theorem 6.3.9 (Rank-nullity Theorem)**.** For a real $m \times n$ matrix A, we have

$$\begin{aligned}\dim(\mathcal{C}(A)) + \dim(\mathcal{N}(A)) &= \text{rank}(A) + \text{nullity}(A) = n \\ &= \dim(\mathcal{R}(A)) + \dim(\mathcal{N}(A))\end{aligned}$$

A notable relationship between row space and null space is that any pair of two vectors coming from the respective subspaces will be orthogonal to each other.

Properties 6.3.10. Given a real matrix A, any vector in its row space $\mathcal{R}(A)$ is orthogonal to all vectors in its null space $\mathcal{N}(A)$ and vice versa.

Proof. Let the shape of A be $m \times n$, we can express A in the form of its row vectors as

$$A = \left[\begin{array}{c} \vec{w}^{(1)T} \\ \hline \vec{w}^{(2)T} \\ \hline \vdots \\ \hline \vec{w}^{(m)T} \end{array}\right]$$

and the corresponding homogeneous system $A\vec{x} = \mathbf{0}$ then can be written as

$$A\vec{x} = \left[\begin{array}{c} \vec{w}^{(1)T} \\ \hline \vec{w}^{(2)T} \\ \hline \vdots \\ \hline \vec{w}^{(m)T} \end{array}\right] \vec{x} = \begin{bmatrix} \vec{w}^{(1)T}\vec{x} \\ \vec{w}^{(2)T}\vec{x} \\ \vdots \\ \vec{w}^{(m)T}\vec{x} \end{bmatrix} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$A_{\text{rref}}\vec{x}_0^{(k+1)} = \mathbf{0}$ then implies that there exists a non-trivial dependence relation between the pivotal column vectors themselves.

where for a solution $\vec{x} = \vec{x}_0$ in the null space of A, each of the dot products $\vec{w}^{(i)T}\vec{x}_0 = 0, i = 1, 2, \ldots, m$, has to be equal to zero. Any vector in the row space of A can be expressed as $\vec{w} = c_1\vec{w}^{(1)} + c_2\vec{w}^{(2)} + \cdots + c_m\vec{w}^{(m)}$ by Definitions 6.3.1 and 6.1.5, and subsequently, its dot product with $\vec{x}_0$

$$\begin{aligned}
\vec{w}^T\vec{x}_0 &= (c_1\vec{w}^{(1)} + c_2\vec{w}^{(2)} + \cdots + c_m\vec{w}^{(m)})^T\vec{x}_0 \\
&= c_1(\vec{w}^{(1)T}\vec{x}_0) + c_2(\vec{w}^{(2)T}\vec{x}_0) + \cdots + c_m(\vec{w}^{(m)T}\vec{x}_0) \\
&= c_1(0) + c_2(0) + \cdots + c_m(0) = 0
\end{aligned}$$

is also zero, therefore they are orthogonal by Properties 4.2.5, which implies that any vector in $\mathcal{R}(A)$ is orthogonal to any another vector in $\mathcal{N}(A)$. □

As a corollary, this is equivalent to all vectors in the generating set or basis for the row space for a matrix being orthogonal to all vectors in those for its null space. The following additional observation will be useful later.

Properties 6.3.11. Non-zero orthogonal vectors are linearly independent.

Proof. We will only prove the case with two vectors in $\mathbb{R}^n$ but those with multiple vectors can be derived in the same essence. Consider $c_1\vec{u}^{(1)} + c_2\vec{u}^{(2)} = \mathbf{0}$ where $\vec{u}^{(1)}$ and $\vec{u}^{(2)}$ are orthogonal, i.e. $\vec{u}^{(1)} \cdot \vec{u}^{(2)} = 0$. Taking dot product with $\vec{u}_1$ on both sides gives

$$\begin{aligned}
\vec{u}^{(1)} \cdot (c_1\vec{u}^{(1)} + c_2\vec{u}^{(2)}) = c_1(\vec{u}^{(1)} \cdot \vec{u}^{(1)}) + c_2(\vec{u}^{(1)} \cdot \vec{u}^{(2)}) &= \vec{u}^{(1)} \cdot \mathbf{0} \\
c_1\left\|\vec{u}^{(1)}\right\|^2 + c_2(0) = c_1\left\|\vec{u}^{(1)}\right\|^2 &= 0
\end{aligned}$$

Since $\vec{u}_1$ is non-zero, $\|\vec{u}_1\|^2 > 0$, and c_1 must be zero. In a similar vein, we can show that c_2 is zero as well. Therefore the only solution to the equation $c_1\vec{u}_1 + c_2\vec{u}_2 = \mathbf{0}$ is the trivial solution $c_1 = c_2 = 0$. By Theorem 6.1.9, the two vectors are linearly independent. □

Example 6.3.2. For the matrix in Example 6.3.1, find its null space and check if Properties 6.3.10 and Theorem 6.3.9 hold.

Solution. The homogeneous system corresponding to the matrix is

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 0 \\ 1 & 2 & 1 & -1 & 0 \\ 1 & 0 & -5 & 3 & 0 \end{array}\right]$$

which can be reduced, following the same steps in Example 6.3.1, to

$$\left[\begin{array}{cccc|c} 1 & 0 & -5 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where there are two non-pivotal columns and hence two free parameters can be assigned to them. Let $x_3 = s$ and $x_4 = t$, then $x_1 = 5s - 3t$ and $x_2 = -3s + 2t$. So the solution to the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5s - 3t \\ -3s + 2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 5 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

and thus a basis for the null space is $\{(5, -3, 1, 0)^T, (-3, 2, 0, 1)^T\}$ where these two vectors are clearly linearly independent (by observing the 0 and 1 of the last two components). As found in Example 6.3.1, the basis for its row space is $\{(1, 1, -2, 1)^T, (1, 2, 1, -1)^T\}$. Subsequently, checking orthogonality between the two bases is straight-forward, and we will only do this for the first vector in the row space basis against the null space basis.

$$(5, -3, 1, 0)^T \cdot (1, 1, -2, 1)^T = (5)(1) + (-3)(1) + (1)(-2) + (0)(1) = 0$$
$$(-3, 2, 0, 1)^T \cdot (1, 1, -2, 1)^T = (-3)(1) + (2)(1) + (0)(-2) + (1)(1) = 0$$

Furthermore, the dimension of null space, or the nullity, is $\dim \mathcal{N}(A) = 2$. Previously we have also found that $\dim \mathcal{C}(A) = \text{rank}(A) = 2$. So $\text{rank}(A) + \text{nullity}(A) = 2 + 2 = 4$, and Theorem 6.3.9 is true. □

Short Exercise: Show that [18]

$$\dim(\mathcal{R}(A)) + \dim(\mathcal{N}(A^T)) = \dim(\mathcal{C}(A)) + \dim(\mathcal{N}(A^T)) = m$$

[18] Replace A by A^T in Theorem 6.3.9 to get $\dim(\mathcal{C}(A^T)) + \dim(\mathcal{N}(A^T)) = m$ and note that $\mathcal{C}(A^T) = \mathcal{R}(A)$.

$\mathcal{N}(A^T)$ is also known as the ***left null space*** of A.

By Properties 6.3.10 and 6.3.11, vectors in the row space and null space of an $m \times n$ matrix A are linearly independent of each other and can form a direct sum $\mathcal{R}(A) \oplus \mathcal{N}(A)$ according to Definition 6.2.9. Note that they are the complement to each other with respect to this direct sum according to Properties 6.2.10. Since $\mathcal{R}(A) \subseteq \mathbb{R}^n$, $\mathcal{N}(A) \subseteq \mathbb{R}^n$ and hence $\mathcal{R}(A) \oplus \mathcal{N}(A) \subseteq \mathbb{R}^n$, from Theorem 6.3.9 and Properties 6.2.5, we conclude that $\mathcal{R}(A) \oplus \mathcal{N}(A)$ is just $\mathbb{R}^n$. In other words, the row space and null space of a matrix can reconstruct the real n-space by forming their direct sum. The similar can be said for its column and left null space. Furthermore, since all vectors in the row space (column space) are orthogonal to those in the (left) null space via Properties 6.3.10, we say that they are actually an ***orthogonal complement*** to each other.

Properties 6.3.12. For a real $m \times n$ matrix A, we have

$$\mathcal{R}(A) \oplus \mathcal{N}(A) = \mathbb{R}^n \qquad\qquad \mathcal{C}(A) \oplus \mathcal{N}(A^T) = \mathbb{R}^m$$

where $\mathcal{R}(A)^\perp = \mathcal{N}(A)$, $\mathcal{N}(A)^\perp = \mathcal{R}(A)$, and $\mathcal{C}(A)^\perp = \mathcal{N}(A^T)$, $\mathcal{N}(A^T)^\perp = \mathcal{C}(A)$, $^\perp$ denotes an orthogonal complement.

We conclude the relationships between the column, row, null, and left null space, a.k.a the ***Four fundamental subspaces*** induced by a matrix, with a diagram (Figure 6.3).

6.4 Python Programming

To check linear independence and find a basis for columns in a matrix, we can use the `columnspace` method in `sympy`. Let's test it with the matrix in Example 6.3.1.

```
import sympy

myMatrix = sympy.Matrix([[1., 1., -2., 1.],
                         [1., 2., 1., -1.],
```

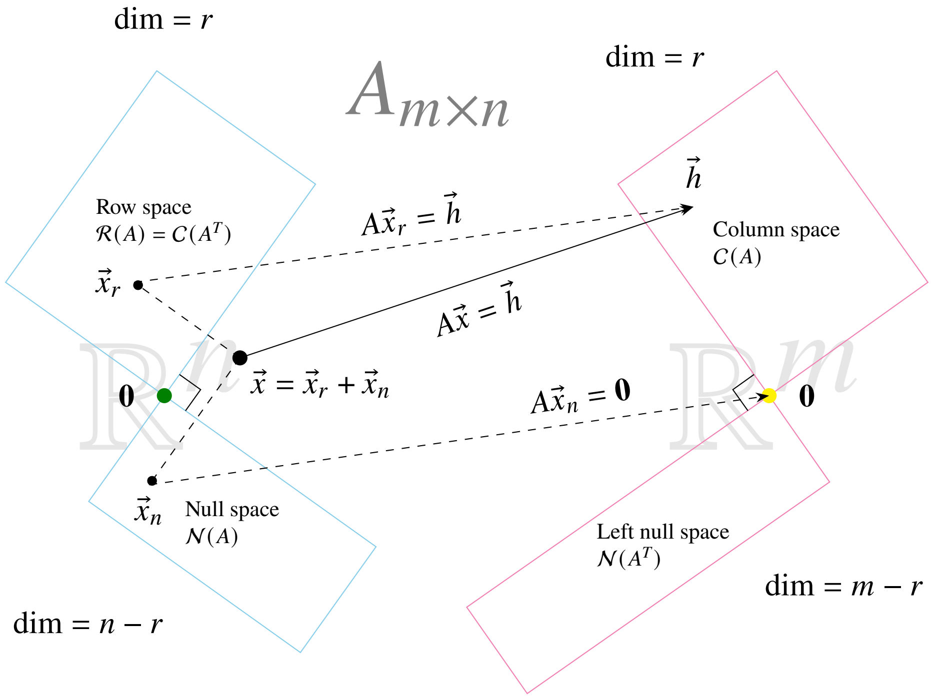


Figure 6.3: The relationships between the four fundamental subspaces for an $m \times n$ real matrix A of rank r: the row space $\mathcal{R}(A) = C(A^T)$, null space $\mathcal{N}(A)$, column space $C(A)$, left null space $\mathcal{N}(A^T)$. Any vector $\vec{x} \in \mathbb{R}^n$ can be partitioned into $\vec{x} = \vec{x}_r + \vec{x}_n$ uniquely, where $\vec{x}_r \in \mathcal{R}(A) \subseteq \mathbb{R}^n$ and $\vec{x}_n \in \mathcal{N}(A) \subseteq \mathbb{R}^n$ are in the row/null space of A respectively. The matrix A maps $\vec{x}_n$ to the zero vector in $\mathbb{R}^m$ and $\vec{x}_r$ to some vector $\vec{h} \in C(A) \subseteq \mathbb{R}^m$ in the column space of A. The total effect on $\vec{x}$ multiplied by A, is the sum of the two responses: $A\vec{x} = A(\vec{x}_r + \vec{x}_n) = A\vec{x}_r + A\vec{x}_n = \vec{h} + \mathbf{0} = \vec{h}$.

```
                          [1., 0., -5., 3.]])
print(myMatrix.columnspace())
```

which gives

```
[Matrix([
[1.0],
[1.0],
[1.0]]),
Matrix([
[1.0],
[2.0],
[  0]])]
```

as expected. The rank can be found in two ways.

```
print(myMatrix.rank()) # or len(myMatrix.columnspace())
```

This returns `2` correctly. We can make a basis for the row space similarly by the `rowspace` method. In the same manner, the null space is computed by the `nullspace` method:

```
print(myMatrix.nullspace())
```

producing an output of

```
[Matrix([
[ 5.0],
[-3.0],
[   1],
[   0]]),
Matrix([
[-3.0],
[ 2.0],
[   0],
[   1]])]
```

The nullity is then simply calculated by `len(myMatrix.nullspace())`, which gives a right answer of `2`.

6.5 Exercises

Exercise 6.1 For $\vec{v}^{(1)} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\vec{v}^{(2)} = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\vec{v}^{(3)} = \begin{bmatrix}0\\1\\1\end{bmatrix}$, find the constants a, b, c such that their linear combination $a\vec{v}^{(1)} + b\vec{v}^{(2)} + c\vec{v}^{(3}$ equals to

(a) $(3, 2, 9)^T$,

(b) $(9, 1, 5)^T$.

Exercise 6.2 Determine if the following sets of vectors are linearly independent.

(a) $\vec{u} = (2, -1)^T$, $\vec{v} = (-4, 2)^T$,

(b) $\vec{u} = (1, 2, 3)^T$, $\vec{v} = (6, 7, 9)^T$, $\vec{w} = (4, 8, 5)^T$, and

(c) $\vec{u} = (1, 3, 3)^T$, $\vec{v} = (3, 2, 9)^T$, $\vec{w} = (1, -4, 3)^T$.

Exercise 6.3 For the basis $\mathcal{B}$: $\vec{v}^{(1)} = \begin{bmatrix}6\\1\\2\end{bmatrix}$, $\vec{v}^{(2)} = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\vec{v}^{(3)} = \begin{bmatrix}2\\3\\3\end{bmatrix}$ (relative to the standard basis $\mathcal{E}$), do the following coordinate conversion.

(a) Find the coordinates of $(5, 2, 3)^T$ in the $\mathcal{B}$ frame,

(b) Transform $(1, -1, 1)_B^T$ from the $\mathcal{B}$ system back to the the standard basis $\mathcal{E}$.

Exercise 6.4 Prove that for any two subspaces $\mathcal{W}_1, \mathcal{W}_2 \subseteq \mathcal{V}$. Their intersection $\mathcal{W}_1 \cap \mathcal{W}_2$ is also a subspace of $\mathcal{V}$. How about their union?

Exercise 6.5 Show that $\mathcal{W}_1 = \text{span}(\{(1, 0, 0, 1)^T, (0, 1, -1, 1)^T\})$ and $\mathcal{W}_2 = \text{span}(\{(1, 0, 1, -1)^T\})$ can be composed to produce a direct sum $\mathcal{W}_1 \oplus \mathcal{W}_2$. Find bases for $\mathcal{W}_1$, $\mathcal{W}_2$ and hence this direct sum. Express $(2, 0, 1, 0)^T$ in the direct sum basis as the combined coordinates of the two smaller subspaces.

Exercise 6.6 Find (bases for) the column, row, null and left null space of

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

and check if Theorem 6.3.9 and Properties 6.3.12 hold in this case.

Chapter 7

More on Coordinate Bases, Linear Transformations

In this chapter we will go deeper about what actually a matrix represents in the big picture. Matrices by nature is a rule of *linear transformation* (or *linear mapping*) between two vector spaces. We are going to study several special types of linear transformations, which ultimately reveals the relationship between any n-dimensional real vector space and the real n-space $\mathbb{R}^n$, as an *isomorphism*. We then move to discuss how a change of coordinates works for vectors and matrices, as well as the *Gram-Schmidt* process to make an *orthonormal* basis.

7.1 About Linear Transformations

7.1.1 Linear Maps between Vector Spaces

Consider two vector spaces, we may want to know if vectors in one of the spaces, let's say $\mathcal{U}$, can be associated to or transformed into those in another vector space, $\mathcal{V}$, according to some rules. This is known as a ***transformation/mapping*** from the vector space $\mathcal{U}$ to $\mathcal{V}$. Of the most concern is the class of ***linear transformations/mappings*** which obeys the two properties listed below.

Definition 7.1.1 (Linear Transformation/Map)**.** A linear transformation (or linear map) from a vector space $\mathcal{U}$ to another vector space $\mathcal{V}$ is a mapping: $T : \mathcal{U} \to \mathcal{V}$, such that for all vectors $\vec{u}_1, \vec{u}_2 \in \mathcal{U}$, and any scalar a, it satisfies:

1. $T(\vec{u}_1 + \vec{u}_2) = T(\vec{u}_1) + T(\vec{u}_2)$ (Additivity), and
2. $T(a\vec{u}_j) = aT(\vec{u}_j)$ (Homogeneity).

These two properties combined are known as *linearity*. An equivalent condition is $T(a\vec{u}_1 + b\vec{u}_2) = aT(\vec{u}_1) + bT(\vec{u}_2)$, where b is any scalar as well.

Notice that if $\mathcal{U}/\mathcal{V}$ coincides with the real n/m-space $\mathbb{R}^n/\mathbb{R}^m$, and we express any vector $\vec{u}$: $[\vec{u}]_B$ in $\mathcal{U}$ with n coordinates using some basis $\mathcal{B}$ of its (similarly for $\vec{v}$: $[\vec{v}]_H$ in $\mathcal{V}$ having m coordinates from some basis $\mathcal{H}$). Then T: $[T]_B^H = A$ where A is any $m \times n$ matrix satisfies the requirements of and is a linear transformation from $\mathcal{U}$ to $\mathcal{V}$ according to the rule $T(\vec{u})$: $A[\vec{u}]_B$. (Short Exercise: show this satisfies the conditions outlined in Definition 7.1.1![1]) This implies that all matrices can be considered as some sort of linear mappings (for now, between $\mathbb{R}^n$ and $\mathbb{R}^m$). In fact, the converse, which states that any linear transformation (between finite-dimensional vector spaces) can be represented by a matrix, is also true as well, and will be discussed in the remaining parts of this section.

Let's now explicitly fix a basis $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ for $\mathcal{U}$ (again, similarly we have $\mathcal{H} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ for $\mathcal{V}$). For each $\vec{u}_j$, denote $\vec{v}^{(j)} = T(\vec{u}_j)$ as the resulting vectors in $\mathcal{V}$ after applying the transformation T over the basis vectors for $\mathcal{U}$. Notice that $[\vec{u}_j]_B = (e_j)_B$ where the j-th basis vector of $\mathcal{B}$ is explicitly represented in a numeric tuple form with the j-th entry being 1 and others being 0 (where the usual hat symbol on e is not present) in the $\mathcal{B}$ system. Due to Definition 6.2.1 and Properties 6.1.12, $T(\vec{u}_j) = \vec{v}^{(j)}$ can be expressed as a unique linear combination as $\vec{v}^{(j)} = a_1^{(j)}\vec{u}_1 + a_2^{(j)}\vec{u}_2 + \cdots a_m^{(j)}\vec{u}_m = \sum_{i=1}^m a_i^{(j)}\vec{u}_i$ of the basis vectors $\vec{v}_i$ from $\mathcal{H}$, i.e.

$$T(\vec{u}_j) = \sum_{i=1}^m a_i^{(j)}\vec{v}_i$$

[1] $T(\vec{u}_1+\vec{u}_2)$: $A([\vec{u}_1]_B+[\vec{u}_2]_B) = A[\vec{u}_1]_B+A[\vec{u}_2]_B$: $T(\vec{u}_1)+T(\vec{u}_2)$ and $T(a\vec{u}_1)$: $A(a[\vec{u}_1]_B) = a(A[\vec{u}_1]_B)$: $aT(\vec{u}_1)$

The matrix formed by the above coefficients $A = a_i^{(j)}$ is then the desired ***matrix representation*** of our linear transformation T. To see this, compare with what we have taken in the last paragraph, $T(\vec{u})\colon\ [\vec{u}]_B$. Subsequently,

$$
\begin{aligned}
T(\vec{u}_j)\colon\ A[\vec{u}_j]_B &= a_i^{(j)}(e_j)_B \\
&= \begin{bmatrix} a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(j)} & \cdots & a_1^{(n)} \\ a_2^{(1)} & a_2^{(2)} & & a_2^{(j)} & & a_2^{(n)} \\ \vdots & & & \vdots & & \vdots \\ a_m^{(1)} & a_m^{(2)} & \cdots & a_m^{(j)} & \cdots & a_m^{(n)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \text{ (the } j\text{-th entry)} \\ \vdots \\ 0 \text{ (the last index is } n) \end{bmatrix} \\
&= \begin{bmatrix} a_1^{(j)} \\ a_2^{(j)} \\ \vdots \\ a_m^{(j)} \end{bmatrix}
\end{aligned}
$$

Due to the structure of $(e_j)_B$, this matrix product yields exactly the j-th column of $A = a_i^{(j)}$ as shown above (see Properties 6.1.4). Moreover, the coordinates of $\vec{v}^{(j)}$ in the $\mathcal{H}$ system

$$
\begin{aligned}
[\vec{v}^{(j)}]_H = [\sum_{i=1}^{m} a_i^{(j)} \vec{v}_i]_H &= \sum_{i=1}^{m} a_i^{(j)} [\vec{v}_i]_H \\
&= \sum_{i=1}^{m} a_i^{(j)} (e_i)_H \\
&= a_1^{(j)} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_2^{(j)} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_m^{(j)} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \text{ (the last index is } m) \end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} a_1^{(j)} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_2^{(j)} \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_m^{(j)} \end{bmatrix} = \begin{bmatrix} a_1^{(j)} \\ a_2^{(j)} \\ \vdots \\ a_m^{(j)} \end{bmatrix}$$

also gives the same j-th column of $A = a_i^{(j)}$. This holds for any j. Hence, the association of the matrix $[A]_B^H = a_i^{(j)}$ to the linear transformation T is consistent, where we have now added the subscript B and superscript H to emphasize the transformation are carried out in reference to the two specific coordinate bases. This reasoning also shows that, to construct the matrix representation of a linear transformation, we compute each of the $T(\vec{u}_j) = \vec{v}^{(j)}$ and find its coordinates in the $\mathcal{H}$ frame, namely $[\vec{v}^{(j)}]_H$, which readily become the j-th column of the matrix to be found. To be more clear, we have

$$\begin{aligned}
[T]_B^H &= \big[[T(\vec{u}_1)]_H|[T(\vec{u}_2)]_H|\cdots|[T(\vec{u}_n)]_H\big] \\
&= \big[[\vec{v}^{(1)}]_H|[\vec{v}^{(2)}]_H|\cdots|[\vec{v}^{(n)}]_H\big] \\
&= \begin{bmatrix} a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(n)} \\ a_2^{(1)} & a_2^{(2)} & & a_2^{(n)} \\ \vdots & & \ddots & \vdots \\ a_m^{(1)} & a_m^{(2)} & \cdots & a_m^{(n)} \end{bmatrix} = a_i^{(j)} = [A]_B^H
\end{aligned}$$

Notice that here the i/j subscript/superscript has been exchanged when compared to like Properties 1.2.3.

Definition 7.1.2 (Matrix Representation of a Linear Transformation)**.** A linear transformation $T : \mathcal{U} \to \mathcal{V}$ as defined in Definition 7.1.1, with respect to the bases $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ for $\mathcal{U}$ and $\mathcal{H} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ for $\mathcal{V}$, has a matrix representation of

$$[T]_B^H = \begin{bmatrix} a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(n)} \\ a_2^{(1)} & a_2^{(2)} & & a_2^{(n)} \\ \vdots & & \ddots & \vdots \\ a_m^{(1)} & a_m^{(2)} & \cdots & a_m^{(n)} \end{bmatrix}$$

where the entries $a_i^{(j)}$ are those according to the relations $T(\vec{u}_j) = \sum_{i=1}^m a_i^{(j)} \vec{v}_i$, or in matrix notation, $[T]_B^H [\vec{u}]_B = [\vec{v}]_H$.

Let's illustrate how it works out using an easy example using the familiar $\mathbb{R}^n$ and $\mathbb{R}^m$.

Example 7.1.1. Let $\mathcal{U} = \mathbb{R}^3$ and $\mathcal{V} = \mathbb{R}^2$, it can be easily verified that $\mathcal{B} = \{(1,2,1)^T, (0,1,-1)^T, (2,-1,1)^T\}$ is a basis for $\mathcal{U}$, and the same goes for $\mathcal{V}$ with a basis $\mathcal{H} = \{(1,2)^T, (2,-1)^T\}$. If a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ obeys the rule $T((x,y,z)^T) = (x+2y, x-y+z)^T$, (By the way, you should verify if this is really a linear transformation.) find its matrix representation $[T]_B^H$ with respect to $\mathcal{B}$ and $\mathcal{H}$. Then, use the results to compute $T((-1,4,-1)^T)$.

Solution. Following Definition 7.1.2, we set out to find how the linear transformation will apply on the basis vectors in $\mathcal{B}$. For the first one, we have

$$T((1,2,1)^T) = ((1) + 2(2), (1) - (2) + (1))^T = (5,0)^T$$

which can be subsequently written as a linear combination of the two basis vectors in $\mathcal{H}$:

$$(5,0)^T = 1(1,2)^T + 2(2,-1)^T$$

Hence $a_1^{(1)} = 1$, $a_2^{(1)} = 2$, and this gives us the first column of $[T]_B^H$ as

$$\begin{bmatrix} 1 & * & * \\ 2 & * & * \end{bmatrix}$$

We repeat the same procedure on the other two basis vectors $(0,1,-1)^T$ and $(2,-1,0)^T$ of $\mathcal{B}$, where it can be shown that

$$\begin{aligned}
T((0,1,-1)^T) &= ((0) + 2(1), (0) - (1) + (-1))^T = (2,-2)^T \\
&= -\frac{2}{5}(1,2)^T + \frac{6}{5}(2,-1)^T \\
T((2,-1,1)^T) &= ((2) + 2(-1), (2) - (-1) + (1))^T = (0,4)^T
\end{aligned}$$

$$= \frac{8}{5}(1,2)^T - \frac{4}{5}(2,-1)^T$$

Therefore, the required matrix representation is

$$[T]_B^H = \begin{bmatrix} 1 & -\frac{2}{5} & \frac{8}{5} \\ 2 & \frac{6}{5} & -\frac{4}{5} \end{bmatrix}$$

For the second part, we start by expressing $(-1,4,1)^T$ in the basis $\mathcal{B}$. As $(-1,4,-1)^T = 1(1,2,1)^T + 1(0,1,-1)^T - 1(2,-1,1)^T$, we have $(-1,4,1)^T$: $(1,1,-1)_B^T$, and then

$$\begin{aligned}
[T((1,1,-1)_B^T)]_H &= [T]_B^H(1,1,-1)_B^T \\
&= \left(\begin{bmatrix} 1 & -\frac{2}{5} & \frac{8}{5} \\ 2 & \frac{6}{5} & -\frac{4}{5} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)_H \\
&= \begin{bmatrix} -1 \\ 4 \end{bmatrix}_H = (-1,4)_H^T
\end{aligned}$$

implying that $T((-1,4,-1)^T) = -1(1,2)^T + 4(2,-1)^T = (7,-6)^T$ in the usual standard basis. This can be cross-checked by directly invoking the given definition of T, where $T((-1,4,-1)^T) = ((-1)+2(4),(-1)-(4)+(-1))^T = (7,-6)^T$ as well. □

Up until now, we have been playing around with the simple real n-space only, but the real (no pun intended) power of the notion of a general vector space lies in its abstraction: Any mathematical object that satisfies the criteria in Definition 6.1.1 is a (real) vector space, and the results that we have already established in the previous parts for the real n-space are readily transferable to them. Two prime examples of abstract vector spaces are the set of (real) polynomials $\mathcal{P}^n$ with a degree up to n and[2] the family of continuous (k-times continuously

[2]We shall argue for some criteria in Definition 6.1.1 for $\mathcal{P}^n$ here. For instances, condition (1) is obvious as adding up two polynomials with a degree up to n can only result in another polynomial with a maximum degree of n. In condition (4), the zero vector for $\mathcal{P}^n$ is simply the constant zero function 0, which is considered to have a degree of -1 by convention.

differentiable) functions C^0 (C^k) over a fixed interval. Now we will see how the concept of linear transformation is laid out when these abstract vector spaces are involved, preparing us for the key insight in the next subsection.

Example 7.1.2. Consider $\mathcal{U} = \mathcal{P}^2$, and $\mathcal{V} = \mathcal{P}^1$, and let the bases for $\mathcal{U}$ and $\mathcal{V}$ be $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{H} = \{1, x\}$. (They are known as the standard bases for $\mathcal{P}^2$ and $\mathcal{P}^1$ respectively. In general the standard basis for $\mathcal{P}^n$ is $\{1, x, x^2, \cdots, x^{n-1}, x^n\}$ and thus $n + 1$-dimensional. Readers are advised to justify why they constitute a basis for the polynomial spaces.) Let $T : \mathcal{U} \to \mathcal{V}$ be $T[p(x)] = p'(x)$ the differentiation operator and find its matrix representation with respect to $\mathcal{B}$ and $\mathcal{H}$.

