The Art of Modelling: Introduction to Physics

Ryan D. Martin

June 7, 2018

Contents

1 Describing motion in one dimension

In this chapter, we will introduce the tools required to describe motion in one dimension. In later chapters, we will use the theories of physics to model the motion of objects, but first, we need to make sure that we have the tools to describe the motion. We generally use the word "kinematics" to label the tools for describing motion (e.g. speed, acceleration, position, etc), whereas we refer to "dynamics" when we use the laws of physics to predict motion (e.g. what motion will occur if a force is applied to an object).

Learning Objectives

- Describe motion in 1D using functions and defining an axis.
- Define position, velocity, speed, and acceleration.
- Use calculus to describe motion.
- Define the meaning of an inertial frame of reference.
- Use Galilean and Lorentz transformations to convert the description of an object's position from one inertial frame to another.

Think About It

You are taking your 6 year old cousin, Lily, to see the aquarium in Toronto. You are sitting on the train in Kingston waiting to leave the station. Lily exclaims that your train is moving, and is excited that you are on your way to Toronto. You wonder why she said this, so you look out the window and notice that train beside you is moving backwards. You conclude that your train is not moving after all. How do you explain this to Lily? Do you know for certain that you aren't moving?

The most simple type of motion to describe is that of a particle that is constrained to move along a straight line (one-dimensional motion); much like a train along a straight piece of track. When we say that we want to describe the motion of the particle (or train), what we mean is that we want to be able to say where it is at what time. Formally, we want to know the particle's **position as a function of time**, which we will label as $x(t)$. The function will only be meaningful if:

- we specify an x-axis and the direction that corresponds to increasing values of x
- we specify an origin where $x = 0$
- we specify the units for the quantity, x.

That is, unless all of these are specified, you would have a hard time describing the motion of an object to one of your friends over the phone.

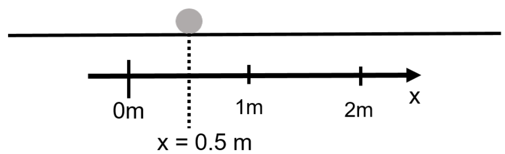


Figure 1.1: In order to describe the motion of the grey ball along a straight line, we introduce the x-axis, represented by an arrow to indicate the direction of increasing x, and the location of the origin, where $x = 0\,\mathrm{m}$. Given our choice of origin, the ball is currently at a position of $x = 0.5\,\mathrm{m}$.

Consider Figure 1.1 where we would like to describe the motion of the grey ball as it moves along a straight line. In order to quantify where the ball is, we introduce the "x-axis", illustrated by the black arrow. The direction of the arrow corresponds to the direction where x increases (i.e. becomes more positive). We have also chosen a point where $x = 0$, and by convention, we choose to express x in units of meters (the S.I. unit for the dimension of length).

Note that we are completely free to choose both the direction of the x-axis and the location of the origin. The x-axis is a mathematical construct that we introduce in order to describe the physical world; we could just as easily have chosen for it to point in the opposite direction with a different origin. Since we are completely free to choose where we define the x-axis, we should choose the option that is most convenient to us.

1.1 Motion with constant speed

Now suppose that the ball in Figure 1.1 is rolling, and that we recorded its x position every second in a table and obtained the values in Table 1.1 (we will ignore measurement uncertainties and pretend that the values are exact). The easiest way to visualize the values in the table is to plot them on a graph. Plotting position as a function of time is one of the most common graphs to make in physics, since it is often a complete description of the motion of an object. We can easily plot these values in Python:

Python Code 1.1: Plotting position versus time

```python
#First, we load pylab module for plotting
import pylab as pl
#We define t as a list of values (note the square brackets):
t = [0.0, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0]
#Similarly, we define the corresponding positions:
x = [0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0]
```

Time [s]	**X position [m]**
0.0 s	0.5 m
1.0 s	1.0 m
2.0 s	1.5 m
3.0 s	2.0 m
4.0 s	2.5 m
5.0 s	3.0 m
6.0 s	3.5 m
7.0 s	4.0 m
8.0 s	4.5 m
9.0 s	5.0 m

Table 1.1: Position of a ball along the x-axis recorded every second.

```
#Define the plot:
pl.plot(t,x,'.')# the '.' means that it will show the actual points instead
    of a line
#Set the range of the axes, add some labels and a grid
pl.ylim(0,6)
pl.xlim(0,10)
pl.xlabel('time [s]')
pl.ylabel('position [m]')
pl.grid()
#Show the plot
pl.show()
```

Output 1.1:

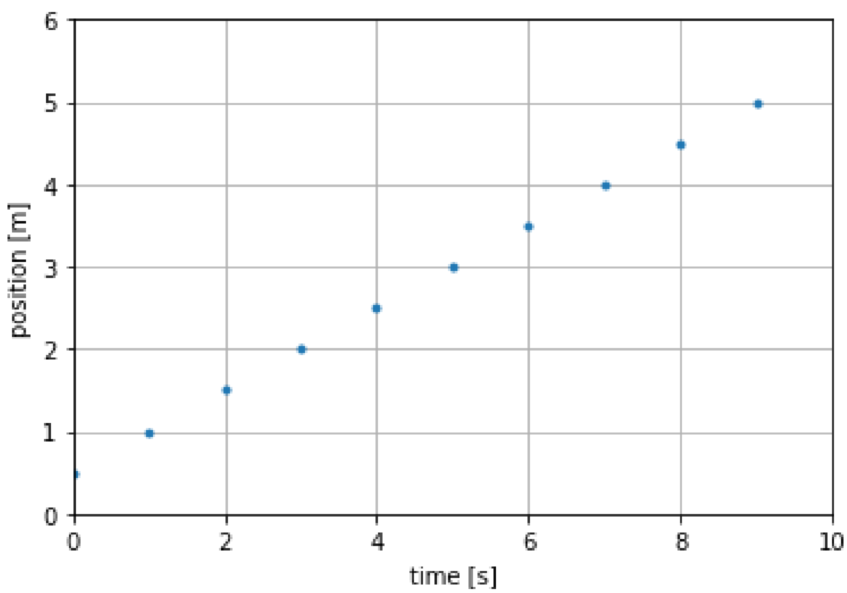


Figure 1.2: Plot of position as a function of time using the values from Table 1.1.

The data plotted in Figure 1.2 show that the x position of the ball increases linearly with time (i.e. it is a straight line). This means that in equal time increments, the ball will cover equal distances. Note that we also had the liberty to choose when we define $t = 0$; in this case, we chose that time is zero when the ball is at $x = 0.5\,\text{m}$.

Checkpoint 1-1

Using the data from Table 1.1, at what position along the x-axis will the ball be when time is $t = 9.5\,\text{s}$, if it continues its motion undisturbed?

Since the position as a function of time for the ball plotted in Figure 1.2 is linear, we can summarize our description of the motion using a function, $x(t)$, instead of having to tabulate the values as we did in Table 1.1. The function will have the functional form:

$$x(t) = a + bt$$

The constant a is the "offset" of the function, the value that the function has at $t = 0\,\text{s}$. The constant b is the slope and gives the rate of change of the position as a function of time. We can determine the values for the constants a and b by choosing any two rows from Table 1.1 (to determine 2 unknown quantities, you need 2 equations), and obtain 2 equations and 2 unknowns. For example, choosing the points where $t = 0\,\text{s}$ and $t = 2.0\,\text{s}$:

$$\begin{aligned}
x(t = 0\,\text{s}) = 0.5\,\text{m} = a + b(0\,\text{s}) \\
x(t = 2.0\,\text{s}) = 1.5\,\text{m} = a + b(2.0\,\text{s})
\end{aligned}$$

The first equation immediately gives $a = 0.5\,\text{m}$, which we can substitute into the second equation to get b:

$$\begin{aligned}
1.5\,\text{m} = a + b(2.0\,\text{s}) = 0.5\,\text{m} + b(2.0\,\text{s}) \\
\therefore b = \frac{(1.5\,\text{m}) - (0.5\,\text{m})}{(2.0\,\text{s})} = 0.5\,\text{m/s}
\end{aligned}$$

which gives us the functional form for $x(t)$:

$$x(t) = (0.5\,\text{m}) + (0.5\,\text{m/s})t$$

where you should note that a and b have different dimensions. Since a is added to something that must then give dimensions of length (for position, x), a has dimensions of length. b is multiplied by time, and that product must have dimensions of length as well; b thus has dimensions of length over time, or "speed" (with S.I. units of m/s).

We can generalize the description of an object whose position increases linearly with time as:

$$\boxed{x(t) = x_0 + v_x t} \tag{1.1}$$

where x_0 is the position of the object at time $t = 0\,\mathrm{s}$ (a from above), and v_x corresponds to the distance that the object covers per unit time (b from above) along the x-axis. We call v_x the "velocity" of the object. If v_x is large, then the object covers more distance in a given time, i.e. it moves faster. If v_x is a negative number, then the object moves in the negative x direction.

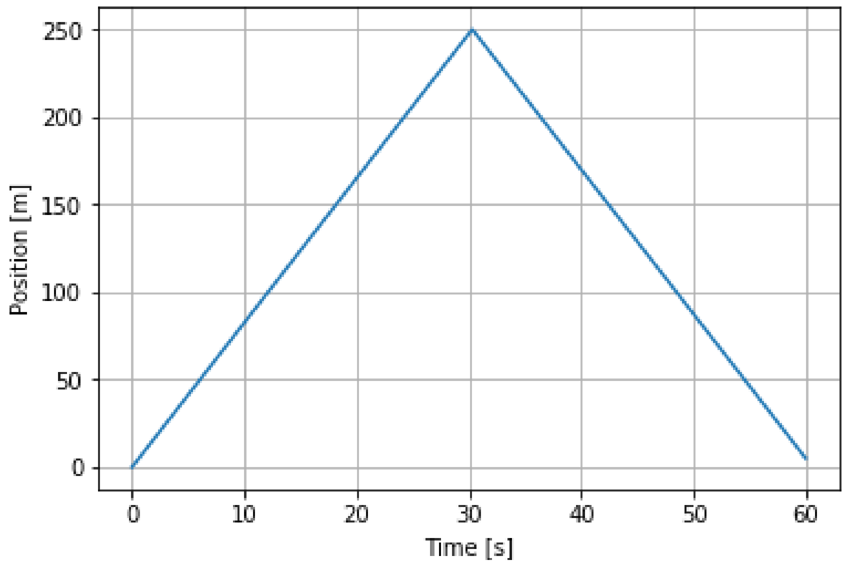


Figure 1.3: Position as a function of time for an object.

Checkpoint 1-2

Referring to Figure 1.3, what can you say about the motion of the object?

A) The object moved faster and faster between $t = 0\,\mathrm{s}$ and $t = 30\,\mathrm{s}$, then slowed down to a stop at $t = 60\,\mathrm{s}$.
B) The object moved in the positive x-direction between $t = 0\,\mathrm{s}$ and $t = 30\,\mathrm{s}$, and then turned around and moved in the negative x-direction between $t = 30\,\mathrm{s}$ and $t = 60\,\mathrm{s}$.
C) The object moved faster between $t = 0\,\mathrm{s}$ and $t = 30\,\mathrm{s}$ than it did between $t = 30\,\mathrm{s}$ and $t = 60\,\mathrm{s}$.

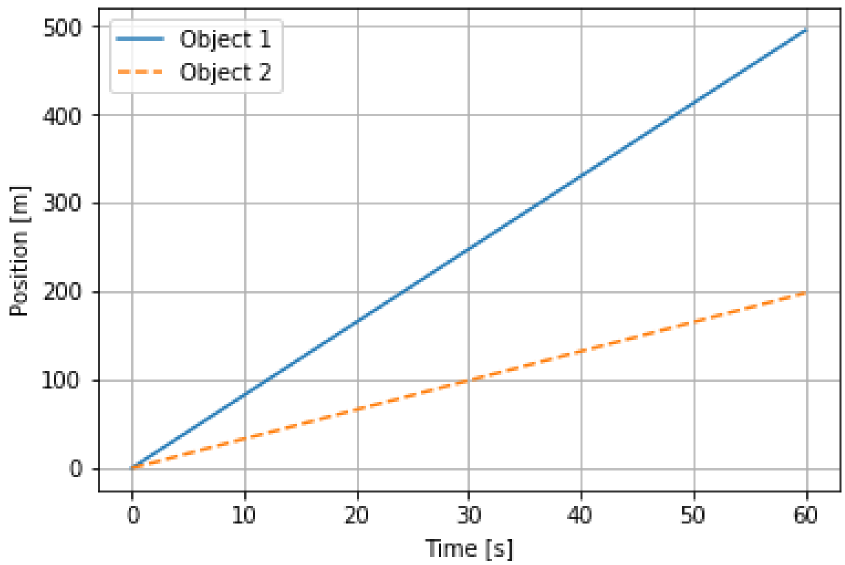


Figure 1.4: Positions as a function of time for two objects.

Checkpoint 1-3

Referring to Figure 1.4, what can you say about the motion of the two objects?

A) Object 1 is slower than Object 2
B) Object 1 is more than twice as fast as Object 2
C) Object 1 is less than twice as fast as Object 2

1.2 Motion with constant acceleration

Until now, we have considered motion where the velocity is a constant (i.e. where velocity does not change with time). Suppose that we wish to describe the position of a falling object that we released from rest at time $t = 0\,\text{s}$. The object will start with a velocity of $0\,\text{m/s}$ and it will **accelerate** as it falls. We say that an object is "accelerating" if its velocity is not constant. As we will see in later chapters, objects that fall near the surface of the Earth experience a constant acceleration (their velocity changes at a constant rate).

Formally, we define acceleration as the rate of change of velocity. Recall that velocity is the rate of change of position, so acceleration is to velocity what velocity is to position. In particular, we saw that if the velocity, v_x, is constant, then position as a function of time is given by:

$$x(t) = x_0 + v_x t \tag{1.1}$$

In analogy, if the acceleration is constant, then the velocity as a function of time is given by:

$$\boxed{v_x(t) = v_{0x} + a_x t} \tag{1.2}$$

where a_x is the "acceleration" and v_{0x} is the velocity of the object at $t = 0$. We can work out the dimensions of acceleration for this equation to make sense. Since we are adding v_{0x} and $a_x t$, we need the dimensions of $a_x t$ to be velocity:

$$\begin{aligned}
[a_x t] &= \frac{L}{T} \\
[a_x][t] &= \frac{L}{T} \\
[a_x]T &= \frac{L}{T} \\
[a_x] &= \frac{L}{T^2}
\end{aligned}$$

Acceleration thus has dimensions of length over time squared, with corresponding S.I. units of $\mathrm{m/s^2}$ (meters per second squared or meters per second per second).

Now that we have an understanding of acceleration, how do we describe the position of an object that is accelerating? We cannot use equation 1.1, since it is only correct if the velocity is constant.

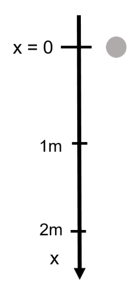


Figure 1.5: X-axis for an object that starts at rest at $x = 0\,\mathrm{m}$ when $t = 0\,\mathrm{s}$ and falls downwards (in the direction of increasing x).

Let us work out the corresponding equation for position as a function of time for accelerated motion using the x-axis depicted in Figure 1.5. We will determine $x(t)$ for the grey ball that starts at rest ($v_{0x} = 0\,\mathrm{m/s}$) at the position $x = 0\,\mathrm{m}$ at time $t = 0\,\mathrm{s}$ with a constant positive acceleration $a_x = 10\,\mathrm{m/s^2}$. We would like to use equation 1.1, but we cannot because it only applies if the velocity is constant. To remedy this, we pretend (we "approximate") that for a very small amount of time, the velocity is almost constant. Let us take a very small interval in time, say $\Delta t = 0.001\,\mathrm{s}$, and approximate that the velocity is constant during that interval.

