1. Let the usual 3-dimensional free space Green function be $G_3(x,y,z;x',y',z')$, and define $G_2(x,y;x,y')$ as

$$G_2(x,y;x',y') \equiv \int_{-Z}^{Z} G_3(x,y,z;x',y',z')d(z-z') \tag{1}$$

We end up with G_2 whose Laplacian with respect to (x',y',z') is

$$\begin{aligned}\nabla'^2 G_2 &= \nabla'^2 \int_{-Z}^{Z} G_3 d(z-z') \\ &= \int_{-Z}^{Z} \nabla'^2 G_3 d(z-z') \\ &= \int_{-Z}^{Z} -4\pi\delta(x-x')\delta(y-y')\delta(z-z')d(z-z') \\ &= -4\pi\delta(x-x')\delta(y-y')\end{aligned} \tag{2}$$

which means G_2 thus obtained is the 2-dimensional free space Green function.
Let's now do the integral explicitly with $G_3 = 1/|\mathbf{x}-\mathbf{x}'|$:

$$G_2(x,y;x',y') = \int_{-Z}^{Z} \frac{d(z-z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \tag{3}$$

Let $\rho = \sqrt{(x-x')^2+(y-y')^2}$ and let $z-z' = \rho\tan\xi$, (3) becomes

$$G_2 = \int_{\xi_0}^{\xi_1} \frac{\frac{\rho}{\cos^2\xi}d\xi}{\frac{\rho}{\cos\xi}} = \int_{\xi_0}^{\xi_1} \frac{d\xi}{\cos\xi} = \frac{1}{2}\ln\left(\frac{1+\sin\xi}{1-\sin\xi}\right)\Bigg|_{\xi_0}^{\xi_1} \tag{4}$$

Apparently

$$\tan\xi_1 = \frac{Z}{\rho} = -\tan\xi_0 \tag{5}$$

hence

$$\begin{aligned}G_2 &= \frac{1}{2}\left[\ln\left(\frac{1+\sin\xi_1}{1-\sin\xi_1}\right) - \ln\left(\frac{1+\sin\xi_0}{1-\sin\xi_0}\right)\right] \\ &= \frac{1}{2}\ln\left[\left(\frac{1+\sin\xi_1}{1-\sin\xi_1}\right)^2\right] \\ &= \ln\left(\frac{\sqrt{Z^2+\rho^2}+Z}{\sqrt{Z^2+\rho^2}-Z}\right) \\ &\equiv \ln X\end{aligned} \tag{6}$$

where

$$\begin{aligned}X = \frac{\sqrt{Z^2+\rho^2}+Z}{\sqrt{Z^2+\rho^2}-Z} &= \frac{1+1+\frac{1}{2}\frac{\rho^2}{Z^2}+O\left(Z^{-4}\right)}{\frac{1}{2}\frac{\rho^2}{Z^2}+O\left(Z^{-4}\right)} \\ &= \left[\frac{2+\frac{1}{2}\frac{\rho^2}{Z^2}+O\left(Z^{-4}\right)}{\frac{1}{2}\frac{\rho^2}{Z^2}}\right]\left[1+O\left(Z^{-2}\right)\right] \\ &= 4Z^2\left[\frac{1}{\rho^2}+O\left(Z^{-2}\right)\right]\end{aligned} \tag{7}$$

Plugging back into (6), we have

$$G_2(x,y,x',y') = \ln\left(4Z^2\right) + \ln\left[\frac{1}{\rho^2} + O\left(Z^{-2}\right)\right] \tag{8}$$

which behaves like $-\ln\rho^2$ when $Z \to \infty$, plus an "inessential" constant $\ln\left(4Z^2\right)$.

It feels a little unsatisfying when we have to swallow the mysterious constant of $\ln\left(4Z^2\right)$ when $Z \to \infty$. So let's explicitly verify that $G_2 = -\ln\rho^2$ is the Green function by definition.

