1. Following the paradignm of problem 2.17, we know

$$G = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} g_m(\rho,\rho') \qquad \text{where} \tag{1}$$

$$\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m}{\partial\rho'}\right) - \frac{m^2}{\rho'^2} g_m = -4\pi\frac{\delta(\rho-\rho')}{\rho} \tag{2}$$

We will write the general solution of $g_m(\rho,\rho')$ as

$$g_m(\rho,\rho') = \begin{cases} \alpha_0 + \beta_0 \ln\rho' & m=0 \\ \alpha_m \rho'^m + \beta_m \rho'^{-m} & m>0 \end{cases} \tag{3}$$

But due to the discontinuity of first-order derivative of g_m in $[\rho-\epsilon,\rho+\epsilon]$

$$\rho'\frac{\partial g_m}{\partial\rho'}\bigg|_{\rho+\epsilon} - \rho'\frac{\partial g_m}{\partial\rho'}\bigg|_{\rho-\epsilon} = -4\pi \tag{4}$$

g_m may have different set of parameters for the range $\rho'<\rho$ and $\rho'>\rho$.
When we enforce the Dirichlet boundary condition on the overall $G(\rho,\phi;\rho'=b,\phi')=0$, since b can be arbitrary positive value, it's clear that for each m, we must have $g_m(\rho,\rho'=b)=0$ independently.

- When $m=0$, let's write $g_0(\rho,\rho')$ as

$$g_0(\rho,\rho') = \begin{cases} \alpha_0 + \beta_0 \ln\rho' & \rho'<\rho \\ \gamma_0 + \delta_0 \ln\rho' & \rho'>\rho \end{cases} \tag{5}$$

  It's clear $\beta_0 = 0$ since otherwise g_0 will diverge at $\rho'=0$. Then the discontinuity relation (4) demands

$$\delta_0 = -4\pi \tag{6}$$

  Boundary condition at $\rho'=b$ requires (since we are considering interior volume, we must take the $\rho'=b>\rho$ case)

$$\gamma_0 + \delta_0 \ln b = \gamma_0 - 4\pi\ln b = 0 \qquad \Longrightarrow \qquad \gamma_0 = 4\pi\ln b \tag{7}$$

  Moreover, the continuity of g_m at $\rho'=\rho$ gives

$$\alpha_0 = \gamma_0 + \delta_0\ln\rho = 4\pi\ln\left(\frac{b}{\rho}\right) \tag{8}$$

  Therefore we obtained the full g_0 as

$$g_0(\rho,\rho') = \begin{cases} 4\pi\ln\left(\dfrac{b}{\rho}\right) & \rho'<\rho \\ 4\pi\ln\left(\dfrac{b}{\rho'}\right) & \rho'>\rho \end{cases}$$
$$= 4\pi\ln\left(\frac{b}{\rho_>}\right) \tag{9}$$

- When $m>0$, let's write $g_m(\rho,\rho')$ as

$$g_m(\rho,\rho') = \begin{cases} \alpha_m\rho'^m + \beta_m\rho'^{-m} & \rho'<\rho \\ \gamma_m\rho'^m + \delta_m\rho'^{-m} & \rho'>\rho \end{cases} \tag{10}$$

  Similar arguments apply to give us the following restrictions

$$g_m \text{ must not diverge at origin}: \qquad \beta_m = 0 \tag{11}$$

$$\text{discontinuity relation (4)}: \qquad m\gamma_m\rho^m - m\delta_m\rho^{-m} - m\alpha_m\rho^m = -4\pi \qquad \Longrightarrow$$
$$\gamma_m\rho^m - \delta_m\rho^{-m} - \alpha_m\rho^m = -\frac{4\pi}{m} \tag{12}$$

$$\text{continuity of } g_m \text{ at } \rho'=\rho: \qquad \alpha_m\rho^m = \gamma_m\rho^m + \delta_m\rho^{-m} \tag{13}$$

$$\text{boundary condition at } \rho'=b: \qquad \gamma_m b^m + \delta_m b^{-m} = 0 \tag{14}$$

Adding (12) and (13) gives

$$\delta_m = \frac{2\pi}{m}\rho^m \tag{15}$$

Then by (14)

$$\gamma_m = -\frac{2\pi}{m}\rho^m b^{-2m} \tag{16}$$

Finally by (13)

$$\alpha_m = \gamma_m + \delta_m \rho^{-2m} = \frac{2\pi}{m}\left(\rho^{-m} - \rho^m b^{-2m}\right) \tag{17}$$

