We will give the most general solution to this three-media reflection/transmission problem, where we assume arbitrary incident angle and permeability of the media.

Let $\mathbf{E}_1/\mathbf{B}_1$ be the incident wave. We expect it to generate a reflected wave $\mathbf{E}_1'/\mathbf{B}_1'$ and a transmitted wave $\mathbf{E}_2/\mathbf{B}_2$ at the n_1/n_2 boundary. Then $\mathbf{E}_2/\mathbf{B}_2$ is going to generate a reflected wave and a transmitted wave $\mathbf{E}_2'/\mathbf{B}_2'$ and $\mathbf{E}_3/\mathbf{B}_3$ at the n_2/n_3 boundary.

All the $\mathbf{E}, \mathbf{B}, \mathbf{k}$ vectors have the wave form

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} \tag{1}$$

$$\mathbf{B}(\mathbf{x}, t) = \sqrt{\mu\epsilon}\frac{\mathbf{k}\times\mathbf{E}(\mathbf{x}, t)}{|\mathbf{k}|} \tag{2}$$

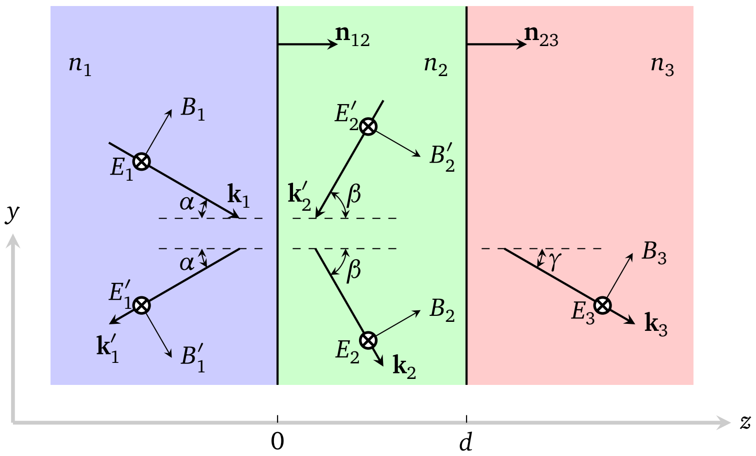
$$|\mathbf{k}| = \omega\sqrt{\mu\epsilon} = \frac{n\omega}{c} \tag{3}$$

The boundary conditions are given by the Maxwell's equations:

n_1/n_2 boundary	n_2/n_3 boundary	
$\left[\epsilon_1\left(\mathbf{E}_1+\mathbf{E}_1'\right)-\epsilon_2\left(\mathbf{E}_2+\mathbf{E}_2'\right)\right]\cdot\mathbf{n}_{12}=0$	$\left[\epsilon_2\left(\mathbf{E}_2+\mathbf{E}_2'\right)-\epsilon_3\mathbf{E}_3\right]\cdot\mathbf{n}_{23}=0$	(4)
$\left(\mathbf{k}_1\times\mathbf{E}_1+\mathbf{k}_1'\times\mathbf{E}_1'-\mathbf{k}_2\times\mathbf{E}_2-\mathbf{k}_2'\times\mathbf{E}_2'\right)\cdot\mathbf{n}_{12}=0$	$\left(\mathbf{k}_2\times\mathbf{E}_2+\mathbf{k}_2'\times\mathbf{E}_2'-\mathbf{k}_3\times\mathbf{E}_3\right)\cdot\mathbf{n}_{23}=0$	(5)
$\left(\mathbf{E}_1+\mathbf{E}_1'-\mathbf{E}_2-\mathbf{E}_2'\right)\times\mathbf{n}_{12}=0$	$\left(\mathbf{E}_2+\mathbf{E}_2'-\mathbf{E}_3\right)\times\mathbf{n}_{23}=0$	(6)
$\left[\frac{1}{\mu_1}\left(\mathbf{k}_1\times\mathbf{E}_1+\mathbf{k}_1'\times\mathbf{E}_1'\right)-\frac{1}{\mu_2}\left(\mathbf{k}_2\times\mathbf{E}_2+\mathbf{k}_2'\times\mathbf{E}_2'\right)\right]\times\mathbf{n}_{12}=0$	$\left[\frac{1}{\mu_2}\left(\mathbf{k}_2\times\mathbf{E}_2+\mathbf{k}_2'\times\mathbf{E}_2'\right)-\frac{1}{\mu_3}\left(\mathbf{k}_3\times\mathbf{E}_3\right)\right]\times\mathbf{n}_{23}=0$	(7)

