In these notes, we first fill the derivation gaps that lead to the Green function expansion (9.98), which gives rise to the exact vector potential equation (9.11), then use the exact equation to justify the two main approximations (9.13), (9.30) in section 9.2 and 9.3.

1. **Green function expansion (9.98)**

The inhomogeneous Helmholtz equation for the Green function (outgoing wave) is given in Chapter 6 (see earlier notes *Helmholtz Equation with Delta Source*)

$$\left(\boldsymbol{\nabla}^2+k^2\right)G\left(\mathbf{x},\mathbf{x}'\right)=-\delta\left(\mathbf{x}-\mathbf{x}'\right) \tag{1}$$

We can expand $G\left(\mathbf{x},\mathbf{x}'\right)$ in spherical harmonics basis,

$$G\left(\mathbf{x},\mathbf{x}'\right)=\sum_{l,m} g_l\left(r,r'\right)Y_{lm}^*\left(\theta',\phi'\right)Y_{lm}\left(\theta,\phi\right) \tag{2}$$

With

$$\boldsymbol{\nabla}^2 f=\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(rf\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \tag{3}$$

and the fact Y_{lm} satisfies the angular part of the Laplace equation,

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{lm}\left(\theta,\phi\right)=-l\left(l+1\right)Y_{lm}\left(\theta,\phi\right) \tag{4}$$

the LHS of (1) becomes

$$\mathrm{LHS}_{(1)}=\sum_{l,m}\left\{\frac{1}{r}\frac{\partial^2\left(rg_l\right)}{\partial r^2}+\left[k^2-\frac{l\left(l+1\right)}{r^2}\right]g_l\right\}Y_{lm}^*\left(\theta',\phi'\right)Y_{lm}\left(\theta,\phi\right) \tag{5}$$

Also by completeness of spherical harmonics (3.56)

$$\mathrm{RHS}_{(1)}=-\delta\left(\mathbf{x}-\mathbf{x}'\right)=-\frac{1}{r^2}\delta\left(r-r'\right)\delta\left(\cos\theta-\cos\theta'\right)\cos\left(\phi-\phi'\right)=-\frac{1}{r^2}\delta\left(r-r'\right)\sum_{l,m}Y_{lm}^*\left(\theta',\phi'\right)Y_{lm}\left(\theta,\phi\right) \tag{6}$$

Thus by orthogonality of spherical harmonics, (1) must imply

$$-\frac{1}{r^2}\delta\left(r-r'\right)=\frac{1}{r}\frac{\partial^2\left(rg_l\right)}{\partial r^2}+\left[k^2-\frac{l\left(l+1\right)}{r^2}\right]g_l=\left[\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+k^2-\frac{l\left(l+1\right)}{r^2}\right]g_l\left(r,r'\right) \tag{7}$$

In regions where $r\neq r'$, (7) is the same differential equation as (9.81), whose solution is a linear combination of two kinds of spherical Bessel functions, i.e.,

$$g_l\left(r,r'\right)=\begin{cases}Aj_l\left(kr\right)+Bn_l\left(kr\right) & \text{for } r<r'\\ Cj_l\left(kr\right)+Dn_l\left(kr\right) & \text{for } r>r'\end{cases} \tag{8}$$

With the asymptotic form of j_l, n_l (9.88), if we reqire g_l to be finite at $r=0$, we must set $B=0$.
Moreover, when $r\to\infty$, $g_l\left(r,r'\right)$ must behave asymptotically as outgoing spherical wave e^{ikr}/r (see the radial part of (9.94)), which is satisfied by the large argument limit of $h_l^{(1)}\left(kr\right)$ (see (9.89)). Thus (8) becomes

$$g_l\left(r,r'\right)=\begin{cases}Aj_l\left(kr\right) & \text{for } r<r'\\ A'h_l^{(1)}\left(kr\right) & \text{for } r>r'\end{cases} \tag{9}$$

Continuity at $r=r'$ requires

$$Aj_l\left(kr'\right)=A'h_l^{(1)}\left(kr'\right) \tag{10}$$

Rearranging (7) gives

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial g_l\left(r,r'\right)}{\partial r}\right]+\left[k^2r^2-l\left(l+1\right)\right]g_l=-\delta\left(r-r'\right) \tag{11}$$

