Introduction to Linear Algebra

for Earth Science Students

Draft Version

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Preface

This is a Linear Algebra textbook specifically designed for students who study any Earth Science related major like Geophysics and Atmospheric Sciences. With these target readers in mind, we set out to provide them with a foundation in Linear Algebra that is adequate to deal with relevant Earth Science problems. We have avoided pedantic mathematical details in the book and are instead focusing on practical methods. Therefore, this book is not suitable for training mathematicians. In each chapter, we first discuss a selected Linear Algebra topic. Then we will move on to some Earth Science examples about that topic if possible. It is followed up by another section which demonstrates how coding in Python may help us solve these Linear Algebra problems. At the end of each chapter, a number of exercises are given for practices, and it can be done either by hand or programming. It is suggested that the readers install the newest version of Python via Anaconda.

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Chapter 1

Introduction to Matrices and Linear Systems

Although the Earth System is well-known to be filled with non-linear processes, we still benefit from learning how to work with linear systems, as which many Earth Science problems can be approximated. This actually works well in a number of cases. For example, in Atmosphere Sciences, we often consider what is called a *perturbation equation*, which assumes that variations from the mean state are small enough to neglect quadratic perturbed terms. *Matrices* are one of the most fundamental objects in Linear Algebra, and we are going to address the basic aspects related to them in the first chapter.

1.1 Definition and Operations of Matrices

1.1.1 Basic Structure of Matrices

Matrices are rectangular arrays of numbers, which can be real or complex. For now we will work with real numbers, and defer the treatment about complex matrices to later chapters. A matrix having m rows and n columns is called an $m \times n$ matrix. A matrix with the same number of rows and columns, i.e. $m = n$, is called a ***Square Matrix***. Below shows some examples of matrices.

$$\begin{bmatrix} 1 & 2 & -2 & 5 \\ 1 & 3 & \sqrt{3} & 7 \end{bmatrix}$$

A 2×4 real matrix.

$$\begin{bmatrix} 2-\imath \\ 0 \\ 1 \\ 3\imath \end{bmatrix}$$

A 4×1 complex matrix.

$$\begin{bmatrix} 3 & 2 & 9 \\ -4 & 0 & 4 \\ 5 & 2 & -3 \end{bmatrix}$$

A 3×3 real, square matrix.

Given any matrix A, its entry at row i and column j will be denoted as A_{ij}. For example,

$$A = \begin{array}{cccc} \text{Col 1} & & & \\ \left[\begin{matrix} 2 \\ 5 \end{matrix}\right. & \begin{matrix} 1 \\ -3 \end{matrix} & \left.\begin{matrix} 9 \\ 7 \end{matrix}\right] & \begin{matrix} \\ \text{Row 2} \end{matrix} \end{array} \qquad\qquad A_{21} = 5$$

Short Exercise: Find A_{13} and A_{22}.[1]

1.1.2 Matrix Operations

Addition and Subtraction

Addition and subtraction between two matrices A and B are carried out *element-wise*, which means that if $C = A \pm B$, then $C_{ij} = A_{ij} \pm B_{ij}$. This implies that the two matrix operands must be of the same shape, and addition/subtraction is not possible for two matrices with different extents. For instance, if we have

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 8 \\ 1 & 7 \end{bmatrix}$$

[1] $A_{13} = 9$, $A_{22} = -3$.

Then

$$
\begin{aligned}
A + B &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 8 \\ 1 & 7 \end{bmatrix} \\
&= \begin{bmatrix} 1+1 & 2+1 \\ 3+0 & 4+8 \\ 5+1 & 6+7 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 3 \\ 3 & 12 \\ 6 & 13 \end{bmatrix}
\end{aligned}
$$

Short Exercise: Find $A - B$.[2]

Scalar Multiplication

Multiplying a matrix by a constant constitutes a ***Scalar Multiplication***, in which all entries are multiplied by the scalar. It is illustrated in the example below.

$$
\begin{aligned}
A &= \begin{bmatrix} 2 & -5 & 6 \\ -1 & 4 & -3 \end{bmatrix} \\
3A &= 3\begin{bmatrix} 2 & -5 & 6 \\ -1 & 4 & -3 \end{bmatrix} \\
&= \begin{bmatrix} 3\times 2 & 3\times(-5) & 3\times 6 \\ 3\times(-1) & 3\times 4 & 3\times(-3) \end{bmatrix} \\
&= \begin{bmatrix} 6 & -15 & 18 \\ -3 & 12 & -9 \end{bmatrix}
\end{aligned}
$$

Short Exercise: Find $\frac{1}{4}A$.[3]

---

[2] $A - B = \begin{bmatrix} 0 & 1 \\ 3 & -4 \\ 4 & -1 \end{bmatrix}$

[3] $A/4 = \begin{bmatrix} 1/2 & -5/4 & 3/2 \\ -1/4 & 1 & -3/4 \end{bmatrix}$

Matrix Multiplication/Matrix Product

Meanwhile, multiplication between two matrices, commonly referred to as ***Matrix Multiplication/Matrix Product***, is not element-wise. It can be only carried out if the number of columns of the first matrix equals to the number of rows of the second matrix, let's say r, in other words, they are of the shape $m \times r$ and $r \times n$ respectively. The resulting matrix is of the shape $m \times n$, which means that the number of rows/columns of the output matrix follows the first/second input matrix respectively. The following two examples explain this requirement.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$$

Since the shapes of A and B are 2×3 and 3×1, the number of columns in A and the number of rows in B are both 3, and hence matrix product between them is possible. The resulting matrix will be of size 2×1.

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 6 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 1 \\ 0 & 5 \\ 3 & 2 \end{bmatrix}$$

As the number of columns in C is 4, which is not equal to the number of rows in D, 3, matrix product is undefined in this case. Now we are ready to see how the entries in matrix product is exactly computed.

Definition 1.1.1 (Matrix Product)**.** Given an $m \times r$ matrix A and an $r \times n$ matrix B, if we denote the matrix product between A and B as $C = AB$, then to calculate any entry in C at row i and column j, we select row i from the first matrix A and column j from the second matrix B. Subsequently, take the products between the r pairs of numbers from that row and column. Their sum will then be the required value of the entry, i.e.

$$C_{ij} = (AB)_{ij} = A_{i1}B_{ij} + A_{i2}B_{2j} + A_{i3}B_{3j} + \ldots + A_{ir}B_{rj}$$

$$= \sum_{k=1}^{r} A_{ik} B_{kj}$$

again, r is the number of columns/rows in the first/second matrix.

Example 1.1.1. Calculate the matrix product $C = AB$, where

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Solution. The output will be a 2×2 matrix. Using the definition, we have

$$\begin{aligned} C_{11} = (AB)_{11} &= A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} \\ &= (1)(1) + (3)(2) + (5)(3) = 22 \\ C_{12} = (AB)_{12} &= A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ &= (1)(4) + (3)(5) + (5)(6) = 49 \end{aligned}$$

Hence the entries along the first row of C will be 22 and 49. The remaining entries at the second row are found in a similar way, and the readers are encouraged to do this themselves. You should be able to get

$$C = \begin{bmatrix} 22 & 49 \\ 28 & 64 \end{bmatrix}$$

□

Matrix product has some important properties, listed as follows.

Properties 1.1.2. If A, B, C are some matrices having compatible shapes so that the matrix multiplication operations below are valid, then

$$\underbrace{A \cdots A}_{k \text{ times}} = A^k \qquad k\text{-th power of a (square) matrix}$$

$(AB)C = A(BC) = ABC$	Associative Property
$(A \pm B)C = AC \pm BC$	Distributive Property
$A(B \pm C) = AB \pm AC$	Distributive Property

Another important observation is that, in general $AB \neq BA$ even if the matrix products AB and BA are both well-defined, so they are not *commutative*. However, as we will see later, there are some distinct exceptions of this.

Example 1.1.2. Calculate $-2A + 3B$, where

$$A = \begin{bmatrix} 1 & 6 & 9 \\ 4 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 8 & 6 \\ -5 & 0 & 3 \end{bmatrix}$$

Solution.

$$\begin{aligned} -2A + 3B &= -2\begin{bmatrix} 1 & 6 & 9 \\ 4 & 4 & 6 \end{bmatrix} + 3\begin{bmatrix} 4 & 8 & 6 \\ -5 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -2 & -12 & -18 \\ -8 & -8 & -12 \end{bmatrix} + \begin{bmatrix} 12 & 24 & 18 \\ -15 & 0 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 12 & 0 \\ -23 & -8 & -3 \end{bmatrix} \end{aligned}$$

□

Example 1.1.3. Compute $(A + 3B)(2A - B)$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 0 \\ 4 & -1 \end{bmatrix}$$

Solution. Using distributive property in Properties 1.1.2, the expression is equivalent to

$$(A + 3B)(2A - B) = A(2A - B) + (3B)(2A - B)$$

$$
\begin{aligned}
&= A(2A) + A(-B) + (3B)(2A) + (3B)(-B) \\
&= 2A^2 - AB + 6BA - 3B^2
\end{aligned}
$$

Bear in mind that $AB \neq BA$. We calculate each term, which gives

$$
\begin{aligned}
A^2 &= \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \\
&= \begin{bmatrix} (1)(1) + (2)(3) & (1)(2) + (2)(5) \\ (3)(1) + (5)(3) & (3)(2) + (5)(5) \end{bmatrix} \\
&= \begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
AB &= \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 4 & -1 \end{bmatrix} \\
&= \begin{bmatrix} (1)(-2) + (2)(4) & (1)(0) + (2)(-1) \\ (3)(-2) + (5)(4) & (3)(0) + (5)(-1) \end{bmatrix} \\
&= \begin{bmatrix} 6 & -2 \\ 14 & -5 \end{bmatrix}
\end{aligned}
$$

Similarly, it is not difficult to obtain

$$
BA = \begin{bmatrix} -2 & -4 \\ 1 & 3 \end{bmatrix} \qquad B^2 = \begin{bmatrix} 4 & 0 \\ -12 & 1 \end{bmatrix}
$$

Hence the final answer will be

$$
\begin{aligned}
2A^2 - AB + 6BA - 3B^2 &= 2\begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix} - \begin{bmatrix} 6 & -2 \\ 14 & -5 \end{bmatrix} + 6\begin{bmatrix} -2 & -4 \\ 1 & 3 \end{bmatrix} - 3\begin{bmatrix} 4 & 0 \\ -12 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 14 & 24 \\ 36 & 62 \end{bmatrix} - \begin{bmatrix} 6 & -2 \\ 14 & -5 \end{bmatrix} + \begin{bmatrix} -12 & -24 \\ 6 & 18 \end{bmatrix} - \begin{bmatrix} 12 & 0 \\ -36 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -16 & 2 \\ 64 & 82 \end{bmatrix}
\end{aligned}
$$

□

Alternatively, one can evaluate $C = A+3B$ and $D = 2A-B$ first, and subsequently calculate the matrix dot product CD. (This is actually somehow easier.) The readers should try this as an exercise.

Matrix Equation Manipulation

For any matrix equation, one can do addition, subtraction and multiplication on both sides of the equation. However, one important note is that multiplying a matrix to an equation requires the matrix to be inserted to the left (or right) on both sides, respecting the order. So, for a matrix equation like (assuming the shapes are compatible),

$$AB - C = DE + F$$

if we want to take the product with a matrix G, then it can be

$$\begin{aligned} G(AB - C) &= G(DE + F) \\ (AB - C)G &= (DE + F)G \end{aligned}$$

but we have, in general

$$\begin{aligned} G(AB - C) &\neq (DE + F)G \\ (AB - C)G &\neq G(DE + F) \end{aligned}$$

Taking successive matrix products follows the same principle, step by step. Using the example above, for another matrix H, we note some possible outcomes.

$$\begin{aligned} HG(AB - C) &= HG(DE + F) \\ (AB - C)GH &= (DE + F)GH \\ G(AB - C)H &= G(DE + F)H \end{aligned}$$

However, be careful that cancellation at both sides may not be correct. If $AB = AC$, then we cannot conclude that $B = C$ for sure, although it is not impossible (and happens quite often).

1.2 Definition of Linear Systems of Equations

The prime application of matrices is to deal with ***Linear Systems (of Equations)***. To understand what a linear system is, we first have to know the definition of a ***Linear Equation*** (possibly in multiple variables, let's say x_1, x_2, ...). For a linear equation, in any additive term, there is at most one variable or unknown, with a power of one, like $3x_1$, $-x_2$. This means that there are no cross-product terms such as $2x_1x_2$, variables with a power that is not one, like x_1^3, or non-linear functions, including $\sin x_1$, e^{x_2}. For n variables, a linear equation has the following definition.

Definition 1.2.1 (Linear Equation)**.** A linear equation has the form of

$$\sum_{j=1}^{n} a_j x_j = a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = h$$

where $x_1, x_2, \ldots$ are the unknowns, while $a_1, a_2, \ldots$ and h are some constants. If $h = 0$, then it is known as a ***Homogeneous Linear Equation***.

Short Exercise: Determine whether the equations below are (a) linear, and if they are linear, then (b) homogeneous or not.[4]

1. $3x + 4y = 5$
2. $\cos x + \ln y = 0$
3. $7x - 5z = 2$
4. $x^2 + y^{-3/2} = 1$
5. $x + 3y + 6z = 0$
6. $xyz = 8$

A system of linear equations are then simply a family of m linear equations, $m \geq 1$.

Definition 1.2.2 (Linear System of Equations)**.** A linear system of size $m \times n$

[4] Linear/Inhomogeneous, Non-linear, Linear/Inhomogeneous, Non-linear, Linear/Homogeneous, Non-linear.

has the form of

$$\begin{cases} \sum_{j=1}^{n} a_j^{(1)} x_j = a_1^{(1)} x_1 + a_2^{(1)} x_2 + a_3^{(1)} x_3 + \cdots + a_n^{(1)} x_n & = h^{(1)} \\ \sum_{j=1}^{n} a_j^{(2)} x_j = a_1^{(2)} x_1 + a_2^{(2)} x_2 + a_3^{(2)} x_3 + \cdots + a_n^{(2)} x_n & = h^{(2)} \\ \vdots & \\ \sum_{j=1}^{n} a_j^{(m)} x_j = a_1^{(m)} x_1 + a_2^{(m)} x_2 + a_3^{(m)} x_3 + \cdots + a_n^{(m)} x_n & = h^{(m)} \end{cases}$$

If $h^{(1)}, h^{(2)}, \ldots$ at the right hand side are all zeros, then the system is called a ***Homogeneous Linear System (of Equations)***. It is not hard to see that for any homogeneous linear system, it always has a trivial solution of $x_j = 0$ for $j = 1, \ldots, n$, or expressed as $\vec{x} = \mathbf{0}$. However, such trivial solution may not be the only solution to the system, as we shall see in !???.

Below shows some examples of linear systems.

$$\begin{cases} 3x + 4y & = 5 \\ 7x + 9y & = 13 \end{cases}$$

A 2×2 linear system with two equations, two unknowns.

$$\begin{cases} x + 2y - 4z & = 3 \\ x - y + 3z & = -4 \end{cases}$$

A 2×3 linear system with two equations, three unknowns.

$$\begin{cases} x + 2y + 3z & = 0 \\ 2x + 3z & = 0 \\ 4x - 5y & = 0 \end{cases}$$

A 3×3 homogeneous linear system (homogeneous as the R.H.S. are all zeros), notice the coefficients for some unknowns in some equations are zeros as well, e.g. in the second equation y has a coefficient of zero and does not appear.

The above formulation of a linear system closely resembles a tabular structure. Therefore, we are motivated to represent such systems with the language of matrices. Indeed, it is possible to rewrite an $m \times n$ linear system as $A\vec{x} = \vec{h}$, where A is an $m \times n$ matrix with entries copied from the coefficients in front of the variables arranged like those in Definition 1.2.2. In this book sometimes we will call it a *coefficient matrix*. Meanwhile, $\vec{x}$ is a column vector (an $n \times 1$ matrix) holding the n unknowns, and $\vec{h}$ is another column vector (an $m \times 1$ matrix) that contains the m constants at the right hand side of the linear system.

Properties 1.2.3. For a linear system like that in Definition 1.2.2, it can be rewritten as $A\vec{x} = \vec{h}$, where $A_{ij} = a_j^{(i)}$, $\vec{x} = x_j$, and $\vec{h} = h^{(i)}$.

Using the second example above as an illustration, we can easily verify that

$$\begin{cases} x + 2y - 4z & = 3 \\ x - y + 3z & = -4 \end{cases}$$

can be expressed as (you should check it yourself)

$$\begin{bmatrix} 1 & 2 & -4 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

An even simpler representation is the ***Augmented Matrix*** which omits the unknowns and concatenates the remaining matrices.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ 1 & -1 & 3 & -4 \end{array}\right]$$

1.3 Elementary Row Operations

When we construct a matrix, it is natural to think about how to utilize its structure. ***Elementary Row Operations*** provide such possibility in three ways, outlined in the following definition.

Definition 1.3.1 (Elementary Row Operations)**.** The three types of elementary row operations are

1. Multiplying a row R_p by any constant $c \neq 0$.
2. Adding a row R_q times any constant $c \neq 0$, to another row R_p, such that the new row becomes $R_p + cR_q$
3. Swapping a row R_p with another row R_q.

To facilitate the operations, we mark these three actions using the following notations.

1. $cR_p \to R_p$,
2. $R_p + cR_q \to R_p$,
3. $R_p \leftrightarrow R_q$

For example, the matrix A

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 11 \end{bmatrix}$$

can be transformed to a new matrix A'

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 5 \end{bmatrix}$$

if we apply the elementary row operation, subtracting $2R_1$ from R_2 (i.e. $R_2 - 2R_1 \to R_2$).
Short Exercise: Find out the resulting matrix A'' if we multiply the first row of A' by 3 and then subtract the second row from the first row.[5]

Attentive readers may have noticed that these three operations are what we have been always doing to equations when solving a linear system, as taught in high

5 $\begin{bmatrix} 0 & 3 & 4 \\ 3 & 3 & 5 \end{bmatrix}$

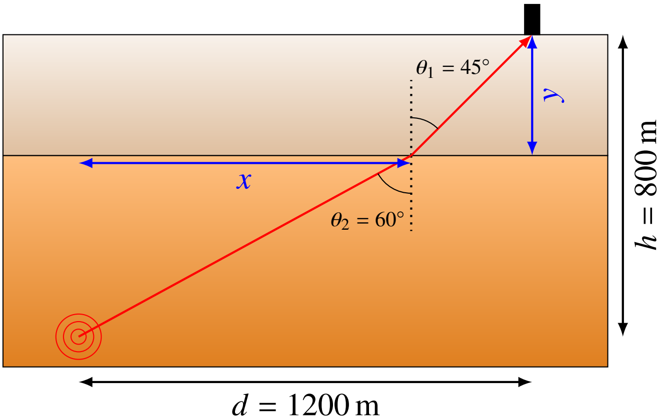


Figure 1.1: The underground schematic for the seismic ray in Example 1.4.1.

school. We re-introduce them as elementary row operations here first as they are fundamental to the treatment of later chapters.

1.4 Earth Science Applications

Example 1.4.1. Seismic wave follows *Snell's Law* like a light ray when it comes to refraction. Assuming the ground can be modelled as a two-layer system (see Figure 1.1), and we know a particular train of seismic wave generated from an underground source that reaches the ground receiver travels at an angle of $\theta_1 = 45°/\theta_2 = 60°$ to the vertical at the top/bottom layer. Given that the horizontal and vertical distance between the seismic source and the surface receiver are $d = 1200$ m and $h = 800$ m. Construct a linear system for this situation in two unknowns: the depth of the top layer y and the horizontal displacement x (in meters) where the wave reaches at the interface relative to the source.

Solution. We can deduce two equations from the given information. Consider the upper portion of the seismic ray, from basic trigonometry, we know that

$$\begin{aligned} \frac{d-x}{y} &= \tan\theta_1 \\ d-x &= (\tan\theta_1)y \\ x+(\tan\theta_1)y &= d \end{aligned}$$

Similarly, for the lower portion of the seismic ray, we have

$$\begin{aligned} \frac{x}{h-y} &= \tan\theta_2 \\ x &= (\tan\theta_2)h-(\tan\theta_2)y \\ x+(\tan\theta_2)y &= (\tan\theta_2)h \end{aligned}$$

The corresponding linear system is

$$\begin{cases} x+(\tan\theta_1)y &= d \\ x+(\tan\theta_2)y &= (\tan\theta_2)h \end{cases}$$

where x and y are the unknowns to be solved. d, h, θ_1 and θ_2 (and hence $\tan\theta_1$ and $\tan\theta_2$) are constants. Expressing the system in matrix form, we have

$$\begin{bmatrix} 1 & \tan\theta_1 \\ 1 & \tan\theta_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d \\ (\tan\theta_2)h \end{bmatrix}$$

Substituting the provided values for the constants ($\tan\theta_1 = \tan(45°) = 1$, $\tan\theta_2 = \tan(60°) = \sqrt{3}$), we have

$$\begin{bmatrix} 1 & 1 \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1200 \\ 800\sqrt{3} \end{bmatrix}$$

□

Example 1.4.2. The radiation transfer across the atmosphere of any planet (including the Earth) in the Solar system can be compared to a *multi-layer model* with fully absorbing layers (note that it is just a simplistic approach). Assume there are N such layers and the total rate of incident Solar radiation reaching the surface is E_{in}. Each of the layers also emits radiation to the other layers directly above/below itself. The rate of emission for the j-th layer that has a temperature T_j is $E_j = \sigma T_j^4$ according to the *Stefan–Boltzmann Law*, with $\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$. The overall scenario can be seen in Figure 1.2. Formulate a linear system that represents the energy equilibrium (incoming radiation = outgoing radiation) of all layers and the surface, with E_j being the unknowns, over $j = 1, 2, \ldots, N, N+1$.

Solution. Considering the energy equilibrium for the first (topmost) layer, we have

$$-2E_1 + E_2 = 0$$

Going down to the second layer, it is

$$E_1 - 2E_2 + E_3 = 0$$

In general, for the j-th layer in the middle, where j runs from 2 to N, we can similarly obtain

$$E_{j-1} - 2E_j + E_{j+1} = 0$$

Finally, for the surface (the $N+1$-th layer), we have

$$\begin{aligned} E_N - E_{N+1} + E_{in} &= 0 \\ E_N - E_{N+1} &= -E_{in} \end{aligned}$$

Summarizing all the $N+1$ equations, they can be expressed in matrix form as

$$\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & & 0 & 0 & 0 \\ 0 & 1 & -2 & & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & & -2 & 1 & 0 \\ 0 & 0 & 0 & & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_{N-1} \\ E_N \\ E_{N+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -E_{in} \end{bmatrix}$$

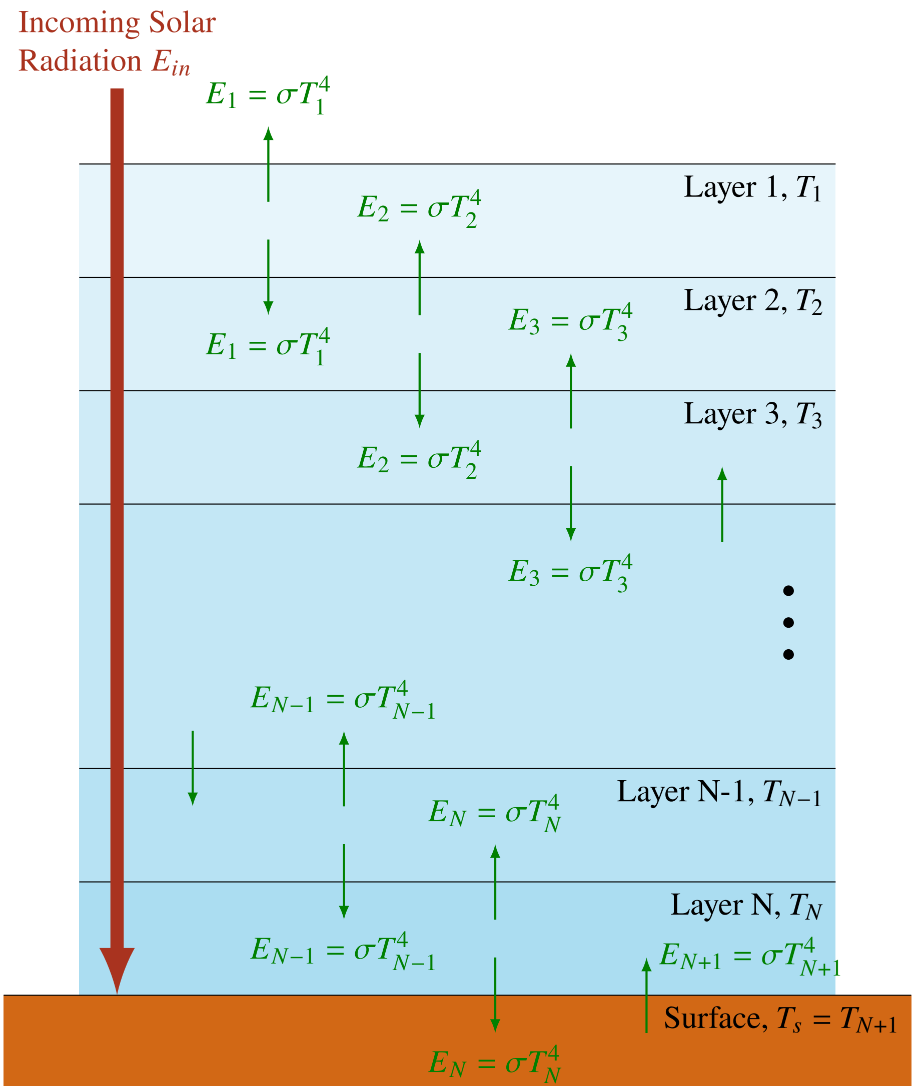


Figure 1.2: The atmospheric profile with multiple (N) absorbing layers in Example 1.4.2. The surface is treated as an extra N+1-th layer.

Particularly, for $N = 4$, it is

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -E_{in} \end{bmatrix}$$

□

We will talk about how to solve the linear systems in these two examples in Section 3.3.

1.5 Python Programming

We will use the package `numpy` and `scipy` throughout the book to solve linear algebra problems via *Python* programming. First, we can define a 2D `numpy` array that works as a matrix.

```
import numpy as np
myMatrix1 = np.array([[1, 4], [5, 3]])
print(myMatrix1)
```

which gives

```
[[1 4]
 [5 3]]
```

representing the matrix

$$\begin{bmatrix} 1 & 4 \\ 5 & 3 \end{bmatrix}$$

We can similarly define another matrix

```
myMatrix2 = np.array([[1, 3], [5, 6]])
```

Addition, subtraction, and scalar multiplication are straight-forward.

```
myMatrix3 = 3*myMatrix1 - 4*myMatrix2
print(myMatrix3)
```

The above code produces

```
[[ -1   0]
 [ -5 -15]]
```

and you can verify the answer by hand. Meanwhile, matrix product is done by the function `np.matmul()`.

```
myMatrix4 = np.matmul(myMatrix1, myMatrix2) # or equivalently
   myMatrix1 @ myMatrix2
print(myMatrix4)
```

gives

```
[[21 27]
 [20 33]]
```

To select a specfic entry, use indexing by square brackets. The first index/second index represents row/column. Beware that each index starts at zero in *Python*. So putting the number 1 in the first/second index actually means the second row/column. So

```
print(myMatrix4[1,0])
```

refers to the entry at row 2, column 1 of `myMatrix4` which is 20. Also, we can select the i-th row (or the j-th column) by `<Matrix>[i-1, :]` (`<Matrix>[:, j-1]`), where the colon `:` implies selecting along the entire row (column). For example,

```
print(myMatrix3[0,:])
print(myMatrix4[:,1])
```

gives `[-1  0]` and `[27 33]` respectively. Now let's see how to perform elementary row operations. It will be easier and less error-prone if we copy the array before performing the operations.

```
myMatrix5 = np.copy(myMatrix4)
myMatrix5[0,:] = myMatrix5[0,:]/3
print(myMatrix5)
```

The lines above, when executed, divide the second row of `myMatrix5` (which is a copy of `myMatrix4`) by 3, and give

```
[[ 7  9]
 [20 33]]
```

Meanwhile, the subsequent lines below

```
myMatrix5[1,:] = myMatrix5[1,:] - 2*myMatrix5[0,:]
print(myMatrix5)
```

proceed to subtract 2 times the first row from the second row, and produce

```
[[ 7  9]
 [ 6 15]]
```

Row interchange is a bit more tricky.

```
myMatrix6 = np.copy(myMatrix4)
myMatrix6[[0, 1],:] = myMatrix6[[1, 0],:]
```

This swaps the first and second row. (You can swap columns in a similar way.) Printing out the new matrix by `print(myMatrix6)` shows

```
[[20 33]
 [21 27]]
```

An important pitfall is that, since our inputs to `np.array` are all integers, the previous arrays will automatically have a data type of `int` (integer). This may produce unexpected errors when the calculation leads to decimals/fractions. If it is the case, then we can avoid such bugs by declaring the array with the keyword `dtype=float` to use *floating point numbers*, like

```
myMatrix1 = np.array([[1, 4], [5, 3]], dtype=float)
```

when printed out via `print(myMatrix1)` it gives

```
[[1. 4.]
 [5. 3.]]
```

Notice the newly appeared decimal points after the original integers. Alternatively, we can add decimal points to the integer entries during the array declaration, as

```
myMatrix1 = np.array([[1., 4.], [5., 3.]])
```

1.6 Exercises

Exercise 1.1 Let

$$A = \begin{bmatrix} 1 & 2 \\ 5 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 3 \\ -2 & 7 \end{bmatrix}$$

Find:

(a) $A + B$,

(b) $2A - \frac{3}{2}B$,

(c) AB,

(d) BA.