Here is the full form of g_m:

$$\begin{aligned} g_m(\rho,\rho') &= \begin{cases} \frac{2\pi}{m}\left(\rho^{-m} - \rho^m b^{-2m}\right)\rho'^m & \rho' < \rho \\ -\frac{2\pi}{m}\rho^m b^{-2m}\rho'^m + \frac{2\pi}{m}\rho^m \rho'^{-m} & \rho' > \rho \end{cases} \\ &= \frac{2\pi}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^m - \left(\frac{\rho\rho'}{b^2}\right)^m\right] \end{aligned} \tag{18}$$

- When $m < 0$, we do this all over again, but with slight changes:

$$\begin{aligned} g_m \text{ must not diverge at origin :} & \qquad \alpha_m = 0 & (19) \\ \text{discontinuity relation (4) :} & \qquad m\gamma_m\rho^m - m\delta_m\rho^{-m} + m\beta_m\rho^{-m} = -4\pi \quad \Longrightarrow & \\ & \qquad \gamma_m\rho^m - \delta_m\rho^{-m} + \beta_m\rho^{-m} = -\frac{4\pi}{m} & (20) \\ \text{continuity of } g_m \text{ at } \rho' = \rho \text{ :} & \qquad \beta_m\rho^{-m} = \gamma_m\rho^m + \delta_m\rho^{-m} & (21) \\ \text{boundary condition at } \rho' = b \text{ :} & \qquad \gamma_m b^m + \delta_m b^{-m} = 0 & (22) \end{aligned}$$

Subtract (21) from (20) gives

$$\gamma_m = -\frac{2\pi}{m}\rho^{-m} \tag{23}$$

By (22):

$$\delta_m = \frac{2\pi}{m}\rho^{-m} b^{2m} \tag{24}$$

By (21):

$$\beta_m = \gamma_m \rho^{2m} + \delta_m = \frac{2\pi}{m}\left(\rho^{-m} b^{2m} - \rho^m\right) \tag{25}$$

And eventually

$$\begin{aligned} g_m(\rho,\rho') &= \begin{cases} \frac{2\pi}{m}\left(\rho^{-m} b^{2m} - \rho^m\right)\rho'^{-m} & \rho' < \rho \\ -\frac{2\pi}{m}\rho^{-m}\rho'^m + \frac{2\pi}{m}\rho^{-m} b^{2m}\rho'^{-m} & \rho' > \rho \end{cases} \\ &= \frac{2\pi}{m}\left[\left(\frac{b^2}{\rho\rho'}\right)^m - \left(\frac{\rho_>}{\rho_<}\right)^m\right] \end{aligned} \tag{26}$$

Notice that (18) and (26) are the same for $\pm m$.

Inserting (9), (18), (26) into (1) gives the series form of the Green function

$$G(\rho,\phi;\rho',\phi') = \ln\left[\left(\frac{b}{\rho_>}\right)^2\right] + 2\sum_{m=1}^{\infty}\frac{1}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^m - \left(\frac{\rho\rho'}{b^2}\right)^m\right]\cos\left[m\left(\phi - \phi'\right)\right] \tag{27}$$

Now we would like to show that (27) has a closed form

$$G(\rho,\phi;\rho',\phi') = \ln\left\{\frac{\rho^2\rho'^2 + b^4 - 2\rho\rho' b^2\cos(\phi-\phi')}{b^2\left[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')\right]}\right\} \tag{28}$$

We can rewrite (28) as

$$\begin{aligned} G &= \ln\left(b^4\right) + \ln\left[1+\left(\frac{\rho\rho'}{b^2}\right)^2 - 2\left(\frac{\rho\rho'}{b^2}\right)\cos\left(\phi-\phi'\right)\right] - \ln\left(b^2\rho_>^2\right) - \ln\left[1+\left(\frac{\rho_<}{\rho_>}\right)^2 - 2\left(\frac{\rho_<}{\rho_>}\right)\cos\left(\phi-\phi'\right)\right] \\ &= \ln\left[\left(\frac{b}{\rho_>}\right)^2\right] + \ln\left[1+\left(\frac{\rho\rho'}{b^2}\right)^2 - 2\left(\frac{\rho\rho'}{b^2}\right)\cos\left(\phi-\phi'\right)\right] - \ln\left[1+\left(\frac{\rho_<}{\rho_>}\right)^2 - 2\left(\frac{\rho_<}{\rho_>}\right)\cos\left(\phi-\phi'\right)\right] \end{aligned} \tag{29}$$

In the solution of problem (2.11), we have shown for $|x| < 1$ the expansion

$$\ln\left(1+x^2-2x\cos\theta\right) = -2\sum_{m=1}^{\infty}\frac{x^m}{m}\cos m\theta \tag{30}$$

Applying this expansion to (29) gets us exactly (27).

