1. **Prob 10.4**

(a) With conductivity σ, the effective permittivity is

$$\widetilde{\epsilon}_r = \epsilon_r + \frac{i\sigma}{\omega\epsilon_0} = \epsilon_r + \frac{iZ_0\sigma}{k} \tag{1}$$

Since the wavelength is much larger than the sphere, it is feasible to use dipole scattering model described in section 10.1, in which case the differential scattering cross section and total cross section are given by (10.10), (10.11):

$$\frac{d\sigma}{d\Omega} = k^4R^6\left|\frac{\widetilde{\epsilon}_r - 1}{\widetilde{\epsilon}_r + 2}\right|^2\left(\frac{1+\cos^2\theta}{2}\right) = k^4R^6\left[\frac{(\epsilon_r-1)^2 + (Z_0\sigma/k)^2}{(\epsilon_r+2)^2 + (Z_0\sigma/k)^2}\right]\left(\frac{1+\cos^2\theta}{2}\right) \tag{2}$$

$$\sigma = \frac{8\pi}{3}k^4R^6\left|\frac{\widetilde{\epsilon}_r - 1}{\widetilde{\epsilon}_r + 2}\right|^2 = \frac{8\pi}{3}k^4R^6\left[\frac{(\epsilon_r-1)^2 + (Z_0\sigma/k)^2}{(\epsilon_r+2)^2 + (Z_0\sigma/k)^2}\right] \tag{3}$$

(b) The absorbed power at the surface of the sphere is

$$P_{\text{abs}} = -\frac{R^2}{2}\,\text{Re}\int \mathbf{n}\cdot\left(\mathbf{E}_{\text{int}}\times\mathbf{H}^*_{\text{int}}\right)d\Omega \tag{4}$$

where $\mathbf{n}$ is pointing outwards, and the subscript "int" indicates the field just inside the spherical interface. Recall the energy conservation relation (6.107)

$$-\int_V \mathbf{J}\cdot\mathbf{E}d^3x = \int_V\left[\frac{\partial u}{\partial t} + \boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H})\right]d^3x \tag{5}$$

The $\partial u/\partial t$ disappears after taking the time average of (5), and invoking the divergence theorem turns (4) into

$$P_{\text{abs}} = \frac{1}{2}\int_V \mathbf{J}\cdot\mathbf{E}^* d^3x \tag{6}$$

By assumption, the skin depth is much larger than the sphere, we can thus use the surface value $\mathbf{J}_{\text{int}}, \mathbf{E}_{\text{int}}$ to evaluate the volume integral, giving

$$P_{\text{abs}} = \frac{1}{2}\cdot\frac{4\pi R^3}{3}\sigma\left|\mathbf{E}_{\text{int}}\right|^2 \tag{7}$$

On the other hand, by (4.55)

$$\mathbf{E}_{\text{int}} = \left(\frac{3}{\widetilde{\epsilon}_r + 2}\right)\mathbf{E}_{\text{inc}} \tag{8}$$

giving the absorption cross section

$$\sigma_{\text{abs}} = \frac{P_{\text{abs}}}{\frac{1}{2Z_0}\left|\mathbf{E}_{\text{inc}}\right|^2} = \frac{12\pi Z_0R^3\sigma}{(\epsilon_r+2)^2 + (Z_0\sigma/k)^2} \tag{9}$$

(c) From the scattered field of dipole (10.2)

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\epsilon_0}k^2\frac{e^{ikr}}{r}\left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n} - \mathbf{n}\times\mathbf{m}/c\right] \tag{10}$$

and the definition of scattering amplitude

$$\mathbf{E}_{\text{sc}} = \frac{e^{ikr}}{r}\mathbf{F}(\mathbf{k},\mathbf{k}_0) \tag{11}$$

we can identify the scattering amplitude as

$$\mathbf{F}(\mathbf{k},\mathbf{k}_0) = \frac{1}{4\pi\epsilon_0}k^2\left[\left(\hat{\mathbf{k}}\times\mathbf{p}\right)\times\hat{\mathbf{k}}\right] \tag{12}$$

where $\mathbf{p}$ is given by (10.5)

$$\mathbf{p} = 4\pi\epsilon_0 \left(\frac{\widetilde{\epsilon}_r - 1}{\widetilde{\epsilon}_r + 2}\right) R^3 \mathbf{E}_{\text{inc}} \tag{13}$$

Note that in the forward direction $\mathbf{k} = \mathbf{k}_0$, $\mathbf{p}$ is perpendicular to $\mathbf{k}_0$, thus

$$\mathbf{F}(\mathbf{k} = \mathbf{k}_0) = k^2 R^3 \left(\frac{\widetilde{\epsilon}_r - 1}{\widetilde{\epsilon}_r + 2}\right) \mathbf{E}_{\text{inc}} \tag{14}$$

