In these notes, we fill in the derivation details for the Cherenkov radiation.

The wave equation for $\Phi(\mathbf{x},t), \mathbf{A}(\mathbf{x},t)$ is

$$\nabla^2\Phi(\mathbf{x},t) - \frac{\epsilon(\omega)}{c^2}\frac{\partial^2\Phi(\mathbf{x},t)}{\partial t^2} = -\frac{4\pi}{\epsilon(\omega)}\rho(\mathbf{x},t) \tag{1}$$

$$\nabla^2\mathbf{A}(\mathbf{x},t) - \frac{\epsilon(\omega)}{c^2}\frac{\partial^2\mathbf{A}(\mathbf{x},t)}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}(\mathbf{x},t) \tag{2}$$

Let $F(\mathbf{x},t) \longleftrightarrow F(\mathbf{k},\omega)$ be a Fourier transform pair, i.e.

$$F(\mathbf{x},t) = \frac{1}{(2\pi)^2}\int d^3k \int d\omega F(\mathbf{k},\omega)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \tag{3}$$

$$F(\mathbf{k},\omega) = \frac{1}{(2\pi)^2}\int d^3x \int dt F(\mathbf{x},t)e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t} \tag{4}$$

The wave equations can be rewritten as

$$\int d^3k \int d\omega\left[k^2 - \frac{\omega^2\epsilon(\omega)}{c^2}\right]\Phi(\mathbf{k},\omega) = \int d^3k \int d\omega \frac{4\pi}{\epsilon(\omega)}\rho(\mathbf{k},\omega) \tag{5}$$

$$\int d^3k \int d\omega\left[k^2 - \frac{\omega^2\epsilon(\omega)}{c^2}\right]\mathbf{A}(\mathbf{k},\omega) = \int d^3k \int d\omega \frac{4\pi}{c}\mathbf{J}(\mathbf{k},\omega) \tag{6}$$

Orthogonality of the Fourier transform requires integrands to be equal for all frequencies, so we have the frequency space wave equation for the potentials

$$\left[k^2 - \frac{\omega^2\epsilon(\omega)}{c^2}\right]\Phi(\mathbf{k},\omega) = \frac{4\pi}{\epsilon(\omega)}\rho(\mathbf{k},\omega) \tag{7}$$

$$\left[k^2 - \frac{\omega^2\epsilon(\omega)}{c^2}\right]\mathbf{A}(\mathbf{k},\omega) = \frac{4\pi}{c}\mathbf{J}(\mathbf{k},\omega) \tag{8}$$

For point charge moving with velocity $\mathbf{v}$, its charge and current density are

$$\rho(\mathbf{x},t) = q\delta(\mathbf{x}-\mathbf{v}t) \qquad \mathbf{J}(\mathbf{x},t) = \mathbf{v}\rho(\mathbf{x}-\mathbf{v}t) = q\mathbf{v}\delta(\mathbf{x}-\mathbf{v}t) \tag{9}$$

The Fourier transform of the charge density is

$$\begin{aligned}\rho(\mathbf{k},\omega) &= \frac{1}{(2\pi)^2}\int d^3x \int dt q\delta(\mathbf{x}-\mathbf{v}t)e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}\\ &= \frac{q}{2\pi}\cdot\underbrace{\frac{1}{2\pi}\int dt e^{i(\omega-\mathbf{k}\cdot\mathbf{v})t}}_{\delta(\omega-\mathbf{k}\cdot\mathbf{v})} = \frac{q}{2\pi}\delta(\omega-\mathbf{k}\cdot\mathbf{v})\end{aligned} \tag{10}$$

(7) and (8) now reads

$$\Phi(\mathbf{k},\omega) = \frac{2q}{\epsilon(\omega)}\left[\frac{\delta(\omega-\mathbf{k}\cdot\mathbf{v})}{k^2 - \frac{\omega^2\epsilon(\omega)}{c^2}}\right] \qquad \mathbf{A}(\mathbf{k},\omega) = \epsilon(\omega)\boldsymbol{\beta}\Phi(\mathbf{k},\omega) \tag{11}$$

