

This is homework number 4 and it is **due on June 21**. You *must* use this template for your homework. You can either print it out and write on it and upload that, or you can use a tablet if you have one. Alternatively, I am providing you with a link to the \LaTeX template. If you type your homework then (1) you'll become accustomed to using \LaTeX which is probably good and (2) you'll get a bonus of 2 points (each HW assignment is 10 points, so this means that your maximum is 12 if you type). Whichever you choose, you will turn it in on Gradescope.

Also, no matter which method you choose, your work must be neat, legible, and must flow clearly. This, of course, includes the requirement that you show your work appropriately. See examples in class of what I am looking for. In particular, there should be nothing that is scratched out (either use Whiteout or something similar or just re-write the whole thing). The work should more or less progress from left to right, top to bottom. In short, imagine that this is a history class or something and you're turning in your *final draft*. Part of what you are being graded on is your ability to communicate well which, at a minimum, means the reader can actually read what you have written.

Three of the problems will be graded for accuracy, the others will be graded for "completion". Here, "completion" means: is it clear the student made an honest attempt at the problem and wrote the solution/attempt up in a neat way?

1. Solve $\partial_t u + u \partial_x u = 0$ on $x \in (-\infty, \infty)$ and $t > 0$ if $u(x, 0) = \begin{cases} 1, & 0 < x < 2 \\ 0, & \text{else} \end{cases}$.

Solution:

$$\begin{aligned} \dot{x}(s) &= z & x(0) &= x_0 \\ \dot{t}(s) &= 1 & y(0) &= 0 & \rightarrow t = s \\ \dot{z}(s) &= 0 & z(0) &= \begin{cases} 1, & 0 < x < 2 \\ 0, & \text{else} \end{cases} \end{aligned}$$

$0 < x_0 < 2$:

$$\dot{x}s = 1 \rightarrow x = s + x_0$$

$$a = s + x_0 \quad b = s$$

$$x_0 = a - b$$

$$\therefore u(x, t) = 1 \in 0 < x - t < 2$$

else:

$$\dot{x}s = 0 \rightarrow x = x_0$$

$$u(x, t) = 0 \in \text{else}$$

$t \geq 0$: Shockwave

$$\dot{x}_s = \frac{\left[\frac{1}{2}u^2\right]}{[u]} = \frac{\frac{1}{2}(1^2 - 0^2)}{1^2 - 0^2} = \frac{1}{2}$$

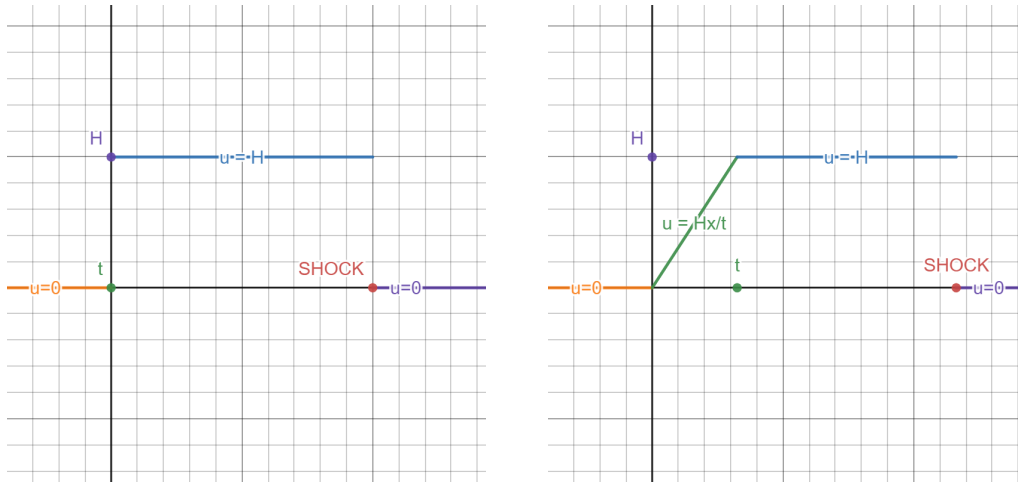
Expansion wave

$$m = \frac{\dot{t}}{\dot{x}} = \frac{1}{z} = \frac{b}{a} \rightarrow z = \frac{x}{t} \in 0 < x < t$$

All resulting in

$$u(x, t) = \begin{cases} \frac{x}{t}, & 0 < x < t \\ 1, & t < x < t + 2 \\ \frac{1}{2}, & x = \frac{1}{2}t + 2 \\ 0, & \text{else} \end{cases}$$

2. For a conservation equation $\partial_t u + \partial_x f(u) = 0$, the solution is shown below for $t = 0$ on the left and for a generic $0 < t < t_1$ on the right:



Use this to answer the following questions:

1. Find a formula for $x_s(t)$ – the shock location (this is the "SHOCK" label in the diagram).
2. At some value of t , this value for the shock will no longer be correct. What is that value?
3. Compute $[q]$ – the jump in q across the shock.