Similar to the text, we are going to treat the two linear polarization cases separately.

1. $\mathbf{E}$ is perpendicular to the plane of incidence.



Consider (6) for the n_2/n_3 boundary,

$$\left(E_2 e^{i\mathbf{k}_2\cdot\mathbf{x}} + E_2' e^{i\mathbf{k}_2'\cdot\mathbf{x}} - E_3 e^{i\mathbf{k}_3\cdot\mathbf{x}}\right)e^{-i\omega t} = 0 \qquad \text{for all } t \text{ and all } \mathbf{x} \text{ on } z = d \text{ plane} \tag{8}$$

This implies

$$E_2 e^{ik_{2z}d} e^{i\left(k_{2x}x+k_{2y}y\right)} + E_2' e^{ik_{2z}'d} e^{i\left(k_{2x}'x+k_{2y}'y\right)} - E_3 e^{ik_{3z}d} e^{i\left(k_{3x}x+k_{3y}y\right)} = 0 \qquad \text{for all } x, y \tag{9}$$

Now for the fixed complex numbers $E_2 e^{ik_{2z}d}, E_2' e^{ik_{2z}'d}, E_3 e^{ik_{3z}d}$, the only way for (9) to be consistent for all x, y is for the phase factors to be identical, i.e.,

$$k_{2x}x + k_{2y}y = k_{2x}'x + k_{2y}'y = k_{3x}x + k_{3y}y \qquad \Longrightarrow$$

$$\mathbf{k}_2\cdot(\mathbf{x}-d\hat{\mathbf{z}}) = \mathbf{k}_2'\cdot(\mathbf{x}-d\hat{\mathbf{z}}) = \mathbf{k}_3\cdot(\mathbf{x}-d\hat{\mathbf{z}}) \qquad \text{for all } \mathbf{x} \text{ on } z = d \text{ plane} \tag{10}$$

This is just the expected reflection symmetry and Snell's law.

(9) is now turned into

$$E_2 e^{ik_2 d\cos\beta} + E_2' e^{-ik_2 d\cos\beta} - E_3 e^{ik_3 d\cos\gamma} = 0 \tag{11}$$

With this, we can rewrite (7) for n_2/n_3 boundary as

$$\sqrt{\frac{\epsilon_2}{\mu_2}}\cos\beta\left(E_2 e^{ik_2 d\cos\beta} - E_2' e^{-ik_2 d\cos\beta}\right) - \sqrt{\frac{\epsilon_3}{\mu_3}}\cos\gamma\left(E_3 e^{ik_3 d\cos\gamma}\right) = 0 \tag{12}$$

(11) and (12) are essentially (7.38) since they describe the same two-media reflection and transmission, aside from the phase factor which is due to the origin shift. This allows us to use (7.39) to express E_2' and E_3 in terms of E_2:

$$E_3 = E_2 \overbrace{\left(\frac{2n_2\cos\beta}{n_2\cos\beta + \frac{\mu_2}{\mu_3}n_3\cos\gamma}\right) e^{i(k_2\cos\beta - k_3\cos\gamma)d}}^{t_{23}} \tag{13}$$

$$E_2' = E_2 \underbrace{\left(\frac{n_2\cos\beta - \frac{\mu_2}{\mu_3}n_3\cos\gamma}{n_2\cos\beta + \frac{\mu_2}{\mu_3}n_3\cos\gamma}\right) e^{i2k_2 d\cos\beta}}_{r_{23}} \tag{14}$$

