INTRODUCTORY

PHYSICS

Building Models to Describe Our World

Ryan Martin • Emma Neary • Olivia Woodman

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Preface

About this textbook

This textbook is written to fill several needs that we believe were not already met by the many existing introductory physics textbooks. First, we wanted to ensure that the textbook is free to use for students and professors. Second, we wanted to design a textbook that is mindful of the new pedagogies being used in introductory physics, by writing it in a way that is adapted to a flipped-classroom approach where students complete readings, think about the readings, and then discuss the material in class. Third, we wanted to create a textbook that also addresses the experimental aspect of physics, by proposing experiments to be conducted at home or in the lab, as well as guidelines for designing and reporting on experimental results. Finally, we wanted to create a textbook that is a sort of "living document", that professors can edit and re-mix for their own needs, and to which students can contribute material as well. The textbook is hosted on GitHub, which allows anyone to make suggestions, point out issues and mistakes, and contribute material.

This textbook is meant to be paired with the accompanying "Question Library", which contains many practice problems, many of which were contributed by students.

This textbook would not have been possible without the support of Queen's University and the Department of Physics, Engineering Physc & Astronomy at Queen's University, as well as the many helpful discussions with the students, technicians and professors at Queen's University.

Hello from the authors

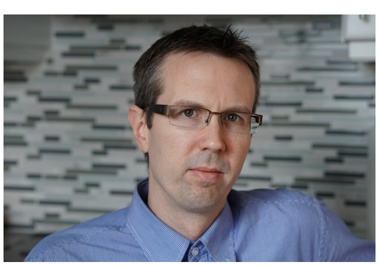

Figure 1: Ryan Martin

Ryan Martin I am a professor of physics at Queen's University. My main research is in the field of particle astrophysics, particularly in studying the properties of neutrinos. I grew up in Switzerland, obtained my Bachelor's, Master's and Ph.D. at Queen's University. I was then a postdoctoral fellow at Lawrence Berkeley National Laboratory, a faculty at the University of South Dakota, before returning to Queen's. I am particularly passionate about education, and I am always seeking opportunities to involve students in helping to make education more accessible. I also like to cook and to play volleyball.

Figure 2: Emma Neary

Emma Neary A bunch of text or the picture does crazy things.A bunch of text or the picture does crazy things.A bunch of text or the picture does crazy things.A bunch of text or the picture does crazy things.A bunch of text or the picture does crazy things.A bunch of text or the picture does crazy things.A bunch of text or the picture does crazy things.A bunch of text or the picture does crazy things.A bunch of text or the picture does crazy things.A bunch of text or the picture does crazy things.A bunch of text or the picture does crazy things.A bunch of text or the picture does crazy things.

Figure 3: Olivia Woodman

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How to use this textbook

This textbook is designed to be used in a flipped-classroom approach, where students complete readings at home, and the material is then discussed in class. The material is thus presented fairly succinctly, and contains **Checkpoint Questions** throughout that are meant to be answered as the students complete the reading. We suggest including these Checkpoint Questions as part of a quiz in a reading assignment (marked based on completion, not correctness), and then using these questions as a starting point for discussions in class.

For topics that are particularly difficult, we have included **Thought Boxes** written by students that try to present the material in a different light. We are always happy if students (or professors) wish to contribute additional thought boxes.

Chapters start with a set of **learning outcomes** and an **opening question** to help students have a sense of the chapter contents. The chapters have **examples** throughout, as well as additional practice problems at the end. The **Question Library** should be consulted for additional practice problems. At the end of the chapter, a **Summary** present the key points from the chapter. We suggest that students carefully read the summaries to make sure that they understand the contents of the chapter (and potentially identify before reading the chapter if the content is only review to them). At the end of the chapters, we also present a section to **think about the material**. This includes questions that can be assigned in reading assignments to research applications of the material or historical context. The thinking about the material section also includes experiments that can be done at home (as part of the reading assignment) or in the lab.

Appendices cover the main background in mathematics (Calculus and Vector), as well as present an introduction to programming in python, which we feel is a useful skill to have in science. There is also an Appendix that is intended to guide work in the lab, by providing examples of how to write experimental proposals and reports, as well as guidelines for reviewing proposals and report. We believe that introductory laboratories should not be be "recipe-based", but rather that students should take an approach similar to that of a researcher in designing (proposing) an experiment, conducting it, and reviewing the proposals and results of their peers.

Credits

This textbook, and especially the many questions in the Question Library would not have been possible without the many contributions from students, teaching assistants and other professors. Below is a list of the people that have contributed material that have made this textbook and Queestion Library possible.

Adam McCaw
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Contents

1 Linear momentum and the centre of mass

In this chapter, we introduce the concepts of linear momentum and of centre of mass. Momentum is a quantity that, like energy, can be defined from Newton's Second Law, to facilitate building models in some cases. Since momentum is often a conserved quantity within a system, it can make calculations much easier than using forces. The concepts of momentum and of centre of mass will also allow us to apply Newton's Second Law to systems comprised of multiple particles including solid objects.

Learning Objectives

- Understand how to calculate linear momentum.
- Understand how to calculate impulse and that it corresponds to a change in momentum.
- Understand when and how to apply conservation of linear momentum to model situations.
- Understand the difference between elastic and inelastic collisions, and when mechanical energy is conserved.
- Understand how to calculate the centre of mass of an object.

Think About It

Rhonda and John are bowling with large watermelons. By mistake, Rhonda lets go of her $3\,\text{kg}$ watermelon at a speed of $10\,\text{m/s}$ and it charges at John's $1\,\text{kg}$ watermelon that was travelling at $2\,\text{m/s}$. If the watermelons hit and start rolling together, at what speed will they be travelling?

A) $2\,\text{m/s}$
B) $4\,\text{m/s}$
C) $8\,\text{m/s}$
D) $-6\,\text{m/s}$

1.1 Momentum

1.1.1 Momentum of a point particle

We can define the momentum, $\vec{p}$, of a particle of mass m and velocity $\vec{v}$ as the vector quantity:

$$\boxed{\vec{p} = m\vec{v}} \tag{1.1}$$

Since this is a vector equation, it corresponds to three equations, one for each component of the momentum vector. It should be noted that the numerical value for the momentum of a particle is arbitrary, as it depends in which frame of reference the velocity of the particle is defined. For example, your velocity with respect to the surface of the Earth is zero, so your momentum relative to the surface of the Earth is zero. However, relative to the surface of the Sun, your velocity, and momentum, are not zero. As we will see, forces are related to a *changes* in momentum, just as they are related to a change in velocity (acceleration).

If the particle has a constant mass, then the time derivative of its momentum is given by:

$$\frac{d}{dt}\vec{p} = \frac{d}{dt}m\vec{v} = m\frac{d}{dt}\vec{v} = m\vec{a}$$

and we can write this as Newton's Second Law, since $m\vec{a}$ must be equal to the vector sum of the forces on the particle of mass m:

$$\boxed{\frac{d}{dt}\vec{p} = \sum \vec{F} = \vec{F}^{net}} \tag{1.2}$$

The equation above is the original form in which Newton first developed his theory. It says that the net force on an object is equal to the rate of change of its momentum. **If the net force on the object is zero, then its momentum is constant, or "conserved"**. In terms of components, Newton's Second Law written for the rate of change of momentum is given by:

$$\begin{aligned}
\frac{dp_x}{dt} &= \sum F_x \\
\frac{dp_y}{dt} &= \sum F_y \\
\frac{dp_z}{dt} &= \sum F_z
\end{aligned}$$

Example 1-1

A particle of mass m is released from rest and allowed to fall freely under the influence of gravity near the Earth's surface (assume that drag is negligible). Is the mechanical energy of the particle conserved? Is the momentum of the particle conserved? If momentum is not conserved, how does momentum change with time? Do your answers change if the force of drag cannot be ignored?

Solution

First, we model the falling particle assuming that there is no force of drag. The only force exerted on the particle is thus its weight.

The mechanical energy of the particle will be conserved only if there are no non-conservative forces doing work on the particle. Since the force of gravity is the only force acting on the particle, its mechanical energy is conserved.

The total momentum of the particle is not conserved, because the sum of the forces on the particle is not zero. Choosing the z axis to be vertical and positive upwards, Newton's Second Law in the z direction is given by:

$$\sum F_z = -mg = \frac{dp_z}{dt}$$

Note that the x and y components of momentum are conserved, since there are no forces with components in that direction. We can find how the z component of the momentum changes with time by taking the anti-derivative of the force with respect to time (from $t = 0$ to $t = T$):

$$\begin{aligned}
\frac{dp_z}{dt} &= -mg \\
\int dp_z &= \int_0^T (-mg)dt \\
p_z(T) - p_z(0) &= -mgT \\
\therefore p_z(T) &= p_z(0) - mgT
\end{aligned}$$

where the z component of momentum, $p_z(T)$ at some time T, is given by its value at time $t = 0$ plus $-mgT$. If the object started at rest ($\vec{v} = 0$), then the magnitude of the momentum, as a function of time, is given by:

$$p(t) = p_z(t) = -mgt$$

and indeed changes with time.

If the force of drag were not negligible, there would be a non-conservative force acting on the particle, so its mechanical energy would no longer be conserved. The particle will accelerate until it reaches terminal velocity. During that phase of acceleration, the net force on the particle is not zero (it is accelerating), so its momentum is not conserved. Once the particle reaches terminal velocity, the net force on the particle is zero, and its momentum is conserved from then on.

Discussion: This simple example highlights the fact that mechanical energy and momentum are conserved under different conditions. Just because one is conserved does not mean that the other is conserved. It also shows that Newton's Second Law is a statement about change in momentum, not momentum itself (just like it is a statement about acceleration, change in velocity, not velocity).

1.1.2 Impulse

When we introduced the concept of energy, we started by calculating the "work", W, done by a force exerted on an object over a specific path between two points:

$$W = \int_A^B \vec{F} \cdot d\vec{l}$$

We then introduced kinetic energy, K, to be that quantity whose change is equal to the net work done on the particle

$$W^{net} = \int_A^B \vec{F}^{net} \cdot d\vec{l} = \Delta K$$

where the net force, $\vec{F}^{net}$, is the vector sum of the forces on the particle.

We can do the same thing, but instead of integrating the force over distance, we can integrate it over time. We thus introduce the concept of "impulse", $\vec{J}$, of a force, as that force integrated from an initial time, t_A, to a final time, t_B:

$$\vec{J} = \int_{t_A}^{t_B} \vec{F} dt \tag{1.3}$$

where it should be clear that impulse is a vector quantity (and the above vector equation thus corresponds to one integral per component). Impulse is, in general, defined as an integral because the force, $\vec{F}$, could change with time. If the force is constant in time (magnitude and direction), then we can define the impulse without using an integral:

$$\vec{J} = \vec{F}\Delta t$$

where Δt is the amount of time over which the force was exerted. Although the force might never be constant, we can sometimes use the above formula to calculate impulse using an average value of the force.

Checkpoint 1-1

What is the SI unit for impulse?

A) kg per m/s^2
B) g per m/s^2
C) kg per m/s
D) kg per s

Example 1-2

Estimate the impulse that is given to someone's head when they are slapped in the face.

Solution

When we slap someone's face with our hand, our hand exerts a force on their face during the period of time, Δt, over which our hand is in contact with their face. During that period of time, the force on their face goes from being 0, to some unpleasantly high value, and then back to zero, so the force cannot be considered constant.

Let us estimate the average magnitude of the slapping force by considering the deceleration of our slapping hand and modelling the motion as one-dimensional. Let us assume that our slapping hand has a mass $m = 1\,\text{kg}$ and that it is has a speed of $2\,\text{m/s}$ just before it makes contact. Furthermore, let us assume that it is in contact with the face for a period of time Δt. This allows us to find the average acceleration of our hand and thus the average force exerted by the face on our hand to stop it:

$$\begin{aligned} a &= \frac{\Delta v}{\Delta t} \\ \therefore F &= ma = m\frac{\Delta v}{\Delta t} \end{aligned}$$

By Newton's Third Law, the force decelerating our hand has the same magnitude as the force that our hand exerts on the face, allowing us to calculate the impulse given to the person's head:

$$\begin{aligned} J = F\Delta t &= \left(m\frac{\Delta v}{\Delta t}\right)\Delta t = m\Delta v \\ &= (1\,\text{kg})(2\,\text{m/s}) = 2\,\text{kgm/s} \end{aligned}$$

Discussion: Note that the impulse given to the head corresponds exactly to the change in momentum of the hand ($\Delta p = m\Delta v$).

So far, we calculated the impulse that is given by a single force. We can also consider the net impulse given to an object by the net force exerted on the object:

$$\vec{J}^{net} = \int_{t_A}^{t_B} \vec{F}^{net} dt$$

Compare this to Newton's Second Law written out using momentum:

$$\begin{aligned} \frac{d}{dt}\vec{p} &= \vec{F}^{net} \\ \int_{\vec{p}_A}^{\vec{p}_B} d\vec{p} &= \int_{t_A}^{t_B} \vec{F}^{net} dt \\ \vec{p}_B - \vec{p}_A &= \int_{t_A}^{t_B} \vec{F}^{net} dt \\ \therefore \Delta\vec{p} &= \int_{t_A}^{t_B} \vec{F}^{net} dt \end{aligned}$$

and we find that the net impulse received by a particle is precisely equal to its change in momentum:

$$\boxed{\Delta\vec{p} = \vec{J}^{net}} \tag{1.4}$$

This is similar to the statement that the net work done on an object corresponds to its change in kinetic energy, although one should keep in mind that momentum is a vector quantity, unlike kinetic energy.

Example 1-3

A car moving with a speed of 100 km/h collides with a building and comes to a complete stop. The driver and passenger each have a mass of 80 kg. The driver wore a seat belt that extended during the collision, so that the force exerted by the seatbelt on the driver acted for about 2.5 s. The passenger did not wear a seat belt and instead was slowed down by the force exerted by the dashboard, over a much smaller amount of time, 0.2 s. Compare the average decelerating force experienced by the driver and the passenger.

Solution

We can calculate the change in momentum of both people, which will be equal to the impulse they received as they collided with the seatbelt or with the dashboard. Since we know the duration in time that the forces were exerted, we can calculate the average force involved in order to give the required impulse. We can assume that this all happens in one dimension, so we use scalar quantities instead of vectors.

The change in momentum along the direction of motion for either the driver or passenger is given by:

$$\Delta p = p_B - p_A = (0) - p_A = -mv_A$$

where v_A is the initial speed of the car, and the final momentum of either person is zero.

The change in momentum is equal to the impulse received by either person during a period of time Δt:

$$J = F\Delta t = \Delta p = -mv_A$$
$$F = -m\frac{v_A}{\Delta t}$$

For the driver, this corresponds:

$$F = (80\,\text{kg})\frac{(27.8\,\text{m/s})}{(2.5\,\text{s})} = 890\,\text{N}$$

and for the passenger:

$$F = (80\,\text{kg})\frac{(27.8\,\text{m/s})}{(0.2\,\text{s})} = 11\,120\,\text{N}$$

The force on the driver is thus comparable to their weight, whereas the passenger experiences an average force that is more than 10 times their weight.

Discussion: Any mechanism that results in a longer collision time will help to reduce the forces that are involved. This is why cars are designed to crumple in head-on collisions. We can understand this in terms of the crumpling of the car absorbing some of the kinetic energy of the car, as well as lengthening the time of the collision so that the forces involved are smaller. Note that we did not need to use impulse to calculate the average force, since we could have just used kinematics to determine the acceleration and Newton's Second Law to calculate the corresponding force. Using impulse is equivalent by construction, but sometimes, it is easier mathematically.

TODO: Question Library question: Give a function F(t), calculate the change in momentum resulting from that force over a certain range of time (i.e. they need to take an integral).

1.1.3 Systems of particles: internal and external forces

So far, we have only used Newton's Second Law to describe the motion of a single point mass particle or to describe the motion of an object whose orientation we did not need to describe (e.g. a block sliding down a hill). In this section, we consider what happens when there are multiple point particles that form a "system".

In physics, we loosely define a system as the ensemble of objects/particles that we wish to describe. So far, we have only described systems made of one particle, so describing the motion of the system was equivalent to describing the motion of that single particle. A system of two particles could be, for example, two billiard balls on a pool table. To describe that system, we would need to provide functions that describe the positions, velocities, and forces exerted on both balls. We can also define functions/quantities that describe the system as a whole, rather than the details. For example, we can define the total kinetic energy of the system, K, corresponding to the sum of kinetic energies of the two balls. We can also define the total momentum of the system, $\vec{P}$, given by the vector sum of the momenta of the two balls.

When considering a system of multiple particles, we distinguish between **internal** and **external** forces. Internal forces are those forces that the particles on the system exert on each other. For example, if the two billiard balls in the system collide with each other, they will each exert a force on the other during the collision; those forces are internal. External forces are all other forces exerted on the particles of the system. For example, the force of gravity and the normal force from the pool table are both external forces exerted on the balls in the system (exerted by the Earth, or by the pool table, neither of which we considered to be part of the system). The force exerted by a person hitting one of the balls with a pool queue is similarly an external force. What we consider to be a system is arbitrary; we could consider the pool table and the Earth to be part of the system along with the two balls; in that case, the normal force and the weight of the balls would become internal forces. The classification of whether a force is internal or external to a system of course depends on what is considered part of the system.

Checkpoint 1-2

Two pool balls crash against each other. Is this (minuscule) force of gravity exerted from one ball to the other an internal or external force?

A) Internal, because it is exerted by a particle in the system on another.
B) External, because one of the particles is not in the system.

The key property of internal forces is that **the vector sum of the internal forces in a system is zero**. Indeed, Newton's Third Law states that for every force exerted by object A on object B, there is a force that is equal in magnitude and opposite in direction exerted by object B on object A. If we consider both objects to be in the same system, then the sum of the internal forces between objects A and B must sum to zero. It is important to note that this is quite different than what we have discussed so far about summing forces. The forces that sum to zero are exerted on *different* objects. Thus far, we had only ever considered summing forces that are exerted on the same object in order to apply Newton's Second Law. We have never encountered a situation where "action" and "reaction" forces are summed together, because they act on different objects.

Emma's Thoughts

Internal vs. External forces - what is the "system" and what forces should we consider?

As discussed above, internal and external forces can only be considered in the context of a specific system. So, how do we define this "system"? How far do we go when defining the system?

For example, let's say that you kick a soccer ball, and it hits a nearby lawn chair, knocking it down. You want to determine what will happen to the soccer ball after it hits the lawn chair. What is defined to be the system here, and how should the forces be classified? Is the force exerted by the soccer ball on the lawn chair an external force? Should we consider the friction between the first foot particle that touches the first soccer ball particle?

The best way to approach "defining the system" is to pin down exactly what you're trying to model. Here, specifically, you are trying to determine the velocity of the ball after it hits the lawn chair. In this situation, thinking about the friction between individual foot and soccer ball particles wouldn't help us to figure out the final velocity of the soccer ball. Rather, thinking of the soccer ball and lawn chair as two giant, continuous particles, colliding and exchanging energy would be helpful. In this situation, it would be useful to consider the "system" to be the soccer ball and lawn chair only.

The force exerted by the soccer ball on the lawn chair would be an internal force, as this gives us information as to the final velocity of the soccer ball and is a force exchanged between the particles within the system. The force that gravity exerts on the lawn chair, normal force of the person's foot and force exerted by the foot on the soccer ball are all forces that we would consider "external", as they are forces that arise from interactions that take place partially outside of our defined "system". Therefore, we neglect them in the context of finding the final velocity of the soccer ball.

Remember - "internal" and "external" are not magical properties of a specific type of force. These definitions are made by us in the quest of building useful models.

1.1.4 Conservation of momentum

Consider a system of two particles with momenta $\vec{p}_1$ and $\vec{p}_2$. Newton's Second Law must hold for each particle:

$$\frac{d\vec{p}_1}{dt} = \sum_k \vec{F}_{1k}$$
$$\frac{d\vec{p}_2}{dt} = \sum_k \vec{F}_{2k}$$

where F_{ik} is the k-th force that is acting on particle i. We can sum these two equations together:

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = \sum_k \vec{F}_{1k} + \sum_k \vec{F}_{2k}$$

The quantity on the right is the sum of the forces exerted on particle 1 plus the sum of the forces exerted on particle 2. In other words, it is the sum of all of the forces exerted on all of the particles in the system, which we can write as a single sum. On the left hand side, we have the sum of the two time derivatives of the momenta, which is equal to the time-derivative of the sum of the momenta. We can thus re-write the equation as:

$$\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = \sum \vec{F}$$

where, again, the sum on the right is the sum over all of the forces exerted on the system. Some of those forces are external (e.g. gravity exerted by Earth on the particles), whereas some of the forces are internal (e.g. a contact force between the two particles). Dividing up the sum into a sum over all external forces ($\vec{F}^{ext}$) and a sum over internal forces ($\vec{F}^{int}$):

$$\sum \vec{F} = \sum \vec{F}^{ext} + \sum \vec{F}^{int}$$

The sum of the internal forces is zero:

$$\sum \vec{F}^{int} = 0$$

because for every force that particle 1 exerts on particle 2, there will be an equal and opposite force exerted by particle 2 on particle 1. We thus have:

$$\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = \sum \vec{F}^{ext}$$

Furthermore, if we introduce the "total momentum of the system", $\vec{P} = \vec{p}_1 + \vec{p}_2$, as the sum of the momenta of the individual particles, we find:

$$\frac{d\vec{P}}{dt} = \sum \vec{F}^{ext}$$

which is the equivalent of Newton's Second Law for a system where, $\vec{P}$, is the total momentum of the system, and the sum of the forces is only over external forces to the system.

