

In these notes, we fill the details omitted in Jackson section 3.13 *Mixed Boundary Conditions; Conducting Plane with a Circular Hole*.

### 1. Proof of equation (3.175)

It was claimed (3.175) that

$$B_l = \frac{1}{l!} \left( -\frac{d}{d|z|} \right)^l \int_0^\infty dk e^{-k|z|} J_0(k\rho) = \frac{1}{l!} \left( -\frac{d}{d|z|} \right)^l \left( \frac{1}{\sqrt{\rho^2 + z^2}} \right) \quad (1)$$

For this, we need to prove the integral identity

$$\int_0^\infty dk e^{-k|z|} J_0(k\rho) = \frac{1}{\sqrt{\rho^2 + z^2}} \quad (2)$$

To see this, insert the expansion of  $J_0(k\rho)$  into the LHS

$$\begin{aligned} \text{LHS}_{(2)} &= \int_0^\infty dk e^{-k|z|} \sum_{j=0}^\infty \frac{(-1)^j}{j!j!} \left( \frac{k\rho}{2} \right)^{2j} \\ &= \sum_{j=0}^\infty \frac{(-1)^j}{j!j!} \left( \frac{\rho}{2} \right)^{2j} \int_0^\infty dk e^{-k|z|} k^{2j} \\ &= \sum_{j=0}^\infty \frac{(-1)^j}{j!j!} \left( \frac{\rho}{2} \right)^{2j} \cdot \left[ \frac{(2j)!}{|z|^{2j+1}} \right] \\ &= \frac{1}{|z|} \sum_{j=0}^\infty \frac{1}{j!} \frac{(-1)^j}{2^j} \left( \frac{\rho}{|z|} \right)^{2j} \frac{(2j)!}{2^j j!} \\ &= \frac{1}{|z|} \sum_{j=0}^\infty \frac{1}{j!} \left[ \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \cdots \left( -\frac{2j-1}{2} \right) \right] \left( \frac{\rho}{|z|} \right)^{2j} \end{aligned} \quad (3)$$

where in the second line, we have used the integral

$$\int_0^\infty e^{-ax} x^n dx = \frac{n!}{a^{n+1}} \quad (4)$$

Then (3) can be readily recognized as the Taylor series for

$$\frac{1}{|z|} \frac{1}{\sqrt{1 + \left( \frac{\rho}{|z|} \right)^2}} = \frac{1}{\sqrt{\rho^2 + z^2}} \quad (5)$$

### 2. Proof of equation (3.176)

Next, we have a claim that

$$B_l = \frac{1}{l!} \left( -\frac{d}{d|z|} \right)^l \left( \frac{1}{\sqrt{\rho^2 + z^2}} \right) = \frac{P_l(|\cos \theta|)}{r^{l+1}} \quad \text{where } \cos \theta = \frac{z}{r}, r = \sqrt{\rho^2 + z^2} \quad (6)$$

We can prove this by induction:

- $l = 0$ , trivially true;
- $l = 1$ :

$$-\frac{d}{d|z|} \frac{1}{\sqrt{\rho^2 + z^2}} = \frac{|z|}{r^3} = \frac{P_1(|\cos \theta|)}{r^2} \quad (7)$$

- By induction, if

$$\frac{1}{(l-1)!} \left( -\frac{d}{d|z|} \right)^{l-1} \left( \frac{1}{r} \right) = \frac{P_{l-1}(|\cos \theta|)}{r^l} \quad (8)$$

Then

$$\begin{aligned}
B_l &= \frac{1}{l} \left( -\frac{d}{d|z|} \right) B_{l-1} \\
&= -\frac{1}{l} \frac{d}{d|z|} \left[ \frac{P_{l-1}(|z|/r)}{r^l} \right] \\
&= -\frac{1}{l} \left[ \frac{1}{r^l} \frac{dP_{l-1}(|z|/r)}{d|z|} + P_{l-1}(|z|/r) \frac{(-l)}{r^{l+1}} \frac{dr}{d|z|} \right] \\
&= -\frac{1}{l} \left[ \frac{1}{r^l} P'_{l-1}(|z|/r) \frac{d(|z|/r)}{d|z|} - l P_{l-1}(|z|/r) \frac{1}{r^{l+1}} \frac{|z|}{r} \right] \\
&= -\frac{1}{l} \left[ \frac{P'_{l-1}(|z|/r)}{r^l} \left( \frac{1}{r} - \frac{|z|}{r^2} \cdot \frac{|z|}{r} \right) - \frac{l P_{l-1}(|z|/r) |z|/r}{r^{l+1}} \right] \\
&= -\frac{1}{r^{l+1}} \left[ \frac{1}{l} P'_{l-1}(|z|/r) \left( 1 - \frac{|z|^2}{r^2} \right) - P_{l-1}(|z|/r) |z|/r \right] \\
&= \frac{P_l(|z|/r)}{r^{l+1}}
\end{aligned} \tag{9}$$