Solution. We essentially do the same thing as in Example 7.1.1 but applied over polynomials now. From elementary calculus, we have

$$\begin{aligned}
T(1) &= \frac{d}{dx}(1) = 0 \\
T(x) &= \frac{d}{dx}(x) = 1 \\
T(x^2) &= \frac{d}{dx}(x^2) = 2x
\end{aligned}$$

and by Definition 7.1.2, the desired matrix representation is

$$[T]_B^H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Notice that we can express, quite trivially

$$\begin{aligned}
1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_H \\
x &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_H
\end{aligned}$$

$$x^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_B$$

using vector notation in the given two standard bases. We can verify the form of $[T]_B^H$ by a test polynomial $c_0 + c_1 x + c_2 x^2$, whose vector representation in $\mathcal{B}$ is clearly $(c_0, c_1, c_2)_B^T$. Then, multiplying $[T]_B^H$ to its left gives

$$[T((c_0, c_1, c_2)_B^T)]_H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}_B^H \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}_B = \begin{bmatrix} c_1 \\ 2c_2 \end{bmatrix}_H$$

which corresponds to the polynomial $c_1 + 2c_2 x$. This coincides with the usual result of differentiation, that is, $\frac{d}{dx}(c_0 + c_1 x + c_2 x^2) = c_1 + 2c_2 x$. □

In each of the previous examples, we consider a linear transformation between two vector spaces that are of the same type (the usual real vectors/polynomials). Below shows what happen when they are mixed together. Actually, due to the abstraction provided by the nature of vector space, the outcome follows easily.

Example 7.1.3. Let $\mathcal{U} = \mathbb{R}^3$ and $\mathcal{V} = \mathcal{P}^2$, while $\mathcal{B} = \{(1,0,0)^T, (0,1,0)^T, (0,0,1)^T\}$ and $\mathcal{H} = \{1, x, x^2\}$ be the standard bases for $\mathcal{U}$ and $\mathcal{V}$ respectively. Show that, the rather trivial linear transformation $T((c_0, c_1, c_2)^T) = c_0 + c_1 x + c_2 x^2$ has a matrix representation of an identity with respect to $\mathcal{B}$ and $\mathcal{H}$.

Solution. Again, we repeat what we have done in the previous two examples. It is apparent that

$$\begin{aligned}
T((1,0,0)_B^T) &= (1) + (0)x + (0)x^2 = 1 = (1,0,0)_H^T \\
T((0,1,0)_B^T) &= (0) + (1)x + (0)x^2 = x = (0,1,0)_H^T \\
T((0,0,1)_B^T) &= (0) + (0)x + (1)x^2 = x^2 = (0,0,1)_H^T
\end{aligned}$$

So by Definition 7.1.2, the desired matrix representation is simply

$$[T]_B^H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is the 3×3 identity matrix. This is expected as the linear transformation is essentially $T[(c_0, c_1, c_2)_B^T] = (c_0, c_1, c_2)_H^T$ where $(c_0, c_1, c_2)_H^T = c_0 + c_1 x + c_2 x^2$, which means that the numeric representation of vectors in the two spaces is preserved under such a linear transformation between them and the only visible change is the subscript. □

Most of the readers should find it boring in the above example as we are just stating the obvious. It is a straight-forward, "one-to-one" association between the standard bases of the real n-space and space of polynomials with degree $n-1$. However, the important message is that given such an association we can always identify any vector of some space as a vector in another space of a completely different class, which is very powerful as many operations become transferable between these two spaces. In this sense, this kind of "one-to-one" mapping is not limited to the identity mapping, or by the bases used for the two vector spaces as we will see in the following subsection.

7.1.2 One-to-one and Onto, Kernel and Range

Continuing our discussion above, to identify a vector (one and only one) from one vector space as another vector in another vector space through a linear mapping, we require it to be ***one-to-one (injective)***. On the other hand, another important property of a linear transformation is that whether it is ***onto (surjective)***, which means that every vector in the latter vector space (*image*) is being mapped onto by some vector(s) in the former vector space (*preimage*). The formal definitions of these two properties are given as below.

Properties 7.1.3 (Injective Transformation)**.** A transformation $T : \mathcal{U} \to \mathcal{V}$ is called one-to-one if for two vectors $\vec{u}_1, \vec{u}_2 \in \mathcal{U}$, $T(\vec{u}_1) = T(\vec{vu}_2)$ implies $\vec{u}_1 = \vec{u}_2$, i.e. an image has one and only one corresponding preimage. Furthermore, if T is linear, then equivalently $T(\vec{u}) = \mathbf{0}$ implies $\vec{u} = \mathbf{0}$ only.

To show the equivalence of the two conditions above, notice that $T(\mathbf{0}) = \mathbf{0}$ if T is linear. (why?)[3] For any $\vec{v}$ such that $T(\vec{v}) = 0$, we have

$$T(\vec{v}) = \mathbf{0} = T(\mathbf{0})$$

and hence $\vec{v}$ must be $\mathbf{0}$ if $T(\vec{v}_1) = T(\vec{v}_2)$ implies $\vec{v}_1 = \vec{v}_2$. The proof of the converse is left as an exercise.

Properties 7.1.4 (Surjective Transformation)**.** A transformation $T : \mathcal{U} \to \mathcal{V}$ is called onto if for any vector $\vec{v} \in \mathcal{V}$ (image), there exists at least one vector(s) $\vec{u} \in \mathcal{U}$ (preimage) such that $T(\vec{u}) = \vec{v}$.

As an illustration, in Example 7.1.2, the differentiation operator $T(p(x)) = p'(x)$ from $\mathcal{P}^2$ to $\mathcal{P}^1$ is onto but not one-to-one. To see these, note that given any image $\vec{v} = d_0 + d_1 x \in \mathcal{P}^1$, all preimages in the form of $\vec{u} = K + d_0 x + \frac{d_1}{2}x^2 \in \mathcal{P}^2$ where K can be any number satisfies $T(\vec{u}) = \vec{v}$ by elementary calculus, and the surjectivity is obvious. To explicitly disprove injectivity, fix an image $\vec{v} = d_0 + d_1 x$ with specific d_0 and d_1, and note that both $\vec{u}_1 = K_1 + d_0 x + \frac{d_1}{2}x^2$ and $\vec{u}_2 = K_2 + d_0 x + \frac{d_1}{2}x^2$ where K_1, K_2 are distinct satisfy $T(\vec{u}_1) = T(\vec{u}_2) = \vec{v}$, but $\vec{u}_1 \neq \vec{u}_2$.

However, in other cases it may not be so easy to check injectivity and surjectivity as directly as above. Therefore, we need a general method to determine if these two properties hold for a transformation between two abstract vector bases. The following theorem links injectivity and surjectivity with their basis vectors, but it requires the transformation to be linear (and here is where the linearity comes to play).

Theorem 7.1.5. A linear transformation $T : \mathcal{U} \to \mathcal{V}$ between two finite-dimensional vector spaces is one-to-one if and only if given any basis $\mathcal{B} =$

[3] $T(\mathbf{0}) = T(0\vec{u}) = 0T(\vec{u}) = \mathbf{0}$ for arbitrary $\vec{v}$ due to the homogeneity property as required in Definition 7.1.1.

$\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ for $\mathcal{U}$, $T(\vec{u}_1), T(\vec{u}_2), \ldots, T(\vec{u}_n) \in \mathcal{V}$ are linearly independent.

Theorem 7.1.6. A linear transformation $T : \mathcal{U} \to \mathcal{V}$ between two finite-dimensional vector spaces is onto if and only if given any basis $\mathcal{H} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ for $\mathcal{V}$, we can find a vector $\vec{u}_i \in \mathcal{U}$ such that $T(\vec{u}_i) = \vec{v}_i$ for each of the $\vec{v}_i$.

Proof. Theorem 7.1.5: The "if" direction is proved by showing $T(\vec{u}_1), T(\vec{u}_2), \ldots, T(\vec{u}_n)$ are linearly independent implies that, if $T(\vec{u}) = \mathbf{0}$ then $\vec{u} = \mathbf{0}$ as suggested by the alternative condition in Properties 7.1.3. By Theorem 6.1.9, the equation $c_1T(\vec{u}_1) + c_2T(\vec{u}_2) + \cdots + c_nT(\vec{u}_n) = \mathbf{0}$ only has $c_j = \mathbf{0}$ as the trivial solution. Now by linearity from Definition 7.1.1, we have

$$\begin{aligned} c_1T(\vec{u}_1) + c_2T(\vec{u}_2) + \cdots + c_nT(\vec{u}_n) &= T(c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n) \\ &= \mathbf{0} \end{aligned}$$

Since $c_j = 0$ is the only possibility, this means that if $T(c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n) = \mathbf{0}$ then $\vec{u} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n$ must be $\mathbf{0}$, hence $T(\vec{u}) = \mathbf{0}$ implies $\vec{u} = 0$ and we are done. The converse is similarly proved, having the argument goes in reverse direction.
Theorem 7.1.6: We compare Theorem 7.1.6 against Properties 7.1.4 to show the part of "if" direction. Since $H = \{\vec{v}_i\}$ is a basis for $\mathcal{V}$, any $\vec{v} \in \mathcal{V}$ can be written as a linear combination of $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_m\vec{v}_m$. If we can find $\vec{u}_i \in \mathcal{U}$ such that $T(\vec{u}_i) = \vec{v}_i$ for all $\vec{v}_i$, then

$$\begin{aligned} \vec{v} &= c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_m\vec{v}_m \\ &= c_1T(\vec{u}_1) + c_2T(\vec{u}_2) + \cdots + c_mT(\vec{u}_m) \\ &= T(c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_m\vec{u}_m) \end{aligned}$$

the last equality uses linearity from Definition 7.1.1 again. This shows that $\vec{u} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_m\vec{u}_m$ is readily one possible vector in $\mathcal{U}$ such that $T(\vec{u}) = \vec{v}$ and the desired result is established. The converse is trivial as we take $\vec{v} = \vec{v}_i$ in Properties 7.1.4 for all possible i. $\square$

Example 7.1.4. Given a linear transformation $T : \mathcal{U} \to \mathcal{V}$ where $\mathcal{U}$ and $\mathcal{V}$ have a dimension of 3 and 4 respectively, if its matrix representation corresponding to some bases $\mathcal{B}$ and $\mathcal{H}$ is

$$[T]_B^H = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

determine whether it is (a) one-to-one, as well as (b) onto, or not.

Solution. (a) By Theorem 7.1.5, we need to check if $T(\vec{u}_1), T(\vec{u}_2), T(\vec{u}_3)$ are linearly independent, where $\vec{u}_1$, $\vec{u}_2, \vec{u}_3$ are the basis vectors from $\mathcal{B}$. Their numeric representation in the $\mathcal{B}$ system is trivially $[\vec{u}_1]_B = (e_1)_B = (1,0,0)_B^T$, $[\vec{u}_2]_B = (e_2)_B = (0,1,0)_B^T$ and $[\vec{u}_3]_B = (e_3)_B = (0,0,1)_B^T$, and hence

$$\begin{aligned}
[T(\vec{u}_1)]_H &= [T]_B^H (e_1)_B \\
&= \left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)_H = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}_H
\end{aligned}$$

which is just the first column of $[T]_B^H$. Similarly, $[T(\vec{u}_2)]_H = (-1,1,0,1)_H^T$, $[T(\vec{u}_3)]_H = (0,1,-1,0)_H^T$ are then the second/third column of $[T]_B^H$. From this we see that in general, the coordinates in $\mathcal{H}$ after transformation $[T(\vec{u}_j)]_H$ is just the j-th column of $[T]_B^H$. (Actually, this has been observed when we are deriving the matrix representation of linear transformations in the beginning of this chapter.) So the problem is reduced to decide whether the column vectors constituting $[T]_B^H$ are linearly independent or not. By Theorem 6.1.9, we can accomplish this by showing if the solution

$[T]_B^H \vec{x} = \mathbf{0}$ is consisted of the trivial solution only, and we have

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \qquad \begin{array}{c} R_3 - R_1 \to R_3 \\ R_4 - R_1 \to R_4 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \qquad \begin{array}{c} R_3 - R_2 \to R_3 \\ R_4 - R_2 \to R_4 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \qquad -\frac{1}{2}R_3 \to R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_4 + 2R_3 \to R_4$$

As every column in this homogeneous system contains a pivot, it demonstrates that $[T]_B^H \vec{x} = \mathbf{0}$ indeed only has the trivial solution $\vec{x} = \mathbf{0}$, and therefore the linear transformation in question is one-to-one.

(b) By Properties 7.1.4, it is equivalent to showing that if the $\{T(\vec{u}_j)\}$s span $\mathcal{W}$, or expressed in terms of the $\mathcal{B}/\mathcal{H}$ coordinates, whether the three transformed vectors $\{[T(\vec{u}_1)]_H, [T(\vec{u}_2)]_H, [T(\vec{u}_3)]_H\}$ span $\mathbb{R}^4$. However, it is apparent that three vectors can never span a four-dimensional vector space as the number of vectors is fewer then the dimension, and thus the linear transformation is not onto.

Notice that in the above arguments we never explicitly say what the vector spaces $\mathcal{U}$ and $\mathcal{V}$ are and only the matrix representation of the linear transformation is involved. However, some may be skeptical as we have fixed bases for the linear transformation and may ask if the results are basis-dependent. We will address this issue in later parts of this chapter. □

Accompanying injectivity and surjectivity is the ideas of ***kernel*** and ***range***. For a (linear) transformation $T : \mathcal{U} \to \mathcal{V}$, its kernel is consisted of vectors in $\mathcal{U}$ that is mapped to the zero vector in $\mathcal{V}$, while its range is made up of all possible vectors in $\mathcal{V}$ that are mapped from $\mathcal{U}$ via T.

Definition 7.1.7. For a (linear) transformation $T : \mathcal{U} \to \mathcal{V}$, its kernel is defined to be

$$\text{Ker}(T) = \{\vec{u} \in \mathcal{U} | T(\vec{u}) = \mathbf{0}_V\}$$

whereas its range is

$$R(T) = \{\vec{v} \in \mathcal{V} | T(\vec{u}) = \vec{v} \text{ for some } \vec{u} \in \mathcal{U}\}$$

Also, notice that the kernel and range are a subspace of $\mathcal{U}$ and $\mathcal{V}$ respectively.[4] Hence it is reasonable to speak of their dimension or basis and we will discuss this matter later. For now, let's look at how to determine the kernel and range of a linear transformation first. For instance, in Example 7.1.2, the kernel is span$(\{1\})$ since the derivative of any constant vanishes, and the range is span$(\{1, x\}) = \mathcal{V} = \mathcal{P}^1$ because we have already shown that every $\mathcal{P}^1$ polynomial in this case have some corresponding preimage in $\mathcal{U} = \mathcal{P}^2$. Here the dimension of kernel/range is 1 and 2.

Example 7.1.5. Given another linear transformation $T : \mathcal{U} \to \mathcal{V}$ where $\mathcal{U}$ and $\mathcal{V}$ are now both having a dimension of 3, if its matrix representation corresponding to some bases $\mathcal{B}$ and $\mathcal{H}$ is

$$[T]_B^H = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

find its kernel and range.

[4] For $\vec{u}_1, \vec{u}_2 \in \text{Ker}(T) \subset \mathcal{U}$, $T(a\vec{u}_1 + b\vec{u}_2) = aT(\vec{u}_1) + bT(\vec{u}_2) = a\mathbf{0}_V + b\mathbf{0}_V = \mathbf{0}_V$ for any scalar a and b so $a\vec{u}_1 + b\vec{u}_2 \in \text{Ker}(T)$ and by Theorem 6.1.2 it is a subspace of $\mathcal{U}$. We leave showing the range is a subspace of $\mathcal{V}$ as an exercise to the readers.

Solution. According to Definition 7.1.7, $\text{Ker}(T)$ is the set of $\vec{u}$ that satisfies $T(\vec{u}) = \mathbf{0}$, or using basis representation, $[T]_B^H[\vec{u}]_B = \mathbf{0}$. Therefore, it is equivalent to finding the null space of $[T]_B^H$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \qquad \begin{array}{c} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \qquad R_2 \leftrightarrow R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_3 + R_2 \to R_3$$

The nullity is 1 and we can let $[u_3]_B = t$ be the free variable, and we have $[u_1]_B = -t$ and $[u_2]_B = 0$ from the first two rows. So the kernel takes the form of

$$\text{Ker}(T) = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}_B = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}_B$$

where $-\infty < t < \infty$, or in other words, $\text{Ker}(T) = \text{span}(\{(-1, 0, 1)_B^T\})$ with a dimension of 1. Similarly, the range of T will be the column space of $[T]_B^H$. From the elimination procedure carried out above, we know that the first two column vectors are linearly independent and the third column is clearly the same as the first column, and thus the range is $R(T) = \text{span}(\{(1, 1, 1)_B^T, (0, -1, 1)_B^T\})$ and has a dimension of 2, which coincides with the rank of the $[T]_B^H$ matrix. Note that we approach the problem with some bases (albeit unknown) fixed to represent the linear transformation in matrix form just like in the last example. Again, we will soon justify that the results are actually unrelated to the choices of bases such that the dimensions of kernel and range are exactly the nullity and rank of any matrix representation of the linear transformation. □

Finally, we can rewrite Properties 7.1.3 and 7.1.4 using the notion of kernel and range.

Properties 7.1.8. A linear transformation $T : \mathcal{U} \to \mathcal{V}$ is one-to-one if and only if the dimension of its kernel $\text{Ker}(T)$ is zero, i.e. $\text{Dim}(\text{Ker}(T)) = 0$. Meanwhile, it is onto if and only if the dimension of range $R(T)$ (rank) is same as the dimension of $\mathcal{V}$.

7.1.3 Vector Space Isomorphism to $\mathbb{R}^n$

A linear transformation where both injectivity and surjectivity hold is known as ***bijective/isomorphic***. As we will immediately see, this property is very central in relating finite-dimensional real vector spaces to the real n-space. Combining Properties 7.1.3 and 7.1.4, for a linear transformation $T : \mathcal{U} \to \mathcal{V}$ to be bijective, every vector $\vec{v} \in \mathcal{V}$ must be an image which there is one and only one preimage $\vec{u} \in \mathcal{U}$ is mapped onto, i.e. there is a unique $\vec{u} \in \mathcal{U}$ that satisfies $T(\vec{u}) = \vec{v}$ for every $\vec{v} \in \mathcal{V}$, which also means that it is *invertible* in the sense that every $\vec{v} \in \mathcal{V}$ can be traced back to one and only one $\vec{u} \in \mathcal{U}$ via the transformation in reverse direction. Hence it makes sense to say a transformation is bijective *between* two vector spaces. There are two major results regarding invertibility. The first one is

Theorem 7.1.9. There always exists a bijective linear mapping between $\mathcal{V}$ itself, i.e. $T : \mathcal{V} \to \mathcal{V}$, that transforms the coordinates of any fixed vector in $\mathcal{V}$ between two different bases (denote them by $\mathcal{B}$ and $\mathcal{B}'$) of its. Such a change of coordinates in $\mathcal{V}$ has a matrix representation $[T]_B^{B'} = P_B^{B'}$ that is invertible.

Proof. Since it is the same vector space $\mathcal{V}$ but just represented in different bases, the number of dimension will stay the same, let's say n, and the bases $\mathcal{B}$ and $\mathcal{B}'$ both are made up of n basis vectors (Properties 6.2.4). Denote them by $\mathcal{B} = \{\vec{v}_{1,B}, \vec{v}_{2,B}, \ldots, \vec{v}_{n,B}\}$ and $\mathcal{B}' = \{\vec{v}_{1,B'}, \vec{v}_{2,B'}, \ldots, \vec{v}_{n,B'}\}$. The desired mapping $T : \mathcal{V} \to \mathcal{V}$ is in fact

$$T(\vec{v}) = \text{id}(\vec{v}) = \vec{v}$$

the ***identity transformation/identity mapping*** as it is just a change of coordinates where the actual vector stays identical. This transformation is then trivially bijective because any vector is just mapped into itself, and is described by $[\vec{v}]'_B = [T]_B^{B'}[\vec{v}]_B$ following Definition 7.1.2 with $\mathcal{U} = \mathcal{V}$ and $\vec{u} = \vec{v}$. Now note that $[\vec{v}]_{B'} = [T]_B^{B'}[\vec{v}]_{B'}$ has a unique solution $[\vec{v}]_B$ for any $[\vec{v}]_{B'}$ as T is bijective and by definition each of $[\vec{v}]_{B'}$ is mapped onto by one and only one $[\vec{v}]_B$. Part (d) to (a) of Theorem 3.2.1 then shows that $[T]_B^{B'}$ is an invertible matrix. According to the discussion prior to Definition 7.1.2, $[T]_B^{B'}$ takes the form of

$$\begin{aligned} P_B^{B'} = [T]_B^{B'} &= \left[[\text{id}(\vec{v}_{1,B})]_{B'}|[\text{id}(\vec{v}_{2,B})]_{B'}|\cdots|[\text{id}(\vec{v}_{n,B})]_{B'}\right] \\ &= \left[[\vec{v}_{1,B}]_{B'}|[\vec{v}_{2,B}]_{B'}|\cdots|[\vec{v}_{n,B}]_{B'}\right] \end{aligned}$$

So we have to find how each of the basis vectors in $\mathcal{B}$ is expressed in the $\mathcal{B}'$ system. Conversely,

$$P_{B'}^{B} = ([T]_B^{B'})^{-1} = [T]_{B'}^{B} = \left[[\vec{v}_{1,B'}]_B|[\vec{v}_{2,B'}]_B|\cdots|[\vec{v}_{n,B'}]_B\right]$$

Be aware that despite it being an identity mapping, the exact matrix representation is dependent on the bases and will usually not be an identity matrix. Nevertheless, such bijectivity between any two coordinate systems of the same vector space implies that all linear transformation from one vector space to another $T : \mathcal{U} \to \mathcal{V}$, together with its (dimensions of) kernel or range, are independent of the choices of bases for either $\mathcal{U}$ or $\mathcal{V}$ and we can pick whatever bases that suit the situation better. The only thing that is dependent on the coordinate systems will be their numeric representation and we will see how it unfolds in the next part. This justify our fixing of bases during several arguments in the last subsection. □

Example 7.1.6. Show that $\mathcal{B} = \{(1,0,1)^T, (0,2,1)^T, (-1,1,2)^T\}$ and $\mathcal{B}' = \{(0,0,1)^T, (2,0,1)^T, (1,-1,0)^T\}$ are both bases for $\mathcal{V} = \mathbb{R}^3$ and find the matrix representation of coordinate conversion between them.

Solution. Just like in Example 6.2.1, we need to check whether the determinants of

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad \text{and} \qquad B' = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

are non-zero or not. A simple computation shows that $\det(B) = 5$ and $\det(B') = -2$ and thus both $\mathcal{B}$ and $\mathcal{B}'$ are bases for $\mathbb{R}^3$. By Theorem 7.1.9, the matrix representation for the change of basis abides

$$[\text{id}]_B^{B'} = [T]_B^{B'} = \left[[\vec{v}_{1,B}]_{B'}|[\vec{v}_{2,B}]_{B'}|[\vec{v}_{3,B}]_{B'}\right]$$

where each of $[\vec{v}_{j,B}]_{B'}$ is found via the equation

$$[(v_{j,B})_1]_{B'}(\vec{v}_{1,B'}) + [(v_{j,B})_2]_{B'}(\vec{v}_{2,B'}) + [(v_{j,B})_3]_{B'}(\vec{v}_{3,B'}) = \vec{v}_{j,B}$$

just as in Example 6.2.1 with $[(v_{j,B})_i]_{B'}$ being the i-th component of $\vec{v}_{j,B}$ in the $\mathcal{B}'$ frame, or equivalently,

$$\begin{aligned}
\left[\vec{v}_{1,B'}|\vec{v}_{2,B'}|\vec{v}_{3,B'}\right]\begin{bmatrix}[(v_{j,B})_1]_{B'}\\ [(v_{j,B})_2]_{B'}\\ [(v_{j,B})_3]_{B'}\end{bmatrix} &= \vec{v}_{j,B} \\
[\vec{v}_{j,B}]_{B'} = \begin{bmatrix}[(v_{j,B})_1]_{B'}\\ [(v_{j,B})_2]_{B'}\\ [(v_{j,B})_3]_{B'}\end{bmatrix} &= \left[\vec{v}_{1,B'}|\vec{v}_{2,B'}|\vec{v}_{3,B'}\right]^{-1}\vec{v}_{j,B} \\
&= B'^{-1}\vec{v}_{j,B}
\end{aligned}$$

Subsequently,

$$\begin{aligned}
[T]_B^{B'} &= \left[[\vec{v}_{1,B}]_{B'}|[\vec{v}_{2,B}]_{B'}|[\vec{v}_{3,B}]_{B'}\right] \\
&= \left[B'^{-1}\vec{v}_{1,B}|B'^{-1}\vec{v}_{2,B}|B'^{-1}\vec{v}_{3,B}\right] \\
&= B'^{-1}\left[\vec{v}_{1,B}|\vec{v}_{2,B}|\vec{v}_{3,B}\right] \\
&= B'^{-1}B
\end{aligned}$$

The readers should verify that we can indeed factor out the B'^{-1} from the columns and put it to the left in the third line, and the required matrix representation for the coordinate change is

$$P_B^{B'} = [T]_B^{B'} = B'^{-1}B = \begin{bmatrix}0 & 2 & 1\\ 0 & 0 & -1\\ 1 & 1 & 0\end{bmatrix}^{-1}\begin{bmatrix}1 & 0 & -1\\ 0 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 2 \\ \frac{1}{2} & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix}$$

Let's take $(2,2,3)^T = 2(1,0,1)^T + 1(0,2,1)^T + 0(-1,1,2)^T = (2,1,0)^T_B$ for double-checking:

$$P_B^{B'}(2,1,0)^T_B = \begin{bmatrix} \frac{1}{2} & 0 & 2 \\ \frac{1}{2} & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix}_B^{B'} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}_B = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}_{B'}$$

and indeed $(2,2,3)^T = 1(0,0,1)^T + 2(2,0,1)^T + (-2)(1,-1,0)^T = (1,2,-2)^T_H$.

□

For other cases of coordinate transformation, more generally, the relation $P_B^{B'} = [\text{id}]_B^{B'} = B'^{-1}B$ still remains valid where B and B' are matrices composed by the basis vectors from the $\mathcal{B}$ and $\mathcal{B}'$ systems, relative to a third basis (without loss of generality we assume it is the standard basis $\mathcal{S}$[5], but the readers are advised to extend this for any other arbitrary basis), that are arranged in columns. To see this from another perspective, take any vector $\vec{v}$ that is expressed in the $\mathcal{B}$ coordinates, $[\vec{v}]_B$. We can view the change in coordinates from $\mathcal{B}$ to $\mathcal{B}'$ in two steps: first from $\mathcal{B}$ to $\mathcal{S}$, and then from $\mathcal{S}$ to $\mathcal{B}'$. From Section 6.2.1, we already know that the former constitutes $[\vec{v}]_S = B[\vec{v}]_B$, and the latter is done by $[\vec{v}]_{B'} = B'^{-1}[\vec{v}]_S$. Combining these two operations together we have $[\vec{v}]_{B'} = B'^{-1}[\vec{v}]_S = B'^{-1}B[\vec{v}]_B$ and hence $[\text{id}]_B^{B'} = B'^{-1}B$.

The second major result in this subsection is

Theorem 7.1.10. There is always a bijective linear mapping between $\mathcal{V}$ and $\mathbb{R}^n$ where $\mathcal{V}$ is any n-dimensional real vector space. In this sense we say $\mathcal{V}$/such a mapping is ***isomorphic***/an ***isomorphism*** to $\mathbb{R}^n$. It has an invertible matrix

[5]Unfortunately, as you may notice, there is actually no satisfying "standard" of what really is a standard basis for (real) finite-dimensional vector space other than the real n-space since any basis can be regarded to be one with respect to itself. Here we just pretend it is available for the sake of reasoning.

representation. Conversely if a matrix representation of a linear transformation is invertible, it is bijective.

Proof. We construct such a mapping explicitly. Note that $\mathcal{V}$ and $\mathbb{R}^n$ are both n-dimensional vector spaces and any of their bases will contain n basis vectors. Denote the basis chosen for $\mathcal{V}$ by $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ and we use the standard basis $\mathcal{S} = \{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$ for $\mathbb{R}^n$. Then the linear mapping $T : \mathcal{V} \to \mathbb{R}^n$ that abides

$$T(\vec{v}_j) = \hat{e}_j$$

where $j = 1, 2, \ldots, n$, is bijective as desired. To see this, by Theorem 7.1.5, as for every $\vec{v}_j$, $T(\vec{v}_j) = \hat{e}_j$ leads to the standard unit vectors that are linearly independent, T is one-to-one. Meanwhile, a direct use of Theorem 7.1.6 over the defined association $T(\vec{v}_j) = \hat{e}_j$ for each of the $\hat{e}_j$ immediately shows that T is onto. Since T is now one-to-one and onto, it is bijective. Again, the bijectivity, in addition to the uniqueness of basis coordinates, implies that for any $\vec{u} \in \mathbb{R}^n$, $[\vec{u}]_S = [T]_B^S[\vec{v}]_B$ has a unique solution $[\vec{v}]_B$, and part (d) to (a) of Theorem 3.2.1 then shows that the matrix representation $[T]_B^S$ is invertible. The converse follows the same argument running in opposite direction. □

This theorem enables us to identify and treat any finite-dimensional real vector space $\mathcal{V}$ as the real n-space $\mathbb{R}^n$ with n being the dimension of $\mathcal{V}$. Thus we can work with $\mathcal{V}$ as if it is $\mathbb{R}^n$ and the results for $\mathbb{R}^n$ derived in this and the last chapter are all applicable on other n-dimensional real vector spaces with an appropriate transformation. Actually, we have been implicitly utilizing this isomorphism relation in many of our previous examples, e.g. writing out the coordinates of a vector from an n-dimensional vector space with n components like an $\mathbb{R}^n$ vector. As a corollary,

Properties 7.1.11. Any two real vector spaces are isomorphic such that there exists a bijective transformation between them, if and only if they have the same number of dimension. Otherwise, there will be no isomorphism between those with different dimensions.

The "if" direction is easy to see because they are both isomorphic to $\mathbb{R}^n$ and bijectivity is transitive. For the "only if" direction, let the two vector spaces $\mathcal{U}$ and $\mathcal{V}$ have dimensions of m and n respectively, and without loss of generality $m < n$. Then they can never be isomorphic since given any transformation $T : \mathcal{U} \to \mathcal{V}$ the m transformed vectors $T(\vec{u}_1), T(\vec{u}_2), \ldots, T(\vec{u}_m)$ will be unable to span the n-dimensional $\mathcal{V}$ and by Properties 7.1.4 all of them are not surjective.

