1. **Prob 7.12**

   (a) Starting with

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t)+\nabla\cdot\mathbf{J}(\mathbf{x},t)=0 \tag{1}$$
$$\epsilon_0\nabla\cdot\mathbf{E}(\mathbf{x},t)=\rho(\mathbf{x},t) \tag{2}$$

Let's write all time-dependent functions in their inverse Fourier integral form:

$$\frac{\partial}{\partial t}\int d\omega\rho(\mathbf{x},\omega)e^{-i\omega t}+\nabla\cdot\int d\omega\mathbf{J}(\mathbf{x},\omega)e^{-i\omega t}=0 \tag{3}$$
$$\epsilon_0\nabla\cdot\int d\omega\mathbf{E}(\mathbf{x},t)e^{-i\omega t}=\int d\omega\rho(\mathbf{x},\omega)e^{-i\omega t} \tag{4}$$

Combining (3) and (4), together with Ohm's law $\mathbf{J}(\mathbf{x},\omega)=\sigma(\omega)\mathbf{E}(\mathbf{x},\omega)$, we have

$$\int d\omega\rho(\mathbf{x},\omega)\left[-i\omega+\frac{\sigma(\omega)}{\epsilon_0}\right]e^{-i\omega t}=0 \tag{5}$$

Then by orthogonality of Fourier basis, we can conclude

$$[\sigma(\omega)-i\omega\epsilon_0]\rho(\mathbf{x},\omega)=0 \tag{6}$$

   (b) If we write

$$\sigma(\omega)=\frac{\sigma_0}{1-i\omega\tau}=\frac{\epsilon_0\omega_p^2\tau}{1-i\omega\tau} \tag{7}$$

then (6) becomes

$$\epsilon_0\left[\frac{\omega_p^2\tau}{1-i\omega\tau}-i\omega\right]\rho(\mathbf{x},\omega)=0 \qquad\Longrightarrow\qquad \underbrace{\left(\omega^2+\frac{i\omega}{\tau}-\omega_p^2\right)}_{f(\omega)}\rho(\mathbf{x},\omega)=0 \tag{8}$$

The inverse Fourier transform of (8) gives

$$\int\left(\omega^2+\frac{i\omega}{\tau}-\omega_p^2\right)\rho(\mathbf{x},\omega)e^{-i\omega t}d\omega=0 \qquad\Longrightarrow\qquad \left(\frac{d^2}{dt^2}+\frac{1}{\tau}\frac{d}{dt}+\omega_p^2\right)\rho(\mathbf{x},t)=0 \tag{9}$$

which is the differential equation $\rho(\mathbf{x},t)$ must satisfy.
Let

$$\widetilde{\omega}_\pm=\frac{1}{2\tau}\left(-i\pm\sqrt{4\omega_p^2\tau^2-1}\right) \tag{10}$$

be the two roots of the equation $f(\omega)=0$, we see that

$$\rho(\mathbf{x},t)=ae^{-i\widetilde{\omega}_+t}+be^{-i\widetilde{\omega}_-t} \tag{11}$$

is the general solution of (9).
When $\omega_p\tau\gg1$, the solution has the form

$$\rho(\mathbf{x},t)=e^{-t/2\tau}\left(ae^{-i\omega_pt}+be^{i\omega_pt}\right) \tag{12}$$

2. **Prob 7.13**

   (a) By (7.61), the wave number in the ionosphere is

$$k'=\frac{\sqrt{\omega^2-\omega_p^2}}{c} \tag{13}$$

Let k'_z and k'_x be the vertical and horizontal component of the wave number which requires

$$k_x'^2 + k_z'^2 = k'^2 \tag{14}$$

Phase continuity along the interface of the ionosphere boundary needs

$$k'_x = k \sin i \tag{15}$$

where k is the wave number of free space, i.e.,

$$k'_x = \frac{\omega}{c} \sin i \tag{16}$$

Thus

$$k_z'^2 = k'^2 - k_x'^2 = \frac{\omega^2}{c^2} \cos^2 i - \frac{\omega_p^2}{c^2} \tag{17}$$

which gives the critical angle

$$i = \cos^{-1} \frac{\omega_p}{\omega} \tag{18}$$

greater than which k'_z will become purely imaginary, hence total reflection.

(b) Plugging in the numbers, we have

$$\cos^2 i = \frac{300^2}{500^2 + 300^2} = \frac{\omega_p^2}{\omega^2} = \frac{ne^2}{\epsilon_0 m \omega^2} \qquad \Longrightarrow \qquad n = \frac{9}{34} \cdot \frac{m\epsilon_0}{e^2} \left(\frac{2\pi c}{\lambda}\right)^2 \approx 6.7 \times 10^{11} \mathrm{m}^{-3} \tag{19}$$