where in the last step, we have used the recurrence relation of Legendre polynomials (reference: [equation 14.10.4 on dlmf.nist.gov](#))

$$\frac{1}{l} P'_{l-1}(x) (1-x^2) - P_{l-1}(x)x = -P_l(x) \tag{10}$$

### 3. Verification that (3.180) is the solution of (3.179) via Weber–Schafheitlin integral

For the dual integral equations (3.179)

$$\int_0^\infty dy y g(y) J_n(yx) = x^n \quad \text{for } 0 \leq x < 1 \tag{11}$$

$$\int_0^\infty dy g(y) J_n(yx) = 0 \quad \text{for } 1 \leq x \tag{12}$$

It was claimed that (3.180) is a solution

$$g(y) = \frac{\Gamma(n+1)}{\sqrt{\pi}\Gamma(n+3/2)} j_{n+1}(y) = \frac{\Gamma(n+1)}{\Gamma(n+3/2)} \frac{J_{n+3/2}(y)}{\sqrt{2y}} \tag{13}$$

Let's verify it via the Weber–Schafheitlin integral (reference [equation 10.22.56 on dlmf.nist.gov](#))

for  $0 < a < b, \operatorname{Re}(\mu + \nu + 1) > \operatorname{Re}(\lambda) > -1$  :

$$\int_0^\infty \frac{J_\mu(at) J_\nu(bt)}{t^\lambda} dt = \frac{a^\mu \Gamma\left(\frac{\nu}{2} + \frac{\mu}{2} - \frac{\lambda}{2} + \frac{1}{2}\right)}{2^\lambda b^{\mu-\lambda+1} \Gamma\left(\frac{\nu}{2} - \frac{\mu}{2} + \frac{\lambda}{2} + \frac{1}{2}\right)} F\left(\frac{\mu + \nu - \lambda + 1}{2}, \frac{\mu - \nu - \lambda + 1}{2}; \mu + 1; \frac{a^2}{b^2}\right) \tag{14}$$

where **F** is the Olver's hypergeometric function

$$F(a, b; c; z) = \frac{1}{\Gamma(c)} F(a, b; c; z) = \frac{1}{\Gamma(c)} \left[ 1 + \frac{ab}{c} z + \frac{1}{2!} \frac{a(a+1)b(b+1)}{c(c+1)} z^2 + \dots \right] \tag{15}$$

- To see (11), insert  $a = x, b = 1, \mu = n, \nu = n + 3/2, \lambda = -1/2$  into (14), we have

$$\int_0^\infty \sqrt{y} J_n(xy) J_{n+3/2}(y) dy = \frac{\sqrt{2} x^n \Gamma(n+3/2)}{\Gamma(1)} \frac{1}{\Gamma(n+1)} \overbrace{F\left(n + \frac{3}{2}, 0, n+1; \frac{a^2}{b^2}\right)}^{=1} \tag{16}$$

Thus we see that (13) satisfies (11).

- For (12), we need  $a = 1, b = x, \mu = n + 3/2, \nu = n, \lambda = 1/2$ , with which the denominator of (14) has  $\Gamma(0) = \infty$ , hence (14) vanishes, which satisfies (12).

#### 4. Steps leading from (3.184) to (3.185)

We are now going to calculate the integration (3.184):

$$\Phi^{(1)}(\rho, z) = \frac{(E_0 - E_1)a^2}{\pi} \underbrace{\int_0^\infty dk j_1(ka) e^{-k|z|} J_0(k\rho)}_I \quad (17)$$

that leads to the claimed result

$$\begin{aligned} \Phi^{(1)}(\rho, z) &= \frac{(E_0 - E_1)a}{\pi} \left[ \sqrt{\frac{R - \lambda}{2}} - \frac{|z|}{a} \tan^{-1} \left( \sqrt{\frac{2}{R + \lambda}} \right) \right] \quad \text{where} \\ \lambda &= \frac{z^2 + \rho^2 - a^2}{a^2} \quad R = \sqrt{\lambda^2 + \frac{4z^2}{a^2}} \end{aligned} \quad (18)$$

