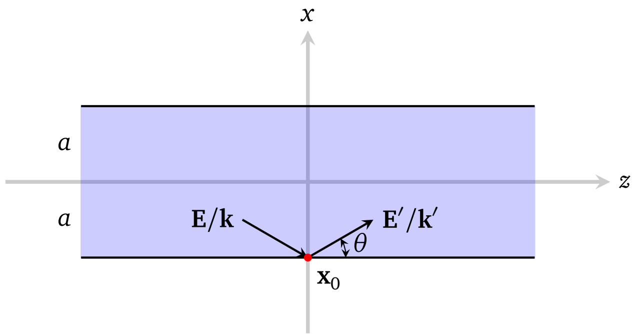
1. Let the two plane waves within the slab waveguide be $\mathbf{E}e^{i\mathbf{k}\cdot\mathbf{x}}$ and $\mathbf{E}'e^{i\mathbf{k}'\cdot\mathbf{x}}$ respectively, where $\mathbf{k} = -k\sin\theta\hat{\mathbf{x}} + k\cos\theta\hat{\mathbf{z}}$ and $\mathbf{k}' = k\sin\theta\hat{\mathbf{x}} + k\cos\theta\hat{\mathbf{z}}$.



Consider the total internal reflection at the point $\mathbf{x}_0 = (x = -a, z = 0)$. Let ϕ be the phase change introduced by the total internal reflection, i.e.,

$$\mathbf{E}'e^{i\mathbf{k}'\cdot\mathbf{x}}\Big|_{\mathbf{x}_0} = e^{i\phi}\cdot\mathbf{E}e^{i\mathbf{k}\cdot\mathbf{x}}\Big|_{\mathbf{x}_0} \quad\Longrightarrow\quad \mathbf{E}' = \mathbf{E}e^{i\phi}e^{i2ka\sin\theta} \quad \text{by (8.121)} \quad\Longrightarrow\quad \mathbf{E}' = \mathbf{E}e^{ip\pi} \tag{1}$$

The field at an arbitrary point $\mathbf{x} = (x, z)$ within the core region $|x| < a$ is thus the superposition of the two plane waves,

$$\begin{aligned}\mathbf{E}_{\text{core}}(x,z) &= \mathbf{E}e^{-ikx\sin\theta + ikz\cos\theta} + \mathbf{E}'e^{ikx\sin\theta+ikz\cos\theta} \\ &= e^{ikz\cos\theta}\mathbf{E}\left(e^{-ikx\sin\theta} + e^{ip\pi}e^{ikx\sin\theta}\right) \\ &= \begin{cases} e^{ikz\cos\theta}\cdot 2\mathbf{E}\cos(kx\sin\theta) & \text{for } p \text{ even} \\ e^{ikz\cos\theta}\cdot(-2i)\mathbf{E}\sin(kx\sin\theta) & \text{for } p \text{ odd}\end{cases}\end{aligned} \tag{2}$$

This shows that the parity of the mode number p agrees with the field's parity with respect to x.

2. For the TE mode, the eigenequation (8.123) has $f = 1$, i.e.,

$$\tan\left(V\xi - \frac{p\pi}{2}\right) = \sqrt{\frac{1}{\xi^2} - 1} \tag{3}$$

For the eigenvalue ξ satisfying $p\pi/2V < \xi < (p+1)\pi/2V$, we can rewrite (3) as

$$V\xi - \frac{p\pi}{2} = \tan^{-1}\sqrt{\frac{1}{\xi^2}-1} = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) \tag{4}$$

For $V \gg 1$ and $p \sim 1$, we expect ξ to be small (see figure 8.14), so the expansion of $\tan^{-1}x$ around infinity is used

$$\tan^{-1}x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} + O\left(\frac{1}{x^5}\right) \tag{5}$$

Applying this to the RHS of (4) up to the $O\left(\xi^3\right)$ order, we get

$$\begin{aligned}\tan^{-1}\sqrt{\frac{1}{\xi^2}-1} &= \frac{\pi}{2} - \frac{\xi}{\sqrt{1-\xi^2}} + \frac{1}{3}\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)^3 + O\left[\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)^5\right] \\ &= \frac{\pi}{2} - \xi\left(1 + \frac{1}{2}\xi^2\right) + \frac{1}{3}\xi^3 + O\left(\xi^5\right) \\ &\approx \frac{\pi}{2} - \xi - \frac{\xi^3}{6}\end{aligned} \tag{6}$$

