1. **Prob 11.30**

   It is known that $\mathbf{E}, \mathbf{B}$ transform between K and K' as

$$\mathbf{E}_{\parallel} = \mathbf{E}'_{\parallel} \qquad \mathbf{E}_{\perp} = \gamma(\mathbf{E}'_{\perp} - \boldsymbol{\beta} \times \mathbf{B}') \qquad \mathbf{B}_{\parallel} = \mathbf{B}'_{\parallel} \qquad \mathbf{B}_{\perp} = \gamma(\mathbf{B}'_{\perp} + \boldsymbol{\beta} \times \mathbf{E}') \tag{1}$$

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \qquad \mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B}) \qquad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \qquad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E}) \tag{2}$$

   Similaryly for $\mathbf{D}, \mathbf{H}$ fields since their tensor $G^{\alpha\beta}$ have the same structure as the electromagnetic tensor $F^{\alpha\beta}$.

$$\mathbf{D}_{\parallel} = \mathbf{D}'_{\parallel} \qquad \mathbf{D}_{\perp} = \gamma(\mathbf{D}'_{\perp} - \boldsymbol{\beta} \times \mathbf{H}') \qquad \mathbf{H}_{\parallel} = \mathbf{H}'_{\parallel} \qquad \mathbf{H}_{\perp} = \gamma(\mathbf{H}'_{\perp} + \boldsymbol{\beta} \times \mathbf{D}') \tag{3}$$

$$\mathbf{D}'_{\parallel} = \mathbf{D}_{\parallel} \qquad \mathbf{D}'_{\perp} = \gamma(\mathbf{D}_{\perp} + \boldsymbol{\beta} \times \mathbf{H}) \qquad \mathbf{H}'_{\parallel} = \mathbf{H}_{\parallel} \qquad \mathbf{H}'_{\perp} = \gamma(\mathbf{H}_{\perp} - \boldsymbol{\beta} \times \mathbf{D}) \tag{4}$$

   With the relations

$$\mathbf{D}' = \epsilon \mathbf{E}' \qquad \mathbf{H}' = \frac{\mathbf{B}'}{\mu} \tag{5}$$

   we can readily get

$$\begin{aligned}
\mathbf{D} = \mathbf{D}_{\parallel} + \mathbf{D}_{\perp} &= \epsilon \mathbf{E}'_{\parallel} + \gamma\left(\epsilon \mathbf{E}'_{\perp} - \boldsymbol{\beta} \times \frac{\mathbf{B}'}{\mu}\right) \\
&= \epsilon \mathbf{E}_{\parallel} + \gamma\left\{\epsilon\gamma(\mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B}) - \frac{\boldsymbol{\beta}}{\mu} \times \left[\mathbf{B}_{\parallel} + \gamma(\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E})\right]\right\} \\
&= \left[\epsilon \mathbf{E}_{\parallel} + \gamma^2 \epsilon \mathbf{E}_{\perp} + \frac{\gamma^2}{\mu} \boldsymbol{\beta} \times (\boldsymbol{\beta} \times \mathbf{E})\right] + \left[\gamma^2 \epsilon \boldsymbol{\beta} \times \mathbf{B} - \frac{\gamma^2}{\mu} \boldsymbol{\beta} \times \mathbf{B}_{\perp}\right] \qquad \text{recall } \boldsymbol{\beta} \times (\boldsymbol{\beta} \times \mathbf{E}) = -\beta^2 \mathbf{E}_{\perp} \\
&= \epsilon \mathbf{E} + \gamma^2\left(\epsilon - \frac{1}{\mu}\right)\left(\beta^2 \mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B}\right)
\end{aligned} \tag{6}$$

