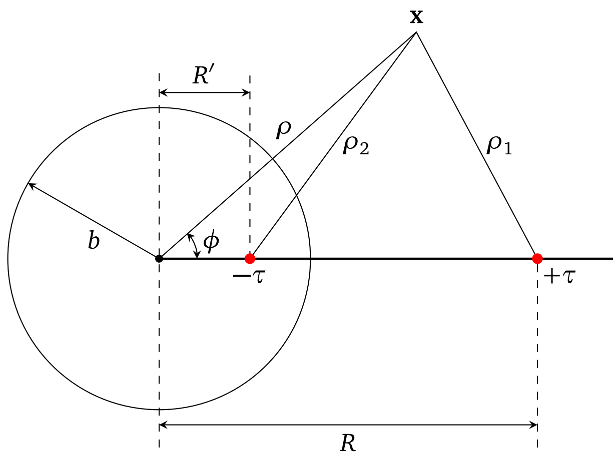
1. We could have used the derivations in problem 2.8 for this, but in the previous notes, we've been using the location of line charge $+\tau$ as the origin, it's worth deriving this with the given cylinder's center as the origin.

   Since in problem 2.8, we have seen that for equal and opposite line charges $\pm\tau$, the equipotential lines are circles, here we start with knowing that the image charge is $-\tau$.



   It's clear that for an arbitrary point $\mathbf{x} = (\rho, \phi, z)$, the potential given by $\pm\tau$ is

$$\Phi(\mathbf{x}) = \frac{\tau}{2\pi\epsilon_0} \ln\left(\frac{\rho_2}{\rho_1}\right) \tag{1}$$

   If this makes an equipotential line $\Phi(\mathbf{x}) = V$ a circle centered at origin with radius b, we will have (temporarily switching to Cartesian coordinates)

$$\begin{aligned}
\frac{\rho_2}{\rho_1} = \frac{\sqrt{(x-R')^2+y^2}}{\sqrt{(x-R)^2+y^2}} = \exp\left(\frac{2\pi\epsilon_0 V}{\tau}\right) \equiv u \qquad &\implies \\
(x-R')^2 + y^2 - u^2\left[(x-R)^2+y^2\right] = 0 \qquad &\implies \\
\left(1-u^2\right)x^2 + \left(1-u^2\right)y^2 - 2\left(R'-u^2R\right)x - \left(u^2R^2 - R'^2\right) = 0 \qquad &\implies \\
x^2 + y^2 - \frac{2\left(R'-u^2R\right)}{1-u^2} - \frac{u^2R^2-R'^2}{1-u^2} = 0 \qquad & \qquad (2)
\end{aligned}$$

   For this to be the given circle, we must have

$$R' = u^2 R \qquad \frac{u^2R^2 - R'^2}{1-u^2} = b^2 \tag{3}$$

   which gives

$$u^2 = \frac{b^2}{R^2} \qquad R' = \frac{b^2}{R} \tag{4}$$

   Due to the definition of u in (2), we must have the positive solution $u = +b/R$, i.e.,

$$V = \frac{\tau}{2\pi\epsilon_0} \ln\left(\frac{b}{R}\right) \qquad R' = \frac{b^2}{R} \tag{5}$$

   Notice here (5) is applicable regardless of whether $b < R$ or $b > R$. The $b < R$ case is depicted in the diagram, where $V < 0$. But for $b > 0$, $V > 0$, and we are simply exchanging the positions of $-\tau \leftrightarrow +\tau$.

