1. Treat the $z=0$ plane as a sphere with infinite radius, we can use equation (2.16) to determine its Green function, where now the image charge is of magnitude -1 at point $\mathbf{x}'_z$ which is $\mathbf{x}'$'s mirrored location across the z-plane, i.e.,

$$\begin{aligned} G(\mathbf{x},\mathbf{x}') &= \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}-\mathbf{x}'_z|} \\ &= \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} \end{aligned} \tag{1}$$

It's easy to see that on the boundary surface S where $z'=0$, G vanishes, and also that for $\mathbf{x} \in V$, any $\mathbf{x}' \in V$ will satisfy

$$\nabla'^2 F(\mathbf{x},\mathbf{x}') = \nabla'^2 \left(\frac{1}{|\mathbf{x}-\mathbf{x}'_z|} \right) = 0. \tag{2}$$

2. Since we are treating the half space $z \geq 0$ as our volume V, then at the boundary, normal direction points to the $-z$ direction, where we have

$$\begin{aligned} -\frac{\partial G}{\partial n'} &= \frac{\partial G}{\partial z'} \\ &= \frac{-(z'-z)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}^3} - \frac{-(z'+z)}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}^3} \end{aligned} \tag{3}$$

Therefore, by (1.44)

$$\begin{aligned} \Phi(\mathbf{x}) &= -\frac{1}{4\pi} \oint_S \Phi(\mathbf{x}') \frac{\partial G}{\partial n'} da' \\ &= -\frac{1}{4\pi} \oint_S \Phi(\mathbf{x}') \left(-\frac{\partial G}{\partial z'} \right) \bigg|_{z'=0} da' \\ &= \frac{1}{4\pi} \oint_S \Phi(\mathbf{x}') \left[\frac{2z}{\sqrt{(x-x')^2+(y-y')^2+z^2}^3} \right] da' \end{aligned} \tag{4}$$

If we now only consider the field point on the z-axis, where $x=y=0$, (4) becomes

$$\begin{aligned} \Phi(\mathbf{x}) &= \frac{z}{2\pi} \oint_S \Phi(\rho',\phi',z'=0) \frac{1}{\sqrt{\rho'^2+z^2}^3} da' \\ &= \frac{z}{2\pi} \int_0^{2\pi} d\phi' \int_0^a \rho' d\rho' V \cdot \frac{1}{\sqrt{\rho'^2+z^2}^3} \qquad \text{let } u \equiv \rho'^2+z^2 \\ &= \frac{zV}{2} \int_{z^2}^{z^2+a^2} u^{-3/2} du \\ &= \frac{zV}{2} \left(-2u^{-1/2} \right) \bigg|_{z^2}^{z^2+a^2} = V \left(1 - \frac{z}{\sqrt{a^2+z^2}} \right) \end{aligned} \tag{5}$$

3. For $\mathbf{x}$ satisfying $\rho^2+z^2 \gg a^2$, (4) gives

$$\begin{aligned} \Phi(\mathbf{x}) &= \frac{1}{4\pi} \oint_S \Phi(\mathbf{x}') \left[\frac{2z}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi'+z^2}^3} \right] da' \\ &= \frac{zV}{2\pi} \int_0^{2\pi} d\phi' \int_0^a \frac{\rho' d\rho'}{(\rho^2+z^2+\rho'^2-2\rho\rho'\cos\phi')^{3/2}} \end{aligned} \tag{6}$$

where the integrand

$$\begin{aligned} \frac{\rho' d\rho'}{(\rho^2+z^2+\rho'^2-2\rho\rho'\cos\phi')^{3/2}} &= \frac{\rho' d\rho'}{\left(\rho^2+z^2\right)^{3/2} \left(1 + \frac{\rho'^2-2\rho\rho'\cos\phi'}{\rho^2+z^2} \right)^{3/2}} \\ &= \frac{1}{(\rho^2+z^2)^{3/2}} d\rho' \cdot A \end{aligned} \tag{7}$$

