1. **Prob 11.27**

(a) In frame K', for the charge distribution $\rho'(\mathbf{x}', t')$ to give total dipole moment $\mathbf{p}'$, by definition we must have

$$\mathbf{p}'(t') = \int \mathbf{x}'\rho'(\mathbf{x}', t')\, d^3x' \tag{1}$$

Note that in order for K' to have zero current, charge conservation requires that ρ' be time independent, i.e. $\rho'(\mathbf{x}', t') = \rho'(\mathbf{x}')$, therefore $\mathbf{p}'$ has no time dependence either.

Switching to the viewpoint of K, we want to find the instantaneous (at time $t = t_0$) magnetic dipole moment. In anticipation of Lorentz transformation, we write it in terms of 4-dimensional integral

$$\mathbf{m}(t_0) = \frac{1}{2}\int \mathbf{x}\times\mathbf{J}(\mathbf{x}, t)\,\delta(t - t_0)\, d^3x\, dt \tag{2}$$

where by Lorentz transformation (11.19) (take $c = 1$),

$$\mathbf{J}(\mathbf{x}, t) = \gamma\boldsymbol{\beta}\rho'(\mathbf{x}', t') + \mathbf{J}'(\mathbf{x}', t) + \frac{(\gamma-1)}{\beta^2}\left[\boldsymbol{\beta}\cdot\mathbf{J}'(\mathbf{x}', t)\right]\boldsymbol{\beta} = \gamma\boldsymbol{\beta}\rho'(\mathbf{x}', t') \tag{3}$$

The dependence on $\mathbf{x}$ and t are implicitly given by the dependence of $\mathbf{x}', t'$ on $\mathbf{x}, t$ through the Lorentz transformation. Since $\mathbf{J}$ is parallel to $\boldsymbol{\beta}$, we have

$$\mathbf{x}\times\mathbf{J} = \mathbf{x}_\perp\times\mathbf{J} = \mathbf{x}'_\perp\times\mathbf{J} = \mathbf{x}'\times\mathbf{J} \tag{4}$$

(2) now becomes

$$\begin{aligned}
\mathbf{m}(t_0) &= \frac{\gamma}{2}\int \mathbf{x}'\times\boldsymbol{\beta}\rho'(\mathbf{x}', t')\,\delta(t - t_0)\, d^3x\, dt && \text{note that } d^3x\,dt = d^3x'\,dt',\ t = \gamma\left(t' + \boldsymbol{\beta}\cdot\mathbf{x}'\right)\\
&= \frac{\gamma}{2}\int \mathbf{x}'\times\boldsymbol{\beta}\rho'(\mathbf{x}', t')\,\delta\left[\gamma\left(t' + \boldsymbol{\beta}\cdot\mathbf{x}'\right) - t_0\right] d^3x'\, dt'\\
&= \frac{\gamma}{2}\int \mathbf{x}'\times\boldsymbol{\beta}\rho(\mathbf{x}', t')\cdot\frac{1}{\gamma}\delta\left[t' - \underbrace{\left(\frac{t_0}{\gamma} - \boldsymbol{\beta}\cdot\mathbf{x}'\right)}_{t'_{0,\mathrm{ret}}}\right] d^3x'\, dt'\\
&= \frac{1}{2}\left[\int \mathbf{x}'\rho'\left(\mathbf{x}', t'_{0,\mathrm{ret}}\right) d^3x'\right]\times\boldsymbol{\beta} && \rho' \text{ is time independent}\\
&= \frac{\mathbf{p}'\times\boldsymbol{\beta}}{2}
\end{aligned} \tag{5}$$

Note that this is an exact relation that is applicable to all orders of β.

Similarly, the electric dipole moment in K is

$$\mathbf{p}(t_0) = \int \mathbf{x}\rho(\mathbf{x}, t)\,\delta(t - t_0)\, d^3x\, dt \tag{6}$$

where by Lorentz transformation,

$$\mathbf{x} = \mathbf{x}' + \frac{(\gamma-1)}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{x}'\right)\boldsymbol{\beta} + \gamma\boldsymbol{\beta}t' \tag{7}$$

$$\rho(\mathbf{x}, t) = \gamma\left[\rho'(\mathbf{x}', t) + \boldsymbol{\beta}\cdot\mathbf{J}'(\mathbf{x}', t')\right] = \gamma\rho'(\mathbf{x}', t') \tag{8}$$

