1. In problem 5.8, we have expressed the internal and external multipole moments in spherical coordinates:

$$\text{internal}: \qquad m_l = -\frac{1}{l(l+1)} \int d^3x' r'^{-(l+1)} P_l^1\left(\cos\theta'\right) J\left(r', \theta'\right) \tag{1}$$

$$\text{external}: \qquad \mu_l = -\frac{1}{l(l+1)} \int d^3x' r'^{l} P_l^1\left(\cos\theta'\right) J\left(r', \theta'\right) \tag{2}$$

For the double-coil setup in problem 5.7, it will be more convenient to write the current density function in spherical coordinates too:

$$J\left(\mathbf{x}'\right) = I\frac{\delta(r'-d)}{d}\left[\delta\left(\theta'-\theta_0\right) + \delta\left(\theta'-\pi+\theta_0\right)\right] \qquad \text{where} \qquad \sin\theta_0 = \frac{a}{\sqrt{a^2+(b/2)^2}} = \frac{a}{d} \tag{3}$$

We can quickly verify the correctness of (3) by doing a space integral to see it recovers the sum of two line integrals along the two loops:

$$\begin{aligned}\int d^3x' J\left(\mathbf{x}'\right) &= I\int_0^{2\pi} d\phi' \int_0^{2\pi} r'^2 dr' \frac{\delta(r'-d)}{d} \int_0^{\pi} \sin\theta' d\theta' \left[\delta\left(\theta'-\theta_0\right) + \delta\left(\theta'-\pi+\theta_0\right)\right] \\ &= 2\pi I \cdot d \cdot 2\sin\theta_0 = 2\cdot 2\pi a I\end{aligned} \tag{4}$$

With (3) inserted into (1),

$$\begin{aligned} m_l &= -\frac{I}{l(l+1)} \int_0^{2\pi} d\phi' \int_0^{\infty} r'^2 dr' r'^{-(l+1)} \frac{\delta(r'-d)}{d} \int_0^{\pi} P_l^1\left(\cos\theta'\right) \sin\theta' d\theta' \left[\delta\left(\theta'-\theta_0\right) + \delta\left(\theta'-\pi+\theta_0\right)\right] \\ &= -\frac{2\pi I}{l(l+1)} d^{-l} \sin\theta_0 \left[P_l^1(\cos\theta_0) + P_l^1(-\cos\theta_0)\right] \end{aligned} \tag{5}$$

By the sign convention of this book (see equation 3.49)

$$P_l^1(\cos\theta_0) + P_l^1(-\cos\theta_0) = -\sin\theta_0\left[P_l'(\cos\theta_0) + P_l'(-\cos\theta_0)\right] \tag{6}$$

(5) turns into

$$m_l = \frac{2\pi I}{l(l+1)} \frac{\sin^2\theta_0}{d^l} \left[P_l'(\cos\theta_0) + P_l'(-\cos\theta_0)\right] = \begin{cases} \dfrac{4\pi I}{l(l+1)} \dfrac{a^2}{d^{l+2}} P_l'\left(\dfrac{b}{2d}\right) & \text{for } l \text{ odd} \\ 0 & \text{for } l \text{ even} \end{cases} \tag{7}$$

Similarly,

$$\mu_l = \begin{cases} \dfrac{4\pi I}{l(l+1)} a^2 d^{l-1} P_l'\left(\dfrac{b}{2d}\right) & \text{for } l \text{ odd} \\ 0 & \text{for } l \text{ even} \end{cases} \tag{8}$$

For $l = 1, \ldots, 5$, these multipole moments are

$$\begin{aligned} & m_2 = \mu_2 = m_4 = \mu_4 = 0 \\ & m_1 = \frac{2\pi I a^2}{d^3} && \mu_1 = 2\pi I a^2 \\ & m_3 = \frac{\pi I a^2}{3d^5}\left[\frac{15}{2}\left(\frac{b}{2d}\right)^2 - \frac{3}{2}\right] = \pi I a^2\left(\frac{b^2-a^2}{2d^7}\right) && \mu_3 = \pi I a^2\left(\frac{b^2-a^2}{2}\right) \\ & m_5 = \frac{2\pi I a^2}{15 d^7}\left[\frac{315}{8}\left(\frac{b}{2d}\right)^4 - \frac{210}{8}\left(\frac{b}{2d}\right)^2 + \frac{15}{8}\right] = \pi I a^2\left(\frac{b^4-6a^2b^2+2a^4}{8d^{11}}\right) \\ & \mu_5 = \pi I a^2\left(\frac{b^4-6a^2b^2+2a^4}{8}\right) \end{aligned} \tag{9}$$