At $t = 0\,\mathrm{s}$, we have $x = 0\,\mathrm{m}$, $v_{0x} = 0\,\mathrm{m/s}$ and $a_x = 10\,\mathrm{m/s^2}$. We can use equation 1.2 to find the velocity at $t = \Delta t$ (at the end of the first interval):

$$\begin{aligned}
v_x(t = \Delta t) &= v_{0x} + a_x \Delta t \\
&= (0\,\mathrm{m/s}) + a_x \Delta t \\
&= a_x \Delta t
\end{aligned}$$

The average velocity during the first interval, v_1^{avg} is then given by averaging the velocity at the beginning and at the end of the interval:

$$\begin{aligned}
v_1^{avg}(t = \Delta t) &= \frac{1}{2}\left[v(t=0) + v(t = \Delta t)\right] \\
&= \frac{1}{2}\left(v_{0x} + a_x \Delta t\right) \\
&= \frac{1}{2}\left((0\,\mathrm{m/s}) + a_x \Delta t\right) \\
&= \frac{1}{2}(10\,\mathrm{m/s^2})(0.001\,\mathrm{s}) \\
&= 0.005\,\mathrm{m/s}
\end{aligned}$$

Using the average velocity during the interval, we can use equation 1.1 to find the position at $t = \Delta t$:

$$\begin{aligned}
x(t = \Delta t) &= x_0 + v_1^{avg} \Delta t \\
&= (0\,\mathrm{m}) + \frac{1}{2} a_x (\Delta t)^2 \\
&= \frac{1}{2}(10\,\mathrm{m/s^2})(0.001\,\mathrm{s})^2 \\
&= 0.000\,005\,\mathrm{m}
\end{aligned}$$

Thus, at time $t = 0.001\,\mathrm{s}$, the object will have a velocity of $v = 0.005\,\mathrm{m/s}$ and be at a position $x = 0.000\,005\,\mathrm{m}$. We can now use these values as the starting velocity and position for the next interval in time. Using variables, at the beginning of the second interval, the velocity is $v(t = \Delta t) = a_x \Delta t$ and at the end of the second interval, it will be $v(t = 2\Delta t) = 2a_x \Delta t$. The average velocity during the second interval is thus given by:

$$\begin{aligned}
v_2^{avg}(t = 2\Delta t) &= \frac{1}{2}\left[v(t = \Delta t) + v(t = 2\Delta t)\right] \\
&= \frac{1}{2}(a_x \Delta t + 2a_x \Delta t) \\
&= \frac{3}{2} a_x \Delta t \\
&= \frac{3}{2}(10\,\mathrm{m/s^2})(0.001\,\mathrm{s}) \\
&= 0.015\,\mathrm{m/s}
\end{aligned}$$

To find the position at the end of the second time interval, when $t = 2\Delta t$, we use equation 1.1 again, but with a different starting position and the average velocity that we just found:

$$\begin{aligned}
x(t = 2\Delta t) &= x(t = \Delta t) + v_2^{avg}\Delta t \\
&= \frac{1}{2}a_x(\Delta t)^2 + \frac{3}{2}a(\Delta t)^2 \\
&= \frac{1}{2}a_x(2\Delta t)^2 \\
&= \frac{1}{2}(10\,\mathrm{m/s^2})(2 \times 0.001\,\mathrm{s})^2 = 0.000\,02\,\mathrm{m}
\end{aligned}$$

You can carry out this exercise to ultimately find the position at any time. However, if you carry it out over a few more intervals, you may notice the following pattern: For the Nth interval when $t = N\Delta t$ at the end of the interval, we have:

$$\begin{aligned}
v(t = (N-1)\Delta t) &= a_x(N-1)\Delta t && \text{(at beginning of interval N)} \\
v(t = N\Delta t) &= a_x N\Delta t && \text{(at end of interval N)} \\
v_N^{avg} &= \frac{1}{2}a_x(2N-1)\Delta t && \text{(average during interval)} \\
x(t = N\Delta t) &= \frac{1}{2}a_x(N\Delta t)^2 && \text{(position at end of interval)}
\end{aligned}$$

The last line gives us exactly what we were after, namely the position as a function of time for a constant acceleration, a_x, when the object started at rest at a position of $x = 0\,\mathrm{m}$:

$$x(t) = \frac{1}{2}a_x t^2 \tag{1.3}$$

If at $t = 0$, the object had an initial position along the x-axis of x_0, then the position $x(t)$ would be shifted by an amount x_0:

$$x(t) = x_0 + \frac{1}{2}a_x t^2 \tag{1.4}$$

Finally, if the object had an initial speed v_{0x} at $t = 0$, one can easily reproduce the iterations above to find that we need to add an additional term to account for this. We arrive at the general description of the position of an object moving in a straight line with acceleration, a_x:

$$\boxed{x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2} \tag{1.5}$$

Note that equation 1.1 is just a special case of the above when $a = 0$.

Example 1-1

A ball is thrown upwards with a velocity of $10\,\mathrm{m/s}$. After what distance will the ball stop before falling back down? Assume that gravity causes a constant downwards acceleration of $9.8\,\mathrm{m/s^2}$.

Solution

We will solve this problem in the following steps:

1. Setup a coordinate system (define the x-axis).
2. Identify the condition that corresponds to the ball stopping its upwards motion and falling back down.
3. Determine the distance at which the ball stopped.

Since we throw the ball upwards with an initial velocity upwards, it makes sense to choose an x-axis that points up and has the origin at the point where we release the ball. With this choice, referring to the variables in equation 1.5, we have:

$$\begin{aligned} x_0 &= 0 \\ v_{0x} &= +10\,\mathrm{m/s} \\ a_x &= -9.8\,\mathrm{m/s^2} \end{aligned}$$

where the initial velocity is in the positive x-direction, and the acceleration, a_x, is in the negative direction (the velocity will be getting smaller and smaller, so its rate of change is negative).

The condition for the ball to stop at the top of the trajectory is that its velocity will be zero (that is what it means to stop). We can use equation 1.2 to find what time that corresponds to:

$$\begin{aligned} v(t) &= v_{0x} + a_x t \\ 0 &= (10\,\mathrm{m/s}) + (-9.8\,\mathrm{m/s^2})t \\ \therefore t &= \frac{(10\,\mathrm{m/s})}{(9.8\,\mathrm{m/s^2})} = 1.02\,\mathrm{s} \end{aligned}$$

Now that we know that it took $1.02\,\mathrm{s}$ to reach the top of the trajectory, we can find how much distance was covered:

$$\begin{aligned} x(t) &= x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \\ x &= (0\,\mathrm{m}) + (10\,\mathrm{m/s})(1.02\,\mathrm{s}) + \frac{1}{2}(-9.8\,\mathrm{m/s^2})(1.02\,\mathrm{s})^2 = 5.10\,\mathrm{m} \end{aligned}$$

and we find that the ball will rise by $5.10\,\mathrm{m}$ before falling back down.

1.2.1 Visualizing motion with constant acceleration

When an object has a constant acceleration, its velocity and position as a function of time are described by the two following equations:

$$v(t) = v_{0x} + a_x t$$
$$x(t) = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$$

where the velocity changes linearly with time, and the position changes quadratically with time (it goes as t^2). Figure 1.6 shows the position and the speed as a function of time for the ball from example 1-1 for the first three seconds of the motion.

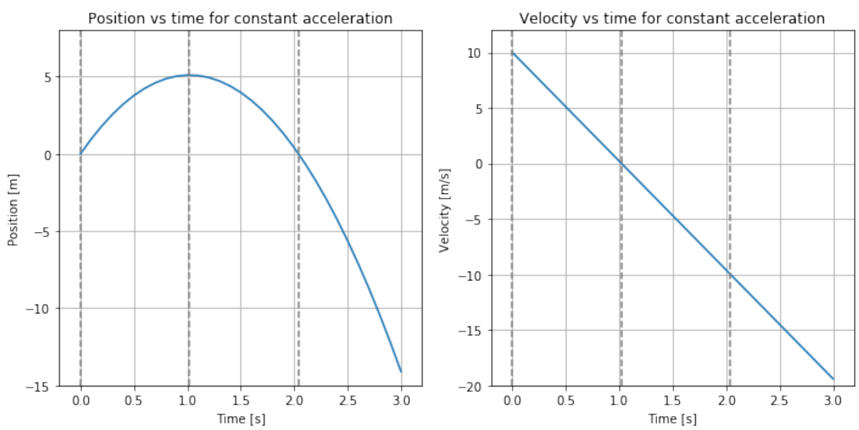


Figure 1.6: Position and speed as a function of time for the ball in example 1-1.

We can divide the motion into three parts (shown by the vertical dashed lines in Figure 1.6):

1) Between $t = 0\,\mathrm{s}$ and $t = 1.02\,\mathrm{s}$

At time $t = 0\,\mathrm{s}$, the ball starts at a position of $x = 0\,\mathrm{m}$ (left) and a speed of $v_{0x} = 10\,\mathrm{m/s}$ (right). During the first second of motion, the position, (t), increases (the ball is moving up), until the position stops increasing at $t = 1.02\,\mathrm{s}$, as found in example 1-1. During that time, the velocity decreases linearly from $10\,\mathrm{m/s}$ to $0\,\mathrm{m/s}$ due to the constant negative acceleration from gravity. At $t = 1.02\,\mathrm{s}$, the velocity is instantaneously $0\,\mathrm{m/s}$ and the ball is momentarily at rest (as it reaches the top of the trajectory before falling back down).

2) Between $t = 1.02\,\mathrm{s}$ and $t = 2.04\,\mathrm{s}$

At $t = 1.02\,\mathrm{s}$, the velocity continues to decrease linearly (it becomes more and more negative) as the ball start to fall back down faster and faster. The position also starts decreasing just after $t = 1.02\,\mathrm{s}$, as the ball returns back down to the point of release. At $t = 2.04\,\mathrm{s}$, the ball returns to the point from which it was thrown, and the ball is going with the same velocity ($10\,\mathrm{m/s}$) as when it was released, but in the opposite direction (downwards).

3) After $t = 2.04\,\text{s}$

If nothing is there to stop the ball, it continues to move downwards with ever increasing velocity. The position continues to become more negative and the velocity continues to become larger in magnitude and more negative.

Checkpoint 1-4

Make a sketch of the acceleration as a function of time corresponding to the position and velocity shown in Figure 1.6.

1.2.2 Speed versus velocity

In the previous example, our language was not quite as precise as it should be when conducting science. Specifically, we need a way to distinguish the situation when the velocity is decreasing (becoming more negative), while the object is actually going faster and faster (after $t = 1.02\,\text{s}$ in Figure 1.6). We will use the term **speed** to refer to how fast an object is moving (how much distance it covers per unit time), and we will use the term **velocity** to also indicate the direction of the motion. In other words, the speed is the absolute value of the velocity[1]. The speed is thus always positive, whereas the velocity can also be negative.

With this vocabulary, the speed of the ball in Figure 1.6 decreases between $t = 0\,\text{s}$ and $t = 1.02\,\text{s}$, and increases thereafter. On the other hand, the velocity continuously decreases (it is always becoming more and more negative). Velocity is thus the more general term since it tells us both the speed and the direction of the motion.

1.3 Using calculus to describe motion

Objects do not necessarily have a constant velocity or acceleration. We thus need to extend our description of the position and velocity of an object to a more general case. This can be done in much the same way as we introduced accelerated motion; namely by pretending that during a very small interval in time, Δt, the velocity and acceleration are constant, and then considering the motion as the sum over many small intervals in time. In the limit that Δt tends to zero, this will be an accurate description.

1.3.1 Instantaneous and average velocity

Suppose that an object is moving with a non constant velocity, and covers a distance Δx in an amount of time Δt. We can define an **average velocity**, v^{avg}:

$$v^{avg} = \frac{\Delta x}{\Delta t}$$

That is, regardless of our choice of time interval, Δt, we can always calculate the average velocity, v^{avg}, over the time interval. That average velocity will be an average over the interval, between some time t and $t + \Delta t$. If we shrink the time interval, and take the limit

[1]This is true for one-dimensional motion, whereas in two or more dimensions, velocity is a vector and speed is the magnitude of that vector.

$\Delta t \to 0$, we can define the **instantaneous velocity**:

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

The instantaneous velocity is the velocity only in that small instant in time where we choose Δx and Δt. Another way to read this equation is that the velocity, v, is the slope of the graph of $x(t)$. Recall that the slope is the "rise over run", in other words, the change in x divided by the corresponding change in t. Indeed, when we had no acceleration, the position as a function of time, equation 1.1, explicitly had the velocity as the slope of a linear function:

$$x(t) = v_{0x} + v_x t$$

If we go back to Figure 1.6, where velocity was no longer constant, we can indeed see that the graph of the velocity versus time, $v(t)$, corresponds to the instantaneous slope of the graph of position versus time, $x(t)$. For $t < 1.02\,\mathrm{s}$, the slope of the $x(t)$ graph is positive but decreasing (as is $v(t)$). At $t = 1.02\,\mathrm{s}$, the slope of $x(t)$ is instantaneously $0\,\mathrm{m/s}$ (as is the velocity). Finally, for $t > 1.02\,\mathrm{s}$, the slope of $x(t)$ is negative and increasing in magnitude, as is $v(t)$.

Leibniz and Newton were the first to develop mathematical tools to deal with calculations that involve quantities that tend to zero, as we have here for our time interval Δt. Nowadays, we call that field of mathematics "calculus", and we will make use of it here. Using the vocabulary of calculus, rather than saying that "instantaneous velocity is the slope of the graph of position versus time at some point in time", we say that "instantaneous velocity is the time derivative of position as a function of time". We also use a slightly different notation so that we do not have to write the limit $\lim_{\Delta t \to 0}$:

$$\boxed{v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \frac{d}{dt} x(t)} \tag{1.6}$$

where we can really think of dt as $\lim_{\Delta t \to 0} \Delta t$, and dx as the corresponding change in position over an *infinitesimally* small time interval dt.

Similarly, we introduce the **instantaneous acceleration**, as the time derivative of $v(t)$:

$$\boxed{a_x(t) = \frac{dv}{dt} = \frac{d}{dt} v(t)} \tag{1.7}$$

Olivia's Thoughts

When looking at a graph of position versus time, it is sometimes hard to tell at first glance whether the speed of the object is increasing or decreasing. This section gives us an easy way to figure it out. The velocity is the instantaneous slope of the graph $x(t)$, so the speed is the "steepness" of that slope. Simply draw a few lines that are tangent to (meaning just touching) the curve, and see what happens as time increases.

If the lines get steeper, the object is speeding up. If they are getting flatter, the object is slowing down.

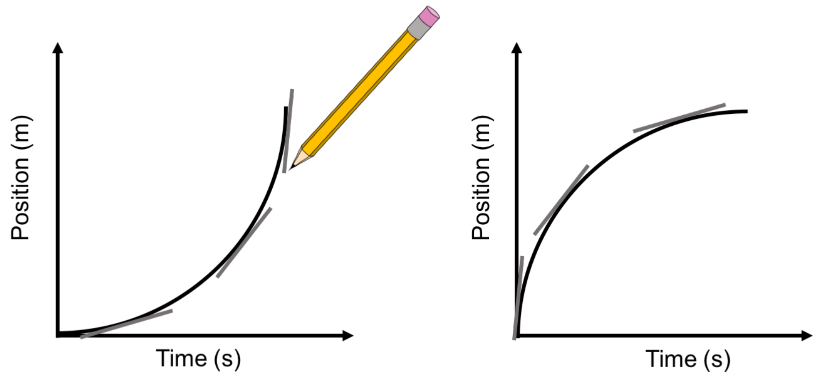


Figure 1.7: Two graphs of $x(t)$ showing tangent lines. Left: The object is speeding up (positive velocity, positive acceleration). Right: The object is slowing down (positive velocity, negative acceleration).

From here, you can also figure out what the direction of the acceleration is. If an object is speeding up, the acceleration and velocity must be in the same direction (i.e. both positive or both negative). If the object is slowing down, they must be in opposite directions. Imagine the graphs in Figure 1.7 are describing the motion of a person running in heavy wind. In the graph on the left, the person is running with the wind ($v(t)$ and $a(t)$ positive), and in the second graph the person is running against the wind ($v(t)$ positive and $a(t)$ negative).

1.3.2 Using calculus to obtain acceleration from position

Suppose that we know the function for position as a function of time, and that it is given by our previous result (for the case when the acceleration a_x is constant):

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$

The velocity is given by taking the derivative of $x(t)$ with respect to time:

$$\begin{aligned}
v(t) &= \frac{dx}{dt} = \frac{d}{dt}\left(x_0 + v_{0x}t + \frac{1}{2}a_x t^2\right) \\
&= v_{0x}t + a_x t
\end{aligned}$$

as we found before, in equation 1.2. The acceleration is then given by the time-derivative of the velocity:

$$\begin{aligned}
a_x &= \frac{dv}{dt} = \frac{d}{dt}\left(v_{0x}t + a_x t\right) \\
&= a_x
\end{aligned}$$

as expected.

Checkpoint 1-5

Chloë has been working on a detailed study of how vicuñas[a] run, and found that their position as a function of time when they start running is well modelled by the function $x(t) = (40\,\mathrm{m/s^2})t^2 + (20\,\mathrm{m/s^3})t^3$. What is the acceleration of the vicuñas?