Coming back to the n_1/n_2 boundary, (6) gives

$$\left(E_1 e^{i\mathbf{k}_1\cdot\mathbf{x}} + E_1' e^{i\mathbf{k}_1'\cdot\mathbf{x}} - E_2 e^{i\mathbf{k}_2\cdot\mathbf{x}} - E_2' e^{i\mathbf{k}_2'\cdot\mathbf{x}}\right) e^{-i\omega t} = 0 \qquad \text{for all } t \text{ and all } \mathbf{x} \text{ on } z=0 \text{ plane} \tag{15}$$

With similar arguments, we obtain

$$\mathbf{k}_1\cdot\mathbf{x} = \mathbf{k}_1'\cdot\mathbf{x} = \mathbf{k}_2\cdot\mathbf{x} = \mathbf{k}_2'\cdot\mathbf{x} \qquad \text{for all } \mathbf{x} \text{ on } z=0 \text{ plane} \tag{16}$$

which again gives the reflection symmetry and Snell's law.

(15) and (7) are equivalent to the restrictions

$$E_1 + E_1' - E_2 - E_2' = 0 \tag{17}$$

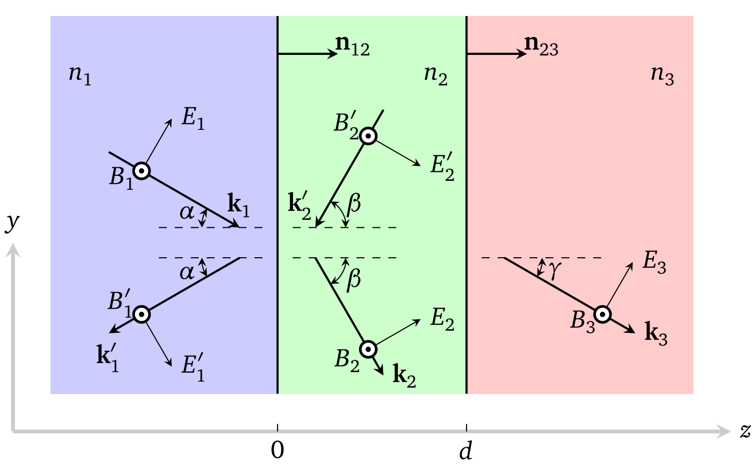
$$\sqrt{\frac{\epsilon_1}{\mu_1}}\cos\alpha\left(E_1 - E_1'\right) - \sqrt{\frac{\epsilon_2}{\mu_2}}\cos\beta\left(E_2 - E_2'\right) = 0 \tag{18}$$

We can get the solution using (14):

$$E_1' = E_1\left[\frac{(1+r_{23})n_1\cos\alpha - \frac{\mu_1}{\mu_2}(1-r_{23})n_2\cos\beta}{(1+r_{23})n_1\cos\alpha + \frac{\mu_1}{\mu_2}(1-r_{23})n_2\cos\beta}\right] \tag{19}$$

$$E_2 = E_1\left[\frac{2n_1\cos\alpha}{(1+r_{23})n_1\cos\alpha + \frac{\mu_1}{\mu_2}(1-r_{23})n_2\cos\beta}\right] \tag{20}$$

2. **E** is parallel to the plane of incidence.



The process that determines reflection symmetry and Snell's law is similar to the perpendicular case above. Then (6) and (7) for the n_2/n_3 boundary require

$$\cos\beta\left(E_2 e^{ik_2 d\cos\beta} - E_2' e^{-ik_2 d\cos\beta}\right) - \cos\gamma E_3 e^{ik_3 d\cos\gamma} = 0 \tag{21}$$

$$\sqrt{\frac{\epsilon_2}{\mu_2}}\left(E_2 e^{ik_2 d\cos\beta} + E_2' e^{-ik_2 d\cos\beta}\right) - \sqrt{\frac{\epsilon_3}{\mu_3}} E_3 e^{ik_3 d\cos\gamma} = 0 \tag{22}$$