Example 7.1.7. Explicitly show that $\mathcal{U} = \mathcal{P}^3$ and $\mathcal{V} = \text{span}(\mathcal{H})$, where $\mathcal{H} = \{e^x, xe^x, x^2e^x, x^3e^x\}$, are isomorphic by considering $T : \mathcal{U} \to \mathcal{V}$, $T[p(x)] = \int_{-\infty}^{x} e^x p(x)dx$.

Solution. It is clear that both $\mathcal{U}$ and $\mathcal{V}$ are four-dimensional and by the above corollary they are isomorphic. Take $\mathcal{B} = \{1, x, x^2, x^3\}$ the standard polynomial basis for $\mathcal{U} = \mathcal{P}^3$ and the linearly independent $\mathcal{H}$ is automatically the basis for $\mathcal{V}$. Now we compute the matrix representation $[T]_B^H$ as follows. By elementary calculus,

$$\begin{aligned}
T(1) &= \int_{-\infty}^{x} e^x dx = e^x \\
T(x) &= \int_{-\infty}^{x} xe^x dx = xe^x - e^x \\
T(x^2) &= \int_{-\infty}^{x} x^2e^x dx = x^2e^x - 2xe^x + 2e^x \\
T(x^3) &= \int_{-\infty}^{x} x^3e^x dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x
\end{aligned}$$

and thus

$$[T]_B^H = \begin{bmatrix} 1 & -1 & 2 & -6 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an upper-triangular matrix and its determinant is simply the product of diagonal entries $(1)^4 = 1 \neq 0$. Therefore, by Theorem 3.2.1, $[T]_B^H$ is invertible and the given transformation, as well as $\mathcal{U}$ and $\mathcal{V}$ themselves, is/are isomorphic according to Theorem 7.1.10. □

Short Exercise: Redo the above example by considering $T[p(x)] = e^x p(x)$ this time.[6]

7.2 More on Coordinate Bases

7.2.1 Linear Change of Coordinates

In previous parts we have already mentioned about change of coordinates between bases for several times, where such a mapping are confined to be linear just like other transformations discussed. In this section we will drive deeper into the details and address two distinct scenarios: change of coordinates for vectors and linear transformations (matrices).

Change of Coordinates for Vectors

The procedure about change of coordinates for vectors have been discussed substantially in Examples 6.2.1, 7.1.6 and explained through Theorem 7.1.9. Here we will focus on its geometric interpretation instead, which will be illustrated by the small example below.

Example 7.2.1. Consider the vector space of $\mathbb{R}^2$ as the x-y plane. Given a basis $\mathcal{B}$ for $\mathbb{R}^2$ that is consisted of two vectors $\vec{u}_1 = (1, 2)^T$ and $\vec{u}_2 = (1, -1)^T$, transform the coordinates of the vector $\vec{v} = (2, 1)^T$ from the standard basis $\mathcal{S}$ to $\mathcal{B}$.

[6] It becomes trivial and the matrix representation is simply the identity matrix.

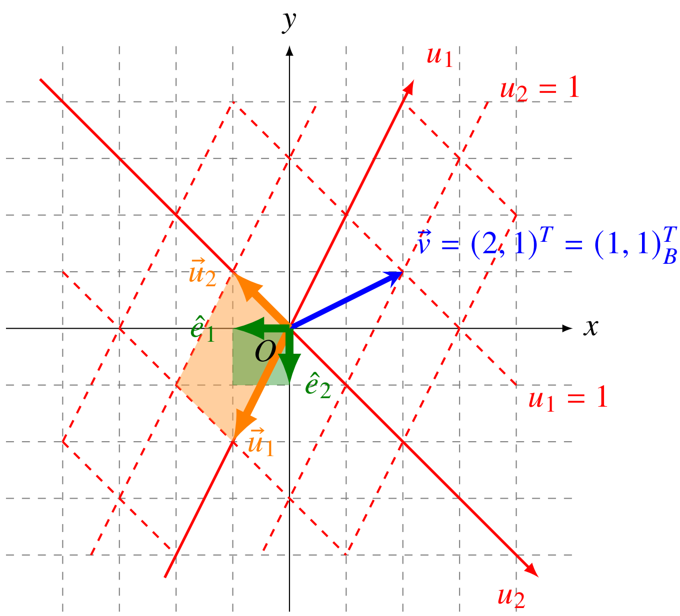
Solution. As before, $P_B^S = [\vec{u}_1|\vec{u}_2]$, and it can be seen that

$$P_B^S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \qquad\qquad P_S^B = (P_B^S)^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Hence the coordinates of $\vec{v}$ in the $\mathcal{B}$ system is

$$[\vec{v}]_B = P_S^B[\vec{v}]_S = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}_S^B \begin{bmatrix} 2 \\ 1 \end{bmatrix}_S = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_B$$

The geometry of this problem is shown in the figure below where each grid line separation represents one unit length of the axis vectors.



□

In this example, we can see that in the two bases $\mathcal{S}$, $\mathcal{B}$, their axis vectors (reversed in the figure) can be transformed via $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(\hat{e}_j) = \vec{u}_j$. The corresponding matrix representation is $\vec{u}_j = [\vec{u}_1|\vec{u}_2]\hat{e}_j = P_B^S\hat{e}_j$. Meanwhile the coordinate transformation follows $[\vec{v}]_B = (P_B^S)^{-1}[\vec{v}]_S = P_S^B[\vec{v}]_S$, where the transformation matrix is the inverse of the former. The former actually alters the vectors themselves and is sometimes known as an ***active (coordinate) transformation***. In contrast, the latter only changes the coordinate frame but keep the vector unchanged and is hence called a ***passive (coordinate) transformation*** (in fact, it is just the identity transformation with a change of basis). We can see that in the example above, after the active transformation the area of square formed by the new two basis vectors is enlarged by a factor of $|\det(P_B^S)| = 3$. Such a magnifying factor is a result of Properties 5.3.4 and the similar holds for cases of any dimension. Oppositely, with the passive transformation we can say that the value of area of an identical square is shrinked to $\left|\det\left(P_S^B\right)\right| = |\det((P_B^S)^{-1})| = |\det(P_B^S)|^{-1} = \frac{1}{3}$ of the original, expressed in the new units. Therefore, the appropriate factors in the two scenarios are the inverse of each other.

Change of Coordinates for Linear Transformations/Matrices

It is also possible to do a change of coordinates for linear transformations and hence the matrices that represent them. Consider a linear transformation $T : \mathcal{U} \to \mathcal{V}$ that has a matrix representation of $[\vec{v}]_H = [T]_B^H[\vec{u}]_B$ where $\mathcal{B}$ and $\mathcal{H}$ are bases for $\mathcal{U}$ and $\mathcal{V}$ respectively. If we want to change the basis for $\mathcal{U}$ from $\mathcal{B}$ to some other basis $\mathcal{B}'$ (and similarly $\mathcal{H}'$ for $\mathcal{V}$), then the new matrix representation of the linear transformation would be $[\vec{v}]_{H'} = [T]_{B'}^{H'}[\vec{u}]_{B'}$. Since they are the same transformation but only expressed in different coordinate systems, these two matrix equations have to be equivalent. Now, the vectors on both sides of the original equation themselves can undergo changes of coordinates according to the previous Theorem 7.1.9 with $[\vec{u}]_B = [\text{id}]_{B'}^B[\vec{u}]_{B'} = P_{B'}^B[\vec{u}]_{B'}$ and $[\vec{v}]_H = [\text{id}]_{H'}^H[\vec{v}]_{H'} = Q_{H'}^H[\vec{v}]_{H'}$, where we denote the change of coordinates matrices from $\mathcal{B}'$ to $\mathcal{B}$ by $P_{B'}^B$ (and similarly $\mathcal{H}'$ to $\mathcal{H}$ by $Q_{H'}^H$). Subsequently,

$$\begin{aligned}
[\vec{v}]_H &= [T]_B^H[\vec{u}]_B \\
Q_{H'}^H[\vec{v}]_{H'} &= [T]_B^H P_{B'}^B[\vec{u}]_{B'} \\
[\vec{v}]_{H'} &= \left((Q_{H'}^H)^{-1}[T]_B^H P_{B'}^B\right)[\vec{u}]_{B'}
\end{aligned}$$

$$\begin{tikzcd}
{[\vec{u}]_B} \arrow[r, "{[T]_B^H}"] & {[\vec{v}]_H} \arrow[d, "{Q_H^{H'} = (Q_{H'}^H)^{-1}}"] \\
{[\vec{u}]_{B'}} \arrow[u, "{P_{B'}^B}"] \arrow[r, "{[T]_{B'}^{H'}}"'] & {[\vec{v}]_{H'}}
\end{tikzcd}$$

Figure 7.1: A schematic showing how the change of coordinate bases works for linear transformation.

Comparing with the latter equation, we can identify $[T]_{B'}^{H'}$ with $(Q_{H'}^H)^{-1}[T]_B^H P_{B'}^B$, and this is the desired formula for change of coordinates over the matrix form of a linear transformation.

Properties 7.2.1. The change of coordinates for the matrix representation of a linear transformation $T : \mathcal{U} \to \mathcal{V}$ from bases $\mathcal{B}$ and $\mathcal{H}$ to $\mathcal{B}'$ and $\mathcal{H}'$ for $\mathcal{U}$ and $\mathcal{V}$ respectively follows the relation

$$[T]_{B'}^{H'} = (Q_{H'}^H)^{-1}[T]_B^H P_{B'}^B$$

where $P_{B'}^B$ and $Q_{H'}^H$ are matrices for change of coordinates on vectors from $\mathcal{B}'$ to $\mathcal{B}$ and $\mathcal{H}'$ to $\mathcal{H}$ individually.

Another way to derive the above formula is to consider the linear transformation with respect to the basis $\mathcal{B}'$ to $\mathcal{H}'$ as three smaller steps: firstly, convert the input vector from the basis $\mathcal{B}'$ back to $\mathcal{B}$ ($P_{B'}^B$); subsequently, carry out the transformation in terms of $\mathcal{B}$ and $\mathcal{H}$ ($[T]_B^H$); finally, map the vector from the basis $\mathcal{H}$ to $\mathcal{H}'$ ($Q_H^{H'} = (Q_{H'}^H)^{-1}$). This flow is illustrated in the schematic of Figure 7.1.

Example 7.2.2. Use Properties 7.2.1 to redo Example 7.1.2 with respect to new bases $\mathcal{B}' = \{1, x-1, (x-1)^2\}$ and $\mathcal{H}' = \{1, x+1\}$.

Solution. First it is instructive to find $P^B_{B'}$ and $Q^H_{H'}$. We leave to the readers to verify that

$$P^B_{B'} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad Q^H_{H'} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and hence by Properties 7.2.1,

$$\begin{aligned}
[T]^{H'}_{B'} &= (Q^H_{H'})^{-1}[T]^H_B P^B_{B'} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -4 \\ 0 & 0 & 2 \end{bmatrix}
\end{aligned}$$

We use a test case to check the answer. Let $p(x) = x^2-3x+1 = (x-1)^2-(x-1)-1$. Then its coordinates in the $\mathcal{B}'$ basis is $(-1,-1,1)^T_{B'}$, and the transformation can be described by

$$[T]^{H'}_{B'}(1,-1,-1)^T_{B'} = \begin{bmatrix} 0 & 1 & -4 \\ 0 & 0 & 2 \end{bmatrix}^{H'}_{B'} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}_{B'} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}_{H'}$$

which corresponds to $-5(1) + 2(x+1) = 2x - 3$, which is consistent with the usual calculation $T(p(x)) = p'(x) = (x^2 - 3x + 1)' = 2x - 3$ from elementary calculus. $\square$

In most of the times, we are interested in the type of linear transformations that are an ***endomorphism*** (sometimes also referred to as a ***linear operator***) in which the mapping is from a vector space $\mathcal{V}$ to itself, i.e. $T : \mathcal{V} \to \mathcal{V}$[7]. Often we also use the same basis $\mathcal{B}$ for the input and output. Subsequently, to change the basis for both of them at the same time, let's say $\mathcal{B}'$, if the matrix for change of coordinates on vectors from $\mathcal{B}'$ to $\mathcal{B}$ is denoted as $P = P_{B'}^{B}$, then Properties 7.2.1 is reduced to $[T]_{B'}^{B'} = (P_{B'}^{B})^{-1}[T]_B^B P_{B'}^B = P^{-1}AP$ where $A = [T]_B^B$ is the original matrix representation of the endomorphism. When it is clear from the context, we will simply write $[T]_B^B$ ($[T]_{B'}^{B'}$) as $[T]_B$ ($[T]_{B'}$).

Properties 7.2.2. For a linear endomorphism $T : \mathcal{V} \to \mathcal{V}$, the change of coordinates for its matrix representation from the old basis $\mathcal{B}$ to the new one $\mathcal{B}'$ is described by the formula

$$[T]_{B'} = (P_{B'}^{B})^{-1}[T]_B P_{B'}^{B}$$

Or speaking loosely, the change of coordinates for a matrix in general takes the form of

$$A' = P^{-1}AP$$

Example 7.2.3. For a two-dimensional vector space $\mathcal{V}$ with a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, if a linear endomorphism $T : \mathcal{V} \to \mathcal{V}$ is defined by $T(\vec{v}_1) = \vec{v}_1$, $T(\vec{v}_2) = \vec{v}_1 + \vec{v}_2$, finds its matrix representation with respect to $\mathcal{B}$. Subsequently, if a new basis $\mathcal{B}'$ is formed by $\{\vec{v}_1', \vec{v}_2'\}$ where $\vec{v}_1' = 2\vec{v}_1 - \vec{v}_2$ and $\vec{v}_2' = -\vec{v}_1 + \vec{v}_2$, use Properties 7.2.2 to compute the matrix representation of the endomorphism with respect to the new basis.

Solution. By Definition 7.1.2, the linear transformation has a matrix representation of

$$\begin{aligned}[T]_B = \big[[T(\vec{v}_1)]_B | [T(\vec{v}_2)]_B\big] &= \big[[\vec{v}_1]_B | [\vec{v}_1 + \vec{v}_2]_B\big] \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

[7] An endomorphism that is at the same time an isomorphism is known as an ***automorphism***, e.g. the linear transformation in Example 7.2.3.

with respect to the old basis $\mathcal{B}$. The appropriate $P_{B'}^{B}$ matrix that will be used for Properties 7.2.2, by Theorem 7.1.9, is

$$\begin{aligned} P_{B'}^{B} = \left[[\vec{v}_1']_B | [\vec{v}_2']_B\right] &= \left[[2\vec{v}_1 - \vec{v}_2]_B | [-\vec{v}_1 + \vec{v}_2]_B\right] \\ &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

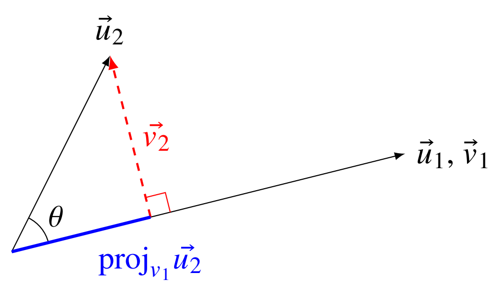
and thus the desired new matrix representation of the endomorphism with respect to $\mathcal{B}'$ is

$$\begin{aligned} [T]_{B'} &= (P_{B'}^{B})^{-1}[T]_B P_{B'}^{B} \\ &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \end{aligned}$$

□

7.2.2 Gram-Schmidt Orthogonalization, QR Decomposition

Sometimes the coordinate basis consists of vectors that are linearly independent but not orthogonal to each other, unlike the standard basis. A common way to create an orthogonal basis from the set is to apply the so-called ***Gram-Schmidt Orthogonalization***. Basically, it is an iterative method. At each step it constructs a vector that are orthogonal to all the previously processed vectors by removing the parallel components projected onto them (blue) while retaining the orthogonal part (red).



Definition 7.2.3 (Algorithm for Gram-Schmidt Orthogonalization)**.** Given a coordinate basis consisted of $\vec{u}_1, \vec{u}_2, \vec{u}_3, \ldots, \vec{u}_n \in \mathbb{R}^m$, Gram-Schmidt Orthogonalization transforms them into $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n \in \mathbb{R}^m$ (m and n are not necessarily equal) according to the following formulae:

$$\begin{aligned}
\vec{v}_1 &= \vec{u}_1 \\
\vec{v}_2 &= \vec{u}_2 - \text{proj}_{v_1}\vec{u}_2 = \vec{u}_2 - \frac{\vec{v}_1 \cdot \vec{u}_2}{\|\vec{v}_1\|^2}\vec{v}_1 \\
\vec{v}_3 &= \vec{u}_3 - \text{proj}_{v_1}\vec{u}_3 - \text{proj}_{v_2}\vec{u}_3 = \vec{u}_3 - \frac{\vec{v}_1 \cdot \vec{u}_3}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{u}_3}{\|\vec{v}_2\|^2}\vec{v}_2 \\
&\vdots \\
\vec{v}_n &= \vec{u}_n - \text{proj}_{v_1}\vec{u}_n - \text{proj}_{v_2}\vec{u}_n - \cdots - \text{proj}_{v_{n-1}}\vec{u}_n \\
&= \vec{u}_n - \frac{\vec{v}_1 \cdot \vec{u}_n}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{u}_n}{\|\vec{v}_2\|^2}\vec{v}_2 - \cdots - \frac{\vec{v}_{n-1} \cdot \vec{u}_n}{\|\vec{v}_{n-1}\|^2}\vec{v}_{n-1}
\end{aligned}$$

In general, for $j \geq 2$, the j-th new vector is computed by

$$\vec{v}_j = \vec{u}_j - \sum_{k=1}^{j-1} \text{proj}_{v_k}\vec{u}_j = \vec{u}_j - \sum_{k=1}^{j-1} \frac{\vec{v}_k \cdot \vec{u}_j}{\|\vec{v}_k\|^2}\vec{v}_k$$

where the expression of projection, Properties 5.2.1, is used.

A variant of Gram-Schmidt Orthogonalization is to normalize every vector at each step immediately, such that $\left\|\hat{v}_j\right\| = 1$ for all j, and the resulted basis is said to be ***orthonormal*** (both orthogonal and of unit length). The formulae in Definition 7.2.3 are then reduced to

Definition 7.2.4 (Gram-Schmidt Orthogonalization with Normalization)**.**

$$\begin{aligned}
\hat{v}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} \\
\hat{v}_2 &= \frac{\vec{u}_2 - (\hat{v}_1 \cdot \vec{u}_2)\hat{v}_1}{\|\vec{u}_2 - (\hat{v}_1 \cdot \vec{u}_2)\hat{v}_1\|}
\end{aligned}$$

$$\begin{aligned}
\hat{v}_3 &= \frac{\vec{u}_3 - (\hat{v}_1 \cdot \vec{u}_3)\hat{v}_1 - (\hat{v}_2 \cdot \vec{u}_3)\hat{v}_2}{\|\vec{u}_3 - (\hat{v}_1 \cdot \vec{u}_3)\hat{v}_1 - (\hat{v}_2 \cdot \vec{u}_3)\hat{v}_2\|} \\
&\vdots \\
\hat{v}_n &= \frac{\vec{u}_n - (\hat{v}_1 \cdot \vec{u}_n)\hat{v}_1 - (\hat{v}_2 \cdot \vec{u}_n)\hat{v}_2 - \cdots - (\hat{v}_{n-1} \cdot \vec{u}_n)\hat{v}_{n-1}}{\|\vec{u}_n - (\hat{v}_1 \cdot \vec{u}_n)\hat{v}_1 - (\hat{v}_2 \cdot \vec{u}_n)\hat{v}_2 - \cdots - (\hat{v}_{n-1} \cdot \vec{u}_n)\hat{v}_{n-1}\|}
\end{aligned}$$

For $j \geq 2$, the general formulae is

$$\hat{v}_j = \frac{\vec{u}_j - \sum_{k=1}^{j-1}(\hat{v}_k \cdot \vec{u}_j)\hat{v}_k}{\left\|\vec{u}_j - \sum_{k=1}^{j-1}(\hat{v}_k \cdot \vec{u}_j)\hat{v}_k\right\|}$$

Example 7.2.4. Perform Gram-Schmidt Orthogonalization with normalization on the coordinate basis for $\mathbb{R}^3$ that is consisted of $\vec{u}_1 = (1, 2, 2)^T$, $\vec{u}_2 = (1, -1, 0)^T$, $\vec{u}_3 = (3, -1, 1)^T$, using the formula in Definition 7.2.4.

Solution. The first vector is

$$\hat{v}_1 = \frac{1}{\sqrt{1^2 + 2^2 + 2^2}} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

The second vector can be found via

$$\begin{aligned}
\vec{u}_2 - (\hat{v}_1 \cdot \vec{u}_2)\hat{v}_1 &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - [(\frac{1}{3})(1) + (\frac{2}{3})(-1) + (\frac{2}{3})(0)] \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - (-\frac{1}{3}) \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ -\frac{7}{9} \\ \frac{2}{9} \end{bmatrix}
\end{aligned}$$

$$\hat{v}_2 = \frac{1}{\sqrt{(\frac{10}{9})^2 + (-\frac{7}{9})^2 + (\frac{2}{9})^2}} \begin{bmatrix} \frac{10}{9} \\ -\frac{7}{9} \\ \frac{2}{9} \end{bmatrix} = \frac{3}{\sqrt{17}} \begin{bmatrix} \frac{10}{9} \\ -\frac{7}{9} \\ \frac{2}{9} \end{bmatrix} = \begin{bmatrix} \frac{10}{3\sqrt{17}} \\ -\frac{7}{3\sqrt{17}} \\ \frac{2}{3\sqrt{17}} \end{bmatrix}$$

By the same essence, we have the third vector as

$$\begin{aligned}
&\vec{u}_3 - (\hat{v}_1 \cdot \vec{u}_3)\hat{v}_1 - (\hat{v}_2 \cdot \vec{u}_3)\hat{v}_2 \\
&= \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} - [(\frac{1}{3})(3) + (\frac{2}{3})(-1) + (\frac{2}{3})(1)] \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \\
&\quad - [(\frac{10}{3\sqrt{17}})(3) + (-\frac{7}{3\sqrt{17}})(-1) + (\frac{2}{3\sqrt{17}})(1)] \begin{bmatrix} \frac{10}{3\sqrt{17}} \\ -\frac{7}{3\sqrt{17}} \\ \frac{2}{3\sqrt{17}} \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} - \color{red}{\frac{13}{\sqrt{17}}} \begin{bmatrix} \frac{10}{3\sqrt{17}} \\ -\frac{7}{3\sqrt{17}} \\ \frac{2}{3\sqrt{17}} \end{bmatrix} = \begin{bmatrix} \frac{2}{17} \\ \frac{2}{17} \\ -\frac{3}{17} \end{bmatrix} \\
\hat{v}_3 &= \frac{1}{\sqrt{(\frac{2}{17})^2 + (\frac{2}{17})^2 + (-(\frac{3}{17}))^2}} \begin{bmatrix} \frac{2}{17} \\ \frac{2}{17} \\ -\frac{3}{17} \end{bmatrix} = \color{blue}{\sqrt{17}} \begin{bmatrix} \frac{2}{17} \\ \frac{2}{17} \\ -\frac{3}{17} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{17}} \\ \frac{2}{\sqrt{17}} \\ -\frac{3}{\sqrt{17}} \end{bmatrix}
\end{aligned}$$

□

Short Exercise: Verify that $\hat{v}_1, \hat{v}_2, \hat{v}_3$ are pairwise orthogonal.[8]

An major application of the Gram-Schmidt process is the ***QR Decomposition***, which factors a matrix into two matrices, one as its orthogonal basis vectors arranged in columns and another one as a upper-triangular matrix (non-zero elements only found along or above the main diagonal) where the elements take the form of $\vec{u}_j \cdot \hat{v}_i$ as shown below. This is very useful in the processing of large matrices and least-square error fitting.

Properties 7.2.5. For a matrix $A = [\vec{u}_1|\vec{u}_2|\vec{u}_3|\cdots|\vec{u}_n]$, and the matrix $Q = [\hat{v}_1|\hat{v}_2|\hat{v}_3|\cdots|\hat{v}_n]$, where the $\hat{v}_j$ are orthonormal vectors that come from carrying

[8]We will only check $\hat{v}_1$ and $\hat{v}_3$ are orthogonal to each other and leave the remaining two pairs to the readers. $\hat{v}_1 \cdot \hat{v}_3 = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})^T \cdot (\frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, -\frac{3}{\sqrt{17}})^T = (\frac{1}{3})(\frac{2}{\sqrt{17}}) + (\frac{2}{3})(\frac{2}{\sqrt{17}}) + (\frac{2}{3})(-\frac{3}{\sqrt{17}}) = 0.$

out Gram-Schmidt orthogonalization on the basis vectors $\vec{u}_j$ according to the Definition 7.2.4, we have $A = QR$, where

$$R = \begin{bmatrix} \hat{v}_1 \cdot \vec{u}_1 & \hat{v}_1 \cdot \vec{u}_2 & \hat{v}_1 \cdot \vec{u}_3 & \cdots & \hat{v}_1 \cdot \vec{u}_n \\ 0 & \hat{v}_2 \cdot \vec{u}_2 & \hat{v}_2 \cdot \vec{u}_3 & & \hat{v}_2 \cdot \vec{u}_n \\ 0 & 0 & \hat{v}_3 \cdot \vec{u}_3 & & \hat{v}_3 \cdot \vec{u}_n \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \hat{v}_n \cdot \vec{u}_n \end{bmatrix}$$

$$\text{i.e. } R_{ij} = \begin{cases} \hat{v}_i \cdot \vec{u}_j & i \le j \\ 0 & i > j \end{cases} \qquad \text{for } 1 \le i, j \le n$$

is an upper triangular $n \times n$ invertible matrix.

Proof. We will show that every column of A and QR coincides. The j-th column of A is simply the j-th vector in the starting basis, $\vec{u}_j$. Meanwhile, the j-th column of QR is Q times the j-th column of R, which is

$$\begin{aligned} QR^{(j)} &= \left[\hat{v}_1 | \hat{v}_2 | \cdots | \hat{v}_j | \cdots | \hat{v}_n\right] \begin{bmatrix} \hat{v}_1 \cdot \vec{u}_j \\ \hat{v}_2 \cdot \vec{u}_j \\ \vdots \\ \hat{v}_j \cdot \vec{u}_j \\ \vdots \\ 0 \end{bmatrix} \\ &= (\hat{v}_1 \cdot \vec{u}_j)\hat{v}_1 + (\hat{v}_2 \cdot \vec{u}_j)\hat{v}_2 + \cdots + (\hat{v}_j \cdot \vec{u}_j)\hat{v}_j + 0 \\ &= \sum_{k=1}^{j} (\hat{v_k} \cdot \vec{u}_j)\hat{v_k} = \sum_{k=1}^{j-1} (\hat{v_k} \cdot \vec{u}_j)\hat{v_k} + (\hat{v_j} \cdot \vec{u}_j)\hat{v_j} \end{aligned}$$

By Definition 7.2.4, we have

$$\hat{v}_j = \frac{\vec{u}_j - \sum_{k=1}^{j-1} (\hat{v}_k \cdot \vec{u}_j)\hat{v}_k}{\left\| \vec{u}_j - \sum_{k=1}^{j-1} (\hat{v}_k \cdot \vec{u}_j)\hat{v}_k \right\|}$$

[9] which after rearrangement, becomes

$$\vec{u}_j = \sum_{k=1}^{j-1}(\hat{v}_k \cdot \vec{u}_j)\hat{v}_k + \left\|\vec{u}_j - \sum_{k=1}^{j-1}(\hat{v}_k \cdot \vec{u}_j)\hat{v}_k\right\|\hat{v}_j$$

Therefore, in order to show that $\vec{u}_j = QR^{(j)}$, by comparing the two expressions, we need to check if

$$\hat{v}_j \cdot \vec{u}_j = \left\|\vec{u}_j - \sum_{k=1}^{j-1}(\hat{v}_k \cdot \vec{u}_j)\hat{v}_k\right\|$$

Consider

$$\begin{aligned}
(\vec{u}_j - \sum_{k=1}^{j-1}(\hat{v}_k \cdot \vec{u}_j)\hat{v}_k) \cdot \hat{v}_j &= \hat{v}_j \cdot \vec{u}_j - \sum_{k=1}^{j-1}(\hat{v}_k \cdot \vec{u}_j)(\hat{v}_k \cdot \hat{v}_j) \\
&= \hat{v}_j \cdot \vec{u}_j
\end{aligned}$$

as $\vec{v}_k \cdot \hat{v}_j = 0$ for $k \neq j$ due to the orthogonality enforced by the Gram-Schmidt process. On the other hand, by Definition 7.2.4 again,

$$\vec{u}_j - \sum_{k=1}^{j-1}(\hat{v}_k \cdot \vec{u}_j)\hat{v}_k = \left\|\vec{u}_j - \sum_{k=1}^{j-1}(\hat{v}_k \cdot \vec{u}_j)\hat{v}_k\right\|\hat{v}_j$$

Therefore,

$$\begin{aligned}
(\vec{u}_j - \sum_{k=1}^{j-1}(\hat{v}_k \cdot \vec{u}_j)\hat{v}_k) \cdot \hat{v}_j &= \left(\left\|\vec{u}_j - \sum_{k=1}^{j-1}(\hat{v}_k \cdot \vec{u}_j\hat{v}_k\right\|\hat{v}_j)\right) \cdot \hat{v}_j \\
&= \left\|\vec{u}_j - \sum_{k=1}^{j-1}(\hat{v}_k \cdot \vec{u}_j)\hat{v}_k\right\|(\hat{v}_j \cdot \hat{v}_j)
\end{aligned}$$

[9] Some may ask if $\left\|\vec{u}_j - \sum_{k=1}^{j-1}(\hat{v}_k \cdot \vec{u}_j)\hat{v}_k\right\|$ can be 0 (or $\vec{u}_j - \sum_{k=1}^{j-1}(\hat{v}_k \cdot \vec{u}_j)\hat{v}_k$ be the zero vector) and $\hat{v}_j$ is not well-defined. However, this will contradict the linear independence of the basis vectors $\vec{u}_k$. We can use induction to show this: (WIP)

$$= \left\| \vec{u}_j - \sum_{k=1}^{j-1} (\hat{v}_k \cdot \vec{u}_j) \hat{v}_k \right\|$$

as $\hat{v}_j \cdot \hat{v}_j = \|\hat{v}_j\|^2 = 1^2 = 1$. The required equality is then established and the result follows. The invertibility of R can be shown by noting that all diagonal elements $\hat{v}_j \cdot \hat{u}_j = \left\| \vec{u}_j - \sum_{k=1}^{j-1} (\hat{v}_k \cdot \vec{u}_j) \hat{v}_k \right\|$ of the upper triangular R matrix are non-zero (see Footnote 9). □

Example 7.2.5. Construct a QR decomposition for the case in Example 7.2.4.

