1. Let's start with the separate-variable ansatz for the $\rho < a$ and $\rho > a$ region respectively:

$$E_z = E(\rho)\,e^{im\phi}e^{ikz} \qquad\qquad H_z = H(\rho)\,e^{im\phi}e^{ikz} \tag{1}$$

Within the core $\rho < a$, we have $\boldsymbol{\nabla}_t n_1^2 = 0$, which turns (8.127) into

$$\nabla_t^2\left[\psi(\rho)\,e^{im\phi}\right] + \gamma^2\psi(\rho)\,e^{im\phi} = 0 \qquad \Longrightarrow \qquad \frac{1}{\rho}\frac{d}{d\rho}(\rho\psi) + \left(\gamma^2 - \frac{m^2}{\rho}\right)\psi = 0 \qquad \gamma^2 = \frac{n_1^2\omega^2}{c^2} - k^2 \tag{2}$$

where $\psi(\rho)$ can be either $E_z(\rho)$ or $H_z(\rho)$. This is the Bessel equation (see (3.75)), whose solution is the linear combination of $J_m(\gamma\rho)$ and $N_m(\gamma\rho)$. Since the core contains the axis where $\rho = 0$, $N_m(\gamma\rho)$ must be rejected, giving the solution form

$$E_z = A_e J_m(\gamma\rho)\,e^{im\phi} \qquad\qquad H_z = A_h J_m(\gamma\rho)\,e^{im\phi} \qquad\qquad \text{for } \rho < a \tag{3}$$

For the cladding region, let

$$\beta^2 = k^2 - \frac{n_2^2\omega^2}{c^2} \tag{4}$$

then for $\rho > a$ where $\boldsymbol{\nabla}_t n_2^2 = 0$, (8.127) has the form

$$\nabla_t^2\left[\psi(\rho)\,e^{im\phi}\right] - \beta^2\psi(\rho)\,e^{im\phi} = 0 \qquad \Longrightarrow \qquad \frac{1}{\rho}\frac{d}{d\rho}(\rho\psi) - \left(\beta^2 + \frac{m^2}{\rho^2}\right)\psi = 0 \tag{5}$$

which is the modified Bessel equation (see (3.98)), whose solution is the linear combination of $I_m(\beta\rho)$ and $K_m(\beta\rho)$. We reject $I_m(\beta\rho)$ for its divergence at the infinity, hence

$$E_z = B_e K_m(\beta\rho)\,e^{im\phi} \qquad\qquad H_z = B_h K_m(\beta\rho)\,e^{im\phi} \qquad\qquad \text{for } \rho > a \tag{6}$$

The corresponding gradients are

$$\text{for } \rho < a \quad \boldsymbol{\nabla}_t E_z = A_e\left[\gamma J_m'(\gamma\rho)\,\hat{\boldsymbol{\rho}} + \frac{im}{\rho}J_m(\gamma\rho)\,\hat{\boldsymbol{\phi}}\right]e^{im\phi} \qquad \boldsymbol{\nabla}_t H_z = A_h\left[\gamma J_m'(\gamma\rho)\,\hat{\boldsymbol{\rho}} + \frac{im}{\rho}J_m(\gamma\rho)\,\hat{\boldsymbol{\phi}}\right]e^{im\phi} \tag{7}$$

$$\text{for } \rho > a \quad \boldsymbol{\nabla}_t E_z = B_e\left[\beta K_m'(\beta\rho)\,\hat{\boldsymbol{\rho}} + \frac{im}{\rho}K_m(\beta\rho)\,\hat{\boldsymbol{\phi}}\right]e^{im\phi} \qquad \boldsymbol{\nabla}_t H_z = B_h\left[\beta K_m'(\beta\rho)\,\hat{\boldsymbol{\rho}} + \frac{im}{\rho}K_m(\beta\rho)\,\hat{\boldsymbol{\phi}}\right]e^{im\phi} \tag{8}$$