2. This is a straightforward application of equation (1.44), given the Green function (28).

$$\begin{aligned} \frac{\partial G}{\partial \rho'} &= \frac{2\rho^2\rho' - 2\rho b^2\cos(\phi-\phi')}{\rho^2\rho'^2 + b^4 - 2\rho\rho' b^2\cos(\phi-\phi')} - \frac{2\rho' - 2\rho\cos(\phi-\phi')}{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')} \qquad \Longrightarrow \\ \frac{\partial G}{\partial n'} = \left.\frac{\partial G}{\partial \rho'}\right|_{\rho'=b} &= \frac{2\rho^2 b - 2\rho b^2\cos(\phi-\phi')}{\rho^2 b^2 + b^4 - 2\rho b^3\cos(\phi-\phi')} - \frac{2b - 2\rho\cos(\phi-\phi')}{\rho^2+b^2-2\rho b\cos(\phi-\phi')} \\ &= \frac{2}{b}\frac{\rho^2-b^2}{\rho^2+b^2-2\rho b\cos(\phi-\phi')} \end{aligned} \tag{31}$$

Then by (1.44)

$$\begin{aligned} \Phi(\mathbf{x}) = -\frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G}{\partial n'}da' &= -\frac{1}{2\pi}\int_0^{2\pi}\Phi(\mathbf{x}')\left.\frac{\partial G}{\partial \rho'}\right|_{\rho'=b}\cdot b\,d\phi' \\ &= \frac{1}{2\pi}\int_0^{2\pi}\Phi(\mathbf{x}')\frac{b^2-\rho^2}{\rho^2+b^2-2\rho b\cos(\phi-\phi')}d\phi' \end{aligned} \tag{32}$$

agreeing with problem 2.12.

3. For the exterior, let's solve these cases again.

- $m = 0$. We will now try to get the parameters in (5). Unlike the free space situation in problem 2.17, here the infinity is the "interior" of the volume, so we must set $\delta_0 = 0$. Other restrictions are listed as usual:

$$\begin{aligned} &\text{discontinuity relation (4)}: & -\beta_0 &= -4\pi & (33)\\ &\text{continuity of } g_m \text{ at } \rho'=\rho: & \alpha_0+\beta_0\ln\rho &= \gamma_0 & (34)\\ &\text{boundary condition at } \rho'=b: & \alpha_0+\beta_0\ln b &= 0 & (35) \end{aligned}$$

  The solutions are

$$\beta_0 = 4\pi \qquad \alpha_0 = -4\pi\ln b \qquad \gamma_0 = -4\pi\ln b + 4\pi\ln\rho \tag{36}$$

  This gives

$$\begin{aligned} g_0(\rho,\rho') &= \begin{cases} -4\pi\ln b + 4\pi\ln\rho' & \rho' < \rho \\ -4\pi\ln b + 4\pi\ln\rho & \rho' > \rho \end{cases} \\ &= 4\pi\ln\left(\frac{\rho_<}{b}\right) \end{aligned} \tag{37}$$

- $m > 0$. Convergence at infinity requires $\gamma_m = 0$ in (10). Other restrictions are now

$$\begin{aligned} &\text{discontinuity relation (4)}: & -m\delta_m\rho^{-m} - m\alpha_m\rho^m + m\beta_m\rho^{-m} &= -4\pi \qquad \Longrightarrow \\ & & \delta_m\rho^{-m} + \alpha_m\rho^m - \beta_m\rho^{-m} &= \frac{4\pi}{m} & (38)\\ &\text{continuity of } g_m \text{ at } \rho'=\rho: & \alpha_m\rho^m + \beta_m\rho^{-m} &= \delta_m\rho^{-m} & (39)\\ &\text{boundary condition at } \rho'=b: & \alpha_m b^m + \beta_m b^{-m} &= 0 & (40) \end{aligned}$$

