1. **Prob 6.18**

(a) For the Dirac string L leading from infinity to the origin along the negative z-axis, the integral can be evaluated directly:

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \frac{g}{4\pi}\int_L \frac{d\mathbf{l}'\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \\
&= \frac{g}{4\pi}\int_{-\infty}^{0}\frac{dz'\hat{\mathbf{z}}\times\left[\rho\hat{\boldsymbol{\rho}}+(z-z')\hat{\mathbf{z}}\right]}{\sqrt{\rho^2+(z-z')^2}^3} \\
&= \frac{g}{4\pi}\rho\hat{\boldsymbol{\phi}}\int_{-\infty}^{0}\frac{dz'}{\sqrt{\rho^2+(z-z')^2}^3} \qquad \text{define } \tan\xi\equiv\frac{z-z'}{\rho} \\
&= \frac{g\rho\hat{\boldsymbol{\phi}}}{4\pi}\int_{\pi/2}^{\tan^{-1}(z/\rho)}\frac{-\rho\frac{1}{\cos^2\xi}d\xi}{\rho^3\frac{1}{\cos^3\xi}} \\
&= \frac{g\hat{\boldsymbol{\phi}}}{4\pi\rho}\sin\xi\Big|_{\tan^{-1}(z/\rho)}^{\pi/2} = \frac{g\hat{\boldsymbol{\phi}}}{4\pi r\sin\theta}(1-\cos\theta)
\end{aligned} \tag{1}$$

(b) By curl formula in spherical coordinates, we have

$$\mathbf{B}=\nabla\times\mathbf{A}=\frac{1}{r\sin\theta}\frac{\partial\left(A_\phi\sin\theta\right)}{\partial\theta}\hat{\mathbf{r}}-\frac{1}{r}\frac{\partial\left(rA_\phi\right)}{\partial r}\hat{\boldsymbol{\theta}}=\frac{g}{4\pi r^2}\hat{\mathbf{r}} \tag{2}$$

which is the point-source field. Note since the line integral is along the $\theta=\pi$ path, (2) should exclude observation points on the negative z-axis where $\mathbf{A}$ is singular.

Here, we also give a general proof that $\nabla\times\mathbf{A}$ agrees with point-source field regardless of the path L we are taking.

Indeed, for $\mathbf{x}\neq\mathbf{x}'$, denote

$$\mathbf{v}\equiv\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}=-\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{3}$$

we would have

$$\nabla\cdot\mathbf{v}=-\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)=0 \qquad \text{for } \mathbf{x}\neq\mathbf{x}' \tag{4}$$

Thus

$$\begin{aligned}
\nabla\times\mathbf{A} &= \frac{g}{4\pi}\int_L \nabla\times\left(d\mathbf{l}'\times\mathbf{v}\right) \\
&= \frac{g}{4\pi}\left[\int_L d\mathbf{l}'(\nabla\cdot\mathbf{v})-\int_L\left(d\mathbf{l}'\cdot\nabla\right)\mathbf{v}\right] \\
&= -\frac{g}{4\pi}\int_L\left(d\mathbf{l}'\cdot\nabla\right)\mathbf{v}
\end{aligned} \tag{5}$$

Using the fact

$$\frac{\partial\mathbf{v}}{\partial x_i}=-\frac{\partial\mathbf{v}}{\partial x_i'} \tag{6}$$

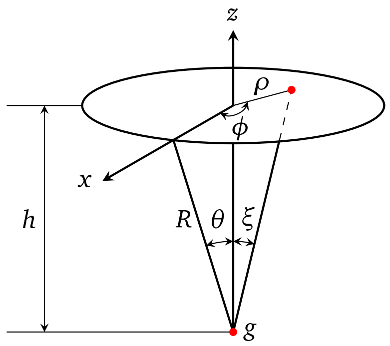
we have

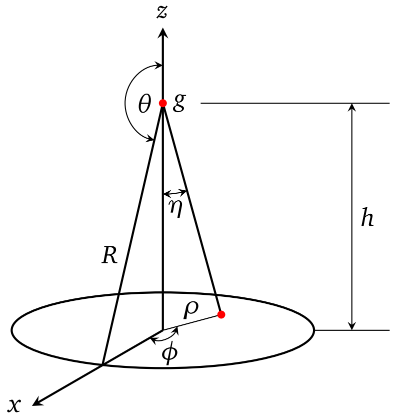
$$\nabla\times\mathbf{A}=\frac{g}{4\pi}\int_L\sum_i dl_i'\frac{\partial\mathbf{v}}{\partial x_i'} \tag{7}$$

where we can see that the integrand is the total differential of $\mathbf{v}$ along an infinitesimal segment of L, thus the entire integral gives the difference of $\mathbf{v}$ at its two endpoints, one at the monopole, the other at infinity, hence

$$\nabla\times\mathbf{A}=\frac{g}{4\pi}\mathbf{v}=\frac{g}{4\pi}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{8}$$

(c) We shall consider separately the cases where $\theta < \pi/2$ and $\theta > \pi/2$.





