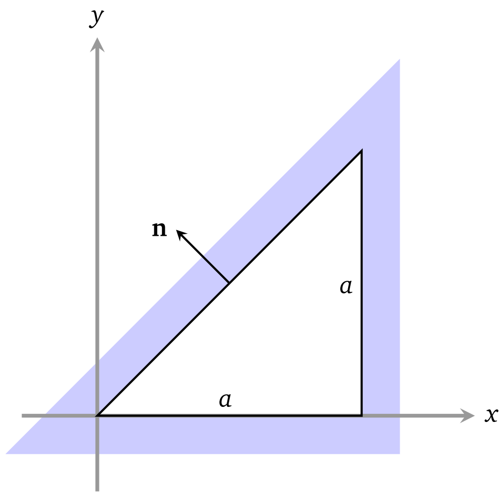


1. Recall the TM/TE mode solution for the $a \times a$ rectangular waveguide is

$$\psi_{mn}^{\mathrm{TM}}(x,y) = \psi_0^{\mathrm{TM}} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) \tag{1}$$

$$\psi_{mn}^{\mathrm{TE}}(x,y) = \psi_0^{\mathrm{TE}} \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right) \tag{2}$$

For this problem, the region of interest is diagonally half of the $a \times a$ rectangular waveguide, which is unlikely to have a separate variable form. Apparently (1) and (2) satisfy the boundary condition of the two orthogonal sides of the triangle. If we can linearly combine two or more solutions to make the hypotenuse meet the boundary condition, we can obtain the solution.

The relevant boundary values for a point (t,t) on the hypotenuse are

$$\text{TM}: \qquad \psi_{mn}^{\mathrm{TM}}(t,t) = \psi_0^{\mathrm{TM}} \sin\left(\frac{m\pi t}{a}\right)\sin\left(\frac{n\pi t}{a}\right) \tag{3}$$

$$\begin{aligned}\text{TE}: \qquad \frac{\partial \psi_{mn}^{\mathrm{TE}}}{\partial n}(t,t) &= \mathbf{n}\cdot\boldsymbol{\nabla}_t\psi_{mn}^{\mathrm{TE}} = \frac{1}{\sqrt{2}}\left(-\frac{\partial \psi_{mn}^{\mathrm{TE}}}{\partial x} + \frac{\partial \psi_{mn}^{\mathrm{TE}}}{\partial y}\right)(t,t) \\ &= \frac{\psi_0^{\mathrm{TE}}}{\sqrt{2}}\left[\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi t}{a}\right)\cos\left(\frac{n\pi t}{a}\right) - \left(\frac{n\pi}{a}\right)\cos\left(\frac{m\pi t}{a}\right)\sin\left(\frac{n\pi t}{a}\right)\right]\end{aligned} \tag{4}$$

We see that $\psi_{mn}^{\mathrm{TM}}(t,t)$ is symmetric in (m,n) and $\partial\psi_{mn}^{\mathrm{TE}}/\partial n$ is antisymmetric in (m,n). Thus we can construct

$$\phi_{mn}^{\mathrm{TM}} = \psi_{mn}^{\mathrm{TM}} - \psi_{nm}^{\mathrm{TM}} = \psi_0^{\mathrm{TM}}\left[\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) - \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\right] \tag{5}$$

$$\phi_{mn}^{\mathrm{TE}} = \psi_{mn}^{\mathrm{TE}} + \psi_{nm}^{\mathrm{TE}} = \psi_0^{\mathrm{TE}}\left[\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right)\right] \tag{6}$$

so that the TM/TE boundary conditions are met on the hypotenuse:

$$\phi_{mn}^{\mathrm{TM}}(t,t) = 0 \qquad\qquad \frac{\partial \phi_{mn}^{\mathrm{TE}}}{\partial n}(t,t) = 0 \tag{7}$$

It is also straightforward to see that $\phi_{mn}^{\mathrm{TM}}, \phi_{mn}^{\mathrm{TE}}$ satisfy eigenequation (8.34)

$$\left(\boldsymbol{\nabla}_t^2 + \gamma^2\right)\phi = 0 \tag{8}$$

with eigenvalue

$$\gamma_{mn}^2 = \frac{\left(m^2+n^2\right)\pi^2}{a^2} \tag{9}$$

and cutoff frequency

$$\omega_{mn} = \frac{\sqrt{m^2+n^2}\,\pi c}{a} \tag{10}$$

For TM, modes with $m > n > 0$ are nontrivial. For TE, modes with $m \geq n > 0$ or $m > n = 0$ are nontrivial. The lowest nontrivial TM mode is $(2,1)$, and the lowest TE mode is $(1,0)$.

