1. In cylindrical coordinates, the current density of the circular loop can be written as

$$\mathbf{J}(\mathbf{x}') = I\delta(\rho'-a)\delta(z')\hat{\boldsymbol{\phi}} \tag{1}$$

From

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' = \frac{\mu_0 I}{4\pi}\int \frac{(-\sin\phi'\hat{\mathbf{x}}+\cos\phi'\hat{\mathbf{y}})\delta(\rho'-a)\delta(z')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{2}$$

we know

$$A_x = -\mathrm{Im}\widetilde{A} \qquad A_y = \mathrm{Re}\widetilde{A} \qquad \text{where} \qquad \widetilde{A}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\int \frac{e^{i\phi'}\delta(\rho'-a)\delta(z')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{3}$$

Recall equation (3.148)

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk e^{im(\phi-\phi')}\cos\left[k(z-z')\right]I_m(k\rho_<)K_m(k\rho_>) \tag{4}$$

where $\rho_>, \rho_<$ are the greater and smaller between ρ, ρ'.
Insert (4) into (3) and perform the selection by the δ-function, we end up with

$$\begin{aligned}\widetilde{A}(\mathbf{x}) &= \frac{\mu_0 I}{4\pi}\cdot\frac{2}{\pi}\cdot a\sum_{m=-\infty}^{\infty}e^{im\phi}\overbrace{\int_0^{2\pi}d\phi' e^{i(1-m)\phi'}}^{2\pi\delta_{m1}}\int_0^\infty dk\cos(kz)I_m(k\rho_<)K_m(k\rho_>)\\ &= \frac{\mu_0 I a}{\pi}e^{i\phi}\int_0^\infty dk\cos(kz)I_1(k\rho_<)K_1(k\rho_>)\end{aligned} \tag{5}$$

(Note $\rho_<, \rho_>$ represent the greater and smaller between ρ, a after the δ selection).
Finally,

$$\begin{aligned}A_\phi &= -\sin\phi A_x + \cos\phi A_y = -\sin\phi\left(-\mathrm{Im}\widetilde{A}\right) + \cos\phi\,\mathrm{Re}\widetilde{A}\\ &= \frac{\mu_0 I a}{\pi}\int_0^\infty dk\cos(kz)I_1(k\rho_<)K_1(k\rho_>)\end{aligned} \tag{6}$$

2. From problem 3.16(b), we have an alternative form of (3)

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{m=-\infty}^{\infty}\int_0^\infty dk e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')e^{-k(z_>-z_<)} \tag{7}$$

Corresponding to (5), the alternative form of $\widetilde{A}$ is

$$\widetilde{A}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\cdot a e^{i\phi}\cdot 2\pi\int_0^\infty dk J_1(ka)J_1(k\rho)e^{-k|z|} \tag{8}$$

which gives

$$A_\phi = \frac{\mu_0 I a}{2}\int_0^\infty dk J_1(ka)J_1(k\rho)e^{-k|z|} \tag{9}$$

3. The field components are given by $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. In cylindrical coordinates, these are (see Wikipedia)

$$B_\rho = \frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z} = -\frac{\partial A_\phi}{\partial z} \tag{10}$$

$$B_\phi = \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho} = 0 \tag{11}$$

$$B_z = \frac{1}{\rho}\left(\frac{\partial\rho A_\phi}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\right) = \frac{1}{\rho}\frac{\partial\rho A_\phi}{\partial\rho} \tag{12}$$

For expansion (6), this gives

$$B_\rho^{(6)}(\rho, z) = \frac{\mu_0 I a}{\pi} \int_0^\infty dk \cdot k \sin(kz) I_1(k\rho_<) K_1(k\rho_>) \tag{13}$$

$$B_z^{(6)}(\rho, z) = \frac{\mu_0 I a}{\pi} \int_0^\infty dk \cos(kz) \frac{1}{\rho} \frac{\partial}{\partial \rho} [\rho I_1(k\rho_<) K_1(k\rho_>)] \tag{14}$$

And for expansion (9),

$$B_\rho^{(9)}(\rho, z) = \frac{\mu_0 I a}{2} \int_0^\infty dk J_1(ka) J_1(k\rho) [k \operatorname{sgn}(z)] e^{-k|z|} \tag{15}$$

$$B_z^{(9)}(\rho, z) = \frac{\mu_0 I a}{2} \int_0^\infty dk J_1(ka) \frac{1}{\rho} \frac{d[\rho J_1(k\rho)]}{d\rho} e^{-k|z|} \tag{16}$$

Also recall in problem 5.7(a), we have solved the field on the axis exactly,

$$\mathbf{B}(0, z) = \frac{\mu_0 I a^2}{2} \frac{\hat{\mathbf{z}}}{\sqrt{a^2 + z^2}^3} \tag{17}$$

On the z-axis, where $\rho = 0$, we have $\rho_< = 0, \rho_> = a$. Since $I_1(0) = J_1(0) = 0$, we see both (13) and (15) vanish on the z-axis, as expected.