Note that the derivation above easily extends to any number, N, of particles, even though we only did it with $N = 2$. In general, for the "ith particle", with momentum $\vec{p}_i$, we can write Newton's Second Law:

$$\frac{d\vec{p}_i}{dt} = \sum_k \vec{F}_{ik}$$

where the sum is over only those forces exerted on particle i. Summing the above equation for all N particles in the system:

$$\frac{d}{dt}\sum_i \vec{p}_i = \sum \vec{F}^{ext} + \sum \vec{F}^{int}$$

where the sum over internal forces will vanish for the same reason as above. Introducing the total momentum of the system, $\vec{P}$:

$$\vec{P} = \sum_i \vec{p}_i$$

We can write an equation for the time-derivative of the total momentum of the system:

$$\boxed{\frac{d\vec{P}}{dt} = \sum \vec{F}^{ext}} \tag{1.5}$$

where the sum of the forces is the sum over all forces external to the system. Thus, **if there are no external forces on a system, then the total momentum of that system is conserved** (if the time-derivative of a quantity is zero then that quantity is constant).

We already argued in the previous section that we can make all forces internal if we choose our system to be large enough. If we make the system be the Universe, then there are no forces external to the Universe, and the total momentum of the Universe must be constant:

$$\frac{d\vec{P}^{Universe}}{dt} = \sum_{Universe} \vec{F}^{ext} = 0$$
$$\therefore \vec{P}^{Universe} = \text{constant}$$

In summary, we saw that:

- If no forces are exerted on a single particle, then the momentum of that particle is constant (conserved).
- In a system of particles, the total momentum of the system is conserved if there are no external forces on the system.
- If there are no non-conservative forces exerted on a particle, then that particle's mechanical energy is constant (conserved).
- In a system of multiple particles, the total mechanical energy of the system will be conserved if there are no non-conservative forces exerted on the system.

When we refer to a force being "exerted on a system", we mean exerted on one or more of the particles in the system. In particular, the sum of the work done by internal forces is not necessarily zero, so **energy and momentum are thus conserved under different conditions**.

Example 1-4

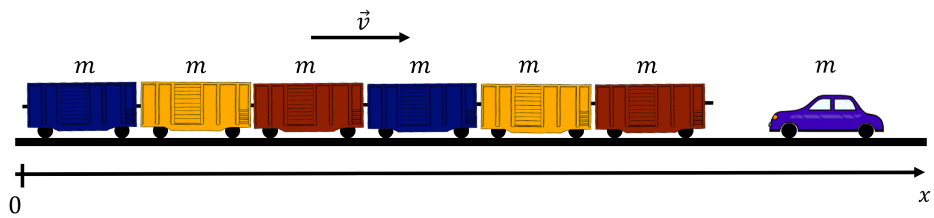

Figure 1.1: A train with N cars of mass m about to collide with a car of mass m that is at rest on the track.

Consider a train made of N cars of equal mass m that is travelling at constant speed v along a straight piece of track where friction and drag are negligible, as depicted in Figure 1.1. An empty car of mass m was left at rest on the track in front of the train. The train collides with the empty car which stays attached to the front of the train. What is the speed of the train after the collision? Is the total mechanical energy of the system conserved?

Solution

When the train collides with the car, it will exert a "collision" force on the car, and the car will exert an opposite force on the train. If we consider both of the train and the car as being part of the same system, then those collision forces will be internal, and the momentum of the system (train + car) will be conserved. The train and car both experience external forces from Earth's gravity and the normal force from the train tracks. However, those two sets of forces cancel each other out, since neither the train nor the car have any acceleration in the vertical direction (the sum of the forces on each object has no net vertical component). Thus, there are no net external forces on the car+train system, and the total momentum of the system is conserved through the collision.

We can model this system in one dimension (along the track), which we call the x axis. We choose the ground as a frame of reference, the positive direction to correspond to the initial velocity of the train, and the origin to be located where the car initially starts. Before the collision, the x component of the momenta of the train (mass Nm) and car (mass m) are:

$$\begin{aligned} p_{train} &= Nmv \\ p_{car} &= 0 \end{aligned}$$

After the collision, the car is attached to the train (and thus has the same speed, v'), so the momenta of the train and car are:

$$\begin{aligned} p'_{train} &= Nmv' \\ p'_{car} &= mv' \end{aligned}$$

where the primes $'$ denote quantities after the collision. Applying conservation of momentum to the system, the total momentum before and after the collision must be equal:

$$\begin{aligned} p_{train} + p_{car} &= p'_{train} + p'_{car} \\ \therefore Nmv &= Nmv' + mv' \\ \therefore v' &= \frac{N}{N+1}v \end{aligned}$$

and the speed of the train with the additional car attached is reduced by a factor $N/(N+1)$ compared to what it was before the collision.

We can check to see if the mechanical energy of the system is conserved, since we know the speeds of the train and car before and after the collision. Since all of the motion is horizontal, gravity and the normal force do no work on either the train or car, so their mechanical energy can be taken as their kinetic energy (their gravitational potential energy does not change after the collision). The total mechanical energy of the system,

E, before the collision is the kinetic energy of the train:

$$E = \frac{1}{2}Nmv^2$$

The total mechanical energy of the system, E', after the collision is:

$$\begin{aligned} E' &= \frac{1}{2}Nmv'^2 + \frac{1}{2}mv'^2 = \frac{1}{2}(N+1)mv'^2 \\ &= \frac{1}{2}(N+1)m\left(\frac{N}{N+1}v\right)^2 \\ &= \frac{1}{2}m\frac{N^2}{N+1}v^2 \end{aligned}$$

and we see that $E' < E$, and thus that the total mechanical energy of the system is not conserved (it is reduced after the collision).

Discussion: We could have solved this problem by carefully modelling the force exerted by the car on the train during the collision, which would have allowed us to find the speed of the train after the collision using its acceleration. This would have required a detailed model for that force, which we do not have. However, by realizing that the train and car could be considered as a system with no net external forces exert on it, we were able to easily find the speed of the train after the collision using conservation of momentum.

We also found that mechanical energy was not conserved. This makes physical sense because, for the car to remain attached to the train, there presumably had to be some significant forces in play that "crushed" the car into the train. Some of the initial kinetic energy of the train was used to deform the train and the car during the collision. We can also think of deforming a material as giving it energy. Sometimes that energy is recoverable (e.g. compressing a spring), sometimes, it is not (e.g. crushing a car).

If the car and train were equipped with large springs to absorb the energy of the impact, the collision could have conserved mechanical energy, as the springs compress and then expand back. The speed of the car and train would then be different after the collision in this case (see example 1-8). It is a feature of collisions where the two bodies remain attached to each other that mechanical energy is not conserved.

1.2 Collisions

In this section we go through a few examples of applying conservation of momentum to model collisions. Collisions can loosely be defined as events where the momenta of individual particles in a system are different before and after the event.

We distinguish between two types of collisions: **elastic** and **inelastic** collisions. Elastic collisions are those for which the total mechanical energy of the system is conserved during the collision (i.e. it is the same before and after the collision). Inelastic collisions are those for which the total mechanical energy of the system is not conserved. In either case, to model the system, one chooses to define the system such that there are no external forces on the system so that total momentum is conserved.

1.2.1 Inelastic collisions

In this section, we give a few examples of modelling inelastic collisions. Inelastic collisions are usually easier to handle mathematically, because one only needs to consider conservation of momentum and does not use conservation of energy (which usually involves equations that are quadratic in the speeds because of the kinetic energy term).

Example 1-5

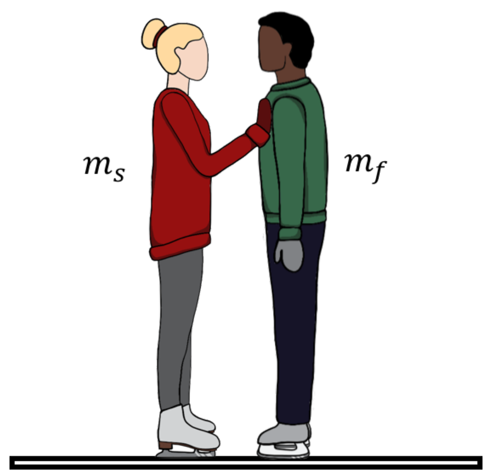

Figure 1.2: One skater pushing another on a frictionless horizontal surface.

You (mass m_s) and your friend (mass m_f) face each other on ice skates on an ice surface that is slippery enough that friction can be considered negligible, as shown in Figure 1.2. You shove your friend away from you so that he moves with velocity $\vec{v}_f$ away from you (the velocity is measured relative to the ice). Is the collision elastic? What is your speed relative to the ice after you shoved your friend?

Solution

We can consider the system as being comprised of you and your friend. There are no net external forces on the system (gravity and normal forces cancel each other), so the momentum of the system will be conserved.

The mechanical energy will not be conserved. You had to use chemical potential energy stored in your muscles to shove your friend. Thus, external energy (i.e. not mechanical energy from you or your friend) was injected into the system, and we should expect the total mechanical energy to be larger after the collision.

Before the collision, both you and your friend have zero speed, and thus zero kinetic energy and zero momentum. After the collision, your friend has a velocity $\vec{v}_f$. We can use conservation of total momentum, $\vec{P}$, to determine your velocity, $\vec{v}_s$, after the collision.

$$\begin{aligned}\vec{P} &= \vec{P}' \\ 0 &= m_s\vec{v}_s + m_f\vec{v}_f \\ \therefore \vec{v}_s &= -\frac{m_f}{m_s}\vec{v}_f\end{aligned}$$

where primes ($'$) denote a quantity after the collision. We find that your velocity is in the opposite direction from that of your friend. Before the collision, the mechanical energy, E, of the system is zero (we can ignore gravitational potential energy, since everything is in the horizontal plane). After the collision, the mechanical energy, E', is:

$$E' = \frac{1}{2}m_s v_s^2 + \frac{1}{2}m_f v_f^2$$

which is clearly bigger than the mechanical energy before the collision (i.e. 0), as we suspected it would be.

Discussion: We find that you recoil in the opposite direction, which makes sense. If you push your friend in one direction, Newton's Third Law says that your friend pushes you in the opposite direction. Your speed furthermore depends on the ratio of your friend's mass to yours. This also makes sense, because if you both feel the same force, the person with the smallest mass will have the highest speed; if your mass is higher than your friend's, then your speed after the collision will be smaller than your friend's.

We also saw that mechanical energy was not conserved. In terms of energy, we can explain this by saying that you burned up chemical potential energy stored in your muscles in order to shove your friend. Because we included both you and your friend in the system, the shove was an internal force and momentum is conserved. Of course, if we had considered only you as the system, then your momentum would not have been conserved during the collision.

The type of collision that we described here is also sometimes called an "explosion". You can imagine all of the parts that make up a bomb as small particles. When the bomb explodes, chemical potential energy is converted into the kinetic energy of the bomb fragments. If you consider all of the particles/fragments of the bomb as a system, then the total momentum of all of the bomb fragments is conserved (and equal to zero if the bomb was initially at rest). Again, mechanical energy would not be conserved (and would increase) as the chemical potential energy is converted into mechanical energy.

Example 1-6

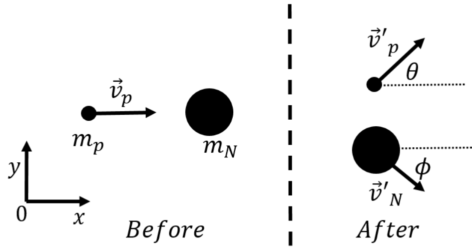

Figure 1.3: A proton of mass m_p colliding inelastically with a nucleus of mass m_N.

A proton of mass m_p and initial velocity $\vec{v}_p$ collides inelastically with a nucleus of mass m_N at rest, as shown in Figure 1.3. A coordinate system is set up as shown, such that the initial velocity of the proton is in the x direction. After the collision, the proton's speed is measured to be v'_p and its velocity vector is found to make an angle θ with the x axis as shown. What is the velocity vector of the nucleus after the collision? Assume that the collision takes place in vacuum.

Solution

As a system, we consider the proton and the nucleus together, so that the total momentum of the system is conserved during the collision, as no other external forces are exerted on the two particles (since they are in vacuum). Because momentum is a vector, each component of the total momentum, $\vec{P}$, is conserved during the collision:

$$\begin{aligned}\vec{P} &= \vec{P}'\\ \therefore P_x &= P'_x\\ \therefore P_y &= P'_y\end{aligned}$$

where, as usual, primes ($'$) denote quantities after the collision. After the collision, both particles will have velocity vectors that have x and y components. Let the velocity vector of the nucleus after the collision be $\vec{v}'_N$ and let ϕ be the angle that it makes with the x axis, as shown in Figure 1.3.

We can start by considering the conservation of the x component of the total momentum. The initial and final momenta in the x direction are given by:

$$\begin{aligned}P_x &= m_p v_p\\ P'_x &= m_p v'_p \cos\theta + m_N v'_N \cos\phi\\ \therefore m_p v_p &= m_p v'_p \cos\theta + m_N v'_N \cos\phi\end{aligned}$$

which gives us a first equation to determine the final velocity of the nucleus.

The y component of the total momentum before the collision is zero since we chose the coordinate system such that the initial velocity of the proton is in the x direction. The initial and final momenta in the y direction are given by:

$$\begin{aligned}
P_y &= 0 \\
P'_y &= m_p v'_p \sin\theta - m_N v'_N \sin\phi \\
\therefore m_p v'_p \sin\theta &= m_N v'_N \sin\phi
\end{aligned}$$

which gives us a second equation to solve for the velocity of the nucleus. With the two equations from momentum conservation, we can solve for the magnitude and direction of the velocity of the nucleus. From the y component of momentum conservation, we can find an expression for the speed of the nucleus:

$$\begin{aligned}
m_p v'_p \sin\theta &= m_N v'_N \sin\phi \\
\therefore v'_N &= \frac{m_p}{m_N} v'_p \sin\theta \frac{1}{\sin\phi}
\end{aligned}$$

which we can substitute into the x equation for momentum conservation to solve for the angle ϕ:

$$\begin{aligned}
m_p v_p &= m_p v'_p \cos\theta + m_N v'_N \cos\phi \\
m_p v_p &= m_p v'_p \cos\theta + m_N \frac{m_p}{m_N} v'_p \sin\theta \frac{\cos\phi}{\sin\phi} \\
v_p &= v'_p \cos\theta + v'_p \sin\theta \frac{1}{\tan\phi} \\
\therefore \tan\phi &= \frac{v'_p \sin\theta}{v_p - v'_p \cos\theta}
\end{aligned}$$

If we were given numbers for the initial and final speed of the proton, as well as the angle θ, we would be able to find a value for the angle ϕ, which we could then use to determine the final speed of the nucleus:

$$v'_N = \frac{m_p}{m_N} v'_p \sin\theta \frac{1}{\sin\phi}$$

Discussion: By using the conservation of momentum equation and writing out the x and y components, we were able to find two equations to determine the magnitude and direction of the nucleus' velocity after the collision. In the limit where $m_N >> m_p$, the final speed of the nucleus would be very small (close to zero).

Example 1-7

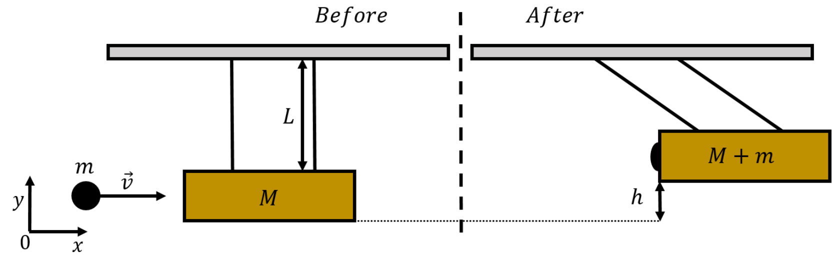

Figure 1.4: A bullet of mass m strikes and embeds itself into a ballistic pendulum of mass M.

A ballistic pendulum is a device that can be built to measure the speed of a projectile. The pendulum is constructed such that the projectile is fired at the bob of the pendulum (typically a block of wood) which then swings as illustrated in Figure 1.4, with the projectile embedded within. By measuring the height that is reached by the pendulum's bob, one can determine the speed of the projectile before it collided with the pendulum. If a ballistic pendulum with a mass M suspended at the end of strings of length L is observed to rise by a height h after being struck by a bullet of mass m, how fast was the bullet moving?

Solution

We can model this situation by dividing it into three phases:

1. Before the bullet collides with the pendulum, only the bullet has momentum in the x direction.
2. Immediately after the **inelastic** collision, the bullet and pendulum form a combined object of mass $M + m$ that has the same momentum as the bullet, in the x direction, before the pendulum starts to swing upwards.
3. The pendulum with the embedded bullet swings upwards until its kinetic energy is zero.

The collision between the bullet and pendulum is inelastic, because some of the kinetic energy of the bullet is used to deform the bullet and the pendulum. In general, any collision where two objects end up "stuck together" is inelastic.

In order to model the pendulum's motion we first apply conservation of momentum to determine the speed, v', of the pendulum and embedded bullet just after the collision. Applying conservation of momentum in the x direction to the system formed by the pendulum and the bullet, just before and after the collision, we have:

$$\begin{aligned} P &= mv \\ P' &= (M+m)v' \\ \therefore mv &= (M+m)v' \\ \therefore v' &= \frac{m}{m+M}v \end{aligned}$$

where P and P' are the initial and final momenta of the system, respectively. The pendulum with the bullet embedded in it will thus have a speed of v' at the bottom of the pendulum's motion, before it swings upwards.

We can now use conservation of energy to model the swinging motion since, at that point, only tension and gravity act on the pendulum, and there are no non-conservative forces. If we choose the origin to be the location of the pendulum at the bottom of its trajectory, its initial gravitational potential energy is zero and its initial mechanical energy, E, is given by:

$$E = \frac{1}{2}(m+M)v'^2$$

At the top of the trajectory, the pendulum with the embedded bullet will stop and have no kinetic energy. The mechanical energy at the top of the trajectory, E', is thus equal to the gravitational potential energy of the pendulum at a height h above the origin:

$$E' = (m+M)gh$$

Applying conservation of mechanical energy allows us to find the initial speed of the bullet:

$$\begin{aligned} E &= E' \\ \frac{1}{2}(m+M)v'^2 &= (m+M)gh \\ v'^2 &= 2gh \\ \left(\frac{m}{m+M}v\right)^2 &= 2gh \\ \therefore v &= \frac{m+M}{m}\sqrt{2gh} \end{aligned}$$

where is the second last line we used the expression for v' that we obtained from conservation of momentum.

Discussion: This example showed a situation in which momentum and energy were both conserved, but not at the same time. This example also highlighted how, by using conservation laws, one can derive models that are much easier to solve mathematically than if one had to model all of the forces involved.

1.2.2 Elastic collisions

In this section, we give a few examples of modelling elastic collisions. Even though it is mechanical energy that is conserved in an elastic collision, one can almost always simplify this to only kinetic energy being conserved. If a collision takes place in a well localized position in space (i.e. before and after the collision are the same point in space), then the potential energies of the objects involved will not change, thus any change in their mechanical energy is due to a change in kinetic energy.

Example 1-8

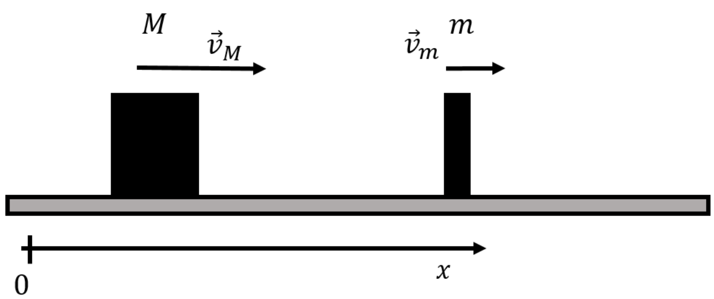

Figure 1.5: Two blocks about to collide elastically.

A block of mass M moves with velocity $\vec{v}_M$ in the x direction, as shown in Figure 1.5. A block of mass m is moving with velocity $\vec{v}_m$ also in the x direction and collides elastically with block M. Both blocks slide with no friction on the horizontal surface. What are the velocities of the two blocks after the collision?

Solution

Because this is an elastic collision, both the total momentum and total mechanical energy are conserved. Equating the total momentum before and after the collision, and considering only the x component gives the following equation:

$$\vec{P} = \vec{P}'$$
$$Mv_M + mv_m = Mv'_M + mv'_m$$

where the primes ($'$) correspond to the quantities after the collision. Note that, in principle, the x components of the velocities (v_M, v'_M, v_m, v'_m) could be negative numbers if the corresponding block is moving in the negative x direction.

For the mechanical energy of the two blocks, we only need to consider their kinetic energy since their gravitational potential energies are the same before and after the collision on the horizontal surface.

$$\begin{aligned}
E &= E' \\
\frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2 &= \frac{1}{2}Mv_M'^2 + \frac{1}{2}mv_m'^2 \\
\therefore Mv_M^2 + mv_m^2 &= Mv_M'^2 + mv_m'^2
\end{aligned}$$

where we cancelled the factor of one half in the last line. This gives two equations (conservation of energy and momentum) and two unknowns (the two speeds after the collision). This is not a linear system of equations, because the equation from conservation of energy is quadratic in the speeds.

