The Art of Modelling: Introduction to Physics

Ryan D. Martin

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1 Potential Energy and Conservation of Energy

In this chapter, we continue to develop the concept of energy in order to introduce a different formulation for Classical Physics that does not use forces. Although we will see that we can describe many phenomena using energy instead of forces, this method is completely equivalent to using Newton's Three Laws, and as such, can be derived from Newton's formulation as we will see. Because energy is a scalar quantity, for many problems, it leads to models that are much easier to develop mathematically than if one had used forces. The chapter will conclude with a presentation of the more modern approach, using "Lagrangian Mechanics", that is currently preferred in physics and forms the basis for extending our description of physics to the microscopic world (e.g. quantum mechanics).

Learning Objectives

- something to learn

Think About It

A question

A) a choice
B) another choice

1.1 Conservative forces

In Chapter **??**, we introduced the concept of work, W, done by a force, $\vec F(\vec r)$, acting on a object as it moves along a path from position A to position B:

$$W = \int_A^B \vec F(\vec r) \cdot d\vec l \tag{1.1}$$

where $\vec F(\vec r)$ is a force vector that, in general, is different in different positions in space ($\vec r$). We can also say that $\vec F$ depends on position by writing $\vec F(\vec r) = \vec F(x, y, z)$, since the position vector $\vec r$, is simply the vector $\vec r = x\hat x + y\hat y + z\hat z$.

The above integral is in general difficult to evaluate, as it depends on the specific path over which the object moved. In Example **??** of Chapter **??**, we calculated the work done by friction on a crate that was slid across the floor along two different paths and indeed found that the work depended on the path that was taken. In Example **??** of the same chapter, we saw that the work done by the force of gravity when moving a box along two different paths did not depend on the path chosen[1].

We call "conservative forces" those force for which the work done only depends on the initial and final positions and not on the path taken. "Non-conservative" forces are those for which the work done does depend on the path taken. The force of gravity is an example of a conservative force, whereas friction is an example of a non-conservative force.

This means that the work done by a conservative force on a "closed path" is zero; that is, the work done by a force on a object that moves along a path that brings the object back to its starting position is zero. Indeed, since the work done by a conservative force only depends on the location of the initial and final positions, and not the path taken between them, the work has to be zero if the object ends back where it started (as a possible path is for the object to not move at all).

Consider the work done by gravity in raising and lowering an object back to its starting position along a vertical path, as depicted in Figure 1.1.

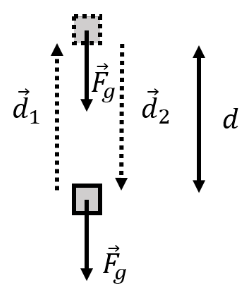

Figure 1.1: An object that has moved up and back down.

The total work done by gravity on this particular closed path is easily shown to be zero, as the work can be broken up into the negative work done as the object moves up (displacement vector $\vec{d}_1$) and the positive work done on the downwards path (displacement vector $\vec{d}_2$):

$$W^{tot} = \vec{F}_g \cdot \vec{d}_1 + \vec{F}_g \cdot \vec{d}_2 = -mgd + mgd = 0$$

In order to write the path integral of the force over a closed path, we introduce a new notation to indicate that the starting and ending position are the same (to avoid writing the integral sign with both limits being the same):

$$\int_A^A \vec{F}(\vec{r}) \cdot d\vec{l} = \oint \vec{F}(\vec{r}) \cdot d\vec{l}$$

[1] At least for those two paths that we tried in the example.

The condition for a force to be conservative is thus:

$$\boxed{\oint \vec{F}(\vec{r}) \cdot d\vec{l} = 0} \tag{1.2}$$

since this means that the work done over a closed path is zero. The condition on the particular form for this integral to be zero can be found by Stokes' Theorem:

$$\oint \vec{F}(\vec{r}) \cdot d\vec{l} = \int_S \left[\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z} \right] \cdot d\vec{A}$$

where the integral on the right is called a "surface integral" over the surface, S, enclosed by the closed path over which the work is being calculated. Do not worry, it is way beyond the scope of this text to understand this integral or Stokes' Theorem in detail! It is however useful in that it gives us the following conditions on the components of a force for that force to be conservative (by requiring the terms in parentheses to be zero):

$$\begin{aligned} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} &= 0 \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} &= 0 \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} &= 0 \end{aligned} \tag{1.3}$$

In general:

1. A force can be conservative if it only depends on position in space, and not speed, time, or any other quantity.
2. A force is conservative if it is constant in magnitude and direction.

