1. **Prob 9.5**

(a) From Chapter 6, the time dependent scalar and vector potential can be written

$$\Phi(\mathbf{x},t)=\frac{1}{4\pi\epsilon_0}\int d^3x'\int dt'\frac{\rho(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}\delta\left(t'+\frac{|\mathbf{x}-\mathbf{x}'|}{c}-t\right) \tag{1}$$

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int d^3x'\int dt'\frac{\mathbf{J}(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}\delta\left(t'+\frac{|\mathbf{x}-\mathbf{x}'|}{c}-t\right) \tag{2}$$

With the harmonic time dependency $e^{-i\omega t}$ of $\rho(\mathbf{x},t)$ and $\mathbf{J}(\mathbf{x},t)$, we can write the potentials as

$$\Phi(\mathbf{x},t)=e^{-i\omega t}\Phi(\mathbf{x})=e^{-i\omega t}\frac{1}{4\pi\epsilon_0}\int\rho\left(\mathbf{x}'\right)\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{3}$$

$$\mathbf{A}(\mathbf{x},t)=e^{-i\omega t}\mathbf{A}(\mathbf{x})=e^{-i\omega t}\frac{\mu_0}{4\pi}\int\mathbf{J}\left(\mathbf{x}'\right)\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{4}$$

Recall the Green function expansion (9.98)

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}=ik\sum_{l=0}^{\infty}j_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l}Y_{lm}^*\left(\theta',\phi'\right)Y_{lm}(\theta,\phi) \tag{5}$$

for observation point outside the source region ($r_>=r, r_<=r'$), the scalar potential becomes

$$\begin{aligned}\Phi(\mathbf{x})&=\frac{ik}{\epsilon_0}\sum_{l=0}^{\infty}h_l^{(1)}(kr)\cdot\int\rho\left(\mathbf{x}'\right)j_l\left(kr'\right)\left[\sum_{m=-l}^{l}Y_{lm}^*\left(\theta',\phi'\right)Y_{lm}(\theta,\phi)\right]d^3x' \quad \text{by Addition Theorem (3.63)}\\&=\frac{ik}{\epsilon_0}\sum_{l=0}^{\infty}h_l^{(1)}(kr)\cdot\left(\frac{2l+1}{4\pi}\right)\int\rho\left(\mathbf{x}'\right)j_l\left(kr'\right)P_l(\cos\gamma)\,d^3x'\end{aligned} \tag{6}$$

where γ is the angle between $\mathbf{x}$ and $\mathbf{x}'$.
For dipole contribution to $\Phi(\mathbf{x})$, we set $l=1$, hence

$$h_1^{(1)}(kr)=-\frac{e^{ikr}}{kr}\left(1+\frac{i}{kr}\right) \tag{7}$$

and with long-wavelength assumption $kr'\to 0$, we have

$$j_1\left(kr'\right)\approx\frac{kr'}{3} \tag{8}$$

This gives

$$\begin{aligned}\Phi^{(1)}(\mathbf{x})&=\frac{ik}{\epsilon_0}\left(-\frac{e^{ikr}}{kr}\right)\left(1+\frac{i}{kr}\right)\cdot\frac{3}{4\pi}\int\rho\left(\mathbf{x}'\right)\frac{kr'}{3}\cos\gamma d^3x'\\&=\frac{e^{ikr}}{4\pi\epsilon_0r^2}(1-ikr)\cdot\overbrace{\int\rho\left(\mathbf{x}'\right)\mathbf{n}\cdot\mathbf{x}'d^3x'}^{\mathbf{n}\cdot\mathbf{p}}\\&=\frac{e^{ikr}}{4\pi\epsilon_0r^2}\mathbf{n}\cdot\mathbf{p}(1-ikr)\end{aligned} \tag{9}$$

For dipole contribution to $\mathbf{A}(\mathbf{x})$, we set $l=0$, thus with $h_0^{(1)}(kr)=e^{ikr}/ikr$ and $j_0(kr')\approx 1$, we have

$$\mathbf{A}^{(0)}(\mathbf{x})=\frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{ikr}\int\mathbf{J}\left(\mathbf{x}'\right)d^3x'=-\frac{i\mu_0\omega}{4\pi}\frac{e^{ikr}}{r}\mathbf{p} \tag{10}$$

where we have used (9.14).
Note in this derivation, the only approximation used is the small argument approximation for $j_l(kr')$, justified by the long-wavelength assumption. In particular, it not assumed that $kr\gg 1$ besides the requirement that the observation point is outside the source region.