From

$$\mathbf{E}(\mathbf{x},t) = -\nabla\Phi(\mathbf{x},t) - \frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t} \qquad \mathbf{B}(\mathbf{x},t) = \nabla\times\mathbf{A}(\mathbf{x},t) \tag{12}$$

we know

$$\mathbf{E}(\mathbf{k},\omega) = -i\mathbf{k}\Phi(\mathbf{k},\omega) + \frac{i\omega}{c}\mathbf{A}(\mathbf{k},\omega) = i\left[\frac{\omega\epsilon(\omega)}{c}\boldsymbol{\beta} - \mathbf{k}\right]\Phi(\mathbf{k},\omega) \tag{13}$$

$$\mathbf{B}(\mathbf{k},\omega) = i\mathbf{k}\times\mathbf{A}(\mathbf{k},\omega) = i\epsilon(\omega)\mathbf{k}\times\boldsymbol{\beta}\Phi(\mathbf{k},\omega) \tag{14}$$

Let $\mathbf{v}$ be aligned with x direction. For an observation point at $(0,b,0)$, the electric field's Fourier transform is

$$\begin{aligned}
\mathbf{E}(\omega) &= \frac{1}{\sqrt{2\pi}}\int \mathbf{E}(b\hat{\mathbf{y}},t)\,e^{i\omega t}dt \\
&= \frac{1}{\sqrt{2\pi}}\int e^{i\omega t}dt\left[\frac{1}{(2\pi)^2}\int d^3k\int d\omega'\mathbf{E}\left(\mathbf{k},\omega'\right)e^{ib\mathbf{k}\cdot\hat{\mathbf{y}}-i\omega' t}\right] \\
&= \frac{1}{(2\pi)^{3/2}}\int d^3k\int d\omega'\mathbf{E}\left(\mathbf{k},\omega'\right)e^{ib\mathbf{k}\cdot\hat{\mathbf{y}}}\overbrace{\left[\frac{1}{2\pi}\int e^{i\left(\omega-\omega'\right)t}dt\right]}^{\delta\left(\omega-\omega'\right)} \\
&= \frac{1}{(2\pi)^{3/2}}\int d^3k\mathbf{E}(\mathbf{k},\omega)\,e^{ibk_2}
\end{aligned} \tag{15}$$

Let's calculate its x component first,

$$\begin{aligned}
E_1(\omega) &= \frac{1}{(2\pi)^{3/2}}\int d^3ki\left[\frac{\omega\epsilon(\omega)}{c}\beta-k_1\right]\cdot\frac{2q}{\epsilon(\omega)}\left[\frac{\overbrace{\delta(\omega-k_1v)}^{\delta(k_1-\omega/v)/v}}{k^2-\dfrac{\omega^2\epsilon(\omega)}{c^2}}\right]e^{ibk_2} \qquad \text{integrate over } k_1 \\
&= \frac{2iq}{(2\pi)^{3/2}}\int dk_2dk_3\left[\frac{\omega\epsilon(\omega)}{c}\beta-\frac{\omega}{v}\right]\frac{1}{v\epsilon(\omega)}\left[\frac{1}{k_2^2+k_3^2+\dfrac{\omega^2}{v^2}-\dfrac{\omega^2}{c^2}\epsilon(\omega)}\right]e^{ibk_2} \\
&= -\frac{2iq\omega}{(2\pi)^{3/2}v^2}\left[\frac{1}{\epsilon(\omega)}-\beta^2\right]\int_{-\infty}^{\infty}dk_2e^{ibk_2}\int_{-\infty}^{\infty}\frac{dk_3}{k_2^2+k_3^2+\lambda^2}
\end{aligned} \tag{16}$$

where we have defined

$$\lambda^2=\frac{\omega^2}{v^2}-\frac{\omega^2}{c^2}\epsilon(\omega)=\frac{\omega^2}{v^2}\left[1-\beta^2\epsilon(\omega)\right] \tag{17}$$

