In these notes, we will derive the Mie scattering solution in details.

Recall in free space, the general multipole solution to the Maxwell equations is given in (9.122)

$$\mathbf{E} = Z_0 \sum_{l,m} \left\{ \frac{i}{k} a_E(l,m) \nabla \times [f_l(kr) \mathbf{X}_{lm}] + a_M(l,m) g_l(kr) \mathbf{X}_{lm} \right\} \tag{1}$$

$$\mathbf{H} = \sum_{l,m} \left\{ a_E(l,m) f_l(kr) \mathbf{X}_{lm} - \frac{i}{k} a_M(l,m) \nabla \times [g_l(kr) \mathbf{X}_{lm}] \right\} \tag{2}$$

where $f_l(kr), g_l(kr)$ are solutions to the radial equation (9.81)

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2} \right] z_l(r) = 0 \tag{3}$$

which are linear combinations of $j_l(kr)$ and $n_l(kr)$. For free space $k^2 = \omega^2/c^2 = \omega^2 \mu_0 \epsilon_0$, and $Z_0 = \sqrt{\mu_0/\epsilon_0}$.

This solution form is easily generalized for any isotropic medium with permittivity ϵ and permeability μ (e.g., for medium with conductivity σ, we can take the permittivity to be complex $\epsilon = \epsilon_0(\epsilon_r + i\sigma/\omega\epsilon_0)$ and treat the medium as free of charge or current). The generalized form is

$$\mathbf{E} = \sum_{l,m} \left\{ a(l,m) g_l(kr) \mathbf{X}_{lm} + \frac{i}{k} b(l,m) \nabla \times [f_l(kr) \mathbf{X}_{lm}] \right\} \tag{4}$$

$$\mathbf{H} = \sqrt{\frac{\epsilon}{\mu}} \sum_{l,m} \left\{ -\frac{i}{k} a(l,m) \nabla \times [g_l(kr) \mathbf{X}_{lm}] + b(l,m) f_l(kr) \mathbf{X}_{lm} \right\} \tag{5}$$

where $k^2 = \omega^2 \mu\epsilon$ can be complex, and $a(l,m), b(l,m)$ will be determined by boundary conditions.

1. **Multipole expansion of plane wave**

   For the Mie scattering problem, consider the incident plane wave (propagating in the $\hat{\mathbf{z}}$ direction) in free space

   $$\mathbf{E} = \boldsymbol{\epsilon} e^{ik_0 z} \qquad\qquad \mathbf{H} = \frac{1}{Z_0} \hat{\mathbf{z}} \times \boldsymbol{\epsilon} e^{ik_0 z} \tag{6}$$

   with $k_0 = \omega\sqrt{\epsilon_0 \mu_0}$ and the polarization vector $\boldsymbol{\epsilon}$ (possibly complex e.g., circularly polarized) perpendicular to $\hat{\mathbf{z}}$.

   We can also set f_l, g_l in (4), (5) to j_l since the plane wave must be regular at the origin.

   Plugging (10.45)

   $$e^{ik_0 z} = \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)} j_l(k_0 r) Y_{l0}(\theta) \tag{7}$$

   into the multipole expansion of $\mathbf{E}$ and $\mathbf{H}$ gives

   $$\boldsymbol{\epsilon} \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)} j_l(k_0 r) Y_{l0}(\theta) = \sum_{l,m} \left\{ a(l,m) j_l(k_0 r) \mathbf{X}_{lm} + \frac{i}{k_0} b(l,m) \nabla \times [j_l(k_0 r) \mathbf{X}_{lm}] \right\} \tag{8}$$

   $$\hat{\mathbf{z}} \times \boldsymbol{\epsilon} \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)} j_l(k_0 r) Y_{l0}(\theta) = \sum_{l,m} \left\{ -\frac{i}{k_0} a(l,m) \nabla \times [j_l(k_0 r) \mathbf{X}_{lm}] + b(l,m) j_l(k_0 r) \mathbf{X}_{lm} \right\} \tag{9}$$

   Recall that

   $$\mathbf{X}_{lm} = \frac{1}{i\sqrt{l(l+1)}} \boldsymbol{\Phi}_{lm} \qquad\qquad \nabla \times [f(r) \boldsymbol{\Phi}_{lm}] = -\frac{l(l+1)}{r} f \mathbf{Y}_{lm} - \frac{1}{r}\frac{d(rf)}{dr} \boldsymbol{\Psi}_{lm} \tag{10}$$

   where $\boldsymbol{\Phi}_{lm}, \boldsymbol{\Psi}_{lm}$ are transverse and $\mathbf{Y}_{lm}$ is radial, and they are orthogonal functions over the solid angles.