Exercise 1.2 Let

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -1 \\ 4 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 & -2 \\ -2 & 1 & 3 \end{bmatrix}$$

Find:

(a) AB,

(b) BA.

Exercise 1.3 Let

$$A = \begin{bmatrix} 4 & 6 \\ 3 & 3 \end{bmatrix}$$
$$B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$
$$C = \begin{bmatrix} 3 & 9 & 1 \\ 4 & 3 & -1 \end{bmatrix}$$

Find:

(a) $(A + B)C$,

(b) $AC + BC$,

(c) $(AB)C$,

(d) $A(BC)$.

Exercise 1.4 Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 7 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 & -2 \\ 4 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

Find:

(a) $(A + B)(2A - B)$,

(b) $(\frac{3}{2}A - B)(-A + \frac{1}{2}B)$.

Exercise 1.5 Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 6 \\ 5 & 2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 3 & 8 \\ 4 & 0 & 7 \end{bmatrix}$$

Find:

(a) A^2,

(b) B^2,

(c) AB,

(d) BA.

Exercise 1.6 Let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

Show that $AB = BA$ in this case.

Exercise 1.7 Rewrite the following system of linear equations in matrix form.

$$\begin{cases} 3y - 4z &= 6 \\ 5x - y + 2z &= 13 \\ 6x + z &= 8 \end{cases}$$

Exercise 1.8 For the following matrix,

$$\begin{bmatrix} 2 & 3 & 5 & 7 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 6 & 10 \end{bmatrix}$$

Find the results if the following elementary row operations are applied on it:

(a) Multiplying the third row by 2, and then subtracting the third row by the second row,

(b) Adding the first row by 3 times the third row, and then interchanging the first and second row.

Exercise 1.9 For the following matrix,

$$\begin{bmatrix} 3 & 0 & 4 & 6 & 9 \\ 5 & 3 & 8 & -1 & 3 \\ 2 & 5 & 4 & 3 & -7 \end{bmatrix}$$

Find the elementary row operations needed to reduce the matrix to

$$\begin{bmatrix} 1 & \frac{5}{2} & 2 & \frac{3}{2} & -\frac{7}{2} \\ 5 & \frac{21}{2} & 10 & -\frac{5}{2} & -\frac{33}{2} \\ 3 & 0 & 4 & 6 & 9 \end{bmatrix}$$

Exercise 1.10 The *dry adiabatic lapse rate*, which is the rate of decrease in air temperature when an unsaturated air parcel rises, is about $\Gamma_{dry} = 9.8\,°\mathrm{C\,km^{-1}}$. When the temperature of the air parcel falls below the *dew point*, the air saturates and condensation occurs. Typically, dew point temperature of an air parcel will decrease at a rate of roughly $\Gamma_{dew} = 2\,°\mathrm{C\,km^{-1}}$. Now, an air parcel with an initial air temperature/dew point temperature of $T_{a,ini} = 25.4\,°\mathrm{C}$ / $T_{dew,ini} = 17.8\,°\mathrm{C}$ at the ground starts to rise. Let z_{cd} and T_{cd} be the height above the ground (in km) and temperature (in °C) of the air parcel when condensation occurs. Construct a linear system with z_{cd} and T_{cd} as the unknowns to represent this situation.

Exercise 1.11 In some ancient Chinese Mathematics texts, the problem of *Chickens and Rabbits in the Same Cage* was posed. *"Now there are some chickens and rabbits placed in the same cage, with a total number of* 35 *heads and* 94 *legs. How many chickens and rabbits are there respectively?"* Given the fact that a chicken (rabbit) has two (four) legs (and obviously only one head), write down the corresponding linear system in terms of the numbers of chickens x and rabbits y.

Chapter 2

Inverses and Determinants

In this chapter, we are going to discuss two important concepts about matrices, which are the *Inverse* and *Determinant*. To derive them, we need to introduce some prerequisite ideas first, including *Identity*, *Transpose*, and the methods of *Gaussian Elimination* and *Laplace Expansion*.

2.1 Identity Matrices and Transpose

2.1.1 Identity Matrices

One important type of matrices is the ***Identity Matrices***. They are $n \times n$ square matrices, where n can be any positive integer, with entries along the ***Main Diagonal*** (where index of row = column) being 1 and other off-diagonal elements being 0. Usually, they are denoted by I_n, or simply I.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad\qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity matrices of size 2×2 and 3×3 with the main diagonal 1s highlighted.

Definition 2.1.1 (Identity Matrix)**.** An identity matrix of the square shape $n \times n$ I_n is defined as $[I_n]_{ij} = 1$, for $i = j$, and $[I_n]_{ij} = 0$, for $i \neq j$, where $1 \leq i, j \leq n$.

Short Exercise: Explicitly write down I_5.[1]

One important property of identity matrices is

Properties 2.1.2. Matrix product between any matrix A with an identity matrix I always produces A whenever the matrix product is defined. If A is of the shape $m \times n$, then $AI_n = I_m A = A$. If A is now a square matrix such that $m = n$ and we let $I_m = I_n = I$, then $AI = IA = A$.

In other words, the identity I can be regarded to be the "1" in the world of matrices. This is one of the cases that $AB = BA$ commutes (if A is a square matrix and $B = I$). Using the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

as an example, the readers can try to computed AI_2 and $I_2 A$ to see if the results are A itself.

2.1.2 Transpose

Transpose of a matrix, denoted by adding the superscript ^{T}, is formed by interchanging its rows and columns. In the special case of square matrix, this operation can be viewed as flipping the elements about the main diagonal.

Definition 2.1.3 (Transpose)**.** The transpose of an $m \times n$ matrix A, denoted as A^T, is formed according to the relation $A^T_{pq} = A_{qp}$, $1 \leq p \leq m$, $1 \leq q \leq n$, i.e. switching the row and column indices. Now A^T is an $n \times m$ matrix.

---

[1] $$I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Two examples are given below to show the outcome of applying transpose on matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -4 & 3 \\ 2 & -2 & 0 \\ -3 & 1 & 4 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 2 & -3 \\ -4 & -2 & 1 \\ 3 & 0 & 4 \end{bmatrix}$$

Particularly, in the second example, we have highlighted the main diagonal of B (as well as B^T) and how the elements flip about it when transpose is carried out. Some useful properties about transpose are listed as follows.

Properties 2.1.4. For two matrices A and B, we have

1. $(cA)^T = cA^T$, where c is any constant,
2. $(A^T)^T = A$, i.e. transposing twice returns the original matrix (quite obvious),
3. $(A \pm B)^T = A^T \pm B^T$, if A and B have the same shape,
4. $(AB)^T = B^T A^T$, if their matrix product is valid,
5. $A_{kk} = A^T_{kk}$ for any k that A_{kk} is defined, i.e. the main diagonal is unaffected by transpose.

Short Exercise: Show that $(ABC)^T = C^T B^T A^T$ if the matrices have compatible shapes for the matrix product.[2]

2.1.3 Symmetric Matrices

A ***Symmetric Matrix*** has its elements mirrored about the main diagonal. Taking transpose of such a matrix will leave it unchanged. Implicitly, it is required to

[2]By (4), $(ABC)^T = ((AB)(C))^T = C^T (AB)^T = C^T B^T A^T$

be a square matrix.

Definition 2.1.5 (Symmetric Matrix)**.** If an $n \times n$ square matrix A and its transpose A^T are the same, i.e. $A_{pq} = A^T_{pq} = A_{qp}$ for all $1 \leq p, q \leq n$, or simply $A = A^T$, then A, and also A^T, are symmetric.

As an example,

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 4 \\ 2 & 4 & 3 \end{bmatrix}$$

is a 3×3 symmetric matrix.

Short Exercise: Show that $Y = XX^T$ and $Z = X^T X$ are symmetric for any matrix X.[3]

In contrast, we have ***Skew-symmetric Matrices*** such that $A^T = -A$. This automatically requires that elements along the main diagonal to be all zeros.

$$\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

A 3×3 skew-symmetric matrix.

2.2 Inverses

2.2.1 Definition and Properties of Inverses

Inverse of a square matrix, denoted by adding the superscript $^{-1}$, is another square matrix such that their matrix product yields an identity matrix.

[3] By Properties 2.1.4, $Y^T = (XX^T)^T = (X^T)^T(X)^T = XX^T = Y$, similar goes for $Z = X^T X$.

Definition 2.2.1 (Inverse)**.** An $n \times n$ square matrix B is said to be the inverse of another $n \times n$ square matrix A if $AB = BA = I_n$. Henceforth, we write $B = A^{-1}$, and $AA^{-1} = A^{-1}A = I$. The opposite direction also holds, i.e. A is the inverse of $B = A^{-1}$. We say that $A = B^{-1}$ and $B = A^{-1}$ are the inverse of each other.

If there exists an inverse A^{-1} for the square matrix A, then both A and A^{-1} are called ***Invertible***. Otherwise, A is said to be ***Singular***. This is another situation in which a matrix product $AB = BA$ (if $B = A^{-1}$) can commute. We can show why $AA^{-1} = I$ implies $A^{-1}A = I$, or vice versa.

Proof. Assume only $AA^{-1} = I$ is true, then multiplying A to the right on both sides of the equation leads to

$$\begin{aligned} AA^{-1}A &= IA \\ A(A^{-1}A) &= A && \text{(Properties 1.1.2 and Properties 2.1.2)} \\ AP &= A \end{aligned}$$

where we write $P = A^{-1}A$. The above implies that multiplying A by P returns A itself. We are tempted to claim that $P = I$ by observing Properties 2.1.2. However, it is not trivial to show that P cannot be any matrix other than I, and to do so we have to wait until later chapters. Nevertheless, it is indeed true as long as A is invertible. Therefore, $P = A^{-1}A = I$. The opposite direction is proved similarly. □

The inverse operation can be somehow viewed as taking the reciprocal in the world of matrices. In addition,

Properties 2.2.2 (Uniqueness of Inverse)**.** If A has an inverse A^{-1}, it is unique.

Proof. This property can be proved easily by first assuming that the invertible matrix A has two different inverses, B and C. Subsequently, by Definition 2.2.1, we have $BA = I$ (and also $AC = I$). Multiplying by C to the right on both sides gives

$$BAC = IC$$

$$
\begin{aligned}
B(AC) &= C && \text{(Properties 1.1.2 and 2.1.2)} \\
B(I) &= C && (AC = I \text{ from assumption}) \\
B &= C && \text{(Properties 2.1.2)}
\end{aligned}
$$

So, B and C are actually the same matrix, implying that the inverse of A is unique. □

Example 2.2.1. Let

$$A = \begin{bmatrix} 4 & 6 \\ 3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{5}{2} & -3 \\ -\frac{3}{2} & 2 \end{bmatrix}$$

Show that A and B are inverse to each other.

Solution.

$$
\begin{aligned}
AB &= \begin{bmatrix} 4 & 6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \frac{5}{2} & -3 \\ -\frac{3}{2} & 2 \end{bmatrix} \\
&= \begin{bmatrix} (4)(\frac{5}{2}) + (6)(-\frac{3}{2}) & (4)(-3) + (6)(2) \\ (3)(\frac{5}{2}) + (5)(-\frac{3}{2}) & (3)(-3) + (5)(2) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2
\end{aligned}
$$

It is easy to verify that $BA = I$ and the readers are invited to do so. Hence $AB = BA = I$, A and B are indeed the inverse of each other. □

The followings are some properties of inverses.

Properties 2.2.3. If a square matrix A is invertible and has an inverse A^{-1}, then

1. $(cA)^{-1} = \frac{1}{c}A^{-1}$, for any constant $c \neq 0$,
2. $(A^{-1})^{-1} = A$, i.e. the inverse of an inverse is the original matrix,
3. $(A^n)^{-1} = (A^{-1})^n$, for any positive integer n,

4. $(AB)^{-1} = B^{-1}A^{-1}$, provided that B is invertible too,
5. $(A^T)^{-1} = (A^{-1})^T$.

However, $(A \pm B)^{-1}$ may not be equal to $A^{-1} \pm B^{-1}$, or even may be singular. We shall briefly prove (4) here.

Proof. It is given that A and B is invertible, and by Definition 2.2.1, we have $AA^{-1} = I$ and

$$BB^{-1} = I$$

Multiplying by A and A^{-1} to the left and right on both sides yields

$$\begin{aligned} ABB^{-1}A^{-1} &= AIA^{-1} \\ AB(B^{-1}A^{-1}) &= (AI)A^{-1} = AA^{-1} && \text{(Properties 1.1.2 and 2.1.2)} \\ &= I && \text{(Definition 2.2.1)} \end{aligned}$$

This shows that multiplying AB by $B^{-1}A^{-1}$ produces an identity matrix, and therefore $(AB)^{-1} = B^{-1}A^{-1}$ is the unique inverse of AB by Definition 2.2.1 and Properties 2.2.2. □

Short Exercise: Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ if A, B and C are invertible.[4]

(4) of Properties 2.2.3 explicitly shows that the product AB is invertible if A and B are themselves invertible. We will show that the converse is true as well. Hence

Properties 2.2.4. For two square matrices A and B, AB is invertible if and only if A and B are invertible.

[4]By (4), $(ABC)^{-1} = ((AB)(C))^{-1} = C^{-1}(AB)^{-1} = C^{-1}B^{-1}A^{-1}$

Proof. The "if" part is just (4) of Properties 2.2.3 and the "only if" part will be proved as follows. Let's assume AB is invertible and has an inverse $C = (AB)^{-1}$, hence we have $(AB)C = I$ by Definition 2.2.1, (notice that A, B, and C are all square matrices of the same extent) and by Properties 1.1.2, $A(BC) = I$. Using Definition 2.2.1 (as well as Properties 2.2.2) again, we immediately identify BC as the inverse of A and A is invertible. The case for B is similarly proved. □

2.2.2 (Reduced) Row Echelon Form

To find the inverse of any matrix, we have to understand a form of matrices called ***(Reduced) Row Echelon Form*** first. The requirements of a matrix being in such a form are shown below.

Definition 2.2.5 ((Reduced) Row Echelon Form)**.** A matrix is in row echelon form if

1. The first non-zero number in every row is 1, which is known as the *"Leading 1"*,
2. *"Leading 1"* of a lower row must appear farther to the right than that of any higher row,
3. Any row consisted of all zeros is placed at the bottom;
4. If additionally, any column containing a leading 1 have zeros elsewhere in that column, then it is in *reduced* row echelon form.

It is apparent that all identity matrices are in (reduced) row echelon form. Examples of row echelon form (but not *reduced*), with the leading 1s highlighted are

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 4 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Meanwhile, examples of *reduced* row echelon form are

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The following matrices are *not* in row echelon form.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

Short Exercise: Decide if the following matrices are in (reduced) row echelon form or not.[5]

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The next goal is to transform the matrix in question to reduced row echelon form by elementary row operations, whose importance will be demonstrated soon. The procedure is comprised of two major parts, *Forward Elimination*, converting the matrix to row echelon form, and *Backward Elimination*, further converting it to reduced row echelon form. The first phase is also named ***Gaussian Elimination***, and together they are called ***Gauss-Jordan Elimination***. We demonstrate the entire procedure using an example.

[5] Yes, Yes (reduced), No.

Example 2.2.2. Carry out Gauss-Jordan Elimination on the following matrix.

$$A = \begin{bmatrix} 2 & 0 & 4 & 6 \\ 3 & 3 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Solution. At each step of the forward phase, the strategy is to look at on the leftmost column that has at least one non-zero entries first (any column consisting of full zeros is ignored). Along that column, we either find an existing leading 1, or create a leading 1 via multiplying a suitable row having a starting entry a (Leading entries selected by the algorithm are known as ***Pivots***, and the process is called ***Pivoting***) that is as large as possible in magnitude, by the constant $1/a$. The row holding the leading 1 is subsequently put at the top, by a row interchange if needed. Such a row is highlighted in red during this example.

$$\begin{bmatrix} 2 & 0 & 4 & 6 \\ 3 & 3 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & 3 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} \qquad \frac{1}{2}R_1 \to R_1$$

We have picked R_1 here but a leading 1 can be obtained from R_2 or R_3 as well. Subsequently, we make all the elements below the leading 1 along that column become zero, by adding the top row (holding the leading 1), times $-a_i$ (where a_i is the corresponding leading entry of row i) to other rows. Those zeros will be highlighted in blue.

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & 3 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} &\to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 3 & -5 & -9 \\ 1 & 2 & 3 & 4 \end{bmatrix} & R_2 - 3R_1 \to R_2 \\ &\to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 3 & -5 & -9 \\ 0 & 2 & 1 & 1 \end{bmatrix} & R_3 - R_1 \to R_3 \end{aligned}$$

The first iteration is finished. We now repeat the same process over the remaining submatrix made up of elements that are not yet highlighted in colour, recursively.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 3 & -5 & -9 \\ 0 & 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -\frac{5}{3} & -3 \\ 0 & 2 & 1 & 1 \end{bmatrix} \qquad \frac{1}{3}R_2 \to R_2$$

$$\to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -\frac{5}{3} & -3 \\ 0 & 0 & \frac{13}{3} & 7 \end{bmatrix} \qquad R_3 - 2R_2 \to R_3$$

$$\to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -\frac{5}{3} & -3 \\ 0 & 0 & 1 & \frac{21}{13} \end{bmatrix} \qquad \frac{3}{13}R_3 \to R_3$$

Now, below every leading 1, all entries are zeros, indicating the forward phase is completed. We have obtained the row echelon form as an intermediate. The backward phase is done similarly but in a bottom up fashion. By adding some multiples of lower rows to higher rows, we turn all the non-zero elements above any leading 1 into zeros.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -\frac{5}{3} & -3 \\ 0 & 0 & 1 & \frac{21}{13} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -\frac{4}{13} \\ 0 & 0 & 1 & \frac{21}{13} \end{bmatrix} \qquad R_2 + \frac{5}{3}R_3 \to R_2$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{13} \\ 0 & 1 & 0 & -\frac{4}{13} \\ 0 & 0 & 1 & \frac{21}{13} \end{bmatrix} \qquad R_1 - 2R_3 \to R_1$$

The matrix is now in reduced row echelon form as required. □

Short Exercise: Repeat the example but start by interchanging R_1 and R_3.[6]

[6]For checking, after the first iteration, it will be

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -8 & -12 \\ 0 & -4 & -2 & -2 \end{bmatrix}$$

and the end result will be the same.

From the short exercise above, we can see that even if we apply different elementary row operations (particularly for the creation of a leading 1) during Gauss-Jordan Elimination, we will acquire the same reduced echelon form in the end. In fact,

Theorem 2.2.6 (Uniqueness of Reduced Row Echelon Form)**.** Reduced row echelon form of a matrix is unique.

We shall omit the proof here. The following properties further reveal how elementary row operations are associated with reduced row echelon form.

Properties 2.2.7. If a matrix can be transformed into another matrix by elementary row operations, they are said to be ***Row Equivalent***.

Since for any pair of row equivalent matrices, either of them can be transformed into the other one by elementary row operations, and hence can be further transformed into the reduced row echelon form of the other matrix, by Theorem 2.2.6, the uniqueness of reduced row echelon form implies that

Properties 2.2.8. Row equivalent matrices have the same reduced row echelon form. Particularly, they are row equivalent to their own reduced row echelon form. If two matrices have different reduced row echelon forms, then they are not row equivalent, and vice versa.

Let's go through one more simple example about Gauss-Jordan Elimination.

Example 2.2.3. Transform the following matrix into reduced row echelon form.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 6 & 4 & 1 \\ 2 & 3 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

Solution. One possible way to do the forward elimination is

$$
\begin{aligned}
\begin{bmatrix} 2 & 2 & 1 \\ 6 & 4 & 1 \\ 2 & 3 & 2 \\ 2 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 6 & 4 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 2 \\ 2 & 1 & 0 \end{bmatrix} & \qquad R_1 \leftrightarrow R_2 \\
\longrightarrow \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{6} \\ 2 & 2 & 1 \\ 2 & 3 & 2 \\ 2 & 1 & 0 \end{bmatrix} & \qquad \frac{1}{6}R_1 \longrightarrow R_1 \\
\longrightarrow \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{6} \\ 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{5}{3} & \frac{5}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} & \qquad R_i - 2R_1 \longrightarrow R_i, \text{for } i = 2, 3, 4 \\
\longrightarrow \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{6} \\ 0 & 1 & 1 \\ 0 & \frac{5}{3} & \frac{5}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} & \qquad \frac{3}{2}R_2 \longrightarrow R_2 \\
\longrightarrow \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{6} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \qquad R_3 - \frac{5}{3}R_2 \longrightarrow R_3, R_4 + \frac{1}{3}R_2 \longrightarrow R_4
\end{aligned}
$$

The backward elimination is simple.

$$
\begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{6} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_1 - \frac{2}{3}R_2 \longrightarrow R_1
$$

□

2.2.3 Finding Inverses by Gaussian Elimination

With Gaussian Elimination, obtaining the inverse A^{-1} of any invertible matrix A is straight-forward. We start by writing out an identity matrix I of the same extent and concatenate this identity matrix to the right of A, leading to a form of $[A|I]$. Then we carry out elementary row operations simultaneously on both sides of $[A|I]$ such that the matrix to the left, originally as A, is reduced to the identity matrix I by Gaussian Elimination. The identity matrix to the right will then be transformed into the desired inverse by the same set of elementary operations, such that the concatenated matrix will appear as $[I|A^{-1}]$,

Example 2.2.4. Find the inverse of

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution. Appending an 3×3 identity matrix to the right, we have

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \qquad R_2 - 2R_3 \to R_2$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right] \qquad R_2 \leftrightarrow R_3$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right] \qquad R_2 - R_3 \to R_2$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1 & -5 & 10 \\ 0 & 1 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right] \qquad R_1 - 5R_3 \to R_1$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right] \qquad R_1 - 4R_2 \to R_1$$

Hence the required inverse is

$$A^{-1} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & 3 \\ 0 & 1 & -2 \end{bmatrix}$$

□

Short Exercise: Find the inverse of A^{-1} above by the same elimination method. [7]

The underlying reason why the above procedure can produce the inverse matrix is the equivalence between elementary row operations and multiplication by appropriate ***Elementary Matrices***.

Theorem 2.2.9 (Elementary Matrices)**.** Any elementary row operation can be represented by multiplying to the left with a suitable elementary matrix. Such a matrix is essentially the one formed after applying that particular elementary row operation on an identity matrix. For the three types of elementary row operations in Definition 1.3.1

1. $cR_p \to R_p$, $c \neq 0$,
2. $R_p + cR_q \to R_p$,
3. $R_p \leftrightarrow R_q$

The corresponding elementary matrices E are square, and *invertible* (see the remark below) matrices with

1. $E_{kk} = 1$ for any k, except $E_{pp} = c$,
2. $E_{kk} = 1$ for all k, with $E_{pq} = c$,
3. $E_{kk} = 1$ for any k, except $E_{pp} = 0$ and $E_{qq} = 0$, with $E_{pq} = E_{qp} = 1$.

Entries not mentioned are all zeros.

[7] You should be able to retrieve the matrix A back.

Since it is quite abstract, it is useful to have some actual examples.

$$\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix} \qquad \text{Multiplying } R_2 \text{ by a factor of } 2, 2R_2 \to R_2$$

$$\begin{bmatrix}1&3&0\\0&1&0\\0&0&1\end{bmatrix} \qquad \text{Adding 3 times } R_2 \text{ to } R_1, R_1 + 3R_2 \to R_1$$

$$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} \qquad \text{Swapping } R_1 \text{ and } R_3, R_1 \leftrightarrow R_3$$

Any elementary row operation can be apparently undone by an inverse elementary row operation (addition vs subtraction, multiplication vs division ($c \neq 0$), swapping twice). Accordingly, any elementary matrix has another corresponding elementary matrix as its inverse, and the readers are invited to think about it in the exercise below.
Short Exercise: Write down the inverses of the three example elementary matrices above.[8]

For instance, consider a matrix

$$\begin{bmatrix}1&4&3\\2&5&1\\-1&0&2\end{bmatrix}$$

then subtracting R_2 from R_3, $R_3 - R_2 \to R_3$. can be written as

$$\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}1&4&3\\2&5&1\\-1&0&2\end{bmatrix} = \begin{bmatrix}1&4&3\\2&5&1\\-3&-5&1\end{bmatrix}$$

[8] $\begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&-3&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$

Short Exercise: Find out the 3×3 elementary matrix for subtracting 4 times the first row from the third row.[9]

Now we are ready to see why finding inverses by Gaussian Elimination works.

Theorem 2.2.10. If a matrix A can be converted to an identity matrix I as its reduced row echelon form by Gaussian Elimination, then it is invertible since the same steps can in turn be applied on I, producing its inverse A^{-1}.

Using the language of Properties 2.2.8, the matrix A has to be row equivalent to I for A^{-1} to exist. This also means if Gaussian Elimination fails to reduce A to I (i.e. the reduced row echelon form of A is some matrix other than the identity), then A^{-1} does not exist.

Proof. Assume A is invertible and hence $AA^{-1} = I$ (Definition 2.2.1). From Theorem 2.2.9, When doing Gaussian Elimination over A, the i-th elementary row operation executed can be represented by an elementary row matrix, denoted as E_i, $i = 1, 2, \ldots, n$ where n is the total number of steps. If we multiply these E_i successively to the left on both sides of the equation $AA^{-1} = I$, we have

$$\begin{aligned}(E_n \cdots E_3 E_2 E_1 A)A^{-1} &= (E_n \cdots E_3 E_2 E_1)I \\ A^{-1} = (I)A^{-1} &= (E_n \cdots E_3 E_2 E_1)I\end{aligned}$$

$E_n \cdots E_3 E_2 E_1 A = I$ because the elementary row operations E_i reduces A to I during Gaussian Elimination as we demand. With $A^{-1} = E_n \cdots E_3 E_2 E_1 I$, we claim that the same set of elementary row operations, equivalent to E_i, $i = 1, 2, \ldots, n$, can also change I to A^{-1} and thus indeed A is invertible. □

As a corollary, because we have $E_n \cdots E_3 E_2 E_1 A = I$ from above, and all E_i are invertible by Theorem 2.2.9, we can multiply their inverses $E_i' = E_i^{-1}$ (which are

[9] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

also elementary matrices), to the left on both sides successively, where i runs backwards from n to 1. This leads to

$$\begin{aligned}(E_1^{-1}E_2^{-1}E_3^{-1}\cdots E_n^{-1})E_n\cdots E_3E_2E_1A &= (E_1^{-1}E_2^{-1}E_3^{-1}\cdots E_n^{-1})I \\ A &= E_1'E_2'E_3'\cdots E_n'\end{aligned}$$

as each of $E_n^{-1}E_n$, $E_{n-1}^{-1}E_{n-1}$, ..., $E_1^{-1}E_1$ cancels out to produce I and hence

Properties 2.2.11. All invertible matrices can be written as a product of some sequence of elementary matrices.

2.3 Determinants

2.3.1 Computing Determinants

Determinant of a *square* matrix A, denoted by $\det(A)$ or $|A|$, is a number linked to certain intrinsic properties of the matrix which can help us to find its inverse (determinant of non-square matrix is undefined). Determinant of a 1×1 matrix is equal to the matrix's only entry. Determinants of 2×2 and 3×3 matrices can be calculated by a trick called ***Sarrus' Rule***.

Sarrus' Rule

Properties 2.3.1 (Sarrus' Rule)**.** Determinants of size 2×2 and 3×3 matrices can be found by the Sarrus' Rule. For a 2×2 matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

Its determinant is computed by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$|A| = a_{11}a_{22} - a_{21}a_{12}$$

which is the product of elements crossed by the red arrow, minus the blue one. Similarly, for a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Its determinant is found by

$$\left| \begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \right.$$

$$\begin{aligned} |A| = &(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) \\ &- (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12}) \end{aligned}$$

Example 2.3.1. Find the determinant of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -5 & 0 & -3 \\ 4 & 3 & 1 \end{bmatrix}$$

Solution. By Sarrus's Rule, we have

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 2 & 4 \\ -5 & 0 & -3 \\ 4 & 3 & 1 \end{vmatrix} \\ &= ((1)(0)(1) + (2)(-3)(4) + (4)(-5)(3)) \\ &\quad - ((4)(0)(4) + (3)(-3)(1) + (1)(-5)(2)) \\ &= (0 - 24 - 60) - (0 - 9 - 10) \end{aligned}$$

$$= -65$$

□

Cofactor Expansion

Another commonly used method to calculate determinants is *Cofactor Expansion*, also known as *Laplace Expansion*. Before discussing cofactor expansion, it is necessary to know what *cofactors* are.

Definition 2.3.2 (Cofactor and Minor). ***Cofactor*** C_{ij} at (i, j) of a matrix A is simply the determinant of the submatrix formed by deleting the i-th row and j-th column of A, which is called ***Minor*** at (i, j) M_{ij}, times the factor of $(-1)^{i+j}$. Mathematically, $C_{ij} = (-1)^{i+j} M_{ij}$.

The $(-1)^{i+j}$ factor can be visualized as a checkerboard pattern like

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

So, for a matrix like

$$\begin{bmatrix} \cancel{1} & 3 & 5 \\ \cancel{2} & \cancel{4} & \cancel{6} \\ \cancel{3} & 5 & 7 \end{bmatrix}$$

Its cofactor at $(2, 1)$ is

$$C_{21} = (-1)^{(2+1)} \begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} \qquad \text{(Definition 2.3.2)}$$

$$= (-1)((3)(7) - (5)(5)) \qquad \text{(Properties 2.3.1)}$$
$$= 4$$

Short Exercise: Find C_{13} and C_{32} for the matrix above.[10]

With ***Cofactor (Laplace) Expansion***, the determinant is computed as the sum of products between each entry and the corresponding cofactor along a picked row/column.