The integration over k_3 is elementary,

$$\int_{-\infty}^{\infty}\frac{dk_3}{k_2^2+k_3^2+\lambda^2}=\frac{\pi}{\sqrt{k_2^2+\lambda^2}} \tag{18}$$

which turns (16) into

$$\begin{aligned}
E_1(\omega) &= -\frac{iq\omega}{\sqrt{2\pi}v^2}\left[\frac{1}{\epsilon(\omega)}-\beta^2\right]\int_{-\infty}^{\infty}\frac{e^{ibk_2}}{\sqrt{k_2^2+\lambda^2}}dk_2 \\
&= -\frac{iq\omega}{v^2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{\epsilon(\omega)}-\beta^2\right]K_0(\lambda b)
\end{aligned} \tag{19}$$

where we have used the integral representation of the modified Bessel function K_0 (DLMF 10.32.E11)

$$\int_{-\infty}^{\infty}\frac{e^{ibk_2}}{\sqrt{k_2^2+\lambda^2}}dk_2=2K_0(\lambda b) \tag{20}$$

The y component of $\mathbf{E}(\omega)$ can also be calculated

$$
\begin{aligned}
E_2(\omega) &= \frac{1}{(2\pi)^{3/2}} \int d^3k (-ik_2) \frac{2q}{\epsilon(\omega)} \left[\frac{\delta(\omega - k_1 v)}{k^2 - \dfrac{\omega^2 \epsilon(\omega)}{c^2}} \right] e^{ibk_2} \\
&= -\frac{2iq}{(2\pi)^{3/2} \epsilon(\omega) v} \int_{-\infty}^{\infty} dk_2 e^{ibk_2} k_2 \int_{\infty}^{\infty} \frac{dk_3}{k_2^2 + k_3^2 + \lambda^2} \\
&= -\frac{2\pi iq}{(2\pi)^{3/2} \epsilon(\omega) v} \int_{-\infty}^{\infty} dk_2 \frac{k_2 e^{ibk_2}}{\sqrt{k_2^2 + \lambda^2}} \\
&= -\frac{iq}{\sqrt{2\pi}\epsilon(\omega) v} \frac{1}{i} \frac{d}{db} \int_{-\infty}^{\infty} dk_2 \frac{e^{ibk_2}}{\sqrt{k_2^2 + \lambda^2}} \\
&= -\frac{2q}{\sqrt{2\pi}\epsilon(\omega) v} \frac{dK_0(\lambda b)}{db} \qquad \text{use } K_0' = -K_1 \\
&= \frac{q}{v} \sqrt{\frac{2}{\pi}} \frac{\lambda}{\epsilon(\omega)} K_1(\lambda b)
\end{aligned}
\tag{21}
$$

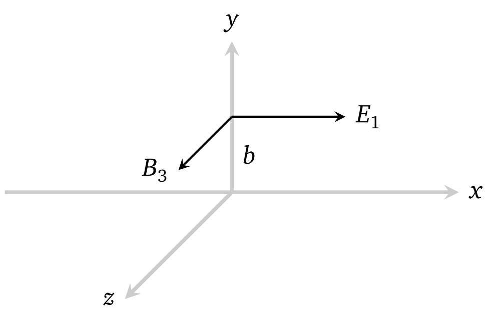
as well as the z component of $\mathbf{B}(\omega)$,

$$
\begin{aligned}
B_3(\omega) &= \frac{1}{(2\pi)^{3/2}} \int d^3k B_3(\mathbf{k}, \omega) e^{ibk_2} \\
&= \frac{1}{(2\pi)^{3/2}} \int d^3k i\epsilon(\omega) (\mathbf{k} \times \boldsymbol{\beta})_3 \Phi(\mathbf{k}, \omega) \\
&= \frac{\epsilon(\omega)\beta}{(2\pi)^{3/2}} \int d^3k (-ik_2) \Phi(\mathbf{k}, \omega) \\
&= \epsilon(\omega) \beta E_2(\omega) \\
&= \frac{q}{c} \sqrt{\frac{2}{\pi}} \lambda K_1(\lambda b)
\end{aligned}
\tag{22}
$$

