1. The definition of the index of refraction $n(\omega)$ comes from the basic Maxwell equations. For example in the isotropic medium without conductivity, the fields $\mathbf{E}, \mathbf{B}$ satisfy the Helmholtz equation (7.3), i.e.

$$\left(\nabla^2 + \omega^2 \mu \epsilon\right) \mathbf{E} = 0 \tag{1}$$

But if, for example, the medium has conductivity σ, and is nonpermeable (see problem 7.4), the equation becomes

$$\left[\nabla^2 + \left(\omega^2 \mu \epsilon + i\omega\mu\sigma\right)\right] \mathbf{E} = 0 \tag{2}$$

Then the plane wave solution is of form

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} \tag{3}$$

where

$$\mathbf{k}\cdot\mathbf{k} = \omega^2 \mu\sigma + i\omega\mu\sigma \tag{4}$$

It is only in this sense that we can define the index of refraction $n(\omega)$ such that

$$n(\omega) = \frac{ck}{\omega} \tag{5}$$

where we see here it involves a square root of a complex number.

For the particular media and its wave vector form (4), we see that

$$n^2(-\omega) = \left[n^2(\omega)\right]^* \tag{6}$$

This gives us two possible conventions to define the index of refraction for negative frequencies, i.e.

$$n(-\omega) = \pm n^*(\omega) \tag{7}$$

In one dimension, for the given frequency ω, the general plane wave solution has the form

$$u_\omega(x, t) \propto e^{-i\omega t}\left[A(\omega) e^{i(\omega/c)n(\omega)x} + B(\omega) e^{-i(\omega/c)n(\omega)x}\right] \tag{8}$$

Then the superposition of all frequencies will produce the general solution

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega t}\left[A(\omega) e^{i(\omega/c)n(\omega)x} + B(\omega) e^{-i(\omega/c)n(\omega)x}\right] \tag{9}$$

2. We do part (c) first.

From (9),

$$u(0, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega t}\left[A(\omega) + B(\omega)\right] \tag{10}$$

$$\frac{\partial u}{\partial x}(0, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega t}\left[\frac{i\omega n(\omega)}{c}\right]\left[A(\omega) - B(\omega)\right] \tag{11}$$

Integrating both sides of (10) with $e^{i\omega' t} dt$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} u(0, t) e^{i\omega' t} dt &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \left[A(\omega) + B(\omega)\right] \int_{-\infty}^{\infty} e^{i\left(\omega' - \omega\right)t} dt \\ &= \sqrt{2\pi} \int_{-\infty}^{\infty} d\omega \left[A(\omega) + B(\omega)\right] \delta\left(\omega' - \omega\right) = \sqrt{2\pi}\left[A\left(\omega'\right) + B\left(\omega'\right)\right]\end{aligned} \tag{12}$$

Similarly for (11), we have

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial x}(0, t) e^{i\omega' t} dt = \sqrt{2\pi}\left[\frac{i\omega' n(\omega')}{c}\right]\left[A\left(\omega'\right) - B\left(\omega'\right)\right] \tag{13}$$

Combining (12) and (13) and relabeling $\omega' \to \omega$, we get

$$A(\omega) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{i\omega t}\left[u(0, t) - \frac{ic}{\omega n(\omega)} \frac{\partial u}{\partial x}(0, t)\right] \tag{14}$$

$$B(\omega) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{i\omega t}\left[u(0, t) + \frac{ic}{\omega n(\omega)} \frac{\partial u}{\partial x}(0, t)\right] \tag{15}$$

3. For part (b), I don't think the claim is generally correct.

   Taking the inverse Fourier transform of (9) and its complex conjugate, we have

$$\int_{-\infty}^{\infty} u(x,t)\, e^{i\omega t} dt = \sqrt{2\pi}\left[A(\omega)\, e^{i(\omega/c)n(\omega)x} + B(\omega)\, e^{-i(\omega/c)n(\omega)x}\right] \tag{16}$$

$$\int_{-\infty}^{\infty} u^*(x,t)\, e^{i\omega t} dt = \sqrt{2\pi}\left[A^*(-\omega)\, e^{i(\omega/c)n^*(-\omega)x} + B^*(-\omega)\, e^{-i(\omega/c)n^*(-\omega)x}\right] \tag{17}$$

   $u(x,t)$ being real is equivalent to the condition

$$A(\omega)\, e^{i(\omega/c)n(\omega)x} + B(\omega)\, e^{-i(\omega/c)n(\omega)x} = A^*(-\omega)\, e^{i(\omega/c)n^*(-\omega)x} + B^*(-\omega)\, e^{-i(\omega/c)n^*(-\omega)x} \tag{18}$$

   for all x.

   But it is not necessary that we must have $n(-\omega) = n^*(\omega)$.

   Certainly, when $n(-\omega) = n^*(\omega)$, by (14) and (15), we have $A(\omega) = A^*(-\omega), B(\omega) = B^*(-\omega)$, which will satisfy (18).

   But alternatively, when $n(-\omega) = -n^*(\omega)$, by (14) and (15), $A(\omega) = B^*(-\omega), B(\omega) = A^*(-\omega)$, which again will satisfy (18).

   So it looks like the relation $n(-\omega) = n^*(\omega)$ can be derived from a specific medium (e.g., (4)) without regarding the reality of $u(x,t)$, and even so, we will have to choose the upper sign from two alternate conventions (7) for negative frequencies.