Solution. The matrix Q is simply

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{10}{3\sqrt{17}} & \frac{2}{\sqrt{17}} \\ \frac{2}{3} & -\frac{7}{3\sqrt{17}} & \frac{2}{\sqrt{17}} \\ \frac{2}{3} & \frac{2}{3\sqrt{17}} & -\frac{3}{\sqrt{17}} \end{bmatrix}$$

And by Properties 7.2.5, the entries in R are

$$\begin{aligned} R &= \begin{bmatrix} \hat{v}_1 \cdot \vec{u}_1 & \hat{v}_1 \cdot \vec{u}_2 & \hat{v}_1 \cdot \vec{u}_3 \\ 0 & \hat{v}_2 \cdot \vec{u}_2 & \hat{v}_2 \cdot \vec{u}_3 \\ 0 & 0 & \hat{v}_3 \cdot \vec{u}_3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -\frac{1}{3} & 1 \\ 0 & \frac{\sqrt{17}}{3} & \frac{13}{\sqrt{17}} \\ 0 & 0 & \frac{1}{\sqrt{17}} \end{bmatrix} \end{aligned}$$

whose values can be readily inferred from the steps during the orthogonalization process itself in Example 7.2.4 (highlighted in red/blue). The readers are encouraged to compute the matrix product QR to see if the original matrix A is recovered.

We conclude this section with a small remark related to the concept of orthogonal complement discussed in Section 6.3.2.

Properties 7.2.6. For an orthogonal(-normal) basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots, \vec{v}_n\}$ for a finite-dimensional vector space $\mathcal{V}$, the subspaces $\mathcal{V}_G$ and $\mathcal{V}_H$ formed by $\mathcal{G} = \{\vec{v}_I\}$ and $\mathcal{H} = \{\vec{v}_J\}$ respectively, where I and J are mutually exclusive indices that together exhaust all integers from 1 to n, are the orthogonal complement to each other, such that $\mathcal{V}_G^\perp = \mathcal{V}_H$ and $\mathcal{V}_G \oplus \mathcal{V}_H = \mathcal{V}$.

□

7.3 Python Programming

We can define a function to a change in coordinates for vectors or matrices. Let's first write a helper function to produce the change of coordinates matrix P proposed in Theorem 7.1.9, which equals to $B'^{-1}B$ as discussed in the end of Example 7.1.6:

```
import numpy as np
from scipy import linalg

def P_matrix(B, B_prime):
    """ Computes the P matrix of change in coordinates. """
    P = linalg.inv(B_prime) @ B
    return(P)
```

Then we use Example 7.2.1 as an illustration for coordinate change for vectors, where regarding $\mathcal{B}$ we have

$$B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

and $B' = I$ as $\mathcal{B}' = \mathcal{S}$ implicitly in this case. We define another function for transforming the coordinates of any given vector as

```
def coord_trans_vector(vec, P):
    """ Transforms the coordinates of a vector. """
    trans_vec = linalg.inv(P) @ vec
    return(trans_vec)
```

Then Example 7.2.1 can be proceeded as follows.

```
B = np.array([[1.,  1.],
              [2., -1.]])

P = P_matrix(B, np.identity(2))
old_v = np.array([2., 1.])
new_v = coord_trans_vector(old_v, P)
print(new_v)
```

which returns `[1. 1.]` correctly. Similarly, according to Properties 7.2.2, we can make a function to carry out the change of coordinates for matrices through

```
def coord_trans_matrix(A, P):
    """ Transforms the coordinates of a matrix. """
    trans_matrix = linalg.inv(P) @ A @ P
    return(trans_matrix)
```

Let's use this to redo Example 7.2.3, where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Subsequently,

```
B = np.array([[2., -1.],
             [-1.,  1.]])
old_A = np.array([[1.,  1.],
              [0.,  1.]])

P = P_matrix(B, np.identity(2))
new_A = coord_trans_matrix(old_A, P)
print(new_A)
```

gives

```
[[ 0.  1.]
 [-1.  2.]]
```

as expected. Meanwhile, to apply Gram-Schmidt Orthogonalization for a basis, in addition to deriving the corresponding QR decomposition, we can use the function `qr` in `scipy.linalg`. Let's we use Examples 7.2.4 and 7.2.5 as a demonstration:

```
A = np.array([[1.,  1.,  3.],
              [2., -1., -1.],
              [2.,  0.,  1.],])
Q, R = linalg.qr(A)
print("Q = ", Q)
print("R = ", R)
```

which yields

```
Q =  [[-0.33333333  0.80845208 -0.48507125]
      [-0.66666667 -0.56591646 -0.48507125]
      [-0.66666667  0.16169042  0.72760688]]
R =  [[-3.          0.33333333 -1.        ]
      [ 0.          1.37436854  3.15296313]
      [ 0.          0.         -0.24253563]]
```

The columns in Q form the desired orthonormal basis. Notice that the signs of the first/third column vectors in Q are flipped when compared to that in Example 7.2.5, which leads to corresponding sign switches in R as well.

7.4 Exercises

Exercise 7.1 Let $\mathcal{V} = \mathcal{W} = \mathbb{R}^3$, and take $\mathcal{B} = \{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\}$ and $\mathcal{H} = \{(1,2,3)^T, (1,-1,0)^T, (2,-1,-1)^T\}$ as bases for $\mathcal{V}$ and $\mathcal{W}$. If a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x,y,z)^T = (x+y+z, 2x+y, x-y-3z)^T$, find its matrix representation and decide if it is one-to-one, onto and hence bijective.

Exercise 7.2 Let $\mathcal{V}$ be the real vector space generated by the basis $\mathcal{B} = \{\cos x, \sin x, x\cos x, x\sin x\}$ and $T : \mathcal{V} \to \mathcal{V}, T[f(x)] = f'(x)$ be the differentiation operator over $\mathcal{V}$. Find the matrix representation of T with respect to $\mathcal{B}$, and determine if T is injective, surjective and hence bijective.

Exercise 7.3 Let $\mathcal{V} = \mathcal{P}^2$, $\mathcal{W} = \mathcal{P}^3$ be the polynomial spaces of degree 2 and 3 respectively. Define $T : \mathcal{V} \to \mathcal{W}$ by

$$T[p(x)] = \int_1^x p(x)dx$$

find its matrix representation with respect to the standard bases and decide if the transformation is isomorphic.

Exercise 7.4 Show that every identity transformation $T : \mathcal{V} \to \mathcal{V}, T(\vec{v}) = \mathrm{id}(\vec{v}) = \vec{v}$ for a finite-dimensional vector space $\mathcal{V}$ with respect to a fixed basis $\mathcal{B}$ throughout always has a matrix representation of an identity matrix such that $[T]_B = I$.

Exercise 7.5 Apply Gram-Schmidt Orthogonalization on the following set of vectors, and then write down their QR Decomposition.

(a) $\vec{u}_1 = (1, 2)^T, \vec{u}_2 = (3, 8)^T$,

(b) $\vec{u}_1 = (1, 2, 1)^T, \vec{u}_2 = (1, 4, 4)^T, \vec{u}_3 = (2, 2, 5)^T$, and

(c) $\vec{u}_1 = (1, -2, 2, 1)^T, \vec{u}_2 = (1, 1, 0, 2)^T, \vec{u}_3 = (2, 3, -1, 0)^T$.

Chapter 8

Complex Vectors/Matrices and Block Form

In this chapter, we will take a detour to talk about two auxiliary topics. The first one is the generalization of vectors and matrices to having complex numbers as entries. Eventually, we will mention about the *complex vector space*, and compare it to the real vector space that we just learnt in the previous chapters. The second one is about *block form* of a matrix (or simply referred to as a *block matrix*) that is composed of smaller *submatrices* as the building blocks. Writing a matrix in block form enables efficient manipulation for many situations that we will encounter in the remaining parts of this book.

8.1 Definition and Operations of Complex Numbers

8.1.1 Basic Structure of Complex Numbers

The idea of complex numbers initially came from some algebra problems that lead to the square root of negative quantities, which was not defined back in the days. Later, mathematicians addressed this issue by introducing the ***imaginary number*** $\imath = \sqrt{-1}$, and $\imath^2 = -1$. For any positive number b, we have $\sqrt{-b^2} = \sqrt{b^2}\sqrt{-1} = b\imath$. ***Complex numbers*** are then quantities in the form of $a + b\imath$, where a and b themselves are real. Here a and b are called the ***real*** and

imaginary part respectively. As a small example of how complex numbers arise, note that the solutions to the quadratic equation $(x+2)^2 = -1$, are $-2 \pm \imath$.

Definition 8.1.1. Complex numbers are scalars in the form of $z = a + b\imath$, where a and b are some real numbers. The real and imaginary part are denoted by $\text{Re}\{z\} = a$ and $\text{Im}\{z\} = b$.

We also need to consider when two complex numbers are equal. This happens when their real parts, as well as imaginary parts, are equal to each other respectively.

Properties 8.1.2. Two complex numbers $z_1 = a + b\imath$, and $z_2 = c + d\imath$, where a, b, c, d are real numbers, are equal if and only if, $\text{Re}\{z_1\} = a = c = \text{Re}\{z_2\}$ and $\text{Im}\{z_1\} = b = d = \text{Im}\{z_2\}$.

For every complex number, there exists a notable complex number associated to it, known as the ***(complex) conjugate***.

Definition 8.1.3. For a complex number $z = a + b\imath$, its complex conjugate is constructed by flipping the sign of the imaginary part, which is denoted as $\overline{z} = a - b\imath$.

8.1.2 Complex Number Operations

Below are some rules about usual operations on two complex numbers.

Addition and Subtraction

Definition 8.1.4. For two complex numbers $z_1 = a + b\imath$, and $z_2 = c + d\imath$, addition and subtraction is carried out over the real parts and the imaginary parts separately, i.e. $z_1 \pm z_2 = (a + b\imath) \pm (c + d\imath) = (a \pm c) + (b \pm d)\imath = (\text{Re}\{z_1\} \pm \text{Re}\{z_2\}) + (\text{Im}\{z_1\} \pm \text{Im}\{z_2\})\imath$.

For instance, adding $1 + 3\imath$ to $2 - 4\imath$ results in $(1 + 2) + (3 - 4)\imath = 3 - \imath$.

Multiplication and Division

Multiplication of two complex numbers simply follows the usual distributive law.

Definition 8.1.5. Given two complex numbers $a + b\imath$, and $c + d\imath$, their product is

$$\begin{aligned}(a + b\imath)(c + d\imath) &= a(c + d\imath) + b\imath(c + d\imath) \\ &= ac + ad\imath + bc\imath + bd\imath^2 \\ &= (ac - bd) + (ad + bc)\imath\end{aligned}$$

Example 8.1.1. Evaluate $(1 + 2\imath)(3 - 4\imath)$.

Solution.

$$\begin{aligned}(1 + 2\imath)(3 - 4\imath) &= ((1)(3) - (2)(-4)) + ((1)(-4) + (2)(3))\imath \\ &= 11 + 2\imath\end{aligned}$$

□

Dividing something by a complex number $a + b\imath$ can be viewed as multiplication by its complex conjugate $a - b\imath$, as

$$\begin{aligned}\frac{1}{a + b\imath} &= \frac{1}{a + b\imath}\frac{a - b\imath}{a - b\imath} \\ &= \frac{a - b\imath}{a^2 - (-b^2) - ab\imath + ba\imath} \\ &= \frac{a - b\imath}{a^2 + b^2}\end{aligned}$$

with an additional factor of $\frac{1}{a^2+b^2}$. It is interesting that this $a^2 + b^2$ term coming from multiplying the complex number by its conjugate over the denominators looks like the square of hypotenuse as in the *Pythagoras' Theorem*. Later on we will see more when we discuss the geometric meaning of complex numbers.

Example 8.1.2. Compute $\frac{1+4\imath}{2+3\imath}$.

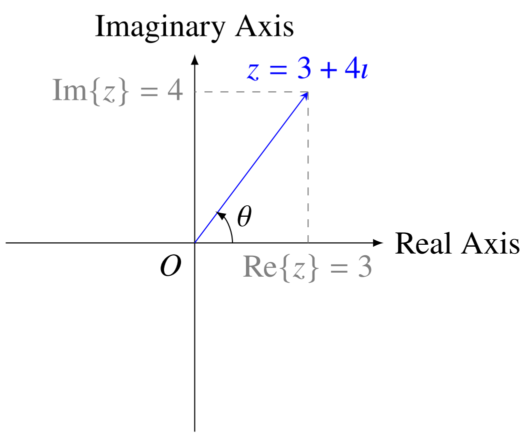
Solution. Following the idea outlined above, we have

$$\begin{aligned}
\frac{1+4\imath}{2+3\imath} &= \frac{1+4\imath}{2+3\imath}\frac{2-3\imath}{2-3\imath} \\
&= \frac{(1+4\imath)(2-3\imath)}{2^2+3^2} \\
&= \frac{((1)(2)-(4)(-3))+((1)(-3)+(4)(2))\imath}{13} = \frac{14}{13}+\frac{5}{13}\imath
\end{aligned}$$

□

8.1.3 Geometric Meaning of Complex Numbers

A complex number can be visualized as a two-dimensional vector, in the so-called ***complex plane*** (or sometimes referred to as the ***Argand plane***), where the x-axis represents the real part and the y-axis represents the imaginary part. These two axes are referred to as the ***real axis*** and ***imaginary axis*** respectively.



A complex number $z = 3 + 4\imath$ represented in the complex plane.

It is obvious that the length of such vector is $|z| = \sqrt{\mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2}$, which is called the ***modulus*** of the corresponding complex number. In the diagram above, the modulus of z is easily seen to be $|z| = 5$.

The angle between the real axis and the complex number is called the ***argument***, shown as $\theta = \arctan(\mathrm{Im}\{z\}/\mathrm{Re}\{z\})$ in the same figure. Since its complex conjugate $\overline{z}$ has the sign of the imaginary part flipped while the real part remains the same, the argument of the complex conjugate is simply the negative of that of the original complex number z. Also, the modulus will be unchanged.

Moreover, from elementary trigonometry, we know that $\mathrm{Re}\{z\} = |z|\cos\theta$ and $\mathrm{Im}\{z\} = |z|\sin\theta$. Hence z can be represented as $z = \mathrm{Re}\{z\} + \imath\,\mathrm{Im}\{z\} = |z|(\cos\theta + \imath\sin\theta)$. We also have the famous ***Euler's Formula***, relating the geometry of any complex number with an exponential raised to an imaginary power.

Definition 8.1.6 (Euler's Formula)**.** An exponential raised to an imaginary power is a complex number such that

$$e^{\imath\theta} = \cos\theta + \imath\sin\theta$$

where θ is taken to be real.

Hence z can be further written as $z = |z|e^{\imath\theta}$, and $\overline{z} = \mathrm{Re}\{z\} - \imath\,\mathrm{Im}\{z\} = |z|(\cos\theta - \imath\sin\theta) = |z|(\cos(-\theta) + \imath\sin(-\theta)) = |z|e^{-\imath\theta}$. Conversely, the quantity $e^{\imath\theta}$ can be regarded as a complex number that has a modulus of 1 and an argument of θ. Additionally, this provides formulae to express sines and cosines with complex exponentials.

Properties 8.1.7. For any θ which is confined to be real,

$$\cos\theta = \frac{e^{\imath\theta} + e^{-\imath\theta}}{2}$$
$$\sin\theta = \frac{e^{\imath\theta} - e^{-\imath\theta}}{2\imath}$$

Proof. By Definition 8.1.6,

$$\begin{aligned}
\frac{e^{\imath\theta}+e^{-\imath\theta}}{2} &= \frac{1}{2}((\cos\theta+\imath\sin\theta)+(\cos(-\theta)+\imath\sin(-\theta))) \\
&= \frac{1}{2}((\cos\theta+\imath\sin\theta)+(\cos\theta-\imath\sin\theta)) \\
&= \cos\theta
\end{aligned}$$

The derivation for $\sin\theta$ is left as an exercise. □

Now we can go back to investigate complex multiplication and division. Multiplication of a complex number z_1 by another complex number z_2, can be viewed as $z_1z_2 = (|z_1|e^{\imath\theta_1})(|z_2|e^{\imath\theta_2}) = |z_1||z_2|e^{\imath(\theta_1+\theta_2)}$.[1] This can be interpreted as, starting with the complex number $z_1 = |z_1|e^{\imath\theta_1}$ on the complex plane, rotating it anti-clockwise by an angle of θ_2, and scaling its modulus by a factor of $|z_2|$.

Similarly, division of z_1 by z_2, is $z_1/z_2 = (|z_1|/|z_2|)e^{\imath(\theta_1-\theta_2)}$. Notice that for a fraction like $1/z = 1/(a+b\imath)$, it can be rewritten as

$$\begin{aligned}
\frac{1}{z} = \frac{1}{|z|e^{\imath\theta}} &= \frac{1}{|z|}e^{-\imath\theta} \\
&= \frac{1}{|z|^2}(|z|e^{-\imath\theta}) \\
&= \frac{1}{|z|^2}\bar{z}
\end{aligned}$$

which is consistent with the discussion about complex division in the last section. In addition, we can observe that $|z|^2 = z\bar{z}$. This is not coincidence, as

$$\begin{aligned}
z\bar{z} &= |z|e^{\imath\theta}|z|e^{-\imath\theta} \\
&= |z|^2e^{\imath(\theta-\theta)} = |z|^2e^0 = |z|^2
\end{aligned}$$

Geometrically, we can think of it as starting with 1 along the real axis in the complex plane, then we scale it by $|z|$ and rotate it by θ, and finally scale it again

[1] We take it for granted that $e^{\imath\theta_1}e^{\imath\theta_2} = e^{\imath(\theta_1+\theta_2)}$.

by $|z|$ but rotate it by $-\theta$, same angle but in opposite direction. The results will be a real number $|z|^2$, since the two opposite rotations cancel out each other.

Below are some properties of modulus and complex conjugate to be remembered.

Properties 8.1.8. For two complex numbers z_1 and z_2, we have

(a) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$,

(b) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$,

(c) $\overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$,

(d) $\overline{\overline{z}} = z$,

(e) $\overline{\overline{z_1} z_2} = z_1 \overline{z_2}$,

(f) $|\overline{z}| = |z|$,

(g) $|z_1 z_2| = |z_1||z_2|$,

(h) $|z_1/z_2| = |z_1|/|z_2|$.

Another very useful result is the ***De Moivre's Formula*** that builds up on the Euler's formula, expressing $e^{\imath\theta}$ raised to an integer power n.

Theorem 8.1.9 (De Moivre's Formula)**.** Given n as an integer, then

$$
\begin{aligned}
(e^{\imath\theta})^n &= e^{\imath(n\theta)} \\
(\cos\theta + \imath\sin\theta)^n &= \cos(n\theta) + \imath\sin(n\theta)
\end{aligned}
$$

8.2 Complex Vectors and Complex Matrices

Our discussion about vectors and matrices in previous chapters is limited to those with real entries. However, we can extend the ideas to include complex elements. A complex vector is simply a vector that have complex number as components. An n-dimensional complex vector can be somehow viewed as a real vector that is $2n$-dimensional, as each complex entry can be expressed in two parts, real and imaginary. This equivalence will be further clarified in the end of this section. A complex matrix is similarly a matrix with complex elements, or from another perspective, formed by complex column vectors.

8.2.1 Operations and Properties of Complex Vectors

Addition and Subtraction for complex vectors are the same as the real counterpart, carried out element-wise. Multiplication by a scalar is also similar, applied to all elements. However, the form of complex dot product is slightly different, as defined below.

Definition 8.2.1. The dot product of two complex vectors $\vec{u}$ and $\vec{v}$ is computed as the sum of products between each pair of elements, but additionally with the conjugate operation applied on the second complex vector beforehand.

$$\begin{aligned}\vec{u}\cdot\vec{v} &= \mathbf{u}^T\overline{\mathbf{v}} \\ &= u_1\overline{v_1} + u_2\overline{v_2} + \cdots + u_n\overline{v_n} = \sum_{k=1}^{n} u_k\overline{v_k}\end{aligned}$$

The bar on $\overline{\mathbf{v}}$ means carrying out conjugate on every entry of $\mathbf{v}$. If $\mathbf{v} = \text{Re}\{\mathbf{v}\} + \imath\,\text{Im}\{\mathbf{v}\}$, where $\text{Re}\{\mathbf{v}\}$ and $\text{Im}\{\mathbf{v}\}$ are the vectors consisted of the real/imaginary part of every element in $\mathbf{v}$, then $\overline{\mathbf{v}} = \text{Re}\{\mathbf{v}\} - \imath\,\text{Im}\{\mathbf{v}\}$.

The Euclidean norm, or length, is defined similarly by

Definition 8.2.2. The length $\|\vec{v}\|$ of a complex vector $\vec{v}$ is calculated as

$$\begin{aligned}\|\vec{v}\| &= \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{\mathbf{v}^T\overline{\mathbf{v}}} \\ &= \sqrt{v_1\overline{v_1} + v_2\overline{v_2} + \cdots + v_n\overline{v_n}}\end{aligned}$$

$$= \sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2} = \sqrt{\sum_{k=1}^{n} |v_k|^2}$$

Properties of complex dot product is hence also varies slightly from its real counterpart, Properties 4.2.3.

Properties 8.2.3. For two complex vectors $\vec{u}$ and $\vec{v}$, we have

$$\begin{aligned}
\vec{u} \cdot \vec{v} &= \overline{\vec{v} \cdot \vec{u}} && \text{Anti-symmetric Property} \\
\vec{u} \cdot (\vec{v} \pm \vec{w}) &= \vec{u} \cdot \vec{v} \pm \vec{u} \cdot \vec{w} && \text{Distributive Property} \\
(\vec{u} \pm \vec{v}) \cdot \vec{w} &= \vec{u} \cdot \vec{w} \pm \vec{v} \cdot \vec{w} && \text{Distributive Property} \\
(a\vec{u}) \cdot (b\vec{v}) &= a\bar{b}(\vec{u} \cdot \vec{v}) && \text{where } a, b \text{ are some complex constants}
\end{aligned}$$

There is no cross product analogous for complex vectors.

Example 8.2.1. Show the anti-symmetric property holds for $\vec{u} = (1 + 2\imath, 3 + \imath)^T$, $\vec{v} = (2 - 5\imath, 1 + 4\imath)^T$.

Solution.

$$\begin{aligned}
\vec{u} \cdot \vec{v} &= (1 + 2\imath)(\overline{2 - 5\imath}) + (3 + \imath)(\overline{1 + 4\imath}) \\
&= (1 + 2\imath)(2 + 5\imath) + (3 + \imath)(1 - 4\imath) \\
&= (-8 + 9\imath) + (7 - 11\imath) \\
&= -1 - 2\imath
\end{aligned}$$

$$\begin{aligned}
\vec{v} \cdot \vec{u} &= (2 - 5\imath)(\overline{1 + 2\imath}) + (1 + 4\imath)(\overline{3 + \imath}) \\
&= (2 - 5\imath)(1 - 2\imath) + (1 + 4\imath)(3 - \imath) \\
&= (-8 - 9\imath) + (7 + 11\imath) \\
&= -1 + 2\imath
\end{aligned}$$

Hence $\vec{u} \cdot \vec{v} = \overline{\vec{v} \cdot \vec{u}}$. □

Short Exercise: Find the norm $\|\vec{u}\|$ and $\|\vec{v}\|$ respectively.[2]

8.2.2 Operations and Properties of Complex Matrices

Matrix multiplication between two complex matrices is carried out in the same way as we have been always doing, according to Definition 1.1.1. However, due to the difference in definition of dot product for real and complex vectors, we can no longer claim like in the discussion of Definition 4.2.1 that the elements of a complex matrix product are complex vector dot products between appropriate rows and columns, which needs a minor modification soon we will see.

Conjugate Transpose

Transpose can be similarly defined for complex matrices. However, there exists a more useful operation that combines transpose and conjugate.

Definition 8.2.4. The ***conjugate transpose*** of a matrix A, denoted as $A^* = \overline{A^T}$, has elements $A^*_{pq} = \overline{A}_{qp}$, where the conjugate of the matrix $\overline{A}$ is produced by changing every element in A to its complex conjugate. Sometimes it is called the *adjoint* or ***Hermitian transpose*** of A, and denoted as A^H.

It means that conjugate transpose is formed by flipping the elements of the matrix about its main diagonal, then subsequently conjugate on all of them. Properties of conjugate transpose are alike to those for real transpose, stated in Properties 2.1.4.

Properties 8.2.5. For two complex matrices A and B, we have

1. $(cA)^* = \overline{c}A^*$, where c is any complex scalar,
2. $(A^*)^* = A$,

[2] $\|\vec{u}\| = \sqrt{(1+2\imath)(1-2\imath)+(3+\imath)(3-\imath)} = \sqrt{(1^2+2^2)+(3^2+1^2)} = \sqrt{15}$. Similarly, $\|\vec{v}\| = \sqrt{46}$.

3. $(A \pm B)^* = A^* \pm B^*$, if A and B have the same shape,
4. $(AB)^* = B^* A^*$, if A and B have compatible shapes.

With complex conjugate of a matrix defined alongside, we can now say that the elements of the complex matrix product $A\overline{B}$, that is, a conjugate has been applied on the latter matrix, are the complex vector dot products between row and column vectors of A and the original B matrix.

Example 8.2.2. For two complex matrices

$$A = \begin{bmatrix} 1 & \imath \\ -\imath & 0 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 1+\imath & 2 \\ 0 & 1-\imath \end{bmatrix}$$

Verify that $(AB)^* = B^* A^*$.

Solution.

$$\begin{aligned}
A^* &= \begin{bmatrix} 1 & \imath \\ -\imath & 0 \end{bmatrix} \\
B^* &= \begin{bmatrix} 1-\imath & 0 \\ 2 & 1+\imath \end{bmatrix} \\
B^* A^* &= \begin{bmatrix} 1-\imath & 0 \\ 2 & 1+\imath \end{bmatrix} \begin{bmatrix} 1 & \imath \\ -\imath & 0 \end{bmatrix} \\
&= \begin{bmatrix} (1-\imath)(1)+(0)(-\imath) & (1-\imath)(\imath)+(0)(0) \\ (2)(1)+(1+\imath)(-\imath) & (2)(\imath)+(1+\imath)(0) \end{bmatrix} = \begin{bmatrix} 1-\imath & 1+\imath \\ 3-\imath & 2\imath \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
AB &= \begin{bmatrix} 1 & \imath \\ -\imath & 0 \end{bmatrix} \begin{bmatrix} 1+\imath & 2 \\ 0 & 1-\imath \end{bmatrix} \\
&= \begin{bmatrix} (1)(1+\imath)+(\imath)(0) & (1)(2)+(\imath)(1-\imath) \\ (-\imath)(1+\imath)+(0)(0) & (-\imath)(2)+(0)(1-\imath) \end{bmatrix} \\
&= \begin{bmatrix} 1+\imath & 3+\imath \\ 1-\imath & -2\imath \end{bmatrix}
\end{aligned}$$

$$(AB)^* = \begin{bmatrix} 1-\imath & 1+\imath \\ 3-\imath & 2\imath \end{bmatrix}$$

□

Determinants and Inverses for complex matrices

Complex matrices also have determinants and inverses, and are calculated in the exact same ways outlined in Sections 2.3 and 2.2. We provide a few examples here.

Example 8.2.3. Calculate the determinant for

$$A = \begin{bmatrix} 1-\imath & 3 & 2 \\ 1+\imath & 0 & \imath \\ 2 & -2\imath & 1 \end{bmatrix}$$

Solution. We apply Cofactor Expansion along the middle row in the way outlined in Properties 2.3.3, the result is

$$\begin{aligned} & -(1+\imath)\begin{vmatrix} 3 & 2 \\ -2\imath & 1 \end{vmatrix} + (0)\begin{vmatrix} 1-\imath & 2 \\ 2 & 1 \end{vmatrix} - (\imath)\begin{vmatrix} 1-\imath & 3 \\ 2 & -2\imath \end{vmatrix} \\ &= -(1+\imath)(3+4\imath) - (\imath)(-8-2\imath) \\ &= -1+\imath \end{aligned}$$

□

Example 8.2.4. Find the inverse of the matrix A in the last example.