Since λ is defined in (17) by its square, we must choose one branch, but which one?
First, due to the large-argument approximation of K_0

$$
K_\nu(\lambda b) \to \sqrt{\frac{\pi}{2}} \frac{e^{-\lambda b}}{\sqrt{\lambda b}} \qquad |\lambda b| \gg 1 \tag{23}
$$

the real part of λ must be positive to ensure the field's decay with distance. On the other hand, if $\epsilon(\omega)$ is to have a non-zero imaginary part, it must be positive which is the physical situation corresponding to energy absorption (see Jackson section 7.5). Therefore by (17) the choice of λ is that it must lie in the fourth quadrant of the complex plane.



Consider the energy emitted out from the cylinder of radius b, it is proportional to the Poynting vector's surface integral ($c/4\pi$ for Gaussian unit)

$$
\left(\frac{dW}{dt}\right)_{\rho > b} = -\frac{c}{4\pi} \int_{-\infty}^{\infty} 2\pi b \,\mathrm{Re}\left[B_3^*(t) E_1(t)\right] dx = -\frac{cb}{2} \mathrm{Re} \int_{-\infty}^{\infty} B_3^*(t) E_1(t) \, dx \tag{24}
$$

Dividing v on both sides, we have

$$\begin{aligned}\left(\frac{dW}{dx}\right)_{\rho>b} = \frac{1}{v}\frac{dW}{dt} &= -\frac{cb}{2}\operatorname{Re}\int_{-\infty}^{\infty} B_3^*(t)E_1(t)dt \\ &= -\frac{cb}{2}\operatorname{Re}\int_{-\infty}^{\infty} dt\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega B_3^*(\omega)e^{i\omega t}\right]\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega' E_1\left(\omega'\right)e^{-i\omega' t}\right] \\ &= -\frac{cb}{2}\operatorname{Re}\int_{-\infty}^{\infty} B_3^*(\omega)E_1(\omega)d\omega \end{aligned} \tag{25}$$

For radiation zone, we take the large argument approximation $|\lambda b| \gg 1$,

$$E_1(\omega) \to \frac{iq\omega}{c^2}\left[1-\frac{1}{\beta^2\epsilon(\omega)}\right]\frac{e^{-\lambda b}}{\sqrt{\lambda b}} \tag{26}$$

$$E_2(\omega) \to \frac{q}{v\epsilon(\omega)}\sqrt{\frac{\lambda}{b}}e^{-\lambda b} \tag{27}$$

$$B_3(\omega) \to \frac{q}{c}\sqrt{\frac{\lambda}{b}}e^{-\lambda b} \tag{28}$$

turning (25) into

$$\left(\frac{dW}{dx}\right)_{\rho>b} \to \operatorname{Re}\int_{-\infty}^{\infty}\frac{q^2}{c^2}\left(-i\sqrt{\frac{\lambda^*}{\lambda}}\right)\omega\left[1-\frac{1}{\beta^2\epsilon(\omega)}\right]e^{-(\lambda+\lambda^*)b}d\omega \tag{29}$$

When λ has a non-zero positive real part, this will decay exponentially with b. But if λ is purely imaginary, there will be no decay. From (17), the condition for λ to be purely imaginary is

$$\beta^2\epsilon(\omega) > 1 \qquad \Longrightarrow \qquad v > \frac{c}{\sqrt{\epsilon(\omega)}} \tag{30}$$

