1. **Center-of-mass frame**

Before going into the solutions of these problems, let's first elaborate on the "center-of-mass" frame – although a more appropriate name is "zero-momentum" frame. Consider a system of n moving particles in the lab frame K, in which the i-th particle has energy E_i and momentum $\mathbf{p}_i$. We wish to find a frame K', moving relative to K with velocity $\mathbf{u}$, such that in K' the total momentum of the system vanishes.

To simplify, let $c = 1$. Consider boosting the total 4-momentum $\left(\sum_{i=1}^n E_i, \sum_{i=1}^n \mathbf{p}_i\right)$ by $\mathbf{u}$. The perpendicular and parallel component of the total 3-momentum are given by

$$\left(\sum_i^n \mathbf{p}_i'\right)_\perp = \left(\sum_{i=1}^n \mathbf{p}_i\right)_\perp \tag{1}$$

$$\left(\sum_i^n \mathbf{p}_i'\right)_\parallel = \gamma\left[\left(\sum_{i=1}^n \mathbf{p}_i\right)_\parallel - \mathbf{u}\sum_{i=1}^n E_i\right] \tag{2}$$

If the sum of $\mathbf{p}_i'$ were to vanish, (1) implies that $\mathbf{u}$ must be parallel or antiparallel to the total 3-momentum $\sum_{i=1}^n \mathbf{p}_i$, and (2) gives its magnitude. In other words, in the "center-of-mass" frame K' which moves relative to K with velocity

$$\mathbf{u} = \frac{\sum_{i=1}^n \mathbf{p}_i}{\sum_{i=1}^n E_i}. \tag{3}$$

the system has no net momentum.

This is true regardless of the rest mass of the particles, since we had no assumption about their rest mass. If all of them have non-zero rest mass, (3) can be rewritten as

$$\mathbf{u} = \frac{\sum_{i=1}^n \gamma_i m_i \mathbf{v}_i}{\sum_{i=1}^n \gamma_i m_i}. \tag{4}$$

which falls back to the non-relativistic case if $\gamma_i \approx 1$.

2. **Prob 11.22**

(a) Let the two photons have energy E_1, E_2 in the lab frame. Since photons have zero rest mass, we have $E_i = |\mathbf{p}_i|$ for $i = 1, 2$. Without loss of generality, we can write

$$\mathbf{p}_1 = E_1\hat{\mathbf{x}} \qquad\qquad \mathbf{p}_2 = E_2\left(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}\right) \tag{5}$$

The "center-of-mass" frame's velocity $\mathbf{u}$ is given by (3),

$$\mathbf{u} = \frac{(E_1 + E_2\cos\phi)\,\hat{\mathbf{x}} + E_2\sin\phi\,\hat{\mathbf{y}}}{E_1 + E_2}. \tag{6}$$

and consequently

$$\beta_u^2 = \frac{E_1^2 + E_2^2 + 2E_1E_2\cos\phi}{(E_1+E_2)^2} \qquad\qquad \gamma_u^2 = \frac{(E_1+E_2)^2}{2E_1E_2(1-\cos\phi)} \tag{7}$$

Transforming the time-component of the 4-momenta to the center-of-mass frame, we have

$$E_1' = \gamma_u(E_1 - \mathbf{u}\cdot\mathbf{p}_1) = \gamma_u\left[\frac{E_1E_2(1-\cos\phi)}{E_1+E_2}\right] \qquad E_2' = \gamma_u(E_2 - \mathbf{u}\cdot\mathbf{p}_2) = \gamma_u\left[\frac{E_1E_2(1-\cos\phi)}{E_1+E_2}\right] \tag{8}$$

Expectedly, they are equal as a result of photons' zero rest mass and zero total momentum in the center-of-mass frame. For the collision to generate electron-positron pair, we must have

$$E_1' = E_2' \geq m_e \tag{9}$$

or

$$\gamma_u^2\left[\frac{E_1E_2(1-\cos\phi)}{E_1+E_2}\right]^2 \geq m_e^2 \qquad \Longrightarrow \qquad \frac{E_1E_2(1-\cos\phi)}{2} \geq m_e^2 \tag{10}$$

With E_1 being the energy of the background photon, the minimum E_2 is achieved with $\phi = \pi$ (head-on collision)

$$E_2 \geq \frac{m_e^2}{E_1} \approx \frac{(0.511\text{MeV})^2}{2.5\times10^{-4}\text{eV}} \approx 1.04\times10^{14}\text{eV} \tag{11}$$

