Here we give a detailed derivation of field-strength transformation under boost (11.149).

The matrix representation of the field-strength tensor $F^{\alpha\beta}$ is given by (11.137)

$$F = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \tag{1}$$

which can be written in block form as

$$F = \begin{bmatrix} 0 & -\mathbf{E}^T \\ \mathbf{E} & \mathbf{B}\cdot\mathbf{S} \end{bmatrix} \tag{2}$$

where S_i's are 3×3 matrices corresponding to the the generators of rotations in the Lorentz group, i.e., the lower-right 3×3 block of (11.91).

To simplify the proof, we take note of the following relations

$$S_i\mathbf{u} = \hat{\mathbf{e}}_i \times \mathbf{u} \tag{3}$$

$$\mathbf{a}\mathbf{b}^T - \mathbf{b}\mathbf{a}^T = -(\mathbf{a}\times\mathbf{b})\cdot\mathbf{S} \tag{4}$$

As is shown in (11.147), under Lorentz transformation A, the field-strength tensor transforms as

$$F' = AFA^T \tag{5}$$

where if we only have boost, A can be written in block form (see 11.98)

$$A = \begin{bmatrix} \gamma & -\gamma\boldsymbol{\beta}^T \\ -\gamma\boldsymbol{\beta} & \underbrace{I + \frac{(\gamma-1)}{\beta^2}\boldsymbol{\beta}\boldsymbol{\beta}^T}_{M} \end{bmatrix} = A^T \tag{6}$$

In these block forms, (5) becomes

$$\begin{aligned} F' = AFA^T &= \begin{bmatrix} \gamma & -\gamma\boldsymbol{\beta}^T \\ -\gamma\boldsymbol{\beta} & M \end{bmatrix}\begin{bmatrix} 0 & -\mathbf{E}^T \\ \mathbf{E} & \mathbf{B}\cdot\mathbf{S} \end{bmatrix}\begin{bmatrix} \gamma & -\gamma\boldsymbol{\beta}^T \\ -\gamma\boldsymbol{\beta} & M \end{bmatrix} \\ &= \begin{bmatrix} -\gamma\boldsymbol{\beta}^T\mathbf{E} & -\gamma\mathbf{E}^T - \gamma\boldsymbol{\beta}^T(\mathbf{B}\cdot\mathbf{S}) \\ M\mathbf{E} & \gamma\boldsymbol{\beta}\mathbf{E}^T + M(\mathbf{B}\cdot\mathbf{S}) \end{bmatrix}\begin{bmatrix} \gamma & -\gamma\boldsymbol{\beta}^T \\ -\gamma\boldsymbol{\beta} & M \end{bmatrix} \\ &= \begin{bmatrix} -\gamma^2\boldsymbol{\beta}^T\mathbf{E} + \gamma^2\mathbf{E}^T\boldsymbol{\beta} + \gamma^2\boldsymbol{\beta}^T(\mathbf{B}\cdot\mathbf{S})\boldsymbol{\beta} & \gamma^2\boldsymbol{\beta}^T\mathbf{E}\boldsymbol{\beta}^T - \gamma\mathbf{E}^T M - \gamma\boldsymbol{\beta}^T(\mathbf{B}\cdot\mathbf{S})M \\ \gamma M\mathbf{E} - \gamma^2\boldsymbol{\beta}\mathbf{E}^T\boldsymbol{\beta} - \gamma M(\mathbf{B}\cdot\mathbf{S})\boldsymbol{\beta} & -\gamma M\mathbf{E}\boldsymbol{\beta}^T + \gamma\boldsymbol{\beta}\mathbf{E}^T M + M(\mathbf{B}\cdot\mathbf{S})M \end{bmatrix} \end{aligned} \tag{7}$$

From (3), we can see $F'^{00} = 0$ as expected. From the fact that $\mathbf{B}\cdot\mathbf{S}$ is antisymmetric, we see that the off-diagonal blocks are antisymmetric too, with the column block $[1:3,0]$ representing the transformed electric field, i.e.,

$$\begin{aligned} \mathbf{E}' = F'^{[1:3,0]} &= \gamma M\mathbf{E} - \gamma^2\boldsymbol{\beta}\mathbf{E}^T\boldsymbol{\beta} - \gamma M(\mathbf{B}\cdot\mathbf{S})\boldsymbol{\beta} \\ &= \gamma\left[I + \frac{(\gamma-1)}{\beta^2}\boldsymbol{\beta}\boldsymbol{\beta}^T\right]\mathbf{E} - \gamma^2\boldsymbol{\beta}\mathbf{E}^T\boldsymbol{\beta} - \gamma\left[I + \frac{(\gamma-1)}{\beta^2}\boldsymbol{\beta}\boldsymbol{\beta}^T\right](\mathbf{B}\cdot\mathbf{S})\boldsymbol{\beta} \qquad \text{using (3)} \\ &= \gamma\mathbf{E} + \gamma\left[\frac{(\gamma-1)}{\beta^2} - \gamma\right]\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E}) - \gamma\overbrace{(\mathbf{B}\cdot\mathbf{S})\boldsymbol{\beta}}^{\mathbf{B}\times\boldsymbol{\beta}} \\ &= \gamma(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E}) \end{aligned} \tag{8}$$

