In Jackson section 3.4, two asymptotic relations were given without proof. The first one is (3.47)

$$P_\nu(\cos\theta) = J_0\left[(2\nu+1)\sin\frac{\theta}{2}\right] \tag{1}$$

and the second one is (3.48b)

$$\nu \approx \left[2\ln\left(\frac{2}{\pi-\beta}\right)\right]^{-1} \tag{2}$$

The former has been shown in my previous notes for Sakurai pp394-misc-bessel-props.pdf (in particular equation (25)). In this doc, we prove the latter.

Let

$$\epsilon = \pi - \beta \tag{3}$$

then following the symbols in the text, we have

$$x = \cos\beta = -\cos\epsilon \approx \frac{\epsilon^2}{2} - 1 \tag{4}$$

$$\xi = \frac{1}{2}(1-x) \approx 1 - \frac{\epsilon^2}{4} \tag{5}$$

We are looking for the order ν which makes

$$P_\nu(\xi) = 1 + \frac{(-\nu)(\nu+1)}{1!1!}\xi + \frac{(-\nu)(-\nu+1)(\nu+1)(\nu+2)}{2!2!}\xi^2 + \cdots \tag{6}$$

vanish.

When $\nu \to 0$, the k-th term of (5) can be approximated by

$$k\text{-th term } \approx -\nu\cdot\frac{(k-1)!k!}{k!k!}\xi^k = -\nu\frac{\xi^k}{k} \tag{7}$$

If (6) were to vanish, we must have

$$1 - \nu\sum_{k=1}^{\infty}\frac{\xi^k}{k} \approx 0 \qquad \Longrightarrow \qquad \frac{1}{\nu} \approx \sum_{k=1}^{\infty}\frac{\xi^k}{k} = -\ln(1-\xi) = \ln\left(\frac{4}{\epsilon^2}\right) = 2\ln\left(\frac{2}{\pi-\beta}\right) \tag{8}$$

which proves (2).