1. **Prob 7.8**

   (a) For $T_{\text{layer}}\left(n_j, t_j\right)$, no interface is involved, the waves $E_\pm$ only have phase change at the two ends, so the transfer matrix is obtained trivially

$$T_{\text{layer}}\left(n_j, t_j\right) = \begin{bmatrix} e^{ik_j t_j} & 0 \\ 0 & e^{-ik_j t_j} \end{bmatrix} = I\cos\left(k_j t_j\right) + i\sigma_3 \sin\left(k_j t_j\right) \tag{1}$$

   (b) At the interface, let $E_\pm$ be the amplitudes of the forward and backward wave on the incident side, and $E'_\pm$ be the amplitudes of the forward and backward wave on the transmitted side, tangential **E** and tangential **H** boundary condition requires (for normal incidence)

$$E_+ + E_- = E'_+ + E'_- \tag{2}$$

$$\frac{k}{\mu}\left(E_+ - E_-\right) = \frac{k'}{\mu'}\left(E'_+ - E'_-\right) \tag{3}$$

   When the materials of the whole stack are nonpermeable ($\mu_j = \mu_0$), we can solve (2) and (3) with

$$\begin{bmatrix} E'_+ \\ E'_- \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{n}{n'}+1 & 1-\frac{n}{n'} \\ 1-\frac{n}{n'} & \frac{n}{n'}+1 \end{bmatrix}\begin{bmatrix} E_+ \\ E_- \end{bmatrix} \quad \Longrightarrow$$

$$T_{\text{interface}}\left(n \to n'\right) = \frac{1}{2}\begin{bmatrix} \frac{n}{n'}+1 & 1-\frac{n}{n'} \\ 1-\frac{n}{n'} & \frac{n}{n'}+1 \end{bmatrix} = I\left(\frac{n/n'+1}{2}\right) - \sigma_1\left(\frac{n/n'-1}{2}\right) \tag{4}$$

   (c) When we apply the transfer matrix to the entire stack, we should have

$$\begin{bmatrix} E_{\text{trans}} \\ 0 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}\begin{bmatrix} E_{\text{inc}} \\ E_{\text{refl}} \end{bmatrix} \tag{5}$$

   The zero below E_{trans} is due to the fact that there is no backward going wave after the last layer. Solving E_{refl} and E_{trans} in (5) gives

$$E_{\text{trans}} = \frac{\det(T)}{t_{22}} E_{\text{inc}} \qquad\qquad E_{\text{refl}} = -\frac{t_{21}}{t_{22}} E_{\text{inc}} \tag{6}$$

2. **Prob 7.9**

   (a) Consider the component of an air gap ($n_2 = 1, t_2$) followed by a layer of material (n, t_1), its transfer matrix is

$$T = T_{\text{interface}}(n \to 1) \cdot T_{\text{layer}}(n, t_1) \cdot T_{\text{interface}}(1 \to n) \cdot T_{\text{layer}}(1, t_2) \tag{7}$$

   The transfer matrix of the overall stack is apparently $T^N \cdot T^{-1}_{\text{layer}}(1, t_2)$, where we have canceled the effect of the first air gap by right multiplying $T^{-1}_{\text{layer}}(1, t_2)$, and by (1)

$$T^{-1}_{\text{layer}}(1, t_2) = T^{*}_{\text{layer}}(1, t_2) = I\cos\alpha_2 - i\sigma_3 \sin\alpha_2 \tag{8}$$

   To calculate T of (7), we will use the following properties of Pauli matrices

$$\sigma_i \sigma_j = \begin{cases} i\epsilon_{ijk}\sigma_k & \text{for } i \neq j \\ I & \text{for } i = j \end{cases} \tag{9}$$

   Then

$$\begin{aligned} T_{\text{interface}}(n \to 1) \cdot T_{\text{layer}}(n, t_1) &= \left[I\left(\frac{n+1}{2}\right) - \sigma_1\left(\frac{n-1}{2}\right)\right] \cdot \left(I\cos\alpha_1 + i\sigma_3 \sin\alpha_1\right) \\ &= I\left(\frac{n+1}{2}\right)\cos\alpha_1 - \sigma_1\left(\frac{n-1}{2}\right)\cos\alpha_1 \\ &\quad + i\sigma_3\left(\frac{n+1}{2}\right)\sin\alpha_1 - \sigma_2\left(\frac{n-1}{2}\right)\sin\alpha_1 \end{aligned} \tag{10}$$