Rearranging the terms into parallel and perpendicular components, we have

$$\mathbf{E}' = \gamma\left(\mathbf{E}_\parallel + \mathbf{E}_\perp\right) + \gamma\boldsymbol{\beta}\times\mathbf{B} - (\gamma-1)\underbrace{\frac{\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E})}{\beta^2}}_{\mathbf{E}_\parallel} = \mathbf{E}_\parallel + \gamma(\mathbf{E}_\perp + \boldsymbol{\beta}\times\mathbf{B}) \quad \Longrightarrow \quad \mathbf{E}'_\parallel = \mathbf{E}_\parallel \quad \mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \boldsymbol{\beta}\times\mathbf{B}) \tag{9}$$

The lower-right 3×3 block of (7) gives us

$$
\begin{aligned}
\mathbf{B}' \cdot \mathbf{S} = F'^{[1:3,1:3]} &= -\gamma M \mathbf{E} \boldsymbol{\beta}^T + \gamma \boldsymbol{\beta} \mathbf{E}^T M + M (\mathbf{B} \cdot \mathbf{S}) M \\
&= -\gamma \left[I + \frac{(\gamma - 1)}{\beta^2} \boldsymbol{\beta} \boldsymbol{\beta}^T \right] \mathbf{E} \boldsymbol{\beta}^T + \gamma \boldsymbol{\beta} \mathbf{E}^T \left[I + \frac{(\gamma - 1)}{\beta^2} \boldsymbol{\beta} \boldsymbol{\beta}^T \right] \\
&\quad + \left[I + \frac{(\gamma - 1)}{\beta^2} \boldsymbol{\beta} \boldsymbol{\beta}^T \right] (\mathbf{B} \cdot \mathbf{S}) \left[I + \frac{(\gamma - 1)}{\beta^2} \boldsymbol{\beta} \boldsymbol{\beta}^T \right] && \text{using (3)} \\
&= -\gamma \left(\mathbf{E} \boldsymbol{\beta}^T - \boldsymbol{\beta} \mathbf{E}^T \right) + \mathbf{B} \cdot \mathbf{S} + \frac{(\gamma - 1)}{\beta^2} \left\{ \boldsymbol{\beta} \left[(\mathbf{B} \cdot \mathbf{S})^T \boldsymbol{\beta} \right]^T + (\mathbf{B} \cdot \mathbf{S}) \boldsymbol{\beta} \boldsymbol{\beta}^T \right\} && \text{using (4), } \mathbf{B} \cdot \mathbf{S} \text{ antisymmetric} \\
&= -\gamma (\boldsymbol{\beta} \times \mathbf{E}) \cdot \mathbf{S} + \mathbf{B} \cdot \mathbf{S} + \frac{(\gamma - 1)}{\beta^2} \left[-\boldsymbol{\beta} (\mathbf{B} \times \boldsymbol{\beta})^T + (\mathbf{B} \times \boldsymbol{\beta}) \boldsymbol{\beta}^T \right] \\
&= -\gamma (\boldsymbol{\beta} \times \mathbf{E}) \cdot \mathbf{S} + \mathbf{B} \cdot \mathbf{S} + \frac{(\gamma - 1)}{\beta^2} \left[-(\mathbf{B} \times \boldsymbol{\beta}) \times \boldsymbol{\beta} \right] \cdot \mathbf{S} \\
&= -\gamma (\boldsymbol{\beta} \times \mathbf{E}) \cdot \mathbf{S} + \mathbf{B} \cdot \mathbf{S} + \frac{(\gamma - 1)}{\beta^2} \left[\beta^2 \mathbf{B} - \boldsymbol{\beta} (\boldsymbol{\beta} \cdot \mathbf{B}) \right] \cdot \mathbf{S} \\
&= \left[\gamma (\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta} (\boldsymbol{\beta} \cdot \mathbf{B}) \right] \cdot \mathbf{S} && (10)
\end{aligned}
$$

allowing us to identify

$$\mathbf{B}' = \gamma (\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta} (\boldsymbol{\beta} \cdot \mathbf{B}) \tag{11}$$

of which the parallel and perpendicular components are

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \qquad \mathbf{B}'_{\perp} = \gamma (\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E}) \tag{12}$$