   Dotting both sides of (8) and (9) with $\mathbf{X}^*_{lm}$ and integrating over the solid angles, we have

   $$a(l,m) j_l(k_0 r) = \sum_{l'=0}^{\infty} i^{l'} \sqrt{4\pi(2l'+1)} j_{l'}(k_0 r) \int \left(\mathbf{X}^*_{lm} \cdot \boldsymbol{\epsilon} \right) Y_{l'0}(\theta) \, d\Omega \tag{11}$$

   $$b(l,m) j_l(k_0 r) = \sum_{l'=0}^{\infty} i^{l'} \sqrt{4\pi(2l'+1)} j_{l'}(k_0 r) \int \left[\mathbf{X}^*_{lm} \cdot (\hat{\mathbf{z}} \times \boldsymbol{\epsilon}) \right] Y_{l'0}(\theta) \, d\Omega \tag{12}$$

Since $\left(\mathbf{X}_{lm}^{*}\cdot\boldsymbol{\epsilon}\right)Y_{l'0}\propto\mathbf{L}^{*}Y_{lm}^{*}\cdot\boldsymbol{\epsilon}Y_{l'0}=\left[\left(\mathbf{L}^{*}\cdot\boldsymbol{\epsilon}\right)Y_{lm}^{*}\right]Y_{l'0}$, and $\mathbf{L}^{*}\cdot\boldsymbol{\epsilon}$ is a linear combination of $L_{\pm}$ and L_z, the integration in (11) vanishes unless $l'=l$ and $m=\pm1$. The same argument applies for (12) because $\hat{\mathbf{z}}\times\boldsymbol{\epsilon}$ is in the x-y plane.

In Jackson section (10.3), the polarization vector $\boldsymbol{\epsilon}$ was chosen to be $\sqrt{2}\boldsymbol{\epsilon}_{\pm}=\hat{\mathbf{x}}\pm i\hat{\mathbf{y}}$ (unnormalized), then (11) gives

$$a\left(l,\pm1\right)j_l\left(k_0r\right)=i^l\sqrt{4\pi\left(2l+1\right)}j_l\left(k_0r\right)\int\frac{1}{\sqrt{l\left(l+1\right)}}\overbrace{L_{\mp}Y_{l,\pm1}^{*}\left(\theta,\phi\right)}^{\sqrt{l(l+1)}Y_{l0}^{*}}Y_{l0}\left(\theta\right)d\Omega\qquad\Longrightarrow$$

$$a\left(l,\pm1\right)=i^l\sqrt{4\pi\left(2l+1\right)}\tag{13}$$

With $\hat{\mathbf{z}}\times\boldsymbol{\epsilon}_{\pm}=\mp i\boldsymbol{\epsilon}_{\pm}$, we have

$$b\left(l,\pm1\right)=\mp ia\left(l,\pm1\right)\tag{14}$$

In summary, for the incident plane wave with circular polarization $\boldsymbol{\epsilon}_{\pm}$,

$$\mathbf{E}_{\text{inc},\pm}=\sqrt{2}\boldsymbol{\epsilon}_{\pm}e^{ik_0z}\qquad\qquad\mathbf{H}_{\text{inc},\pm}=\mp\frac{i}{Z_0}\mathbf{E}_{\text{inc}}\tag{15}$$

its multipole expansion is

$$\mathbf{E}_{\text{inc},\pm}=\sum_{l=1}^{\infty}i^l\sqrt{4\pi\left(2l+1\right)}\left\{j_l\left(k_0r\right)\mathbf{X}_{l,\pm1}\pm\frac{1}{k_0}\boldsymbol{\nabla}\times\left[j_l\left(k_0r\right)\mathbf{X}_{l,\pm1}\right]\right\}\tag{16}$$

