1. **Prob 7.16**

 (a) For now, let's follow the hint and assume that the solutions of displacement $\mathbf{D}$ and electric field $\mathbf{E}$ have the form

$$\mathbf{D}(\mathbf{x},t) = \mathbf{D}_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \qquad \mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \tag{1}$$

We will later justify this ansatz form by proving its consistency with the Maxwell equations.
The Maxwell equations

$$\boldsymbol{\nabla}\times\mathbf{E} - i\omega\mathbf{B} = 0 \qquad \boldsymbol{\nabla}\times\mathbf{H} + i\omega\mathbf{D} = 0 \tag{2}$$

will require

$$\boldsymbol{\nabla}\times\left(\frac{\boldsymbol{\nabla}\times\mathbf{E}}{i\omega\mu_0}\right) + i\omega\mathbf{D} = 0 \qquad \Longrightarrow \qquad \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) - \omega^2\mu_0\mathbf{D} = 0 \tag{3}$$

Replacing (1) into (3) gives

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_0) + \omega^2\mu_0\mathbf{D}_0 = 0 \tag{4}$$

In the subsequent discussion, we will use $\mathbf{E},\mathbf{D}$ to represent the amplitudes $\mathbf{E}_0,\mathbf{D}_0$.

 (b) With the vector identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$, (4) can be written as

$$(\mathbf{k}\cdot\mathbf{E})\mathbf{k} - k^2\mathbf{E} + \omega^2\mu_0\mathbf{D} = 0 \qquad \Longrightarrow \qquad (\mathbf{n}\cdot\mathbf{E})\mathbf{n} - \left(\mathbf{E} - \frac{\omega^2\mu_0}{k^2}\mathbf{D}\right) = 0 \tag{5}$$

In each principal direction i, (5) turns into

$$\left(\sum_{j=1}^{3} n_j \frac{D_j}{\mu_0\epsilon_j}\right) n_i - \left(\frac{1}{\mu_0\epsilon_j} - \frac{\omega^2}{k^2}\right) D_i = 0 \qquad \Longrightarrow \qquad \left(\sum_{j=1}^{3} n_j v_j^2 D_j\right) n_i - \left(v_i^2 - v^2\right) D_i = 0 \tag{6}$$

which is a homogeneous system of equations

$$\begin{bmatrix} n_1^2v_1^2 - \left(v_1^2 - v^2\right) & n_1n_2v_2^2 & n_1n_3v_3^2 \\ n_2n_1v_1^2 & n_2^2v_2^2 - \left(v_2^2 - v^2\right) & n_2n_3v_3^2 \\ n_3n_1v_1^2 & n_3n_2v_2^2 & n_3^2v_3^2 - \left(v_3^2 - v^2\right) \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = 0 \tag{7}$$

or, in eigenequation form

$$\left(v^2 I - A\right)\mathbf{D} = 0 \tag{8}$$

where

$$A = \begin{bmatrix} \left(1-n_1^2\right)v_1^2 & -n_1n_2v_2^2 & -n_1n_3v_3^2 \\ -n_2n_1v_1^2 & \left(1-n_2^2\right)v_2^2 & -n_2n_3v_3^2 \\ -n_3n_1v_1^2 & -n_3n_2v_2^2 & \left(1-n_3^2\right)v_3^3 \end{bmatrix} \tag{9}$$

For any non-zero $\mathbf{D}$ to be possible at all, the determinant of the matrix $v^2I - A$ must vanish.
It is obvious that v^2 can take three values (the eigenvalues of A), with possibility of degeneracy.
Denoting

$$a_i = n_i^2 v_i^2 \qquad b_i = v_i^2 - v^2 \tag{10}$$

then $\det\left(v^2 I - A\right) = 0$ implies

$$(a_1 - b_1)(a_2 - b_2)(a_3 - b_3) + 2a_1a_2a_3 - a_1a_3(a_2 - b_2) - a_1a_2(a_3 - b_3) - a_2a_3(a_1 - b_1) = 0$$