With $\hat{\mathbf{z}} \times \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}} \times \hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\rho}}$ and (8.126), we have for the core region $\rho < a$,

$$\begin{aligned}
\mathbf{E}_t &= \frac{i}{\gamma^2}\left(k\boldsymbol{\nabla}_t E_z - \omega\mu_0\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t H_z\right) \\
&= \frac{i}{\gamma^2}\left\{kA_e\left[\gamma J_m'(\gamma\rho)\,\hat{\boldsymbol{\rho}} + \frac{im}{\rho}J_m(\gamma\rho)\,\hat{\boldsymbol{\phi}}\right]e^{im\phi} - \omega\mu_0 A_h\left[\gamma J_m'(\gamma\rho)\,\hat{\boldsymbol{\phi}} - \frac{im}{\rho}J_m(\gamma\rho)\,\hat{\boldsymbol{\rho}}\right]e^{im\phi}\right\} \\
&= \hat{\boldsymbol{\rho}}\left[\frac{ikA_e}{\gamma}J_m'(\gamma\rho) - \frac{m\omega\mu_0 A_h}{\gamma^2\rho}J_m(\gamma\rho)\right]e^{im\phi} - \hat{\boldsymbol{\phi}}\left[\frac{mkA_e}{\gamma^2\rho}J_m(\gamma\rho) + \frac{i\omega\mu_0 A_h}{\gamma}J_m'(\gamma\rho)\right]e^{im\phi}
\end{aligned} \tag{9}$$

$$\begin{aligned}
\mathbf{H}_t &= \frac{i}{\gamma^2}\left(k\boldsymbol{\nabla}_t H_z + \omega\epsilon_0 n_1^2\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t E_z\right) \\
&= \frac{i}{\gamma^2}\left\{kA_h\left[\gamma J_m'(\gamma\rho)\,\hat{\boldsymbol{\rho}} + \frac{im}{\rho}J_m(\gamma\rho)\,\hat{\boldsymbol{\phi}}\right]e^{im\phi} + \omega\epsilon_0 n_1^2 A_e\left[\gamma J_m'(\gamma\rho)\,\hat{\boldsymbol{\phi}} - \frac{im}{\rho}J_m(\gamma\rho)\,\hat{\boldsymbol{\rho}}\right]e^{im\phi}\right\} \\
&= \hat{\boldsymbol{\rho}}\left[\frac{ikA_h}{\gamma}J_m'(\gamma\rho) + \frac{m\omega\epsilon_0 n_1^2 A_e}{\gamma^2\rho}J_m(\gamma\rho)\right]e^{im\phi} - \hat{\boldsymbol{\phi}}\left[\frac{mkA_h}{\gamma^2\rho}J_m(\gamma\rho) - \frac{i\omega\epsilon_0 n_1^2 A_e}{\gamma}J_m'(\gamma\rho)\right]e^{im\phi}
\end{aligned} \tag{10}$$

Note that the definition of β^2 differs from γ^2 by a minus sign, we must replace $\gamma^2 \to -\beta^2$ in (8.126) to obtain the cladding's transverse fields, hence for $\rho > a$,

$$\mathbf{E}_t = -\hat{\boldsymbol{\rho}}\left[\frac{ikB_e}{\beta}K_m'(\beta\rho) - \frac{m\omega\mu_0 B_h}{\beta^2\rho}K_m(\beta\rho)\right]e^{im\phi} + \hat{\boldsymbol{\phi}}\left[\frac{mkB_e}{\beta^2\rho}K_m(\beta\rho) + \frac{i\omega\mu_0 B_h}{\beta}K_m'(\beta\rho)\right]e^{im\phi} \tag{11}$$

$$\mathbf{H}_t = -\hat{\boldsymbol{\rho}}\left[\frac{ikB_h}{\beta}K_m'(\beta\rho) + \frac{m\omega\epsilon_0 n_2^2 B_e}{\beta^2\rho}K_m(\beta\rho)\right]e^{im\phi} + \hat{\boldsymbol{\phi}}\left[\frac{mkB_h}{\beta^2\rho}K_m(\beta\rho) - \frac{i\omega\epsilon_0 n_2^2 B_e}{\beta}K_m'(\beta\rho)\right]e^{im\phi} \tag{12}$$

