These notes are more detailed derivation and explanation for Jackson section 6.5 which I found to be confusing. Especially the symbols like $\mathbf{R}, \hat{\mathbf{R}}, R$, most critically $[]_{\text{ret}}$, have subtly changed meaning when introducing Heaviside-Feynman formula.

In section 6.4, we have established that the Green function

$$G_{\pm}\left(\mathbf{x}, t; \mathbf{x}', t'\right) = \frac{\delta\left[t' - \left(t \mp \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{|\mathbf{x}-\mathbf{x}'|} \tag{1}$$

is the solution to the differential equation

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G_{\pm}\left(\mathbf{x}, t; \mathbf{x}', t'\right) = -4\pi\delta\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(t-t'\right) \tag{2}$$

Then we arrived at equation (6.47), that for a given source function $f(\mathbf{x}, t)$, the solution to the wave equation

$$\boldsymbol{\nabla}^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi f(\mathbf{x}, t) \tag{3}$$

can be expressed using the retarded Green function G_+ for the commonest physical problems

$$\Psi(\mathbf{x}, t) = \int \frac{[f(\mathbf{x}', t')]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{4}$$

where

$$\left[f\left(\mathbf{x}', t'\right)\right]_{\text{ret}} = f\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right) \tag{5}$$

The way to look at (5) is this, given a function $f(\mathbf{x}', t')$ with free variables $\mathbf{x}'$ and t', and another two parameters $\mathbf{x}$ and t, the retardation $[f]_{\text{ret}}$ turns it into a function of three free variable $\mathbf{x}', \mathbf{x}, t$, where the freedom of choosing t' disappears (since we must choose $t' = t - |\mathbf{x}-\mathbf{x}'|/c$).

1. **Wave equation for the fields, i.e., equation (6.49), (6.50)**

   Compare equation (6.15), (6.16), i.e., the wave equation for the scalar and vector potentials in the Lorenz gauge:

$$\boldsymbol{\nabla}^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{6}$$

$$\boldsymbol{\nabla}^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} \tag{7}$$

   with the standard form (3), we can identify the source functions as

$$f_{\Phi}(\mathbf{x}, t) = \frac{\rho(\mathbf{x}, t)}{4\pi\epsilon_0} \qquad \mathbf{f}_{\mathbf{A}}(\mathbf{x}, t) = \frac{\mu_0\mathbf{J}(\mathbf{x}, t)}{4\pi} \tag{8}$$

   Taking the gradient of (6) gives

$$\boldsymbol{\nabla}\left(\boldsymbol{\nabla}^2\Phi\right) - \frac{1}{c^2}\frac{\partial^2(\boldsymbol{\nabla}\Phi)}{\partial t^2} = -\frac{\boldsymbol{\nabla}\rho}{\epsilon_0} \quad\Longrightarrow\quad \boldsymbol{\nabla}^2(\boldsymbol{\nabla}\Phi) - \frac{1}{c^2}\frac{\partial^2(\boldsymbol{\nabla}\Phi)}{\partial t^2} = -\frac{\boldsymbol{\nabla}\rho}{\epsilon_0} \tag{9}$$

   With the Maxwell equation

$$\boldsymbol{\nabla}\Phi = -\mathbf{E} - \frac{\partial\mathbf{A}}{\partial t} \tag{10}$$

   (9) becomes

$$\begin{aligned}
\boldsymbol{\nabla}^2\left(-\mathbf{E} - \frac{\partial\mathbf{A}}{\partial t}\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(-\mathbf{E} - \frac{\partial\mathbf{A}}{\partial t}\right) &= -\frac{\boldsymbol{\nabla}\rho}{\epsilon_0} && \Longrightarrow \\
\boldsymbol{\nabla}^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + \frac{\partial}{\partial t}\left(\boldsymbol{\nabla}^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}\right) &= \frac{\boldsymbol{\nabla}\rho}{\epsilon_0} && \text{by (7)}
\end{aligned}$$

$$\boldsymbol{\nabla}^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = -\frac{1}{\epsilon_0}\left(-\boldsymbol{\nabla}\rho - \frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t}\right) \tag{11}$$

which is (6.49).

