1. **Prob 8.9**

   (a) With the simplifying notion,

$$\langle \mathbf{a}, \mathbf{b} \rangle = \int_V \mathbf{a}^* \cdot \mathbf{b} d^3x \tag{1}$$

define the functional

$$f[\mathbf{X}] \equiv \frac{\langle \mathbf{X}, \nabla \times (\nabla \times \mathbf{X}) \rangle}{\langle \mathbf{X}, \mathbf{X} \rangle} \tag{2}$$

When $\mathbf{E}$ is the exact solution of the vector Helmholtz equation

$$\nabla \times (\nabla \times \mathbf{E}) = k^2 \mathbf{E} \tag{3}$$

we have

$$f[\mathbf{E}] = k^2 \tag{4}$$

But for the perturbed field $\mathbf{E} + \delta\mathbf{E}$, up to first order of $\delta\mathbf{E}$,

$$\begin{aligned}
f[\mathbf{E}+\delta\mathbf{E}] &= \frac{\langle \mathbf{E}+\delta\mathbf{E}, \nabla \times [\nabla \times (\mathbf{E}+\delta\mathbf{E})] \rangle}{\langle \mathbf{E}+\delta\mathbf{E}, \mathbf{E}+\delta\mathbf{E} \rangle} \\
&\approx \frac{\langle \mathbf{E}, \nabla \times (\nabla \times \mathbf{E}) \rangle + \langle \delta\mathbf{E}, \nabla \times (\nabla \times \mathbf{E}) \rangle + \langle \mathbf{E}, \nabla \times (\nabla \times \delta\mathbf{E}) \rangle}{\langle \mathbf{E}, \mathbf{E} \rangle + \langle \mathbf{E}, \delta\mathbf{E} \rangle + \langle \delta\mathbf{E}, \mathbf{E} \rangle} \\
&= \left[k^2 + k^2 \frac{\langle \delta\mathbf{E}, \mathbf{E} \rangle}{\langle \mathbf{E}, \mathbf{E} \rangle} + \frac{\langle \mathbf{E}, \nabla \times (\nabla \times \delta\mathbf{E}) \rangle}{\langle \mathbf{E}, \mathbf{E} \rangle} \right] \left(1 + \frac{\langle \mathbf{E}, \delta\mathbf{E} \rangle}{\langle \mathbf{E}, \mathbf{E} \rangle} + \frac{\langle \delta\mathbf{E}, \mathbf{E} \rangle}{\langle \mathbf{E}, \mathbf{E} \rangle} \right)^{-1} \\
&\approx \left[k^2 + k^2 \frac{\langle \delta\mathbf{E}, \mathbf{E} \rangle}{\langle \mathbf{E}, \mathbf{E} \rangle} + \frac{\langle \mathbf{E}, \nabla \times (\nabla \times \delta\mathbf{E}) \rangle}{\langle \mathbf{E}, \mathbf{E} \rangle} \right] \left(1 - \frac{\langle \mathbf{E}, \delta\mathbf{E} \rangle}{\langle \mathbf{E}, \mathbf{E} \rangle} - \frac{\langle \delta\mathbf{E}, \mathbf{E} \rangle}{\langle \mathbf{E}, \mathbf{E} \rangle} \right) \\
&\approx k^2 + \frac{\langle \mathbf{E}, \nabla \times (\nabla \times \delta\mathbf{E}) \rangle}{\langle \mathbf{E}, \mathbf{E} \rangle} - k^2 \frac{\langle \mathbf{E}, \delta\mathbf{E} \rangle}{\langle \mathbf{E}, \mathbf{E} \rangle}
\end{aligned} \tag{5}$$

