1. **Problem 9.2**

Let the initial position of the positive charges be $(\pm a/\sqrt{2}, 0)$ and that of the negative charges be $(0, \pm a/\sqrt{2})$. The charges and their time-dependent positions are

$$\begin{aligned}
+q &: \qquad \frac{a}{\sqrt{2}}(\ \ \cos\omega t,\ \ \ \sin\omega t, 0) \\
+q &: \qquad \frac{a}{\sqrt{2}}(-\cos\omega t, -\sin\omega t, 0) \\
-q &: \qquad \frac{a}{\sqrt{2}}(-\sin\omega t,\ \ \ \cos\omega t, 0) \\
-q &: \qquad \frac{a}{\sqrt{2}}(\ \ \sin\omega t, -\cos\omega t, 0)
\end{aligned} \tag{1}$$

The quadrupole components can be calculated from

$$Q_{\alpha\beta} = \sum_i q_i \left(3x_{i\alpha}x_{i\beta} - r^2\delta_{\alpha\beta}\right) \tag{2}$$

Apparently $Q_{\alpha 3} = Q_{3\beta} = 0$. Explicit calculation produces

$$Q_{11} = 3qa^2(\cos^2\omega t - \sin^2\omega t) = 3qa^2\cos 2\omega t \tag{3}$$

$$Q_{12} = Q_{21} = 3qa^2(\cos\omega t\sin\omega t + \sin\omega t\cos\omega t) = 3qa^2\sin 2\omega t \tag{4}$$

$$Q_{22} = 3qa^2\left(\sin^2\omega t - \cos^2\omega t\right) = -3qa^2\cos 2\omega t \tag{5}$$

In complex convention, these components are

$$Q_{11} = 3qa^2 e^{-i2\omega t} = -Q_{22} \qquad\qquad Q_{12} = Q_{21} = i3qa^2e^{-i2\omega t} \tag{6}$$

Then

$$\mathbf{Q}(\mathbf{n}) = 3qa^2e^{-i2\omega t}\begin{bmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} = 3qa^2e^{-i2\omega t}\begin{bmatrix} \sin\theta e^{i\phi} \\ i\sin\theta e^{i\phi} \\ 0 \end{bmatrix} \tag{7}$$

and

$$\mathbf{n}\times\mathbf{Q}(\mathbf{n}) = 3qa^2e^{-i2\omega t}\begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}\times\begin{bmatrix} \sin\theta e^{i\phi} \\ i\sin\theta e^{i\phi} \\ 0 \end{bmatrix} = 3qa^2e^{-i2\omega t}\begin{bmatrix} -i\sin\theta\cos\theta e^{i\phi} \\ \sin\theta\cos\theta e^{i\phi} \\ i\sin^2\theta e^{i2\phi} \end{bmatrix} \tag{8}$$

and by (9.44), in the long wavelength approximation and in the far zone,

$$\mathbf{H} = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\mathbf{n}\times\mathbf{Q}(\mathbf{n}) = -\frac{ick^3qa^2}{8\pi}\frac{e^{ikr}}{r}\left(-\hat{\mathbf{x}}i\sin\theta\cos\theta e^{i\phi} + \hat{\mathbf{y}}\sin\theta\cos\theta e^{i\phi} + \hat{\mathbf{z}}i\sin^2\theta e^{i2\phi}\right)e^{-i2\omega t} \tag{9}$$

while the electric field $\mathbf{E}$ is related to $\mathbf{H}$ via

$$\mathbf{E} = Z_0\mathbf{H}\times\mathbf{n} \tag{10}$$

It should be noted that in this configuration, all of electric monopole, dipole and magnetization vanish, so the electric quadrupole accounts for the lowest order of field.

With (9.46)

$$|[\mathbf{n}\times\mathbf{Q}(\mathbf{n})]\times\mathbf{n}|^2 = \mathbf{Q}^*\cdot\mathbf{Q} - |\mathbf{n}\cdot\mathbf{Q}|^2 = 9q^2a^4\left(2\sin^2\theta - \sin^4\theta\right) = 9q^2a^4\left(1-\cos^4\theta\right) \tag{11}$$

then the angular distribution of power is given by (9.45)

$$\frac{dP}{d\Omega} = \frac{c^2Z_0}{1152\pi^2}k^6|[\mathbf{n}\times\mathbf{Q}(\mathbf{n})]\times\mathbf{n}|^2 = \frac{c^2Z_0}{128\pi^2}k^6q^2a^4\left(1-\cos^4\theta\right) = \frac{Z_0\omega^6}{2\pi^2c^4}q^2a^4\left(1-\cos^4\theta\right) \tag{12}$$

where we should take care to note $k = 2\omega/c$ since the quadrupole is radiating at 2ω frequency.