Solution. The computation of inverse follows Properties 2.3.11. First, we note that

$$\frac{1}{\det(A)} = \frac{1}{-1+\imath}$$

$$
\begin{aligned}
&= \frac{1}{-1+\imath}\frac{-1-\imath}{-1-\imath} \\
&= \frac{-1-\imath}{1+1} = -\frac{1+\imath}{2}
\end{aligned}
$$

Then, we proceed to compute the cofactor matrix for A, which is

$$
\begin{aligned}
C &= \begin{bmatrix}
\begin{vmatrix} 0 & \imath \\ -2\imath & 1 \end{vmatrix} & -\begin{vmatrix} 1+\imath & \imath \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1+\imath & 0 \\ 2 & -2\imath \end{vmatrix} \\
-\begin{vmatrix} 3 & 2 \\ -2\imath & 1 \end{vmatrix} & \begin{vmatrix} 1-\imath & 2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1-\imath & 3 \\ 2 & -2\imath \end{vmatrix} \\
\begin{vmatrix} 3 & 2 \\ 0 & \imath \end{vmatrix} & -\begin{vmatrix} 1-\imath & 2 \\ 1+\imath & \imath \end{vmatrix} & \begin{vmatrix} 1-\imath & 3 \\ 1+\imath & 0 \end{vmatrix}
\end{bmatrix} \\
&= \begin{bmatrix}
-2 & -1+\imath & 2-2\imath \\
-3-4\imath & -3-\imath & 8+2\imath \\
3\imath & 1+\imath & -3-3\imath
\end{bmatrix}
\end{aligned}
$$

Thus, by Properties 2.3.11, the inverse of A is

$$
\begin{aligned}
A^{-1} &= \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{\det(A)}C^T \\
&= -\frac{1+\imath}{2}\begin{bmatrix}
-2 & -3-4\imath & 3\imath \\
-1+\imath & -3-\imath & 1+\imath \\
2-2\imath & 8+2\imath & -3-3\imath
\end{bmatrix} \\
&= \begin{bmatrix}
1+\imath & -\frac{1}{2}+\frac{7}{2}\imath & \frac{3}{2}-\frac{3}{2}\imath \\
1 & 1+2\imath & -\imath \\
-2 & -3-5\imath & 3\imath
\end{bmatrix}
\end{aligned}
$$

□

Short Exercise: Find A^{-1} via Gaussian Elimination.[3]

Below are some useful properties of determinant and inverse for complex matrices, that can be compared to Properties 2.3.9 and 2.2.3.

[3]Notice that we will now need to multiply rows with complex constants instead when doing elementary row operations. You should be able to get the same answer.

Properties 8.2.6. If A is a complex matrix, then

1. $\det(A^T) = \det(A)$,
2. $\det(A^*) = \overline{\det(A)}$,
3. $\det(kA) = k^n \det(A)$, for any complex constant k,
4. $\det(AB) = \det(A)\det(B)$, and
5. $\det(A^{-1}) = \frac{1}{\det(A)}$, given A is invertible.

Additionally, if A is non-singular, then

1. $(cA)^{-1} = \frac{1}{c}A^{-1}$, for any complex scalar $c \neq 0$,
2. $(A^{-1})^{-1} = A$,
3. $(A^n)^{-1} = (A^{-1})^n$, for any positive integer n,
4. $(AB)^{-1} = B^{-1}A^{-1}$, provided that B is invertible too,
5. $(A^T)^{-1} = (A^{-1})^T$,
6. $(A^*)^{-1} = (A^{-1})^*$.

8.2.3 The Complex n-space $\mathbb{C}^n$

Similar to the real n-space $\mathbb{R}^n$ brought up in Definition 4.1.2, the set of all complex vectors, now with n complex components, forms the ***complex n-space*** $\mathbb{C}^n$ as follows.

Definition 8.2.7 (The Complex n-space $\mathbb{C}^n$)**.** The complex n-space $\mathbb{C}^n$ is defined as the set of all possible n-tuples in the form of $\vec{v} = (v_1, v_2, v_3, \ldots, v_n)^T$, where v_i can be any complex numbers, for $i = 1, 2, 3, \ldots, n$. They are known as n-dimensional complex vectors.

A very interesting (and perhaps quite confusing) fact about the complex n-space $\mathbb{C}^n$, or an n-dimensional complex vector, is that it can be considered as $2n$-dimensional when put in the frame of a real vector space. The key lies in Definition 6.1.1 where if the underlying scalar is set to $\mathbb{R}$ or $\mathbb{C}$ so that it becomes a real/complex vector space. Notice the subtle difference between a real/complex vector (that is indicative of its components being real/complex) and real/complex vector space (concerning the underlying scalar used in scalar multiplication). We take $\mathbb{C}$ as a vector space here for illustration. If $\mathbb{C}$ is treated as a complex vector space, i.e. over $\mathbb{C}$ itself, then $\{1\}$ is a basis for $\mathbb{C}$ since the scalar multiplication of 1 by any arbitrary complex scalar can generate all complex numbers. Hence, the dimension of $\mathbb{C}$ is 1 over $\mathbb{C}$ (Properties 6.2.4 still holds for complex vector spaces). Otherwise, if $\mathbb{C}$ is taken as a real vector space, then $\{1\}$ is not sufficient to be a basis for $\mathbb{C}$ since multiplication by any real scalar a can never produce complex numbers with a non-zero imaginary part. Instead, $\{1, \imath\}$ can be a basis for $\mathbb{C}$ over $\mathbb{R}$ as linear combinations of 1 and $\imath$ with real coefficients can produce all complex numbers. So by Properties 6.2.4, the dimension of $\mathbb{C}$ over $\mathbb{R}$ is 2, and with Theorem 7.1.10 it is isomorphic to $\mathbb{R}^2$ in this situation. An explicit isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$ over $\mathbb{R}$ is simply

$$T(a + b\imath) = (a, b)^T$$

Extending this observation, $\mathbb{C}^n$ can either be treated as n-dimensional over $\mathbb{C}$ or $2n$-dimensional over $\mathbb{R}$ (and is isomorphic to $\mathbb{R}^{2n}$). However, unless mentioned otherwise, we consider any $\mathbb{C}^n$ vector is taken over $\mathbb{C}$ (the former case) onwards.

8.3 Manipulating Block Matrices

Moving to our second topic, a ***block matrix*** is a matrix written in smaller *submatrices* as if they are usual entries. For example, a 2×2 block matrix has the form of

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C, D are themselves matrices having the shapes of $m \times r$, $m \times q$, $n \times r$, $n \times q$, and m, n, p, q can be any positive integer. As a more concrete example, we have

$$M = \left[\begin{array}{ccc|cc} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ \hline 2 & -1 & 0 & 1 & -2 \end{array}\right]$$

being a 3×5 matrix at the same time a 2×2 block matrix where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 4 \\ 2 & -1 \end{bmatrix}$$
$$C = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & -2 \end{bmatrix}$$

are of the shapes 2×3, 2×2, 1×3 and 1×2. We can extend this for block matrices of any partition. For instance, a 3×4 block matrix will be in the form of

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \end{bmatrix}$$

where the M_{ij}s are submatrices, and for a fixed i (j), M_{ij} has the same number of rows (columns).

8.3.1 Block Matrix Multiplication

With the structure of a block matrix explained, we can now examine how matrix multiplication between two block matrices is done. Let's take a look at the easiest case of two 2×2 block matrices:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad N = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$$

Of course, from the very beginning (Section 1.1), we know that M and N themselves have to be of the shapes $m \times r$ and $r \times n$ as an ordinary matrix, but how about the submatrices? In fact, we just carry out the multiplication as if each of them are a single entry, such that

$$MN = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \begin{bmatrix} AX+BZ & AY+BW \\ CX+DZ & CY+DW \end{bmatrix}$$

Then, for each resulting block to be valid, the number of columns in A and C (B and D) must be the same as that of rows in X and Y (Z and W). So that A and C will have the shapes of $m_1 \times r_1$, $m_2 \times r_1$, X and Y will have the shapes of $r_1 \times n_1$, $r_1 \times n_2$, $m_1 + m_2 = m$, $n_1 + n_2 = n$. Similarly, B and D need to have the shapes of $m_1 \times r_2$, $m_2 \times r_2$, Z and W need to have the shapes of $r_2 \times n_1$, $r_2 \times n_2$, and $r = r_1 + r_2$. In short, the position of cut along the column direction of M must coincide with that along the row direction of N. Below is a walk-through example.

Example 8.3.1. Given

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad N = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$$

as a 3×3 and 3×2 matrix respectively, with

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 \end{bmatrix}$$
$$X = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$Z = \begin{bmatrix} -1 \end{bmatrix} \qquad W = \begin{bmatrix} 1 \end{bmatrix}$$

such that the partitions look like

$$M = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -2 \\ \hline 0 & -1 & 1 \end{array}\right] \qquad N = \left[\begin{array}{c|c} 0 & 1 \\ 2 & -1 \\ \hline -1 & 1 \end{array}\right]$$

Use block matrix multiplication to compute MN.

Solution. Note that the cuts along the column/row direction in M and N are both located in-between the 2nd/3rd index. Consequentially, we can use the formula above:

$$MN = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \begin{bmatrix} AX + BZ & AY + BW \\ CX + DZ & CY + DW \end{bmatrix}$$

which requires us to compute

$$\begin{aligned}
AX &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} & BZ &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{bmatrix} -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\
AY &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} & BW &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
CX &= \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \end{bmatrix} & DZ &= \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} -1 \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \\
CY &= \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} & DW &= \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
MN &= \begin{bmatrix} AX + BZ & AY + BW \\ CX + DZ & CY + DW \end{bmatrix} \\
&= \left[\begin{array}{c|c} \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} & \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ \hline \begin{bmatrix} -2 \end{bmatrix} + \begin{bmatrix} -1 \end{bmatrix} & \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} \end{array}\right] = \left[\begin{array}{c|c} 3 & 0 \\ 4 & -3 \\ \hline -3 & 2 \end{array}\right]
\end{aligned}$$

The readers can check the answer by computing the matrix product in the usual way. □

For multiplication involving block matrices with more blocks, the two block matrices M and N must have a partition of $m \times r$ and $r \times n$ blocks and the block multiplication is carried out as if they are individual entries in usual matrix multiplication as well. Particularly, the positions where the r column/row partition of M/N occurs must align exactly. Given

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & \cdots & M_{1r} \\ M_{21} & M_{22} & M_{23} & & M_{2r} \\ M_{31} & M_{32} & M_{33} & & M_{3r} \\ \vdots & & & \ddots & \vdots \\ M_{m1} & M_{m2} & M_{m3} & & M_{mr} \end{bmatrix} \quad \text{and } N = \begin{bmatrix} N_{11} & N_{12} & N_{13} & \cdots & N_{1n} \\ N_{21} & N_{22} & N_{23} & & N_{2n} \\ N_{31} & N_{32} & N_{33} & & N_{3n} \\ \vdots & & & \ddots & \vdots \\ N_{r1} & N_{r2} & N_{r3} & & N_{rn} \end{bmatrix}$$

this means that the numbers of columns and rows in M_{ik} and N_{kj} for any fixed k should be equal, such that M_{ik} and N_{kj} are of the shapes $m_i \times r_k$ and $r_k \times n_j$.

8.3.2 Inverse and Determinant of a Block Matrix

To properly utilize block matrices, we also need to know how to compute some basic quantities related to them, like inverse and determinant. Since most of the situations involve 2×2 block matrices only, we will handle them exclusively. Specifically, we consider 2×2 block matrices in the form of

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C, D are submatrices of the shapes $p \times p$, $p \times q$, $q \times p$ and $q \times q$, such that A, D and thus M are square. To proceed, we need the following observations.

Properties 8.3.1. Denote the $p \times p$ and $q \times q$ identity matrices by I_p and I_q. Then the matrix

$$\begin{bmatrix} I_p & 0_{p\times q} \\ -C & I_q \end{bmatrix}$$

is invertible, particularly having a determinant of 1. If furthermore, A is invertible, then

$$\begin{bmatrix} A^{-1} & 0_{p\times q} \\ 0_{q\times p} & I_q \end{bmatrix}$$

is also invertible with a determinant of $\det\left(A^{-1}\right) = (\det(A))^{-1}$.

Proof. For the first matrix, simply note that it is a lower-triangular matrix with all diagonal elements being 1 and therefore it has a determinant of 1. Therefore, by Properties 2.3.8, it is invertible. Similarly, by repeated cofactor expansions along the bottommost row for q times, the determinant of the second matrix can be seen to be $\det\left(A^{-1}\right) = (\det(A))^{-1}$ (Properties 2.3.9). If A is invertible, then $\det\left(A^{-1}\right) = (\det(A))^{-1}$ is nonzero by Properties 2.3.8 again, and

$$\begin{bmatrix} A^{-1} & 0_{p\times q} \\ 0_{q\times p} & I_q \end{bmatrix}$$

will also be invertible. □

The above properties imply that these two matrices are the results from elementary row operations (Properties 2.2.11), and therefore, their product

$$\begin{aligned}
\begin{bmatrix} I_p & 0_{p\times q} \\ -C & I_q \end{bmatrix}\begin{bmatrix} A^{-1} & 0_{p\times q} \\ 0_{q\times p} & I_q \end{bmatrix} &= \begin{bmatrix} I_pA^{-1} + 0_{p\times q}0_{q\times p} & I_p0_{p\times q} + 0_{p\times q}I_q \\ (-C)A^{-1} + I_q0_{q\times p} & -C0_{p\times q} + I_qI_q \end{bmatrix} \\
&= \begin{bmatrix} A^{-1} & 0_{p\times q} \\ -CA^{-1} & I_q \end{bmatrix}
\end{aligned}$$

can also be arrived via elementary row operations and is invertible as well. By multiplying this matrix to M, we have

$$\begin{aligned}
\begin{bmatrix} A^{-1} & 0_{p\times q} \\ -CA^{-1} & I_q \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \begin{bmatrix} A^{-1}A + 0_{p\times q}C & A^{-1}B + 0_{p\times q}D \\ -CA^{-1}A + I_qC & -CA^{-1}B + I_qD \end{bmatrix} \\
&= \begin{bmatrix} I_p & A^{-1}B \\ -C + C & -CA^{-1}B + D \end{bmatrix} \\
&= \begin{bmatrix} I_p & A^{-1}B \\ 0_{q\times p} & D - CA^{-1}B \end{bmatrix}
\end{aligned}$$

The bottom right block, $D - CA^{-1}B$, is known as the ***Schur complement*** of block A in M, denoted as M/A and has the same shape $q \times q$ as D. The above block multiplication constitutes a *block Gaussian Elimination* over the matrix M to make it *block upper-triangular*. It is not hard to see that

$$\begin{bmatrix} I_p & A^{-1}B \\ 0_{q\times p} & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I_p & -A^{-1}B \\ 0_{q\times p} & I_q \end{bmatrix} = \begin{bmatrix} I_p & 0_{p\times q} \\ 0_{q\times p} & D - CA^{-1}B \end{bmatrix}$$

Therefore,

$$\begin{aligned} &\begin{bmatrix} A^{-1} & 0_{p\times q} \\ -CA^{-1} & I_q \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_p & -A^{-1}B \\ 0_{q\times p} & I_q \end{bmatrix} \\ = &\begin{bmatrix} I_p & A^{-1}B \\ 0_{q\times p} & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I_p & -A^{-1}B \\ 0_{q\times p} & I_q \end{bmatrix} = \begin{bmatrix} I_p & 0_{p\times q} \\ 0_{q\times p} & D - CA^{-1}B \end{bmatrix} \end{aligned}$$

According to the above equation, if the Schur complement $M/A = D - CA^{-1}B$ is also invertible, then the inverse of M will exist, because

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left(\begin{bmatrix} A^{-1} & 0_{p\times q} \\ -CA^{-1} & I_q \end{bmatrix} \right)^{-1} \begin{bmatrix} I_p & 0_{p\times q} \\ 0_{q\times p} & D - CA^{-1}B \end{bmatrix} \left(\begin{bmatrix} I_p & -A^{-1}B \\ 0_{q\times p} & I_q \end{bmatrix} \right)^{-1}$$

where the three matrices on R.H.S. are all invertible.[4] By Properties 2.2.3, we arrive at

$$\begin{aligned} M^{-1} &= \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \\ &= \left(\begin{bmatrix} A^{-1} & 0_{p\times q} \\ -CA^{-1} & I_q \end{bmatrix}^{-1} \begin{bmatrix} I_p & 0_{p\times q} \\ 0_{q\times p} & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I_p & -A^{-1}B \\ 0_{q\times p} & I_q \end{bmatrix}^{-1} \right)^{-1} \\ &= \begin{bmatrix} I_p & -A^{-1}B \\ 0_{q\times p} & I_q \end{bmatrix} \begin{bmatrix} I_p & 0_{p\times q} \\ 0_{q\times p} & (D - CA^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} A^{-1} & 0_{p\times q} \\ -CA^{-1} & I_q \end{bmatrix} \end{aligned}$$

[4]The invertibility of the first and last matrix follows the same arguments in Properties 8.3.1, while for the matrix in the middle we have required that $D - CA^{-1}B$ has to be invertible, and its inverse can be readily seen to be

$$\begin{bmatrix} I_p & 0_{p\times q} \\ 0_{q\times p} & D - CA^{-1}B \end{bmatrix}^{-1} = \begin{bmatrix} I_p & 0_{p\times q} \\ 0_{q\times p} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} I_p & -A^{-1}B(D-CA^{-1}B)^{-1} \\ 0_{q\times p} & (D-CA^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} A^{-1} & 0_{p\times q} \\ -CA^{-1} & I_q \end{bmatrix}$$
$$= \begin{bmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} A^{-1}+A^{-1}B(M/A)^{-1}CA^{-1} & -A^{-1}B(M/A)^{-1} \\ -(M/A)^{-1}CA^{-1} & (M/A)^{-1} \end{bmatrix}$$

To summarize, we have the following statements.

Properties 8.3.2. For the 2×2 block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A and D are square submatrices, if A and its Schur complement $M/A = D - CA^{-1}B$ of block A are both invertible, then M is invertible with

$$M^{-1} = \begin{bmatrix} A^{-1}+A^{-1}B(M/A)^{-1}CA^{-1} & -A^{-1}B(M/A)^{-1} \\ -(M/A)^{-1}CA^{-1} & (M/A)^{-1} \end{bmatrix}$$

Properties 8.3.3. The determinant of the 2×2 block matrix in Properties 8.3.2 is

$$\det(M) = \det(A)\det\left(D - CA^{-1}B\right) = \det(A)\det(M/A)$$

if A^{-1} is well-defined.

Proof. From the derivation above, we have

$$\begin{bmatrix} A^{-1} & 0_{p\times q} \\ -CA^{-1} & I_q \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_p & -A^{-1}B \\ 0_{q\times p} & I_q \end{bmatrix} = \begin{bmatrix} I_p & 0_{p\times q} \\ 0_{q\times p} & D - CA^{-1}B \end{bmatrix}$$

Evaluating the determinants of both sides leads to

$$\det\left(\begin{bmatrix} A^{-1} & 0_{p\times q} \\ -CA^{-1} & I_q \end{bmatrix}\right)\det(M)\det\left(\begin{bmatrix} I_p & -A^{-1}B \\ 0_{q\times p} & I_q \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} I_p & 0_{p\times q} \\ 0_{q\times p} & D - CA^{-1}B \end{bmatrix}\right)$$

In the same vein of Properties 8.3.1, we then have

$$(\det(A))^{-1}\det(M)(1) = \det\left(D - CA^{-1}B\right)$$
$$\det(M) = \det(A)\det\left(D - CA^{-1}B\right) = \det(A)\det(M/A)$$

□

Example 8.3.2. Use Properties 8.3.2 and 8.3.3 to compute the inverse and determinant of the following matrix

$$M = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 1 \\ \hline -2 & -1 & 2 \end{array}\right]$$

via the partition above.

Solution. To use Properties 8.3.2, we need to first compute A^{-1} and $M/A = D - CA^{-1}B$. We leave to the readers for verifying that

$$A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$
$$M/A = D - CA^{-1}B = [2] - [-2 \quad -1]\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = [-1]$$

Then by Properties 8.3.2, we have

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(M/A)^{-1}CA^{-1} & -A^{-1}B(M/A)^{-1} \\ -(M/A)^{-1}CA^{-1} & (M/A)^{-1} \end{bmatrix}$$
$$= \left[\begin{array}{c|c} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}[-1]\,[-2 \quad -1]\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} & -\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}[-1] \\ \hline -[-1]\,[-2 \quad -1]\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} & [-1] \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -3 & 4 & -2 \\ 2 & -2 & 1 \\ \hline -2 & 3 & -1 \end{array}\right]$$

Meanwhile, by Properties 8.3.3,

$$\begin{aligned} \det(M) &= \det(A)\det(M/A) \\ &= \det\left(\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\right)\det([-1]) = (1)(-1) = -1 \end{aligned}$$

□

Similar results are also available in terms of the Schur complement using block D instead of A.

Properties 8.3.4. For the 2×2 block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A and D are square submatrices, if D and its Schur complement $M/D = A - BD^{-1}C$ of block D are both invertible, then M is invertible with

$$M^{-1} = \begin{bmatrix} (M/D)^{-1} & -(M/D)^{-1}BD^{-1} \\ -D^{-1}C(M/D)^{-1} & D^{-1} + D^{-1}C(M/D)^{-1}BD^{-1} \end{bmatrix}$$

and its determinant can be computed by

$$\det(M) = \det(D)\det\left(A - BD^{-1}C\right) = \det(D)\det(M/D)$$

Proof. See Exercise 8.6. □

8.3.3 Restriction of a Linear Transformation, Direct Sum of a Matrix

In the last chapter we have discussed about linear transformations between two vector spaces, let's say, from $\mathcal{U}$ to $\mathcal{V}$. Sometimes we only care about how the linear transformation works on some specific subspace $\mathcal{W}$ of $\mathcal{U}$. This leads to the idea of ***restriction*** of a linear transformation as follows.

Definition 8.3.5. Given a linear transformation $T : \mathcal{U} \to \mathcal{V}$ and a proper subspace $\mathcal{W} \subset \mathcal{U}$, the restriction of T to $\mathcal{W}$ is defined as

$$T|_W : \mathcal{W} \to \mathcal{V}, T|_W(\vec{w}) = T(\vec{w}) \text{ for any } \vec{w} \in \mathcal{W}.$$

In simpler terms, $T|_W$ works exactly as T but only defined on $\mathcal{W}$. Assume the vector spaces involved are all finite-dimensional, and $\dim(\mathcal{W}) = r < n = \dim(\mathcal{U})$. $\mathcal{W}$ then has a basis $\mathcal{B}_W$ with r generating vectors, which by part (c) of Properties 6.2.7 can be extended to a new basis $\mathcal{B}' = \mathcal{B}_W \cup \mathcal{G}$ for $\mathcal{U}$, where $\mathcal{G}$ contains $n - r$ vectors and $\mathcal{B}' = \mathcal{B}_W \cup \mathcal{G}$ has exactly n linearly independent vectors by construction. Some may wonder why we suddenly talk about the restriction of a linear transformation here and the reason is that its related principles can be viewed from the stand point of a block matrix.

To see this, let $\mathcal{B}_W = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\}$ and $\mathcal{G} = \{\vec{u}_{r+1}, \ldots, \vec{u}_n\}$, and thus $\mathcal{B}' = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r, \vec{u}_{r+1}, \ldots, \vec{u}_n\}$. By Definition 6.2.9, since the vectors in $\mathcal{B}' = \mathcal{B}_W \cup \mathcal{G}$ are designed to be linear independent, the subspace $\mathcal{W}^C$ generated by $\mathcal{G}$ will be the complement of $\mathcal{W}$ as a result of $\mathcal{W} \oplus \mathcal{W}^C$ being a direct sum that produces $\mathcal{U}$. Any $\vec{w} \in \mathcal{W} \subset \mathcal{U}$ will then have a coordinate representation of

$$(w_1, w_2, \ldots, w_r, 0, \ldots, 0)^T$$

in the $\mathcal{B}$' basis where components beyond the r-th index are all zeros. From the perspective of direct sum, it is the same as $\vec{w} \oplus \mathbf{0}_{n-r} = (w_1, w_2, \ldots, w_r)^T_{B_W} \oplus (0, \ldots, 0)^T_G$, i.e. $\vec{w}$ has no components in $\mathcal{W}^C$. By Definition 7.1.2, writing out the matrix representation of $[T]_{B'}^{H}$ where H is an arbitrary basis for $\mathcal{V}$ results in

$$[T]_{B'}^{H} = \begin{bmatrix} a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(r)} & a_1^{(r+1)} & \cdots & a_1^{(n)} \\ a_2^{(1)} & a_2^{(2)} & & a_2^{(r)} & a_2^{(r+1)} & & a_2^{(n)} \\ \vdots & & & \vdots & \vdots & & \vdots \\ a_m^{(1)} & a_m^{(2)} & \cdots & a_m^{(r)} & a_m^{(r+1)} & \cdots & a_m^{(n)} \end{bmatrix}$$

Since we are only concerned about $\vec{w} \in \mathcal{W} \subset \mathcal{U}$ (or $\vec{w} \oplus \mathbf{0} \in \mathcal{W} \oplus \mathcal{W}^C$) when dealing with $T|_W$, when we apply T on $\vec{w}$, which is

$$\begin{aligned} [T]_{B'}^{H}[\vec{w}]_{B'} &= \left[\begin{array}{cccc|ccc} a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(r)} & a_1^{(r+1)} & \cdots & a_1^{(n)} \\ a_2^{(1)} & a_2^{(2)} & & a_2^{(r)} & a_2^{(r+1)} & & a_2^{(n)} \\ \vdots & & & \vdots & \vdots & & \vdots \\ a_m^{(1)} & a_m^{(2)} & \cdots & a_m^{(r)} & a_m^{(r+1)} & \cdots & a_m^{(n)} \end{array}\right]_{B_W+G}^{H} \left[\begin{array}{c} w_1 \\ w_2 \\ \vdots \\ w_r \\ \hline 0 \\ \vdots \\ 0 \end{array}\right]_{B_W+G} \\ &= [T|_W \,|\, T|_{W^C}]_{B_W+G}^{H}[\vec{w} \oplus \mathbf{0}]_{B_W+G} \end{aligned}$$

We can simply ignore $[T|_{W^C}]_G^H$, the block at the right of the $[T]_{B'}^{H}$ partition as well as discard the all-zero components of $[\vec{w}]_{B'}$ starting from the $(r+1)$-th index, and keep only the other block $[T|_W]_{B_W}^{H}$ at the left and the $[\vec{w}]_{B_W}$ part. The output of the truncated multiplication

$$[T|_W]_{B_W}^{H}[\vec{w}]_{B_W} = \begin{bmatrix} a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(r)} \\ a_2^{(1)} & a_2^{(2)} & & a_2^{(r)} \\ \vdots & & & \vdots \\ a_m^{(1)} & a_m^{(2)} & \cdots & a_m^{(r)} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_r \end{bmatrix}$$

will be the same as that coming from the full form above.

Properties 8.3.6. For a linear transformation $T : \mathcal{U} \to \mathcal{V}$ between two finite-dimensional spaces, if a proper subspace $\mathcal{W}$ of $\mathcal{U}$ is generated by a basis $\mathcal{B}_W = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_r\}$, then the matrix representation of the restriction of T to $\mathcal{W}$ with respect to $\mathcal{B}_W$ and $\mathcal{H}$ will be given by

$$[T|_W]^H_{B_W} = \big[[T(\vec{w}_1)]_H|[T(\vec{w}_2)]_H|\cdots|[T(\vec{w}_r)]_H\big]$$

where $\mathcal{H}$ is any basis for $\mathcal{V}$. (This can be compared to Definition 7.1.2.)

In general, the effect of a linear transformation $T : \mathcal{U} \to \mathcal{V}$ applied to $\vec{u} \in \mathcal{U}$ is equivalent to the sum of responses from the restrictions of T to a set of subspaces $\mathcal{W}_1, \mathcal{W}_2, \cdots, \mathcal{W}_s$ where they constitute a direct sum $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots \oplus \mathcal{W}_s = \mathcal{U}$, applied on the corresponding components $\vec{w}_1 \in \mathcal{W}_1, \vec{w}_2 \in \mathcal{W}_2, \cdots, \vec{w}_s \in \mathcal{W}_s$ of $\vec{u} = \vec{w}_1 \oplus \vec{w}_2 \oplus \cdots \oplus \vec{w}_s$ in these smaller subspaces: $T(\vec{u}) = T(\vec{w}_1) \oplus T(\vec{w}_2) \oplus \cdots \oplus T(\vec{w}_s)$.

Example 8.3.3. Given a linear transformation $T : \mathcal{U} \to \mathcal{V}$ that has a matrix representation of

$$[T]^H_B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

with respect to some bases $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and $\mathcal{H} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for $\mathcal{U}$ and $\mathcal{V}$, find the restriction of T to $\mathcal{W}$, where $\mathcal{W} \subset \mathcal{U}$ has a basis of $B_W = \{\vec{w}_1, \vec{w}_2\}$, with $\vec{w}_1 = \vec{u}_1 + \vec{u}_2$ and $\vec{w}_2 = \vec{u}_1 + \vec{u}_2 + \vec{u}_3$.

Solution. We will take an indirect approach of reconstructing the basis first by finding a third vector generating $\mathcal{W}^C$ and producing the direct sum $\mathcal{W} \oplus \mathcal{W}^C = \mathcal{U}$. The change of coordinates matrix $P^B_{B_W}$ from B_W to B as devised in Theorem 7.1.9 appropriate in this situation is a 3×2 matrix instead since there are only two basis vectors in $\mathcal{B}_W$, and it can be easily seen to be

$$\begin{aligned} P^B_{B_W} &= \big[[\vec{w}_1]_B|[\vec{w}_2]_B\big] \\ &= \big[[\vec{u}_1 + \vec{u}_2]_B|[\vec{u}_1 + \vec{u}_2 + \vec{u}_3]_B\big] \end{aligned}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and we are to find $[\vec{w}_3]_B$ to complete a basis $\mathcal{B}' = \mathcal{B}_W \cup \{\vec{w}_3\}$ and hence $P_{B'}^B$. An algorithm to do so, motivated by Footnote 12 in Chapter 6, is to apply Gaussian Elimination to $P_{B_W}^B$ and then append $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to the right of it to make an identity matrix, and reverse the entire reduction procedure as follows.

$$\begin{aligned}
\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} &\to \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} & R_2 - R_1 \to R_2 \\
&\to \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} & R_2 \leftrightarrow R_3 \\
&\to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} & R_1 - R_2 \to R_1
\end{aligned}$$

$$\begin{aligned}
\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] &\to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] & R_1 + R_2 \to R_1 \\
&\to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] & R_2 \leftrightarrow R_3 \\
&\to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{array}\right] & R_2 + R_1 \to R_2
\end{aligned}$$

So $[\vec{w}_3]_B = (0, 1, 0)_B^T$ is a possible choice. While in this case the algorithm looks like an overkill, it can be very powerful when the number of dimensions and vectors to be appended become much larger.[5] Now

$$P^B_{B'} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and by Properties 7.2.1

$$\begin{aligned} [T]^H_{B'} &= [T]^H_B P^B_{B'} \\ &= \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & -1 \end{bmatrix} \end{aligned}$$

The matrix representation of the restriction of T to $\mathcal{W}$ with respect to $\mathcal{B}_W$ agrees with the first two columns of $[T]^H_{B'}$. The third column of $[T]^H_{B'}$ that characterizes the action of $T|_{W_C}$, is removed. These lead to

$$[T|_W]^H_{B_W} = \begin{bmatrix} 2 & 4 \\ 2 & 3 \\ 0 & 1 \end{bmatrix}$$

□

Short Exercise: Directly apply Properties 8.3.6 to redo the example above.[6]

With the concept of restriction, we can now introduce the matrix analogous of a direct sum. For a linear transformation $T : \mathcal{U} \to \mathcal{V}$, if the vector spaces

[5] In fact, we only need to keep track of the row swapping operations with full-zero rows.

