Python For Finance

M1 - Economie & Finance - Risk Measures

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Roadmap

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What is Value-at-Risk (VaR)?

Definition (one-day α-level VaR)

For an asset / portfolio return r_{t+1}, the one-day α-VaR at time t is

$$\mathrm{VaR}_t(\alpha) = F^{-1}_{r_{t+1}|\mathcal{F}_t}(\alpha),$$

where F^{-1} is the conditional α-quantile of the return distribution.

- Value at Risk is defined as the quantile of the conditional distribution of portfolio losses, given information available at time t.
- Typically: $\alpha = 1\%, 2.5\%, 5\%$; horizon: 1 day or 10 days (Basel).
- Used by banks, asset managers, regulators (Basel II/III) for capital requirements.
- Key questions:
  - How do we estimate VaR from historical data?
  - How do we know if a VaR model is *good*?

Historical Simulation: Idea

- Non-parametric approach: use past returns as an empirical distribution of future returns.
- Suppose we have T past daily portfolio returns

$$r_{t-T+1}, \dots, r_t.$$

- Sort them from smallest to largest and take the empirical α-quantile:

$$\text{VaR}_t^{\text{HS}}(\alpha) = F_{r_t}^{-1}(\alpha) = \text{empirical } \alpha\text{-quantile of } \{r_{t-T+1}, \dots, r_t\}.$$

- Interpretation: "If the future looks like the recent past, this is the loss we exceed in only α% of days."

Historical Simulation: Pros and Cons

Advantages

- Very simple, easy to explain.
- No assumption on returns.
- Automatically captures
  - fat tails,
  - skewness,
  - nonlinearities.

Limitations

- Uses a fixed historical window $\Rightarrow$ slow to react to regime changes.
- Assumes that past is representative of the future.
- For very low α (e.g., 0.1%), we may have very few (or no) extreme observations.
- No explicit modeling of time-varying volatility.

Rolling Historical VaR

- In practice we recompute historical VaR *every day* using a fixed-size rolling window.
- For each day t:
  1. Take last T returns $r_{t-T+1}, \dots, r_t$,
  2. Compute empirical α-quantile,
  3. Report $\mathrm{VaR}_t^{\mathrm{roll}}(\alpha)$.
- This generate a *time series* of VaR estimates.

Why Model-Based (GARCH) VaR?

- Historical VaR is backward-looking and slow in reacting to volatility changes.
- Financial returns exhibit:
  - volatility clustering,
  - time-varying conditional variance.
- GARCH-type models explicitly capture time-varying volatility:

$$r_{t+1} = \mu_t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} = \sigma_{t+1} z_{t+1},$$

  where σ_{t+1}^2 follows a GARCH recursion.
- Once σ_{t+1} is forecasted, VaR is obtained from the conditional distribution of z_{t+1}.

Example: GARCH(1,1) VaR

GARCH(1,1) volatility dynamics

$$\sigma_{t+1}^2 = \omega + \alpha\varepsilon_t^2 + \beta\sigma_t^2, \quad \omega > 0,\ \alpha \geq 0,\ \beta \geq 0,\ \alpha + \beta < 1.$$

- z_{t+1} are i.i.d. with CDF F_z (Gaussian or Student-t).
- Then

$$\mathrm{VaR}_t^{\mathrm{GARCH}}(\alpha) = \mu_t + \sigma_{t+1} F_z^{-1}(\alpha).$$

- Intuition:
  - VaR reacts quickly to volatility spikes (e.g., crisis days).
  - Allows scenario analysis and stress testing under different volatility regimes.
- In the notebook: *Predicted VaR* is computed from the fitted GARCH model and compared to historical VaR.

What Does it Mean to "Backtest" VaR?

Goal: We want to check if our VaR model actually works in practice.

- Each day, the model gives a forecast:

$$\text{VaR}_t(\alpha) \quad = \text{the predicted worst loss for day } t+1.$$

- The next day, we observe the realized return: r_{t+1}.