Taking the curl of (7) yields (6.50)

$$\nabla\times\left(\nabla^2\mathbf{A}\right)-\frac{1}{c^2}\frac{\partial^2(\nabla\times\mathbf{A})}{\partial t^2}=-\mu_0\nabla\times\mathbf{J}\qquad\Longrightarrow\qquad\nabla^2\mathbf{B}-\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}=-\mu_0\nabla\times\mathbf{J}\tag{12}$$

2. **Jefimenko's generalization (6.55), (6.56)**

First a few words on (6.53)

$$\left[\nabla'\rho\right]_{\text{ret}}=\nabla'[\rho]_{\text{ret}}-\left[\frac{\partial\rho}{\partial t'}\right]_{\text{ret}}\nabla'\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)=\nabla'[\rho]_{\text{ret}}-\frac{\mathbf{x}-\mathbf{x}'}{c\,|\mathbf{x}-\mathbf{x}'|}\left[\frac{\partial\rho}{\partial t'}\right]_{\text{ret}}\tag{13}$$

where we have discarded the use of symbols like $\mathbf{R},\hat{\mathbf{R}},R$ to avoid abuse when we come to Heaviside-Feynman later. Note this is true because $\rho\left(\mathbf{x}',t'\right)$ is a function of two free variables $\mathbf{x}'$ and t', where $\mathbf{x}'$ is not dependent on t' in any way. Then (13) is nothing but the chain rule and the definition of the $[]_{\text{ret}}$ from (5).

(6.54) is obtained the same way, now formulated without symbols $\mathbf{R}$, etc.

$$\left[\nabla'\times\mathbf{J}\right]_{\text{ret}}=\nabla'\times[\mathbf{J}]_{\text{ret}}+\frac{1}{c}\left[\frac{\partial\mathbf{J}}{\partial t'}\right]_{\text{ret}}\times\left(\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\right)\tag{14}$$

To see (6.55), we can apply the retarded solution form (4) on the wave equation for $\mathbf{E}$ (11), which gives

$$\begin{aligned}\mathbf{E}(\mathbf{x},t)&=\frac{1}{4\pi\epsilon_0}\int\frac{\left[-\nabla'\rho-\frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t'}\right]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|}d^3x'\\&=\frac{1}{4\pi\epsilon_0}\left\{\int\frac{-\nabla'[\rho]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|}d^3x'+\int\frac{\mathbf{x}-\mathbf{x}'}{c\,|\mathbf{x}-\mathbf{x}'|^2}\left[\frac{\partial\rho}{\partial t'}\right]_{\text{ret}}d^3x'-\int\frac{1}{c^2\,|\mathbf{x}-\mathbf{x}'|}\left[\frac{\partial\mathbf{J}}{\partial t'}\right]_{\text{ret}}d^3x'\right\}\end{aligned}\tag{15}$$

where the first integral

$$\begin{aligned}\int\frac{-\nabla'[\rho]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|}d^3x'&=\overbrace{\int-\nabla'\left(\frac{[\rho]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|}\right)d^3x'}^{0}+\int\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)[\rho]_{\text{ret}}\,d^3x'\\&=\int\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}[\rho]_{\text{ret}}\,d^3x'\end{aligned}\tag{16}$$

Hence the field

$$\mathbf{E}(\mathbf{x},t)=\frac{1}{4\pi\epsilon_0}\int\left\{\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}[\rho]_{\text{ret}}+\frac{\mathbf{x}-\mathbf{x}'}{c\,|\mathbf{x}-\mathbf{x}'|^2}\left[\frac{\partial\rho}{\partial t'}\right]_{\text{ret}}-\frac{1}{c^2\,|\mathbf{x}-\mathbf{x}'|}\left[\frac{\partial\mathbf{J}}{\partial t'}\right]_{\text{ret}}\right\}d^3x'\tag{17}$$

This is (6.55) written explicitly as a function of $\mathbf{x}$ and t, because the t' dependency disappeared due to the $[]_{\text{ret}}$ (see comments after equation (5) above), and the $\mathbf{x}'$ dependency disappeared after the integral.

