1. Let the space Fourier transform and inverse transform for $\mathbf{A}(\mathbf{x},t)$ be

$$\widetilde{\mathbf{A}}(\mathbf{k},t)=\frac{1}{(2\pi)^{3/2}}\int d^3x\mathbf{A}(\mathbf{x},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{1}$$

$$\mathbf{A}(\mathbf{x},t)=\frac{1}{(2\pi)^{3/2}}\int d^3k\widetilde{\mathbf{A}}(\mathbf{k},t)\,e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2}$$

Applying the diffusion equation

$$\nabla^2\mathbf{A}(\mathbf{x},t)-\mu\sigma\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t}=0 \tag{3}$$

to (2) gives

$$\int d^3k\left(k^2+\mu\sigma\frac{\partial}{\partial t}\right)\widetilde{\mathbf{A}}(\mathbf{k},t)\,e^{i\mathbf{k}\cdot\mathbf{x}}=0 \tag{4}$$

Integrating (4) with $e^{-i\mathbf{k}'\cdot\mathbf{x}}d^3x$ yields

$$\int d^3k\left(k^2+\mu\sigma\frac{\partial}{\partial t}\right)\widetilde{\mathbf{A}}(\mathbf{k},t)\underbrace{\int d^3x e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}}_{\propto\delta(\mathbf{k}-\mathbf{k}')}=0 \tag{5}$$

From which we see for any $\mathbf{k}$,

$$\left(k^2+\mu\sigma\frac{\partial}{\partial t}\right)\widetilde{\mathbf{A}}(\mathbf{k},t)=0 \qquad\Longrightarrow\qquad \widetilde{\mathbf{A}}(\mathbf{k},t)=\widetilde{\mathbf{A}}(\mathbf{k},0)\,e^{-k^2t/\mu\sigma} \tag{6}$$

where

$$\widetilde{\mathbf{A}}(\mathbf{k},0)=\frac{1}{(2\pi)^{3/2}}\int d^3x'\mathbf{A}\left(\mathbf{x}',0\right)e^{-i\mathbf{k}\cdot\mathbf{x}'} \tag{7}$$

With (6) and (7) plugged back into (2), we get

$$\begin{aligned}\mathbf{A}(\mathbf{x},t)&=\frac{1}{(2\pi)^3}\int d^3ke^{-k^2t/\mu\sigma}\left[\int d^3x'\mathbf{A}\left(\mathbf{x}',0\right)e^{-i\mathbf{k}\cdot\mathbf{x}'}\right]e^{i\mathbf{k}\cdot\mathbf{x}}\\&=\int d^3x'\left[\frac{1}{(2\pi)^3}\int d^3ke^{-k^2t/\mu\sigma}e^{i\mathbf{k}\cdot\left(\mathbf{x}-\mathbf{x}'\right)}\right]\mathbf{A}\left(\mathbf{x}',0\right)\\&=\int d^3x'G\left(\mathbf{x}-\mathbf{x}',t\right)\mathbf{A}\left(\mathbf{x}',0\right)\end{aligned} \tag{8}$$

where we have identified the content in the bracket as the Green function

$$G\left(\mathbf{x}-\mathbf{x}',t\right)=\frac{1}{(2\pi)^3}\int d^3ke^{-k^2t/\mu\sigma}e^{i\mathbf{k}\cdot\left(\mathbf{x}-\mathbf{x}'\right)} \qquad \text{for } t>0 \tag{9}$$

2. Let $\widetilde{G}(\mathbf{k},\omega)$ be the Fourier transform in both space and time for $G(\mathbf{x}-\mathbf{x}',t)$, i.e.,

$$G\left(\mathbf{x}-\mathbf{x}',t\right)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}d\omega\int d^3k\widetilde{G}(\mathbf{k},\omega)\,e^{i\mathbf{k}\cdot\left(\mathbf{x}-\mathbf{x}'\right)}e^{i\omega t} \tag{10}$$

Equating (10) with (9) gives

$$\int_{-\infty}^{\infty}d\omega\widetilde{G}(\mathbf{k},\omega)\,e^{i\omega t}=\frac{1}{2\pi}e^{-k^2t/\mu\sigma}\cdot\Theta(t) \tag{11}$$

Integrating both sides of (11) with $e^{-i\omega' t}dt$, we have

$$\int_{-\infty}^{\infty} d\omega \widetilde{G}(\mathbf{k},\omega) \overbrace{\int_{-\infty}^{\infty} e^{i(\omega-\omega')t} dt}^{2\pi\delta(\omega-\omega')} = \frac{1}{2\pi}\int_0^{\infty} e^{-k^2 t/\mu\sigma} e^{-i\omega' t} dt \qquad \Longrightarrow \qquad \widetilde{G}(\mathbf{k},\omega') = \frac{1}{(2\pi)^2}\frac{1}{\dfrac{k^2}{\mu\sigma}+i\omega'} \tag{12}$$

Then applying $\partial/\partial t - (1/\mu\sigma)\nabla^2$ to (10) yields

$$\begin{aligned}
\frac{\partial G}{\partial t} - \frac{1}{\mu\sigma}\nabla^2 G &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega \int d^3k \left(i\omega + \frac{k^2}{\mu\sigma}\right)\widetilde{G}(\mathbf{k},\omega)\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{i\omega t} \qquad \text{use (12)} \\
&= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} d\omega\right)\left[\frac{1}{(2\pi)^3}\int d^3k e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\right] \\
&= \delta(\mathbf{x}-\mathbf{x}')\,\delta(t)
\end{aligned} \tag{13}$$

