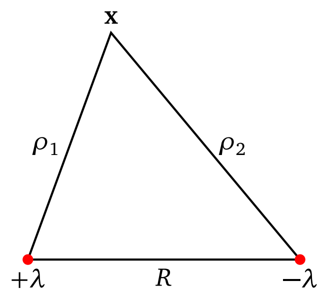
1. For line charge λ, the potential for a point $\mathbf{x} = (\rho, \phi, z)$ is

$$\Phi(\mathbf{x}) = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{\rho_0}{\rho}\right) \tag{1}$$

where ρ_0 is an arbitrary reference radius where the potential is zero.



For two line charges $\pm\lambda$ (see diagram above), the potential is then obviously

$$\Phi(\mathbf{x}) = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{\rho_2}{\rho_1}\right) \tag{2}$$

The equipotential line for $\Phi(\mathbf{x}) = V$ imposes the restriction of ρ_1, ρ_2 as

$$V = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{\rho_2}{\rho_1}\right) \implies \frac{\rho_2}{\rho_1} = \exp\left(\frac{2\pi\epsilon_0 V}{\lambda}\right) \equiv u \tag{3}$$

Let $\mathbf{x} = (x, y, z)$, then (3) implies (assuming origin is where $+\lambda$ is at and x-direction points to $-\lambda$)

$$\begin{aligned}
\frac{\sqrt{(x-R)^2 + y^2}}{\sqrt{x^2 + y^2}} &= u && \implies \\
x^2 - 2Rx + R^2 + y^2 - u^2x^2 - u^2y^2 &= 0 && \implies \\
\left(1-u^2\right)x^2 - 2Rx + \left(1-u^2\right)y^2 + R^2 &= 0 && \implies \\
\left(x - \frac{R}{1-u^2}\right)^2 + y^2 + \frac{R^2}{1-u^2} - \frac{R^2}{(1-u^2)^2} &= 0 && \implies \\
\left(x - \frac{R}{1-u^2}\right)^2 + y^2 - \frac{R^2u^2}{(1-u^2)^2} &= 0 && (4)
\end{aligned}$$

I.e., the equipotential line of $\Phi(\mathbf{x}) = V$ is a circle centered at $x = R/(1-u^2)$, with radius $Ru/|(1-u^2)|$.
Note that

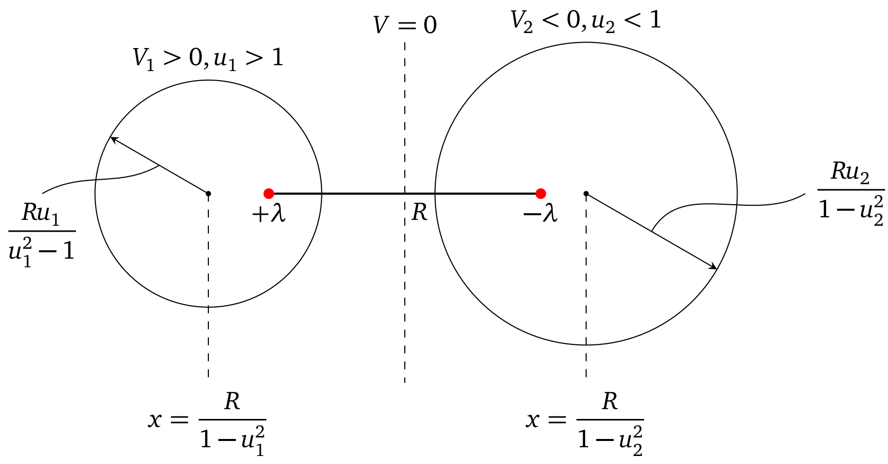
- When $V = 0$, $u = 1$, the circle has infinite radius, which degenerates into the middle line.
- When $V > 0$, $u > 1$, the circle's center has negative x-coordinate. The circle's left-most and right-most point are

$$\frac{R}{1-u^2} \mp \frac{Ru}{u^2-1} = \frac{R}{1 \mp u}$$

  We can see that the circle is completely to the left of the $x = R/2$ middle line, and encloses the charge $+\lambda$. Also, lower-potential circle encloses higher-potential circle.
- When $V < 0$,$u < 1$, the circle's center's x-coordinate is greater than R. The circle's left-most and right-most point are

$$\frac{R}{1-u^2} \mp \frac{Ru}{1-u^2} = \frac{R}{1 \pm u}$$

  We can see that the circle is completely to the right of the $x = R/2$ middle line, and encloses the charge $-\lambda$. Also higher-potential (less negative) circle encloses lower-potential (more negative) circle.