2. For any point $\mathbf{x} = (\rho, \phi, z)$, by (1), the potential is

$$\Phi(\mathbf{x}) = \frac{\tau}{2\pi\epsilon_0} \ln\left(\frac{\sqrt{\rho^2 + R'^2 - 2\rho R' \cos\phi}}{\sqrt{\rho^2 + R^2 - 2\rho R \cos\phi}}\right) \tag{6}$$

To see the $\rho \to \infty$ asymptotic behavior, we can expand

$$
\begin{aligned}
\Phi(\mathbf{x}) &= \frac{\tau}{4\pi\epsilon_0} \ln\left(\frac{1+\dfrac{b^4}{\rho^2 R^2}-\dfrac{2b^2\cos\phi}{\rho R}}{1+\dfrac{R^2}{\rho^2}-\dfrac{2R\cos\phi}{\rho}}\right) \\
&= \frac{\tau}{4\pi\epsilon_0}\left[\ln\left(1+\underbrace{\frac{b^4}{\rho^2R^2}-\frac{2b^2\cos\phi}{\rho R}}_{A}\right)-\ln\left(1+\underbrace{\frac{R^2}{\rho^2}-\frac{2R\cos\phi}{\rho}}_{B}\right)\right]
\end{aligned} \tag{7}
$$

using

$$
\ln(1+x) = x - \frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \tag{8}
$$

We now collect terms up to the ρ^{-3} order:

(a) The ρ^{-1} order term has contributions from A and B:

$$
-2\left(\frac{b^2\cos\phi}{\rho R}-\frac{R\cos\phi}{\rho}\right) = 2\cdot\frac{R^2-b^2}{\rho R}\cos\phi \tag{9}
$$

(b) The ρ^{-2} order term has contributions from all of A, A^2, B, B^2:

$$
\begin{aligned}
\frac{b^4}{\rho^2R^2}-\frac{1}{2}\left(\frac{4b^4\cos^2\phi}{\rho^2R^2}\right)-\frac{R^2}{\rho^2}+\frac{1}{2}\left(\frac{4R^2\cos^2\phi}{\rho^2}\right) &= \frac{b^4\left(1-2\cos^2\phi\right)}{\rho^2R^2}-\frac{R^4\left(1-2\cos^2\phi\right)}{\rho^2R^2} \\
&= 2\cdot\frac{1}{2}\frac{R^4-b^4}{\rho^2R^2}\cos 2\phi
\end{aligned} \tag{10}
$$

(c) The ρ^{-3} order term has contributions from A^2, A^3, B^2, B^3:

$$
\begin{aligned}
&\left(-\frac{1}{2}\right)\left(\frac{-4b^6\cos\phi}{\rho^3R^3}\right)+\frac{1}{3}\left(\frac{-8b^6\cos^3\phi}{\rho^3R^3}\right)+\frac{1}{2}\left(\frac{-4R^3\cos\phi}{\rho^3}\right)-\frac{1}{3}\left(\frac{-8R^3\cos^3\phi}{\rho^3}\right) \\
=&\frac{6b^6\cos\phi-8b^6\cos^3\phi}{3\rho^3R^3}-\frac{6R^6\cos\phi-8R^6\cos^3\phi}{3\rho^3R^3} \\
=&2\cdot\frac{1}{3}\left[\frac{\left(b^6-R^6\right)\overbrace{\left(3\cos\phi-4\cos^3\phi\right)}^{-\cos 3\phi}}{\rho^3R^3}\right] \\
=&2\cdot\frac{1}{3}\frac{R^6-b^6}{\rho^3R^3}\cos 3\phi
\end{aligned} \tag{11}
$$

