Here we fill in some derivation details in section 8.5.

1. **Equation (8.48)**

For TM mode, $\psi e^{ikz} = E_z$ and $H_z = 0$. Then by (8.33) and (8.31)

$$\mathbf{E}_t = \frac{ik}{\gamma^2}\boldsymbol{\nabla}_t E_z \qquad \mathbf{H}_t = \frac{\epsilon\omega}{k}\hat{\mathbf{z}}\times\mathbf{E}_t \tag{1}$$

Then

$$\begin{aligned}
\mathbf{S} &= \frac{1}{2}\mathbf{E}\times\mathbf{H}^* = \frac{1}{2}\left(\mathbf{E}_t + E_z\hat{\mathbf{z}}\right)\times\mathbf{H}_t^* \\
&= \frac{1}{2}\left(\frac{ik}{\gamma^2}\boldsymbol{\nabla}_t E_z + E_z\hat{\mathbf{z}}\right)\times\left[\frac{\epsilon\omega}{k}\hat{\mathbf{z}}\times\left(\frac{-ik}{\gamma^2}\boldsymbol{\nabla}_t E_z^*\right)\right] \\
&= \frac{1}{2}\left[\frac{k\epsilon\omega}{\gamma^4}\boldsymbol{\nabla}_t E_z\times\left(\hat{\mathbf{z}}\times\boldsymbol{\nabla}_t E_z^*\right) - \frac{i\epsilon\omega}{\gamma^2}E_z\hat{\mathbf{z}}\times\left(\hat{\mathbf{z}}\times\boldsymbol{\nabla}_t E_z^*\right)\right] \\
&= \frac{1}{2}\left(\frac{k\epsilon\omega}{\gamma^4}\hat{\mathbf{z}}\,|\boldsymbol{\nabla}_t E_z|^2 + \frac{i\epsilon\omega}{\gamma^2}E_z\boldsymbol{\nabla}_t E_z^*\right)
\end{aligned} \tag{2}$$

which is the upper line of (8.48).

For TE mode, $\psi e^{ikz} = H_z, E_z = 0$,

$$\mathbf{H}_t = \frac{ik}{\gamma^2}\boldsymbol{\nabla}_t H_z \qquad \mathbf{E}_t = -\frac{\mu\omega}{k}\hat{\mathbf{z}}\times\mathbf{H}_t \tag{3}$$

which gives

$$\mathbf{S} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^* = \left(-\frac{1}{2}\mathbf{H}\times\mathbf{E}^*\right)^* = \left\{\frac{1}{2}\left(\frac{ik}{\gamma^2}\boldsymbol{\nabla}_t H_z + H_z\hat{\mathbf{z}}\right)\times\left[\frac{\mu\omega}{k}\hat{\mathbf{z}}\times\left(\frac{-ik}{\gamma^2}\boldsymbol{\nabla}_t H_z^*\right)\right]\right\}^* \tag{4}$$

Comparing this with the second line of (2), evidently we can compute (4) by substituting $\mu \longleftrightarrow \epsilon, E_z \longleftrightarrow H_z$ in (2) followed by taking the complex conjugate, i.e.,

$$\mathbf{S} = \frac{1}{2}\left(\frac{k\mu\omega}{\gamma^4}\hat{\mathbf{z}}\,|\boldsymbol{\nabla}_t H_z|^2 - \frac{i\mu\omega}{\gamma^2}H_z^*\boldsymbol{\nabla}_t H_z\right) \tag{5}$$

which is the lower line of (8.48).

2. **Equation (8.52)**

For TM mode, the energy density is

$$\begin{aligned}
u &= \frac{\epsilon}{4}\left(\mathbf{E}_t\cdot\mathbf{E}_t^* + E_zE_z^*\right) + \frac{\mu}{4}\left(\mathbf{H}_t\cdot\mathbf{H}_t^*\right) \\
&= \frac{\epsilon}{4}\left(\frac{k^2}{\gamma^4}|\boldsymbol{\nabla}_t E_z|^2 + E_zE_z^*\right) + \frac{\mu}{4}\left(\frac{\epsilon^2\omega^2}{\gamma^4}|\boldsymbol{\nabla}_t E_z|^2\right) \\
&= \frac{\epsilon}{4\gamma^4}\left(k^2 + \mu\epsilon\omega^2\right)|\boldsymbol{\nabla}_t E_z|^2 + \frac{\epsilon}{4}E_zE_z^*
\end{aligned} \tag{6}$$

Using

$$\gamma^2 = \mu\epsilon\omega_\lambda^2 \qquad k^2 = \mu\epsilon\left(\omega^2 - \omega_\lambda^2\right) \tag{7}$$

as well as (see equation (8.49), (8.50) and the eigenequation (8.34))

$$\int_A |\boldsymbol{\nabla}_t\psi|^2\,da = -\int_A \psi^*\boldsymbol{\nabla}_t^2\psi\,da = \gamma^2\int_A \psi^*\psi\,da \tag{8}$$

then we have

$$U = \int_A u\,da = \left[\frac{\epsilon}{4\gamma^2}\cdot\mu\epsilon\left(2\omega^2 - \omega_\lambda^2\right) + \frac{\epsilon}{4}\right]\int_A E_zE_z^*\,da = \frac{\epsilon}{2}\left(\frac{\omega}{\omega_\lambda}\right)^2\int_A E_zE_z^*\,da \tag{9}$$

