1. **General solution with arbitrary incident angle and ϵ, μ**

From problem 7.4, we have established that inside the metal medium, the permittivity is to be considered complex

$$\widetilde{\epsilon}_2 = \epsilon_2 + \frac{i\sigma}{\omega} \tag{1}$$

Let $\mathbf{E}_1/\mathbf{B}_1/\mathbf{k}_1$ and $\mathbf{E}_1'/\mathbf{B}_1'/\mathbf{k}_1'$ be the incident wave and the reflected wave at the 1/2 boundary. Let $\mathbf{E}_2/\mathbf{B}_2/\mathbf{k}_2$ be the transmitted wave at 1/2 boundary. Let $\mathbf{E}_2'/\mathbf{B}_2'/\mathbf{k}_2'$ and $\mathbf{E}_3/\mathbf{B}_3/\mathbf{k}_3$ be the reflected and transmitted wave at the 2/3 bounary.

All fields shall have the plane wave form

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \qquad \mathbf{B}(\mathbf{x},t) = \frac{\mathbf{k}\times\mathbf{E}}{\omega}e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \qquad \text{where } \mathbf{k}\cdot\mathbf{k} = \omega^2\mu\epsilon \tag{2}$$

We will give the most general solution to this three-media reflection/transmission problem.

Let the boundary normals $\mathbf{n}_{12}, \mathbf{n}_{23}$ be pointing along the $\hat{\mathbf{z}}$ direction and let the plane of incidence be the x-z plane. Then the boundary conditions are given by the Maxwell's equations:

$$\begin{array}{rr} \text{1/2 boundary} & \text{2/3 boundary} \\ \left[\epsilon_1\left(\mathbf{E}_1+\mathbf{E}_1'\right) - \widetilde{\epsilon}_2\left(\mathbf{E}_2+\mathbf{E}_2'\right)\right]\cdot\hat{\mathbf{z}} = 0 & \left[\widetilde{\epsilon}_2\left(\mathbf{E}_2+\mathbf{E}_2'\right) - \epsilon_3\mathbf{E}_3\right]\cdot\hat{\mathbf{z}} = 0 \quad (3) \\ \left(\mathbf{k}_1\times\mathbf{E}_1 + \mathbf{k}_1'\times\mathbf{E}_1' - \mathbf{k}_2\times\mathbf{E}_2 - \mathbf{k}_2'\times\mathbf{E}_2'\right)\cdot\hat{\mathbf{z}} = 0 & \left(\mathbf{k}_2\times\mathbf{E}_2 + \mathbf{k}_2'\times\mathbf{E}_2' - \mathbf{k}_3\times\mathbf{E}_3\right)\cdot\hat{\mathbf{z}} = 0 \quad (4) \\ \left(\mathbf{E}_1+\mathbf{E}_1'-\mathbf{E}_2-\mathbf{E}_2'\right)\times\hat{\mathbf{z}} = 0 & \left(\mathbf{E}_2+\mathbf{E}_2'-\mathbf{E}_3\right)\times\hat{\mathbf{z}} = 0 \quad (5) \\ \left[\frac{1}{\mu_1}\left(\mathbf{k}_1\times\mathbf{E}_1+\mathbf{k}_1'\times\mathbf{E}_1'\right) - \frac{1}{\mu_2}\left(\mathbf{k}_2\times\mathbf{E}_2+\mathbf{k}_2'\times\mathbf{E}_2'\right)\right]\times\hat{\mathbf{z}} = 0 & \left[\frac{1}{\mu_2}\left(\mathbf{k}_2\times\mathbf{E}_2+\mathbf{k}_2'\times\mathbf{E}_2'\right) - \frac{1}{\mu_3}\left(\mathbf{k}_3\times\mathbf{E}_3\right)\right]\times\hat{\mathbf{z}} = 0 \quad (6) \end{array}$$

