Let $\rho(\beta)$ be the uniform charge density parameterized with β, then

$$\begin{aligned}
Q &= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{R(\theta)} \rho(\beta) r^2 dr = 2\pi\rho(\beta) \int_0^{\pi} \sin\theta d\theta \frac{[R(\theta)]^3}{3} \\
&= 2\pi\rho(\beta) \int_{-1}^{1} d(\cos\theta) \frac{R^3 [1+\beta P_2(\cos\theta)]^3}{3} \\
&= \frac{2\pi\rho(\beta) R^3}{3} \int_{-1}^{1} d(\cos\theta) \left[1 + 3\beta P_2(\cos\theta) + O\left(\beta^2\right)\right] && \text{orthogonality of Legendre polynomials} \\
&= \frac{4\pi\rho(\beta) R^3}{3} \left[1 + O\left(\beta^2\right)\right] && (1)
\end{aligned}$$

So up to $O(\beta)$, we have a constant charge density

$$\rho(\beta) \approx \frac{3Q}{4\pi R^3} \tag{2}$$

At this order, the electric multipole moments are

$$\begin{aligned}
q_{lm}(\beta) &= \int r^l Y_{lm}^*(\theta,\phi) \rho(\beta) d^3x \\
&= \delta_{m0} \cdot 2\pi \sqrt{\frac{(2l+1)}{4\pi}} \int_0^{\pi} \sin\theta d\theta P_l(\cos\theta) \int_0^{R(\theta)} r^{l+2} \cdot \frac{3Q}{4\pi R^3} dr \\
&= \delta_{m0} \cdot 2\pi \sqrt{\frac{(2l+1)}{4\pi}} \cdot \frac{3Q}{4\pi R^3} \int_{-1}^{1} dx P_l(x) \left\{ \frac{R^{l+3} [1+\beta P_2(x)]^{l+3}}{l+3} \right\} \\
&\approx \delta_{m0} \cdot \frac{3QR^l}{2(l+3)} \sqrt{\frac{2l+1}{4\pi}} \left[\overbrace{\int_{-1}^{1} P_l(x) dx}^{2\delta_{l0}} + \overbrace{\int_{-1}^{1} (l+3)\beta P_l(x) P_2(x) dx}^{(l+3)\beta\delta_{l2}\cdot 2/5} \right] \\
&= \delta_{l0}\delta_{m0} \cdot \sqrt{\frac{1}{4\pi}} Q + \delta_{l2}\delta_{m0} \cdot \frac{3QR^2}{\sqrt{20\pi}} \beta && (3)
\end{aligned}$$

The first term is the monopole which does not have β dependency and does not radiate. The second term corresponds to quadrupole moment, which will be harmonic if β is harmonically oscillating.

We must also consider the magnetic multipole moment due to the current (see 9.172). But given the azimuthal symmetry, $\mathbf{J}(\mathbf{x})$ has only $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ component, and they do not depend on ϕ, thus in (9.172)

$$\nabla \cdot (\mathbf{x} \times \mathbf{J}) = \nabla \cdot \left\{ \mathbf{x} \times \left[J_r(r,\theta) \hat{\mathbf{r}} + J_\theta(r,\theta) \hat{\boldsymbol{\theta}} \right] \right\} = \nabla \cdot \left[r J_\theta(r,\theta) \hat{\boldsymbol{\phi}} \right] = 0 \tag{4}$$

Converting spherical tensor q_{20} to Cartesian tensor Q_{ij} with (4.6), we have

$$Q_{33} = 2\sqrt{\frac{4\pi}{5}} q_{20} = 2\sqrt{\frac{4\pi}{5}} \cdot \frac{3QR^2}{\sqrt{20\pi}} \beta = \frac{6QR^2}{5} \beta \tag{5}$$

Then by (9.51) and (9.52), the angular distribution of power and total power are

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{c^2 Z_0 k^6}{512\pi^2} Q_{33}^2 \sin^2\theta \cos^2\theta = \frac{9c^2 Z_0 k^6 \beta^2 Q^2 R^4}{3200\pi^2} \sin^2\theta \cos^2\theta \\
P &= \frac{c^2 Z_0 k^6 Q_{33}^2}{960\pi} = \frac{3c^2 Z_0 k^6 \beta^2 Q^2 R^4}{2000\pi} && (6)
\end{aligned}$$