2. To relate the capacitance of the two-cylinder system with the line charges, consider the following configurations:
   (a) the electrostatic problem with Dirichlet boundary condition that the two insulated conducting cylinders have potential $V_1 > 0$ and $V_2 < 0$ respectively;
   (b) two equal but opposite line charges $\pm\lambda$ separated by R, which yield equipotential lines of V_1 and V_2 that align with the given cylinders' profiles (but no conductors).

   By uniqueness of boundary problem, the electric field solution for the volume will be the same in both configurations. In particular, $\int_S \mathbf{E} \cdot d\mathbf{a}$ over a unit z-length of the cylinder will be the same for both configurations. Therefore in configuration (a), the cylinders will have a total charge of $\pm\lambda$ per unit length (which will be distributed unevenly on the surface of the cylinder of course). Then by definition of capacitance, we have

$$C = \frac{\lambda}{V_1 - V_2} \qquad \text{by (3)} \implies \qquad C = \frac{2\pi\epsilon_0}{\ln\left(\frac{u_1}{u_2}\right)} \tag{5}$$

   Now compare the goal claim with (5), we only need to prove

$$\ln\left(\frac{u_1}{u_2}\right) = \cosh^{-1}\left(\frac{d^2 - a^2 - b^2}{2ab}\right) \qquad \text{or equivalently}$$
$$\frac{u_1}{u_2} + \frac{u_2}{u_1} = \frac{d^2 - a^2 - b^2}{ab} \tag{6}$$

   From the diagram above, we have the following relations

$$\frac{Ru_1}{u_1^2 - 1} = a \qquad \frac{Ru_2}{1 - u_2^2} = b \qquad \frac{R}{1 - u_2^2} - \frac{R}{1 - u_1^2} = d \tag{7}$$

   (6) can be proved by brute force, but it will make our life a little easier if we define

$$u_1^2 = 1 + \alpha \qquad u_2^2 = 1 - \beta \tag{8}$$

   Then our goal (6) is equivalent to

$$\frac{u_1^2 + u_2^2}{u_1 u_2} = \frac{2 + \alpha - \beta}{\sqrt{1+\alpha}\sqrt{1-\beta}} = \frac{d^2 - a^2 - b^2}{ab} \tag{9}$$

   Indeed, from (7)

$$a = \frac{R\sqrt{1+\alpha}}{\alpha} \qquad b = \frac{R\sqrt{1-\beta}}{\beta} \qquad d = R\left(\frac{1}{\beta} + \frac{1}{\alpha}\right) \tag{10}$$

Therefore

$$
\begin{aligned}
\frac{d^2-a^2-b^2}{ab} &= \frac{\left(\dfrac{1}{\alpha}+\dfrac{1}{\beta}\right)^2-\dfrac{1+\alpha}{\alpha^2}-\dfrac{1-\beta}{\beta^2}}{\dfrac{\sqrt{1+\alpha}\sqrt{1-\beta}}{\alpha\beta}} \\
&= \frac{\dfrac{\alpha^2+2\alpha\beta+\beta^2}{\alpha^2\beta^2}-\dfrac{(1+\alpha)\,\beta^2}{\alpha^2\beta^2}-\dfrac{(1-\beta)\,\alpha^2}{\alpha^2\beta^2}}{\dfrac{\sqrt{1+\alpha}\sqrt{1+\beta}}{\alpha\beta}} \\
&= \frac{\dfrac{2\alpha\beta-\alpha\beta^2-\beta\alpha^2}{\alpha^2\beta^2}}{\dfrac{\sqrt{1+\alpha}\sqrt{1+\beta}}{\alpha\beta}} = \frac{2-\alpha-\beta}{\sqrt{1+\alpha}\sqrt{1+\beta}} \qquad (11)
\end{aligned}
$$

which is exactly (9).

3. From part 2, we know that

$$
C = \frac{2\pi\epsilon_0}{\cosh^{-1}\left(\dfrac{d^2-a^2-b^2}{2ab}\right)} \qquad (12)
$$

Now let

$$
s = \frac{d^2-a^2-b^2}{2ab} \gg 1 \qquad \text{and} \qquad t = \cosh^{-1} s \qquad (13)
$$

Then

$$
\begin{aligned}
\frac{e^t+e^{-t}}{2} &= s && \Longrightarrow \\
\left(e^t\right)^2-2s\cdot e^t+1 &= 0 && \Longrightarrow \\
e^t &= \frac{2s\pm\sqrt{4s^2-4}}{2} = s\pm\sqrt{s^2-1} && (14)
\end{aligned}
$$

It's clear that in (12) we are looking for a positive value of $t=\cosh^{-1} s$, so we will take the positive sign in (14) so $e^t>1$.