$$\mathbf{H}_{\text{inc},\pm}=\sqrt{\frac{\epsilon_0}{\mu_0}}\sum_{l=1}^{\infty}i^l\sqrt{4\pi\left(2l+1\right)}\left\{-\frac{i}{k_0}\boldsymbol{\nabla}\times\left[j_l\left(k_0r\right)\mathbf{X}_{l,\pm1}\right]\mp ij_l\left(k_0r\right)\mathbf{X}_{l,\pm1}\right\}\tag{17}$$

2. **Scattered and internal wave**

Without loss of generality, let the incident wave be polarized corresponding to $m=1$ or $m=-1$ (arbitrary polarization is the linear superposition of these two orthogonal cases). Let $\mathbf{E}_{\text{sc}},\mathbf{H}_{\text{sc}}$ be the scattered wave, and let $\mathbf{E}_{\text{int}},\mathbf{H}_{\text{int}}$ be the internal wave within the spherical boundary. For them to satisfy Maxwell equations in the corresponding medium, they necessarily have form (4) and (5). Since internal region contains the origin, its radial funciton must be $j_l\left(nk_0r\right)$, where $n=\sqrt{\mu\epsilon}/\sqrt{\mu_0\epsilon_0}$ is the index of refraction of the sphere. The expected far-zone behavior of the scattered wave is an outgoing wave e^{ik_0r}/r, so its radial function must be $h_l^{(1)}\left(k_0r\right)$. Therefore, we can write

$$\mathbf{E}_{\text{sc},\pm}=\sum_{l,m}i^l\sqrt{4\pi\left(2l+1\right)}\left\{a_{lm}h_l^{(1)}\left(k_0r\right)\mathbf{X}_{lm}+\frac{i}{k_0}b_{lm}\boldsymbol{\nabla}\times\left[h_l^{(1)}\left(k_0r\right)\mathbf{X}_{lm}\right]\right\}\tag{18}$$

$$\mathbf{H}_{\text{sc},\pm}=\sqrt{\frac{\epsilon_0}{\mu_0}}\sum_{l,m}i^l\sqrt{4\pi\left(2l+1\right)}\left\{-\frac{i}{k_0}a_{lm}\boldsymbol{\nabla}\times\left[h_l^{(1)}\left(k_0r\right)\mathbf{X}_{lm}\right]+b_{lm}h_l^{(1)}\left(k_0r\right)\mathbf{X}_{lm}\right\}\tag{19}$$

$$\mathbf{E}_{\text{int},\pm}=\sum_{l,m}i^l\sqrt{4\pi\left(2l+1\right)}\left\{c_{lm}j_l\left(nk_0r\right)\mathbf{X}_{lm}+\frac{i}{nk_0}d_{lm}\boldsymbol{\nabla}\times\left[j_l\left(nk_0r\right)\mathbf{X}_{lm}\right]\right\}\tag{20}$$

$$\mathbf{H}_{\text{int},\pm}=\sqrt{\frac{\epsilon}{\mu}}\sum_{l,m}i^l\sqrt{4\pi\left(2l+1\right)}\left\{-\frac{i}{nk_0}c_{lm}\boldsymbol{\nabla}\times\left[j_l\left(nk_0r\right)\mathbf{X}_{lm}\right]+d_{lm}j_l\left(nk_0r\right)\mathbf{X}_{lm}\right\}\tag{21}$$

where we have inserted the factor $i^l\sqrt{4\pi\left(2l+1\right)}$ for each l-term for later convenience.

The boundary condition at the interface requires normal $\mathbf{B},\mathbf{D}$ to be continuous, and tangential $\mathbf{E},\mathbf{H}$ to be continuous. Let's consider tangential requirement for now. At $r=R$, due to the orthonality of VSH, the coefficients for the tangential component $\boldsymbol{\Phi}_{lm}$ and $\boldsymbol{\Psi}_{lm}$ must match between $\mathbf{E}_{\text{inc}}+\mathbf{E}_{\text{sc}}/\mathbf{E}_{\text{int}}$ as well as between $\mathbf{H}_{\text{inc}}+\mathbf{H}_{\text{sc}}/\mathbf{H}_{\text{int}}$ (recall (10)).