Properties 2.3.3 (Cofactor/Laplace Expansion)**.** The determinant of a $n \times n$ square matrix A, $|A|$, can be found by selecting either a fixed row i, or column j, and adding up the products of every element-cofactor pair along that row/column. For the former case, the determinant is computed as

$$\begin{aligned} |A| &= A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in} \\ &= \sum_{k=1}^{n} A_{ik}C_{ik} \end{aligned}$$

For the latter case, the determinant is similarly found by

$$\begin{aligned} |A| &= A_{1j}C_{1j} + A_{2j}C_{2j} + \cdots + A_{nj}C_{nj} \\ &= \sum_{k=1}^{n} A_{kj}C_{kj} \end{aligned}$$

where C is defined as in Definition 2.3.2. Regardless of the row or column chosen, the result is always the same.

Example 2.3.2. Again, for the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$

[10] $C_{13} = -2$, $C_{32} = 4$

Find its determinant via cofactor expansion.

Solution. According to Properties 2.3.3, if we choose the first row, its determinant is

$$\begin{aligned}|A| &= A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} \\ &= (1)((-1)^{1+1}\begin{vmatrix}4 & 6\\5 & 7\end{vmatrix}) + (3)((-1)^{1+2}\begin{vmatrix}2 & 6\\3 & 7\end{vmatrix}) \\ &\quad + (5)((-1)^{1+3}\begin{vmatrix}2 & 4\\3 & 5\end{vmatrix}) && \text{(Definition 2.3.2)} \\ &= (1)(-2) + (3)(4) + (5)(-2) = 0 && \text{(Properties 2.3.1)}\end{aligned}$$

□

Short Exercise: Confirm the answer by carrying out cofactor expansion on another row or column.[11]

Example 2.3.3. Find the determinant of

$$A = \begin{bmatrix}1 & 4 & 4 & 4\\2 & 0 & 4 & 6\\2 & 1 & 1 & 0\\6 & 2 & 3 & 1\end{bmatrix}$$

Solution. It is a 4×4 matrix and we have to apply cofactor expansion. We can choose row or column that contains zero to reduce the computation. Here we pick the second column and by Properties 2.3.3, we have

$$|A| = (-1)^{1+2}(4)\begin{vmatrix}2 & 4 & 6\\2 & 1 & 0\\6 & 3 & 1\end{vmatrix} + (-1)^{2+2}(0)\begin{vmatrix}1 & 4 & 4\\2 & 1 & 0\\6 & 3 & 1\end{vmatrix}$$

[11]You should able to get $|A| = 0$, no matter which row/column is selected.

$$+(-1)^{3+2}(1)\begin{vmatrix}1 & 4 & 4\\ 2 & 4 & 6\\ 6 & 3 & 1\end{vmatrix} + (-1)^{4+2}(2)\begin{vmatrix}1 & 4 & 4\\ 2 & 4 & 6\\ 2 & 1 & 0\end{vmatrix}$$

By Sarrus' Rule (Properties 2.3.1), we have (the detailed calculations are omitted)

$$|A| = (-4)(-6) + 0 + (-1)(50) + (2)(18) = 10$$

□

Finally, we can derive two simple results about determinants from the perspective of cofactor expansion.

Properties 2.3.4. If a matrix have a row/column with full zeros, or two identical/proportional rows/columns, then it has a determinant of zero.

The first case is trivial (just do the expansion along the row/column with full zeros). We will prove the second case alongside the introduction of the properties of determinants coming up in the next subsection.

2.3.2 Properties of Determinants

There are some important results about determinants. First of all, it is very easy to see that determinants for any $n \times n$ identity matrix I_n is just 1. Second, there is a close relation between elementary row operations/elementary matrices and (their effects on) determinants.

Properties 2.3.5. The three types of elementary row operations in Definition 1.3.1, when applied on some matrix A,

1. $cR_p \to R_p$, $c \neq 0$,
2. $R_p + cR_q \to R_p$,
3. $R_p \leftrightarrow R_q$,

change the determinant of A by a factor of c, 1, and -1, respectively.

Properties 2.3.6. For the three types of elementary matrices E in Theorem 2.2.9 that correspond to the elementary row operations in Definition 1.3.1,

1. $E_{kk} = 1$ for any k, except $E_{pp} = c$ ($cR_p \to R_p$, $c \neq 0$),
2. $E_{kk} = 1$ for all k, with $E_{pq} = c$ ($R_p + cR_q \to R_p$),
3. $E_{kk} = 1$ for any k, except $E_{pp} = 0$ and $E_{qq} = 0$, with $E_{pq} = E_{qp} = 1$ ($R_p \leftrightarrow R_q$),

their determinants are c, 1, and -1, respectively.

We will prove the above properties for the second kind of elementary row operations/elementary matrices (corresponding to addition/subtraction). The properties for the two other types of elementary matrices are easy to show and we will take them for granted, such that we can complete the second case in Properties 2.3.4 first, which is in turn used for demonstrating the final result.

Proof. Consider an $n \times n$ square matrix A that has two identical and adjacent rows with indices i_1 and i_2, where $i_2 = i_1 + 1$ (hence one of the indices is odd and another is even), then cofactor expansion along the odd row (let's say i_1) will give

$$\begin{aligned}
|A| &= \sum_{k=1}^{n} A_{i_1 k} C_{i_1 k} \\
&= \sum_{k=1}^{n} (-1)^{i_1+k} A_{i_1 k} M_{i_1 k}
\end{aligned}$$

by Properties 2.3.3 and Definition 2.3.2. Similarly by considering the even row, we have

$$\begin{aligned}
|A| &= \sum_{k=1}^{n} A_{i_2 k} C_{i_2 k} \\
&= \sum_{k=1}^{n} (-1)^{i_2+k} A_{i_2 k} M_{i_2 k}
\end{aligned}$$

But since the i_1-th and i_2-th row are identical, $A_{i_1 k} = A_{i_2 k}$. Furthermore, as these two identical rows are also adjacent, the minors $M_{i_1 k} = M_{i_2 k}$ are also equal. The only difference between the two expressions for $|A|$ above is the $(-1)^{i+j}$ factor. And because i_1 is odd and i_2 is even, they are differed by a negative sign only. Explicitly, we have

$$\begin{aligned}
|A| &= \sum_{k=1}^{n} (-1)^{i_1+k} A_{i_1 k} M_{i_1 k} \\
&= \sum_{k=1}^{n} (-1)^{(i_2-1)+k} A_{i_2 k} M_{i_2 k} \\
&= (-1) \sum_{k=1}^{n} (-1)^{i_2+k} A_{i_2 k} M_{i_2 k} \\
&= -|A|
\end{aligned}$$

Therefore $|A| = -|A|$ and $|A| = 0$ must equal to zero. Now we can generalize the results to non-adjacent, proportional rows by doing the first and third kind (multiplication and swapping) of elementary row operations when appropriate, and the second case in Properties 2.3.4 is completed. Subsequently, for the addition/subtraction type of elementary row operations, let's say $R_p + cR_q \to R_p$ is applied on some matrix A (this A is not the same one in the first part) to produce A', then

$$A' = \begin{bmatrix}
\vdots & \vdots & & \vdots \\
A_{p1} + cA_{q1} & A_{p2} + cA_{q2} & \cdots & A_{pn} + cA_{qn} \\
\vdots & \vdots & & \vdots \\
A_{q1} & A_{q2} & \cdots & A_{qn} \\
\vdots & \vdots & & \vdots
\end{bmatrix}$$

where we have only written out the rows R_p and R_q. By applying cofactor expansion along R_p following Properties 2.3.3, we have

$$|A'| = \sum_{k=1}^{n} [(A_{pk} + cA_{qk}) C_{pk}]$$

$$= \sum_{k=1}^{n} A_{pk} C_{pk} + c \sum_{k=1}^{n} A_{qk} C_{pk}$$

We identify the first term with $|A|$ that is computed from using cofactor expansion on the row R_p of A. The second term can be thought as the determinant of a matrix $\tilde{A}$ that is formed by replacing R_p by R_q in A and subsequently expanded along that row. So $\tilde{A}$ practically has two identical rows $R_p = R_q$ and by the previous result the value of $|\tilde{A}|$ is zero. Therefore $|A'| = |A| + c|\tilde{A}| = |A| + c(0) = |A|$, implying that the addition/subtraction type of elementary row operations does not affect the value of determinant. □

Since the values of determinants for elementary matrices, by Properties 2.3.6, coincide exactly with the factors by how the determinant of some other matrix changes when the corresponding elementary row operations are applied on it (represented by multiplication to the left by these elementary matrices) as shown in Properties 2.3.5, we can conclude

Theorem 2.3.7. For any elementary matrix E and another arbitrary matrix A, we have

$$\det(EA) = \det(E)\det(A)$$

This theorem will be of use when we later prove other properties of determinant. However, before doing so, we will demonstrate how to utilize Properties 2.3.5 (or equivalently 2.3.6) to ease the calculation of determinants.

Example 2.3.4. Re-do Example 2.3.3 utilizing Properties 2.3.5.

Solution. We can factor out the 2 in second row and subtract 3 times the third row from the fourth row. By Properties 2.3.5, we have

$$|A| = \begin{vmatrix} 1 & 4 & 4 & 4 \\ 2 & 0 & 4 & 6 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 3 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 4 & 4 & 4 \\ 1 & 0 & 2 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 3 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 4 & 4 & 4 \\ 1 & 0 & 2 & 3 \\ 2 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix}$$

The determinant in the last line can be computed by doing cofactor expansion along the fourth row which now contains two zeros. With Properties 2.3.3 and 2.3.1, it is

$$\begin{vmatrix} 1 & 4 & 4 & 4 \\ 1 & 0 & 2 & 3 \\ 2 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix} = 0 + (-1)^{4+2}(-1)\begin{vmatrix} 1 & 4 & 4 \\ 1 & 2 & 3 \\ 2 & 1 & 0 \end{vmatrix} + 0 + (-1)^{4+4}(1)\begin{vmatrix} 1 & 4 & 4 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{vmatrix}$$
$$= 0 + (-1)(9) + 0 + (1)(14) = 5$$

and hence $|A| = 2(5) = 10$. □

Finally, it is the time to show the following properties about determinants.

Properties 2.3.8. An invertible matrix has a non-zero determinant. Otherwise, a singular matrix has a determinant of zero.

Proof. Let's denote the matrix in question as A. Assume that A is invertible and hence by Properties 2.2.11 it can be written as the product of some elementary matrices $E_1, E_2, \ldots, E_{n-1}, E_n$, i.e.

$$A = E_1 E_2 \cdots E_{n-1} E_n$$

Taking the determinant of both sides, we have

$$\det(A) = \det(E_1 E_2 \cdots E_{n-1} E_n)$$

By repetitively using Theorem 2.3.7, we have

$$\det(A) = \det(E_1(E_2 \cdots E_{n-1} E_n))$$

$$\begin{aligned}
&= \det(E_1)\det(E_2 \cdots E_{n-1}E_n) \\
&= \det(E_1)\det(E_2)\det(\cdots E_{n-1}E_n) \\
&= \det(E_1)\det(E_2)\cdots\det(E_{n-1})\det(E_n)
\end{aligned}$$

Since by Properties 2.3.6, all elementary matrices have a non-zero determinant (particularly we have required $c \neq 0$ in Properties 2.3.6), and all $\det(E_i) \neq 0$, we have $\det(A) \neq 0$. To finish the part about singular matrices, we note that by Theorem 2.2.10, singular matrices have reduced row echelon forms that are not the identity. Furthermore, we have the observation that all other square reduced row echelon forms that are not the identity has a determinant of zero. (Why?)[12] With these two pieces of information, we leave the remaining proof to interested readers. □

Properties 2.3.9. For any $n \times n$ square matrices A and B, we have

1. $\det(A^T) = \det(A)$,
2. $\det(kA) = k^n \det(A)$, for any constant k,
3. $\det(AB) = \det(A)\det(B)$, and
4. $\det(A^{-1}) = \frac{1}{\det(A)}$, if A is invertible.

By extension, $\det(A_1A_2\cdots A_n) = \det(A_1)\det(A_2)\cdots\det(A_n)$.

For instance, if

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 5 \\ 1 & 0 \end{bmatrix}$$

then

$$|A| = (2)(9) - (3)(5) = 3 \qquad |B| = (4)(0) - (5)(1) = -5$$

$$AB = \begin{bmatrix} 2 & 3 \\ 5 & 9 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 1 & 0 \end{bmatrix}$$

[12] Such matrices must have at least one row of full zeros, and by Properties 2.3.4 we are done.

$$
\begin{aligned}
&= \begin{bmatrix} (2)(4)+(3)(1) & (2)(5)+(3)(0) \\ (5)(4)+(9)(1) & (5)(5)+(9)(0) \end{bmatrix} \\
&= \begin{bmatrix} 11 & 10 \\ 29 & 25 \end{bmatrix} \\
|AB| &= (11)(25) - (10)(29) \\
&= -15 = (3)(-5) = |A||B|
\end{aligned}
$$

So we can see in this case, $\det(AB) = \det(A)\det(B)$ indeed. Now we will formally prove this ((3) of Properties 2.3.9).

Proof. There are two cases to consider, A being invertible or singular. If A is singular, then by Properties 2.2.4, AB is also singular. And by Properties 2.3.8, both $\det(A)$ and $\det(AB)$ will be zero, and the equality holds trivially. Otherwise, if A is invertible, then we can follow the idea in the proof of Properties 2.3.8, and let $A = E_1E_2\cdots E_{n-1}E_n$ as a product of elementary matrices in sequence. By using Theorem 2.3.7 back and forth, we can readily show

$$
\begin{aligned}
\det(AB) &= \det(E_1E_2\cdots E_{n-1}E_nB) \\
&= \det(E_1)\det(E_2)\cdots\det(E_{n-1})\det(E_n)\det(B) \\
&= (\det(E_1)\det(E_2)\cdots\det(E_{n-1})\det(E_n))\det(B) = \det(A)\det(B)
\end{aligned}
$$

So the equality is true in both cases. □

Short Exericse: Prove (4) of Properties 2.3.9.[13]

[13]Consider $A^{-1}A = I$, and take determinant on both sides. By (3), we have

$$
\begin{aligned}
\det\left(A^{-1}A\right) &= \det(I) \\
\det\left(A^{-1}\right)\det(A) &= 1 \quad \text{(The identity always has a determinant of 1)} \\
\det\left(A^{-1}\right) &= \frac{1}{\det(A)}
\end{aligned}
$$

2.3.3 Finding Inverses by Adjugate

An alternative method to compute the inverse of a matrix is by using its ***Adjugate***, which is the transpose of its associated cofactor matrix.

Definition 2.3.10 (Adjugate)**.** For a matrix A, its adjugate is defined as

$$[\text{adj}(A)]_{pq} = (C_{pq})^T = C_{qp}$$

where C is formulated in Definition 2.3.2.

Properties 2.3.11. The inverse of a matrix A can be computed from its adjugate by

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

From this formula, obviously, a singular matrix that has a determinant of zero does not has an inverse.

Example 2.3.5. For a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

It is not difficult to see that the determinant is $ad - bc$, and the adjugate matrix is

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So the inverse, if $ad - bc \neq 0$, is

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 2.3.6. Find the inverse of the following matrix by evaluating its adjugate.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 4 & 11 \end{bmatrix}$$

Solution. First of all, by Sarrus' Rule (Properties 2.3.1)

$$\begin{aligned} |A| &= ((1)(3)(11) + (2)(5)(1) + (3)(1)(4)) \\ &\quad - ((3)(3)(1) + (1)(5)(4) + (2)(1)(11)) \\ &= (33 + 10 + 12) - (9 + 20 + 22) \\ &= 4 \end{aligned}$$

The adjugate matrix is

$$\begin{aligned} \operatorname{adj}(A) &= \begin{bmatrix} \begin{vmatrix} 3 & 5 \\ 4 & 11 \end{vmatrix} & -\begin{vmatrix} 1 & 5 \\ 1 & 11 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 4 & 11 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 11 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \end{bmatrix}^T \\ &= \begin{bmatrix} 13 & -6 & 1 \\ -10 & 8 & -2 \\ 1 & -2 & 1 \end{bmatrix}^T \\ &= \begin{bmatrix} 13 & -10 & 1 \\ -6 & 8 & -2 \\ 1 & -2 & 1 \end{bmatrix} \end{aligned}$$

(be careful not to forget the transpose!) Putting the pieces together according to the formula in Properties 2.3.11, we have

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$$= \frac{1}{4}\begin{bmatrix} 13 & -10 & 1 \\ -6 & 8 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{13}{4} & -\frac{5}{2} & \frac{1}{4} \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

□

A summarizing point to be emphasized is that

Theorem 2.3.12 (Equivalence Statement)**.** For a square matrix A, the followings are equivalent:

(a) A is invertible, i.e. A^{-1} exists,

(b) $\det(A) \neq 0$,

(c) The reduced row echelon form of A is I.

which is just a rephrasing of Properties 2.3.8 and Theorem 2.2.10. Particularly, invertibility is equivalent to a non-zero determinant. We will see the expansion of this equivalence statement in later chapters.

2.4 Python Programming

To create an identity matrix of size n, we use `np.identity(n)`. For example,

```
import numpy as np
I4 = np.identity(4)
print(I4)
```

returns

```
[[1. 0. 0. 0.]
 [0. 1. 0. 0.]
 [0. 0. 1. 0.]
 [0. 0. 0. 1.]]
```

Applying transpose on a matrix is simple where we just add `.T` after the array variable, like

```
myMatrix1 = np.array([[1., 0., 3.],
                      [1., 4., 1.],
                      [-1., 2., 4.]])
print(myMatrix1)
print(myMatrix1.T)
```

yields

```
[[ 1.  0.  3.]
 [ 1.  4.  1.]
 [-1.  2.  4.]]
[[ 1.  1. -1.]
 [ 0.  4.  2.]
 [ 3.  1.  4.]]
```

Finding the inverse of a matrix requires the `scipy.linalg` library and call the `inv` function.

```
from scipy import linalg
myMatrix2 = linalg.inv(myMatrix1)
print(myMatrix2)
print(myMatrix1@myMatrix2) # Check: should give the identity
```

gives the expected results of

```
[[ 0.4375   0.1875  -0.375  ]
 [-0.15625  0.21875  0.0625 ]
 [ 0.1875  -0.0625   0.125  ]]
[[1. 0. 0.]
 [0. 1. 0.]
 [0. 0. 1.]]
```

Meanwhile, we can use the `det` function to calculate the determinant of a matrix as follows. First,

```
print(linalg.det(myMatrix1))
```

gives the expected output of `32.0`. As another example,

```
myMatrix3 = np.array([[3., 1., 3., 2.],
                      [0., -1., -3., 1.],
                      [1., -1., -2., 0.],
```

```
                       [2., 0., 1., 0.]])
print(linalg.det(myMatrix3))
```

produces an extremely small value of `1.1102230246251562e-16`. In fact, the matrix

$$\begin{bmatrix} 3 & 1 & 3 & 2 \\ 0 & -1 & -3 & 1 \\ 1 & -1 & -2 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix}$$

has a determinant of exactly zero. It is an artifact of numerical error when using floating point numbers. If we keep going ahead and computes its inverse by `linalg.inv(myMatrix3)`, we will obtain an absurd output of

```
[[ 1.200959e+15 -2.401919e+15  3.602879e+15 -3.602879e+15]
 [ 6.004799e+15 -1.200959e+16  1.801439e+16 -1.801439e+16]
 [-2.401919e+15  4.803839e+15 -7.205759e+15  7.205759e+15]
 [-1.200959e+15  2.401919e+15 -3.602879e+15  3.602879e+15]]
```

that have entries of extremely large magnitude. This phenomenon is due to the extremely small "determinant", through Properties 2.3.11, magnifies the adjugate by being in the denominator. (The actual computation does not use Properties 2.3.11 directly but this is one perspective to view the problem.) To prevent this, we can add a `if` condition to look for singularity, defining a function like

```
def safe_inv(matrix):
    if np.abs(linalg.det(matrix)) < np.finfo(float).eps:
        print("Warning: The matrix is highly singular!")
        return(np.nan)
    else:
        return(linalg.inv(matrix))
```

where `np.finfo(float).eps` gives the so-called *machine epsilon* ϵ (the order of relative round-off error) of `float` and we want the absolute value of the determinant be larger than that. Subsequently, calling `safe_inv(myMatrix3)` will print a warning. Finally, we note that we can use `sympy` to acquire the reduced row echelon form of a matrix. Let's use the matrix in Example 2.2.3 for demonstration.

```
import sympy

myMatrix4 = np.array([[2., 2., 1.],
                      [6., 4., 1.],
                      [2., 3., 2.],
                      [2., 1., 0.]])
myMatrix4_sympy = sympy.Matrix(myMatrix4) # Convert the numpy
    array to a sympy matrix
print(myMatrix4_sympy.rref())
```

then returns two objects

```
(Matrix([
[1, 0, -0.5],
[0, 1,  1.0],
[0, 0,    0],
[0, 0,    0]]), (0, 1))
```

The first one is the reduced row echelon form we want, and the second is a tuple which keeps the column indices of the pivots. `sympy` also does *zero testing* such that

```
myMatrix3_sympy = sympy.Matrix(myMatrix3)
print(myMatrix3_sympy**(-1))
```

raises properly the error of

```
NonInvertibleMatrixError("Matrix det == 0; not invertible.")
    sympy.matrices.common.NonInvertibleMatrixError: Matrix det
    == 0; not invertible.
```

2.5 Exercises

Exercise 2.1 Find the determinant of the matrix below by inspection.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 10 & 11 & 12 \\ 0 & 0 & 0 & 13 & 14 \\ 0 & 0 & 0 & 0 & 15 \end{bmatrix}$$

Exercise 2.2 Let

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 6 \\ 0 & 1 \end{bmatrix}$$

Verify:

(a) $(AB)^T = B^T A^T$,

(b) $(AB)^{-1} = B^{-1} A^{-1}$, and

(c) $\det(AB) = \det(A)\det(B)$.

for this particular case.

Exercise 2.3 If

$$A = \begin{bmatrix} 3 & 2 & 9 \\ 1 & 2 & 3 \\ 4 & 0 & 4 \end{bmatrix}$$

Find its inverse by

(a) Gaussian Elimination, and

(b) Determinant and adjugate.

Exercise 2.4 Let

$$A = \begin{bmatrix} 0 & 2 & 5 \\ 0 & 4 & 9 \\ 1 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 4 & 6 \\ 3 & 5 & 8 \end{bmatrix}$$

Verify:

(a) $(AB)^T = B^T A^T$,

(b) $(AB)^{-1} = B^{-1} A^{-1}$, and

(c) $\det(AB) = \det(A)\det(B)$.

for this particular case.

Exercise 2.5 Show that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix}$$

is singular.

Exercise 2.6 Given

$$A = \begin{bmatrix} 1 & 9 & 1 & 4 \\ 0 & 6 & 2 & 8 \\ 1 & 9 & 3 & 9 \\ 0 & 9 & 0 & 1 \end{bmatrix}$$

Find its determinant, inverse, and determinant of the inverse.

Exercise 2.7 For the following matrix,

$$A = \begin{bmatrix} p & 1 & 2 \\ 0 & 2 & p \\ 4 & -2 & 0 \end{bmatrix}$$

Find the values of p such that A is invertible.

Exercise 2.8 Show that for any square matrix A, $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric. Hence show with an explicit formula that any square matrix A can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

Exercise 2.9 Prove that if A is an invertible $n \times n$ matrix, $|A| \neq 0$, then we have

$$\det(\text{adj}(A)) = (\det(A))^{n-1}$$

using Properties 2.3.9 and 2.3.11.

Chapter 3

Solutions for Linear Systems

The last chapter has introduced the necessary machinery for solving linear systems and now we are going to see how to apply them under suitable circumstances. Remember, in the first chapter, we have formulated some problems about linear systems of equations appearing in the Earth System, and they will be solved accordingly.

3.1 Number of Solutions for Linear Systems

Before tackling any linear system, we may like to know there are how many solutions. In fact, there are only three possibilities.

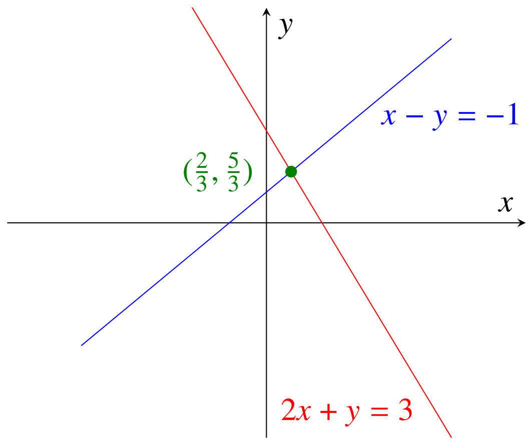
Theorem 3.1.1 (Number of Solutions for a Linear System)**.** For a system of linear equations $A\vec{x} = \vec{h}$ (recall Definition 1.2.2 and Properties 1.2.3), it has either:

1. No solution,
2. An unique solution, or
3. Infinitely many solutions.

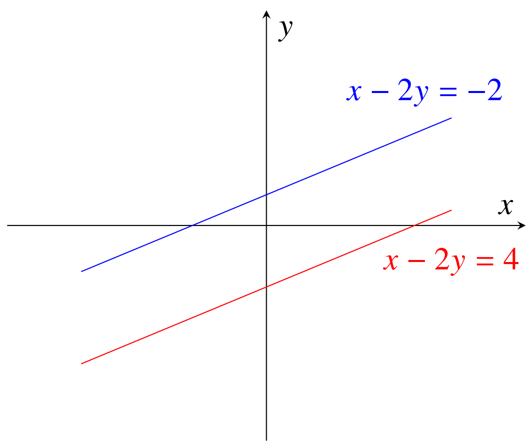
for the unknowns $\vec{x}$.

This can be illustrated by considering a linear system with two equations and two unknowns, with each equation representing a line. There are three types of scenarios.

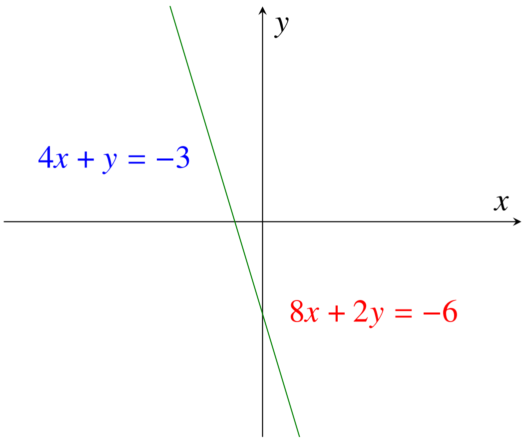
$$\begin{cases} a_1x + b_1y & = h_1 \\ a_2x + b_2y & = h_2 \end{cases}$$



One Solution: Two non-parallel lines (red/blue) intersecting at one point (green).



No Solution: Two parallel lines never touch each other.



Infinitely Many Solutions: Two parallel lines overlap each other.

It goes similarly for any linear system of three unknowns in which equations represent planes instead. The readers can try to imagine and visualize the possibilities. (The intersection of two non-parallel planes will be a line.) In fact, this theorem about the existence of solutions is true for any number of variables and equations. Some readers may think if there can be finitely many solutions only. Unfortunately, it is impossible. Assume there are at least two distinct solutions $\vec{x}_1, \vec{x}_2$ to the system $A\vec{x} = \vec{h}$, then it is easy to show by construction all $\vec{x}_t = t\vec{x}_1 + (1-t)\vec{x}_2$ for any t will be valid solutions which are infinitely many.

Naturally, the next question is about how to find out which case the linear system belongs to. The following theorem reveals the relation between the number of solutions for a *square* linear system and the determinant of its coefficient matrix.

Theorem 3.1.2. For a square linear system $A\vec{x} = \vec{h}$, if the coefficient matrix A is invertible, i.e. $\det(A) \neq 0$, there is always only one unique solution. However, if A is singular, $\det(A) = 0$, then it has either no solution, or infinitely many solutions.

As a consequence, if the homogeneous linear system $A\vec{x} = \mathbf{0}$ is singular with $\det(A) = 0$, since it always has a trivial solution of $\vec{x} = \mathbf{0}$, the above theorem

implies that the homogeneous system must have infinitely many solutions (since it does not have no solution). We defer the proof of Theorem 3.1.2, as well as the discussion about non-square systems, until we start actually solving linear systems in the next subsection.
Short Exercise: By inspection, determine the number of solutions for the following linear systems.[1]

$$\begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 4 \\ 1 & 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & 3 \\ 1 & 5 & 2 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3.2 Solving Linear Systems

Finally it is the time to get down to solving linear systems (preferably written in form of matrices), and we have two methods to choose.

1. By Gaussian Elimination, for linear system in any shape, or
2. By Inverse, which is apparently only applicable for square, invertible coefficient matrices.

3.2.1 Solving Linear Systems by Gaussian Elimination

Like in Section 2.2.3, applying Gaussian Elimination on the augmented matrix (introduced at the end of Section 1.2) of a linear system can yield the solution at right hand side. The principles involving elementary row operations are the same as stated in Theorems 2.2.9 and 2.2.10, but with $A\vec{x} = \vec{h}$ instead of $AA^{-1} = I$. In addition, the coefficient matrix A can be non-square, but we will look at the easier case of a coefficient matrix A first.

[1] These two homogeneous linear system has a determinant of -1 and 0, and hence by Theorem 3.1.2 the first system has a unique solution and the second one has infinitely many solutions.