[6] $[T(\vec{w}_1)]_H = [T(\vec{u}_1 + \vec{u}_2)]_H = [T]^H_B (1,1,0)^T_B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}^H_B \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_B = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}_H$ and this will be the first column of $[T|_W]^H_{B_W}$. The second column is derived similarly by evaluating $[T(\vec{w}_2)]_H$.

(finite-dimensional) involved are direct sums such that $\mathcal{U} = \mathcal{W} \oplus \mathcal{W}_C$ and $\mathcal{V} = \mathcal{Y} \oplus \mathcal{Y}^C$, and the ranges

$$R(T|_W) \in \mathcal{Y} \qquad\qquad R(T|_{W^C}) \in \mathcal{Y}^C$$

of the two restrictions are such that vectors in $\mathcal{W}$ and $\mathcal{W}^C$ are mapped by T to vectors in $\mathcal{Y}$ and $\mathcal{Y}^C$ respectively, then $T = T|_W \oplus T|_{W^C}$ is a ***matrix direct sum*** in the sense that the linear transformation T maps each of the complement subspaces of $\mathcal{U}$ into complement subspaces of $\mathcal{V}$. If we write the input vector $\vec{u} = \vec{w} \oplus \vec{w}^C$ as a direct sum where $\vec{w} \in \mathcal{W}$ and $\vec{w}^C \in \mathcal{W}^C$, then the output vector will also become a direct sum $\vec{v} = \vec{y} \oplus \vec{y}^C$ where $\vec{y} = T|_W(\vec{w}) = T(\vec{w})$ and $\vec{y}^C = T|_{W^C}(\vec{w}^C) = T(\vec{w}^C)$, which can be obtained by first computing $T(\vec{w})$ and $T(\vec{w}^C)$ individually, and then directly concatenating them together.

Definition 8.3.7 (Matrix Direct Sum)**.** The direct sum of two matrices acting as linear transformations $T_1 : \mathcal{U}_1 \to \mathcal{V}_1$ and $T_2 : \mathcal{U}_2 \to \mathcal{V}_2$ is $T = T_1 \oplus T_2$ such that for any vector direct sum $\vec{u} = \vec{u}_1 \oplus \vec{u}_2$ in $\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2$, applying T on $\vec{u}$ will yield an output of a vector direct sum $T(\vec{u}) = \vec{v} = \vec{v}_1 \oplus \vec{v}_2$ in $\mathcal{V}_1 \oplus \mathcal{V}_2$ as well, where $\vec{v}_1 = T_1(\vec{u}_1) = T|_{U_1}(\vec{u}_1) \in \mathcal{V}_1$ and $\vec{v}_2 = T_2(\vec{u}_2) = T|_{U_2}(\vec{u}_2) \in \mathcal{V}_2$. The matrix direct sum is then the matrix representation of $T = T_1 \oplus T_2$ with respect to the direct sum bases for $\mathcal{U}_1 \oplus \mathcal{U}_2$ and $\mathcal{V}_1 \oplus \mathcal{V}_2$.

Using the above definition, if $\mathcal{U}_1$ and $\mathcal{U}_2$ has a basis $\mathcal{B}_1 = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_r\}$ and $\mathcal{B}_2 = \{\vec{w}_{r+1}, \vec{w}_{r+2}, \dots, \vec{w}_n\}$, and $\mathcal{V}_1$ and $\mathcal{V}_2$ has a basis $\mathcal{H}_1 = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_s\}$ and $\mathcal{H}_2 = \{\vec{v}_{s+1}, \vec{v}_{s+2}, \dots, \vec{v}_m\}$, where r, n, s, m are some integers, then $T = T_1 \oplus T_2$ will have a *block diagonal* matrix representation of

$$[T]^{H_1+H_2}_{B_1+B_2} = \begin{bmatrix} ([T_1]^{H_1}_{B_1})_{s\times r} & 0_{s\times(n-r)} \\ 0_{(m-s)\times r} & ([T_2]^{H_2}_{B_2})_{(m-s)\times(n-r)} \end{bmatrix}^{H_1+H_2}_{B_1+B_2}$$

with respect to $\mathcal{B}_1 \oplus \mathcal{B}_2$ and $\mathcal{H}_1 \oplus \mathcal{H}_2$. To see this, let $\vec{u} = \vec{u}_1 \oplus \vec{u}_2$, $\vec{u}_1 \in \mathcal{U}_1$ and $\vec{u}_2 \in \mathcal{U}_2$, then

$$\begin{aligned} T(\vec{u}) &= [T]^{H_1+H_2}_{B_1+B_2}[\vec{u}]_{B_1+B_2} \\ &= \begin{bmatrix} ([T_1]^{H_1}_{B_1})_{s\times r} & 0_{s\times(n-r)} \\ 0_{(m-s)\times r} & ([T_2]^{H_2}_{B_2})_{(m-s)\times(n-r)} \end{bmatrix}^{H_1+H_2}_{B_1+B_2} \begin{bmatrix} [\vec{u}_1]_{B_1,r} \\ [\vec{u}_2]_{B_2,n-r} \end{bmatrix}_{H_1+H_2} \end{aligned}$$

$$= \begin{bmatrix} ([T_1]_{B_1}^{H_1})_{s\times r}[\vec{u}_1]_{B_1,r} + 0_{s\times(n-r)}[\vec{u}_2]_{B_2,n-r} \\ 0_{(m-s)\times r}[\vec{u}_1]_{B_1,r} + ([T_2]_{B_2}^{H_2})_{(m-s)\times(n-r)}[\vec{u}_2]_{B_2,n-r} \end{bmatrix}_{H_1+H_2}$$
$$= \begin{bmatrix} ([T_1]_{B_1}^{H_1}[\vec{u}_1]_{B_1})_s \\ ([T_2]_{B_2}^{H_2}[\vec{u}_2]_{B_2})_{m-s} \end{bmatrix}_{H_1+H_2}$$
$$= T_1(\vec{u}_1) \oplus T_2(\vec{u}_2)$$

where the image is a direct sum composed of $T_1(\vec{u}_1) : [T_1]_{B_1}^{H_1}[\vec{u}_1]_{B_1} \in \mathcal{V}_1$ and $T_2(\vec{u}_2) : [T_2]_{B_2}^{H_2}[\vec{u}_2]_{B_2} \in \mathcal{V}_2$ from applying T_1 and T_2 separately to the preimages $\vec{u}_1 \in \mathcal{U}_1$ and $\vec{u}_2 \in \mathcal{U}_2$ in the two subspaces. For example, the matrix direct sum of $A \oplus B$ given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 8 \\ 1 & 1 \\ 4 & 0 \end{bmatrix}$$

is

$$A \oplus B = \left[\begin{array}{ccc|cc} 1 & 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right]$$

in which A and B are matrices representing linear transformations of $\mathcal{U}_1 \to \mathcal{V}_1$ and $\mathcal{U}_2 \to \mathcal{V}_2$, where $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{V}_1$, $\mathcal{V}_2$ have dimensions of $3, 2, 2, 3$. Subsequently, $A \oplus B$ is a matrix corresponding to a mapping from $\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2$ to $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$. Finally, the matrix direct sum of more than two matrices $A_1, A_2, A_3, \ldots, A_{n-1}, A_n$ are defined recursively just like a vector direct sum as

$$A_1 \oplus A_2 \oplus A_3 \oplus \cdots A_{n-1} \oplus A_n$$
$$= (\cdots((A_1 \oplus A_2) \oplus A_3) \oplus \cdots A_{n-1}) \oplus A_n$$

As another example, sometimes we may regard a matrix that does not look like a direct sum to be effectively one with respect to appropriate coordinate systems in a broader sense.

Example 8.3.4. For a linear transformation $T : \mathcal{U} \to \mathcal{V}$ that has a matrix representation of

$$[T]_B^H = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 \end{bmatrix}$$

with respect to some bases $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$, $\mathcal{H} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, show that it can turn into a matrix direct sum if the coordinate systems are changed according to $\mathcal{B}' = \{\vec{u}'_1, \vec{u}'_2, \vec{u}'_3, \vec{u}'_4\}$, $\mathcal{H}' = \{\vec{v}'_1, \vec{v}'_2, \vec{v}'_3\}$, where

$$\begin{aligned} \vec{u}'_1 &= \vec{u}_1 & \vec{v}'_1 &= \vec{v}_1 + \vec{v}_2 \\ \vec{u}'_2 &= \vec{u}_3 & \vec{v}'_2 &= -\vec{v}_2 + \vec{v}_3 \\ \vec{u}'_3 &= \vec{u}_1 + \vec{u}_2 & \vec{v}'_3 &= \vec{v}_1 - \vec{v}_3 \\ \vec{u}'_4 &= \vec{u}_1 + \vec{u}_4 \end{aligned}$$

Solution. The change of coordinate matrices for Properties 7.2.1 are

$$P_{B'}^B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{B'}^{B} \qquad Q_{H'}^H = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}_{H'}^{H}$$

and the new matrix representation of T is

$$\begin{aligned} [T]_{B'}^{H'} &= (Q_{H'}^H)^{-1}[T]_B^H P_{B'}^B \\ &= \left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}_{H'}^{H} \right)^{-1} \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 \end{bmatrix}_B^H \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{B'}^{B} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}_H^{H'} \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 \end{bmatrix}_B^H \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{B'}^{B} \end{aligned}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}_{B'}^{H'}$$

where

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & -1 \end{bmatrix}$$

□

8.4 Python Programming

Complex numbers in Python are written as `a+bj`. For example,

```
z_1 = 1 - 2j
z_2 = 3 + 1j
print(z_1, z_2)
```

returns `(1-2j) (3+1j)`. Conjugate, modulus and argument can be found by

```
import numpy as np

print(np.conjugate(z_1))
print(np.abs(z_2))
print(np.angle(z_1))
```

which yields `(1+2j)`, `3.162278` ($\sqrt{10}$), and `-1.10715` (in radians). Addition, subtraction, multiplication and division of complex numbers in Python are coded just like if they are ordinary numbers.

```
print(3*z_1 + z_2) # (6-5j)
print(z_1 - 2*z_2) # (-5-4j)
print(z_1 * z_2) # (5-5j)
print(z_1 / z_2) # (0.1-0.7000000000000001j), floating-point
    error
```

The same goes for complex matrices and their multiplication. As an example,

```
A = np.array([[4.   , 2.+1.j],
              [-3.j, 1-2.j]])
B = np.array([[3-1.j, 0],
              [2-5.j, 4.j]])
print(A @ B)
```

produces

```
[[ 21.-12.j  -4. +8.j]
 [-11.-18.j   8. +4.j]]
```

Conjugate and Hermitian transpose of a complex matrix is retrieved by

```
print(np.conjugate(A))
print(np.conjugate(B).T) # Hermitian transpose = conjugate +
    transpose, or just .H if np.matrix is used instead of np.
    array
```

resulting in

```
[[ 4.-0.j  2.-1.j]
 [-0.+3.j  1.+2.j]]
[[3.+1.j 2.+5.j]
 [0.-0.j 0.-4.j]]
```

The usual functions for inverse and determinant also work on complex matrices. The lines

```
print(linalg.det(A))
print(linalg.inv(B))
```

give

```
(1-2j)
[[3.00000000e-01+0.1j   0.00000000e+00+0.j   ]
 [3.25000000e-01+0.275j 1.38777878e-17-0.25j ]] # Again, round
     -off error
```

We can use 1D complex matrices as complex vectors.

```
u = np.array([1+1.j, -3.j, 2])
v = np.array([5, 1+2.j, 1-4.j])
```

The complex dot product between two complex vectors are then found by `vdot`

```
print(np.vdot(u,v).conj())
```

notice that a conjugate is needed since `numpy` defines complex dot product with a different convention such that the first complex vector is conjugated instead of the second one. It then outputs the correct answer of `(1+10j)`. The `norm` function still works fine, e.g. `print(linalg.norm(u))` gives `3.87298` ($\sqrt{(1+\imath)(1-\imath)+(-3\imath)(3\imath)+(2)^2}=\sqrt{15}$). Finally, for the discussion in the last section, to build a block matrix using submatrices, we can use the `block` function as

```
C = np.array([1, 3+2.j])
D = np.array([-1.j, 2])

print(np.block([[A,B],
                [C,D]]))
```

which outputs

```
[[ 4.+0.j  2.+1.j  3.-1.j  0.+0.j]
 [-0.-3.j  1.-2.j  2.-5.j  0.+4.j]
 [ 1.+0.j  3.+2.j -0.-1.j  2.+0.j]]
```

Another example of constructing a block diagonal matrix is

```
print(np.block([[A, np.zeros([2,3])],
                [np.zeros([3,2]), np.identity(3)]]))
```

generating

```
[[ 4.+0.j  2.+1.j  0.+0.j  0.+0.j  0.+0.j]
 [-0.-3.j  1.-2.j  0.+0.j  0.+0.j  0.+0.j]
 [ 0.+0.j  0.+0.j  1.+0.j  0.+0.j  0.+0.j]
 [ 0.+0.j  0.+0.j  0.+0.j  1.+0.j  0.+0.j]
 [ 0.+0.j  0.+0.j  0.+0.j  0.+0.j  1.+0.j]]
```

8.5 Exercises

Exercise 8.1 By considering Euler's formula stated in Definition 8.1.6, we have for any θ, ϕ

$$e^{\imath\theta} = \cos\theta + \imath\sin\theta$$

$$
\begin{aligned}
e^{\imath\phi} &= \cos\phi + \imath\sin\phi \\
e^{\imath(\theta+\phi)} &= \cos(\theta+\phi) + \imath\sin(\theta+\phi)
\end{aligned}
$$

If we take the product of the first two equations, we also have

$$
e^{\imath(\theta+\phi)} = (\cos\theta + \imath\sin\theta)(\cos\phi + \imath\sin\phi)
$$

By equating the two expressions of $e^{\imath(\theta+\phi)}$, expand and compare the real and imaginary parts, prove the famous angle sum identities, which are

$$
\begin{aligned}
\cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\
\sin(\theta+\phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi
\end{aligned}
$$

Hence, by either using the results above, or the De Moivre's Formula, prove the double angle formula shown below.

$$
\begin{aligned}
\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\
\sin(2\theta) &= 2\sin\theta\cos\theta
\end{aligned}
$$

Exercise 8.2 Evaluate

(a) $(1+\imath)(3-2\imath)$,

(b) $\overline{(2-\imath)/(4+\imath)}$,

(c) $(3+5\imath)\overline{(1+\imath)/(2-3\imath)}$

as well as their modulus and argument.

Exercise 8.3 For $\vec{u} = (1+\imath, 2-\imath, 3)^T$, $\vec{v} = (2+\imath, 1-2\imath, \imath)^T$, and $\vec{w} = (-\imath, 3, 1-\imath)^T$, find

(a) $\vec{u}\cdot\vec{v}$,

(b) $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{w})$,

(c) $\|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{w}$.

Exercise 8.4 For the two complex matrices below,

$$A = \begin{bmatrix} 1+\imath & -\imath & 3 \\ 0 & 2-\imath & 1 \\ -1 & \imath & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2-\imath & \imath \\ -\imath & 3+\imath & 1-\imath \\ 0 & 1 & 2\imath \end{bmatrix}$$

compute AB, and verify $(AB^*)^* = BA^*$.

Exercise 8.5 For the matrix

$$A = \begin{bmatrix} 1-4\imath & -3\imath & 2+\imath \\ 1-\imath & 0 & 3\imath \\ -2 & 1 & 3+\imath \end{bmatrix}$$

find its determinant and inverse.

Exercise 8.6 Prove the formulae in Properties 8.3.4, by noting that

$$\begin{bmatrix} I_p & 0_{p\times q} \\ -D^{-1}C & D^{-1} \end{bmatrix} = \begin{bmatrix} I_p & 0_{p\times q} \\ 0_{q\times p} & D^{-1} \end{bmatrix} \begin{bmatrix} I_p & 0_{p\times q} \\ -C & I_q \end{bmatrix}$$

and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_p & 0_{p\times q} \\ -D^{-1}C & D^{-1} \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & BD^{-1} \\ 0_{q\times p} & I_q \end{bmatrix}$$

Exercise 8.7 Write down the direct sum of the following three matrices.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \\ -1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 4 & 0 & -3 \\ 0 & 2 & -1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \end{bmatrix}$$

Exercise 8.8 Show that given two bases $\mathcal{B} = \{\cos x, \sin x, 1, x, x^2\}$ and $\mathcal{H} = \{\cos x, \sin x, 1, x\}$ which generate vector spaces $\mathcal{U}$ and $\mathcal{V}$ respectively, the differentiation operator $T(f(x)) = f'(x) : \mathcal{U} \to \mathcal{V}$ has a 2×2 block matrix direct sum representation.

Chapter 9

Eigenvalues and Eigenvectors

In this section we will discuss a very important topic in Linear Algebra, the *eigenvalue-eigenvector* problem. By finding the eigenvectors of a square matrix which span subspaces that are *invariant* under the corresponding linear operator, it is sometimes possible to obtain a coordinate basis such that the matrix can be *diagonalized*, i.e. become a diagonal matrix under that particular change of coordinates. One of the practical usages of *diagonalization* is to solve systems of linear ordinary differential equations (ODEs) which is also commonly seen in many areas of Earth Science. In the end of this chapter, we will build up from the idea of invariant subspaces and introduce the concept of *cyclic subspaces*, leading to a famous related result called the *Cayley-Hamilton Theorem*.

9.1 Eigenvalues and Eigenvectors of a Square Matrix

9.1.1 Definition of Eigenvalues and Eigenvectors

Consider a linear operator/endomorphism $T : \mathcal{V} \to \mathcal{V}$, an interesting question is about if a vector $\vec{v} \in \mathcal{V}$ under this mapping will remain stationary in direction such that the image $T(\vec{v}) = \lambda v$ is a scalar multiple of the original vector, or in other words, the effect of T on $\vec{v}$ is simply a rescaling. In this situation, the vector $\vec{v}$ is known as an ***eigenvector*** of T and the factor λ is the corresponding ***eigenvalue***. Since a linear operator is a mapping between a vector space itself, it has a square matrix representation under any basis. This fact extends the ideas of eigenvalues and eigenvectors to square matrices.

Definition 9.1.1. Given a linear operator $T : \mathcal{V} \to \mathcal{V}$, we call λ and $\vec{v}_\lambda$ its eigenvalue and eigenvector if

$$T(\vec{v}_\lambda) = \lambda\vec{v}_\lambda$$

Similarly, given an $n \times n$ square matrix A, λ and $\vec{v}_\lambda$ will be an eigenvalue and eigenvector for it when

$$A\vec{v}_\lambda = \lambda\vec{v}_\lambda$$

This is a special case in which a vector space $\mathcal{V}$ is finite-dimensional, $\dim(\mathcal{V}) = n$, and $A = [T]_B$ is just the matrix representation of T with respect to some basis $\mathcal{B}$.

Notice that there can be more than one eigenvalues and eigenvectors. An example is given by the matrix

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$$

It can be seen that the vector $\vec{v}_1 = (1, 2)^T$ is an eigenvector of A, as

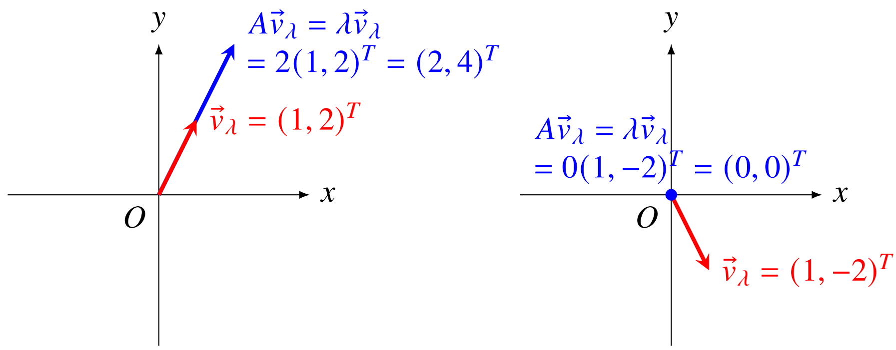
$$\begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

that corresponds to an eigenvalue of $\lambda = 2$. Meanwhile, $\vec{v}_2 = (1, -2)^T$ is another eigenvector that has an eigenvalue of $\lambda = 0$, since

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

We emphasize that a zero eigenvalue is perfectly valid.

Short Exercise: Prove that all vectors in form of $s(1,2)^T$, where s is any number, are eigenvectors for the matrix A above with $\lambda = 2$.[1]



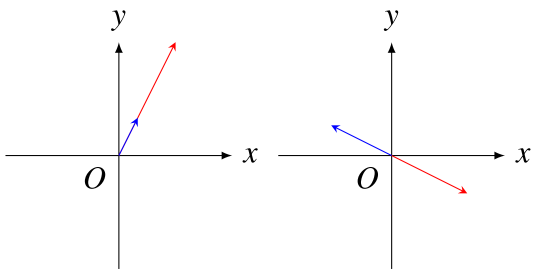
Illustrations for the example above with $\lambda = 2 > 1$ (Extension), and $\lambda = 0$ (Vanished).

There are infinitely many eigenvectors which are oriented in the same direction for a single eigenvalue as seen in the remark of the last short exercise. Particularly, they are actually the span of any one of these eigenvectors that is non-zero. Thus, along a single direction, only one of them is needed for representation, and its span is at the same time a subspace by Properties 6.1.6. This subspace is known as the ***eigenspace*** corresponding to that eigenvalue. Moreover, there may be more than one linearly independent eigenvectors for the same eigenvalue, and the dimension of eigenspace generated by them will be greater than one as well. In addition, the zero vector, technically, can be the eigenvectors of any matrix since $A\vec{0} = \vec{0} = \lambda\vec{0}$ for any matrix A and scalar λ. However, it is a trivial solution, plus more importantly the zero vector is always linearly dependent by definition, and will not be taken into consideration (unlike the totally fine eigenvalue of zero).

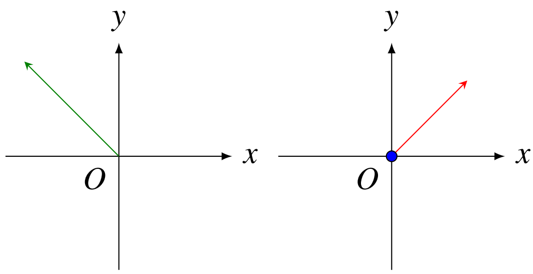
Below is the visualization of some other possibilities of eigenvector rescaling.

---

[1] $\begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}\left(s\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = s\begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = s\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\left(s\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$. In general, if $A\vec{v}_\lambda = \lambda\vec{v}_\lambda$ so that $\vec{v}_\lambda$ is some non-zero eigenvector, then $A(s\vec{v}_\lambda) = sA\vec{v}_\lambda = s\lambda\vec{v}_\lambda = \lambda(s\vec{v}_\lambda)$ and therefore all of its non-zero scalar multiples $s\vec{v}_\lambda$ is also an eigenvector.



Contraction ($0 < \lambda < 1$), Reversal ($\lambda < 0$).



Unchanged ($\lambda = 1$), Vanished ($\lambda = 0$).

The eigenspace actually belongs to a broader class of subspaces known as the invariant subspaces.

9.1.2 Finding Eigenvalues and Eigenvectors with Characteristic Polynomials

To find eigenvalues, rearrange the equation in Definition **??** relating eigenvalues and eigenvectors to obtain

$$\begin{aligned}
A\vec{v}_\lambda &= \lambda\vec{v}_\lambda \\
A\vec{v}_\lambda &= \lambda I\vec{v}_\lambda && (I\vec{u} = \vec{u} \text{ for any } \vec{u}) \\
(A - \lambda I)\vec{v}_\lambda &= \vec{0}
\end{aligned}$$

The last line constitutes a homogeneous linear system $B\vec{v}_\lambda = \vec{0}$ where $B = A - \lambda I$. For this system to have a non-trivial solution and hence an eigenvector, it is required that $\det(B) = \det(A - \lambda I) = 0$ from Theorem **??**. The relationship

$\det(A - \lambda I) = 0$ is called the characteristic equation. The roots for λ of the characteristic polynomial are then the desired eigenvalues.
Short Exercise: By inspection, find all three eigenvalues of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

For each eigenvalue there corresponds at least one eigenvectors. The number of eigenvectors for an eigenvalue depend on the number of the root appearing in the characteristic equation. The eigenvectors are then the general solution of the matrix equation $B\vec{v}_\lambda = (A - \lambda I)\vec{v}_\lambda = \vec{0}$, which exists because of the condition $\det(A - \lambda I) = 0$. The number of eigenvectors obeys the following theorem.

Theorem 9.1.2. For each particular eigenvalue λ_j for a matrix A computed from its characteristic equation $\det(A - \lambda I) = 0$, the amount of λ_j appearing as its root, or equivalently the power n of the factor $(x - \lambda_j)^n$ in the characteristic polynomial, is called the algebraic multiplicity.

Meanwhile, the amount of eigenvectors $\vec{v}_{\lambda_j}$ corresponding to λ_j is called the geometric multiplicity. With these two quantities defined, we have

$$1 \leq \text{Geometric Multiplicity} \leq \text{Algebraic Multiplicity}$$

for every eigenvalue λ_j.

Example 9.1.1. Find all eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

The characteristic equation is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1 - \lambda & -1 \\ 0 & 1 - \lambda \end{vmatrix} \\ &= (1 - \lambda)^2 = 0 \end{aligned}$$

Apparently, there is only one eigenvalue $\lambda = 1$, which has an algebraic multiplicity of 2. Possible eigenvectors are then found by solving

$$\left[\begin{array}{cc|c} 1-1 & -1 & 0 \\ 0 & 1-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

where the general solution is easily seen to be $t(1,0)^T$. So for the eigenvalue $\lambda = 1$, there is only one eigenvector $(1,0)^T$, which implies a geometric multiplicity of 1.

Example 9.1.2. For the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

the characteristic polynomial is

$$\begin{aligned} \begin{vmatrix} 1-\lambda & 3 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} &= (1-\lambda)(1-\lambda)(2-\lambda) - (1)(1-\lambda)(1) \\ &= (1-\lambda)((2-3\lambda+\lambda^2)-1) \\ &= (1-\lambda)(1-3\lambda+\lambda^2) \end{aligned}$$

The roots and thus eigenvalues are $\lambda = 1$, as well as

$$\begin{aligned} \lambda &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(1)}}{2} \\ &= \frac{3}{2} \pm \frac{\sqrt{5}}{2} \end{aligned}$$

Particularly, for the eigenvalue $\lambda = \frac{3}{2} + \frac{\sqrt{5}}{2}$, the eigenvector is inferred from the homogeneous system

$$\left[\begin{array}{ccc|c} -\frac{1}{2} - \frac{\sqrt{5}}{2} & 3 & 1 & 0 \\ 0 & -\frac{1}{2} - \frac{\sqrt{5}}{2} & 0 & 0 \\ 1 & 0 & \frac{1}{2} - \frac{\sqrt{5}}{2} & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} - \frac{\sqrt{5}}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} - \frac{\sqrt{5}}{2} & 3 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2}-\frac{\sqrt{5}}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

whose general solution and hence the eigenvector is $(-\frac{1}{2}+\frac{\sqrt{5}}{2}, 0, 1)^T$ for $\lambda = \frac{3}{2}+\frac{\sqrt{5}}{2}$. Short Exercise: Find the eigenvectors for other remaining eigenvalues.

This section is ended with some notable properties of eigenvalue and eigenvector.

Properties 9.1.3. For a square matrix A,

1. A^T shares the same eigenvalues, but for each eigenvalue the eigenvector is not guaranteed to be the same,
2. The eigenvalues for the inverse A^{-1}, provided that it exists, are the reciprocals of the eigenvalues of A, but the eigenvectors are the same. This can be proved by starting with $A\vec{v}_\lambda = \lambda\vec{v}_\lambda$, and multiplying to the left on both sides by A^{-1}.

Cayley-Hamilton Theorem

Here we introduce an important theorem, Cayley-Hamilton Theorem, stating that, for every square matrix, it satisfies its own characteristic equation, which means that substituting the matrix into the characteristic polynomial as the variable results in a zero matrix.

Theorem 9.1.4. By Cayley-Hamilton Theorem, for any $n \times n$ square matrix A, if the characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = \sum_{k=0}^{n} c_k \lambda^k$$

then we have

$$p(A) = \sum_{k=0}^{n} c_k A^k = [\mathbf{0}]$$

which is a $n \times n$ zero matrix.

One may be tempted to substitute $\lambda = A$ into $\det(A - \lambda I)$ to prove the Cayley-Hamilton Theorem. However, since λ is a scalar but A is a matrix, it is not a rigorous proof. Correct proofs require advanced knowledge, which will not be presented here.

Example 9.1.3. With the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$$

verify the Cayley-Hamilton Theorem, and use the Cayley-Hamilton Theorem to evaluate $A^2 - 7A + 6I$.