(b) In this part, the background photon has energy $E_1 = 500\text{eV}$, hence

$$E_2 \approx \frac{(0.511\text{MeV})^2}{500\text{eV}} \approx 5.2 \times 10^8\text{eV} \tag{12}$$

3. **Prob 11.23**

(a) The invariance of the norm of the total 4-momentum gives

$$W^2 - \underbrace{\left|\mathbf{p}_1' + \mathbf{p}_2'\right|^2}_{0} = (E_{\text{lab}} + m_2)^2 - |\mathbf{p}_{\text{lab}}|^2 = m_2^2 + 2m_2E_{\text{lab}} + \overbrace{\left(E_{\text{lab}}^2 - |\mathbf{p}_{\text{lab}}|^2\right)}^{m_1^2} = m_1^2 + m_2^2 + 2m_2E_{\text{lab}}. \tag{13}$$

By (3), the center-of-mass frame's velocity is given by

$$\mathbf{u} = \frac{\mathbf{p}_{\text{lab}}}{E_{\text{lab}} + m_2} \tag{14}$$

with Lorentz factor

$$\gamma_u = \sqrt{\frac{1}{1-u^2}} = \frac{E_{\text{lab}} + m_2}{\sqrt{(E_{\text{lab}} + m_2)^2 - |\mathbf{p}_{\text{lab}}|^2}} = \frac{E_{\text{lab}} + m_2}{W} \tag{15}$$

Applying the Lorentz transformation to $\mathbf{p}_{\text{lab}}$ gives

$$\mathbf{p}' = \gamma_u (\mathbf{p}_{\text{lab}} - \mathbf{u}E_{\text{lab}}) = \frac{m_2\mathbf{p}_{\text{lab}}}{W} \tag{16}$$

(b) This is already obtained in (14) and (15), which came from the center-of-mass frame derivation in (3).

(c) In the non-relativistic limit, the kinetic energy of particle 1 is approximately $p_{\text{lab}}^2/2m_1$, so

$$E_{\text{lab}} \approx m_1 + \frac{p_{\text{lab}}^2}{2m_1} \tag{17}$$

thus

$$\begin{aligned} W = \left(m_1^2 + m_2^2 + 2m_2E_{\text{lab}}\right)^{1/2} &= \left[(m_1 + m_2)^2 + \frac{m_2 p_{\text{lab}}^2}{m_1}\right]^{1/2} && \text{ignoring } O\left(p_{\text{lab}}^4\right) \\ &\approx m_1 + m_2 + \left(\frac{m_2}{m_1 + m_2}\right)\frac{p_{\text{lab}}^2}{2m_1} \end{aligned} \tag{18}$$

Putting (18) into (16) and keeping up to $O(p_{\text{lab}})$, we have

$$\mathbf{p}' \approx \left(\frac{m_2}{m_1 + m_2}\right)\mathbf{p}_{\text{lab}} \tag{19}$$

Taking similar approximations to (14) gives

$$\boldsymbol{\beta}_{\text{cm}} \approx \frac{\mathbf{p}_{\text{lab}}}{m_1 + m_2} \tag{20}$$

4. **Prob 11.24**

From (13), we can get the minimum total energy of the incoming particle, then we can obtain its kinetic energy,

$$T_{\text{th}} = E_{\text{lab}} - m_1 = \frac{W^2 - m_1^2 - m_2^2}{2m_2} - m_1 \tag{21}$$

where W is the sum of the rest mass of the resulting particles. Plugging in the numerical values, we get

(a) For $\gamma p \to \pi^0 p$, $T_{\text{th}} = 144.7\text{MeV}$.

(b) For $pp \to ppp\bar{p}$, $T_{\text{th}} = 5.63\text{GeV}$.

(c) For $e^-e^- \to e^-e^-p\bar{p}$, $T_{\text{th}} = 3.451\text{TeV}$. For $e^+e^- \to p\bar{p}$, $T_{\text{th}} = 3.447\text{TeV}$.