For the case where $\mathbf{E}_1$'s polarization is perpendicular to the incident plane, (5) requires

$$\left(E_1e^{i\mathbf{k}_1\cdot\mathbf{x}} + E_1'e^{i\mathbf{k}_1'\cdot\mathbf{x}} - E_2e^{i\mathbf{k}_2\cdot\mathbf{x}} - E_2'e^{i\mathbf{k}_2'\cdot\mathbf{x}}\right)e^{-i\omega t} = 0 \qquad \text{for all } t \text{ and } \mathbf{x} \text{ on the } z=0 \text{ plane} \tag{7}$$

$$\left(E_2e^{i\mathbf{k}_2\cdot\mathbf{x}} + E_2'e^{i\mathbf{k}_2'\cdot\mathbf{x}} - E_3e^{i\mathbf{k}_3\cdot\mathbf{x}}\right)e^{-i\omega t} = 0 \qquad \text{for all } t \text{ and } \mathbf{x} \text{ on the } z=D \text{ plane} \tag{8}$$

which established the usual restrictions

$$\mathbf{k}_1\cdot\mathbf{x} = \mathbf{k}_1'\cdot\mathbf{x} = \mathbf{k}_2\cdot\mathbf{x} = \mathbf{k}_2'\cdot\mathbf{x} \qquad \text{for all } \mathbf{x} \text{ on the } z=0 \text{ plane} \tag{9}$$

$$\mathbf{k}_2\cdot(\mathbf{x}-D\hat{\mathbf{z}}) = \mathbf{k}_2'\cdot(\mathbf{x}-D\hat{\mathbf{z}}) = \mathbf{k}_3\cdot(\mathbf{x}-D\hat{\mathbf{z}}) \qquad \text{for all } \mathbf{x} \text{ on the } z=D \text{ plane} \tag{10}$$

For the case where $\mathbf{E}_1$'s polarization is parallel to the incident plane, (6) can be used to produce the same restrictions (9) and (10).

In the most general case, we should write $\mathbf{k}_2, \mathbf{k}_2', \mathbf{k}_3$ as complex vectors, i.e.,

$$\mathbf{k}_2 = \mathbf{k}_{2R} + i\mathbf{k}_{2I} \qquad \mathbf{k}_2' = \mathbf{k}_{2R}' + i\mathbf{k}_{2I}' \qquad \mathbf{k}_3 = \mathbf{k}_{3R} + i\mathbf{k}_{3I} \tag{11}$$

subject to the restriction for the corresponding medium

$$k_R^2 - k_I^2 = \mathrm{Re}\left(\omega^2\mu\epsilon\right) \qquad 2\mathbf{k}_R\cdot\mathbf{k}_I = \mathrm{Im}\left(\omega^2\mu\epsilon\right) \tag{12}$$

Then treating the real and imaginary parts of (9) and (10) separately will reach the expected reflection symmetry and Snell's law.

$$\mathbf{k}_1\cdot\hat{\mathbf{x}} = \mathbf{k}_1'\cdot\hat{\mathbf{x}} = \mathbf{k}_{2R}\cdot\hat{\mathbf{x}} = \mathbf{k}_{2R}'\cdot\hat{\mathbf{x}} = \mathbf{k}_{3R}\cdot\hat{\mathbf{x}} \qquad \mathbf{k}_{2I}\cdot\hat{\mathbf{x}} = \mathbf{k}_{2I}'\cdot\hat{\mathbf{x}} = \mathbf{k}_{3I}\cdot\hat{\mathbf{x}} = 0 \tag{13}$$

(12) and (13) allow us to completely determine all the wave vectors. The details for the 1/2 boundary was done in problem 7.4, and relationship between medium 1 and 3 is the same as the reflection and refraction between two non-metal media.

Here are the results for $\mathbf{k}_2, \mathbf{k}_2'$ (α is the incident angle):

$$k_{2I}^2 = k_{2I}'^2 = \frac{k_1^2}{2}\left\{\left[\sin^2\alpha - \left(\frac{n_2}{n_1}\right)^2\right] + \sqrt{\left[\left(\frac{n_2}{n_1}\right)^2 - \sin^2\alpha\right] + \left(\frac{n_2}{n_1}\right)^4\left(\frac{\sigma}{\omega\epsilon_2}\right)^2}\right\} \tag{14}$$

$$k_{2R}^2 = k_{2R}'^2 = \frac{k_1^2}{2}\left\{\left[\sin^2\alpha + \left(\frac{n_2}{n_1}\right)^2\right] + \sqrt{\left[\left(\frac{n_2}{n_1}\right)^2 - \sin^2\alpha\right] + \left(\frac{n_2}{n_1}\right)^4\left(\frac{\sigma}{\omega\epsilon_2}\right)^2}\right\} \tag{15}$$

