The text has given the general solution of two dimensional problem using separation of variables (equation (2.71)):

$$\Phi(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi + \beta_n) \tag{1}$$

When we consider the interior volume $\rho < b$, it includes the origin where $\rho = 0$. This requires all the $b_i, i = 0, 1, \cdots$ to vanish.

Instead of having a_n and α_n as parameters, we convert (1) into an equivalent form

$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \sin n\phi + \sum_{n=1}^{\infty} c_n \rho^n \cos n\phi \tag{2}$$

For the surface $\rho = b$, it's clear that a_0 is the average potential on the surface, i.e.,

$$a_0 = \langle \Phi(b,\phi) \rangle = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi) d\phi \tag{3}$$

Now the Fourier coefficients can be obtained readily

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\Phi(b,\phi')}{b^n} \sin n\phi' d\phi' \tag{4}$$

$$c_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\Phi(b,\phi')}{b^n} \cos n\phi' d\phi' \tag{5}$$

Inserting these into (2), we get

$$\begin{aligned}
\Phi(\rho,\phi) &= a_0 + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\int_0^{2\pi} d\phi' \Phi(b,\phi') \frac{\rho^n}{b^n} \overbrace{\left(\sin n\phi \sin n\phi' + \cos n\phi \cos n\phi'\right)}^{\cos n(\phi-\phi')}\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\phi' \Phi(b,\phi')\left[1 + 2\sum_{n=1}^{\infty} \frac{\rho^n}{b^n} \cos n(\phi - \phi')\right]
\end{aligned} \tag{6}$$

Let $\gamma = \phi - \phi'$, we recognize that the sum in (6) is the real part of the sum of the geometric series., which is

$$\mathrm{Re}\sum_{n=1}^{\infty}\left(\frac{\rho e^{i\gamma}}{b}\right)^n \tag{7}$$

Recall for $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n = \frac{1}{1-x} \tag{8}$$

This turns (7) into

$$\begin{aligned}
\mathrm{Re}\left(\frac{1}{1 - \frac{\rho e^{i\gamma}}{b}}\right) - 1 &= \mathrm{Re}\left(\frac{\rho e^{i\gamma}}{b - \rho e^{i\gamma}}\right) \\
&= \mathrm{Re}\left(\frac{\rho\cos\gamma + i\rho\sin\gamma}{b - \rho\cos\gamma - i\rho\sin\gamma}\right) \\
&= \frac{\mathrm{Re}\left[(\rho\cos\gamma + i\rho\sin\gamma)(b - \rho\cos\gamma + i\rho\sin\gamma)\right]}{(b - \rho\cos\gamma)^2 + \rho^2\sin^2\gamma} \\
&= \frac{b\rho\cos\gamma - \rho^2}{b^2 + \rho^2 - 2b\rho\cos\gamma}
\end{aligned} \tag{9}$$

Thus the bracket of (6) becomes

$$1 + 2\cdot\frac{b\rho\cos\gamma - \rho^2}{b^2 + \rho^2 - 2b\rho\cos\gamma} = \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos\gamma} \tag{10}$$

which gives

$$\Phi(\rho,\phi)=\frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi-\phi')}d\phi' \tag{11}$$

For the exterior volume of the cylinder, all of $b_0, a_1, a_2, \cdots$ must vanish to make $\rho\to\infty$ not diverge, which means the admissible solution will have similar form to (2) except all ρ^n will be replaced by ρ^{-n}. Then the same argument will lead to (6) with the $\rho\longleftrightarrow b$ exchange, which then leads to (11) with the same exchange $\rho\longleftrightarrow b$.