TODO: Checkpoint: Is the force exerted by a person when pushing a crate conservative?

Example 1-1

Is the force of gravity on an object of mass m, near the surface of the Earth, given by:

$$\vec{F}(x, y, z) = 0\hat{x} + 0\hat{y} - mg\hat{z}$$

conservative? Note that we have defined the z axis to be vertical and positive upwards.

Solution

The force is expected to be conservative since it is constant in magnitude and direction.

We can verify this explicitly using the conditions in Equation 1.3:

$$\begin{aligned}
\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} &= \frac{\partial}{\partial y}(-mg) - 0 &&= 0\\
\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} &= 0 - \frac{\partial}{\partial x}(-mg) &&= 0\\
\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} &= 0 - 0 &&= 0
\end{aligned}$$

and the force is indeed conservative since all three conditions are zero.

Example 1-2

Is the force given by:

$$\vec{F}(x,y,z) = \frac{-k}{r^3}\vec{r} = \frac{-kx}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{x} + \frac{-ky}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{y} + \frac{-kz}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{z}$$

conservative?

Solution

Since the force only depends on position, it could be conservative, so we must check using the conditions from Equation 1.3:

$$\begin{aligned}
\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} &= \frac{\partial}{\partial y}\left(\frac{-kz}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) - \frac{\partial}{\partial z}\left(\frac{-ky}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)\\
&= \frac{3kz(2y)}{2(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{3ky(2z)}{2(x^2+y^2+z^2)^{\frac{5}{2}}} = 0\\
\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} &= \frac{\partial}{\partial z}\left(\frac{-kx}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) - \frac{\partial}{\partial x}\left(\frac{-kz}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)\\
&= \frac{3kx(2z)}{2(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{3kz(2x)}{2(x^2+y^2+z^2)^{\frac{5}{2}}} = 0\\
\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} &= \frac{\partial}{\partial x}\left(\frac{-ky}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) - \frac{\partial}{\partial y}\left(\frac{-kx}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)\\
&= \frac{3ky(2x)}{2(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{3kx(2y)}{2(x^2+y^2+z^2)^{\frac{5}{2}}} = 0
\end{aligned}$$

where we used the Chain Rule to take the derivatives. Since all of the conditions are zero, the force is conservative. As we will see, the force represented here is similar mathematically to both the force that Newton introduced in his Universal Theory of Gravity, and the force introduced by Coulomb as the electric force, which are both conservative.

TODO: Make taking one of the above partial derivatives a problem in the Math section of the question library!

1.2 Potential energy

In this section, we introduce the concept of "potential energy". Potential energy is a scalar function of position that can be defined for any conservative force in a way to make it easy to calculate the work done by that force over any path. Since the work done by a conservative force in going from position A to position B does not depend on the particular path taken, but only on the end points, we can write the work done by a conservative force in terms of a "potential energy function", $U(\vec r)$, that can be evaluated at the end points:

$$\boxed{-W = -\int_A^B \vec F(\vec r)\cdot d\vec l = U(\vec r_B) - U(\vec r_A) = \Delta U} \tag{1.4}$$

where we have have chosen to define the function $U(\vec r)$ so that it relates to the **negative** of the work done for reasons that will be apparent in the next section. Figure 1.2 shows an example of an arbitrary path between two points A and B in two dimensions for which one could calculate the work done by a conservative force.