$$\begin{aligned}
\mathbf{H} = \mathbf{H}_{\parallel} + \mathbf{H}_{\perp} &= \frac{\mathbf{B}'_{\parallel}}{\mu} + \gamma\left(\frac{\mathbf{B}'_{\perp}}{\mu} + \epsilon \boldsymbol{\beta} \times \mathbf{E}'\right) \\
&= \frac{\mathbf{B}_{\parallel}}{\mu} + \gamma\left\{\frac{\gamma}{\mu}(\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E}) + \epsilon \boldsymbol{\beta} \times \left[\mathbf{E}_{\parallel} + \gamma(\mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B})\right]\right\} \\
&= \left[\frac{\mathbf{B}_{\parallel}}{\mu} + \frac{\gamma^2}{\mu} \mathbf{B}_{\perp} + \epsilon\gamma^2 \boldsymbol{\beta} \times (\boldsymbol{\beta} \times \mathbf{B})\right] + \left[\epsilon\gamma^2 \boldsymbol{\beta} \times \mathbf{E}_{\perp} - \frac{\gamma^2}{\mu} \boldsymbol{\beta} \times \mathbf{E}\right] \\
&= \frac{\mathbf{B}}{\mu} + \gamma^2\left(\epsilon - \frac{1}{\mu}\right)\left(-\beta^2 \mathbf{B}_{\perp} + \boldsymbol{\beta} \times \mathbf{E}\right)
\end{aligned} \tag{7}$$

2. **Prob 11.31**

   With the assumption that $\mathbf{B} = B\hat{\mathbf{z}}, \mathbf{E} = E\hat{\boldsymbol{\rho}}$, we can rewrite (6) and (7) as

$$\mathbf{D} = \left[\epsilon E + \gamma^2\left(\epsilon - \frac{1}{\mu}\right)\left(\beta^2 E + \beta B\right)\right]\hat{\boldsymbol{\rho}} = \left\{\overbrace{\left[\mu\epsilon + \gamma^2\beta^2(\mu\epsilon - 1)\right]}^{p} E + \overbrace{\gamma^2\beta(\mu\epsilon - 1)}^{q} B\right\}\frac{\hat{\boldsymbol{\rho}}}{\mu} \tag{8}$$

$$\mathbf{H} = \left[\frac{B}{\mu} - \gamma^2\left(\epsilon - \frac{1}{\mu}\right)\left(\beta^2 B + \beta E\right)\right]\hat{\mathbf{z}} = \left\{\underbrace{\left[1 - \gamma^2\beta^2(\mu\epsilon - 1)\right]}_{r} B \underbrace{-\gamma^2\beta(\mu\epsilon - 1)}_{-q} E\right\}\frac{\hat{\mathbf{z}}}{\mu} \tag{9}$$

   In the lab frame, we have

$$\nabla \times \mathbf{H} = 0 \qquad \Longrightarrow \qquad rB - qE = C_1 \tag{10}$$

$$\nabla \cdot \mathbf{D} = 0 \qquad \Longrightarrow \qquad (pE + qB)\rho = C_2 \tag{11}$$

   where C_1, C_2 are constants to be determined by the boundary conditions.

   Let's consider the boundary value at the inner (or outer) edge of the cylinder at $\rho = a$ or $\rho = b$. Since the medium is moving, we can only use the *local* relation $\mathbf{D} = \epsilon \mathbf{E}, \mathbf{H} = \mathbf{B}/\mu$ in the rest frame co-moving with the cylinder's edge. For such a frame K', we can apply Lorentz transformation to obtain $\mathbf{E}'_{\text{out}}, \mathbf{B}'_{\text{out}}$ which are the electric and magnetic field just outside the medium.

$$\mathbf{E}'_{\text{out}} = \gamma(\boldsymbol{\beta} \times \mathbf{B}) = \gamma\beta B_0 \hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \gamma\beta B_0 \hat{\boldsymbol{\rho}} \tag{12}$$

$$\mathbf{B}'_{\text{out}} = \gamma \mathbf{B} = \gamma B_0 \hat{\mathbf{z}} \tag{13}$$

It is understood that γ, β in (12) and (13) are at $\rho = a$ or $\rho = b$.

