Solving Biological Stochastic Optimal Control Problems using Physics Informed Neural Networks

An Annotated Bibliography

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1 Background on Biological Control Applications

Control problems are of interest in a wide range of biological contexts, including ecology, epidemiology, and biomedicine. Given any control system, it is natural to consider the investigation of optimal control policies, which reflect tradeoffs between the application of costly controls and minimization of undesirable outcomes. Ecological applications of optimal control are commonly in stabilization, harvesting, or extinction of species [10, 5, 24, 40, 23, 27, 25, 49, 34] . Epidemiological applications commonly are designed to model the tradeoffs between the impact of the epidemic and the costs of the intervention [35, 2, 45, 32, 8]. Biomedical applications have primarily considered treatment application scheduling, such as for radio or chemotherapy in control of cancer [38, 46, 43, 16, 18, 31, 26, 1].

2 Background on Optimal Control Methods

Optimal control problems are generally solved by Pontryagin's maximum principle, which results in an optimal open-loop control law [29, 30, 48]. However, the dynamic programming approach by the Hamilton-Jacobi-Bellman equation can also be used to solve optimal control problems [30, 48, 12]. While the HJB approach requires solution of an partial differential equation, which presents significant theoretical and numerical difficulties, doing so can yield an optimal feedback-control law which is generally more robust than an open-loop control law [48, 12, 20].

2.1 Dynamic Programming

The dynamic programming approach to optimal control originates from Bellman's principle of optimality [4], and can enable the determination of optimal feedback controls through solution of the resulting HJB equation. However, the method of dynamic programming is applicable to a optimization problems in a wide range of settings. A good general introduction to dynamic programming can be found in the classic text by Bellman [4]. More recent overviews, with an emphasis on computational applications, are provided by volumes 1 and 2 of Bertsekas [6, 7]. General algorithm texts often provide simple introductions to dynamic programming, one such text is Cormen et al. [13].

2.2 Applied Introductions to Optimal Control Theory

There are many introductory texts which provide applied introductions to traditional optimal control problems, so we consider only a few [29, 30, 48]. Kamien and Schwartz [29] provide a very accessible introduction to optimal control problems within the context of economics and management problems. Kirk [30] provides a similar introduction aimed at engineering problems, and includes a section on basic algorithms for solving optimal control problems numerically. Stengel [48] provides an extensive for deterministic and stochastic engineering applications of control, but is primarily concerned with linear or weakly nonlinear systems from physical problems, and is relatively ill suited as a sole introductory text.

2.3 Expositions of the Theory

Optimal control theory forms an intersection of functional analysis and optimization theory; a reasonable grasp of mathematical results in each of those fields is necessary to develop a theoretical understanding of optimal control. A gentle introduction to the necessary theory is provided in Liberzon [33]. Luenberger [36] provides a slightly terse introduction to functional analysis and optimization theory. The mathematical theory behind optimal control is thoroughly, if less accessibly, elaborated in Fleming and Rishel [20], Clarke [12].

Most optimal control texts are concerned with various versions of the Pontryagin's maximum principle (PMP) as opposed to the Hamilton-Jacobi-Bellman equation approach. This is in part due to the challenges in the well-posedness of the HJB equation. Developments in mathematical theory in the 1980s worked to resolve these challenges through the development of viscosity solutions [14, 3]. Classic literature on viscosity solutions includes the work by Crandall and Lions [14]. A more recent summary of this literature can be found in Bardi et al. [3]. Evans [17] is a good general text on the theory of partial differential equations.

3 Background on Stochastic Optimal Control

3.1 Stochastic Dynamics

Stochastic dynamical systems are one method to attempt to incorporate unmodelled complexity into mathematical models. Stochastic dynamics commonly but not exclusively refers to the use of Ito stochastic differential equations to model a phenomena [28]. The relevant Ito calculus is well introduced in Jacobs [28] at a basic level, or Oksendal [42] at a more advanced level.

3.2 Applied Stochastic Optimal Control

Few or no standard resources exist to introduce stochastic optimal control generally to applied practitioners at a level only slightly more advanced than Jacobs [28]. Stengel [48] provides some introduction to stochastic optimal control in the linear-quadratic case. Kamien and Schwartz [29] devote several pages to considering simple time discounted infinite horizon problems.

Much of the difficulty in introducing stochastic optimal control at an introductory level seems to stem from a desire to provide a mathematical rather than a computational introduction. Mathematically, stochastic optimal control is fairly complex, even in simple cases. However, the subject can be significantly simplified at a computational setting by taking existence and uniqueness as assumed-assumptions which are often implicit in applied computational settings.

3.3 Theory of Stochastic Control

The requisite mathematics for stochastic optimal control is fairly advanced, and resources are commonly written to make free use of concepts in graduate probability, functional analysis, and partial differential equations. Classic overviews of stochastic optimal control problems are provided in Fleming [19] and Fleming and Rishel [20]. Standard textbooks are Fleming and Soner [21] and Yong and Zhou [50].

We largely do not consider the forward-backward stochastic differential equation technique towards deriving stochastic optimal controls, but that technique is becoming more common in both the stochastic maximum principle and dynamic programming formalism. FBSDEs are exposited in Ma and Yong [37] at a level similar to Fleming and Rishel [20].

4 Background on Physics Informed Neural Networks

4.1 Neural Networks

Neural networks are an increasingly standard approximation and surrogate method within applied mathematics. The mathematical and computational theory underlying neural networks is well exposited by Simon [47]. Data driven techniques more generally are well introduced and contextualized for applied mathematics in Brunton and Kutz [9].

4.2 Physics Informed Neural Networks

Physics informed neural networks are a relatively recent technique which translates problem of solving a partial differential equation to a problem of training a neural network [44, 15]. A key advantage of this technique is that it is mesh-free, which allows it to avoid the curse of dimensionality faced by most PDE solvers in the cases of moderate to high dimensional spaces [15]. Physics informed neural networks have seen success in application to a wide range of problems, including fluid dynamics [11], power systems [39], material science [51], and optimal control, including of PDEs [41] and optimal feedback control in aerospace settings [22].

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