The characteristic polynomial is

$$\begin{aligned} \begin{vmatrix} 1-\lambda & -1 \\ 3 & 5-\lambda \end{vmatrix} &= (1-\lambda)(5-\lambda) - (3)(-1) \\ &= 5 - 6\lambda + \lambda^2 + 3 \\ &= \lambda^2 - 6\lambda + 8 \end{aligned}$$

Replacing all λ^k terms in the characteristic polynomial with A^k (Notice that the constant term c_0 becomes $c_0 I$), we have

$$\begin{aligned} A^2 - 6A + 8I &= \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}^2 - 6\begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & -6 \\ 18 & 22 \end{bmatrix} + \begin{bmatrix} -6 & 6 \\ -18 & -30 \end{bmatrix} + \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

So Cayley-Hamilton Theorem holds in this case. We can quickly compute $A^2 - 7A + 6I$ by

$$\begin{aligned} A^2 - 7A + 6I &= (A^2 - 7A + 6I) - (A^2 - 6A + 8I) \\ &= -A - 2I \\ &= -\begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 1 \\ -3 & -7 \end{bmatrix} \end{aligned}$$

since $A^2 - 6A + 8I$ is a zero matrix.

9.2 Diagonalization

9.2.1 Ideas and Properties of Diagonalization

The properties of eigenvectors allow us to carry out diagonalization which helps us to solve linear algebra related problem. A matrix P is said to diagonalize another matrix A if the product $P^{-1}AP$ results in a diagonal matrix D, where non-zero entries are only found along the main diagonal.

Definition 9.2.1. A square matrix A is diagonalizable, if there exists some invertible square matrix P, such that

$$P^{-1}AP = D$$

where D is a diagonal matrix.

Particularly, the matrix P required for diagonalizing matrix A is formed by combining all the eigenvectors of A column by column. This is only possible if the amount of distinct eigenvectors is equal to the size of A.

Properties 9.2.2. A $n \times n$ square matrix A can be diagonalized by another matrix P, if A has n linearly independent eigenvectors, and the column vectors of P are those eigenvectors. A equivalent condition is that for every eigenvalue, its geometric multiplicity is equal to the algebraic multiplicity. The diagonal entries of $P^{-1}AP = D$, are the eigenvalues λ_j corresponding to the eigenvectors $\vec{v}_{\lambda_j}$ in the same column of P.

Proof Consider two matrix dot products AP and PD, where $P = [\vec{v}_{\lambda_1}|\cdots|\vec{v}_{\lambda_n}]$

$$\begin{aligned} AP &= A[\vec{v}_{\lambda_1}|\cdots|\vec{v}_{\lambda_n}] \\ &= [A\vec{v}_{\lambda_1}|\cdots|A\vec{v}_{\lambda_n}] \\ &= [\lambda_1\vec{v}_{\lambda_1}|\cdots|\lambda_n\vec{v}_{\lambda_n}] \end{aligned}$$

where we have used the Definition **??** and the second step can be compared to the last paragraph in Section **??**. Also

$$\begin{aligned} PD &= [\vec{v}_{\lambda_1}|\cdots|\vec{v}_{\lambda_n}]\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \\ &= [\lambda_1\vec{v}_{\lambda_1}|\cdots|\lambda_n\vec{v}_{\lambda_n}] \end{aligned}$$

So $AP = PD$. Notice that P is invertible by Theorem **??**, since P is made up of linearly independent eigenvectors, and thus $P^{-1}AP = D$.

The original matrix A and its diagonalization form $P^{-1}AP$ share some similarities sometimes called invariants.

Properties 9.2.3. If there are a diagonalizable matrix A, and its diagonalization form $D = P^{-1}AP$, then A and D have the

1. Same determinant,
2. Same trace,
3. Same eigenvalues,

4. Same characteristic equation.

Short Exercise: Prove the invariant property for determinant.

9.2.2 Diagonalization for Real Eigenvalues

For a diagonalizable matrix with real eigenvalues, diagonalization is straight forward by the use of Properties 9.2.2. Below shows a simple example.

Example 9.2.1. For the matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -2 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$$

It is given that its eigenvectors are $(1, 1, 0)^T, (1, 0, 1)^T, (0, 1, 1)^T$ for $\lambda = 2, 4, 6$ respectively. Concatenating them column by column yields

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The matrix product

$$\begin{aligned} D &= P^{-1}AP \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & -1 & 1 \\ -2 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 4 & 0 \\ 2 & 0 & 6 \\ 0 & 4 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \end{aligned}$$

This is expected from Properties 9.2.2.
Short Exercise: Confirm the provided eigenvalues and eigenvectors. Also, reverse the diagonalization to recover the original matrix.
Short Exercise: Verify the invariant properties for this example.

9.2.3 Diagonalization for Complex Eigenvalues

It is not uncommon for a real square matrix A to have complex eigenvalues. In such cases, to perform diagonalization, one possible approach is following what we have done in the last section to generate $D = P^{-1}AP$, which can be useful sometimes, but the downsides are that P and D are comprised of complex numbers, despite A being a real matrix. There is, indeed, another method uses the property of complex numbers introduced in the last chapter, that avoids the appearance of complex numbers. But first of all, we need to introduce a basic theorem about complex roots of an equation.

Theorem 9.2.4. For a real polynomial equation with order n

$$p(x) = \sum_{k=0}^{n} c_k x^k$$

If $x_0 = a + b\imath$ is a complex root so that $p(x_0) = \sum_{k=0}^{n} c_k x_0^k = 0$, where a and b are real constants, then $\overline{x_0} = a - b\imath$ is also a root for the equation.

Proof By Properties **??**, if we take the complex conjugate on both sides of the polynomial equation with $x = x_0$, then

$$\begin{aligned}
\overline{p(x_0)} &= \overline{\sum_{k=0}^{n} c_k x_0^k} = \overline{0} \\
p(\overline{x_0}) &= \sum_{k=0}^{n} c_k \overline{x_0}^k = 0
\end{aligned}$$

Since the characteristic equation is a real polynomial equation for the real matrix A, by the theorem we have just proved, we know that complex roots for the characteristic equation and hence complex eigenvalues always come in a conjugate pair. For a pair of complex eigenvalues, their eigenvectors are also the conjugate of each other.

Properties 9.2.5. If λ_0 is an eigenvalue for a real matrix A with an eigenvector of $\vec{v}_{\lambda_0}$, then A also has $\overline{\lambda_0}$ and $\overline{\vec{v}_{\lambda_0}}$ as another eigenvalue and eigenvector.

Proof By definition,

$$A\vec{v}_{\lambda_0} = \lambda_0\vec{v}_{\lambda_0}$$

Taking complex conjugate on both sides, we have

$$\begin{aligned}\overline{A\vec{v}_{\lambda_0}} &= \overline{\lambda_0\vec{v}_{\lambda_0}} \\ A\overline{\vec{v}_{\lambda_0}} &= \overline{\lambda_0}\,\overline{\vec{v}_{\lambda_0}}\end{aligned}$$

with the use of Properties **??**, and noting that $\overline{A} = A$ as A is a real matrix.

Now as we know that complex eigenvalues and eigenvectors always appear as a pair of complex conjugates, and conjugates share the same real and imaginary part except a sign difference, we are encouraged to use their real and imaginary part like two eigenvalues and eigenvectors for making up. The following theorem shows that it is possible to do so with some tweaks.

Theorem 9.2.6. The procedure in Properties 9.2.2 can be extended for complex eigenvalue $\lambda_0 = \mathrm{Re}\{\lambda_0\} + \imath\,\mathrm{Im}\{\lambda_0\}$, the corresponding eigenvector $\vec{v}_{\lambda_0} = \mathrm{Re}\{\vec{v}_{\lambda_0}\} + \imath\,\mathrm{Im}\{\vec{v}_{\lambda_0}\}$, as well as their complex conjugates, by replacing the corresponding columns in

$$P = [\cdots|\vec{v}_{\lambda_0}|\overline{\vec{v}_{\lambda_0}}|\cdots] \qquad\qquad D = \begin{bmatrix} \ddots & 0 & 0 & \\ & \lambda_0 & 0 & \\ & 0 & \overline{\lambda_0} & \\ & 0 & 0 & \ddots \end{bmatrix}$$

by

$$P = [\cdots|\operatorname{Re}\{\vec{v}_{\lambda_0}\}|\operatorname{Im}\{\vec{v}_{\lambda_0}\}|\cdots] \quad D = \begin{bmatrix} \ddots & 0 & 0 & \\ & \operatorname{Re}\{\lambda_0\} & -\operatorname{Im}\{\lambda_0\} & \\ & \operatorname{Im}\{\lambda_0\} & \operatorname{Re}\{\lambda_0\} & \\ & 0 & 0 & \ddots \end{bmatrix}$$

Proof As in the proof for Properties 9.2.2, we set to prove that $AP = PD$ for the columns concerned. To make it easier to read, denote $\operatorname{Re}\{\lambda_0\} = a$ and $\operatorname{Im}\{\lambda_0\} = b$. It is easy to see that

$$\begin{aligned} PD &= [\cdots|\operatorname{Re}\{\vec{v}_{\lambda_0}\}|\operatorname{Im}\{\vec{v}_{\lambda_0}\}|\cdots] \begin{bmatrix} \ddots & 0 & 0 & \\ & a & b & \\ & -b & a & \\ & 0 & 0 & \ddots \end{bmatrix} \\ &= [\cdots|a\operatorname{Re}\{\vec{v}_{\lambda_0}\} - b\operatorname{Im}\{\vec{v}_{\lambda_0}\}|b\operatorname{Re}\{\lambda_0\} + a\operatorname{Im}\{\vec{v}_{\lambda_0}\}|\cdots] \end{aligned}$$

Notice that the relation between eigenvalue and eigenvector can be expanded as

$$\begin{aligned} A\vec{v}_{\lambda_0} &= \lambda_0\vec{v}_{\lambda_0} \\ A(\operatorname{Re}\{\vec{v}_{\lambda_0}\} + \imath\operatorname{Im}\{\vec{v}_{\lambda_0}\}) &= (a + b\imath)(\operatorname{Re}\{\vec{v}_{\lambda_0}\} + \imath\operatorname{Im}\{\vec{v}_{\lambda_0}\}) \\ A\operatorname{Re}\{\vec{v}_{\lambda_0}\} + \imath A\operatorname{Im}\{\vec{v}_{\lambda_0}\} &= (a\operatorname{Re}\{\vec{v}_{\lambda_0}\} - b\operatorname{Im}\{\vec{v}_{\lambda_0}\}) + \imath(b\operatorname{Re}\{\vec{v}_{\lambda_0}\} + a\operatorname{Im}\{\vec{v}_{\lambda_0}\}) \end{aligned}$$

By equating real and imaginary parts, we have

$$A\operatorname{Re}\{\vec{v}_{\lambda_0}\} = a\operatorname{Re}\{\vec{v}_{\lambda_0}\} - b\operatorname{Im}\{\vec{v}_{\lambda_0}\}$$

$$A\,\mathrm{Im}\{\vec{v}_{\lambda_0}\} = b\,\mathrm{Re}\{\vec{v}_{\lambda_0}\} + a\,\mathrm{Im}\{\vec{v}_{\lambda_0}\}$$

Hence

$$\begin{aligned}
AP &= A[\cdots|\,\mathrm{Re}\{\vec{v}_{\lambda_0}\}|\,\mathrm{Im}\{\vec{v}_{\lambda_0}\}|\cdots] \\
&= [\cdots|A\,\mathrm{Re}\{\vec{v}_{\lambda_0}\}|A\,\mathrm{Im}\{\vec{v}_{\lambda_0}\}|\cdots] \\
&= [\cdots|a\,\mathrm{Re}\{\vec{v}_{\lambda_0}\} - b\,\mathrm{Im}\{\vec{v}_{\lambda_0}\}|b\,\mathrm{Re}\{\lambda_0\} + a\,\mathrm{Im}\{\vec{v}_{\lambda_0}\}|\cdots] \\
&= PD
\end{aligned}$$

The only caveat is to prove that P is invertible, or equivalently $\mathrm{Re}\{\vec{v}_{\lambda_0}\}$ and $\mathrm{Im}\{\vec{v}_{\lambda_0}\}$ are linearly independent. We can prove that by assuming they are linearly dependent, so $\mathrm{Re}\{\vec{v}_{\lambda_0}\} = k\,\mathrm{Im}\{\vec{v}_{\lambda_0}\}$ for some k, and plugging this into the expressions of $A\,\mathrm{Re}\{\vec{v}_{\lambda_0}\}$ and $A\,\mathrm{Im}\{\vec{v}_{\lambda_0}\}$ to derive a contradiction.

The block in the form

$$\begin{bmatrix}
\ddots & 0 & 0 & \\
 & a & b & \\
 & -b & a & \\
 & 0 & 0 & \ddots
\end{bmatrix}$$

generally represents a rotation by a degree of $\theta = \arctan(b/a)$. It means that a complex eigenvalue entails a rotation. This will be discussed more thoroughly in the next chapter.

Example 9.2.2. Using Theorem 9.2.6 to convert

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

into the form $D = P^{-1}AP$.
The characteristic equation is

$$\begin{bmatrix} 1-\lambda & 1 \\ -2 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 2$$

$$= \lambda^2 - 2\lambda + 3 = 0$$

The roots are

$$\begin{aligned}\lambda &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2} \\ &= 1 \pm \sqrt{2}\imath\end{aligned}$$

Since the eigenvectors occur in a conjugate pair, we only need to find one of them. We can find the eigenvector for $\lambda = 1 - \sqrt{2}\imath$, by solving the system

$$\left[\begin{array}{cc|c} \sqrt{2}\imath & 1 & 0 \\ -2 & \sqrt{2}\imath & 0 \end{array}\right] \to \left[\begin{array}{cc|c} \sqrt{2}\imath & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So the eigenvector for $\lambda = 1 - \sqrt{2}\imath$ is $(\imath, \sqrt{2})^T$, and the eigenvector for $\lambda = 1 + \sqrt{2}\imath$ is $(-\imath, \sqrt{2})^T$. Applying Theorem 9.2.6, with $a = 1$, $b = \sqrt{2}$, $\mathrm{Re}\{\vec{v}_{\lambda_0}\} = (0, \sqrt{2})^T$, $\mathrm{Im}\{\vec{v}_{\lambda_0}\} = (-1, 0)^T$, we have

$$\begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \sqrt{2} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ \sqrt{2} & 0 \end{bmatrix}$$

9.3 System of Ordinary Differential Equations

Ordinary Differential Equations appear frequently in the area of Physics and Earth Science. The easiest class of Ordinary Differential Equations is the first-order, linear, homogeneous Ordinary Differential Equations.

Definition 9.3.1. A first-order linear ordinary differential equation is a differential equation in the form of

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where x is the indepedent variable, $y(x)$ is the dependent variable, $P(x)$ and

$Q(x)$ is some function of x.

First-order means that the highest order derivative involved is the first derivative. Linearity means that there are no cross product terms between the dependent variable and its derivatives, e.g. $y\frac{dy}{dx}$, y^2. Finally, if it is of constant coefficients, and homogeneous, then it implies that $P(x)$ is a constant and $Q(x) = 0$ respectively.

Theorem 9.3.2. For a first-order, linear, constant coefficients, homogenenous ordinary differential equation in the form of

$$\frac{dy}{dx} = \beta y$$

where β is a constant, the general solution is

$$y(x) = ce^{\beta x}$$

with c as the integration constant. A direct substitution can verify the solution.

However, for some real-life problems, there are multiple ordinary differential equations that are related to each other to be solved. This especially happens in the area of Earth System Science where we have to consider the interactions, such as mass flow, between different units. This gives rise to a system of ordinary differential equations.

Definition 9.3.3. A system of ordinary differential equations that are first-order, linear, constant coefficients, and homogeneous has the form of

$$\begin{cases} dy_1/dx & = \alpha_1 y_1 + \beta_1 y_2 + \gamma_1 y_3 + \cdots \\ dy_2/dx & = \alpha_2 y_1 + \beta_2 y_2 + \gamma_2 y_3 + \cdots \\ dy_3/dx & = \alpha_3 y_1 + \beta_3 y_2 + \gamma_3 y_3 + \cdots \\ \cdots & = \cdots \end{cases}$$

and so on, up to dy_n/dx, or more compactly

$$\frac{dy_i}{dx} = \alpha_i y_1 + \beta_i y_2 + \gamma_i y_3 + \cdots$$

or in matrix notation

$$\mathbf{y}' = A\mathbf{y}$$

where $\alpha_i, \beta_i, \gamma_i, \cdots$ are constants, A is a square $n \times n$ matrix, and

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \mathbf{y}' = \begin{bmatrix} dy_1/dx \\ dy_2/dx \\ dy_3/dx \end{bmatrix}$$

$$A = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \cdots \\ \alpha_2 & \beta_2 & \gamma_2 & \\ \alpha_3 & \beta_3 & \gamma_3 & \\ \vdots & & & \ddots \end{bmatrix}$$

An example is the system

$$\begin{cases} dy_1/dx & = 3y_1 - y_2 \\ dy_2/dx & = 2y_1 \end{cases}$$

which can be rewritten into

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \mathbf{y}$$

Since each ordinary differential equations for dy_i/dx now involves multiple dependent variables y_j, where $j = 1, 2, 3, \cdots$, at the right hand side, we cannot directly use the result from Theorem 9.3.2 to solve the system, unless we can find a way to transform the system so that each equation is in terms of a single dependent variable only. Notice that if we make a change of variables such that $\mathbf{y} = P\mathbf{z}$ and hence $\mathbf{y}' = P\mathbf{z}'$, then the system $\mathbf{y}' = A\mathbf{y}$ becomes

$$P\mathbf{z}' = AP\mathbf{z}$$

$$\mathbf{z}' = (P^{-1}AP)\mathbf{z}$$

if we assume P is invertible. The term $P^{-1}AP$ immediately tells us a hint in the sense that it resembles a diagonalization described in 9.2.2. If $P^{-1}AP = D$, which is a diagonal matrix, then the system becomes $\mathbf{z}' = D$z, written more explicitly, is

$$\begin{cases} dz_1/dx & = D_{11}z_1 \\ dz_2/dx & = D_{22}z_2 \\ dz_3/dx & = D_{33}z_3 \\ \cdots & = \cdots \end{cases}$$

which are all solvable in their own. Subsequently, we have the following conclusion.

Theorem 9.3.4. For a system of ordinary differential equations in the form of

$$\mathbf{y}' = A\mathbf{y}$$

where A is a square $n \times n$ matrix with constant entries, it is solvable if A is diagonalizable, and we make the change of variables

$$\mathbf{y} = P\mathbf{z}$$

where P is the eigenvector matrix outlined in Properties 9.2.2. So that the system becomes

$$\mathbf{z}' = (P^{-1}AP)\mathbf{z} = D\mathbf{z}$$

After solving for $\mathbf{z}$, we can recover the required solution by computing $\mathbf{y} = P\mathbf{z}$.

Example 9.3.1. Solve the following system of differential equations.

$$\begin{cases} dy_1/dx &= \frac{5}{2}y_1 + y_2 + \frac{1}{2}y_3 \\ dy_2/dx &= \frac{3}{2}y_1 + 3y_2 - \frac{1}{2}y_3 \\ dy_3/dx &= \frac{3}{2}y_1 + y_2 + \frac{3}{2}y_3 \end{cases}$$

The system written in matrix notation is

$$\begin{bmatrix} dy_1/dx \\ dy_2/dx \\ dy_3/dx \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & 1 & \frac{1}{2} \\ \frac{3}{2} & 3 & -\frac{1}{2} \\ \frac{3}{2} & 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

We leave to the readers to check that the eigenvectors for the coefficient matrix are $(-1, 1, 1)^T$, $(1, -1, 1)^T$, $(1, 1, 1)^T$ for $\lambda = 1, 2, 4$. Hence by the theorem we have just derived, we can make the change of variables

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

so that the system becomes

$$\begin{aligned} \begin{bmatrix} dz_1/dx \\ dz_2/dx \\ dz_3/dx \end{bmatrix} &= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{5}{2} & 1 & \frac{1}{2} \\ \frac{3}{2} & 3 & -\frac{1}{2} \\ \frac{3}{2} & 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \end{aligned}$$

The solution in terms of z_i is $z_1 = c_1 e^x$, $z_2 = c_2 e^{2x}$, $z_3 = c_3 e^{4x}$. The integration constants c_i can be determined by the so-called initial condition. Assume that the initial condition reads $y_1 = 3$, $y_2 = 1$, $y_3 = 5$, for $x = 0$, then substitution into the change of variables gives

$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \end{bmatrix}$$

$$\begin{bmatrix}3\\1\\5\end{bmatrix}=\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$$

It is not hard to see that $c_1 = 1$, $c_2 = 2$, $c_3 = 2$, $z_1 = e^x$, $z_2 = 2e^{2x}$, $z_3 = 2e^{4x}$. So the full solution for y_i is

$$\begin{cases} y_1 &= -e^x + 2e^{2x} + 2e^{4x} \\ y_2 &= e^x - 2e^{2x} + 2e^{4x} \\ y_3 &= e^x + 2e^{2x} + 2e^{4x} \end{cases}$$

or in vector notation

$$\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=e^x\begin{bmatrix}-1\\1\\1\end{bmatrix}+e^{2x}\begin{bmatrix}2\\-2\\2\end{bmatrix}+e^{4x}\begin{bmatrix}2\\2\\2\end{bmatrix}$$

Short Exercise: Derive another solution if the initial condition is $y_1(x = 0) = 0$, $y_2(x = 0) = 4$, $y_3(x = 0) = 6$ instead.

Example 9.3.2. Solve the system $\mathbf{y}' = A\mathbf{y}$, where

$$A=\begin{bmatrix}1&1\\-2&1\end{bmatrix}$$

is given in Example 9.2.2.

From the previous work we know that the eigenvectors are $(\imath, \sqrt{2})^T$ and $(-\imath, \sqrt{2})^T$ for $\lambda = 1 - \sqrt{2}\imath, 1 + \sqrt{2}\imath$ respectively. In this situation it is advantageous to work with complex numbers as we will see soon. Similar to the example above, letting $\mathbf{y} = P\mathbf{z}$, where

$$P=\begin{bmatrix}\imath&-\imath\\\sqrt{2}&\sqrt{2}\end{bmatrix}$$

then we have

$$\mathbf{z}' = (P^{-1}AP)\mathbf{z} = D\mathbf{z}$$

with

$$D = \begin{bmatrix} 1-\sqrt{2}\imath & 0 \\ 0 & 1+\sqrt{2}\imath \end{bmatrix}$$

The solution in Theorem 9.3.2 is valid even when β is complex. Hence the solution for z will be

$$\begin{cases} z_1 &= c_1 e^{(1-\sqrt{2}\imath)x} = c_1 e^x \left(\cos\left(\sqrt{2}x\right) - \imath \sin\sqrt{2}x\right) \\ z_2 &= c_2 e^{(1+\sqrt{2}\imath)x} = c_2 e^x \left(\cos\left(\sqrt{2}x\right) + \imath \sin\sqrt{2}x\right) \end{cases}$$

where Euler's formula from Definition **??** is applied. Hence

$$\begin{aligned} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} \imath & -\imath \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} c_1 e^x \left(\cos\left(\sqrt{2}x\right) - \imath \sin\left(\sqrt{2}x\right)\right) \\ c_2 e^x \left(\cos\left(\sqrt{2}x\right) + \imath \sin\left(\sqrt{2}x\right)\right) \end{bmatrix} \\ &= \begin{bmatrix} (c_1+c_2)e^x \sin\left(\sqrt{2}x\right) + \imath(c_1-c_2)e^x\cos\left(\sqrt{2}x\right) \\ \sqrt{2}(c_1+c_2)e^x\cos\left(\sqrt{2}x\right) - \imath\sqrt{2}(c_1-c_2)e^x\sin\left(\sqrt{2}x\right) \end{bmatrix} \\ &= \begin{bmatrix} C_1 e^x \sin\left(\sqrt{2}x\right) + \imath C_2 e^x \cos\left(\sqrt{2}x\right) \\ \sqrt{2}C_1 e^x \cos\left(\sqrt{2}x\right) - \imath\sqrt{2}C_2 e^x \sin\left(\sqrt{2}x\right) \end{bmatrix} \end{aligned}$$

if we write $C_1 = c_1 + c_2$, $C_2 = c_1 - c_2$. Both the real and imaginary parts of $\mathbf{y}$ will satisfy the system, and they form the general solution. It is because A is a real matrix, hence we can rewrite

$$\begin{aligned} \vec{y}' &= A\vec{y} \\ \operatorname{Re}\left\{\vec{y}'\right\} + \imath \operatorname{Im}\left\{\vec{y}'\right\} &= A(\operatorname{Re}\{\vec{y}\} + \imath \operatorname{Im}\{\vec{y}\}) \\ \operatorname{Re}\left\{\vec{y}'\right\} + \imath \operatorname{Im}\left\{\vec{y}'\right\} &= A\operatorname{Re}\{\vec{y}\} + \imath A \operatorname{Im}\{\vec{y}\} \end{aligned}$$

Equating the real and imaginary parts gives

$$\begin{aligned}\vec{y}'_{Re} &= A\vec{y}_{Re}\\ \vec{y}'_{Im} &= A\vec{y}_{Im}\end{aligned}$$

So the final answer, expressed in real values, is

$$\begin{cases} y_1 &= C_1 e^x \sin\left(\sqrt{2}x\right) + C_2 e^x \cos\left(\sqrt{2}x\right) \\ y_2 &= \sqrt{2}C_1 e^x \cos\left(\sqrt{2}x\right) - \sqrt{2}C_2 e^x \sin\left(\sqrt{2}x\right) \end{cases}$$

where C_1, C_2 are to be decided by initial condition.

This will be the same solution basis we will obtain if we consider only one eigenvalue λ_0 out of the conjugate pair and compute $\vec{y} = \vec{v}_{\lambda_0} e^{\lambda_0 x}$.

9.4 Invariant Subspaces, Cayley-Hamilton Theorem

Remark If the coefficient matrix for a system of ordinary differential equation is not diagonalizable, then we need to employ other methods to solve the system. The most common method, generalized from diagonalization, is the Jordan Normal Form. Interested readers are invited to search about it.

9.5 Earth Science Applications

9.6 Python Programming

9.7 Exercises

Exercise 9.1 If a matrix A has a determinant of zero, show that it must have $\lambda = 0$ as an eigenvalue.

Exercise 9.2 Show that if the eigenvalues of a matrix A all have an algebraic multiplicity of 1, or in other words, no repeated root for the characteristic equation, then A is diagonalizable.

Exercise 9.3 Find the eigenvalues of the following matrices.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 4 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

as well as their transpose.

Exercise 9.4 Find the eigenvalues and corresponding eigenvectors for

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Repeat the calculation for its inverse.

Exercise 9.5 Find the eigenvalues and corresponding eigenvectors for

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

and perform Gram-Schmidt orthogonalization on the eigenvectors found.

Exercise 9.6 For the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

compute its characteristic polynomial. By applying Cayley-Hamilton Theorem,

(a) Find A^3,

(b) Prove

$$A^n = \begin{bmatrix} 1 & -(2^n - 1) & 2^n - 1 \\ 1 & 1 & -1 \\ 1 & -(2^n - 1) & 2^n - 1 \end{bmatrix}$$

for $n > 3$ by Mathematical Induction (Assume this holds for $n = k$, then prove for $n = k + 1$), and

(c) Prove inverse of A does not exist (Hint: Assume A^{-1} exists and multiply both sides by the inverse on the equation given by Cayley-Hamilton Theorem).

Exercise 9.7 Apply diagonalization for the following matrix, and prove that the characteristic polynomial remains unchanged, if possible.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 5 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Exercise 9.8 Solve the following system of ordinary differential equations.

$$\begin{cases} y_1' &= -y_1 + y_2 + 3y_3 \\ y_2' &= -2y_1 + 3y_2 + 2y_3 \\ y_3' &= -2y_1 + y_2 + 4y_3 \end{cases}$$

with the initial condition $y_1(0) = 3$, $y_2(0) = 2$, $y_3(0) = 9$.

Exercise 9.9 Given an idealized situation where there are three chemical gases P, Q, R. Denote their concentrations by $[P], [Q], [R]$ respectively. If they undergo reactions in a closed pathway $P \to Q \to R \to P$ that can be regarded as first-order, so that the governing equations about their concentrations are

$$\begin{cases} d[P]/dt &= k_{31}[R] - k_{12}[P] \\ d[Q]/dt &= k_{12}[P] - k_{23}[Q] \\ d[R]/dt &= k_{23}[Q] - k_{31}[R] \end{cases}$$

where k_{mn} are all constants. Derive the time evolution of the concentrations of the three gases. What happens when $t \to \infty$?

