1. From Prob 3.23, we have the following alternative forms of the Green function

$$G(\mathbf{x},\mathbf{x}') = \frac{4}{a}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sum_{n=1}^{\infty}\frac{J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{J_{m+1}^2(x_{mn})\,x_{mn}\sinh\left(\frac{x_{mn}L}{a}\right)}\sinh\left(\frac{x_{mn}z_<}{a}\right)\sinh\left[\frac{x_{mn}(L-z_>)}{a}\right] \tag{1}$$

$$G(\mathbf{x},\mathbf{x}') = \frac{4}{L}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)\frac{I_m\left(\frac{n\pi\rho_<}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)}$$
$$\times\left[I_m\left(\frac{n\pi a}{L}\right)K_m\left(\frac{n\pi\rho_>}{L}\right)-I_m\left(\frac{n\pi\rho_>}{L}\right)K_m\left(\frac{n\pi a}{L}\right)\right] \tag{2}$$

$$G(\mathbf{x},\mathbf{x}') = \frac{8}{La^2}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sum_{k=1}^{\infty}\sin\left(\frac{k\pi z}{L}\right)\sin\left(\frac{k\pi z'}{L}\right)\sum_{n=1}^{\infty}\frac{J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{\left[\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2\right]J_{m+1}^2(x_{mn})} \tag{3}$$

Given the setup of this problem, any interior point's potential is given by

$$\begin{aligned}\Phi(\rho,\phi,z) &= -\frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G}{\partial n'}da' \\ &= -\frac{1}{4\pi}\int_0^{2\pi}d\phi'\int_0^b V\rho' d\rho'\left(\frac{\partial G}{\partial z'}\right)\bigg|_{z'=L}\end{aligned} \tag{4}$$

For all 3 alternative forms, the integration in $d\phi'$ will eliminate all ms but $m=0$. Carrying out the remaining steps gives us the following

(a) For form (1):

$$\begin{aligned}\Phi(\rho,\phi,z) &= -\frac{1}{4\pi}\cdot\frac{4}{a}\cdot 2\pi\int_0^b V\rho' d\rho'\sum_{n=1}^{\infty}\frac{J_0\left(\frac{x_{0n}\rho}{a}\right)J_0\left(\frac{x_{0n}\rho'}{a}\right)}{J_1^2(x_{0n})\,x_{0n}\sinh\left(\frac{x_{0n}L}{a}\right)}\sinh\left(\frac{x_{0n}z}{a}\right)\left(-\frac{x_{0n}}{a}\right)\cosh 0 \\ &= \frac{2V}{a^2}\sum_{n=1}^{\infty}\frac{J_0\left(\frac{x_{0n}\rho}{a}\right)}{J_1^2(x_{0n})}\frac{\sinh\left(\frac{x_{0n}z}{a}\right)}{\sinh\left(\frac{x_{0n}L}{a}\right)}\int_0^b J_0\left(\frac{x_{0n}\rho'}{a}\right)\rho' d\rho' \qquad \left(\text{use}\int J_0(x)\,x\,dx = xJ_1(x)\right) \\ &= \frac{2bV}{a}\sum_{n=1}^{\infty}\frac{J_0\left(\frac{x_{0n}\rho}{a}\right)J_1\left(\frac{x_{0n}b}{a}\right)}{J_1^2(x_{0n})\,x_{0n}}\frac{\sinh\left(\frac{x_{0n}z}{a}\right)}{\sinh\left(\frac{x_{0n}L}{a}\right)}\end{aligned} \tag{5}$$

(b) For form (2), denote

$$g(\rho,\rho') \equiv \frac{I_0\left(\frac{n\pi\rho_<}{L}\right)}{I_0\left(\frac{n\pi a}{L}\right)}\left[I_0\left(\frac{n\pi a}{L}\right)K_0\left(\frac{n\pi\rho_>}{L}\right)-I_0\left(\frac{n\pi\rho_>}{L}\right)K_0\left(\frac{n\pi a}{L}\right)\right] \tag{6}$$

then,

$$\begin{aligned}\Phi(\rho,\phi,z) &= -\frac{1}{4\pi}\cdot\frac{4}{L}\cdot 2\pi\int_0^b V\rho' d\rho'\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\left(\frac{n\pi}{L}\right)\cos(n\pi)\,g(\rho,\rho') \\ &= -\frac{2\pi V}{L^2}\sum_{n=1}^{\infty}(-1)^n n\sin\left(\frac{n\pi z}{L}\right)\underbrace{\int_0^b g(\rho,\rho')\rho' d\rho'}_{I}\end{aligned} \tag{7}$$

The evaluation of integral I depends on whether $\rho \le b$ or $\rho > b$. If $\rho \le b$, the integral must be done in two segments $\int_0^\rho + \int_\rho^b$. We will not derive the general form here but will calculate in the next part with specific values of ρ, z.