- A **VaR exception** (also called a *violation*) occurs when the actual loss is worse than the VaR prediction:

$$I_{t+1} = \mathbb{1}\{\, r_{t+1} < \text{VaR}_t(\alpha)\,\}.$$

- Intuition:
    - If VaR is a 1% prediction, we expect exceptions on about 1 out of 100 days.
    - If exceptions happen too often, the model underestimates risk.
    - If exceptions cluster together (many in a row), the model does not react to volatility.
- Backtesting = **compare predicted risk vs actual losses and test if exceptions behave as they should**.

Unconditional Coverage: Kupiec (1995) POF Test

- Kupiec's test asks one simple question: Did the VaR model produce the right number of exceptions?
- Null Hypothesis H_0 : The model's predicted VaR violation rate is correct. In other words: The observed proportion of exceptions = expected proportion (α).
- Alternative Hypothesis H_1: The model's predicted VaR violation rate is not correct. In other words: The observed exception rate differs from .
- Cases:
  - Too many violations ($x/N \gg \alpha$) $\Rightarrow$ VaR is *underestimated*.
  - Too few violations ($x/N \ll \alpha$) $\Rightarrow$ VaR is *too conservative*.
- Limitation: ignores time pattern of violations (only counts how many).

POF Test

- Let $x = \sum_{t=1}^{N} I_t$ = number of exceptions over N days.
- Under H_0: $x \sim \text{Binomial}(N, p)$ with $p = \alpha$.
- Kupiec's Proportion of Failures (POF) likelihood ratio statistic:

$$LR_{\text{POF}} = -2 \log \left(\frac{(1-p)^{N-x} p^x}{\left(1 - \frac{x}{N}\right)^{N-x} \left(\frac{x}{N}\right)^x} \right).$$

- Asymptotically, $LR_{\text{POF}} \sim \chi^2(1)$ under the null. (Wilks' Theorem: Under very general conditions, likelihood ratio tests follow a chi-square distribution asymptotically)
- For a 95% confidence level, the critical value is: 3.84

Independence: Why is it important?

- Kupiec's POF test only checks the *number* of violations, not their *timing*.
- Even if the average number of violations is correct, they may be *clustered*.
- Clustering means: when one violation happens, another is more likely soon after.
- In risk management:
  - clustered violations indicate a model that misses volatility dynamics;
  - such a model underestimates risk during crises (when it matters most).
- We therefore test whether $\{I_t\}$ behaves like i.i.d. Bernoulli(α) to see if the model is well suited.

Christoffersen (1998) Independence Test

- Model the exception indicators $\{I_t\}$ as a first-order Markov chain.
- Count the observed transitions:
  - n_{00}: no failure $\to$ no failure,
  - n_{01}: no failure $\to$ failure,
  - n_{10}: failure $\to$ no failure,
  - n_{11}: failure $\to$ failure.
- Define transition probabilities:

$$\pi_0 = \frac{n_{01}}{n_{00} + n_{01}}, \quad \pi_1 = \frac{n_{11}}{n_{10} + n_{11}}, \quad \pi = \frac{n_{01} + n_{11}}{n_{00} + n_{01} + n_{10} + n_{11}}.$$

- Null and alternative:
  - H_0: independence $\Rightarrow \pi_0 = \pi_1 = \pi$,
  - H_1: dependence (clustering) $\Rightarrow \pi_0 \neq \pi_1$.
- Likelihood ratio statistic:

$$LR_{\text{CCI}} = -2 \log \left(\frac{(1-\pi)^{n_{00}+n_{10}} \pi^{n_{01}+n_{11}}}{(1-\pi_0)^{n_{00}} \pi_0^{n_{01}} (1-\pi_1)^{n_{10}} \pi_1^{n_{11}}} \right).$$

- Under H_0: $LR_{\text{CCI}} \sim \chi^2(1)$ asymptotically.

Why Should VaR Violations Be Independent?

- A correct VaR model must satisfy:
  1. **Correct number** of violations (Kupiec test),
  2. **Independence** of violations (Christoffersen test).
- Independence means:

$$P(I_t = 1 \mid I_{t-1}) = P(I_t = 1) = \alpha.$$

- Interpretation: a violation today should not make another violation tomorrow more (or less) likely.

Why Independence Matters for GARCH VaR

- GARCH models explicitly capture *volatility clustering*.
- If the model is well specified:
  - increases in volatility $\Rightarrow$ VaR automatically rises,
  - the model is not "surprised" by consecutive large losses.
- Independent violations therefore indicate:
  - the GARCH volatility dynamics are correctly estimated,
  - the model adapts quickly to changing market conditions.