Answers to Exercises

Exercise 1.1

(a) $\begin{bmatrix} -3 & 5 \\ 3 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 8 & -\frac{1}{2} \\ 13 & -\frac{25}{2} \end{bmatrix}$

(c) $\begin{bmatrix} -8 & 17 \\ -18 & 8 \end{bmatrix}$

(d) $\begin{bmatrix} 11 & -11 \\ 33 & -11 \end{bmatrix}$

Exercise 1.2

(a) $\begin{bmatrix} -2 & 1 & 3 \\ -1 & -1 & -9 \\ -8 & 2 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} -8 & -5 \\ 15 & 3 \end{bmatrix}$

Exercise 1.3

(a) $\begin{bmatrix} 42 & 72 & 0 \\ 32 & 51 & -1 \end{bmatrix}$

(b) Same as above

(c) $\begin{bmatrix} 90 & 162 & 2 \\ 51 & 99 & 3 \end{bmatrix}$

(d) Same as above

Exercise 1.4

(a) $\begin{bmatrix} 16 & 23 & 129 \\ 133 & 33 & 102 \\ 27 & 9 & 128 \end{bmatrix}$

(b) $\begin{bmatrix} -\frac{233}{4} & -\frac{19}{4} & \frac{69}{2} \\ -\frac{339}{4} & -16 & 31 \\ \frac{109}{4} & \frac{33}{4} & -\frac{289}{4} \end{bmatrix}$

Exercise 1.5

(a) $\begin{bmatrix} 16 & 6 & 3 \\ 34 & 13 & 12 \\ 9 & 2 & 27 \end{bmatrix}$

(b) $\begin{bmatrix} 27 & 15 & 69 \\ 37 & 12 & 85 \\ 36 & 12 & 69 \end{bmatrix}$

(c) $\begin{bmatrix} 14 & 3 & 26 \\ 29 & 9 & 60 \\ 12 & 21 & 41 \end{bmatrix}$

(d) $\begin{bmatrix} 33 & 13 & 24 \\ 47 & 19 & 21 \\ 39 & 14 & 12 \end{bmatrix}$

Exercise 1.6

$$\begin{bmatrix} 0 & 3 & -4 \\ 5 & -1 & 2 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 13 \\ 8 \end{bmatrix}$$

or

$$\left[\begin{array}{ccc|c} 0 & 3 & -4 & 6 \\ 5 & -1 & 2 & 13 \\ 6 & 0 & 1 & 8 \end{array}\right]$$

Exercise 1.7

(a) $\begin{bmatrix} 2 & 3 & 5 & 7 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 8 & 12 \end{bmatrix}$

(b) (WIP)

Exercise 1.8 The air temperature/dew point at any height z before saturation is $T_a = T_{a,ini} - (\Gamma_{dry})z$ and $T_{dew} = T_{dew,ini} - (\Gamma_{dew})z$ respectively. At the condensation level $z = z_{cd}$, the air temperature equals to the dew point temperature $T_a = T_{dew} = T_{cd}$, and hence we have

$$T_{a,ini} - \Gamma_{dry}(z_{cd}) = T_{dew,ini} - \Gamma_{dew}(z_{cd}) = T_{cd}$$

which can be separated into two equations

$$\begin{cases} T_{a,ini} - \Gamma_{dry}(z_{cd}) & = T_{cd} \\ T_{dew,ini} - \Gamma_{dew}(z_{cd}) & = T_{cd} \end{cases}$$

Rearranging to put the unknowns z_{cd} and T_{cd} to L.H.S., we obtain

$$\begin{cases} T_{cd} + \Gamma_{dry}(z_{cd}) & = T_{a,ini} \\ T_{cd} + \Gamma_{dew}(z_{cd}) & = T_{dew,ini} \end{cases}$$

or, in matrix form

$$\begin{bmatrix} 1 & \Gamma_{dry} \\ 1 & \Gamma_{dew} \end{bmatrix} \begin{bmatrix} T_{cd} \\ z_{cd} \end{bmatrix} = \begin{bmatrix} T_{a,ini} \\ T_{dew,ini} \end{bmatrix}$$

Plugging in the lapse rates, we have

$$\begin{bmatrix} 1 & 9.8 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} T_{cd} \\ z_{cd} \end{bmatrix} = \begin{bmatrix} 25.4 \\ 17.8 \end{bmatrix}$$

Exercise 1.9 Obviously, there are 35 chickens and rabbits in total, and $x + y = 35$. Considering the total amount of legs, we also have $2x + 4y = 94$. Hence the required linear system is

$$\begin{cases} x + y & = 35 \\ 2x + 4y & = 94 \end{cases}$$

In matrix form, it is

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 35 \\ 94 \end{bmatrix}$$

Exercise 2.1 (Applying cofactor expansion along the leftmost column recursively) The determinant is just the product of the diagonal elements = $(1)(6)(10)(13)(15) = 11700$.

Exercise 2.2

(a) $\begin{bmatrix} 8 & 20 \\ 15 & 37 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} -\frac{37}{4} & \frac{15}{4} \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$

(c) $\begin{vmatrix} 8 & 15 \\ 20 & 37 \end{vmatrix} = -4 = (-1)(4) = \begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} \begin{vmatrix} 4 & 6 \\ 0 & 1 \end{vmatrix}$

Exercise 2.3

(a)

$$
\left[\begin{array}{ccc|ccc} 3 & 2 & 9 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 4 & 0 & 4 & 0 & 0 & 1 \end{array}\right]
$$

$$
\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 2 & 9 & 1 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \qquad R_1 \leftrightarrow R_2
$$

$$
\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & -4 & 0 & 1 & -3 & 0 \\ 0 & -8 & -8 & 0 & -4 & 1 \end{array}\right] \qquad R_2 - 3R_1 \longrightarrow R_2, R_3 - 4R_1 \longrightarrow R_3
$$

$$
\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & -\frac{1}{8} \end{array}\right] \qquad -\frac{1}{4}R_2 \longrightarrow R_2, -\frac{1}{8}R_3 \longrightarrow R_3
$$

$$
\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{8} \end{array}\right] \qquad R_3 - R_2 \longrightarrow R_3
$$

$$
\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{4} & \frac{3}{8} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{8} \end{array}\right] \qquad R_1 - 3R_3 - 2R_2 \longrightarrow R_1
$$

(b) $\det(A) = -32$ and

$$
\operatorname{adj}(A) = \begin{bmatrix} \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & 9 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 9 \\ 4 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 4 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & 9 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 9 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \end{bmatrix}^T
= \begin{bmatrix} 8 & 8 & -8 \\ -8 & -24 & 8 \\ -12 & 0 & 4 \end{bmatrix}^T
$$

$$= \begin{bmatrix} 8 & -8 & -12 \\ 8 & -24 & 0 \\ -8 & 8 & 4 \end{bmatrix}$$

Hence

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)}\mathrm{adj}(A) \\ &= -\frac{1}{32}\begin{bmatrix} 8 & -8 & -12 \\ 8 & -24 & 0 \\ -8 & 8 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{3}{8} \\ -\frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{8} \end{bmatrix} \end{aligned}$$

Exercise 2.4

(a) $\begin{bmatrix} 19 & 35 & 9 \\ 33 & 61 & 16 \\ 52 & 96 & 24 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 4 & 2 \\ 5 & 9 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 18 & -10 & 1 \\ -6 & 3 & 1 \\ -\frac{11}{4} & \frac{7}{4} & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 2 & -2 \\ -1 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 7 & -4 & 1 \\ -\frac{9}{2} & \frac{5}{2} & 0 \\ 2 & -1 & 0 \end{bmatrix}$

(c) $\begin{vmatrix} 19 & 33 & 52 \\ 35 & 61 & 96 \\ 9 & 16 & 24 \end{vmatrix} = -4 = (-2)(2) = \begin{vmatrix} 0 & 2 & 5 \\ 0 & 4 & 9 \\ 1 & 2 & 1 \end{vmatrix} \begin{vmatrix} 2 & 3 & 4 \\ 2 & 4 & 6 \\ 3 & 5 & 8 \end{vmatrix}$

Exercise 2.5 Either by evaluating the determinant to show that $|A| = 0$, or find its reduced row echelon form which is

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

and not equal to the identity.

Exercise 2.6

$$\det(A) = -42$$
$$\det\left(A^{-1}\right) = -\frac{1}{42}$$
$$A^{-1} = \begin{bmatrix} \frac{9}{7} & -\frac{3}{14} & -\frac{2}{7} & -\frac{6}{7} \\ -\frac{1}{21} & -\frac{1}{21} & \frac{1}{21} & \frac{1}{7} \\ -\frac{11}{7} & -\frac{15}{14} & \frac{11}{7} & \frac{5}{7} \\ \frac{3}{7} & \frac{3}{7} & -\frac{3}{7} & -\frac{2}{7} \end{bmatrix}$$

Exercise 2.7 By cofactor expansion along the first column, we can obtain the determinant of A as

$$|A| = 2p^2 + 4p - 16$$

which has two roots, $p = -4$ and $p = 2$ such that $|A| = 0$ and A is not invertible. All values of p other than $p = -4$ and $p = 2$ make A invertible.

Exercise 2.8 $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A^T + A$,
and $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$. We can split A into

$$\begin{aligned} A &= A + \frac{1}{2}(A^T - A^T) \\ &= \frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}A^T - \frac{1}{2}A^T \\ &= \frac{1}{2}A + \frac{1}{2}A^T + \frac{1}{2}A - \frac{1}{2}A^T \\ &= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) \end{aligned}$$

where the first term is symmetric and the second term is skew-symmetric.

Exercise 2.9

$$\begin{aligned}
A^{-1} &= \frac{1}{\det(A)}\operatorname{adj}(A) \\
\det\left(A^{-1}\right) &= \det\left(\frac{1}{\det(A)}\operatorname{adj}(A)\right) \qquad \text{(Notice that } \frac{1}{\det(A)} \text{ is now a scalar)} \\
\frac{1}{\det(A)} &= (\frac{1}{\det(A)})^n \det(\operatorname{adj}(A)) \\
\det(\operatorname{adj}(A)) &= (\det(A))^{n-1}
\end{aligned}$$

Exercise 3.1

$$A^{-1} = \begin{bmatrix} \frac{1}{21} & \frac{5}{21} & \frac{2}{21} \\ \frac{11}{42} & -\frac{29}{42} & \frac{1}{42} \\ \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{h} = \begin{bmatrix} \frac{1}{21} & \frac{5}{21} & \frac{2}{21} \\ \frac{11}{42} & -\frac{29}{42} & \frac{1}{42} \\ \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 6 \\ \frac{7}{2} \\ -\frac{13}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{3}{2} \end{bmatrix}$$

or

$$\left[\begin{array}{ccc|c} 5 & 1 & 3 & 6 \\ 2 & -1 & 1 & \frac{7}{2} \\ 3 & 2 & -4 & -\frac{13}{2} \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & \frac{1}{5} & \frac{3}{5} & \frac{6}{5} \\ 2 & -1 & 1 & \frac{7}{2} \\ 3 & 2 & -4 & -\frac{13}{2} \end{array}\right] \qquad \frac{1}{5}R_1 \to R_1$$

$$\to \left[\begin{array}{ccc|c} 1 & \frac{1}{5} & \frac{3}{5} & \frac{6}{5} \\ 0 & -\frac{7}{5} & -\frac{1}{5} & \frac{11}{10} \\ 0 & \frac{7}{5} & -\frac{29}{5} & -\frac{101}{10} \end{array}\right] \qquad R_2 - 2R_1 \to R_2, R_3 - 3R_1 \to R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & \frac{1}{5} & \frac{3}{5} & \frac{6}{5} \\ 0 & -\frac{7}{5} & -\frac{1}{5} & \frac{11}{10} \\ 0 & 0 & -6 & -9 \end{array}\right] \qquad R_3 + R_2 \to R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & \frac{1}{5} & \frac{3}{5} & \frac{6}{5} \\ 0 & 1 & \frac{1}{7} & -\frac{11}{14} \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right] \qquad -\frac{5}{7}R_2 \to R_2, -\frac{1}{6}R_3 \to R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & \frac{1}{5} & 0 & \frac{3}{10} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right] \qquad R_1 - \frac{3}{5}R_3 \to R_1, R_2 - \frac{1}{7}R_3 \to R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right] \qquad R_1 - \frac{1}{5}R_2 \to R_1$$

Exercise 3.2

$$A^{-1} = \begin{bmatrix} -\frac{1}{8} & 0 & \frac{7}{8} \\ \frac{3}{16} & -\frac{1}{2} & -\frac{5}{16} \\ \frac{1}{16} & \frac{1}{2} & -\frac{7}{16} \end{bmatrix}$$

$$\vec{x}_1 = A^{-1}\vec{h}_1 = \begin{bmatrix} -\frac{1}{8} & 0 & \frac{7}{8} \\ \frac{3}{16} & -\frac{1}{2} & -\frac{5}{16} \\ \frac{1}{16} & \frac{1}{2} & -\frac{7}{16} \end{bmatrix}\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

$$\vec{x}_2 = A^{-1}\vec{h}_2 = \begin{bmatrix} -\frac{1}{8} & 0 & \frac{7}{8} \\ \frac{3}{16} & -\frac{1}{2} & -\frac{5}{16} \\ \frac{1}{16} & \frac{1}{2} & -\frac{7}{16} \end{bmatrix}\begin{bmatrix} \frac{19}{4} \\ 1 \\ \frac{5}{4} \end{bmatrix}, \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{4} \end{bmatrix}$$

Exercise 3.3

$$\left[\begin{array}{ccc|c} 3 & 0 & 4 & 2 \\ 1 & 1 & 2 & -1 \\ 1 & -2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 1 & 1 & 2 & -1 \\ 3 & 0 & 4 & 2 \end{array}\right] \qquad R_1 \leftrightarrow R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 3 & 2 & -1 \\ 0 & 6 & 4 & 2 \end{array}\right] \qquad R_2 - R_1 \to R_2, R_3 - 3R_1 \to R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 6 & 4 & 2 \end{array}\right] \qquad \frac{1}{3}R_2 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 4 \end{array}\right] \qquad R_3 - 6R_2 \rightarrow R_3$$

The last row is inconsistent and the system has no solution.
Note: You may get, to the right of the last row, some number other than 4, but this is possible and not wrong. (Why?)

Exercise 3.4

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -3 & 2 \\ 1 & 0 & 0 & -1 & 5 \\ 3 & 2 & -2 & -7 & 9 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 5 \\ 1 & 1 & -1 & -3 & 2 \\ 3 & 2 & -2 & -7 & 9 \end{array}\right] \qquad R_1 \leftrightarrow R_2$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 5 \\ 0 & 1 & -1 & -2 & -3 \\ 0 & 2 & -2 & -4 & -6 \end{array}\right] \qquad R_2 - R_1 \rightarrow R_2, R_3 - 3R_1 \rightarrow R_3$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 5 \\ 0 & 1 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad R_3 - 2R_2 \rightarrow R_3$$

Let $p = s$, $q = t$ as the two free variables. Substituting them back into the equations, we have $m - t = 5$ and $n - s - 2t = -3$, hence $m = 5 + t$ and $n = -3 + s + 2t$, and

$$\begin{bmatrix} m \\ n \\ p \\ q \end{bmatrix} = \begin{bmatrix} 5+t \\ -3+s+2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Exercise 3.5 The determinant of the coefficient matrix can be found to be

$$\begin{vmatrix} 1 & 0 & \alpha \\ 0 & \alpha & 0 \\ \alpha & 0 & 1 \end{vmatrix} = -\alpha^3 + \alpha$$
$$= -\alpha(\alpha - 1)(\alpha + 1)$$

The system will have no solution or infinitely many of them only when the determinant equals to zero, which gives us three possible values of $\alpha = -1, 0, 1$. When $\alpha = -1$, the system is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right] \qquad R_3 + R_1 \to R_3$$

where the last row is inconsistent and there is no solution. When $\alpha = 0$, it becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

It is obvious that $x = z = 0$, and $y = t$ is a free variable, so the solution is infinitely many and is in the form of

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The last case, $\alpha = 1$, gives rise to the system of

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_3 - R_1 \to R_3$$

such that $y = 0$ and $z = t$ can be set to be a free variable and there are infinitely many solutions in the form of

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Exercise 3.6 The first two equations below come from the left inner loop and right inner loop, but one of them can be replaced by the outer loop as well.

$$\begin{aligned} -4I_1 + 6I_2 &= 6 \\ -6I_2 + 9I_3 &= -12 \\ I_1 + I_2 + I_3 &= 0 \end{aligned}$$

and the solution is $I_1 = -\frac{3}{19}$, $I_2 = \frac{17}{19}$, $I_3 = -\frac{14}{19}$ (in Amperes).

Exercise 3.7 Substituting the given wave solution forms into the equation, we have

$$\begin{aligned} &\omega\tilde{\eta}\sin(kx + ly - \omega t) + H(-k\tilde{U}\sin(kx + ly - \omega t) \\ &- l\tilde{V}\sin(kx + ly - \omega t)) = 0 \\ &\omega\tilde{U}\sin(kx + ly - \omega t) = gk\tilde{\eta}\sin(kx + ly - \omega t) \\ &\omega\tilde{V}\sin(kx + ly - \omega t) = gl\tilde{\eta}\sin(kx + ly - \omega t) \end{aligned}$$

Cancelling out all the sine factors, we arrive at the linear system displayed in the question

$$\begin{cases} \omega\tilde{\eta} - kH\tilde{U} - lH\tilde{V} &= 0 \\ \omega\tilde{U} - kg\tilde{\eta} &= 0 \\ \omega\tilde{V} - lg\tilde{\eta} &= 0 \end{cases}$$

For $\tilde{U}$, $\tilde{V}$, $\tilde{\eta}$ to have a non-trivial solution other than all zeros, we require the determinant of the corresponding coefficient matrix to be zero according to Theorem 3.1.2, which leads to

$$\begin{aligned} \begin{vmatrix} \omega & -kH & -lH \\ -kg & \omega & 0 \\ -lg & 0 & \omega \end{vmatrix} &= 0 \\ \omega^3 - gHk^2\omega - gHl^2\omega &= 0 \\ \omega^2 - gH(k^2 + l^2) &= 0 \end{aligned}$$

as the dispersion relation of gravity wave.

Exercise 3.8 $T_{cd} \approx 15.9\,°\mathrm{C}$, $z_{cd} \approx 0.97\,\mathrm{km}$.

Exercise 3.9 $x = 23$, $y = 12$. For the extra part, the new system of equations become (denote the number of third species as z)

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 48 \\ 122 \end{bmatrix}$$

By Gaussian Elimination, we have

$$\begin{aligned}
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 48 \\ 2 & 4 & 3 & 122 \end{array}\right] &\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 48 \\ 0 & 2 & 1 & 26 \end{array}\right] & & R_2 - 2R_1 \to R_2 \\
&\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 48 \\ 0 & 1 & \frac{1}{2} & 13 \end{array}\right] & & \frac{1}{2}R_2 \to R_2 \\
&\to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 35 \\ 0 & 1 & \frac{1}{2} & 13 \end{array}\right] & & R_1 - R_2 \to R_1
\end{aligned}$$

Let $z = t$ as the free variable, then we have $y = 13 - \frac{1}{2}t$ and $x = 35 - \frac{1}{2}t$, and hence

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35 - \frac{1}{2}t \\ 13 - \frac{1}{2}t \\ t \end{bmatrix} = \begin{bmatrix} 35 \\ 13 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Since the numbers of species must be a non-negative integer, the solution can be expressed in a more good-looking form of

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35 \\ 13 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

where $s = \frac{t}{2}$, and the range of s is $0, 1, \dots, 13$ (when s reaches 13 there is no chicken remained).

Exercise 4.1

(a) $(2, 5, 5, 12)^T$

(b) $(1, \frac{7}{2}, \frac{7}{2}, 8)^T$

(c) $(1)(1) + (3)(2) + (3)(2) + (7)(5) = 48$

(d) $(1)(1) + (2)(3) + (2)(3) + (5)(7) = 48$

(e) $\vec{u} - 2\vec{v} = (-1, -1, -1, -3)^T$, $2\vec{u} + \vec{v} = (3, 8, 8, 19)^T$, $(\vec{u} - 2\vec{v}) \cdot (2\vec{u} + \vec{v}) = (-1)(3) + (-1)(8) + (-1)(8) + (-3)(19) = -76$

Exercise 4.2

(a)

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 4 & 1 \\ 8 & 1 & 1 \end{vmatrix} = 3\hat{i} + \hat{j} - 25\hat{k} = (3, 1, -25)^T$$

$$\vec{v} \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & 1 & 1 \\ 7 & 4 & 1 \end{vmatrix} = -3\hat{i} - \hat{j} + 25\hat{k} = (-3, -1, 25)^T$$

(b)

$$\begin{aligned} A\vec{v} &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 1 \end{bmatrix} \\ \vec{u} \cdot (\vec{Av}) &= (7, 4, 1)^T \cdot (10, 2, 1)^T \\ &= (7)(10) + (4)(2) + (1)(1) \\ &= 79 \\ A^T\vec{u} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 12 \end{bmatrix} \\ (A^T\vec{u}) \cdot \vec{v} &= (7, 11, 12)^T \cdot (8, 1, 1)^T \\ &= (7)(8) + (11)(1) + (12)(1) \end{aligned}$$

$$= 79$$

(c) By (a), $\vec{u} \times \vec{v} = (3, 1, -25)^T$ and $(3\vec{u} - 4\vec{v}) = (-11, 8, -1)^T$, then

$$\begin{aligned}(3\vec{u} - 4\vec{v}) \cdot (\vec{u} \times \vec{v}) &= (-11, 8, -1)^T \cdot (3, 1, -25)^T \\ &= (-11)(3) + (8)(1) + (-1)(-25) = 0\end{aligned}$$

This makes sense as we have shown that $\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\vec{u} \times \vec{v}) = 0$, and therefore by distributive property $(\alpha\vec{u} + \beta\vec{v}) \cdot (\vec{u} \times \vec{v}) = 0$ for any α and β.

Exercise 4.3

(a)

$$\begin{aligned}\|\vec{u}\| &= \sqrt{1^2 + (-3)^2 + 9^2} = \sqrt{91} \\ \hat{u} &= (\frac{1}{\sqrt{91}}, -\frac{3}{\sqrt{91}}, \frac{9}{\sqrt{91}})^T \\ \|\vec{v}\| &= \sqrt{1^2 + (-2)^2 + 4^2} = \sqrt{21} \\ \hat{v} &= (\frac{1}{\sqrt{21}}, -\frac{2}{\sqrt{21}}, \frac{4}{\sqrt{21}})^T\end{aligned}$$

(b)

$$\begin{aligned}\vec{u} \cdot \vec{v} = (1)(1) + (-3)(-2) + (9)(4) &= 43 \\ \cos\theta = \frac{43}{\sqrt{21}\sqrt{91}} &\approx 0.9836 \\ \theta &\approx 0.181 \text{ rad}\end{aligned}$$

(c) $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 9 \\ 1 & -2 & 4 \end{vmatrix} = 6\hat{i} + 5\hat{j} + \hat{k} = (6, 5, 1)^T$

(d) $\vec{u} \cdot (\vec{u} \times \vec{v}) = (1, -3, 9)^T \cdot (6, 5, 1)^T = (1)(6) + (-3)(5) + (9)(1) = 0$,
$\vec{v} \cdot (\vec{u} \times \vec{v}) = (1)(6) + (-2)(5) + (4)(1) = 0$

Exercise 4.4

Typhoon Name	Time	Speed	Direction	Vector Form
Nuri	2008/08/22, 08:00	$13\,\mathrm{km\,h^{-1}}$	315°	$(-9.192, 9.192)$
Vicente	2012/07/24, 02:00	$18\,\mathrm{km\,h^{-1}}$	299°	$(-15.743, 8.727)$
Linfa	2015/07/09, 23:00	$15\,\mathrm{km\,h^{-1}}$	245°	$(-13.595, -6.339)$
Mangkhut	2018/09/16, 22:00	$25\,\mathrm{km\,h^{-1}}$	288°	$(-23.776, 7.725)$

Exercise 4.5

$$\begin{aligned}
\|\vec{u} + \vec{v}\|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \\
&= \|\vec{u}\|^2 + 2(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2 \\
&\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 \\
&= (\|\vec{u}\| + \|\vec{v}\|)^2
\end{aligned}$$

Exercise 4.6

$$\begin{aligned}
\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) + (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\
&= (\|\vec{u}\|^2 + 2(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2) + (\|\vec{u}\|^2 - 2(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2) \\
&= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2
\end{aligned}$$

Exercise 4.7 In Example 4.3.1, we have

$$\overrightarrow{F_{\text{cor}}} = (2\Omega(v \sin\varphi - w\cos\varphi))\hat{\imath} + (-2\Omega u \sin\varphi)\hat{\jmath} + (2\Omega u \cos\varphi)\hat{k}$$

and hence the rate of work done is

$$\overrightarrow{F_{\text{cor}}} \cdot \vec{v}$$

$$= [(2\Omega(v \sin\varphi - w\cos\varphi))\hat{\imath} + (-2\Omega u \sin\varphi)\hat{\jmath} + (2\Omega u \cos\varphi)\hat{k}] \cdot (u\hat{\imath} + v\hat{\jmath} + w\hat{k})$$
$$= (2\Omega(v\sin\varphi - w\cos\varphi))u + (-2\Omega u \sin\varphi)v + (2\Omega u\cos\varphi)w$$
$$= 2\Omega uv \sin\varphi - 2\Omega uw \sin\varphi - 2\Omega uv\sin\varphi + 2\Omega uw \sin\varphi = 0$$

Alternatively, note that $\overrightarrow{F_{\text{cor}}} = -2\overrightarrow{\Omega} \times \vec{v}$ and $(\Omega \times \vec{v}) \cdot \vec{v} = 0$ always holds.

Exercise 5.1

(a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -9 \\ 0 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{9}{2} \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$

Exercise 5.2

(a) Normal vector to the line is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Equation: $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 9 \end{bmatrix} \rightarrow x - y = -7$

(b) Normal vector to the plane is $\begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 8 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ -27 \\ -32 \end{bmatrix}$.

Equation: $\begin{bmatrix} 20 \\ -27 \\ -32 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ -27 \\ -32 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \rightarrow 20x - 27y - 32z = -25$

Exercise 5.3 Part 1: Choose $(0,0,2)^T$ as a reference point on the plane. Projection of the vector from $(0,0,2)^T$ to $(3,2,9)^T$: $(3-0)\hat{\imath}+(2-0)\hat{\jmath}+(9-2)\hat{k} = 3\hat{\imath}+2\hat{\jmath}+7\hat{k}$ onto the normal vector $\hat{\imath}+2\hat{\jmath}+5\hat{k}$ of the plane is

$$\frac{(3)(1)+(2)(2)+(7)(5)}{\sqrt{1^2+2^2+5^2}} = \frac{42}{\sqrt{30}}$$

which is the required distance.
Part 2: Choose $(0,1,2)^T$ as a reference point along the line. Find the projection of $(3,2,9)^T - (0,1,2)^T = 3\hat{\imath}+1\hat{\jmath}+7\hat{k}$ onto the direction vector $\hat{\jmath}+2\hat{k}$, which is

$$\frac{(3)(0)+(1)(1)+(7)(2)}{0^2+1^2+2^2}(\hat{\jmath}+2\hat{k}) = 3(\hat{\jmath}+2\hat{k}) = 3\hat{\jmath}+6\hat{k}$$

The displacement vector between the point and line (which is orthogonal to the line) is then $(3\hat{\imath}+1\hat{\jmath}+7\hat{k})-(3\hat{\jmath}+6\hat{k}) = 3\hat{\imath}-2\hat{\jmath}+\hat{k}$ and the required distance equals to $\sqrt{3^2+(-2)^2+1^2} = \sqrt{14}$.

Exercise 5.4 Using the hints, we have the distance as

$$\begin{aligned}
\frac{(\vec{v}-\vec{u})\cdot(\hat{l}\times\hat{m})}{\|\hat{l}\times\hat{m}\|} &= \frac{[(\vec{b}+\hat{m}t)-(\vec{a}+\hat{l}s)]\cdot(\hat{l}\times\hat{m})}{\|\hat{l}\times\hat{m}\|} \\
&= \frac{(\vec{b}-\vec{a})\cdot(\hat{l}\times\hat{m})+[\hat{m}\cdot(\hat{l}\times\hat{m})]t-[\hat{l}\cdot(\hat{l}\times\hat{m})]s}{\|\hat{l}\times\hat{m}\|}
\end{aligned}$$

Notice that $\hat{l}\times\hat{m}$ is orthogonal to both $\hat{l}$ and $\hat{m}$, and thus $\hat{l}\cdot(\hat{l}\times\hat{m}) = \hat{m}\cdot(\hat{l}\times\hat{m}) = 0$ both vanish. Therefore we are left with

$$\frac{(\vec{b}-\vec{a})\cdot(\hat{l}\times\hat{m})}{\|\hat{l}\times\hat{m}\|}$$

If $\vec{a}\cdot(\hat{l}\times\hat{m}) = \vec{b}\cdot(\hat{l}\times\hat{m})$, then the numerator $(\vec{b}-\vec{a})\cdot(\hat{l}\times\hat{m}) = 0$ equals to zero such that the two lines intersect. In this case, the values of s or t at the point of intersection ($\vec{u}=\vec{v}$) can be found by applying a cross product with $\hat{m}$ on $\vec{u} = \vec{a}+\hat{l}s = \vec{b}+\hat{m}s = \vec{v}$ and note that $\hat{m}\times\hat{m} = \vec{0}$, and hence

$$(\vec{a}+\hat{l}s)\times\hat{m} = (\vec{b}+\hat{m}s)\times\hat{m}$$

$$\vec{a} \times \hat{m} + s(\hat{l} \times \hat{m}) = \vec{b} \times \hat{m} + s(\hat{m} \times \hat{m}) = \vec{b} \times \hat{m} + s\vec{0}$$
$$s(\hat{l} \times \hat{m}) = (\vec{b} - \vec{a}) \times \hat{m}$$

s is then inferred from the scaling ratio of $(\vec{b} - \vec{a}) \times \hat{m}$ to $(\hat{l} \times \hat{m})$. t is found similarly.

Exercise 5.5

$$\frac{1}{2}\left\|\vec{a} \times \vec{b}\right\| = \frac{1}{2}\left\|\vec{b} \times \vec{c}\right\| = \frac{1}{2}\|\vec{c} \times \vec{a}\|$$
$$\rightarrow \frac{1}{2}\|\vec{a}\|\left\|\vec{b}\right\| \sin C = \frac{1}{2}\left\|\vec{b}\right\|\|\vec{c}\| \sin A = \frac{1}{2}\|\vec{c}\|\|\vec{a}\| \sin B$$
$$\rightarrow \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

where we divide the entire equality by $abc = \|\vec{a}\|\left\|\vec{b}\right\|\|\vec{c}\|$).

Exercise 5.6 It is just $\frac{1}{6}|(\vec{u} \times \vec{v}) \cdot \vec{w}|$.

Exercise 5.7

(a) $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & 1 & 5 \end{vmatrix} = 7\hat{i} + \hat{j} - 3\hat{k}$
Area $= \sqrt{7^2 + 1^2 + (-3)^2} = \sqrt{59}$

(b) Volume is the absolute value of $|\vec{u} \times \vec{v}| \cdot \vec{w} = |(7\hat{i} + \hat{j} - 3\hat{k}) \cdot (\hat{i} + 4\hat{j})| = |(7)(1) + (1)(4) + (-3)(0)| = 11$

(c) Volume = abs $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 5 \\ 1 & 5 & 4 \end{vmatrix} = 0.$
So the three vectors are co-planar.

Exercise 5.8

(a) The solution refers to the point $(1, 1, 0)$.

(b) By Gaussian Elimination, one possible form of general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{5}{3} \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{3} \\ -\frac{5}{3} \\ 1 \end{bmatrix}$$

Therefore, the solution space is a line parallel to $-\frac{1}{3}\hat{\imath} - \frac{5}{3}\hat{\jmath} + \hat{k}$ and passing through the point $(\frac{2}{3}, -\frac{5}{3}, 0)^T$.

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