The total power of radiation is given by (9.49)

$$P = \frac{c^2Z_0k^6}{1440\pi}\sum_{\alpha,\beta}\left|Q_{\alpha\beta}\right|^2 = \frac{8Z_0\omega^6}{5\pi c^4}q^2a^4 \tag{13}$$

2. **Problem 9.3**

If the potential does not change with time, this configuration is the electrostatic example discussed in section 3.3, in particular, the static potential is given by (3.36)

$$\Phi(r,\theta) = V\left[\frac{3}{2}\left(\frac{R}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\left(\frac{R}{r}\right)^4 P_3(\cos\theta) + \frac{11}{16}\left(\frac{R}{r}\right)^6 P_5(\cos\theta) + \cdots\right] \qquad \text{for } r > R \tag{14}$$

These three terms correspond to the contributions from the $l = 1, 3, 5$ multipole moments.

The radiation is thus dominated by the dipole contribution $l = 1$.

Comparing the dipole potential (1.24)

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{x}}{|\mathbf{x}|^3} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} \tag{15}$$

with the dipole contribution of (14) gives the equivalent dipole

$$p = 4\pi\epsilon_0 r^2 \cdot V\frac{3}{2}\left(\frac{R}{r}\right)^2 \qquad \Longrightarrow \qquad \mathbf{p} = 6\pi\epsilon_0 V R^2 \hat{\mathbf{z}} \tag{16}$$

Then we can use (9.19) to calculate the dominating dipole contribution of the radiation fields

$$\mathbf{H} = \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r} = -\frac{3\epsilon_0 c k^2 V R^2}{2}\frac{e^{ikr}}{r}\sin\theta\,\hat{\boldsymbol{\phi}} \tag{17}$$

$$\mathbf{E} = Z_0\mathbf{H}\times\mathbf{n} = -\frac{3k^2VR^2}{2}\frac{e^{ikr}}{r}\sin\theta\,\hat{\boldsymbol{\theta}} \tag{18}$$

where the time dependency $e^{-i\omega t}$ is understood.

The angular distribution of power by (9.22) is

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}k^4\left|(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\right|^2 = \frac{c^2 Z_0}{32\pi^2}k^4 p^2\sin^2\theta = \frac{9}{8}\sqrt{\frac{\epsilon_0}{\mu_0}}k^4V^2R^4\sin^2\theta \tag{19}$$

The total power is given by (9.24)

$$P = \frac{c^2 Z_0 k^4}{12\pi}|\mathbf{p}|^2 = 3\pi\sqrt{\frac{\epsilon_0}{\mu_0}}k^4V^2R^4 \tag{20}$$

It is clear from (14) that the next order of electric moment is $l = 3$, i.e., there is no contribution from electric quadrupoles. The only remaining $l < 3$ contribution is the magnetization $\mathbf{m}$. The following calculation shows it also vanishes.

Indeed, the surface charge is

$$\sigma = -\epsilon_0\mathbf{n}\cdot\nabla\Phi\Big|_{r=R} = -\epsilon_0\mathbf{n}\cdot\left(\frac{\partial\Phi}{\partial r}\mathbf{n} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\boldsymbol{\theta}}\right)\Big|_{r=R} = \frac{3\epsilon_0 V}{R}\cos\theta \tag{21}$$

Then by charge continuity, the surface current density $\mathbf{K} = K\hat{\boldsymbol{\theta}}$ satisfies

$$\nabla\cdot\mathbf{K} - i\omega\sigma = 0 \qquad \Longrightarrow \qquad \frac{1}{R\sin\theta}\frac{\partial(K\sin\theta)}{\partial\theta} = i\omega\cdot\frac{3\epsilon_0 V}{R}\cos\theta \qquad \Longrightarrow \qquad \mathbf{K} = \frac{i3\omega\epsilon_0 V\sin\theta}{2}\hat{\boldsymbol{\theta}} \tag{22}$$

which gives the magnetization

$$\mathbf{m} \propto \int \mathbf{x}\times\mathbf{K}da \propto \int_0^\pi \sin^2\theta d\theta\int_0^{2\pi}\hat{\boldsymbol{\phi}}d\phi = \int_0^\pi \sin^2\theta d\theta\int_0^{2\pi}(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}})d\phi = 0 \tag{23}$$