A) $a_x(t) = 40\,\mathrm{m/s^2}$
B) $a_x(t) = 80\,\mathrm{m/s^2}$
C) $a_x(t) = 40\,\mathrm{m/s^2} + (20\,\mathrm{m/s^3})t$
D) $a_x(t) = 80\,\mathrm{m/s^2} + (120\,\mathrm{m/s^3})t$

[a] Never heard of vicuñas? Internet!

1.3.3 Using calculus to obtain position from acceleration

Now that we saw that we can use derivatives to determine acceleration from position, we will see how to do the reverse and use acceleration to determine position. Let us suppose that we have a constant acceleration, $a_x(t) = a_x$, and that we know that at time $t = 0\,\mathrm{s}$, the object had a speed of v_{0x} and was located at a position x_0.

Since we only know the acceleration as a function of time, we first need to find the velocity as a function of time. We start with:

$$a_x(t) = a_x = \frac{d}{dt}v(t)$$

which tells us that we know the slope (derivative) of the function $v(t)$, but not the actual function. In this case, we must do the opposite of taking the derivative, which in calculus is called taking the "anti-derivative" with respect to t and has the symbol $\int dt$. In other words, if:

$$\frac{d}{dt}v(t) = a_x(t)$$

then:

$$v(t) = \int a_x(t)dt$$

Since in this case, $a_x(t)$ is a constant, a_x, the anti-derivative is easily found:

$$\int a_x dt = a_x t + C$$

The velocity is thus given by:

$$v(t) = \int a_x dt = a_x t + C$$

The constant C is determined by what we call our "initial conditions". In this case, we stated that at time $t = 0$, the velocity should be v_{0x}. The constant C is thus v_{0x}:

$$v(t) = C + a_x t = v_{0x} + a_x t$$

and we recover the formula for velocity when the acceleration is constant. Now that we know the velocity as a function of time, we can take one more anti-derivative with respect to time to obtain the position:

$$\begin{aligned} v(t) &= \frac{dx}{dt} \\ \therefore x(t) &= \int v(t)dt \end{aligned}$$

In the case where acceleration is constant, this gives:

$$\begin{aligned} x(t) &= \int v(t)dt \\ &= \int (v_{0x} + a_x t)dt \\ &= v_{0x}t + \frac{1}{2}a_x t^2 + C' \end{aligned}$$

where C' is a different constant than the one we had when determining velocity. The constant is given by our initial conditions. If the object was located at position $x = x_0$ at time $t = 0$, then $C' = x_0$ and we recover the equation for position as a function of time for constant acceleration:

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$

Checkpoint 1-6

The acceleration of a cricket jumping sideways is observed to increase linearly with time, that is, $a_x(t) = a_0 + jt$, where a_0 and j are constants. What can you say about the velocity of the cricket as a function of time?

A) it is constant
B) it increases linearly with time ($v(t) \propto t$)
C) it increases quadratically with time ($v(t) \propto t^2$)
D) it increases with the cube of time ($v(t) \propto t^3$)

Checkpoint 1-7

Choose the graph of $x(t)$ for the case when acceleration is given by $\cos(\omega t)$, where ω is a constant. The velocity and position are zero at $t = 0$

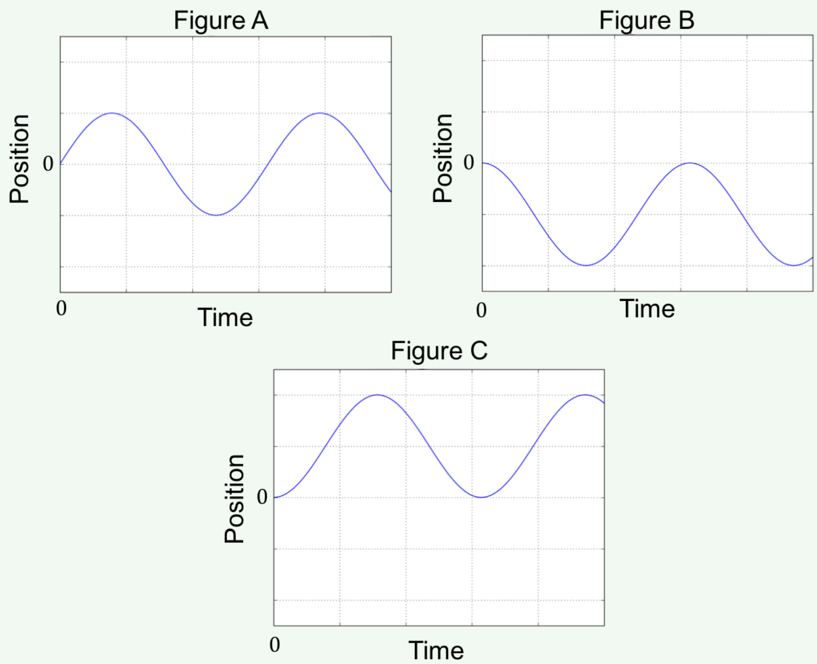


Figure 1.8: Choose the correct position versus time graph.

A) Figure A
B) Figure B
C) Figure C

1.4 Relative motion

In order to describe the motion of an object confined to a straight line, we introduced an axis (x) with a specified direction (in which x increases) and an origin (where $x = 0$). Sometimes, it can be more convenient to use an axis that is *moving*. For example, consider a person, Alice, moving inside of a train headed for the French town of Nice. The train is moving with a constant speed, v'^B as measured from the ground. Suppose that another person, Brice, describes Alice's position using the function $x^A(t)$ using an x-axis defined inside of the train car ($x = 0$ where Brice is sitting, and positive x is in the direction of the train's motion), as depicted in Figure 1.9 below. As long as any person is in the train with Brice, they will easily be able to describe Alice's motion using the x-axis that is moving with the train. Suppose that the train goes through the French town of Hossegor, where a surfer, Igor, watches the train go by. If Igor wishes to describe Alice's motion, it is easier for him to use a different axis, say x', that is fixed to the ground and not moving with the train.

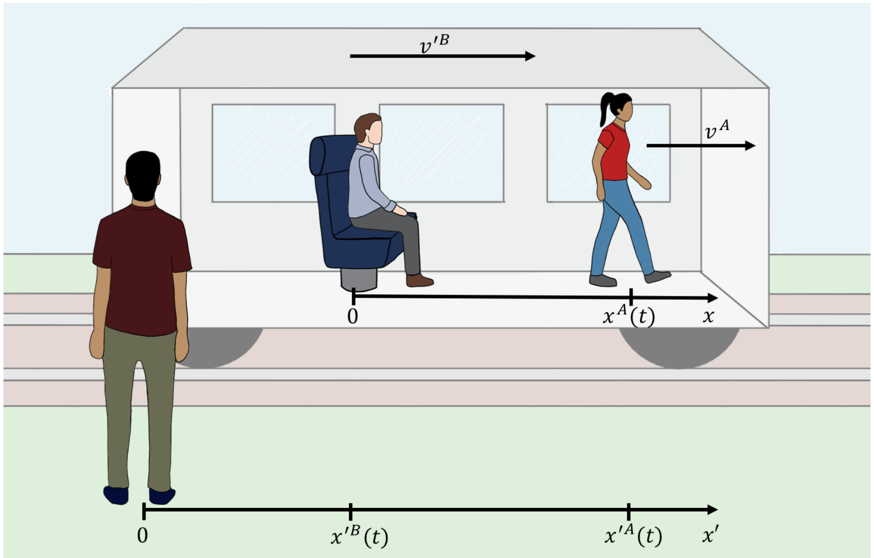


Figure 1.9: Alice is walking in the train and her position is described by both Brice, who is sitting in the train (using the x axis), and Igor, who is at rest on the ground (using the x' axis).

Since Brice already went through the work of determining the function $x^A(t)$ in the **reference frame** of the train, we wish to determine how to *transform* $x^A(t)$ into the reference frame of the train station, $x'^A(t)$, so that Igor can also describe Alice's motion. In other words, we wish to describe Alice's motion in two different *reference frames*.

A reference frame is simply a choice of coordinates, in this case, a choice of x-axis. Ideally, in physics, we prefer to use *inertial* reference frames, which are reference frames that are either "at rest" or that are moving at a constant speed relative to a frame that we consider at rest.

In principle, if you blocked out all of the windows in the train, it would not be possible for Alice and Brice to determine if the train is moving at constant speed or if it is stopped. Thus, the concept of a "rest frame" is itself arbitrary. It is not possible to define a frame of reference that is truly at rest. Even Igor's frame of reference, the train station, is on the planet Earth, which is moving around the Sun with a speed of 108 000 km/h.

Not only is it impossible to define a frame of reference that is truly at rest, the rules from transforming from one frame to the other depend on the speed between the reference frames. Our common experience is described by what we call "Galilean Relativity", but if the speed between trains is very large, close to the speed of light, then we need to use Einstein's Special Theory of Relativity.

Referring to Figure 1.9, we wish to use Brice's description of Alice's motion, $x^A(t)$, and convert it into a description, $x'^A(t)$ that Igor can use in the train station. Since Brice is at rest in the train, the speed of Brice *relative* to Igor is $v'^B(t)$. The first step is for Igor to describe Brice's position, $x'^B(t)$, (that is, the position of Brice's origin). Assume that we choose $t = 0$ to be the point in time where the two origins are aligned. Since the train is moving at a constant speed, v_B (as measured by Brice), then the position of Brice's origin as measured from Igor's origin is given by:

$$x'^B(t) = v'^B t$$

Now that Igor can describe the position of the origin of Brice's coordinate system, he can use Brice's description of Alice's motion. Recall that $x^A(t)$ is Brice's measure of Alice's distance from his origin. Similarly, $x'^B(t)$, is Igor's measure of the distance from his origin to Brice's origin. Thus, to obtain Alice's distance from Igor's origin, we simply add the distance, $x'^B(t)$, from Igor's origin to Brice's origin, and then add, $x^A(t)$, the distance from Brice's origin to Alice. Thus:

$$\boxed{x'^A(t) = x'^B(t) + x^A(t) = v'^B t + x^A(t)} \tag{1.8}$$

which tells us how to obtain the position of object A in the x' reference frame, when $x^A(t)$ is the description the object's position in the x reference frame which is moving with a velocity v'^B relative to the x' reference frame.

Since we know the position of Alice as measured in Igor's frame of reference, we can now easily find her velocity and her acceleration, as measured by Igor. Her velocity as measured by Igor, v'^A, is given by the time-derivative of her position measured in Igor's frame of reference:

$$\begin{aligned}
v'^A(t) &= \frac{d}{dt}x'^A(t) && (1.9)\\
&= \frac{d}{dt}(v'^B t + x^A(t)) && (1.10)\\
&= v'^B + \frac{d}{dt}x^A(t) && (1.11)\\
&= v'^B + v^A(t) && (1.12)
\end{aligned}$$

where $v^A(t) = \frac{d}{dt}x^A(t)$ is Alice's speed as measured by Brice, in the train. That is, the velocity of Alice as measured by Igor is the sum of the velocity of the train relative to the ground and the velocity of Alice relative to the train, which makes sense. If we now determine Alice's acceleration, $a'^A(t)$, as measured by Igor, we find:

$$\begin{aligned}
a'^A(t) &= \frac{d}{dt}v'^A(t) && (1.13)\\
&= \frac{d}{dt}(v'^B + v^A) && (1.14)\\
&= 0 + \frac{d}{dt}v^A(t) && (1.15)\\
&= a^A && (1.16)
\end{aligned}$$

where we have explicitly used the fact that the train is moving at constant velocity ($\frac{d}{dt}v'^B = 0$). Here we find that both Brice and Igor will measure the same number when referring to

Alice's acceleration (if the train is moving at a constant velocity). This is a particularity of "inertial" frame of references: accelerations do not depend on the reference frame, as long as the reference frames are moving with a constant velocity relative to each other. As we will see later, forces exerted on an object are directly related to the acceleration experienced by that object. Thus, the forces on an object do not depend on the choice of inertial reference frame.

Example 1-2

A large boat is sailing North at a speed of $v'^B = 15\,\text{m/s}$ and a restless passenger is walking about on the deck. Chloë, another passenger on the boat, finds that the passenger is walking at a constant speed of $v^A = 3\,\text{m/s}$ towards the South (opposite the direction of the boat's motion). Marcel is watching the boat pass by from the shore. What velocity (magnitude and direction) does Marcel measure for the restless passenger?

Solution

First, we must choose coordinate systems in the boat and on the shore. On the boat, let us define an x axis that is positive in the North direction and has an origin such that the position of the restless passenger was $x^A(t=0) = 0$ at time $t = 0$. In Chloë's reference frame, the passenger is thus described by:

$$x^A(t) = v^A t = (-3\,\text{m/s})t$$

where we note that v^A is negative since the passenger is moving the in negative x direction (the passenger is walking towards the South, but we chose positive x to be in the North direction). On shore, we choose an x' axis that also is positive in the North direction. We can choose the origin such that the origin of the boat's coordinate system was $x' = 0$. The origin of the boat's coordinate system as measured by Marcel (on shore) is thus:

$$x'^B(t) = v'^B t = (15\,\text{m/s})t$$

The position of the passenger, $x'^A(t)$, as measured by Marcel, is then given by adding the position of the boat's origin and the position of the passenger as measured from the boat's origin:

$$\begin{aligned}
x'^A(t) &= x'^B(t) + x^A(t) \\
&= v'^B t + v^A t \\
&= (v'^B + v^A)t \\
&= ((15\,\text{m/s}) + (-3\,\text{m/s}))t \\
&= (12\,\text{m/s})t
\end{aligned}$$

To find the velocity of the passenger as measured by Marcel, we take the time derivative:

$$\begin{aligned}
v'^A &= \frac{d}{dt}x'^A(t) \\
&= \frac{d}{dt}\Big((v'^B + v^A)t\Big) \\
&= (v'^B + v^A) \\
&= ((15\,\text{m/s}) + (-3\,\text{m/s})) \\
&= 12\,\text{m/s}
\end{aligned}$$

Since this is a positive number, Marcel still sees the passenger moving in the North direction (the direction of his positive x' axis), but with a speed of $12\,\text{m/s}$, which is less than that of the boat. On the boat, the passenger appears to be walking towards the South, but the net motion of the passenger relative to the ground is still in the North direction, as their speed is less than that of the boat.

1.5 Summary

Key Takeaways

To describe motion in one dimension, we must define an axis with:

1. An origin (where $x = 0$)
2. A direction (the direction in which x increases).

We describe the position of an object with a function $x(t)$ that *depends* on time. The rate of change of position is called "velocity", $v_x(t)$, and the rate of change of velocity is called "acceleration", $a_x(t)$:

$$\begin{aligned} v_x(t) &= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \\ a_x(t) &= \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv_x}{dt} \end{aligned}$$

Given the acceleration, one can find the velocity and position:

$$\begin{aligned} v_x(t) &= \int a_x(t)dt \\ x(t) &= \int v_x(t)dt \end{aligned}$$

With a constant acceleration, $a_x(t) = a_x$, if the object had velocity v_{0x} and position x_0 at $t = 0$:[a]

$$\begin{aligned} v_x(t) &= v_{0x}t + a_x t \\ x(t) &= x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \\ v^2 - v_0^2 &= 2a\Delta x \end{aligned}$$

An inertial frame of reference is one that is moving with a constant velocity. It is impossible to define a frame of reference that is truly "at rest", so we consider inertial frames of reference only relative to other frames of reference that we also consider to be inertial. If an object has position x^A as measured in a frame of reference x that is moving at constant speed v'^B as measured in a second frame of reference x', then in the x' reference frame, the kinematic quantities for the object are obtained by the Galilean transformation:

$$\begin{aligned} x'^A(t) &= v'^B t + x^A(t) \\ v'^A(t) &= v'^B + v^A(t) \\ a'^A(t) &= a(t) \end{aligned}$$

[a] We did not derive the third of these kinematic equations in this chapter, but it is derived in problem 1-1.