This gives the Frank-Tamm formula

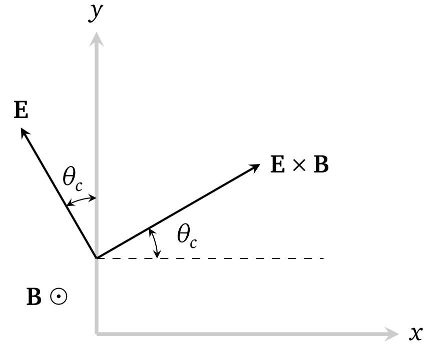
$$\frac{dW}{dx} = \frac{q^2}{c^2}\int_{\{\omega:\epsilon(\omega)>1/\beta^2\}}\omega\left[1-\frac{1}{\beta^2\epsilon(\omega)}\right]d\omega \tag{31}$$

When $\epsilon(\omega)$ is not too much different for different frequencies, we can see that the radiation has higher weight for high frequencies. Also the human eyes are more sensitive to blue than ultraviolet, so the radiation appears blue.

To see the polarization, from (26), (27),

$$\begin{aligned}\frac{E_1}{E_2} &= \frac{\dfrac{iq\omega}{c^2}\left[1-\dfrac{1}{\beta^2\epsilon(\omega)}\right]\dfrac{e^{-\lambda b}}{\sqrt{-i|\lambda|b}}}{\dfrac{q}{v\epsilon(\omega)}\sqrt{\dfrac{-i|\lambda|}{b}}e^{-\lambda b}} \\ &= -\frac{\omega v\epsilon(\omega)}{c^2}\frac{1}{|\lambda|}\left[1-\frac{1}{\beta^2\epsilon(\omega)}\right] && \text{use } |\lambda| = \frac{\omega}{v}\sqrt{\beta^2\epsilon(\omega)-1} \\ &= -\sqrt{\beta^2\epsilon(\omega)-1}\end{aligned} \tag{32}$$

which is a negative real number. This means the electric field is linearly polarized.



By (26) – (28), E_2 and B_3 have the same phase, but E_1 has the opposite phase. With the help of the diagram above, we see that the Poynting vector points to the direction forming a angle θ_c with the x axis such that

$$\tan\theta_c = \sqrt{\beta^2\epsilon(\omega)-1} \qquad \Longrightarrow \qquad \cos\theta_c = \frac{1}{\sqrt{\beta^2\epsilon(\omega)}} \tag{33}$$

The radiation condition (30) guarantees the existence of this angle.

Let's calculate the vector potential $\mathbf{A}(\mathbf{x},t)$,

$$\begin{aligned}
\mathbf{A}(\mathbf{x},t) &= \frac{1}{(2\pi)^2}\int d^3k \int d\omega \mathbf{A}(\mathbf{k},\omega)\, e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \\
&= \frac{1}{(2\pi)^2}\int d^3k \int d\omega \epsilon(\omega)\, \boldsymbol{\beta}\frac{2q}{\epsilon(\omega)}\left[\frac{\delta(\omega-\mathbf{k}\cdot\mathbf{v})}{k^2-\dfrac{\omega^2\epsilon(\omega)}{c^2}}\right] e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \\
&= \frac{2q}{(2\pi)^2}\boldsymbol{\beta}\underbrace{\int d^3k\left[\frac{e^{ik_1(x-vt)}e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}}}{k_1^2(1-\beta^2\epsilon)+k_\perp^2}\right]}_{I}
\end{aligned} \tag{34}$$

where $\mathbf{k}_\perp, \boldsymbol{\rho}$ are the transverse component of the wave and position vector.