we get

$$\begin{aligned}
\mathbf{p}(t_0) &= \int\left[\mathbf{x}' + \frac{(\gamma-1)}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{x}'\right)\boldsymbol{\beta} + \gamma\boldsymbol{\beta}t'\right]\cdot\gamma\rho'(\mathbf{x}', t')\cdot\frac{1}{\gamma}\delta\left(t' - t'_{0,\mathrm{ret}}\right) d^3x'\, dt'\\
&= \int \mathbf{x}'\rho'\left(\mathbf{x}', t'_{0,\mathrm{ret}}\right) d^3x' + \frac{(\gamma-1)}{\beta^2}\int\left(\boldsymbol{\beta}\cdot\mathbf{x}'\right)\boldsymbol{\beta}\rho'\left(\mathbf{x}', t'_{0,\mathrm{ret}}\right) d^3x' + \gamma\boldsymbol{\beta}t'_{0,\mathrm{ret}}\underbrace{\int\rho'\left(\mathbf{x}', t'_{0,\mathrm{ret}}\right) d^3x'}_{Q_{\mathrm{total}}=0}\\
&= \mathbf{p}' + \frac{(\gamma-1)}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{p}'\right)\boldsymbol{\beta}
\end{aligned} \tag{9}$$

Again, this relation is exact. If we stop at first order in β, the second term can be ignored.

(b) If in K', there is no charge density but only current density, we have

$$\mathbf{m}'\left(t_0'\right) = \frac{1}{2}\int \mathbf{x}' \times \mathbf{J}'\left(\mathbf{x}', t'\right) d^3x' \tag{10}$$

Charge conservation requires $\mathbf{J}'$ to be time independent, as well as

$$\boldsymbol{\nabla}' \cdot \mathbf{J}'\left(\mathbf{x}'\right) = 0 \tag{11}$$

The charge density and current density in K are related by

$$\rho(\mathbf{x}, t) = \gamma\boldsymbol{\beta} \cdot \mathbf{J}'\left(\mathbf{x}', t'\right) \qquad \mathbf{J}(\mathbf{x}, t) = \mathbf{J}'\left(\mathbf{x}', t'\right) + \frac{(\gamma-1)}{\beta^2}\left[\boldsymbol{\beta} \cdot \mathbf{J}'\left(\mathbf{x}', t'\right)\right]\boldsymbol{\beta} \tag{12}$$

The electric dipole moment in K is then

$$\begin{aligned}
\mathbf{p}(t_0) &= \int \mathbf{x}\rho(\mathbf{x}, t)\,\delta(t - t_0)\,d^3x\,dt \\
&= \int \left[\mathbf{x}' + \frac{(\gamma-1)}{\beta^2}\left(\boldsymbol{\beta} \cdot \mathbf{x}'\right)\boldsymbol{\beta} + \gamma\boldsymbol{\beta}t'\right] \cdot \gamma\left[\boldsymbol{\beta} \cdot \mathbf{J}'\left(\mathbf{x}', t'\right)\right] \cdot \frac{1}{\gamma}\delta\left(t' - t'_{0,\text{ret}}\right) d^3x'\,dt' \\
&= \underbrace{\int \mathbf{x}'\left[\boldsymbol{\beta} \cdot \mathbf{J}'\left(\mathbf{x}', t'_{0,\text{ret}}\right)\right] d^3x'}_{X} + \underbrace{\frac{(\gamma-1)}{\beta^2}\int \left(\boldsymbol{\beta} \cdot \mathbf{x}'\right)\left[\boldsymbol{\beta} \cdot \mathbf{J}'\left(\mathbf{x}', t'_{0,\text{ret}}\right)\right]\boldsymbol{\beta}\,d^3x'}_{Y} + \\
&\quad \underbrace{\gamma\boldsymbol{\beta}t'_{0,\text{ret}}\left[\boldsymbol{\beta} \cdot \int \mathbf{J}'\left(\mathbf{x}', t'_{0,\text{ret}}\right) d^3x'\right]}_{Z}
\end{aligned} \tag{13}$$

Recall the derivation from Jackson (5.52) that if $\boldsymbol{\nabla}' \cdot \mathbf{J}' = 0$,

$$\int \mathbf{J}'\left(\mathbf{x}'\right) d^3x' = 0 \qquad \text{and} \qquad \int \left(x_i'J_j' + x_j'J_i'\right) d^3x' = 0 \tag{14}$$

