This is homework number 3 and it is **due on June 14**. You *must* use this template for your homework. You can either print it out and write on it and upload that, or you can use a tablet if you have one. Alternatively, I am providing you with a link the the LaTeXtemplate. If you type your homework then (1) you'll become accustomed to using LaTeXwhich is probably good and (2) you'll get a bonus of 2 points (each HW assignment is 10 points, so this means that your maximum is 12 if you type). Whichever you choose, you will turn it in on Gradescope.

Also, no matter which method you choose, your work must be neat, legible, and must flow clearly. See examples in class of what I am looking for. In particular, there should be nothing that is scratched out (either use Whiteout or something similar or just re-write the whole thing). The work should more or less progress from left to right, top to bottom. In short, imagine that this is a history class or something and you're turning in your *final draft*. Part of what you are being graded on is your ability to communicate well which, at a minimum, means the reader can actually read what you have written.

Each problem as an associated point value that is given below. Three of the problems will be graded for accuracy, the other seven will be graded for "completion". Here, "completion" means: is it clear the student made an honest attempt at the problem and wrote the solution/attempt up in a neat way?

1. Solve $\partial_t u + u\partial_x u = 2$ for $t \geq 0$ if $u(x, 0) = 3x$.

Solution:

$$\begin{array}{lll} \dot{t}(s) = 1 & t(0) = 0 & \to t = s \\ \dot{x}(s) = z & x(0) = x_0 & \\ \dot{z}(s) = 2 & z(0) = 3x_0 & \end{array}$$

$$z(s) = 2s + 3x_0$$

$$\dot{x}(s) = 2s + 3x_0 \to x = s^2 + 3sx_0 + c$$

$$x(0) = x_0 = c \to x = s^2 + 3sx_0 + x_0$$

a,b subsitution

$$a = s^2 + 3sx_0 + x_0 \quad b = s$$

$$\frac{a}{3b+1} - b^2 = x_0$$

$$\therefore u(x,t) = 2t + 3\left(\frac{x}{3t+1} - t^2\right)$$

2. Solve $(y+u)\partial_x u + y\partial_y u = x - y$ with $u(x,1) = 1 + x$.

Solution:

$$\begin{array}{lll} \dot{x}(s) = y + z & x(0) = x_0 & \\ \dot{y}(s) = y & y(0) = 1 & \to y = e^s \\ \dot{z}(s) = x - y & z(0) = 1 + x_0 & \end{array}$$

$$\ddot{z} + \dot{y} = x = y + \dot{z}$$

$$\ddot{z} = z$$

$$z = Ce^s$$

$$z(0) = 1 + x_0 \to z = (1 + x_0)e^s$$

$$\dot{x} = e^s + (1 + x_0)e^s$$

$$x = 2e^s + x_0 e^s - 2|_{x_0 = 2 + x_0 + c \to c = -2}$$

a,b subsitution

$$a = 2e^s + x_0 e^s - 2 \quad b = e^s$$

$$x_0 = \frac{a + 2 - 2b}{b}$$

$$\therefore u(x,y) = x - y + 2$$

3. Solve $\partial_y u + 2(1+u)\partial_x u = 0$ with $u(x,0) = \begin{cases} 1, & x < 0 \\ 3, & x > 0 \end{cases}$.

Solution:

$$\begin{aligned} \dot{x}(s) &= 2(1+z) & x(0) &= x_0 & & \\ \dot{y}(s) &= 1 & y(0) &= 0 & &\to y = s \\ \dot{z}(s) &= 0 & z(0) &= \begin{cases} 1, & x_0 < 0 \\ 3, & x_0 > 0 \end{cases} & &\to z = \begin{cases} 1, & x_0 < 0 \\ 3, & x_0 > 0 \end{cases} \end{aligned}$$

for $x < 0$:

$$\begin{aligned} \dot{x} &= 4 & x &= 4s + x_0 \\ a &= 4s + x_0 & b &= s \\ x_0 &= a - 4b \end{aligned}$$

for $x > 0$:

$$\begin{aligned} \dot{x} &= 8 & x &= 8s + x_0 \\ a &= 8s + x_0 & b &= s \\ x_0 &= a - 8b \end{aligned}$$