We see that if $\langle \mathbf{E}, \nabla \times (\nabla \times \delta\mathbf{E}) \rangle = k^2 \langle \mathbf{E}, \delta\mathbf{E} \rangle$, or

$$\int_V \mathbf{E}^* \cdot \nabla \times (\nabla \times \delta\mathbf{E}) d^3x = k^2 \int_V \mathbf{E}^* \cdot \delta\mathbf{E} d^3x \tag{6}$$

then $f[\mathbf{E}+\delta\mathbf{E}] = k^2 + O\left[(\delta\mathbf{E})^2\right]$, i.e., $f[\mathbf{E}]$ is a variational principle for (3). We will now prove (6).
With the vector identity $\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$ and the vanishing divergence of $\delta\mathbf{E}$, the LHS of (6) becomes

$$\mathrm{LHS}_{(6)} = -\int_V \mathbf{E}^* \cdot \left(\nabla^2 \delta\mathbf{E}\right) d^3x = -\int_V E_j^* \left(\nabla^2 \delta E_j\right) d^3x \tag{7}$$

(where repeated index is implicitly summed over here and below).
Invoking Green's theorem

$$\int_V \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) d^3x = \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathbf{n} da \tag{8}$$

with ϕ identified to E_j^* and ψ identified to δE_j, we have

$$\int_V \left[\mathbf{E}^* \cdot \left(\nabla^2 \delta\mathbf{E}\right) - \delta\mathbf{E} \cdot \left(\nabla^2 \mathbf{E}^*\right)\right] d^3x = \int_S \left(E_j^* \nabla \delta E_j - \delta E_j \nabla E_j^*\right) \cdot \mathbf{n} da \tag{9}$$

This turns (7) into

$$\begin{aligned}
\mathrm{LHS}_{(6)} &= -\int_V \delta\mathbf{E} \cdot \left(\nabla^2 \mathbf{E}^*\right) d^3x - \int_S \left(E_j^* \nabla \delta E_j - \delta E_j \nabla E_j^*\right) \cdot \mathbf{n} da \\
&= \int_V \delta\mathbf{E} \cdot \nabla \times (\nabla \times \mathbf{E}^*) d^3x - \int_S \left(E_j^* \nabla \delta E_j - \delta E_j \nabla E_j^*\right) \cdot \mathbf{n} da \qquad \text{assuming } k^2 \text{ is real} \\
&= \mathrm{RHS}_{(6)} - \int_S \left(E_j^* \nabla \delta E_j - \delta E_j \nabla E_j^*\right) \cdot \mathbf{n} da
\end{aligned} \tag{10}$$

Thus to prove (6), it suffices to show

$$\int_S \left(E_j^* \nabla \delta E_j - \delta E_j \nabla E_j^*\right) \cdot \mathbf{n} da = 0 \tag{11}$$

which is rephrased and proved as the following lemma.

Lemma 1. *Let* $\mathbf{a}, \mathbf{b}$ *be two vector fields satisfying* $\nabla \cdot \mathbf{a} = \nabla \cdot \mathbf{b} = 0$ *in* V *and* $\mathbf{n} \times \mathbf{a} = \mathbf{n} \times \mathbf{b} = 0$ *on* S*. Then*

$$\int_S \left(b_j \nabla a_j - a_j \nabla b_j\right) \cdot \mathbf{n} da = 0 \tag{12}$$

Proof. Since on S, $\mathbf{n} \times \mathbf{b}$=0, then

$$\begin{aligned}
0 &= (\nabla \times \mathbf{a}) \cdot (\mathbf{n} \times \mathbf{b}) \\
&= \left(\epsilon_{ijk} \frac{\partial a_j}{\partial x_i}\right)(\epsilon_{lmk} n_l b_m) \\
&= \frac{\partial a_j}{\partial x_i} n_l b_m \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) \\
&= \frac{\partial a_j}{\partial x_i} n_i b_j - \frac{\partial a_j}{\partial x_i} n_j b_i \\
&= b_j \nabla a_j \cdot \mathbf{n} - \nabla(\mathbf{a} \cdot \mathbf{n}) \cdot \mathbf{b} && \text{use } \nabla\psi \cdot \mathbf{b} = \nabla \cdot (\psi \mathbf{b}) - \psi \nabla \cdot \mathbf{b} \\
&= b_j \nabla a_j \cdot \mathbf{n} - \left\{ \nabla \cdot [(\mathbf{a} \cdot \mathbf{n})\mathbf{b}] - (\mathbf{a} \cdot \mathbf{n}) \overbrace{\nabla \cdot \mathbf{b}}^{0} \right\} \\
&= b_j \nabla a_j \cdot \mathbf{n} - \nabla \cdot [(\mathbf{a} \cdot \mathbf{n})\mathbf{b}]
\end{aligned} \tag{13}$$