- For $\theta < \pi/2$ (see figure on the left), for any point on the disc at $z = h$, the z-component of the magnetic induction is

$$B_z = \frac{g\cos\xi}{4\pi\,(h/\cos\xi)^2} = \frac{g}{4\pi h^2}\cos^3\xi \tag{9}$$

Thus the total flux through the upper disc is

$$\begin{aligned}
\Phi_{\text{upper-disc}} &= \int_0^{h\tan\theta} \rho\, d\rho \int_0^{2\pi} d\phi B_z \qquad\qquad \text{where } \rho = h\tan\xi \\
&= 2\pi \int_0^{\theta} (h\tan\xi)\frac{h}{\cos^2\xi}d\xi \cdot \frac{g}{4\pi h^2}\cos^3\xi \\
&= \frac{g}{2}\int_0^{\theta} \sin\xi\, d\xi = \frac{g}{2}(1-\cos\theta)
\end{aligned} \tag{10}$$

- For $\theta > \pi/2$ (see figure on the right), for any point on the disc at $z = -h$, the z-component of the magnetic induction is

$$B_z = -\frac{g\cos\eta}{4\pi\,(h/\cos\eta)^2} = -\frac{g}{4\pi h^2}\cos^3\eta \tag{11}$$

and the total flux through the lower disc is

$$\begin{aligned}
\Phi_{\text{lower-disc}} &= \int_0^{h\tan(\pi-\theta)} \rho\, d\rho \int_0^{2\pi} d\phi B_z \qquad\qquad \text{where } \rho = h\tan\eta \\
&= -2\pi \int_0^{\pi-\theta} (h\tan\eta)\frac{h}{\cos^2\eta}d\eta \cdot \frac{g}{4\pi h^2}\cos^3\eta \\
&= -\frac{g}{2}\int_0^{\pi-\theta} \sin\eta\, d\eta = -\frac{g}{2}(1+\cos\theta)
\end{aligned} \tag{12}$$

(d) With (1), we have

$$\oint \mathbf{A}\cdot d\mathbf{l} = A_\phi \cdot 2\pi R\sin\theta = \frac{g}{4\pi R}\,\frac{1-\cos\theta}{\sin\theta}\cdot 2\pi R\sin\theta = \frac{g}{2}(1-\cos\theta) \tag{13}$$

which agrees with (10) but differs from (12) by g.
The difference is a result of the fact that the vector potential $\mathbf{A}$ in (1) is not defined on the Dirac string, in our case the negative z-axis. I.e.,

$$\mathbf{B}_{\text{monopole}} = \nabla\times\mathbf{A} - \mathbf{B}' \qquad\qquad \text{where } \mathbf{B}' \text{ vanishes except on } L \tag{14}$$

The correction term $\mathbf{B}'$ is necessary so $\nabla\cdot(\nabla\times\mathbf{A}) = 0$ for all points in space. In other words, $\mathbf{B}'$ provides the "return flux" so

$$\oint_S \mathbf{B}'\cdot\mathbf{n}da = -\oint_S \mathbf{B}_{\text{monopole}}\cdot\mathbf{n}da = -g \qquad\qquad \text{for any sphere enclosing } g \tag{15}$$

Now we can see why $\oint \mathbf{A}\cdot d\mathbf{l}$ along upper loop agrees with the flux $\Phi_{\text{upper-disc}}$ because this is just the Stokes theorem for virtue of $\mathbf{B}_{\text{monopole}} = \nabla\times\mathbf{A}$ in the northern hemisphere. On the other hand when $\theta > \pi/2$, applying Stokes theorem on the lower disc gives

$$\oint_{\text{lower-loop}} \mathbf{A}\cdot d\mathbf{l} = \int_{\text{lower-disc}} (\nabla\times\mathbf{A})\cdot\mathbf{n}da = \int_{\text{lower-disc}} \mathbf{B}_{\text{monopole}}\cdot\mathbf{n}da + \int_{\text{lower-disc}} \mathbf{B}'\cdot\mathbf{n}da \tag{16}$$

We see that due to the spherical symmetry of $\mathbf{B}_{\text{monopole}}$, the first term in (16) is the same as (10), the monopole flux through the upper disc, and the second term is just $-g$, the "return flux" concentrated on the Dirac string, needed to make the divergence of $\nabla\times\mathbf{A}$ vanish. This is the same $-g$ difference between (12) and (10).