Square Systems

Example 3.2.1. Solve the following linear system by Gaussian Elimination.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 4 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 8 \end{bmatrix}$$

Solution. We re-write the system in augmented form and apply Gaussian Elimination, aiming to reduce the matrix to the left into the identity.

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 1 & 1 & 4 & 10 \\ 2 & 0 & 3 & 8 \end{array}\right] &\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad R_2 - R_1 \to R_2, R_3 - 2R_1 \to R_3 \\ &\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad R_2 - 2R_3 \to R_2, R_1 - R_3 \to R_1 \end{aligned}$$

which translates to

$$\begin{cases} x = 1 \\ y = 1 \\ z = 2 \end{cases} \quad \text{or} \qquad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Note that we have successfully converted the coefficient matrix to the identity along the way, which by Theorem 2.3.12 the coefficient matrix is invertible. This explains the first part of Theorem 3.1.2 as in this case every unknown is associated only to a leading 1 in the corresponding column and a unique solution can always be derived. □

Example 3.2.2. Solve the linear system of

$$\begin{bmatrix} 3 & 7 & 2 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$$

Solution. Again, we apply Gaussian Elimination on the augmented matrix to obtain

$$\begin{aligned}
\left[\begin{array}{ccc|c} 3 & 7 & 2 & 8 \\ 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 2 \end{array}\right] &\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 3 & 7 & 2 & 8 \\ 0 & 2 & 1 & 2 \end{array}\right] && R_1 \leftrightarrow R_2 \\
&\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 4 & 2 & 2 \\ 0 & 2 & 1 & 2 \end{array}\right] && R_2 - 3R_1 \to R_2 \\
&\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 1 & 2 \end{array}\right] && \frac{1}{4}R_2 \to R_2 \\
&\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{array}\right] && R_3 - 2R_2 \to R_3
\end{aligned}$$

The last row corresponds to $0 = 1$ which is contradictory. As a consequence, the system is inconsistent, no solution exists. □

Example 3.2.3. Find the solution for the following linear system.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Solution. Gaussian Elimination leads to

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 5 & 3 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \qquad R_2 - 2R_1 \to R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_3 - R_2 \to R_3, R_1 - 2R_2 \to R_1$$

Now, the last row corresponds to $0 = 0$, implying one equation is spurious. This also means that it has a ***Free Variable***, which means that we can assign one unknown as a parameter for expressing other variables. We choose such unknowns according to the rule that they should not be fixed to a pivot in the reduced coefficient matrix. As the variables x and y already correspond to the two pivots in the first/second columns, we can only let $z = t$. From the first row and second row, we obtain $x = 1 + t$, $y = -t$ respectively. Therefore,

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

where $-\infty < t < \infty$ is any scalar. The first column vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is the so-called ***Particular Solution***. When it is complemented by the second column vector which is multiplied by the free parameter t

$$t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

they constitute the entire set of ***General Solution***. □

Short Exercise: Try plugging in any number t to the general solution and verify the consistency.[2]

The general solution encompasses all possible solutions to the linear system. For broader situations, it can contain more than one pairs of free parameter and column vector (or none, for the rather trivial cases of zero or a unique solution). The amount of free variables can be seen to be the number of columns in the coefficient matrix, minus the number of pivots in the reduced row echelon form. In case of multiple free variables, we assign the corresponding amount of free parameters to the non-pivots and apply the same procedure to get a set of general solution. Any column vector of a free parameter can be scaled as we desire.[3]

Meanwhile, the particular solution can be set to any valid solution to the system (the choice does not affect the structure of any column vector that comes along with a free parameter, see the footnote to the short exercise above). If the linear system is homogeneous, then the zero vector will always be a possible particular solution.

---

[2]Let's say $t = 1$ and $\tilde{x} = \begin{bmatrix}1\\0\\0\end{bmatrix} + (1)\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}2\\-1\\1\end{bmatrix}$, then clearly $A\tilde{x} = \begin{bmatrix}1&2&1\\2&5&3\\0&1&1\end{bmatrix}\begin{bmatrix}2\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix}$. It can become a new particular solution by noting that the original solution can be rewritten as

$$\vec{x} = \begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}1\\-1\\1\end{bmatrix} + (t-1)\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}2\\-1\\1\end{bmatrix} + t'\begin{bmatrix}1\\-1\\1\end{bmatrix} = \tilde{x} + t'\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

where we "extract" $\tilde{x}$ from generating a shifted free parameter $t' = t - 1$ and according to this relation, it represents the same set of general solution as the original expression.

[3]Using the last example as a demonstration,

$$\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{t}{2}\begin{bmatrix}2\\-2\\2\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} + s\begin{bmatrix}2\\-2\\2\end{bmatrix}$$

where we use $s = \frac{t}{2}$ as a new free parameter and the column vector associated to that free parameter is now scaled by a factor of 2.

We have seen in the previous two examples that if the reduced row echelon form of the square coefficient matrix has some row of full zeros, then it either leads to no solution (if inconsistent) or infinitely many solutions (if consistent). Since such a matrix at the same time has a determinant of zero (by Properties 2.3.4) and is singular. This establishes the second part of Theorem 3.1.2.

For non-square coefficient matrices, two cases occur.

1. There are more equations (rows) than unknowns (columns). The system is ***Overdetermined***. The reduced row echelon form then must have at least one row of full zeros. If any one of them is inconsistent, then contradiction will arise just like in Example 3.2.2 and there will be no solution. However, if all zero rows are consistent (i.e. $0 = 0$), then there still can be a unique solution or infintely many of them.

2. There are fewer equations (rows) than unknowns (columns). The system is said to be ***Underdetermined***. There must be unknowns that are non-pivots in the reduced row echelon form of the coefficient matrix. Hence free variables, and infinitely many solutions ensue if there is no *inconsistent* row of full zeros (then there is no solution). The calculation is similar to that in Example 3.2.3.

Let's see some examples for non-square linear systems.

Overdetermined Systems

Example 3.2.4. Find the solution to the following overdetermined system, if any.

$$\begin{bmatrix} 1 & 4 & 0 \\ 2 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 3 \\ 5 \end{bmatrix}$$

Solution.

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 4 \\ 2 & 2 & 3 & 8 \\ 1 & 1 & 2 & 3 \\ 0 & 3 & 1 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 0 & 4 \\ 0 & -6 & 3 & 0 \\ 0 & -3 & 2 & -1 \\ 0 & 3 & 1 & 5 \end{array}\right] \qquad R_2 - 2R_1 \to R_2, R_3 - R_1 \to R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & 4 & 0 & 4 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -3 & 2 & -1 \\ 0 & 3 & 1 & 5 \end{array}\right] \qquad -\frac{1}{6}R_2 \to R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 4 & 0 & 4 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & \frac{5}{2} & 5 \end{array}\right] \qquad R_3 + 3R_2 \to R_3, R_4 - 3R_2 \to R_4$$

$$\to \left[\begin{array}{ccc|c} 1 & 4 & 0 & 4 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & \frac{5}{2} & 5 \end{array}\right] \qquad 2R_3 \to R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & 4 & 0 & 4 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 10 \end{array}\right] \qquad R_4 - \frac{5}{2}R_3 \to R_4$$

The last row is inconsistent and hence the overdetermined system has no solution. □

Example 3.2.5. Show that there are infinitely many solution to the following overdetermined system.

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \end{bmatrix}$$

Solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 1 & 2 & 5 & 3 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -3 & -1 \\ 0 & -1 & -3 & -1 \end{array}\right] \qquad \begin{array}{r} R_2 - R_1 \to R_2, R_3 - 2R_1 \to R_3 \\ R_4 - R_1 \to R_4 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_3 + R_2 \to R_3, R_4 + R_2 \to R_4$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_1 - R_2 \to R_1$$

Two out of the four equations are redundant and there are effectively two constraints only, over the three variables. We can let the non-pivot unknown $z = t$ be a free variable like in Example 3.2.3, and derive $x = 1 + t$, $y = 1 - 3t$ from the first two rows. Thus the general solution is

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+t \\ 1-3t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

where $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is a particular solution. $\square$

Underdetermined Systems

Example 3.2.6. Solve the following underdetermined system.

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 3 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Solution.

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 2 & 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 1 \end{array}\right] \qquad R_2 - 2R_1 \to R_2$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & 0 & 1 \end{array}\right] \qquad R_2 \leftrightarrow R_3$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{array}\right] \qquad R_3 + R_2 \to R_3$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \qquad \frac{1}{2}R_3 \to R_3$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \qquad R_2 - 2R_3 \to R_2,\, R_1 - R_3 \to R_1$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \qquad R_1 - R_2 \to R_1$$

From the third row, we have $v = 1$ immediately. The only unknown that is not associated to a pivot is u and we can let $u = t$ be a free variable. From the first two equations, we retrieve $y = -1 - t$ and $x = -t$, and therefore the general solution is

$$\vec{x} = \begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} = \begin{bmatrix} -t \\ -1-t \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

with $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ as a particular solution. □

3.2.2 Solving Linear Systems by Inverse

For a square linear system $A\vec{x} = \vec{h}$, if A has a non-zero determinant and is invertible, then we can utilize its inverse to recover the solution. Remember that multiplying the inverse to a matrix returns an identity matrix, it is possible to multiply the inverse A^{-1} to the left on both sides of the equation $A\vec{x} = \vec{h}$ to cancel out the A on the L.H.S., which leads to

$$\begin{aligned} A^{-1}A\vec{x} = (A^{-1}A)\vec{x} &= A^{-1}\vec{h} \\ \vec{x} = I\vec{x} &= A^{-1}\vec{h} \qquad \text{(Definition 2.2.1 and Properties 2.1.2)} \end{aligned}$$

This solution is unique, guaranteed by Theorem 3.1.2.

Example 3.2.7. Solve the linear system $A\vec{x} = \vec{h}$ by the inverse method, where

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix} \qquad \vec{h} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

Solution. It can be checked that the inverse of the coefficient matrix is

$$A^{-1} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

The readers are encouraged to verify the inverse. Subsequently, we have the solution to the linear system as

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\vec{h} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

□

Doing Gaussian Elimination to find the inverse and then compute the solution by $\vec{x} = A^{-1}\vec{h}$ in Section 3.2.2 is somehow the same as using Gaussian Elimination directly to solve the linear system suggested by Section 3.2.1. Hypothetically, if there are a large amount of linear systems which all share the same coefficient matrix A, but different $\vec{h}_k$ to be solved, then the former approach may be more efficient at first sight. However, in computer, calculation of inverse can be unstable (see Section 2.4) and there are some other practical reasons not to do so, as we shall see in Section 3.4. Besides, Theorem 2.3.12 can be extended as below by incorporating Theorem 3.1.2:

Theorem 3.2.1. [Equivalence Statement, ver. 2] For a square matrix A, the followings are equivalent:

(a) A is invertible, i.e. A^{-1} exists,

(b) $\det(A) \neq 0$,

(c) The reduced row echelon form of A is I,

(d) The linear system $A\vec{x} = \vec{h}$ has a unique solution, particularly $A\vec{x} = \mathbf{0}$ has only the trivial solution $\vec{x} = \mathbf{0}$.

3.3 Earth Science Applications

Now we are going to revisit and find the solutions to the two linear system problems in Section 1.4.

Example 3.3.1. Solve for the horizontal displacement x and depth of top layer y in the seismic ray problem of Example 1.4.1.

Solution. The linear system is

$$\begin{bmatrix} 1 & 1 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1200 \\ 800\sqrt{3} \end{bmatrix}$$

Since it is just a 2×2 coefficient matrix, we can directly use the expression in Example 2.3.5 to find its inverse, which is

$$\frac{1}{\sqrt{3}-1} \begin{bmatrix} \sqrt{3} & -1 \\ -1 & 1 \end{bmatrix} = \frac{1+\sqrt{3}}{2} \begin{bmatrix} \sqrt{3} & -1 \\ -1 & 1 \end{bmatrix}$$

and solve the system by multiplying the inverse following the method demonstrated in Section 3.2.2, leading to

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1+\sqrt{3}}{2} \begin{bmatrix} \sqrt{3} & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1200 \\ 800\sqrt{3} \end{bmatrix} = \begin{bmatrix} 600 + 200\sqrt{3} \\ 600 - 200\sqrt{3} \end{bmatrix}$$

Therefore the required horizontal displacement and depth of top layer are about 946.4 m and 253.6 m respectively. □

Example 3.3.2. Find the radiative loss E_j and hence temperature T_j in each layer of the multi-layer model in Example 1.4.2. In particular, what is the temperature at the surface ($j = N + 1$)?

Solution. The linear system is

$$\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & & 0 & 0 & 0 \\ 0 & 1 & -2 & & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & & -2 & 1 & 0 \\ 0 & 0 & 0 & & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_{N-1} \\ E_N \\ E_{N+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -E_{in} \end{bmatrix}$$

where N is any positive integer. Since N can be arbitrarily large, we may wish to avoid the direct computation of a massive inverse. Instead, we resort to a tactful way of row reduction to reveal the pattern of R_j. Rather than starting the reduction at the top as usual, we build up at the bottom, subtracting the lower row from the row directly above it and then moving up a row, repeated until we reach the top.

$$\left[\begin{array}{ccccccc|c}
-2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & & 0 & 0 & 0 & 0 \\
\vdots & & & \ddots & & & \vdots & \vdots \\
0 & 0 & 0 & & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & -1 & -E_{in}
\end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccccc|c}
-2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & & 0 & 0 & 0 & 0 \\
\vdots & & & \ddots & & & \vdots & \vdots \\
0 & 0 & 0 & & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & & 1 & -1 & 0 & -E_{in} \\
0 & 0 & 0 & \cdots & 0 & 1 & -1 & -E_{in}
\end{array}\right] \qquad R_N + R_{N+1} \rightarrow R_N$$

$$\rightarrow \left[\begin{array}{ccccccc|c}
-2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & & 0 & 0 & 0 & 0 \\
\vdots & & & \ddots & & & \vdots & \vdots \\
0 & 0 & 0 & & -1 & 0 & 0 & -E_{in} \\
0 & 0 & 0 & & 1 & -1 & 0 & -E_{in} \\
0 & 0 & 0 & \cdots & 0 & 1 & -1 & -E_{in}
\end{array}\right] \qquad R_{N-1} + R_N \rightarrow R_{N-1}$$

$$\rightarrow \vdots \qquad \text{(Keep going up)}$$

$$\rightarrow \left[\begin{array}{ccccccc|c} -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & & 0 & 0 & 0 & -E_{in} \\ \vdots & & & \ddots & & & \vdots & \vdots \\ 0 & 0 & 0 & & -1 & 0 & 0 & -E_{in} \\ 0 & 0 & 0 & & 1 & -1 & 0 & -E_{in} \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 & -R_{in} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccccc|c} -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & & 0 & 0 & 0 & -E_{in} \\ 0 & 1 & -1 & & 0 & 0 & 0 & -E_{in} \\ \vdots & & & \ddots & & & \vdots & \vdots \\ 0 & 0 & 0 & & -1 & 0 & 0 & -E_{in} \\ 0 & 0 & 0 & & 1 & -1 & 0 & -E_{in} \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 & -E_{in} \end{array}\right] \qquad R_2 + R_3 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccccccc|c} -1 & 0 & 0 & \cdots & 0 & 0 & 0 & -E_{in} \\ 1 & -1 & 0 & & 0 & 0 & 0 & -E_{in} \\ 0 & 1 & -1 & & 0 & 0 & 0 & -E_{in} \\ \vdots & & & \ddots & & & \vdots & \vdots \\ 0 & 0 & 0 & & -1 & 0 & 0 & -E_{in} \\ 0 & 0 & 0 & & 1 & -1 & 0 & -E_{in} \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 & -E_{in} \end{array}\right] \qquad R_1 + R_2 \rightarrow R_1$$

From the first row, we readily obtain $E_1 = E_{in}$. The second row yields the equation

$$\begin{aligned} E_1 - E_2 &= -E_{in} \\ E_2 &= E_1 + E_{in} = E_{in} + E_{in} = 2E_{in} \end{aligned}$$

Similarly, the subsequent rows are all in the form of $E_{j+1} = E_j + E_{in}$, and inductively we have $E_j = jE_{in}$. $E_1 = E_{in}$ is the emission of radiation from Earth as a whole as viewed from the space, and the *emission temperature* is $T_1 = \sqrt[4]{E_1/\sigma} = \sqrt[4]{E_{in}/\sigma}$ by Stefan–Boltzmann Law. The surface releases terrestrial radiation at the rate of $E_{N+1} = (N+1)E_{in}$ and has a temperature of $T_{N+1} = \sqrt[4]{E_{N+1}/\sigma} = \sqrt[4]{(N+1)E_{in}/\sigma} = (N+1)^{1/4}\sqrt[4]{E_{in}/\sigma} = (N+1)^{1/4}T_1$, i.e.

the surface temperature is $(N+1)^{1/4}$ times the emission temperature. Our earth has an emission temperature of 255 K and a surface temperature of 288 K on average (notice that we have to use Kelvin instead of degree Celsius!), which leads to an effective number of absorbing layers $N = (288/255)^4 - 1 = 0.627$. □

3.4 Python Programming

For solving square linear systems in the form of $A\vec{x} = \vec{h}$, we can again import the `scipy.linalg` library and call the `solve` function with the coefficient matrix A as the first argument and $\vec{h}$ placed in the second one.

```
import numpy as np
from scipy import linalg

A = np.array([[1., 0., 1.],
              [2., 2., 3.],
              [1., 2., 0.]])
h = np.array([0., -1., 1.])
x = linalg.solve(A,h)
```

This corresponds to the linear system

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

which has a solution of

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

`print(x)` then gives the correct output of `[ 1. -0. -1.]`. However, if A is a singular matrix like the one shown in Section 2.4

```
A = np.array([[3., 1., 3., 2.],
              [0., -1., -3., 1.],
              [1., -1., -2., 0.],
```

```
                [2., 0., 1., 0.]]) # "myMatrix3" in the last
                    chapter
h = np.array([0., 1., 1., -1.])
x = linalg.solve(A,h)
print(x)
```

raises a warning and an unreasonable output of

```
LinAlgWarning: Ill-conditioned matrix
(rcond=3.42661e-18): result may not be accurate.
  x = linalg.solve(A,h)
[ 4.803839e+15  2.401919e+16 -9.607679e+15 -4.803839e+15]
```

Again, we can use the `sympy` package for the rescue as follows.

```
import sympy

A_sympy = sympy.Matrix(A)
h_sympy = sympy.Matrix(h)
A_sympy.solve(h_sympy)
```

which raises the same "not invertible" error as in Section 2.4. We note that, unfortunately, there is no simple way to deal with over/under-determined systems using either `scipy` or `sympy`. Moreover, there are two questions that may come to the curious readers when reading the programming sections of these two chapters. First, which of `scipy` and `sympy` should we choose over another? Second, why we don't compute the inverse of A and solve the system by something along the line of `x = linalg.inv(A) @ h`? For the first question, we note that `scipy` is numerical while `sympy` is symbolic, which means that if we are dealing with real data we may find `scipy` adequate and more efficient, while if we are focusing on the theoretical part of Mathematics we can obtain a more analytical solution with `sympy`. To the second question, we refer the readers to this excellent Stack Overflow post (31256252).

3.5 Exercises

Exercise 3.1 Solve the following linear system.

$$\begin{cases} 5x + y + 3z &= 6 \\ 2x - y + z &= \frac{7}{2} \\ 3x + 2y - 4z &= -\frac{13}{2} \end{cases}$$

Exercise 3.2 Solve $A\vec{x} = \vec{h}_k$, where

$$A = \begin{bmatrix} 6 & 7 & 7 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\vec{h}_1 = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \qquad \vec{h}_2 = \begin{bmatrix} 19/4 \\ 1 \\ 5/4 \end{bmatrix}$$

Exercise 3.3 Derive the solution to the following linear system.

$$\begin{cases} 3x + 4z &= 2 \\ x + y + 2z &= -1 \\ x - 2y &= 0 \end{cases}$$

Exercise 3.4 Solve the following linear system.

$$\begin{cases} m + n - p - 3q &= 2 \\ m - q &= 5 \\ 3m + 2n - 2p - 7q &= 9 \end{cases}$$

How about if the R.H.S. of the third equation is equal to 3 instead?

Exercise 3.5 For the following linear system,

$$\begin{bmatrix} 1 & 0 & \alpha \\ 0 & \alpha & 0 \\ \alpha & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ \alpha \end{bmatrix}$$

Find the values of α so that the system has no solution, or infinitely many solutions.

Exercise 3.6 In a geology field trip, an outcrop is examined. It is observed that the rock mainly consists of crystals of three distinct colors (gray/pink/black). Assume that crystal of each color corresponds to exactly one type of mineral. Three samples are gathered, have their densities measured and composition percentages of the three types of crystal analyzed. The data are as follows:

	gray	pink	black	density (g/cm^3)
Sample A	40%	50%	10%	2.645
Sample B	55%	40%	5%	2.6325
Sample C	45%	45%	10%	2.65

From the data, infer the densities of the constituent minerals.

Exercise 3.7 *Ohm's law* relates voltage drop of a current due to resistance by $V = IR$. In addition, *Kirchhoff's Second Law* states that: The voltage gain balances the voltage drop around any closed loop (net voltage change must be zero). The clockwise convention is adopted, i.e. around a loop, a battery with its positive terminal facing the clockwise direction is considered a voltage gain, and clockwise current passing through a resistor is deemed as a voltage drop. Together with the knowledge that current at a junction must conserve (*Kirchhoff's First Law*), find I_1, I_2, I_3 (assumed flowing in the direction as indicated) for the circuit in Figure 3.1.
You will obtain two equations by considering any two loops with Kirchhoff's Second Law, and one from Kirchhoff's First Law. So, there are three equations, for the three unknown currents.

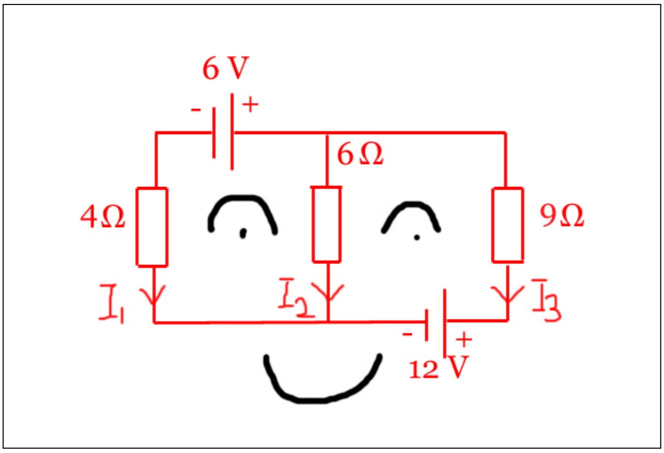


Figure 3.1: The circuit for Exercise 3.7

Exercise 3.8 The *shallow water equations* (see Figure 3.2) describe the evolution of gravity wave under some approximations such as *hydrostatic balance* and a sufficiently shallow fluid depth, and has the form of

$$\begin{cases} \frac{\partial \eta}{\partial t} + H(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) & = 0 \\ \frac{\partial u}{\partial t} & = -g\frac{\partial \eta}{\partial x} \\ \frac{\partial v}{\partial t} & = -g\frac{\partial \eta}{\partial y} \end{cases}$$

when the Coriolis effect is ignored. By assuming a travelling wave solution

$$u = \tilde{U}\cos(kx + ly - \omega t)$$
$$v = \tilde{V}\cos(kx + ly - \omega t)$$
$$\eta = \tilde{\eta}\cos(kx + ly - \omega t)$$

where $\tilde{U}$, $\tilde{V}$, $\tilde{\eta}$ are some constants to be determined, show that the equations become

$$\begin{cases} \omega\tilde{\eta} - kH\tilde{U} - lH\tilde{V} & = 0 \\ \omega\tilde{U} - kg\tilde{\eta} & = 0 \\ \omega\tilde{V} - lg\tilde{\eta} & = 0 \end{cases}$$

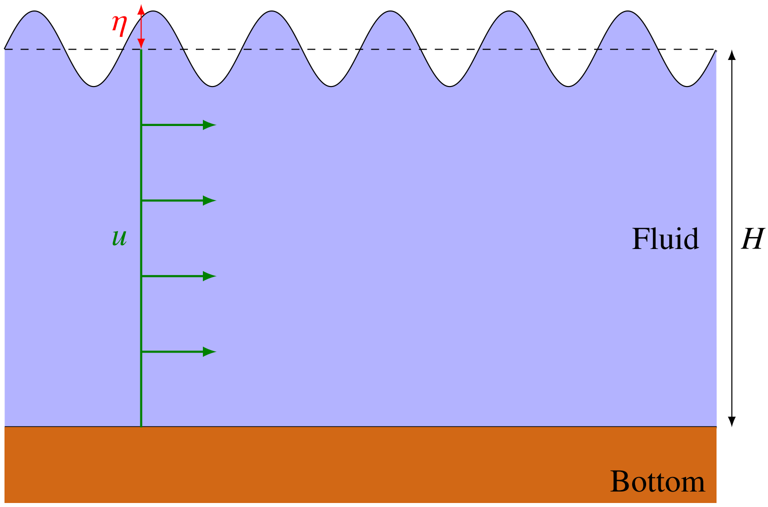


Figure 3.2: The x-z cross-section of shallow water system in Exercise 3.8. η is the height of free surface, H is the mean depth of the fluid, and u is the fluid velocity along x-axis.

By requiring that $\tilde{U}, \tilde{V}, \tilde{\eta}$ have a non-trivial solution so that they are not all zeros, derive the dispersion relation of gravity wave, which is

$$\omega^2 = gH(k^2 + l^2)$$
$$\omega = c\kappa$$

where $c = \sqrt{gH}$ is the wave speed, and $\kappa = \sqrt{k^2 + l^2}$ is the total wavenumber.

Exercise 3.9 Solve for the condensation height and temperature z_{cd} and T_{cd} in Exercise 1.10.

Exercise 3.10 Solve the *Chickens and Rabbits in the Same Cage* problem in Exercise 1.11. If we now introduce a new type of mystical creature who has one head and three legs, and throw them in another cage along with some chickens and rabbits, find all possible numbers of the three species if the cage now has 48 heads and 122 legs.

Chapter 4

Introduction to Vectors

After three chapters of discussion about matrices, it is the time to talk about another closely related concept in linear algebra, that is, vectors. While *vectors* and *vector spaces* have strictly mathematical definitions which make them abstract, we will take a more physical point of view with the special case of (finite-dimensional) geometric vectors first.

4.1 Definition and Operations of Geometric Vectors

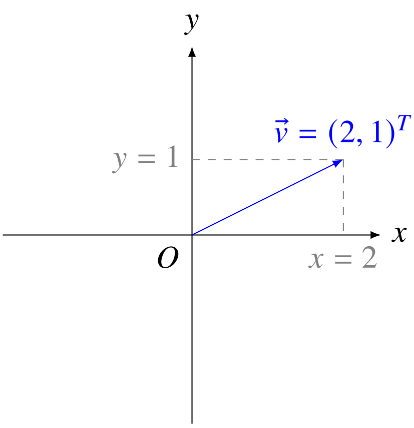
4.1.1 Basic Structure of Vectors in the Real n-space $\mathbb{R}^n$

A ***(Geometric) Vector*** is a physical quantity represented by an ordered tuple of components (numbers), e.g. $(1, 8, 7, 4)$, $(1 - \imath, 1 + 3\imath, 2)$. It has a *magnitude (length)* and *direction*, resembling an arrow. Some real-life examples are: two-dimensional flow velocity (u, v), relative position of an airplane to a ground radar (x, y, z).

Definition 4.1.1 (n-dimensional Vector)**.** A n-dimensional vector consists of n ordered elements called ***Components*** and are denoted by either an arrow or boldface, like $\vec{v}$ or $\mathbf{v}$. It is expressed as

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = (v_1, v_2, v_3, \ldots, v_n)^T$$

A n-dimensional vector can be regarded to be an $n \times 1$ (***Column Vector***) or $1 \times n$ matrix (***Row Vector***) and vice versa, depending on the situation. Usually the form of a column vector is more commonly taken than row vector and the column form is assumed throughout the book if it is not further specified.



A 2D vector drawn in an x-y plane.

Movement 移動速度和方向	1-min Average Strength 一分鐘平均強度		Distance/Bearing from HK 與香港的距離和方位角
WNW 西北偏西 (288°) 18 km/h	70 kt (130 km/h)	TY (Cat. 1) 一級颱風	SSE 東南偏南 116 km
WNW 西北偏西 (289°) 20 km/h	70 kt (130 km/h)	TY (Cat. 1) 一級颱風	WSW 西南偏西 178 km

Forecast for *Typhoon Higos*. (taken from Hong Kong Weather Watch) Its horiztonal movement is a two-dimensional vector, even though the speed and direction are given instead of the velocities in x and y-direction (they can be converted to each other).

Implicit in the definition of n-dimensional vectors is the n-dimensional space they are residing in. Assume the components of those vectors are all real, then the set of all such vectors constitutes the ***Real n-space*** $\mathbb{R}^n$.

