

1. The Proca field equation is given in (12.92)

$$\partial^\beta F_{\beta\alpha} + \mu^2 A_\alpha = \frac{4\pi}{c} J_\alpha \quad (1)$$

Taking ∂^α on both sides yields

$$\partial^\alpha \partial^\beta F_{\beta\alpha} + \mu^2 \partial^\alpha A_\alpha = \frac{4\pi}{c} \partial^\alpha J_\alpha \quad \implies \quad \partial_\alpha A^\alpha = 0 \quad (2)$$

I.e., it is necessary for the Proca field to satisfy the Lorenz gauge condition.

With the spatial component of the Proca field given as plane wave

$$\mathbf{A} = \epsilon_0 e^{ikz - i\omega t} \quad (3)$$

Lorenz gauge condition requires

$$\frac{\partial A^0}{\partial t} + ik\hat{\mathbf{z}} \cdot \epsilon_0 e^{ikz - i\omega t} = 0 \quad (4)$$

If A^0 has harmonic time dependence $e^{-i\omega t}$, then the above gives the constraint

$$A^0 = \frac{k}{\omega} \hat{\mathbf{z}} \cdot \epsilon_0 e^{ikz - i\omega t} \quad (5)$$

With the Lorenz gauge condition, (1) is equivalent to

$$\square A_\alpha + \mu^2 A_\alpha = \frac{4\pi}{c} J_\alpha \quad (6)$$

For a source-free field, this gives the constraint

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + \mu^2 \right) A_\alpha = 0 \quad \implies \quad \omega^2 = k^2 + \mu^2 \quad (7)$$

With these, the field strengths can be calculated

$$\mathbf{B} = \nabla \times \mathbf{A} = ik\hat{\mathbf{z}} \times \epsilon_0 e^{ikz - i\omega t} \quad (8)$$

$$\mathbf{E} = -\nabla A^0 - \frac{\partial \mathbf{A}}{\partial t} = i \left[-\frac{k^2}{\omega} \hat{\mathbf{z}} (\hat{\mathbf{z}} \cdot \epsilon_0) + \omega \epsilon_0 \right] e^{ikz - i\omega t} \quad (9)$$

From problem 12.16, we have obtained the symmetric stress tensor

$$\Theta^{00} = \frac{1}{8\pi} [\mathbf{E}^2 + \mathbf{B}^2 + \mu^2 (A^0 A^0 + \mathbf{A} \cdot \mathbf{A})] \quad (10)$$

$$\Theta^{i0} = \frac{1}{4\pi} [(\mathbf{E} \times \mathbf{B})_i + \mu^2 A^i A^0] \quad (11)$$

Using the general rule of time average of product of complex quantities,

$$\langle XY \rangle = \frac{1}{2} \text{Re}(XY^*) \quad (12)$$

we obtain

$$\begin{aligned} \langle \Theta^{00} \rangle &= \frac{1}{16\pi} \left\{ \left| -\frac{k^2}{\omega} \hat{\mathbf{z}} (\hat{\mathbf{z}} \cdot \epsilon_0) + \omega \epsilon_0 \right|^2 + k^2 |\hat{\mathbf{z}} \times \epsilon_0|^2 + \mu^2 \left[\left(\frac{k}{\omega} \hat{\mathbf{z}} \cdot \epsilon_0 \right)^2 + 1 \right] \right\} \quad \text{by (7)} \\ &= \frac{\omega^2 - k^2 (\hat{\mathbf{z}} \cdot \epsilon_0)^2}{8\pi} \end{aligned} \quad (13)$$

$$\begin{aligned} \langle \Theta^{30} \rangle &= \frac{1}{8\pi} \left\{ \omega k [1 - (\hat{\mathbf{z}} \cdot \epsilon_0)^2] + \mu^2 (\hat{\mathbf{z}} \cdot \epsilon_0) \cdot \frac{k}{\omega} (\hat{\mathbf{z}} \cdot \epsilon_0) \right\} \\ &= \left(\frac{k}{\omega} \right) \left[\frac{\omega^2 - k^2 (\hat{\mathbf{z}} \cdot \epsilon_0)^2}{8\pi} \right] \end{aligned} \quad (14)$$

It is then straightforward to see that

- For transverse wave $\hat{\mathbf{z}} \cdot \epsilon_0 = 0$, $F_t = \langle \Theta^{30} \rangle = \omega k / 8\pi$.
- For longitudinal wave $\hat{\mathbf{z}} \cdot \epsilon_0 = 1$, $F_l = \langle \Theta^{30} \rangle = \mu^2 k / 8\pi \omega = (\mu/\omega)^2 F_t$.
- For arbitrary polarization, $\langle \Theta^{30} \rangle / \langle \Theta^{00} \rangle = k/\omega$.