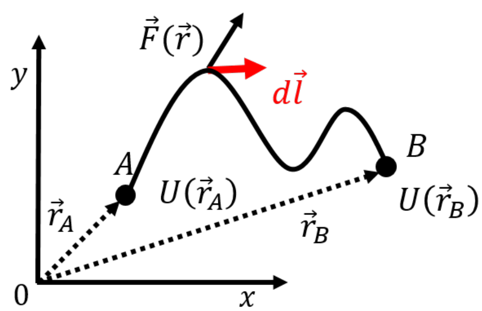

Figure 1.2: Illustration of calculating the work of a conservative function along an arbitrary path.

Once we know the function for the potential energy, $U(\vec r)$, we can calculate the work done by the associated force along any path. In order to determine the function, $U(\vec r)$, we can calculate the work that is done along a path over which the integral for work is easy (usually, a straight line).

For example, near the surface of the Earth, the force of gravity on an object of mass, m, is given by:

$$\vec F_g = -mg\hat z$$

where we have defined the z axis to be vertical and positive upwards. We already showed in Example 1-1 that this force is conservative and that we can thus calculate a potential energy function. To do so, we can calculate the work done by the force of gravity over a straight vertical path, from position A to position B, as shown in Figure 1.3.

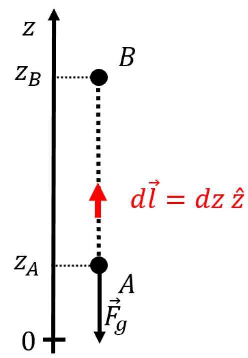

Figure 1.3: A vertical path for calculating the work done by gravity.

The work done by gravity from position A to position B is:

$$\begin{aligned}
W &= \int_A^B \vec{F}(\vec{r}) \cdot d\vec{l} \\
&= \int_{z_A}^{z_B} (-mg\hat{z}) \cdot (dz\hat{z}) \\
&= -mg \int_{z_A}^{z_B} dz \\
&= -mg(z_B - z_A)
\end{aligned}$$

By inspection, we can now identify the functional form for the potential energy function, $U(\vec{r})$. We require that:

$$-W = U(\vec{r}_B) - U(\vec{r}_A) = U(z_B) - U(z_A)$$

where we replaced the position vector $\vec{r}$, by the z coordinate, since this is a one dimensional situation. Therefore:

$$\begin{aligned}
-W = mg(z_B - z_A) &= U(z_B) - U(z_A) \\
\therefore U(z) &= mgz + C
\end{aligned}$$

and we have found that, for the force of gravity near the surface of the Earth, one can define a potential energy function, $U(z)mgz + C$.

It is important to note that, since it is only the **difference** in potential energy that matters when calculating the work done, the potential energy function can have an arbitrary constant, C, added to it. Thus, **the value of the potential energy function is meaningless, and only differences in potential energy are meaningful and related to the work done on an object**. In other words, it does not matter where the potential energy is equal to zero, and by choosing C, we can therefore choose a convenient location where the potential function is zero.

Example 1-3

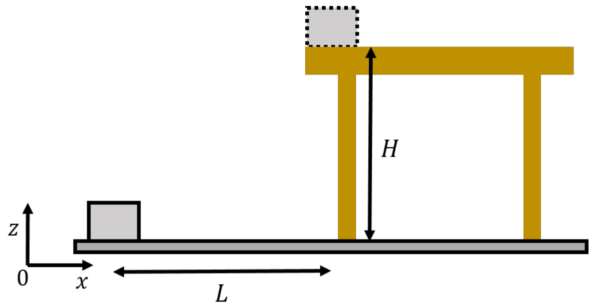

Figure 1.4: A box moved from the ground up onto a table.

Calculate the work done **by gravity** when a box of mass, m, is moved from the ground up onto a table that is a distance L away horizontally and H vertically, as illustrated in Figure 1.4. How much work must be done by a person moving the box?

