1. We can write the potential of the interior in the form

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}\left(\frac{r}{a}\right)^{l}Y_{lm}(\theta,\phi) \tag{1}$$

where the coefficients A_{lm} can be determined by

$$\begin{aligned}A_{lm}&=\int d\Omega\Phi(R,\theta,\phi)Y_{lm}^{*}(\theta,\phi)\\&=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int_0^{\pi}P_l^m(\cos\theta)\sin\theta\,d\theta\underbrace{\int_0^{2\pi}\Phi(R,\theta,\phi)e^{-im\phi}d\phi}_{I}\end{aligned} \tag{2}$$

By the problem statement, we can write the integral I as $I=I_++I_-$, where

$$I_+=\sum_{j=0}^{n-1}\int_{(2j)(2\pi/2n)}^{(2j+1)(2\pi/2n)}(+V)e^{-im\phi}d\phi \tag{3}$$

$$I_-=\sum_{j=0}^{n-1}\int_{(2j+1)(2\pi/2n)}^{(2j+2)(2\pi/2n)}(-V)e^{-im\phi}d\phi \tag{4}$$

Carrying out the integrals $I_\pm$ explicitly, we have

$$I_+=V\frac{1}{(-im)}\sum_{j=0}^{n-1}\left[e^{-im(2j+1)(2\pi/2n)}-e^{-im(2j)(2\pi/2n)}\right] \tag{5}$$

$$I_-=-V\frac{1}{(-im)}\sum_{j=0}^{n-1}\left[e^{-im(2j+2)(2\pi/2n)}-e^{-im(2j+1)(2\pi/2n)}\right] \tag{6}$$

which gives

$$\begin{aligned}I&=\frac{V}{(-im)}\sum_{j=0}^{n-1}\left[2e^{-im(2j+1)\pi/n}-2e^{-im2j\pi/n}\right]\\&=\frac{2V}{(-im)}\sum_{j=0}^{n-1}e^{-im2j\pi/n}\left(e^{-im\pi/n}-1\right)\\&=\frac{2V}{(-im)}\left(e^{-im\pi/n}-1\right)\sum_{j=0}^{n-1}\omega^j \qquad \text{where } \omega=e^{-im2\pi/n}\end{aligned} \tag{7}$$

Notice from

$$\sum_{j=0}^{n-1}\omega^j=\begin{cases}\dfrac{1-\omega^n}{1-\omega}=0 & \text{for } \omega\neq 1\\ n & \text{for } \omega=1\end{cases} \tag{8}$$

we know the sum will vanish unless m is a integer multiple of n. But if m/n is an even integer, I will vanish due to the factor $e^{-im\pi/n}-1$ in front of the sum. So the only way for I not to vanish is when $m=(2k+1)n$ for some k, in which case

$$I=\frac{2V}{(-im)}\left[e^{-i(2k+1)\pi}-1\right]\cdot n=\frac{4Vn}{im} \qquad \text{for } m=(2k+1)n \tag{9}$$

2. With $n=1$, only $m=\pm1,\pm3,\pm5,\ldots$ will contribute. Thus up to $l=3$, the potential can be expressed as

$$\begin{aligned}\Phi(r,\theta,\phi)\approx&\left[A_{1,1}Y_{1,1}(\theta,\phi)+A_{1,-1}Y_{1,-1}(\theta,\phi)\right]\left(\frac{r}{a}\right)+\\&\left[A_{2,1}Y_{2,1}(\theta,\phi)+A_{2,-1}Y_{2,-1}(\theta,\phi)\right]\left(\frac{r}{a}\right)^2+\\&\left[A_{3,1}Y_{3,1}(\theta,\phi)+A_{3,-1}Y_{3,-1}(\theta,\phi)+A_{3,3}Y_{3,3}(\theta,\phi)+A_{3,-3}Y_{3,-3}(\theta,\phi)\right]\left(\frac{r}{a}\right)^3\end{aligned} \tag{10}$$