Important Equations

Position, Velocity, and Acceleration:

$$v_x(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

$$a_x(t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv_x}{dt}$$

$$v_x(t) = \int a_x(t) dt$$

$$x(t) = \int v_x(t) dt$$

Kinematic Equations:

$$v_x(t) = v_{0x} t + a_x t$$

$$x(t) = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$$

$$v^2 - v_0^2 = 2a\Delta x$$

Relative Motion:

$$x'^A(t) = v'^B t + x^A(t)$$

$$v'^A(t) = v'^B + v^A(t)$$

$$a'^A(t) = a(t)$$

1.6 Thinking about the material

1.6.1 Reflect and research

1. Look up the depth of a competition diving pool. What is the relationship between the height of the diving platform and the minimum pool depth? Why? If the designers of the pool assumed that every diver drops straight down off the diving board, would the pool still be safe for divers that jump up first?
2. When did Galileo Galilei first describe his principles of Galilean Relativity?
3. In Galileo's "Dialogue Concerning the Two Chief World Systems", what example did he use to describe relative motion?
4. Imagine that you are a judge, trying to charge an irresponsible driver for speeding on the highway. In the courtroom, he argues that in his own frame of reference, he was sitting still with respect to his car. In fact, he says that it was the officer, parked on the side of the highway that was speeding. You realize that in his reference frame, he is indeed correct - but that's not what matters! How do you explain the relative motion of driving laws to this sneaky offender, in order to serve him justice?

1.6.2 To try at home

Design an experiment to find the acceleration due to gravity: One of the important equations in this chapter was $x(t) = x_0 + v_{0x} + \frac{1}{2}at^2$. If you drop an object from rest, the distance it falls can be represented by $x = \frac{1}{2}gt^2$. Design an experiment to find the value of the acceleration due to gravity, g. Hint: This equation can be viewed as a linear relationship between x and t^2.

1.6.3 To try in the lab

1.7 Sample Problems and Solutions

1.7.1 Problems

Problem 1-1: Derive a kinematic equation that is independent of time. Specifically, derive: $v^2 - v_0^2 = 2ax$, starting with equations 1.2 and 1.5.

Problem 1-2: Rob is riding is bike at a speed of $8\,\text{m/s}$. He passes by a velociraptor, as one often does, who is eating by the side of the road. The velociraptor begins chasing him. The velociraptor accelerates from rest at a rate of $4\,\text{m/s}^2$.

a) Assuming it takes 3 seconds for the velociraptor to react, how long does it take from the moment Rob passes by for the velicoraptor to catch up to him?
b) If there is a safe place 70 metres from where Rob passes the velociraptor, will Rob make it there in time to escape being eaten?

Problem 1-3: Figure 1.10 shows a graph of the acceleration, $a(t)$, of a particle moving in one dimension. Draw the corresponding velocity and position graphs. Assume that $v(0) = 0$ and $x(0) = 0$.

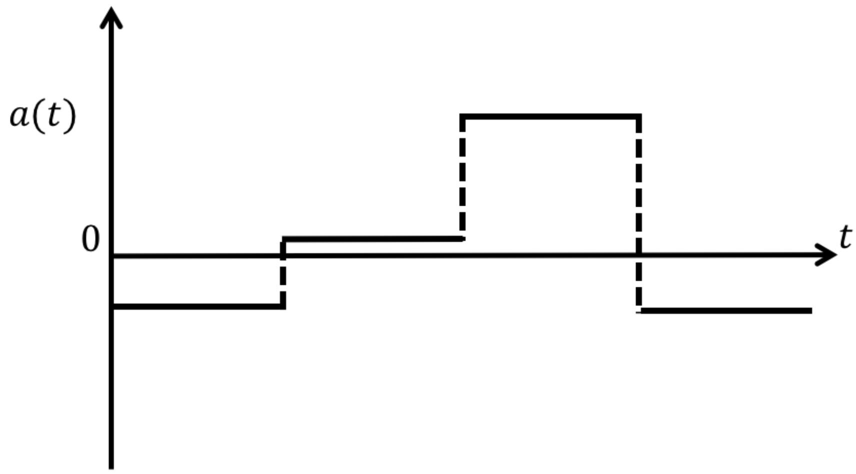


Figure 1.10: A graph of acceleration as a function of time. The scale and units are arbitrary.

1.7.2 Solutions

Solution to problem 1-1: We start with the equations for position and velocity that we derived in this chapter:

$$\begin{aligned} x &= x_0 + v_0 t + \frac{1}{2}at^2 \\ v &= v_0 + at \end{aligned}$$

The first equation can be written as:

$$\Delta x = v_0 t + \frac{1}{2}at^2$$

Our goal is to find an equation that is independent of time t. We start by isolating t in our equation for velocity:

$$\begin{aligned} v &= v_0 + at \\ t &= \frac{v - v_0}{a} \end{aligned}$$

We then substitute this value of t into our equation for Δx:

$$\begin{aligned} \Delta x &= v_0 t + \frac{1}{2}at^2 \\ \Delta x &= v_0\left(\frac{v - v_0}{a}\right) + \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2 \end{aligned}$$

We want the left hand side to be $2a\Delta x$, so we multiply each term by $2a$:

$$\begin{aligned} 2a\Delta x &= (2a)v_0\left(\frac{v - v_0}{a}\right) + (2a)\frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2 \\ 2a\Delta x &= (2v_0)a\left(\frac{v - v_0}{a}\right) + a^2\left(\frac{v - v_0}{a}\right)^2 \\ 2a\Delta x &= 2v_0(v - v_0) + (v - v_0)^2 \end{aligned}$$

We distribute $2v_0$ into the brackets. Then we expand the third term and get:

$$\begin{aligned} 2a\Delta x &= (2v_0 v - 2v_0^2) + (v_0 - v^2)(v_0 - v^2) \\ 2a\Delta x &= (2v_0 v - 2v_0^2) + (v_0^2 - 2v_0 v + v^2) \end{aligned}$$

All that's left to do is collect like terms, and we get the formula we are looking for:

$$\begin{aligned} 2a\Delta x &= 2v_0 v - 2v_0^2 + v_0^2 - 2v_0 v + v^2 \\ 2a\Delta x &= (v^2) + (2v_0 v - 2v_0 v) + (v_0^2 - 2v_0^2) \\ 2a\Delta x &= v^2 - v_0^2 \\ v^2 - v_0^2 &= 2a\Delta x \\ &\therefore \text{QED} \end{aligned}$$

If you choose a coordinate system such that x_0, this equation becomes $v^2 - v_0^2 = 2ax$.

Solution to problem 1-2: We start by choosing our coordinate system. The solution is simplest if the x axis is positive in the direction of motion and has an origin at the point where Rob passes the velociraptor. We set $t = 0$ to be the moment the velociraptor starts running.

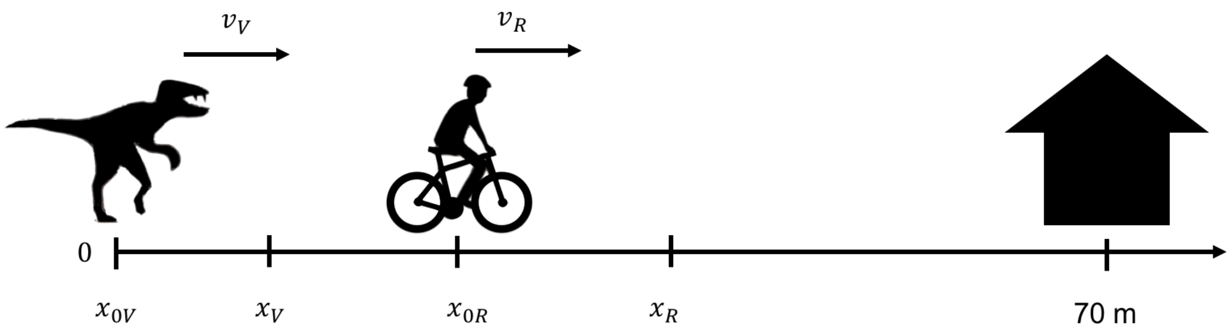


Figure 1.11: Rob is being chased by a velociraptor. At $t = 0$, Rob is a distance x_{0R} from the velociraptor. Safety is 70 m away from the origin.

(a) What do we mean by "catch up"? It means that Rob and the velociraptor are in the same place at the same time. So, we are interested in the value of t when $x_R = x_V$. We need two equations, one describing Rob's position and one describing the position of the velociraptor. Rob is moving at a constant velocity, so his position is described by:

$$x_R = x_{0R} + v_R t$$

The velociraptor has a constant acceleration, so its position is described by:

$$x_V = x_{0V} + v_{0V} t + \frac{1}{2} a_V t^2$$

We can use a table to take stock of our known values:

Rob	**Velociraptor**
$x_{0R} = ?$	$x_{0V} = 0\,\mathrm{m}$
$v_R = 8\,\mathrm{m/s}$	$v_{0V} = 0\,\mathrm{m/s}$
	$a_V = 4\,\mathrm{m/s^2}$

x_{0R} is Rob's position at the instant the velociraptor starts running. The value of x_{0R} is unknown but can be easily solved for. It takes 3 seconds for the velociraptor to react, so at $t = 0$, Rob has moved $(8\,\mathrm{m/s}) \times (3\,\mathrm{s}) = 24\,\mathrm{m} = x_{0R}$ (where we used the formula $x = vt$).

Since $v_{0V} = 0$ (the velociraptor starts running from rest) and $x_{0V} = 0$, we can write our equations as:

$$x_R = x_{0R} + v_R t$$
$$x_V = \frac{1}{2} a_V t^2$$

Remember that we want to find t when $x_R = x_V$. Setting the above equations equal to one another gives:

$$x_{0R} + v_R t = \frac{1}{2} a_V t^2$$

that we can rearrange to get the quadratic:

$$0 = \frac{1}{2} a_V t^2 - v_R t - x_{0R}$$

Solving the quadratic gives $t = 6\,\mathrm{s}$. This doesn't quite give us the answer we want. We want to know how long it takes the velociraptor to catch up *from the moment Rob passes by*, so we have to add on the 3 s reaction time, giving a total time of 9 s.

(b) We can use this solution to figure out whether Rob makes it to safety. The velociraptor catches up after 9 seconds. In 9 seconds, Rob has travelled a distance of $(8\,\mathrm{m/s}) \times (9\,\mathrm{s}) = 72\,\mathrm{m}$. The shelter is only 70 m away, so Rob gets to safety in time!

Solution to problem 1-3:

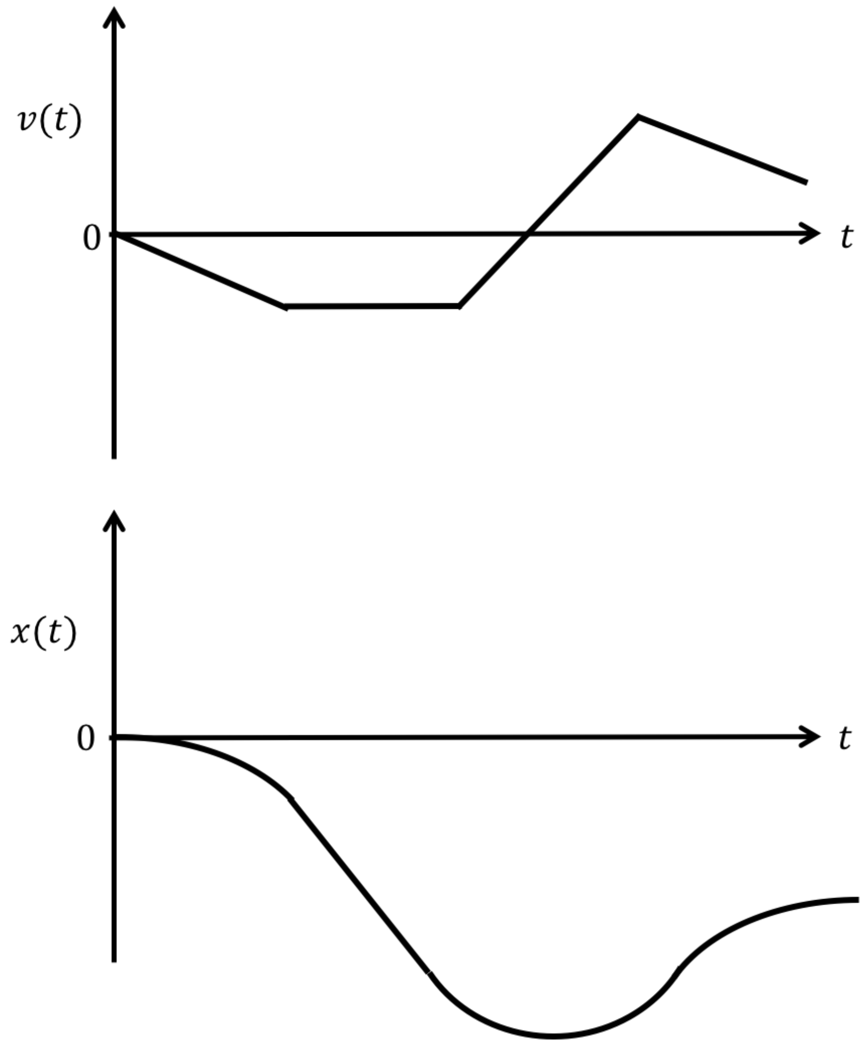


Figure 1.12: Graphs of $v(t)$ and $x(t)$ corresponding to the accleration versus time graph given in the question.

We start by drawing the graph of $v(t)$ from the graph of $a(t)$. Solutions may vary, but a few key features must be present:

- Velocity is zero at $t = 0$.
- When acceleration is negative, the velocity is decreasing. When the acceleration is positive, the velocity is increasing.
- When the acceleration is zero, the graph of $v(t)$ is a horizontal line

We can get the graph of $x(t)$ from the graph of $v(t)$. The graph of $x(t)$ should have these features:

- Position is zero at $x = 0$

- When the velocity is negative, $x(t)$ is decreasing. When the velocity is positive, $x(t)$ is increasing
- The particle turns around (the position goes from decreasing to increasing) when the velocity changes sign.
- When the velocity is negative and decreasing, or if it is positive and increasing, the magnitude of the slope of $x(t)$ increases. When velocity is positive and decreasing or negative and increasing, the magnitude of the slope decreases. When velocity is constant, the slope of $x(t)$ does not change.
  - Note:This is the same as saying that when the velocity and acceleration are both negative or both positive (when they are in the same direction), the slope of $x(t)$ increases in magnitude; when the acceleration and velocity are in opposite directions, the slope of $x(t)$ decreases in magnitude.

2 Describing motion in multiple dimensions

In this chapter, we will learn how to extend our description of an object's motion to two and three dimensions by using vectors. We will also consider the specific case of an object moving along the circumference of a circle.

Learning Objectives

- Describe motion in a 2D plane.
- Describe motion in 3D space.
- Describe motion along the circumference of a circle.

Think About It

Jake and Madi are riding a carousel that spins at a constant rate. Madi is closer to the centre of the carousel than Jake is. What can you say about their accelerations?

A) Both of their accelerations are zero
B) Madi's acceleration is greater than Jake's
C) Jake's acceleration is greater than Madi's
D) Madi and Jake have the same non-zero acceleration

2.1 Motion in two dimensions

2.1.1 Using vectors to describe motion in two dimensions

We can specify the location of an object with its coordinates, and we can describe any displacement by a vector. First, consider the case of an object moving with a constant velocity in a particular direction. We can specify the position of the object at any time, t, using its position vector, $\vec{r}(t)$, which is a function of time. We can describe the x and y components of the position vector with independent functions, $x(t)$, and $y(t)$, respectively:

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = x(t)\hat{x} + y(t)\hat{y}$$

Suppose that in a period of time Δt, the object goes from a position described by the position vector $\vec{r}_1$ to a position described by the position vector $\vec{r}_2$, as illustrated in Figure ??.

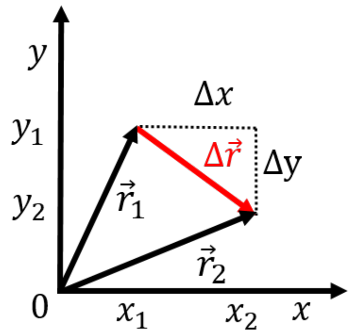


Figure 2.1: Illustration of a displacement vector, $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$, for an object that was located at position $\vec{r}_1$ at time t_1 and at position $\vec{r}_2$ at time $t_2 = t_1 + \Delta t$.

We can define a **displacement vector**, $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$, and by analogy to the one dimensional case, we can define an **average** velocity vector, $\vec{v}$ as:

$$\vec{v} = \frac{\Delta\vec{r}}{\Delta t} \tag{2.1}$$

The average velocity vector will have the same direction as $\Delta\vec{r}$, since it is the displacement vector divided by a scalar (Δt). The magnitude of the velocity vector, which we call "speed", will be proportional to the length of the displacement vector. If the object moves a large distance in a small amount of time, it will thus have a large velocity vector. This definition of the velocity vector thus has the correct intuitive properties (points in the direction of motion, is larger for faster objects).