or in component forms (β is the angle between $\mathbf{k}_{2R}$ and $\mathbf{k}_{2I}$)

$$\mathbf{k}_2 = \hat{\mathbf{x}}k_{2x} + \hat{\mathbf{z}}k_{2z} \qquad \mathbf{k}_2' = \hat{\mathbf{x}}k_{2x} - \hat{\mathbf{z}}k_{2z} \qquad \text{where} \tag{16}$$

$$k_{2x} = k_{2R}\sin\beta = k_1 \sin\alpha \qquad k_{2z} = k_{2R}\cos\beta + ik_{2I} \qquad \cos\beta = \left(\frac{k_1^2}{k_{2R}k_{2I}}\right)\left(\frac{n_2}{n_1}\right)^2\left(\frac{\sigma}{2\omega\epsilon_2}\right) \tag{17}$$

And for $\mathbf{k}_3$:

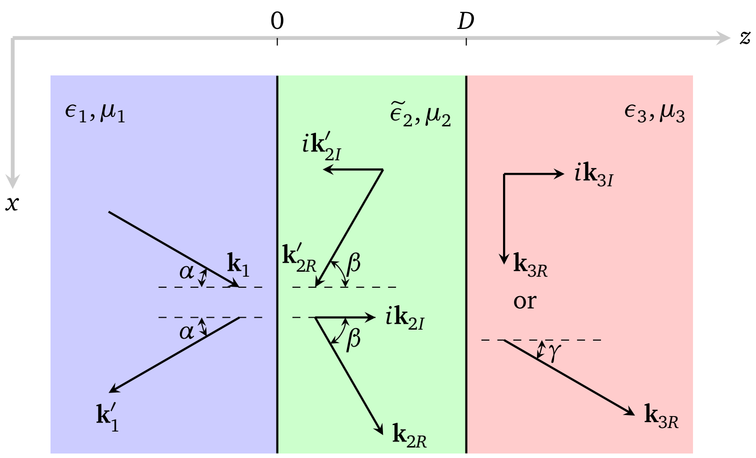
$$k_{3I} = \begin{cases} 0 & \text{for } \sin\alpha < \dfrac{n_3}{n_1} \\ k_1\sqrt{\sin^2\alpha - \left(\dfrac{n_3}{n_1}\right)^2} & \text{for } \sin\alpha \geq \dfrac{n_3}{n_1} \end{cases} \qquad k_{3R} = \begin{cases} k_1 \cdot \dfrac{n_3}{n_1} & \text{for } \sin\alpha < \dfrac{n_3}{n_1} \\ k_1 \sin\alpha & \text{for } \sin\alpha \geq \dfrac{n_3}{n_1} \end{cases} \tag{18}$$

or in component forms

$$\mathbf{k}_3 = \hat{\mathbf{x}}k_{3x} + \hat{\mathbf{z}}k_{3z} \qquad \text{where} \tag{19}$$

$$k_{3x} = k_1 \sin\alpha \qquad k_{3z} = k_1\sqrt{\left(\frac{n_3}{n_1}\right)^2 - \sin^2\alpha} \tag{20}$$

A geometric representation of the wave vectors is given below. Note in medium 3, depending on whether $\sin\alpha > n_3/n_1$, we may end up with two different configurations (which is complex analytic continuation of each other).



For general solutions of the amplitudes, let's consider the two different polarization modes separately.

(a) $\mathbf{E}_1$ is perpendicular to the plane of incidence.
Given the polarization, all $\mathbf{E}$'s are in the $\hat{\mathbf{y}}$ direction. On the 2/3 boundary, (5) is turned into

$$E_2 e^{ik_{2z}D} + E_2' e^{-ik_{2z}D} - E_3 e^{ik_{3z}D} = 0 \tag{21}$$

(6) becomes

$$\frac{k_{2z}}{\mu_2}\left(E_2 e^{ik_{2z}D} - E_2' e^{-ik_{2z}D}\right) - \frac{k_{3z}}{\mu_3} E_3 e^{ik_{3z}D} = 0 \tag{22}$$