2. Now we calculate the z-direction field B_z for points on the axis with small z. Recall from problem 5.8, for interior region ($r < r'$):

$$\mathbf{A} = -\hat{\boldsymbol{\phi}}\frac{\mu_0}{4\pi}\sum_{l=1}^{\infty} P_l^1(\cos\theta)\, r^l m_l \tag{10}$$

The dipole contribution ($l = 1$) to A_ϕ is thus:

$$\begin{aligned} A_\phi^{(1)} &= -\frac{\mu_0}{4\pi}\left[P_1^1(\cos\theta)\, r\right] m_1 = \frac{\mu_0}{4\pi}\rho m_1 \qquad \Longrightarrow \\ B_z^{(1)}(\rho=0,z) &= \left[\boldsymbol{\nabla}\times\mathbf{A}^{(1)}\right]_z(0,z) = \frac{1}{\rho}\frac{\partial \rho A_\phi^{(1)}}{\partial\rho}\bigg|_{\rho=0} = \frac{\mu_0}{2\pi}m_1 = \frac{\mu_0 I a^2}{d^3} \end{aligned}\tag{11}$$

Similarly, contribution from $l = 3$ is

$$\begin{aligned} A_\phi^{(3)} &= -\frac{\mu_0}{4\pi}\left[P_3^1(\cos\theta)\, r^3\right] m_3 = \frac{\mu_0}{4\pi}\left[\sin\theta\left(\frac{15}{2}\cos^2\theta - \frac{3}{2}\right) r^3\right] m_3 \\ &= \frac{\mu_0}{4\pi}\left(\frac{15}{2}\rho z^2 - \frac{3}{2}\rho r^2\right) m_3 \\ &= \frac{\mu_0}{4\pi}\left[\frac{15}{2}\rho z^2 - \frac{3}{2}\rho\left(\rho^2+z^2\right)\right] m_3 \\ &= \frac{\mu_0}{4\pi}\left(6\rho z^2 - \frac{3}{2}\rho^3\right) m_3 \qquad \Longrightarrow \\ B_z^{(3)}(\rho=0,z) &= \frac{1}{\rho}\frac{\partial \rho A_\phi^{(3)}}{\partial\rho}\bigg|_{\rho=0} = \frac{3\mu_0 z^2}{\pi}m_3 = \frac{\mu_0 I a^2}{d^7}\left[\frac{3\left(b^2-a^2\right)z^2}{2}\right] \end{aligned}\tag{12}$$

Lastly, for $l = 5$:

$$\begin{aligned} A_\phi^{(5)} &= -\frac{\mu_0}{4\pi}\left[P_5^1(\cos\theta)\, r^5\right] m_5 = \frac{\mu_0}{4\pi}\left[\sin\theta\left(\frac{315}{8}\cos^4\theta - \frac{210}{8}\cos^2\theta + \frac{15}{8}\right) r^5\right] m_5 \\ &= \frac{\mu_0}{4\pi}\left(\frac{315}{8}\rho z^4 - \frac{210}{8}\rho z^2 r^2 + \frac{15}{8}\rho r^4\right) m_5 \\ &= \frac{\mu_0}{4\pi}\left[\frac{315}{8}\rho z^4 - \frac{210}{8}\rho z^2\left(\rho^2+z^2\right) + \frac{15}{8}\rho\left(\rho^2+z^2\right)^2\right] m_5 \\ &= \frac{\mu_0}{4\pi}\left[15\rho z^4 + O\left(\rho^3\right)\right] m_5 \qquad \Longrightarrow \\ B_z^{(5)}(\rho=0,z) &= \frac{1}{\rho}\frac{\partial \rho A_\phi^{(5)}}{\partial\rho}\bigg|_{\rho=0} = \frac{15\mu_0 z^4}{2\pi}m_5 = \frac{\mu_0 I a^2}{d^{11}}\left[\frac{15\left(b^4-6a^2b^2+2a^4\right)z^4}{16}\right] \end{aligned}\tag{13}$$

As expected, (11)-(13) agree with the result obtained in problem 5.7(b).