2. The attenuation constant β_{mn} can be calculated via (see (8.51), (8.59), (8.57))

$$P = \frac{1}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_{mn}}\right)^2\sqrt{1-\frac{\omega_{mn}^2}{\omega^2}}\begin{Bmatrix}\epsilon\\ \mu\end{Bmatrix}\int_A \phi\phi^* da \tag{11}$$

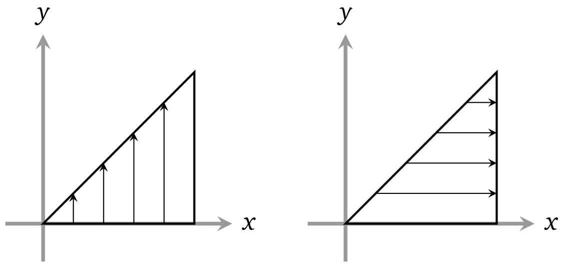
$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2\oint_C \left\{\begin{array}{c}\dfrac{1}{\mu^2\omega_{mn}^2}\left|\dfrac{\partial\phi}{\partial n}\right|^2 \\ \dfrac{1}{\mu\epsilon\omega_{mn}^2}\left(1-\dfrac{\omega_{mn}^2}{\omega^2}\right)|\mathbf{n}\times\boldsymbol{\nabla}_t\phi|^2+\dfrac{\omega_{mn}^2}{\omega^2}|\phi|^2\end{array}\right\} dl \tag{12}$$

$$\beta_{mn} = -\frac{1}{2P}\frac{dP}{dz} \tag{13}$$

We can simplify the calculation of the integral of (11) by noticing that for a function $g(x,y)$ satisfying $g(x,y) = g(y,x)$, we have

$$\int_A g(x,y)\,da = \frac{1}{2}\int_0^a dx\int_0^a dy\, g(x,y) \tag{14}$$

for A the triangular region depicted in the figure below.



This is because the $\int_A$ integral can be calculated with two orders, therefore

$$\begin{aligned}\int_A g(x,y)\,da &= \int_0^a dx\int_0^x dy\, g(x,y) = \int_0^a dy\int_y^a dx\, g(x,y)\\
&= \frac{1}{2}\left[\int_0^a dx\int_0^x dy\, g(x,y)+\int_0^a dy\int_y^a dx\, g(x,y)\right] \qquad \text{use } g(x,y)=g(y,x)\\
&= \frac{1}{2}\left[\int_0^a dx\int_0^x dy\, g(x,y)+\int_0^a dy\int_y^a dx\, g(y,x)\right]\\
&= \frac{1}{2}\int_0^a dx\int_0^a dy\, g(x,y)\end{aligned} \tag{15}$$

In the following calculation of attenuation constant, we take $\psi_0^{\mathrm{TM/TE}} = 1$, since it will be canceled in (13).

(a) **TM mode**

In the TM mode, only modes with $m > n > 0$ are nontrivial. Applying (14) to $g(x,y) = |\phi|^2$, we have

$$\int_A \phi\phi^* da = \frac{a^2}{4} \qquad \Longrightarrow \qquad P = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_{mn}}\right)^2\sqrt{1-\frac{\omega_{mn}^2}{\omega^2}}\cdot\left(\frac{a^2}{4}\right) \tag{16}$$