(b) To get the field from $\Phi(\mathbf{x})$, denote

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} f(r)\,\mathbf{x}\cdot\mathbf{p} \qquad \text{where} \qquad f(r) = \frac{e^{ikr}}{r^3}(1-ikr) \tag{11}$$

Note that

$$\begin{aligned}
\boldsymbol{\nabla} f(r) &= \frac{d}{dr}\left[\frac{e^{ikr}}{r^3}(1-ikr)\right]\mathbf{n} \\
&= \left[ik\frac{e^{ikr}}{r^3}(1-ikr) - \frac{3e^{ikr}}{r^4}(1-ikr) - ik\frac{e^{ikr}}{r^3}\right]\mathbf{n} \\
&= \frac{e^{ikr}}{r^3}\left[k^2 r - \frac{3(1-ikr)}{r}\right]\mathbf{n}
\end{aligned} \tag{12}$$

Then

$$\begin{aligned}
\boldsymbol{\nabla}\left[f(r)\,\mathbf{x}\cdot\mathbf{p}\right] &= \boldsymbol{\nabla} f(r)(\mathbf{x}\cdot\mathbf{p}) + f(r)\,\boldsymbol{\nabla}(\mathbf{x}\cdot\mathbf{p}) \\
&= \frac{e^{ikr}}{r^3}\left[k^2 r - \frac{3(1-ikr)}{r}\right]\mathbf{n}(\mathbf{x}\cdot\mathbf{p}) + \frac{e^{ikr}}{r^3}(1-ikr)\,\mathbf{p} \\
&= \left[\mathbf{p} - 3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\right]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr} + k^2\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\frac{e^{ikr}}{r}
\end{aligned} \tag{13}$$

In Lorenz gauge, the electric field is given by

$$\begin{aligned}
\mathbf{E}(\mathbf{x}) = -\boldsymbol{\nabla}\Phi(\mathbf{x}) + i\omega\mathbf{A}(\mathbf{x}) &= -\frac{1}{4\pi\epsilon_0}\boldsymbol{\nabla}\left[f(r)\,\mathbf{x}\cdot\mathbf{p}\right] + \frac{k^2}{4\pi\epsilon_0}\mathbf{p}\frac{e^{ikr}}{r} \\
&= \frac{1}{4\pi\epsilon_0}\left\{\left[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p}) - \mathbf{p}\right]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr} + k^2\overbrace{\left[\mathbf{p} - \mathbf{n}(\mathbf{n}\cdot\mathbf{p})\right]}^{(\mathbf{n}\times\mathbf{p})\times\mathbf{n}}\frac{e^{ikr}}{r}\right\}
\end{aligned} \tag{14}$$

which agrees with (9.18).

$\mathbf{H}$ is given by the usual relation $\mathbf{H} = \boldsymbol{\nabla}\times\mathbf{A}/\mu_0$ which, of course, would agree with (9.18).