The following method allows many models for elastic collisions between two particles to be solved easily by converting the quadratic equation from energy conservation into an equation that is linear in the speeds. First, write both equations so that the quantities related to each particle are on opposite sides of the equation. For momentum, this gives:

$$\begin{aligned}
Mv_M + mv_m &= Mv_M' + mv_m' \\
\therefore M(v_M - v_M') &= m(v'm - v_m) \qquad (1.6)
\end{aligned}$$

For conservation of energy, this gives:

$$\begin{aligned}
Mv_M^2 + mv_m^2 &= Mv_M'^2 + mv_m'^2 \\
\therefore M(v_M^2 - v_M'^2) &= M(v_m'^2 - v_m^2)
\end{aligned}$$

which we can re-write as:

$$\begin{aligned}
M(v_M^2 - v_M'^2) &= M(v_m'^2 - v_m^2) \\
M(v_M - v_M')(v_M + v_M') &= M(v_m' - v_m)(v_m' + v_m)
\end{aligned}$$

We can then divide this by the momentum equation (Equation 1.6):

$$\begin{aligned}
\frac{M(v_M - v_M')(v_M + v_M')}{M(v_M - v_M')} &= \frac{M(v_m' - v_m)(v_m' + v_m)}{m(v'm - v_m)} \\
\therefore v_M + v_M' &= v_m' + v_m
\end{aligned}$$

which gives us an equation that is much easier to work with, since it is linear in the speeds. If we re-arrange this last equation back so that quantities before and after the collision are on different sides of the equality:

$$\boxed{v_M - v_m = -(v_M' - v_m')}$$

we can see that the relative speed between M and m is the same before and after the collision. That is, if block M "saw" block m approaching with a speed of $3\,\mathrm{m/s}$ before the collision, it would “see” block m moving *away* with speed $3\,\mathrm{m/s}$ after the collision, regardless of the actual directions and velocities of the block, if the collision was elastic.

By using this equation with the original conservation of momentum equation, we now have two equations and two unknowns that are easy to solve:

$$\begin{aligned}
v_M - v_m &= -(v'_M - v'_m)\\
Mv_M + mv_m &= Mv'_M + mv'_m
\end{aligned}$$

Solving for v'_m in both equations gives:

$$\begin{aligned}
v_M - v_m &= -(v'_M - v'_m)\\
\therefore v'_m &= v_M + v'_M - v_m\\
Mv_M + mv_m &= Mv'_M + mv'_m\\
\therefore v'_m &= \frac{1}{m}(Mv_M + mv_m - Mv'_M)
\end{aligned}$$

Equating the two expressions for v'_m allows us to solve for v'_M:

$$\begin{aligned}
\frac{1}{m}(Mv_M + mv_m - Mv'_M) &= v_M + v'_M - v_m\\
Mv_M + mv_m - Mv'_M &= mv_M + mv'_M - mv_m\\
(M-m)v_M + 2mv_m &= (M+m)v'_M\\
\therefore v'_M &= \frac{M-m}{M+m}v_M + \frac{2m}{M+m}v_m
\end{aligned}$$

One can easily solve for the other speed, v'_m:

$$\therefore v'_m = \frac{m-M}{M+m}v_m + \frac{2M}{M+m}v_M$$

And writing these together:

$$\begin{aligned}
v'_M &= \frac{M-m}{M+m}v_M + \frac{2m}{M+m}v_m\\
v'_m &= \frac{m-M}{M+m}v_m + \frac{2M}{M+m}v_M
\end{aligned}$$

Discussion: The formulas that we obtained above are valid for any one dimensional elastic collision.

Checkpoint 1-3

Two trains of equal masses collide elastically on a track. If train A was at rest and train B had a speed v, what are the speeds of the trains after the collision?

A) Both trains A and B travel away from each other with speeds $\frac{1}{2}$v.
B) Train A will be at rest and train B will move away with a speed v.
C) Both trains A and B will stick together and move at a speed of v.
D) Train B will be at rest and train A will move away at a speed of v.

Example 1-9

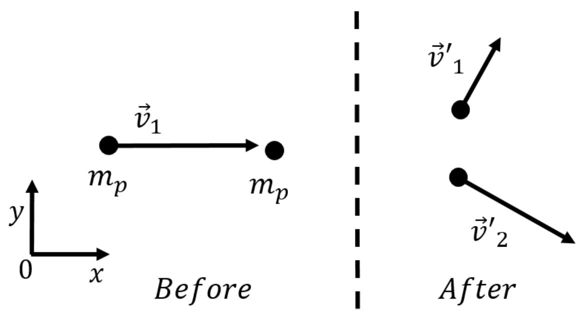

Figure 1.6: A proton elastically collides with a proton at rest.

A proton of mass m and initial velocity $\vec{v}_1$ collides elastically with a second proton that is at rest. After the collision, the two protons have velocities $\vec{v}'_1$ and $\vec{v}'_2$, as shown in Figure 1.6. Show that the velocity vectors of the two protons are perpendicular after the collision.

Solution

This example highlights a particular feature of elastic collisions when the two objects have the same mass and one of the objects is initially at rest. The conservation of momentum for the system comprised of the two protons can be written as:

$$m\vec{v}_1 = m\vec{v}'_1 + m\vec{v}'_2$$
$$\vec{v}_1 = \vec{v}'_1 + \vec{v}'_2$$

where the left hand side corresponds to the initial total momentum and the right hand side to the total momentum after the collision. In the second line, we cancelled out the mass, and obtained a vector relation between the velocity vectors. We can graphically illustrate the vector relation as in Figure 1.7 which shows the triangle that is formed by adding the two outgoing velocity vectors to obtain the initial velocity vector.

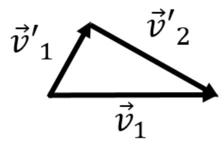

Figure 1.7: Graphical illustration of the relation between the initial and final velocity vectors as a vector sum.

Conservation of kinetic energy for the collision can be written as:

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2$$
$$v_1^2 = v_1'^2 + v_2'^2$$

where the left hand side corresponds to the initial kinetic energy and the right hand side to the final kinetic energy. We cancelled the mass and factor of one half in the second line. This last equation gives a relation between the magnitudes of the velocity vectors. By comparing the equation above to Pythagoras' theorem, and by inspecting the triangle in Figure 1.7, it is clear that the triangle must be a right angle triangle, and thus that $\vec{v}'_1$ and $\vec{v}'_2$ must be perpendicular.

1.2.3 Frames of reference

Review Topics

Before proceeding, you may wish to review sections **??** and **??** on expressing velocities in different frames of reference.

Because the momentum of a particle is defined using the velocity of the particle, its value depends on the reference frame that we chose to measure that velocity. In some cases, it is useful to apply momentum conservation in a frame of reference where the total momentum of the system is zero. For example, consider two particles of mass m_1 and m_2, moving towards each other with velocities $\vec{v}_1$ and $\vec{v}_2$, respectively, as measured in a frame of reference S, as illustrated in Figure 1.8.

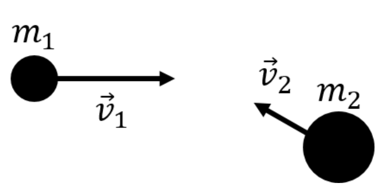

Figure 1.8: Two particles moving towards each other.

In the frame of reference S, the total momentum, $\vec{P}$, of the two particles can be written:

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2$$

Consider a frame of reference, S', that is moving with velocity $\vec{v}_{CM}$ relative to the frame of reference S. In that frame of reference, the velocities of the two particles are different and given by:

$$\begin{aligned}\vec{v}'_1 &= \vec{v}_1 - \vec{v}_{CM}\\ \vec{v}'_2 &= \vec{v}_2 - \vec{v}_{CM}\end{aligned}$$

The total momentum, $\vec{P}'$, in the frame of reference S' is then given by[1]:

$$\begin{aligned}\vec{P}' &= m_1\vec{v}'_1 + m_2\vec{v}'_2\\ &= m_1(\vec{v}_1 - \vec{v}_{CM}) + m_2(\vec{v}_2 - \vec{v}_{CM})\\ &= m_1\vec{v}_1 + m_2\vec{v}_2 - (m_1+m_2)\vec{v}_{CM}\end{aligned}$$

We can choose the velocity of the frame, $\vec{v}_{CM}$ such that the total momentum in that frame of reference is zero:

$$\begin{aligned}\vec{P}' &= 0\\ m_1\vec{v}_1 + m_2\vec{v}_2 - (m_1+m_2)\vec{v}_{CM} &= 0\\ \therefore \vec{v}_{CM} &= \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}\end{aligned}$$

This "special" frame of reference, in which the total momentum of the system is zero, is called the "centre of mass frame of reference". The velocity of centre of mass frame of reference can easily be obtained if there are N particles involved instead of two:

$$\boxed{\therefore \vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \dots}{m_1+m_2+m_3+\dots} = \frac{\sum m_i\vec{v}_i}{\sum m_i}} \tag{1.7}$$

Again, you should note that because the above equation is a vector equation, it represents one equation per component of the vectors. For example, the x component of the velocity of the centre of mass frame of reference is given by:

$$\therefore v_{CMx} = \frac{m_1v_{1x} + m_2v_{2x} + m_3v_{3x} + \dots}{m_1+m_2+m_3+\dots} = \frac{\sum m_iv_{ix}}{\sum m_i}$$

Example 1-10

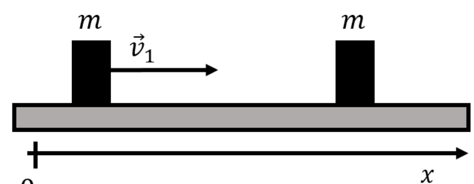

Figure 1.9: One block approaching another identical block at rest, as seen in the lab frame of reference.

[1]Note that we are using primes ($'$) to denote quantities in a different reference frame, not after a collision.

In the frame of reference of a lab, a block of mass m has a velocity $\vec{v}_1$ directed along the positive x axis and is approaching a second block of mass m that is at rest ($\vec{v}_2 = 0$), as shown in Figure 1.9. What is the velocity of the centre of mass frame? What is the velocity of each block in the centre of mass frame? Verify that the total momentum is zero in the centre of mass frame.

Solution

Since this is a one dimensional situation, we only need to evaluate the x component of the velocity of the centre of mass:

$$\begin{aligned}\vec{v}_{CM} &= \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}\\ \therefore v_{CMx} &= \frac{m_1 v_{1x} + m_2 v_{2x}}{m_1+m_2}\\ &= \frac{mv_1 + m(0)}{m+m}\\ &= \frac{1}{2}v_1\end{aligned}$$

The centre of mass frame of reference is thus also moving along the positive direction of the x axis, but with a speed that is half of that of the moving block. In the centre of mass frame of reference, it appears that the block on the left is slower than in the lab frame and that the block on the right is moving in the negative x direction. The velocities of the two blocks in the centre of mass frame of reference are given by:

$$\begin{aligned}v_1' &= v_1 - v_{CMx} = \frac{1}{2}v_1\\ v_2' &= (0) - v_{CMx} = -\frac{1}{2}v_1\end{aligned}$$

Thus, in the reference frame of the centre of mass, the two block are approaching each other with the same speed ($v_1/2$), which is only the case because the two blocks have the same mass. The blocks, as viewed in the centre of mass frame of reference, are shown in Figure 1.10.

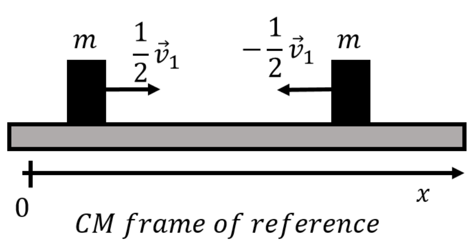

Figure 1.10: In the centre of mass frame of reference, the block approach each other with the same speed, because they have the same mass.

Clearly, the total momentum is zero in the centre of mass frame of reference:

$$\vec{P}' = m\vec{v}'_1 + m\vec{v}'_2 = m\left(\frac{1}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_1\right) = 0$$

Discussion: As we have seen, in the centre of mass frame of reference the total momentum is zero. If there are only two particles, and they have the same mass, then, in the centre of mass frame of reference, they both have the same speed and move either towards or away from each other.

Example 1-11

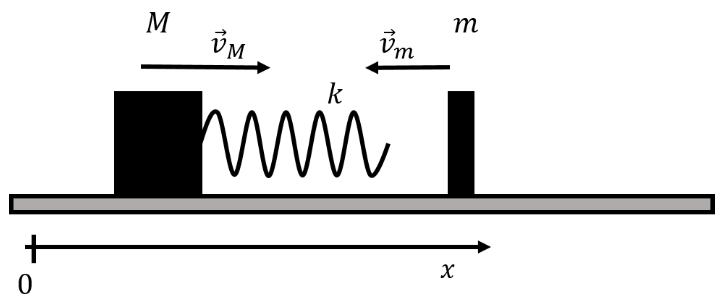

Figure 1.11: One block attached to a spring about to collide with another block.

A block of mass M with a spring of spring constant k attached to it is sliding on a frictionless surface with velocity $\vec{v}_M$ in the x direction. A second block of mass m has velocity $\vec{v}_m$ also in the x direction (shown above in the negative x direction, but let us assume that we do not necessarily know the direction, only that the two blocks will collide). During the collision between the blocks, what is the maximum amount by which the spring is compressed?

Solution

The collision is elastic because the energy used to compress the spring is "given back" when the spring extends again, since the spring force is conservative.

They key to modelling the compression of the spring is to identify the condition under which the spring is maximally compressed. This will occur at the point during the collision where the two masses will have exactly the same velocity, momentarily moving in unison as the spring is maximally compressed. Because, instantaneously, the masses have the same velocity, there is a frame of reference in which the two masses are at rest, and the momentum is zero. Of course, that frame of reference is the centre of mass frame of reference.

Because the collision is one-dimensional, we can calculate the velocity of the centre of mass as:

$$v_{CM}=\frac{Mv_M+mv_m}{m+M}$$

where we note that v_m is a negative number, since the block of mass m is moving in the negative x direction. The total momentum, $\vec{P}^{CM}$, in the centre of mass frame of reference must be zero. Writing this out for the x component and transforming the velocities of the two blocks into the centre of mass frame of reference:

$$\begin{aligned}P_x^{CM}=M(v_M-v_{CM})+m(v_m-v_{CM})=0\\ \therefore (v_m-v_{CM})=-\frac{M}{m}(v_M-v_{CM})\end{aligned}$$

Also note that we can write the velocity difference v_M-v_{CM} without using the centre of mass velocity:

$$\begin{aligned}v_M-v_{CM}&=v_M-\frac{Mv_M+mv_m}{m+M}=\frac{1}{m+M}(v_M(m+M)-Mv_M-mv_m)\\&=\frac{m}{m+M}(v_M-v_m)\end{aligned}$$

We can then use conservation of energy in the centre of mass frame to determine the maximal compression of the spring. Before the collision, the total mechanical energy in the system, E, is the sum of the kinetic energies of the two blocks (as the spring is not compressed):

$$\begin{aligned}E&=\frac{1}{2}m(v_m-v_{CM})^2+\frac{1}{2}M(v_M-v_{CM})^2\\&=\frac{1}{2}\frac{M^2}{m}(v_M-v_{CM})^2+\frac{1}{2}M(v_M-v_{CM})^2\\&=\frac{1}{2}M\left(1+\frac{M}{m}\right)(v_M-v_{CM})^2\\&=\frac{1}{2}M\left(\frac{m+M}{m}\right)(v_M-v_{CM})^2\\&=\frac{1}{2}M\left(\frac{m+M}{m}\right)\left(\frac{m}{m+M}(v_M-v_m)\right)^2\\&=\frac{1}{2}\left(\frac{mM}{m+M}\right)(v_M-v_m)^2\end{aligned}$$

where we used our expressions above to simplify the expression. When the spring is maximally compressed, the two blocks are at rest and the mechanical energy of the system, E', is all "stored" as spring potential energy:

$$E'=\frac{1}{2}kx^2$$

where x is the distance by which the spring is compressed. Equating the two allows us to determine the maximal compression of the spring:

$$\begin{aligned}E &= E'\\ \frac{1}{2}\left(\frac{mM}{m+M}\right)(v_M - v_m)^2 &= \frac{1}{2}kx^2\\ \therefore x &= \sqrt{\frac{1}{k}\left(\frac{mM}{m+M}\right)}(v_M - v_m)\end{aligned}$$

Discussion: By modelling the collision in the centre of mass frame of reference, we were easily able to determine the maximal compression of the spring. This would have been more difficult in the lab frame of reference, because the two blocks would still be moving when the spring is maximally compressed, so we would have needed to determine their speeds to determine the total mechanical energy when the spring is compressed.

When we calculated the initial kinetic energy, we found that it was given by:

$$E = \frac{1}{2}\left(\frac{mM}{m+M}\right)(v_M - v_m)^2 = \frac{1}{2}M_{red}(v_M - v_m)^2$$

The combination of masses in parentheses is called the "reduced mass" of the system, and is a sort of effective mass that can be used to model the system as a whole.

1.3 The centre of mass

In this section, we show how to generalize Newton's Second Law so that it may describe the motion of an object that is not a point particle. Any object can be described as being made up of point particles; for example, those particles could be the atoms that make up regular matter. We can thus use the same terminology as in the previous sections to describe a complicated object as a "system" comprised of many point particles, themselves described by Newton's Second Law. A system could be a rigid object where the point particles cannot move relative to each other, such as atoms in a solid[2]. Or, the system could be a gas, made of many atoms moving around, or it could be a combination of many solid objects moving around.

In the previous section, we saw how the total momentum and the total mechanical energy of the system could be used to describe the system as a whole. In this section, we will define the centre of mass which will allow us to describe the position of the system as a whole.

Consider a system comprised of N point particles. Each point particle i, of mass m_i, can be described by a position vector, $\vec{r}_i$, a velocity vector, $\vec{v}_i$, and an acceleration vector, $\vec{a}_i$, relative to some coordinate system. Newton's Second Law can be applied to any one of the

[2] In reality, even atoms in a solid can move relative to each other, but they do not move by large amounts compared to the object.

particles in the system:

$$\sum_k \vec{F}_{ik} = m_i\vec{a}_i$$

where $\vec{F}_{ik}$ is the k-th force exerted on particle i. We can write Newton's Second Law once for each of the N particles, and we can sum those N equations together:

$$\sum_k \vec{F}_{1k} + \sum_k \vec{F}_{2k} + \sum_k \vec{F}_{3k} + \ldots = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots$$
$$\sum \vec{F} = \sum_i m_i\vec{a}_i$$

where the sum on the left is the sum of all of the forces exerted on all of the particles in the system[3] and the sum over i on the right is over all of the N particles in the system. As we have already seen, the sum of all of the forces exerted on the system can be divided into separate sums over external and internal forces:

$$\sum \vec{F} = \sum \vec{F}^{ext} + \sum \vec{F}^{int}$$

and the sum over the internal forces is zero[4]. We can thus write that the sum of the external forces exerted on the system is given by:

$$\sum \vec{F}^{ext} = \sum_i m_i\vec{a}_i \tag{1.8}$$

We would like this equation to resemble Newton's Second Law, but for the system as a whole. Suppose that the system has a total mass, M:

$$M = m_1 + m_2 + m_3 + \cdots = \sum_i m_i$$

we would like to have an equation of the form:

$$\sum \vec{F}^{ext} = M\vec{a}_{CM} \tag{1.9}$$

to describe the system as a whole. However, it is not clear what the acceleration, $\vec{a}_{CM}$ refers to, since the particles in the system could all be moving in different directions. Suppose that there is a point in the system, whose position is given by the vector, $\vec{r}_{CM}$, in such a way that the acceleration above is the second time-derivative of that position vector:

$$\vec{a}_{CM} = \frac{d^2}{dt^2}\vec{r}_{CM}$$

[3]Again, we are summing together forces that are acting on **different** particles

[4]Recall, the internal forces are those forces that particles in the system are exerting on one another. Because of Newton's Third Law, these will sum to zero.

We can compare Equations 1.8 and 1.9 to determine what the position vector $\vec{r}_{CM}$ corresponds to:

$$\begin{aligned}
\sum \vec{F}^{ext} &= \sum_i m_i \vec{a}_i = \sum_i m_i \frac{d^2}{dt^2}\vec{r}_i \\
\sum \vec{F}^{ext} &= M\vec{a}_{CM} = M\frac{d^2}{dt^2}\vec{r}_{CM} \\
\therefore M\frac{d^2}{dt^2}\vec{r}_{CM} &= \sum_i m_i \frac{d^2}{dt^2}\vec{r}_i
\end{aligned}$$

Re-arranging, and noting that the masses are constant in time, and so they can be factored into the derivatives:

$$\begin{aligned}
\frac{d^2}{dt^2}\vec{r}_{CM} &= \frac{1}{M}\sum_i m_i \frac{d^2}{dt^2}\vec{r}_i \\
\frac{d^2}{dt^2}\vec{r}_{CM} &= \frac{d^2}{dt^2}\left(\frac{1}{M}\sum_i m_i \vec{r}_i\right) \\
\therefore \vec{r}_{CM} &= \frac{1}{M}\sum_i m_i \vec{r}_i
\end{aligned}$$

where in the last line we set the quantities that have the same time derivative equal to each other[5]. $\vec{r}_{CM}$ is the vector that describes the position of the "centre of mass" (CM). The position of the centre of mass is described by Newton's Second Law applied to the system as a whole:

$$\boxed{\sum \vec{F}^{ext} = M\vec{a}_{CM}} \tag{1.10}$$

where M is the total mass of the system, and the sum of the forces is the sum over only external forces on the system.