First, the relation

$$j_1(x) = -j'_0(x) \quad (19)$$

will turn the integral  $I$  in (17) into

$$\begin{aligned} I &= \int_0^\infty dk [-j'_0(ka)] e^{-k|z|} J_0(k\rho) \\ &= \int_0^\infty dk \left[ -\frac{dj_0(ka)}{adk} \right] e^{-k|z|} J_0(k\rho) \\ &= -\frac{1}{a} j_0(ka) e^{-k|z|} J_0(k\rho) \Big|_0^\infty + \frac{1}{a} \int_0^\infty dk j_0(ka) \frac{d}{dk} [e^{-k|z|} J_0(k\rho)] \\ &= \frac{1}{a} \left[ 1 + \int_0^\infty j_0(ka) (-|z|) e^{-k|z|} J_0(k\rho) dk + \int_0^\infty j_0(ka) e^{-k|z|} J'_0(k\rho) \rho dk \right] \\ &= \frac{1}{a} \left[ 1 - |z| \underbrace{\int_0^\infty j_0(ka) e^{-k|z|} J_0(k\rho) dk}_{K_0} - \rho \underbrace{\int_0^\infty j_0(ka) e^{-k|z|} J_1(k\rho) dk}_{K_1} \right] \end{aligned} \quad (20)$$

where in the last step, we have used

$$J'_0(x) = -J_1(x) \quad (21)$$

##### (a) Calculation of $K_1$

Since  $j_0(ka) = \sin ka / (ka)$ , then

$$\begin{aligned} K_1 &= \int_0^\infty \frac{\sin ka}{ka} e^{-k|z|} J_1(k\rho) dk \\ &= \frac{1}{a} \operatorname{Im} \left[ \int_0^\infty e^{ika} e^{-k|z|} \frac{J_1(k\rho)}{k} dk \right] \quad \text{define } s \equiv |z| - ia \\ &= \frac{1}{a} \operatorname{Im} \left[ \int_0^\infty e^{-sk} \frac{J_1(k\rho)}{k} dk \right] \end{aligned} \quad (22)$$

The content of the square bracket is the Laplace transform of  $J_1(k\rho)/k$ , denoted  $\mathcal{L}\{J_1(k\rho)/k\}(s)$ . But before calculating it, we have to take a detour.

It is clear that the proof that established (2) can be used to produce

$$\mathcal{L}\{J_0(k\rho)\}(s) = \frac{1}{\sqrt{\rho^2 + s^2}} \quad (23)$$

With (21) and the Laplace transform property

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0^-) \quad (24)$$

we have

$$\mathcal{L}\{J_1(k\rho)\}(s) = \mathcal{L}\{-J'_0(k\rho)\}(s) = \mathcal{L}\left\{-\frac{dJ_0(k\rho)}{\rho dk}\right\}(s) = -\frac{1}{\rho}\left(\frac{s}{\sqrt{s^2+\rho^2}} - 1\right) = \frac{1}{\rho}\left(1 - \frac{s}{\sqrt{\rho^2+s^2}}\right) \quad (25)$$

Note the recurrence relation of Bessel functions (reference [equation 10.6.2 on dlmf.nist.gov](http://dlmf.nist.gov/10.6.2))

$$J'_\nu(x) = J_{\nu-1}(x) - \frac{\nu}{x}J_\nu(x) \quad (26)$$

Apply the Laplace transform to (26) with  $\nu = 1$ ,

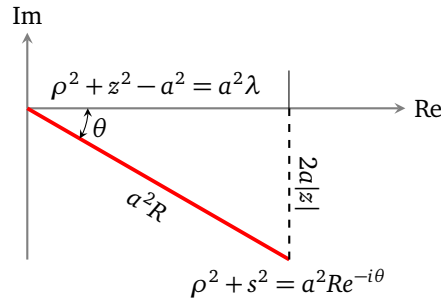
$$\begin{aligned} \mathcal{L}\left\{\frac{J_1(k\rho)}{k\rho}\right\}(s) &= \mathcal{L}\{J_0(k\rho)\}(s) - \mathcal{L}\{J'_1(k\rho)\}(s) \\ &= \mathcal{L}\{J_0(k\rho)\}(s) - \mathcal{L}\left\{\frac{dJ_1(k\rho)}{\rho dk}\right\}(s) && \text{by (23),(24),(25)} \\ &= \frac{1}{\sqrt{\rho^2+s^2}} - \frac{1}{\rho}\left[\frac{s}{\rho}\left(1 - \frac{s}{\sqrt{\rho^2+s^2}}\right)\right] \\ &= \frac{\rho^2 - s\sqrt{\rho^2+s^2} + s^2}{\rho^2\sqrt{\rho^2+s^2}} = \frac{1}{\rho^2}(\sqrt{\rho^2+s^2} - s) \quad \Rightarrow \\ \mathcal{L}\left\{\frac{J_1(k\rho)}{k}\right\}(s) &= \frac{1}{\rho}(\sqrt{\rho^2+s^2} - s) \end{aligned} \quad (27)$$

In order to get  $K_1$  in (22), it remains to extract the imaginary part of (27). Apparently

$$\text{Im}\left[\frac{1}{\rho}(\sqrt{\rho^2+s^2} - s)\right] = \frac{1}{\rho}(\text{Im}\sqrt{\rho^2+s^2} - \text{Im}s) = \frac{1}{\rho}(\text{Im}\sqrt{\rho^2+s^2} + a) \quad (28)$$

