1. Prob 11.8

By (11.29), the frequency of the light as measured from the moving liquid satisfies

$$\omega' = \gamma \left(\omega \mp \nu k\right) \tag{1}$$

where *k* is the wave number of light in the liquid measured from the lab frame, i.e.,

$$k = \frac{n(\omega)\,\omega}{c} \tag{2}$$

and the \mp sign is taken when the liquid is moving parallel or antiparallel with the light's direction of propagation. Up to first order of v, (1) yields

$$\omega' = \gamma \omega \left[1 \mp \frac{n(\omega) \nu}{c} \right] \approx \omega \left(1 + \frac{\nu^2}{2c^2} \right) \left[1 \mp \frac{n(\omega) \nu}{c} \right] \approx \omega \mp \frac{n(\omega) \nu \omega}{c}$$
(3)

With velocity addition law, the speed of light as measured in the lab frame is

$$c_{\text{lab}} = \frac{\pm \nu + u'}{1 \pm \frac{\nu u'}{c^2}} = \frac{\pm \nu + \frac{c}{n(\omega')}}{1 \pm \frac{\nu}{n(\omega')c}} \approx \left[\frac{c}{n(\omega')} \pm \nu \right] \left[1 \mp \frac{\nu}{n(\omega')c} \right] \approx \frac{c}{n(\omega')} \pm \nu \mp \frac{\nu}{n^2(\omega')}$$
(4)

Expanding $n(\omega'), n^2(\omega')$ around ω and up to first order of v, the above gives

$$c_{\text{lab}} \approx c \left[\frac{1}{n(\omega)} - \frac{1}{n^2(\omega)} \frac{dn}{d\omega} \Big|_{\omega} \Delta \omega \right] \pm v \mp \frac{v}{n^2(\omega)}$$

$$= \frac{c}{n(\omega)} \pm v \left[1 - \frac{1}{n^2(\omega)} + \frac{\omega}{n(\omega)} \frac{dn}{d\omega} \Big|_{\omega} \right]$$
(5)

2. Prob 11.9

(a) From the given transforms

$$x^{\prime \alpha} = \left(g^{\alpha \beta} + \epsilon^{\alpha \beta}\right) x_{\beta} \tag{6}$$

$$x^{\alpha} = \left(g^{\alpha\beta} + \epsilon'^{\alpha\beta}\right) x_{\beta}' \tag{7}$$

and the relation between contravariant and covariant vector

$$x_{\beta} = g_{\beta\gamma} x^{\gamma} \tag{8}$$

we have (plugging (8) into (6))

$$\begin{split} x'^{\alpha} &= \left(g^{\alpha\beta} + \epsilon^{\alpha\beta}\right) g_{\beta\gamma} x^{\gamma} & \text{expand } x^{\gamma} \text{ using (7)} \\ &= \left(g^{\alpha\beta} + \epsilon^{\alpha\beta}\right) g_{\beta\gamma} \left(g^{\gamma\delta} + \epsilon'^{\gamma\delta}\right) x_{\delta}' \\ &= \left(g^{\alpha\beta} + \epsilon^{\alpha\beta}\right) \left(\delta_{\beta}^{\ \delta} + g_{\beta\gamma} \epsilon'^{\gamma\delta}\right) x_{\delta}' & \text{ignore } O\left(\epsilon^{2}\right) \text{ for infinitesimal } \epsilon \\ &= g^{\alpha\beta} \delta_{\beta}^{\ \delta} x_{\delta}' + \epsilon^{\alpha\beta} \delta_{\beta}^{\ \delta} x_{\delta}' + g^{\alpha\beta} g_{\beta\gamma} \epsilon'^{\gamma\delta} x_{\delta}' \\ &= \underbrace{g^{\alpha\beta} x_{\beta}'}_{x'^{\alpha}} + \underbrace{\epsilon^{\alpha\beta} x_{\beta}' + \epsilon'^{\alpha\delta} x_{\delta}'}_{\left(\epsilon^{\alpha\beta} + \epsilon'^{\alpha\beta}\right) x_{\beta}'} \end{split}$$