5. **Prob 11.25**

(a) As usual, the invariance of scalar product requires

$$W^2 = (E_1+E_2)^2 - |\mathbf{p}_1+\mathbf{p}_2|^2 = \left(E_1^2-p_1^2\right) + \left(E_2^2-p_2^2\right) + 2\left(E_1E_2 - \mathbf{p}_1\cdot\mathbf{p}_2\right) \tag{22}$$

Since θ is the angle between two "head-on" colliding beams, we have $\mathbf{p}_1\cdot\mathbf{p}_2 = -p_1p_2\cos\theta$. Then the above becomes

$$\begin{aligned}
W^2 &= m_1^2 + m_2^2 + 2\left(E_1E_2 + p_1p_2\cos\theta\right) \\
&= m_1^2 + m_2^2 + 2\left(\sqrt{p_1^2+m_1^2}\sqrt{p_2^2+m_2^2} + p_1p_2\cos\theta\right) \qquad \text{up to } O\left(\frac{m^2}{p^2}\right) \\
&\approx m_1^2 + m_2^2 + 2p_1p_2\left(1+\cos\theta\right) + 2\left(\frac{p_2m_1^2}{2p_1} + \frac{p_1m_2^2}{2p_2}\right) \\
&= 4p_1p_2\cos^2\frac{\theta}{2} + (p_1+p_2)\left(\frac{m_1^2}{p_1} + \frac{m_2^2}{p_2}\right)
\end{aligned} \tag{23}$$

(b) By (3), the velocity of the center-of-mass frame is given by

$$\mathbf{u} = \frac{\mathbf{p}_1+\mathbf{p}_2}{E_1+E_2} = \left(\frac{p_1-p_2}{E_1+E_2}\right)\cos\frac{\theta}{2}\hat{\mathbf{x}} + \left(\frac{p_1+p_2}{E_1+E_2}\right)\sin\frac{\theta}{2}\hat{\mathbf{y}} \tag{24}$$

where $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ are the horizontal and vertical direction shown in the problem's diagram. Also from the diagram, it is clear that

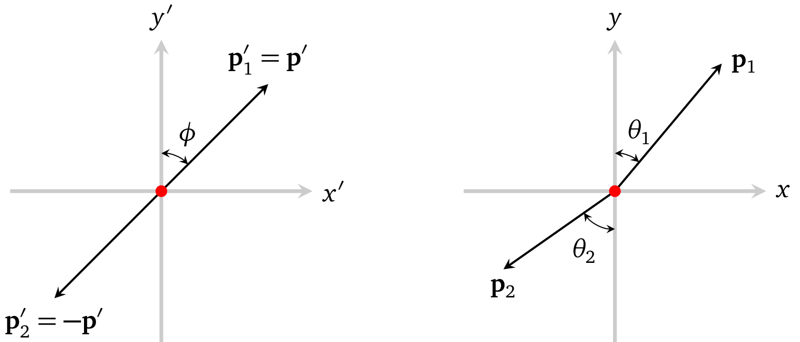
$$\tan\alpha = \frac{u_y}{u_x} = \left(\frac{p_1+p_2}{p_1-p_2}\right)\tan\frac{\theta}{2} \qquad\qquad \beta = \frac{u_y}{\sin\alpha} = \frac{(p_1+p_2)}{(E_1+E_2)\sin\alpha}\sin\frac{\theta}{2} \tag{25}$$

(c) Omitted.

(d) With $p_1 = p_2 = 100\text{GeV}$, $m_1 = m_2 = 938.5\text{MeV}$, we see that the contribution of the proton's rest mass to the total energy is almost negligible. When $p_1 = p_2$, by (25), we have $\alpha = \pi/2$, and $\beta \approx \sin\theta/2 \approx 0.17$ which is not a negligiblely low speed, so the lab frame is not a good approximation of the center-of-mass frame.

Now consider the resulting pions in the center-of-mass frame K'. If they have equal and opposite momenta $p' = 10\text{GeV}$, they also have equal energy

$$E' = \sqrt{p'^2 + m_\pi^2} = \sqrt{10^2 + 0.135^2}\text{GeV} \approx 10.001\text{GeV} \tag{26}$$



Without specifying the direction of the resulting pions, we don't have enough information to calculate the definite momenta in the lab frame. In K', let ϕ be the angle between $\mathbf{p}_1'$ and y-axis. Transforming the two 3-momenta back to the lab frame gives

