In section 3.11, equation (3.147) is claimed without detailed explanation:

$$W\left[I_m(x), K_m(x)\right] = -\frac{1}{x} \tag{1}$$

It was briefly mentioned that this came from the asymptotic forms of $I_m(x), K_m(x)$ according to (3.102),(3.103), and the fact that the Wronskian is proportional to $1/x$ by Sturm-Liouville theory. We elaborate on this argument in these notes.

Recall that the modified Bessel functions are

$$I_\nu(x) = i^{-\nu} J_\nu(ix) \tag{2}$$

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) \tag{3}$$

where

$$H_\nu^{(1)}(y) = J_\nu(y) + iN_\nu(y) \tag{4}$$

$$N_\nu(y) = \frac{J_\nu(y)\cos\nu\pi - J_{-\nu}(y)}{\sin\nu\pi} \tag{5}$$

Let $y = ix$, then the Wronskian is

$$\begin{aligned} W\left[I_\nu(x), K_\nu(x)\right] &= I_\nu(x)K_\nu'(x) - K_\nu(x)I_\nu'(x) \\ &= \left[i^{-\nu}J_\nu(y)\right]\left[\frac{\pi}{2}i^{\nu+1}\cdot i\frac{dH_\nu^{(1)}(y)}{dy}\right] - \left[\frac{\pi}{2}i^{\nu+1}H_\nu^{(1)}(y)\right]\left[i^{-\nu}\cdot i\frac{dJ_\nu(y)}{dy}\right] \\ &= -\frac{\pi}{2}\left[J_\nu\left(J_\nu' + iN_\nu'\right) - \left(J_\nu + iN_\nu\right)J_\nu'\right] \\ &= \frac{i\pi}{2}\left(J_\nu' N_\nu - J_\nu N_\nu'\right) \end{aligned} \tag{6}$$

where J_ν', N_ν' are with respect to their argument y.

Plugging (5) into (6), we get

$$\begin{aligned} W &= \frac{i\pi}{2}\left[J_\nu'\left(\frac{J_\nu\cos\nu\pi - J_{-\nu}}{\sin\nu\pi}\right) - J_\nu\left(\frac{J_\nu'\cos\nu\pi - J_{-\nu}'}{\sin\nu\pi}\right)\right] \\ &= \frac{i\pi}{2\sin\nu\pi}\left(\underbrace{J_\nu J_{-\nu}' - J_\nu' J_{-\nu}}_{U}\right) \end{aligned} \tag{7}$$

Recall that both J_ν and $J_{-\nu}$ are solutions to the order-ν Bessel equation

$$J_\nu'' + \frac{1}{y}J_\nu' + \left(1 - \frac{\nu^2}{y^2}\right)J_\nu = 0 \tag{8}$$

$$J_{-\nu}'' + \frac{1}{y}J_{-\nu}' + \left(1 - \frac{\nu^2}{y^2}\right)J_{-\nu} = 0 \tag{9}$$

Multiplying (8) with $J_{-\nu}$ and (9) with J_ν and then subtracting the results, we get

$$\begin{aligned} J_{-\nu}J_\nu'' + \frac{1}{y}J_{-\nu}J_\nu' - J_\nu J_{-\nu}'' - \frac{1}{y}J_\nu J_{-\nu}' = 0 \quad &\Longrightarrow \quad J_{-\nu}J_\nu'' - J_\nu J_{-\nu}'' = \frac{1}{y}\left(J_\nu J_{-\nu}' - J_{-\nu}J_\nu'\right) \quad \Longrightarrow \\ -U' = \frac{U}{y} \quad &\Longrightarrow \quad U = \frac{c}{y} \end{aligned} \tag{10}$$

Plugging (10) back to (7), we have

$$W = \frac{i\pi}{2\sin\nu\pi}\frac{c}{y} = \frac{c\pi}{2\sin\nu\pi}\frac{1}{x} \tag{11}$$

which proves the claim that W is proportional to $1/x$.

Now we can use the the $\lim_{y\to 0}$ behavior of $J_\nu(y)$ to determine c, because

$$c = yU \quad \Longrightarrow \quad c = \lim_{y\to 0} y\left(J_\nu J_{-\nu}' - J_\nu' J_{-\nu}\right) \tag{12}$$

By definition

$$J_\nu(y) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+\nu+1)} \left(\frac{y}{2}\right)^{2j+\nu} \tag{13}$$

$$J_{-\nu}(y) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j-\nu+1)} \left(\frac{y}{2}\right)^{2j-\nu} \tag{14}$$

which turns the limit in (12) into

$$\begin{aligned} c &= y\left[\frac{1}{\Gamma(\nu+1)}\left(\frac{y}{2}\right)^{\nu} \cdot \frac{1}{\Gamma(1-\nu)}\left(\frac{-\nu}{2}\right)\left(\frac{y}{2}\right)^{-\nu-1} - \frac{1}{\Gamma(\nu+1)}\left(\frac{\nu}{2}\right)\left(\frac{y}{2}\right)^{\nu-1} \cdot \frac{1}{\Gamma(1-\nu)}\left(\frac{y}{2}\right)^{-\nu}\right] \\ &= \frac{-2\nu}{\Gamma(\nu+1)\Gamma(1-\nu)} \\ &= -\frac{2}{\Gamma(\nu)\Gamma(1-\nu)} = -\frac{2\sin\nu\pi}{\pi} \end{aligned} \tag{15}$$

where in the last step, we have used the Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin z\pi$.

Together with (11), this gives us the desired results (1).