Aero 455 **Name: Benjamin Tollison**

1. Using Bernoulli's equation instead of general energy equation, prove that the induced velocity in the fully contracted wake of a rotor climbing vertically is twice the induced velocity in the rotor plane. You could use the mass and momentum equations.

Solution:

$$\begin{cases} P_0 + \frac{1}{2}\rho v_0^2 = P_1 + \frac{1}{2}\rho v_1^2 \\ P_2 + \frac{1}{2}\rho v_2^2 = P_3 + \frac{1}{2}\rho v_3^2 \\ v_0 = 0 \\ v_1 = v_2 = v_i \\ v_3 = w \end{cases}$$

$$\begin{cases} \rho v_i \sigma_i = \rho w \sigma_w \\ T = \int_{\sigma_3} \rho v_3 \left(v_3 \cdot n\right) d\sigma_3 - \int_{\sigma_0} \rho \cancelto{0}{v_0} \left(\cancelto{0}{v_0} \cdot n\right) d\sigma_0 \end{cases}$$

Simplifying the thrust equation produces

$$T = \rho v_3^2 \sigma_3 = \dot{m} w = \rho w^2 \sigma_3 \tag{1}$$

Using the relation of $P_0 = P_3$ and plugging back into Bernoulli's produces

$$P_0 + \cancelto{0}{\frac{1}{2}\rho v_0^2} = P_1 + \frac{1}{2}\rho v_i^2$$

$$P_0 = P_1 + \frac{1}{2}\rho v_i^2$$

$$P_2 + \cancel{\frac{1}{2}\rho v_i^2} = P_1 + \cancel{\frac{1}{2}\rho v_i^2} + \frac{1}{2}\rho w^2$$

$$P_2 - P_0 = \frac{1}{2}\rho w^2$$

and the difference of pressure can be defined as the thrust across the blade membrane