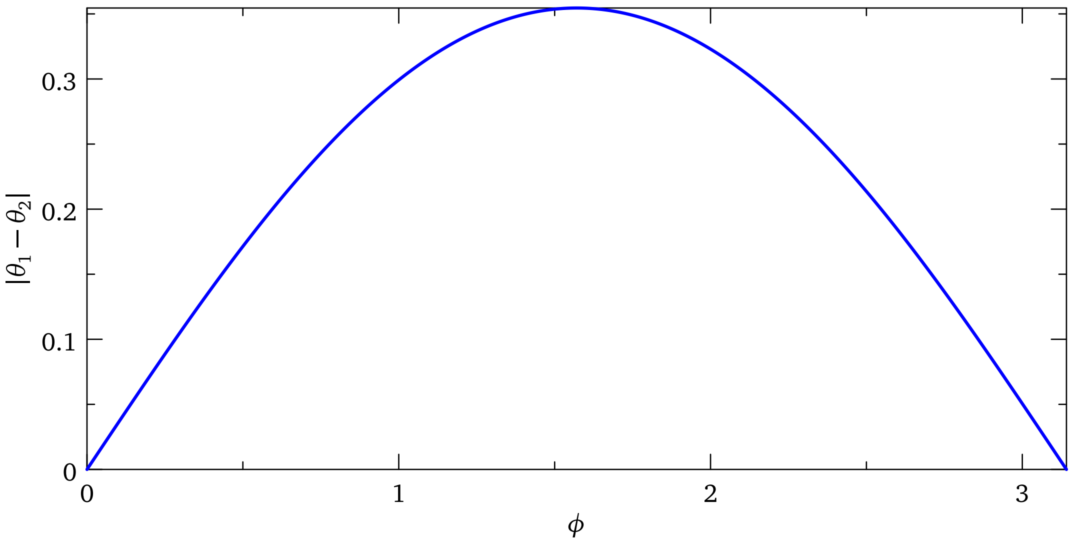
$$p_{1x} = p_{1x}' = p'\sin\phi \qquad\qquad p_{1y} = \gamma\left(p_{1y}' + \beta E_1'\right) = \gamma\left(p'\cos\phi + \beta E'\right) \tag{27}$$

$$p_{2x} = p_{2x}' = -p'\sin\phi \qquad\qquad p_{2y} = \gamma\left(p_{2y}' + \beta E_2'\right) = -\gamma\left(p'\cos\phi - \beta E'\right) \tag{28}$$

The deviation from collinearity in the lab frame is measured by

$$|\theta_1 - \theta_2| = \left|\tan^{-1}\left[\frac{p'\sin\phi}{\gamma\left(p'\cos\phi + \beta E'\right)}\right] - \tan^{-1}\left[\frac{p'\sin\phi}{\gamma\left(p'\cos\phi - \beta E'\right)}\right]\right| \tag{29}$$

With the numerical values plugged in, the graph $|\theta_1-\theta_2| \sim \phi$ is plotted below (with necessary modification so when $|\theta_1-\theta_2| > \pi/2$, the plot takes value $\pi - |\theta_1-\theta_2|$ so the deviation angle is always acute):



The maximum deviation from collinearity is about 0.355 radian, or 20.3°. This is a significant angle, so the lab frame is not a good approximation of the center-of-mass frame.

6. **Prob 11.26**

(a) i. For the first relation, with $m_3 = m_1, m_4 = m_2$, in the center-of-mass frame, we can write

$$q'^2 + m_1^2 = E_3'^2 \tag{30}$$
$$q'^2 + m_2^2 = E_4'^2 \tag{31}$$

where $E_3' + E_4' = W$.
Solving for E_3', E_4' and q' gives (while using (13) to simplify)

$$E_3' = \frac{W^2 + m_1^2 - m_2^2}{2W} = \frac{m_1^2 + m_2 E_1}{W} \tag{32}$$
$$E_4' = \frac{W^2 + m_2^2 - m_1^2}{2W} = \frac{m_2^2 + m_2 E_1}{W} \tag{33}$$
$$q' = \sqrt{E_4'^2 - m_2^2} = \frac{m_2 p_1}{W} \tag{34}$$