What Clustering of Violations Reveals

- Clustering of VaR failures implies:
  - volatility increased faster than the model anticipated,
  - VaR remained too low for several days,
  - the model repeatedly underestimates risk during stress periods.
- This is the worst possible failure: insufficient capital exactly when risk is highest.

Interpretation: Why Independence is a Good Sign

- If violations are independent:
  - the model is **risk-sensitive**: it scales VaR up when volatility rises;
  - violations occur only by random chance, not by structural weakness;
  - no systematic "surprise" losses during crises.
- Independence $\Rightarrow$ VaR is well calibrated and responsive.
- Result: the VaR model provides reliable protection when it matters most.

Christoffersen Conditional Coverage (CC) Test

- A good VaR model should have:
  1. Correct unconditional coverage,
  2. Independent violations.
- Christoffersen's Conditional Coverage test combines both into one statistic:

$$LR_{\text{CC}} = LR_{\text{POF}} + LR_{\text{CCI}}.$$

- Under the joint null (correct coverage *and* independence):

$$LR_{\text{CC}} \sim \chi^2(2).$$

- If LR_{CC} exceeds the critical value, the VaR model is rejected.

From Individual Risk to Systemic Risk

- VaR is a *marginal* risk measure: focuses on one asset / portfolio /institution.
- Systemic risk: risk of distress of the *system as a whole* (market, banking sector).
- Key question:

> How much does the distress of one institution increase the risk of the rest of the system?

- Adrian & Brunnermeier (2014) propose CoVaR to quantify this contribution.

CoVaR: Definition

- Let r_{it} be the return (or loss) of institution i and r_{mt} the return of the "system" (e.g., market index).
- CoVaR of the market, given that institution i is in distress:

$$\mathbb{P}\Big(r_{mt} \leq \mathrm{CoVaR}_t^{m|r_{it}=\mathrm{VaR}_{it}(\alpha)} \,\Big|\, r_{it} = \mathrm{VaR}_{it}(\alpha)\Big) = \alpha.$$

- Interpretation:
  - Standard VaR: risk of i alone.
  - CoVaR: risk of the system *conditional* on i being in its own VaR event.

CoVaR: Contribution to Systemic Risk

- Define CoVaR in a "normal" state:

$$\text{CoVaR}_t^{m|r_{it}=\text{Median}(r_{it})},$$

  i.e., when institution i is in a median state (not distressed).
- Then

$$\Delta\text{CoVaR}_{it}(\alpha) = \text{CoVaR}_t^{m|r_{it}=\text{VaR}_{it}(\alpha)} - \text{CoVaR}_t^{m|r_{it}=\text{Median}(r_{it})}.$$

- Interpretation:
  - ΔCoVaR measures how much the system's tail risk increases when i is in distress.
  - Higher ΔCoVaR $\Rightarrow$ stronger systemic impact of i.

Gaussian Linear Approximation

- Under a bivariate Gaussian (or linear) model for (r_{mt}, r_{it}):
  - conditional expectation is linear,
  - conditional variance depends on correlation and marginal volatilities.
- Let $\rho_{m,i,t+1}$ be the conditional correlation, and $\sigma_{m,t+1}, \sigma_{i,t+1}$ the conditional volatilities.
- Then (under suitable assumptions):

$$\Delta\mathrm{CoVaR}_{\alpha}^{m|i,t+1} = \rho_{m,i,t+1} \times \frac{\sigma_{m,t+1}}{\sigma_{i,t+1}} \times \Big(\mathrm{VaR}_{\alpha}^{i,t+1} - \mathrm{Median}(r_{it})\Big).$$

Implications of the Gaussian Linear Formula

- ΔCoVaR scales with:
  - the **correlation** between institution *i* and the system,
  - the ratio of their **volatilities**,
  - the severity of *i*'s own tail loss ($\mathrm{VaR}_{\alpha}^{i,t+1}$).
- Intuition:
  - A large, volatile and highly correlated institution has high ΔCoVaR.
  - An idiosyncratic (weakly correlated) institution has limited systemic impact.