For the $\mathbf{B}$ field solution, we have

$$\begin{aligned}\mathbf{B}(\mathbf{x},t)&=\frac{\mu_0}{4\pi}\int\frac{[\nabla'\times\mathbf{J}]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|}d^3x'\\&=\frac{\mu_0}{4\pi}\left\{\int\frac{\nabla'\times[\mathbf{J}]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|}d^3x'+\int\left[\frac{\partial\mathbf{J}}{\partial t'}\right]_{\text{ret}}\times\left(\frac{\mathbf{x}-\mathbf{x}'}{c\,|\mathbf{x}-\mathbf{x}'|^2}\right)d^3x'\right\}\end{aligned}\tag{18}$$

where the first integral

$$\begin{aligned}\int\frac{\nabla'\times[\mathbf{J}]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|}d^3x'&=\int\nabla'\times\left(\frac{[\mathbf{J}]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|}\right)d^3x'-\int\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\times[\mathbf{J}]_{\text{ret}}\,d^3x'\\&=\underbrace{\oint_{\infty}\mathbf{n}'\times\left(\frac{[\mathbf{J}]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|}\right)da'}_{0}-\int\left(\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right)\times[\mathbf{J}]_{\text{ret}}\,d^3x'\end{aligned}\tag{19}$$

This gives

$$\mathbf{B}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int\left\{[\mathbf{J}]_{\text{ret}}\times\left(\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right)+\left[\frac{\partial\mathbf{J}}{\partial t'}\right]_{\text{ret}}\times\left(\frac{\mathbf{x}-\mathbf{x}'}{c\,|\mathbf{x}-\mathbf{x}'|^2}\right)\right\}d^3x' \tag{20}$$

3. **Heaviside-Feynman expression, i.e., problem 6.2**

When a point charge q is moving along a trajectory $\mathbf{r}(\tau)$, the charge density function $\rho(\mathbf{x}',t')$ and the current density function $\mathbf{J}(\mathbf{x}',t')$ can be written as

$$\rho\left(\mathbf{x}',t'\right)=q\delta\left[\mathbf{x}'-\mathbf{r}\left(t'\right)\right] \tag{21}$$

$$\mathbf{J}\left(\mathbf{x}',t'\right)=\rho\left(\mathbf{x}',t'\right)\mathbf{v}\left(t'\right)=q\delta\left[\mathbf{x}'-\mathbf{r}\left(t'\right)\right]\left(\frac{d\mathbf{r}}{d\tau}\right)\left(t'\right) \tag{22}$$

(a) This part asks to prove

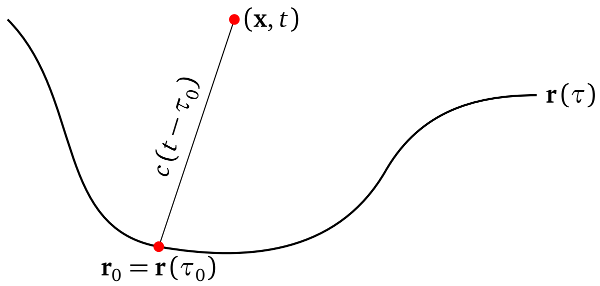
$$\int d^3x'\delta\left[\mathbf{x}'-\mathbf{r}\left(t_{\text{ret}}\right)\right]=\frac{1}{\kappa} \tag{23}$$

where

$$\kappa=1-\frac{\mathbf{v}\cdot(\mathbf{x}-\mathbf{x}')}{c\,|\mathbf{x}-\mathbf{x}'|} \tag{24}$$

is "evaluated" at the retarded time.

