1. We just need to verify that the alleged Green function

$$G(x,y;x',y') = 2\sum_{n=1}^{\infty} g_n(y,y')\sin(n\pi x)\sin(n\pi x') \tag{1}$$

satisfies

(a) $G = 0$ on the boundary (as function of $\mathbf{x}'$);
(b) $\nabla'^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$.

(a) is obviously satisfied by definition of G and g_n. To see (b), notice

$$\frac{\partial^2 G}{\partial x'^2} = -2\sum_{n=1}^{\infty} n^2\pi^2 g_n \sin(n\pi x)\sin(n\pi x') \tag{2}$$

$$\frac{\partial^2 G}{\partial y'^2} = 2\sum_{n=1}^{\infty} \frac{\partial^2 g_n}{\partial y'^2}\sin(n\pi x)\sin(n\pi x') \tag{3}$$

Thus

$$\begin{aligned}\nabla'^2 G &= 2\sum_{n=1}^{\infty}\left(\frac{\partial^2 g_n}{\partial y'^2} - n^2\pi^2 g_n\right)\sin(n\pi x)\sin(n\pi x')\\ &= \left[-4\pi\delta\left(y'-y\right)\right]\cdot 2\sum_{n=1}^{\infty}\sin(n\pi x)\sin(n\pi x')\end{aligned} \tag{4}$$

To get the correct overall factor, let $U_n(x) = a\sin(n\pi x)$ be the set or orthogonal basis over range $[0,1]$ (not counting cosines for now). Normality requires

$$\int_0^1 |U_n(x)|^2 dx = \int_0^1 a^2\sin^2(n\pi x)\,dx = 1 \qquad \Longrightarrow \qquad a = \sqrt{2} \tag{5}$$

Hence over the range $[0,1]$, the completeness relation is given by

$$\sum_{n=1}^{\infty} a^2\sin(n\pi x)\sin(n\pi x') = \delta\left(x'-x\right) \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty}\sin(n\pi x)\sin(n\pi x') = \frac{1}{2}\delta\left(x'-x\right) \tag{6}$$

Thus (4) becomes

$$\nabla'^2 G = -4\pi\delta\left(y'-y\right)\delta\left(x'-x\right) = -4\pi\delta\left(\mathbf{x}-\mathbf{x}'\right) \tag{7}$$

2. Given that $g_n(y,y')$ satisfies

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right) g_n(y,y') = -4\pi\delta\left(y'-y\right) \tag{8}$$

We will try the following ansatz:

$$g_n(y,y') = \begin{cases} A\sinh(n\pi y') + B\cosh(n\pi y') & \text{for } y' < y \\ C\sinh[n\pi(1-y')] + D\cosh[n\pi(1-y')] & \text{for } y' > y\end{cases} \tag{9}$$

Boundary condition $g_n(y,0) = 0$ requires $B = 0$, and similarly $g_n(y,1) = 0$ requires $D = 0$. Now integrate (8) with respect to dy' over the infinitesimal range $[y-\epsilon, y+\epsilon]$, we get

$$\begin{aligned}\int_{y-\epsilon}^{y+\epsilon}\left[\frac{\partial}{\partial y'}\left(\frac{\partial g_n}{\partial y'}\right) - n^2\pi^2 g_n\right]dy' = \int_{y-\epsilon}^{y+\epsilon} -4\pi\delta\left(y'-y\right)dy' \qquad &\Longrightarrow\\ \left.\frac{\partial g_n}{\partial y'}\right|_{y+\epsilon} - \left.\frac{\partial g_n}{\partial y'}\right|_{y-\epsilon} = -4\pi \qquad &\Longrightarrow\\ -n\pi\cdot C\cosh[n\pi(1-y)] - n\pi\cdot A\cosh(n\pi y) = -4\pi \qquad &\Longrightarrow\\ C\cosh[n\pi(1-y)] + A\cosh(n\pi y) = \frac{4}{n}\end{aligned} \tag{10}$$

Recall hyperbolic sum of angle formula

$$\sinh(\eta+\xi) = \cosh\eta\sinh\xi + \sinh\eta\cosh\xi \tag{11}$$

gives

$$\cosh[n\pi(1-y)]\sinh(n\pi y) + \sinh[n\pi(1-y)]\cosh(n\pi y) = \sinh(n\pi) \tag{12}$$

Compare (12) with (10), we can identify (up to an overall factor)

$$A = \frac{4}{n\sinh(n\pi)}\sinh[n\pi(1-y)] \tag{13}$$

$$C = \frac{4}{n\sinh(n\pi)}\sinh(n\pi y) \tag{14}$$

which gives

$$\begin{aligned} g_n(y,y') &= \begin{cases} \dfrac{4}{n\sinh(n\pi)}\sinh[n\pi(1-y)]\sinh\left(n\pi y'\right) & \text{for } y' < y \\ \dfrac{4}{n\sinh(n\pi)}\sinh(n\pi y)\sinh\left[n\pi\left(1-y'\right)\right] & \text{for } y' > y \end{cases} \\ &= \frac{4}{n\sinh(n\pi)}\sinh(n\pi y_<)\sinh[n\pi(1-y_>)] \end{aligned} \tag{15}$$

And the full Green function is

$$G(\mathbf{x},\mathbf{x}') = 8\sum_{n=1}^{\infty}\frac{1}{n\sinh(n\pi)}\sin(n\pi x)\sin\left(n\pi x'\right)\sinh(n\pi y_<)\sinh[n\pi(1-y_>)] \tag{16}$$