Solving the expansion wave:

$$\dot{x_s} = \frac{\dot{y}}{\dot{x} = \frac{b}{a}}$$

$$z = \frac{2a}{b} - 1$$

$$\therefore u(x,y) = \begin{cases} 1, y > \frac{x}{4} \\ \frac{2x}{y} - 1, \frac{x}{4} < y < \frac{x}{8} \\ 3, y < \frac{x}{8} \end{cases}$$

4. Solve $\partial_t u - u\partial_x u = -2u$ with $u(x,0) = x$.

Solution:

$$\begin{array}{lll} \dot{t}(s) = 1 & t(0) = 0 & \to t = s \\ \dot{x}(s) = z & x(0) = x_0 & \\ \dot{z}(s) = -2z & z(0) = x_0 & \end{array}$$

$$z = x_0 e^{-2s}$$

$$\dot{x}(s) = x_0 e^{-2s} \to \frac{-1}{2} x_0 e^{-2s} + c|_{x(0)=x_0}$$

$$\to x = \frac{3}{2} x_0 e^{-2s} - \frac{-1}{2} x_0$$

a,b subsitution

$$a = \tfrac{3}{2} x_0 e^{-2s} - \tfrac{-1}{2} x_0 \quad b = x_0$$

$$x_0 = \frac{2a}{3 - e^{2b}}$$

$$u(x,y) = \left(\frac{2x}{3 - e^{-2y}}\right) e^{-2y}$$

5. Solve $\partial_y u + u\partial_x u = 0$ if $u(x,0) = \begin{cases} 1, & x \leq 0 \\ 1-x, & 0 \leq x \leq 1 \\ 0, & 1 \leq x \end{cases}$. First solve this for $y \leq 1$ and then for $y \geq 1$.

Solution:

$$\begin{aligned} &\dot{x}(s) = z \quad x(0) = x_0 \\ &\dot{y}(s) = 1 \quad y(0) = 0 \qquad\qquad \to y = s \\ &\dot{z}(s) = 0 \qquad z(0) \qquad \to z = \begin{cases} 1, & x_0 \geq 0 \\ 1 - x_0, & 0 \leq x_0 \geq 1 \\ 0, & 1 \leq x_0 \end{cases} \end{aligned}$$

for $x \leq 0$:

$$x = s + x_0$$

$$y = x - x_0$$

for $0 \leq x_0 \leq 1$:

$$x = s - x_0 s + c$$

$$x(0) = x_0 = c$$

$$x = s - x_0 s + x_0$$

$$y = \frac{x - x_0}{1 - x_0}$$

for $x \geq 1$:

$$x = x_0$$

$y \leq 1$ is an expansion wave

$$\dot{x_s} = \frac{1}{z} = \frac{b}{a}$$

$$z = \frac{a}{b}$$

$y \geq 1$ is an compression wave

$$f(u) = \frac{1}{2}u^2$$

$$\dot{x_s} = \frac{[f]}{[u]} = 1 - \frac{x_0}{2}$$

$y \leq 1$:

$$\therefore u(x,y) \begin{cases} 1, & x \leq y \\ 1 - \frac{x-y}{1-y}, & y \leq x \leq 1+y \\ 1 - \frac{x-y}{2-2y}, & y+1 \leq x \leq 1 \\ 0, & x \geq 1 \end{cases}$$

$y \geq 1$:

$$\therefore u(x,y) \begin{cases} 1, & x \leq 1 \\ \frac{x}{y}, & 1 \leq x \leq y \\ 1 - \frac{x-y}{1-y}, & y \leq x \leq 1+y \\ 0, & x \geq 1 \end{cases}$$