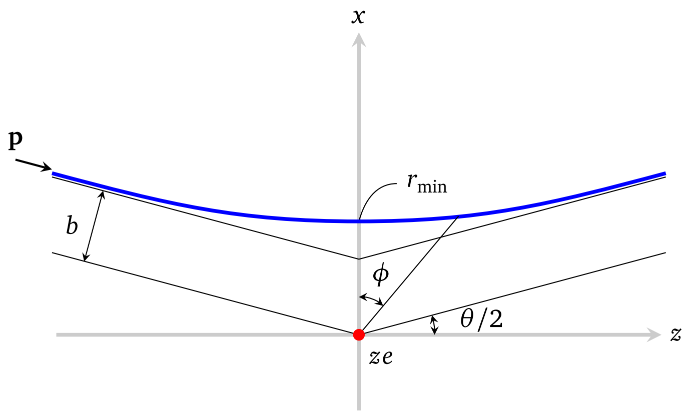
This is the (failed) attempt to derive the relativistic Rutherford scattering formula (Jackson equation 13.1).

We are working in the rest frame of the massive particle with mass $M \to \infty$ and charge ze. The incident electron (mass m, charge $-e$) comes from infinity with initial momentum $\mathbf{p}$, and impact parameter b. The following calculation strongly suggests (although not rigorously and conclusively) that (13.1) is not exact for all scattering angles and all initial velocities (maybe it is an approximation without explicitly stated conditions?).



Let the electron have conserved angular momentum $L = pb$, conserved energy $E = \sqrt{p^2c^2 + m^2c^4}$. Its potential energy at distance r is $V(r) = -ze^2/r$.

With polar coordinates (r, ϕ), conservation of energy and angular momentum require

$$E = \gamma mc^2 + V(r) \quad \Longrightarrow \quad \gamma = \frac{E - V(r)}{mc^2} \tag{1}$$

$$L = \gamma mr^2\dot{\phi} \quad \Longrightarrow \quad \dot{\phi} = \frac{L}{\gamma mr^2} \tag{2}$$

On the other hand, by definition,

$$\frac{1}{\gamma^2} = 1 - \left(\frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right) = 1 - \frac{1}{c^2}\left[\left(\frac{dr}{d\phi}\dot{\phi}\right)^2 + r^2\dot{\phi}^2\right] = 1 - \frac{1}{c^2}\left(\frac{L}{\gamma mr^2}\right)^2\left[\left(\frac{dr}{d\phi}\right)^2 + r^2\right] \tag{3}$$

Combining (3) with (1), we obtain the orbit equation

$$\left(\frac{L}{mr^2}\right)^2\left[\left(\frac{dr}{d\phi}\right)^2 + r^2\right] = \gamma^2c^2\left(1 - \frac{1}{\gamma^2}\right) = \frac{[E - V(r)]^2}{m^2c^2} - c^2 \tag{4}$$

With $u \equiv 1/r$, (4) can be rewritten as

$$\left(\frac{L}{m}\right)^2 u^4\left[\frac{1}{u^4}\left(\frac{du}{d\phi}\right)^2 + \frac{1}{u^2}\right] = \frac{\left(E + ze^2u\right)^2}{m^2c^2} - c^2 \quad \Longrightarrow$$

$$\left(\frac{du}{d\phi}\right)^2 = \frac{\left(E + ze^2u\right)^2}{c^2L^2} - \frac{m^2c^2}{L^2} - u^2 = \frac{\left(z^2e^4 - c^2L^2\right)u^2 + 2Eze^2u + \overbrace{E^2 - m^2c^4}^{p^2c^2}}{c^2L^2} = Au^2 + Bu + C \tag{5}$$

where

$$A = \frac{z^2e^4 - c^2L^2}{c^2L^2} \qquad B = \frac{2Eze^2}{c^2L^2} \qquad C = \frac{p^2c^2}{c^2L^2} \tag{6}$$