To see this, let $G_a = -\ln\left(\rho^2 + a\right)$, we will show when $a \to 0$, we have

$$\nabla'^2 G_a = 0 \qquad\qquad \text{when } \rho \neq 0 \tag{9}$$

$$\int_0^{2\pi} d\phi \int_0^{\infty} \rho\, d\rho\, \nabla'^2 G_a = -4\pi \tag{10}$$

which together ensure that G_2 is the Green function. Indeed,

$$\begin{aligned}
-\nabla'^2 G_a &= \frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d\ln(\rho^2+a)}{d\rho}\right] \\
&= \frac{1}{\rho}\frac{d}{d\rho}\left(\frac{2\rho^2}{\rho^2+a}\right) \\
&= \frac{1}{\rho}\frac{\left(\rho^2+a\right)4\rho - 2\rho^2\cdot 2\rho}{(\rho^2+a)^2} = \frac{4a}{(\rho^2+a)^2}
\end{aligned} \tag{11}$$

which satisfies (9) with $a \to 0$, and

$$\begin{aligned}
\int_0^{2\pi} d\phi \int_0^{\infty} \rho\, d\rho\, \nabla'^2 G_a &= 2\pi\int_0^{\infty} \frac{-4a\rho\, d\rho}{(\rho^2+a)^2} \\
&= -4\pi a\left(-\frac{1}{\rho^2+a}\right)\Bigg|_0^{\infty} = -4\pi
\end{aligned} \tag{12}$$

which satisfies (10).

2. From now on, we need to change the meaning of ρ to mean $|\mathbf{x}|$, and ρ' to mean $|\mathbf{x}'|$. First let's take a note that in 2d cylindrical coordinates, $\delta(\mathbf{x}-\mathbf{x}')$ has the form

$$\delta\left(\mathbf{x}-\mathbf{x}'\right) = \frac{\delta\left(\rho-\rho'\right)}{\rho}\delta(\phi-\phi') \tag{13}$$

so the integral $\int d\phi' \int \rho' d\rho' \delta(\mathbf{x}-\mathbf{x}') = 1$.

Then it's straightforward to verify that

$$G = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} g_m(\rho,\rho') \qquad \text{where} \tag{14}$$

$$\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m}{\partial\rho'}\right) - \frac{m^2}{\rho'^2} g_m = -4\pi\frac{\delta(\rho-\rho')}{\rho} \tag{15}$$

is the Green function.

Surely, applying

$$\nabla'^2 = \frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right) + \frac{1}{\rho'^2}\frac{\partial^2}{\partial\phi'^2} \tag{16}$$

to G gives

$$\begin{aligned}
\nabla'^2 G &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\left[\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m}{\partial\rho'}\right) - \frac{m^2}{\rho'^2} g_m\right] e^{im(\phi-\phi')} \\
&= -4\pi\frac{\delta(\rho-\rho')}{\rho}\left[\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\right] \\
&= -4\pi\frac{\delta\left(\rho-\rho'\right)}{\rho}\delta\left(\phi-\phi'\right) = -4\pi\delta(\mathbf{x}-\mathbf{x}')
\end{aligned} \tag{17}$$

3. On either side $\rho' < \rho$ or $\rho' > \rho$, equation (15) is the same as (2.68) in the text, which has general solution

$$g_m(\rho,\rho') = \begin{cases} a_0 + b_0 \ln\rho' & m = 0 \\ a_m \rho'^m + b_m \rho'^{-m} & m \neq 0 \end{cases} \tag{18}$$

where a_0, b_0, a_m, b_m can be functions of ρ.

Integrate (15) with $\rho' d\rho'$ in the infinitesimal range of $[\rho - \epsilon, \rho + \epsilon]$, we get

$$\rho' \frac{\partial g_m}{\partial \rho'}\bigg|_{\rho+\epsilon} - \rho' \frac{\partial g_m}{\partial \rho'}\bigg|_{\rho-\epsilon} = -4\pi \tag{19}$$

- First consider $m > 0$. For the range $\rho' < \rho$, b_m must vanish since the region included the origin. Similarly for $\rho' > \rho$, $a_m = 0$. Thus the discontinuity in first-order derivative in (19) gives

$$-m b_m \rho^{-m} - m a_m \rho^m = -4\pi \qquad \Longrightarrow \qquad b_m \rho^{-m} + a_m \rho^m = \frac{4\pi}{m} \tag{20}$$