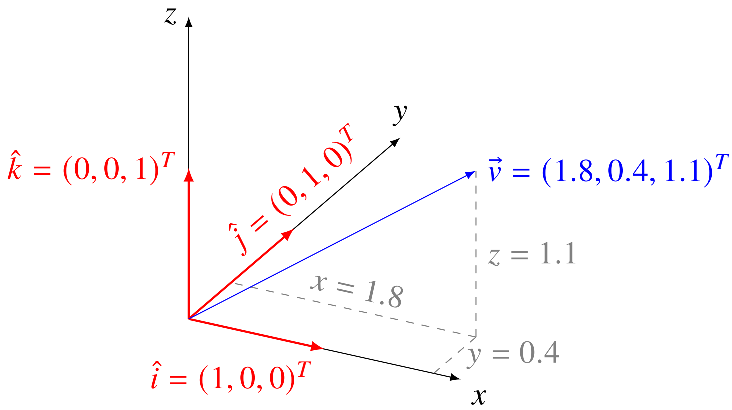
Definition 4.1.2 (The Real n-space $\mathbb{R}^n$)**.** The Real n-space $\mathbb{R}^n$ is defined as the set of all possible vectors (or "points") $\vec{v} = (v_1, v_2, v_3, \ldots, v_n)^T$ as defined in Definition 4.1.1, where v_i can take any *real* value, for $i = 1, 2, 3, \ldots, n$. The member vectors in $\mathbb{R}^n$ are known as n-dimensional *real* vectors.

While we have not clearly defined what a vector space is, we note that $\mathbb{R}^n$ fulfills the requirements of being a vector space. The detailed discussion of this aspect will be deferred to Chapter ??. Meanwhile, the complex counterpart will be explored in Chapter ??.

An n-dimensional real geometric vectors as in Definition 4.1.1 and 4.1.2 can be written as the sum of n ***Standard Unit Vectors*** that have a magnitude of 1, denoted by $\hat{e}_p$, oriented in the positive direction along the p-th coordinates axes, $p = 1, 2, \ldots, n$. Particularly in the three-dimensional (real) space $\mathbb{R}^3$, $\hat{e}_1 = \hat{i} = (1, 0, 0)^T$, $\hat{e}_2 = \hat{j} = (0, 1, 0)^T$, $\hat{e}_3 = \hat{k} = (0, 0, 1)^T$ corresponding to x, y, z axes respectively.

Definition 4.1.3 (Standard Unit Vector)**.** A standard unit vector $\hat{e}_p$ consists of 1 at the p-th entry and 0 elsewhere. Mathematically, $[\hat{e}_p]_q = 1$ when $q = p$ and $[\hat{e}_p]_q = 0$ when $q \neq p$.

Below is an example of a vector in 3D x-y-z space ($\mathbb{R}^3$).



$$\vec{v} = \begin{bmatrix} 1.8 \\ 0.4 \\ 1.1 \end{bmatrix} = 1.8 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0.4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1.1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1.8\hat{\imath} + 0.4\hat{\jmath} + 1.1\hat{k}$$
$$= (1.8, 0.4, 1.1)^T$$

where we have written $\vec{v}$ in two forms, as a tuple and sum of the standard unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$.

4.1.2 Fundamental Vector Operations

Addition and Subtraction

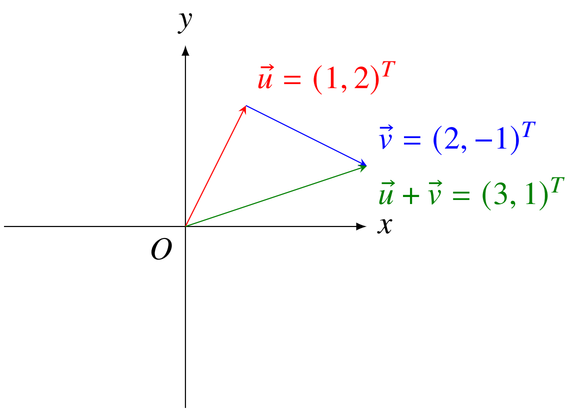
Same as their matrix counterpart, addition and subtraction between vectors is element-wise. Again, they are only valid for vectors of the same dimension. For $\vec{w} = \vec{u} \pm \vec{v}$, we have $w_i = u_i \pm v_i$. If

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

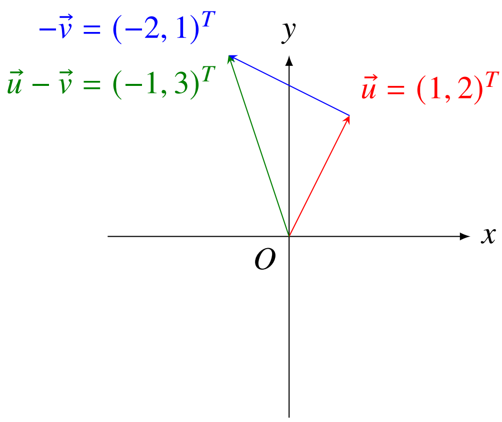
then

$$\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\vec{u} - \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$



Addition: The tail of the blue vector is placed to the head of the red vector, and the resultant green vector is from the origin to the head of blue vector.



Subtraction: Similar to addition but with the blue vector oriented in the opposite direction.

Scalar Multiplication

Multiplying a scalar (number) to a vector means that all components are multiplied by that scalar.

$$2\begin{bmatrix}2\\0\\1\\9\end{bmatrix}=\begin{bmatrix}4\\0\\2\\18\end{bmatrix}$$

Looking back at vector subtraction, it can be viewed as addition with a factor of -1.

$$\begin{bmatrix}7\\5\\9\end{bmatrix}-\begin{bmatrix}3\\6\\9\end{bmatrix}=\begin{bmatrix}7\\5\\9\end{bmatrix}+(-1)\begin{bmatrix}3\\6\\9\end{bmatrix}=\begin{bmatrix}7\\5\\9\end{bmatrix}+\begin{bmatrix}-3\\-6\\-9\end{bmatrix}=\begin{bmatrix}4\\-1\\0\end{bmatrix}$$

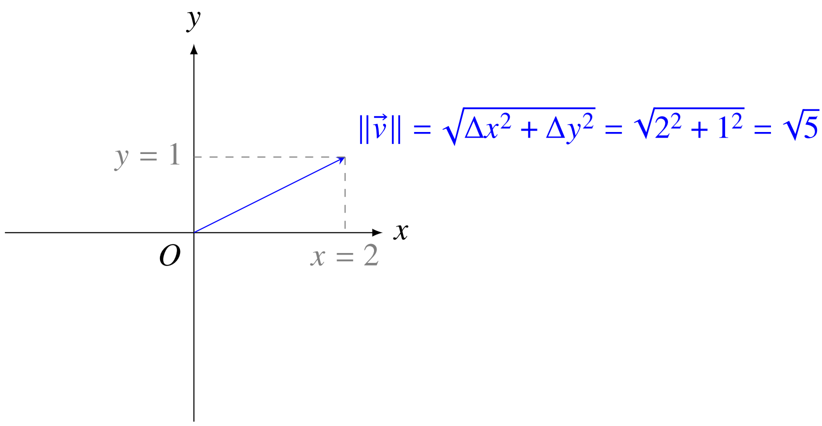
Length and Unit Vector

Length (Magnitude), or more formally ***Euclidean Norm***, of a vector $\vec{v}$ is based on a generalized version of ***Pythagoras' Theorem***, and is evaluated to be the square root of the sum of squares of components.

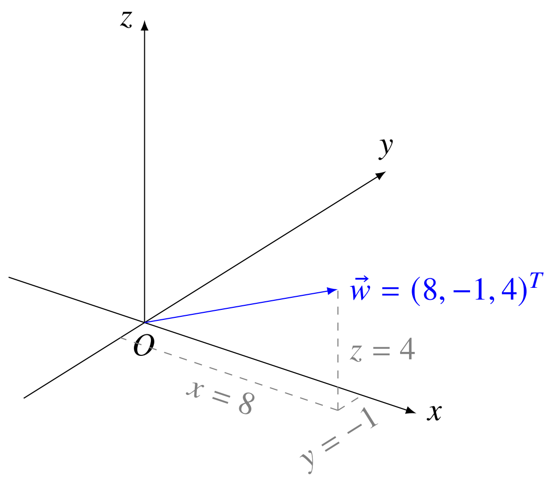
Definition 4.1.4 (Vector Length)**.** Length, or magnitude of a n-dimensional *real* vector $\vec{v}$, denoted by $\|\vec{v}\|$, is given by

$$\begin{aligned}\|\vec{v}\| &= \sqrt{v_1^2+v_2^2+v_3^2+\cdots+v_n^2}\\ &= \sqrt{\sum_{k=1}^{n} v_k^2}\end{aligned}$$

For instance, the length of a two-dimensional vector follows the usual Pythagoras' Theorem as below.



Here is another example which is three-dimensional.



$$\vec{w} = \begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix} \qquad \|\vec{w}\| = \sqrt{8^2 + (-1)^2 + 4^2} = 9$$

We can create a ***Unit Vector*** from some vector $\vec{v}$ that has a length of 1 and orients in the same direction as $\vec{v}$ is simply produced by dividing (normalizing) $\vec{v}$ by its distance $\|\vec{v}\|$.

Definition 4.1.5 (Unit Vector)**.** The unit vector corresponding to a non-zero vector $\vec{v}$ is denoted as $\hat{v}$ and is given by

$$\hat{v} = \frac{1}{\|\vec{v}\|}\vec{v}$$

where $\|\vec{v}\|$ is defined as in Definition 4.1.4.

Note that despite vectors can carry physical units, unit vectors are all *dimensionless* when formulated in this way.
Short Exercise: Find a unit vector for $\vec{w} = (8, -1, 4)^T$ in the previous example, and verify that it has a length of 1.[1]

4.2 Special Vector Operations

Now we are going to introduce two special types of vector operations: *dot product*, and *cross product*.

4.2.1 Dot Product

(Real) Dot Product (or ***Scalar Product***) is defined for two (real) vectors that have the same number of dimension. Its value is the sum of products of paired components between the two vectors. In other words, it can be regarded as the matrix product between a row vector ($1 \times m$ matrix) and a column vector ($m \times 1$ matrix).

Definition 4.2.1 (Dot Product (Real))**.** The dot product between two n-dimensional *real* vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are denoted as either $\vec{u} \cdot \vec{v}$, or by matrix notation $\mathbf{u}^T\mathbf{v}$. They are defined as

$$\vec{u} \cdot \vec{v} = \mathbf{u}^T\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n$$

[1] $\|\vec{w}\| = 9$, $\hat{w} = \vec{w}/\|\vec{w}\| = \frac{1}{9}(8, -1, 4)^T = (\frac{8}{9}, -\frac{1}{9}, \frac{4}{9})^T$, $\|\hat{w}\| = \sqrt{(\frac{8}{9})^2 + (-\frac{1}{9})^2 + (\frac{4}{9})^2} = 1$.

$$= \sum_{k=1}^{n} u_k v_k$$

which is a scalar quantity.

Conversely, it can be said that entries of a matrix product are vector dot products between the corresponding row and column. It is emphasized that we are restricting ourselves to real entries since complex vectors introduce extra complications. Then, for two *real* matrices expressed in the form of combined row/column vectors,

$$A = [\vec{u}^{(1)}|\vec{u}^{(2)}|\cdots|\vec{u}^{(m)}]^T \qquad B = [\vec{v}^{(1)}|\vec{v}^{(2)}|\cdots|\vec{v}^{(m)}]$$

$$= \begin{bmatrix} \vec{u}_1^{(1)} & \vec{u}_2^{(1)} & \cdots & \vec{u}_n^{(1)} \\ \vec{u}_1^{(2)} & \vec{u}_2^{(2)} & \cdots & \vec{u}_n^{(2)} \\ \vdots & \vdots & & \vdots \\ \vec{u}_1^{(m)} & \vec{u}_2^{(m)} & \cdots & \vec{u}_n^{(m)} \end{bmatrix} \qquad = \begin{bmatrix} \vec{v}_1^{(1)} & \vec{v}_1^{(2)} & \cdots & \vec{v}_1^{(m)} \\ \vec{v}_2^{(1)} & \vec{v}_2^{(2)} & \cdots & \vec{v}_2^{(m)} \\ \vdots & \vdots & & \vdots \\ \vec{v}_n^{(1)} & \vec{v}_n^{(2)} & \cdots & \vec{v}_n^{(m)} \end{bmatrix}$$

(notice that the expression of A has a transpose) their matrix product AB can be written as

$$AB = \begin{bmatrix} \vec{u}^{(1)} \cdot \vec{v}^{(1)} & \vec{u}^{(1)} \cdot \vec{v}^{(2)} & \cdots & \vec{u}^{(1)} \cdot \vec{v}^{(m)} \\ \vec{u}^{(2)} \cdot \vec{v}^{(1)} & \vec{u}^{(2)} \cdot \vec{v}^{(2)} & \cdots & \vec{u}^{(2)} \cdot \vec{v}^{(m)} \\ \vdots & \vdots & & \vdots \\ \vec{u}^{(m)} \cdot \vec{v}^{(1)} & \vec{u}^{(m)} \cdot \vec{v}^{(2)} & \cdots & \vec{u}^{(m)} \cdot \vec{v}^{(m)} \end{bmatrix}$$

In addition, it is easy to see that

Properties 4.2.2. The length of a vector, as defined in Definition 4.1.4, can be written using its dot product between itself as

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} \qquad \text{or} \qquad ||\vec{v}||^2 = \vec{v} \cdot \vec{v}$$

Notice that $\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2 \geq 0$. This quantity is always strictly greater than zero ($\vec{v} \cdot \vec{v} > 0$) unless $\vec{v} = \mathbf{0}$ is the zero vector (then $\vec{v} \cdot \vec{v} = 0$), which makes sense physically given that it represents length.

Example 4.2.1. If $\vec{u} = (1, 2, 3, 4, 5)^T$ and $\vec{v} = (-1, 0, 1, 0, -1)^T$, find the dot product $\vec{u} \cdot \vec{v} = \mathbf{u}^T\mathbf{v}$.

Solution.

$$\vec{u} \cdot \vec{v} = (1)(-1) + (2)(0) + (3)(1) + (4)(0) + (5)(-1) = -3$$

Alternatively,

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = -3$$

□

Here are some properties of dot product.

Properties 4.2.3. For three n-dimensional vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, the following establishes.

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \vec{v} \cdot \vec{u} && \text{Commutative Property} \\ \vec{u} \cdot (\vec{v} \pm \vec{w}) &= \vec{u} \cdot \vec{v} \pm \vec{u} \cdot \vec{w} && \text{Distributive Property} \\ (\vec{u} \pm \vec{v}) \cdot \vec{w} &= \vec{u} \cdot \vec{w} \pm \vec{v} \cdot \vec{w} && \text{Distributive Property} \\ (a\vec{u}) \cdot (b\vec{v}) &= ab(\vec{u} \cdot \vec{v}) && \text{where } a, b \text{ are some constants} \end{aligned}$$

Additionally, if A is an $n \times n$ square matrix, then

$$\begin{aligned} \vec{u} \cdot (A\vec{v}) &= \mathbf{u}^T(A\mathbf{v}) = (A^T\mathbf{u})^T\mathbf{v} = (A^T\vec{u}) \cdot \vec{v} \\ (A\vec{u}) \cdot \vec{v} &= (A\mathbf{u})^T\mathbf{v} = \mathbf{u}^T(A^T\mathbf{v}) = \vec{u} \cdot (A^T\vec{v}) \end{aligned}$$

where we have used Definition 4.2.1 and Properties 2.1.4.

Example 4.2.2. For $\vec{u} = (1,3,1)^T$ and $\vec{v} = (2,-1,1)^T$, find $\|(\vec{u}+\vec{v})\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$.

Solution. By Properties 4.2.3, we can rewrite the expression as

$$\begin{aligned}(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) &= \vec{u}\cdot(\vec{u}+\vec{v}) + \vec{v}\cdot(\vec{u}+\vec{v}) \\ &= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} \\ &= \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}\end{aligned}$$

Subsequently,

$$\begin{aligned}&\vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} \\ &= (1,3,1)^T\cdot(1,3,1)^T + 2((1,3,1)^T\cdot(2,-1,1)^T) + (2,-1,1)^T\cdot(2,-1,1)^T \\ &= (1^2+3^2+1^2) + 2((1)(2)+(3)(-1)+(1)(1)) + (2^2+(-1)^2+1^2) \\ &= 11 + 2(0) + 6 \\ &= 17\end{aligned}$$

Alternatively, one can calculate $\vec{w} = \vec{u}+\vec{v} = (1,3,1)^T + (2,-1,1)^T = (3,2,2)^T$ and find $\vec{w}\cdot\vec{w} = \|\vec{w}\|^2$ instead. (which is easier) □

Example 4.2.3. Given $\vec{u}$ and $\vec{v}$ as defined in the example above, if

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & -1 \end{bmatrix}$$

verify that $\vec{u}\cdot(A\vec{v}) = (A^T\vec{u})\cdot\vec{v}$.

Solution.

$$A\vec{v} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
&= \begin{bmatrix} (1)(2) + (2)(-1) + (1)(1) \\ (2)(2) + (0)(-1) + (3)(1) \\ (1)(2) + (1)(-1) + (-1)(1) \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} \\
\vec{u} \cdot (A\vec{v}) &= (1, 3, 1)^T \cdot (1, 7, 0)^T \\
&= (1)(1) + (3)(7) + (1)(0) \\
&= 22
\end{aligned}$$

On the other hand,

$$\begin{aligned}
A^T\vec{u} &= \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} (1)(1) + (2)(3) + (1)(1) \\ (2)(1) + (0)(3) + (1)(1) \\ (1)(1) + (3)(3) + (-1)(1) \end{bmatrix} \\
&= \begin{bmatrix} 8 \\ 3 \\ 9 \end{bmatrix} \\
(A^T\vec{u}) \cdot \vec{v} &= (8, 3, 9)^T \cdot (2, -1, 1)^T \\
&= (8)(2) + (3)(-1) + (9)(1) \\
&= 22
\end{aligned}$$

□

Geometric Meaning of Dot Product

The geometric meaning of dot product is embedded in the relation below.

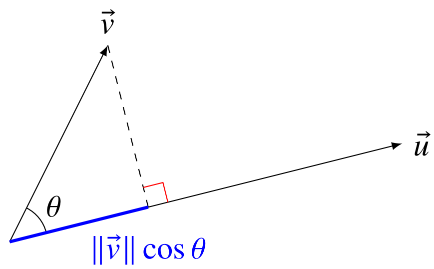
Properties 4.2.4. For two vectors $\vec{u}$ and $\vec{v}$ that are of the same dimension, we have

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$. Furthermore, if $\hat{u}$ and $\hat{v}$ are unit vectors (Definition 4.1.5), it reduces to

$$\hat{u} \cdot \hat{v} = \cos\theta$$

This means that the dot product between two vectors $\vec{u}$ and $\vec{v}$ is geometrically the signed product between $\vec{u}$ and the parallel component (projection) of $\vec{v}$ onto $\vec{u}$ (or vice versa), which is illustrated in the figure below. While an angle has a clear physical meaning only in a two/three-dimensional space, such relation generalizes the idea of an angle to higher dimensions.



Example 4.2.4. Find the angle between $\vec{u}$ and $\vec{v}$ in Example 4.2.1.

Solution. From Example 4.2.1, we have $\vec{u} \cdot \vec{v} = -3$, and

$$\begin{aligned}
\|\vec{u}\| &= \sqrt{1^2 + 2^2 + 3^2 + 4^2 + 5^2} = \sqrt{55} \\
\|\vec{v}\| &= \sqrt{(-1)^2 + 0^2 + 1^2 + 0^2 + (-1)^2} = \sqrt{3}
\end{aligned}$$

By Properties 4.2.4, we have

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$$
\begin{aligned}
&= \frac{-3}{(\sqrt{55})(\sqrt{3})} \\
&\approx -0.2335 \\
\theta &\approx 1.806\,\text{rad}
\end{aligned}
$$

□

By Properties 4.2.4, if the absolute value of the dot product $|\vec{u} \cdot \vec{v}|$ is equal to $\|\vec{u}\|\|\vec{v}\|$, where $\vec{u}$ and $\vec{v}$ are non-zero vectors, then it implies that $\cos\theta = \pm 1$, θ is either 0 or π, and hence the two vectors are parallel. On the other hand, we have the following observation.

Properties 4.2.5. If the dot product between two vectors $\vec{u}$ and $\vec{v}$ is zero ($\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} = 0$), then by Properties 4.2.4, $\cos\theta = 0$ and the angle θ between $\vec{u}$ and $\vec{v}$ is $\frac{\pi}{2}$. In this case, $\vec{u}$ and $\vec{v}$ are said to be perpendicular, or *orthogonal*. The converse is also true.

From this, the concept of "***Orthogonal***" becomes an extension of "perpendicular" in higher dimensions. Note that *the zero vector is regarded to be orthogonal to any vector*, so even if $\vec{u}$ or $\vec{v}$ is a zero vector, this properties still hold.

Some may notice that as $-1 \leq \cos\theta \leq 1$, if $|\vec{u} \cdot \vec{v}| > \|\vec{u}\|\|\vec{v}\|$, then θ will be undefined in Properties 4.2.4. However, the ***Cauchy–Schwarz Inequality*** ensures this will not happen.

Theorem 4.2.6 (Cauchy–Schwarz Inequality)**.** Given two *real* vectors $\vec{u}$ and $\vec{v}$ that are n-dimensional ($\mathbb{R}^n$), the following inequality holds.

$$
\begin{aligned}
|\vec{u} \cdot \vec{v}| &\leq \|\vec{u}\|\|\vec{v}\| \\
|u_1v_1 + u_2v_2 + \cdots + u_nv_n| &\leq \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}
\end{aligned}
$$

Proof. Consider $\vec{w} = \vec{u} + t\vec{v}$, where t is any scalar, then $\|\vec{w}\|^2 = \vec{w} \cdot \vec{w}$ can be written as a quadratic polynomial as

$$(\vec{u} + t\vec{v}) \cdot (\vec{u} + t\vec{v}) = \|\vec{u}\|^2 + 2t(\vec{u} \cdot \vec{v}) + t^2\|\vec{v}\|^2$$

using Properties 4.2.3. Now notice that from Properties 4.2.2, we have

$$\|\vec{u}\|^2 + 2t(\vec{u}\cdot\vec{v}) + t^2\|\vec{v}\|^2 = (\vec{u}+t\vec{v})\cdot(\vec{u}+t\vec{v}) = \|\vec{u}+t\vec{v}\|^2 \geq 0$$

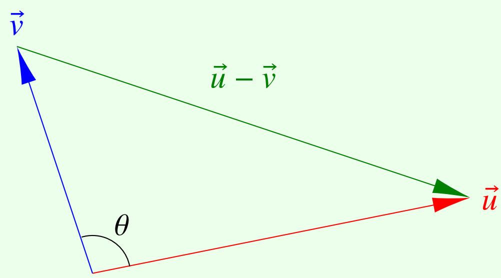
Since it is a quadratic polynomial, and we have shown that it is always greater than or equal to zero, i.e. has no root or a repeated root, it means that the discriminant must be negative or zero. So,

$$\begin{aligned}
\Delta = b^2 - 4ac &\leq 0 \\
(2(\vec{u}\cdot\vec{v}))^2 - 4\|\vec{u}\|^2\|\vec{v}\|^2 &\leq 0 \\
(\vec{u}\cdot\vec{v})^2 - \|\vec{u}\|^2\|\vec{v}\|^2 &\leq 0 \\
(\vec{u}\cdot\vec{v})^2 &\leq \|\vec{u}\|^2\|\vec{v}\|^2 \\
|\vec{u}\cdot\vec{v}| &\leq \|\vec{u}\|\|\vec{v}\|
\end{aligned}$$

□

Short Exercise: Think about under what circumstances the Cauchy–Schwarz Inequality turns into an equality (i.e. $|\vec{u}\cdot\vec{v}| = \|\vec{u}\|\|\vec{v}\|$).[2]

Example 4.2.5. Prove the ***Cosine Law*** by considering the triangle below



and expanding the dot product $\|(\vec{u}-\vec{v})\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$.

Solution. Let denote the lengths $\|\vec{u}\|$, $\|\vec{v}\|$, $\|(\vec{u}-\vec{v})\|$ be a, b, c, then

$$c^2 = \|(\vec{u}-\vec{v})\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \qquad \text{(Properties 4.2.2)}$$

[2] When $\vec{u}$ and $\vec{v}$ are parallel, i.e. $\vec{u} = k\vec{v}$ for some scalar k, or $\vec{v} = \mathbf{0}$.

$$
\begin{aligned}
&= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} && \text{(Properties 4.2.3)}\\
&= \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 && \text{(Properties 4.2.2 and 4.2.3)}\\
&= \|\vec{u}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta + \|\vec{v}\|^2 && \text{(Properties 4.2.4)}\\
&= a^2 - 2ab\cos\theta + b^2
\end{aligned}
$$

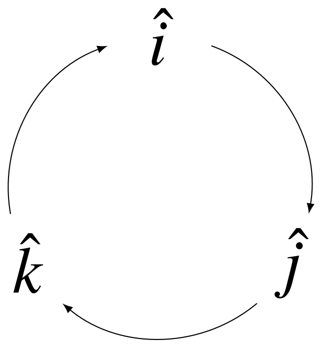
□

4.2.2 Cross Product

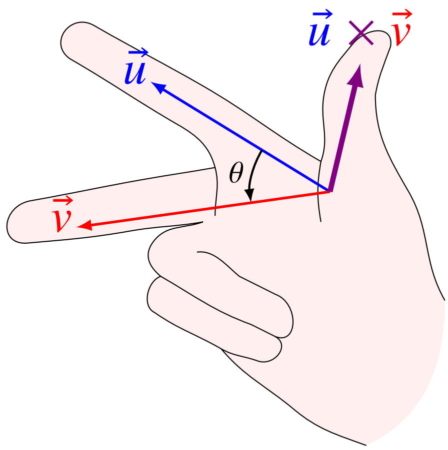
Another important type of vector product is the ***Cross Product*** (or sometimes just ***Vector Product***), which returns a three-dimensional vector from two other three-dimensional vectors as inputs. *The output vector has to be orthogonal to the two input vectors*, and the direction is determined by the ***Right Hand Rule***. Motivated by these requirements, we have the following basic definitions of cross product between the three standard unit vectors in $\mathbb{R}^3$.

Definition 4.2.7. The computation of cross products (denoted by $\times$) involving the standard unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ in $\mathbb{R}^3$ obeys the following rules.

1. $\hat{\imath}\times\hat{\jmath} = \hat{k}$, $\hat{\jmath}\times\hat{\imath} = -\hat{k}$,
2. $\hat{\jmath}\times\hat{k} = \hat{\imath}$, $\hat{k}\times\hat{\jmath} = -\hat{\imath}$,
3. $\hat{k}\times\hat{\imath} = \hat{\jmath}$, $\hat{\imath}\times\hat{k} = -\hat{\jmath}$, and
4. $\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = \mathbf{0}$



A cyclic diagram for memorizing Definition 4.2.7. A clockwise / anti-clockwise permutation produces a positive / negative unit vector of the third.



Demonstration of the right hand rule.

Cross product shares pretty much the same properties with dot product (compared to Properties 4.2.3), except the (anti-)commutative part as the right hand rule specifically requires chirality that is compatible with Definition 4.2.7. Also, be aware of the difference that cross product produces another three-dimensional vector, but dot product yields a scalar.

Properties 4.2.8. For two three-dimensional vectors $\vec{u}$ and $\vec{v}$, we have

$$\begin{aligned}
\vec{u} \times \vec{v} &= -\vec{v} \times \vec{u} && \text{Anti-commutative Property} \\
\vec{u} \times (\vec{v} \pm \vec{w}) &= \vec{u} \times \vec{v} \pm \vec{u} \times \vec{w} && \text{Distributive Property} \\
(\vec{u} \pm \vec{v}) \times \vec{w} &= \vec{u} \times \vec{w} \pm \vec{v} \times \vec{w} && \text{Distributive Property} \\
(a\vec{u}) \times (b\vec{v}) &= ab(\vec{u} \times \vec{v}) && \text{where } a, b \text{ are some constants}
\end{aligned}$$

The calculation of cross product then follows from the rules above, leading to the determinant shorthand below.

Properties 4.2.9. For $\vec{u} = (u_1, u_2, u_3)^T$ and $\vec{v} = (v_1, v_2, v_3)^T$ in $\mathbb{R}^3$, their cross product $\vec{u} \times \vec{v}$ can be written in the form of a determinant as

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Proof. Starting from Definition 4.2.7 and Properties 4.2.8, we have

$$\begin{aligned}
\vec{u} \times \vec{v} &= (u_1\hat{i} + u_2\hat{j} + u_3\hat{k}) \times (v_1\hat{i} + v_2\hat{j} + v_3\hat{k}) \\
&= u_1v_1(\hat{i} \times \hat{i}) + u_1v_2(\hat{i} \times \hat{j}) + u_1v_3(\hat{i} \times \hat{k}) \\
&\quad + u_2v_1(\hat{j} \times \hat{i}) + u_2v_2(\hat{j} \times \hat{j}) + u_2v_3(\hat{j} \times \hat{k}) \\
&\quad + u_3v_1(\hat{k} \times \hat{i}) + u_3v_2(\hat{k} \times \hat{j}) + u_3v_3(\hat{k} \times \hat{k}) && \text{(Properties 4.2.8)} \\
&= u_1v_1(\mathbf{0}) + u_1v_2(\hat{k}) - u_1v_3(\hat{j}) \\
&\quad - u_2v_1(\hat{k}) + u_2v_2(\mathbf{0}) + u_2v_3(\hat{i}) \\
&\quad + u_3v_1(\hat{j}) - u_3v_2(\hat{i}) + u_3v_3(\mathbf{0}) && \text{(Definition 4.2.7)} \\
&= (u_2v_3 - u_3v_2)\hat{i} + (u_3v_1 - u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}
\end{aligned}$$

Meanwhile, expanding along the first row of the determinant form

$$\begin{aligned}
\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} &= \hat{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\
&= (u_2v_3 - u_3v_2)\hat{i} + (u_3v_1 - u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}
\end{aligned}$$

yields the identical results. □

Example 4.2.6. Given two vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

Find $\vec{u} \times \vec{v}$.