Plugging (6) back into (4) yields an approximate equation for ξ:

$$\frac{\xi^3}{6} + (V+1)\xi - \frac{(p+1)\pi}{2} = 0 \qquad \text{or} \qquad \xi^3 + \overbrace{6(V+1)}^{a}\xi - \overbrace{3(p+1)\pi}^{b} = 0 \tag{7}$$

We have essentially approximated a trancedental equation (3) by a cubic equation (7), accurate to the order $O(\xi^3)$. The cubic equation (7) happens to be in the "depressed cubic" form and can be solved by Cardano's formula:

$$\xi = \sqrt[3]{u_1} + \sqrt[3]{u_2} \qquad \text{where} \qquad u_{1,2} = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}} \tag{8}$$

It follows from the assumption $V \gg 1 \sim p$ that $a \gg b$, therefore

$$d \equiv \sqrt{\frac{b^2}{4} + \frac{a^3}{27}} = \left(\frac{a}{3}\right)^{3/2} \left(\frac{27}{4}\frac{b^2}{a^3} + 1\right)^{1/2} = \left(\frac{a}{3}\right)^{3/2} \left[1 + \frac{27}{8}\frac{b^2}{a^3} + O\left(\frac{b^4}{a^6}\right)\right] \tag{9}$$

Also with

$$(1+x)^{1/3} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} + O\left(x^4\right) \tag{10}$$

we can obtain the approximate form of ξ,

$$\begin{aligned}
\xi &= \left(\frac{b}{2} + d\right)^{1/3} + \left(\frac{b}{2} - d\right)^{1/3} \\
&= d^{1/3}\left[\left(1 + \frac{b}{2d}\right)^{1/3} - \left(1 - \frac{b}{2d}\right)^{1/3}\right] \\
&= d^{1/3}\left[\frac{1}{3}\frac{b}{d} + \frac{10}{81\cdot 8}\frac{b^3}{d^3} + O\left(\frac{b^5}{d^5}\right)\right] \\
&= \frac{1}{3} b d^{-2/3} + \frac{10}{81\cdot 8} b^3 d^{-8/3} + O\left(b^5 d^{-14/3}\right)
\end{aligned} \tag{11}$$

From (9), we see that

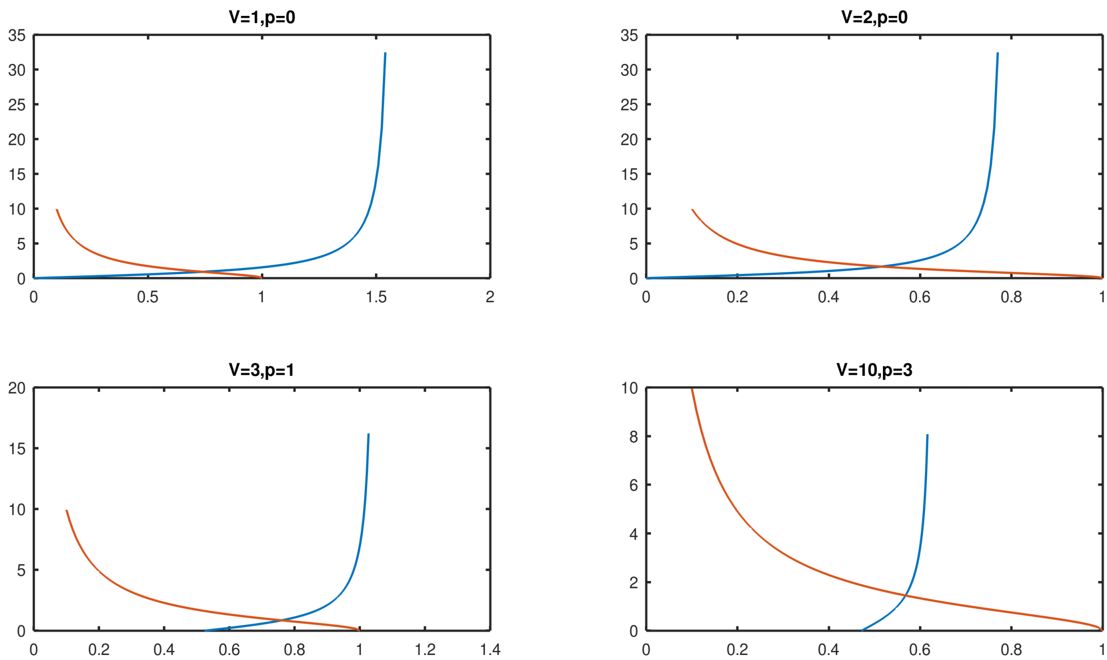
$$b d^{-2/3} = b\left(\frac{a}{3}\right)^{-1}\left[1 - \frac{9}{4}\frac{b^2}{a^3} + O\left(\frac{b^4}{a^6}\right)\right] = 3\cdot\frac{b}{a}\left[1 - \frac{9}{4}\frac{b^2}{a^3} + O\left(\frac{b^4}{a^6}\right)\right] = 3\cdot\frac{b}{a} - \frac{3\cdot 9}{4}\frac{b^3}{a^4} + O\left(\frac{b^5}{a^7}\right) \tag{12}$$