or,

$$
\begin{aligned}
a_1 b_2 b_3 + a_2 b_1 b_3 + a_3 b_1 b_2 = b_1 b_2 b_3 & \qquad \Longrightarrow \\
\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} = 1 & \qquad \Longrightarrow \\
\frac{n_1^2 v_1^2}{v_1^2 - v^2} + \frac{n_2^2 v_2^2}{v_2^2 - v^2} + \frac{n_3^2 v_3^2}{v_3^2 - v^2} = 1 = n_1^2 + n_2^2 + n_3^2 & \qquad \Longrightarrow \\
\sum_{i=1}^{3} \frac{n_i^2 v^2}{v_i^2 - v^2} = 0 & \qquad (11)
\end{aligned}
$$

Immediately, we see that $v^2 = 0$ is an eigenvalue, which corresponds to the trivial case of non-propagating fields. By (11), the non-zero eigenvalues satisfy

$$\sum_{i=1}^{3} \frac{n_i^2}{v_i^2 - v^2} = 0 \tag{12}$$

which is equivalent to the quadratic equation in v^2:

$$n_1^2 \left(v_2^2 - v^2\right)\left(v_3^2 - v^2\right) + n_2^2 \left(v_1^2 - v^2\right)\left(v_3^2 - v^2\right) + n_3^2 \left(v_1^2 - v^2\right)\left(v_2^2 - v^2\right) = 0 \tag{13}$$

which usually has two distinct roots.

We must take care of the possibility of $b_1 b_2 b_3 = 0$, the opposite of which was assumed in deriving (11). It's easy to see that the only situation for $b_1 b_2 b_3 = 0$ to happen while still satisfying the eigenequation is when $v^2 = v_1^2 = v_2^2 = v_3^2$, i.e., the dielectric is isotropic and v^2 is the triple degenerated eigenvalue which is the usual speed of light in such a medium.

We cannot help noticing that in deriving (11) or (12), we have not used the divergence equation $\boldsymbol{\nabla} \cdot \mathbf{D} = 0$ at all. The divergence equation is equivalent to $\mathbf{k} \cdot \mathbf{D} = 0$, which represents an addition of a fourth row $[n_1 n_2 n_3]$ to the matrix $v^2 I - A$ on the LHS of (7). In other words, to prove that the ansatz (1) satisfies all the Maxwell equations, it is sufficient to prove that the new row $[n_1 n_2 n_3]$ is a linear combination of the three rows in $v^2 I - A$. This is actually straightforward to verify: multiply the second row by n_2/n_1, and multiply the third row by n_3/n_1, and add them to the first row, we have a new row whose columns are

$$
\begin{aligned}
\text{1st column}: & \qquad n_1^2 v_1^2 - v_1^2 + v^2 + n_2^2 v_1^2 + n_3^2 v_1^2 = v^2 \\
\text{2nd column}: & \qquad n_1 n_2 v_2^2 + \frac{n_2^3}{n_1} v_2^2 - \frac{n_2}{n_1} v_2^2 + \frac{n_2}{n_1} v^2 + \frac{n_3^2 n_2}{n_1} v_2^2 = \frac{n_2}{n_1} v^2 \\
\text{3rd column}: & \qquad n_1 n_3 v_3^2 + \frac{n_2^2 n_3}{n_1} v_3^2 + \frac{n_3^3}{n_1} v_3^2 - \frac{n_3}{n_1} v_3^2 + \frac{n_3}{n_1} v^2 = \frac{n_3}{n_1} v^2
\end{aligned}
$$

which is proportional to the desired row $[n_1 n_2 n_3]$.

We should also note that $\mathbf{D}$ is a transverse wave because $\mathbf{n} \cdot \mathbf{D} = 0$, but $\mathbf{E}$ is not in general a transverse wave due to the anisotropy.

(c) From the eigenequation perspective, for the two non-zero eigenvalues $v^2 = \lambda$ and $v^2 = \mu$, the two corresponding displacement vectors $\mathbf{D}_\lambda$ and $\mathbf{D}_\mu$ are known to be orthogonal to each other by virtue of being the eigenvectors for λ and μ (when there is no degeneracy).