(c) For form (3):

$$\Phi(\rho,\phi,z)=-\frac{1}{4\pi}\cdot\frac{8}{La^2}\cdot 2\pi\int_0^b V\rho' d\rho' \sum_{k=1}^{\infty}\sin\left(\frac{k\pi z}{L}\right)\left(\frac{k\pi}{L}\right)\cos(k\pi)\sum_{n=1}^{\infty}\frac{J_0\left(\frac{x_{0n}\rho}{a}\right)J_0\left(\frac{x_{0n}\rho'}{a}\right)}{\left[\left(\frac{x_{0n}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2\right]J_1^2(x_{0n})}$$

$$=-\frac{4\pi V}{L^2a^2}\sum_{k=1}^{\infty}(-1)^k k\sin\left(\frac{k\pi z}{L}\right)\sum_{n=1}^{\infty}\frac{J_0\left(\frac{x_{0n}\rho}{a}\right)}{\left[\left(\frac{x_{0n}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2\right]J_1^2(x_{0n})}\int_0^b J_0\left(\frac{x_{0n}\rho'}{a}\right)\rho' d\rho'$$

$$=-\frac{4b\pi V}{L^2 a}\sum_{n=1}^{\infty}\frac{J_0\left(\frac{x_{0n}\rho}{a}\right)J_1\left(\frac{x_{0n}b}{a}\right)}{J_1^2(x_{0n})x_{0n}}\sum_{k=1}^{\infty}\frac{(-1)^k k\sin\left(\frac{k\pi z}{L}\right)}{\left(\frac{x_{0n}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2} \tag{8}$$

2. For the given parameters $\rho=0, z=L/2, b=L/4=a/2$, we can simplify the potential expression as the following

(a) For form (5):

$$\Phi(\rho=0,\phi,z=a)=V\sum_{n=1}^{\infty}\frac{J_1\left(\frac{x_{0n}}{2}\right)}{J_1^2(x_{0n})x_{0n}}\frac{\sinh(x_{0n})}{\sinh(2x_{0n})} \tag{9}$$

(b) For form (7), first note

$$g(\rho=0,\rho')=K_0\left(\frac{n\pi\rho'}{L}\right)-\frac{K_0\left(\frac{n\pi}{2}\right)}{I_0\left(\frac{n\pi}{2}\right)}I_0\left(\frac{n\pi\rho'}{L}\right) \tag{10}$$

which gives the integral

$$I=\int_0^b K_0\left(\frac{n\pi\rho'}{L}\right)\rho' d\rho'-\frac{K_0\left(\frac{n\pi}{2}\right)}{I_0\left(\frac{n\pi}{2}\right)}\int_0^b I_0\left(\frac{n\pi\rho'}{L}\right)\rho' d\rho' \tag{11}$$

We will make use of equation 10.43.1 on nist.gov

$$\int z^{\nu+1}Z_\nu(z)\,dz=z^{\nu+1}Z_{\nu+1}(z) \qquad \text{for } Z_\nu(z)=I_\nu(z) \text{ or } e^{i\nu\pi}K_\nu(z) \tag{12}$$

which gives

$$I=T_1(\rho)\Big|_{\rho=0}^{\rho=b}-T_2(\rho)\Big|_{\rho=0}^{\rho=b} \qquad \text{where}$$

$$T_1(\rho)=\left(\frac{L}{n\pi}\right)^2\left(\frac{n\pi\rho}{L}\right)\left[-K_1\left(\frac{n\pi\rho}{L}\right)\right] \tag{13}$$

$$T_2(\rho)=\frac{K_0\left(\frac{n\pi}{2}\right)}{I_0\left(\frac{n\pi}{2}\right)}\left(\frac{L}{n\pi}\right)^2\left(\frac{n\pi\rho}{L}\right)I_1\left(\frac{n\pi\rho}{L}\right) \tag{14}$$

While we can easily see $T_2(0)=0$, an easy *mistake* is to declare $T_1(0)=0$ due to its ρ factor. Because $K_1(x)$ diverges at 0, we must evaluate $T_1(0)$ with a limiting procedure. Denote $x=n\pi\rho/L$, by the asymptotic form (3.103), we have

$$T_1(0)=\left(\frac{L}{n\pi}\right)^2[-xK_1(x)]\longrightarrow-\left(\frac{L}{n\pi}\right)^2 \qquad \text{as } x\to 0 \tag{15}$$