For example, if the object went from position (x_1, y_1) to position (x_2, y_2) in an amount of time Δt, the average velocity vector is given by:

$$\begin{aligned}
\vec{v} &= \frac{\Delta\vec{r}}{\Delta t}\\
&= \frac{1}{\Delta t}\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}\\
&= \frac{1}{\Delta t}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}\\
&= \begin{pmatrix} \frac{\Delta x}{\Delta t} \\ \frac{\Delta y}{\Delta t} \end{pmatrix}\\
&= \begin{pmatrix} v_x \\ v_y \end{pmatrix}\\
\therefore \vec{v} &= v_x\hat{x} + v_y\hat{y}
\end{aligned}$$

That is, the x and y components of the average velocity vector can be found by separately determining the average velocity in each direction. For example, $v_x = \frac{\Delta x}{\Delta t}$ corresponds to the average velocity in the x direction, and can be considered independent from the velocity in the y direction, v_y. The magnitude of the average velocity vector (i.e. the average speed), is given by:

$$||\vec{v}|| = \sqrt{v_x^2 + v_y^2} = \frac{1}{\Delta t}\sqrt{\Delta x^2 + \Delta y^2} = \frac{\Delta r}{\Delta t}$$

where Δr is the magnitude of the displacement vector. Thus, the average speed is given by the distance covered divided by the time taken to cover that distance, in analogy to the one dimensional case.

Checkpoint 2-1

A llama runs in a field from a position $(x_1, y_1) = (2\,\text{m}, 5\,\text{m})$ to a position $(x_2, y_2) = (6\,\text{m}, 8\,\text{m})$ in a time $\Delta t = 0.5\,\text{s}$, as measured by Marcel, a llama farmer standing at the origin of the Cartesian coordinate system. What is the average speed of the llama?

A) $1\,\text{m/s}$
B) $5\,\text{m/s}$
C) $10\,\text{m/s}$
D) $15\,\text{m/s}$

If the velocity of the object is not constant, then we define the **instantaneous velocity vector** by taking the limit $\Delta t \to 0$:

$$\vec{v}(t) = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \tag{2.2}$$

which gives us the time derivative of the position vector (in one dimension, it was the time derivative of position). Writing the components of the position vector as functions $x(t)$ and $y(t)$, the instantaneous velocity becomes:

$$\begin{aligned}\boxed{\vec v(t) = \frac{d}{dt}\vec r(t)} \quad (2.3)\\ &= \frac{d}{dt}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}\\ &= \begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{pmatrix}\\ &= \begin{pmatrix}v_x(t)\\ v_y(t)\end{pmatrix}\\ \therefore \vec v(t) &= v_x(t)\hat x + v_y(t)\hat y\end{aligned}$$

where, again, we find that the components of the velocity vector are simply the velocities in the x and y direction. This means that we can treat motion in two dimensions as having two independent components: a motion along x and a separate motion along y. This highlights the usefulness of the vector notation for allowing us to use one vector equation ($\vec v = \frac{d}{dt}\Delta\vec r$) to represent two equations (one for x and one for y).

Similarly the acceleration vector is given by:

$$\begin{aligned}\boxed{\vec a(t) = \frac{d}{dt}\vec v(t)} \quad (2.4)\\ &= \begin{pmatrix}\frac{dv_x}{dt}\\ \frac{dv_y}{dt}\end{pmatrix}\\ &= \begin{pmatrix}a_x(t)\\ a_y(t)\end{pmatrix}\\ \therefore \vec a(t) &= a_x(t)\hat x + a_y(t)\hat y\end{aligned}$$

For example, if an object is at position $\vec r_0 = (x_0, y_0)$ with a velocity vector $\vec v_0 = v_{0x}\hat x + v_{0y}\hat y$ at time $t = 0$, and has a constant acceleration vector, $\vec a = a_x\hat x + a_y\hat y$, then the velocity vector at some later time t, $\vec v(t)$, is given by:

$$\vec v(t) = \vec v_0 + \vec a t$$

Or, if we write out the components explicitly:

$$\begin{pmatrix}v_x(t)\\ v_y(t)\end{pmatrix} = \begin{pmatrix}v_{0x}\\ v_{0y}\end{pmatrix} + \begin{pmatrix}a_x t\\ a_y t\end{pmatrix}$$

which really can be considered as two independent equations for the components of the velocity vector:

$$\begin{aligned} v_x(t) &= v_{0x} + a_x t \\ v_y(t) &= v_{0y} + a_y t \end{aligned}$$

which is the same equation that we had for one dimensional kinematics, but once for each coordinate. The position vector is given by:

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

with components:

$$\begin{aligned} x(t) &= x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \\ y(t) &= y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \end{aligned}$$

which again shows that two dimensional motion can be considered as separate and independent motions in each direction.

Example 2-1

An object starts at the origin of a coordinate system at time $t = 0\,\text{s}$, with an initial velocity vector $\vec{v}_0 = (10\,\text{m/s})\hat{x} + (15\,\text{m/s})\hat{y}$. The acceleration in the x direction is $0\,\text{m/s}^2$ and the acceleration in the y direction is $-10\,\text{m/s}^2$.

(a) Write an equation for the position vector as a function of time.
(b) Determine the position of the object at $t = 10\,\text{s}$.
(c) Plot the trajectory of the object for the first $5\,\text{s}$ of motion.

Solution

a)We can consider the motion in the x and y direction separately. In the x direction, the acceleration is 0, and the position is thus given by:

$$\begin{aligned} x(t) &= x_0 + v_{0x}t \\ &= (0\,\text{m}) + (10\,\text{m/s})t \\ &= (10\,\text{m/s})t \end{aligned}$$

In the y direction, we have a constant acceleration, so the position is given by:

$$\begin{aligned} y(t) &= y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \\ &= (0\,\text{m}) + (15\,\text{m/s})t + \frac{1}{2}(-10\,\text{m/s}^2)t^2 \\ &= (15\,\text{m/s})t - \frac{1}{2}(10\,\text{m/s}^2)t^2 \end{aligned}$$

The position vector as a function of time can thus be written as:

$$\begin{aligned} \vec{r}(t) &= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\ &= \begin{pmatrix} (10\,\text{m/s})t \\ (15\,\text{m/s})t - \frac{1}{2}(10\,\text{m/s}^2)t^2 \end{pmatrix} \end{aligned}$$

b) Using $t = 10\,\text{s}$ in the above equation gives:

$$\begin{aligned} \vec{r}(t = 10\,\text{s}) &= \begin{pmatrix} (10\,\text{m/s})(10\,\text{s}) \\ (15\,\text{m/s})(10\,\text{s}) - \frac{1}{2}(10\,\text{m/s}^2)(10\,\text{s})^2 \end{pmatrix} \\ &= \begin{pmatrix} (100\,\text{m}) \\ (-350\,\text{m}) \end{pmatrix} \end{aligned}$$

c) We can plot the trajectory using python:

Python Code 2.1: Trajectory in xy plane

```
#import modules that we need
import numpy as np #for arrays of numbers
import pylab as pl #for plotting

#define functions for the x and y positions:
def x(t):
    return 10*t

def y(t):
    return 15*t-0.5*10*t**2

#define 10 values of t from 0 to 5 s:
tvals = np.linspace(0,5,10)

#calculate x and y at those 10 values of t using the functions
#we defined above:
xvals = x(tvals)
```

```
yvals = y(tvals)

#plot the result:
pl.plot(xvals,yvals, marker='o')
pl.xlabel("x [m]",fontsize=14)
pl.ylabel("y [m]",fontsize=14)
pl.title("Trajectory in the xy plane",fontsize=14)
pl.grid()
pl.show()
```

Output 2.1:

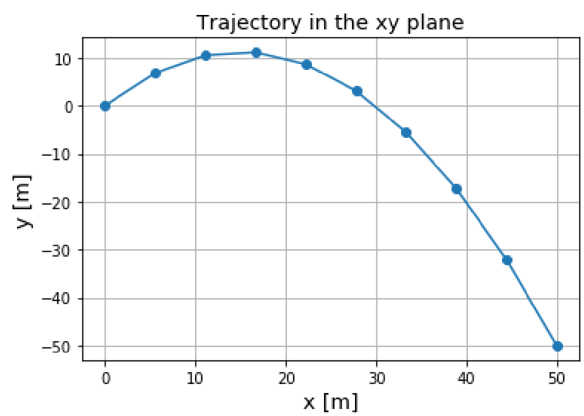


Figure 2.2: Parabolic trajectory of an object with no acceleration in the x direction and a negative acceleration in the y direction.

As you can see, the trajectory is a parabola, and corresponds to what you would get when throwing an object with an initial velocity with upwards (positive y) and horizontal (positive x) components. If you look at only the y axis, you will see that the object first goes up, then turns around and goes back down. This is exactly what happens when you throw a ball upwards, independently of whether the object is moving in the x direction. In the x direction, the object just moves with a constant velocity. The points on the graph are drawn for constant time intervals (the time between each point, Δt is constant). If you look at the distance between points projected onto the x axis, you will see that they are all equidistant and that along x, the motion corresponds to that of an object with constant velocity.

Checkpoint 2-2

In example **??**, what is the velocity vector exactly at the top of the parabola in Figure **??**?

A) $\vec{v} = (10\,\text{m/s})\hat{x} + (15\,\text{m/s})\hat{y}$
B) $\vec{v} = (15\,\text{m/s})\hat{y}$
C) $\vec{v} = (10\,\text{m/s})\hat{x}$
D) none of the above

Example 2-2

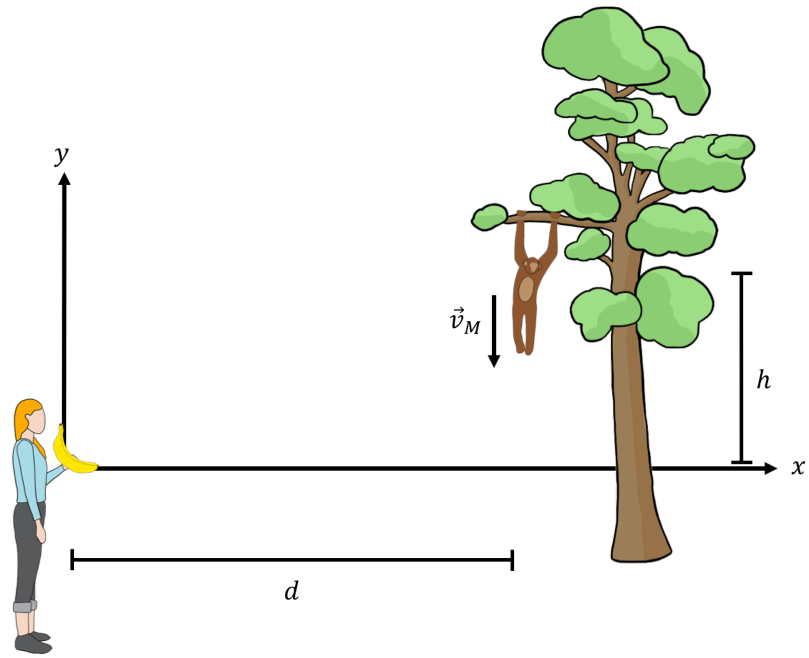


Figure 2.3: Feeding a monkey in a tree.

A monkey is hanging from a tree branch and you want to feed the monkey by throwing it a banana (Figure **??**). You know that the monkey is easily frightened and will let go of the tree branch the instant you throw the banana. The monkey is a horizontal distance d away and a height h above the point from which you release the banana when you throw it. At what angle with respect to the horizontal should you throw the banana so that the banana reaches the monkey?

Solution

This question is asking us to find what the angle, θ, of the banana's initial velocity, v_{0B}, should be if we want to hit the monkey with the banana. This angle can be found by finding the ratio of the initial vertical velocity to the initial horizontal velocity:

$$\tan\theta = \frac{v_{0By}}{v_{Bx}}$$

Now, how do we know if we hit the monkey? A "hit" means that the monkey and the banana are **in the same place at the same time** at some time, t. So, our approach will be as follows: We will start by finding equations that describe the motion of the monkey and of the banana. Then, we will use our conditions for a successful "hit" to find the ratio $\frac{v_{0By}}{v_{Bx}}$ that we want for our initial throw, and use that to find θ.

First, we need to set up our coordinate system. We set the origin to be where you release the banana; the origin is thus the initial position of the banana. We let y be in the vertical direction and let x be in the horizontal direction. We let "up" be positive and let the values of x increase as you get closer to the tree, as shown in Figure **??**.

We can treat the x and y components of the banana's and of the monkey's velocity and position as independent. The monkey's motion has only a vertical component. The acceleration of the monkey in the vertical direction is the acceleration due to gravity, $a_y = -g$, which points in the $-y$ direction. The initial vertical position of the monkey is h and its initial vertical velocity is 0, so we can describe the vertical position of the monkey, $y_M(t)$, by:

$$y_M(t) = y_0 + v_{yo}t + \frac{1}{2}a_y t^2 = h + (0) - \frac{1}{2}gt^2$$

The horizontal position of the monkey is constant, and is equal to $x_M(t) = d$.

The banana's motion has both x and y components. There is no acceleration in the x direction, and we defined the banana's initial horizontal position to be $x_{0B} = 0$, so the horizontal position of the banana, x_B is given by:

$$x_B(t) = x_0 + v_{x0} = (0) + v_{xB}t$$

We defined the initial vertical position of the banana to be $y_{0B} = 0$. The vertical position as a function of time, $y_B(t)$, can thus be described by:

$$y_B(t) = y_0 + v_{y0}t + \frac{1}{2}a_y t^2 = (0) + v_{0yB}t - \frac{1}{2}gt^2$$

Now that we have equations that describe the position of both the banana and the monkey, we can use our conditions for a successful hit. For the monkey and the banana to be in the same position, we need $y_M(t) = y_B(t)$ and $x_B(t) = d$ at some time t. Setting our equations for $y_M(t)$ and $y_B(t)$ equal to one another gives:

$$h - \frac{1}{2}gt^2 = v_{0yB}t - \frac{1}{2}gt^2$$
$$\therefore h = v_{0yB}t$$

And setting $X_M(t) = d$ equal to $x_B(t)$ gives:

$$\therefore d = v_{xB}$$

We can just divide one equation by the other to find:

$$\begin{aligned}\frac{h}{d} &= \frac{v_{0By}t}{v_{xB}t}\\ \frac{h}{d} &= \frac{v_{0By}}{v_{xB}}\end{aligned}$$

This gives us the ratio we are looking for, so we now know that

$$\begin{aligned}\tan\theta &= \frac{h}{d}\\ \therefore \theta &= \tan^{-1}\left(\frac{h}{d}\right)\end{aligned}$$

Some people make the mistake of thinking that you should aim below the monkey. This equation is telling us that what we should actually do is aim directly at the monkey! This works because the x and y components of the motion are independent, and the acceleration of the banana is the same as the acceleration of the monkey.

2.1.2 Relative motion

In the previous chapter, we examined how to convert the description of motion from one reference frame to another. Recall the one dimensional situation where we described the position of an object, A, using an axis x as $x^A(t)$. Suppose that the reference frame, x, is moving with a constant speed, v'^B, relative to a second reference frame, x'. We found that the position of the object is described in the x' reference frame as:

$$x'^A(t) = v'^B t + x^A(t)$$

if the origins of the two systems coincided at $t = 0$. The equation above simply states that the distance of the object to the x' origin is the sum of the distance from the x' origin to the x origin **and** the distance from the x origin to the object.

In two dimensions, we proceed in exactly the same way, but use vectors instead:

$$\vec{r}'^A(t) = \vec{v}'^B t + \vec{r}^A(t)$$

where $r^A(t)$ is the position of the object as described in the xy reference frame, $\vec{v}'^B$, is the velocity vector describing the motion of the origin of the xy coordinate system relative to an $x'y'$ coordinate system and $\vec{r}'^A(t)$ is the position of the object in the $x'y'$ coordinate system. We have assumed that the origins of the two coordinate systems coincided at $t = 0$ and that the axes of the coordinate systems are parallel (x parallel to x' and y parallel to y').

Note that the velocity of the object in the $x'y'$ system is found by adding the velocity of xy relative to $x'y'$ and the velocity of the object in the xy frame ($\vec{v}^A(t)$):

$$\begin{aligned}\frac{d}{dt}\vec{r}'^A(t) &= \frac{d}{dt}(\vec{v}'^B t + \vec{r}^A(t))\\ &= \vec{v}'^B + \vec{v}^A(t)\end{aligned}$$

As an example, consider the situation depicted in Figure **??**. Brice is on a boat off the shore of Nice, with a coordinate system xy, and is describing the position of a boat carrying Alice. He describes Alice's position as $\vec{r}^A(t)$ in the xy coordinate system. Igor is on the shore and also wishes to describe Alice's position using the work done by Brice. Igor sees Brice's boat move with a velocity $\vec{v}'^B$ as measured in his $x'y'$ coordinate system. In order to find the vector pointing to Alice's position $\vec{r}'^A(t)$, he adds the vector from his origin to Brice's origin ($\vec{v}'^B t$) and the vector from Brice's origin to Alice $\vec{r}^A(t)$.