2. The "efficiency" is defined relative to the scattering of massless electromagnetic field. In both cases, the electric field \mathbf{E} induces oscillation of the electron, which in turn generates outgoing waves. A massless field is always transverse, so the amplitude of the electric field is equal to ω (assuming the incident wave of \mathbf{A} has unit amplitude), which is the same for a transverse Proca field (see (9)). However for longitudinal Proca field, the amplitude of the electric field is $\omega - k^2/\omega$ (again, see (9)), which is reduced from transverse amplitude by a factor of μ^2/ω^2 . Therefore the scattering efficiency, being proportional to the square of the amplitude, is

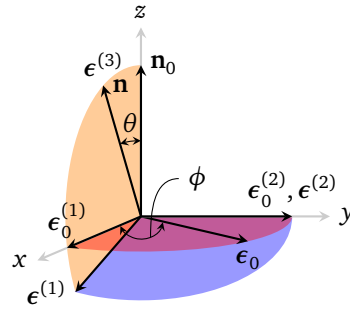
$$E_0 = \begin{cases} 1 & \text{for transverse incident Proca field} \\ \left(\frac{\mu}{\omega}\right)^4 & \text{for longitudinal incident Proca field} \end{cases} \quad (15)$$

The differential cross section is thus defined by the usual way with this additional efficiency factor

$$\frac{d\sigma}{d\Omega} = r_0^2 E_0 |\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0|^2 \frac{F_{\text{out}}}{F_{\text{in}}} \quad (16)$$

3. When the incident wave is transverse, its polarization $\boldsymbol{\epsilon}_0$ is in the x-y plane (see diagram below), which can be parameterized as

$$\boldsymbol{\epsilon}_0 = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}} \quad (17)$$



The three orthogonal outgoing polarizations are

$$\boldsymbol{\epsilon}^{(1)} = \cos \theta \hat{\mathbf{x}} - \sin \theta \hat{\mathbf{z}} \quad \boldsymbol{\epsilon}^{(2)} = \hat{\mathbf{y}} \quad \boldsymbol{\epsilon}^{(3)} = \sin \theta \hat{\mathbf{x}} + \cos \theta \hat{\mathbf{z}} \quad (18)$$

The differential cross section, averaged over all incident polarizations, and summed over all outgoing polarizations, is given by

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left\langle \frac{d\sigma_1}{d\Omega} + \frac{d\sigma_2}{d\Omega} + \frac{d\sigma_3}{d\Omega} \right\rangle_{\text{avg over } \phi} \quad (19)$$

where by (16),

$$\frac{d\sigma_1}{d\Omega} = r_0^2 \cdot 1 \cdot |\boldsymbol{\epsilon}^{(1)*} \cdot \boldsymbol{\epsilon}_0|^2 \frac{F_t}{F_t} = r_0^2 \cos^2 \phi \cos^2 \theta \quad (20)$$

$$\frac{d\sigma_2}{d\Omega} = r_0^2 \cdot 1 \cdot |\boldsymbol{\epsilon}^{(2)*} \cdot \boldsymbol{\epsilon}_0|^2 \frac{F_t}{F_t} = r_0^2 \sin^2 \phi \quad (21)$$

$$\frac{d\sigma_3}{d\Omega} = r_0^2 \cdot 1 \cdot |\boldsymbol{\epsilon}^{(3)*} \cdot \boldsymbol{\epsilon}_0|^2 \frac{F_l}{F_t} = r_0^2 \left(\frac{\mu}{\omega}\right)^2 \cos^2 \phi \sin^2 \theta \quad (22)$$

Here we used $E_0 = 1$ for the transverse incident wave.

Taking the average in ϕ gives

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle_t = \frac{r_0^2}{2} \left[1 + \cos^2 \theta + \left(\frac{\mu}{\omega}\right)^2 \sin^2 \theta \right] \quad (23)$$

4. If the incident wave is longitudinal, we take $\boldsymbol{\epsilon}_0 = \hat{\mathbf{z}}$. The calculation of the differential cross section is similar to the transverse case, except with $E_0 = (\mu/\omega)^4$ and $F_{\text{in}} = F_l, F_{\text{out}} = F_t$, giving

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle_l = \left(\frac{\mu}{\omega}\right)^4 r_0^2 \left[\left(\frac{\omega}{\mu}\right)^2 \sin^2 \theta + \cos^2 \theta \right] = \left(\frac{\mu}{\omega}\right)^2 r_0^2 \left[\sin^2 \theta + \left(\frac{\mu}{\omega}\right)^2 \cos^2 \theta \right] \quad (24)$$