Applying the Lorentz transformation, we can get E_4

$$\begin{aligned} E_4 &= \gamma\left[E_4' + \boldsymbol{\beta}\cdot(-\mathbf{q}')\right] \qquad \text{using (14), (15) for } \boldsymbol{\beta}, \gamma \\ &= \gamma\left[m_2\gamma - \frac{p_1^2 m_2 \cos\theta'}{W(m_2+E_1)}\right] \\ &= m_2\left[\gamma^2 - \frac{p_1^2\cos\theta'}{W^2}\right] \end{aligned} \tag{35}$$

from which we can get the first form of the energy loss

$$\Delta E = E_4 - m_2 = \frac{m_2 p_1^2}{W^2}\left(1-\cos\theta'\right) \tag{36}$$

ii. For the second relation, we can start from the conservation of momentum relation

$$p_1 = p_3\cos\theta_3 + p_4\cos\theta_4 \qquad p_3\sin\theta_3 = p_4\sin\theta_4 \tag{37}$$

Eliminating θ_3 gives

$$\begin{aligned} \left(p_1 - p_4\cos\theta_4\right)^2 + p_4^2\sin^2\theta_4 = p_3^2 = E_3^2 - m_1^2 \qquad &\Longrightarrow \\ p_1^2 + p_4^2 - 2p_1p_4\cos\theta_4 = E_3^2 - m_1^2 \end{aligned} \tag{38}$$

By energy conservation, we can replace E_3 with $E_1 + m_2 - E_4$, also we can replace p_4 with $\sqrt{E_4^2 - m_2^2}$. This simplifies (38) to

$$\begin{aligned} p_1\sqrt{E_4^2 - m_2^2}\cos\theta_4 = (E_1+m_2)\left(E_4 - m_2\right) \qquad &\text{or} \\ p_1\sqrt{\Delta E\left(\Delta E + 2m_2\right)}\cos\theta_4 = (E_1+m_2)\,\Delta E \end{aligned} \tag{39}$$

from which we can obtain

$$\Delta E = \frac{2m_2 p_1^2 \cos^2\theta_4}{(E_1+m_2)^2 - p_1^2\cos^2\theta_4} = \frac{2m_2 p_1^2\cos^2\theta_4}{W^2 + p_1^2 \sin^2\theta_4} \tag{40}$$

iii. For the third relation, note that via $\mathbf{p}_1 = \mathbf{p}_3 + \mathbf{p}_4$ and $E_1 + m_2 = E_3 + E_4$, we have

$$\begin{aligned} Q^2 &= (\mathbf{p}_1 - \mathbf{p}_3)^2 - (E_1 - E_3)^2 \\ &= p_4^2 - \left(E_4 - m_2\right)^2 \\ &= \left(E_4^2 - m_2^2\right) - \Delta E^2 \\ &= \Delta E\,(\Delta E + 2m_2) - \Delta E^2 = 2m_2\Delta E \end{aligned} \tag{41}$$

so finally

$$\Delta E = \frac{Q^2}{2m_2} \tag{42}$$

(b) From the second relation (40), we see that the maximum energy loss is achieved at $\theta_4 = 0$ or $\theta_4 = \pi$, in which case

$$\Delta E_{\text{max}} = 2m_2 \frac{p_1^2}{W^2} = 2\gamma^2\beta^2 m_e \tag{43}$$

Note here γ, β are given in (14) and (15) which are the γ, β for the movement of center-of-mass frame, not for the movement of the incident particle's rest frame. But if the incident particle is much heavier than the target electron, its rest frame can be approximately treated as the center-of-mass frame.

Viewed from the rest frame of the incident particle, the electron is coming in with 4-momentum $\left(E_e', -\gamma\beta m_e \hat{\mathbf{x}}\right)$, and bounced back with 4-momentum $\left(E_e', \gamma\beta m_e\hat{\mathbf{x}}\right)$ after the collision. Transforming the difference of these two 4-vectors back to the lab frame gives

$$\Delta E = \gamma\left(\Delta E_e' + \beta\Delta\mathbf{p}_e'\right) = 2\gamma^2\beta^2 m_e \tag{44}$$

(c) For electron-electron collision, we can use the first relation (36), which takes maximum value for $\theta' = \pi$, giving

$$\Delta E_{\text{max}} = \frac{2m_e p_1^2}{W^2} = \frac{2m_e\,(\gamma\beta m_e)^2}{2m_e^2 + 2m_e E_1} = \frac{\gamma^2\beta^2 m_e}{1+\gamma} = (\gamma - 1)\,m_e \tag{45}$$