2. **Prob 7.17**

(a) The equation of motion for an electron is given in (7.63)

$$m\ddot{\mathbf{x}} - e\mathbf{B}_0 \times \dot{\mathbf{x}} = -e\mathbf{E}e^{-i\omega t} \tag{14}$$

Assuming the displacement $\mathbf{x}$ is harmonic $\mathbf{x} = \mathbf{x}e^{-i\omega t}$, then (14) becomes

$$-m\omega^2 \mathbf{x} + i\omega e\mathbf{B}_0 \times \mathbf{x} = -e\mathbf{E} \tag{15}$$

(Note from now on, we adopt the Einstein summation convention.)

The k-th component is thus given as

$$
\begin{aligned}
-m\omega^2 x_k + i\omega e \epsilon_{ijk} B_i x_j = -eE_k & \qquad \Longrightarrow \\
-m\omega^2 \left(-e x_k\right) + i\omega e \epsilon_{ijk} B_i \left(-e x_j\right) = e^2 E_k & \qquad \Longrightarrow \qquad (16) \\
m\omega^2 p_k - i\omega e \epsilon_{ijk} B_i p_j = -e^2 E_k & \qquad (17)
\end{aligned}
$$

This gives the linear relation between one electron's dipole moment and electric field

$$\begin{bmatrix} m\omega^2 & i\omega eB_3 & -i\omega eB_2 \\ -i\omega eB_3 & m\omega^2 & i\omega eB_1 \\ i\omega eB_2 & -i\omega eB_1 & m\omega^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = -e^2 \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{18}$$

whose inverse matrix is calculated explicitly

$$\frac{1}{m^3\omega^6 - m\omega^4 e^2 B_0^2} \begin{bmatrix} m^2\omega^4 - \omega^2 e^2 B_1^2 & -\omega^2 e^2 B_1 B_2 - im\omega^3 eB_3 & -\omega^2 e^2 B_1 B_3 + im\omega^3 eB_2 \\ -\omega^2 e^2 B_2 B_1 + im\omega^3 eB_3 & m^2\omega^4 - \omega^2 e^2 B_2^2 & -\omega^2 e^2 B_2 B_3 - im\omega^3 eB_1 \\ -\omega^2 e^2 B_3 B_1 - im\omega^3 eB_2 & -\omega^2 e^2 B_3 B_2 + im\omega^3 eB_1 & m^2\omega^4 - \omega^2 e^2 B_3^2 \end{bmatrix} \tag{19}$$

We can recognize the jk-th element as

$$\begin{aligned} \frac{\delta_{jk} m^2\omega^4 - \omega^2 e^2 B_j B_k - im\omega^3 e\epsilon_{jkl} B_l}{m^3\omega^6 - m\omega^4 e^2 B_0^2} &= \frac{\delta_{jk} m^2\omega^4 - m^2\omega^2\omega_B^2 b_j b_k - im^2\omega^3\omega_B \epsilon_{jkl} b_l}{m^3\omega^6 - m^3\omega^4\omega_B^2} \\ &= \frac{\delta_{jk}\omega^2 - \omega_B^2 b_j b_k - i\omega\omega_B\epsilon_{jkl} b_l}{m\omega^2\left(\omega^2 - \omega_B^2\right)} \end{aligned} \tag{20}$$

Aggregating all the N electrons in a unit volume gives the linear relationship between polarization P_j and E_k:

$$P_j = \frac{-Ne^2}{m\omega^2\left(\omega^2 - \omega_B^2\right)} \left(\delta_{jk}\omega^2 - \omega_B^2 b_j b_k - i\omega\omega_B\epsilon_{jkl} b_l\right) E_k \tag{21}$$