In summary,

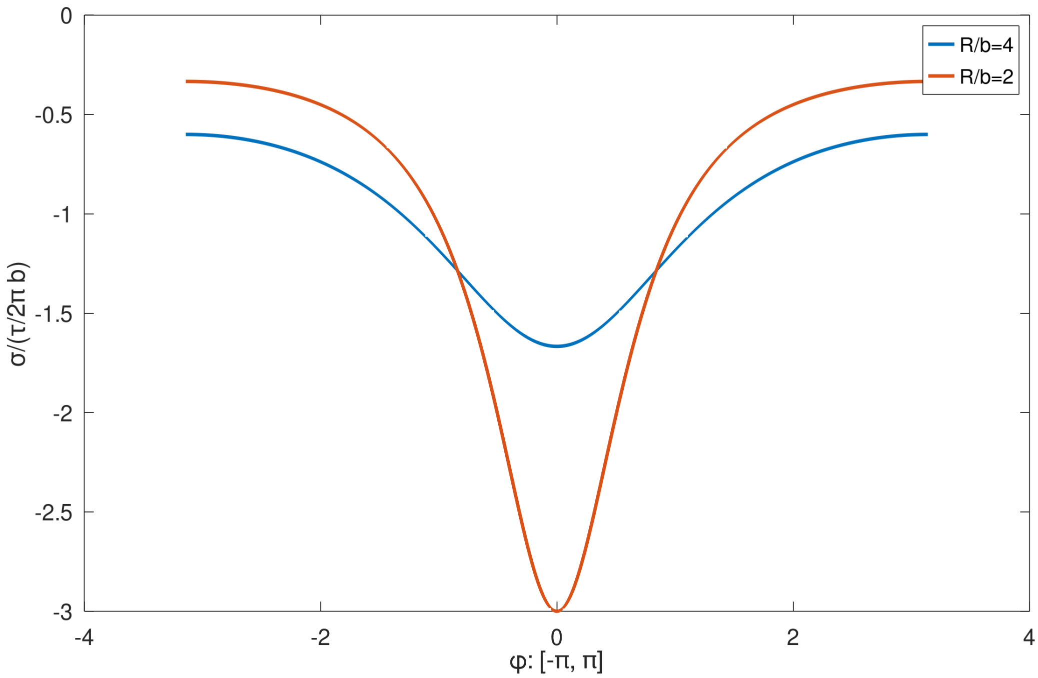
$$
\Phi(\mathbf{x}) = \frac{\tau}{2\pi\epsilon_0}\left(\frac{R^2-b^2}{\rho R}\cos\phi+\frac{1}{2}\frac{R^4-b^4}{\rho^2R^2}\cos 2\phi+\frac{1}{3}\frac{R^6-b^6}{\rho^3R^3}\cos 3\phi+\cdots\right) \tag{12}
$$

3. By (6)

$$
\begin{aligned}
\frac{\partial\Phi}{\partial\rho} &= \frac{\tau}{4\pi\epsilon_0}\frac{\partial}{\partial\rho}\left[\ln\left(\rho^2+\frac{b^4}{R^2}-\frac{2\rho b^2\cos\phi}{R}\right)-\ln\left(\rho^2+R^2-2\rho R\cos\phi\right)\right] \\
&= \frac{\tau}{4\pi\epsilon_0}\left(\frac{2\rho-\dfrac{2b^2\cos\phi}{R}}{\rho^2+\dfrac{b^4}{R^2}-\dfrac{2\rho b^2\cos\phi}{R}}-\frac{2\rho-2R\cos\phi}{\rho^2+R^2-2\rho R\cos\phi}\right)
\end{aligned} \tag{13}
$$

Therefore

$$
\begin{aligned}
\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial \rho}\Big|_{\rho=b} &= -\frac{\tau}{2\pi}\left[\frac{b-\dfrac{b^2\cos\phi}{R}}{\dfrac{b^2}{R^2}\left(b^2+R^2-2bR\cos\phi\right)} - \frac{b-R\cos\phi}{b^2+R^2-2bR\cos\phi}\right] \\
&= -\frac{\tau}{2\pi b}\left(\frac{R^2-b^2}{b^2+R^2-2bR\cos\phi}\right) \qquad (14)
\end{aligned}
$$



4. The force per length between the cylinder and the line charge $+\tau$ is the same as the force between the two line charges $-\tau$ and $+\tau$ separated by $R - R' = R - b^2/R$, which is

$$
F = \frac{\tau^2}{2\pi\epsilon_0}\frac{1}{R-\dfrac{b^2}{R}} = \frac{\tau^2}{2\pi\epsilon_0}\frac{R}{R^2-b^2} \qquad (15)
$$