This can be solved using (7.41)

$$E_3 = E_2 \overbrace{\left(\frac{2n_2\cos\beta}{\frac{\mu_2}{\mu_3}n_3\cos\beta + n_2\cos\gamma}\right) e^{i(k_2\cos\beta - k_3\cos\gamma)d}}^{t_{23}} \tag{23}$$

$$E_2' = E_2 \underbrace{\left(\frac{\frac{\mu_2}{\mu_3}n_3\cos\beta - n_2\cos\gamma}{\frac{\mu_2}{\mu_3}n_3\cos\beta + n_2\cos\gamma}\right) e^{i2k_2 d\cos\beta}}_{r_{23}} \tag{24}$$

For the n_1/n_2 boundary, (6) and (7) require

$$\cos\alpha(E_1 - E_1') - \cos\beta\left(E_2 - E_2'\right) = 0 \tag{25}$$

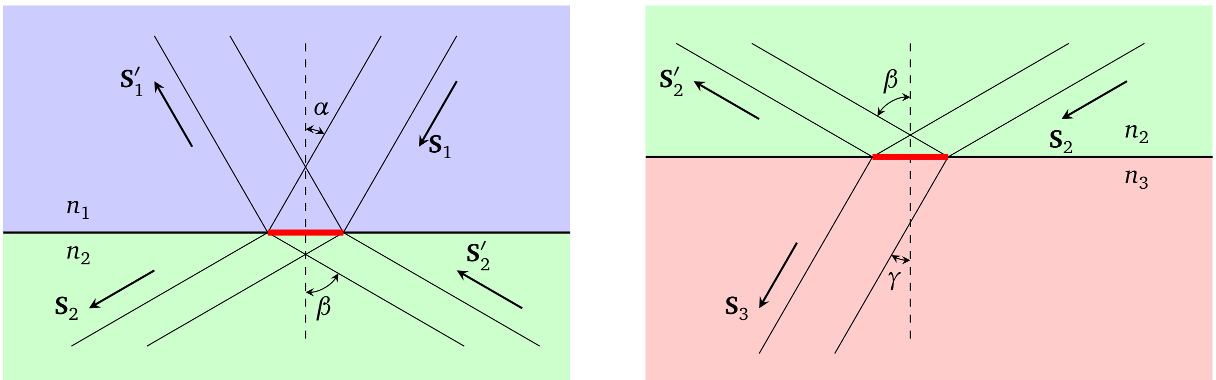
$$\sqrt{\frac{\epsilon_1}{\mu_1}}\left(E_1 + E_1'\right) - \sqrt{\frac{\epsilon_2}{\mu_2}}\left(E_2 + E_2'\right) = 0 \tag{26}$$

of which the solution is

$$E_1' = E_1\left[\frac{\frac{\mu_1}{\mu_2}(1+r_{23})n_2\cos\alpha - (1-r_{23})n_1\cos\beta}{\frac{\mu_1}{\mu_2}(1+r_{23})n_2\cos\alpha + (1-r_{23})n_1\cos\beta}\right] \tag{27}$$

$$E_2 = E_1\left[\frac{2n_1\cos\alpha}{\frac{\mu_1}{\mu_2}(1+r_{23})n_2\cos\alpha + (1-r_{23})n_1\cos\beta}\right] \tag{28}$$

3. We will now show how the Poynting flux is involved in the energy conservation of the reflection and transmission.



Let A be an area patch on the n_1/n_2 boundary. The energy incoming onto and outgoing from A are

incoming from n_1 as incident wave:	$\lvert\mathbf{S}_1\rvert A\cos\alpha$
outgoing to n_1 as reflected wave:	$\lvert\mathbf{S}_1'\rvert A\cos\alpha$
incoming from n_2 as reflected wave:	$\lvert\mathbf{S}_2'\rvert A\cos\beta$
outgoing to n_2 as transmitted wave:	$\lvert\mathbf{S}_2\rvert A\cos\beta$

where **S** is the Poynting vector of the corresponding wave

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* = \frac{1}{2c}\frac{n}{\mu}|E|^2\,\hat{\mathbf{k}} \tag{29}$$