The meaning of this statement needs a lot of explanation. First we take t_{ret} to mean $t'=t-|\mathbf{x}-\mathbf{x}'|/c$. The definition of κ does not make sense since if (23) is correct, the integration of d^3x' should have eliminated $\mathbf{x}'$, so κ shouldn't contain that symbol. This is a result of abuse of symbols. A better way to describe the claim (23) is the following.



For any given observation point $\mathbf{x}$ and any arbitrary time t, and an arbitrarily defined trajectory $\mathbf{r}(\tau)$, we now ask the question, what is an earlier time $\tau_0<t$, such that the time for light to travel between $\mathbf{r}(\tau_0)$ and $\mathbf{x}$ is exactly $t-\tau_0$. In other words, we are looking for the zero τ_0 of the equation

$$\tau_0=t-\frac{|\mathbf{x}-\mathbf{r}(\tau_0)|}{c} \tag{25}$$

with the given $\mathbf{x},t,\mathbf{r}(\tau)$.

Assuming there is a unique zero τ_0 satisfying (25) (more discussions about the existence and uniqueness at the end of this document). This will imply a unique zero, denoted $\mathbf{r}_0=\mathbf{r}(\tau_0)$, for the equation

$$g\left(\mathbf{x}'\right)\equiv\mathbf{x}'-\mathbf{r}\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)=0 \tag{26}$$

Thus by the property of δ function (see Jackson page 26), the integrand of (23) is

$$\delta\left[g\left(\mathbf{x}'\right)\right]=\frac{1}{|(\boldsymbol{\nabla}'g)(\mathbf{r}_0)|}\delta\left(\mathbf{x}'-\mathbf{r}_0\right) \tag{27}$$

and the integration of (27) throughout space gives just

$$\int d^3x'\delta\left[g\left(\mathbf{x}'\right)\right]=\frac{1}{|(\boldsymbol{\nabla}'g)(\mathbf{r}_0)|}\qquad\text{where}$$

$$\begin{aligned}\left(\boldsymbol{\nabla}'g\right)(\mathbf{r}_0)&=\left[1-\frac{d\mathbf{r}}{d\tau}\cdot\left(\frac{-\boldsymbol{\nabla}'|\mathbf{x}-\mathbf{x}'|}{c}\right)\right](\mathbf{r}_0)\\&=1-\mathbf{v}(\tau_0)\cdot\frac{\mathbf{x}-\mathbf{r}_0}{c\,|\mathbf{x}-\mathbf{r}_0|}\\&=1-\frac{\mathbf{v}(\tau_0)\cdot[\mathbf{x}-\mathbf{r}(\tau_0)]}{c\,|\mathbf{x}-\mathbf{r}(\tau_0)|}=\kappa\end{aligned} \tag{28}$$

(28) is the unambiguous interpretation of κ given earlier, for which we can see now there is no symbol $\mathbf{x}'$ but a well defined quantity depending on τ_0 which is the unique zero of (25).

(b) Before proving this part, let's comment on (6.57)

$$\left[\frac{\partial f(\mathbf{x}', t')}{\partial t'}\right]_{\text{ret}} = \frac{\partial}{\partial t}\left[f(\mathbf{x}', t')\right]_{\text{ret}} \tag{29}$$

This is straightforward to prove but two points are worth noting: 1) on both sides the t' dependency has disappeared, which is consistent with the comments earlier after (5); and 2) that this works only when $\mathbf{x}'$ and t' are independent, free variables, in particular, $\mathbf{x}'$ must have no dependency on t'.

Insert (21) and (22) into (17), and apply (29) to eliminate partial derivatives in t', we get

$$\mathbf{E}(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0}\left\{\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}[\rho]_{\text{ret}}\, d^3x' + \frac{\partial}{\partial t}\int \frac{\mathbf{x}-\mathbf{x}'}{c\,|\mathbf{x}-\mathbf{x}'|^2}[\rho]_{\text{ret}}\, d^3x' - \frac{\partial}{\partial t}\int \frac{1}{c^2\,|\mathbf{x}-\mathbf{x}'|}[\mathbf{J}]_{\text{ret}}\, d^3x'\right\} \tag{30}$$