Then we can simplify the terms in (13)

$$X = \hat{\mathbf{e}}_i \int x_i'\beta_j J_j' d^3x' = \hat{\mathbf{e}}_i\beta_j\left[\overbrace{\frac{1}{2}\int \left(x_i'J_j' + x_j'J_i'\right) d^3x'}^{0} + \overbrace{\frac{1}{2}\int \left(x_i'J_j' - x_j'J_i'\right) d^3x'}^{\epsilon_{ijk}m_k'}\right] = \boldsymbol{\beta} \times \mathbf{m}' \tag{15}$$

$$\begin{aligned}
Y &= \frac{(\gamma-1)}{\beta^2}\boldsymbol{\beta}\int \beta_i x_i'\beta_j J_j' d^3x' = \frac{(\gamma-1)}{\beta^2}\boldsymbol{\beta}\left(\beta_i\beta_j\epsilon_{ijk}m_k'\right) = 0 \\
Z &= 0
\end{aligned} \tag{16}$$

giving the exact relation

$$\mathbf{p} = \boldsymbol{\beta} \times \mathbf{m}' \tag{17}$$

We can also calculate the magnetic dipole moment in K,

$$\begin{aligned}
\mathbf{m}(t_0) &= \frac{1}{2}\int \mathbf{x} \times \mathbf{J}(\mathbf{x}, t)\,\delta(t - t_0)\,d^3x\,dt \\
&= \frac{1}{2}\int \left[\mathbf{x}' + \frac{(\gamma-1)}{\beta^2}\left(\boldsymbol{\beta} \cdot \mathbf{x}'\right)\boldsymbol{\beta} + \gamma\boldsymbol{\beta}t'\right] \times \left\{\mathbf{J}'\left(\mathbf{x}', t'\right) + \frac{(\gamma-1)}{\beta^2}\left[\boldsymbol{\beta} \cdot \mathbf{J}'\left(\mathbf{x}', t'\right)\right]\boldsymbol{\beta}\right\} \cdot \frac{1}{\gamma}\delta\left(t' - t'_{0,\text{ret}}\right) d^3x'\,dt' \\
&= \frac{1}{2\gamma}\left\{\int \mathbf{x}' \times \mathbf{J}' d^3x' + \frac{(\gamma-1)}{\beta^2}\int \left[\left(\boldsymbol{\beta} \cdot \mathbf{x}'\right)\left(\boldsymbol{\beta} \times \mathbf{J}'\right) + \left(\boldsymbol{\beta} \cdot \mathbf{J}'\right)\left(\mathbf{x}' \times \boldsymbol{\beta}\right)\right] d^3x'\right\} \\
&= \frac{1}{2\gamma}\left\{2\mathbf{m}' + \frac{(\gamma-1)}{\beta^2}\boldsymbol{\beta} \times \overbrace{\int \left[\left(\boldsymbol{\beta} \cdot \mathbf{x}'\right)\mathbf{J}' - \left(\boldsymbol{\beta} \cdot \mathbf{J}'\right)\mathbf{x}'\right] d^3x'}^{\boldsymbol{\beta} \times \int \left(\mathbf{J}' \times \mathbf{x}'\right) d^3x' = 2\mathbf{m}' \times \boldsymbol{\beta}}\right\} \\
&= \mathbf{m}' - \frac{(\gamma-1)}{\gamma\beta^2}\left(\boldsymbol{\beta} \cdot \mathbf{m}'\right)\boldsymbol{\beta}
\end{aligned} \tag{18}$$

which again, is exact. The second term can be ignored for first order approximation in β.

2. **Prob 11.28**

   (a) From problem 11.27, up to first order in β, frame K sees a dipole $\mathbf{p} \approx \mathbf{p}'$ (see equation (9) with second term dropped) moving with velocity $\mathbf{v}$. This moving dipole gives rise to an scalar potential

   $$\Phi = \frac{\mathbf{p}\cdot\mathbf{R}}{R^3} \tag{19}$$

   From the first form of problem 6.22 (a), the vector potential is

   $$\mathbf{A} = \boldsymbol{\beta}\frac{(\mathbf{p}\cdot\mathbf{R})}{R^3} \tag{20}$$

   (b) From the 4-potential $(\Phi', \mathbf{A}' = 0)$ in K', applying the Lorentz transformation back to K gives

   $$\Phi = \gamma\Phi' \qquad\qquad \mathbf{A} = \gamma\boldsymbol{\beta}\Phi' \tag{21}$$

   which agrees with (19) and (20) up to first order in β (where $\gamma = 1 + O\left(\beta^2\right) \approx 1$).