By switching $\mathbf{a}$ with $\mathbf{b}$ and taking the difference with (12), we know at any point on S,

$$\begin{aligned}
0 &= (\nabla \times \mathbf{a}) \cdot (\mathbf{n} \times \mathbf{b}) - (\nabla \times \mathbf{b}) \cdot (\mathbf{n} \times \mathbf{a}) \\
&= \left(b_j \nabla a_j - a_j \nabla b_j\right) \cdot \mathbf{n} - \nabla \cdot [(\mathbf{a} \cdot \mathbf{n})\mathbf{b} - (\mathbf{b} \cdot \mathbf{n})\mathbf{a}] \\
&= \left(b_j \nabla a_j - a_j \nabla b_j\right) \cdot \mathbf{n} - \nabla \cdot [\mathbf{n} \times (\mathbf{b} \times \mathbf{a})]
\end{aligned} \tag{14}$$

It is very tempting to drop the divergence term in (14) based on the fact that both $\mathbf{a}$ and $\mathbf{b}$ are along the normal direction on the boundary hence $\mathbf{b} \times \mathbf{a} = 0$. But this is incorrect because the vanishing field at a point does not imply vanishing divergence at that point.

In fact, with the vector identity $\nabla \cdot (\mathbf{x} \times \mathbf{y}) = \mathbf{y} \cdot (\nabla \times \mathbf{x}) - \mathbf{x} \cdot (\nabla \times \mathbf{y})$, this divergence term can be written

$$\nabla \cdot [\mathbf{n} \times (\mathbf{b} \times \mathbf{a})] = (\mathbf{b} \times \mathbf{a}) \cdot \overbrace{(\nabla \times \mathbf{n})}^{0} - \mathbf{n} \cdot \nabla \times (\mathbf{b} \times \mathbf{a}) = -\mathbf{n} \cdot \nabla \times (\mathbf{b} \times \mathbf{a}) \tag{15}$$

With this substituted into (14) and taking the surface itegral, we have

$$\begin{aligned}
0 &= \int_S \left(b_j \nabla a_j - a_j \nabla b_j\right) \cdot \mathbf{n} da + \int_S \mathbf{n} \cdot \nabla \times (\mathbf{b} \times \mathbf{a}) da && \text{by the Divergence theorem} \\
&= \int_S \left(b_j \nabla a_j - a_j \nabla b_j\right) \cdot \mathbf{n} da + \int_V \nabla \cdot [\nabla \times (\mathbf{b} \times \mathbf{a})] d^3x && \text{curl has vanishing divergence} \\
&= \int_S \left(b_j \nabla a_j - a_j \nabla b_j\right) \cdot \mathbf{n} da
\end{aligned} \tag{16}$$

□

(b) For the TM_{010} mode, $E_z \hat{\mathbf{z}}$ is the entire electric field, so if we use the trial longitudinal field (setting $E_0 = 1$)

$$E_z = \cos\left(\frac{\pi\rho}{2R}\right) \tag{17}$$

we will have

$$\nabla \times \mathbf{E} = -\frac{\partial E_z}{\partial \rho} \hat{\boldsymbol{\phi}} = \left(\frac{\pi}{2R}\right) \sin\left(\frac{\pi\rho}{2R}\right) \hat{\boldsymbol{\phi}} \qquad \Longrightarrow$$

$$\nabla \times (\nabla \times \mathbf{E}) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho \left(\frac{\pi}{2R}\right) \sin\left(\frac{\pi\rho}{2R}\right)\right] \hat{\mathbf{z}} = \left[\frac{1}{\rho}\left(\frac{\pi}{2R}\right) \sin\left(\frac{\pi\rho}{2R}\right) + \left(\frac{\pi}{2R}\right)^2 \cos\left(\frac{\pi\rho}{2R}\right)\right] \hat{\mathbf{z}} \tag{18}$$

