

For a real-valued electric field $\mathbf{E}(\mathbf{x}, t)$ that is a free space solution of the Maxwell equation, its plane wave decomposition has the form

$$\mathbf{E}(\mathbf{x}, t) = \text{Re} \int \frac{d^3k}{(2\pi)^3} \mathbf{E}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} - i\mathbf{k}t} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} [\mathbf{E}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} - i\mathbf{k}t} + \mathbf{E}^*(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}t}] \quad (1)$$

where

$$\mathbf{E}(\mathbf{k}) = \int d^3x \mathbf{E}(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}t} \quad (2)$$

This can be verified by plugging (2) into (1) and use $\int d^3p e^{i\mathbf{p}\cdot\mathbf{q}} = (2\pi)^3 \delta(\mathbf{q})$.

Same applies for $\mathbf{B}(\mathbf{x}, t)$, i.e.,

$$\mathbf{B}(\mathbf{x}, t) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} [\mathbf{B}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} - i\mathbf{k}t} + \mathbf{B}^*(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}t}] \quad \text{where} \quad \mathbf{B}(\mathbf{k}) = \int d^3x \mathbf{B}(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}t} \quad (3)$$

The $\mathbf{E}(\mathbf{k}), \mathbf{B}(\mathbf{k})$ amplitudes satisfy the usual orthogonality relation

$$\mathbf{B}(\mathbf{k}) = \frac{\hat{\mathbf{k}} \times \mathbf{E}(\mathbf{k})}{c} \quad (4)$$

By (1) and (3), for $\mathbf{Y} = \mathbf{E}$ or \mathbf{B} , we have

$$\begin{aligned} \mathbf{Y}(\mathbf{x}, t) \cdot \mathbf{Y}(\mathbf{x}, t) = & \frac{1}{4} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \left[\mathbf{Y}(\mathbf{k}) \cdot \mathbf{Y}(\mathbf{k}') e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x} - i\mathbf{c}(\mathbf{k}+\mathbf{k}')t} + \right. \\ & \mathbf{Y}^*(\mathbf{k}) \cdot \mathbf{Y}^*(\mathbf{k}') e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x} + i\mathbf{c}(\mathbf{k}+\mathbf{k}')t} + \\ & \mathbf{Y}(\mathbf{k}) \cdot \mathbf{Y}^*(\mathbf{k}') e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x} - i\mathbf{c}(\mathbf{k}-\mathbf{k}')t} + \\ & \left. \mathbf{Y}^*(\mathbf{k}) \cdot \mathbf{Y}(\mathbf{k}') e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x} + i\mathbf{c}(\mathbf{k}-\mathbf{k}')t} \right] \quad (5) \end{aligned}$$

Thus at t , the instantaneous total energy of the field is

$$\begin{aligned} U(t) = & \int d^3x \left[\frac{\epsilon_0}{2} \mathbf{E}(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}, t) + \frac{1}{2\mu_0} \mathbf{B}(\mathbf{x}, t) \cdot \mathbf{B}(\mathbf{x}, t) \right] \\ = & \frac{1}{8} \int d^3x \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \left\{ \left[\epsilon_0 \mathbf{E}(\mathbf{k}) \cdot \mathbf{E}(\mathbf{k}') + \frac{\mathbf{B}(\mathbf{k}) \cdot \mathbf{B}(\mathbf{k}')}{\mu_0} \right] e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x} - i\mathbf{c}(\mathbf{k}+\mathbf{k}')t} + \right. \\ & \left[\epsilon_0 \mathbf{E}^*(\mathbf{k}) \cdot \mathbf{E}^*(\mathbf{k}') + \frac{\mathbf{B}^*(\mathbf{k}) \cdot \mathbf{B}^*(\mathbf{k}')}{\mu_0} \right] e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x} + i\mathbf{c}(\mathbf{k}+\mathbf{k}')t} + \\ & \left[\epsilon_0 \mathbf{E}(\mathbf{k}) \cdot \mathbf{E}^*(\mathbf{k}') + \frac{\mathbf{B}(\mathbf{k}) \cdot \mathbf{B}^*(\mathbf{k}')}{\mu_0} \right] e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x} - i\mathbf{c}(\mathbf{k}-\mathbf{k}')t} + \\ & \left. \left[\epsilon_0 \mathbf{E}^*(\mathbf{k}) \cdot \mathbf{E}(\mathbf{k}') + \frac{\mathbf{B}^*(\mathbf{k}) \cdot \mathbf{B}(\mathbf{k}')}{\mu_0} \right] e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x} + i\mathbf{c}(\mathbf{k}-\mathbf{k}')t} \right\} \\ = & \frac{1}{8} \int \frac{d^3k}{(2\pi)^3} \left\{ \left[\epsilon_0 \mathbf{E}(\mathbf{k}) \cdot \mathbf{E}(-\mathbf{k}) + \frac{\mathbf{B}(\mathbf{k}) \cdot \mathbf{B}(-\mathbf{k})}{\mu_0} \right] + \left[\epsilon_0 \mathbf{E}^*(\mathbf{k}) \cdot \mathbf{E}^*(-\mathbf{k}) + \frac{\mathbf{B}^*(\mathbf{k}) \cdot \mathbf{B}^*(-\mathbf{k})}{\mu_0} \right] + \right. \\ & \left. 2 \left[\epsilon_0 \mathbf{E}(\mathbf{k}) \cdot \mathbf{E}^*(\mathbf{k}) + \frac{\mathbf{B}(\mathbf{k}) \cdot \mathbf{B}^*(\mathbf{k})}{\mu_0} \right] \right\} \quad (6) \end{aligned}$$