Since

$$\left[\rho(\mathbf{x}', t')\right]_{\text{ret}} = q\delta\left[\mathbf{x}' - \mathbf{r}\left(t - \frac{\mathbf{x}-\mathbf{x}'}{c}\right)\right] = q\delta\left[g(\mathbf{x}')\right] = \frac{q\delta(\mathbf{x}'-\mathbf{r}_0)}{\kappa} \tag{31}$$

$$\left[\mathbf{J}(\mathbf{x}', t')\right]_{\text{ret}} = \frac{q\delta(\mathbf{x}'-\mathbf{r}_0)}{\kappa}\mathbf{v}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right) \tag{32}$$

all three integrands in (30) have support only at $\mathbf{x}' = \mathbf{r}_0$, thus

$$\mathbf{E}(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{x}-\mathbf{r}_0}{\kappa\,|\mathbf{x}-\mathbf{r}_0|^3} + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mathbf{x}-\mathbf{r}_0}{\kappa\,|\mathbf{x}-\mathbf{r}_0|^2}\right) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\mathbf{v}}{\kappa\,|\mathbf{x}-\mathbf{r}_0|}\right)\right] \tag{33}$$

We have discarded the use of $[\,]_{\text{ret}}$ because it contradicts with the original definition which applies to a function with t' as a free variable. Instead, we make it clear here that $\mathbf{r}_0, \mathbf{v}$ and κ are all evaluated at τ_0, while τ_0 is the zero of (25) hence is a function of $\mathbf{x}$ and t.

Similarly for (20), we will end up with

$$\mathbf{B}(\mathbf{x}, t) = \frac{\mu_0 q}{4\pi}\left\{\frac{\mathbf{v}\times(\mathbf{x}-\mathbf{r}_0)}{\kappa\,|\mathbf{x}-\mathbf{r}_0|^3} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}\times(\mathbf{x}-\mathbf{r}_0)}{\kappa\,|\mathbf{x}-\mathbf{r}_0|^2}\right]\right\} \tag{34}$$

(c) Here we will derive an alternate form of (33) and (34) (i.e., (6.60) and (6.61)). Define

$$\mathbf{R}(\mathbf{x}, \tau_0) \equiv \mathbf{x} - \mathbf{r}(\tau_0) \tag{35}$$

Then (33) and (34) are simplified into

$$\mathbf{E}(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{R}}}{\kappa R^2} + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\hat{\mathbf{R}}}{\kappa R}\right) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\mathbf{v}}{\kappa R}\right)\right] \tag{36}$$

$$\mathbf{B}(\mathbf{x}, t) = \frac{\mu_0 q}{4\pi}\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R^2} + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R}\right)\right] \tag{37}$$

The following relations will be used

$$\frac{\partial \mathbf{R}}{\partial \tau_0} = -\frac{d\mathbf{r}}{d\tau_0} = -\mathbf{v} \tag{38}$$

$$\frac{\partial R}{\partial \tau_0} = \frac{\partial}{\partial \tau_0}(\mathbf{R}\cdot\mathbf{R})^{1/2} = \frac{1}{R}\left(\frac{\partial \mathbf{R}}{\partial \tau_0}\right)\cdot\mathbf{R} = -\mathbf{v}\cdot\hat{\mathbf{R}} = -(1-\kappa)c \tag{39}$$

$$\frac{\partial \hat{\mathbf{R}}}{\partial \tau_0} = \frac{\partial}{\partial \tau_0}\left(\frac{\mathbf{R}}{R}\right) = -\frac{\mathbf{v}}{R} - \frac{\mathbf{R}}{R^2}\frac{\partial R}{\partial \tau_0} = -\frac{\mathbf{v}}{R} + c(1-\kappa)\frac{\hat{\mathbf{R}}}{R} \tag{40}$$

