Stochastic Analysis Note

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Chapter 1 stochastic calculus

An ito integral is a martingale

1.1. Feynman-Kac formula

Theorem Feynman-Kac： There are two parallel threads, the first is the following partial differential equation

$$\frac{\partial v}{\partial t}+\mu(x,t)\frac{\partial v}{\partial x}+\frac{1}{2}\sigma^2(x,t)\frac{\partial^2 v}{\partial x^2}-rv=0,\quad v(x,T)=\varphi(x)$$

The second thread is

$$\begin{gathered}v(x,t)=e^{-r(T-t)}E[\varphi(X_T)|\mathcal{F}_t]\\ dX_t=\mu(X_t,t)dt+\sigma(X_t,t)dW_t\end{gathered}$$

We say

可以看到，Feynman-Kac 定理说明了以上两种定价方法之间的联系。

除此之外，当我们正向使用 Feynman-Kac 定理时，可以用期望来简化计算 PDE；当我们逆向使用 Feynman-Kac 定理时，它给我们提供了波动率不是常数的假设下，资产价格的 PDE 解法

Chapter 2 Change of measure and Girsanov theorem

2.1. Change of measure

Definition : In $(\Omega, \mathcal{F}, P)$, Z is a almost everywhere nonnegative random variable, and $E(Z) = 1$. Define $\tilde{P}(A) = \int_A Z(\omega)dP(\omega),\ \forall A \in \mathcal{F}$, we can proof $\tilde{P}(A)$ is a probability measure.

In the above definition, $Z = Z(\omega)$, we want to extend it : for a filtration $\mathcal{F}(t)$

$$Z(t) = E[Z|\mathcal{F}(t)] \quad 0 \le t \le T$$

using expectation conditioned on σ algebra, we convert $Z(\omega)$ into $Z(t, \omega)$ making it a stochastic process. Since the original Z is Radon-Nikodym derivative, we call the process Radon-Nikodym derivative process.

We can prove the process is a martingale, for $0 \le s \le t \le T$

$$E[Z(t)|\mathcal{F}(s)] = E[E[Z|\mathcal{F}(t)]|\mathcal{F}(s)] = Z(s)$$

Lemma : 5.2.1 and 5.2.2, $Z(t)$ is the direct cause of measure change, so it is the direct cause of $\tilde{E}$, these two lemmas tell us how to covert one from another, just like $\tilde{E}[X] = E[XZ]$ for random variable.

2.2. Girsanov theorem

The following definition partly is from Sherev

Girsanov Theorem : Let $W(t), 0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}(t), 0 \le t \le T$, be a filtration for this Brownian motion. Let $\Theta(t), 0 \le t \le T$, be an adapted process.

Define a new process

$$\begin{aligned}
\tilde{W}(t) &= W(t) + \int_0^t \Theta(u)du \\
&= W(t) - \int_0^t -\Theta(u)(dW(u))^2 \\
&= W(t) - \int_0^t dX(u)dW(u) \\
&= W(t) - [X, W]_t \\
or& \\
d\tilde{W}(t) &= dW(t) + \Theta(t)dt
\end{aligned}$$

Define a new measure

$$
\begin{aligned}
Z(t) &= exp\left\{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\right\} \\
&= exp\left\{\int_0^t -\Theta(u)dW(u) - \frac{1}{2}\int_0^t (-\Theta(u)dW(u))^2\right\} \\
&= exp\left\{\int_0^t dX(u) - \frac{1}{2}\int_0^t (dX(u))^2\right\} \\
&= exp\left\{X(t) - \frac{1}{2}[X,X]_t\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow Z(\omega) = Z(T,\omega) \\
&\Rightarrow \tilde{P}(A) = \int_A Z(\omega)dP(\omega),\ \forall A \in \mathcal{F}
\end{aligned}
$$

$Z(t)$ is called exponential martingale process, meaning it is a martingale. then under the probability measure $\tilde{P}$, the process $\tilde{W}(t), 0 \le t \le T$ is a Brownian motion

Remark : How to use this theorem, or what is the motivation for inventing this theorem? First we have the basic Brownian motion, which can model stock price and has many wonderful properties (e.g. it is a martingale) . But stock price may have other components besides simple Brownian motion (e.g. a drift part). It will make the process not be a martingale. We can not change the new process, so we find a new measure using some features from the new process and make the process a martingle under the new measure. It's all about accommdating to the new process $\tilde{W}(t)$

$$
\begin{aligned}
&(\Theta\ or\ X) \overset{W}{\Leftarrow} \tilde{W} \\
&\Downarrow \\
&(\Theta\ or\ X) \overset{W}{\Rightarrow} Z \overset{P}{\Rightarrow} \tilde{P}
\end{aligned}
$$

$$
then\ \tilde{W}\ is\ a\ Brownian\ motion\ under\ measure\ \tilde{P}
$$

In terms of finance, using this theorem, we can make a process, which is not a martingale under measure P , a martingale uner masure Q

2.3. Martingale representation theorem

定理（鞅表示）：设 $\{W_t, \mathcal{F}_t^{\mathcal{W}}, t \in [0,T]\}$ 是 $(\Omega, \mathcal{F}, P)$ 上的布朗运动，而 $\{M_t, t \in [0,T]\}$ 为 $\mathcal{F}_t^{\mathcal{W}}$-鞅，且满足 $M_t \in \mathcal{L}^2(\Omega), t \in [0,T]$ ，则存在一个$\mathcal{F}_t^{\mathcal{W}}$-适应的过程 $\{\Gamma_t, t \in [0,T]\} \in \mathcal{V}[0,T]$ ，使得 $M_t - M_0 = \int_0^t \Gamma_u dW_u$ 成立。