Then the functional $f[\mathbf{E}]$ can be computed

$$f[\mathbf{E}] = \frac{\displaystyle\int_V \mathbf{E}^* \cdot \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) d^3x}{\displaystyle\int_V \mathbf{E}^* \cdot \mathbf{E} d^3x} = \frac{\displaystyle 2\pi d \int_0^R \left[\frac{1}{\rho} \left(\frac{\pi}{2R} \right) \sin\left(\frac{\pi\rho}{2R} \right) \cos\left(\frac{\pi\rho}{2R} \right) + \left(\frac{\pi}{2R} \right)^2 \cos^2\left(\frac{\pi\rho}{2R} \right) \right] \rho\, d\rho}{\displaystyle 2\pi d \int_0^R \cos^2\left(\frac{\pi\rho}{2R} \right) \rho\, d\rho} \tag{19}$$

With some explicit integrations, we will obtain

$$\int_0^R \left(\frac{\pi}{2R} \right) \sin\left(\frac{\pi\rho}{2R} \right) \cos\left(\frac{\pi\rho}{2R} \right) d\rho = \frac{1}{2} \qquad \int_0^R \cos^2\left(\frac{\pi\rho}{2R} \right) \rho\, d\rho = R^2 \left(\frac{1}{4} - \frac{1}{\pi^2} \right) \tag{20}$$

hence

$$f[\mathbf{E}] = k^2 = \frac{\frac{1}{2} + \left(\frac{\pi^2}{16} - \frac{1}{4} \right)}{R^2 \left(\frac{1}{4} - \frac{1}{\pi^2} \right)} \qquad \Longrightarrow \qquad kR = \frac{\pi}{2} \sqrt{\frac{\pi^2 + 4}{\pi^2 - 4}} \approx 2.415 \tag{21}$$

which is slightly greater than $x_{01} = 2.405$.

(c) Now with another trial field

$$E_z = 1 + \alpha \left(\frac{\rho}{R} \right)^2 - (1 + \alpha) \left(\frac{\rho}{R} \right)^4 \tag{22}$$

we have

$$\boldsymbol{\nabla} \times \mathbf{E} = \left[-\frac{2\alpha}{R^2} \rho + 4 \left(\frac{1+\alpha}{R^4} \right) \rho^3 \right] \hat{\boldsymbol{\phi}} \qquad \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \left[-\frac{4\alpha}{R^2} + 16 \left(\frac{1+\alpha}{R^4} \right) \rho^2 \right] \hat{\mathbf{z}} \tag{23}$$