$$\frac{T}{\sigma_2} = \frac{1}{2}\rho w^2 \tag{2}$$

Plugging equation 1 into 2 gives

$$\frac{\rho w^2 \sigma_3}{\sigma_2} = \frac{1}{2}\rho w^2$$

$$\frac{\cancel{\rho w^2} \sigma_3}{\sigma_2} = \frac{1}{2}\cancel{\rho w^2}$$

$$\frac{\sigma_3}{\sigma_2} = \frac{1}{2} \tag{3}$$

Plugging equation 3 into the conservation of mass

$$\frac{\sigma_2}{\sigma_3} = \frac{w}{v_i}$$

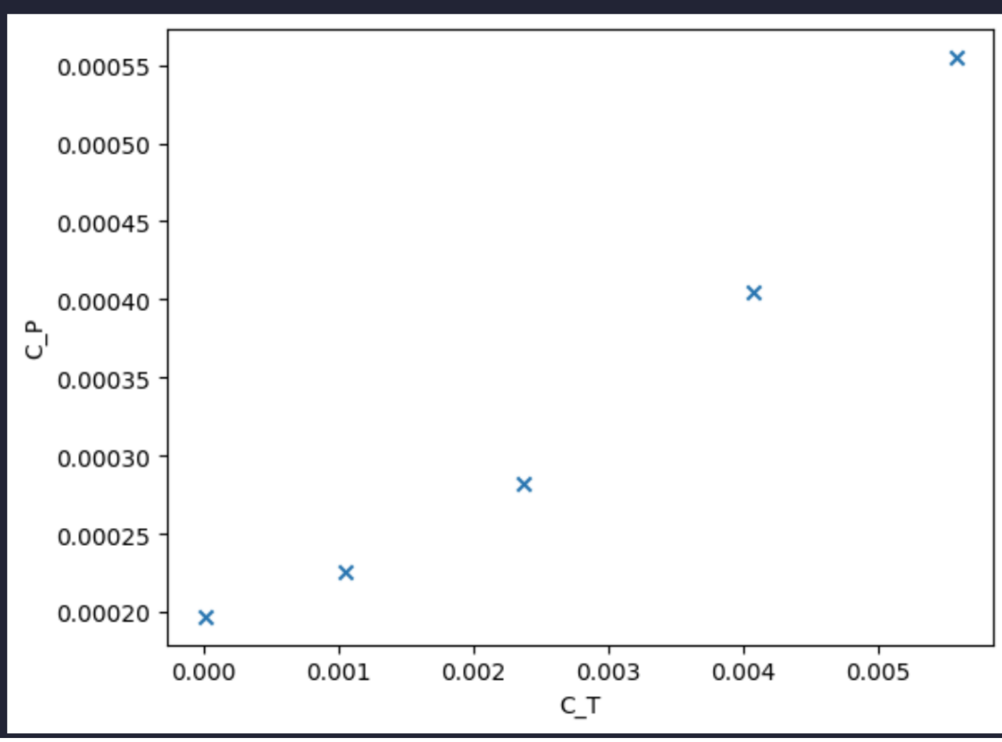
$$2 = \frac{w}{v_i}$$

$$\therefore w = 2v_i$$

2. Measurements have been made of rotor performance at a fixed rotor speed for a series of blade pitch angles. The values of C_T that were measured were 6.0000E-06, 0.0010490, 0.0023760, 0.0040760 and 0.0055810, and the corresponding values of C_P were 0.000197, 0.000226, 0.000282, 0.000405 and 0.000555, respectively. Plot this data in the form of a power polar (C_T vs. C_P). Explain (and show in a chart) how to extract induced power factor (κ) and zero thrust power (profile power) for the rotor from these measured data. Then, to the experimental power polar chart add the analytical power polar curve predicted by modified momentum theory

Solution:

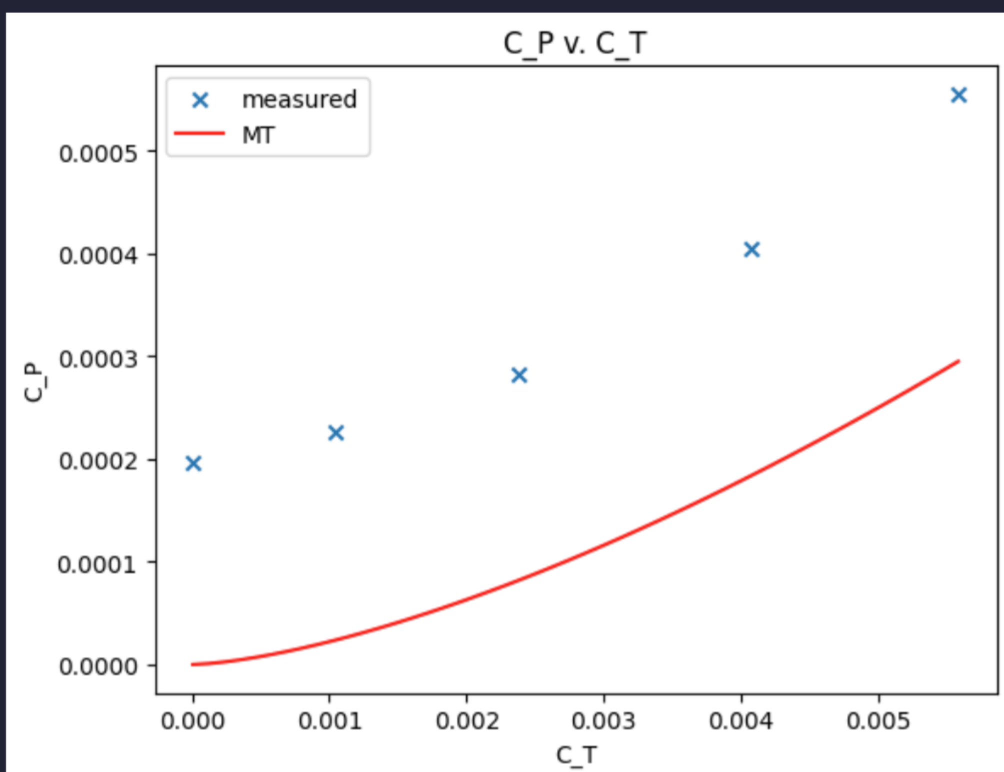
	C_Tmeasured	C_Pmeasured
0	0.000006	0.000197
1	0.001049	0.000226
2	0.002376	0.000282
3	0.004076	0.000405
4	0.005581	0.000555