Then we can express E_2', E_3 in terms of E_2:

$$E_2' = E_2 \overbrace{\left(\frac{\dfrac{k_{2z}}{\mu_2} - \dfrac{k_{3z}}{\mu_3}}{\dfrac{k_{2z}}{\mu_2} + \dfrac{k_{3z}}{\mu_3}}\right)}^{r_{23}} e^{i2k_{2z}D} \qquad E_3 = E_2 \overbrace{\left(\frac{2\cdot\dfrac{k_{2z}}{\mu_2}}{\dfrac{k_{2z}}{\mu_2} + \dfrac{k_{3z}}{\mu_3}}\right)}^{t_{23}} e^{i(k_{2z}-k_{3z})D} \tag{23}$$

(5) and (6) for the 1/2 boundary are now

$$E_1 + E_1' - E_2 - E_2' = 0 \tag{24}$$

$$\frac{k_{1z}}{\mu_1}\left(E_1 - E_1'\right) - \frac{k_{2z}}{\mu_2}\left(E_2 - E_2'\right) = 0 \tag{25}$$

which finally gives

$$E_1' = E_1 \left[\frac{(1+r_{23})\dfrac{k_{1z}}{\mu_1} - (1-r_{23})\dfrac{k_{2z}}{\mu_2}}{(1+r_{23})\dfrac{k_{1z}}{\mu_1} + (1-r_{23})\dfrac{k_{2z}}{\mu_2}} \right] \qquad E_2 = E_1 \left[\frac{2\cdot\dfrac{k_{1z}}{\mu_1}}{(1+r_{23})\dfrac{k_{1z}}{\mu_1} + (1-r_{23})\dfrac{k_{2z}}{\mu_2}} \right] \tag{26}$$

(b) **E** is parallel to the plane of incidence.
We need to write every amplitude in component forms, i.e, $\mathbf{E} = \hat{\mathbf{x}}E_x + \hat{\mathbf{z}}E_z$. Transverse wave condition entails

$$\mathbf{k}\cdot\mathbf{E} = k_x E_x + k_z E_z = 0 \tag{27}$$

also take note of the component identity

$$k_x^2 + k_z^2 = \mathbf{k}\cdot\mathbf{k} = \omega^2\mu\epsilon \tag{28}$$

At the 2/3 boundary, (3) turns into

$$\widetilde{\epsilon}_2\left(E_{2z}e^{ik_{2z}D} + E_{2z}'e^{-ik_{2z}D}\right) - \epsilon_3 E_{3z}e^{ik_{3z}D} = 0 \tag{29}$$

and (5) implies

$$E_{2x}e^{ik_{2z}D} + E_{2x}'e^{-ik_{2z}D} - E_{3x}e^{ik_{3z}D} = 0 \qquad \Longrightarrow \qquad k_{2z}\left(E_{2z}e^{ik_{2z}D} - E_{2z}'e^{-ik_{2z}D}\right) - k_{3z}E_{3z}e^{ik_{3z}D} = 0 \tag{30}$$

With (29) and (30), we can express E_{3z}, E_{2z}' in terms of E_{2z}:

$$E_{2z}' = E_{2z}\overbrace{\left(\frac{\dfrac{k_{2z}}{\widetilde{\epsilon}_2} - \dfrac{k_{3z}}{\epsilon_3}}{\dfrac{k_{2z}}{\widetilde{\epsilon}_2} + \dfrac{k_{3z}}{\epsilon_3}}\right)}^{r_{23}} e^{i2k_{2z}D} \qquad E_{3z} = E_{2z}\overbrace{\left(\frac{2\cdot\dfrac{k_{2z}}{\epsilon_3}}{\dfrac{k_{2z}}{\widetilde{\epsilon}_2} + \dfrac{k_{3z}}{\epsilon_3}}\right)}^{t_{23}} e^{i(k_{2z}-k_{3z})D} \tag{31}$$

(3) and (5) for the 1/2 boundary are

$$\epsilon_1\left(E_{1z} + E_{1z}'\right) - \widetilde{\epsilon}_2\left(E_{2z} + E_{2z}'\right) = 0 \tag{32}$$
$$k_{1z}\left(E_{1z} - E_{1z}'\right) - k_{2z}\left(E_{2z} - E_{2z}'\right) = 0 \tag{33}$$

which gives

$$E_{1z}' = E_{1z} \left[\frac{(1+r_{23})\dfrac{k_{1z}}{\epsilon_1} - (1-r_{23})\dfrac{k_{2z}}{\widetilde{\epsilon}_2}}{(1+r_{23})\dfrac{k_{1z}}{\epsilon_1} + (1-r_{23})\dfrac{k_{2z}}{\widetilde{\epsilon}_2}} \right] \qquad E_{2z} = E_{1z} \left[\frac{2\cdot\dfrac{k_{1z}}{\widetilde{\epsilon}_2}}{(1+r_{23})\dfrac{k_{1z}}{\epsilon_1} + (1-r_{23})\dfrac{k_{2z}}{\widetilde{\epsilon}_2}} \right] \tag{34}$$

