Python For Finance

M1 - Economie & Finance - Modern Portfolio Theory (MPT)

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Part I — Introduction

Financial Portfolio Management theory development took place through three historical phases:

- Traditional Portfolio Theory (TPT): Before 1952
- Modern Portfolio Theory (MPT) 1952 - 1990,
- Post-Modern Portfolio Theory (PMPT) After 1990.

Part I — Introduction

Traditional Portfolio Theory (TPT):

- Before 1938: Stock-market information was fragmented, non-uniform, and slow to disseminate. Investment practice centered on picking individual stocks without a precise, shared analytical framework.
- 1938 — John Burr Williams: In *The Theory of Investment Value*, Williams introduced the dividend-discount model, providing the first modern investment framework. Concurrent improvements in financial-reporting standards enabled more rigorous analysis of company financials.
- Portfolio construction then: portfolios were assembled security by security. Investors bought instruments whose discounted cash flows looked attractive relative to current prices. Little attention was paid to portfolio-level characteristics.

Part I — Introduction

Modern Portfolio Theory (MPT): *Portfolio Selection, Markowitz, 1952*

For investor following the following preference structure:

- Monotonicity: Between portfolios with the same risk, the investor prefers the one with higher expected return.
- Risk aversion: Between portfolios with the same expected return, the investor prefers the one with lower variance.

Preferences can be represented by:

$$U(\mathrm{w}) = \mathbb{E}[R_p] - \frac{\lambda}{2}\mathrm{Var}(R_p) = \mathrm{w}^\top \mu - \frac{\lambda}{2}\mathrm{w}^\top \Sigma\, \mathrm{w}, \qquad \lambda > 0.$$

with μ the vector of expected asset returns, Σ , the covariance matrix of asset returns and λ, the risk–aversion parameter

$\implies$ Introduction of a risk–return trade-off in the investment framework

Part I — Introduction

Under the risk averse preference structure and assuming that investor are rational, the key idea of markowitz is : what matters for risk assessment is not only how volatile each asset of a portfolio is, but how assets co-move.

For a given investment universe, each investor will choose a portfolio that maximize the expected return for a given level of risk / minimize the risk given an expected return.

The implication is that an investor will not invest in a portfolio if a second portfolio exists with a more favorable risk vs expected return profile.

The subset containing all the portfolio minimizing the risk vs the expected return is called **the efficient frontier.**

Notation

- We consider a universe of n assets.
- $x = (x_1, \ldots, x_n)$ is the vector of weights in the portfolio
- The portfolio is fully invested: $\sum_{i=1}^{n} x_i = 1$
- $R = (R_1, \ldots, R_n)$ is the vector of asset return with R_i the return of asset i.
- Portfolio return: $R(x) = \sum_{i=1}^{n} x_i R_i = x^\top R$.
- $\mu = \mathbb{E}[R]$ and $\Sigma = \mathrm{Cov}(R) = \mathbb{E}\big[(R-\mu)(R-\mu)^\top\big]$ are the vector of expected returns and the covariance matrix of asset returns.

Portfolio Expected Return and Variance

The expected return of the portfolio is:

$$\mu(x) \;=\; \mathbb{E}[R(x)] \;=\; \mathbb{E}[x^\top R] \;=\; x^\top \mathbb{E}[R] \;=\; x^\top \mu.$$

whereas its variance is equal to:

$$\begin{aligned}
\sigma^2(x) &= \mathrm{Var}\big(R(x)\big) = \mathbb{E}\big[(R(x) - \mu(x))^2\big] \\
&= \mathbb{E}\big[(x^\top R - x^\top \mu)\,(x^\top R - x^\top \mu)\big] \\
&= \mathbb{E}\big[x^\top (R-\mu)(R-\mu)^\top x\big] \\
&= x^\top\, \mathbb{E}\big[(R-\mu)(R-\mu)^\top\big]\, x \\
&= x^\top \Sigma x\,.
\end{aligned}$$

Two Equivalent Optimization Problems

1) Max expected return with a volatility cap (σ-problem):

$$\begin{aligned} \max_{x} \quad & \mu(x) \\ \text{s.t.} \quad & \sigma(x) \le \sigma^\star \end{aligned}$$

2) Min volatility with a return floor (μ-problem):

$$\begin{aligned} \min_{x} \quad & \sigma(x) \\ \text{s.t.} \quad & \mu(x) \ge \mu^\star \end{aligned}$$

The objective function is quadratic and constraints are linear. This optimization problem has a closed-form solution via Lagrange multipliers when short-sale constraints are absent.

Two Mutual Fund Theorem

Statement

In the absence of a risk-free asset, any portfolio on the efficient frontier can be formed as a combination of *any two* efficient portfolios (the "mutual funds").