Solution

Since gravity is a conservative force, we can use the potential energy function $U(z) = mgz + C$ to calculate the work done by gravity when the box is moved. The work done by gravity will only depend on the change in height, H, as the potential energy function only depends on the z coordinate of an object. We can choose the origin of our coordinate system to be the ground and choose the constant $C = -0$, so that the potential energy function at the starting position of the box is:

$$U(z_A = 0) = mgz + C = 0$$

The potential energy function when the box is on the table, with $z = H$, is given by:

$$U(z_B = H) = mgz + C = mgH$$

The change in potential energy, $\Delta U = U(z_B) - U(z_A)$ is equal to the negative of the work done by gravity. The work done by gravity, W_g, is thus:

$$\begin{aligned}
&-W_g = U(z_B) - U(z_A) = mgH - 0 \\
&\therefore W = -mgH
\end{aligned}$$

which is the same that we found in Example **??** of Chapter **??**. The work done by gravity is negative, as we found previously, which makes sense since gravity has a component in the opposite direction of motion.

The work done by a person, W_p, to move the box can easily be found by considering the net work done on the box. While the box is moving, only the person and gravity are exerting forces on the box, so those are the only two forces performing work. Since the box starts and ends at rest, the net work done on the box must be zero (no change in kinetic energy):

$$W^{net} = 0 = W_g + W_p$$
$$\therefore W_p = -W_g = mgH$$

Discussion: We find that the person had to do positive work, which makes sense, since they had to exert a force with a component in the direction of motion (upwards). It is also interesting to note that it does not matter if the person exerted a constant force or whether they varied the force that they exerted on the box as they moved it: the amount of work done by the person is fixed to be the negative of the work done by gravity.

Example 1-4

The force exerted by a spring that is extended or compressed by a distance, x, is given by Hookes' Law:

$$\vec{F}(x) = -kx\hat{x}$$

where the x axis is defined to be colinear with the spring with and the origin is located at the rest position of the spring. Show that: (a) the force exerted by the spring onto an object is conservative, and (b), determine the corresponding potential energy function.

Solution

Since the force depends on position, it could be conservative, which we can check with the conditions from Equation 1.3:

$$\begin{aligned}
\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} &= 0 - 0 && = 0 \\
\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} &= \frac{\partial}{\partial z}(-kx)) - 0 && = 0 \\
\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} &= 0 - \frac{\partial}{\partial y}(-kx)) && = 0
\end{aligned}$$

and the force is indeed conservative. To determine the potential energy function, let us calculate the work done by the spring from position x_A to position x_B:

$$\begin{aligned} W &= \int_A^B \vec{F}(\vec{r}) \cdot d\vec{l} \\ &= \int_A^B (-kx\hat{x}) \cdot dx\hat{x} \\ &= \int_{x_A}^{x_B} (-kx)dx = \left[-\frac{1}{2}kx^2\right]_{x_A}^{x_B} \\ &= -\left(\frac{1}{2}kx_B^2 - \frac{1}{2}kx_A^2\right) \end{aligned}$$

Again, comparing with:

$$-W = U(\vec{r}_B) - U(\vec{r}_A) = U(x_B) - U(x_A)$$

We can identify the potential energy for a spring:

$$U(x) = \frac{1}{2}kx^2 + C$$

where, in general, the constant C can take any value. If we choose $C = 0$, then the potential energy is zero when the spring is at rest, although it is not important what choice is made.

1.2.1 Recovering the force from potential energy

Given a (scalar) potential energy function, $U(\vec{r})$, it is possible to determine the (vector) force that is associated with it. Take for example the potential energy from a spring (Example 1-4):

$$U(x) = \frac{1}{2}kx^2 + C$$

In one dimension, this is simply the negative of the anti-derivative of the x component of the spring force:

$$\begin{aligned} F(x) &= -kx \\ U(x) &= -\int F(x)dx = \int (kx)dx = \frac{1}{2}kx^2 + C \\ \therefore F(x) &= -\frac{d}{dx}U(x) \end{aligned}$$

Thus, the force can be obtained by taking the derivative with respect to position of the negative of potential energy function.