where

$$
\begin{aligned}
A &= \rho' \left(1 + \frac{\rho'^2 - 2\rho\rho' \cos\phi'}{\rho^2 + z^2} \right)^{-3/2} \\
&= \rho' \left\{ 1 - \frac{3}{2} \left(\frac{\rho'^2 - 2\rho\rho' \cos\phi'}{\rho^2 + z^2} \right) + \frac{1}{2!} \left(-\frac{3}{2} \right) \left(-\frac{5}{2} \right) \left(\frac{\rho'^2 - 2\rho\rho' \cos\phi'}{\rho^2 + z^2} \right)^2 + O\left[\left(\rho^2 + z^2 \right)^{-3} \right] \right\} \\
&\approx \rho' \left[1 - \frac{3}{2} \frac{\rho'^2}{\rho^2 + z^2} + \frac{15}{8} \frac{\rho'^4 + 4\rho^2\rho'^2 \cos^2\phi'}{(\rho^2 + z^2)^2} + X \cdot \cos\phi' \right] \\
&= \rho' \left[1 - \frac{3}{2} \frac{\rho'^2}{\rho^2 + z^2} + \frac{15}{8} \frac{\rho'^4 + 2\rho^2\rho'^2}{(\rho^2 + z^2)^2} + X \cdot \cos\phi' + Y \cdot \cos 2\phi' \right]
\end{aligned} \tag{8}
$$

The $\cos\phi'$ and $\cos 2\phi'$ terms in A will vanish after the integration over $d\phi'$, so combining (6)-(8) we finally have

$$
\begin{aligned}
\Phi(\mathbf{x}) &\approx \frac{zV}{2\pi} \frac{1}{(\rho^2 + z^2)^{3/2}} \cdot 2\pi \int_0^a \left[\rho' - \frac{3}{2} \frac{\rho'^3}{\rho^2 + z^2} + \frac{15}{8} \frac{\rho'^5 + 2\rho^2\rho'^3}{(\rho^2 + z^2)^2} \right] d\rho' \\
&= \frac{zV}{(\rho^2 + z^2)^{3/2}} \left[\frac{a^2}{2} - \frac{3}{2} \frac{1}{4} \frac{a^4}{\rho^2 + z^2} + \frac{15}{8} \frac{1}{6} \frac{a^6}{(\rho^2 + z^2)^2} + \frac{15}{8} \frac{2}{4} \frac{\rho^2 a^4}{(\rho^2 + z^2)^2} \right] \\
&= \frac{Va^2}{2} \frac{z}{(\rho^2 + z^2)^{3/2}} \left[1 - \frac{3a^2}{4(\rho^2 + z^2)} + \frac{5a^4}{8(\rho^2 + z^2)^2} + \frac{15\rho^2 a^2}{8(\rho^2 + z^2)^2} \right]
\end{aligned} \tag{9}
$$

In the common applicable range of (b) and (c) where $\rho = 0$ and $z \gg a$, (9) gives

$$
\Phi(\mathbf{x}) \approx \frac{Va^2}{2} \frac{1}{z^2} \left(1 - \frac{3a^2}{4z^2} + \frac{5a^4}{8z^4} \right) = V \left(\frac{a^2}{2z^2} - \frac{3a^4}{8z^4} + \frac{5a^6}{16z^6} \right) \tag{10}
$$

But on the other hand, (5) gives

$$
\begin{aligned}
\Phi(\mathbf{x}) = V \left[1 - \left(1 + \frac{a^2}{z^2} \right)^{-1/2} \right] &\approx V \left[1 - \left(1 - \frac{1}{2} \frac{a^2}{z^2} + \frac{1}{2!} \frac{1}{2} \frac{3}{2} \frac{a^4}{z^4} - \frac{1}{3!} \frac{1}{2} \frac{3}{2} \frac{5}{2} \frac{a^6}{z^6} \right) \right] \\
&= V \left(\frac{a^2}{2z^2} - \frac{3a^4}{8z^4} + \frac{5a^6}{16z^6} \right)
\end{aligned} \tag{11}
$$

which agrees with (10).