$$\psi(\rho,\phi) = J_m\left(\frac{x_{mn}\rho}{R}\right)e^{i(m\phi-\omega t)} \qquad \gamma = \frac{x_{mn}}{R} \tag{1}$$

$$\nabla_t\psi = \overbrace{\gamma J_m'(\gamma\rho)}^{a}e^{i(m\phi-\omega t)}\hat{\boldsymbol{\rho}} + \overbrace{\frac{m}{\rho}J_m(\gamma\rho)}^{b}e^{i(m\phi-\omega t+\pi/2)}\hat{\boldsymbol{\phi}} \tag{2}$$

$$\hat{\mathbf{z}}\times\nabla_t\psi = ae^{i(m\phi-\omega t)}\hat{\boldsymbol{\phi}} - be^{i(m\phi-\omega t+\pi/2)}\hat{\boldsymbol{\rho}} \tag{3}$$

At $\rho = 0$:

1. For $m = 0$, $J_0'(0) = 0$ so $a = b = 0$.
2. For $m = 1$, $J_1'(0) = J_0(0)/2 = 1/2$, $J_1(\gamma\rho) \to \gamma\rho/2$ as $\rho \to 0$, so $a = b = \gamma/2$.
3. For $m > 1$, $J_m'(0) = 0$ and $J_m(\gamma\rho) \to (\gamma\rho/2)^m/m!$ as $\rho \to 0$, so $a = b = 0$.

In TM mode

$$E_z = \psi\cos\left(\frac{p\pi z}{d}\right) = J_m(\gamma\rho)\cos\left(\frac{p\pi z}{d}\right)e^{i(m\phi-\omega t)} \tag{4}$$

$$\operatorname{Re}E_z = J_m(\gamma\rho)\cos\left(\frac{p\pi z}{d}\right)\cos(m\phi-\omega t) \tag{5}$$

$$\mathbf{E}_t = -\overbrace{\frac{p\pi}{d\gamma^2}\sin\left(\frac{p\pi z}{d}\right)}^{e_t}\nabla_t\psi = -e_t a e^{i(m\phi-\omega t)}\hat{\boldsymbol{\rho}} - e_t b e^{i(m\phi-\omega t+\pi/2)}\hat{\boldsymbol{\phi}} \tag{6}$$

$$E_x = -e_t a e^{i(m\phi-\omega t)}\cos\phi + e_t b e^{i(m\phi-\omega t+\pi/2)}\sin\phi \tag{7}$$

$$\operatorname{Re}E_x = -e_t a\cos(m\phi-\omega t)\cos\phi - e_t b\sin(m\phi-\omega t)\sin\phi \tag{8}$$

$$E_y = -e_t a e^{i(m\phi-\omega t)}\sin\phi - e_t b e^{i(m\phi-\omega t+\pi/2)}\cos\phi \tag{9}$$

$$\operatorname{Re}E_y = -e_t a\cos(m\phi-\omega t)\sin\phi + e_t b\sin(m\phi-\omega t)\cos\phi \tag{10}$$

Setting the factor $\epsilon\omega/\gamma^2 = 1$,

$$\mathbf{H}_t = \overbrace{i\cos\left(\frac{p\pi z}{d}\right)}^{h_t}\hat{\mathbf{z}}\times\nabla_t\psi = h_t a e^{i(m\phi-\omega t+\pi/2)}\hat{\boldsymbol{\phi}} + h_t b e^{i(m\phi-\omega t)}\hat{\boldsymbol{\rho}} \tag{11}$$

$$H_x = h_t b e^{i(m\phi-\omega t)}\cos\phi - h_t a e^{i(m\phi-\omega t+\pi/2)}\sin\phi \tag{12}$$

$$\operatorname{Re}H_x = h_t b\cos(m\phi-\omega t)\cos\phi + h_t a\sin(m\phi-\omega t)\sin\phi \tag{13}$$

$$H_y = h_t b e^{i(m\phi-\omega t)}\sin\phi + h_t a e^{i(m\phi-\omega t+\pi/2)}\cos\phi \tag{14}$$

$$\operatorname{Re}H_y = h_t b\cos(m\phi-\omega t)\sin\phi - h_t a\sin(m\phi-\omega t)\cos\phi \tag{15}$$

In TE mode

$$H_z = \psi \sin\left(\frac{p\pi z}{d}\right) = J_m(\gamma\rho)\sin\left(\frac{p\pi z}{d}\right)e^{i(m\phi-\omega t)} \tag{16}$$

$$\operatorname{Re} H_z = J_m(\gamma\rho)\sin\left(\frac{p\pi z}{d}\right)\cos(m\phi-\omega t) \tag{17}$$

$$\mathbf{H}_t = \overbrace{\frac{p\pi}{d\gamma^2}\cos\left(\frac{p\pi z}{d}\right)}^{h_t}\boldsymbol{\nabla}_t\psi = h_t a e^{i(m\phi-\omega t)}\hat{\boldsymbol{\rho}} + h_t b e^{i(m\phi-\omega t+\pi/2)}\hat{\boldsymbol{\phi}} \tag{18}$$

$$H_x = h_t a e^{i(m\phi-\omega t)}\cos\phi - h_t b e^{i(m\phi-\omega t+\pi/2)}\sin\phi \tag{19}$$

$$\operatorname{Re} H_x = h_t a\cos(m\phi-\omega t)\cos\phi + h_t b\sin(m\phi-\omega t)\sin\phi \tag{20}$$

$$H_y = h_t a e^{i(m\phi-\omega t)}\sin\phi + h_t b e^{i(m\phi-\omega t+\pi/2)}\cos\phi \tag{21}$$

$$\operatorname{Re} H_y = h_t a\cos(m\phi-\omega t)\sin\phi - h_t b\sin(m\phi-\omega t)\cos\phi \tag{22}$$

$$\mathbf{E}_t = \overbrace{-i\sin\left(\frac{p\pi z}{d}\right)}^{e_t}\hat{\mathbf{z}}\times\boldsymbol{\nabla}_t\psi = e_t a e^{i(m\phi-\omega t-\pi/2)}\hat{\boldsymbol{\phi}} - e_t b e^{i(m\phi-\omega t)}\hat{\boldsymbol{\rho}} \tag{23}$$

$$E_x = -e_t b e^{i(m\phi-\omega t)}\cos\phi - e_t a e^{i(m\phi-\omega t-\pi/2)}\sin\phi \tag{24}$$

$$\operatorname{Re} E_x = -e_t b\cos(m\phi-\omega t)\cos\phi - e_t a\sin(m\phi-\omega t)\sin\phi \tag{25}$$

$$E_y = -e_t b e^{i(m\phi-\omega t)}\sin\phi + e_t a e^{i(m\phi-\omega t-\pi/2)}\cos\phi \tag{26}$$

$$\operatorname{Re} E_y = -e_t b\cos(m\phi-\omega t)\sin\phi + e_t a\sin(m\phi-\omega t)\cos\phi \tag{27}$$

Note that in the special case at $\rho = 0$ with $m = 1$, E_x, E_y, H_x, H_y have no ϕ dependency as indicated by (8), (10) etc, so it's ok to pass $\phi = 0$ when $\rho = 0$.