Hinge model

To investigate the hinge model, we'll look at the deterministic part, given by

$$\ln(SSB_t) = \begin{cases} \alpha_1 + \beta_1 t & \text{if } t \leq C \ , \\ \alpha_2 + \beta_2 t & \text{if } t > C. \end{cases} \tag{1}$$

For $\ln(SSB_C)$ to be continuous requires that

$$\begin{aligned} \alpha_1 + \beta_1 C &= \alpha_2 + \beta_2 C & (2) \\ \alpha_2 &= \alpha_1 + \beta_1 C - \beta_2 C & (3) \\ &= \alpha_1 + C(\beta_1 - \beta_2) & (4) \end{aligned}$$

Substituting back into Equation (1)

$$\ln(SSB_t) = \begin{cases} \alpha_1 + \beta_1 t & \text{if } t \leq C \ , \\ \alpha_1 + C(\beta_1 - \beta_2) + \beta_2 t & \text{if } t > C. \end{cases} \tag{5}$$

Define

$$\begin{aligned} \beta_2 &= \beta_1 + \delta_\beta & (6) \\ \delta_\beta &= \beta_2 - \beta_1 & (7) \end{aligned}$$

Substitute in Equation (5)

$$\begin{aligned} \ln(SSB_t) &= \begin{cases} \alpha_1 + \beta_1 t & \text{if } t \leq C \ , \\ \alpha_1 + C(-\delta_\beta) + (\beta_1 + \delta_\beta)t & \text{if } t > C. \end{cases} & (8) \\ &= \begin{cases} \alpha_1 + \beta_1 t & \text{if } t \leq C \ , \\ \alpha_1 + \beta_1 t + \delta_\beta(t - C) & \text{if } t > C. \end{cases} & (9) \end{aligned}$$

Define

$$\eta_t = \begin{cases} 0 & \text{if } t \leq C \ , \\ 1 & \text{if } t > C. \end{cases} \tag{10}$$

$$\ln(SSB_t) = \alpha_1 + \beta_1 t + \eta_t \delta_\beta (t - C). \tag{11}$$