3. Let's write the d^3k integral of (9) in spherical coordinates. For $t>0$,

$$\begin{aligned}
G(\mathbf{x}-\mathbf{x}',t) &= \frac{1}{(2\pi)^3}\int d^3k e^{-k^2 t/\mu\sigma} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \\
&= \frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\infty} dk\cdot k^2 e^{-k^2 t/\mu\sigma}\int_0^{\pi}\sin\theta\, d\theta\, e^{ik|\mathbf{x}-\mathbf{x}'|\cos\theta} \\
&= \frac{1}{(2\pi)^2}\int_0^{\infty} dk\cdot k^2 e^{-k^2 t/\mu\sigma}\left(\frac{e^{ik|\mathbf{x}-\mathbf{x}'|} - e^{-ik|\mathbf{x}-\mathbf{x}'|}}{ik|\mathbf{x}-\mathbf{x}'|}\right) \\
&= \frac{1}{(2\pi)^2}(I_+ - I_-)
\end{aligned} \tag{14}$$

where if we denote $r \equiv |\mathbf{x}-\mathbf{x}'|$,

$$\begin{aligned}
I_\pm &= \frac{1}{ir}\int_0^{\infty} dk\cdot k\exp\left(-\frac{k^2 t}{\mu\sigma} \pm ikr\right) \\
&= \frac{1}{ir}\int_0^{\infty} dk\cdot k\exp\left[-\frac{t}{\mu\sigma}\left(k\mp\frac{i\mu\sigma r}{2t}\right)^2\right]\exp\left(-\frac{\mu\sigma r^2}{4t}\right) \\
&= \frac{1}{ir}\exp\left(-\frac{\mu\sigma r^2}{4t}\right)\left\{\int_0^{\infty} dk\cdot\left(k\mp\frac{i\mu\sigma r}{2t}\right)\exp\left[-\frac{t}{\mu\sigma}\left(k\mp\frac{i\mu\sigma r}{2t}\right)^2\right]\pm\right. \\
&\qquad\qquad \left.\int_0^{\infty} dk\left(\frac{i\mu\sigma r}{2t}\right)\exp\left[-\frac{t}{\mu\sigma}\left(k\mp\frac{i\mu\sigma r}{2t}\right)^2\right]\right\}
\end{aligned} \tag{15}$$

The first integral in (15) is elementary and is the same for I_+ and I_- hence they cancel in (14), and the second integral can be found via the standard form Gaussian integral

$$\int_{-\infty}^{\infty} e^{-p(x+c)^2}dx = \sqrt{\frac{\pi}{p}} \qquad \text{for } p,c\in\mathbb{C}, \operatorname{Re}p>0 \tag{16}$$

Putting everything together, including the $\Theta(t)$ factor to accommodate the $t<0$ case, we have

$$\begin{aligned}
G(\mathbf{x}-\mathbf{x}',t) &= \Theta(t)\cdot\frac{1}{(2\pi)^2}2\cdot\frac{1}{ir}\exp\left(-\frac{\mu\sigma r^2}{4t}\right)\frac{1}{2}\left(\frac{i\mu\sigma r}{2t}\right)\sqrt{\frac{\pi\mu\sigma}{t}} \\
&= \Theta(t)\cdot\left(\frac{\mu\sigma}{4\pi t}\right)^{3/2}\exp\left(-\frac{\mu\sigma|\mathbf{x}-\mathbf{x}'|^2}{4t}\right)
\end{aligned} \tag{17}$$

4. Let's analyze this part with an extremely localized initial vector potential $\mathbf{A}(\mathbf{x}',0)=\mathbf{A}_0\delta(\mathbf{x}')$. Then by (17) and (8), the vector potential of the remote observation point is

$$\mathbf{A}(\mathbf{x},t) = \int d^3x'\left(\frac{\mu\sigma}{4\pi t}\right)^{3/2}\exp\left(-\frac{\mu\sigma|\mathbf{x}-\mathbf{x}'|^2}{4t}\right)\mathbf{A}_0\delta(\mathbf{x}') = \mathbf{A}_0\left(\frac{\mu\sigma}{4\pi t}\right)^{3/2}\exp\left(\frac{-\mu\sigma|\mathbf{x}|^2}{4t}\right) \tag{18}$$

When t is close to zero, $\mathbf{A}(\mathbf{x})$ is close to zero because exponential drop is faster than any power of t. This corresponds to the phase where the initial potential has not propagated to the remote observation point.

Define $\alpha = 4t/\mu\sigma$, then it is an easy calculation that when

$$\alpha = \frac{2|\mathbf{x}|^2}{3} \tag{19}$$

the magnitude of $\mathbf{A}(\mathbf{x})$ reaches maximum. This corresponds to the phase where the "peak" of the initial local potential is propagating through $\mathbf{x}$.

When $t \to \infty$, the magnitude of the potential $\mathbf{A}(\mathbf{x})$ drops as $t^{-3/2}$.

The plot of (18) is shown below.