For the integral in (12), note that

$$\text{horizontal side}: \quad \left|\frac{\partial\phi}{\partial n}\right| = \left|\frac{\partial\phi}{\partial y}(x,0)\right| = \left|\left(\frac{n\pi}{a}\right)\sin\left(\frac{m\pi x}{a}\right)-\left(\frac{m\pi}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\right| \tag{17}$$

$$\text{vertical side}: \quad \left|\frac{\partial\phi}{\partial n}\right| = \left|\frac{\partial\phi}{\partial x}(a,y)\right| = \left|\left(\frac{m\pi}{a}\right)(-1)^m\sin\left(\frac{n\pi y}{a}\right)-\left(\frac{n\pi}{a}\right)(-1)^n\sin\left(\frac{m\pi y}{a}\right)\right| \tag{18}$$

$$\begin{aligned}\text{hypotenus}: \quad \left|\frac{\partial\phi}{\partial n}\right| &= \frac{1}{\sqrt{2}}\left|\left(-\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\right)(t,t)\right|\\
&= \frac{1}{\sqrt{2}}\left|-2\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi t}{a}\right)\sin\left(\frac{n\pi t}{a}\right)+2\left(\frac{n\pi}{a}\right)\cos\left(\frac{n\pi t}{a}\right)\sin\left(\frac{m\pi t}{a}\right)\right|\\
&= \frac{1}{\sqrt{2}}\left|\left(\frac{\pi}{a}\right)\left\{(m+n)\sin\left[\frac{(m-n)\pi t}{a}\right]-(m-n)\sin\left[\frac{(m+n)\pi t}{a}\right]\right\}\right|\end{aligned} \tag{19}$$

Then we have

$$\int_{\text{hor}} \left|\frac{\partial \phi}{\partial n}\right|^2 dl = \int_{\text{ver}} \left|\frac{\partial \phi}{\partial n}\right|^2 dl = \frac{\left(m^2+n^2\right)\pi^2}{2a} \qquad \int_{\text{hyp}} \left|\frac{\partial \phi}{\partial n}\right|^2 dl = \frac{\left(m^2+n^2\right)\pi^2}{\sqrt{2}a} \tag{20}$$

Plugging these back into (12), we get

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(\frac{1}{\mu^2\omega_{mn}^2}\right)\cdot\left(1+\frac{1}{\sqrt{2}}\right)a\gamma_{mn}^2 = \frac{a}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1+\frac{1}{\sqrt{2}}\right)\left(\frac{\epsilon}{\mu}\right) \tag{21}$$

then to (13), we get

$$\beta_{mn}^{\text{TM}} = \left(\frac{2+\sqrt{2}}{\sigma\delta a}\right)\sqrt{\frac{\epsilon}{\mu}}\left(\sqrt{1-\frac{\omega_{mn}^2}{\omega^2}}\right)^{-1} \qquad \Longrightarrow \qquad \xi_{mn}^{\text{TM}} = 1 \tag{22}$$

(b) **TE mode**

In TE mode, we have the following:

$$\text{horizontal side :} \quad |\mathbf{n}\times\nabla_t\phi| = \left|\frac{\partial\phi}{\partial x}(x,0)\right| = \left|\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi x}{a}\right)+\left(\frac{n\pi}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\right| \tag{23}$$

$$|\phi| = \left|\cos\left(\frac{m\pi x}{a}\right)+\cos\left(\frac{n\pi x}{a}\right)\right| \tag{24}$$

$$\text{vertical side :} \quad |\mathbf{n}\times\nabla_t\phi| = \left|\frac{\partial\phi}{\partial y}(a,y)\right| = \left|\left(\frac{n\pi}{a}\right)(-1)^m\sin\left(\frac{n\pi y}{a}\right)+\left(\frac{m\pi}{a}\right)(-1)^n\sin\left(\frac{m\pi y}{a}\right)\right| \tag{25}$$

$$|\phi| = \left|(-1)^m\cos\left(\frac{n\pi y}{a}\right)+(-1)^n\cos\left(\frac{m\pi y}{a}\right)\right| \tag{26}$$

$$\begin{aligned}\text{hypotenuse :} \quad |\mathbf{n}\times\nabla_t\phi| &= \frac{1}{\sqrt{2}}\left|\left(\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\right)(t,t)\right| \\ &= \frac{1}{\sqrt{2}}\left|2\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi t}{a}\right)\cos\left(\frac{n\pi t}{a}\right)+2\left(\frac{n\pi}{a}\right)\sin\left(\frac{n\pi t}{a}\right)\cos\left(\frac{m\pi t}{a}\right)\right| \\ &= \frac{1}{\sqrt{2}}\left|\left(\frac{\pi}{a}\right)\left\{(m+n)\sin\left[\frac{(m+n)\pi t}{a}\right]+(m-n)\sin\left[\frac{(m-n)\pi t}{a}\right]\right\}\right|\end{aligned} \tag{27}$$