Solution.

$$
\begin{aligned}
\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 3 & -1 & 1 \end{vmatrix} \\
&= \hat{i} \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} && \text{(Cofactor Expansion along the first row)} \\
&= 2\hat{i} + 5\hat{j} - \hat{k} = (2, 5, -1)^T
\end{aligned}
$$

□

Short Exercise: Check if $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$ by finding the corresponding dot products.[3]
Short Exercise: Following the short exercise above, show in general, $\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\vec{u} \times \vec{v}) = 0$.[4]

Geometric Meaning of Cross Product

Similar to vector dot product, vector cross product has a geometric interpretation.

Properties 4.2.10. Given two vectors $\vec{u}$ and $\vec{v}$ which are both three-dimensional, the magnitude (length) of $\vec{u} \times \vec{v}$ is related to the angle between $\vec{u}$ and $\vec{v}$ as

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin \theta$$

Immediately, we know that if $\vec{u}$ and $\vec{v} = k\vec{u}$, where k is some constant, are parallel, their cross product will be a zero vector as $\theta = 0$ (or π) and $\sin \theta = 0$.

[3] $\vec{u} \cdot (\vec{u} \times \vec{v}) = (1, 0, 2)^T \cdot (2, 5, -1)^T = (1)(2) + (0)(5) + (2)(-1) = 0$, $\vec{v} \cdot (\vec{u} \times \vec{v}) = (3, -1, 1)^T \cdot (2, 5, -1)^T = (3)(2) + (-1)(5) + (1)(-1) = 0$. In both cases the zero dot product shows they are orthogonal.

[4] From the derivation of Properties 4.2.9, $\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\hat{i} + (u_3v_1 - u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$, and $\vec{u} \cdot (\vec{u} \times \vec{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0$ where all terms cancel out, similar for $\vec{v}$.

This is equivalent to the statement of $\vec{u} \times \vec{u} = \mathbf{0}$. (You can also arrive at this conclusion with Properties 4.2.8.[5])

Example 4.2.7. If $\vec{u} = (1, 2, 3)^T$, and $\vec{v} = (-1, 1, 0)^T$, find $(\vec{u} + 2\vec{v}) \times (\vec{u} - \vec{v})$.

Solution. Observe that

$$\begin{aligned}
(\vec{u} + 2\vec{v}) \times (\vec{u} - \vec{v}) &= \vec{u} \times (\vec{u} - \vec{v}) + 2\vec{v} \times (\vec{u} - \vec{v}) \\
&= \vec{u} \times \vec{u} - \vec{u} \times \vec{v} + 2\vec{v} \times \vec{u} - 2\vec{v} \times \vec{v} \\
&= \mathbf{0} - \vec{u} \times \vec{v} - 2\vec{u} \times \vec{v} - 2(\mathbf{0}) \\
&= -3\vec{u} \times \vec{v}
\end{aligned}$$

where the fact that $\vec{u} \times \vec{u} = 0$, $\vec{v} \times \vec{v} = 0$ and Properties 4.2.8 are used. Now, with Properties 4.2.9, we have

$$\begin{aligned}
-3\vec{u} \times \vec{v} &= -3 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{vmatrix} \\
&= -3 \left(\hat{i} \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} \right) \\
&= -3(-3\hat{i} - 3\hat{j} + 3\hat{k}) \\
&= 9\hat{i} + 9\hat{j} - 9\hat{k} = (9, 9, -9)^T
\end{aligned}$$

The readers can try the alternative of computing $\vec{u} + 2\vec{v}$ and $\vec{u} - \vec{v}$ first and then finally their cross product. □

Finally, cancellation of dot product or cross product at both sides of an equation is generally not correct, and here is a table summarizing the inputs and outputs of dot/cross product for clarification.

[5] The anti-commutative property requires $\vec{u} \times \vec{u} = -\vec{u} \times \vec{u}$ and hence $2(\vec{u} \times \vec{u}) = \mathbf{0}$.

	Input	Output
Dot Product, or Scalar Product ($\cdot$)	Two vectors of the same dimension, the order does not matter (commutative)	A scalar
Cross Product, or Vector Product ($\times$)	Two three-dimensional vectors ($\mathbb{R}^3$), the order is important (anti-commutative)	Another three-dimensional vector

4.3 Earth Science Applications

Example 4.3.1. The *Coriolis Effect* is a phenomenon describing the deflection of motion due to rotation of the Earth. It introduces an apparent force known as *Coriolis Force* which is given by $\overrightarrow{F_{\text{cor}}} = -2\vec{\Omega} \times \vec{v}$ where $\Omega = \left\|\vec{\Omega}\right\| = 7.292 \times 10^{-5}\,\text{rad}\,\text{s}^{-1}$ represents the angular speed of Earth's rotation, and $\vec{\Omega}$ is oriented in the direction of the North Pole. Define the local frame of reference (see Figure 4.1) with the x-direction being the zonal direction, y-direction being the meridional direction, and z-direction being the zenith direction (normal to the Earth's surface), then we have $\vec{v} = (u, v, w) = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$ as the flow velocity in this local Cartesian coordinate system with unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ along the x, y, z axes. It can be seen that $\vec{\Omega} = (\Omega \cos\varphi)\hat{\jmath} + (\Omega \sin\varphi)\hat{k}$ where φ is the latitude. Now by expanding $\overrightarrow{F_{\text{cor}}} = -2\vec{\Omega} \times \vec{v}$ show that the components of Coriolis Force along the local x, y, z directions are

$$\begin{aligned}
F_{\text{cor},x} &= 2\Omega(v \sin\varphi - w \cos\varphi) \\
F_{\text{cor},y} &= -2\Omega u \sin\varphi \\
F_{\text{cor},z} &= 2\Omega u \cos\varphi
\end{aligned}$$

The *Coriolis Parameter* f is usually used to denote the factor $2\Omega \sin\varphi$.

Solution. Using Properties 4.2.9 to expand $\overrightarrow{F_{\text{cor}}}$ gives

$$-2\vec{\Omega} \times \vec{v} = -2((\Omega \cos\varphi)\hat{\jmath} + (\Omega \sin\varphi)\hat{k}) \times (u\hat{\imath} + v\hat{\jmath} + w\hat{k})$$

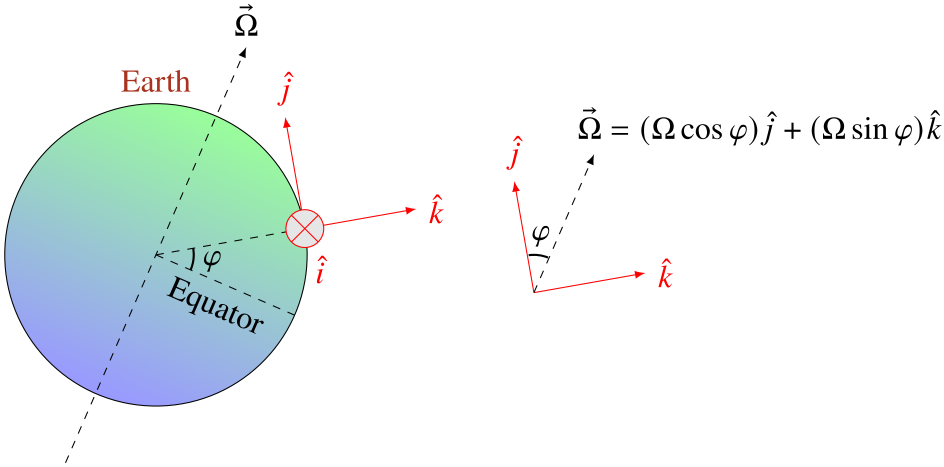


Figure 4.1: An illustration of the coordinate frame in Example 4.3.1.

$$\begin{aligned}
&= -2\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & \Omega\cos\varphi & \Omega\sin\varphi \\ u & v & w \end{vmatrix} \\
&= -2[(w\Omega\cos\varphi - v\Omega\sin\varphi)\hat{\imath} + (u\Omega\sin\varphi)\hat{\jmath} - (u\Omega\cos\varphi)\hat{k}] \\
&= [2\Omega(v\sin\varphi - w\cos\varphi)]\hat{\imath} + (-2\Omega u\sin\varphi)\hat{\jmath} + (2\Omega u\cos\varphi)\hat{k}
\end{aligned}$$

The $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ components correspond to $F_{\mathrm{cor},x}$, $F_{\mathrm{cor},y}$, $F_{\mathrm{cor},z}$ respectively. Assume w is negligible, then $F_{\mathrm{cor},x} = fv$ and $F_{\mathrm{cor},y} = -fu$. □

4.4 Python Programming

We can use one-dimensional **numpy** arrays as vectors.

```
import numpy as np

myVec1 = np.array([-1., 2., 4.])
myVec2 = np.array([2., 1., 3.])
```

Addition, subtraction, and scalar multiplication works just like for matrices.

```
myVec3 = -myVec1 + 2*myVec2
print(myVec3)
```

gives the expected output of `[5. 0. 2.]`. We can select a component of any vector by indexing. Again, remember that indices in *Python* start from zero. `print(myVec3[1])` then returns `0.0`. The magnitude of a vector can be checked with `np.linalg.norm`. For example,

```
print(np.linalg.norm(myVec1))
```

produces `4.58257569495584` ($\sqrt{(-1)^2+2^2+4^2} = \sqrt{21}$). Dot product is computed via `np.dot` as follows.

```
myDot = np.dot(myVec1, myVec2)
print(myDot)
```

which outputs `12.0` (as $(-1)(2)+(2)(1)+(4)(3) = 12$). Similarly, cross product is found by `np.cross`.

```
myCross = np.cross(myVec1, myVec2)
print(myCross)
```

then gives

```
[ 2. 11. -5.]
```

and we can check if the cross product is orthogonal to the two input vectors.

```
# All lines below return zero.
print(np.dot(myVec1, myCross))
print(np.dot(myVec2, myCross))
print(np.dot(myVec3, myCross))
```

Dot product is defined for any two vectors with the same dimension, but cross product is only defined for three-dimensional vectors (or in some other sense two-dimensional), so

```
myVec4 = np.array([1., 3., 2., 0.])
myVec5 = np.array([2., 1., 0., -1.])
print(np.dot(myVec4, myVec5))
```

yields a valid output of 5.0, but

```
print(np.cross(myVec4, myVec5))
```

raises the error of

```
ValueError: incompatible dimensions for cross product
(dimension must be 2 or 3)
```

Finally, we note that following this Stack Overflow post (2827393), we can compute the unit vector of any given vector and angle between two vectors (based from the second observation in Properties 4.2.4, $\theta = \cos^{-1}(\hat{u} \cdot \hat{v})$).

```
def unit_vector(vector):
    """ Returns the unit vector of the vector.  """
    return vector / np.linalg.norm(vector)

def angle_between(v1, v2):
    """ Returns the angle in radians between vectors 'v1' and
        'v2'. """
    v1_u = unit_vector(v1)
    v2_u = unit_vector(v2)
    return np.arccos(np.clip(np.dot(v1_u, v2_u), -1.0, 1.0))
```

The `np.clip` is to avoid numerical round-off error that causes the dot product of the two normalized input vectors to just fall outside (e.g. `1.0000000000000002`) the valid range $[-1, 1]$ of $\cos^{-1}$. The naive way of (here the lists will be cast to one-dimensional arrays automatically during calculation.)

```
np.arccos(np.dot([1., 0, 0], [2., 0, 0]))
```

leads to the warning of

```
RuntimeWarning: invalid value encountered in arccos
nan
```

but

```
angle_between([1., 0, 0], [2., 0, 0])
```

gives `0.0` properly. Trying this on `myVec4` and `myVec5` with

```
print(unit_vector(myVec4))
print(angle_between(myVec4, myVec5))
```

produces a unit vector of `[0.267 0.802 0.535 0.    ]`, and an angle of `0.993757` (in radians).

4.5 Exercises

Exercise 4.1 For $\vec{u} = (1, 3, 3, 7)^T$ and $\vec{v} = (1, 2, 2, 5)^T$, find

(a) $\vec{u} + \vec{v}$,

(b) $\frac{3}{2}\vec{u} - \frac{1}{2}\vec{v}$,

(c) $\vec{u} \cdot \vec{v}$,

(d) $\vec{v} \cdot \vec{u}$,

(e) $(\vec{u} - 2\vec{v}) \cdot (2\vec{u} + \vec{v})$.

Exercise 4.2 For $\vec{u} = (7, 4, 1)^T$, $\vec{v} = (8, 1, 1)^T$, and

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Verify that

(a) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$,

(b) $\vec{u} \cdot (\vec{Av}) = (A^T\vec{u}) \cdot \vec{v}$,

(c) Compute $(3\vec{u} - 4\vec{v}) \cdot (\vec{u} \times \vec{v})$, is the answer what you expect?

Exercise 4.3 For $\vec{u} = (1, -3, 9)^T$ and $\vec{v} = (1, -2, 4)^T$, find

(a) Their unit vectors $\hat{u}$ and $\hat{v}$,

(b) The angle between them, by calculating their dot product,

(c) The cross product $\vec{u} \times \vec{v}$, and

(d) Show that the vector obtained from the cross product above is orthogonal (perpendicular) to $\vec{u}$ and $\vec{v}$, by calculating the corresponding dot products.

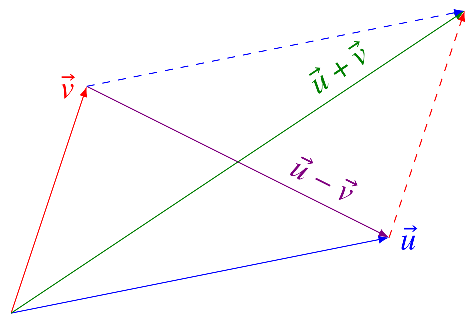


Figure 4.2: The parallelogram constructed by vectors for Exercise 4.6.

Exercise 4.4 The following table contains incomplete data about the movement of several typhoons at some moments. Complete the table by filling in the blanks. The first one has been done as an example.

Typhoon Name	Time	Speed	Direction	Vector Form
Nuri	2008/08/22, 08:00	$13\,\mathrm{km\,h^{-1}}$	315°	$(-9.192, 9.192)$
Vicente	2012/07/24, 02:00	$18\,\mathrm{km\,h^{-1}}$	299°	
Linfa	2015/07/09, 23:00			$(-13.595, -6.339)$
Mangkhut	2018/09/16, 22:00		288°	$(\quad, 7.725)$

Exercise 4.5 Prove the Triangular Inequality.

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Exercise 4.6 Prove the Parallelogram Law. (See Figure 4.2)

$$2\|\vec{u}\|^2 + 2\|\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2$$

Exercise 4.7 Show that Coriolis Force derived in Example 4.3.1 does zero work and hence is consistent with the fact that it is an apparent force and never produces/consumes energy by itself.

Chapter 5

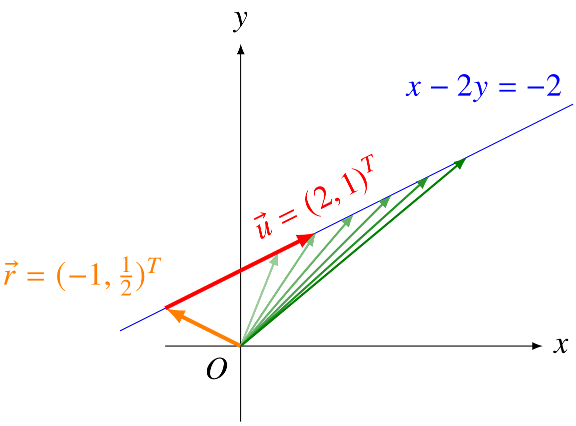
More on Vector Geometry

Vectors provides valuable assistance when it comes to describing geometric objects. In this chapter we are going to exploit the knowledge learnt in the previous chapters to solve geometry problems and inspect more deeply the intimate relationship between vectors, dot/cross products, and geometry.

5.1 Lines and Planes

(Straight) lines and *planes* are geometric shapes of importance in two/three-dimensional spaces ($\mathbb{R}^2$ and $\mathbb{R}^3$) and due to their simplicity they will be the first to be discussed. They can be expressed either in terms of (a) an equation, and (b) vectors. We will start from the easier case of a line.

Since a straight line is a one-dimensional object, the vector form of such a line can be expressed by a fixed vector that points to its initial position, plus another vector oriented along the line's direction, times an arbitrary parameter which controls its extension or contraction, so that it traces out the line when changed continuously.



The graph of $x - 2y = -2$ can take the vector form of $\overrightarrow{OP} = \vec{r} + t\vec{u} = (-1, \frac{1}{2})^T + t(2, 1)^T$. The orange/red arrow represents the initial position/direction, and the locus of green arrow is controlled by t like a slider. The cases for $t = 0.75, 1, 1.25, 1.5, 1.75, 2$ are shown.

Short Exercise: Choose any value of t and substitute that value into the expression of $\overrightarrow{OP}$ above to see if the x and y-components satisfy the starting equation. Also, try to increase/decrease the value of t to observe how the vector traces out the desired straight line.[1]

5.1.1 Translating Equation Form to Vector Form

The general equation form of a line on an x-y plane is $ax + by = h$, resembling a linear system of one equation with two unknowns. From Section 3.2.1, it can be observed that it has infinitely many solutions and possesses a free variable. Let $y = t$, then rearranging the equation we have $x = (h - bt)/a$ where t is any scalar. Denote the origin as O and any point on the line as P, then

$$\overrightarrow{OP} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{h}{a} - \frac{b}{a}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{h}{a} \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{b}{a} \\ 1 \end{bmatrix}$$

[1] Let's say $t = -0.25$, $\overrightarrow{OP} = (-1, 0.5)^T + (-0.25)(2, 1)^T = (-1.5, 0.25)^T$, $x - 2y = (-1.5) - 2(0.25) = -2$.

This is one possible vector form of the line. The ideas behind can be borrowed from Example 3.2.3, with $(\frac{h}{a}, 0)^T$ being the particular solution/initial position, and $(-\frac{b}{a}, 1)^T$ as the direction of that line, multiplied by a free parameter to complete the general solution. For example, if we have $3x - 2y = 5$, then by the same method, we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{3} + \frac{2}{3}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Bear in mind that the direction vector from the general solution can be scaled freely. In addition, any initial position vector (particular solution) can be chosen as long as it links to a point on the line and satisfies the equation. (Refer to the discussion about particular/general solution in Section 3.2.1) Hence there is no unique vector form for a line. For instance,

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + t_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

is equivalent to

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

for the line equation $2x - y = -1$.
Short Exercise: Check the equivalence of the two vector forms above by choosing a value for t_1 and finding the corresponding t_2 so that the vector points to the same position.[2]
Short Exercise: What is the vector form of the equation $ax + by = h$ for the degenerate case $a = 0$?[3]

---

[2]For $t_1 = 1$, we have $(1, 3)^T + (1)(2, 4)^T = (3, 7)^T$ as a point on the line, and for the another vector form $(-1, -1)^T + t_2(1, 2)^T = (3, 7)^T$ to coincide we have $t_2 = 4$. In this case, it can be shown that the general relation between the two forms is $t_2 = 2t_1 + 2$, as

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + t_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \left(\begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right) + 2t_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + (2t_1 + 2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

[3]The equation is reduced to $y = \frac{h}{b}$ and we select $x = t$ as the free variable instead.
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ \frac{h}{b} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{h}{b} \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

5.1.2 Recovering Equation Form from Vector Form

On the other hand, inferring line equation from the vector form is not straightforward at first sight. Since the vector form of a line always contains an arbitrary parameter, which is absent in the equation form, the motivation is to remove the parameter through some manipulation.

Remember that from Properties 4.2.5 the dot product between orthogonal (perpendicular) vectors returns zero. This means that by carrying out dot product with the ***Normal Vector*** which is orthogonal to the direction vector, on both sides of the vector form will eliminate the parameter and recover the line equation. For example, given that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

We know that $(4, -1)^T$ is a normal vector orthogonal to the direction vector (see the short exercise below). So, by taking dot product with $(4, -1)^T$ on both sides, we have

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix} \\ 4x - y &= 1 + (0)t = 1 \end{aligned}$$

Notice that the coefficients of the equation are the same as the components of the normal vector.
Short Exercise: Verify that (a, b) is always orthogonal to $(b, -a)$, and vice versa.[4]

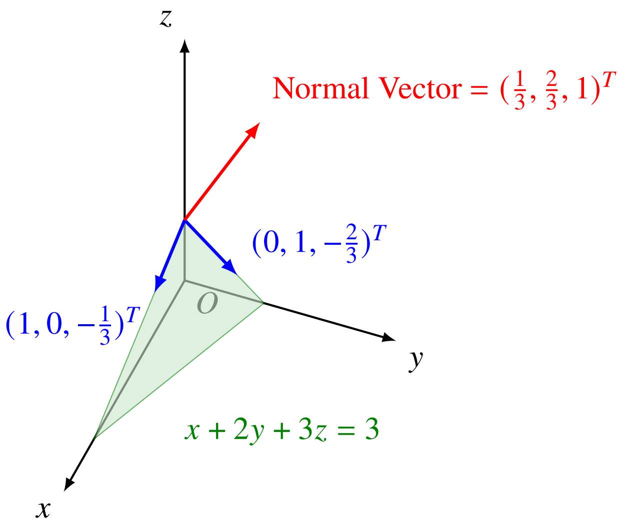
5.1.3 Generalizing to Higher Dimension

Similar concepts can be applied on the equation and vector form for planes. General form of equation of a plane in 3D space is $ax + by + cz = h$, which is a

[4] $(a, b)^T \cdot (b, -a)^T = (a)(b) + (b)(-a) = 0$

linear system of one equation with three unknowns, from the demonstration in Section 3.2.1 we know there are two free variables and two direction vectors for such a plane. By assigning the free variables to non-pivots, we obtain the vector form of the plane.

Recall from Section 4.2.2, cross product of any two non-parallel vectors on the plane will give a third vector normal to the plane. Subsequently, we can take the dot product with this newly obtained normal vector to convert the vector form back to a plane equation. Again, the coefficients of the plane equation match the components of the normal vector, differed at most by a multiplicative factor.



The plane represented by the equation $x + 2y + 3z = 3$. Notice that the normal vector can be found via computing $(1, 0, -\frac{1}{3})^T \times (0, 1, -\frac{2}{3})^T = (\frac{1}{3}, \frac{2}{3}, 1)^T$. The normal vector is magnified for the purpose of illustration.

Example 5.1.1. Transform the plane equation $2x + 3y + z = 4$ to vector form and convert the acquired vector form back to the starting equation to check consistency.

Solution. For the first part, we can let $y = s$, $z = t$, then from the plane equation we have $x = \frac{1}{2}(4 - 3s - t)$ and hence

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(4-3s-t) \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

where $-\infty < s < \infty$, $-\infty < t < \infty$ are some free parameters. To recover the original equation, we can find the normal vector by doing cross product on the two direction vectors obtained above. By Properties 4.2.9, it is

$$\begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{vmatrix} = \hat{\imath} + \frac{3}{2}\hat{\jmath} + \frac{1}{2}\hat{k}$$

The next step is to take the dot product with the normal vector just retrieved.

$$\begin{aligned}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} &= \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} + s\begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} + t\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} \\
x + \frac{3}{2}y + \frac{1}{2}z &= 2 + s(0) + t(0) = 2 \\
\rightarrow 2x + 3y + z &= 4
\end{aligned}$$

$\square$

The correspondence between the coefficients of a linear equation and components of its normal vector is not a coincidence. In fact, even for higher dimensional cases, where there is no intuitive geometric interpretation, we still have the following results.

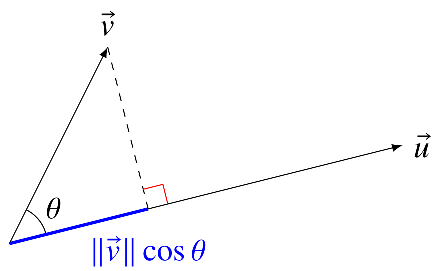
Properties 5.1.1. For an equation in the form of $a_1x_1+a_2x_2+a_3x_3+\cdots+a_nx_n = h$, it has a normal vector of $(a_1, a_2, a_3, \ldots, a_n)^T$.

The procedures carried in the last example can be similarly applied to higher dimensional situations where the equation now represents a ***Hyperplane***.

5.2 More on Geometric Applications of Dot Product

5.2.1 Projection

We have mentioned in Properties 4.2.4 that dot product between two vectors is related to the projection of one vector onto another. By rearranging the formula of Properties 4.2.4, we can derive the length of projection as follows.



Properties 5.2.1. For two vectors $\vec{u}$ and $\vec{v}$ having the same dimension, denote the ***(signed) Scalar Projection*** of $\vec{v}$ onto $\vec{u}$ by $\widetilde{\text{proj}}_u v$. It is computed according to

$$\widetilde{\text{proj}}_u v = \|\vec{v}\| \cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}$$

If we want to supply the projection along $\vec{u}$ with directionality, then we can utilize its unit vector $\hat{u}$ to make it a ***Vector Projection***:

$$\overrightarrow{\text{proj}}_u v = (\widetilde{\text{proj}}_u v)\hat{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}\hat{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{u}$$

$$= (\widetilde{\text{proj}}_u v)\frac{\vec{u}}{\|\vec{u}\|}$$

where we have used Definition 4.1.5 to write out the unit vector.

Example 5.2.1. Find the projection of $\vec{v} = -2\hat{i} + 3\hat{j} - \hat{k}$ onto $\vec{u} = 4\hat{i} + \hat{j} - 3\hat{k}$ using Properties 5.2.1.

Solution. The signed scalar projection of $\vec{v}$ into $\vec{u}$ is

$$\begin{aligned}\widetilde{\text{proj}}_u v &= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}\\ &= \frac{(-2)(4)+(3)(1)+(-1)(-3)}{\sqrt{(4)^2+(1)^2+(-3)^2}}\\ &= -\frac{2}{\sqrt{26}} = -\frac{\sqrt{26}}{13}\end{aligned}$$

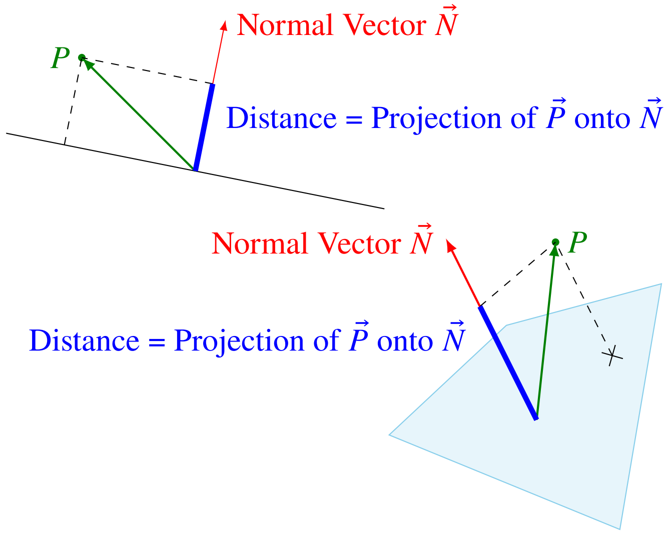
and the vector projection is

$$\begin{aligned}\overrightarrow{\text{proj}}_u v &= (\widetilde{\text{proj}}_u v)\frac{\vec{u}}{\|\vec{u}\|}\\ &= (-\frac{\sqrt{26}}{13})\left(\frac{4\hat{i}+\hat{j}-3\hat{k}}{\sqrt{26}}\right)\\ &= -\frac{1}{13}(4\hat{i}+\hat{j}-3\hat{k}) = (-\frac{4}{13}, -\frac{1}{13}, \frac{3}{13})^T\end{aligned}$$

□

5.2.2 Distance

Distance of a point to a line/plane (in $\mathbb{R}^2/\mathbb{R}^3$ respectively) can be found by the projection of any vector starting somewhere from the line/plane to the point, onto the normal vector of that line/plane, as shown in the figures below.



Example 5.2.2. Find the distance from the plane $x - 2y + 3z = 6$ to the point $(3, 3, 6)^T$.

Solution. From the equation of the plane, and by Properties 5.1.1, it can be inferred that the normal vector of the plane is $\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$. We can select any point on the plane as we wish, let's say $(4, 2, 2)^T$, and the vector from there to the point $(3, 3, 6)^T$ is computed as their difference $(3, 3, 6)^T - (4, 2, 2)^T = -\hat{\imath} + \hat{\jmath} + 4\hat{k}$. Then the distance is found from the length of the projection of $-\hat{\imath} + \hat{\jmath} + 4\hat{k}$ onto the normal vector of the plane $\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$. By Properties 5.2.1, it is

$$\frac{(-\hat{\imath} + \hat{\jmath} + 4\hat{k}) \cdot (\hat{\imath} - 2\hat{\jmath} + 3\hat{k})}{\|\hat{\imath} - 2\hat{\jmath} + 3\hat{k}\|} = \frac{(-1)(1) + (1)(-2) + (4)(3)}{\sqrt{(1)^2 + (-2)^2 + (3)^2}} = \frac{9}{\sqrt{14}}$$

□

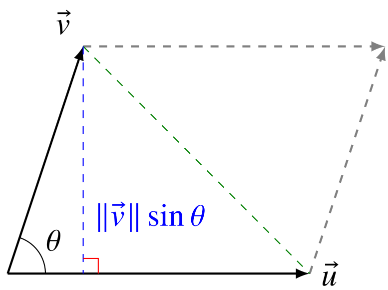
Sometimes the calculation may lead to a negative value for the projection and we may want to take the absolute value. The case of finding the distance of a point to a line of $\mathbb{R}^3$ is considered in Exercise 5.3.