Although we have formally derived Newton's Second Law for a system of particles, we really have been using this result throughout the text. For example, when we modelled a block sliding down an incline, we never worried that the block was made of many atoms all interacting with each other and the surroundings. Instead, we only considered the external forces on the block, namely, the normal force from the incline, any frictional forces, and the total weight of the object (the force exerted by gravity). Technically, the force of gravity is not exerted on the block as a whole, but on each of the atoms. However, when we sum the force of gravity exerted on each atom:

$$m_1\vec{g} + m_2\vec{g} + m_3\vec{g} + \dots = (m_1 + m_2 + m_3 + \dots)\vec{g} = M\vec{g}$$

we find that it can be modelled by considering the block as a single particle of mass M upon which gravity is exerted. The centre of mass is sometimes described as the "centre

[5]Technically, the terms in the derivatives are only equal to within two constants of integration, $\vec{r}_{CM} = \frac{1}{M}\sum_i m_i\vec{r}_i + at + b$, which we can set to zero

of gravity", because it **corresponds to the location where we can model the total force of gravity, $M\vec{g}$, as being exerted**. When we applied Newton's Second Law to the block, we then described the motion of the block as a whole (and not the motion of the individual atoms). Specifically, we modelled the motion of the centre of mass of the block.

The position of the centre of mass is a vector equation that is true for each coordinate:

$$\begin{aligned}
\vec{r}_{CM} &= \frac{1}{M}\sum_i m_i\vec{r}_i \\
\therefore x_{CM} &= \frac{1}{M}\sum_i m_i x_i \\
\therefore y_{CM} &= \frac{1}{M}\sum_i m_i y_i \\
\therefore z_{CM} &= \frac{1}{M}\sum_i m_i z_i \qquad (1.11)
\end{aligned}$$

The centre of mass is that **position in a system that is described by Newton's Second Law when it is applied to the system as a whole**. The centre of mass can be thought of as an average position for the system (it is the average of the positions of the particles in the system, weighted by their mass). By describing the position of the centre of mass, we are not worried about the detailed positions of the all of the particles in the system, but rather only the average position of the system as a whole. In other words, this is equivalent to viewing the whole system as a single particle of mass M located at the position of the centre of mass.

Consider, for example, a person throwing a dumbbell that is made from two spherical masses connected by a rod, as illustrated in Figure 1.12. The dumbbell will rotate in a complex manner as it moves through the air. However, the centre of mass of the dumbbell will travel along a parabolic trajectory (projectile motion), because the only external force exerted on the dumbbell during its trajectory is gravity.

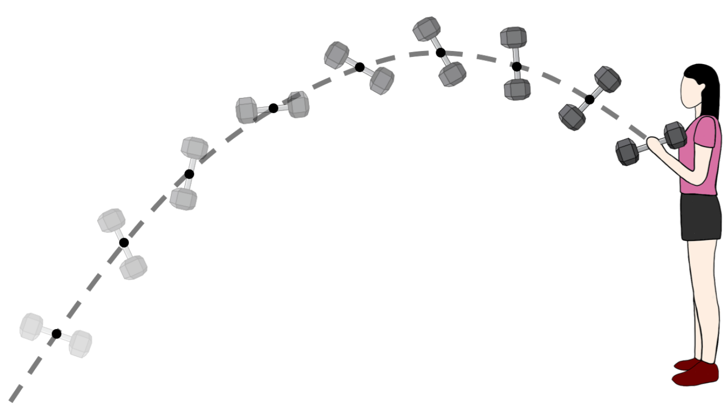

Figure 1.12: The motion of the centre of mass of a dumbbell is described by Newton's Second Law, even if the motion of the rotating dumbbell is more complex.

If we take the derivative with respect to time of the centre of mass position, we obtain the velocity of the centre of mass, and its components, which allow us to describe how the system is moving as a whole:

$$\begin{aligned}
\vec{v}_{CM} &= \frac{d}{dt}\vec{r}_{CM} = \frac{1}{M}\sum_i m_i \frac{d}{dt}\vec{r}_i = \frac{1}{M}\sum_i m_i\vec{v}_i \\
\therefore v_{CMx} &= \frac{1}{M}\sum_i m_i v_{ix} \\
\therefore v_{CMy} &= \frac{1}{M}\sum_i m_i v_{iy} \\
\therefore v_{CMz} &= \frac{1}{M}\sum_i m_i v_{iz}
\end{aligned} \tag{1.12}$$

Note that this is the same velocity that we found earlier for the velocity of the centre of mass frame of reference. In the centre of mass frame of reference, the total momentum of the system is zero. This makes sense, because the centre of mass represents the average position of the system; if we move "with the system", then the system appears to have zero momentum.

We can also define the total momentum of the system, $\vec{P}$, in terms of the total mass, M, of the system and the velocity of the centre of mass:

$$\begin{aligned}
\vec{P} &= \sum m_i\vec{v}_i = \frac{M}{M}\sum m_i\vec{v}_i \\
&= M\vec{v}_{CM}
\end{aligned}$$

which we can also use in Newton's Second Law:

$$\frac{d}{dt}\vec{P} = \sum \vec{F}^{ext}$$

and again, see that the total momentum of the system is conserved if the net external force on the system is zero. In other words, the centre of mass of the system will move with constant velocity when momentum is conserved.

Finally, we can also define the acceleration of the centre of mass by taking the time derivative of the velocity:

$$\begin{aligned}
\vec{a}_{CM} &= \frac{d}{dt}\vec{v}_{CM} = \frac{1}{M}\sum_i m_i \frac{d}{dt}\vec{v}_i = \frac{1}{M}\sum_i m_i\vec{a}_i \\
\therefore a_{CMx} &= \frac{1}{M}\sum_i m_i a_{ix} \\
\therefore a_{CMy} &= \frac{1}{M}\sum_i m_i a_{iy} \\
\therefore a_{CMz} &= \frac{1}{M}\sum_i m_i a_{iz}
\end{aligned} \tag{1.13}$$

Example 1-12

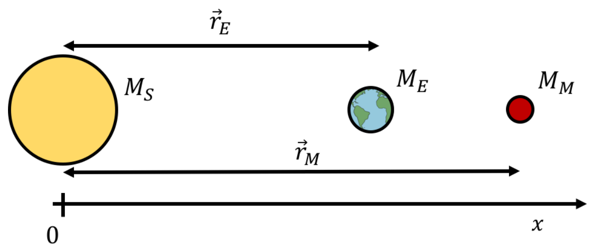

Figure 1.13: A syzygy between the Sun, Earth, and Mars.

In astronomy, a syzygy is defined as the event in which three bodies are all lined up along a straight line. For example, a syzygy occurs when the Sun (mass $M_S = 2.00 \times 10^{30}\,\text{kg}$), Earth (mass $M_E = 5.97 \times 10^{24}\,\text{kg}$), and Mars (mass $M_M = 6.39 \times 10^{23}\,\text{kg}$) are all lined up, as in Figure 1.13. How far from the centre of the Sun is the centre of mass of the Sun, Earth, Mars system during a syzygy?

Solution

Since this is a one-dimensional problem, we can define an x axis that is co-linear with the three bodies, and find only the x coordinate of the position of the centre of mass. We are free to choose the origin of the coordinate system, so we choose the origin to be located at the centre of the Sun. This way, the position of the centre of mass along the x axis will directly correspond to its distance from the centre of the Sun.

The Sun, Earth, and Mars are not point particles. However, because they are spherically symmetric, their centres of mass correspond to their geometric centres. We can thus model them as point particles with the mass of the body located at the corresponding geometric centre. If $r_E = 1.50 \times 10^{11}\,\text{m}$ ($r_M = 2.28 \times 10^{11}\,\text{m}$) is the distance from the centre of the Earth (Mars) to the centre of the Sun, then the position of the centre of mass is given by:

$$\begin{aligned}
x_{CM} &= \frac{1}{M}\sum_i m_i x_i \\
&= \frac{M_S(0) + M_E r_E + M_M r_M}{M_S + M_E + M_M} \\
&= \frac{(2.00 \times 10^{30}\,\text{kg})(0) + (5.97 \times 10^{24}\,\text{kg})(1.50 \times 10^{11}\,\text{m}) + (6.39 \times 10^{23}\,\text{kg})(2.28 \times 10^{11}\,\text{m})}{(2.00 \times 10^{30}\,\text{kg}) + (5.97 \times 10^{24}\,\text{kg}) + (6.39 \times 10^{23}\,\text{kg})} \\
&= 5.21 \times 10^{5}\,\text{m}
\end{aligned}$$

The centre of mass of the Sun-Earth-Mars system during a syzygy is located approximately 500 km from the centre of the Sun.

Discussion: The radius of the Sun is approximately 700 000 km, so the centre of mass of the system is well inside of the Sun. The Sun is so much more massive than either of the Earth or Mars, that the two planets hardly contribute to shifting the centre of mass away from the centre of the Sun. We would generally consider the masses of the two planets to be negligible if one wanted to model how the solar system itself moves around the Milky Way galaxy.

Example 1-13

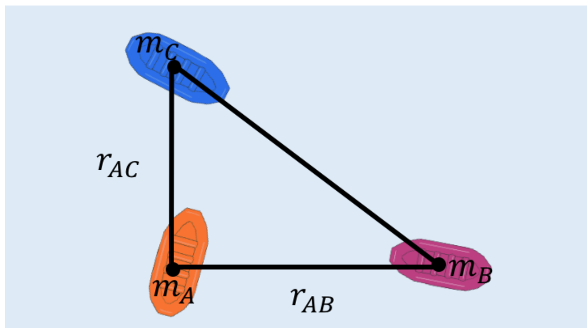

Figure 1.14: Three people on rafts on a lake.

Alice (mass m_A), Brice (mass m_B), and Chloë (mass m_C) are stranded on individual rafts of negligible mass on a lake, off of the coast of Nyon. The rafts are located at the corners of a right-angle triangle, as illustrated in Figure 1.14, and are connected by ropes. The distance between Alice and Brice is r_{AB} and the distance between Alice and Chloë is r_{AC}, as illustrated. Alice decides to pull on the rope that connects her to Chloë , while Brice decide to pull on the rope that connects him to Alice. Where will the three rafts meet?

Solution

We consider the system comprised of the three people and their rafts and model each person and their raft as a point particle with the mass concentrated at the centre of the raft. The forces exerted by pulling on the ropes are internal forces (one particle on the other), and will thus have no impact on the motion of the centre of mass of the system. There are no net external forces exerted on the system (the forces of gravity are balanced out by the forces of buoyancy from the rafts). The centre of mass of the system does not move when the people are pulling on the ropes, so they must ultimately meet at the centre of mass.

We can define a coordinate system such that the origin is located where Alice is initially located, the x axis is in the direction from Alice to Brice, and the y axis is in the direction from Alice to Chloë. The initial positions of Alice, Brice, and Chloë are thus:

$$\begin{aligned}
\vec{r}_A &= 0\hat{x} + 0\hat{y}\\
\vec{r}_B &= r_{AB}\hat{x} + 0\hat{y}\\
\vec{r}_C &= 0\hat{x} + r_{AC}\hat{y}
\end{aligned}$$

respectively. The x and y coordinates of the centre of mass are thus:

$$\begin{aligned}
x_{CM} &= \frac{1}{M}\sum_i m_i x_i = \frac{m_A(0) + m_B r_{AB} + m_C(0)}{m_A + m_B + m_C} = \left(\frac{m_B}{m_A + m_B + m_C}\right) r_{AB}\\
y_{CM} &= \frac{1}{M}\sum_i m_i y_i = \frac{m_A(0) + m_B(0) + m_C r_{AC}}{m_A + m_B + m_C} = \left(\frac{m_C}{m_A + m_B + m_C}\right) r_{AC}
\end{aligned}$$

which corresponds to the position where the three rafts will meet, relative to the initial position of Alice.

Discussion: By using the centre of mass, we easily found where the three rafts would meet. If we had used Newton's Second Law on the three rafts individually, the model would have been complicated by the fact that the forces exerted by Alice and Brice on the ropes change direction as the rafts begin to move, which would have required the use of integrals to determine the motion of each person.

TODO: Question library problem, projectile that splits in flight, where does it land? See Giancolli example 9-18.

1.3.1 The centre of mass for a continuous object

So far, we have considered the centre of mass for a system made of point particles. In this section, we show how one can determine the centre of mass for a "continuous object"[6]. We previously argued that if an object is uniform and symmetric, its centre of mass will be located at the centre of the object. Let us show this explicitly for a uniform rod of total mass M and length L, as depicted in Figure 1.15.

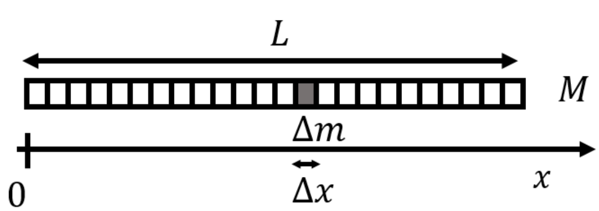

Figure 1.15: A rod of length L and mass M.

In order to determine the centre of mass of the rod, we first model the rod as being made

[6] In reality, there are of course no continuous objects since, at the atomic level, everything is made of particles.

of N small "mass elements" each of equal mass, Δm, and of length Δx, as shown in Figure 1.15. If we choose those mass elements to be small enough, we can model them as point particles, and use the same formulas as above to determine the centre of mass of the rod.

We define the x axis to be co-linear with the rod, such that the origin is located at one end of the rod. We can define the "linear mass density" of the rod, λ, as the mass per unit length of the rod:

$$\lambda = \frac{M}{L}.$$

A small mass element of length Δx, will thus have a mass, Δm, given by:

$$\Delta m = \lambda \Delta x$$

If there are N mass elements that make up the rod, the x position of the centre of mass of the rod is given by:

$$\begin{aligned} x_{CM} &= \frac{1}{M}\sum_i^N m_i x_i = \frac{1}{M}\sum_i^N \Delta m x_i \\ &= \frac{1}{M}\sum_i^N \lambda \Delta x x_i \end{aligned}$$

where x_i is the x coordinate of the i-th mass element. Of course, we can take the limit over which the length, Δx, of each mass element goes to zero to obtain an integral:

$$x_{CM} = \lim_{\Delta x \to 0} \frac{1}{M}\sum_i^N \lambda \Delta x x_i = \frac{1}{M}\int_0^L \lambda x dx$$

where the discrete variable x_i became the continuous variable x, and Δx was replaced by dx (which is the same, but indicates that we are taking the limit of $\Delta x \to 0$). The integral is easily found:

$$\begin{aligned} x_{CM} &= \frac{1}{M}\int_0^L \lambda x dx = \frac{1}{M}\lambda \left[\frac{1}{2}x^2\right]_0^L \\ &= \frac{1}{M}\lambda\frac{1}{2}L^2 = \frac{1}{M}\left(\frac{M}{L}\right)\frac{1}{2}L^2 \\ &= \frac{1}{2}L \end{aligned}$$

where we substituted the definition of λ back in to find, as expected, that the centre of mass of the rod is half its length away from one of the ends.

Suppose that the rod was instead not uniform and that its linear density depended on the position x along the rod:

$$\lambda(x) = 2a + 3bx$$

We can still find the centre of mass by considering an infinitesimally small mass element of mass dm, and length dx. In terms of the linear mass density and length of the mass element, dx, the mass dm is given by:

$$dm = \lambda(x)dx$$

The x position of the centre of mass is thus found the same way as before, except that the linear mass density is now a function of x:

$$\begin{aligned}
x_{CM} &= \frac{1}{M}\int_0^L \lambda(x)xdx = \frac{1}{M}\int_0^L (2a+3bx)xdx = \frac{1}{M}\int_0^L (2ax+3bx^2)dx \\
&= \frac{1}{M}\left[ax^2+bx^3\right]_0^L \\
&= \frac{1}{M}(aL^2+bL^3)
\end{aligned}$$

In general, for a continuous object, the position of the centre of mass is given by:

$$\begin{aligned}
\vec r_{CM} &= \frac{1}{M}\int \vec r dm \\
\therefore x_{CM} &= \frac{1}{M}\int xdm \\
\therefore y_{CM} &= \frac{1}{M}\int ydm \\
\therefore z_{CM} &= \frac{1}{M}\int zdm \qquad (1.14) \\
& \qquad (1.15)
\end{aligned}$$

where in general, one will need to write dm in terms of something that depends on position (or a constant) so that the integrals can be evaluated over the spatial coordinates (x,y,z) over the range that describe the object. In the above, we wrote $dm = \lambda dx$ to express the mass element in terms of spatial coordinates.

Example 1-14

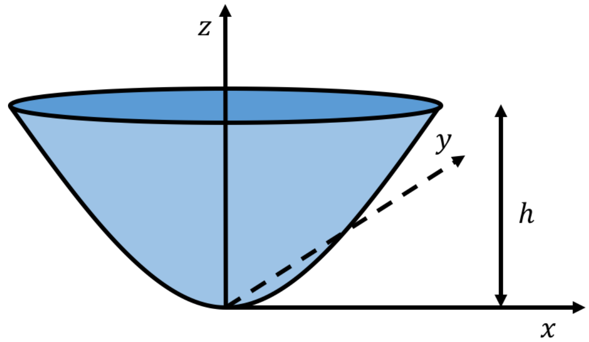

Figure 1.16: A symmetric bowl with parabolic sides is completely filled with water. The bowl has a height h.

A bowl of height h has parabolic sides and a circular cross-section, as illustrated in Figure 1.16. The bowl is filled with water. The bowl itself has a negligible mass and thickness, so that the mass of the full bowl is dominated by the mass of the water. Where is the centre of mass of the full bowl?

Solution

We can define a coordinate system such that the origin is located at the bottom of the bowl and the z axis corresponds to the axis of symmetry of the bowl. Because the bowl is full of water, and the bowl itself has negligible mass, we can model the full bowl as a uniform body of water with the same shape as the bowl and (volume) mass density ρ equal to the density of water. Furthermore, by symmetry, the centre of mass of the bowl will be on the z axis.

Because the bowl has a circular cross-section, we can divide it up into disk-shaped mass elements, dm, that have an infinitesimally small height dz, and a radius $r(z)$, that depends on their z coordinate (Figure 1.16).

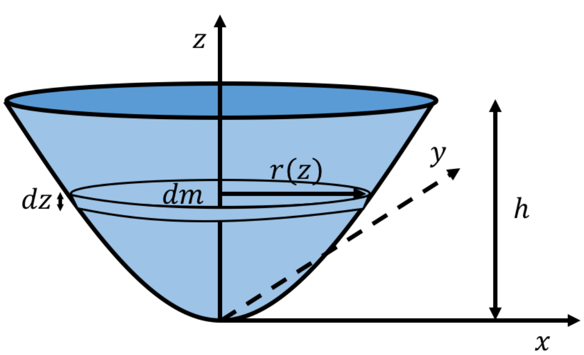

Figure 1.17: The parabolic bowl divided up into disk-shaped mass elements, dm, that have an infinitesimally small height dz, and a radius $r(z)$, that depends on their z coordinate.

The centre of mass of each disk-shaped mass element will be located where the corresponding disk intersects the z axis. The mass of one disk element is given by:

$$dm = \rho dV = \rho\pi r^2(z)dz$$

where $dV = \pi r(z)^2 dz$ is the volume of the disk with radius $r(z)$ and thickness dz. The radius of the infinitesimal disk depends on its z position, since the radii of the different disks must describe a parabola:

$$\begin{aligned} z(r) &= \frac{1}{a^2}r^2 \\ r(z) &= a\sqrt{z} \\ \therefore dm &= \rho\pi r^2(z)dz = \rho\pi a^2 z dz \end{aligned}$$

where we introduced the constant a so that the dimensions are correct. The constant a determines how "steep" the parabolic sides are. The z coordinate of the centre of mass is thus given by:

$$\begin{aligned} z_{CM} &= \frac{1}{M}\int z dm = \frac{1}{M}\int_0^h z(\rho\pi a^2 z dz) = \frac{\rho\pi a^2}{M}\int_0^h z^2 dz \\ &= \frac{\rho\pi a^2}{M}\left[\frac{1}{3}z^3\right]_0^h \\ &= \frac{\rho\pi a^2}{3M}h^3 \end{aligned}$$

However, we are not quite done, since we do not know the total mass, M, of the water. To find the total mass of water, M, we proceed in an analogous way, and determine the value of the sum (integral) of all of the mass elements:

$$M = \int dm = \int_0^h \rho\pi a^2 z dz = \rho\pi a^2\left[\frac{1}{2}z^2\right]_0^h = \frac{1}{2}\rho\pi a^2 h^2$$

Substituting this value for M, we can determine the z coordinate of the centre of mass of the full bowl:

$$z_{CM} = \frac{\rho\pi a^2}{3M}h^3 = \frac{2\rho\pi a^2}{3\rho\pi a^2 h^2}h^3 = \frac{2}{3}h$$

Regardless of the actual shape of the parabola (the parameter a), the centre of mass will always be two thirds of the way up from the bottom of the bowl.

Discussion: In determining the centre of mass of a three dimensional object, we used symmetry to argue that the x and y coordinates would be zero. We then found the z position of the centre of mass by dividing up the bowl into infinitesimally small mass elements (disks) along the direction in which we needed to find the centre of mass coordinate.

Checkpoint 1-4

True or False: The centre of mass of a continuous object is always located within the object.

A) True
B) False

TODO: For above, redo the figure, and split into a figure with just the bowl in the question and a figure showing, dm, dz, etc, in the solution.

1.4 Summary

Key Takeaways

The momentum vector, $\vec{p}$, of a point particle of mass m with velocity $\vec{v}$ is defined as:

$$\vec{p} = m\vec{v}$$

We can write Newton's Second Law for a point particle in term of its momentum:

$$\frac{d}{dt}\vec{p} = \sum \vec{F} = \vec{F}^{net}$$

where the net force on the particle determines the rate of change of its momentum. In particular, if there is no net force on a particle, its momentum will not change.