Then

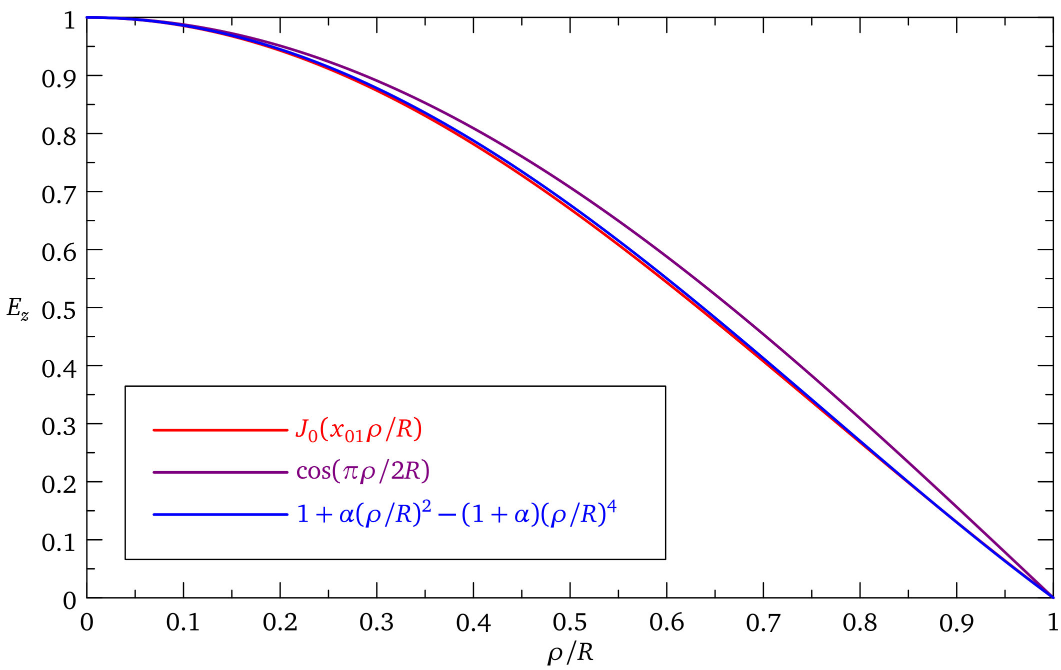
$$f[\mathbf{E}] = \frac{\displaystyle\int_0^R \left[1 + \alpha \left(\frac{\rho}{R} \right)^2 - (1+\alpha) \left(\frac{\rho}{R} \right)^4 \right] \left[-\frac{4\alpha}{R^2} + 16 \left(\frac{1+\alpha}{R^4} \right) \rho^2 \right] \rho\, d\rho}{\displaystyle\int_0^R \left[1 + \alpha \left(\frac{\rho}{R} \right)^2 - (1+\alpha) \left(\frac{\rho}{R} \right)^4 \right]^2 \rho\, d\rho} = \frac{1}{R^2} \left(\frac{\frac{1}{3}\alpha^2 + \frac{4}{3}\alpha + 2}{\frac{1}{60}\alpha^2 + \frac{7}{60}\alpha + \frac{4}{15}} \right) \tag{24}$$

Tedious calculations show that $f[\mathbf{E}]$ achieves minimum at $\alpha = \left(-10 + \sqrt{34}\right)/3$, which corresponds to

$$kR = \sqrt{80 \left(\frac{17 - 2\sqrt{34}}{68 + \sqrt{34}} \right)} \approx 2.405 \tag{25}$$

which is closer than the previous trial function, but the calculation is much more tedious.

The exact solution and the two optimal trial solutions are plotted below.



2. **Prob 8.10**

(a) It suffices to prove for any electric field $\mathbf{E}$ satisfying the boundary condition $\mathbf{n}\times\mathbf{E}\big|_S = 0$,

$$\int_V \mathbf{E}^*\cdot\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E})d^3x = \int_V (\boldsymbol{\nabla}\times\mathbf{E}^*)\cdot(\boldsymbol{\nabla}\times\mathbf{E})\,d^3x \tag{26}$$

We have already seen that the LHS is

$$\text{LHS}_{(26)} = -\int_V \mathbf{E}^*\cdot\boldsymbol{\nabla}^2\mathbf{E}d^3x = -\int_V E_j^*\boldsymbol{\nabla}^2E_j d^3x \tag{27}$$

For the RHS of (26), let's expand the curls

$$\begin{aligned}(\boldsymbol{\nabla}\times\mathbf{E}^*)\cdot(\boldsymbol{\nabla}\times\mathbf{E}) &= \left(\epsilon_{ijk}\frac{\partial E_j^*}{\partial x_i}\right)\left(\epsilon_{lmk}\frac{\partial E_m}{\partial x_l}\right)\\ &= \frac{\partial E_j^*}{\partial x_i}\frac{\partial E_m}{\partial x_l}\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)\\ &= \frac{\partial E_j^*}{\partial x_i}\frac{\partial E_j}{\partial x_i} - \frac{\partial E_j^*}{\partial x_i}\frac{\partial E_i}{\partial x_j}\\ &= \boldsymbol{\nabla}E_j^*\cdot\boldsymbol{\nabla}E_j - \boldsymbol{\nabla}E_j^*\cdot\frac{\partial\mathbf{E}}{\partial x_j}\end{aligned} \tag{28}$$