2. **Solution to problem 7.5**

(a) With the simplifying assumption given in problem 7.5, by (14), (15) we have

$$k_{2I}^2 = \frac{k_1^2}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon_0}\right)^2} - 1\right] = \frac{\omega^2\mu_0\epsilon_0}{2}\cdot\left(\frac{\sigma}{\omega\epsilon_0}\right)\left[\sqrt{1+\left(\frac{\omega\epsilon_0}{\sigma}\right)^2} - \frac{\omega\epsilon_0}{\sigma}\right] = \frac{1}{\delta^2}\left[1 + O\left(\frac{\omega\epsilon_0}{\sigma}\right)\right] \tag{35}$$

hence

$$k_{2I} \approx \frac{1}{\delta} \qquad \text{and similarly} \qquad k_{2R} \approx \frac{1}{\delta} \tag{36}$$

By (17)

$$k_{2z} = \frac{1+i}{\delta} \tag{37}$$

With

$$\gamma = \frac{\omega\delta}{c}(1-i) \qquad \lambda = (1-i)\frac{D}{\delta} \tag{38}$$

substituted into the definition of r_{23} in (23), we have

$$r_{23} = \left(\frac{1 - \frac{\gamma}{2}}{1 + \frac{\gamma}{2}} \right) e^{-2\lambda} \tag{39}$$

Then using (26), we get

$$\frac{E_1'}{E_1} = \frac{\left(1+\frac{\gamma}{2}\right)\left(\frac{\gamma}{2}-1\right) + e^{-2\lambda}\left(1-\frac{\gamma}{2}\right)\left(\frac{\gamma}{2}+1\right)}{\left(1+\frac{\gamma}{2}\right)\left(\frac{\gamma}{2}+1\right) + e^{-2\lambda}\left(1-\frac{\gamma}{2}\right)\left(\frac{\gamma}{2}-1\right)} \approx \frac{-1+e^{-2\lambda}}{1+\gamma - e^{-2\lambda}(1-\gamma)} = \frac{-\left(1-e^{-2\lambda}\right)}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})} \tag{40}$$

where we have dropped $O\left(\gamma^2\right)$ in the approximation.
Then using (26) and (23) for E_3 gives

$$\frac{E_3}{E_1} = e^{-i\omega D/c} \left[\frac{2\gamma e^{-\lambda}}{\left(1+\frac{\gamma}{2}\right)\left(\frac{\gamma}{2}+1\right) + e^{-2\lambda}\left(1-\frac{\gamma}{2}\right)\left(\frac{\gamma}{2}-1\right)} \right] \approx e^{-i\omega D/c} \left[\frac{2\gamma e^{-\lambda}}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})} \right] \tag{41}$$

which agrees with the claim up to a global phase factor (due to the shift of origin by $D\hat{\mathbf{z}}$).

(b) When $D \to 0$, $E_3/E_1 \to 1$, and when $D \to \infty$, $E_3/E_1 \to 0$ as expected.

(c) From (41), if we ignore the $\gamma\left(1+e^{-2\lambda}\right)$ term from the denominator, and notice $|\gamma|^2 = 2(\mathrm{Re}\,\gamma)^2$, we end up with

$$T = \frac{|E_3|^2}{|E_1|^2} \approx \frac{8(\mathrm{Re}\,\gamma)^2 e^{-2D/\delta}}{1 - 2e^{-2D/\delta}\cos(2D/\delta) + e^{-4D/\delta}} \tag{42}$$

This approximation is good until

$$\left|1 - e^{-2\lambda}\right| \approx \left|\gamma\left(1+e^{-2\lambda}\right)\right| \tag{43}$$

For small $|\lambda|$, this condition is roughly

$$|2\lambda| \approx |2\gamma| \qquad \text{or} \qquad D \approx \frac{\omega\delta^2}{c} \tag{44}$$