1. In this part, we shall first fill the derivation details of section 6.13 (Hertz vectors).

   We start from the macroscopic relations

$$\mathbf{D} = \epsilon\mathbf{E} + \mathbf{P}_{\text{ext}} \qquad\qquad \mathbf{B} = \mu\mathbf{H} + \mu_0\mathbf{M}_{\text{ext}} \tag{1}$$

   Since

$$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \qquad \Longrightarrow \qquad \nabla\times\left(\mathbf{E} + \frac{\partial\mathbf{A}}{\partial t}\right) = 0 \tag{2}$$

   we can write

$$\mathbf{E} + \frac{\partial\mathbf{A}}{\partial t} = -\nabla\Phi \tag{3}$$

   Then

$$\nabla\cdot\mathbf{D} = 0 \qquad \Longrightarrow \qquad \nabla\cdot(\epsilon\mathbf{E} + \mathbf{P}_{\text{ext}}) = 0 \qquad \Longrightarrow \qquad -\epsilon\nabla^2\Phi - \epsilon\frac{\partial(\nabla\cdot\mathbf{A})}{\partial t} + \nabla\cdot\mathbf{P}_{\text{ext}} = 0 \tag{4}$$

   Also

$$\begin{aligned}
\nabla\times\mathbf{H} &= \frac{\partial\mathbf{D}}{\partial t} & \Longrightarrow \\
\nabla\times(\mathbf{B} - \mu_0\mathbf{M}_{\text{ext}}) &= \mu\frac{\partial}{\partial t}(\epsilon\mathbf{E} + \mathbf{P}_{\text{ext}}) & \Longrightarrow \\
\nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} - \mu_0\nabla\times\mathbf{M}_{\text{ext}} &= \mu\epsilon\left[-\nabla\frac{\partial\Phi}{\partial t} - \frac{\partial^2\mathbf{A}}{\partial t^2}\right] + \mu\frac{\partial\mathbf{P}_{\text{ext}}}{\partial t} & \Longrightarrow \\
\mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} + \nabla\left(\nabla\cdot\mathbf{A} + \mu\epsilon\frac{\partial\Phi}{\partial t}\right) &= \mu\frac{\partial\mathbf{P}_{\text{ext}}}{\partial t} + \mu_0\nabla\times\mathbf{M}_{\text{ext}} & (5)
\end{aligned}$$

   If we adopt the Lorenz gauge

$$\nabla\cdot\mathbf{A} + \mu\epsilon\frac{\partial\Phi}{\partial t} = 0 \tag{6}$$

   (4) and (5) take the standard wave equation form

$$\mu\epsilon\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi = -\frac{1}{\epsilon}\nabla\cdot\mathbf{P}_{\text{ext}} \tag{7}$$

$$\mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} = \mu\frac{\partial\mathbf{P}_{\text{ext}}}{\partial t} + \mu_0\nabla\times\mathbf{M}_{\text{ext}} \tag{8}$$

   Now introduce two vector polarization potentials $\mathbf{\Pi}_{\text{e}}, \mathbf{\Pi}_{\text{m}}$, such that

$$\mathbf{A} = \mu\frac{\partial\mathbf{\Pi}_{\text{e}}}{\partial t} + \mu_0\nabla\times\mathbf{\Pi}_{\text{m}} \tag{9}$$

$$\Phi = -\frac{1}{\epsilon}\nabla\cdot\mathbf{\Pi}_{\text{e}} \tag{10}$$

   Replacing (10) into (7) and (9) into (8), we have

$$\nabla\cdot\left[-\mu\epsilon\frac{\partial^2\mathbf{\Pi}_{\text{e}}}{\partial t^2} + \nabla^2\mathbf{\Pi}_{\text{e}} + \mathbf{P}_{\text{ext}}\right] = 0 \tag{11}$$

$$\mu\frac{\partial}{\partial t}\left[\mu\epsilon\frac{\partial^2\mathbf{\Pi}_{\text{e}}}{\partial t^2} - \nabla^2\mathbf{\Pi}_{\text{e}} - \mathbf{P}_{\text{ext}}\right] = \mu_0\nabla\times\left[-\mu\epsilon\frac{\partial^2\mathbf{\Pi}_{\text{m}}}{\partial t^2} + \nabla^2\mathbf{\Pi}_{\text{m}} + \mathbf{M}_{\text{ext}}\right] \tag{12}$$