The net impulse vector, $\vec{J}^{net}$, given over a period of time is defined as the net force exerted on a particle integrated from a time t_A to a time t_B:

$$\vec{J}^{net} = \int_{t_A}^{tB} \vec{F}^{net} dt$$

The net impulse vector is also equal to the change in momentum of the particle in that same period of time:

$$\vec{J}^{net} = \Delta\vec{p} = \vec{p}_B - \vec{p}_A$$

When we define a system of particles, we can distinguish between internal and external forces. Internal forces are those forces exerted by the particles in the system on each other. External forces are those forces on the particles in the system that are not exerted by the particles on each other. The sum over all of the forces on all of the particles in the system will be equal to the sum over the external forces, because the sum over all internal forces on a system is always zero (Newton's Third Law).

The total momentum of a system, $\vec{P}$, is the sum of the momenta, $\vec{p}_i$, of all of the particles in the system:

$$\vec{P} = \sum \vec{p}_i$$

The rate of change of the momentum of a system is equal to the sum of the external forces on the system:

$$\frac{d}{dt}\vec{P} = \sum \vec{F}^{ext}$$

which can be thought of as an equivalent description as Newton's Second Law, but for the system as a whole. If the net (external) force on a system is zero, then the total momentum of the system is conserved.

Collisions are those events when the particles in a system interact (e.g. by colliding) and change their momenta. When modelling collisions, it is usually beneficial to first define a system for which the total momentum is conserved before and after the collision. One can then write the total momentum of the system in terms of the momenta of the individual particles before and after the collision and equate them to determine the momenta of the individual particles.

Collisions can be elastic or inelastic. Elastic collisions are those where, in addition to the total momentum, the total mechanical energy of the system is conserved. The total mechanical energy can usually be taken as the sum of the kinetic energies of the particles in the system.

Inelastic collisions are those in which the total mechanical energy of the system is not conserved. One can usually identify if mechanical energy was introduced or removed from the system and determine if the collision is elastic. It is important to identify when momentum and mechanical energy are conserved. Momentum is conserved if no net force is exerted on the system, whereas mechanical energy is conserved if no net work was done on the system by non-conservative forces.

We can always choose in which frame of reference to model a collision. In some cases, it is convenient to use the frame of reference of the centre of mass of the system, because in that frame of reference, the total momentum of the system is zero.

If a system has a total mass M, then one can use Newton's Second Law to describe its motion:

$$\begin{aligned}\sum \vec{F}^{ext} &= M\vec{a}_{CM} \\ \sum \vec{F}^{ext} &= \frac{d}{dt}\vec{P}\end{aligned}$$

where the sum of the forces is over all of the external forces on the system. The acceleration vector, $\vec{a}_{CM}$, describes the motion of the "centre of mass" of the system. $\vec{P} = M\vec{v}_{CM}$ is the total momentum of the system.

The centre of mass of a system is a mass-weighted average of the positions of all of the particles of mass m_i and position $\vec{r}_i$ that comprise the system:

$$\vec{r}_{CM} = \frac{1}{M}\sum_i m_i\vec{r}_i$$

The vector equation can be broken into components to find the x, y, and z component of the position of the centre of mass. Similarly, one can also define the velocity of the centre of mass of the system, in terms of the individual velocities, $\vec{v}_i$, of the particles in the system:

$$\vec{v}_{CM} = \frac{1}{M}\sum_i m_i\vec{v}_i$$

Finally, one can define the acceleration of the centre of mass of the system, in terms of the individual accelerations, $\vec{a}_i$, of the particles in the system:

$$\vec{a}_{CM} = \frac{1}{M}\sum_i m_i\vec{a}_i$$

If the system is a continuous object, we can find its centre of mass using a sum (integral) of infinitesimally small mass elements, dm, weighted by their position:

$$\begin{aligned}
\vec{r}_{CM} &= \frac{1}{M}\int \vec{r}dm \\
\therefore x_{CM} &= \frac{1}{M}\int xdm \\
\therefore y_{CM} &= \frac{1}{M}\int ydm \\
\therefore z_{CM} &= \frac{1}{M}\int zdm
\end{aligned}$$

The strategy to set up the integrals above is usually to express the mass element, dm, in terms of the position and density of the material of which the object is made. One can then integrate over position in the range defined by the dimensions of the object.

Important Equations

Momentum of a point particle:

$$\vec{p} = m\vec{v}$$
$$\frac{d}{dt}\vec{p} = \sum \vec{F} = \vec{F}^{net}$$

Impulse:

$$\vec{J}^{net} = \int_{t_A}^{tB} \vec{F}^{net} dt$$
$$\vec{J}^{net} = \Delta\vec{p} = \vec{p}_B - \vec{p}_A$$

Momentum of a system:

$$\vec{P} = \sum \vec{p}_i$$
$$\frac{d}{dt}\vec{P} = \sum \vec{F}^{ext}$$

Newton's Second Law for a system:

$$\sum \vec{F}^{ext} = M\vec{a}_{CM}$$
$$\sum \vec{F}^{ext} = \frac{d}{dt}\vec{P}$$

Position of the Centre of Mass of a system:

$$\vec{r}_{CM} = \frac{1}{M}\sum_i m_i\vec{r}_i$$

Velocity of the Centre of Mass of a system:

$$\vec{v}_{CM} = \frac{1}{M}\sum_i m_i\vec{v}_i$$

Acceleration of the Centre of Mass of a system:

$$\vec{a}_{CM} = \frac{1}{M}\sum_i m_i\vec{a}_i$$

Position of the Centre of Mass for a continuous object:

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}dm$$
$$\therefore x_{CM} = \frac{1}{M}\int xdm$$
$$\therefore y_{CM} = \frac{1}{M}\int ydm$$
$$\therefore z_{CM} = \frac{1}{M}\int zdm$$

1.5 Thinking about the material

1.5.1 Reflect and research

1. Explain how Newton's Cradle illustrates the conservation of momentum. Are the collisions in Newton's Cradle elastic? Explain!
2. Gymnasts have specially engineered "crash mats" for landing after doing spins and flips in the air. Why do these crash mats have to be specially engineered, and why can't the gymnast just use a big pile of blankets?
3. Give 2 examples where the centre of mass of a system is not physically inside of the system.
4. The Volvo XC60 is supposedly the safest car in the world that money can buy. Why is this?
5. In the boxing world, boxers use principles of momentum to "ride the punch". Research and explain how this method helps boxers to minimize injuries.

1.5.2 To try at home

Activity 1-1: Grab two or three of your friends and ask them to hold a bed sheet. Throw an egg at full speed onto the bed sheet. What happens to the egg, and why?

1.5.3 To try in the lab

1.6 Sample problems and solutions

1.6.1 Problems

Problem 1-1: Find the centre of mass of a uniform half-disk. (Solution)

1.6.2 Solutions

Solution to problem ??: First, we should draw a diagram with appropriate axis.

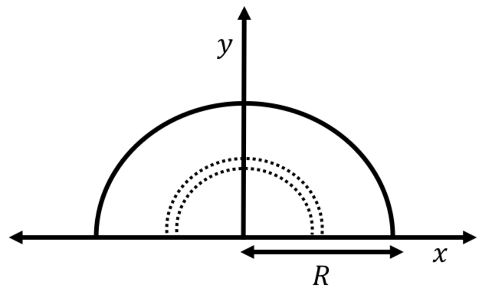

Figure 1.18: A uniform half-disk.

Let us say that the mass of a semicircular ring is m. Considering a slice of this half-disk with a width of dy and mass dm,

$$\sigma = \frac{mass}{area}$$
$$\sigma = \frac{m}{\frac{1}{2}\pi r^2}$$
$$\therefore \sigma = \frac{2m}{\pi r^2}$$

$$\begin{aligned} dm &= \sigma dA \\ &= \frac{2m}{\pi r^2}\pi y dy \\ \therefore &= \frac{2m}{r^2} y dy \end{aligned}$$

The y coordinate of the position of the centre of mass will be located at $x = 0$ and $y = \frac{2r}{\pi}$ by symmetry.

Using this information, we can use the centre of mass formula for the y coordinate to determine the centre of mass for the half-disk.

$$
\begin{aligned}
y_{CM} &= \frac{1}{m}\int y dm \\
&= \frac{1}{m}\int_0^r (\frac{2y}{\pi})\frac{2m}{r^2} y dy \\
&= \frac{4}{\pi r^2}\int_0^r y^2 dy \\
&= \frac{4}{\pi r^2}(\frac{R^3}{3}) \\
&= \frac{4r}{3\pi} \\
\therefore y_{CM} &= \frac{4r}{3\pi}
\end{aligned}
$$

Therefore, the centre of mass of a uniform half-disk is $x_{CM} = 0$, $y_{CM} = \frac{4r}{3\pi}$.

2 Rotational dynamics

In this Chapter, we use Newton's Second Law to develop a formalism to describe how objects rotate. In particular, we will introduce the concept of torque which plays a similar role to that of force in non-rotational dynamics. We will also introduce the concept of moment of inertia to describe how objects resist rotational motion.

Learning Objectives

- Understand how to use vector quantities for describing the kinematics of rotations.
- Understand how to use torque to determine the angular acceleration of an object.
- Understand conditions for static and dynamic equilibrium.
- Understand how to determine the moment of inertia of an object.

Think About It

A construction worker has to turn a lever at an angle of $55°$. The length of the lever is $0.61\,\mathrm{m}$ long and is a force of $25\,\mathrm{N}$ is exerted to turn it. What is the net torque on the lever?

A) $9.0\,\mathrm{Nm}$
B) $34\,\mathrm{Nm}$
C) $12\,\mathrm{Nm}$
D) $22\,\mathrm{Nm}$

2.1 Rotational kinematic vectors

TODO: Review box to rotational kinematics, and vector product. (Note: The vector section on the right hand rule is labelled subsec:Vectors:rotationalmotion)

2.1.1 Scalar rotational kinematic quantities

Recall that we can describe the motion of a particle along a circle of radius R by using its angular position, θ, its angular velocity, ω, and its angular acceleration, α. With a suitable choice of coordinate systen, the angular position can be defined as the angle made by the position vector of the particles, $\vec{r}$, and the x axis of a coordinate system whose origin is the centre of the circle and for which the motion is around the z axis, as shown in Figure 2.1.

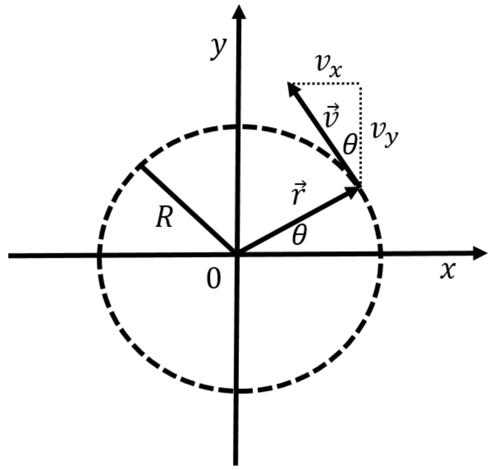

Figure 2.1: Angular position for a particle moving around the z axis (out of the page), along a circle of radius R centre at the origin.

The angular velocity, ω, is the rate of the change of the angular position, and the angular acceleration, α, is the rate of change of the angular velocity:

$$\omega = \frac{d}{dt}\theta$$
$$\alpha = \frac{d}{dt}\omega$$

If the angular acceleration is constant, then angular velocity and position as a function of time are given by:

$$\omega(t) = \omega_0 + \alpha t$$
$$\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

where θ_0 and ω_0 are the angular position and velocity, respectively, at $t = 0$.

We can also describe the motion of the particle in terms of "linear" quantities (as opposed to "angular" quantities) along a one-dimensional axis that is curved along the circle. If s is the distance along the circumference of the circle, measured counter-clockwise from where the circle intersects the x axis, then it is related to the angular displacement:

$$s = R\theta$$

if θ is expressed in radians. Similarly, the linear velocity along the s axis, v_s, and the corresponding acceleration, a_s, are given by:

$$v_s = \frac{ds}{dt} = \frac{d}{dt}R\theta = R\omega$$
$$a_s = \frac{dv}{dt} = \frac{d}{dt}R\omega = R\alpha$$

where the radius of the circle, R, is a constant that can be taken out of the time derivatives. For motion along a circle, the velocity vector, $\vec{v}$, of the particle is always tangent to the circle (Figure 2.1), so v_s corresponds to the speed of the particle. The acceleration vector, $\vec{a}$, is in general not tangent to the circle; a_s represents the component of the acceleration vector that is tangent to the circle. If $a_s = 0$, then $\alpha = 0$, and the particle is moving with a constant speed (uniform circular motion), and the acceleration vector points towards the centre of the circle.

Checkpoint 2-1

Which of the following statements correctly describes the speeds at points A and B on the following rotating disk?

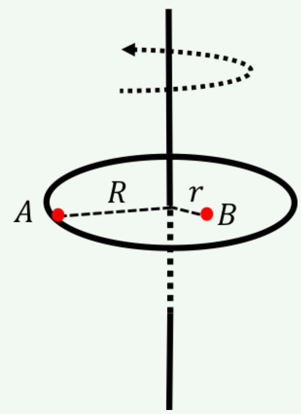

Figure 2.2: Two points at different radii on a rotating disk.

A) Both points A and B have the same angular and linear speeds.
B) Both points A and B have the same linear speed but they have different angular speeds.
C) Both points A and B have the same angular speed but they have different linear speeds.

2.1.2 Vector rotational kinematic quantities

In the previous section, we defined angular quantities to describe the motion of a particle about the z axis along a circle of radius R that lies in the xy plane. By using vectors, we can define the angular quantities for rotation about an **axis that can point in any direction**. Given an axis of rotation, the path of any particle rotating about that axis can be described by a circle that lies in the plane perpendicular to that axis of rotation, as illustrated in Figure 2.3.

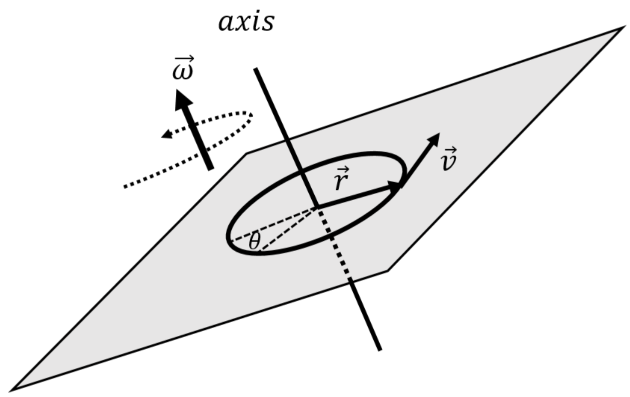

Figure 2.3: Defining the vector $\vec{r}$ and the angular velocity, $\vec{\omega}$ for a particle with velocity $\vec{v}$ rotating about an axis in a general direction.

We define the vector, $\vec{r}$, for a particle to be the vector that goes from the axis of rotation to the particle and is in a plane perpendicular to the axis of rotation, as in Figure 2.3. Given the velocity vector of the particle, $\vec{v}$, we define its angular velocity vector, $\vec{\omega}$, **about the axis of rotation**, as:

$$\boxed{\vec{\omega} = \frac{1}{r^2}\vec{r} \times \vec{v}} \tag{2.1}$$

The angular velocity vector is perpendicular to both the velocity vector and the vector $\vec{r}$, since it is defined as their cross-product. Thus, the **angular velocity vector is co-linear with the axis of rotation**. By using the angular velocity vector, we can specify **the direction of the axis of rotation as well as the direction in which the particle is rotating about that axis**. The direction of rotation is given by the right hand rule for axial vectors: when you point your thumb in the same direction as the angular velocity vector, the direction of rotation is the direction that your fingers point when you curl them, as illustrated in Figure. 2.4.

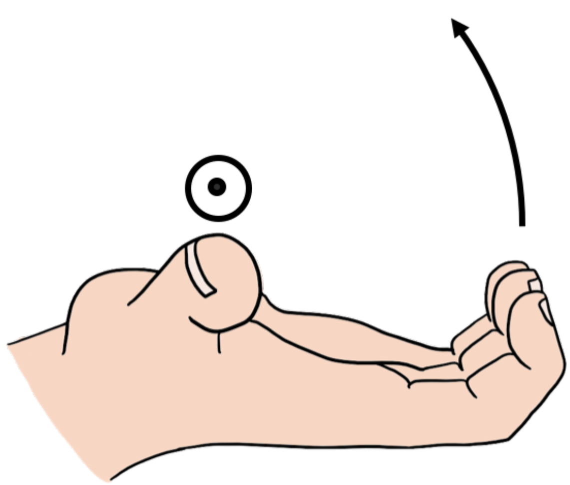

Figure 2.4: Using the right hand rule for axial quantities. In this case, the direction of rotation is counter clockwise when looking at the page (the direction that the fingers curl), so the rotation vector points out of the page (the direction of the thumb).

This definition of the angular velocity is consistent with the description from the previous section for motion about a circle of radius R that lies in the xy plane, as in Figure 2.1. In that case, the magnitude of the angular velocity is given by:

$$\begin{aligned}
\omega &= \frac{1}{r^2}||\vec{r}\times\vec{v}|| = \frac{1}{r^2}rv\sin\phi = \frac{v}{R} \\
\therefore v &= R\omega
\end{aligned}$$

where ϕ is the angle between the vectors $\vec{r}$ and $\vec{v}$ ($90°$ for motion around a circle). The direction of the angular velocity in Figure 2.1 is in the positive z direction, which corresponds to counter-clockwise rotation about the z axis.

Checkpoint 2-2

You push on the right-hand side of a door to open it, as the door's hinges are on the left. The angular velocity of the door is:

A) Upwards
B) Downwards
C) Forwards
D) Backwards

One can always define an angular velocity vector **relative to a point**, even if the particle is not moving along a circle. If we define the vector $\vec{r}$ to be the vector from the point of rotation to the particle, then the angular velocity vector describes the motion of the particle as if it were instantaneously moving in a circle centred at the point of rotation, in a plane given by the vectors $\vec{r}$ and $\vec{v}$.

Consider, for example, the particle in Figure 2.5 which is moving in a straight line with a velocity vector in the xy plane at a position $\vec{r}$ relative to the origin. We can define its angular velocity vector relative to the origin, which will be in the positive z direction.

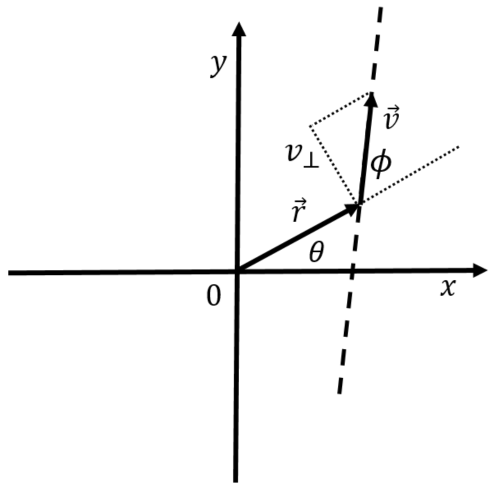

Figure 2.5: Angular position for a particle moving in a straight line.

The angular velocity describes the motion of the particle as if it were **instantaneously moving along a circle of radius r centred about the origin**. The angular velocity is related to the component of $\vec{v}$, $v_\perp$, that is perpendicular to $\vec{r}$ (which is the component tangent to the circle of radius r, in Figure 2.5):

$$||\vec{\omega}|| = \frac{1}{r^2}||\vec{r} \times \vec{v}|| = \frac{v \sin\phi}{r} = \frac{v_\perp}{r} \tag{2.2}$$

where ϕ is the angle between $\vec{r}$ and $\vec{v}$.

Similarly, we can define the angular acceleration vector, $\vec{\alpha}$, about an axis of rotation:

$$\vec{\alpha} = \frac{1}{r^2}\vec{r} \times \vec{a} \tag{2.3}$$

where $\vec{a}$ is the particle's acceleration vector, and $\vec{r}$ is the vector from the axis of rotation to the particle. The direction of the angular acceleration is co-linear with the axis of rotation and the right-hand rule gives the rotational direction of the angular acceleration. We can also define the angular acceleration about a point; in that case, the direction of the vector will define an instantaneous axis of rotation about a circle of radius r centred at the point as well as the direction of the angular acceleration about that axis.

Finally, we can define an angular displacement vector, $\vec{\theta}$, relative to an axis of rotation. The direction of the angular displacement vector will be co-linear with the axis of rotation, its direction will indicate the direction of rotation about that axis, and its magnitude (in radians) will correspond to the angular displacement (as shown in Figure 2.3). We can only relate the angular displacement vector to an infinitesimal linear displacement vector, $d\vec{s}$, since the position vector $\vec{r}$ from the axis of rotation will be different at each end of the displacement vector, if the displacement is large:

$$d\vec{\theta} = \frac{1}{r^2}\vec{r} \times d\vec{s}$$

TODO: Checkpoint question: An ant on a disk that is rotating slower and slower as illustrated. MC: the angular velocity vector is into the page and the angular acceleration is out of the page, etc... (they will be in opposite directions)

The angular velocity vector is the rate of change of the angular displacement vector:

$$\begin{aligned}\vec{\omega} &= \frac{d\vec{\theta}}{dt} = \frac{d}{dt}\frac{1}{r^2}\vec{r}\times d\vec{s} = \frac{1}{r}\frac{d\vec{s}}{dt} = \frac{1}{r}\vec{v}_s \\ \therefore v_s &= r\omega\end{aligned}$$

where $\vec{v}_s$ is the (instantaneous) tangential velocity around the circle. The angular acceleration vector is the rate of change of the angular velocity vector:

$$\vec{\alpha} = \frac{d}{dt}\vec{\omega}$$

Given the angular kinematic quantities, the related linear quantities at a position $\vec{r}$ from the axis of rotation are given by:

$$\begin{aligned}d\vec{s} &= d\vec{\theta}\times\vec{r} \\ \vec{v}_s &= \vec{\omega}\times\vec{r} \\ \vec{a}_s &= \vec{\alpha}\times\vec{r}\end{aligned} \tag{2.4}$$

where the linear quantities are always in the direction perpendicular to $\vec{r}$ (tangent to the circle, for motion around a circle). In other words, one cannot, say, take the acceleration vector, obtain the angular acceleration vector, and then get back the original acceleration vector - one will only get back the component of the acceleration vector that is perpendicular to $\vec{r}$.

