Lecture 3

Dispersion

A typical wave:

$$u(x,t) = A\cos(kx - \omega t)$$

Normally for waves in real media there will be a specific relationship between ω and k

$$\omega = f(k)$$

This is known as the **dispersion relation**.

Example: Sound Waves

$$\omega = k\sqrt{\frac{B}{\rho}}$$

where B is the bulk modulus of air, and ρ is the density of the air. So,

$$v = \frac{\omega}{k} = \sqrt{\frac{B}{\rho}}$$

$\rightarrow$ v is independent of ω and k
$\rightarrow$ $\omega \propto k$

The group velocity:

$$\frac{\partial \omega}{\partial k} = \sqrt{\frac{B}{\rho}}$$
$$\text{i.e.} \quad v = v_g$$

Sound waves are therefore **non-dispersive**.

Example: Deep Water Waves

$$\omega = \text{const.} \times \sqrt{gk}$$
$$v = \text{const.} \times \sqrt{\frac{g}{k}}$$

Therefore deep water waves **are dispersive**, as v depends on k.

$$v_g = \frac{\partial \omega}{\partial k} = \frac{\text{const.}}{2}\sqrt{\frac{g}{k}} = \frac{v}{2}$$
$$v_g \neq v$$

Brief comments about light

We tend to think of light as travelling at c ($3 \times 10^8 \mathrm{ms}^{-1}$). This is only really true in a vacuum, where:

$$\omega = ck \quad \text{non-dispersive}$$

In most media the speed of light is less than c. For example, in glass:

$$v \approx \frac{2}{3}c$$

A slightly odd example: electromagnetic waves in Earth's ionosphere:

$$\text{dispersion relation:} \quad \omega^2 = c^2k^2 + \omega_p^2$$

where ω_p is a constant - the frequency of natural circular oscillation of electrons in the ionosphere. So,

$$\begin{aligned} v &= \sqrt{c^2 + \frac{\omega_p^2}{k^2}} \\ v_g &= \frac{c^2 k}{\sqrt{c^2k^2 + \omega_p^2}} \end{aligned}$$

Note that the velocity of the light waves is actually *faster* than the speed of light, but that the group velocity is *slower* than the speed of light, and it is the group velocity that matters here.

[Image]

The energy travels at v_g, and it is always the case that $v_g < c$.

The Wave Equation

Take a general form for a travelling wave:

$$\begin{aligned} y(x,t) &= f(x - vt) \\ &= f(g(x,t)) \end{aligned}$$

We're heading for a second order differential equation. Differentiation using the chain rule:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x} \\ &= \frac{\mathrm{d}f}{\mathrm{d}g} \times 1 \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}f}{\mathrm{d}g} \times \frac{\mathrm{d}g}{\mathrm{d}t} \\ &= \frac{\mathrm{d}f}{\mathrm{d}g} \times -v \\ &= -v\frac{\mathrm{d}y}{\mathrm{d}x} \end{aligned}$$

Note, for a left-travelling wave:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = +v\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2 f(g)}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 f}{\mathrm{d}g^2} \times \left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}f}{\mathrm{d}g} \times \frac{\mathrm{d}^2 g}{\mathrm{d}x^2}$$

We find that:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 f}{\mathrm{d}g^2}$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = v^2\frac{\mathrm{d}^2 f}{\mathrm{d}g^2}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = v^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$$

This is the Wave Equation.