In the ideal case where $\epsilon(\omega) = \epsilon(k_1 v)$ is constant, the integral I can be written as

$$I = \int d^2k_\perp e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}} \underbrace{\int_{-\infty}^{\infty} \frac{e^{ik_1(x-vt)}dk_1}{(1-\beta^2\epsilon)\left(k_1+\dfrac{k_\perp}{\sqrt{\beta^2\epsilon-1}}\right)\left(k_1-\dfrac{k_\perp}{\sqrt{\beta^2\epsilon-1}}\right)}}_{J} \tag{35}$$

The two poles of J are at

$$k_\pm = \pm\frac{k_\perp}{\sqrt{\beta^2\epsilon-1}} \tag{36}$$

In the physical situation where ϵ has a small positive imaginary part (absorbing medium), these two poles approaches the real axis from below.

For observation point $x > vt$, we close the contour in the upper half-plane, which does not include any poles. The arc integral vanishes due to $x > vt$, so J also vanishes by Cauchy's theorem.

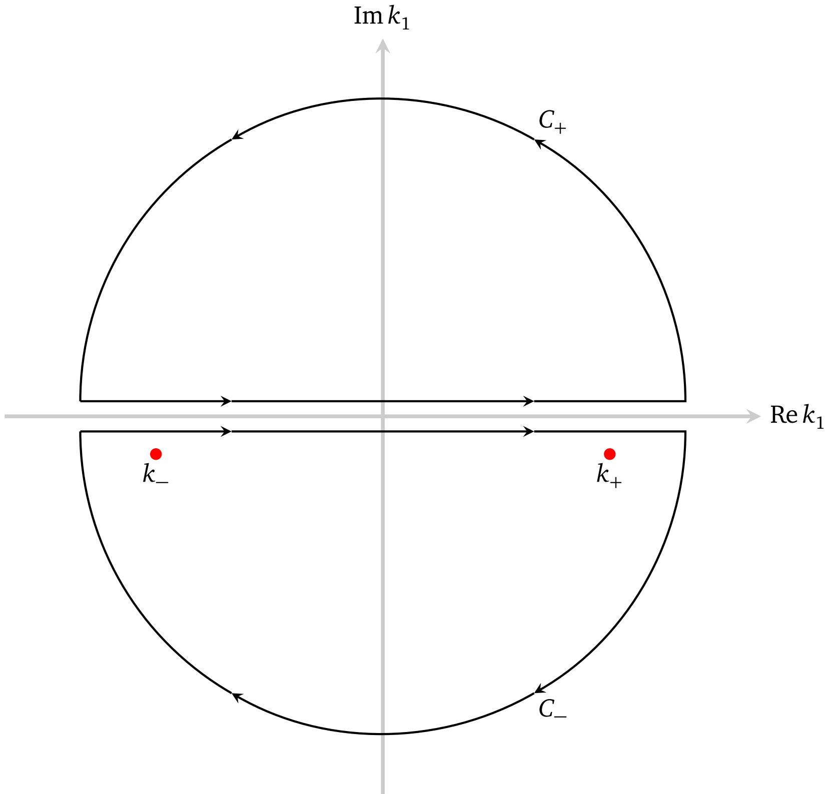
But for observation point $x < vt$, we close the contour in the lower half-plane, and by Cauchy's residue theorem, we have

$$\begin{aligned}
J + \overbrace{\int_{\text{lower-arc}}}^{=0} = \oint_{C_-} &= -2\pi i\left[\operatorname{Res}(k_-)+\operatorname{Res}(k_+)\right] \qquad \Longrightarrow \\
J &= -2\pi i\left(\frac{1}{1-\beta^2\epsilon}\right)\left[\frac{e^{ik_-(x-vt)}}{k_--k_+}+\frac{e^{ik_+(x-vt)}}{k_+-k_-}\right] \\
&= -2\pi i\left(\frac{1}{1-\beta^2\epsilon}\right)\left(\frac{k_\perp}{\sqrt{\beta^2\epsilon-1}}\right)^{-1}\cdot i\sin\left[\frac{k_\perp(x-vt)}{\sqrt{\beta^2\epsilon-1}}\right] \\
&= \frac{2\pi}{\sqrt{\beta^2\epsilon-1}}\frac{1}{k_\perp}\sin\left[\frac{k_\perp(vt-x)}{\sqrt{\beta^2\epsilon-1}}\right]
\end{aligned} \tag{37}$$