We can readily get the solutions

$$\alpha_m = \frac{2\pi}{m}\rho^{-m} \qquad \beta_m = -\frac{2\pi}{m}\rho^{-m}b^{2m} \qquad \delta_m = \frac{2\pi}{m}\left(\rho^m - \rho^{-m}b^{2m}\right) \tag{41}$$

hence

$$\begin{aligned} g_m(\rho,\rho') &= \begin{cases} \frac{2\pi}{m}\left(\rho^{-m}\rho'^m - \rho^{-m}b^{2m}\rho'^{-m}\right) & \rho' < \rho \\ \frac{2\pi}{m}\left(\rho^m - \rho^{-m}b^{2m}\right)\rho'^{-m} & \rho' > \rho \end{cases} \\ &= \frac{2\pi}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^m - \left(\frac{b^2}{\rho\rho'}\right)^m\right] \end{aligned} \tag{42}$$

- $m < 0$. Similarly, here $\delta_m = 0$, and the restrictions are

$$\begin{aligned} \text{discontinuity relation (4)}: &\qquad m\gamma_m\rho^m - m\alpha_m\rho^m + m\beta_m\rho^{-m} = -4\pi \qquad \Longrightarrow \\ &\qquad \gamma_m\rho^m - \alpha_m\rho^m + \beta_m\rho^{-m} = -\frac{4\pi}{m} && (43) \\ \text{continuity of } g_m \text{ at } \rho' = \rho: &\qquad \alpha_m\rho^m + \beta_m\rho^{-m} = \gamma_m\rho^m && (44) \\ \text{boundary condition at } \rho' = b: &\qquad \alpha_m b^m + \beta_m b^{-m} = 0 && (45) \end{aligned}$$

The solutions are

$$\beta_m = -\frac{2\pi}{m}\rho^m \qquad \alpha_m = \frac{2\pi}{m}\rho^m b^{-2m} \qquad \gamma_m = \frac{2\pi}{m}\left(\rho^m b^{-2m} - \rho^{-m}\right) \tag{46}$$

hence

$$\begin{aligned} g_m(\rho,\rho') &= \begin{cases} \frac{2\pi}{m}\left(\rho^m b^{-2m}\rho'^m - \rho^m\rho'^{-m}\right) & \rho' < \rho \\ \frac{2\pi}{m}\left(\rho^m b^{-2m} - \rho^{-m}\right)\rho'^m & \rho' > \rho \end{cases} \\ &= \frac{2\pi}{m}\left[\left(\frac{\rho\rho'}{b^2}\right)^m - \left(\frac{\rho_>}{\rho_<}\right)^m\right] \end{aligned} \tag{47}$$

With (37), (42), (47), for the exterior volume, we have the series representation of Green function

$$G = \ln\left[\left(\frac{\rho_<}{b}\right)^2\right] + 2\sum_{m=1}^{\infty}\frac{1}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^m - \left(\frac{b^2}{\rho\rho'}\right)^m\right]\cos\left[m\left(\phi - \phi'\right)\right] \tag{48}$$

By (30),

$$\begin{aligned} G &= \ln\left[\left(\frac{\rho_<}{b}\right)^2\right] + \ln\left[1 + \left(\frac{b^2}{\rho\rho'}\right)^2 - 2\left(\frac{b^2}{\rho\rho'}\right)\cos\left(\phi-\phi'\right)\right] - \ln\left[1 + \left(\frac{\rho_<}{\rho_>}\right)^2 - 2\left(\frac{\rho_<}{\rho_>}\right)\cos\left(\phi-\phi'\right)\right] \\ &= \ln\left[\left(\rho\rho'\right)^2\right] + \ln\left[1 + \left(\frac{b^2}{\rho\rho'}\right)^2 - 2\left(\frac{b^2}{\rho\rho'}\right)\cos\left(\phi-\phi'\right)\right] - \\ &\quad \ln\left[(\rho_> b)^2\right] - \ln\left[1 + \left(\frac{\rho_<}{\rho_>}\right)^2 - 2\left(\frac{\rho_<}{\rho_>}\right)\cos\left(\phi-\phi'\right)\right] \\ &= \ln\left\{\frac{\rho^2\rho'^2 + b^4 - 2\rho\rho' b^2\cos\left(\phi-\phi'\right)}{b^2\left[\rho^2 + \rho'^2 - 2\rho\rho'\cos\left(\phi-\phi'\right)\right]}\right\} \end{aligned} \tag{49}$$

which has identical form as (28).

When we calculate $\partial G/\partial n'$, because we are dealing with exterior volume, this derivative is now $-\partial G/\partial\rho'|_{\rho'=b}$. So the potential calculated using (1.44) must have a flipped sign, also agreeing with 2.12's exterior case.