$$|\phi| = \left|2\cos\left(\frac{m\pi t}{a}\right)\cos\left(\frac{n\pi t}{a}\right)\right| = \left|\cos\left[\frac{(m+n)\pi t}{a}\right]+\cos\left[\frac{(m-n)\pi t}{a}\right]\right| \tag{28}$$

Since $n=0$ or $m=n$ is allowed in TE, the integral of (11) and (12) will have to be treated differently than the general $m>n>0$ case.

i. $m>n=0$

Using (14), we get

$$\int_A \phi\phi^* da = \frac{a^2}{2} \qquad \Longrightarrow \qquad P = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega}{\omega_{m0}}\right)^2\sqrt{1-\frac{\omega_{m0}^2}{\omega^2}}\cdot\left(\frac{a^2}{2}\right) \tag{29}$$

Integral in (12) can be obtained via the following results

$$\int_{\text{hor}} |\mathbf{n}\times\nabla_t\phi|^2 dl = \int_{\text{ver}} |\mathbf{n}\times\nabla_t\phi|^2 dl = \frac{m^2\pi^2}{2a} \qquad \int_{\text{hyp}} |\mathbf{n}\times\nabla_t\phi|^2 dl = \frac{\sqrt{2}m^2\pi^2}{a} \tag{30}$$

$$\int_{\text{hor}} |\phi|^2 dl = \int_{\text{ver}} |\phi|^2 dl = \frac{3a}{2} \qquad \int_{\text{hyp}} |\phi|^2 dl = 2\sqrt{2}a \tag{31}$$

which gives

$$\begin{aligned}-\frac{dP}{dz} &= \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{m0}}\right)^2\left[\frac{1}{\mu\epsilon\omega_{m0}^2}\left(1-\frac{\omega_{m0}^2}{\omega^2}\right)\left(1+\sqrt{2}\right)a\gamma_{m0}^2+\left(\frac{\omega_{m0}^2}{\omega^2}\right)\left(3+2\sqrt{2}\right)a\right] \\ &= \frac{a}{2\sigma\delta}\left(\frac{\omega}{\omega_{m0}}\right)^2\left[\left(1+\sqrt{2}\right)+\left(2+\sqrt{2}\right)\left(\frac{\omega_{m0}^2}{\omega^2}\right)\right]\end{aligned} \tag{32}$$

Hence

$$\beta_{m0}^{\mathrm{TE}} = \frac{1}{\sigma\delta a}\sqrt{\frac{\epsilon}{\mu}}\left(\sqrt{1-\frac{\omega_{m0}^2}{\omega^2}}\right)^{-1}\left[\left(1+\sqrt{2}\right)+\left(2+\sqrt{2}\right)\left(\frac{\omega_{m0}^2}{\omega^2}\right)\right] \qquad \Longrightarrow \tag{33}$$

$$\xi_{m0}^{\mathrm{TE}} = \frac{1+\sqrt{2}}{2+\sqrt{2}} \qquad \eta_{m0}^{\mathrm{TE}} = 1 \tag{34}$$

ii. $m = n > 0$
Calculation shows that (29) applies in this case too.
But

$$\int_{\mathrm{hor}} |\mathbf{n}\times\boldsymbol{\nabla}_t\phi|^2\,dl = \int_{\mathrm{ver}} |\mathbf{n}\times\boldsymbol{\nabla}_t\phi|^2\,dl = \frac{2m^2\pi^2}{a} \qquad \int_{\mathrm{hyp}} |\mathbf{n}\times\boldsymbol{\nabla}_t\phi|^2\,dl = \frac{\sqrt{2}m^2\pi^2}{a} \tag{35}$$

$$\int_{\mathrm{hor}} |\phi|^2\,dl = \int_{\mathrm{ver}} |\phi|^2\,dl = 2a \qquad \int_{\mathrm{hyp}} |\phi|^2\,dl = \frac{3\sqrt{2}a}{2} \tag{36}$$

