1. Let function $B_\rho(\rho,z)$ and $B_z(\rho,z)$ be expanded near $\rho=0$ in increasing orders of ρ:

$$B_\rho(\rho,z)=\sum_{n=0}^{\infty}a_n(z)\rho^n \tag{1}$$

$$B_z(\rho,z)=\sum_{n=0}^{\infty}b_n(z)\rho^n \tag{2}$$

The basic restrictions for these coefficients come from

$$\nabla\cdot\mathbf{B}=0 \quad\text{or}\quad \frac{1}{\rho}\frac{\partial(\rho B_\rho)}{\partial\rho}+\frac{\partial B_z}{\partial z}=0 \tag{3}$$

$$\nabla\times\mathbf{B}=0 \quad\text{or}\quad \frac{\partial B_\rho}{\partial z}-\frac{\partial B_z}{\partial\rho}=0 \tag{4}$$

hence

$$\frac{1}{\rho}\sum_{n=0}^{\infty}(n+1)a_n\rho^n+\sum_{n=0}^{\infty}b_n'\rho^n=0 \tag{5}$$

$$\sum_{n=0}^{\infty}a_n'\rho^n-\sum_{n=0}^{\infty}nb_n\rho^{n-1}=0 \tag{6}$$

Matching the coefficients, we obtain

$$a_0=0 \qquad a_n=-\frac{b_{n-1}'}{n+1} \qquad b_n=\frac{a_{n-1}'}{n} \qquad \text{for } n\geq 1 \tag{7}$$

By Taylor expansion, and by successively applying (7), we get

$$b_0(z)=B_z(0,z) \tag{8}$$

$$b_1(z)=a_0'(z)=0 \tag{9}$$

$$a_1(z)=-\frac{b_0'(z)}{2}=-\frac{1}{2}\left[\frac{\partial B_z(0,z)}{\partial z}\right] \tag{10}$$

$$b_2(z)=\frac{a_1'(z)}{2}=-\frac{1}{4}\left[\frac{\partial^2 B_z(0,z)}{\partial z^2}\right] \tag{11}$$

$$a_2(z)=-\frac{b_1'(z)}{3}=0 \tag{12}$$

$$b_3(z)=\frac{a_2'(z)}{3}=0 \tag{13}$$

$$a_3(z)=-\frac{b_2'(z)}{4}=\frac{1}{16}\left[\frac{\partial^3 B_z(0,z)}{\partial z^3}\right] \tag{14}$$

$$\cdots$$

which give the desired form

$$B_\rho(\rho,z)=-\frac{\rho}{2}\left[\frac{\partial B_z(0,z)}{\partial z}\right]+\frac{\rho^3}{16}\left[\frac{\partial^3 B_z(0,z)}{\partial z^3}\right]+\cdots \tag{15}$$

$$B_z(\rho,z)=B_z(0,z)-\frac{\rho^2}{4}\left[\frac{\partial^2 B_z(0,z)}{\partial z^2}\right]+\cdots \tag{16}$$

2. We can drop higher terms only when the n-th derivative of $B_z(0,z)$ diminishes much more quickly than $1/\rho^n$, which is typically true for the field with "slow" variance in z.