Boundary conditions dictate that normal **D** and **B**, as well as tangential **E** and **H** be continuous across $\rho = a$

$$\text{normal } \mathbf{B}: \quad \frac{ikA_h}{\gamma}J_m'(\gamma a) + \frac{m\omega\epsilon_0 n_1^2 A_e}{\gamma^2 a}J_m(\gamma a) = -\left[\frac{ikB_h}{\beta}K_m'(\beta a) + \frac{m\omega\epsilon_0 n_2^2 B_e}{\beta^2 a}K_m(\beta a)\right] \tag{13}$$

$$\text{normal } \mathbf{D}: \quad n_1^2\left[\frac{ikA_e}{\gamma}J_m'(\gamma a) - \frac{m\omega\mu_0 A_h}{\gamma^2 a}J_m(\gamma a)\right] = -n_2^2\left[\frac{ikB_e}{\beta}K_m'(\beta a) - \frac{m\omega\mu_0 B_h}{\beta^2 a}K_m(\beta a)\right] \tag{14}$$

$$\text{tangential } (\hat{\mathbf{z}})\ \mathbf{H}: \quad A_h J_m(\gamma a) = B_h K_m(\beta a) \tag{15}$$

$$\text{tangential } (\hat{\boldsymbol{\phi}})\ \mathbf{H}: \quad \frac{mkA_h}{\gamma^2 a}J_m(\gamma a) - \frac{i\omega\epsilon_0 n_1^2 A_e}{\gamma}J_m'(\gamma a) = -\left[\frac{mkB_h}{\beta^2 a}K_m(\beta a) - \frac{i\omega\epsilon_0 n_2^2 B_e}{\beta}K_m'(\beta a)\right] \tag{16}$$

$$\text{tangential } (\hat{\mathbf{z}})\ \mathbf{E}: \quad A_e J_m(\gamma a) = B_e K_m(\beta a) \tag{17}$$

$$\text{tangential } (\hat{\boldsymbol{\phi}})\ \mathbf{E}: \quad \frac{mkA_e}{\gamma^2 a}J_m(\gamma a) + \frac{i\omega\mu_0 A_h}{\gamma}J_m'(\gamma a) = -\left[\frac{mkB_e}{\beta^2 a}K_m(\beta a) + \frac{i\omega\mu_0 B_h}{\beta}K_m'(\beta a)\right] \tag{18}$$

Eliminating the B's in (16) and (18) using (15) and (17), we get

$$\frac{mkA_h}{\gamma^2 a}J_m - \frac{i\omega\epsilon_0 n_1^2 A_e}{\gamma}J_m' = -\left[\frac{mkA_h}{\beta^2 a}J_m - \frac{i\omega\epsilon_0 n_2^2 A_e}{\beta}J_m\frac{K_m'}{K_m}\right] \implies$$

$$\left(\frac{1}{\gamma^2} + \frac{1}{\beta^2}\right)\frac{mkA_h}{a} = \left(\frac{n_1^2}{\gamma}\frac{J_m'}{J_m} + \frac{n_2^2}{\beta}\frac{K_m'}{K_m}\right)i\omega\epsilon_0 A_e \tag{19}$$

$$\frac{mkA_e}{\gamma^2 a}J_m + \frac{i\omega\mu_0 A_h}{\gamma}J_m' = -\left[\frac{mkA_e}{\beta^2 a}J_m + \frac{i\omega\mu_0 A_h}{\beta}J_m\frac{K_m'}{K_m}\right] \implies$$

$$\left(\frac{1}{\gamma^2} + \frac{1}{\beta^2}\right)\frac{mkA_e}{a} = -\left(\frac{1}{\gamma}\frac{J_m'}{J_m} + \frac{1}{\beta}\frac{K_m'}{K_m}\right)i\omega\mu_0 A_h \tag{20}$$

