- **Prob 3.9**

  Since the potential at $z=0$ and $z=L$ are zero, this suggests we use the second form of separation of variable scheme, with $\Phi(\rho,\phi,z)=R(\rho)Q(\phi)Z(z)$ where

$$\frac{d^2Z}{dz^2}+k^2Z=0 \tag{1}$$

$$\frac{d^2Q}{d\phi^2}+m^2Q=0 \tag{2}$$

$$\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}-\left(1+\frac{m^2}{x^2}\right)R=0 \qquad \text{where } x=k\rho \tag{3}$$

  With the boundary condition at the end caps, as well as the consideration that $R(\rho)$ must converge at the origin, we have the general series solution form

$$\Phi(\rho,\phi,z)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}I_m\left(\frac{n\pi\rho}{L}\right)(A_{nm}\sin m\phi+B_{nm}\cos m\phi)\sin\left(\frac{n\pi z}{L}\right) \tag{4}$$

  By the boundary condition on the side

$$V(\phi,z)=\sum_{n=1}^{\infty}\underbrace{\left[\sum_{m=0}^{\infty}I_m\left(\frac{n\pi b}{L}\right)(A_{nm}\sin m\phi+B_{nm}\cos m\phi)\right]}_{T_n}\sin\left(\frac{n\pi z}{L}\right) \tag{5}$$

  We recognize the outer sum as the Fourier expansion of $V(\phi,z)$ in the orthonormal basis of $\sqrt{2/L}\sin(n\pi z/L)$, thus

$$T_n=\frac{2}{L}\int_0^L V(\phi,z)\sin\left(\frac{n\pi z}{L}\right)dz \tag{6}$$

  The inner sum is a similar Fourier series, except this time with cosine terms.

$$T_n=\sum_{m=0}^{\infty}I_m\left(\frac{n\pi b}{L}\right)(A_{nm}\sin m\phi+B_{nm}\cos m\phi) \qquad \Longrightarrow$$

$$A_{nm}=\frac{1}{I_m\left(\frac{n\pi b}{L}\right)}\cdot\frac{1}{\pi}\int_0^{2\pi}T_n\sin m\phi\, d\phi \tag{7}$$

$$B_{nm}=\frac{1}{I_m\left(\frac{n\pi b}{L}\right)}\cdot\frac{1}{\pi}\int_0^{2\pi}T_n\cos m\phi\, d\phi \tag{8}$$

  where by the usual Fourier transform rules, $B_{n,m=0}$ will need to be further divided by two.

- **Prob 3.10**

  With the given surface potential

$$V(\phi,z)=V(\phi)=\begin{cases} V & \text{for } -\frac{\pi}{2}\le\phi\le\frac{\pi}{2} \\ -V & \text{for } \frac{\pi}{2}\le\phi\le\frac{3\pi}{2}\end{cases} \tag{9}$$

  we have

$$T_n=\frac{2}{L}V(\phi)\int_0^L\sin\left(\frac{n\pi z}{L}\right)dz=\frac{2V(\phi)}{n\pi}[1-(-1)^n]=\begin{cases}\frac{4V(\phi)}{(2k+1)\pi} & n=2k+1\\ 0 & n=2k\end{cases} \tag{10}$$

For $n = 2k+1$, by (7),(8)

$$A_{nm} = \frac{1}{I_m\left(\frac{n\pi b}{L}\right)} \frac{1}{\pi} \frac{4V}{n\pi} \left(\int_{-\pi/2}^{\pi/2} \sin m\phi \, d\phi - \int_{\pi/2}^{3\pi/2} \sin m\phi \, d\phi \right) = 0$$

$$\begin{aligned} B_{nm} &= \frac{1}{I_m\left(\frac{n\pi b}{L}\right)} \frac{1}{\pi} \frac{4V}{n\pi} \left(\int_{-\pi/2}^{\pi/2} \cos m\phi \, d\phi - \int_{\pi/2}^{3\pi/2} \cos m\phi \, d\phi \right) \\ &= \frac{4V}{n\pi^2 I_m\left(\frac{n\pi b}{L}\right)} \frac{1}{m} \left[\sin\left(\frac{m\pi}{2}\right) - \sin\left(-\frac{m\pi}{2}\right) + \sin\left(\frac{m\pi}{2}\right) - \sin\left(\frac{3m\pi}{2}\right) \right] \\ &= \begin{cases} \dfrac{16V}{nm\pi^2 I_m\left(\frac{n\pi b}{L}\right)} \sin\left(\dfrac{m\pi}{2}\right) & m, n \text{ odd} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{11}$$

In summary, for $n = 2k+1, m = 2l+1$, we can write (4) as

$$\Phi(\rho,\phi,z) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{I_{2l+1}\left[\frac{(2k+1)\pi\rho}{L}\right]}{I_{2l+1}\left[\frac{(2k+1)\pi b}{L}\right]} \frac{16V}{(2k+1)(2l+1)\pi^2} \sin\left[\frac{(2l+1)\pi}{2}\right] \cos\left[(2l+1)\phi\right] \sin\left[\frac{(2k+1)\pi z}{L}\right] \tag{12}$$

Recall the asymptotic behavior of $I_m(x)$ (equation (3.102))

$$I_m(x) \to \frac{1}{m!}\left(\frac{x}{2}\right)^m \qquad \text{as } x \to 0 \tag{13}$$

Together with $z = L/2$, this turns (12) into

$$\begin{aligned} \Phi\left(\rho,\phi,\frac{L}{2}\right) &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \left(\frac{\rho}{b}\right)^{2l+1} \frac{16V}{(2k+1)(2l+1)\pi^2} (-1)^l \cos\left[(2l+1)\phi\right] (-1)^k \\ &= \frac{16V}{\pi^2} \left[\overbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}}^{\pi/4} \right] \left[\sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{\rho}{b}\right)^{2l+1} \cos\left[(2l+1)\phi\right] \right] \\ &= \frac{4V}{\pi} \operatorname{Re}\left[\sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} z^{2l+1} \right] \qquad \text{where } z \equiv \frac{\rho}{b} e^{i\phi} \\ &= \frac{4V}{\pi} \operatorname{Re}\left(\tan^{-1} z\right) \end{aligned} \tag{14}$$

Notice for $z = x + iy$, if we write

$$\alpha + i\beta = \tan^{-1}(x+iy) \qquad \text{and thus} \qquad \alpha - i\beta = \tan^{-1}(x-iy) \tag{15}$$

we have

$$\begin{aligned} 2\alpha &= \tan^{-1}(x+iy) + \tan^{-1}(x-iy) && \Longrightarrow \\ \tan 2\alpha &= \frac{x+iy+x-iy}{1-(x+iy)(x-iy)} && \Longrightarrow \\ \operatorname{Re}\left[\tan^{-1}(x+iy)\right] &= \alpha = \frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right) \end{aligned} \tag{16}$$

which turns (14) into

$$\begin{aligned} \Phi\left(\rho,\phi,\frac{L}{2}\right) &= \frac{4V}{\pi}\frac{1}{2}\tan^{-1}\left(\frac{\frac{2\rho\cos\phi}{b}}{1-\frac{\rho^2}{b^2}}\right) \\ &= \frac{2V}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right) \end{aligned} \tag{17}$$

agreeing with Prob 2.13.