The discriminant is

$$\Delta = B^2 - 4AC = \frac{4\left(p^2c^2 + m^2c^4\right)z^2e^4 - 4\left(z^2e^4 - c^2L^2\right)p^2c^2}{c^4L^4} = \frac{4\left(m^2z^2e^4 + p^2L^2\right)}{L^4} > 0 \tag{7}$$

So we have two real roots of the quadratic equation $Au^2 + Bu + C = 0$

$$u_1 = \frac{-B + \sqrt{\Delta}}{2A} \qquad u_2 = \frac{-B - \sqrt{\Delta}}{2A} \tag{8}$$

Obviously, $C > 0$. The sign of B is the same as z.

1. When $A < 0$, regardless of the sign of B, we have $u_1 < 0 < u_2$. The motion is allowed in the range $u \in [0, u_2]$, i.e., an electron starting from infinity moves all the way to the stationary point at minimum distance $r_{\min} = 1/u_2$, then returns to infinity. This corresponds to the scattering process that works for both repulsive and attractive forces.
2. If $A > 0$, and $B > 0$ (i.e., $z > 0$), we see that $u_2 < u_1 < 0$. By (5), the admissible range for u is $[0, \infty)$. If the electron starts at infinity $u = 0$, it will keep going until $u \to \infty$ ($r = 0$) since the stationary point u_1, u_2 are not in the admissible range. This represents a capture orbit. Qualitatively, the condition $z > 0$ means this is an attractive force for the incident electron, and the condition $A > 0$, or $z^2e^4 > c^2p^2b^2$, indicates the electron is either moving too closely to the axis, or its initial momentum is too small, resulting it eventually being captured by the target particle.
3. When $A > 0$, but $B < 0$, we have $0 < u_2 < u_1$. The motion is allowed in the range $u \in [0, u_2]$, the closest distance is $r_{\min} = 1/u_2$. This corresponds to a scattering orbit in a repulsive force.

For the scattering process as a whole, the change in momentum is equal to the impulse (time integration of the Coulomb force),

$$\Delta \mathbf{p} = \int_{-\infty}^{\infty} \frac{-ze^2}{r^2} (\cos\phi\, \hat{\mathbf{x}} + \sin\phi\, \hat{\mathbf{z}})\, dt \tag{9}$$

By symmetry, the z component vanishes on both sides, leaving the x component equation

$$\begin{aligned}
2p \sin\frac{\theta}{2} &= |\Delta \mathbf{p}| = -ze^2 \int_{-\infty}^{\infty} \frac{\cos\phi}{r^2} dt && \text{by (2)} \\
&= -\frac{mze^2}{L} \int_{-\infty}^{\infty} \gamma \cos\phi\, \dot{\phi}\, dt = -\frac{mze^2}{L} \overbrace{\int_{-\phi_\infty}^{\phi_\infty} \gamma \cos\phi\, d\phi}^{I}
\end{aligned} \tag{10}$$

In non-relativistic limit where $\gamma = 1$, we can obtain the Rutherford scattering formula

$$\begin{aligned}
2p \sin\frac{\theta}{2} &= -\frac{mze^2}{L} \cdot 2\sin\phi_\infty && \text{note } \phi_\infty = \frac{\pi}{2} - \frac{\theta}{2} && \Longrightarrow \\
2p \sin\frac{\theta}{2} &= -\frac{mze^2}{pb} \cdot 2\cos\frac{\theta}{2} &&&& \Longrightarrow \\
b &= -\frac{mze^2}{p^2} \cot\frac{\theta}{2} = -\frac{ze^2}{pv} \cot\frac{\theta}{2} &&&& \Longrightarrow \\
\frac{d\sigma}{d\Omega} &= \frac{b}{\sin\theta} \left|\frac{db}{d\theta}\right| = \left(\frac{ze^2}{2pv}\right)^2 \frac{1}{\sin^4\theta/2}
\end{aligned} \tag{11}$$