Right multiplying

$$T_{\text{interface}}\left(1 \to n\right) = I\left(\frac{1/n+1}{2}\right) - \sigma_1\left(\frac{1/n-1}{2}\right) = I\left(\frac{n+1}{2n}\right) + \sigma_1\left(\frac{n-1}{2n}\right) \tag{11}$$

to (10) yields

$$\begin{aligned}T_{\text{interface}}\left(n \to 1\right) \cdot T_{\text{layer}}\left(n, t_1\right) \cdot T_{\text{interface}}\left(1 \to n\right) = I &\left[\frac{(n+1)^2}{4n}\cos\alpha_1 - \frac{(n-1)^2}{4n}\cos\alpha_1\right] \\ &+ i\sigma_3\left[\frac{(n+1)^2}{4n}\sin\alpha_1 + \frac{(n-1)^2}{4n}\sin\alpha_1\right] \\ &- \sigma_2\left[\frac{2\left(n^2-1\right)}{4n}\sin\alpha_1\right]\end{aligned} \tag{12}$$

Finally right multiplying

$$T_{\text{layer}}\left(1, t_2\right) = I\cos\alpha_2 + i\sigma_3\sin\alpha_2 \tag{13}$$

to (12) gives

$$\begin{aligned}T = \frac{1}{4n}\Bigg\{ I &\left[\overbrace{(n+1)^2\cos\alpha_1\cos\alpha_2 - (n-1)^2\cos\alpha_1\cos\alpha_2 - (n+1)^2\sin\alpha_1\sin\alpha_2 - (n-1)^2\sin\alpha_1\sin\alpha_2}^{(n+1)^2\cos(\alpha_1+\alpha_2)-(n-1)^2\cos(\alpha_1-\alpha_2)}\right] + \\ i\sigma_3 &\left[\underbrace{(n+1)^2\cos\alpha_1\sin\alpha_2 - (n-1)^2\cos\alpha_1\sin\alpha_2 + (n+1)^2\sin\alpha_1\cos\alpha_2 + (n-1)^2\sin\alpha_1\cos\alpha_2}_{(n+1)^2\sin(\alpha_1+\alpha_2)+(n-1)^2\sin(\alpha_1-\alpha_2)}\right] + \\ &2\sigma_1\left(n^2-1\right)\sin\alpha_1\sin\alpha_2 - 2\sigma_2\left(n^2-1\right)\sin\alpha_1\cos\alpha_2\Bigg\}\end{aligned} \tag{14}$$

(b) It seems the term "optical thickness" has multiple definitions depending on the context. Here, we are going to interpret it as the physical thickness times the corresponding index of refraction, which means $\alpha_1 = \alpha_2 = \pi/2$. Thus by (14)

$$T = \frac{1}{4n}\left[-2\left(n^2+1\right)I + 2\left(n^2-1\right)\sigma_1\right] = -\left[\frac{1}{2}\left(n+\frac{1}{n}\right)I - \frac{1}{2}\left(n-\frac{1}{n}\right)\sigma_1\right] \tag{15}$$

Note this form is equivalent to $T = -e^{-\lambda\sigma_1}$ since

$$e^{-\lambda\sigma_1} = \sum_{k=0}^{\infty}\frac{(-\lambda\sigma_1)^k}{k!} = I\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{(2k)!} - \sigma_1\sum_{k=0}^{\infty}\frac{\lambda^{2k+1}}{(2k+1)!} = I\cosh\lambda - \sigma_1\sinh\lambda \tag{16}$$

with $\lambda = \ln n$, we can identify

$$\cosh\lambda = \frac{1}{2}\left(n+\frac{1}{n}\right) \qquad \sinh\lambda = \frac{1}{2}\left(n-\frac{1}{n}\right) \tag{17}$$

For the whole stack

$$\begin{aligned}T_{\text{stack}} = T^N\left(-i\sigma_3\right) &= -i\left(-1\right)^N e^{-\lambda N\sigma_1}\sigma_3 = \\ &= -i\left(-1\right)^N\begin{bmatrix}\cosh\lambda N & -\sinh\lambda N \\ -\sinh\lambda N & \cosh\lambda N\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix} \\ &= -i\left(-1\right)^N\begin{bmatrix}\cosh\lambda N & \sinh\lambda N \\ -\sinh\lambda N & -\cosh\lambda N\end{bmatrix}\end{aligned} \tag{18}$$

Thus by (6)

$$\frac{|E_{\text{trans}}|^2}{|E_{\text{inc}}|^2} = \frac{1}{\cosh^2\lambda N} = \frac{4n^{2N}}{\left(n^{2N}+1\right)^2} \qquad \longrightarrow \qquad 4\exp\left(-N\ln n^2\right) \qquad \text{when} \qquad n^{2N} \gg 1 \tag{19}$$