Graphing momentum theory of

$$C_P = \frac{C_T^{\frac{3}{2}}}{\sqrt{2}}$$

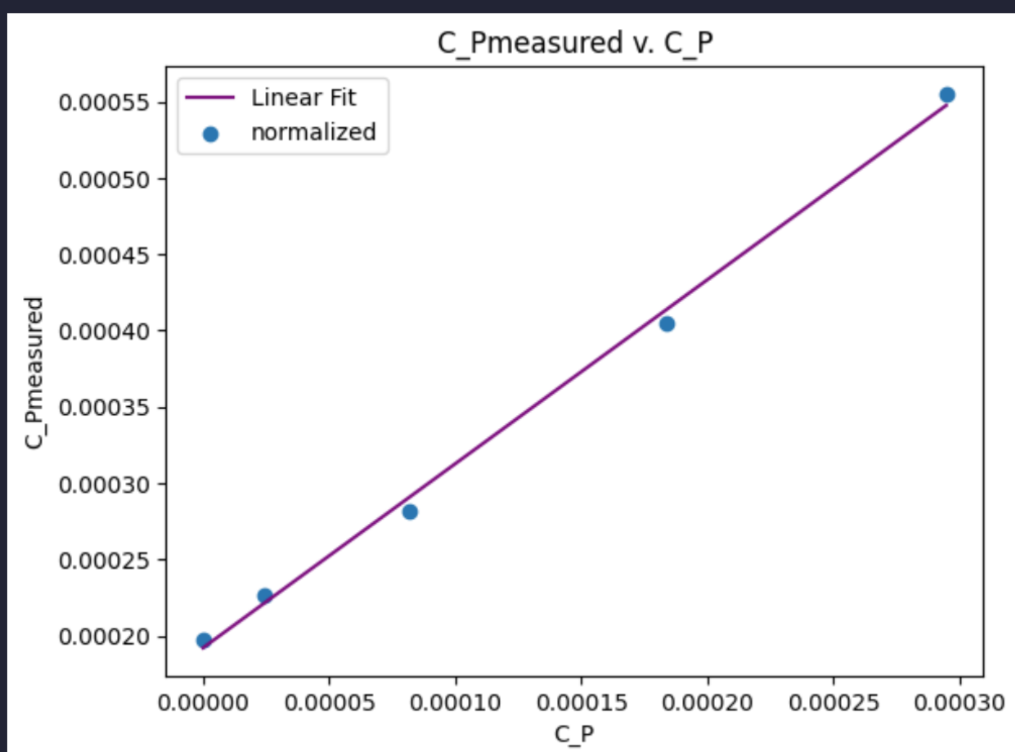
	C_Tmeasured	C_Pmeasured	C_P_ideal
0	0.000006	0.000197	1.039230e-08
1	0.001049	0.000226	2.402416e-05
2	0.002376	0.000282	8.189449e-05
3	0.004076	0.000405	1.840078e-04
4	0.005581	0.000555	2.948173e-04



Normalizing the $C_{P,\text{ideal}}$ with the measured values produces the following graph and then finding the linear fit produces the following.

$\kappa = 1.20573759600615$

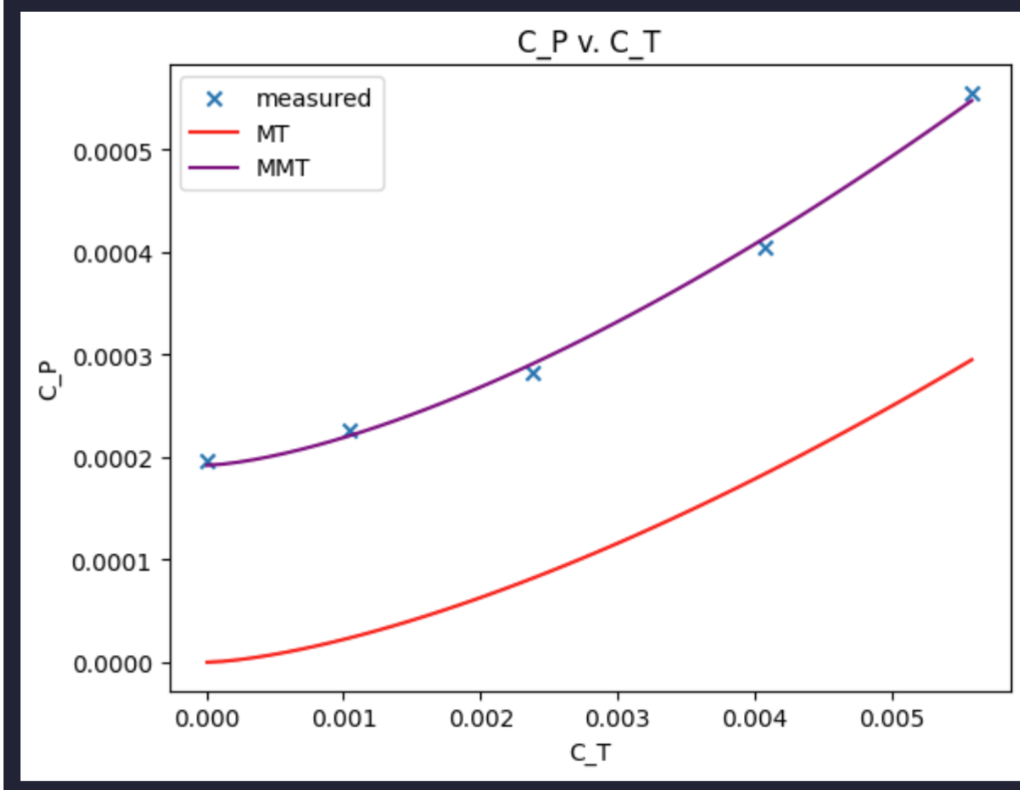
$C_{P0} = 0.000191987975132399$



Creating the following modified momentum theory of

$$C_P = \kappa \frac{C_T^{\frac{3}{2}}}{\sqrt{2}} + C_{P0}$$

	C_Tmeasured	C_Pmeasured	C_P_ideal	C_P_MMT
0	0.000006	0.000197	1.039230e-08	0.000192
1	0.001049	0.000226	2.402416e-05	0.000221
2	0.002376	0.000282	8.189449e-05	0.000291
3	0.004076	0.000405	1.840078e-04	0.000414
4	0.005581	0.000555	2.948173e-04	0.000547