TODO: Checkpoint MC: A particle has an angular velocity in the negative z direction. In which way is the particle's velocity vector at a point in its trajectory when it is on the positive y axis? (positive x direction)

2.2 Rotational dynamics for a single particle

Suppose that a single force, $\vec{F}$, is acting on a particle of mass m. Newton's Second Law for the particle is then given by:

$$\vec{F} = m\vec{a}$$

We can define a point of rotation such that $\vec{r}$ is the position of the particle relative to that point. We can take the cross-product of $\vec{r}$ with both sides of the equation in Newton's Second Law:

$$\vec{r}\times\vec{F} = m\vec{r}\times\vec{a}$$

The left hand-side of the equation is called "the torque of $\vec{F}$ relative to the point of rotation", and is usually denoted by $\vec{\tau}$:

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}} \tag{2.5}$$

The right-hand side of the equation is related to the angular acceleration vector, $\vec{\alpha}$, about that point of rotation:

$$m\vec{r} \times \vec{a} = mr^2\vec{\alpha}$$

Putting this altogether, we get:

$$\vec{\tau} = mr^2\vec{\alpha}$$

If more than one force is exerted on the particle, it is easy to show that the **net torque** from the net force on the particle **is equal to the sum of the torques on the particle**:

$$\begin{aligned}\vec{r} \times (\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots) = (\vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \vec{r} \times \vec{F}_3 + \dots) \\ \therefore \vec{r} \times \sum \vec{F} = \sum \vec{\tau} = \vec{\tau}^{net}\end{aligned}$$

We can write "Newton's Second Law for the rotational dynamics of a particle":

$$\boxed{\sum \vec{\tau} = \vec{\tau}^{net} = mr^2\vec{\alpha}} \tag{2.6}$$

This equation provides us an alternate formulation to Newton's Second Law that is useful for describing the motion of a particle that is rotating. The left-hand side of the equation corresponds to the "causes of motion" (much like the sum of the forces in Newton's Second Law), and the right-hand side of the equation to the inertia and the kinematics. A few things to note when comparing to Newton's Second Law:

1. The rotational quantities, torque and angular acceleration, **are only defined with respect to a point or axis of rotation** (as this determines the vector $\vec{r}$). If one chooses a different point of rotation, then the torque and angular acceleration will be different.
2. The angular acceleration of a particle is proportional to the net torque exerted on it, much like the linear acceleration is proportional to the net force exerted on the particle.
3. Torque about a centre of rotation can be thought of as the equivalent of a force that causes things rotate about an axis that goes through the point of rotation and that is parallel to the torque/angular acceleration vectors.
4. Instead of mass, it is mass times r^2 that plays the role of inertia and determines how large of an angular acceleration a particle will experience for a given net torque.

Example 2-1

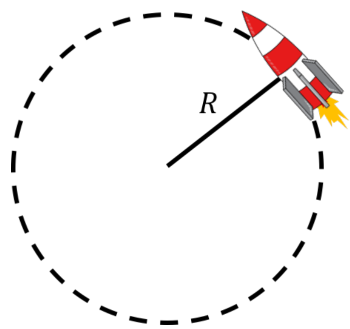

Figure 2.6: A toy rocket accelerating around a circle of radius R, as seen from above.

A toy rocket is attached to a string on a horizontal frictionless table, as shown in Figure 2.6. The rocket has a mass m and produces a constant force of thrust with a magnitude F that accelerates the rocket along a circle of radius R (the length of the string). If the rocket starts at rest, what distance along the circumference of the circle will the rocket have travelled after a time, t?

Solution

We can model the rocket as a point particle of mass m with the following forces exerted on it:

1. $\vec{F}$, the thrust of the rocket, always acting tangent to the circle.
2. $\vec{T}$, the force of tension in the string, always acting towards the centre of the circle.
3. $\vec{F}_g$, the rocket's weight, acting into the page, with magnitude mg.
4. $\vec{N}$, a normal force exerted by the table, out of the page, with magnitude mg.

Because the normal force and the weight are equal in magnitude and opposite in direction, the net force will be the sum of the force of thrust and the force of tension, which are always perpendicular to each other. Thinking about this with Newton's Second Law, we could model the force of thrust as increasing the speed of the particle, while the force of tension keeps the rocket moving in a circle (it can do no work to increase the speed, since it is always perpendicular to the motion).

We can also think about this in terms of torques and angular acceleration about the centre of the circle. The thrust will result in a net torque about the centre of rotation, which will lead to the rocket having an angular acceleration. By determining the angular acceleration, we can then model the displacement at some time, t, using kinematics. The force of tension will create no torque about the centre of the circle because the force of tension is always co-linear with the position vector, $\vec{r}$ (the cross-product of co-linear vectors is always zero).

We introduce a coordinate system whose origin coincides with the centre of the circle, as shown in Figure 2.7, so that $\vec{r}$ corresponds to the position of the rocket relative to the origin. We have not shown the weight and normal forces on the diagram.

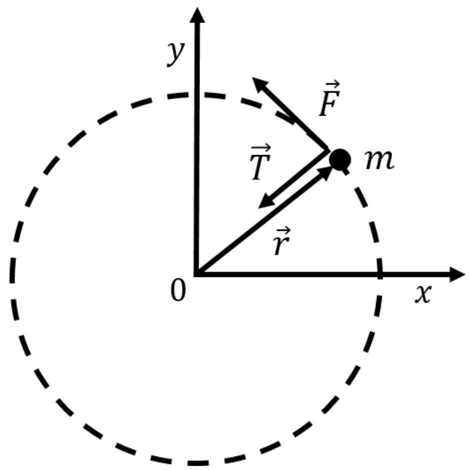

Figure 2.7: Coordinate system to describe the motion of the rocket.

The net torque on the rocket about the point of rotation is given by the cross-product between the thrust force, $\vec{F}$, and the position vector, $\vec{r}$:

$$\vec{\tau}^{net} = \vec{r} \times \vec{F}$$

and will point in the positive z direction (as given by the right hand rule). $\vec{r}$ and $\vec{F}$ are perpendicular, so the magnitude of the net torque is given by:

$$\tau^{net} = rF\sin(90^\circ) = RF$$

where R is the magnitude of $\vec{r}$. The net torque vector is thus:

$$\vec{\tau}^{net} = RF\hat{z}$$

Applying the rotational version of Newton's Second Law allows us to determine the angular acceleration:

$$\begin{aligned}\vec{\tau}^{net} &= mr^2\vec{\alpha}\\ RF\hat{z} &= mR^2\vec{\alpha}\\ \therefore \vec{\alpha} &= \frac{F}{mR}\hat{z}\end{aligned}$$

The angular acceleration vector points in the positive z direction (as does the net torque), and indicates that the rocket is accelerating in the counter-clockwise direction about the z axis.

After a period of time t, the rocket will have covered an angular displacement, $\Delta\theta$, given by:

$$\begin{aligned}\Delta\theta = \theta(t) - \theta_0 &= \omega_0 t + \frac{1}{2}\alpha t^2 \\ &= \frac{1}{2}\frac{F}{mR}t^2\end{aligned}$$

The linear displacement, Δs, that corresponds to this angular displacement is:

$$\Delta s = R\Delta\theta = \frac{1}{2}\frac{F}{m}t^2$$

Discussion: The formula that we found for the total linear displacement is the same that we would have found if the particle were moving in a straight line with a net force F applied to it (as the particle would have a constant acceleration given by F/m).

2.3 Torque

The torque associated with a force is a mathematical tool to describe how much a particular force will cause a particle (or solid) object to rotate about a given point or a given axis of rotation. A torque is **only defined relative to an axis or point of rotation**. It never makes sense to say "the torque is ...", and one should always say "the torque about this axis/point of rotation is ... ". Angular quantities (torque, angular velocity, angular displacement, etc) are only ever defined relative to a specific axis or point of rotation.

Mathematically, the torque vector from a force $\vec{F}$ exerted at a position $\vec{r}$ relative to the axis or point of rotation is defined as:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Note that the torque from a given force increases if that force is further from the axis of rotation (if $\vec{r}$ has a bigger magnitude).

Consider the solid disk of radius $\vec{r}$ depicted in Figure 2.8. The disk can rotate about an axis that passes through the centre of the disk and that is perpendicular to the plane of the disk. A force, $\vec{F}$ is exerted on the edge of the disk as shown.

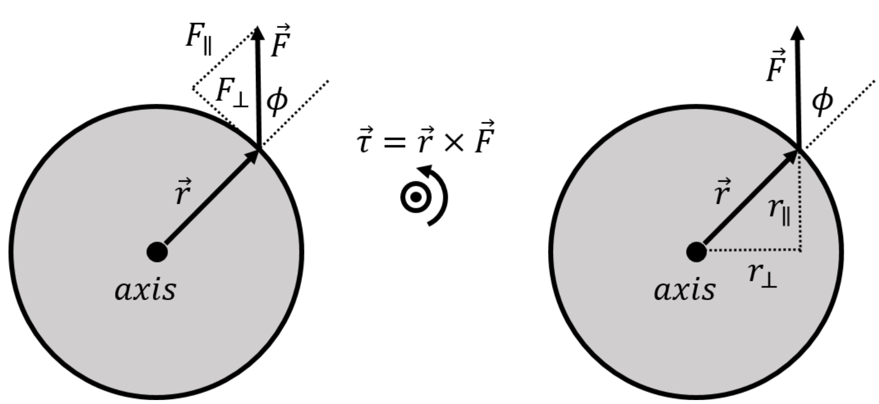

Figure 2.8: A force exerted on the perimeter of a disk that can rotate about an axis that is perpendicular to the disk and passes through its centre. We can determine the resulting torque by considering either the component of $\vec{F}$ that is perpendicular to $\vec{r}$ (left panel) or the component of $\vec{r}$ that is perpendicular to $\vec{F}$ (right panel). The torque vector, $\vec{\tau}$, is out of the page, as illustrated in the centre.

Intuitively, that force will cause the disk to rotate in the counter-clockwise direction. The torque from the force $\vec{F}$ about the axis as rotation is given by:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

where the vector $\vec{r}$ is perpendicular to the axis of rotation and goes from the axis of rotation to the point where $\vec{F}$ is exerted. The direction of the torque vector is out of the page (right hand rule, see Figure 2.8), and will thus lead to an angular acceleration that is also out of the page, which corresponds to the counter-clockwise direction, as anticipated.

We can break up the force into components that are parallel ($F_{\parallel}$) and perpendicular ($F_{\perp}$) to the vector $\vec{r}$, as shown on the left panel of Figure 2.8. Only the component of the force that is perpendicular to $\vec{r}$ will contribute to rotating the disk. Imagine that the force is from a string that you have attached to the perimeter of the disk; if you pull on the string such that the force is parallel to $\vec{r}$, the disk would not rotate. The magnitude of the torque is given by:

$$\tau = rF\sin\phi \tag{2.7}$$

where ϕ is the angle between $\vec{r}$ and $\vec{F}$, as shown in Figure 2.8. $F\sin\phi$ is precisely the component of $\vec{F}$ that is perpendicular to $\vec{r}$, so we could also write the magnitude of the torque as:

$$\tau = rF_{\perp}$$

which highlights that only the component of the force that is perpendicular to $\vec{r}$ contributes to the torque. Instead of combining the $\sin\phi$ with F to obtain $F_{\perp}$, the component of $\vec{F}$ perpendicular to $\vec{r}$, we can instead combine the $\sin\phi$ with r in Equation 2.7 to obtain $r_{\perp}$, the component of $\vec{r}$ that is perpendicular to $\vec{F}$. This is illustrated in the right panel of Figure 2.8. The magnitude of the torque is thus also given by:

$$\tau = r_{\perp}F$$

The quantity $r_{\perp}$ is called the "lever arm" of the force about a specific axis of rotation.

Checkpoint 2-3

Why is the handle of a door placed on the side of the door that is opposite to the hinges?

A) Because it increases the lever arm of a force used to rotate the door about the handle.
B) Because it increases the perpendicular component of force used to rotate the door about the hinges.
C) Because it increases the lever arm of a force used to rotate the door about the hinges.
D) Because it would be inconvenient if the handle were next to the hinges.

2.4 Rotation about an axis versus rotation about a point

When defining angular quantities (torque, angular acceleration, etc.), it is important to identify whether these are defined relative to an axis or to a point of rotation. This, in turn, determines the vector $\vec r$ that is involved in the definition of the angular quantities.

Consider a disk of radius r with a force, $\vec F$ exerted on its perimeter, as illustrated in Figure 2.9. The disk can only rotate about an axis that is perpendicular to the disk and that goes through the centre of the disk, like a wheel mounted on an axle. The force has a component, $\vec F_{plane}$, that lies in the plane perpendicular to the axis of rotation, and a component, $\vec F_{axis}$, that is parallel to axis of rotation.

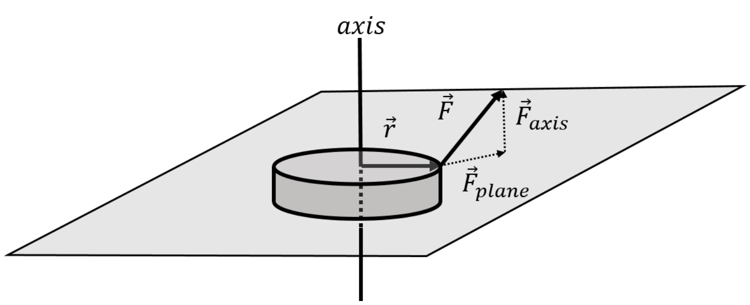

Figure 2.9: A force exerted on disk that can only rotate about an axis through its centre and perpendicular to its plane. Only the component of $\vec F$ that is in the plane perpendicular to the axis of rotation, $\vec F_{plane}$, will contribute to the torque about the axis of rotation.

The vector $\vec r$ is **always defined to be perpendicular to the axis of rotation and to go from the axis of rotation to the point where the force $\vec F$ is exerted**, as illustrated. The torque obtained by taking the cross product:

$$\vec\tau = \vec r \times \vec F$$

will be perpendicular to both $\vec r$ and $\vec F$, and will thus not be parallel to the axis of rotation. **Only the component of the torque that is parallel to the axis of rotation** will contribute to rotating the disk about the axis. Only the component of the force that lies in the plane perpendicular to the axis of rotation, $\vec{F}_{plane}$, will contribute to the component of the torque about that axis of rotation. Thus, when we need to determine the torque about an axis of rotation, we can **consider vectors $\vec{r}$ and $\vec{F}$ that lie in the plane perpendicular to the axis of rotation**. The torque of $\vec{F}$ relative to the axis of rotation is thus:

$$\vec{\tau}_{axis} = \vec{r} \times \vec{F}_{plane}$$

Furthermore, only the component of $\vec{F}_{plane}$ that is perpendicular to $\vec{r}$ will contribute to that torque, as we saw in the previous section.

In general, solid objects such as a disk can only rotate about an axis. In that case, one can consider only the components of forces that lie in the plane perpendicular to the axis of rotation in order to calculate the components of the torques about that axis that are parallel to that axis.

A point particle may be able to rotate about any axis that goes through a point of rotation. The net torque vector on the particle about that point will indicate the direction of the axis about which the particle would rotate. This is illustrated in the left panel of Figure 2.10.

Instead, if the particle were constrained to rotate about the z axis (e.g. if the particle is on a track), then we would use the component of the torque vector that is parallel to the z axis to describe its motion, as illustrated in the right panel. The z component of the torque could be determined by using only the components of the forces that lie in the plane perpendicular to the axis, and defining the vector $\vec{r}$ from the axis to the particle rather than from the point of rotation to the particle.

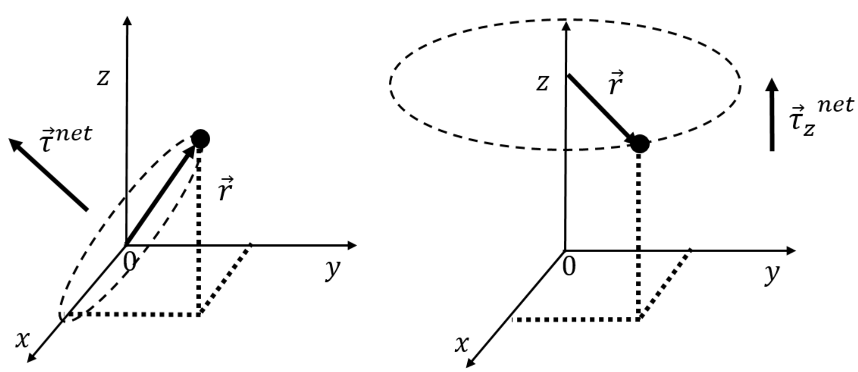

Figure 2.10: Left panel: a particle rotating about a circle centred at the origin with an axis determined from the net torque vector. Right panel: a particle that is constrained to rotate about the z axis.

Example 2-2

A force given by $\vec{F} = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}$ is exerted at a position $\vec{r} = r_x\hat{x} + r_y\hat{y} + r_z\hat{z}$. Calculate the torque about the z axis as well as the torque about the origin.

Solution

To calculate the torque about the z axis, we need to take the components of the vectors $\vec{r}$ and $\vec{F}$ that lie in the $x - y$ plane, since that is the plane perpendicular to the axis of rotation (the z axis). This gives:

$$\vec{\tau}_z = (r_x\hat{x} + r_y\hat{y}) \times (F_x\hat{x} + F_y\hat{y}) = (r_xF_y - r_yF_y)\hat{z}$$

If instead we want to calculate the torque about the origin, we take the cross-product between the two vectors:

$$\begin{aligned}\vec{\tau} &= (r_x\hat{x} + r_y\hat{y} + r_z\hat{z}) \times (F_x\hat{x} + F_y\hat{y} + F_z\hat{z})\\ &= (r_yF_z - r_zF_y)\hat{x} + (r_zF_x - r_xF_z)\hat{y} + (r_xF_y - r_yF_y)\hat{z}\end{aligned}$$

If a particle were located at the given position, the force would cause the particle to (instantaneously) rotate about an axis that goes through the origin and is parallel to the torque vector.

Discussion: This example highlights the difference between calculating the torque about an axis of rotation that goes through the origin and determining the torque about the origin. When calculating the torque about an axis that goes through the origin, we only consider the components of the vectors $\vec{r}$ and $\vec{F}$ that are in the plane perpendicular to the axis of rotation. This would correspond to a situation in which the particle is constrained to move in a plane that is perpendicular to the axis of rotation. Instead, if we calculate the torque about the origin, then the torque vector determines the axis of rotation through the origing about which the particle would rotate.

2.5 Rotational dynamics for a solid object

We now consider the rotational dynamics for a solid object about a specific axis of rotation. Just as we did in Chapter **??**, we model a solid object as a system made of many particles of mass m_i. Because all of the points in a solid must move in unison, they all **rotate about an axis of rotation instead of a point**. We describe the position of each particle i by a vector $\vec{r}_i$ that is **perpendicular to the axis of rotation and goes from the axis to the corresponding particle**, as shown in Figure 2.11.

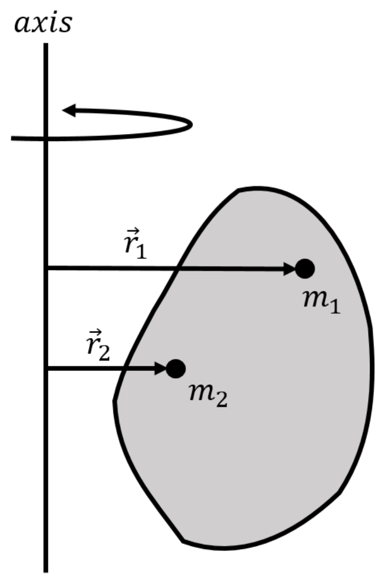

Figure 2.11: Two point particles that are part of a large solid object and their position vectors relative to an axis of rotation.

We wish to model the motion of the object as it rotates about a specific axis. Thus, when considering the net torque on any particle i, we only consider the component of the particle's net torque that is parallel to the axis of rotation (that component of torque that comes from forces that are in the plane perpendicular to the rotation axis).

We can write the rotational version of Newton's Second Law for particle, i, with mass m_i, and position vector $\vec{r}_i$ relative to the rotation axis:

$$\sum_k \vec{\tau}_{ik} = \vec{\tau}_i^{net} = m_i r_i^2 \vec{\alpha}_i$$

where $\vec{\tau}_{ik}$ is the k-th torque on particle i. $\vec{\tau}_i^{net}$ is the net torque on the particle **about the axis of rotation** and $\vec{\alpha}_i$ is the particle's angular acceleration about that axis.

We can divide the torques exerted on a particle into internal and external torques. Internal torques are those exerted by another particle in the system, whereas external torques are exerted by something external to the system. If particle 1 exerts a torque $\vec{tau}$ on particle 2, particle 2 will exert an equal and opposite torque, $-\vec{\tau}$ on particle 1.