When $m \neq 0$, we can further eliminate A_e, A_h from (19) and (20)

$$\left(\frac{1}{\gamma^2} + \frac{1}{\beta^2}\right)^2\frac{m^2}{a^2}k^2 = \left(\frac{n_1^2}{\gamma}\frac{J_m'}{J_m} + \frac{n_2^2}{\beta}\frac{K_m'}{K_m}\right)\left(\frac{1}{\gamma}\frac{J_m'}{J_m} + \frac{1}{\beta}\frac{K_m'}{K_m}\right)\frac{\omega^2}{c^2} \tag{21}$$

Finally, from (2) and (4), we can express $k^2, \omega^2/c^2$ in terms of γ^2, β^2,

$$k^2 = \frac{n_2^2\gamma^2 + n_1^2\beta^2}{n_1^2 - n_2^2} \qquad \frac{\omega^2}{c^2} = \frac{\gamma^2 + \beta^2}{n_1^2 - n_2^2} \tag{22}$$

and (21) is turned into the desired form

$$\frac{m^2}{a^2}\left(\frac{1}{\gamma^2} + \frac{1}{\beta^2}\right)\left(\frac{n_1^2}{\gamma^2} + \frac{n_2^2}{\beta^2}\right) = \left(\frac{n_1^2}{\gamma}\frac{J_m'}{J_m} + \frac{n_2^2}{\beta}\frac{K_m'}{K_m}\right)\left(\frac{1}{\gamma}\frac{J_m'}{J_m} + \frac{1}{\beta}\frac{K_m'}{K_m}\right) \tag{23}$$

It is worth noting that we have not used (13) and (14) when we derive (21), this is because they are redundant as a result of the ansatz satisfying the Maxwell equations. To verify this, (13) and (14) can be rewritten

$$ikA_h\left(\frac{1}{\gamma}\frac{J_m'}{J_m} + \frac{1}{\beta}\frac{K_m'}{K_m}\right) = -\frac{m\omega\epsilon_0 A_e}{a}\left(\frac{n_1^2}{\gamma^2} + \frac{n_2^2}{\beta^2}\right) \tag{24}$$

$$ikA_e\left(\frac{n_1^2}{\gamma}\frac{J_m'}{J_m} + \frac{n_2^2}{\beta}\frac{K_m'}{K_m}\right) = \frac{m\omega\mu_0 A_h}{a}\left(\frac{n_1^2}{\gamma^2} + \frac{n_2^2}{\beta^2}\right) \tag{25}$$

which would have required

$$k^2\left(\frac{1}{\gamma}\frac{J_m'}{J_m} + \frac{1}{\beta}\frac{K_m'}{K_m}\right)\left(\frac{n_1^2}{\gamma}\frac{J_m'}{J_m} + \frac{n_2^2}{\beta}\frac{K_m'}{K_m}\right) = \frac{m^2}{a^2}\frac{\omega^2}{c^2}\left(\frac{n_1^2}{\gamma^2} + \frac{n_2^2}{\beta^2}\right)^2 \tag{26}$$

which is equivalent to the eigenequation (23).

2. For $m = 0$, (19) and (20) still hold. We have a TM mode and TE mode:

$$\text{TE mode}: \quad A_h \neq 0 \implies \frac{1}{\gamma}\frac{J_0'}{J_0} + \frac{1}{\beta}\frac{K_0'}{K_0} = 0 \qquad \text{TM mode}: \quad A_e \neq 0 \implies \frac{n_1^2}{\gamma}\frac{J_0'}{J_0} + \frac{n_2^2}{\beta}\frac{K_0'}{K_0} = 0 \tag{27}$$

At cutoff, we require $\beta = 0$ so there is no radiation outside of the core. For the eigenequation (27) to make sense, we must have $J_0(\gamma a) = 0$. It is easy to verify that when $\beta = 0$ and $\gamma a = x_{01}$, $V = n_1\omega a\sqrt{2\Delta}/c = x_{01} = 2.405$.

3. Omitted (this problem is covered in details in section 12-9 in *Optical Waveguide Theory* by A.W. Snyder and J.D. Love).