Solution:

3. Show (by differentiating) that the Alembert Solution solves the wave equation. For ease of notation, take $c^2 = 1$.

Solution:

$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds|_{c=1}$$

$$\partial_x u = \frac{1}{2}[\phi'(x + t) + \phi'(x - t)] + \frac{1}{2}[\psi(x + t) - \psi(x - t)]$$

$$\partial_{xx} u = \frac{1}{2}[\phi''(x + t) + \phi''(x - t) + \psi'(x + t) - \psi'(x - t)]$$

$$\partial_t u = \frac{1}{2}[\phi'(x + t) - \phi'(x - t)] + \frac{1}{2}[\psi(x + t) + \psi(x - t)]$$

$$\partial_{tt} u = \frac{1}{2}[\phi''(x + t) + \phi''(x - t) + \psi'(x + t) - \psi'(x - t)]$$

$$\partial_{xx} u = \partial_{tt} u$$

4. Solve $\partial_{tt}u = \partial_{xx}u$ if $u(x, 0) = \sin x$ and $\partial_t u(x, 0) = x$.

Solution:

$$c^2 = 1 \quad \phi(x) = \sin x \quad \phi(0) = x$$

$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds|_{c=1}$$

$$u(x, t) = \frac{1}{2}[\sin(x + t) + \sin(x - t)] + \left[\frac{1}{4}s^2 \right]_{x-ct}^{x+ct}$$

Using $\sin(a + b) = \sin a \cos b + \cos a \sin b$:

$$\sin(x + t) = \sin x \cos t + \cos x \sin t$$

$$\sin(x - t) = \sin x \cos t - \cos x \sin t$$

$$\rightarrow \sin(x + t) + \sin(x - t) = 2 \sin x \cos t$$

Simplifying $\left[\frac{1}{4}s^2 \right]_{x-ct}^{x+ct}$:

$$\frac{1}{4}[(x + t)^2 - (x - t)^2] = \frac{1}{4}[x^2 + 2xt + t^2 - x^2 + 2xt - t^2]$$

$$\rightarrow xt$$

$$\therefore u(x, t) = \sin x \cos t + xt$$

5. Solve $\partial_{xx}u - 3\partial_x\partial_tu - 4\partial_{tt}u = 0$ if $u(x, 0) = x^2$ and $\partial_tu(x, 0) = e^x$.

Solution:

$$(\partial_xu - 4\partial_tu)(\partial_xu + \partial_tu) = 0$$

Solving where $\partial_xu - 4\partial_tu = 0$:

$$\begin{aligned} \dot{x}(s) &= 1 & x(0) &= x_0 \rightarrow x = s + x_0 \\ \dot{t}(s) &= -4 & y(0) &= 0 \rightarrow t = -4s \\ \dot{z}(s) &= 0 & z(0) &= x_0^2 \rightarrow z = x_0^2 \end{aligned}$$

$$\begin{aligned} a &= s + x_0 & b &= -4s \\ a &= x_0 - \frac{1}{4}b & s &= \frac{-1}{4}b \\ x_0 &= a + \frac{1}{4}b \end{aligned}$$

$$u(x, t) = \left(x + \frac{1}{4}t\right)^2$$

Solving where $\partial_xu + \partial_tu = 0$:

$$\begin{aligned} \dot{x}(s) &= 1 & x(0) &= x_0 \rightarrow x = s + x_0 \\ \dot{t}(s) &= 1 & y(0) &= 0 \rightarrow t = s \\ \dot{z}(s) &= 0 & z(0) &= x_0^2 \rightarrow z = x_0^2 \end{aligned}$$

$$\begin{aligned} a &= s + x_0 & b &= s \\ a &= x_0 + b \\ x_0 &= a - b \end{aligned}$$

$$u(x, t) = (x - t)^2$$

I can't figure out how to implement the $\psi(x) = e^x$.