CS314 Spring 2023
Homework 7 Solutions

Problem 1 – Reversible Functions

Are the following functions reversible? Justify your answer.

1. $f(x) = x$
   Answer: Yes. The identity function is always reversible.

2. $f(x) = 2 * x + 1$
   Answer: Yes. Given output y, I can find $x = \frac{y-1}{2}$.

3. $f(x) = x \bmod 5$
   Answer: No. Given output 1, for example, the values of x satisfying $1 = x \bmod 5$ are $x = 1, 6, 11, \ldots$ etc.

4. $f(x) = x^2$
   Answer: No. Given output 1, for example, the values of x satisfying $1 = x^2$ are $x = 1$ and -1.

5. $f(x, y) = x + y$
   Answer: No. Given output 6, for example, there's more than one pair of (x, y) satisfying $6 = x + y$, e.g. $(2, 4)$ or $(4, 2)$, etc.

Problem 2 – Hadamard Gate

Show that the Hadamard Gate H is reversible.

Answer: Given input $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, for $||\alpha||^2 + ||\beta||^2 = 1$, we show that $HH|\psi\rangle = |\psi\rangle$, i.e. we can apply H again to the output $H|\psi\rangle$ to retrieve the input $|\psi\rangle$. Applying H once to $|\psi\rangle$ yields:

$$H|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix}.$$

Applying H again yields

$$HH|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \alpha + \beta + \alpha - \beta \\ \alpha + \beta + -\alpha + \beta \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2\alpha \\ 2\beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = |\psi\rangle.$$

You can also show H is reversible by showing $HH = I$, where I is the identity matrix, since $I|\psi\rangle = |\psi\rangle$.

Problem 3 – Deutsch-Jozsa Algorithm

Documentation for the Deutsch-Jozsa Algorithm, including some examples, is given in the following link: `https://quantum-computing.ibm.com/composer/docs/iqx/guide/deutsch-jozsa-algorithm`.

Please give screenshots of your constructed circuit for the following problems:

1. The circuit for the Deutsch-Jozsa Algorithm with number of bits $n = 4$ for the constant function.

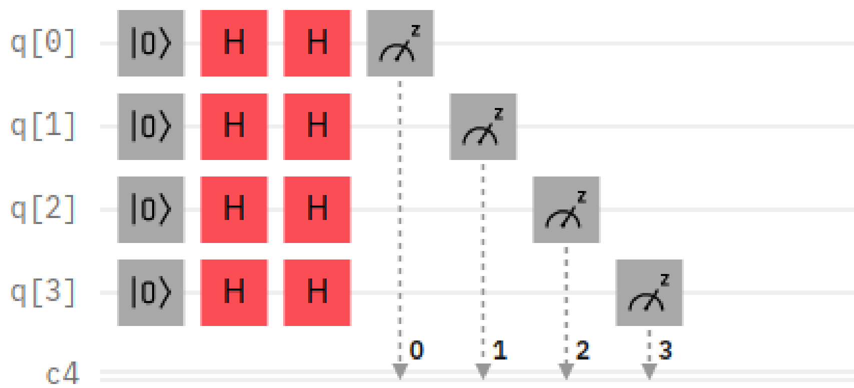

Figure 1: Solution for Problem 3

Note that the black box is supposed to be between the Hadamard gates. Also, the output for a constant function in this case is $|0000\rangle$ for this circuit. In the lecture notes, the output is $|1111\rangle$ because of the X gates before the two Hadamard gates, like in Figure 2.

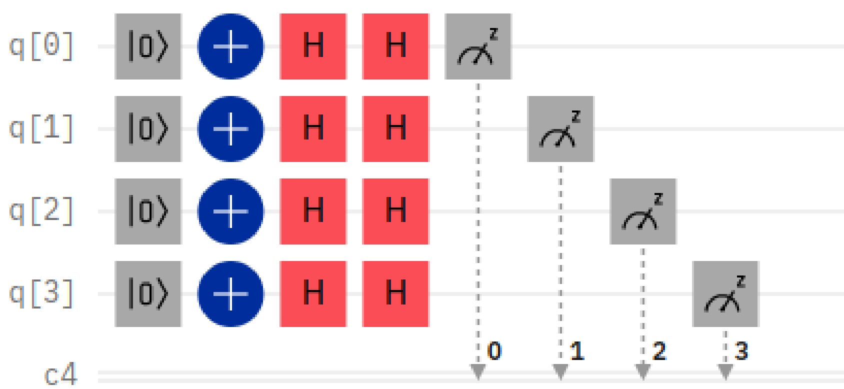

Figure 2: Solution for Problem 3