In three dimensions, the situation is similar, although the potential energy function (and the components of the force vector) will generally depend on all three position coordinates, x, y, and z. In three dimensions, the force is given by the gradient of the negative of potential energy function[2]:

$$
\begin{aligned}
\vec{F}(\vec{r}) &= -\vec{\nabla}U(\vec{r}) = -\vec{\nabla}U(x,y,z)\\
\therefore F_x(x,y,z) &= -\frac{\partial}{\partial x}U(x,y,z)\\
\therefore F_y(x,y,z) &= -\frac{\partial}{\partial y}U(x,y,z)\\
\therefore F_z(x,y,z) &= -\frac{\partial}{\partial z}U(x,y,z) \qquad (1.5)
\end{aligned}
$$

1.3 Mechanical energy and conservation of energy

Recall the Work-Energy Theorem, relating the net work done on an object to its change in kinetic energy along a path:

$$W^{net} = \Delta K = K_B - K_A$$

where K_B (K_A) is the final (initial) kinetic energy of the object along some path from A to B over which the net work was done. Generally, the net work done is the sum of the work done by conservative forces, W^C, and the work done by none conservative forces, W^{NC}:

$$W^{net} = W^C + W^{NC}$$

The work done by conservative forces can be expressed in terms of changes in potential energy functions. For example, suppose that two conservative forces, $\vec{F}_1$ and $\vec{F}_2$, are exerted on the object. The work done by those two forces in terms of the change in potential energy is:

$$
\begin{aligned}
W_1 &= -\Delta U_1\\
W_2 &= -\Delta U_2
\end{aligned}
$$

where U_1 and U_2 are the change in potential energy associated with forces $\vec{F}_1$ and $\vec{F}_2$, respectively. We can re-arrange the Work-Energy Theorem as follows[3]:

$$
\begin{aligned}
W^{net} = W^C + W^{NC} &= -\Delta U_1 - \Delta U_2 + W^{NC} = \Delta K\\
\therefore W^{NC} &= \Delta U_1 + \Delta U_2 + \Delta K
\end{aligned}
$$

That is, the work done by non-conservative forces is equal to the sum of the change in potential and kinetic energies. In general, if we write ΔU as the total change in potential energy of the object (the sum of the changes in potential energies associated with each conservative force), we can write this in a more general form:

$$\boxed{W^{NC} = \Delta U + \Delta K} \qquad (1.6)$$

[2]As you may recall from Appendix **??**, the gradient is a vector that points towards the direction of maximal increase in a multi-variate function.

[3]This is why we defined potential energy as negative of the work; it becomes a positive term when we move it to the same side of the equation as the kinetic energy!

This fundamental result is what we call the "conservation of mechanical energy". In particular, note that if there are no non-conservative forces doing work on the object:

$$\boxed{\Delta K + \Delta U = 0} \quad \text{if no non-conservative forces} \tag{1.7}$$
$$-\Delta U = \Delta K$$

That is, the sume of the change in potential and kinetic energies of the object is always zero. If the potential energy of the object increases, then the kinetic energy of the object must decrease by the same amount, if there are no non-conservative forces doing work on the object.

We can introduce the "mechanical energy", E, of an object as the sum of potential and kinetic energies of the object:

$$\boxed{E = U + K} \tag{1.8}$$

If the object started at position A, with potential energy U_A and kinetic energy K_A, and ended up at position B with potential energy U_B and kinetic energy K_B, then we can write the mechanical energy at both positions, and its change as:

$$\begin{aligned} E_A &= U_A + K_A \\ E_B &= U_B + K_B \\ \therefore \Delta E &= E_B - E_A = \Delta U + \Delta K \end{aligned}$$

Thus, the change in mechanical energy of the object is equal to the work done by none-conservative forces:

$$W^{NC} = \Delta U + \Delta K = \Delta E$$

and if there is no work done by non-conservative forces on the object, then the mechanical energy of the object does not change:

$$\begin{aligned} \Delta E &= 0 \quad \text{if no non-conservative forces} \\ \therefore E &= \text{constant} \end{aligned}$$

The introduction of mechanical energy gives us a completely different way to think about mechanics. We can now think of an object as having "energy" (potential or kinetic), and we can think of forces as changing the energy of the object.

