The general solution of free space real vector potential $\mathbf{A}(\mathbf{x},t)$ is a superposition of plane waves:

$$\mathbf{A}(\mathbf{x},t)=\int\frac{d^3k}{(2\pi)^3}\left[\mathbf{A}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-ickt}+\mathbf{A}^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+ickt}\right]\tag{1}$$

where the amplitude $\mathbf{A}(\mathbf{k})$ can be obtained by

$$\mathbf{A}(\mathbf{k})=\frac{1}{2}\int d^3x\mathbf{A}(\mathbf{x},t)e^{-i\mathbf{k}\cdot\mathbf{x}+ickt}\tag{2}$$

This is justified since plugging (2) into the RHS of (1) gives

$$\int\frac{d^3k}{(2\pi)^3}\frac{1}{2}\left[\int d^3x'\mathbf{A}(\mathbf{x}',t)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}+\int d^3x'\mathbf{A}(\mathbf{x}',t)e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\right]=\int d^3x'\mathbf{A}(\mathbf{x}',t)\delta(\mathbf{x}-\mathbf{x}')=\mathbf{A}(\mathbf{x},t)\tag{3}$$

Similarly for the $\mathbf{E},\mathbf{B}$ fields,

$$\mathbf{E}(\mathbf{x},t)=\int\frac{d^3k}{(2\pi)^3}\left[\mathbf{E}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-ickt}+\mathbf{E}^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+ickt}\right]\qquad\mathbf{E}(\mathbf{k})=\frac{1}{2}\int d^3x\mathbf{E}(\mathbf{x},t)e^{-i\mathbf{k}\cdot\mathbf{x}+ickt}\tag{4}$$

$$\mathbf{B}(\mathbf{x},t)=\int\frac{d^3k}{(2\pi)^3}\left[\mathbf{B}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-ickt}+\mathbf{B}^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+ickt}\right]\qquad\mathbf{B}(\mathbf{k})=\frac{1}{2}\int d^3x\mathbf{B}(\mathbf{x},t)e^{-i\mathbf{k}\cdot\mathbf{x}+ickt}\tag{5}$$

For each plane wave $\mathbf{k}$, by $\mathbf{E}=-\partial\mathbf{A}/\partial t$ and $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$, we can write

$$\mathbf{E}(\mathbf{k})=ick\mathbf{A}(\mathbf{k})\qquad\mathbf{B}(\mathbf{k})=i\mathbf{k}\times\mathbf{A}(\mathbf{k})\tag{6}$$

We can further decompose $\mathbf{A}(\mathbf{k})$ into the circular polarization basis $\boldsymbol{\epsilon}_\pm(\mathbf{k})$,

$$\mathbf{A}(\mathbf{k})=\sum_\lambda\boldsymbol{\epsilon}_\lambda(\mathbf{k})A_\lambda(\mathbf{k})\tag{7}$$

which gives

$$\mathbf{E}(\mathbf{x},t)=\sum_\lambda\int\frac{d^3k}{(2\pi)^3}\left[\mathbf{E}_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-ickt}+\mathbf{E}_\lambda^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+ickt}\right]\qquad\text{where }\mathbf{E}_\lambda(k)=ickA_\lambda(\mathbf{k})\boldsymbol{\epsilon}_\lambda(\mathbf{k})\tag{8}$$

$$\mathbf{B}(\mathbf{x},t)=\sum_\lambda\int\frac{d^3k}{(2\pi)^3}\left[\mathbf{B}_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-ickt}+\mathbf{B}_\lambda^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+ickt}\right]\qquad\text{where }\mathbf{B}_\lambda(k)=iA_\lambda(\mathbf{k})[\mathbf{k}\times\boldsymbol{\epsilon}_\lambda(\mathbf{k})]\tag{9}$$

For the plane wave in the $(\mathbf{k},\lambda)$ mode, its energy density is

$$u_\lambda(\mathbf{k})=\frac{\epsilon_0}{2}\left[\frac{\mathbf{E}_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-ickt}+\mathbf{E}_\lambda^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+ickt}}{(2\pi)^3}\right]^2+\frac{1}{2\mu_0}\left[\frac{\mathbf{B}_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-ickt}+\mathbf{B}_\lambda^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+ickt}}{(2\pi)^3}\right]^2\tag{10}$$

Using $\boldsymbol{\epsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\epsilon}_\lambda(\mathbf{k})=0$ and $\boldsymbol{\epsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\epsilon}_\lambda^*(\mathbf{k})=1$, this turns into

$$u_\lambda(\mathbf{k})=\frac{1}{(2\pi)^6}\left[\epsilon_0\mathbf{E}_\lambda(\mathbf{k})\cdot\mathbf{E}_\lambda^*(\mathbf{k})+\frac{1}{\mu_0}\mathbf{B}_\lambda(\mathbf{k})\cdot\mathbf{B}_\lambda^*(\mathbf{k})\right]=\frac{2\epsilon_0c^2k^2}{(2\pi)^6}|A_\lambda(\mathbf{k})|^2\tag{11}$$

According to the problem statement, the number of photons for each mode $(\mathbf{k},\lambda)$ is to be defined as this plane wave's energy divided by $\hbar ck$. But clearly the energy of the whole plane wave is infinity. Let's use the "Big Box" trick. Consider a big box L^3 where the waves have cyclic boundary conditions, i.e.,

$$k_iL=2n_i\pi\qquad\text{for some integer }n_i\tag{12}$$

Knowing we will eventually take the $L\to\infty$ limit, we can write the number of photons of mode $(\mathbf{k},\lambda)$ as

$$N_\lambda(\mathbf{k})=\frac{L^3u_\lambda(\mathbf{k})}{\hbar ck}=\frac{L^3}{(2\pi)^6}\frac{2\epsilon_0ck}{\hbar}|A_\lambda(\mathbf{k})|^2\tag{13}$$