By the optical theorem, the total cross section is

$$\sigma_{\text{tot}} = \frac{\dfrac{2\pi}{kZ_0} \operatorname{Im}\left[\mathbf{E}_{\text{inc}}^* \cdot \mathbf{F}(\mathbf{k} = \mathbf{k}_0)\right]}{\dfrac{1}{2Z_0} \left|\mathbf{E}_{\text{inc}}\right|^2} = 4\pi k R^3 \operatorname{Im}\left(\frac{\widetilde{\epsilon}_r - 1}{\widetilde{\epsilon}_r + 2}\right) = \frac{12\pi Z_0 \sigma R^3}{(\epsilon_r + 2)^2 + (Z_0\sigma/k)^2} \tag{15}$$

which only matches the absorption cross section (9). This is expected since the scattering cross section (3) is of $O\left(k^3 R^3\right)$ order higher than the absorption cross section.

2. **Prob 10.5**

(a) For this part, let the direction of $\mathbf{B}_{\text{inc}}$ be the $\hat{\mathbf{z}}$ direction. Applying the integral form of Faraday's law to the disc at r, θ, we have

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int_{\text{disc}} \frac{\partial \mathbf{B}_{\text{inc}}}{\partial t} \cdot d\mathbf{a} \quad \Longrightarrow \quad \frac{J}{\sigma} \cdot 2\pi r \sin\theta = i\omega B_{\text{inc}} \pi r^2 \sin^2\theta \quad \Longrightarrow \quad \mathbf{J} = \frac{i\omega\sigma B_{\text{inc}} r \sin\theta}{2} \hat{\boldsymbol{\phi}} \tag{16}$$

Thus the magnetic dipole is

$$\begin{aligned}
\mathbf{m} &= \frac{1}{2}\int_V \mathbf{r} \times \mathbf{J} d^3x = \frac{1}{2}\frac{i\omega\sigma B_{\text{inc}}}{2} \int \left(-\hat{\boldsymbol{\theta}}\right) r^2 \sin\theta \, d^3x = \\
&= \hat{\mathbf{z}} \frac{1}{2}\frac{i\omega\sigma B_{\text{inc}}}{2} \overbrace{\int_0^R r^4 dr}^{R^5/5} \overbrace{\int_0^\pi \sin^3\theta \, d\theta}^{4/3} \int_0^{2\pi} d\phi \\
&= \frac{i2\pi\sigma\omega R^5 \mathbf{B}_{\text{inc}}}{15} \\
&= \frac{i4\pi\sigma Z_0}{k\mu_0} (kR)^2 \frac{R^3}{30} \mathbf{B}_{\text{inc}}
\end{aligned} \tag{17}$$

(b) With this magnetic dipole, from (10) and (11), the scattering amplitude is

$$\mathbf{F}(\mathbf{k}, \mathbf{k}_0) = \frac{1}{4\pi\epsilon_0} k^2 \left[\left(\hat{\mathbf{k}} \times \mathbf{p}\right) \times \hat{\mathbf{k}} - \hat{\mathbf{k}} \times \mathbf{m}/c\right] \tag{18}$$

For forward direction $\mathbf{k} = \mathbf{k}_0$, recall that $\mathbf{B}_{\text{inc}} = \hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}/c$, thus

$$\hat{\mathbf{k}}_0 \times \mathbf{m}/c = -\frac{1}{c^2}\frac{i4\pi\sigma Z_0 R^3}{30k\mu_0} (kR)^2 \mathbf{E}_{\text{inc}} = -4\pi\epsilon_0 \cdot \frac{i\sigma Z_0 R^3}{30k} (kR)^2 \mathbf{E}_{\text{inc}} \tag{19}$$

and

$$\mathbf{F}(\mathbf{k} = \mathbf{k}_0) = \mathbf{E}_{\text{inc}} k^2 R^3 \left[\left(\frac{\widetilde{\epsilon}_r - 1}{\widetilde{\epsilon}_r + 2}\right) + \frac{i\sigma Z_0}{30k} (kR)^2\right] \tag{20}$$

By optical theorem, the total cross section is

$$\sigma_{\text{tot}} = 4\pi k R^3 \operatorname{Im}\left[\left(\frac{\widetilde{\epsilon}_r - 1}{\widetilde{\epsilon}_r + 2}\right) + \frac{i\sigma Z_0}{30k} (kR)^2\right] = 12\pi R^3 \sigma Z_0 \left[\frac{1}{(\epsilon_r + 2)^2 + (Z_0\sigma/k)} + \frac{(kR)^2}{90}\right] \tag{21}$$