Integrating from $r' - \epsilon$ to $r' + \epsilon$ with $\epsilon \to 0$ while using (9) to eliminate A', we get

$$\frac{1}{r'^2} = kAj_l' - kA\frac{j_l h_l^{(1)\prime}}{h_l^{(1)}} \qquad \Longrightarrow$$

$$A = \frac{1}{kr'^2}\left[\frac{h_l^{(1)}}{j_l' h_l^{(1)} - j_l h_l^{(1)\prime}}\right] = -\frac{1}{kr'^2}\frac{h_l^{(1)}}{W\left(j_l, h_l^{(1)}\right)} = -\frac{1}{kr'^2}\frac{h_l^{(1)}}{i/(kr')^2} = ikh_l^{(1)}\left(kr'\right) \tag{12}$$

where we have used the definition and property of the Wronskian $W\left(j_l, h_l^{(1)}\right)$ given in (9.91).

With (12) substituted into (9), we obtain (9.97)

$$g_l\left(r, r'\right) = ikj_l\left(kr_<\right)h_l^{(1)}\left(kr_>\right) \tag{13}$$

and the Green function expansion (9.98)

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} = ik\sum_{l=0}^{\infty} j_l\left(kr_<\right)h_l^{(1)}\left(kr_>\right)\sum_{m=-l}^{l} Y_{lm}^*\left(\theta', \phi'\right)Y_{lm}\left(\theta, \phi\right) \tag{14}$$

For the field point outside the source region ($r_> = r, r_< = r'$), plugging (14) into (9.3) yields (9.11)

$$\mathbf{A}(\mathbf{x}) = \mu_0 ik\sum_{l,m} h_l^{(1)}(kr)\,Y_{lm}(\theta, \phi)\int \mathbf{J}\left(\mathbf{x}'\right)j_l\left(kr'\right)Y_{lm}^*\left(\theta', \phi'\right)d^3x' \tag{15}$$

which is an *exact* formula.

2. **Justifying (9.13) and (9.30)**

   I think it is rather confusing for Jackson to keep referring back to (9.9) in section 9.2 and 9.3 since (9.9) is derived with the far zone assumption ($kr \to \infty$), but in section 9.2 and 9.3, near zone behaviors do get discussed (e.g., (9.20)). In fact, (9.13) and (9.30) should not be stated as the result of keeping more terms in (9.9) but instead, they are the result of keeping only contributions from $l = 0$ and $l = 1$ in (15), with the additional assumption that $kr' \ll 1$ (i.e., source distribution is well within a wavelength's range), and the order $(kr')^l$ approximation.

   - For $l = 0$,

$$h_0^{(1)}(kr) = \frac{e^{ikr}}{ikr} \qquad Y_{00} = \sqrt{\frac{1}{4\pi}} \qquad j_0\left(kr'\right) \approx 1 \qquad \Longrightarrow \qquad \mathbf{A}^{(l=0)}(\mathbf{x}) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \mathbf{J}\left(\mathbf{x}'\right)d^3x' \tag{16}$$

   - For $l = 1$,

$$h_1^{(1)}(kr) = -\frac{e^{ikr}}{kr}\left(1 + \frac{i}{kr}\right) \qquad j_1\left(kr'\right) \approx \frac{kr'}{3}$$

$$Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} \qquad Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

   If we take $\mathbf{x}$ to be on the z axis, i.e., $\theta = 0$, only $m = 0$ will contribute, giving

$$\begin{aligned}\mathbf{A}^{(l=1)}(\mathbf{x}) &\approx \mu_0 ik\cdot\left[-\frac{e^{ikr}}{kr}\left(1 + \frac{i}{kr}\right)\right]\sqrt{\frac{3}{4\pi}}\int \mathbf{J}\left(\mathbf{x}'\right)\frac{kr'}{3}\sqrt{\frac{3}{4\pi}}\cos\theta' d^3x' \\ &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\int \mathbf{J}\left(\mathbf{x}'\right)\left(\mathbf{n}\cdot\mathbf{x}'\right)d^3x'\end{aligned} \tag{17}$$