Also note by (7)

$$\mathbf{A}(\mathbf{k})\cdot\mathbf{A}^*(\mathbf{k}) = |A_+(\mathbf{k})|^2 + |A_-(\mathbf{k})|^2 \tag{14}$$

hence

$$\begin{aligned}
N(\mathbf{k}) = N_+(\mathbf{k}) + N_-(\mathbf{k}) &= \frac{L^3}{(2\pi)^6}\frac{2\epsilon_0 ck}{\hbar}\mathbf{A}(\mathbf{k})\cdot\mathbf{A}^*(\mathbf{k}) \qquad \text{by (6)} \implies \\
&= \frac{L^3}{(2\pi)^6}\frac{2\epsilon_0 ck}{\hbar}\cdot\frac{1}{2c^2k^2}\left[\mathbf{E}(\mathbf{k})\cdot\mathbf{E}^*(\mathbf{k}) + c^2\mathbf{B}(\mathbf{k})\cdot\mathbf{B}^*(\mathbf{k})\right] \\
&= \frac{L^3}{(2\pi)^6}\frac{\epsilon_0}{\hbar ck}\left[\mathbf{E}(\mathbf{k})\cdot\mathbf{E}^*(\mathbf{k}) + c^2\mathbf{B}(\mathbf{k})\cdot\mathbf{B}^*(\mathbf{k})\right]
\end{aligned} \tag{15}$$

For this big box, the total number of photons for all wave vectors will be $\sum_i N(\mathbf{k}_i)$, where the sum goes over all the integer grid points defined in (12).

When $L\to\infty$, the sum becomes integral, but due to (12), the integral measure will be $d^3k\cdot(2\pi/L)^3$ (which is necessary to fix the dimensions), i.e.,

$$\begin{aligned}
N = \lim_{L\to\infty}\int d^3k\left(\frac{2\pi}{L}\right)^3 N(\mathbf{k}) &= \lim_{L\to\infty}\int d^3k\left(\frac{2\pi}{L}\right)^3\cdot\frac{L^3}{(2\pi)^6}\frac{\epsilon_0}{\hbar ck}\left[\mathbf{E}(\mathbf{k})\cdot\mathbf{E}^*(\mathbf{k}) + c^2\mathbf{B}(\mathbf{k})\cdot\mathbf{B}^*(\mathbf{k})\right] \\
&= \frac{1}{(2\pi)^3}\frac{\epsilon_0}{\hbar c}\underbrace{\int\frac{d^3k}{k}\left[\mathbf{E}(\mathbf{k})\cdot\mathbf{E}^*(\mathbf{k}) + c^2\mathbf{B}(\mathbf{k})\cdot\mathbf{B}^*(\mathbf{k})\right]}_{I}
\end{aligned} \tag{16}$$

where we see the parameter L has dropped out.

Using the integral representation of $\mathbf{E}(\mathbf{k})$ and $\mathbf{B}(\mathbf{k})$ in (4) and (5), we have

$$\begin{aligned}
I &= \frac{1}{4}\int\frac{d^3k}{k}\int d^3x\int d^3x'\left[\mathbf{E}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x}',t) + c^2\mathbf{B}(\mathbf{x},t)\cdot\mathbf{B}(\mathbf{x}',t)\right]e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \\
&= \frac{1}{4}\int d^3x\int d^3x'\left[\mathbf{E}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x}',t) + c^2\mathbf{B}(\mathbf{x},t)\cdot\mathbf{B}(\mathbf{x}',t)\right]\underbrace{\int d^3k\left[\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k}\right]}_{J}
\end{aligned} \tag{17}$$

where J is the famous Fourier transform of the Coulomb potential, evaluated below

$$\begin{aligned}
\int d^3k\left[\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k}\right] &= \int_0^\infty kdk\int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\phi e^{-ik|\mathbf{x}-\mathbf{x}'|\cos\theta} \\
&= 2\pi\int_0^\infty kdk\left(\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}-e^{-ik|\mathbf{x}-\mathbf{x}'|}}{ik|\mathbf{x}-\mathbf{x}'|}\right) \\
&= \frac{4\pi}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{Im}\int_0^\infty e^{ik|\mathbf{x}-\mathbf{x}'|}dk \\
&= \frac{4\pi}{|\mathbf{x}-\mathbf{x}'|}\lim_{\mu\to 0}\left(\mathrm{Im}\int_0^\infty e^{ik|\mathbf{x}-\mathbf{x}'|}e^{-\mu k}\right) \\
&= \frac{4\pi}{|\mathbf{x}-\mathbf{x}'|}\lim_{\mu\to 0}\left[\mathrm{Im}\left(\frac{1}{\mu-i|\mathbf{x}-\mathbf{x}'|}\right)\right] \\
&= \frac{4\pi}{|\mathbf{x}-\mathbf{x}'|}\lim_{\mu\to 0}\left(\frac{|\mathbf{x}-\mathbf{x}'|}{\mu^2+|\mathbf{x}-\mathbf{x}'|^2}\right) \\
&= \frac{4\pi}{|\mathbf{x}-\mathbf{x}'|^2}
\end{aligned} \tag{18}$$

Putting everything back together in (16), we finally get

$$N = \frac{\epsilon_0}{8\pi^2\hbar c}\int d^3x\int d^3x'\left[\frac{\mathbf{E}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x}',t) + c^2\mathbf{B}(\mathbf{x},t)\cdot\mathbf{B}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|^2}\right] \tag{19}$$

This is $1/2$ of the desired result, I cannot find where in my derivation I have mistakenly generated this factor.