(9)

Thus for inversion to hold, we must require

$$\epsilon^{\alpha\beta} = -\epsilon^{\prime\alpha\beta} \tag{10}$$

(b) Writing the squared distance of the prime frame in terms of the unprimed frame, we get

$$x'^{a}x'_{\alpha} = \left[\left(g^{\alpha\beta} + \epsilon^{\alpha\beta} \right) x_{\beta} \right] \left(g_{\alpha\delta} x'^{\delta} \right)$$

$$= \left[\left(g^{\alpha\beta} + \epsilon^{\alpha\beta} \right) g_{\beta\gamma} x^{\gamma} \right] \left[g_{\alpha\delta} \left(g^{\delta\mu} + \epsilon^{\delta\mu} \right) x_{\mu} \right]$$

$$= \left(\delta^{\alpha}{}_{\gamma} + \epsilon^{\alpha\beta} g_{\beta\gamma} \right) \left(\delta_{\alpha}{}^{\mu} + g_{\alpha\delta} \epsilon^{\delta\mu} \right) x^{\gamma} x_{\mu} \qquad \text{ignore } O\left(\epsilon^{2} \right) \text{ for infinitesimal } \epsilon$$

$$= x^{\alpha} x_{\alpha} + \epsilon^{\alpha\beta} \underbrace{g_{\beta\gamma} x^{\gamma}}_{x_{\beta}} x_{\alpha} + \epsilon^{\delta\mu} \underbrace{g_{\alpha\delta} x^{\alpha}}_{x_{\delta}} x_{\mu}$$

$$= x^{\alpha} x_{\alpha} + 2\epsilon^{\alpha\beta} x_{\alpha} x_{\beta} \qquad (11)$$

If norm is preserved, the second term must vanish for any covariant vector x. Keep in mind that we are using repeated-index convention, the condition that the second term vanishes is equivalent to requiring ϵ being antisymmetric, i.e.,

$$\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} \tag{12}$$

This can be seen by, for example, setting x=(0,1,0,0), the second term is $2\epsilon^{11}$, so all of the diagonal components of ϵ are clearly zero. Subsequently by setting, e.g., x=(0,1,0,1), the second term yields $2(\epsilon^{13}+\epsilon^{31}+\epsilon^{11}+\epsilon^{33})$, so we must have $\epsilon^{13}=-\epsilon^{31}$, and so on. Thus, condition (12) is necessary for the norm to be preserved.

(c) (6) can be rewritten as transform between contravariant vectors

$$x^{\prime \alpha} = \left(g^{\alpha \beta} + \epsilon^{\alpha \beta}\right) g_{\beta \gamma} x^{\gamma} = \left(\delta^{\alpha}_{\ \gamma} + \epsilon^{\alpha \beta} g_{\beta \gamma}\right) x^{\gamma} \tag{13}$$

Since $\epsilon^{\alpha\beta}$ is antisymmetric, we can write its matrix representation as

$$\epsilon = \begin{bmatrix} 0 & \zeta_1 & \zeta_2 & \zeta_3 \\ -\zeta_1 & 0 & -\omega_3 & \omega_2 \\ -\zeta_2 & \omega_3 & 0 & -\omega_1 \\ -\zeta_3 & -\omega_2 & \omega_1 & 0 \end{bmatrix}$$
 (14)

with infinitesimal ζ and ω , the matrix representation of the transform in (13) can be explicitly written out

$$\delta^{\alpha}{}_{\gamma} + \epsilon^{\alpha\beta} g_{\beta\gamma} = \begin{bmatrix} 1 & -\zeta_1 & -\zeta_2 & -\zeta_3 \\ -\zeta_1 & 1 & \omega_3 & -\omega_2 \\ -\zeta_2 & -\omega_3 & 1 & \omega_1 \\ -\zeta_3 & \omega_2 & -\omega_1 & 1 \end{bmatrix} = I - \boldsymbol{\zeta} \cdot \mathbf{K} - \boldsymbol{\omega} \cdot \mathbf{S} = I + L \approx e^L$$
 (15)