Rewrite (13) as

$$
s = \underbrace{\frac{d^2}{2ab}}_{s_0}\left(1-\overbrace{\frac{a^2+b^2}{d^2}}^{\equiv\epsilon}\right) \qquad (15)
$$

then

$$
\begin{aligned}
e^t = s+\sqrt{s^2-1} &= s_0(1-\epsilon)+s_0\sqrt{(1-\epsilon)^2-\frac{1}{s_0^2}} \\
&= s_0(1-\epsilon)+s_0\left[1-2\epsilon+O\left(\frac{1}{d^4}\right)\right]^{1/2} \\
&\approx s_0(1-\epsilon)+s_0\,(1-\epsilon) \\
&= 2s_0\,(1-\epsilon) && \Longrightarrow \quad (16) \\
C = \frac{2\pi\epsilon_0}{t} &\approx \frac{2\pi\epsilon_0}{\ln(2s_0)+\ln(1-\epsilon)} = \frac{2\pi\epsilon_0}{\ln(2s_0)\left[1+\dfrac{\ln(1-\epsilon)}{\ln(2s_0)}\right]} \\
&\approx \frac{2\pi\epsilon_0}{\ln(2s_0)}\left[1-\frac{\ln(1-\epsilon)}{\ln(2s_0)}\right] \approx \frac{2\pi\epsilon_0}{\ln(2s_0)}\left[1+\frac{\epsilon}{\ln(2s_0)}\right] && (17)
\end{aligned}
$$

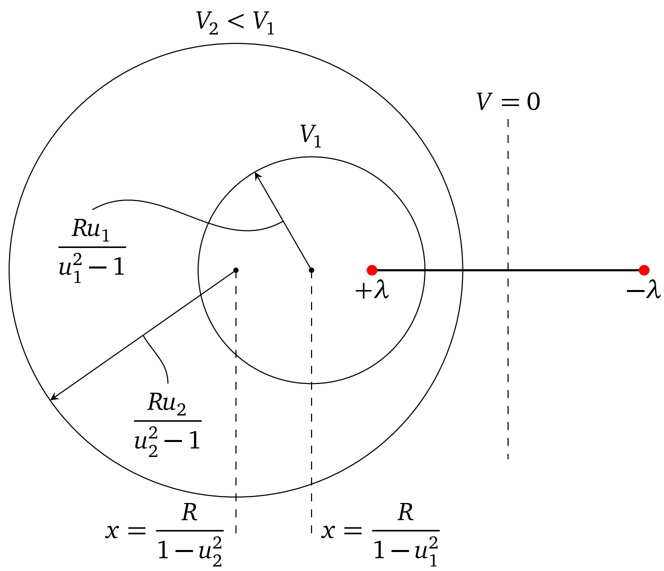
Approximation to the order of ϵ^0 gives

$$C^{(0)} = \frac{2\pi\epsilon_0}{\ln(2s_0)} = \frac{2\pi\epsilon_0}{\ln\left(\dfrac{d^2}{ab}\right)} = \pi\epsilon_0\left[\ln\left(\frac{d}{\sqrt{ab}}\right)\right]^{-1} \tag{18}$$

The ϵ^1 order solution is

$$C^{(1)} = \pi\epsilon_0\left[\ln\left(\frac{d}{\sqrt{ab}}\right)\right]^{-1}\left[1 + \frac{a^2+b^2}{d^2}\frac{1}{\ln\left(\dfrac{d^2}{ab}\right)}\right] \tag{19}$$

4. When one circle is enclosing another, they are necessarily on the same half plane as depicted below (due to the geometric symmetry, we can assume they are on the positive half plane without loss of generality).



At first sight, it might look like the argument about these conductors having equal and opposite charge per length breaks down since they both seem to enclose the same $+\lambda$ charge line. But in this case, the volume we are interested in is the space between the two cylinders. Then for the outer cylinder, its surface normal is pointing inward. Thus when applying Gauss's theorem to the outer cylinder, we have to count the negative charge $-\lambda$ as being enclosed by the surface, hence the previous argument still holds, i.e., equation (5) is still applicable in this case.

Unlike before, here $u_2 > 1$, hence the only thing that changes in the proof (7) - (11) is that in (7), b now needs to be changed to $-b$, which gives the capacitance

$$C = \frac{2\pi\epsilon_0}{\cosh^{-1}\left(\dfrac{a^2+b^2-d^2}{2ab}\right)} \tag{20}$$

When the cylinders are concentric, $d = 0$. Hence if we let

$$t = \cosh^{-1}\left(\frac{a^2+b^2}{2ab}\right) \tag{21}$$

we will have

$$e^t + e^{-t} = \frac{a^2+b^2}{ab} = \frac{b}{a} + \frac{a}{b} \tag{22}$$

which yields a positive solution for t:

$$t = \ln\left(\frac{b}{a}\right) \qquad \text{hence} \qquad C = \frac{2\pi\epsilon_0}{\ln\left(\dfrac{b}{a}\right)} \tag{23}$$