Example 1-5

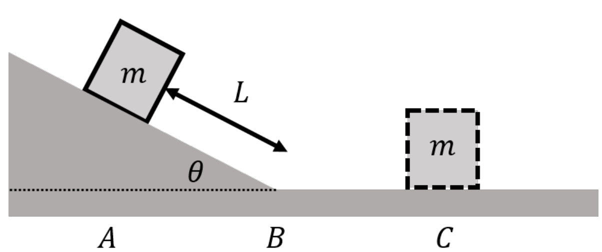

Figure 1.5: A block slides down an incline before sliding on a flat surface and stopping.

A block of mass m is placed at rest on an incline that makes an angle θ with respect to the horizontal, as shown in Figure 1.5. The block is nudged slightly so that the force of static friction is overcome and the block starts to accelerate down the incline. At the bottom of the incline, the block slides on a horizontal surface. The coefficient of kinetic friction between the block and the incline is μ_{k1}, and the coefficient of kinetic friction between the block and horizontal surface is μ_{k2}. If one assumes that the block started at rest a distance L from the horizontal surface, how far along the horizontal surface will the block slide before stopping?

Solution

This is the same problem that we solved in Chapter ref:applyingnewtonslaws Example **??**. In that case, we solved for the acceleration of the block using Newton's Second Law and then used kinematics to find how far the block goes. We can solve this problem in a much easier way using conservation of energy.

It is still a good idea to think about what forces are applied on the object in order to determine if there are non-conservative forces doing work. In this case, the forces on the block are:

1. The normal force, which does no work, as it is always perpendicular to the motion.
2. Weight, which does work when the height of the object changes, which we can model with a potential energy function.
3. Friction, which is a non-conservative force, whose work we must determine.

Let us divide the motion into two segments: (1) a segment along the incline (positions A to B in Figure 1.5, where gravitational potential energy changes, and (2), the horizontal segment from positions B to position C on the figure. We can then apply conservation of energy for each segment.

Starting with the first segment, we can choose the gravitational potential energy to be zero when the block is at the bottom of the incline. The block starts at a height $h = L\sin\theta$ above the bottom of the incline. The gravitational potential energy for the beginning and end of the first segment are thus:

$$\begin{aligned}
U_A &= mgL\sin\theta \\
U_B &= 0 \\
\Delta U_1 &= U_B - U_A = -mgH
\end{aligned}$$

Since the block starts at rest, its kinetic energy is zero at position A, and if the speed of the box is v_B at position B, we can write its kinetic energy at both positions as:

$$\begin{aligned}
K_A &= 0 \\
K_B &= \frac{1}{2}mv_B^2 \\
\Delta K_1 &= \frac{1}{2}mv_B^2
\end{aligned}$$

The mechanical energy of the object at positions A and B is thus:

$$\begin{aligned}
E_A &= U_A + K_A = mgL\sin\theta \\
E_B &= U_B + K_B = \frac{1}{2}mv_B^2 \\
\Delta E &= E_B - E_A = \frac{1}{2}mv_B^2 - mgL\sin\theta
\end{aligned}$$

Finally, since we have a non-conservative force, the force of kinetic friction acting on the first segment, we need to calculate the work done by that force. We found in Example **??** that the force of friction had magnitude $f_k = \mu_k N = \mu_k mg\cos\theta$. Since the force of friction is anti-parallel to the displacement vector of length L down the incline, the work done by friction is:

$$W^{NC} = W_f = -f_k L = -\mu_k mg\cos\theta L$$

Applying conservation of energy along the first segment

1.4 Potential energy diagrams and equilibria

1.5 The Lagrangian formulation of classical physics

1.6 Summary

Key Takeaways

Something that was learned

Important Equations

This is an important equation

$$E = mc^2$$

1.7 Thinking about the material

1.7.1 Reflect and research

1. Something to research more.

1.7.2 To try at home

Activity 1-1: Try doing this

1.7.3 To try in the lab

1.8 Sample problems and solutions

1.8.1 Problems

Problem 1-1: A question

a) How close can he get to the hurdle before he has to jump?
b) What maximum height does he reach?

1.8.2 Solutions

Solution to problem 1-1: the solution