2. **Prob 9.6**

(a) Up to first order of $|\mathbf{x}'|/r$, we have

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} \approx \frac{1}{r}\left(1-\frac{2\mathbf{x}\cdot\mathbf{x}'}{r^2}\right)^{-1/2} \approx \frac{1}{r}\left(1+\frac{\mathbf{n}\cdot\mathbf{x}'}{r}\right) \qquad \left|\mathbf{x}-\mathbf{x}'\right| \approx r\left(1-\frac{\mathbf{n}\cdot\mathbf{x}'}{r}\right) \tag{15}$$

Thus from (1),

$$\begin{aligned}\Phi(\mathbf{x},t) &= \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho\left(\mathbf{x}', t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x}-\mathbf{x}'|} \\ &\approx \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{1}{r}\left(1+\frac{\mathbf{n}\cdot\mathbf{x}'}{r}\right)\cdot\left[\rho\left(\mathbf{x}', t-\frac{r}{c}\right)+\frac{1}{c}\frac{\partial\rho(\mathbf{x}',t')}{\partial t'}\bigg|_{t'=t-r/c}\cdot\left(\mathbf{n}\cdot\mathbf{x}'\right)\right]\end{aligned} \tag{16}$$

Up to first order of $|\mathbf{x}'|/r$, the integral has three parts,

1. $$\frac{1}{r}\int \rho\left(\mathbf{x}', t-\frac{r}{c}\right)d^3x' = \frac{Q_{\mathrm{ret}}}{r}$$
2. $$\frac{1}{r^2}\mathbf{n}\cdot\int \mathbf{x}'\rho\left(\mathbf{x}', t-\frac{r}{c}\right)d^3x' = \frac{\mathbf{n}\cdot\mathbf{p}_{\mathrm{ret}}}{r^2}$$
3. $$\frac{1}{cr}\mathbf{n}\cdot\frac{\partial}{\partial t'}\left[\left[\int \mathbf{x}'\rho\left(\mathbf{x}',t'\right)d^3x'\right]\right]\bigg|_{t'=t-r/c} = \frac{1}{cr}\mathbf{n}\cdot\frac{\partial\mathbf{p}(t')}{\partial t'}\bigg|_{t-r/c} = \frac{1}{cr}\mathbf{n}\cdot\frac{\partial\mathbf{p}_{\mathrm{ret}}}{\partial t} \tag{17}$$

The first part is the monopole contribution and can be ignored for our purpose. And in the third part, $\partial\mathbf{p}_{\mathrm{ret}}/\partial t$ is understood to represent the time derivative of the dipole evaluated at the retarded time.
In summary, the dipole contribution to the scalar potential is

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{r^2}\mathbf{n}\cdot\mathbf{p}_{\mathrm{ret}}+\frac{1}{cr}\mathbf{n}\cdot\frac{\partial\mathbf{p}_{\mathrm{ret}}}{\partial t}\right) \tag{18}$$

For vector potential, we can replace ρ with $\mathbf{J}$ in (16), i.e.,

$$\mathbf{A}(\mathbf{x},t) \approx \frac{\mu_0}{4\pi}\int d^3x' \frac{1}{r}\left(1+\frac{\mathbf{n}\cdot\mathbf{x}'}{r}\right)\cdot\left[\mathbf{J}\left(\mathbf{x}', t-\frac{r}{c}\right)+\frac{1}{c}\frac{\partial\mathbf{J}(\mathbf{x}',t')}{\partial t'}\bigg|_{t'=t-r/c}\cdot\left(\mathbf{n}\cdot\mathbf{x}'\right)\right] \tag{19}$$

But from (9.14), we see the integral

$$\frac{1}{r}\int \mathbf{J}d^3x' = -\frac{1}{r}\int \mathbf{x}'\left(\boldsymbol{\nabla}'\cdot\mathbf{J}\right)d^3x' = \frac{1}{r}\frac{\partial}{\partial t}\int \mathbf{x}'\rho d^3x' \tag{20}$$

is of order $|\mathbf{x}'|/r$, so all the $\mathbf{n}\cdot\mathbf{x}'$ terms in (19) can be ignored. Thus the vector potential becomes

$$\mathbf{A}(\mathbf{x},t) \approx \frac{\mu_0}{4\pi r}\frac{\partial}{\partial t'}\left[\left[\int \mathbf{x}'\rho\left(\mathbf{x}',t'\right)d^3x'\right]\right]\bigg|_{t'=t-r/c} = \frac{\mu_0}{4\pi r}\frac{\partial\mathbf{p}_{\mathrm{ret}}}{\partial t} \tag{21}$$