大白话解释是，如果 M_t 是鞅，那么 M_t 可以被表示为一个伊藤积分的形式，即没有 dt 项而仅仅只有 dW_t 项。

Chapter 3 Jump processes

3.1. Notation

Poisson Process

$$
N(t)
$$

Compound Poisson Process

$$Q(t)=\sum_{i=1}^{N(t)} Y_i$$

Compensated Poisson Process

$$M(t)=N(t)-\lambda t$$

Pure Jump Process: Poisson process and a compound Poisson process belong to this catagory

$$J(t)$$

Jump Process

$$X(t)=X(0)+I(t)+R(t)+J(t)$$

3.2. Outline

在Jump Proces被定义后，我们需要研究

微分

$$dX(t)$$

积分

$$\int_0^t \Phi(s)dX(s)$$

Ito-Doeblin公式（$X(t)$ 函数 $f(X(t))$ 的微积分）

$$f(X(t))=f(X(0))+\int_0^t f'(X(s))dX^c(s)+\frac{1}{2}\int_0^t f''(X(s))dX^c(s)dX^c(s)+\sum_{0<s\leq t}[f(X(s))-f($$

测度变换

3.3. Definition, Differential and Integral

$$X(t)=X(0)+I(t)+R(t)+J(t)$$
$$dX(s)=\Gamma(s)dW(s)+\Theta(s)ds+\Delta J(s)$$
$$\int_0^t \Phi(s)dX(s)=\int_0^t \Phi(s)\Gamma(s)dW(s)+\int_0^t \Phi(s)\Theta(s)ds+\sum_{0<s\leq t}\Phi(s)\Delta J(s)$$

这里的定义和 Ito Process 定义目的是一致的，因为单纯的布朗运动太过简单或太特例，我们加更多的东西使它一般化。它的地位是几何布朗运动中指数上的部分，和 t 一样是随机分析中的基本元素。

3.4. Quadratic Variation

定义 $X(t)$ 为自变量的函数 $f(X(t))$ ，要讨论它的微积分，则必须讨论 $X(t)$ 的二次变差

3.5. Differential and Integral for $f(X(t))$

Ito-Doeblin formula for one jump process：对于一个跳过程 $X(t)$ 以及一个一二阶导都存在的函数 $f(x)$ ，有

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX^c(s) + \frac{1}{2}\int_0^t f''(X(s))dX^c(s)dX^c(s) + \sum_{0<s\leq t}[f(X(s)) - f($$

它的微分形式我认为是

$$\begin{aligned} f(X(t)) &= f(X(0)) + \int_0^t f'(X(s))dX^c(s) + \frac{1}{2}\int_0^t f''(X(s))dX^c(s)dX^c(s) + \sum_{0<s\leq t}[f(X(s)) - f \\ &= f(X(0)) + \int_0^t f'(X(s))dX^c(s) + \frac{1}{2}\int_0^t f''(X(s))dX^c(s)dX^c(s) \\ &\quad + \int_0^t [f(X(s)) - f(X(s-))]dN(t) \end{aligned}$$
$$df(X(t)) = f'(X(t))dX^c(t) + \frac{1}{2}f''(X(t))dX^c(t)dX^c(t) + [f(X(t)) - f(X(t-))]dN(t)$$

Shreve说虽然积分形式很容易给出，但是微分形式比较难给出，言下之意就是上面我写出的这种微分形式肯定是错的，这是因为微分形式一般是

$$df(t) = a(t)dt + b(t)dW(t)$$

也就是等式右边的每一项肯定是一个微分乘一个系数，而这个系数应该是某种整体性的东西，而不是这里无论怎么处理都是两个随机过程相减。换句话说，只有在 $f(X(t)) - f(X(t-))$ 化成某种整体性的 $g(t)\ or\ g(t-)$ 的时候，微分形式才存在，也就是一个跳点前后的值存在关系，我能够用一个表示另一个。Shreve给出了一个能找到微分形式的特例，Example11.5.2中的几何泊松过程，它的跳点满足 $S(u) = (\sigma+1)S(u-)$ ，因此能找到微分形式。

3.6. Change of Measure

3.7. Martingale Representation Theorem

假设Jump Process $X(t) = X(0) + I(t) + R(t) + J(t)$ 是鞅， $\Phi(s)$ 满足

1. 左连续
2. 适应
3. $E\int_0^t \Gamma^2(s)\Phi^2(s)ds < \infty\ for\ all\ t \geq 0$

下述积分是鞅

$$\int_0^t \Phi(s)dX(s)$$

Shreve给出积分例子

$$X(t) = M(t) = N(t) - \lambda t, \ \ \Phi(s) = \Delta N(s)$$
$$\int_0^t \Phi(s) dX(s) = \int_0^t \Delta N(s) dM(s) = N(t)$$

我们知道compensated Poisson process $M(t)$ 已经是鞅了，但是当 $\Phi(s) = \Delta N(s)$ 积分出来很明显不是鞅，导致积分不是鞅的原因是 $\Phi(s) = \Delta N(s)$ 不为左连续，他分析从实际来说，一个人股票的持仓一定只能依据之前的所有信息，而没有办法在看到当前股价后同时改变持仓，也就是 $\Phi(s) = \Delta N(s)$ 一定要为左连续。