For TM mode, we only need to replace the density with

$$u = \frac{\mu}{4}\left(\mathbf{H}_t \cdot \mathbf{H}_t^* + H_z H_z^*\right) + \frac{\epsilon}{4}\left(\mathbf{E}_t \cdot \mathbf{E}_t^*\right) = \frac{\mu}{4}\left(\frac{k^2}{\gamma^4}\left|\boldsymbol{\nabla}_t H_z\right|^2 + H_z H_z^*\right) + \frac{\epsilon}{4}\left(\frac{\mu^2\omega^2}{\gamma^4}\left|\boldsymbol{\nabla}_t H_z\right|^2\right) \tag{10}$$

The remaining steps are similar to (6)-(9), with the substitution $\mu \longleftrightarrow \epsilon, E_z \longleftrightarrow H_z$,

$$U = \frac{\mu}{2}\left(\frac{\omega}{\omega_\lambda}\right)^2 \int_A H_z H_z^* da \tag{11}$$

3. **Equation (8.59)**

For TM mode, by (1),

$$\mathbf{H}_t = \frac{i\epsilon\omega}{\gamma^2}\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t E_z = \frac{i\epsilon\omega}{\gamma^2}\hat{\mathbf{z}} \times \left(\frac{\partial E_z}{\partial n}\mathbf{n} + \frac{\partial E_z}{\partial m}\mathbf{m}\right) \tag{12}$$

where $\mathbf{m} = \hat{\mathbf{z}} \times \mathbf{n}$ is a unit vector in the transverse plane orthogonal to $\mathbf{n}$.

Thus applying (8.58) gives

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\oint_C \left|\mathbf{n} \times \mathbf{H}\right|^2 dl = \frac{1}{2\sigma\delta}\frac{\epsilon^2\omega^2}{\gamma^4}\oint_C \left|\frac{\partial E_z}{\partial n}\right|^2 dl = \frac{1}{2\sigma\delta}\frac{\omega^2}{\omega_\lambda^4\mu^2}\oint_C \left|\frac{\partial E_z}{\partial n}\right|^2 dl \tag{13}$$

For TE mode,

$$\mathbf{H} = \mathbf{H}_t + H_z\hat{\mathbf{z}} = \frac{ik}{\gamma^2}\boldsymbol{\nabla}_t H_z + H_z\hat{\mathbf{z}} \tag{14}$$

So,

$$\begin{aligned} -\frac{dP}{dz} &= \frac{1}{2\sigma\delta}\oint_C \left|\mathbf{n} \times \mathbf{H}\right|^2 dl = \frac{1}{2\sigma\delta}\oint_C \left(\frac{k^2}{\gamma^4}\left|\mathbf{n} \times \boldsymbol{\nabla}_t H_z\right|^2 + \left|H_z\right|^2\right) dl \\ &= \frac{1}{2\sigma\delta}\oint_C \left[\left(\frac{\omega^2 - \omega_\lambda^2}{\mu\epsilon\omega_\lambda^4}\right)\left|\mathbf{n} \times \boldsymbol{\nabla}_t H_z\right|^2 + \left|H_z\right|^2\right] dl \end{aligned} \tag{15}$$

4. **Geometric factor ξ_{m0}, η_{m0} of TE mode rectangular waveguide**

For TE-$m0$ of the rectangular waveguide,

$$H_z = H_0\cos\left(\frac{m\pi x}{a}\right)e^{ikz} \implies \text{on the contour } \left|\mathbf{n} \times \boldsymbol{\nabla}_t H_z\right|^2 = \begin{cases} \left(\frac{m\pi H_0}{a}\right)^2 \sin^2\left(\frac{m\pi x}{a}\right) & \text{for } y = 0 \text{ or } y = b \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

With

$$\mu\epsilon\omega_{m0}^2 = \gamma_{m0}^2 = \frac{\pi^2 m^2}{a^2} \tag{17}$$

(8.59) yields

$$\begin{aligned} -\frac{dP}{dz} &= \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{m0}}\right)^2 \left|H_0\right|^2 \left[2\left(1 - \frac{\omega_{m0}^2}{\omega^2}\right)\overbrace{\int_0^a \sin^2\left(\frac{m\pi x}{a}\right)dx}^{a/2} + \left(\frac{\omega_{m0}^2}{\omega^2}\right)\overbrace{\oint_C \cos^2\left(\frac{m\pi x}{a}\right)dl}^{a+2b}\right] \\ &= \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{m0}}\right)^2 \left|H_0\right|^2 \left[a + 2b\left(\frac{\omega_{m0}^2}{\omega^2}\right)\right] \end{aligned} \tag{18}$$

On the other hand, by (8.51)

$$P = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega}{\omega_{m0}}\right)^2 \sqrt{1 - \frac{\omega_{m0}^2}{\omega^2}}\left|H_0\right|^2 \overbrace{\int_0^b dy \int_0^a \cos^2\left(\frac{m\pi x}{a}\right)dx}^{ab/2} \tag{19}$$

Matching $\beta_{m0} = -(dP/dz)/2P$ with (8.63) yields

$$\xi_{m0} = \frac{a}{a+b} \qquad \eta_{m0} = \frac{2b}{a+b} \tag{20}$$