1. **Prob 11.5**

From the velocity addition formula (11.31)

$$u_{\parallel} = \frac{u'_{\parallel} + v}{1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}} \tag{1}$$

$$\mathbf{u}_{\perp} = \frac{\mathbf{u}'_{\perp}}{\gamma\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)} \tag{2}$$

and the Lorentz transform

$$t = \gamma\left(t' + \frac{\mathbf{v}\cdot\mathbf{x}'}{c^2}\right) \tag{3}$$

we can calculate the K-frame acceleration via

$$a_{\parallel} = \frac{du_{\parallel}}{dt} \qquad\qquad \mathbf{a}_{\perp} = \frac{d\mathbf{u}_{\perp}}{dt} \tag{4}$$

where

$$\frac{d}{dt} = \frac{1}{\gamma\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)}\frac{d}{dt'} \tag{5}$$

With $u'_{\parallel} = \mathbf{v}\cdot\mathbf{u}'/v$, taking the differential of (1) gives

$$du_{\parallel} = \left(\frac{du'_{\parallel}}{1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}}\right) - \frac{\left(u'_{\parallel} + v\right)\left(\frac{\mathbf{v}\cdot d\mathbf{u}'}{c^2}\right)}{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^2} = \frac{du'_{\parallel} + \overbrace{du'_{\parallel}\left(\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right) - u'_{\parallel}\left(\frac{\mathbf{v}\cdot d\mathbf{u}'}{c^2}\right)}^{0} - v\left(\frac{\mathbf{v}\cdot d\mathbf{u}'}{c^2}\right)}{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^2} = \frac{du'_{\parallel}\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^2} \tag{6}$$

so by (5),

$$a_{\parallel} = \frac{du_{\parallel}}{dt} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3} a'_{\parallel} \tag{7}$$

Similarly, with $\mathbf{u}'_{\perp} = \mathbf{u}' - (\mathbf{v}\cdot\mathbf{u}')\mathbf{v}/v^2$, taking the differential of (2) gives

$$\begin{aligned}
d\mathbf{u}_{\perp} &= \frac{1}{\gamma}\left[\left(\frac{d\mathbf{u}'_{\perp}}{1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}}\right) - \frac{\mathbf{u}'_{\perp}\left(\frac{\mathbf{v}\cdot d\mathbf{u}'}{c^2}\right)}{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^2}\right] = \frac{d\mathbf{u}'_{\perp}\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right) - \mathbf{u}'_{\perp}\left(\frac{\mathbf{v}\cdot d\mathbf{u}'}{c^2}\right)}{\gamma\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^2} = \frac{d\mathbf{u}'_{\perp} + d\mathbf{u}'_{\perp}\left(\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right) - \mathbf{u}'_{\perp}\left(\frac{\mathbf{v}\cdot d\mathbf{u}'}{c^2}\right)}{\gamma\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^2} \\
&= \frac{d\mathbf{u}'_{\perp} + \left[d\mathbf{u}' - \frac{(\mathbf{v}\cdot d\mathbf{u}')\mathbf{v}}{v^2}\right]\left(\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right) - \left[\mathbf{u}' - \frac{(\mathbf{v}\cdot\mathbf{u}')\mathbf{v}}{v^2}\right]\left(\frac{\mathbf{v}\cdot d\mathbf{u}'}{c^2}\right)}{\gamma\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^2} = \frac{d\mathbf{u}'_{\perp} + \left[\frac{d\mathbf{u}'(\mathbf{v}\cdot\mathbf{u}') - \mathbf{u}'(\mathbf{v}\cdot d\mathbf{u}')}{c^2}\right]}{\gamma\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^2} \\
&= \frac{d\mathbf{u}'_{\perp} + \frac{\mathbf{v}\times(d\mathbf{u}'\times\mathbf{u}')}{c^2}}{\gamma\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^2}
\end{aligned} \tag{8}$$

therefore

$$\mathbf{a}_{\perp} = \frac{d\mathbf{u}_{\perp}}{dt} = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\left[\mathbf{a}'_{\perp} + \frac{\mathbf{v}\times(\mathbf{a}'\times\mathbf{u}')}{c^2}\right] \tag{9}$$

2. **Prob 11.6**

Since the frame of the rocket is not inertial, we must be careful to designate our reference frames. From the earth frame K, let the accelerating rocket have instantaneous velocity $v(t)$ at earth time t. We can always set up an *inertial* frame $K'(t)$ such that it moves with velocity $v(t)$ relative to earth. From $K'(t)$'s point of view, the rocket's relative velocity is $u' = 0$, but its acceleration is a' (where $a' = \pm g$ depending on the phase of the trip).

It is between K and $K'(t)$ – both of which are inertial – that we apply the acceleration transform (7) and (9) to obtain the acceleration $a(t)$ as measured from K,

$$\frac{dv(t)}{dt} = a(t) = \left[1 - \frac{v^2(t)}{c^2}\right]^{3/2} a' \tag{10}$$

This differential equation holds for each of the accelerating and decelerating phase of the rocket's trip at the corresponding K-frame time measurement (to be determined).

Separating variables for (10) and integrating both sides, we have

$$\int \frac{dv}{\sqrt{1 - \frac{v^2}{c^2}}^{\,3}} = \int a' dt \tag{11}$$

The LHS of (11) can be integrated by the substitution $v/c = \sin\theta$ to give

$$\int \frac{c\cos\theta\, d\theta}{\cos^3\theta} = c\tan\theta + C = \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} + C \tag{12}$$

For the first accelerating phase ($a' = g$), the RHS of (11) is gt, together with the initial condition $v(0) = 0$, we can write

$$v(t) = \frac{gt}{\sqrt{1 + \frac{g^2t^2}{c^2}}} \qquad \text{for the 1st accelerating phase} \tag{13}$$

If this first phase lasts $T' = 5$ years in the rocket's frame, we can relate this time measurement to the earth frame measurement T by

$$T' = \int_0^T \frac{dt}{\gamma_{v(t)}} = \int_0^T \sqrt{\frac{1}{1 + \frac{g^2t^2}{c^2}}}\, dt = \frac{c}{g}\sinh^{-1}\left(\frac{gT}{c}\right) \tag{14}$$

or

$$T = \frac{c}{g}\sinh\left(\frac{gT'}{c}\right) \approx 84 \text{ years} \tag{15}$$

The decelerating phase is symmetric to the accelerating phase in that (13) now reads

$$v(t) = \overbrace{\frac{gT}{\sqrt{1 + \frac{g^2T^2}{c^2}}}}^{v_{\max}} - \frac{g(t-T)}{\sqrt{1 + \frac{g^2(t-T)^2}{c^2}}} \qquad \text{for the 1st decelerating phase} \tag{16}$$

It is then easy to show that it takes another $T = 84$ years to come to a stop, and when the rockets comes back to earth, $84 \times 4 = 336$ years would have passed on earth.

The furthest distance traveled can be obtained by

$$2 \times \int_0^T \frac{gt}{\sqrt{1 + \frac{g^2t^2}{c^2}}}\, dt = \frac{2c^2}{g}\left[\sqrt{1 + \frac{g^2T^2}{c^2}} - 1\right] \approx 166 \text{ light years} \tag{17}$$