Applying the boundary condition (that tangential **H** and normal **D** are continuous across the boundary) in K', we get the fields just inside the medium

$$\text{tangential } \mathbf{H}: \qquad \mathbf{B}'_{\text{in}} = \mu \mathbf{B}'_{\text{out}} = \mu\gamma B_0 \hat{\mathbf{z}} \tag{14}$$

$$\text{normal } \mathbf{D}: \qquad \mathbf{E}'_{\text{in}} = \frac{\mathbf{E}'_{\text{out}}}{\epsilon} = \frac{\gamma\beta}{\epsilon} B_0 \hat{\boldsymbol{\rho}} \tag{15}$$

Transforming these back to the lab frame yields

$$\mathbf{E}_{\text{in}} = \gamma\left(\mathbf{E}'_{\text{in}} - \boldsymbol{\beta} \times \mathbf{B}'_{\text{in}}\right) = \gamma\left(\frac{\gamma\beta}{\epsilon} B_0 \hat{\boldsymbol{\rho}} - \mu\gamma B_0 \boldsymbol{\beta} \times \hat{\mathbf{z}}\right) = -\mu B_0 \left(1 - \frac{1}{\mu\epsilon}\right)\gamma^2 \beta \hat{\boldsymbol{\rho}} \tag{16}$$

$$\mathbf{B}_{\text{in}} = \gamma\left(\mathbf{B}'_{\text{in}} + \boldsymbol{\beta} \times \mathbf{E}'_{\text{in}}\right) = \gamma\left(\mu\gamma B_0 \hat{\mathbf{z}} + \frac{\gamma\beta}{\epsilon} B_0 \boldsymbol{\beta} \times \hat{\boldsymbol{\rho}}\right) = \mu B_0 \left(\gamma^2 - \frac{\gamma^2\beta^2}{\mu\epsilon}\right)\hat{\mathbf{z}} \tag{17}$$

Plugging (16), (17) for $\rho = a$ or $\rho = b$ into (10), (11) establishes the constants

$$C_1 = \mu B_0 \qquad\qquad C_2 = 0 \tag{18}$$

In fact, it is easy to see that with these constants, (16) and (17) are the general solution for all $\rho \in [a, b]$, i.e.,

$$\mathbf{E}_{\text{in}}(\rho) = -\mu B_0 \left(1 - \frac{1}{\mu\epsilon}\right)\left(\frac{\omega\rho/c}{1 - \omega^2\rho^2/c^2}\right)\hat{\boldsymbol{\rho}} \tag{19}$$

$$\mathbf{B}_{\text{in}}(\rho) = \mu B_0 \left[\frac{1 - \omega^2\rho^2/\left(c^2\mu\epsilon\right)}{1 - \omega^2\rho^2/c^2}\right]\hat{\mathbf{z}} \tag{20}$$

The voltage difference between the two edges is obtained by the integral

$$\begin{aligned}
V = \int_a^b |E(\rho)|\, d\rho &= \frac{\mu\omega B_0}{c}\left(1 - \frac{1}{\mu\epsilon}\right)\int_a^b \frac{\rho\, d\rho}{1 - \omega^2\rho^2/c^2} \\
&= \frac{\mu\omega B_0}{c}\left(1 - \frac{1}{\mu\epsilon}\right)\frac{c^2}{2\omega^2}\ln\left(\frac{1 - \omega^2 a^2/c^2}{1 - \omega^2 b^2/c^2}\right) \qquad \omega a/c < \omega b/c \ll 1 \\
&\approx \frac{\mu\omega B_0}{c}\left(1 - \frac{1}{\mu\epsilon}\right)\frac{c^2}{2\omega^2}\left(\omega^2 b^2/c^2 - \omega^2 a^2/c^2\right) \\
&= \frac{\mu\omega B_0}{2c}\left(1 - \frac{1}{\mu\epsilon}\right)\left(b^2 - a^2\right)
\end{aligned} \tag{21}$$