Also notice

$$\tau_0 = t - \frac{R(\mathbf{x}, \tau_0)}{c} \quad\Longrightarrow\quad \frac{\partial \tau_0}{\partial t} = 1 - \frac{1}{c}\frac{\partial R}{\partial \tau_0}\frac{\partial \tau_0}{\partial t} = 1 + (1-\kappa)\frac{\partial \tau_0}{\partial t} \quad\Longrightarrow\quad \frac{\partial \tau_0}{\partial t} = \frac{1}{\kappa} \tag{41}$$

Finally for any function $f(\tau_0)$,

$$\frac{\partial f(\tau_0)}{\partial t} = \frac{\partial f(\tau_0)}{\partial \tau_0}\frac{\partial \tau_0}{\partial t} = \frac{1}{\kappa}\frac{\partial f(\tau_0)}{\partial \tau_0} \qquad \Longrightarrow \qquad \frac{\partial^2}{\partial t^2}[f(\tau_0)] = \frac{\partial}{\partial t}\left[\frac{1}{\kappa}\frac{\partial f(\tau_0)}{\partial \tau_0}\right] \tag{42}$$

By (40), the third term in the bracket of (36) can be broken into:

$$\begin{aligned}
-\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\mathbf{v}}{\kappa R}\right) &= \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{1}{\kappa}\frac{\partial \hat{\mathbf{R}}}{\partial \tau_0} - \frac{c(1-\kappa)}{\kappa}\frac{\hat{\mathbf{R}}}{R}\right] \\
&= \underbrace{\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{1}{\kappa}\frac{\partial \hat{\mathbf{R}}}{\partial \tau_0}\right)}_{\text{apply (42)}} - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\hat{\mathbf{R}}}{\kappa R}\right) + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\hat{\mathbf{R}}}{R}\right) \\
&= \frac{1}{c^2}\frac{\partial^2 \hat{\mathbf{R}}}{\partial t^2} - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\hat{\mathbf{R}}}{\kappa R}\right) + \frac{1}{c}\left[R\frac{\partial}{\partial t}\left(\frac{\hat{\mathbf{R}}}{R^2}\right) + \frac{\hat{\mathbf{R}}}{R^2}\frac{\partial R}{\partial t}\right] \\
&= \frac{1}{c^2}\frac{\partial^2 \hat{\mathbf{R}}}{\partial t^2} - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\hat{\mathbf{R}}}{\kappa R}\right) + \frac{R}{c}\frac{\partial}{\partial t}\left(\frac{\hat{\mathbf{R}}}{R^2}\right) - \left(\frac{1-\kappa}{\kappa}\right)\frac{\hat{\mathbf{R}}}{R^2}
\end{aligned} \tag{43}$$

Adding the other two terms in (36) yields

$$\mathbf{E}(\mathbf{x},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{R}}}{R^2} + \frac{R}{c}\frac{\partial}{\partial t}\left(\frac{\hat{\mathbf{R}}}{R^2}\right) + \frac{1}{c^2}\frac{\partial^2 \hat{\mathbf{R}}}{\partial t^2}\right] \tag{44}$$

The second term in the bracket of (37) can be broken into

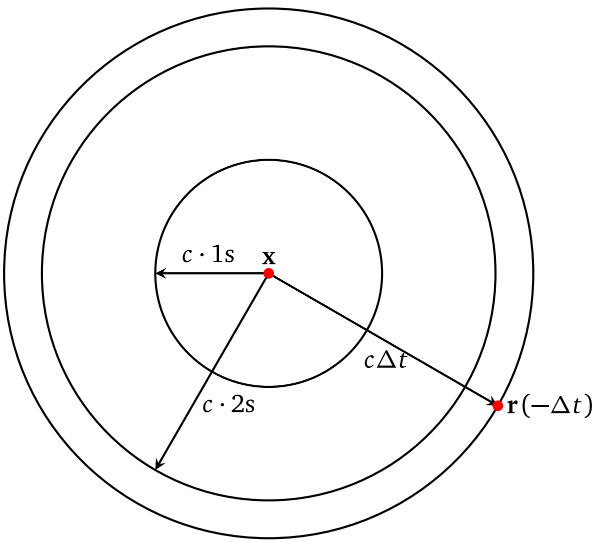
$$\begin{aligned}
\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{R}\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa}\right) &= \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{R}\right)\left(\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa}\right) + \frac{1}{cR}\frac{\partial}{\partial t}\left(\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa}\right) \\
&= -\frac{1}{c}\frac{1}{R^2}\frac{\partial R}{\partial t}\left(\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa}\right) + \frac{1}{cR}\frac{\partial}{\partial t}\left(\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa}\right) \\
&= \left(\frac{1-\kappa}{\kappa}\right)\left(\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R^2}\right) + \frac{1}{cR}\frac{\partial}{\partial t}\left(\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa}\right)
\end{aligned}$$