$$b^3 d^{-8/3} = b^3\left(\frac{a}{3}\right)^{-4}\left[1 + O\left(\frac{b^2}{a^3}\right)\right] = 81\cdot\frac{b^3}{a^4} + O\left(\frac{b^5}{a^7}\right) \tag{13}$$

$$O\left(b^5 d^{-14/3}\right) = O\left(\frac{b^5}{a^7}\right) \tag{14}$$

Plugging (12)-(14) into (11), and keeping the lowest two orders, we have

$$\xi \approx \frac{1}{3}\left(3\cdot\frac{b}{a} - \frac{3\cdot 9}{4}\frac{b^3}{a^4}\right) + \frac{10}{81\cdot 8}\cdot 81\cdot\frac{b^3}{a^4} = \frac{b}{a} - \frac{b^3}{a^4} = \frac{(p+1)\pi}{2(V+1)}\left[1 - \frac{(p+1)^2\pi^2}{24(V+1)^3}\right] \tag{15}$$



The plot for various V, p combinations are shown above. The following table shows the approximated ξ by (15) v.s. numerical solutions.

Method	$V=1, p=0$	$V=2, p=0$	$V=3, p=1$	$V=10, p=3$
by (15)	0.745025	0.515624	0.765212	0.568375
Numerical	0.739085	0.514933	0.759621	0.567921

3. For TE mode, let $\mathbf{E}$ and $\mathbf{E}'$ be in the $\hat{\mathbf{y}}$ direction (coming out of paper in the diagram above). Treating the amplitude of $\mathbf{E}$ as unity, we can write the fields of the two waves for the core region $|x| < a$:

$$\mathbf{E}(x,z) = \hat{\mathbf{y}} e^{-ikx\sin\theta + ikz\cos\theta} \tag{16}$$

$$\begin{aligned}\mathbf{H}(x,z) &= \sqrt{\frac{\epsilon_1}{\mu_1}} \hat{\mathbf{k}} \times \mathbf{E} = \sqrt{\frac{\epsilon_1}{\mu_1}} (-\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}) \times \hat{\mathbf{y}} e^{-ikx\sin\theta + ikz\cos\theta} \\ &= \sqrt{\frac{\epsilon_1}{\mu_1}} e^{ikz\cos\theta} (-\sin\theta\hat{\mathbf{z}} - \cos\theta\hat{\mathbf{x}}) e^{-ikx\sin\theta}\end{aligned} \tag{17}$$

$$\mathbf{E}'(x,z) = \hat{\mathbf{y}} e^{ip\pi} e^{ikx\sin\theta + ikz\cos\theta} \tag{18}$$

$$\begin{aligned}\mathbf{H}'(x,z) &= \sqrt{\frac{\epsilon_1}{\mu_1}} \hat{\mathbf{k}}' \times \mathbf{E}' = \sqrt{\frac{\epsilon_1}{\mu_1}} (\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}) \times \hat{\mathbf{y}} e^{ip\pi} e^{ikx\sin\theta + ikz\cos\theta} \\ &= \sqrt{\frac{\epsilon_1}{\mu_1}} e^{ikz\cos\theta} (\sin\theta\hat{\mathbf{z}} - \cos\theta\hat{\mathbf{x}}) e^{ip\pi} e^{ikx\sin\theta}\end{aligned} \tag{19}$$

which gives the superposition of the fields within the core region

$$\mathbf{E}_{\text{core}}(x,z) = e^{ikz\cos\theta} \hat{\mathbf{y}} \left(e^{ip\pi} e^{ikx\sin\theta} + e^{-ikx\sin\theta}\right) \tag{20}$$

