(*The calculation is extremely tedious, we will only work in details for the axial field so we can verify part (b).*)

1. From problem 5.3, we have

$$B_z(0,z) = \frac{\mu_0 NI}{2}(\cos\theta_1 + \cos\theta_2) = \frac{\mu_0 NI}{2}\left[\frac{L/2+z}{\sqrt{a^2+(L/2+z)^2}} + \frac{L/2-z}{\sqrt{a^2+(L/2-z)^2}}\right] \tag{1}$$

Observe that

$$\begin{aligned}\frac{d}{dz}\left[\frac{L/2\pm z}{\sqrt{a^2+(L/2\pm z)^2}}\right] &= \frac{\pm 1}{\sqrt{a^2+(L/2\pm z)^2}} + \frac{(L/2\pm z)(-1/2)(\pm 2)(L/2\pm z)}{\sqrt{a^2+(L/2\pm z)^2}^3}\\ &= \frac{\pm\left[a^2+(L/2\pm z)^2\right] \mp (L/2\pm z)^2}{\sqrt{a^2+(L/2\pm z)^2}^3} = \frac{\pm a^2}{\sqrt{a^2+(L/2\pm z)^2}^3} \end{aligned}\tag{2}$$

$$\frac{d^2}{dz^2}\left[\frac{L/2\pm z}{\sqrt{a^2+(L/2\pm z)^2}}\right] = \frac{\pm a^2(-3/2)(\pm 2)(L/2\pm z)}{\sqrt{a^2+(L/2\pm z)^2}^5} = \frac{-3a^2(L/2\pm z)}{\sqrt{a^2+(L/2\pm z)^2}^5} \tag{3}$$

By problem 5.4, we can estimate the axial component at $(\rho = a^-, z)$ to be

$$\begin{aligned} B_z\left(a^-,z\right) \approx & B_z(0,z) - \frac{a^2}{4}\left[\frac{\partial^2 B_z(0,z)}{\partial z^2}\right]\\ =&\frac{\mu_0 NI}{2}\overbrace{\left[\frac{L/2+z}{\sqrt{a^2+(L/2+z)^2}} + \frac{L/2-z}{\sqrt{a^2+(L/2-z)^2}}\right]}^{X}\\ &+\frac{\mu_0 NI}{2}\cdot\frac{3a^4}{4}\underbrace{\left[\frac{L/2+z}{\sqrt{a^2+(L/2+z)^2}^5} + \frac{L/2-z}{\sqrt{a^2+(L/2-z)^2}^5}\right]}_{Y} \end{aligned}\tag{4}$$

In anticipation of the result in (b) of this problem, we are going to expand (4) up to $O\left(1/L^4\right)$. Recall that the Taylor expansion

$$(1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \frac{35x^4}{128} + \cdots \tag{5}$$

gives us (yes, it's absolutely necessary to work on all these terms to get to the $O(1/L^4)$ accuracy)

$$\begin{aligned}\frac{1}{\sqrt{a^2+(L/2\pm z)^2}} &= \frac{1}{L/2}\left[1 \pm \frac{4z}{L} + \frac{4\left(a^2+z^2\right)}{L^2}\right]^{-1/2}\\ &= \frac{2}{L}\left\{1 - \frac{1}{2}\left[\pm\frac{4z}{L} + \frac{4\left(a^2+z^2\right)}{L^2}\right] + \frac{3}{8}\left[\pm\frac{4z}{L} + \frac{4\left(a^2+z^2\right)}{L^2}\right]^2\right.\\ &\qquad \left. - \frac{5}{16}\left[\pm\frac{4z}{L} + \frac{4\left(a^2+z^2\right)}{L^2}\right]^3 + \frac{35}{128}\left[\pm\frac{4z}{L} + \frac{4\left(a^2+z^2\right)}{L^2}\right]^4 + \cdots\right\}\\ &= \frac{2}{L}\left\{1 \mp \frac{2z}{L} - \frac{2\left(a^2+z^2\right)}{L^2} + \frac{3}{8}\left[\frac{16z^2}{L^2} \pm \frac{32z\left(a^2+z^2\right)}{L^3} + \frac{16\left(a^2+z^2\right)^2}{L^4}\right]\right.\\ &\qquad \left. - \frac{5}{16}\left[\pm\frac{64z^3}{L^3} + \frac{192z^2\left(a^2+z^2\right)}{L^4}\right] + \frac{35}{128}\frac{256z^4}{L^4} + O\left(\frac{1}{L^5}\right)\right\}\\ &\approx \frac{2}{L}\left(1 \mp \frac{2z}{L} + \frac{4z^2-2a^2}{L^2} \pm \frac{12za^2-8z^3}{L^3} + \frac{6a^4-48a^2z^2+16z^4}{L^4}\right) \end{aligned}\tag{6}$$

This gives

$$X \approx 2\left(1+\frac{4z^2-2a^2}{L^2}+\frac{6a^4-48a^2z^2+16z^4}{L^4}\right)+\frac{4z}{L}\left(-\frac{2z}{L}+\frac{12za^2-8z^3}{L^3}\right)$$
$$=2-\frac{4a^2}{L^2}+\frac{12a^4-48a^2z^2}{L^4} \tag{7}$$

For approximating $Y$ up to $O(1/L^4)$, we only need to use

$$\frac{1}{\sqrt{a^2+(L/2\pm z)^2}^5}\approx\frac{32}{L^5} \tag{8}$$

which gives

$$Y\approx\frac{32}{L^4} \tag{9}$$

Thus finally

$$B_z\left(a^-,z\right)\approx\mu_0 NI\left(1-\frac{2a^2}{L^2}+\frac{6a^4-24a^2z^2}{L^4}+\frac{12a^4}{L^4}\right)$$
$$=\mu_0 NI\left(1-\frac{2a^2}{L^2}-\frac{24a^2z^2}{L^4}+\frac{18a^4}{L^4}\right) \tag{10}$$

2. Applying Ampère's law to the rectangular loop with long sides along the $z$ direction above and below the solenoid wall, we have

$$B_z\left(a^-,z\right)-B_z\left(a^+,z\right)=\mu_0 NI \tag{11}$$

which gives the "outside" axial field

$$B_z\left(a^+,z\right)=B_z\left(a^-,z\right)-\mu_0 NI\approx\frac{-2\mu_0 NIa^2}{L^2}\left(1+\frac{12z^2}{L^2}-\frac{9a^2}{L^2}\right) \tag{12}$$

3. When $z=\pm L/2$, applying (1) to the 0th order will give us

$$B_z\left(\rho,\pm\frac{L}{2}\right)\approx\frac{\mu_0 NI}{2} \tag{13}$$

Then applying the radial formula from problem 5.4 to the 1st order while using (2) gives

$$B_\rho\left(\rho,\pm\frac{L}{2}\right)=-\left(\frac{\rho}{2}\right)\left[\frac{\partial B_z(0,z)}{\partial z}\right]=-\left(\frac{\rho}{2}\right)\cdot\frac{\mu_0 NI}{2}\left(\mp\frac{1}{a}\right)=\pm\frac{\mu_0 NI}{4}\left(\frac{\rho}{a}\right) \tag{14}$$