If the particle is moving at relativistic speed, we can rewrite (10) as

$$\begin{aligned}
2p \sin\frac{\theta}{2} &= -\frac{mze^2}{L} \left(\gamma \sin\phi \Big|_{-\phi_\infty}^{\phi_\infty} - \int_{-\phi_\infty}^{\phi_\infty} \frac{d\gamma}{d\phi} \sin\phi\, d\phi \right) \\
&= -\frac{mze^2}{L} \cdot 2\gamma_\infty \sin\phi_\infty + \underbrace{\frac{mze^2}{L} \int_{-\phi_\infty}^{\phi_\infty} \frac{d\gamma}{d\phi} \sin\phi\, d\phi}_{K}
\end{aligned} \tag{12}$$

Now if term K vanishes, we would end up with

$$2p \sin\frac{\theta}{2} = -\frac{2\gamma_\infty mze^2}{pb} \cos\frac{\theta}{2} \qquad \Longrightarrow \qquad b = -\frac{ze^2}{pv} \cot\frac{\theta}{2} \tag{13}$$

which would give (11) as desired.

But it is clear that K does not vanish, since both $d\gamma/d\phi$ and $\sin\phi$ are odd in ϕ. *This is a strong hint that (11) does not hold in relativistic regime.*

We can also try to calculate integral I directly. By (1), the integral I becomes

$$I = \int_{-\phi_\infty}^{\phi_\infty} \left(\frac{E + ze^2 u}{mc^2} \right) \cos\phi\, d\phi = \frac{2E}{mc^2} \sin\phi_\infty + \frac{ze^2}{mc^2} \overbrace{\int_{-\phi_\infty}^{\phi_\infty} u \cos\phi\, d\phi}^{J} \tag{14}$$

Differentiating (5) on both sides yields

$$2u'u'' = 2Auu' + Bu' \qquad \Longrightarrow \qquad u'' - Au = \frac{B}{2} \tag{15}$$

Then integrating with $\cos\phi\, d\phi$ gives

$$\begin{aligned}
B\sin\phi_\infty = \int_{-\phi_\infty}^{\phi_\infty} \frac{B}{2}\cos\phi\, d\phi = \int_{-\phi_\infty}^{\phi_\infty} \left(u'' - Au\right)\cos\phi\, d\phi & \qquad \text{integrate by parts} \\
= u'\cos\phi\Big|_{-\phi_\infty}^{\phi_\infty} + \int_{-\phi_\infty}^{\phi_\infty} u'\sin\phi\, d\phi - AJ & \qquad \text{integrate by parts again} \\
= u'\cos\phi\Big|_{-\phi_\infty}^{\phi_\infty} + \underbrace{u\sin\phi\Big|_{-\phi_\infty}^{\phi_\infty}}_{0} - (A+1)J & \qquad (16)
\end{aligned}$$

By the sign convention indicated by the diagram, we have $u' = du/d\phi > 0$ for $\phi < 0$ (left half of diagram, as ϕ increases, u increases), and $u' < 0$ for $\phi > 0$ (right half of diagram, as ϕ increases, u decreases). With the correct sign, (5) gives

$$u'(-\phi_\infty) = \sqrt{C} = -u'(\phi_\infty) \qquad \Longrightarrow \qquad u'\cos\phi\Big|_{-\phi_\infty}^{\phi_\infty} = -2\sqrt{C}\cos\phi_\infty \tag{17}$$

hence by (16)

$$J = -\frac{2\sqrt{C}\cos\phi_\infty}{A+1} - \frac{B\sin\phi_\infty}{A+1} \tag{18}$$

This looks promising except that after putting (18) into (14) to obtain I, and then putting I back to (10), one would yield the useless identity

$$2p\sin\frac{\theta}{2} = 2p\sin\frac{\theta}{2}$$

This is not a surprise at all since in a central field, energy conservation implies impulse-momentum relation, so this way of calculating integral I (by using orbit equation (5) which is derived using energy conservation) yields no new relationship between b and θ.