

1. Using Bernoulli's equation instead of general energy equation, prove that the induced velocity in the fully contracted wake of a rotor climbing vertically is twice the induced velocity in the rotor plane. You could use the mass and momentum equations.

Solution:

$$\begin{cases} P_0 + \frac{1}{2}\rho v_0^2 = P_1 + \frac{1}{2}\rho v_1^2 \\ P_2 + \frac{1}{2}\rho v_2^2 = P_3 + \frac{1}{2}\rho v_3^2 \\ v_0 = 0 \\ v_1 = v_2 = v_i \\ v_3 = w \end{cases}$$

$$\begin{cases} \rho v_i \sigma_i = \rho w \sigma_w \\ T = \int_{\sigma_3} \rho v_3 (v_3 \cdot \mathbf{n}) d\sigma_3 - \int_{\sigma_0} \rho v_0 (v_0 \cdot \mathbf{n}) d\sigma_0 \end{cases}$$

Simplifying the thrust equation produces

$$T = \rho v_3^2 \sigma_3 = \dot{m} w = \rho w^2 \sigma_3 \quad (1)$$

Using the relation of $P_0 = P_3$ and plugging back into Bernoulli's produces

$$\begin{aligned} P_0 + \cancel{\frac{1}{2}\rho v_0^2} &= P_1 + \frac{1}{2}\rho v_i^2 \\ P_0 &= P_1 + \frac{1}{2}\rho v_i^2 \\ P_2 + \cancel{\frac{1}{2}\rho v_i^2} &= P_1 + \cancel{\frac{1}{2}\rho v_i^2} + \frac{1}{2}\rho w^2 \\ P_2 - P_0 &= \frac{1}{2}\rho w^2 \end{aligned}$$

and the difference of pressure can be defined as the thrust across the blade membrane

$$\frac{T}{\sigma_2} = \frac{1}{2}\rho w^2 \quad (2)$$

Plugging equation 1 into 2 gives

$$\frac{\rho w^2 \sigma_3}{\sigma_2} = \frac{1}{2} \rho w^2$$

$$\frac{\cancel{\rho w^2} \sigma_3}{\sigma_2} = \frac{1}{2} \cancel{\rho w^2}$$

$$\frac{\sigma_3}{\sigma_2} = \frac{1}{2} \quad (3)$$

Plugging equation 3 into the conservation of mass

$$\frac{\sigma_2}{\sigma_3} = \frac{w}{v_i}$$

$$2 = \frac{w}{v_i}$$

$$\therefore w = 2v_i$$

2. Measurements have been made of rotor performance at a fixed rotor speed for a series of blade pitch angles. The values of C_T that were measured were $6.0000\text{E-}06$, 0.0010490 , 0.0023760 , 0.0040760 and 0.0055810 , and the corresponding values of C_P were 0.000197 , 0.000226 , 0.000282 , 0.000405 and 0.000555 , respectively. Plot this data in the form of a power polar (C_T vs. C_P). Explain (and show in a chart) how to extract induced power factor (κ) and zero thrust power (profile power) for the rotor from these measured data. Then, to the experimental power polar chart add the analytical power polar curve predicted by modified momentum theory

Solution:

3. Find all eigenvalues and eigenfunctions for the equation: $\varphi'' + \lambda\varphi = 0$ with $\varphi(0) = 0$ and $\varphi'(\pi) = 0$.

Solution:

4. Solve

$$\begin{cases} \frac{\partial^2}{\partial t^2} u = \frac{\partial^2}{\partial x^2} u \\ u(0, t) = u(\pi, t) = 0 \\ u(x, 0) = f(x), \frac{\partial}{\partial t} u(x, 0) = 0, \end{cases}$$

where $f(x)$ is the “hat function”: it’s 0 and 0 and π and 1 and $\pi/2$ and linear in between.

Solution:

5. This is another perspective on the SOV method. Consider the problem

$$\begin{cases} \frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u \\ u(0, t) = u(\pi, t) = 0 \\ u(x, 0) = g(x) \end{cases}.$$

For each fixed x , assume that the solution can be written as $\sum_{k=1}^{\infty} B_k \sin(kx)$. Note that the B_k depend on t so a better way to write it is $\sum_{k=1}^{\infty} B_k(t) \sin(kx)$. Starting from this point, find the SOV solution. (Note: I had a typo that said $B_k(x)$ before; it should be $B_k(t)$ SORRY!!)

Solution: Plugging the function into the PDE we get:

$$\begin{aligned} \sum_{k=1}^{\infty} B'_k(t) \sin kx &= \sum_{k=1}^{\infty} B_k(t) \frac{d^2}{dx^2} \sin kx \\ &= \sum_{k=1}^{\infty} B_k(t) (-k^2 \sin kx) \\ &= - \sum_{k=1}^{\infty} B_k(t) k^2 \sin kx. \end{aligned}$$

Subtracting the left from the right:

$$0 = \sum_{k=1}^{\infty} (B'_k(t) + k^2 B_k(t)) \sin kx.$$

Since the $\sin kx$ functions are linearly independent, this implies that for all k , $B'_k(t) = -k^2 B_k(t)$. So we get an ODE for B_k and the solution is $B_k(t) = B_k(0)e^{-k^2 t}$. To find the values of the constants note that:

$$g(x) = u(x, 0) = \sum_{k=1}^{\infty} B_k(0) \sin kx.$$

That is, as before, $B_k(0) = \frac{2}{\pi} \int_0^{\pi} g(y) \sin ky dy$.