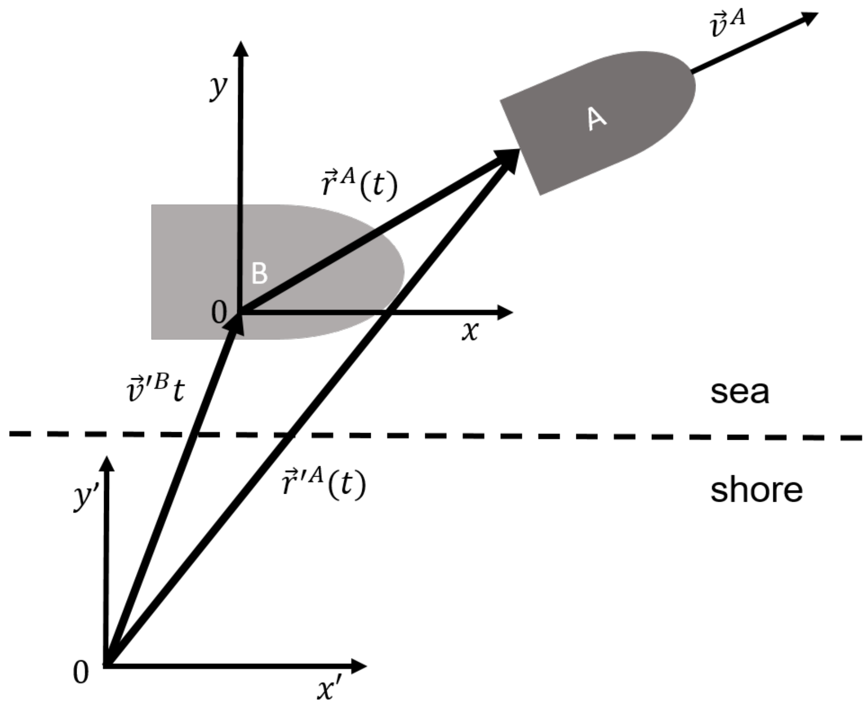


Figure 2.4: Example of converting from one reference frame to another in two dimensions using vector addition.

Writing this out by coordinate, we have:

$$x'^A(t) = v_x'^B t + x^A(t)$$
$$y'^A(t) = v_y'^B t + y^A(t)$$

and for the velocities:

$$v_x'^A(t) = v_x'^B + v_x^A(t)$$
$$v_y'^A(t) = v_y'^B + v_y^A(t)$$

Checkpoint 2-3

You are on a boat and crossing a North-flowing river, from the East bank to the West bank. You point your boat in the West direction and cross the river. Chloë is watching your boat cross the river from the shore, in which direction does she measure your velocity vector to be?

A) in the North direction
B) in the West direction
C) a combination of North and West directions

2.2 Motion in three dimensions

The big challenge was to expand our description of motion from one dimension to two. Adding a third dimension ends up being trivial now that we know how to use vectors. In three dimensions, we describe the position of a point using three coordinates, so all of the vectors simply have three independent components, but are treated in exactly the same way as in the two dimensional case. The position of an object is now described by three independent functions, $x(t)$, $y(t)$, $z(t)$, that make up the three components of a position vector $\vec r(t)$:

$$\vec r(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$
$$\therefore \vec r(t) = x(t)\hat x + y(t)\hat y + z(t)\hat z$$

The velocity vector now has three components and is defined analogously to the 2D case:

$$\vec v(t) = \frac{d\vec r}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{pmatrix}$$
$$\therefore \vec v(t) = v_x(t)\hat x + v_y(t)\hat y + v_z(t)\hat z$$

and the acceleration is defined in a similar way:

$$\vec a(t) = \frac{d\vec v}{dt} = \begin{pmatrix} \frac{dv_x}{dt} \\ \frac{dv_y}{dt} \\ \frac{dv_z}{dt} \end{pmatrix} = \begin{pmatrix} a_x(t) \\ a_y(t) \\ a_z(t) \end{pmatrix}$$
$$\therefore \vec a(t) = a_x(t)\hat x + a_y(t)\hat y + a_z(t)\hat z$$

In particular, if an object has a constant acceleration, $\vec a = a_x\hat x + a_y\hat y + a_z\hat z$, and started at

$t = 0$ with a position $\vec{r}_0$ and velocity $\vec{v}_0$, then its velocity vector is given by:

$$\vec{v}(t) = \vec{v}_0 + \vec{a}t = \begin{pmatrix} v_{0x} + a_x t \\ v_{0y} + a_y t \\ v_{0z} + a_z t \end{pmatrix}$$

and the position vector is given by:

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 = \begin{pmatrix} x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \\ y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \\ z_0 + v_{0z}t + \frac{1}{2}a_z t^2 \end{pmatrix}$$

where again, we see how writing a single vector equation (e.g. $\vec{v}(t) = \vec{v}_0 + \vec{a}t$) is really just a way to write the three independent equations that are true for each component.

2.3 Accelerated motion when the velocity vector changes direction

One key difference with one dimensional motion is that, in two dimensions, it is possible to have a non-zero acceleration even when the speed is constant. Recall, the acceleration **vector** is defined as the time derivative of the velocity **vector** (equation **??**). This means that if the velocity vector changes with time, then the acceleration vector is non-zero. If the length of the velocity vector (speed) is constant, it is still possible that the **direction** of the velocity vector changes with time, and thus, that the acceleration vector is non-zero. This is, for example, what happens when an object goes around in a circle with a constant speed (the direction of the velocity vector changes).

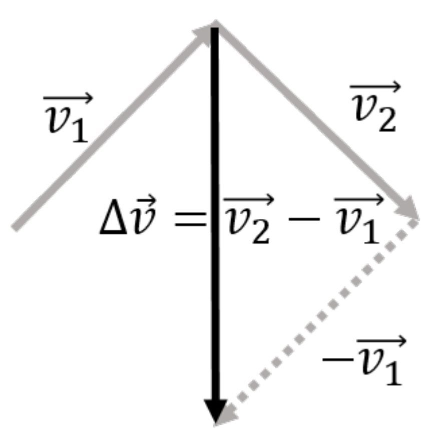


Figure 2.5: Illustration of how the direction of the velocity vector can change when speed is constant.

Figure **??** shows an illustration of a velocity vector, $\vec{v}(t)$, at two different times, $\vec{v}_1$ and $\vec{v}_2$, as well as the vector difference, $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$, between the two. In this case, the length of the velocity vector did not change with time ($||\vec{v}_1|| = ||\vec{v}_2||$). The acceleration vector is given by:

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t}$$

and will have a direction parallel to $\Delta\vec{v}$, and a magnitude that is proportional to Δv. Thus, even if the velocity vector does not change amplitude (speed is constant), the acceleration vector can be non-zero if the velocity vector changes *direction*.

Let us write the velocity vector, $\vec{v}$, in terms of its magnitude, v, and a unit vector, $\hat{v}$, in the direction of $\vec{v}$:

$$\begin{aligned}
\vec{v} &= v_x\hat{x} + v_y\hat{y} = v\hat{v} \\
v &= ||\vec{v}|| = \sqrt{v_x^2 + v_y^2} \\
\hat{v} &= \frac{v_x}{v}\hat{x} + \frac{v_y}{v}\hat{y}
\end{aligned}$$

In the most general case, both the magnitude of the velocity and its direction can change with time. That is, both the direction and the magnitude of the velocity vector are functions of time:

$$\vec{v}(t) = v(t)\hat{v}(t)$$

When we take the time derivative of $\vec{v}(t)$ to obtain the acceleration vector, we need to take the derivative of a product of two functions of time, $v(t)$ and $\hat{v}(t)$. Using the rules for taking the derivative of a product, the acceleration vector is given by:

$$\begin{aligned}
\vec{a} &= \frac{d}{dt}\vec{v}(t) = \frac{d}{dt}v(t)\hat{v}(t) \\
&\boxed{\vec{a} = \frac{dv}{dt}\hat{v}(t) + v(t)\frac{d\hat{v}}{dt}} \qquad (2.5)
\end{aligned}$$

and has two terms. The first term, $\frac{dv}{dt}\hat{v}(t)$, is zero if the speed is constant ($\frac{dv}{dt} = 0$). The second term, $v(t)\frac{d\hat{v}}{dt}$, is zero if the direction of the velocity vector is constant ($\frac{d\hat{v}}{dt} = 0$). In general though, the acceleration vector has two terms corresponding to the change in speed, and to the change in the direction of the velocity, respectively.

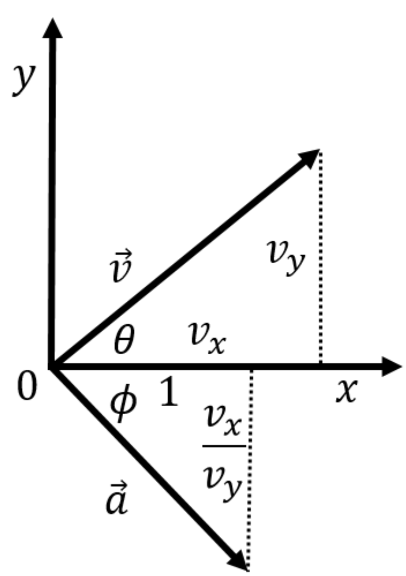
The specific functional form of the acceleration vector will depend on the path being taken by the object. If we consider the case where speed is constant, then we have:

$$\begin{aligned}
v(t) &= v \\
\frac{dv}{dt} &= 0 \\
v_x^2(t) + v_y^2(t) &= v^2 \\
\therefore v_y(t) &= \sqrt{v^2 - v_x(t)^2}
\end{aligned}$$

In other words, if the magnitude of the velocity is constant, then the x and y components are no longer independent (if the x component gets larger, then the y component must get smaller so that the total magnitude remains unchanged). If the speed is constant, then the acceleration vector is given by:

$$\begin{aligned}
\vec{a} &= \frac{dv}{dt}\hat{v}(t) + v\frac{d\hat{v}}{dt} \\
&= 0 + v\frac{d}{dt}\hat{v}(t) \\
&= v\frac{d}{dt}\left(\frac{v_x(t)}{v}\hat{x} + \frac{v_y(t)}{v}\hat{y})\right) \\
&= \frac{dv_x}{dt}\hat{x} + \frac{d}{dt}\sqrt{v^2 - v_x(t)^2}\hat{y} \\
&= \frac{dv_x}{dt}\hat{x} + \frac{1}{2\sqrt{v^2 - v_x(t)^2}}(-2v_x(t))\frac{dv_x}{dt}\hat{y} \\
&= \frac{dv_x}{dt}\hat{x} - \frac{v_x(t)}{\sqrt{v^2 - v_x(t)^2}}\frac{dv_x}{dt}\hat{y} \\
&= \frac{dv_x}{dt}\hat{x} - \frac{v_x(t)}{v_y(t)}\frac{dv_x}{dt}\hat{y} \\
\therefore \quad & \boxed{\vec{a} = \frac{dv_x}{dt}\left(\hat{x} - \frac{v_x(t)}{v_y(t)}\hat{y}\right)}
\end{aligned} \tag{2.6}$$

where most of the algebra that we did was to separate the x and y components of the acceleration vector, and we used the Chain Rule to take the derivative of the square root. The resulting acceleration vector is illustrated in Figure ?? along with the velocity vector[1].



[1]Rather, it is a vector parallel to the acceleration vector that is illustrated, as the factor of $\frac{dv_x}{dt}$ was omitted (as you recall, multiplying by a scalar only changes the length, not the direction)

Figure 2.6: Illustration that the acceleration vector is perpendicular to the velocity vector if speed is constant.

The velocity vector has components v_x and v_y, which allows us to calculate the angle, θ that it makes with the x axis:

$$\tan(\theta) = \frac{v_y}{v_x}$$

Similarly, the vector that is parallel to the acceleration has components of 1 and $-\frac{v_x}{v_y}$, allowing us to determine the angle, ϕ, that it makes with the x axis:

$$\tan(\phi) = \frac{v_x}{v_y}$$

Note that $\tan(\theta)$ is the inverse of $\tan(\phi)$, or in other words, $\tan(\theta) = \cot(\phi)$, meaning that θ and ϕ are complementary and thus must sum to $\frac{\pi}{2}$ (90°). This means that **the acceleration vector is perpendicular to the velocity vector if the speed is constant and the direction of the velocity changes**.

In other words, when we write the acceleration vector, we can identify two components, $\vec{a}_{\parallel}(t)$ and $\vec{a}_{\perp}(t)$:

$$\begin{aligned}
\vec{a} &= \frac{dv}{dt}\hat{v}(t) + v(t)\frac{d\hat{v}}{dt} \\
&= \vec{a}_{\parallel}(t) + \vec{a}_{\perp}(t) \\
\therefore \vec{a}_{\parallel}(t) &= \frac{dv}{dt}\hat{v}(t) \\
\therefore \vec{a}_{\perp}(t) &= v\frac{d\hat{v}}{dt} = \frac{dv_x}{dt}\left(\hat{x} - \frac{v_x(t)}{v_y(t)}\hat{y}\right)
\end{aligned}$$

where $\vec{a}_{\parallel}(t)$ is the component of the acceleration that is parallel to the velocity vector, and is responsible for changing its magnitude, and $\vec{a}_{\perp}(t)$, is the component that is perpendicular to the velocity vector and is responsible for changing the direction of the motion.

Checkpoint 2-4

A satellite moves in a circular orbit around the Earth with a constant speed. What can you say about its acceleration vector?

A) it has a magnitude of zero.
B) it is perpendicular to the velocity vector.
C) it is parallel to the velocity vector.
D) it is in a direction other than parallel or perpendicular to the velocity vector.

2.4 Circular motion

We often consider the motion of an object around a circle of fixed radius, R. In principle, this is motion in two dimensions, as a circle is necessarily in a two dimensional plane.

However, since the object is constrained to move along the circumference of the circle, it can be thought of (and treated as) motion along a one dimensional axis that is curved.

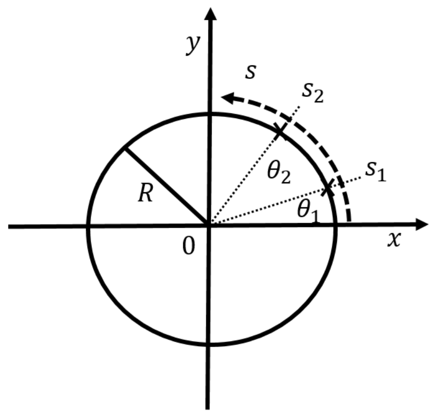


Figure 2.7: Describing the motion of an object around a circle of radius R.

Figure **??** shows how we can describe motion on a circle. We could use $x(t)$ and $y(t)$ to describe the position on the circle, however, $x(t)$ and $y(t)$ are no longer independent since they have to correspond to the coordinates of points on a circle:

$$x^2(t) + y^2(t) = R^2$$

Instead of using x and y, we could think of an axis that is bent around the circle (as shown by the curved arrow in Figure **??**, the s axis). The s axis is such that $s = 0$ where the circle intersects the x axis, and the value of s increases as we move counter-clockwise along the circle. Distance along the s axis thus corresponds to the distance along the circumference of the circle.

Another variable that could be used for position instead of s is the angle, θ, between the position vector of the object and the x axis, as illustrated in Figure **??**. If we express the angle θ in radians, then it is easy to convert between s and θ. Recall, an angle in radians is defined as the length of an arc subtended by that angle divided by the radius of the circle. We thus have:

$$\boxed{\theta(t) = \frac{s(t)}{R}} \tag{2.7}$$

In particular, if the object has gone around the whole circle, then $s = 2\pi R$ (the circumference of a circle), and the corresponding angle is, $\theta = \frac{2\pi R}{R} = 2\pi$, namely 360°.