With the definition of plasma frequency

$$\omega_p^2 = \frac{Ne^2}{\epsilon_0 m} \tag{22}$$

we have

$$P_j = \epsilon_0 \left[-\frac{\omega_p^2}{\omega^2\left(\omega^2 - \omega_B^2\right)} \left(\delta_{jk}\omega^2 - \omega_B^2 b_j b_k - i\omega\omega_B\epsilon_{jkl} b_l\right) \right] E_k \tag{23}$$

which enables us to identify the content of the square bracket as the susceptibility tensor

$$\chi_{jk} = -\frac{\omega_p^2}{\omega^2\left(\omega^2 - \omega_B^2\right)} \left(\delta_{jk}\omega^2 - \omega_B^2 b_j b_k - i\omega\omega_B\epsilon_{jkl} b_l\right) \tag{24}$$

(b) The dielectric tensor ϵ_{jk} is given by

$$\epsilon_{jk} = \epsilon_0\left(\delta_{jk} + \chi_{jk}\right) = \frac{\epsilon_0}{\omega^2\left(\omega^2 - \omega_B^2\right)} \left[\delta_{jk}\omega^2\left(\omega^2 - \omega_B^2 - \omega_p^2\right) + \omega_p^2\omega_B^2 b_j b_k + i\omega\omega_p^2\omega_B\epsilon_{jkl} b_l\right] \tag{25}$$

Denoting

$$\alpha \equiv \omega^2\left(\omega^2 - \omega_B^2 - \omega_p^2\right) \qquad \beta \equiv \omega_p^2\omega_B^2 \qquad \gamma \equiv \omega\omega_p^2\omega_B \tag{26}$$

the eigenvalues of ϵ_{jk} is thus $\epsilon_0/\left[\omega^2\left(\omega^2 - \omega_B^2\right)\right]$ times the root of

$$\begin{vmatrix} \alpha + \beta b_1^2 - \lambda & \beta b_1 b_2 + i\gamma b_3 & \beta b_1 b_3 - i\gamma b_2 \\ \beta b_2 b_1 - i\gamma b_3 & \alpha + \beta b_2^2 - \lambda & \beta b_2 b_3 + i\gamma b_1 \\ \beta b_3 b_1 + i\gamma b_2 & \beta b_3 b_2 - i\gamma b_1 & \alpha + \beta b_3^2 - \lambda \end{vmatrix} = 0 \tag{27}$$

We solve this equation explicitly:

$$\begin{aligned} 0 = \; & \left(\alpha + \beta b_1^2 - \lambda\right)\left(\alpha + \beta b_2^2 - \lambda\right)\left(\alpha + \beta b_3^2 - \lambda\right) \\ & + (\beta b_1 b_2 + i\gamma b_3)(\beta b_2 b_3 + i\gamma b_1)(\beta b_3 b_1 + i\gamma b_2) \\ & + (\beta b_2 b_1 - i\gamma b_3)(\beta b_3 b_2 - i\gamma b_1)(\beta b_1 b_3 - i\gamma b_2) \\ & - \left(\alpha + \beta b_1^2 - \lambda\right)(\beta b_2 b_3 + i\gamma b_1)(\beta b_3 b_2 - i\gamma b_1) \\ & - \left(\alpha + \beta b_2^2 - \lambda\right)(\beta b_1 b_3 - i\gamma b_2)(\beta b_3 b_1 + i\gamma b_2) \\ & - \left(\alpha + \beta b_3^2 - \lambda\right)(\beta b_1 b_2 + i\gamma b_3)(\beta b_2 b_1 - i\gamma b_3) \end{aligned} \tag{28}$$

