Lecture 4

Comments on the wave equation

- It's a one-dimensional wave equation.
- It applies to left and right travelling waves.
- It also applies to standing waves.

So,

$$y(x,t) = Ae^{i(kx-\omega t)}$$

This is a solution to the wave equation, and we find that:

$$v^2 = \frac{\omega^2}{k^2}$$
$$v = \frac{\omega}{k}$$

For light:

$$\frac{\mathrm{d}^2 E}{\mathrm{d}x^2} = \mu_0 \epsilon_0 \frac{\mathrm{d}^2 E}{\mathrm{d}t^2}$$

where E is the electric field, μ_0 is the permeability of free space, and ϵ_0 is the permittivity of free space.

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Examples of mechanical waves

Waves on a taut string

[Image]

Consider a transverse wave travelling left to right along a taut string. The individual particles of the string only move along the y axis and not along the x axis.

Consider a small element of the string.

[Image]

Tension in the string means that the forces are acting along the string itself.

$$F_{1,x} = F_{2,x} = T$$
$$F_{1,y} = F_{1,x} \tan\theta_1 = F_{1,x} \frac{\mathrm{d}y}{\mathrm{d}x}[x_1]$$
$$F_{2,y} = F_{2,x} \tan\theta_2 = F_{2,x} \frac{\mathrm{d}y}{\mathrm{d}x}[x_1 + \Delta x]$$

There is a net upward force.

$$F_y = F_{2,y} - F_{1,y} = T\left(\frac{\mathrm{d}y}{\mathrm{d}x}[x_1] - \frac{\mathrm{d}y}{\mathrm{d}x}[x_1 + \Delta x]\right)$$

Newton's Second Law is

$$F = ma$$

The mass of this small element of the string is given by

$$m = \mu \Delta x$$

where μ is the mass per unit length.

$$\begin{aligned} F_y = \mu \Delta x \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= T \left(\frac{\mathrm{d}y}{\mathrm{d}x} [x_1] - \frac{\mathrm{d}y}{\mathrm{d}x} [x_1 + \Delta x] \right) \\ \mu \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= \frac{T \left(\frac{\mathrm{d}y}{\mathrm{d}x} [x_1] - \frac{\mathrm{d}y}{\mathrm{d}x} [x_1 + \Delta x] \right)}{\Delta x} \end{aligned}$$

Take the limit of $\Delta x \to 0$

$$\begin{aligned} \mu \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= T \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \\ v &= \sqrt{\frac{T}{\mu}} \end{aligned}$$

Calculating how much energy is transmitted by a wave

Transfer of power

We know that waves transfer energy. It is useful to consider the power - the rate of energy transfer.

Consider a wave travelling from left to right again.

[Image]

We need to find out how much force is being applied to the piece of string to the right of point A.

$$\begin{aligned} F_y &= -F_x \frac{\mathrm{d}y}{\mathrm{d}x} \\ P &= F_x v \\ P &= -F_x \frac{\mathrm{d}y}{\mathrm{d}x} \frac{\mathrm{d}y}{\mathrm{d}t} \end{aligned}$$

This is instantaneous power transfer at time t and position x. For a sinusoidal wave:

$$\begin{aligned} y &= A \cos(kx - \omega t) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -Ak \sin(kx - \omega t) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= A\omega \sin(kx - \omega t) \\ P(x,t) &= F_x A^2 k\omega \sin^2(kx - \omega t) \end{aligned}$$