1. **Prob 5.26**

Consider the vector potential $\mathbf{A} = A\hat{\mathbf{z}}$ generated from one wire of radius R. For $\rho > R$, the magnetic induction is given by

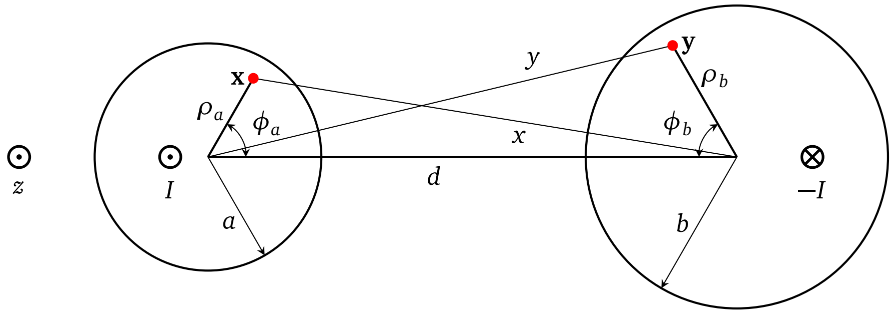
$$B_{\phi,\text{out}} = \frac{\mu_0 I}{2\pi\rho} = -\frac{\partial A_{\text{out}}}{\partial \rho} \qquad \Longrightarrow \qquad A_{\text{out}} = -\frac{\mu_0 I}{2\pi}(\ln\rho + C) \tag{1}$$

For $\rho < a$:

$$B_{\phi,\text{in}} = \frac{\mu_0 I\rho}{2\pi R^2} = -\frac{\partial A_{\text{in}}}{\partial \rho} \qquad \Longrightarrow \qquad A_{\text{in}} = -\frac{\mu_0 I}{4\pi}\left(\frac{\rho}{R}\right)^2 \tag{2}$$

If we demand A to be continuous at $\rho = R$, we must have

$$A_{\text{out}} = -\frac{\mu_0 I}{4\pi}\left[\ln\left(\frac{\rho}{R}\right)^2 + 1\right] \tag{3}$$



From the diagram above, it's clear that the vector potentials at point $\mathbf{x}, \mathbf{y}$ are

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \left\{-\frac{\mu_0 I}{4\pi}\left(\frac{\rho_a}{a}\right)^2 - \frac{\mu_0(-I)}{4\pi}\left[\ln\left(\frac{x}{b}\right)^2 + 1\right]\right\}\hat{\mathbf{z}} \\
&= -\frac{\mu_0 I}{4\pi}\left[\left(\frac{\rho_a}{a}\right)^2 - \ln\left(\frac{x}{b}\right)^2 - 1\right]\hat{\mathbf{z}}
\end{aligned} \tag{4}$$

Similarly

$$\begin{aligned}
\mathbf{A}(\mathbf{y}) &= \left\{-\frac{\mu_0(-I)}{4\pi}\left(\frac{\rho_b}{b}\right)^2 - \frac{\mu_0 I}{4\pi}\left[\ln\left(\frac{y}{a}\right)^2 + 1\right]\right\}\hat{\mathbf{z}} \\
&= \frac{\mu_0 I}{4\pi}\left[\left(\frac{\rho_b}{b}\right)^2 - \ln\left(\frac{y}{a}\right)^2 - 1\right]\hat{\mathbf{z}}
\end{aligned} \tag{5}$$

Therefore

$$\begin{aligned}
L = \frac{2W}{I^2} = \frac{1}{I^2}\int \mathbf{J}\cdot\mathbf{A}d^3x = -\frac{\mu_0}{4\pi}\Bigg\{ &\frac{1}{\pi a^2}\int_{\text{wire-}a}\left[\left(\frac{\rho_a}{a}\right)^2 - \ln\left(\frac{x}{b}\right)^2 - 1\right]d^3x + \\
&\frac{1}{\pi b^2}\int_{\text{wire-}b}\left[\left(\frac{\rho_b}{b}\right)^2 - \ln\left(\frac{y}{a}\right)^2 - 1\right]d^3x\Bigg\}
\end{aligned} \tag{6}$$

where per unit length in the z direction,

$$\begin{aligned}
\int_{\text{wire-}a}\left[\left(\frac{\rho_a}{a}\right)^2 - \ln\left(\frac{x}{b}\right)^2 - 1\right]d^3x &= \int_0^a \rho_a d\rho_a \int_0^{2\pi} d\phi_a\left[\left(\frac{\rho_a}{a}\right)^2 - \ln\left(\frac{x}{b}\right)^2 - 1\right] \\
&= 2\pi\int_0^a \rho_a d\rho_a\left[\left(\frac{\rho_a}{a}\right)^2 - 1\right] - \int_0^a \rho_a d\rho_a \int_0^{2\pi} d\phi_a \ln\left(\frac{x}{b}\right)^2 \\
&= 2\pi\left(-\frac{a^2}{4}\right) - \int_0^a \rho_a d\rho_a \underbrace{\int_0^{2\pi} d\phi_a \ln\left(\frac{x}{b}\right)^2}_{X}
\end{aligned} \tag{7}$$