On the RHS, the coefficients for λ powers are

$$\begin{aligned}
\lambda^3: &\quad -1\\
\lambda^2: &\quad \left(\alpha+\beta b_1^2\right)+\left(\alpha+\beta b_2^2\right)+\left(\alpha+\beta b_3^2\right)=3\alpha+\beta\\
\lambda^1: &\quad -\left(\alpha+\beta b_1^2\right)\left(\alpha+\beta b_2^2\right)-\left(\alpha+\beta b_2^2\right)\left(\alpha+\beta b_3^2\right)-\left(\alpha+\beta b_3^2\right)\left(\alpha+\beta b_1^2\right)\\
&\quad +\left(\beta^2 b_2^2 b_3^2+\gamma^2 b_1^2\right)+\left(\beta^2 b_1^2 b_3^2+\gamma^2 b_2^2\right)+\left(\beta^2 b_1^2 b_2^2+\gamma^2 b_3^2\right)=\gamma^2-3\alpha^2-2\alpha\beta\\
\lambda^0: &\quad \left(\alpha+\beta b_1^2\right)\left(\alpha+\beta b_2^2\right)\left(\alpha+\beta b_3^2\right)\\
&\quad +2\left[\beta^3 b_1^2 b_2^2 b_3^2-\beta\gamma^2\left(b_1^2 b_2^2+b_2^2 b_3^2+b_3^2 b_1^2\right)\right]\\
&\quad -\left(\alpha+\beta b_1^2\right)\left(\beta^2 b_2^2 b_3^2+\gamma^2 b_1^2\right)\\
&\quad -\left(\alpha+\beta b_2^2\right)\left(\beta^2 b_1^2 b_3^2+\gamma^2 b_2^2\right)\\
&\quad -\left(\alpha+\beta b_3^2\right)\left(\beta^2 b_1^2 b_2^2+\gamma^2 b_3^2\right)=\left(\alpha^2-\gamma^2\right)(\alpha+\beta)
\end{aligned}$$

Then we have the cubic equation for λ:

$$\begin{aligned}
\lambda^3-(3\alpha+\beta)\lambda^2+\left(3\alpha^2+2\alpha\beta-\gamma^2\right)\lambda-(\alpha-\gamma)(\alpha+\gamma)(\alpha+\beta)=0 \qquad \Longrightarrow\\
[\lambda-(\alpha-\gamma)][\lambda-(\alpha+\gamma)][\lambda-(\alpha+\beta)]=0
\end{aligned} \tag{29}$$

which apparently has three roots

$$\lambda_1=\alpha-\gamma \qquad \lambda_2=\alpha+\gamma \qquad \lambda_3=\alpha+\beta \tag{30}$$

Thus the three eigenvalues for ϵ_{jk} are

$$\epsilon_1=\frac{\epsilon_0}{\omega^2\left(\omega^2-\omega_B^2\right)}(\alpha-\gamma)=\epsilon_0\left[1-\frac{\omega_p^2}{\omega\left(\omega-\omega_B\right)}\right] \tag{31}$$

$$\epsilon_2=\frac{\epsilon_0}{\omega^2\left(\omega^2-\omega_B^2\right)}(\alpha+\gamma)=\epsilon_0\left[1-\frac{\omega_p^2}{\omega\left(\omega+\omega_B\right)}\right] \tag{32}$$

$$\epsilon_3=\frac{\epsilon_0}{\omega^2\left(\omega^2-\omega_B^2\right)}(\alpha+\beta)=\epsilon_0\left(1-\frac{\omega_p^2}{\omega^2}\right) \tag{33}$$

Note these eigenvalues are independent of $\mathbf{B}_0$'s direction (but they do depend on $|\mathbf{B}_0|$ due to the involvement of ω_B). This is reasonable since without the magnetic field, the medium is isotropic, whichever the direction of $\mathbf{B}_0$ is, the generated anisotropy will be with respect to this direction, but we should expect the principal values to remain independent of that direction. From hindsight, we could have an easier method to solve λ in (27) by selecting a simpler $\mathbf{b}$ (e.g., $\hat{\mathbf{z}}$), in which case the determinant in (27) becomes

$$\begin{vmatrix}\alpha-\lambda & i\gamma & 0\\ -i\gamma & \alpha-\lambda & 0\\ 0 & 0 & \alpha+\beta-\lambda\end{vmatrix}=0 \tag{34}$$

which gives the same eigenvalues (30).

More rigorously, we can prove that the matrix $\epsilon_{jk}(\mathbf{b})$ and $\epsilon_{jk}(\hat{\mathbf{z}})$ are related by a similar transformation via the rotation $R:\mathbf{b}\to\hat{\mathbf{z}}$, so they must produce the same set of eigenvalues.