Hence

$$\begin{aligned}-\frac{dP}{dz} &= \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mm}}\right)^2\left[\frac{1}{\mu\epsilon\omega_{mm}^2}\left(1-\frac{\omega_{mm}^2}{\omega^2}\right)\left(2+\frac{1}{\sqrt{2}}\right)a\gamma_{mm}^2+\left(\frac{\omega_{mm}^2}{\omega^2}\right)\left(4+\frac{3\sqrt{2}}{2}\right)a\right]\\ &= \frac{a}{2\sigma\delta}\left(\frac{\omega}{\omega_{mm}}\right)^2\left[\left(2+\frac{1}{\sqrt{2}}\right)+\left(2+\sqrt{2}\right)\left(\frac{\omega_{mm}^2}{\omega^2}\right)\right]\end{aligned} \tag{37}$$

giving the attenuation constant and geometric factors

$$\beta_{mm}^{\mathrm{TE}} = \frac{1}{\sigma\delta a}\sqrt{\frac{\epsilon}{\mu}}\left(\sqrt{1-\frac{\omega_{mm}^2}{\omega^2}}\right)^{-1}\left[\left(2+\frac{1}{\sqrt{2}}\right)+\left(2+\sqrt{2}\right)\left(\frac{\omega_{mm}^2}{\omega^2}\right)\right] \qquad \Longrightarrow \tag{38}$$

$$\xi_{mm}^{\mathrm{TE}} = \frac{2\sqrt{2}+1}{2\sqrt{2}+2} \qquad \eta_{mm}^{\mathrm{TE}} = 1 \tag{39}$$

iii. m>n>0
In this case

$$\int_A \phi\phi^* = \frac{a^2}{4} \qquad \Longrightarrow \qquad P = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega}{\omega_{mn}}\right)^2\sqrt{1-\frac{\omega_{mn}^2}{\omega^2}}\cdot\left(\frac{a^2}{4}\right) \tag{40}$$

and

$$\int_{\mathrm{hor}} |\mathbf{n}\times\boldsymbol{\nabla}_t\phi|^2\,dl = \int_{\mathrm{ver}} |\mathbf{n}\times\boldsymbol{\nabla}_t\phi|^2\,dl = \frac{\left(m^2+n^2\right)\pi^2}{2a} \qquad \int_{\mathrm{hyp}} |\mathbf{n}\times\boldsymbol{\nabla}_t\phi|^2\,dl = \frac{\left(m^2+n^2\right)\pi^2}{\sqrt{2}a} \tag{41}$$

$$\int_{\mathrm{hor}} |\phi|^2\,dl = \int_{\mathrm{ver}} |\phi|^2\,dl = a \qquad \int_{\mathrm{hyp}} |\phi|^2\,dl = \sqrt{2}a \tag{42}$$

Thus

$$\begin{aligned}-\frac{dP}{dz} &= \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2\left[\frac{1}{\mu\epsilon\omega_{mn}^2}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)\cdot\left(1+\frac{1}{\sqrt{2}}\right)a\gamma_{mn}^2+\left(\frac{\omega_{mn}^2}{\omega^2}\right)\left(2+\sqrt{2}\right)a\right]\\ &= \frac{a}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2\left[\left(1+\frac{1}{\sqrt{2}}\right)+\left(1+\frac{1}{\sqrt{2}}\right)\left(\frac{\omega_{mn}^2}{\omega^2}\right)\right]\end{aligned} \tag{43}$$

And finally

$$\beta_{mn}^{\mathrm{TE}} = \frac{1}{\sigma\delta a}\sqrt{\frac{\epsilon}{\mu}}\left(\sqrt{1-\frac{\omega_{mn}^2}{\omega^2}}\right)^{-1}\left[\left(2+\sqrt{2}\right)+\left(2+\sqrt{2}\right)\left(\frac{\omega_{mn}^2}{\omega^2}\right)\right] \qquad \Longrightarrow \tag{44}$$

$$\xi_{mn}^{\mathrm{TE}} = 1 \qquad \eta_{mn}^{\mathrm{TE}} = 1 \tag{45}$$