Indeed, consider the two particles that exert an equal and opposite force (Newton's Third Law), $\vec{F}$, on each other, and an arbitrary point/axis of rotation, as illustrated in Figure 2.12. The torque on particle 1 from the force exerted by particle 2 will have the same magnitude as the torque on particle 2 from the force by particle 1. This is because both forces have the same magnitude and they are co-linear, which results in them having the same lever arm. The torque vector from each force will be in opposite directions, because the forces are in opposite direction. Newton's Third Law thus also holds for torques.

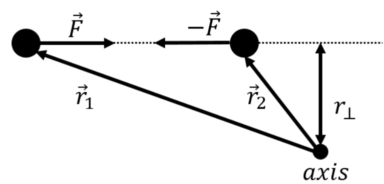

Figure 2.12: Two particles will exert equal and opposite torques on each other due to Newton's Third Law; the forces exerted by each particle on the other are co-linear and will thus have the same lever arm relative to any point/axis of rotation.

We can sum together the equations for each particle i:

$$\begin{aligned}\vec{\tau}_1^{net} + \vec{\tau}_2^{net} + \vec{\tau}_3^{net} + \ldots &= m_1 r_1^2 \vec{\alpha}_1 + m_2 r_2^2 \vec{\alpha}_2 + m_3 r_3^2 \vec{\alpha}_3 + \ldots \\ \sum_i \vec{\tau}_i^{net} &= \sum_i m_i r_i^2 \vec{\alpha}_i\end{aligned}$$

where the sum over all of the torques exerted on each particle will be equal to the net external torque exerted on all of the particles, since the sum of the internal torques, $\vec{\tau}_i^{int}$, will be zero:

$$\sum_i \vec{\tau}_i^{net} = \sum_i \vec{\tau}_i^{int} + \sum_i \vec{\tau}_i^{ext} = \sum_i \vec{\tau}_i^{ext} = \vec{\tau}^{ext}$$

where $\vec{\tau}^{ext}$ is the net external torque on the system.

All of the particles are part of the same rigid body, and cannot move relative to each other. Furthermore, they must all move around circles that are centred about the axis of rotation and in a plane perpendicular to that axis. They must thus all have the same angular acceleration[1], $\vec{\alpha}_i = \vec{\alpha}_1 = \vec{\alpha}_2 = \cdots = \vec{\alpha}$. We can thus factor the angular acceleration, $\vec{\alpha}$, out of the sum.

We can thus write Newton's Second Law for rotational dynamics of a solid object as:

$$\begin{aligned}\sum_i \vec{\tau}_i^{net} &= \sum_i m_i r_i^2 \vec{\alpha}_i \\ \therefore \vec{\tau}^{ext} &= \left(\sum_i m_i r_i^2\right) \vec{\alpha}\end{aligned}$$

The term in parentheses describes how the various masses are distributed relative to the axis of rotation. The term in parenthesis is called the **moment of inertia of the object**, and usually denoted with the letter, I:

$$\boxed{I = \sum_i m_i r_i^2} \tag{2.8}$$

[1] They will have different linear accelerations, but the angular acceleration (and velocity) will be the same for all particles if they are moving in unison.

The moment of inertia is a property of the object **relative to a specific axis of rotation**. Re-writing Newton's Second Law for the rotational dynamics of solid objects using the moment of inertia:

$$\boxed{\vec{\tau}^{ext} = I\vec{\alpha}} \tag{2.9}$$

The net torque exerted on an object in the direction of the axis of rotation is thus equal to its moment of inertia about that axis multiplied by its angular acceleration about that axis. In other words, the moment of inertia describes how the object will resist rotational motion given a net torque. An object with a smaller moment of inertia will have a larger angular acceleration for a given torque. Again, this is analogous to the linear case, where the acceleration of an object given a net force is determined by its inertial mass.

Example 2-3

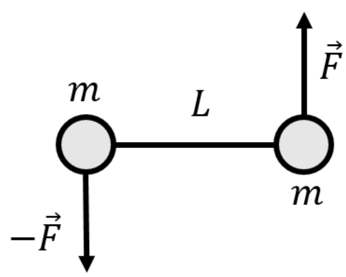

Figure 2.13: A dumbbell made of two small identical masses separated by a distance L.

Two small point masses, m, are connected by a mass-less rod of length L to form a dumbbell, as illustrated in Figure 2.13. A net force of magnitude F is exerted on each mass, in opposite directions, as illustrated in the Figure. What is the linear acceleration of the centre of mass of the dumbbell? What is the angular acceleration of the dumbbell relative to an axis that goes through its centre of mass and is perpendicular to the page? What is the angular acceleration of the dumbbell relative to an axis that goes through one of the masses and is perpendicular to the page?

Solution

We model the dumbbell as a rigid body made of two point masses held at a fixed distance. The linear acceleration of the centre of mass must be zero, because the net force on the dumbbell is zero. However, just because the centre of mass does not move does not mean that all parts of the dumbbell are immobile.

First, we calculate the angular acceleration relative to an axis that is perpendicular the page and goes through the centre of mass. The centre of mass is located midway between the two masses, as illustrated in Figure 2.14. We also define a coordinate system as shown, such that the z axis is out of the page.

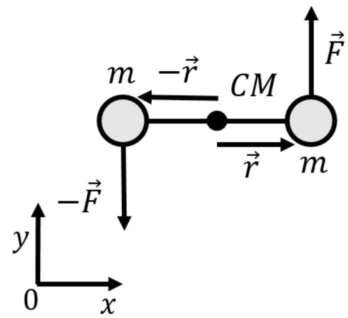

Figure 2.14: The dumbbell rotating about the centre of mass.

The vector from the axis of rotation to each mass will have the same magnitude, r, but different directions. The net external torque on the dumbbell relative to the axis that goes through the centre of mass, $\vec{\tau}^{ext}$, which is equal to the sum of the torques from each force:

$$\begin{aligned}\vec{\tau}^{ext} &= \vec{r} \times \vec{F} + (-\vec{r}) \times (-\vec{F}) \\ &= 2(\vec{r} \times \vec{F}) = 2(r\hat{x} \times F\hat{y}) = 2rF(\hat{x} \times \hat{y}) = 2rF\hat{z} \\ &= LF\hat{z}\end{aligned}$$

where we used the fact that $2r = L$. The net torque is thus non zero and in the positive z direction; the dumbbell will have an angular acceleration that is parallel to the net torque, and thus will accelerate in the counter-clockwise direction.

The moment of inertia of the dumbbell relative to the axis through the centre of mass is given by:

$$I = \sum_i m_i r_i^2 = mr^2 + mr^2 = 2mr^2 = \frac{1}{2}mL^2$$

Using Newton's Second Law for rotational dynamics, we find the angular acceleration to be:

$$\begin{aligned}\vec{\tau}^{ext} &= I\vec{\alpha} \\ LF\hat{z} &= \frac{1}{2}mL^2\vec{\alpha} \\ \therefore \vec{\alpha} &= \frac{2F}{mL}\hat{z}\end{aligned}$$

Because the centre of mass is fixed (the sum of the forces is zero), the two ends of the dumbbell will rotate about an axis that goes through the centre of mass. This is a feature of all situations in which the net force on an object is zero and the net torque about an axis that goes through the centre of mass is non-zero.

Let us now calculate the angular acceleration of the dumbbell about an axis that goes through one of the masses, as illustrated in Figure 2.15.

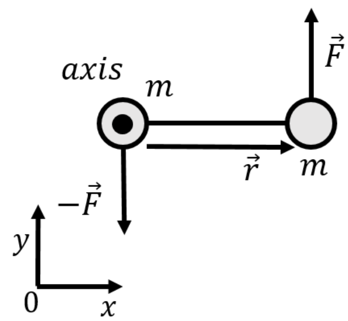

Figure 2.15: The dumbbell rotating about one of its ends.

We first calculate the net torque on the dumbbell. The vector that goes from the axis of rotation to the force exerted on the mass that coincides with the rotation axis is zero. Thus, only the force exerted on the mass that is not at the rotation axis contributes to the net torque:

$$\vec{\tau}^{ext} = \vec{r} \times \vec{F} = LF\hat{z}$$

The moment of inertia of the dumbbell about this axis is:

$$I = \sum_i m_i r_i^2 = m(0)^2 + m(r^2) = mL^2$$

which is larger than it was about the centre of mass. Again, the angular acceleration is found using Newton's Second Law for rotational dynamics:

$$\begin{aligned}
\vec{\tau}^{ext} &= I\vec{\alpha} \\
LF\hat{z} &= mL^2\vec{\alpha} \\
\therefore \vec{\alpha} &= \frac{F}{mL}\hat{z}
\end{aligned}$$

We find that the angular acceleration is smaller about an axis that goes through one of the mass than it is about an axis through the centre of mass. Because the centre of mass of the dumbbell is fixed, we can only think of the dumbbell as instantaneously rotating about one of its ends; that is, the motion of the dumbbell will not be such that one mass rotates about the other; this is only true instantaneously.

Discussion: This simple example illustrates several key features about rotational dynamics:

- If the sum of the forces on an object is zero, it does not mean that the entire object is stationary; it only implies that the centre of mass is stationary (or rather, moving with a constant velocity, but we can always choose to model the system in a frame of reference where the centre of mass is stationary).

- If the sum of the forces on an object is zero, and the sum of the external torques is non-zero, the object will rotate about an axis that goes through the centre of mass. That is, all points on the object will move along circles that are centred on an axis that goes through the centre of mass.
- We can model the rotating object about any axis that we choose. In general, the net external torque and the moment of inertia will depend on the choice of axis, as will the resulting angular acceleration.
- When determining the motion of the centre of mass, we can draw a free-body diagram, and the location of where the forces are exerted do not matter.
- When determining how the object rotates, we cannot use a free-body diagram, because it matters where the forces are applied (as the torque from a given force depends on the location where the force is applied relative to the axis of rotation).

2.6 Equilibrium

In this section, we consider the conditions under which an object is in static or dynamic equilibrium. An object is in equilibrium if it does not rotate when viewed in a frame of reference where the object's centre of mass is stationary (or moving at constant velocity).

2.6.1 Static equilibrium

An object is in static equilibrium, if **both the sum of the external forces exerted on the object and the sum of the external torques (about any axis) are zero**. If the object is in static equilibrium the centre of mass will have no acceleration and the object will have no angular acceleration. In the centre of mass frame of reference, the object is immobile.

Example 2-4

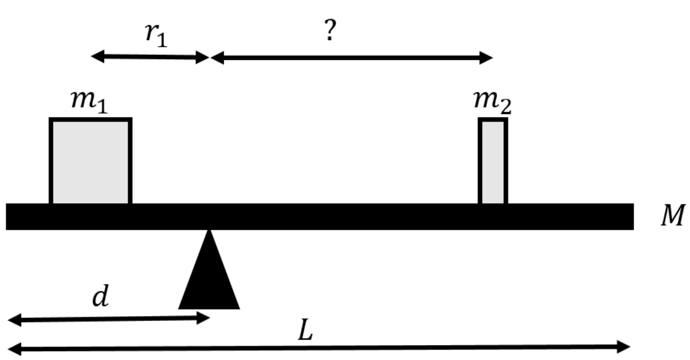

Figure 2.16: Two masses on a balance.

Two masses, m_1 and m_2 are placed on a balance as shown in Figure 2.16. The balance is made of a plank of mass M and length L that is placed on a fulcrum that is a distance d from one of the edges of the plank. If mass m_1 is placed at a distance r_1 from the fulcrum, how far should mass m_2 be placed on the other side of the plank in order for the balance to be in equilibrium?

Solution

We can consider the plank as the object that is in static equilibrium. Thus, the sum of the forces and the sum of the torques on the plank must be zero. We first start by identifying the forces that are exerted on the plank; these are:

1. $\vec{F}_g$, the weight of the plank, exerted at the centre of mass of the plank.
2. $\vec{F}_1$, a force equal to the weight of mass m_1, exerted at the location of m_1.
3. $\vec{F}_2$, a force equal to the weight of mass m_2, exerted at the location of m_2.
4. $\vec{N}$, a normal force exerted by the fulcrum.

The forces are illustrated in Figure 2.17 along with our choice of coordinate system. The z axis is not illustrated, and comes out of the page.

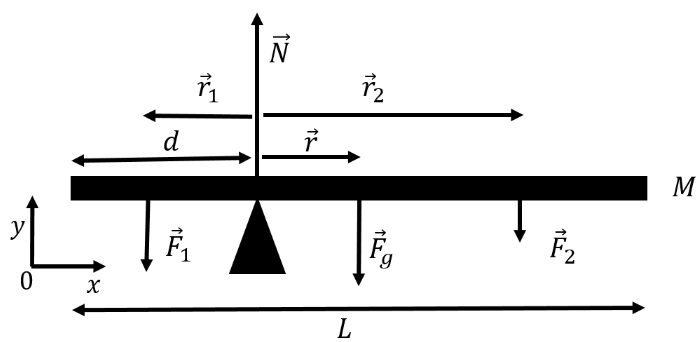

Figure 2.17: Forces exerted on the plank.

All of the forces are in the y direction, so we only write the y component of Newton's Second Law (with zero acceleration), which allows us to determine the magnitude of the normal force:

$$\sum F_y = N - Mg - m_1 g - m_2 g = 0$$
$$\therefore N = (M + m_1 + m_2)g$$

Because the plank is in static equilibrium, the sum of the torques must also be zero. We can choose the axis of rotation about which to calculate the torques. We choose an axis that is parallel to the z axis (out of the page) and goes through the fulcrum. In general, since we can choose the axis of rotation, it is usually convenient to choose an axis that goes through a point where at least one force is being exerted, because the torque from that force will be zero (its lever arm will be zero). Furthermore, since all of the forces are in the xy plane, the net torque on the plank will be in the z direction, so it makes sense to choose an axis in that direction.

The torques from the weight of the plank and from the force exerted by mass m_2 will be in the negative z direction, and the torque from the force exerted by mass m_1 will be in the positive z direction. The normal force will not result in any torque, because it is exerted at the axis of rotation and has a lever arm of zero.

We define $\vec{r}_1$ as the vector from the fulcrum to mass m_1. The torque, $\vec{\tau}_1$, from the force exerted by mass m_1 is given by:

$$\begin{aligned}\vec{\tau}_1 &= \vec{r}_1 \times \vec{F}_1 = (-r_1\hat{x}) \times (-F_1\hat{y})\\ &= r_1F_1(\hat{x}\times\hat{y}) = r_1F_1\hat{z} = r_1m_1g\hat{z}\end{aligned}$$

where we used the fact that the magnitude of $\vec{F}_1$ is m_1g. Similarly, the torques from the force exerted by m_2, $\vec{\tau}_2$, and by the weight, $\vec{\tau}_g$, are given by:

$$\begin{aligned}\vec{\tau}_2 &= \vec{r}_2 \times \vec{F}_2 = -m_2gr_2\hat{z}\\ \vec{\tau}_g &= \vec{r} \times \vec{F}_g = -rMg\hat{z} = -\left(\frac{L}{2} - d\right)Mg\hat{z}\end{aligned}$$

where $\frac{L}{2} - d$ is the distance between the fulcrum and where the weight of the plank is exerted. We require that the z component of the net torque be equal to zero (since all of the torques are in the z direction), which allows us to determine r_2:

$$\begin{aligned}\sum \tau_z &= \tau_{1z} + \tau_{2z} + \tau_{gz} = 0\\ m_1gr_1 - m_2gr_2 - \left(\frac{L}{2} - d\right)Mg &= 0\\ \therefore r_2 &= \frac{1}{m_2}\left(m_1r_1 - \left(\frac{L}{2} - d\right)M\right)\end{aligned}$$

Note that because we chose to calculate the torques about a point that goes through the fulcrum, in this case, we did not need to determine the value of the normal force which we obtained from Newton's Second Law.

Discussion: This example highlights the fact that when an object is in static equilibrium, we can choose a convenient axis about which to calculate the torques. In this case, by calculating the torques about the fulcrum, we did not need to consider the torque from the normal force. If we had chosen a different point, then the torque from the normal force would have been non-zero, and we would have used Newton's Second Law to express the normal force in terms of the other quantities. Physically, if we had placed the fulcrum at the centre of the plank $d = L/2$, then we would have found that $m_1r_1 = m_2r_2$, the well known equation for a balance. This equation, of course, comes from requiring that the torques from the forces exerted by m_1 and m_2 are equal in magnitude and opposite in direction.

Example 2-5

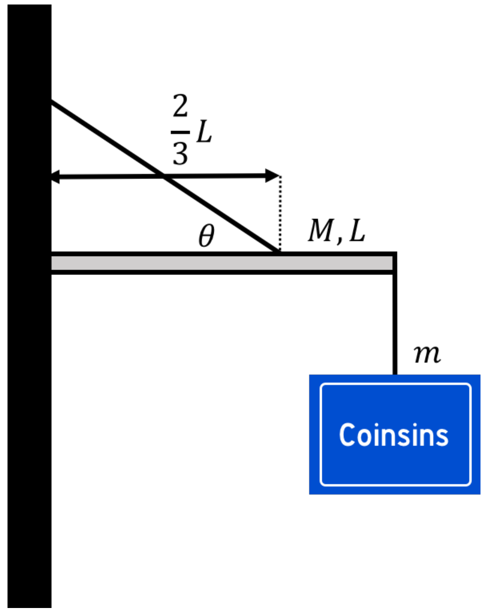

Figure 2.18: A sign is suspended on a horizontal bar of mass M and length L.

A sign holder is built by attaching a bar of mass M and length L to a wall using a hinge that allows the bar to rotate in the vertical plane. The sign of mass m is attached to the end of the bar that is opposite fo the wall. The bar is held up by a rope that is attached to the wall on one end and to the bar on the other end, two thirds of the length of the bar from the wall, as illustrated in Figure 2.18. The rope makes an angle θ with respect to the horizontal bar. Find the tension in the rope and the magnitude of the force exerted by the hinge onto the bar.

Solution

The whole system does not move and so it is in static equilibrium. In order to determine the forces exerted on the bar by the rope and the hinge, we model the bar as being in static equilibrium. The forces exerted on the bar are:

1. $\vec{F}_g$, the weight of the bar, with magnitude Mg, exerted at the bar's centre of mass.
2. $\vec{F}_m$, a downwards forced exerted by the sign at the end of the bar, with magnitude mg.
3. $\vec{T}$, a force of tension exerted by the rope at a distance $2/3L$ from the wall.
4. $\vec{R}$, a force exerted by the hinge on the bar at the end next to the wall[a]. We expect that the force from the hinge will have both a horizontal component, R_x, and a vertical component, R_y, in order for the net force on the bar to be zero.

The forces are illustrated in Figure 2.19 along with our choice of coordinate system (and the z axis, not shown, points out of the page).

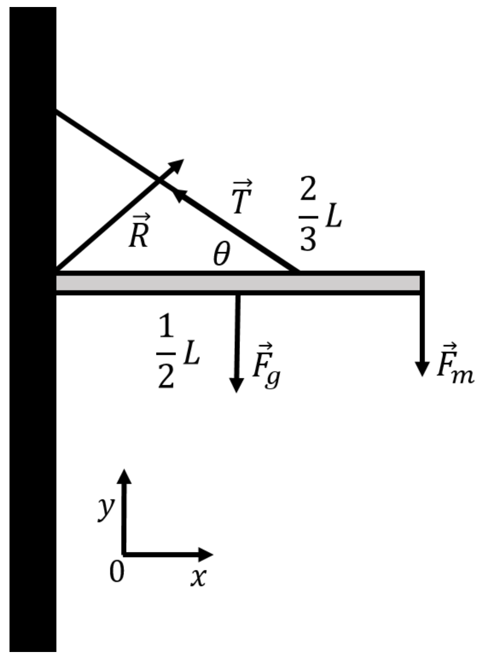

Figure 2.19: Forces on the bar that is holding the sign of mass m.

We start by writing out the x and y components of Newton's Second Law (with zero acceleration):

$$\sum F_x = R_x - T\cos\theta = 0$$
$$\sum F_y = R_y + T\sin\theta - Mg - mg = 0$$

We can choose the axis about which to calculate the torques. Since all of the forces are in the xy plane, we choose to calculate the torques about an axis parallel to the z axis that goes through the hinge on the wall. The force from the hinge, $\vec{R}$, will thus result in a torque of zero (since it has a lever arm of zero). The torque from each force about the hinge is given by:

$$\vec{\tau}_M = \vec{r}_M \times \vec{F}_g = \left(\frac{L}{2}\hat{x}\right) \times (-Mg\hat{y}) = -Mg\frac{L}{2}\hat{z}$$
$$\vec{\tau}_T = \vec{r}_T \times \vec{T} = \left(\frac{L}{3}\hat{x}\right) \times (-T\cos\theta\hat{x} + T\sin\theta\hat{y}) = T\sin\theta\frac{L}{3}\hat{z}$$
$$\vec{\tau}_m = \vec{r}_m \times \vec{F}_m = (L\hat{x}) \times (-mg\hat{y}) = -mgL\hat{z}$$

The sum of the torques in the z direction must be zero for static equilibrium, which allows us to determine the magnitude of the force of tension:

$$\begin{aligned}\sum \tau_z = \tau_{Mz} + \tau_{Tz} + \tau_{mz} = 0\\ -Mg\frac{L}{2} + T\sin\theta\frac{L}{3} - mgL = 0\\ -Mg\frac{1}{2} + T\sin\theta\frac{1}{3} - mg = 0\\ \therefore T = \frac{3g}{\sin\theta}\left(m + \frac{M}{2}\right)\end{aligned}$$

Using the x and y components of Newton's Second Law, we can now use the tension to determine the x and y components of the force exerted by the hinge:

$$\begin{aligned}R_x &= T\cos\theta = \frac{3g}{\tan\theta}\left(m + \frac{M}{2}\right)\\ R_y &= (M+m)g - T\sin\theta = (M+m)g - 3g\left(m + \frac{M}{2}\right) = -\left(2m + \frac{M}{2}\right)g\end{aligned}$$

We find that the y component from the hinge is in the negative y direction, so **our diagram in Figure 2.19 is wrong**! If you removed the hinge on the wall and instead held that end of the bar with your hand, you would feel that the end of the bar is trying to go into the wall and upwards, as the bar tries to rotate with the opposite end moving downwards due to the weight of the sign. You would have to push in the positive x and negative y direction to keep the bar from moving.