Now using the diagram below, we obtain

$$\sqrt{\rho^2+s^2} = (\rho^2 + z^2 - a^2 - 2a|z|i)^{1/2} = a\sqrt{R}e^{-i\theta/2} \quad (29)$$



hence

$$\text{Im}\sqrt{\rho^2+s^2} = -a\sqrt{R}\sin\frac{\theta}{2} \quad (30)$$

$\sin\theta/2$  can be found via

$$1 - 2\sin^2\frac{\theta}{2} = \cos\theta = \frac{\lambda}{R} \quad \Rightarrow \quad \sin\frac{\theta}{2} = \sqrt{\frac{R-\lambda}{2R}} \quad (31)$$

Inserting (30), (31) into (28) gives

$$\text{Im}\mathcal{L}\left\{\frac{J_1(k\rho)}{k}\right\}(s) = \frac{a}{\rho}\left(1 - \sqrt{\frac{R-\lambda}{2}}\right) \quad (32)$$

which reduces (20) into

$$I = \frac{1}{a}\left(\sqrt{\frac{R-\lambda}{2}} - |z|K_0\right) \quad (33)$$

Compare with the desired result (18), it remains to show

$$K_0 = \int_0^\infty \frac{\sin ka}{ka} e^{-k|z|} J_0(k\rho) dk = \frac{1}{a} \tan^{-1} \left( \sqrt{\frac{2}{R+\lambda}} \right) \quad \text{or}$$

$$\int_0^\infty \frac{\sin ka}{k} e^{-k|z|} J_0(k\rho) dk_{F(|z|)} = \tan^{-1} \left( \sqrt{\frac{2}{R+\lambda}} \right) \quad (34)$$

(b) **Calculation of  $K_0$**

Similar to the  $K_1$  case, define

$$F(s) \equiv \int_0^\infty \frac{e^{-sk}}{k} J_0(k\rho) dk, \quad s = |z| - ia \quad (35)$$

we just need to show

$$\text{Im}[F(s)] = \tan^{-1} \left( \sqrt{\frac{2}{R+\lambda}} \right) \quad (36)$$

Notice by (23)

$$F'(s) = - \int_0^\infty e^{-sk} J_0(k\rho) dk = - \frac{1}{\sqrt{\rho^2 + s^2}} \quad (37)$$

Thus  $F(s)$  is readily solvable as (reference [WolframAlpha](#))

$$\begin{aligned} F(s) &= -\tanh^{-1} \left( \frac{s}{\sqrt{\rho^2 + s^2}} \right) \\ &= \frac{1}{2} \ln \left( 1 - \frac{s}{\sqrt{\rho^2 + s^2}} \right) - \frac{1}{2} \ln \left( 1 + \frac{s}{\sqrt{\rho^2 + s^2}} \right) \\ &= \frac{1}{2} \ln \left( \frac{\sqrt{\rho^2 + s^2} - s}{\sqrt{\rho^2 + s^2} + s} \right) \\ &= \frac{1}{2} \ln \left[ \frac{(\sqrt{\rho^2 + s^2} - s)^2}{\rho^2} \right] \\ &= \ln \left( \frac{\sqrt{\rho^2 + s^2} - s}{\rho} \right) \end{aligned} \quad (38)$$

The imaginary part of  $F(s)$  is just the argument of the complex number  $\sqrt{\rho^2 + s^2} - s$ , i.e.,

$$\begin{aligned} \text{Im}[F(s)] &= \text{Arg} \left( \sqrt{\rho^2 + s^2} - s \right) \quad (\text{see diagram above}) \\ &= \text{Arg} \left[ a\sqrt{R} \left( \cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \right) - |z| + ia \right] \\ &= \tan^{-1} \left( \frac{a - a\sqrt{R} \sin \frac{\theta}{2}}{a\sqrt{R} \cos \frac{\theta}{2} - |z|} \right) \end{aligned} \quad (39)$$

From (31), we have

$$\cos \frac{\theta}{2} = \sqrt{\frac{R+\lambda}{2R}} \quad (40)$$

hence

$$\text{Im}[F(s)] = \tan^{-1} \left( \frac{a - a\sqrt{R} \sqrt{\frac{R-\lambda}{2R}}}{a\sqrt{R} \sqrt{\frac{R+\lambda}{2R}} - |z|} \right) = \tan^{-1} \left( \frac{a - a\sqrt{\frac{R-\lambda}{2}}}{a\sqrt{\frac{R+\lambda}{2}} - |z|} \right) = \tan^{-1} \left( \sqrt{\frac{2}{R+\lambda}} \right) \quad (41)$$