Thus the desired identity to prove, (26), is equivalent to

$$\int_V\left(E_j^*\boldsymbol{\nabla}^2E_j + \boldsymbol{\nabla}E_j^*\cdot\boldsymbol{\nabla}E_j\right)d^3x = \int_V \boldsymbol{\nabla}E_j^*\cdot\frac{\partial\mathbf{E}}{\partial x_j}d^3x \tag{29}$$

By Green's first identity,

$$\text{LHS}_{(29)} = \int_S E_j^*\mathbf{n}\cdot\boldsymbol{\nabla}E_j da \tag{30}$$

The integrand of the RHS of (29) is a divergence, because

$$\boldsymbol{\nabla}E_j^*\cdot\frac{\partial\mathbf{E}}{\partial x_j} = \boldsymbol{\nabla}\cdot\left(E_j^*\frac{\partial\mathbf{E}}{\partial x_j}\right) - E_j^*\boldsymbol{\nabla}\cdot\frac{\partial\mathbf{E}}{\partial x_j} = \boldsymbol{\nabla}\cdot\left(E_j^*\frac{\partial\mathbf{E}}{\partial x_j}\right) - E_j^*\frac{\partial}{\partial x_j}\overbrace{(\boldsymbol{\nabla}\cdot\mathbf{E})}^{0} = \boldsymbol{\nabla}\cdot\left(E_j^*\frac{\partial\mathbf{E}}{\partial x_j}\right) \tag{31}$$

which allows us to turn its volume integral into a surface integral

$$\text{RHS}_{(29)} = \int_V \boldsymbol{\nabla}\cdot\left(E_j^*\frac{\partial\mathbf{E}}{\partial x_j}\right)d^3x = \int_S E_j^*\frac{\partial\mathbf{E}}{\partial x_j}\cdot\mathbf{n}da = \int_S \boldsymbol{\nabla}(\mathbf{E}\cdot\mathbf{n})\cdot\mathbf{E}^*da \tag{32}$$

Now the equivalent claim to prove becomes

$$\int_S E_j^*\mathbf{n}\cdot\boldsymbol{\nabla}E_j da = \int_S \boldsymbol{\nabla}(\mathbf{E}\cdot\mathbf{n})\cdot\mathbf{E}^*da \tag{33}$$

This is indeed true by applying (13) with $\mathbf{a}=\mathbf{E},\mathbf{b}=\mathbf{E}^*$ (third to last line of (13)).

(b) With

$$B_z = \overbrace{B_0\frac{\rho}{R}\left(1-\frac{\rho}{2R}\right)\cos\phi}^{\psi}\sin\left(\frac{\pi z}{d}\right) \tag{34}$$

we have

$$\begin{aligned}\boldsymbol{\nabla}_t\psi &= \frac{\partial\psi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}}\\ &= B_0\left[\frac{1}{R}\left(1-\frac{\rho}{2R}\right)+\frac{\rho}{R}\left(-\frac{1}{2R}\right)\right]\cos\phi\,\hat{\boldsymbol{\rho}} - B_0\frac{1}{R}\left(1-\frac{\rho}{2R}\right)\sin\phi\,\hat{\boldsymbol{\phi}}\\ &= \frac{B_0}{R}\left[\left(1-\frac{\rho}{R}\right)\cos\phi\,\hat{\boldsymbol{\rho}} - \left(1-\frac{\rho}{2R}\right)\sin\phi\,\hat{\boldsymbol{\phi}}\right]\end{aligned} \tag{35}$$