5.3 More on Geometric Applications of Cross Product

Unless specified, all vectors in this section is assumed to be of $\mathbb{R}^3$.

5.3.1 Area

The area of the parallelogram formed by two vectors $\vec{u}$, $\vec{v}$ are simply the absolute value of their cross product.



Properties 5.3.1. Directly from Properties 4.2.10, the area of the parallelogram formed by two vectors $\vec{u}$, $\vec{v}$ is

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$$

Similarly, the area of triangle outlined by $\vec{u}$, $\vec{v}$ is half of the quantity above:

$$\frac{1}{2}\|\vec{u} \times \vec{v}\| = \frac{1}{2}\|\vec{u}\|\|\vec{v}\| \sin\theta$$

Example 5.3.1. Find the area of the parallelogram formed by $\vec{u} = (-1, -2, 4)^T$ and $\vec{v} = (3, 0, 1)^T$.

Solution. By Properties 4.2.9, the cross product between the two given vectors is

$$\begin{aligned}\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & 4 \\ 3 & 0 & 1 \end{vmatrix} \\ &= -2\hat{i} + 13\hat{j} + 6\hat{k}\end{aligned}$$

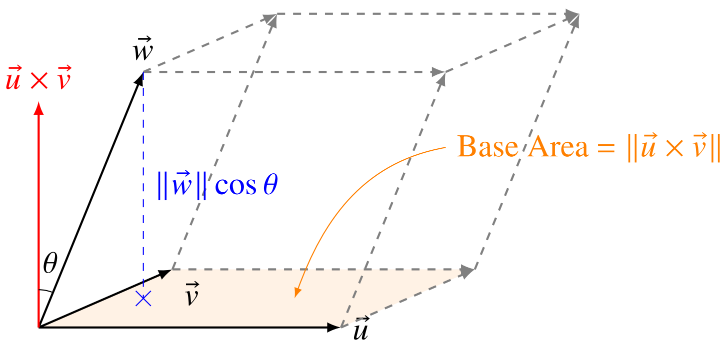
Therefore, as suggested by Properties 5.3.1, the required area is

$$\begin{aligned}\|\vec{u} \times \vec{v}\| &= \sqrt{(-2)^2 + (13)^2 + (6)^2} \\ &= \sqrt{209}\end{aligned}$$

□

5.3.2 Volume

Meanwhile, volume of parallelepiped (see the figure below) formed by three vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ is given by the absolute value of the so-called ***Scalar Triple Product*** as follows.



Properties 5.3.2 (Scalar Triple Product)**.** The volume of parallelepiped con-

structed by three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is calculated as

$$\|\vec{u}\times\vec{v}\|\|\vec{w}\|\cos\theta = |(\vec{u}\times\vec{v})\cdot\vec{w}| = \text{abs}\left(\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\right)$$

where

$$(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

is the scalar triple product of $\vec{u}$, $\vec{v}$, and $\vec{w}$. Also, this determinant form along with Properties 2.3.5 indicates that

$$\begin{aligned} &(\vec{u}\times\vec{v})\cdot\vec{w} = (\vec{v}\times\vec{w})\cdot\vec{u} = (\vec{w}\times\vec{u})\cdot\vec{v} \\ &= -(\vec{v}\times\vec{u})\cdot\vec{w} = -(\vec{w}\times\vec{v})\cdot\vec{u} = -(\vec{u}\times\vec{w})\cdot\vec{v} \end{aligned}$$

Proof. We will prove the determinant formula shown above for $(\vec{u}\times\vec{v})\cdot\vec{w}$ briefly. By Properties 4.2.9, we have

$$\vec{u}\times\vec{v} = (u_2v_3 - u_3v_2)\hat{\imath} + (u_3v_1 - u_1v_3)\hat{\jmath} + (u_1v_2 - u_2v_1)\hat{k}$$

and then according to Definition 4.2.1

$$\begin{aligned} (\vec{u}\times\vec{v})\cdot\vec{w} &= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)^T \cdot (w_1, w_2, w_3)^T \\ &= (u_2v_3 - u_3v_2)(w_1) + (u_3v_1 - u_1v_3)(w_2) + (u_1v_2 - u_2v_1)(w_3) \end{aligned}$$

which is equal to

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = w_1(u_2v_3 - u_3v_2) - w_2(u_1v_3 - u_3v_1) + w_3(u_1v_2 - u_2v_1)$$

where we expand along the third row of the determinant by Properties 2.3.3. □

If the volume of parallelepiped evaluated from the scalar triple product is zero, it implies that the three composing vectors are ***Co-planar***, i.e. lying on the same plane.

Properties 5.3.3. Given three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, if their scalar triple product $(\vec{u} \times \vec{v}) \cdot \vec{w} = 0$, then $\vec{u}$, $\vec{v}$, and $\vec{w}$ are co-planar and all lie on the same plane, and vice versa.

Note that if $\vec{w} = \alpha\vec{u} + \beta\vec{v}$, where α and β are some scalars, then $\vec{u}$, $\vec{v}$, $\vec{w}$ are co-planar, and $(\vec{u} \times \vec{v}) \cdot \vec{w} = 0$. (compare this to the explanation of answer to part (c) of Exercise 4.2)

Example 5.3.2. Find the volume of the parallelepiped formed by $\vec{u} = (1, -2, 2)^T$, $\vec{v} = (-1, -1, 1)^T$ and $\vec{w} = (2, 1, 0)^T$.

Solution. By Properties 5.3.2, the triple scalar product of the three given vectors is

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} 1 & -2 & 2 \\ -1 & -1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = -3$$

and the volume is $|-3| = 3$. □

Remarks

The solution of a linear system can be considered as a point/line/plane/hyperplane, depending on the number of free variables (0/1/2 or more). We may also like to call it a *solution space*. However, while such shapes surely occupy space geometrically, we have been shying away from defining what really constitutes a *vector (sub)space* mathematically, which will be the main point of discussion in the next chapter.

5.4 Useful Vector Identities

In this section, we will prove some key vector identities that may be of utilities to some readers.

Properties 5.4.1 (Vector Triple Product)**.** The ***Vector Triple Product*** of three vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ is defined as

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

Proof. By Properties 4.2.9, the L.H.S. can be expanded into

$$\begin{aligned}
&\vec{u} \times (\vec{v} \times \vec{w}) \\
&= (u_1\hat{i} + u_2\hat{j} + u_3\hat{k}) \\
&\quad \times [(v_2w_3 - v_3w_2)\hat{i} + (v_3w_1 - v_1w_3)\hat{j} + (v_1w_2 - v_2w_1)\hat{k}] \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_2w_3 - v_3w_2 & v_3w_1 - v_1w_3 & v_1w_2 - v_2w_1 \end{vmatrix}
\end{aligned}$$

The $\hat{i}$ component along the x-direction is

$$\begin{aligned}
&u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3) \\
&= u_2w_2v_1 + u_3w_3v_1 - u_2v_2w_1 - u_3v_3w_1 \\
&= u_1w_1v_1 + u_2w_2v_1 + u_3w_3v_1 - u_1v_1w_1 - u_2v_2w_1 - u_3v_3w_1 \\
&= (u_1w_1 + u_2w_2 + u_3w_3)v_1 - (u_1v_1 + u_2v_2 + u_3v_3)w_1 \\
&= (\vec{u} \cdot \vec{w})v_1 - (\vec{u} \cdot \vec{v})w_1
\end{aligned}$$

which is equal to the $\hat{i}$ component on the R.H.S. and the same can be shown for the $\hat{j}$, $\hat{k}$ components similarly, so the equality establishes. □

Properties 5.4.2 (Jacobi Identity)**.**

$$\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = \mathbf{0}$$

Proof. By Properties 5.4.1, we have

$$
\begin{aligned}
& \vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) \\
= & \; [(\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}] \\
& + [(\vec{v} \cdot \vec{u})\vec{w} - (\vec{v} \cdot \vec{w})\vec{u}] \\
& + [(\vec{w} \cdot \vec{v})\vec{u} - (\vec{w} \cdot \vec{u})\vec{v}] \\
= & \; [(\vec{u} \cdot \vec{w})\vec{v} - (\vec{w} \cdot \vec{u})\vec{v}] \\
& + [(\vec{v} \cdot \vec{u})\vec{w} - (\vec{u} \cdot \vec{v})\vec{w}] \\
& + [(\vec{w} \cdot \vec{v})\vec{u} - (\vec{v} \cdot \vec{w})\vec{u}] \\
= & \; 0\vec{v} + 0\vec{w} + 0\vec{u} = \mathbf{0}
\end{aligned}
$$

□

Properties 5.4.3 (Lagrange's Identity)**.**

$$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$$

Proof. Manipulating the geometric formulae of dot/cross product, we have

$$
\begin{aligned}
\|\vec{u} \times \vec{v}\|^2 &= \|\vec{u}\|^2 \|\vec{v}\|^2 \sin^2\theta && \text{(Properties 4.2.10)} \\
&= \|\vec{u}\|^2 \|\vec{v}\|^2 (1 - \cos^2\theta) \\
&= \|\vec{u}\|^2 \|\vec{v}\|^2 - \|\vec{u}\|^2 \|\vec{v}\|^2 \cos^2\theta \\
&= \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2 && \text{(Properties 4.2.4)}
\end{aligned}
$$

□

The last identity is the ***Cosine Law for Spherical Trigonometry***.

Properties 5.4.4 (Cosine Law for Spherical Trigonometry)**.**

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

where a, b, c are the (subtended angle of) three arcs (in radians) of a spherical

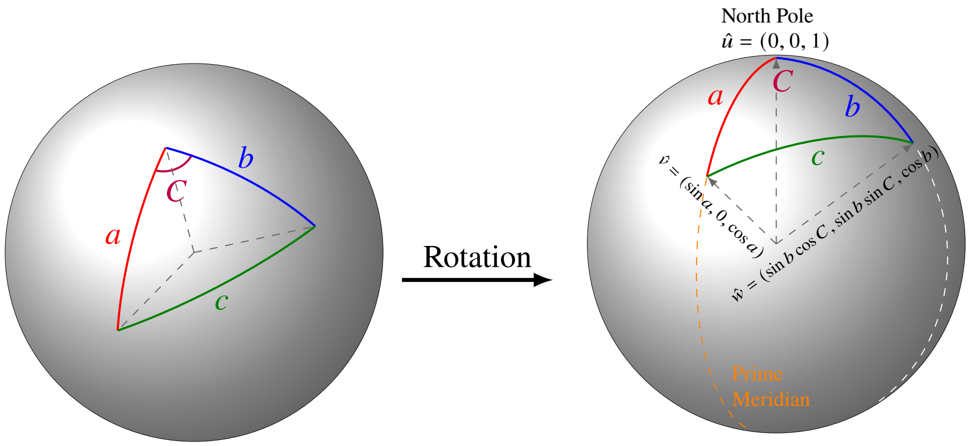


Figure 5.1: The spherical triangle on a unit sphere as described in Properties 5.4.4.

triangle on a unit sphere and C is the angle between the two arcs a and b, as shown in Figure 5.1.

Proof. For the given spherical triangle, we can always rotate the coordinate system (see Figure 5.1) while keeping its shape intact, such that the corner C is positioned exactly at the north pole ($\hat{u} = (0, 0, 1)^T$) and one of the two arcs starting from corner C (let's say a) lies along the Prime Meridian (angle from the x-axis is $0°$, i.e. $y = 0$). The vector $\hat{v}$ at the end of arc a will then have a direction of $(\sin a, 0, \cos a)^T$. The vector $\hat{w}$ to the remaining corner at the intersection of arcs b and c will similarly have a z-component of $\cos b$, and its projection on x-y plane will be $\sin b$ and the x/y-component will then be $\sin b \cos C$ and $\sin b \sin C$, i.e. $\hat{w} = (\sin b \cos C, \sin b \sin C, \cos b)^T$. Now consider the dot product $\hat{v} \cdot \hat{w}$. The geometric meaning of dot product (Properties 4.2.4) implies that it is the angle between $\hat{v}$ and $\hat{w}$, that is, $\hat{v} \cdot \hat{w} = \cos c$. On the other hand,

$$\begin{aligned}
\hat{v} \cdot \hat{w} &= (\sin a, 0, \cos a)^T \cdot (\sin b \cos C, \sin b \sin C, \cos b)^T \\
&= (\sin a)(\sin b \cos C) + (0)(\sin b \sin C) + (\cos a)(\cos b) \\
&= \cos a \cos b + \sin a \sin b \cos C
\end{aligned}$$

Therefore, equaling the two expressions of $\hat{v} \cdot \hat{w}$ gives the desired formula of $\cos c = \cos a \cos b + \sin a \sin b \cos C$. □

5.5 Earth Science Applications

Example 5.5.1. Derive the *Haversine Formula* for finding the great-circle distance between any two points on a sphere with their latitudes/longitudes provided. Hence find the distance between New York (40.73 °N, 73.94 °W) and Warsaw (52.24 °N, 21.02 °E).

Solution. Denote the latitudes/longitudes of the two locations by $\varphi_{1,2}$ and $\lambda_{1,2}$. Staring from the Cosine Law for Spherical Trigonometry (Properties 5.4.4) with corner C still fixed at north pole but arc a not necessarily along the Prime Meridian, we have $C = \lambda_2 - \lambda_1$, $a = \frac{\pi}{2} - \varphi_1$, $b = \frac{\pi}{2} - \varphi_2$, and

$$\begin{aligned}
\cos c &= \cos a \cos b + \sin a \sin b \cos C \\
\cos c &= \cos\left(\frac{\pi}{2} - \varphi_1\right) \cos\left(\frac{\pi}{2} - \varphi_2\right) + \sin\left(\frac{\pi}{2} - \varphi_1\right) \sin\left(\frac{\pi}{2} - \varphi_2\right) \cos(\lambda_2 - \lambda_1) \\
\cos c &= \sin \varphi_1 \sin \varphi_2 + \cos \varphi_1 \cos \varphi_2 \cos(\lambda_2 - \lambda_1)
\end{aligned}$$

The *haversine* of an angle θ is $\text{hav}(\theta) = \sin^2(\frac{\theta}{2}) = \frac{1}{2}(1 - \cos\theta)$ and therefore $\cos\theta = 1 - 2\,\text{hav}(\theta)$. Subsequently,

$$\begin{aligned}
\cos c &= \sin \varphi_1 \sin \varphi_2 + \cos \varphi_1 \cos \varphi_2 (1 - 2\,\text{hav}(\lambda_2 - \lambda_1)) \\
\cos c &= \sin \varphi_1 \sin \varphi_2 + \cos \varphi_1 \cos \varphi_2 - 2 \cos \varphi_1 \cos \varphi_2 \,\text{hav}(\lambda_2 - \lambda_1) \\
\cos c &= \cos(\varphi_2 - \varphi_1) - 2 \cos \varphi_1 \cos \varphi_2 \,\text{hav}(\lambda_2 - \lambda_1) \\
(1 - 2\,\text{hav}(c)) &= (1 - 2\,\text{hav}(\varphi_2 - \varphi_1)) - 2 \cos \varphi_1 \cos \varphi_2 \,\text{hav}(\lambda_2 - \lambda_1) \\
\text{hav}(c) &= \text{hav}(\varphi_2 - \varphi_1) + \cos \varphi_1 \cos \varphi_2 \,\text{hav}(\lambda_2 - \lambda_1)
\end{aligned}$$

where we have used the trigonometric identity $\cos(\theta - \phi) = \cos\theta \cos\phi + \sin\theta \sin\phi$ in the middle. The Haversine Formula is now established and we can use it to calculate the angle c subtended by the arc between two locations and hence their distance by $d = rc$ where r is the radius (of the Earth, 6370 km). For New York (40.73 °N, 73.94 °W) and Warsaw (52.24 °N, 21.02 °E), $\lambda_1 = -73.94°$, $\lambda_2 = 21.02°$, $\varphi_1 = 40.73°$, $\varphi_2 = 52.24°$, and

$$\begin{aligned}
\operatorname{hav}(c) &= \operatorname{hav}(52.24° - 40.73°) \\
&\qquad + \cos(40.73°)\cos(52.24°)\operatorname{hav}(21.02° - (-73.94°)) \\
&= \operatorname{hav}(11.51°) + \cos(40.73°)\cos(52.24°)\operatorname{hav}(94.96°) \\
&= \sin^2(\frac{11.51°}{2}) + \cos(40.73°)\cos(52.24°)\sin^2(\frac{94.96°}{2}) \\
\sin^2(\frac{c}{2}) &\approx 0.26214 \\
c &\approx 61.6° = 1.075\,\text{rad}
\end{aligned}$$

and therefore the required distance is $d = rc = (6370\,\text{km})(1.075\,\text{rad}) \approx 6848\,\text{km}$. The value computed by the Haversine Formula will be slightly off from the true value since the Earth is not a perfect sphere but rather an oblate one. □

Example 5.5.2. The Earth's magnetic field can be approximated by a magnetic dipole, so that the magnetic field lines on the Earth's surface are oriented from the geomagnetic North Pole to geomagnetic South Pole (like longitudinal lines but for the geomagnetic dipole). In 2020, the geomagnetic North Pole is at 80.7 °N, 72.7 °W. Find the magnetic declination (angle from the geographic North to geomagnetic North) at Tokyo (35.65 °N, 139.84 °E) according to this *geomagnetic dipole model*.

Solution. To find the magnetic declination we need to calculate the three arcs of the spherical triangle with its three corners at the geographic/geomagnetic North Pole and Tokyo. The arc distance between geographic/geomagnetic North Pole d is simply $90° - 80.7° = 9.3°$. Similarly, the arc from the geographic North Pole to Tokyo is $a = 90° - 35.65° = 54.35°$. We can use the Haversine Formula derived in the last example to obtain the arc from the geomagnetic North Pole to Tokyo, which yields

$$\operatorname{hav}(t) = \operatorname{hav}(80.7° - 35.65°)$$

$$
\begin{aligned}
&+ \cos(35.65^\circ)\cos(80.7^\circ)\operatorname{hav}((-72.7^\circ) - 139.84^\circ) \\
&= \operatorname{hav}(45.05^\circ) + \cos(35.65^\circ)\cos(80.7^\circ)\operatorname{hav}(-212.54^\circ) \\
&\approx 0.26777 \\
c &\approx 62.3^\circ
\end{aligned}
$$

Denote the declination angle by D. By Properties 5.4.4, we have

$$
\begin{aligned}
\cos d &= \cos a \cos t + \sin a \sin t \cos D \\
\cos(9.3^\circ) &= \cos(54.35^\circ)\cos(62.3^\circ) + \sin(54.35^\circ)\sin(62.3^\circ)\cos D \\
\cos D &\approx 0.9951 \\
D &\approx \pm 5.7^\circ
\end{aligned}
$$

To determine the sign, we note that concluded from the longitudes of Tokyo and geomagnetic North, the geomagnetic North is located to the east of Tokyo, and hence $D = 5.7\,^\circ$E. However, we note that the actual declination is $7.8\,^\circ$W which has an opposite sign and is far from our answer (you can extract the value from https://www.ngdc.noaa.gov/geomag/calculators/magcalc.shtml). The reason is that the geomagnetic dipole is only a rough first-order approximation, while in reality the Earth's magnetic field has a much more complex structure. □

5.6 Python Programming

Projection as in Properties 5.2.1 can be calculated by `numpy` functions and let's wrap them up in our self-defined function as below.

```
def scalar_projection(u, v):
    """ Calculates the scalar projection of v onto u. """
    return np.dot(u,v) / np.linalg.norm(u)
```

This computes the scalar projection of $\vec{v}$ onto $\vec{u}$. Testing with Example 5.2.1 shows

```
u = np.array([4., 1., -3.])
v = np.array([-2., 3., -1.])
print(scalar_projection(u, v))
```

a consistent output of `-0.39223`. Incorporating the unit vector function (`unit_vector()`) defined in the last chapter's programming section, we obtain the vector projection.

```
def vector_projection(u, v):
    """ Calculates the vector projection of v onto u. """
    return scalar_projection(u, v) * unit_vector(u)

print(vector_projection(u, v))
```

This results in `[-0.3077 -0.0769 0.2308]` which matches the example's answer. Area of parallelogram formed by two vectors is the magnitude of their cross product and the corresponding function is typed below.

```
def area_parallelogram(u, v):
    """ Calculate the area of parallelogram formed by two
       vectors u and v. """
    return np.linalg.norm(np.cross(u,v))
```

`print(area_parallelogram(u, v))` then gives `18.974`. Meanwhile, the function to compute volume of parallelepiped made up of three vectors can be defined such that it uses the determinant formula in Properties 5.3.2.

```
def volume_parallelepiped(u, v, w):
    """ Calculate the volume of parallelepiped formed by two
       vectors u, v, w. """
    return np.abs(np.linalg.det(np.c_[u,v,w]))

w = np.array([1., 2., -3.])
print(volume_parallelepiped(u, v, w))
```

(`np.c_[]` is a short hand of combining arrays column by column) produces `14.00000...04` due to numerical round-off error (the analytical anawer would be just 14). Finally, let's conclude this section by defining the Haversine Formula in Example 5.5.1.

```
def Haversine_dist(latlon1, latlon2):
    """ Haversine Formula for computing the great-circle
        distance between two places on the Earth.
        Input: (lat1, lon1), (lat2, lon2) in degrees.
        Output: Great-circle distance in km.
    """
```

```
    R_Earth = 6370 # Earth's Radius
    lat1, lon1 = latlon1[0], latlon1[1]
    lat2, lon2 = latlon2[0], latlon2[1]
    # Converting degree to radian
    lat1_rad, lon1_rad, lat2_rad, lon2_rad = np.deg2rad(lat1),
        np.deg2rad(lon1), np.deg2rad(lat2), np.deg2rad(lon2)
    # Haversine's Formula
    hav_c = np.sin((lat2_rad-lat1_rad)/2)**2 + np.cos(lat1_rad
       )*np.cos(lat2_rad)*np.sin((lon2_rad-lon1_rad)/2)**2
    arc_c = 2*np.arcsin(np.sqrt(hav_c)) # Inverting to get the
        great-circle arc angle
    return(R_Earth*arc_c) # Arc angle to arc length
```

Using the latitudes and longitudes of New York and Warsaw in Example 5.5.1 for testing, `Haversine_dist((40.73, -73.94), (52.24, 21.02))` outputs `6847.76`.

5.7 Exercises

Exercise 5.1 Parameterize the following equations into vector form.

(a) $6x + 8y = 9$,

(b) $x + 9y + 9z = 7$,

(c) $y = 3, -\infty < x < \infty$, and

(d) $2x + z = 9, -\infty < y < \infty$.

Exercise 5.2 Eliminate the parameters and find the direct equation.

(a)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} + s \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 8 \\ 0 \\ 5 \end{bmatrix}$$

where $-\infty < s, t < \infty$.

Exercise 5.3 Find the distance of the point $(3, 2, 9)^T$ to the plane $x+2y+5z = 10$, as well as the distance of the point $(3, 2, 9)^T$ to the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

where $-\infty < t < \infty$.

Exercise 5.4 Prove that the shortest distance between two lines, $\vec{u} = \vec{a} + s\hat{l}$ and $\vec{v} = \vec{b} + t\hat{m}$, where $-\infty < s, t < \infty$, $\vec{a}, \vec{b}$ are some arbitrary vectors and $\hat{l}, \hat{m}$ are some fixed, non-parellel unit vectors representing direction of the two lines, is

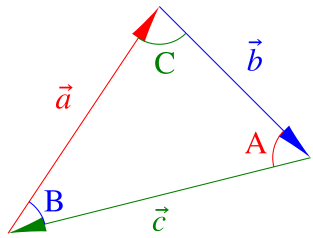
$$\text{Dist}(u, v) = \frac{(\hat{a} - \hat{b}) \cdot (\hat{l} \times \hat{m})}{\|\hat{l} \times \hat{m}\|}$$

Hints: Geometrically, the distance between these two lines is the projection of any vector from one line to another onto the vector normal to the plane made by $\hat{l}$ and $\hat{m}$.

$$\frac{(\vec{v} - \vec{u}) \cdot (\hat{l} \times \hat{m})}{\|\hat{l} \times \hat{m}\|}$$

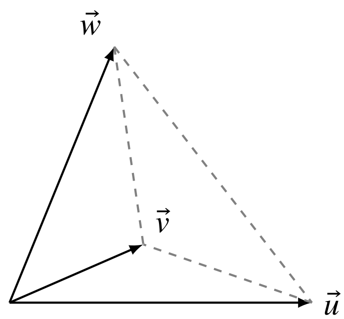
Draw a diagram to convince yourself it is true. What does it imply if $\vec{a} \cdot (\hat{l} \times \hat{m}) = \vec{b} \cdot (\hat{l} \times \hat{m})$?

Exercise 5.5 Prove Sine Law with vector notation by considering the triangle below



and equating three expressions of its area $\frac{1}{2}\left\|\vec{a}\times\vec{b}\right\| = \frac{1}{2}\left\|\vec{b}\times\vec{c}\right\| = \frac{1}{2}\|\vec{c}\times\vec{a}\|$. Properties 5.3.1 will be useful.

Exercise 5.6 By extending Properties 5.3.2, derive a vector formula for the volume of a tetrahedron (pyramid).



Exercise 5.7 For $\vec{u} = (1,2,3)^T$, $\vec{v} = (2,1,5)^T$, $\vec{w} = (1,4,0)^T$, find

(a) Area of the parallelogram formed by $\vec{u}$ and $\vec{v}$,

(b) Volume of the parallelepiped formed by $\vec{u}$, $\vec{v}$ and $\vec{w}$,

(c) Redo the above for $\vec{w} = (1,5,4)^T$, what does the result tell you?

Exercise 5.8 Find the geometric interpretation of solutions of the following systems of linear equations.

(a)

$$\begin{cases} x + 2y + 2z &= 3 \\ 3x - y + 3z &= 2 \\ x - 2y - z &= -1 \end{cases}$$

(b)

$$\begin{cases} 2x - y - z &= 3 \\ x + y + 2z &= -1 \\ x + 4y + 7z &= -6 \end{cases}$$

Answers to Exercises

Exercise 1.1

(a) $\begin{bmatrix} -3 & 5 \\ 3 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 8 & -\frac{1}{2} \\ 13 & -\frac{25}{2} \end{bmatrix}$

(c) $\begin{bmatrix} -8 & 17 \\ -18 & 8 \end{bmatrix}$

(d) $\begin{bmatrix} 11 & -11 \\ 33 & -11 \end{bmatrix}$

Exercise 1.2

(a) $\begin{bmatrix} -2 & 1 & 3 \\ -1 & -1 & -9 \\ -8 & 2 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} -8 & -5 \\ 15 & 3 \end{bmatrix}$

Exercise 1.3

(a) $\begin{bmatrix} 42 & 72 & 0 \\ 32 & 51 & -1 \end{bmatrix}$

(b) Same as above

(c) $\begin{bmatrix} 90 & 162 & 2 \\ 51 & 99 & 3 \end{bmatrix}$

(d) Same as above

Exercise 1.4

(a) $\begin{bmatrix} 16 & 23 & 129 \\ 133 & 33 & 102 \\ 27 & 9 & 128 \end{bmatrix}$

(b) $\begin{bmatrix} -\frac{233}{4} & -\frac{19}{4} & \frac{69}{2} \\ -\frac{339}{4} & -16 & 31 \\ \frac{109}{4} & \frac{33}{4} & -\frac{289}{4} \end{bmatrix}$

Exercise 1.5

(a) $\begin{bmatrix} 16 & 6 & 3 \\ 34 & 13 & 12 \\ 9 & 2 & 27 \end{bmatrix}$

(b) $\begin{bmatrix} 27 & 15 & 69 \\ 37 & 12 & 85 \\ 36 & 12 & 69 \end{bmatrix}$

(c) $\begin{bmatrix} 14 & 3 & 26 \\ 29 & 9 & 60 \\ 12 & 21 & 41 \end{bmatrix}$

(d) $\begin{bmatrix} 33 & 13 & 24 \\ 47 & 19 & 21 \\ 39 & 14 & 12 \end{bmatrix}$

Exercise 1.6

$$AB = BA = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

Exercise 1.7

$$\begin{bmatrix} 0 & 3 & -4 \\ 5 & -1 & 2 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 13 \\ 8 \end{bmatrix}$$

or

$$\left[\begin{array}{ccc|c} 0 & 3 & -4 & 6 \\ 5 & -1 & 2 & 13 \\ 6 & 0 & 1 & 8 \end{array}\right]$$

Exercise 1.8

(a) $\begin{bmatrix} 2 & 3 & 5 & 7 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 8 & 12 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 5 & 12 & 23 & 37 \\ 1 & 3 & 6 & 10 \end{bmatrix}$

Exercise 1.9

1. Interchange Row 1 and Row 3 ($R_1 \leftrightarrow R_3$),
2. Multiply Row 1 by $\frac{1}{2}$ ($\frac{1}{2}R_1 \to R_1$),
3. Subtract Row 3 from Row 2 ($R_2 - R_3 \to R_2$),
4. Add 3 times Row 1 to Row 2 ($R_2 + 3R_1 \to R_2$).

The order of step 1 and 2, as well as step 3 and 4, can be interchanged.