Indeed, from (25), we see the matrix $\epsilon_{jk}(\mathbf{b})$ has the form

$$\epsilon_{jk}(\mathbf{b})=A\delta_{jk}+Bb_jb_k+C\epsilon_{jkl}b_l \tag{35}$$

Sandwiching this between R and R^{-1}, the pq-th element of the product matrix is

$$\begin{aligned}
R_{pj}\epsilon_{jk}(\mathbf{b})\left(R^{-1}\right)_{kq}&=A\cdot R_{pj}\delta_{jk}\left(R^{-1}\right)_{kq}+B\cdot\left(R_{pj}b_j\right)\left[b_k\left(R^{-1}\right)_{kq}\right]+C\cdot\left[R_{pj}\epsilon_{jkl}b_l\left(R^{-1}\right)_{kq}\right]\\
&=A\delta_{pq}+B\hat{z}_p\hat{z}_q+C\cdot\left(R_{pj}\epsilon_{jkl}R_{qk}\right)b_l
\end{aligned} \tag{36}$$

As a tensor, ϵ_{jkl} satisfies the rotation invariance

$$R_{pj}R_{qk}R_{rl}\epsilon_{jkl}=\epsilon_{pqr} \tag{37}$$

Multiplying both sides by $R_{rl} = \left(R^{-1}\right)_{lr}$ and sum over r, we have

$$R_{pj}R_{qk}\left[\left(R^{-1}\right)_{lr}R_{rl}\right]\epsilon_{jkl} = R_{pj}R_{qk}\epsilon_{jkl} = R_{rl}\epsilon_{pqr} \tag{38}$$

Thus the RHS of (36) becomes

$$A\delta_{pq} + B\hat{z}_p\hat{z}_q + C\cdot\left(R_{rl}b_l\right)\epsilon_{pqr} = A\delta_{pq} + B\hat{z}_p\hat{z}_q + C\epsilon_{pqr}\hat{z}_r = \epsilon_{pq}\left(\hat{\mathbf{z}}\right) \tag{39}$$

proving the similar transformation.

(c) From (5) and the relation between **D** and **E**, we have

$$\begin{aligned}
(\mathbf{k}\cdot\mathbf{E})\,k_j - k^2E_j + \omega^2\mu_0\epsilon_0\left(\delta_{jk} + \chi_{jk}\right)E_k = 0 & \qquad \Longrightarrow \\
(1-\xi)\,E_j + \xi\,(\mathbf{n}\cdot\mathbf{E})\,n_j + \chi_{jk}E_k = 0 & \qquad \text{where } \xi = \left(\frac{ck}{\omega}\right)^2
\end{aligned} \tag{40}$$

For the last part, I don't really understand what it means by "effective dielectric constant for propagation of the plane wave with positive and negative helicity". A precise definition of such effective dielectric constant would be helpful. We have seen that for an anisotropic medium, **D** and **E** are not aligned, and that **E** is not even a transverse wave. We have also seen from problem 7.16 that there are two phase velocities for the two transverse components of **D**. I don't even seen an intuitive way to define helicity in such anisotropic medium (e.g., are we talking about the circular polarization for **D** or **E**?)

An observation that may be relavant to this question is that in (34), we can easily calculate the three eigenvectors

$$\alpha - \gamma \longleftrightarrow \hat{\mathbf{x}} + i\hat{\mathbf{y}} \qquad\qquad \alpha + \gamma \longleftrightarrow \hat{\mathbf{x}} - i\hat{\mathbf{y}} \qquad\qquad \alpha + \beta \longleftrightarrow \hat{\mathbf{z}} \tag{41}$$

where $\hat{\mathbf{z}}$ is the rotated **b** direction. These forms are familiar with how the helicities are defined, e.g., $\boldsymbol{\epsilon}_\pm = \boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2$. What is more, corresponding to $\alpha \mp \gamma$, (31) and (32) will be approximately proportional to $1 - \omega_p^2/\omega^2 \mp \omega_p^2\omega_B/\omega^3$, which match the problem statement when $\mathbf{b} = \mathbf{n}$.