1. By (6.117)

$$
\begin{aligned}
\mathbf{P}_{\text{field}} &= \mu_0 \epsilon_0 \int \mathbf{E} \times \mathbf{H} d^3 x \\
&= \frac{1}{c^2} \int (-\boldsymbol{\nabla} \Phi) \times \mathbf{H} d^3 x \\
&= \frac{1}{c^2} \int \left[-\boldsymbol{\nabla} \times (\Phi \mathbf{H}) + \Phi \boldsymbol{\nabla} \times \mathbf{H} \right] d^3 x \\
&= -\frac{1}{c^2} \oint_{\infty} \mathbf{n} \times (\Phi \mathbf{H}) \, da + \frac{1}{c^2} \int \Phi \mathbf{J} d^3 x \qquad \text{if } |\mathbf{x}|^2 \Phi \mathbf{H} \to 0 \text{ as } |\mathbf{x}| \to \infty \\
&= \frac{1}{c^2} \int \Phi \mathbf{J} d^3 x
\end{aligned} \tag{1}
$$

2. Expanding $\Phi(\mathbf{x})$ around the origin, we have

$$
\Phi(\mathbf{x}) = \Phi(0) + \mathbf{x} \cdot (\boldsymbol{\nabla} \Phi)(0) + O\left(|\mathbf{x}|^2\right) \tag{2}
$$

Then (1) can be approximated by

$$
\mathbf{P}_{\text{field}} \approx \frac{\Phi(0)}{c^2} \int \mathbf{J} d^3 x - \frac{1}{c^2} \int \left[\mathbf{E}(0) \cdot \mathbf{x} \right] \mathbf{J} d^3 x \tag{3}
$$

Now we can use the two corollaries after equation (5.52) in section 5.6, namely

$$
\int \mathbf{J} d^3 x = 0 \tag{4}
$$

$$
\int \left(x_i J_j + x_j J_i \right) d^3 x = 0 \tag{5}
$$

on (3) to get

$$
\mathbf{P}_{\text{field}} \approx -\frac{1}{c^2} \int \left[\mathbf{E}(0) \cdot \mathbf{x} \right] \mathbf{J} d^3 x \tag{6}
$$

Consider its k-th component

$$
\begin{aligned}
P_{\text{field},k} &\approx -\frac{1}{c^2} \sum_i E_{0,i} \int x_i J_k d^3 x \\
&= -\frac{1}{c^2} \sum_i E_{0,i} \frac{1}{2} \int \left(x_i J_k - x_k J_i \right) d^3 x \\
&= -\frac{1}{c^2} \sum_i E_{0,i} \sum_j \epsilon_{ikj} \frac{1}{2} \int (\mathbf{x} \times \mathbf{J})_j \, d^3 x \\
&= -\frac{1}{c^2} \sum_i E_{0,i} \epsilon_{ikj} m_j \\
&= \frac{1}{c^2} \left[\mathbf{E}(0) \times \mathbf{m} \right]_k
\end{aligned} \tag{7}
$$

Therefore

$$
\mathbf{P}_{\text{field}} \approx \frac{1}{c^2} \mathbf{E}(0) \times \mathbf{m} \tag{8}
$$

3. When $\mathbf{E}(\mathbf{x}) = \mathbf{E}_0$ is a uniform field, the infinite surface integral term in (1) does not vanish and must be calculated explicitly.

$$
\Delta \mathbf{P}_{\text{field}} = -\frac{1}{c^2} \oint_{\infty} \mathbf{n} \times (\Phi \mathbf{H}) \, da \tag{9}
$$

At infinity, the $\mathbf{H}$ field is dominated by the contribution from dipole moment given in (5.56)

$$\mathbf{H} = \frac{1}{4\pi}\left[\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{m}) - \mathbf{m}}{|\mathbf{x}|^3}\right] \tag{10}$$

which turns (9) into

$$\begin{aligned}
\Delta \mathbf{P}_{\text{field}} &= \frac{1}{c^2}\frac{1}{4\pi}\oint_\infty \frac{\mathbf{n}\times(\Phi\mathbf{m})}{|\mathbf{x}|^3} da \\
&= \lim_{r\to\infty}\frac{1}{c^2}\frac{1}{4\pi}\int \frac{\mathbf{x}\times(\Phi\mathbf{m})}{r^4} r^2 d\Omega \\
&= -\lim_{r\to\infty}\frac{1}{c^2}\frac{1}{4\pi}\int \frac{\mathbf{x}\times[(\mathbf{E}_0\cdot\mathbf{x})\mathbf{m}]}{r^2} d\Omega
\end{aligned} \tag{11}$$

The k-th component of the integral is

$$\int \frac{1}{r^2}\sum_{i,j}\epsilon_{ijk}x_i m_j(\mathbf{E}_0\cdot\mathbf{x})\,d\Omega = \sum_{i,j,l}\epsilon_{ijk}m_j E_{0,l}\int \frac{x_i x_l}{r^2}d\Omega = \sum_{i,j,l}\epsilon_{ijk}m_j E_{0,l}\cdot\frac{1}{3}\delta_{il}4\pi = \frac{4\pi}{3}(\mathbf{E}_0\times\mathbf{m})_k \tag{12}$$

Plugging (12) back into (11) gives

$$\Delta\mathbf{P}_{\text{field}} = -\frac{1}{3c^2}\mathbf{E}_0\times\mathbf{m} \qquad \Longrightarrow \qquad \mathbf{P}_{\text{field}} = \frac{2}{3c^2}\mathbf{E}_0\times\mathbf{m} \tag{13}$$