$$\mathbf{H}_{\text{core}}(x,z) = \sqrt{\frac{\epsilon_1}{\mu_1}} e^{ikz\cos\theta} \left[\hat{\mathbf{z}}\sin\theta \left(e^{ip\pi} e^{ikx\sin\theta} - e^{-ikx\sin\theta}\right) - \hat{\mathbf{x}}\cos\theta \left(e^{ip\pi} e^{ikx\sin\theta} + e^{-ikx\sin\theta}\right)\right] \tag{21}$$

hence

$$\mathbf{E}_{\text{core}} \times \mathbf{H}^*_{\text{core}} = \sqrt{\frac{\epsilon_1}{\mu_1}} (\hat{\mathbf{x}} S_x + \hat{\mathbf{z}} S_z) \tag{22}$$

where

$$S_x = \sin\theta \left(e^{ip\pi} e^{ikx\sin\theta} + e^{-ikx\sin\theta}\right)\left(e^{-ip\pi} e^{-ikx\sin\theta} - e^{ikx\sin\theta}\right) = \sin\theta \left(e^{-ip\pi} e^{-i2kx\sin\theta} - e^{ip\pi} e^{i2kx\sin\theta}\right) \tag{23}$$

$$S_z = \cos\theta \left(e^{ip\pi} e^{ikx\sin\theta} + e^{-ikx\sin\theta}\right)\left(e^{-ip\pi} e^{-ikx\sin\theta} + e^{ikx\sin\theta}\right) = \cos\theta \left(2 + e^{-ip\pi} e^{-i2kx\sin\theta} + e^{ip\pi} e^{i2kx\sin\theta}\right) \tag{24}$$

Clearly, S_x is purely imaginary, and S_z is real. So there is only energy flow in the z direction in the core, which can be obtained by the integral

$$\begin{aligned}P_{\text{core}} &= \int_{-a}^{a} \frac{1}{2} \operatorname{Re}\left(\mathbf{E}_{\text{core}} \times \mathbf{H}^*_{\text{core}}\right) dx \\ &= \sqrt{\frac{\epsilon_1}{\mu_1}} \cos\theta \int_{-a}^{a} \left[1 + \cos(2kx\sin\theta + p\pi)\right] dx \\ &= \sqrt{\frac{\epsilon_1}{\mu_1}} \cos\theta \int_{-a}^{a} \left[1 \pm \cos(2kx\sin\theta)\right] dx && + \text{ for even } p, - \text{ for odd } p \\ &= \sqrt{\frac{\epsilon_1}{\mu_1}} \cos\theta \left[2a \pm \frac{\sin(2ka\sin\theta)}{k\sin\theta}\right] \\ &= 2\sqrt{\frac{\epsilon_1}{\mu_1}} a\cos\theta \left[1 \pm \frac{\sin(2ka\sin\theta)}{2ka\sin\theta}\right] && \text{recall } ka\sin\theta = V\xi \\ &= 2\sqrt{\frac{\epsilon_1}{\mu_1}} a\cos\theta \left[1 \pm \frac{\sin(2V\xi)}{2V\xi}\right]\end{aligned} \tag{25}$$

Now let's compute the energy flow in the cladding region.

At the $x = -a$ interface, the total internal reflection generates an evanescent wave into the $x < -a$ region, whose complex wavevector is

$$\mathbf{k}_{\text{eva}} = \hat{\mathbf{z}} k \cos\theta - i\hat{\mathbf{x}} k \sqrt{\cos^2\theta - \frac{n_2^2}{n_1^2}} \qquad \Longrightarrow \qquad \hat{\mathbf{k}}_{\text{eva}} = \frac{\mathbf{k}_{\text{eva}}}{k(n_2/n_1)} = \hat{\mathbf{z}} \cdot \frac{n_1}{n_2}\cos\theta - i\hat{\mathbf{x}}\sqrt{\frac{n_1^2}{n_2^2}\cos^2\theta - 1} \tag{26}$$