By using the angle, θ, instead of x and y, we are effectively using polar coordinates, with a fixed radius. As we already saw, the x and y positions are related to θ by:

$$x(t) = R\cos(\theta(t))$$
$$y(t) = R\sin(\theta(t))$$

where R is a constant. For an object moving along the circle, we can write its position vector, $\vec{r}(t)$, as:

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = R \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix}$$

and the velocity vector is thus given by:

$$\begin{aligned}
\vec{v}(t) = \frac{d}{dt}\vec{r}(t) &= \frac{d}{dt} R \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix} \\
&= R \begin{pmatrix} \frac{d}{dt}\cos(\theta(t)) \\ \frac{d}{dt}\sin(\theta(t)) \end{pmatrix} \\
&= R \begin{pmatrix} -\sin(\theta(t))\frac{d\theta}{dt} \\ \cos(\theta(t))\frac{d\theta}{dt} \end{pmatrix}
\end{aligned}$$

where we used the Chain Rule to calculate the time derivatives of the trigonometric functions (since $\theta(t)$ is function of time). The magnitude of the velocity vector is given by:

$$\begin{aligned}
||\vec{v}|| &= \sqrt{v_x^2 + v_y^2} \\
&= \sqrt{\left(-R\sin(\theta(t))\frac{d\theta}{dt}\right)^2 + \left(R\cos(\theta(t))\frac{d\theta}{dt}\right)^2} \\
&= \sqrt{R^2\left(\frac{d\theta}{dt}\right)^2 [\sin^2(\theta(t)) + \cos^2(\theta(t)]} \\
&= R\left|\frac{d\theta}{dt}\right|
\end{aligned}$$

The position and velocity vectors are illustrated in Figure **??** for an angle θ in the first quadrant ($0 < \theta < \frac{\pi}{2}$).

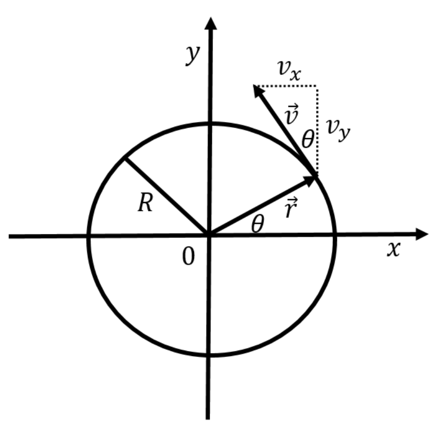


Figure 2.8: The position vector, $\vec{r}(t)$ is always perpendicular to the velocity vector, $\vec{v}(t)$, for motion on a circle.

In this case, you can note that the x component of the velocity is negative (in the equation above, and in the Figure). From the equation above, you can also see that $\frac{|v_x|}{|v_y|} = \tan(\theta)$, which is illustrated in Figure **??**, showing that **the velocity vector is tangent to the circle** and perpendicular to the position vector. This is always the case for motion along a circle.

We can simplify our description of motion along the circle by using either $s(t)$ or $\theta(t)$ instead of the vectors for position and velocity. If we use $s(t)$ to represent position along the circumference ($s = 0$ where the circle intersects the x axis), then the velocity along the s axis is:

$$\begin{aligned}
v_s(t) &= \frac{d}{dt}s(t) \\
&= \frac{d}{dt}R\theta(t) \\
&= R\frac{d\theta}{dt}
\end{aligned}$$

where we used the fact that $\theta = s/R$ to convert from s to θ. The velocity along the s axis is thus precisely equal to the magnitude of the two-dimensional velocity vector (derived above), which makes sense since the velocity vector is tangent to the circle (and thus in the s "direction").

If the object has a **constant speed**, v_s, along the circle and started at a position along the circumference $s = s_0$, then its position along the s axis can be described using 1D kinematics:

$$s(t) = s_0 + v_s t$$

or, in terms of θ:

$$\begin{aligned}
\theta(t) &= \frac{s(t)}{R} = \frac{s_0}{R} + \frac{v_s}{R}t \\
&= \theta_0 + \frac{d\theta}{dt}t \\
&= \theta_0 + \omega t \\
&\boxed{\therefore \omega = \frac{d\theta}{dt}}
\end{aligned}$$

where we introduced θ_0 as the angle corresponding to the position s_0, and we introduced $\omega = \frac{d\theta}{dt}$, which is analogous to velocity, but for an angle. ω is called the **angular velocity** and is a measure of the rate of change of the angle θ (as it is the time derivative of the angle). The relation between the "linear" velocity v_s (the magnitude of the velocity vector, which corresponds to the velocity in the direction tangent to the circle) and ω is:

$$\boxed{v_s = R\frac{d\theta}{dt} = R\omega}$$

Similarly, if the object is accelerating, we can define an **angular acceleration**, $\alpha(t)$, as the rate of change of the angular velocity:

$$\alpha(t) = \frac{d\omega}{dt}$$

which can directly be related to the acceleration in the s direction, $a_s(t)$:

$$\begin{aligned} a_s(t) &= \frac{d}{dt}v_s \\ &= \frac{d}{dt}\omega R = R\frac{d\omega}{dt} \\ \boxed{a_s(t) = R\alpha} \end{aligned}$$

Thus, the linear quantities (those along the s axis) can be related to the angular quantities by multiplying the angular quantities by R:

$$s = R\theta \tag{2.8}$$
$$v_s = R\omega \tag{2.9}$$
$$a_s = R\alpha \tag{2.10}$$

If the object started at $t = 0$ with a position $s = s_0$ ($\theta = \theta_0$), and an initial linear velocity v_{0s} (angular velocity ω_0), and has a **constant linear acceleration** around the circle, a_s (angular acceleration, α), then the position of the object can be described using either the linear or the angular quantities:

$$\begin{aligned} s(t) &= s_0 + v_{s0}t + \frac{1}{2}a_s t^2 \\ \theta(t) &= \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \end{aligned}$$

As you recall from section **??**, we can compute the acceleration **vector** and identify components that are parallel and perpendicular to the velocity vector:

$$\begin{aligned} \vec{a} &= \vec{a}_{\parallel}(t) + \vec{a}_{\perp}(t) \\ &= \frac{dv}{dt}\hat{v}(t) + v\frac{d\hat{v}}{dt} \end{aligned}$$

The first term, $\vec{a}_{\parallel}(t) = \frac{dv}{dt}\hat{v}(t)$, is parallel to the velocity vector $\hat{v}$, and has a magnitude given by:

$$||\vec{a}_{\parallel}(t)|| = \frac{dv}{dt} = \frac{d}{dt}v(t) = \frac{d}{dt}R\omega = R\alpha$$

That is, the component of the acceleration vector that is parallel to the velocity is precisely the acceleration in the s direction (the linear acceleration). This component of the acceleration is responsible for increasing (or decreasing) the speed of the object and is zero if the object goes around the circle with a constant speed (linear or angular).

As we saw earlier, the perpendicular component of the acceleration, $\vec{a}_{\perp}(t)$, is responsible for changing the direction of the velocity vector (as the object continuously changes direction when going in a circle). When the motion is around a circle, this component of the acceleration vector is called "centripetal" acceleration (i.e. acceleration pointing towards the centre of the circle, as we will see). We can calculate the centripetal acceleration in terms of our angular variables, noting that the unit vector in the direction of the velocity is $\hat{v} = -\sin(\theta)\hat{x} + \cos(\theta)\hat{y}$:

$$\begin{aligned}
\vec{a}_{\perp}(t) &= v\frac{d\hat{v}}{dt} \\
&= (\omega R)\frac{d}{dt}\left[-\sin(\theta)\hat{x} + \cos(\theta)\hat{y}\right] \\
&= \omega R\left[-\frac{d}{dt}\sin(\theta)\hat{x} + \frac{d}{dt}\cos(\theta)\hat{y}\right] \\
&= \omega R\left[-\cos(\theta)\frac{d\theta}{dt}\hat{x} - \sin(\theta)\frac{d\theta}{dt}\hat{y}\right] \\
&= \omega R[-\cos(\theta)\omega\hat{x} - \sin(\theta)\omega\hat{y}] \\
\Aboxed{\vec{a}_{\perp}(t) &= \omega^2 R[-\cos(\theta)\hat{x} - \sin(\theta)\hat{y}]}
\end{aligned} \tag{2.11}$$

where you can easily verify that the vector $[-\cos(\theta)\hat{x} - \sin(\theta)\hat{y}]$ has unit length and points towards the centre of the circle (when the tail is placed on a point on the circle at angle θ). The centripetal acceleration thus points towards the centre of the circle and has magnitude:

$$a_c(t) = ||\vec{a}_{\perp}(t)|| = \omega^2(t)R = \frac{v^2(t)}{R} \tag{2.12}$$

where in the last equal sign, we wrote the centripetal acceleration in terms of the speed around the circle ($v = ||\vec{v}|| = v_s$).

If an object goes around a circle, it will always have a centripetal acceleration (since its velocity vector must change direction). In addition, if the object's speed is changing, it will also have a linear acceleration, which points in the same direction as the velocity vector (it changes the velocity vector's length but not its direction).

Checkpoint 2-5

A vicuña is going clockwise around a circle that is centred at the origin of an xy coordinate system that is in the plane of the circle. The vicuña runs faster and faster around the circle. In which direction does its acceleration vector point just as the vicuña is at the point where the circle intersects the positive y axis?

A) In the negative y direction
B) In the positive y direction
C) A combination of the positive y and positive x directions
D) A combination of the negative y and positive x directions
E) A combination of the negative y and negative x directions

2.4.1 Period and frequency

When an object is moving around in a circle, it will typically complete more than one revolution. If the object is going around the circle with a constant speed, we call the motion "uniform circular motion", and we can define the **period and frequency** of the motion.

The period, T, is defined to be the time that it takes to complete one revolution around the circle. If the object has constant angular speed ω, we can find the time, T, that it takes to complete one full revolution, from $\theta = 0$ to $\theta = 2\pi$:

$$\omega = \frac{\Delta\theta}{T} = \frac{2\pi}{T}$$
$$\boxed{\therefore T = \frac{2\pi}{\omega}} \tag{2.13}$$

We would obtain the same result using the linear quantities; in one revolution, the object covers a distance of $2\pi R$ at a speed of v:

$$v = \frac{2\pi R}{T}$$
$$T = \frac{2\pi R}{v} = \frac{2\pi R}{\omega R} = \frac{2\pi}{\omega}$$

The frequency, f, is defined to be the inverse of the period:

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

and has SI units of $\text{Hz} = \text{s}^{-1}$. Think of frequency as the number of revolutions completed per second. Thus, if the frequency is $f = 1\,\text{Hz}$, the object goes around the circle once per second. Given the frequency, we can of course obtain the angular velocity:

$$\omega = 2\pi f$$

which is sometimes called the "angular frequency" instead of the angular velocity. The angular velocity can really be thought of as a frequency, as it represents the "amount of angle" per second that an object covers when going around a circle. The angular velocity does not tell us anything about the actual speed of the object, which depends on the radius $v = \omega R$. This is illustrated in Figure **??**, where two objects can be travelling around two circles of radius R_1 and R_2 with the same angular velocity ω. If they have the same angular velocity, then it will take them the same amount of time to complete a revolution. However, the outer object has to cover a much larger distance (the circumference is larger), and thus has to move with a larger linear speed.

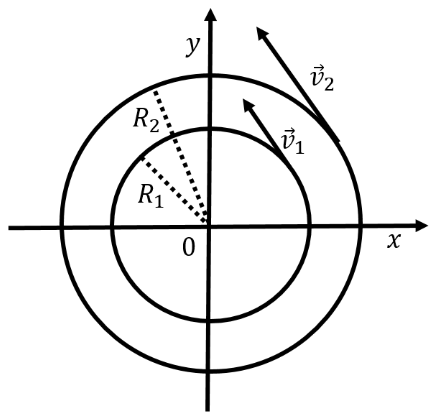


Figure 2.9: For a given angular velocity, the linear velocity will be larger on a larger circle ($v = \omega R$).

Checkpoint 2-6

A motor is rotating at 3000 rpm, what is the corresponding frequency in Hz?

A) 5 Hz
B) 50 Hz
C) 500 Hz

Olivia's Thoughts

There's a trick I like to use to remember how linear and angular velocities work. Figure **??** shows your hand in two positions, which we call (1) and (2).

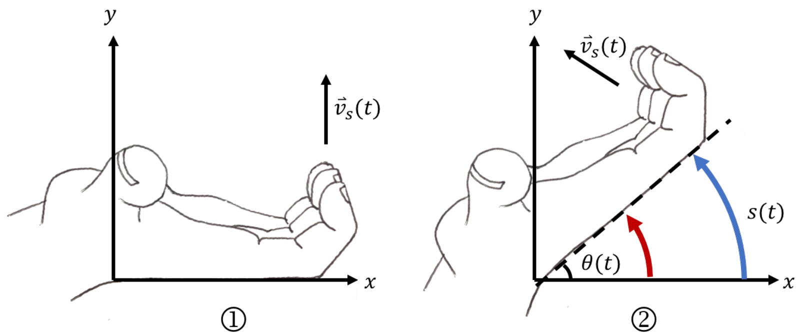


Figure 2.10: How to use your hand to better understand circular motion

Let's say you want to describe the location of your fingers in (2). Start by putting your hand in position (1). This is the position where $\theta = 0$ and $s = 0$. Imagine that your wrist (or your thumb, whichever you prefer) is fixed at the origin. If you keep your fingers perpendicular to your hand, they will always point in the positive s direction.

Imagine that you have a blue glob of paint on the back of your pinky. Rotate your hand until it is in position (2). The length of the curve that the paint makes is the value of s. The angle between the back of your hand and the positive x-axis is θ. Now, imagine that there is a red glob of paint at your palm. It takes the same amount of time for your palm to get from position (1) to position (2) as it does for your fingers. Since they both go through the same angle θ in the same amount of time, the **angular velocity**, ω must be the same for both. However, the blue line left by your fingers will be much longer than the red line left by your palm. Your fingers travelled a greater distance than your palm in the same amount of time, so they must have a greater **linear velocity**, v_s. The further you are from your thumb, the greater the linear velocity will be, which we know from the formula $v_s = R\omega$.

If you kept rotating your hand around the circle, you would see that your fingers always point in the same direction as your linear velocity. This means that if you are using cartesian coordinates, the direction of your linear velocity is always changing.

There are a couple of limitations to this trick. Remember that this only works for circular motion (the radius R must be constant) and that if you are moving in the negative s direction, your fingers will point antiparallel to the linear velocity.

2.5 Summary

Key Takeaways

When the motion of an object is in more than one dimension, we describe the position of the object using a vector, $\vec r$. We can treat motion in two dimensions as having two independent components, a motion along x and a separate motion along y:

$$\vec r(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = x(t)\hat x + y(t)\hat y$$

This can be extended for motion in N dimensions.

The instantaneous velocity vector and the acceleration vector are given by:

$$\begin{aligned}\vec v(t) &= \frac{d}{dt}\vec r(t)\\ \vec a(t) &= \frac{d}{dt}\vec v(t)\end{aligned}$$

If an object has position $\vec r^A$ as measured in a frame of reference xy that is moving at constant speed $\vec v'^B$ as measured in a second frame of reference $x'y'$, then in the $x'y'$ reference frame:

$$\begin{aligned}\vec r'^A(t) &= \vec v'^B t + \vec r^A(t)\\ \vec v'^A(t) &= \vec v'^B + \vec v^A(t)\\ \vec a'^A(t) &= \vec a^A(t)\end{aligned}$$

An acceleration can change the magnitude and/or the direction of the velocity vector.

1. The component of the acceleration that is parallel to the velocity vector changes the magnitude of the velocity.
2. The component of the acceleration that is perpendicular to the velocity vector changes the direction of the velocity.