Now we can evaluate I correctly by

$$I=T_1(b)-T_1(0)-T_2(b)$$

$$=\left(\frac{L}{n\pi}\right)^2-\frac{L^2}{4n\pi}\left[K_1\left(\frac{n\pi}{4}\right)+\frac{K_0\left(\frac{n\pi}{2}\right)}{I_0\left(\frac{n\pi}{2}\right)}I_1\left(\frac{n\pi}{4}\right)\right] \tag{16}$$

Plugging this into (7), we end up with

$$
\begin{aligned}
\Phi(\rho=0,\phi,z=L/2) &= -\frac{2\pi V}{L^2}\sum_{n=1}^{\infty}(-1)^n\, n\sin\left(\frac{n\pi}{2}\right)\cdot I \qquad \text{(only odd ns remain)}\\
&= \frac{2\pi V}{L^2}\sum_{n\text{ odd}} n\sin\left(\frac{n\pi}{2}\right)\left\{\left(\frac{L}{n\pi}\right)^2-\frac{L^2}{4n\pi}\left[K_1\left(\frac{n\pi}{4}\right)+\frac{K_0\left(\frac{n\pi}{2}\right)}{I_0\left(\frac{n\pi}{2}\right)}I_1\left(\frac{n\pi}{4}\right)\right]\right\}\\
&= \frac{2V}{\pi}\overbrace{\sum_{n\text{ odd}}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)}^{\pi/4}+\frac{V}{2}\sum_{n\text{ odd}}\sin\left(\frac{n\pi}{2}\right)\left[K_1\left(\frac{n\pi}{4}\right)+\frac{K_0\left(\frac{n\pi}{2}\right)}{I_0\left(\frac{n\pi}{2}\right)}I_1\left(\frac{n\pi}{4}\right)\right]\\
&= \frac{V}{2}+\frac{V}{2}\sum_{n=0}^{\infty}(-1)^n\left\{K_1\left[\frac{(2n+1)\pi}{4}\right]+\frac{K_0\left[\frac{(2n+1)\pi}{2}\right]}{I_0\left[\frac{(2n+1)\pi}{2}\right]}I_1\left[\frac{(2n+1)\pi}{4}\right]\right\} \qquad (17)
\end{aligned}
$$

(c) For form (8):

$$
\begin{aligned}
\Phi(\rho=0,\phi,z=a) &= -\frac{2\pi V}{L^2}\sum_{n=1}^{\infty}\frac{J_1\left(\frac{x_{0n}}{2}\right)}{J_1(x_{0n})\,x_{0n}}\sum_{k=1}^{\infty}\frac{(-1)^k k\sin\left(\frac{k\pi}{2}\right)}{\left(\frac{2x_{0n}}{L}\right)^2+\left(\frac{k\pi}{L}\right)^2}\\
&= V\sum_{n=1}^{\infty}\frac{J_1\left(\frac{x_{0n}}{2}\right)}{J_1^2(x_{0n})\,x_{0n}}\sum_{k=1}^{\infty}\frac{(-2\pi)(-1)^k k\sin\left(\frac{k\pi}{2}\right)}{(2x_{0n})^2+(k\pi)^2} \qquad (18)
\end{aligned}
$$

Again, the relation between (18) and (9) is given by the Fourier transform

$$
\frac{\sinh(x_{0n})}{\sinh(2x_{0n})}=\sum_{k=1}^{\infty}\frac{(-2\pi)(-1)^k k\sin\left(\frac{k\pi}{2}\right)}{(2x_{0n})^2+(k\pi)^2} \qquad (19)
$$

which is provable from equation (33) of my notes for Prob 3.23, by taking the derivative with respect to z' at $z'=L$.

Apparently form (18) will converge much slower than (9) or (17) since it involves the extra infinite sum in k.

A simple python script is used to calculate (9) and (17) for 10 iterations. The result is shown below.

Iteration	$\Phi(\rho,\phi,z)$ by (9)	$\Phi(\rho,\phi,z)$ by (17)
0	0.0689326815	0.0328306826
1	0.0715821867	0.0773298612
2	0.0715328168	0.0705415934
3	0.0715293025	0.0717072322
4	0.0715293676	0.0714963211
5	0.0715293730	0.0715356408
6	0.0715293729	0.0715281674
7	0.0715293729	0.0715296072
8	0.0715293729	0.0715293270
9	0.0715293729	0.0715293819