In other words,

$$\boldsymbol{\Phi}_{lm}\text{ for }\mathbf{E}_{\text{tan}}:\qquad\delta_{m,\pm1}j_l\left(k_0R\right)+a_{lm}h_l^{(1)}\left(k_0R\right)=c_{lm}j_l\left(nk_0R\right)\tag{22}$$

$$\boldsymbol{\Psi}_{lm}\text{ for }\mathbf{E}_{\text{tan}}:\qquad\pm\delta_{m,\pm1}\left.\frac{d\left[rj_l\left(k_0r\right)\right]}{dr}\right|_{r=R}+ib_{lm}\left.\frac{d\left[rh_l^{(1)}\left(k_0r\right)\right]}{dr}\right|_{r=R}=\frac{id_{lm}}{n}\left.\frac{d\left[rj_l\left(nk_0r\right)\right]}{dr}\right|_{r=R}\tag{23}$$

$$\boldsymbol{\Phi}_{lm}\text{ for }\mathbf{H}_{\text{tan}}:\qquad\sqrt{\frac{\epsilon_0}{\mu_0}}\left[\mp i\delta_{m,\pm1}j_l\left(k_0R\right)+b_{lm}h_l^{(1)}\left(k_0R\right)\right]=\sqrt{\frac{\epsilon}{\mu}}d_{lm}j_l\left(nk_0R\right)\tag{24}$$

$$\boldsymbol{\Psi}_{lm}\text{ for }\mathbf{H}_{\text{tan}}:\qquad\sqrt{\frac{\epsilon_0}{\mu_0}}\left\{\delta_{m,\pm1}\left.\frac{d\left[rj_l\left(k_0r\right)\right]}{dr}\right|_{r=R}+a_{lm}\left.\frac{d\left[rh_l^{(1)}\left(k_0r\right)\right]}{dr}\right|_{r=R}\right\}=\sqrt{\frac{\epsilon}{\mu}}\frac{c_{lm}}{n}\left.\frac{d\left[rj_l\left(nk_0r\right)\right]}{dr}\right|_{r=R}\tag{25}$$

For $m \neq \pm 1$, the unknowns $a_{lm}, b_{lm}, c_{lm}, d_{lm}$ satisfy a homogeneous system of equations, which has only the trivial solution for general values of ϵ, μ, R. We see the conclusion, that the scattered and internal wave only have m component matching that of the incident wave, is a consequence of the boundary condition about tangential $\mathbf{E}, \mathbf{H}$ continuity.

With the simplifying constants

$$J_l \equiv j_l(k_0 R) \qquad H_l \equiv h_l^{(1)}(k_0 R) \qquad N_l \equiv j_l(nk_0 R) \tag{26}$$

$$J_l' \equiv \left.\frac{d\left[k_0 r j_l(k_0 r)\right]}{d(k_0 r)}\right|_{r=R} \qquad H_l' \equiv \left.\frac{d\left[k_0 r h_l^{(1)}(k_0 r)\right]}{d(k_0 r)}\right|_{r=R} \qquad N_l' \equiv \left.\frac{d\left[nk_0 r j_l(nk_0 r)\right]}{d(nk_0 r)}\right|_{r=R} \tag{27}$$

the inhomogeneous system of equations for $m = \pm 1$ can be written as

$$J_l + a_{l,\pm1} H_l = c_{l,\pm1} N_l \tag{28}$$

$$\pm J_l' + i b_{l,\pm1} H_l' = \frac{i d_{l,\pm1}}{n} N_l' \tag{29}$$

$$\mp i J_l + b_{l,\pm1} H_l = \frac{n\mu_0}{\mu} d_{l,\pm1} N_l \tag{30}$$

$$J_l' + a_{l,\pm1} H_l' = \frac{\mu_0}{\mu} c_{l,\pm1} N_l' \tag{31}$$

for which the solutions are

$$a_{l,\pm1} = \frac{\mu_0 J_l N_l' - \mu J_l' N_l}{\mu H_l' N_l - \mu_0 H_l N_l'} \tag{32}$$