The acceleration vector for motion in two dimensions can be written as:

$$\vec a = \frac{dv}{dt}\hat v(t) + v(t)\frac{d\hat v}{dt}$$

If the speed is constant, then the acceleration vector is given by:

$$\vec a = \frac{dv_x}{dt}\left(\hat x - \frac{v_x(t)}{v_y(t)}\hat y\right)$$

If the position of an object moving in a circle of radius R is described by its position along the curved axis s, then its position along the circle can be described using an angle, θ, in radians:

$$\theta(t) = \frac{s(t)}{R}$$

For an object moving along a circle, we can write its position vector, $\vec{r}(t)$, as:

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = R \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix}$$

The angular velocity, ω, is the rate of change of the angle. The angular acceleration, α, is the rate of change of the angular velocity:

$$\omega = \frac{d\theta}{dt}$$
$$\alpha = \frac{d\omega}{dt}$$

The linear kinematic quantities can be found from the angular quantities:

$$s = R\theta$$
$$v_s = R\omega$$
$$a_s = R\alpha$$

For circular motion, the velocity vector is tangent to the circle and the perpendicular component of the acceleration is called the centripetal acceleration. The centripetal acceleration points towards the centre of the circle and has a magnitude of:

$$a_c(t) = \omega^2(t)R = \frac{v^2(t)}{R}$$

The centripetal acceleration vector can be written as:

$$\vec{a}_\perp(t) = \omega^2 R[-\cos(\theta)\hat{x} - \sin(\theta)\hat{y}]$$

Uniform circular is when an object is going around a circle with a constant speed. The period, T, is the time that it takes for the object to complete one revolution of the circle. The frequency, f, is the inverse of the period, and can be thought of as the number of revolutions completed per second:

$$T = \frac{2\pi}{\omega}$$
$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Important Equations

Motion in 2D:

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = x(t)\hat{x} + y(t)\hat{y}$$

$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t)$$

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t)$$

Relative Motion 2D:

$$\vec{r}'^{A}(t) = \vec{v}'^{B}t + \vec{r}^{A}(t)$$

$$\vec{v}'^{A}(t) = \vec{v}'^{B} + \vec{v}^{A}(t)$$

$$\vec{a}'^{A}(t) = \vec{a}^{A}(t)$$

Acceleration Vector 2D:

$$\vec{a} = \frac{dv}{dt}\hat{v}(t) + v(t)\frac{d\hat{v}}{dt}$$

$$\text{(constant speed)} \quad \vec{a} = \frac{dv_x}{dt}\left(\hat{x} - \frac{v_x(t)}{v_y(t)}\hat{y}\right)$$

Circular Motion:

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = R\begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix}$$

$$\omega = \frac{d\theta}{dt}$$

$$\alpha = \frac{d\omega}{dt}$$

$$s = R\theta$$

$$v_s = R\omega$$

$$a_s = R\alpha$$

$$a_c(t) = \omega^2(t)R = \frac{v^2(t)}{R}$$

$$\vec{a}_\perp(t) = \omega^2 R[-\cos(\theta)\hat{x} - \sin(\theta)\hat{y}]$$

$$T = \frac{2\pi}{\omega}$$

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

2.6 Thinking about the material

2.6.1 Reflect and research

1. Why do we observe the sun to be “changing directions” in the sky throughout the year?
2. It was once believed that there was an absolute space and time called the “luminiferous aether” which all objects moved relative to. What was the name of the experiment that disproved this belief?
3. Find the centripetal acceleration of the Earth around the Sun.
4. Your friend William believes that the earth is flat. Name two relative motion phenomena that would falsify William’s belief.

2.6.2 To try at home

Activity 2-1: All you need for this exercise is two identical objects that you can throw, one other person, and something tall that you can drop the objects from. Both people should hold the objects at the exact same height. One person will drop their object, and the other person will give their object an initial horizontal velocity by throwing it straight forward (make sure they don’t throw it up or down!). Which object do you expect to land first? Why? If you do not see the results you expect, think of reasons why that might be.

2.6.3 To try in the lab

2.7 Sample problems and solutions

2.7.1 Problems

Problem 2-1: Ethan is jumping hurdles. He gets a running start, moving with a velocity of 3 m/s [E], and will not slow down before jumping. The hurdle is 0.5 m high and the maximum speed he can have when he leaves the ground is 5 m/s. (You can assume Ethan is a point particle, and ignore air resistance).

a) How close can he get to the hurdle before he has to jump?
b) What maximum height does he reach?

Problem 2-2: A cowboy swings a lasso above his head. The lasso moves at a constant speed in a circle of radius 1.5 m in the horizontal plane. A hawk flies toward the lasso at 50 km/h. The hawk sees the end of the lasso moving at 60 km/h when the lasso is directly in front of it (see Figure **??**). In the reference frame of the cowboy ...

a) How long does it take for the lasso to complete one revolution?
b) What is the centripetal acceleration of the end of the lasso?
c) What is the angular acceleration of the lasso?

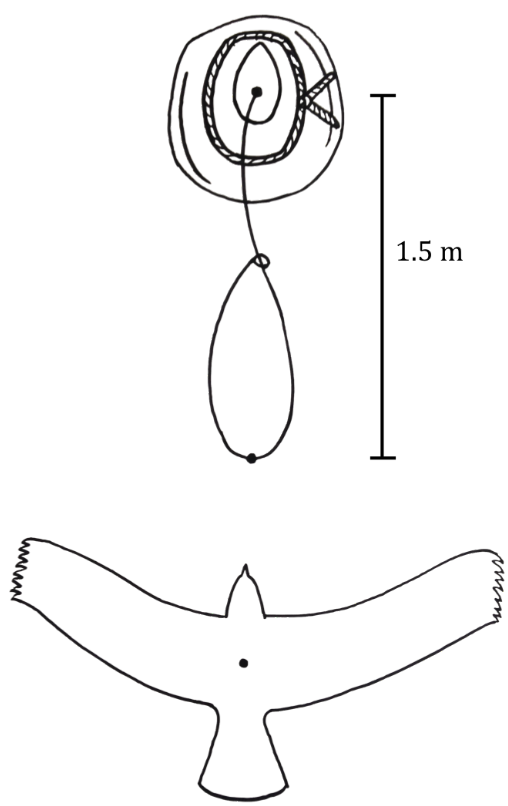


Figure 2.11: The problem as viewed from above. This diagram depicts the moment that the end of the lasso passes in front of the hawk.

2.7.2 Solutions

Solution to problem ??: This is a question about motion in two dimensions. Our approach will be to consider the x and y components of the motion separately. We start by drawing a diagram and setting up our coordinate axes. We will set "up" to be in the positive y direction and East to be in the positive x direction, as in Figure **??**. We set the origin to be where Ethan leaves the ground.

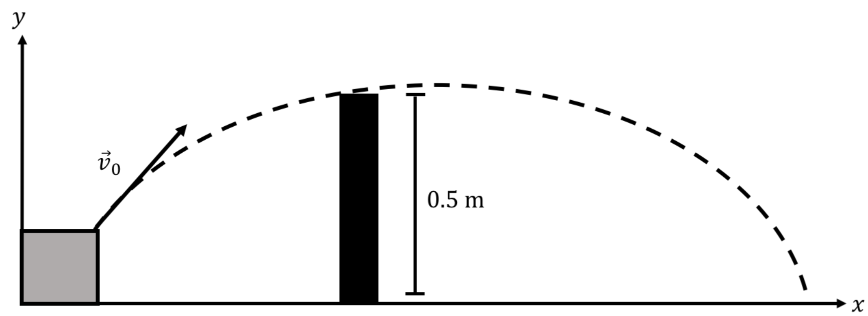


Figure 2.12: Ethan wants to clear a 0.5 m *hurdle and has an initial velocity* $\vec{v}_0$ *with* x *and* y *components.*

a) We are given that Ethan's maximum initial speed is 5 m/s and that the horizontal component of his velocity must be at least 3 m/s. The first question we need to answer is whether we want the horizontal component to be equal to or greater than 3 m/s. This question is asking us to minimize the horizontal distance Ethan must travel to reach a height of 0.5 m. Since the magnitude of our initial velocity cannot exceed 5 m/s, we want as little of our initial velocity as possible to be in the x direction, so we want $v_x = 3\,\text{m/s}$. To find the y component of the initial velocity, we use the Pythagorean theorem:

$$\begin{aligned} v_x^2 + v_{0y}^2 &= v_0^2 \\ v_{0y} &= \sqrt{v_0^2 - v_x^2} \\ v_{0y} &= \sqrt{(5\,\text{m/s})^2 - (3\,\text{m/s})^2} \\ v_{0y} &= 4\,\text{m/s} \end{aligned}$$

We set Ethan's takeoff point to be at the origin, so we know that $x_0 = 0$ and $y_0 = 0$. Once Ethan is in the air, there will be no acceleration in the x direction, and the only acceleration in the y direction will be due to gravity. So, Ethan's position at any time t can be described by the following equations:

$$\begin{aligned} x &= v_x t \\ y &= v_{0y} t - \frac{1}{2} g t^2 \end{aligned}$$

where g is the acceleration due to gravity, $g = 9.8\,\text{m/s}^2$.

We are interested in the value of x when the vertical displacement y is equal to the height of the hurdle. So we will find the value of t when $y = 0.5\,\text{m}$ and find the value of x at this time.

First, rearrange the equation for y and solve the quadratic:

$$\begin{aligned}0 &= -\frac{1}{2}gt^2 + v_{0y}t - y\\ 0 &= \frac{1}{2}(-9.8\,\text{m/s}^2)t^2 + (4\,\text{m/s})t - 0.5\,\text{m}\\ t &= 0.15\,\text{s}, \quad 0.66\,\text{s}\end{aligned}$$

We want to know when Ethan reaches 0.5 m for the first time, so $t = 0.15\,\text{s}$. All that's left is to find the horizontal displacement at this time:

$$\begin{aligned}x &= v_x t\\ &= (3\,\text{m/s})(0.15\,\text{s})\\ &= 0.45\,\text{m}\end{aligned}$$

$\therefore$ he can get as close as 0.45 m from the hurdle before he has to jump, if his initial horizontal velocity is 3 m/s.

b) Ethan's motion follows a parabolic shape. At the maximum height, Ethan's vertical velocity is equal to zero. Our approach will be to solve for the value of y when $v_y = 0$. Our known values are,

$$\begin{aligned}v_{0y} &= 4\,\text{m/s}\\ v_y &= 0\,\text{m/s}\\ g &= 9.8\,\text{m/s}^2\end{aligned}$$

The easiest way to solve this problem is to use the formula,

$$v_y^2 = v_{0y}^2 - 2gy$$

$$\text{So,} \qquad y = \frac{v_y^2 - v_{0y}^2}{(-2g)}$$

Substituting our values for v_y, v_{0y}, and g, we get:

$$\begin{aligned}y_{max} &= \frac{(-4\,\text{m/s})^2}{(2)(-9.8\,\text{m/s}^2)}\\ y_{max} &= 0.82\,\text{m}\end{aligned}$$

$\therefore$ Ethan reaches a maximum height of 0.82 m.

(Note: there is more than one way to solve for y_{max})

Solution to problem ??:

a) Our goal is the find the period of the lasso's motion. To do this, we can use the formula:

$$T = \frac{2\pi}{\omega}$$

for which we need the angular velocity, ω. We know the radius of the lasso, so if we find the linear velocity of the end point of the lasso, we can find the angular velocity using:

$$\omega = \frac{v}{R}$$

We start by using what we know about relative motion to find the linear velocity of the lasso in the cowboy's reference frame. First, we need to set up our coordinate systems. We assign the xy coordinate system to the hawk's reference frame and we assign the $x'y'$ system to the cowboy's reference frame. The solution will be simplest if we align the coordinate systems so that positive y and positive y' are in the same direction, as in Figure **??**. When we are talking about the velocity of the hawk, we will denote it with the superscript "H", and when we are talking about the lasso, we will use "L".

We want the velocity of the **lasso** in the **cowboy's reference frame**, so we want v'^L. To find this, we start with the velocity of the lasso in the hawk's reference frame,v^L and then take into account that the hawk is moving relative to the cowboy. We do this by adding the velocity of the hawk in the cowboy's reference frame, v'^H, to v^L. So, our equation is,

$$v^L + v'^H = v'^L$$

We are adding velocities, which have both a magnitude and a direction. However, we were not given any directions in the problem, so we describe the directions with respect to our coordinate system. The way we have set up our axes, the velocity of the hawk in the cowboy's reference frame is simply $50\,\text{km/h}$ in the positive y' direction.

Now here's the key to solving this problem: We don't know the speed of the lasso in the cowboy's reference frame, but we do know something about its direction. Since the motion of the lasso is circular, its velocity must be tangent to the circle. This means that when the lasso is directly in front of the hawk, its velocity must be in either the $+x'$ or $-x'$ direction. In this case, we can just choose one, so we will choose the $+x'$ direction.

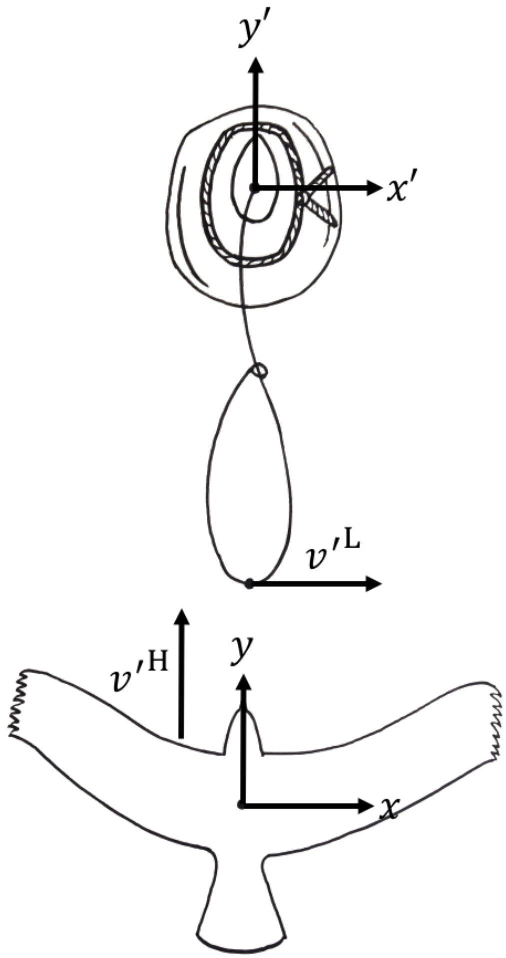


Figure 2.13: The two coordinate systems are aligned so that positive y' and positive y are in the same direction. The velocity vectors of the hawk and the lasso in the reference frame of the cowboy are shown.

The velocity vectors v'^H and v'^L are shown in Figure **??**. Remember that when we add two vectors they must be lined up so that the "head" of one touches the "tail" of the other, so there can only be one direction for v^L, as shown in Figure **??**.

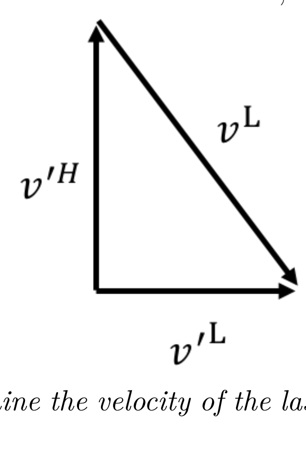


Figure 2.14: Vector addition to determine the velocity of the lasso in the cowboy's reference frame.

This is a right angle triangle, so we use the Pythagorean theorem so solve for v'^L:

$$\begin{aligned} v'^{L^2} + v'^{H^2} &= v^{L^2} \\ &= \sqrt{v^{L^2} - v'^{H^2}} \\ &= \sqrt{(60\,\text{km/h})^2 - (50\,\text{km/h})^2} \\ v'^L &= 33\,\text{km/h} \end{aligned}$$

The linear velocity of the end of the lasso at this moment is 33 km/h in the positive x direction. To find the angular velocity, first convert the linear velocity from km/h to m/s:

$$\frac{33\,\text{km}}{1\,\text{h}} \times \frac{1000\,\text{m}}{1\,\text{km}} \times \frac{1\,\text{h}}{3600\,\text{s}} = 9.2\,\text{m/s}$$

Now we can substitute $\omega = \frac{v}{R}$ into $T = \frac{2\pi}{\omega}$ and solve for T:

$$\begin{aligned} T &= 2\pi\frac{R}{v} \\ &= 2\pi\frac{1.5\,\text{m}}{9.2\,\text{m/s}} \\ &= 2\pi\frac{1.5\,\text{m}}{9.2\,\text{m/s}} \\ T &= 1.0\,\text{s} \end{aligned}$$

$\therefore$ it takes 1.0 s for the lasso to complete one revolution.

b) The motion is uniform circular motion, so it has a centripetal acceleration given by

$$a_c(t) = \frac{v^2(t)}{R}$$

To find the centripetal acceleration of the end of the lasso, we just substitute in our values for v and R.

$$\begin{aligned} a_c(t) &= \frac{(9.2\,\text{m/s})^2}{1.5\,\text{m}} \\ a_c(t) &= 56\,\text{m/s}^2 \end{aligned}$$

$\therefore$ the centripetal acceleration of the end of the lasso is $56\,\text{m/s}^2$ towards the centre of the circle.

c) You may be tempted to divide the centripetal acceleration by R to find the angular acceleration α. However, the angular acceleration is the rate of change of the angular velocity. The angular velocity is constant, so the angular acceleration is zero. (Remember that in the equation $a_s = R\alpha$, a_s refers to the component of acceleration that is **parallel** to the velocity.)