3. A helicopter with a gross weight of 3,000 lb (1,360.5 kg), a main rotor radius of 13.2 ft (4.0 m), a rotor tip speed of 680 ft/s (207.3 m/s), and has 275 hp (205 kW) delivered to the main rotor shaft. For hovering at sea level conditions, compute: (a) the rotor disk loading, (b) the ideal power loading, (c) the thrust and torque coefficients, (d) the figure of merit and actual power loading.

Solution: a)

$$T = mg$$

$$\sigma = \pi R^2$$

$$DL = \frac{T}{\sigma}$$

$$DL = 265.520$$

b)

$$PL_{ideal} = \frac{T}{Tv_i} = \frac{1}{v_i} = \sqrt{\frac{2\rho}{DL}}$$

$$PL_{ideal} = 0.096058$$

c)

$$C_T = \frac{T}{\rho\sigma(\Omega R)^2}$$

$$C_{P,ideal} = \frac{P_h}{\rho\sigma(\Omega R)^3}, C_{P,measured} = \frac{P}{\rho\sigma(\Omega R)^3}$$

$$C_Q = C_P$$

$$C_T = 0.0050439$$

$$C_{P,ideal} = 0.0002533$$

$$C_{P,actual} = 0.0003737$$

d)

$$FM = \frac{P_h}{P_{actual}} = \frac{C_{P,ideal}}{C_{P,measured}}$$

$$PL_{actual} = \frac{PL_{ideal}}{FM}$$

$$FM = 0.67777$$

$$PL_{actual} = 0.141727$$

4. For the helicopter described in the previous question, the tail rotor radius is 2.3 ft (0.701 m) and the tail rotor is located 15.3 ft (4.66 m) from the main rotor shaft. Calculate the thrust and power required by the tail rotor when helicopter is hovering at sea level. Assume that the figure of merit of the tail rotor is 0.7.

Solution:

$$\tau = F \times r$$

$$Q = T_{tail} \times r$$

$$Q = C_{P,actual} \rho \sigma \left(\Omega R\right)^2 R$$

$$\therefore T_{tail} = \frac{Q}{r}$$

$$P = \frac{T_{tail}^{\frac{3}{2}}}{\sqrt{2\rho\sigma}}$$

$$T_{tail} = 848.8455[\mathrm{N}]$$

$$P_{tail} = 12716.4812[\mathrm{W}]$$

5. This is another perspective on the SOV method. Consider the problem

$$\begin{cases} \frac{\partial}{\partial t}u = \frac{\partial^2}{\partial x^2}u \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = g(x) \end{cases} .$$

For each fixed x, assume that the solution can be written as $\sum_{k=1}^{\infty} B_k \sin(kx)$. Note that the B_k depend on t so a better way to write it is $\sum_{k=1}^{\infty} B_k(t)\sin(kx)$. Starting from this point, find the SOV solution. (Note: I had a typo that said $B_k(x)$ before; it should be $B_k(t)$ SORRY!!)

Solution: Plugging the function into the PDE we get:

$$\begin{aligned} \sum_{k=1}^{\infty} B_k'(t)\sin kx &= \sum_{k=1}^{\infty} B_k(t)\frac{d^2}{dx^2}\sin kx \\ &= \sum_{k=1}^{\infty} B_k(t)(-k^2\sin kx) \\ &= -\sum_{k=1}^{\infty} B_k(t)k^2\sin kx. \end{aligned}$$

Subtracting the left from the right:

$$0 = \sum_{k=1}^{\infty}(B_k'(t) + k^2B_k(t))\sin kx.$$

Since the $\sin kx$ functions are linearly independent, this implies that for all k, $B_k'(t) = -k^2B_k(t)$. So we get an ODE for B_k and the solution is $B_k(t) = B_k(0)e^{-k^2t}$. To find the values of the constants note that:

$$g(x) = u(x,0) = \sum_{k=1}^{\infty} B_k(0)\sin kx.$$

That is, as before, $B_k(0) = \frac{2}{\pi}\int_0^{\pi} g(y)\sin ky\,dy$.