The transverse electric field is obtained via (8.77)

$$\mathbf{E}_t = -\frac{i\omega\mu}{\gamma^2}\sin\left(\frac{\pi z}{d}\right)\hat{\mathbf{z}}\times\nabla_t\psi = -\overbrace{\frac{i\omega\mu}{\gamma^2}\frac{B_0}{R}}^{E_0}\sin\left(\frac{\pi z}{d}\right)\left[\left(1-\frac{\rho}{R}\right)\cos\phi\,\hat{\boldsymbol{\phi}} + \left(1-\frac{\rho}{2R}\right)\sin\phi\,\hat{\boldsymbol{\rho}}\right] \tag{36}$$

with

$$E_\rho = E_0\left(1-\frac{\rho}{2R}\right)\sin\phi\,\sin\left(\frac{\pi z}{d}\right) \qquad E_\phi = E_0\left(1-\frac{\rho}{R}\right)\cos\phi\,\sin\left(\frac{\pi z}{d}\right) \tag{37}$$

(c) With the curl formula in cylindrical coordinates, we have (setting $E_0 = 1$)

$$\begin{aligned}
\nabla\times\mathbf{E} &= -\frac{\partial E_\phi}{\partial z}\hat{\boldsymbol{\rho}} + \frac{\partial E_\rho}{\partial z}\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left[\frac{\partial(\rho E_\phi)}{\partial\rho} - \frac{\partial E_\rho}{\partial\phi}\right]\hat{\mathbf{z}} \\
&= -\left(\frac{\pi}{d}\right)\left(1-\frac{\rho}{R}\right)\cos\phi\cos\left(\frac{\pi z}{d}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\pi}{d}\right)\left(1-\frac{\rho}{2R}\right)\sin\phi\cos\left(\frac{\pi z}{d}\right)\hat{\boldsymbol{\phi}} + \\
&\quad \frac{1}{\rho}\left[\left(1-\frac{2\rho}{R}\right)\cos\phi\sin\left(\frac{\pi z}{d}\right) - \left(1-\frac{\rho}{2R}\right)\cos\phi\sin\left(\frac{\pi z}{d}\right)\right]\hat{\mathbf{z}} \\
&= -\left(\frac{\pi}{d}\right)\left(1-\frac{\rho}{R}\right)\cos\phi\cos\left(\frac{\pi z}{d}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\pi}{d}\right)\left(1-\frac{\rho}{2R}\right)\sin\phi\cos\left(\frac{\pi z}{d}\right)\hat{\boldsymbol{\phi}} - \frac{3}{2R}\cos\phi\sin\left(\frac{\pi z}{d}\right)\hat{\mathbf{z}}
\end{aligned} \tag{38}$$

Carrying out the integrals in the functional, we get

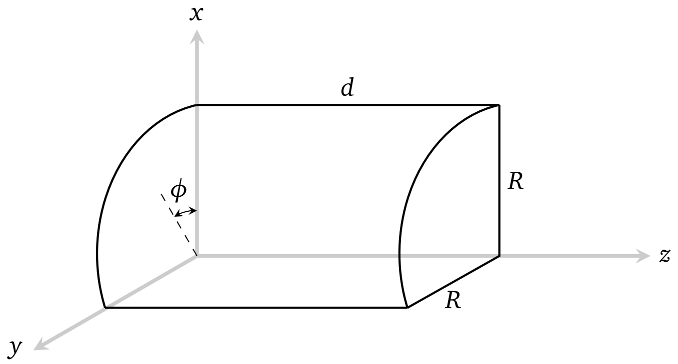
$$k^2 = f[\mathbf{E}] = \frac{\int_V (\nabla\times\mathbf{E}^*)\cdot(\nabla\times\mathbf{E})\,d^3x}{\int_V \mathbf{E}^*\cdot\mathbf{E}d^3x} = \frac{\left(\frac{\pi}{d}\right)^2\frac{R^2}{12} + \left(\frac{\pi}{d}\right)^2\frac{11R^2}{48} + \frac{9}{8}}{\frac{R^2}{12} + \frac{11R^2}{48}} = \frac{\pi^2}{d^2} + \frac{18}{5R^2} \tag{39}$$

Recall that the variational principle gets the overall eigenvalue $\omega_\lambda^2\mu\epsilon$, this implies a γ value of (see (8.79))

$$\sqrt{k^2 - \frac{\pi^2}{d^2}} = \frac{1}{R}\sqrt{\frac{18}{5}} \approx \frac{1.897}{R} \tag{40}$$

where the exact solution is $1.841/R$.