  But continuity of g_m at $\rho' = \rho$ requires

$$a_m \rho^m = b_m \rho^{-m} \tag{21}$$

  Combining (20) and (21) gives

$$a_m = \frac{2\pi}{m}\rho^{-m} \qquad b_m = \frac{2\pi}{m}\rho^m \tag{22}$$

  So

$$g_m(\rho,\rho') = \begin{cases} \dfrac{2\pi}{m}\left(\dfrac{\rho'}{\rho}\right)^m & \rho' < \rho \\ \dfrac{2\pi}{m}\left(\dfrac{\rho}{\rho'}\right)^m & \rho' > \rho \end{cases} \tag{23}$$

$$= \frac{2\pi}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \tag{24}$$

- When $m < 0$, a_m must vanish for $\rho' < \rho$, and b_m vanishes for $\rho' > \rho$, in which case (19) now gives

$$m a_m \rho^m + m b_m \rho^{-m} = -4\pi \qquad \Longrightarrow \qquad a_m \rho^m + b_m \rho^{-m} = \frac{-4\pi}{m} \tag{25}$$

  Continuity at $\rho' = \rho$ gives

$$b_m \rho^{-m} = a_m \rho^m \tag{26}$$

  Finally

$$a_m = -\frac{2\pi}{m}\rho^{-m} \qquad b_m = -\frac{2\pi}{m}\rho^m \tag{27}$$

  and

$$g_m(\rho,\rho') = \begin{cases} -\dfrac{2\pi}{m}\left(\dfrac{\rho}{\rho'}\right)^m & \rho' < \rho \\ -\dfrac{2\pi}{m}\left(\dfrac{\rho'}{\rho}\right)^m & \rho' > \rho \end{cases} \tag{28}$$

$$= -\frac{2\pi}{m}\left(\frac{\rho_>}{\rho_<}\right)^m \tag{29}$$

  Notice that (24) and (29) are the same for $\pm m$.

- When $m = 0$, let the general solution be

$$g_0(\rho,\rho') = \begin{cases} a_0 + b_0 \ln\rho' & \rho' < \rho \\ c_0 + d_0 \ln\rho' & \rho' > \rho \end{cases} \tag{30}$$

For the interior, we must set $b_0 = 0$ since otherwise when $\rho' \to 0$, g_0 will diverge. But it's ok to have $d_0 \neq 0$ in case of Neumann boundary condition at infinity, where we only require $\oint_S \partial G_N/\partial n' da' = -4\pi$ (see equation 1.45), which a non-zero d_0 can satisfy (considering in 2d problem $da' = \rho' d\rho'$).
Then applying (19) to this case will give

$$d_0 = -4\pi \tag{31}$$

and continuity at $\rho' = \rho$ gives

$$a_0 = c_0 - 4\pi \ln \rho \tag{32}$$

If we set the arbitrary constant c_0 to zero, then we have $a_0 = -4\pi \ln \rho$, hence

$$\begin{aligned} g_0(\rho, \rho') &= \begin{cases} -4\pi \ln \rho & \rho' < \rho \\ -4\pi \ln \rho' & \rho' > \rho \end{cases} \\ &= -4\pi \ln \rho_> \end{aligned} \tag{33}$$

Finally, plugging (24), (29), (33) into (14) gives

$$\begin{aligned} G(\rho, \phi; \rho', \phi') &= \frac{1}{2\pi} \left\{ -4\pi \ln(\rho_>) + \sum_{m=-\infty}^{-1} \left[-\frac{2\pi}{m} \left(\frac{\rho_>}{\rho_<} \right)^m e^{im(\phi-\phi')} \right] + \sum_{m=1}^{\infty} \left[\frac{2\pi}{m} \left(\frac{\rho_<}{\rho_>} \right)^m e^{im(\phi-\phi')} \right] \right\} \\ &= -\ln\left(\rho_>^2\right) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_<}{\rho_>} \right)^m \cos\left[m(\phi - \phi') \right] \end{aligned} \tag{34}$$

It's worth emphasizing that we have the $\ln \rho_>$ term in the final result because we are dealing with 2d problem, where we can afford to let $\partial G/\partial n'$ behave like $1/\rho$ at infinity so when it's multiplied with the area element $da' \propto \rho' d\rho'$ it can still satisfy the Neumann boundary condition. This is not allowed in 3d, since the area element is $r^2 dr$.

On the other hand, if we are dealing with the Dirichlet boundary condition, the $\ln \rho$ term will not be allowed, as is shown in the example in the text from page 77.