Thus for patch A, energy conservation means

$$\frac{n_1}{\mu_1}|E_1|^2\cos\alpha + \frac{n_2}{\mu_2}\left|E_2'\right|^2\cos\beta = \frac{n_1}{\mu_1}\left|E_1'\right|^2\cos\alpha + \frac{n_2}{\mu_2}|E_2|^2\cos\beta \tag{30}$$

Similarly for a patch A' on the n_2/n_3 boundary, three waves are involved, and energy conservation means

$$\frac{n_2}{\mu_2}|E_2|^2\cos\beta = \frac{n_2}{\mu_2}\left|E_2'\right|^2\cos\beta + \frac{n_3}{\mu_3}|E_3|^2\cos\gamma \tag{31}$$

To see that our calculations agree with energy conservation, let's consider the perpendicular polarization case. Rewriting (17) and (18), and taking the complex conjugation for one of them, we get

$$E_1 + E_1' = E_2 + E_2' \tag{32}$$

$$\frac{n_1}{\mu_1}\cos\alpha\left(E_1^* - E_1'^*\right) = \frac{n_2}{\mu_2}\cos\beta\left(E_2^* - E_2'^*\right) \tag{33}$$

Multiplying (32) and (33) gives

$$\frac{n_1}{\mu_1}\cos\alpha\left(|E_1|^2 - \left|E_1'\right|^2 + E_1'E_1^* - E_1E_1'^*\right) = \frac{n_2}{\mu_2}\cos\beta\left(|E_2|^2 - \left|E_2'\right|^2 + E_2'E_2^* - E_2E_2'^*\right) \tag{34}$$

Since $E_1'E_1^* - E_1E_1'^*$ and $E_2'E_2^* - E_2E_2'^*$ are purely imaginary, (30) is obtained by taking the real part of (34).
The parallel polarization case can be proved similarly by rewriting (25) and (26). Same for the n_2/n_3 conservation equation (31).
Lastly, adding (30) and (31) will give

$$\frac{n_1}{\mu_1}|E_1|^2\cos\alpha = \frac{n_1}{\mu_1}\left|E_1'\right|^2\cos\alpha + \frac{n_3}{\mu_3}|E_3|^2\cos\gamma \tag{35}$$

This is just a claim about the overall energy conservation while we ignore the role of the middle layer.

4. Solution to problem 7.2.

   (a) Here we apply the simplifying assumption that $\mu_i = 1$ and $\alpha = \beta = \gamma = 0$, i.e., normal incidence.

      i. For the perpendicular polarization (use (13)-(20)):

$$r_{23} = \frac{n_2 - n_3}{n_2 + n_3}e^{i2n_2\omega/\omega_0} \qquad \text{where } \omega_0 \equiv \frac{c}{d} \tag{36}$$

$$\left|E_1'\right|^2 = |E_1|^2\left|\frac{(1+r_{23})n_1 - (1-r_{23})n_2}{(1+r_{23})n_1 + (1-r_{23})n_2}\right|^2 \implies$$

$$R = \frac{\left|E_1'\right|^2}{|E_1|^2} = \left|\frac{(1+r_{23})n_1 - (1-r_{23})n_2}{(1+r_{23})n_1 + (1-r_{23})n_2}\right|^2 \tag{37}$$

$$\left|E_3\right|^2 = |E_1|^2\left(\frac{2n_2}{n_2+n_3}\right)^2\left|\frac{2n_1}{(1+r_{23})n_1 + (1-r_{23})n_2}\right|^2 \implies$$

$$T = \frac{n_3|E_3|^2}{n_1|E_1|^2} = \left(\frac{2n_2}{n_2+n_3}\right)^2\frac{4n_1n_3}{|(1+r_{23})n_1 + (1-r_{23})n_2|^2} \tag{38}$$