   (c) In K', there is only electric field $\mathbf{E}'$ generated by the dipole. With Lorentz transformation into K and up to first order in β, we have

   $$\mathbf{E} = \mathbf{E}'_{\|} + \gamma\left(\mathbf{E}'_{\perp} - \boldsymbol{\beta}\times\mathbf{B}'\right) = \mathbf{E}'_{\|} + \gamma\mathbf{E}'_{\perp} \approx \mathbf{E}' \tag{22}$$

   $$\mathbf{B} = \mathbf{B}'_{\|} + \gamma\left(\mathbf{B}'_{\perp} + \boldsymbol{\beta}\times\mathbf{E}'\right) = \gamma\boldsymbol{\beta}\times\mathbf{E}' \approx \boldsymbol{\beta}\times\mathbf{E} \tag{23}$$

   Note in (22), in frame K', $\mathbf{E}'$ is only the function of $\mathbf{x}'$, thus we can write $\mathbf{E}$ as explicit function of $\mathbf{x}, t$,

   $$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}'\left(\mathbf{x}'(\mathbf{x}, t)\right) \tag{24}$$

   In K, when the origin coincides with the dipole's position $\mathbf{x}_0$, $\mathbf{E}$ is apparently a dipole field just as in K'. But for any fixed point as the origin, the function $\mathbf{x}'(\mathbf{x}, t)$ is time-dependent, and the field will have contribution from a time-dependent higher multipole moment, e.g., the quadrupole contribution given in problem 6.21 part (c).

3. **Prob 11.29**

   (a) From part (b) of problem 11.27, up to first order in β, the electric and magnetic dipole in frame K are

   $$\mathbf{p} = \boldsymbol{\beta}\times\mathbf{m}' \qquad\qquad \mathbf{m} \approx \mathbf{m}' \tag{25}$$

   which give rise to the scalar and vector potential

   $$\Phi = \frac{\mathbf{p}\times\mathbf{R}}{R^3} = \frac{(\boldsymbol{\beta}\times\mathbf{m}')\cdot\mathbf{R}}{R^3} \qquad\qquad \mathbf{A} = \frac{\mathbf{m}\times\mathbf{R}}{R^3} \approx \frac{\mathbf{m}'\times\mathbf{R}}{R^3} \tag{26}$$

   It is routine verification that the 4-potential $(\Phi, \mathbf{A})$ is the Lorentz transformation of $(\Phi' = 0, \mathbf{A}')$ in the limit $\gamma \to 1$.

   (b) Transforming the electric and magnetic field using Lorentz transformation from K' to K, we have

   $$\mathbf{B} = \mathbf{B}'_{\|} + \gamma\left(\mathbf{B}'_{\perp} + \boldsymbol{\beta}\times\mathbf{E}'\right) = \mathbf{B}'_{\|} + \gamma\mathbf{B}'_{\perp} \approx \mathbf{B}' \tag{27}$$

   $$\mathbf{E} = \mathbf{E}'_{\|} + \gamma\left(\mathbf{E}'_{\perp} - \boldsymbol{\beta}\times\mathbf{B}'\right) = -\gamma\boldsymbol{\beta}\times\mathbf{B}' \approx \mathbf{B}\times\boldsymbol{\beta} \tag{28}$$

   i.e., the third form of the electric field.
   From (26), we can also calculate the electric field via

   $$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial\mathbf{A}}{\partial t} \tag{29}$$

   where

   $$-\boldsymbol{\nabla}\Phi = \frac{3(\mathbf{p}\cdot\mathbf{n})\mathbf{n} - \mathbf{p}}{R^3} \tag{30}$$

   is the electric field due to the dipole $\mathbf{p} = \boldsymbol{\beta}\times\mathbf{m}'$. With $\partial\mathbf{R}/\partial t = -\boldsymbol{\beta}$, we have

   $$-\frac{\partial\mathbf{A}}{\partial t} = -\mathbf{m}'\times\frac{\partial}{\partial t}\left(\frac{\mathbf{R}}{R^3}\right) = -\mathbf{m}'\times\left[-\frac{\boldsymbol{\beta}}{R^3} + \frac{3\mathbf{R}(\mathbf{R}\cdot\boldsymbol{\beta})}{R^5}\right] = -\mathbf{m}'\times\left[\frac{3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta}) - \boldsymbol{\beta}}{R^3}\right] \tag{31}$$

   This gives the first desired form.
   Taking half of (30) and adding it to (31), we can obtain the second form via simple vector identities

   $$\mathbf{E} = \mathbf{E}_{\text{dipole}}\left(\mathbf{p}_{\text{eff}} = \frac{\boldsymbol{\beta}\times\mathbf{m}'}{2}\right) + \frac{3}{2}\mathbf{n}\times\left[\frac{\mathbf{m}(\mathbf{n}\cdot\boldsymbol{\beta}) + \boldsymbol{\beta}(\mathbf{n}\cdot\mathbf{m})}{R^3}\right] \tag{32}$$

   This form can be interpreted as the E/M dual of problem 11.28, where in K' we have only electric dipole, whose K-frame effective magnetic dipole has a factor of 1/2 according to (5). Then (32) is simply the dual of the magnetic field calculated in problem 6.22 (just compare the second term of (32) with the dual of problem 6.22 (b)).