Exercise 1.10 The air temperature/dew point at any height z before saturation is $T_a = T_{a,ini} - (\Gamma_{dry})z$ and $T_{dew} = T_{dew,ini} - (\Gamma_{dew})z$ respectively. At the condensation level $z = z_{cd}$, the air temperature equals to the dew point temperature $T_a = T_{dew} = T_{cd}$, and hence we have

$$T_{a,ini} - \Gamma_{dry}(z_{cd}) = T_{dew,ini} - \Gamma_{dew}(z_{cd}) = T_{cd}$$

which can be separated into two equations

$$\begin{cases} T_{a,ini} - \Gamma_{dry}(z_{cd}) & = T_{cd} \\ T_{dew,ini} - \Gamma_{dew}(z_{cd}) & = T_{cd} \end{cases}$$

Rearranging to put the unknowns z_{cd} and T_{cd} to the L.H.S., we obtain

$$\begin{cases} T_{cd} + \Gamma_{dry}(z_{cd}) & = T_{a,ini} \\ T_{cd} + \Gamma_{dew}(z_{cd}) & = T_{dew,ini} \end{cases}$$

or, in matrix form

$$\begin{bmatrix} 1 & \Gamma_{dry} \\ 1 & \Gamma_{dew} \end{bmatrix} \begin{bmatrix} T_{cd} \\ z_{cd} \end{bmatrix} = \begin{bmatrix} T_{a,ini} \\ T_{dew,ini} \end{bmatrix}$$

Plugging in the lapse rates, we have

$$\begin{bmatrix} 1 & 9.8 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} T_{cd} \\ z_{cd} \end{bmatrix} = \begin{bmatrix} 25.4 \\ 17.8 \end{bmatrix}$$

Exercise 1.11 Obviously, there are 35 chickens and rabbits in total, and $x+y = 35$. Considering the total amount of legs, we also have $2x + 4y = 94$. Hence the required linear system is

$$\begin{cases} x + y & = 35 \\ 2x + 4y & = 94 \end{cases}$$

In matrix form, it is

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 35 \\ 94 \end{bmatrix}$$

Exercise 2.1 (Applying cofactor expansion along the leftmost column recursively) The determinant is just the product of the diagonal elements = $(1)(6)(10)(13)(15) = 11700$.

Exercise 2.2

(a) $\begin{bmatrix} 8 & 20 \\ 15 & 37 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} -\frac{37}{4} & \frac{15}{4} \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$

(c) $\begin{vmatrix} 8 & 15 \\ 20 & 37 \end{vmatrix} = -4 = (-1)(4) = \begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} \begin{vmatrix} 4 & 6 \\ 0 & 1 \end{vmatrix}$

Exercise 2.3

(a)

$$
\begin{aligned}
&\left[\begin{array}{ccc|ccc} 3 & 2 & 9 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 4 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \\
\rightarrow &\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 2 & 9 & 1 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 & 1 \end{array}\right] && R_1 \leftrightarrow R_2 \\
\rightarrow &\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & -4 & 0 & 1 & -3 & 0 \\ 0 & -8 & -8 & 0 & -4 & 1 \end{array}\right] && R_2 - 3R_1 \rightarrow R_2, R_3 - 4R_1 \rightarrow R_3 \\
\rightarrow &\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & -\frac{1}{8} \end{array}\right] && -\frac{1}{4}R_2 \rightarrow R_2, -\frac{1}{8}R_3 \rightarrow R_3 \\
\rightarrow &\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{8} \end{array}\right] && R_3 - R_2 \rightarrow R_3
\end{aligned}
$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{4} & \frac{3}{8} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{8} \end{array}\right] \qquad R_1 - 3R_3 - 2R_2 \rightarrow R_1$$

(b) $\det(A) = -32$ and

$$\operatorname{adj}(A) = \begin{bmatrix} \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & 9 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 9 \\ 4 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 4 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & 9 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 9 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 8 & 8 & -8 \\ -8 & -24 & 8 \\ -12 & 0 & 4 \end{bmatrix}^T$$

$$= \begin{bmatrix} 8 & -8 & -12 \\ 8 & -24 & 0 \\ -8 & 8 & 4 \end{bmatrix}$$

Hence

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$$= -\frac{1}{32} \begin{bmatrix} 8 & -8 & -12 \\ 8 & -24 & 0 \\ -8 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{3}{8} \\ -\frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{8} \end{bmatrix}$$

Exercise 2.4

(a) $\begin{bmatrix} 19 & 35 & 9 \\ 33 & 61 & 16 \\ 52 & 96 & 24 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 4 & 2 \\ 5 & 9 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 18 & -10 & 1 \\ -6 & 3 & 1 \\ -\frac{11}{4} & \frac{7}{4} & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 2 & -2 \\ -1 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 7 & -4 & 1 \\ -\frac{9}{2} & \frac{5}{2} & 0 \\ 2 & -1 & 0 \end{bmatrix}$

(c) $\begin{vmatrix} 19 & 33 & 52 \\ 35 & 61 & 96 \\ 9 & 16 & 24 \end{vmatrix} = -4 = (-2)(2) = \begin{vmatrix} 0 & 2 & 5 \\ 0 & 4 & 9 \\ 1 & 2 & 1 \end{vmatrix} \begin{vmatrix} 2 & 3 & 4 \\ 2 & 4 & 6 \\ 3 & 5 & 8 \end{vmatrix}$

Exercise 2.5 Either by evaluating the determinant to show that $|A| = 0$, or find its reduced row echelon form which is

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

and not equal to the identity.

Exercise 2.6

$$\det(A) = -42$$
$$\det\left(A^{-1}\right) = -\frac{1}{42}$$
$$A^{-1} = \begin{bmatrix} \frac{9}{7} & -\frac{3}{14} & -\frac{2}{7} & -\frac{6}{7} \\ -\frac{1}{21} & -\frac{1}{21} & \frac{1}{21} & \frac{1}{7} \\ -\frac{11}{7} & -\frac{15}{14} & \frac{11}{7} & \frac{5}{7} \\ \frac{3}{7} & \frac{3}{7} & -\frac{3}{7} & -\frac{2}{7} \end{bmatrix}$$

Exercise 2.7 By cofactor expansion along the first column, we can obtain the determinant of A as

$$|A| = 2p^2 + 4p - 16$$

which has two roots, $p = -4$ and $p = 2$ such that $|A| = 0$ and A is not invertible. All values of p other than $p = -4$ and $p = 2$ make A invertible.

Exercise 2.8 $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A^T + A$,
and $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$. We can split A into

$$\begin{aligned}
A &= A + \frac{1}{2}(A^T - A^T) \\
&= \frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}A^T - \frac{1}{2}A^T \\
&= \frac{1}{2}A + \frac{1}{2}A^T + \frac{1}{2}A - \frac{1}{2}A^T \\
&= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)
\end{aligned}$$

where the first term is symmetric and the second term is skew-symmetric.

Exercise 2.9

$$\begin{aligned}
A^{-1} &= \frac{1}{\det(A)}\mathrm{adj}(A) \\
\det\left(A^{-1}\right) &= \det\left(\frac{1}{\det(A)}\mathrm{adj}(A)\right) \qquad \text{(Notice that } \frac{1}{\det(A)} \text{ is now a scalar)} \\
\frac{1}{\det(A)} &= (\frac{1}{\det(A)})^n \det(\mathrm{adj}(A)) \\
\det(\mathrm{adj}(A)) &= (\det(A))^{n-1}
\end{aligned}$$

Exercise 3.1

$$A^{-1} = \begin{bmatrix} \frac{1}{21} & \frac{5}{21} & \frac{2}{21} \\ \frac{11}{42} & -\frac{29}{42} & \frac{1}{42} \\ \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{h} = \begin{bmatrix} \frac{1}{21} & \frac{5}{21} & \frac{2}{21} \\ \frac{11}{42} & -\frac{29}{42} & \frac{1}{42} \\ \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 6 \\ \frac{7}{2} \\ -\frac{13}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{3}{2} \end{bmatrix}$$

or

$$\left[\begin{array}{ccc|c} 5 & 1 & 3 & 6 \\ 2 & -1 & 1 & \frac{7}{2} \\ 3 & 2 & -4 & -\frac{13}{2} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{5} & \frac{3}{5} & \frac{6}{5} \\ 2 & -1 & 1 & \frac{7}{2} \\ 3 & 2 & -4 & -\frac{13}{2} \end{array}\right] \qquad \frac{1}{5}R_1 \rightarrow R_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{5} & \frac{3}{5} & \frac{6}{5} \\ 0 & -\frac{7}{5} & -\frac{1}{5} & \frac{11}{10} \\ 0 & \frac{7}{5} & -\frac{29}{5} & -\frac{101}{10} \end{array}\right] \qquad R_2 - 2R_1 \rightarrow R_2, R_3 - 3R_1 \rightarrow R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{5} & \frac{3}{5} & \frac{6}{5} \\ 0 & -\frac{7}{5} & -\frac{1}{5} & \frac{11}{10} \\ 0 & 0 & -6 & -9 \end{array}\right] \qquad R_3 + R_2 \rightarrow R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{5} & \frac{3}{5} & \frac{6}{5} \\ 0 & 1 & \frac{1}{7} & -\frac{11}{14} \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right] \qquad -\frac{5}{7}R_2 \rightarrow R_2, -\frac{1}{6}R_3 \rightarrow R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{5} & 0 & \frac{3}{10} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right] \qquad R_1 - \frac{3}{5}R_3 \rightarrow R_1, R_2 - \frac{1}{7}R_3 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right] \qquad R_1 - \frac{1}{5}R_2 \rightarrow R_1$$

Exercise 3.2

$$A^{-1} = \begin{bmatrix} -\frac{1}{8} & 0 & \frac{7}{8} \\ \frac{3}{16} & -\frac{1}{2} & -\frac{5}{16} \\ \frac{1}{16} & \frac{1}{2} & -\frac{7}{16} \end{bmatrix}$$

$$\vec{x}_1 = A^{-1}\vec{h}_1 \qquad \vec{x}_2 = A^{-1}\vec{h}_2$$

$$= \begin{bmatrix} -\frac{1}{8} & 0 & \frac{7}{8} \\ \frac{3}{16} & -\frac{1}{2} & -\frac{5}{16} \\ \frac{1}{16} & \frac{1}{2} & -\frac{7}{16} \end{bmatrix} \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \qquad = \begin{bmatrix} -\frac{1}{8} & 0 & \frac{7}{8} \\ \frac{3}{16} & -\frac{1}{2} & -\frac{5}{16} \\ \frac{1}{16} & \frac{1}{2} & -\frac{7}{16} \end{bmatrix} \begin{bmatrix} \frac{19}{4} \\ 1 \\ \frac{5}{4} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{4} \end{bmatrix}$$

Exercise 3.3

$$\left[\begin{array}{ccc|c} 3 & 0 & 4 & 2 \\ 1 & 1 & 2 & -1 \\ 1 & -2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 1 & 1 & 2 & -1 \\ 3 & 0 & 4 & 2 \end{array}\right] \qquad R_1 \leftrightarrow R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 3 & 2 & -1 \\ 0 & 6 & 4 & 2 \end{array}\right] \qquad R_2 - R_1 \to R_2, R_3 - 3R_1 \to R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 6 & 4 & 2 \end{array}\right] \qquad \frac{1}{3}R_2 \to R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 4 \end{array}\right] \qquad R_3 - 6R_2 \to R_3$$

The last row is inconsistent and the system has no solution.
Note: You may get, to the right of the last row, some number other than 4, but this is possible and not wrong. (Why?)

Exercise 3.4

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -3 & 2 \\ 1 & 0 & 0 & -1 & 5 \\ 3 & 2 & -2 & -7 & 9 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 5 \\ 1 & 1 & -1 & -3 & 2 \\ 3 & 2 & -2 & -7 & 9 \end{array}\right] \qquad R_1 \leftrightarrow R_2$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 5 \\ 0 & 1 & -1 & -2 & -3 \\ 0 & 2 & -2 & -4 & -6 \end{array}\right] \qquad R_2 - R_1 \to R_2, R_3 - 3R_1 \to R_3$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 5 \\ 0 & 1 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad R_3 - 2R_2 \to R_3$$

Let $p = s$, $q = t$ as the two free variables. Substituting them back into the equations, we have $m - t = 5$ and $n - s - 2t = -3$, hence $m = 5 + t$ and $n = -3 + s + 2t$, and

$$\begin{bmatrix} m \\ n \\ p \\ q \end{bmatrix} = \begin{bmatrix} 5 + t \\ -3 + s + 2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Exercise 3.5 The determinant of the coefficient matrix can be found to be

$$\begin{aligned} \begin{vmatrix} 1 & 0 & \alpha \\ 0 & \alpha & 0 \\ \alpha & 0 & 1 \end{vmatrix} &= -\alpha^3 + \alpha \\ &= -\alpha(\alpha - 1)(\alpha + 1) \end{aligned}$$

The system will have no solution or infinitely many of them only when the determinant equals to zero, which gives us three possible values of $\alpha = -1, 0, 1$. When $\alpha = -1$, the system is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right] \qquad R_3 + R_1 \to R_3$$

where the last row is inconsistent and there is no solution. When $\alpha = 0$, it becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

It is obvious that $x = z = 0$, and $y = t$ is a free variable, so the solution is infinitely many and is in the form of

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The last case, $\alpha = 1$, gives rise to the system of

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_3 - R_1 \rightarrow R_3$$

such that $y = 0$ and $z = t$ can be set to be a free variable and there are infinitely many solutions in the form of

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Exercise 3.6 The system can be written as

$$\begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.55 & 0.4 & 0.05 \\ 0.45 & 0.45 & 0.1 \end{bmatrix} \begin{bmatrix} gy \\ pk \\ bk \end{bmatrix} = \begin{bmatrix} 2.645 \\ 2.6325 \\ 2.65 \end{bmatrix}$$

and has a unique solution of $gy = 2.65$ (quartz), $pk = 2.55$ (feldspar), $bk = 3.1$ (biotite).

Exercise 3.7 The first two equations below come from the left inner loop and right inner loop, but one of them can be replaced by the outer loop as well.

$$\begin{aligned} -4I_1 + 6I_2 &= 6 \\ -6I_2 + 9I_3 &= -12 \\ I_1 + I_2 + I_3 &= 0 \end{aligned}$$

and the solution is $I_1 = -\frac{3}{19}$, $I_2 = \frac{17}{19}$, $I_3 = -\frac{14}{19}$ (in Amperes).

Exercise 3.8 Substituting the given wave solution forms into the equation, we have

$$\begin{aligned} &\omega\tilde{\eta}\sin(kx + ly - \omega t) + H(-k\tilde{U}\sin(kx + ly - \omega t) \\ &- l\tilde{V}\sin(kx + ly - \omega t)) = 0 \end{aligned}$$

$$
\begin{aligned}
\omega\tilde{U}\sin(kx+ly-\omega t) &= gk\tilde{\eta}\sin(kx+ly-\omega t)\\
\omega\tilde{V}\sin(kx+ly-\omega t) &= gl\tilde{\eta}\sin(kx+ly-\omega t)
\end{aligned}
$$

Cancelling out all the sine factors, we arrive at the linear system displayed in the question

$$
\begin{cases}
\omega\tilde{\eta} - kH\tilde{U} - lH\tilde{V} & = 0\\
\omega\tilde{U} - kg\tilde{\eta} & = 0\\
\omega\tilde{V} - lg\tilde{\eta} & = 0
\end{cases}
$$

For $\tilde{U}$, $\tilde{V}$, $\tilde{\eta}$ to have a non-trivial solution other than all zeros, we require the determinant of the corresponding coefficient matrix to be zero according to Theorem 3.1.2, which leads to

$$
\begin{aligned}
\begin{vmatrix}
\omega & -kH & -lH\\
-kg & \omega & 0\\
-lg & 0 & \omega
\end{vmatrix} &= 0\\
\omega^3 - gHk^2\omega - gHl^2\omega &= 0\\
\omega^2 - gH(k^2+l^2) &= 0
\end{aligned}
$$

as the dispersion relation of gravity wave.

Exercise 3.9 $T_{cd} \approx 15.9\,°\mathrm{C}$, $z_{cd} \approx 0.97\,\mathrm{km}$.

Exercise 3.10 $x = 23$, $y = 12$. For the extra part, the new system of equations become (denote the number of third species as z)

$$
\begin{bmatrix}
1 & 1 & 1\\
2 & 4 & 3
\end{bmatrix}
\begin{bmatrix}
x\\ y\\ z
\end{bmatrix}
=
\begin{bmatrix}
48\\ 122
\end{bmatrix}
$$

By Gaussian Elimination, we have

$$
\left[\begin{array}{ccc|c}
1 & 1 & 1 & 48\\
2 & 4 & 3 & 122
\end{array}\right]
\to
\left[\begin{array}{ccc|c}
1 & 1 & 1 & 48\\
0 & 2 & 1 & 26
\end{array}\right]
\qquad R_2 - 2R_1 \to R_2
$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 48 \\ 0 & 1 & \frac{1}{2} & 13 \end{array}\right] \qquad \frac{1}{2}R_2 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 35 \\ 0 & 1 & \frac{1}{2} & 13 \end{array}\right] \qquad R_1 - R_2 \rightarrow R_1$$

Let $z = t$ as the free variable, then we have $y = 13 - \frac{1}{2}t$ and $x = 35 - \frac{1}{2}t$, and hence

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35 - \frac{1}{2}t \\ 13 - \frac{1}{2}t \\ t \end{bmatrix} = \begin{bmatrix} 35 \\ 13 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Since the numbers of species must be a non-negative integer, the solution can be expressed in a more good-looking form of

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35 \\ 13 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

where $s = \frac{t}{2}$, and the range of s is $0, 1, \dots, 13$ (when s reaches 13 there is no chicken remained).

Exercise 4.1

(a) $(2, 5, 5, 12)^T$

(b) $(1, \frac{7}{2}, \frac{7}{2}, 8)^T$

(c) $(1)(1) + (3)(2) + (3)(2) + (7)(5) = 48$

(d) $(1)(1) + (2)(3) + (2)(3) + (5)(7) = 48$

(e) $\vec{u} - 2\vec{v} = (-1, -1, -1, -3)^T$, $2\vec{u} + \vec{v} = (3, 8, 8, 19)^T$, $(\vec{u} - 2\vec{v}) \cdot (2\vec{u} + \vec{v}) = (-1)(3) + (-1)(8) + (-1)(8) + (-3)(19) = -76$

Exercise 4.2

(a)

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 7 & 4 & 1 \\ 8 & 1 & 1 \end{vmatrix} = 3\hat{\imath} + \hat{\jmath} - 25\hat{k} = (3, 1, -25)^T$$

$$\vec{v} \times \vec{u} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 8 & 1 & 1 \\ 7 & 4 & 1 \end{vmatrix} = -3\hat{\imath} - \hat{\jmath} + 25\hat{k} = (-3, -1, 25)^T$$

(b)

$$\begin{aligned} A\vec{v} &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 1 \end{bmatrix} \\ \vec{u} \cdot (\vec{A}v) &= (7, 4, 1)^T \cdot (10, 2, 1)^T \\ &= (7)(10) + (4)(2) + (1)(1) \\ &= 79 \\ A^T\vec{u} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 12 \end{bmatrix} \\ (A^T\vec{u}) \cdot \vec{v} &= (7, 11, 12)^T \cdot (8, 1, 1)^T \\ &= (7)(8) + (11)(1) + (12)(1) \\ &= 79 \end{aligned}$$

(c) By (a), $\vec{u} \times \vec{v} = (3, 1, -25)^T$ and $(3\vec{u} - 4\vec{v}) = (-11, 8, -1)^T$, then

$$\begin{aligned} (3\vec{u} - 4\vec{v}) \cdot (\vec{u} \times \vec{v}) &= (-11, 8, -1)^T \cdot (3, 1, -25)^T \\ &= (-11)(3) + (8)(1) + (-1)(-25) = 0 \end{aligned}$$

This makes sense as we have shown that $\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\vec{u} \times \vec{v}) = 0$ in a previous short exercise, and therefore by distributive property $(\alpha\vec{u} + \beta\vec{v}) \cdot (\vec{u} \times \vec{v}) = 0$ for any α and β.

Exercise 4.3

(a)

$$\|\vec{u}\| = \sqrt{1^2 + (-3)^2 + 9^2} = \sqrt{91}$$
$$\hat{u} = (\frac{1}{\sqrt{91}}, -\frac{3}{\sqrt{91}}, \frac{9}{\sqrt{91}})^T$$
$$\|\vec{v}\| = \sqrt{1^2 + (-2)^2 + 4^2} = \sqrt{21}$$
$$\hat{v} = (\frac{1}{\sqrt{21}}, -\frac{2}{\sqrt{21}}, \frac{4}{\sqrt{21}})^T$$

(b)

$$\vec{u} \cdot \vec{v} = (1)(1) + (-3)(-2) + (9)(4) = 43$$
$$\cos\theta = \frac{43}{\sqrt{21}\sqrt{91}} \approx 0.9836$$
$$\theta \approx 0.181 \text{ rad}$$

(c) $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 9 \\ 1 & -2 & 4 \end{vmatrix} = 6\hat{i} + 5\hat{j} + \hat{k} = (6, 5, 1)^T$

(d) $\vec{u} \cdot (\vec{u} \times \vec{v}) = (1, -3, 9)^T \cdot (6, 5, 1)^T = (1)(6) + (-3)(5) + (9)(1) = 0$, $\vec{v} \cdot (\vec{u} \times \vec{v}) = (1)(6) + (-2)(5) + (4)(1) = 0$

Exercise 4.4

Typhoon Name	Time	Speed	Direction	Vector Form
Nuri	2008/08/22, 08:00	$13\,\text{km}\,\text{h}^{-1}$	315°	$(-9.192, 9.192)$
Vicente	2012/07/24, 02:00	$18\,\text{km}\,\text{h}^{-1}$	299°	$(-15.743, 8.727)$
Linfa	2015/07/09, 23:00	$15\,\text{km}\,\text{h}^{-1}$	245°	$(-13.595, -6.339)$
Mangkhut	2018/09/16, 22:00	$25\,\text{km}\,\text{h}^{-1}$	288°	$(-23.776, 7.725)$

Exercise 4.5

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})\\ &= \|\vec{u}\|^2 + 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2\\ &\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2\\ &= (\|\vec{u}\| + \|\vec{v}\|)^2\end{aligned}$$

Exercise 4.6

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})\\ &= (\|\vec{u}\|^2 + 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2) + (\|\vec{u}\|^2 - 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2)\\ &= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2\end{aligned}$$

Exercise 4.7 In Example 4.3.1, we have

$$\overrightarrow{F_{\text{cor}}} = (2\Omega(v\sin\varphi - w\cos\varphi))\hat{\imath} + (-2\Omega u\sin\varphi)\hat{\jmath} + (2\Omega u\cos\varphi)\hat{k}$$

and hence the rate of work done is

$$\begin{aligned}&\overrightarrow{F_{\text{cor}}}\cdot\vec{v}\\ &= [(2\Omega(v\sin\varphi - w\cos\varphi))\hat{\imath} + (-2\Omega u\sin\varphi)\hat{\jmath} + (2\Omega u\cos\varphi)\hat{k}]\cdot(u\hat{\imath} + v\hat{\jmath} + w\hat{k})\\ &= (2\Omega(v\sin\varphi - w\cos\varphi))u + (-2\Omega u\sin\varphi)v + (2\Omega u\cos\varphi)w\\ &= 2\Omega uv\sin\varphi - 2\Omega uw\sin\varphi - 2\Omega uv\sin\varphi + 2\Omega uw\sin\varphi = 0\end{aligned}$$

Alternatively, note that $\overrightarrow{F_{\text{cor}}} = -2\vec{\Omega}\times\vec{v}$ and $(\Omega\times\vec{v})\cdot\vec{v} = 0$ always holds.

Exercise 5.1

(a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -9 \\ 0 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{9}{2} \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$

Exercise 5.2

(a) Normal vector to the line is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Equation: $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 9 \end{bmatrix} \to x - y = -7$

(b) Normal vector to the plane is $\begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 8 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ -27 \\ -32 \end{bmatrix}$.

Equation: $\begin{bmatrix} 20 \\ -27 \\ -32 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ -27 \\ -32 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \to 20x - 27y - 32z = -25$

Exercise 5.3 Part 1: Choose $(0, 0, 2)^T$ as a reference point on the plane. Projection of the vector from $(0, 0, 2)^T$ to $(3, 2, 9)^T$: $(3-0)\hat{\imath}+(2-0)\hat{\jmath}+(9-2)\hat{k} = 3\hat{\imath} + 2\hat{\jmath} + 7\hat{k}$ onto the normal vector $\hat{\imath} + 2\hat{\jmath} + 5\hat{k}$ of the plane is

$$\frac{(3)(1) + (2)(2) + (7)(5)}{\sqrt{1^2 + 2^2 + 5^2}} = \frac{42}{\sqrt{30}}$$

which is the required distance.
Part 2: Choose $(0, 1, 2)^T$ as a reference point along the line. Find the projection

of $(3, 2, 9)^T - (0, 1, 2)^T = 3\hat{\imath} + 1\hat{\jmath} + 7\hat{k}$ onto the direction vector $\hat{\jmath} + 2\hat{k}$, which is

$$\frac{(3)(0) + (1)(1) + (7)(2)}{0^2 + 1^2 + 2^2}(\hat{\jmath} + 2\hat{k}) = 3(\hat{\jmath} + 2\hat{k}) = 3\hat{\jmath} + 6\hat{k}$$

The displacement vector between the point and line (which is orthogonal to the line) is then $(3\hat{\imath} + 1\hat{\jmath} + 7\hat{k}) - (3\hat{\jmath} + 6\hat{k}) = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$ and the required distance equals to $\sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$.

Exercise 5.4 Using the hints, we have the distance as

$$\begin{aligned}\frac{(\vec{v} - \vec{u}) \cdot (\hat{l} \times \hat{m})}{\|\hat{l} \times \hat{m}\|} &= \frac{[(\vec{b} + \hat{m}t) - (\vec{a} + \hat{l}s)] \cdot (\hat{l} \times \hat{m})}{\|\hat{l} \times \hat{m}\|} \\ &= \frac{(\vec{b} - \vec{a}) \cdot (\hat{l} \times \hat{m}) + [\hat{m} \cdot (\hat{l} \times \hat{m})]t - [\hat{l} \cdot (\hat{l} \times \hat{m})]s}{\|\hat{l} \times \hat{m}\|}\end{aligned}$$

Notice that $\hat{l} \times \hat{m}$ is orthogonal to both $\hat{l}$ and $\hat{m}$, and thus $\hat{l} \cdot (\hat{l} \times \hat{m}) = \hat{m} \cdot (\hat{l} \times \hat{m}) = 0$ both vanish. Therefore we are left with

$$\frac{(\vec{b} - \vec{a}) \cdot (\hat{l} \times \hat{m})}{\|\hat{l} \times \hat{m}\|}$$

If $\vec{a} \cdot (\hat{l} \times \hat{m}) = \vec{b} \cdot (\hat{l} \times \hat{m})$, then the numerator $(\vec{b} - \vec{a}) \cdot (\hat{l} \times \hat{m}) = 0$ equals to zero such that the two lines intersect. In this case, the values of s or t at the point of intersection ($\vec{u} = \vec{v}$) can be found by applying a cross product with $\hat{m}$ on $\vec{u} = \vec{a} + \hat{l}s = \vec{b} + \hat{m}s = \vec{v}$ and note that $\hat{m} \times \hat{m} = \vec{0}$, and hence

$$\begin{aligned}(\vec{a} + \hat{l}s) \times \hat{m} &= (\vec{b} + \hat{m}s) \times \hat{m} \\ \vec{a} \times \hat{m} + s(\hat{l} \times \hat{m}) &= \vec{b} \times \hat{m} + s(\hat{m} \times \hat{m}) = \vec{b} \times \hat{m} + s\vec{0} \\ s(\hat{l} \times \hat{m}) &= (\vec{b} - \vec{a}) \times \hat{m}\end{aligned}$$

s is then inferred from the scaling ratio of $(\vec{b} - \vec{a}) \times \hat{m}$ to $(\hat{l} \times \hat{m})$. t is found similarly.

Exercise 5.5

$$
\begin{aligned}
&\frac{1}{2}\left\|\vec{a}\times\vec{b}\right\| = \frac{1}{2}\left\|\vec{b}\times\vec{c}\right\| = \frac{1}{2}\|\vec{c}\times\vec{a}\| \\
\rightarrow\ &\frac{1}{2}\|\vec{a}\|\left\|\vec{b}\right\|\sin C = \frac{1}{2}\left\|\vec{b}\right\|\|\vec{c}\|\sin A = \frac{1}{2}\|\vec{c}\|\|\vec{a}\|\sin B \\
\rightarrow\ &\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\end{aligned}
$$

where we divide the entire equality by $abc = \|\vec{a}\|\left\|\vec{b}\right\|\|\vec{c}\|$).

Exercise 5.6 It is just $\frac{1}{6}(\vec{u}\times\vec{v})\cdot\vec{w}$.

Exercise 5.7

(a) $\vec{u}\times\vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 2 & 1 & 5 \end{vmatrix} = 7\hat{\imath}+\hat{\jmath}-3\hat{k}$

Area $= \sqrt{7^2+1^2+(-3)^2} = \sqrt{59}$

(b) Volume is the absolute value of $|\vec{u}\times\vec{v}|\cdot\vec{w} = |(7\hat{\imath}+\hat{\jmath}-3\hat{k})\cdot(\hat{\imath}+4\hat{\jmath})| = |(7)(1)+(1)(4)+(-3)(0)| = 11$

(c) Volume = abs $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 5 \\ 1 & 5 & 4 \end{vmatrix} = 0.$

So the three vectors are co-planar.

Exercise 5.8

(a) The solution refers to the point $(1, 1, 0)$.

(b) By Gaussian Elimination, one possible form of general solution is

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{5}{3} \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{1}{3} \\ -\frac{5}{3} \\ 1 \end{bmatrix}
$$

Therefore, the solution space is a line parallel to $-\frac{1}{3}\hat{\imath} - \frac{5}{3}\hat{\jmath} + \hat{k}$ and passing through the point $(\frac{2}{3}, -\frac{5}{3}, 0)^T$.

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