Adding the first term in (37) yields

$$\mathbf{B}(\mathbf{x},t) = \frac{\mu_0 q}{4\pi}\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa^2 R^2} + \frac{1}{cR}\frac{\partial}{\partial t}\left(\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa}\right)\right] \tag{45}$$

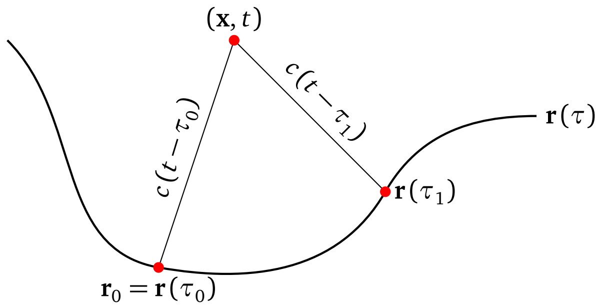
4. **On the existence and uniqueness of equation (25)'s zeros**

   Now we come back to take a closer look at the assumption of existence and uniqueness of equation (25)'s zeros. I.e., for any given $\mathbf{x}, t$ and a charge's trajectory $\mathbf{r}(\tau)$, is there always a time in the past $\tau_0 < t$, such that (25) holds; and if it exists, is it unique?



Without loss of generality, let t be "now". For the given $\mathbf{x}$, imagine a sphere centered at $\mathbf{x}$ with radius $c \times 1$s. Now if the charge happened to be on this sphere 1 second ago, then we know that $\tau_0 = t - 1$s will be the desired zero of (25). If not, was it on the sphere of raiuds $c \times 2$s 2 seconds ago? So on and so forth. We can see that the existence question is equivalent to the question that as we play the charge's trajectory backward in time, and let the sphere's radius expand with speed c, will the sphere ever catch up with the reverse-going charge? It is apparent that if the charge's reverse motion is always slower than c, the expanding sphere is bound to catch up with the charge at some distant past. So the zero τ_0 is guaranteed to exist if the charge moves with speed lower than c (which implies its reverse motion is slower than c). On the contrary, if the charge can move superluminally, τ_0 may not exist (i.e., the reverse motion of the charge moves faster than c, so the expanding sphere will never catch up).

It is also easy to see that if the charge can move superluminally, τ_0 may not be unique - after all, if it is allowed to move faster than the expanding sphere, it can choose its course to meet the sphere multiple times.



It is then worth asking whether uniqueness is guaranteed if the charge always moves slower than c. The answer is yes.

Assume we have two zeros of (25) τ_0 and τ_1 where $\tau_0 < \tau_1$. Triangle inequality requires

$$|\mathbf{r}(\tau_0) - \mathbf{r}(\tau_1)| > |c(t-\tau_0) - c(t-\tau_1)| = c(\tau_1 - \tau_0) \tag{46}$$

But this means the charge must be moving superluminally between τ_0 and τ_1, contrary to the assumption.

In summary, if the charge always moves with speed less than c, (25) is guaranteed to have one and only one zero τ_0, which justifies the usage of δ function property (27). On the contrary, if the charge can move superluminally, τ_0 may not exist, or there may be multiple τ_0's. In the former case, the charge will not have any effect for the observation point $\mathbf{x}$, while in the latter case, theory of "shockwave" must be used.