Consider the complex function

$$f(z) = f\left(\rho_a e^{i\phi_a}\right) = \ln\left(\frac{x}{b}\right)^2 = \ln\left(\frac{\rho_a^2 + d^2 - 2\rho_a d\cos\phi_a}{b^2}\right) \tag{8}$$

Since it is the vector potential (3) for the part of space without current distribution, it must be harmonic (i.e., its Laplacian vanishes). Then by the mean value theorem, its mean value along a circle is its value at the center of circle, i.e.,

$$X = \int_0^{2\pi} d\phi_a \ln\left(\frac{x}{b}\right)^2 = 2\pi\ln\left(\frac{d}{b}\right)^2 \tag{9}$$

This reduces (7) as

$$\int_{\text{wire-}a} d^3x = -\frac{\pi a^2}{2} - \pi a^2 \ln\left(\frac{d}{b}\right)^2 \tag{10}$$

And by $a \longleftrightarrow b$ exchange,

$$\int_{\text{wire-}b} d^3x = -\frac{\pi b^2}{a} - \pi b^2 \ln\left(\frac{d}{a}\right)^2 \tag{11}$$

Finally with (10), (11) plugged back into (6), we obtain

$$L = \frac{\mu_0}{4\pi}\left[1 + 2\ln\left(\frac{d^2}{ab}\right)\right] \tag{12}$$

2. **Prob 5.27**

Let's use two methods to calculate the self-inductance.

(a) **Using vector potential A.**

When the inner wire is solid, by (3), it generates a vector potential on the outer shell

$$A_{\text{in-on-out}} = -\frac{\mu_0 I}{4\pi}\left[\ln\left(\frac{a}{b}\right)^2 + 1\right] \tag{13}$$

But the outer shell generates a vector potential on itself

$$A_{\text{out-on-out}} = -\frac{\mu_0(-I)}{4\pi} \tag{14}$$

Combining the two gives the net vector potential on the outer shell

$$A_{\text{out}} = -\frac{\mu_0 I}{4\pi}\ln\left(\frac{a}{b}\right)^2 \tag{15}$$

The outer shell contributes no **B** fields for its interior $\rho < a$, therefore its contribution of vector potential to the its region is constant, which is equal to the value at $\rho = a$ by continuity, i.e.,

$$A_{\text{out-on-in}} = \frac{\mu_0 I}{4\pi} \tag{16}$$

For the region $\rho < b$ though, the vector potential contribution from the inner wire is

$$A_{\text{in-on-in}} = -\frac{\mu_0 I}{4\pi}\left(\frac{\rho}{b}\right)^2 \tag{17}$$

Thus the vector potential on the inner wire is

$$A_{\text{in}} = -\frac{\mu_0 I}{4\pi}\left[\left(\frac{\rho}{b}\right)^2 - 1\right] \tag{18}$$

The self inductance is

$$\begin{aligned} L_{\text{solid}} = \frac{1}{I^2}\int \mathbf{J}\cdot\mathbf{A}d^3x &= \frac{1}{I^2}\left[\int_{\text{out}} J_{\text{out}}A_{\text{out}}d^3x + \int_{\text{in}} J_{\text{in}}A_{\text{in}}d^3x\right] \\ &= \frac{\mu_0}{4\pi}\ln\left(\frac{a}{b}\right)^2 + \frac{1}{\pi b^2}\left(-\frac{\mu_0}{4\pi}\right)\cdot 2\pi\int_0^b \rho d\rho\left[\left(\frac{\rho}{b}\right)^2 - 1\right] \\ &= \frac{\mu_0}{4\pi}\left[\ln\left(\frac{a}{b}\right)^2 + \frac{1}{2}\right] \end{aligned} \tag{19}$$

When inner wire is hallow, (13)-(16) still hold, but (17) becomes

$$A_{\text{in-on-in}} = -\frac{\mu_0 I}{4\pi} \tag{20}$$

which makes the combined A_{in} vanish, so only the first integral in (19) survives, which gives the self-inductance

$$L_{\text{hallow}} = \frac{\mu_0}{4\pi} \ln\left(\frac{a}{b}\right)^2 \tag{21}$$

(b) **Using magnetic induction B.**
When inner wire is solid, there is **B** field in both regions $\rho < b$ and $b < \rho < a$, but none in $\rho > a$.

$$B_{\rho<b} = \frac{\mu_0 I \rho}{2\pi b^2} \qquad\qquad B_{b<\rho<a} = \frac{\mu_0 I}{2\pi\rho} \tag{22}$$

Thus

$$\begin{aligned} L_{\text{solid}} = \frac{1}{I^2}\int \frac{B^2}{\mu_0} d^3x &= 2\pi\mu_0 \cdot \left[\int_0^b \rho\, d\rho \left(\frac{\rho}{2\pi b^2}\right)^2 + \int_b^a \rho\, d\rho \left(\frac{1}{2\pi\rho}\right)^2\right] \\ &= \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi}\ln\left(\frac{a}{b}\right) \end{aligned} \tag{23}$$

agreeing with (19).
For hollow inner wire, $B_{\rho<b} = 0$ in (22), and only the second integral in (23) contributes, thus

$$L_{\text{hallow}} = \frac{\mu_0}{2\pi} \ln\left(\frac{a}{b}\right) \tag{24}$$