2. **Prob 6.19**

(a) Under space inversion, the spherical coordinates will transform as

$$r\to r \qquad \theta\to\pi-\theta \qquad \phi\to\pi+\phi \tag{17}$$

This means

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \qquad \to \qquad \sin\phi\,\hat{\mathbf{x}} - \cos\phi\,\hat{\mathbf{y}} = -\hat{\boldsymbol{\phi}} \tag{18}$$

Thus the vector potential in (1) will transform as

$$\mathbf{A} = \frac{g\hat{\boldsymbol{\phi}}}{4\pi r\sin\theta}(1-\cos\theta) \qquad \to \qquad \mathbf{A}' = -\frac{g\hat{\boldsymbol{\phi}}}{4\pi r\sin\theta}(1+\cos\theta) \tag{19}$$

(b) Clearly

$$\delta\mathbf{A} = \mathbf{A}'-\mathbf{A} = -\frac{g\hat{\boldsymbol{\phi}}}{2\pi r\sin\theta} = -\frac{g}{2\pi}\left(-\frac{y\hat{\mathbf{x}}}{x^2+y^2} + \frac{x\hat{\mathbf{y}}}{x^2+y^2}\right) = -\frac{g}{2\pi}\nabla\left[\tan^{-1}\left(\frac{y}{x}\right)\right] \tag{20}$$

(c) Let's use L to denote the Dirac string leading from infinity to g along the negative z-axis, as in problem 6.18, and use L' to denote the space-inverted L, i.e., the Dirac string leading from infinity to g along the positive z-axis. On the one hand, by (20),

$$\mathbf{A}_{L'} - \mathbf{A}_L = -\frac{g}{2\pi}\nabla\left[\tan^{-1}\left(\frac{y}{x}\right)\right] \tag{21}$$

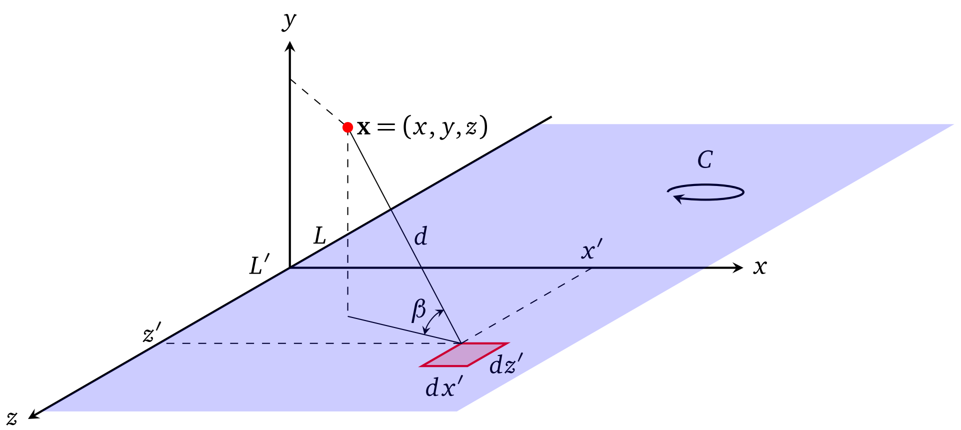
on the other hand by Jackson 6.162,

$$\mathbf{A}_{L'} - \mathbf{A}_L = \frac{g}{4\pi}\nabla\Omega_C(\mathbf{x}) \tag{22}$$

We shall show the two expressions are compatible, i.e.,

$$\Omega_C(\mathbf{x}) = -2\tan^{-1}\left(\frac{y}{x}\right) + \text{constant} \tag{23}$$

Here, the contour C is the half plane $y=0$ with z axis as one side, and all the other three sides at infinity. C runs along with the $-\hat{\mathbf{z}}$ direction (since we chose the difference $\mathbf{A}_{L'}-\mathbf{A}_L$), hence its positive normal direction is $-\hat{\mathbf{y}}$.



With the sign convention of Ω_C stated in problem 5.1, we see that at $\mathbf{x}$, the differential solid angle subtended by the patch $(x', x'+dx') \times (z', z'+dz')$ is

$$d\Omega_C\left(x', z'\right) = \frac{dx'dz'\sin\beta}{d^2} = \frac{y\,dx'dz'}{\sqrt{(x-x')^2 + y^2 + (z-z')^2}^3} \tag{24}$$

whose integral over the half plane gives the total solid angle

$$\begin{aligned}
\Omega_C &= \int_0^\infty dx' \int_{-\infty}^\infty dz' \frac{y}{\sqrt{(x-x')^2+y^2+(z-z')^2}^3} && \text{let } \rho^2 \equiv \left(x-x'\right)^2 + y^2, \tan\xi \equiv \frac{z-z'}{\rho} \\
&= \int_0^\infty y\,dx' \int_{\pi/2}^{-\pi/2} \frac{-\rho\frac{1}{\cos^2\xi}d\xi}{\rho^3\frac{1}{\cos^3\xi}} \\
&= \int_0^\infty \frac{2y\,dx'}{\rho^2} && \text{let } \tan\eta \equiv \frac{x-x'}{y} \\
&= \int_{\tan^{-1}(x/y)}^{-\pi/2} \frac{-2y^2\frac{1}{\cos^2\eta}d\eta}{y^2\frac{1}{\cos^2\eta}} \\
&= -2\left[-\frac{\pi}{2} - \tan^{-1}\left(\frac{x}{y}\right)\right] = 2\left[\pi - \tan^{-1}\left(\frac{y}{x}\right)\right]
\end{aligned} \tag{25}$$

which proved (23).