Dynamic Conditional Correlation (DCC): Time-Varying Correlations?

- So far, we allowed **volatility** (risk of each asset) to change over time using GARCH.
- But portfolio risk also depends on **how assets move together**.
- This "moving together" is measured by **correlation**.
- **Key idea:** in the real world, correlations are **not** constant.
  - In calm times: correlations are often low $\Rightarrow$ diversification helps.
  - In crises: correlations become high $\Rightarrow$ many assets fall together.
- We would like a model where correlations can **change over time**.

Correlation and Systemic Risk (Intuition)

- **Systemic risk**: risk that **many institutions are in trouble at the same time**.
- Think of several banks at once:
  - If their returns are **highly correlated**, then their losses tend to happen together.
  - If correlations **increase**, joint losses become more likely.
- Risk measures:
  - **VaR (Value-at-Risk)**: a threshold such that large losses beyond it are rare.
  - **CoVaR**: VaR of the whole system **given** that one bank is already in distress.
- When correlations go up:
  - portfolio VaR increases,
  - CoVaR can jump, because bad news for one bank is more likely to be bad for others too.

Why Constant Correlation Is Not Enough

- A **constant correlation model** assumes one fixed number ρ forever.
- But in reality:
  - Calm period: low correlation.
  - Crisis period: high correlation.
- **Problem:** a fixed ρ cannot capture these changes.
- **Solution:** Dynamic Conditional Correlation model DCC(1,1) (Engle, 2002):
  - mathematically simple (similar spirit to GARCH),
  - allows correlations to **evolve over time**.

Simple Example: Two-Bank Portfolio

Two-Bank Portfolio

- We hold an equally weighted portfolio of Bank A and Bank B.
- Case 1: $\rho_{A,B} = 0$ (no correlation)
  - Losses of A and B are independent.
  - Diversification works well.
- Case 2: $\rho_{A,B} = 0.9$ (very high correlation)
  - A and B tend to lose money at the same time.
  - Diversification works poorly.
- **Goal of DCC:** model how we move from something like Case 1 (low correlation) to Case 2 (high correlation) and back over time.

Step 1: Covariance–Correlation Decomposition

- Let ε_t be the vector of returns **innovations** (shocks) at time t.
- H_t is the **conditional covariance matrix** of ε_t:

$$H_t = D_t \, R_t \, D_t.$$

- With:
  - $D_t = \mathrm{diag}\big(\sqrt{h_{1t}}, \ldots, \sqrt{h_{Nt}}\big)$
    - diagonal matrix with the **volatility** of each asset at time t,
    - each h_{it} can come from a **univariate GARCH** model.
  - R_t is the **time-varying correlation matrix** between the assets.
- Interpretation:
  - D_t controls the **size** of shocks for each asset.
  - R_t controls **how shocks move together**.

Step 2: Standardized Residuals

- We “remove” the individual volatilities using D_t.
- Define the **standardized residuals**:

$$u_t = D_t^{-1}\varepsilon_t = \left(u_{1t}, \ldots, u_{Nt}\right)^\top, \qquad u_{it} = \frac{\varepsilon_{it}}{\sqrt{h_{it}}}.$$

- Intuition:
  - ε_{it} is the shock to asset i.
  - $\sqrt{h_{it}}$ is its current volatility.
  - u_{it} is the **scaled shock**: “how big is the shock compared to its usual risk?”.
- Let $\bar{Q} = \mathbb{E}[u_t u_t^\top]$ be the **unconditional covariance** of u_t.
  - In practice, we estimate $\bar{Q}$ from the data as the sample covariance (or correlation) of u_t.
  - We assume $\bar{Q}$ is **positive definite** (no degenerate combinations).

Step 3: DCC(1,1) Dynamics for Q_t

- Introduce a matrix Q_t that will capture the **time-varying dependence** of u_t.
- Q_t follows a GARCH-type recursion:

$$\boxed{Q_t = (1-\alpha-\beta)\,\bar{Q} + \alpha\, u_{t-1}u_{t-1}^{\top} + \beta\, Q_{t-1}, \qquad \alpha \geq 0,\ \beta \geq 0,\ \alpha+\beta < 1.}$$

- Interpretation:
  - $(1-\alpha-\beta)\,\bar{Q}$: long-run average dependence.
  - $\alpha\, u_{t-1}u_{t-1}^{\top}$: impact of the **most recent shocks**.
  - $\beta\, Q_{t-1}$: **persistence** of past dependence.
- The conditions $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$ help keep the process stable.