If we consider the vector potential of the same form

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} [\mathbf{A}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} - ickt} + \mathbf{A}^*(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x} + ickt}] \quad (7)$$

and the relations $\mathbf{E} = -\partial\mathbf{A}/\partial t$, $\mathbf{B} = \nabla \times \mathbf{A}$, we have

$$\mathbf{E}(\mathbf{k}) = ick\mathbf{A}(\mathbf{k}) \implies \mathbf{E}(\mathbf{k}) \cdot \mathbf{E}(-\mathbf{k}) = [ick\mathbf{A}(\mathbf{k})] \cdot [ick\mathbf{A}(-\mathbf{k})] = -c^2k^2\mathbf{A}(\mathbf{k}) \cdot \mathbf{A}(-\mathbf{k}) \quad (8)$$

$$\mathbf{B}(\mathbf{k}) = i\mathbf{k} \times \mathbf{A}(\mathbf{k}) \implies \mathbf{B}(\mathbf{k}) \cdot \mathbf{B}(-\mathbf{k}) = [i\mathbf{k} \times \mathbf{A}(\mathbf{k})] \cdot [-i\mathbf{k} \times \mathbf{A}(-\mathbf{k})] = k^2\mathbf{A}(\mathbf{k}) \cdot \mathbf{A}(-\mathbf{k}) \quad (9)$$

We see that (8) and (9), and their complex conjugates, render the first two brackets of (6) zero, leaving

$$U(t) = \frac{\epsilon_0}{4} \int \frac{d^3k}{(2\pi)^3} [\mathbf{E}(\mathbf{k}) \cdot \mathbf{E}^*(\mathbf{k}) + c^2\mathbf{B}(\mathbf{k}) \cdot \mathbf{B}^*(\mathbf{k})] \quad (10)$$

The total number of photons can be obtained by dividing the integrand of (10) by $\hbar ck$ before the integration, i.e.,

$$\begin{aligned} N(t) &= \frac{\epsilon_0}{4} \int \frac{d^3k}{(2\pi)^3} \left[\frac{\mathbf{E}(\mathbf{k}) \cdot \mathbf{E}^*(\mathbf{k}) + c^2\mathbf{B}(\mathbf{k}) \cdot \mathbf{B}^*(\mathbf{k})}{\hbar ck} \right] \quad \text{by (2)} \\ &= \frac{\epsilon_0}{4\hbar c} \frac{1}{(2\pi)^3} \int \frac{d^3k}{k} \int d^3x \int d^3x' [\mathbf{E}(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}', t) + c^2\mathbf{B}(\mathbf{x}, t) \cdot \mathbf{B}(\mathbf{x}', t)] e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \\ &= \frac{\epsilon_0}{4\hbar c} \frac{1}{(2\pi)^3} \int d^3x \int d^3x' [\mathbf{E}(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}', t) + c^2\mathbf{B}(\mathbf{x}, t) \cdot \mathbf{B}(\mathbf{x}', t)] \underbrace{\int d^3k \left[\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k} \right]}_{\frac{4\pi}{|\mathbf{x}-\mathbf{x}'|^2}} \\ &= \frac{\epsilon_0}{8\pi^2\hbar c} \int d^3x \int d^3x' \left[\frac{\mathbf{E}(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}', t) + c^2\mathbf{B}(\mathbf{x}, t) \cdot \mathbf{B}(\mathbf{x}', t)}{|\mathbf{x}-\mathbf{x}'|^2} \right] \end{aligned} \quad (11)$$

which has an additional 1/2 factor compared to the claim given by Jackson.

Note, the inner-most integral of (11) is calculated via

$$\begin{aligned} \int d^3k \left[\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k} \right] &= \int_0^\infty k dk \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi e^{-ik|\mathbf{x}-\mathbf{x}'|\cos\theta} \\ &= 2\pi \int_0^\infty k dk \left(\frac{e^{ik|\mathbf{x}-\mathbf{x}'|} - e^{-ik|\mathbf{x}-\mathbf{x}'|}}{ik|\mathbf{x}-\mathbf{x}'|} \right) \\ &= \frac{4\pi}{|\mathbf{x}-\mathbf{x}'|} \text{Im} \int_0^\infty e^{ik|\mathbf{x}-\mathbf{x}'|} dk \\ &= \frac{4\pi}{|\mathbf{x}-\mathbf{x}'|} \lim_{\mu \rightarrow 0} \left(\text{Im} \int_0^\infty e^{ik|\mathbf{x}-\mathbf{x}'|} e^{-\mu k} dk \right) \\ &= \frac{4\pi}{|\mathbf{x}-\mathbf{x}'|} \lim_{\mu \rightarrow 0} \left[\text{Im} \left(\frac{1}{\mu - i|\mathbf{x}-\mathbf{x}'|} \right) \right] \\ &= \frac{4\pi}{|\mathbf{x}-\mathbf{x}'|} \lim_{\mu \rightarrow 0} \left(\frac{|\mathbf{x}-\mathbf{x}'|}{\mu^2 + |\mathbf{x}-\mathbf{x}'|^2} \right) \\ &= \frac{4\pi}{|\mathbf{x}-\mathbf{x}'|^2} \end{aligned} \quad (12)$$