$$b_{l,\pm1} = \pm \frac{i\left(\mu J_l N_l' - n^2 \mu_0 J_l' N_l\right)}{\mu H_l N_l' - n^2 \mu_0 H_l' N_l} \tag{33}$$

$$c_{l,\pm1} = \frac{\mu\left(J_l H_l' - J_l' H_l\right)}{\mu H_l' N_l - \mu_0 H_l N_l'} \tag{34}$$

$$d_{l,\pm1} = \pm \frac{in\mu\left(J_l H_l' - J_l' H_l\right)}{\mu H_l N_l' - n^2 \mu_0 H_l' N_l} \tag{35}$$

With this solution, we find that the normal $\mathbf{B}, \mathbf{D}$ continuity constraints, i.e.,

$$\mathbf{Y}_{lm} \text{ for } \mathbf{D}_{\text{norm}}: \qquad \epsilon_0\left(\pm\delta_{m,\pm1} J_l + i b_{lm} H_l\right) = \epsilon \frac{i d_{lm}}{n} N_l \tag{36}$$

$$\mathbf{Y}_{lm} \text{ for } \mathbf{B}_{\text{norm}}: \qquad \mu_0 \sqrt{\frac{\epsilon_0}{\mu_0}}\left(\delta_{m,\pm1} J_l + a_{lm} H_l\right) = \mu \sqrt{\frac{\epsilon}{\mu}} \frac{c_{lm}}{n} N_l \tag{37}$$

are automatically satisfied (e.g., for $m = \pm 1$, (36), (37) are equivalent to (30), (28) respectively). The seemingly redundant normal constraints should be no surprise at all, since the general form (1) and (2) were derived using the divergence equations.

In summary, the scattered and internal wave are

$$\mathbf{E}_{\text{sc},\pm} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left\{ a_{l,\pm1} h_l^{(1)}(k_0 r) \mathbf{X}_{l,\pm1} + \frac{i}{k_0} b_{l,\pm1} \boldsymbol{\nabla} \times \left[h_l^{(1)}(k_0 r) \mathbf{X}_{l,\pm1}\right] \right\} \tag{38}$$

$$\mathbf{H}_{\text{sc},\pm} = \sqrt{\frac{\epsilon_0}{\mu_0}} \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left\{ -\frac{i}{k_0} a_{l,\pm1} \boldsymbol{\nabla} \times \left[h_l^{(1)}(k_0 r) \mathbf{X}_{l,\pm1}\right] + b_{l,\pm1} h_l^{(1)}(k_0 r) \mathbf{X}_{l,\pm1} \right\} \tag{39}$$

$$\mathbf{E}_{\text{int},\pm} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left\{ c_{l,\pm1} j_l(nk_0 r) \mathbf{X}_{l,\pm1} + \frac{i}{nk_0} d_{l,\pm1} \boldsymbol{\nabla} \times \left[j_l(nk_0 r) \mathbf{X}_{l,\pm1}\right] \right\} \tag{40}$$

$$\mathbf{H}_{\text{int},\pm} = \sqrt{\frac{\epsilon}{\mu}} \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left\{ -\frac{i}{nk_0} c_{l,\pm1} \boldsymbol{\nabla} \times \left[j_l(nk_0 r) \mathbf{X}_{l,\pm1}\right] + d_{l,\pm1} j_l(nk_0 r) \mathbf{X}_{l,\pm1} \right\} \tag{41}$$

with the coefficients given in (32)–(35).

It is worth mentioning that when ϵ is complex (e.g., the medium has a non-zero conductivity σ), the solutions above keep their forms while allowing a complex n hence consequently complex argument in the spherical Bessel function $j_l(nk_0 r)$ thanks to the analyticity of j_l. When nk_0 is complex, its imaginary part dictates the attenuation of the wave in the medium, i.e., skin effect, although in a much more complicated manner than the simple reflection/transmission by planar interface.