Step 4: From Q_t to the Correlation Matrix R_t

- Q_t is not yet a **correlation matrix** (its diagonal is not necessarily 1).
- To obtain the correlation matrix, we **rescale** Q_t to force ones on the diagonal:

$$\boxed{R_t = \mathrm{diag}(Q_t)^{-1/2}\ Q_t\ \mathrm{diag}(Q_t)^{-1/2}.}$$

- Here:
  - $\mathrm{diag}(Q_t)$ is the diagonal matrix made of the diagonal elements of Q_t.
  - $\mathrm{diag}(Q_t)^{-1/2}$ denotes taking the inverse square root of each diagonal element.
- Result:
  - R_t has 1's on the diagonal,
  - off-diagonal entries of R_t are the **time-varying correlations** between assets.

Step 5: Pairwise Correlation $\rho_{ij,t}$

- Let $q_{ij,t}$ be the (i,j) entry of Q_t.
- The DCC **correlation** between series i and j at time t is:

$$\boxed{\rho_{ij,t} = \frac{q_{ij,t}}{\sqrt{q_{ii,t}\, q_{jj,t}}}.}$$

- Intuition:
  - $q_{ij,t}$ measures the "raw" co-movement.
  - $q_{ii,t}$ and $q_{jj,t}$ measure the "raw variance" of each series.
  - Dividing by the square root of the product of the variances turns this into a **correlation**, between -1 and 1.

Step 6: Bivariate DCC(1,1) (Two-Asset Case)

Consider $N = 2$ and

$$\bar{Q} = \begin{pmatrix} \bar{q}_{11} & \bar{q}_{12} \\ \bar{q}_{12} & \bar{q}_{22} \end{pmatrix}.$$

The recursion for each element of Q_t is:

$$\begin{aligned} q_{12,t} &= (1-\alpha-\beta)\,\bar{q}_{12} + \alpha\, u_{1,t-1}u_{2,t-1} + \beta\, q_{12,t-1}, \\ q_{11,t} &= (1-\alpha-\beta)\,\bar{q}_{11} + \alpha\, u_{1,t-1}^2 + \beta\, q_{11,t-1}, \\ q_{22,t} &= (1-\alpha-\beta)\,\bar{q}_{22} + \alpha\, u_{2,t-1}^2 + \beta\, q_{22,t-1}. \end{aligned}$$

Then the DCC(1,1) correlation between the two series is:

$$\boxed{\rho_{12,t} = \frac{(1-\alpha-\beta)\bar{q}_{12} + \alpha u_{1,t-1}u_{2,t-1} + \beta q_{12,t-1}}{\sqrt{\left((1-\alpha-\beta)\bar{q}_{11} + \alpha u_{1,t-1}^2 + \beta q_{11,t-1}\right)\left((1-\alpha-\beta)\bar{q}_{22} + \alpha u_{2,t-1}^2 + \beta q_{22,t-1}\right)}}}$$

Key Takeaways

- **VaR** summarizes tail risk of a portfolio and is central in regulatory capital.
- Different estimation methods:
  - historical/rolling,
  - model-based (GARCH).
- **Backtesting** (Kupiec POF, Christoffersen independence, CC):
  - ensures VaR is statistically consistent with realized losses,
  - detects underestimation or clustering of risk.
- **CoVaR and ΔCoVaR** extend VaR to systemic risk:
  - measure impact of one institution's distress on the whole system.
- **DCC(1,1)**:
  - provides time-varying correlations,
  - crucial for understanding contagion and computing dynamic CoVaR.

From Theory to Practice

- The Jupyter notebook implements:
  - VaR estimation (historical, rolling, GARCH),
  - VaR backtesting (POF, independence, CC),
  - DCC(1,1) estimation,
  - CoVaR and ΔCoVaR computation.
- Use the slides to understand *why* each step matters.
- Use the notebook to see *how* it is done in code.