Discussion: In this example, we saw that we needed to use both the sum of the forces and the sum of the torques in order to determine the forces on the bar.

[a] We chose the letter R for "Reaction", as this is the force of reaction from the hinge as the bar pushes against the hinge.

2.6.2 Dynamic equilibrium

TODO: Review box on inertial forces

When an object is in dynamic equilibrium, its centre of mass is accelerating, but the object is not rotating when viewed from its centre of mass frame of reference. Thus, the sum of the external forces exerted on the object is not zero, while the net external torque exerted on the object is zero, in the frame of reference of the centre of mass.

Consider, for example, a speed skater going around a circular track or radius R, and leaning into the centre making an angle θ with the ice, as depicted in Figure 2.20. The skater's centre of mass is accelerating, because she is going around a circle, so the net force on the skater is not zero. However, in the reference frame of the skater, the skater is not rotating; she is thus in dynamic equilibrium.

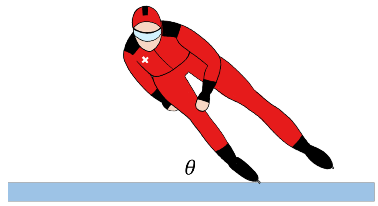

Figure 2.20: A speed skater leaning in as she goes around a circle.

The forces on the skater are:

1. $\vec{F}_g$, her weight, exerted at her centre of mass with magnitude Mg.
2. $\vec{N}$, a normal force, exerted by the ice upwards on her skates.
3. $\vec{f}_s$, a force of static friction, exerted towards the centre of the circle, by the ice on her skates.

The forces are illustrated in Figure 2.21 along with our choice of coordinate system.

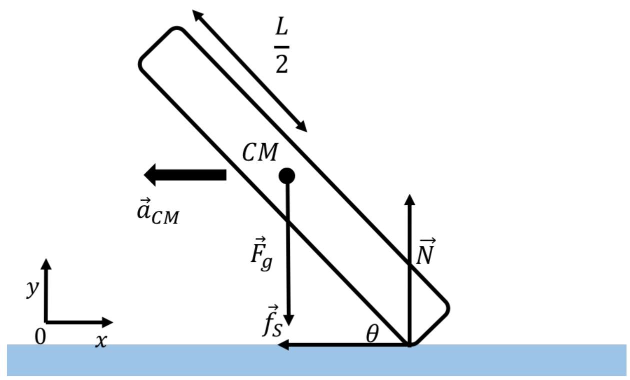

Figure 2.21: Forces on the speed skater from Figure 2.20.

The sum of the forces exerted on the skater must be towards the centre of the circle and equal to the mass of the skater times her centripetal acceleration (which is the acceleration of her centre of mass, $\vec{a}_{CM}$). The x and y components of Newton's Second Law are thus given by:

$$\sum F_x = -f_s = -ma_{CM} - m\frac{v^2}{R}$$
$$\sum F_y = N - mg = 0$$

All of the forces exerted on the skater are in the xy plane, so we consider torques about an axis that is co-linear with the z axis. Consider the torques about an axis through the point of contact between the skates and the ice; there is a net torque in the counter-clockwise direction due to the weight of the skater (the weight is the only force that can result in a torque about the point of contact with the ice). We expect that the skater would topple over, however, this must not be a correct model for the skater, since we know that it is possible for her to lean in without falling.

Consider, instead, the sum of the torques about an axis through her centre of mass. If the skater has a length L and the centre of mass is in the middle of the skater, the sum of the torques about the centre of mass is given by the torques from the normal forces and the force of friction:

$$\sum \tau = \tau_{Nz} + \tau_{f_s z} = \frac{L}{2}\cos\theta N - \frac{L}{2}\sin\theta f_s$$

About the centre of mass, the torques must be zero for the skater not to rotate, and this would give a relation between the force of static friction and the normal force.

Why do we get an incorrect model when we take the torques about the point of contact between the ice and the skater? In order to determine if the skater is rotating, we need to be in the same reference frame as the skater. However, the frame of reference of the skater is not an inertial frame of reference, since the skater is accelerating. We can still model the forces on the skater in the non-accelerating frame of reference, **as long as we include the inertial force** $-m\vec{a}_{CM}$ in that frame of reference. In the frame of reference of the skater, there is an additional inertial force $-m\vec{a}_{CM}$ in order for the sum of the forces to be zero (in the frame of reference of the skater, the sum of the forces must be zero since the skater is not accelerating in that frame of reference). The additional inertial force is exerted at the centre of mass, as illustrated in Figure 2.22.

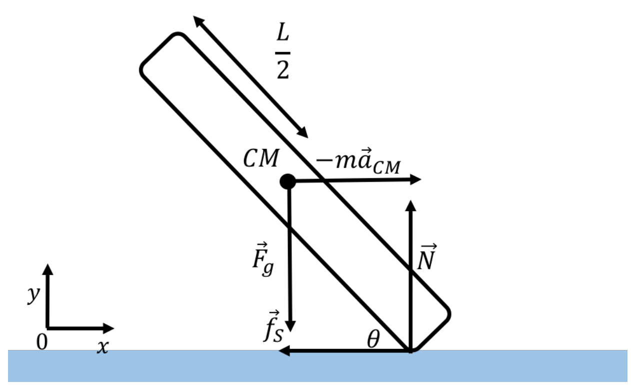

Figure 2.22: Forces on the speed skater from Figure 2.20 as seen in the accelerating frame of reference of the centre of mass.

The reason that our model worked when taking the torques about the centre of mass is that the inertial force, exerted at the centre of mass, does not result in a torque (since it has a lever arm of zero). Our model was technically wrong, but if we take the torques about the centre of mass, then we do not need to worry about the inertial force. If we include the additional inertial force, then we can take the torques about any point, just as in the static equilibrium case.

2.7 Moment of inertia

In order to model how an object rotates about an axis, we can use Newton's Second Law for rotational dynamics:

$$\vec{\tau}^{ext} = I\vec{\alpha}$$

where $\vec{\tau}^{ext}$ is the net external torque exerted on the object about the axis of rotation, $\vec{\alpha}$ is the angular acceleration of the object, and I is the moment of inertia of the object (about the axis). If we consider the object as being made of many particles of mass m_i each located at a position $\vec{r}_i$ relative to the axis of rotation, the moment of inertia is defined as:

$$I = \sum_i m_i r_i^2$$

Consider, for example, the moment of inertia of a uniform rod of mass M and length L that is rotated about an axis perpendicular to the rod that pass through one of the ends of the rod, as depicted in Figure 2.23.

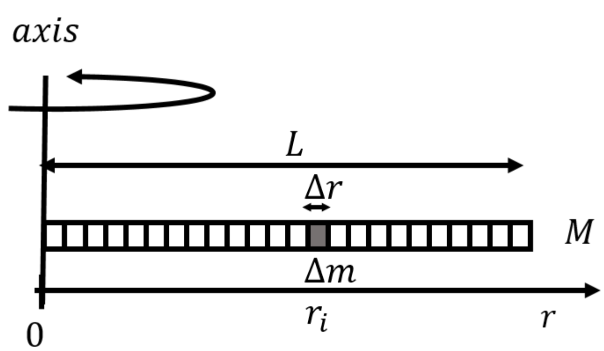

Figure 2.23: A rod of length L and mass M being rotated about an axis perpendicular to the rod that goes through one of its ends.

We introduce the linear mass density of the rod, λ, as the mass per unit length:

$$\lambda = \frac{M}{L}$$

We model the rod as being made of many small mass elements of mass Δm, of length Δr, at a location r_i, as illustrated in Figure 2.23. Using the linear mass density, the mass element has a mass of:

$$\Delta m = \lambda \Delta r$$

The rod is made of many such mass elements, and the moment of inertia of the rod is thus given by:

$$I = \sum_i \Delta m r_i^2 = \sum_i \lambda \Delta r r_i^2$$

If we take the limit in which the length of the mass element is infinitesimally small ($\Delta r \to dr$) the sum can be written as an integral over the dimension of the rod:

$$\begin{aligned} I &= \int_0^L \lambda r_i^2 dr = \frac{1}{3}\lambda L^3 = \frac{1}{3}\left(\frac{M}{L}\right) L^3 \\ &= \frac{1}{3} ML^2 \end{aligned}$$

where we re-expressed the linear mass density in terms of the mass and length of the rod. In general, we can write the moment of inertia of a continuous object as:

$$I = \int r^2 dm$$

where dm is a small mass element that makes up the object, r is the distance from that mass element to the axis of rotation, and the integral is over the dimension of the object. As we did above, we would usually set up this integral so that dm is expressed in terms of r so that we can take an integral over r.

Example 2-6

Calculate the moment of inertia of a uniform thin ring of mass M and radius R, rotated about an axis that goes through its centre and is perpendicular to the disk.

Solution

We take a small mass element dm of the ring, as shown in Figure 2.24.

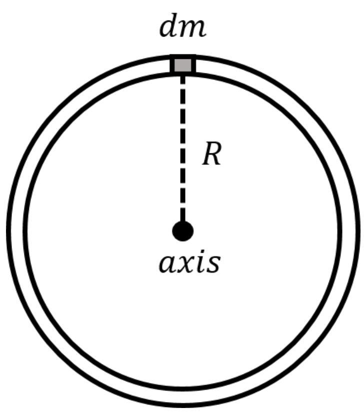

Figure 2.24: A small mass element on a ring.

The moment of inertia is given by:

$$I = \int dmr^2$$

In this case, each mass element around the ring will be the same distance away from the axis of rotation. The value r^2 in the integral is a constant over the whole ring, and so can be taken out of the integral:

$$I = \int dmr^2 = R^2 \int dm$$

where we used the fact that the ring has a radius R, so the distance r of each mass element to the axis of rotation is R. The integral:

$$\int dm$$

just means "sum all of the mass elements, dm", and is thus equal to M, the total mass of the ring. The moment of inertia of the ring is thus:

$$I = R^2 \int dm = MR^2$$

Example 2-7

Calculate the moment of inertia of a uniform disk of mass M and radius R, rotated about an axis that goes through its centre and is perpendicular to the disk.

Solution

We need to split up the disk into mass elements, dm, that we can sum together to obtain the moment of inertia of the disk. We can choose a ring of radius r and radial thickness dr for the shape of our mass element, as depicted in Figure 2.25.

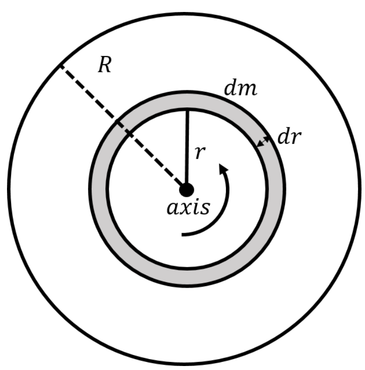

Figure 2.25: A mass element, dm, in the shape of a ring of radius r and radial thickness dr.

We can define a surface mass density, σ, equal to the mass per unit area of the disk:

$$\sigma = \frac{M}{\pi R^2}$$

The mass of the ring shaped element is thus given by:

$$dm = \sigma 2\pi r dr$$

where $2\pi r dr$ is the area of the mass element. You can imagine unfolding the mass element into a rectangle of height dr and of length $2\pi r$ to obtain its area. Now that we have expressed the mass element in terms of r, we can proceed to calculate the moment of inertia of the disk.

The moment of inertia is given by:

$$\begin{aligned}
I &= \int dmr^2 = \int_0^R \sigma 2\pi r dr r^2 = 2\pi\sigma \int_0^R r^3 dr \\
&= 2\pi\sigma \frac{1}{4} R^4 = 2\pi \left(\frac{M}{\pi R^2}\right) \frac{1}{4} R^4 \\
&= \frac{1}{2} M R^2
\end{aligned}$$

where we removed the surface mass density by expressing it in term of the total mass and radius of the disk. **Discussion:** The moment of inertia of a disk of mass M and radius R is half of that of a ring of radius R and mass M. It is thus easier to rotate the disk than the ring.

2.7.1 The parallel axis theorem

The moment of inertia of a solid object can be difficult to calculate, especially if the object is not symmetric. The parallel axis theorem allows us to determine the moment of inertia of an object about an axis, if we already know the moment of inertia of the object about an axis that is parallel and goes through the centre of mass of the object.

Consider an object for which we know the moment of inertia, I_{CM}, about an axis that goes through the object's centre of mass. We define a coordinate system such that the origin is located at the centre of mass, and the z axis is parallel to the axis about which we know the moment of inertia, as illustrated in Figure 2.26.

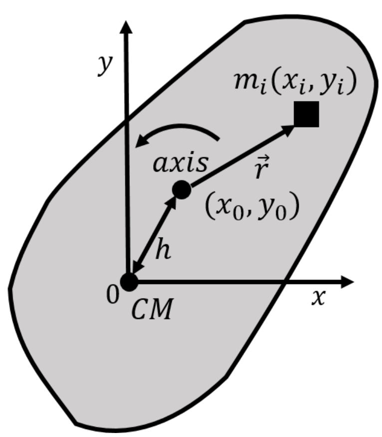

Figure 2.26: An object with a coordinate system whose origin is at the object's centre of mass. We wish to determine the object's moment of inertia through a second, parallel, axis located a distance h away from the centre of mass.

We wish to determine the moment of inertia for the object for an axis that is parallel to the z axis, but goes through a point with coordinates (x_0, y_0) located a distance h away from the centre of mass. The moment of inertia about an axis parallel to the z axis and that goes through that point, I_h is given by:

$$I_h = \sum_i m_i r_i^2$$

where m_i is a mass element of the object located at a distance r_i from the axis of rotation. If the mass element is located at a position (x_i, y_i) relative to the centre of mass, we can write the distance r_i in terms of the position of the mass element, and of the position of the axis of rotation:

$$r_i^2 = (x_i - x_0)^2 + (y_i - y_0)^2 = x_i^2 - 2x_i x_0 + x_0^2 + y_i^2 - 2y_i y_0 + y_0^2$$

Note that:

$$x_0^2 + y_0^2 = h^2$$

The moment of inertia, I_h, can thus be written as:

$$\begin{aligned}
I_h &= \sum_i m_i r_i^2 = \sum_i (m_i(x_i^2 + y_i^2) - 2x_0 m_i x_i - 2y_0 m_i y_i + m_i h^2) \\
&= \sum_i m_i(x_i^2 + y_i^2) + h^2 \sum_i m_i - 2x_0 \sum_i m_i x_i - 2y_0 \sum_i m_i y_i
\end{aligned}$$

where we broke the sum up into several sums, and factored constant terms (h, x_0, y_0) out of the sums, since these constants do not depend on which mass element we are considering. The first term is the moment of inertia about the centre of mass, since $x_i^2 + y_i^2$ is the distance to the centre of mass. The second term is h^2 times the total mass of the object, since the sum of all the m_i is just the mass, M, of the object. Now consider the term:

$$-2x_0 \sum_i m_i x_i$$

The sum, $\sum m_i x_i$ is the numerator in the definition of the x coordinate of the centre of mass! The sum is thus zero, because we choose the origin to be located at the centre of mass. The last two terms in the sum are thus identically zero, because they correspond to the x and y coordinates of the centre of mass!

We can thus write the parallel axis theorem:

$$\boxed{I_h = I_{CM} + Mh^2} \tag{2.10}$$

where I_{CM} is the moment of inertia of an object of mass M about an axis that goes through the centre of mass and, I_h, is the moment of inertia about a second axis that is parallel to the first and a distance h away.

Example 2-8

In the previous section, we calculated the moment of inertia of a rod of length L and mass M through an axis that is perpendicular to the rod and through one of its ends, and found that it was given by:

$$I = \frac{1}{3}ML^2$$

What is the moment of inertia of the rod about an axis that is perpendicular to the rod and goes through its centre of mass?

Solution

In this case, we know the moment of inertia through an axis that does not go through the centre of mass. The centre of mass is located a distance $h = L/2$ away from the point about which we know the moment of inertia, I_h.

Using the parallel axis theorem, we can find the moment of inertia through the centre of mass:

$$\begin{aligned} I_{CM} &= I_h - Mh^2 \\ &= \frac{1}{3}ML^2 - M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 \end{aligned}$$

Discussion: We find that the moment of inertia about the centre of mass is smaller than when the rod is rotated about its end. This makes sense because when rotating the rod about its end, more of its mass is further away from the axis of rotation, which results in a larger moment of inertia.

2.8 Summary

Key Takeaways

We can describe the kinematics of rotational motion using vectors to indicate both an axis of rotation and the direction of rotation about that axis. If a particle with velocity vector, $\vec{v}$, is rotating about an axis, then its angular velocity vector, $\vec{\omega}$, relative to that axis is defined as:

$$\vec{\omega} = \frac{1}{r^2}\vec{r} \times \vec{v}$$

where $\vec{r}$ is a vector from the axis of rotation to the particle. The particle rotates in a circle that lies in the plane defined by $\vec{r}$ and $\vec{v}$, perpendicular to the axis of rotation. The direction of the angular velocity vector is co-linear with the axis of rotation and the direction of rotation is given by the right-hand rule for rotational quantities.

One can define the angular velocity of a particle relative to a point of rotation, even if the particle is not moving in a circle. In that case, the angular velocity corresponds to the angular velocity of the particle as if it were instantaneously moving about a circle.

If a particle moving around a circle has a tangential acceleration, $\vec{a}_\perp$, then its angular acceleration vector is defined as:

$$\vec{\alpha} = \frac{1}{r^2}\vec{r} \times \vec{a}_\perp$$

The torque from a force, $\vec{F}$, exerted at a position $\vec{r}$, relative to an axis (or point) of rotation is defined as:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Torque is analogous to force in that it is used to model the causes of motion. Torques are only ever defined relative to an axis or point of rotation. The torque vector will be co-linear with the axis about which the object on which the force is exerted would rotate as a result of that force.

The magnitude of the torque can be written using either the component of the force, $F_\perp$ perpendicular to the vector $\vec{r}$, or the lever arm, $r_\perp$, of the force relative to the axis of rotation:

$$\begin{aligned}\tau &= rF\sin\phi\\ &= rF_\perp\\ &= r_\perp F\end{aligned}$$

where ϕ is the angle between the vectors $\vec{r}$ and $\vec{F}$ when these are placed "tail to tail".

Using rotational/angular quantities, we can modify Newton's Second Law to describe rotational dynamics about a given axis (or point) of rotation. For a point particle, this gives:

$$\vec{\tau}^{net} = mr^2\vec{\alpha}$$

where $\vec{\tau}^{net}$ is the net torque on the particle (the sum of the torques from each force exerted on the particle) about the axis, and $\vec{\alpha}$ is the resulting angular acceleration about that axis.

For an object (either continuous or made of point particles), the rotational version of Newton's Second Law for rotation about a specific axis is given by:

$$\vec{\tau}^{net} = I\vec{\alpha}$$

where I is the moment of inertia of the object about that axis.

Objects are in equilibrium if they are not rotating when viewed in their centre of mass frame of reference. Thus, for an object to be in equilibrium, the sum of the torques on the object, in the centre of mass reference frame, must be zero.

An object is in static equilibrium if the centre of mass is not accelerating, and thus the sum of the external forces on the object is zero. To model the torques on an object in static equilibrium, one can choose the axis about which to calculate the torques. A good choice is to choose an axis that is perpendicular to the plane in which the forces on the object are exerted (if such a plane exists), and to choose the axis to go through a point where at least one force is exerted (so that torques exerted at that point are identically zero).

An object is in dynamic equilibrium if the centre of mass is accelerating, but the object does not rotate when viewed in the frame of reference of its centre of mass. In dynamic equilibrium, if one models the torques exerted on the object about an axis that does not go through the centre of mass, then one must remember to include an inertial force exerted at the centre of mass.

The moment of inertia of an object is given by

$$I = \sum_i m_i r_i^2$$

if the object is modelled as a system of point particles of mass m_i each a distance r_i

from the axis of rotation. For a continuous object, the moment of inertia is given by:

$$I = \int r^2 dm$$

where dm is a small mass element a distance r from the axis of rotation and the integral is over the dimension of the object. Generally, one can set up the integral by expressing dm in terms of r using the density of the object, and then integrating r over the dimension of the object.

If the moment of inertia of an object of mass M about an axis that goes through the centre of mass is given by I_{CM}, then the moment of inertia, I_h, of the object through an axis that is parallel and a distance h from the centre of mass is given by the parallel axis theorem:

$$I_h = I_{CM} + Mh^2 \quad \text{Parallel axis theorem}$$

Important Equations

This is an important equation

$$E = mc^2$$

2.9 Thinking about the material

2.9.1 Reflect and research

1. Something to research more.

2.9.2 To try at home

Activity 2-1: Take a large textbook and consider the 3 axes that are parallel to the sides of the textbook and go through the centre of mass. By rotating the book along the three axes successively, determine the axis about which the moment of inertia of the textbook is the largest.

Activity 2-2: Confirm that the moment of inertia of a rod is smaller if the rod is rotated about its centre of mass than if it is rotated by one of its ends.

2.9.3 To try in the lab

Measure the moment of inertia of a disk, and compare with a prediction (e.g. build a yoyo...)

2.10 Sample problems and solutions

2.10.1 Problems

2.10.2 Solutions