Therefore the I integral when $x < vt$ becomes

$$\begin{aligned}
I &= \frac{2\pi}{\sqrt{\beta^2\epsilon-1}}\int d^2k_\perp\frac{e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}}}{k_\perp}\sin\left[\frac{k_\perp(vt-x)}{\sqrt{\beta^2\epsilon-1}}\right] \\
&= \frac{2\pi}{\sqrt{\beta^2\epsilon-1}}\int_0^\infty k_\perp dk_\perp\frac{1}{k_\perp}\sin\left[\frac{k_\perp(vt-x)}{\sqrt{\beta^2\epsilon-1}}\right]\underbrace{\int_0^{2\pi} d\phi\, e^{ik_\perp\rho\cos\phi}}_{2\pi J_0(k_\perp\rho)} \\
&= \frac{(2\pi)^2}{\sqrt{\beta^2\epsilon-1}}\int_0^\infty J_0(k_\perp\rho)\sin\left[\frac{k_\perp(vt-x)}{\sqrt{\beta^2\epsilon-1}}\right]dk_\perp
\end{aligned} \tag{38}$$

where we have used the integral representation of the Bessel function of the first kind $J_0(z)$ (DLMF 10.9.E2)



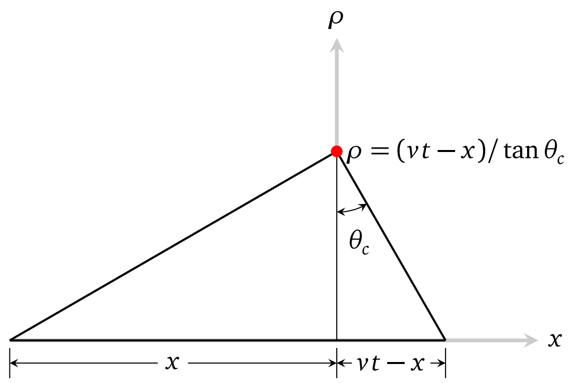
This remaining integral can be looked up from 6.671 of *I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals,Series and Products Eighth Edition*

$$\int_0^\infty J_\nu(ax)\sin(bx)\,dx = \begin{cases} \dfrac{\sin\left[\nu \sin^{-1}\left(\dfrac{b}{a}\right)\right]}{\sqrt{a^2-b^2}} & \text{for } a > b \\ \dfrac{a^\nu \cos\left(\dfrac{\nu\pi}{2}\right)}{\sqrt{b^2-a^2}\left(b+\sqrt{b^2-a^2}\right)^\nu} & \text{for } a < b \end{cases} \tag{39}$$

In our case

$$a > b \qquad \Longrightarrow \qquad \rho > \frac{vt-x}{\sqrt{\beta^2\epsilon - 1}} = \frac{vt-x}{\tan\theta_c} \tag{40}$$

and by the geometry of the shockwave cone



we see that $a > (<) b$ corresponds to the case where observation point is outside (inside) the shockwave cone. Using (39) with $\nu = 0$ and together with (34), we get the closed form of $\mathbf{A}(\mathbf{x}, t)$

$$\mathbf{A}(\mathbf{x}, t) = \begin{cases} \dfrac{2q\boldsymbol{\beta}}{\sqrt{(x-vt)^2 - (\beta^2\epsilon - 1)\rho^2}} & \text{for } x < vt \text{ and } \rho < \dfrac{vt-x}{\tan\theta_c} \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

A is singular on the boundary of the shockwave cone, well defined inside and vanishes outside.