Let's first notice for odd m, $A_{lm} = A_{l,-m}$ since

$$A_{lm} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int_0^\pi P_l^m(\cos\theta)\sin\theta\, d\theta \cdot \frac{4V}{im} \tag{11}$$

$$\begin{aligned} A_{l,-m} &= \sqrt{\frac{2l+1}{4\pi}\frac{(l+m)!}{(l-m)!}}\int_0^\pi (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)\sin\theta\, d\theta \cdot \frac{4V}{(-im)} \\ &= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int_0^\pi P_l^m(\cos\theta)\sin\theta\, d\theta \cdot \frac{4V}{im} = A_{lm} \end{aligned} \tag{12}$$

therefore

$$\begin{aligned} T_{lm} \equiv A_{lm}Y_l^m(\theta,\phi) + A_{l,-m}Y_{l,-m}(\theta,\phi) &= A_{lm}\left[\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(\cos\theta) + \sqrt{\frac{2l+1}{4\pi}\frac{(l+m)!}{(l-m)!}}e^{-im\phi}P_l^{-m}(\cos\theta)\right] \\ &= A_{lm}\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\left(e^{im\phi} - e^{-im\phi}\right) \\ &= \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]\left[\int_0^\pi P_l^m(\cos\theta)\sin\theta\, d\theta\right]P_l^m(\cos\theta)\cdot\frac{4V}{im}\cdot 2i\sin m\phi \\ &= \left[\frac{2(2l+1)V}{m\pi}\frac{(l-m)!}{(l+m)!}\right]\left[\int_0^\pi P_l^m(\cos\theta)\sin\theta\, d\theta\right]P_l^m(\cos\theta)\sin m\phi \end{aligned} \tag{13}$$

With the well known associated Legendre functions (reference Wolfram)

$$\begin{aligned} P_1^1(\cos\theta) &= -\sin\theta & P_2^1(\cos\theta) &= -3\sin\theta\cos\theta \\ P_3^1(\cos\theta) &= -\frac{3}{2}\left(5\cos^2\theta - 1\right)\sin\theta & P_3^3(\cos\theta) &= -15\sin^3\theta \end{aligned} \tag{14}$$

we can calculate T_{lm} as the following

$$T_{11} = \left(\frac{6V}{\pi}\frac{1}{2}\right)\left(\overbrace{\int_0^\pi -\sin^2\theta\, d\theta}^{-\pi/2}\right)(-\sin\theta\sin\phi) = \frac{3V}{2}\sin\theta\sin\phi \tag{15}$$

$$T_{21} = \left(\frac{10V}{\pi}\frac{1}{6}\right)\left(\overbrace{\int_0^\pi -3\sin^2\theta\cos\theta\, d\theta}^{0}\right)(-3\sin\theta\cos\theta\sin\phi) = 0 \tag{16}$$

$$\begin{aligned} T_{31} &= \left(\frac{14V}{\pi}\frac{2}{24}\right)\left[\overbrace{\int_0^\pi -\frac{3}{2}\left(5\cos^2\theta - 1\right)\sin^2\theta\, d\theta}^{-3\pi/16}\right]\left[-\frac{3}{2}\left(5\cos^2\theta - 1\right)\sin\theta\sin\phi\right] \\ &= \frac{21V}{64}\left(5\cos^2\theta - 1\right)\sin\theta\sin\phi \end{aligned} \tag{17}$$

$$\begin{aligned} T_{33} &= \left(\frac{14V}{3\pi}\frac{1}{720}\right)\left[\overbrace{\int_0^\pi -15\sin^4\theta\, d\theta}^{-45\pi/8}\right]\left(-15\sin^3\theta\sin 3\phi\right) \\ &= \frac{35V}{64}\sin^3\theta\sin 3\phi \end{aligned} \tag{18}$$

It seems quite messy, but recall that in this problem, when $n = 1$, our positively and negatively charged hemispheres are in the y_+ and y_- half spaces, while in section 3.3, they are in the z_+ and z_- half spaces. This suggests a coordinate change that maps

$$y = \sin\theta\sin\phi \qquad \longrightarrow \qquad z' = \cos\theta' \tag{19}$$

Then simple algebra will show the equivalence of these two solutions:

$$\begin{aligned} T_{11}\left(\frac{r}{a}\right) + T_{31}\left(\frac{r}{a}\right)^3 + T_{33}\left(\frac{r}{a}\right)^3 &= \frac{3V}{2}\left(\frac{r}{a}\right)\cos\theta' - \frac{7V}{8}\left(\frac{r}{a}\right)^3\left(\frac{5}{2}\cos^3\theta' - \frac{3}{2}\cos\theta'\right) \\ &= \frac{3V}{2}\left(\frac{r}{a}\right)P_1\left(\cos\theta'\right) - \frac{7V}{8}\left(\frac{r}{a}\right)^3 P_3\left(\cos\theta'\right) \end{aligned} \tag{20}$$