We should verify that the longitudinal and transverse current density

$$\mathbf{J}_l(\mathbf{x},t) = -\frac{1}{4\pi}\boldsymbol{\nabla}\int \frac{\boldsymbol{\nabla}'\cdot\mathbf{J}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{1}$$

$$\mathbf{J}_t(\mathbf{x},t) = \frac{1}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int \frac{\mathbf{J}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{2}$$

satisfy

$$\mathbf{J}_l + \mathbf{J}_t = \mathbf{J} \tag{3}$$

$$\boldsymbol{\nabla}\cdot\mathbf{J}_t = 0 \tag{4}$$

$$\boldsymbol{\nabla}\times\mathbf{J}_l = 0 \tag{5}$$

(4) and (5) are obvious since $\mathbf{J}_t$ is a curl and $\mathbf{J}_l$ is a gradient.

Moving the first curl into the integral in (2) and using the vector identity

$$\boldsymbol{\nabla}\times(\psi\mathbf{a}) = \boldsymbol{\nabla}\psi\times\mathbf{a} + \psi\boldsymbol{\nabla}\cdot\mathbf{a} \tag{6}$$

we have

$$\boldsymbol{\nabla}\times\int \frac{\mathbf{J}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d^3x' = \int \boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\times\mathbf{J}\left(\mathbf{x}',t\right)d^3x' \tag{7}$$

Applying curl to (7) while using the vector identity

$$\boldsymbol{\nabla}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\boldsymbol{\nabla}\cdot\mathbf{b}) - \mathbf{b}(\boldsymbol{\nabla}\cdot\mathbf{a}) + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} - (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b} \tag{8}$$

we have

$$\begin{aligned}
\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int \frac{\mathbf{J}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d^3x' &= \int \left\{-\mathbf{J}\left(\mathbf{x}',t\right)\boldsymbol{\nabla}^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) + \left[\mathbf{J}\left(\mathbf{x}',t\right)\cdot\boldsymbol{\nabla}\right]\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right\}d^3x' \\
&= \int -\mathbf{J}\left(\mathbf{x}',t\right)(-4\pi)\,\delta\left(\mathbf{x}-\mathbf{x}'\right)d^3x' + \int \left[\mathbf{J}\left(\mathbf{x}',t\right)\cdot\boldsymbol{\nabla}\right]\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3x' \\
&= 4\pi\cdot\mathbf{J}(\mathbf{x},t) + \int \left[\mathbf{J}\left(\mathbf{x}',t\right)\cdot\boldsymbol{\nabla}\right]\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3x'
\end{aligned} \tag{9}$$

Denote

$$\frac{1}{r} \equiv \frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{10}$$

Thus to prove (3), it's equivalent to prove

$$0 = \int \left\{-\left[\boldsymbol{\nabla}'\cdot\mathbf{J}\left(\mathbf{x}',t\right)\right]\boldsymbol{\nabla}\left(\frac{1}{r}\right) + \left[\mathbf{J}\left(\mathbf{x}',t\right)\cdot\boldsymbol{\nabla}\right]\boldsymbol{\nabla}\left(\frac{1}{r}\right)\right\}d^3x \tag{11}$$

Firstly, it's easy to see

$$\left[\boldsymbol{\nabla}'\cdot\mathbf{J}\left(\mathbf{x}',t\right)\right]\boldsymbol{\nabla}\left(\frac{1}{r}\right) = \boldsymbol{\nabla}\left[\frac{\boldsymbol{\nabla}'\cdot\mathbf{J}(\mathbf{x}',t)}{r}\right] \tag{12}$$

Then, using the vector identity

$$\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b} + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} + \mathbf{a}\times(\boldsymbol{\nabla}\times\mathbf{b}) + \mathbf{b}\times(\boldsymbol{\nabla}\times\mathbf{a}) \tag{13}$$

and identifying

$$\mathbf{a} \longleftrightarrow \mathbf{J}\left(\mathbf{x}',t\right) \qquad\qquad \mathbf{b} \longleftrightarrow \boldsymbol{\nabla}'\left(\frac{1}{r}\right) \tag{14}$$

we know

$$\left[\mathbf{J}\left(\mathbf{x}',t\right)\cdot\boldsymbol{\nabla}\right]\boldsymbol{\nabla}\left(\frac{1}{r}\right) = \left[\mathbf{J}\left(\mathbf{x}',t\right)\cdot\boldsymbol{\nabla}\right]\left[-\boldsymbol{\nabla}'\left(\frac{1}{r}\right)\right] = -\boldsymbol{\nabla}\left[\mathbf{J}\left(\mathbf{x}',t\right)\cdot\boldsymbol{\nabla}'\left(\frac{1}{r}\right)\right] \tag{15}$$

Now with (12) and (15) plugged back into (11), we finally get

$$\begin{aligned}\mathrm{RHS}_{(11)} &= -\int \boldsymbol{\nabla}\left[\frac{\boldsymbol{\nabla}'\cdot\mathbf{J}(\mathbf{x}',t)}{r} + \mathbf{J}\big(\mathbf{x}',t\big)\cdot\boldsymbol{\nabla}'\left(\frac{1}{r}\right)\right] d^3x' \\ &= -\boldsymbol{\nabla}\int \boldsymbol{\nabla}'\cdot\left[\frac{\mathbf{J}(\mathbf{x}',t)}{r}\right] d^3x' \\ &= -\boldsymbol{\nabla}\oint_{\infty} \frac{\mathbf{J}(\mathbf{x}',t)}{r}\cdot d\mathbf{a} = 0 \end{aligned} \tag{16}$$

provided $\mathbf{J}(\mathbf{x}',t)/r \to 0$ as $\mathbf{x}' \to \infty$.