(b) The fields can be evaluated routinely,

$$\begin{aligned}\mathbf{B}(\mathbf{x},t) = \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x},t) &= \frac{\mu_0}{4\pi}\left[\boldsymbol{\nabla}\left(\frac{1}{r}\right)\times\frac{\partial\mathbf{p}_{\mathrm{ret}}}{\partial t}+\frac{1}{r}\boldsymbol{\nabla}\times\frac{\partial\mathbf{p}_{\mathrm{ret}}}{\partial t}\right] \\ &= \frac{\mu_0}{4\pi}\left\{-\frac{1}{r^2}\mathbf{n}\times\frac{\partial\mathbf{p}_{\mathrm{ret}}}{\partial t}+\frac{1}{r}\hat{\mathbf{e}}_k\epsilon_{ijk}\frac{\partial}{\partial x_i}\left[\frac{\partial p_j(t')}{\partial t'}\bigg|_{t-r/c}\right]\right\} \\ &= \frac{\mu_0}{4\pi}\left\{-\frac{1}{r^2}\mathbf{n}\times\frac{\partial\mathbf{p}_{\mathrm{ret}}}{\partial t}+\frac{1}{r}\hat{\mathbf{e}}_k\epsilon_{ijk}\left[\frac{\partial^2 p_j(t')}{\partial t'^2}\right]\bigg|_{t-r/c}\cdot\left(-\frac{x_i}{cr}\right)\right\} \\ &= \frac{\mu_0}{4\pi}\left(-\frac{1}{r^2}\mathbf{n}\times\frac{\partial\mathbf{p}_{\mathrm{ret}}}{\partial t}-\frac{1}{cr}\mathbf{n}\times\frac{\partial^2\mathbf{p}_{\mathrm{ret}}}{\partial t^2}\right)\end{aligned} \tag{22}$$

For electric field,

$$
\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= -\nabla\Phi(\mathbf{x},t) - \frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t} \\
&= -\frac{1}{4\pi\epsilon_0}\left[\underbrace{\nabla\left(\frac{1}{r^2}\mathbf{n}\cdot\mathbf{p}_{\text{ret}}\right)}_{\mathbf{X}} + \underbrace{\nabla\left(\frac{1}{cr}\mathbf{n}\cdot\frac{\partial \mathbf{p}_{\text{ret}}}{\partial t}\right)}_{\mathbf{Y}}\right] - \frac{\mu_0}{4\pi r}\frac{\partial^2 \mathbf{p}_{\text{ret}}}{\partial t^2} \\
&= -\frac{1}{4\pi\epsilon_0}\left(\mathbf{X}+\mathbf{Y}+\frac{1}{c^2 r}\frac{\partial^2 \mathbf{p}_{\text{ret}}}{\partial t^2}\right)
\end{aligned}
\tag{23}
$$