3. **Prob 8.11**



If we take the trial field

$$E_z = \left(\frac{\rho}{R}\right)^{\nu}\left(1-\frac{\rho}{R}\right)\sin 2\phi \tag{41}$$

as the only component of the field, then

$$\begin{aligned}
\int_V \mathbf{E}^* \cdot \mathbf{E} d^3x &= d\int_0^{\pi/2} d\phi \int_0^R \rho d\rho \left(\frac{\rho}{R}\right)^{2\nu}\left(1-\frac{\rho}{R}\right)^2 \sin^2 2\phi \\
&= \frac{\pi d}{4}\int_0^R \left(\frac{\rho^{2\nu+1}}{R^{2\nu}} - \frac{2\rho^{2\nu+2}}{R^{2\nu+1}} + \frac{\rho^{2\nu+3}}{R^{2\nu+2}}\right) d\rho \\
&= \frac{\pi d R^2}{4}\left(\frac{1}{2\nu+2} - \frac{2}{2\nu+3} + \frac{1}{2\nu+4}\right) \\
&= \frac{\pi d R^2}{4}\frac{2}{(2\nu+2)(2\nu+3)(2\nu+4)}
\end{aligned} \tag{42}$$

Also

$$\begin{aligned}
\nabla \times \mathbf{E} &= \frac{1}{\rho}\frac{\partial E_z}{\partial \phi}\hat{\boldsymbol{\rho}} - \frac{\partial E_z}{\partial \rho}\hat{\boldsymbol{\phi}} \\
&= 2\frac{\rho^{\nu-1}}{R^{\nu}}\left(1-\frac{\rho}{R}\right)\cos 2\phi\, \hat{\boldsymbol{\rho}} - \left[\frac{\nu\rho^{\nu-1}}{R^{\nu}}\left(1-\frac{\rho}{R}\right) - \frac{\rho^{\nu}}{R^{\nu+1}}\right]\sin 2\phi\, \hat{\boldsymbol{\phi}} \\
&= \frac{2}{R^{\nu}}\left(\rho^{\nu-1} - \frac{\rho^{\nu}}{R}\right)\cos 2\phi\, \hat{\boldsymbol{\rho}} - \frac{1}{R^{\nu}}\left[\nu\rho^{\nu-1} - \frac{(\nu+1)\rho^{\nu}}{R}\right]\sin 2\phi\, \hat{\boldsymbol{\phi}}
\end{aligned} \tag{43}$$

hence

$$\begin{aligned}
\int_V (\nabla \times \mathbf{E}^*) \cdot (\nabla \times \mathbf{E})\, d^3x &= \frac{\pi d}{4} \cdot 4\left(\frac{1}{2\nu} - \frac{2}{2\nu+1} + \frac{1}{2\nu+2}\right) + \frac{\pi d}{4}\left[\frac{\nu^2}{2\nu} - \frac{2\nu(\nu+1)}{2\nu+1} + \frac{(\nu+1)^2}{2\nu+2}\right] \\
&= \frac{\pi d}{4}\left[\frac{2\nu^2 + 2\nu + 8}{(2\nu)(2\nu+1)(2\nu+2)}\right]
\end{aligned} \tag{44}$$

which gives the eigenvalue

$$k^2R^2 = \frac{\displaystyle\int_V (\nabla \times \mathbf{E}^*) \cdot (\nabla \times \mathbf{E})\, d^3x}{\displaystyle\int_V \mathbf{E}^* \cdot \mathbf{E} d^3x} = \frac{(\nu+2)(2\nu+3)\left(\nu^2+\nu+4\right)}{\nu(2\nu+1)} \tag{45}$$

Numerical calculation shows that the minimum is achieved at $\nu \approx 1.568$ with minimum value $kR_{\text{min}} \approx 5.205$ where the exact eigenvalue is $kR = 5.136$.