   We can identify the bracket of (11) as a curl, i.e.,

$$\mu\epsilon\frac{\partial^2\mathbf{\Pi}_{\text{e}}}{\partial t^2} - \nabla^2\mathbf{\Pi}_{\text{e}} = \mathbf{P}_{\text{ext}} - \frac{\mu_0}{\mu}\nabla\times\mathbf{V} \tag{13}$$

which turns (12) into

$$-\mu_0 \boldsymbol{\nabla} \times \frac{\partial \mathbf{V}}{\partial t} = \mu_0 \boldsymbol{\nabla} \times \left[ -\mu\epsilon \frac{\partial^2 \boldsymbol{\Pi}_\mathrm{m}}{\partial t^2} + \boldsymbol{\nabla}^2 \boldsymbol{\Pi}_\mathrm{m} + \mathbf{M}_\mathrm{ext} \right] \tag{14}$$

This means the bracket on the RHS of (14) differs $-\partial \mathbf{V}/\partial t$ by a gradient of a scalar field, i.e.,

$$\mu\epsilon \frac{\partial^2 \boldsymbol{\Pi}_\mathrm{m}}{\partial t^2} - \boldsymbol{\nabla}^2 \boldsymbol{\Pi}_\mathrm{m} - \mathbf{M}_\mathrm{ext} = \frac{\partial \mathbf{V}}{\partial t} + \boldsymbol{\nabla} \frac{\partial \xi}{\partial t} \tag{15}$$

In the next part (problem 6.23), we will prove $\mathbf{V}$ and $\xi$ are arbitrary up to a gauge transform, hence we can set them to zero, which will turn (13) and (15) into the standard wave function form

$$\mu\epsilon \frac{\partial^2 \boldsymbol{\Pi}_\mathrm{e}}{\partial t^2} - \boldsymbol{\nabla}^2 \boldsymbol{\Pi}_\mathrm{e} = \mathbf{P}_\mathrm{ext} \tag{16}$$

$$\mu\epsilon \frac{\partial^2 \boldsymbol{\Pi}_\mathrm{m}}{\partial t^2} - \boldsymbol{\nabla}^2 \boldsymbol{\Pi}_\mathrm{m} = \mathbf{M}_\mathrm{ext} \tag{17}$$

In combination with (3), (9), (10), we can write the fields

$$\mathbf{E} = \frac{1}{\epsilon} \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_\mathrm{e}) - \mu \frac{\partial^2 \boldsymbol{\Pi}_\mathrm{e}}{\partial t^2} - \mu_0 \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{\Pi}_\mathrm{m}}{\partial t} \tag{18}$$

$$\mathbf{B} = \mu \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{\Pi}_\mathrm{e}}{\partial t} + \mu_0 \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\Pi}_\mathrm{m}) \tag{19}$$

In addition, in places where $\mathbf{P}_\mathrm{ext} = 0$, we can combine (16) and (18) to obtain

$$\begin{aligned} \mathbf{E} &= \frac{1}{\epsilon} \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_\mathrm{e}) - \frac{1}{\epsilon} \boldsymbol{\nabla}^2 \boldsymbol{\Pi}_\mathrm{e} - \mu_0 \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{\Pi}_\mathrm{m}}{\partial t} \\ &= \frac{1}{\epsilon} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\Pi}_\mathrm{e}) - \mu_0 \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{\Pi}_\mathrm{m}}{\partial t} \end{aligned} \tag{20}$$

which has similar form as (19).

2. **Prob 6.23**

   (a) With the transformation

$$\boldsymbol{\Pi}'_\mathrm{e} = \boldsymbol{\Pi}_\mathrm{e} + \mu_0 \boldsymbol{\nabla} \times \mathbf{G} - \boldsymbol{\nabla} g \tag{21}$$

$$\boldsymbol{\Pi}'_\mathrm{m} = \boldsymbol{\Pi}_\mathrm{m} - \mu \frac{\partial \mathbf{G}}{\partial t} \tag{22}$$

where $\mathbf{G}, g$ satisfy

$$\mu\epsilon \frac{\partial^2 \mathbf{G}}{\partial t^2} - \boldsymbol{\nabla}^2 \mathbf{G} = \frac{\mathbf{V} + \boldsymbol{\nabla}\xi}{\mu} \tag{23}$$

$$\mu\epsilon \frac{\partial^2 g}{\partial t^2} - \boldsymbol{\nabla}^2 g = 0 \tag{24}$$

let's calculate the wave equation form (13), (15) for $\boldsymbol{\Pi}'_\mathrm{e}$ and $\boldsymbol{\Pi}'_\mathrm{m}$.