The evanescent electric field for $x < -a$ is given by (7.39)

$$\begin{aligned}
\mathbf{E}_{\text{eva}}(x,z) &= \mathbf{E}(-a,0)\left(\frac{2n_1\sin\theta}{n_1\sin\theta + i\sqrt{n_1^2\cos^2\theta - n_2^2}}\right) e^{i\mathbf{k}_{\text{eva}}\cdot[(x+a)\hat{\mathbf{x}} + z\hat{\mathbf{z}}]} \\
&= \hat{\mathbf{y}} e^{ika\sin\theta}\left(\frac{2\sin\theta}{\sin\theta + i\sqrt{\cos^2\theta - \frac{n_2^2}{n_1^2}}}\right) e^{ikz\cos\theta} e^{k(x+a)\sqrt{\cos^2\theta - n_2^2/n_1^2}} \\
&= \underbrace{\hat{\mathbf{y}}\, e^{ika\sin\theta}\left(\frac{2\xi}{\xi + i\sqrt{1-\xi^2}}\right) e^{ikz\cos\theta} e^{V[(x+a)/a]\sqrt{1-\xi^2}}}_{E_{\text{eva}}}
\end{aligned} \tag{27}$$

This gives the magnetic field for $x < -a$

$$\mathbf{H}_{\text{eva}} = \sqrt{\frac{\epsilon_2}{\mu_2}}\hat{\mathbf{k}}_{\text{eva}} \times \mathbf{E}_{\text{eva}} = \sqrt{\frac{\epsilon_2}{\mu_2}} E_{\text{eva}}\left(-\hat{\mathbf{x}} \cdot \frac{n_1}{n_2}\cos\theta - i\hat{\mathbf{z}}\sqrt{\frac{n_1^2}{n_2^2}\cos^2\theta - 1}\right) \tag{28}$$

and thus

$$\begin{aligned}
\operatorname{Re}\left(\mathbf{E}_{\text{eva}} \times \mathbf{H}^*_{\text{eva}}\right) &= \sqrt{\frac{\epsilon_2}{\mu_2}}\left|E_{\text{eva}}\right|^2 \frac{n_1}{n_2}\cos\theta\,\hat{\mathbf{z}} \qquad\qquad \text{assuming } \mu_1 = \mu_2 = \mu_0 \\
&= \sqrt{\frac{\epsilon_1}{\mu_1}}\left|E_{\text{eva}}\right|^2 \cos\theta\,\hat{\mathbf{z}}
\end{aligned} \tag{29}$$

Similarly for the $x > a$ region

$$\mathbf{E}'_{\text{eva}} = \overbrace{\hat{\mathbf{y}}\, e^{ip\pi} e^{ika\sin\theta}\left(\frac{2\xi}{\xi + i\sqrt{1-\xi^2}}\right) e^{ikz\cos\theta} e^{-V[(x-a)/a]\sqrt{1-\xi^2}}}^{E'_{\text{eva}}} \tag{30}$$

and

$$\operatorname{Re}\left(\mathbf{E}'_{\text{eva}} \times \mathbf{H}'^*_{\text{eva}}\right) = \sqrt{\frac{\epsilon_1}{\mu_1}}\left|E'_{\text{eva}}\right|^2 \cos\theta\,\hat{\mathbf{z}} \tag{31}$$

The total energy flow in the cladding can be obtained

$$\begin{aligned}
P_{\text{clad}} &= \int_{-\infty}^{-a} \frac{1}{2}\operatorname{Re}\left(\mathbf{E}_{\text{eva}} \times \mathbf{H}^*_{\text{eva}}\right) dx + \int_a^{\infty} \frac{1}{2}\operatorname{Re}\left(\mathbf{E}'_{\text{eva}} \times \mathbf{H}'^*_{\text{eva}}\right) dx \\
&= \sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta \int_a^{\infty}\left|\frac{2\xi}{\xi + i\sqrt{1-\xi^2}}\right|^2 e^{-2V[(x-a)/a]\sqrt{1-\xi^2}} dx \\
&= 2\sqrt{\frac{\epsilon_1}{\mu_1}} a\cos\theta \frac{\xi^2}{V\sqrt{1-\xi^2}}
\end{aligned} \tag{32}$$

Notice (32) does not depend on the parity of p. This may look different than the claim, but it is easy to show that when p is even, the equation (8.123) implies $\xi^2 = \cos^2(V\xi)$.

Putting (25) and (32) together, the fractions of energy flow within the core and that within the cladding are

$$F_{\text{core}} = \frac{1}{S}\left[1 \pm \frac{\sin(2V\xi)}{2V\xi}\right] \qquad F_{\text{clad}} = \frac{1}{S}\frac{\xi^2}{V\sqrt{1-\xi^2}} \qquad + \text{ for even } p, - \text{ for odd } p \tag{33}$$

with

$$S = \left[1 \pm \frac{\sin(2V\xi)}{2V\xi}\right] + \frac{\xi^2}{V\sqrt{1-\xi^2}} \tag{34}$$