Evaluation of $\mathbf{X}$ and $\mathbf{Y}$ is tedious but straightforward,

$$
\begin{aligned}
\mathbf{X} &= \nabla\left(\frac{1}{r^3}\mathbf{x}\cdot\mathbf{p}_{\text{ret}}\right) = \nabla\left(\frac{1}{r^3}\right)(\mathbf{x}\cdot\mathbf{p}_{\text{ret}}) + \frac{1}{r^3}\nabla(\mathbf{x}\cdot\mathbf{p}_{\text{ret}}) \\
&= -\frac{3}{r^3}\mathbf{n}(\mathbf{n}\cdot\mathbf{p}_{\text{ret}}) + \frac{1}{r^3}\hat{\mathbf{e}}_j\frac{\partial}{\partial x_j}\left[x_i p_i(t-r/c)\right] \\
&= -\frac{3}{r^3}\mathbf{n}(\mathbf{n}\cdot\mathbf{p}_{\text{ret}}) + \frac{1}{r^3}\hat{\mathbf{e}}_j\left[\delta_{ij}p_i(t-r/c) + x_i\left.\frac{\partial p_i(t')}{\partial t'}\right|_{t-r/c}\cdot\left(-\frac{x_j}{cr}\right)\right] \\
&= \frac{1}{r^3}\left[\mathbf{p}_{\text{ret}} - 3\mathbf{n}(\mathbf{n}\cdot\mathbf{p}_{\text{ret}})\right] - \frac{1}{cr^2}\mathbf{n}\left(\mathbf{n}\cdot\frac{\partial \mathbf{p}_{\text{ret}}}{\partial t}\right)
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\mathbf{Y} &= \nabla\left(\frac{1}{cr^2}\mathbf{x}\cdot\frac{\partial \mathbf{p}_{\text{ret}}}{\partial t}\right) = \nabla\left(\frac{1}{cr^2}\right)\left(\mathbf{x}\cdot\frac{\partial \mathbf{p}_{\text{ret}}}{\partial t}\right) + \frac{1}{cr^2}\nabla\left(\mathbf{x}\cdot\frac{\partial \mathbf{p}_{\text{ret}}}{\partial t}\right) \\
&= -\frac{2}{cr^2}\mathbf{n}\left(\mathbf{n}\cdot\frac{\partial \mathbf{p}_{\text{ret}}}{\partial t}\right) + \frac{1}{cr^2}\hat{\mathbf{e}}_j\frac{\partial}{\partial x_j}\left[x_i\left.\frac{\partial p_i(t')}{\partial t'}\right|_{t-r/c}\right] \\
&= -\frac{2}{cr^2}\mathbf{n}\left(\mathbf{n}\cdot\frac{\partial \mathbf{p}_{\text{ret}}}{\partial t}\right) + \frac{1}{cr^2}\hat{\mathbf{e}}_j\left[\delta_{ij}\left.\frac{\partial p_i(t')}{\partial t'}\right|_{t-r/c} + x_i\left.\frac{\partial^2 p_i(t')}{\partial t'^2}\right|_{t-r/c}\left(-\frac{x_j}{cr}\right)\right] \\
&= \frac{1}{cr^2}\frac{\partial}{\partial t}\left[\mathbf{p}_{\text{ret}} - 2\mathbf{n}(\mathbf{n}\cdot\mathbf{p}_{\text{ret}})\right] - \frac{1}{c^2 r}\mathbf{n}\left(\mathbf{n}\cdot\frac{\partial^2 \mathbf{p}_{\text{ret}}}{\partial t^2}\right)
\end{aligned}
\tag{25}
$$

Putting everything back to (23) yields the desired identity

$$
\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= \frac{1}{4\pi\epsilon_0}\left\{\left(1+\frac{r}{c}\frac{\partial}{\partial t}\right)\left[\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{p}_{\text{ret}}) - \mathbf{p}_{\text{ret}}}{r^3}\right] + \frac{1}{c^2 r}\left[\mathbf{n}\left(\mathbf{n}\cdot\frac{\partial^2 \mathbf{p}_{\text{ret}}}{\partial t^2}\right) - \frac{\partial^2 \mathbf{p}_{\text{ret}}}{\partial t^2}\right]\right\} \\
&= \frac{1}{4\pi\epsilon_0}\left\{\left(1+\frac{r}{c}\frac{\partial}{\partial t}\right)\left[\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{p}_{\text{ret}}) - \mathbf{p}_{\text{ret}}}{r^3}\right] + \frac{1}{c^2 r}\mathbf{n}\times\left(\mathbf{n}\times\frac{\partial^2 \mathbf{p}_{\text{ret}}}{\partial t^2}\right)\right\}
\end{aligned}
\tag{26}
$$

(c) If the source has harmonic time dependency, the following substitutions hold

$$
-i\omega \longleftrightarrow \frac{\partial}{\partial t} \qquad\qquad \mathbf{p}e^{ikr-i\omega t} \longleftrightarrow \mathbf{p}_{\text{ret}}(t')
\tag{27}
$$

It is then straightforward to veriy they connect the fields of arbitrary time dependency (22), (26) to the harmonic time dependency (9.18).