In section 12.10, Jackson uses free field Lagrangian density to define the canonical stress tensor $T^{\alpha\beta}$ and proves its conservation law $\partial_\alpha T^{\alpha\beta} = 0$. This agrees with the source-free Poynting theorem (6.108). In these notes, we give a detailed derivation for the general scenario where there exists external current J^λ.

Recall that the full Lagrangian density with current coupling is given in (12.85)

$$\mathscr{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\lambda A^\lambda \tag{1}$$

The important difference from the free field Lagrangian density is that $\mathscr{L}$'s dependence on the coordinates x^α is now through one more explicit variable J_λ.

From this, the canonical stress tensor is defined similarly following (12.102) and (12.103)

$$T^{\alpha\beta} = \frac{\partial \mathscr{L}}{\partial\left(\partial_\alpha A^\lambda\right)} \partial^\beta A^\lambda - g^{\alpha\beta} \mathscr{L} \tag{2}$$

It is clear that the additional $-J_\lambda A^\lambda / c$ term in (1) does not contribute to the partial derivative in (2), so (12.104) still applies (with $\mathscr{L}_{\text{em}}$ replaced by $\mathscr{L}$):

$$\begin{aligned} T^{\alpha\beta} &= -\frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} \partial^\beta A^\lambda - g^{\alpha\beta} \mathscr{L} \\ &= -\frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} \partial^\beta A^\lambda - g^{\alpha\beta} \mathscr{L}_{\text{em}} + \frac{1}{c} g^{\alpha\beta} J_\lambda A^\lambda \end{aligned} \tag{3}$$

In proving (12.107), Jackson uses the general Lagrangian density $\mathscr{L}\left(\phi_k, \partial^\alpha \phi_k\right)$, so the derivation is valid until the second equation after (12.107)

$$\partial_\alpha T^{\alpha\beta} = \overbrace{\sum_k \left[\frac{\partial \mathscr{L}}{\partial \phi_k} \partial^\beta \phi_k + \frac{\partial \mathscr{L}}{\partial\left(\partial_\alpha \phi_k\right)} \partial^\beta\left(\partial_\alpha \phi_k\right)\right]}^{X} - \partial^\beta \mathscr{L} \tag{4}$$

But when $\mathscr{L}$ now includes the external current term $-J_\lambda A^\lambda / c$, the derivative of $\mathscr{L}$ with respect to x_β is not fully described by X above, but is instead

$$\partial^\beta \mathscr{L} = X + \frac{\partial \mathscr{L}}{\partial J_\lambda} \partial^\beta J_\lambda = X - \frac{1}{c} A^\lambda \partial^\beta J_\lambda \tag{5}$$

generalizing (12.107) to

$$\partial_\alpha T^{\alpha\beta} = \frac{1}{c} A^\lambda \partial^\beta J_\lambda \tag{6}$$

With the additional term $g^{\alpha\beta} J_\lambda A^\lambda / c$ in (3), the identity (12.111) now reads

$$\begin{aligned} T^{\alpha\beta} &= \overbrace{\frac{1}{4\pi}\left(g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu}\right)}^{\Theta^{\alpha\beta}} - \frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} \partial^\lambda A^\beta + \frac{1}{c} g^{\alpha\beta} J_\lambda A^\lambda \\ &= \Theta^{\alpha\beta} + \underbrace{\frac{1}{4\pi} \partial_\lambda\left(F^{\lambda\alpha} A^\beta\right)}_{T_D^{\alpha\beta}} - \frac{1}{4\pi} A^\beta \partial_\lambda F^{\lambda\alpha} + \frac{1}{c} g^{\alpha\beta} J_\lambda A^\lambda \end{aligned} \tag{7}$$

However here, we cannot drop the third term since there is now source current. Instead, we can use (12.89) to get

$$T^{\alpha\beta} = \Theta^{\alpha\beta} + T_D^{\alpha\beta} - \frac{1}{c} A^\beta J^\alpha + \frac{1}{c} g^{\alpha\beta} J_\lambda A^\lambda \tag{8}$$

Putting (6) and (8) together and using the fact $\partial_\alpha T_D^{\alpha\beta} = 0$, we have

$$\begin{aligned} \frac{1}{c} A^\lambda \partial^\beta J_\lambda = \partial_\alpha T^{\alpha\beta} &= \partial_\alpha \Theta^{\alpha\beta} - \frac{1}{c} \partial_\alpha\left(A^\beta J^\alpha\right) + \frac{1}{c} \partial^\beta\left(J_\lambda A^\lambda\right) \qquad \Longrightarrow \\ \partial_\alpha \Theta^{\alpha\beta} &= \frac{1}{c}\left(A^\lambda \partial^\beta J_\lambda + J^\alpha \partial_\alpha A^\beta - A^\lambda \partial^\beta J_\lambda - J_\lambda \partial^\beta A^\lambda\right) \\ &= \frac{1}{c}\left(J_\lambda \partial^\lambda A^\beta - J_\lambda \partial^\beta A^\lambda\right) = -\frac{1}{c} J_\lambda F^{\beta\lambda} \end{aligned} \tag{9}$$

which agrees with (12.118), whose component form is Poynting theorem (6.108).

Note that only the conservation law of *symmetrized* $\Theta^{\alpha\beta}$ will recover Poynting theorem that has source current. For the non-symmetrized $T^{\alpha\beta}$, (6) is the general "conservation law", whose component form is not identical to (6.108).