      ii. For the parallel polarization (use (23)-(28)):

$$r_{23} = \frac{n_3 - n_2}{n_2 + n_3}e^{i2n_2\omega/\omega_0} \tag{39}$$

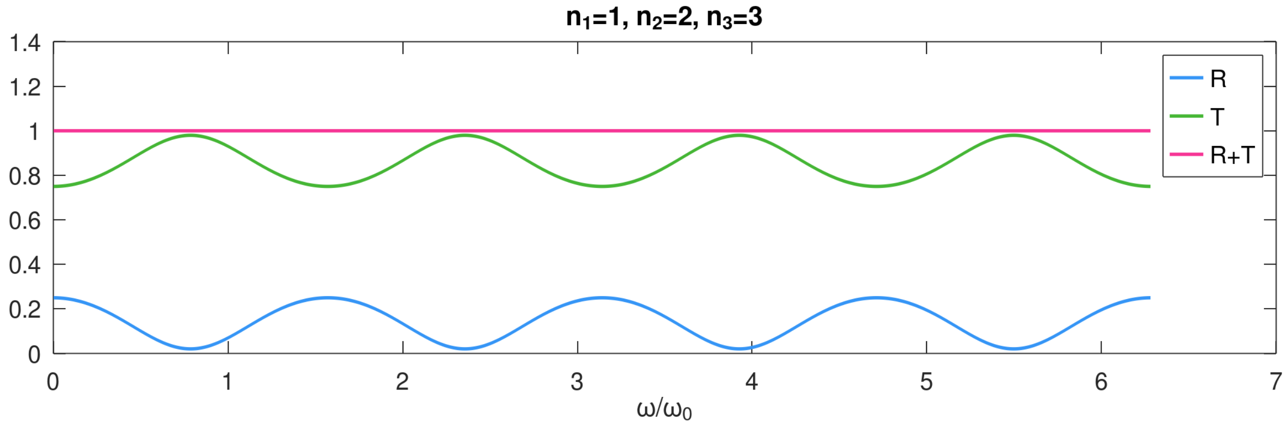
$$\left|E_1'\right|^2 = |E_1|^2\left|\frac{(1+r_{23})n_2 - (1-r_{23})n_1}{(1+r_{23})n_2 + (1-r_{23})n_1}\right|^2 \implies$$

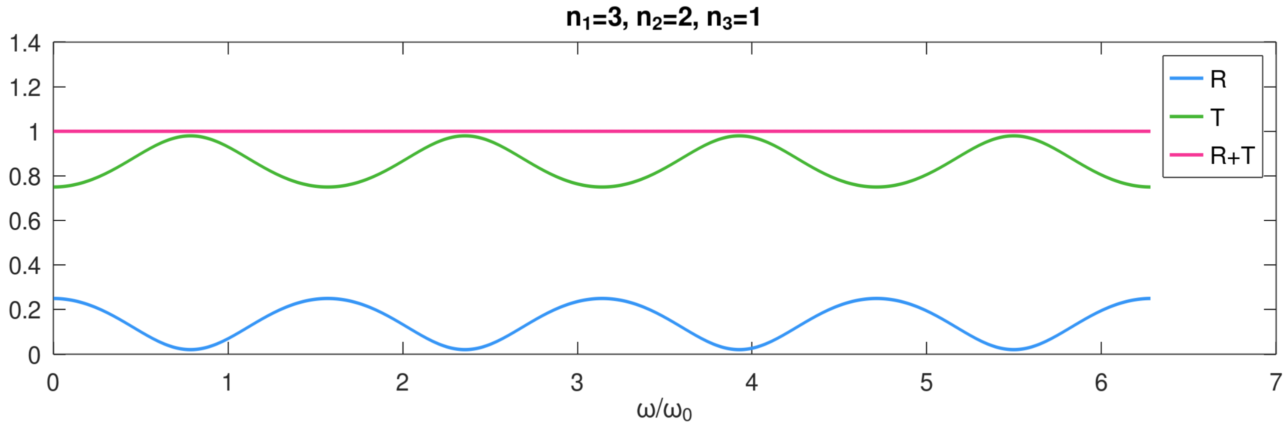
$$R = \frac{\left|E_1'\right|^2}{|E_1|^2} = \left|\frac{(1+r_{23})n_2 - (1-r_{23})n_1}{(1+r_{23})n_2 + (1-r_{23})n_1}\right|^2 \tag{40}$$