(a) Now let's evaluate (14) as $\rho \to 0$.

$$\begin{aligned} B_z^{(6)}(0, z) &= \frac{\mu_0 I a}{\pi} \int_0^\infty dk \cos(kz) K_1(ka) \frac{1}{\rho} \left[I_1(0) + \rho k I_1'(0) \right] \Big|_{\rho=0} \\ &= \frac{\mu_0 I a}{\pi} \underbrace{\int_0^\infty dk \cos(kz) K_1(ka) k}_{X} \end{aligned} \tag{18}$$

where

$$X = \frac{d}{dz} \int_0^\infty dk \sin(kz) K_1(ka) \tag{19}$$

By equation (10.29.3) on nist.gov, $K_1(x) = -K_0'(x)$, gives

$$\begin{aligned} X &= \frac{d}{dz} \int_0^\infty dk \sin(kz) \left[-K_0'(ka) \right] \\ &= \frac{d}{dz} \int_0^\infty dk \sin(kz) \left[-\frac{1}{a} \frac{dK_0(ka)}{dk} \right] \\ &= -\frac{1}{a} \frac{d}{dz} \left[\underbrace{\sin(kz) K_0(ka)}_{\equiv f(k)} \Big|_{k=0}^{k=\infty} - \int_0^\infty z \cos(kz) K_0(ka) \, dk \right] \end{aligned} \tag{20}$$

Apparently $\lim_{k\to\infty} f(k) = 0$. Also by equation (3.103)

$$\lim_{k\to 0} f(k) = \lim_{k\to 0} (kz) \cdot \left[-\ln\left(\frac{ka}{2} \right) - 0.5772 \right] = 0 \tag{21}$$

This gives

$$X = \frac{1}{a} \frac{d}{dz} \left[z \overbrace{\int_0^\infty \cos(kz) K_0(ka) \, dk}^{Y} \right] \tag{22}$$

By equation (10.43.20) on nist.gov

$$\int_0^\infty \cos(\alpha t) K_0(t) \, dt = \frac{\pi}{2\sqrt{1 + \alpha^2}} \tag{23}$$

we can calculate

$$Y = \frac{\pi}{2\sqrt{a^2+z^2}} \tag{24}$$

Plugging (24), (22) into (18), we finally get

$$B_z^{(6)}(0,z) = \frac{\mu_0 I a}{\pi} \cdot \frac{1}{a} \cdot \frac{\pi}{2} \frac{d}{dz}\left(\frac{z}{\sqrt{a^2+z^2}}\right) = \frac{\mu_0 I a^2}{2} \frac{1}{\sqrt{a^2+z^2}^3} \tag{25}$$

agreeing with the exact solution (17).

(b) Next, let's evaluate (16) as $\rho \to 0$.

$$\begin{aligned} B_z^{(9)}(0,z) &= \frac{\mu_0 I a}{2} \int_0^\infty dk J_1(ka) e^{-k|z|} \frac{1}{\rho}\left[J_1(0) + \rho k J_1'(0)\right]\Bigg|_{\rho=0} \\ &= \frac{\mu_0 I a}{2} \underbrace{\int_0^\infty dk J_1(ka) k e^{-k|z|}}_{W} \end{aligned} \tag{26}$$

where the integral W is recognized as

$$\begin{aligned} W &= -\frac{d}{d|z|} \int_0^\infty dk J_1(ka) e^{-k|z|} \\ &= -\frac{d}{d|z|} \mathscr{L}\{J_1(ka)\}(|z|) \end{aligned} \tag{27}$$

The Laplace transform of $J_1(ka)$ has already been worked out in previous notes (see equation (25) here), which is quoted below

$$\mathscr{L}\{J_1(ka)\}(s) = \frac{1}{a}\left(1 - \frac{s}{\sqrt{a^2+s^2}}\right) \tag{28}$$

This gives

$$W = -\frac{d}{d|z|}\left(\frac{1}{a} - \frac{|z|}{a\sqrt{a^2+z^2}}\right) = \frac{a}{\sqrt{a^2+z^2}^3} \tag{29}$$

Thus

$$B_z^{(9)}(0,z) = \frac{\mu_0 I a^2}{2} \frac{1}{\sqrt{a^2+z^2}^3} \tag{30}$$