Let x_A and x_B be two distinct efficient frontier portfolios. Then any efficient portfolio can be written as

$$x(\alpha) \;=\; \alpha\, x_A \;+\; (1-\alpha)\, x_B, \qquad 1^\top x(\alpha) = 1.$$

Implications

- If the desired return μ_t lies between μ_A and μ_B, then $0 \le \alpha \le 1$ and both funds are held long.
- If μ_t lies outside $[\mu_B, \mu_A]$, then $\alpha \notin [0,1]$: one fund is shorted (negative weight) and the other is leveraged.

In the presence of a risk-free asset: CAPM

Risk-free asset: an asset that pays/ return a risk-free rate R_f with certainty at maturity.

In practice: short-term government securities. **Key properties:**

- If held to maturity, the risk-free asset has **zero variance** in returns (hence "risk-free").
- It is **uncorrelated with any other asset** (by definition, since its variance is zero).

Given the two funds theorem, investors will allocate their capital by combining the risk free rate with the portfolio in the efficient frontier that maximize the risk return ration (Sharpe ratio). Thus the efficient frontier becomes a straight line : the capital allocation line / the secutiry market line.

From the Efficient Frontier to CAPM

Bridging the Concepts:

- The Capital Asset Line combines the risk-free asset with the optimal risky portfolio, providing the best trade-off between risk and return.
- All investors will choose some combination of:
  - Risk-free asset (with return = r_f)
  - Market portfolio (highest Sharpe ratio among risky assets)

Key Insight:

- The Market Portfolio, which lies at the tangency point between the efficient frontier and the Capital Asset Line, is the optimal risky portfolio.
- This leads to the CAPM, stating that an asset's expected return depends on:
  - the risk-free rate (r_f)
  - compensation for systematic risk, measured by β

What CAPM says in one line

The expected return on any asset equals the risk-free rate plus a premium for bearing systematic risk:

$$\mathbb{E}[R_i] = r_f + \beta_i \big(\mathbb{E}[R_m] - r_f\big)$$

- r_f: risk-free rate.
- $\mathbb{E}[R_m] - r_f$: risk premium .
- β_i: asset's sensitivity to market moves (systematic risk).
  - $\beta = 1$: moves like the market
  - $\beta > 1$: more volatile
  - $\beta < 1$: less volatile
  - $\beta < 0$: hedges market

Limits of the Markowitz (Mean–Variance) Model

Modeling assumptions

- Risk is fully captured by variance.
- Single-period, static setup; parameters (μ, Σ) are known and stationary.
- Frictionless markets: no taxes/transaction costs, perfect divisibility, no shorting/borrowing cost.

Practical weaknesses

- *Estimation error*: frontier is highly sensitive to μ.
- Ignores higher moments and tail risk (skewness, kurtosis, drawdowns).
- High turnover from small input changes; ignores liquidity and market impact.
- Adding realistic constraints (ex: lot sizes) makes the problem harder to solve.

TP tool tips

- For descriptive statistics, use the methods already implemented in pandas.
- For geometric annual returns, use :

$$g = \left(\prod_{t=1}^{T}(1 + r_t)\right)^{252/T} - 1.$$

- $\beta_i = cov(R_i, R_m)/var_{R_m}$
- expected return $= R_f + \beta *$risk premium
- for vector/matrix computation use np.dot(). For example, given a matrix A and a matrix B, the product of A x B is np.dot(A, B)
- to transpose a vector/matrix use .T . Example : A.T is the transpose of matrix A.

Appendix

Co-movement is captured by **covariance**. Covariance is a measure of the joint variability of two random variables.The sign of the covariance, therefore, shows the tendency in the linear relationship between the variables.

For two random returns R_i, R_j with expected return μ_i, μ_j:

$$\mathrm{Cov}(R_i, R_j) \;=\; \mathbb{E}\big[(R_i - \mu_i)(R_j - \mu_j)\big]\,.$$

Interpretation:

- Cov > 0: when R_i is above average, R_j tends to be above average too.
- Cov < 0: they tend to move in opposite directions.
- Cov ≈ 0: little linear co-movement.

$$\rho_{ij} = \frac{\mathrm{Cov}(R_i, R_j)}{\sigma_i \sigma_j}, \qquad -1 \leq \rho_{ij} \leq 1,$$

And

$$\mathrm{Cov}(R_i, R_j) = \rho_{ij}\, \sigma_i\, \sigma_j.$$

Cauchy–Schwarz inequality:

$$|\mathrm{Cov}(X, Y)| \leq \sigma_X\, \sigma_Y,$$

Sample estimators (from data):

$$\widehat{\mathrm{Cov}}(R_i, R_j) = \frac{1}{T-1} \sum_{t=1}^{T} \left(R_{i,t} - \bar{R}_i\right)\left(R_{j,t} - \bar{R}_j\right),$$

$$\widehat{\rho}_{ij} = \frac{\widehat{\mathrm{Cov}}(R_i, R_j)}{\hat{\sigma}_i \hat{\sigma}_j}.$$