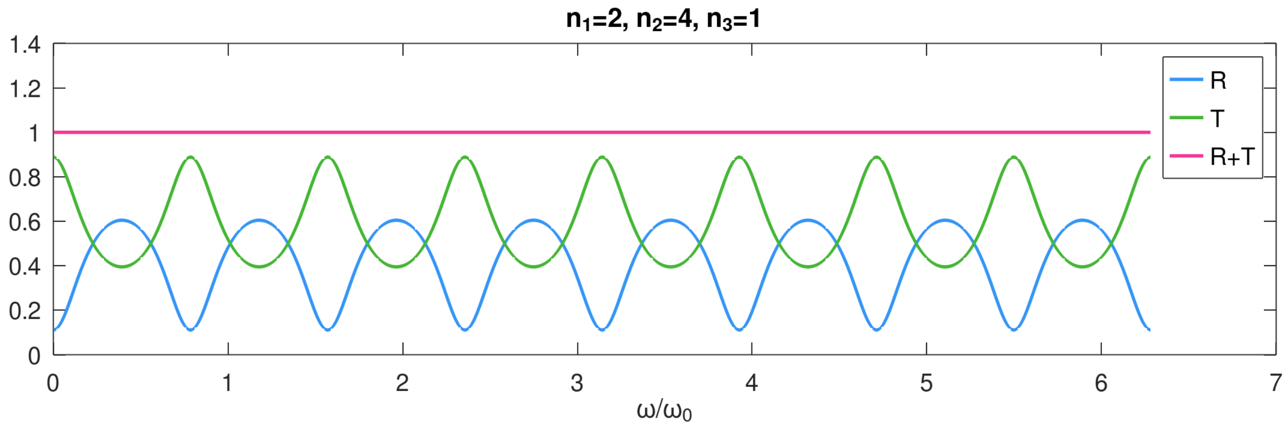
$$|E_3|^2 = |E_1|^2\left(\frac{2n_2}{n_2+n_3}\right)^2\left|\frac{2n_1}{(1+r_{23})n_2 + (1-r_{23})n_1}\right|^2 \implies$$

$$T = \frac{n_3|E_3|^2}{n_1|E_1|^2} = \left(\frac{2n_2}{n_2+n_3}\right)^2\frac{4n_1n_3}{|(1+r_{23})n_2 + (1-r_{23})n_1|^2} \tag{41}$$

Numerically, the two polarizations produce the same R and T because r_{23}'s in each case have opposite signs. The plots for different combination of refraction indices are shown below.







(b) Using the perpendicular polarization formula (36)-(38), in order for R to be zero, we need

$$(1+r_{23})n_1 = (1-r_{23})n_2 \qquad \Longrightarrow \qquad \frac{n_2-1}{n_2+1}e^{i2n_2\omega/\omega_0} = r_{23} = \frac{n_2-n_1}{n_2+n_1} \tag{42}$$

For this to be true, we must have

$$\frac{n_2-1}{n_2+1} = \pm\frac{n_2-n_1}{n_2+n_1} \tag{43}$$

The $+$ sign is untenable since it entails $n_2 = 0$. The $-$ sign yields

$$n_2 = \sqrt{n_1} \qquad \text{and} \qquad \frac{2n_2\omega d}{c} = (2k+1)\pi \qquad \Longrightarrow \qquad d = \frac{(2k+1)\pi c}{2\sqrt{n_1}\omega} \tag{44}$$