$$\begin{aligned} \mu\epsilon \frac{\partial^2 \boldsymbol{\Pi}'_\mathrm{e}}{\partial t^2} - \boldsymbol{\nabla}^2 \boldsymbol{\Pi}'_\mathrm{e} &= \mu\epsilon \frac{\partial^2}{\partial t^2} (\boldsymbol{\Pi}_\mathrm{e} + \mu_0 \boldsymbol{\nabla} \times \mathbf{G} - \boldsymbol{\nabla} g) - \boldsymbol{\nabla}^2 (\boldsymbol{\Pi}_\mathrm{e} + \mu_0 \boldsymbol{\nabla} \times \mathbf{G} - \boldsymbol{\nabla} g) \\ &= \mu\epsilon \frac{\partial^2 \boldsymbol{\Pi}_\mathrm{e}}{\partial t^2} - \boldsymbol{\nabla}^2 \boldsymbol{\Pi}_\mathrm{e} + \mu_0 \boldsymbol{\nabla} \times \left( \mu\epsilon \frac{\partial^2 \mathbf{G}}{\partial t^2} - \boldsymbol{\nabla}^2 \mathbf{G} \right) - \boldsymbol{\nabla} \left( \mu\epsilon \frac{\partial^2 g}{\partial t^2} - \boldsymbol{\nabla}^2 g \right) \\ &= \mathbf{P}_\mathrm{ext} \end{aligned} \tag{25}$$

$$\begin{aligned} \mu\epsilon \frac{\partial^2 \boldsymbol{\Pi}'_\mathrm{m}}{\partial t^2} - \boldsymbol{\nabla}^2 \boldsymbol{\Pi}'_\mathrm{m} &= \mu\epsilon \frac{\partial^2}{\partial t^2} \left( \boldsymbol{\Pi}_\mathrm{m} - \mu \frac{\partial \mathbf{G}}{\partial t} \right) - \boldsymbol{\nabla}^2 \left( \boldsymbol{\Pi}_\mathrm{m} - \mu \frac{\partial \mathbf{G}}{\partial t} \right) \\ &= \mu\epsilon \frac{\partial^2 \boldsymbol{\Pi}_\mathrm{m}}{\partial t^2} - \boldsymbol{\nabla}^2 \boldsymbol{\Pi}_\mathrm{m} - \mu \frac{\partial}{\partial t} \left( \mu\epsilon \frac{\partial^2 \mathbf{G}}{\partial t^2} - \boldsymbol{\nabla}^2 \mathbf{G} \right) \\ &= \mathbf{M}_\mathrm{ext} \end{aligned} \tag{26}$$

I.e., the transformed $\boldsymbol{\Pi}_\mathrm{e}, \boldsymbol{\Pi}_\mathrm{m}$ satisfy the wave equation with $\mathbf{V} = 0, \xi = 0$.

(b) With the transform $\mathbf{\Pi}_\mathrm{e} \to \mathbf{\Pi}'_\mathrm{e}, \mathbf{\Pi}_\mathrm{m} \to \mathbf{\Pi}'_\mathrm{m}$, by (9), the vector potential is going to transform from $\mathbf{A} \to \mathbf{A}'$, where

$$\begin{aligned}
\mathbf{A}' &= \mu \frac{\partial \mathbf{\Pi}'_\mathrm{e}}{\partial t} + \mu_0 \nabla \times \mathbf{\Pi}'_\mathrm{m} \\
&= \mu \frac{\partial}{\partial t} (\mathbf{\Pi}_\mathrm{e} + \mu_0 \nabla \times \mathbf{G} - \nabla g) + \mu_0 \nabla \times \left( \mathbf{\Pi}_\mathrm{m} - \mu \frac{\partial \mathbf{G}}{\partial t} \right) \\
&= \mu \frac{\partial \mathbf{\Pi}_\mathrm{e}}{\partial t} + \mu_0 \nabla \times \mathbf{\Pi}_\mathrm{m} - \nabla \left( \mu \frac{\partial g}{\partial t} \right) \\
&= \mathbf{A} - \nabla \left( \mu \frac{\partial g}{\partial t} \right) = \mathbf{A} + \nabla \Lambda
\end{aligned} \tag{27}$$

which is a gauge transform with gauge $\Lambda = -\mu \partial g / \partial t$.