1. Without loss of generality, let the charge be rotating around z axis with angular velocity ω_0. Let $\rho_0(\mathbf{x})$ be the charge distribution at $t=0$ and let $\mathbf{x}=(r,\theta,\phi)$ be the spherical coordinates. Then at time t, the charge distribution is

$$\rho(\mathbf{x},t)=\rho_0(r,\theta,\phi-\omega_0 t) \tag{1}$$

By definition (see (4.3)) the multipole moment $q_{lm}(t)$ is given by

$$\begin{aligned}
q_{lm}(t) &= \int Y^*_{lm}(\theta',\phi')\, r'^l \rho_0(r',\theta',\phi'-\omega_0 t)\, d^3x' \qquad \text{variable change } \phi'\to\phi'+\omega_0 t\\
&= \int Y^*_{lm}(\theta',\phi'+\omega_0 t)\, r'^l \rho_0(r',\theta',\phi')\, d^3x'\\
&= \int \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta')\, e^{-im\phi'}e^{-im\omega_0 t} r'^l \rho_0(\mathbf{x}')\, d^3x'\\
&= e^{-im\omega_0 t}\int Y^*_{lm}(\theta',\phi')\, r'^l \rho_0(\mathbf{x}')\, d^3x'\\
&= e^{-im\omega_0 t} q_{lm}(0)
\end{aligned} \tag{2}$$

This shows that for a rotating source, its multipole expansion's lm-th component is oscillating with frequency $m\omega_0$.

2. For a periodically changing source $\rho(\mathbf{x},t)$, it can be expanded into Fourier series (note n ranges from all integers)

$$\rho(\mathbf{x},t)=\sum_{n=-\infty}^{\infty}\rho_n(\mathbf{x})\, e^{-in\omega_0 t} \tag{3}$$

where

$$\rho_n(\mathbf{x})=\frac{1}{T}\int_0^T \rho(\mathbf{x},t')\, e^{in\omega_0 t'}dt' \tag{4}$$

To verify this, substitute (4) into the RHS of (3),

$$\text{RHS}_{(3)}=\sum_{n=-\infty}^{\infty}\frac{1}{T}\int_0^T\rho(\mathbf{x},t')\, e^{in\omega_0(t'-t)}dt' = \frac{1}{T}\int_0^T \rho(\mathbf{x},t')\, dt' \overbrace{\sum_{n=-\infty}^{\infty} e^{in\omega_0(t'-t)}}^{T\delta(t'-t)} = \rho(\mathbf{x},t) \tag{5}$$

where we have used

$$\delta(x)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}e^{inx} \qquad\Longrightarrow\qquad \sum_{n=-\infty}^{\infty}e^{in\omega_0(t'-t)}=2\pi\delta\left[\omega_0(t'-t)\right]=\frac{2\pi}{\omega_0}\delta(t'-t)=T\delta(t'-t) \tag{6}$$

(3) can be written in identical form that has no negative frequencies

$$\rho(\mathbf{x},t)=\rho_0(\mathbf{x})+\sum_{n=1}^{\infty}2\,\mathrm{Re}\left[\rho_n(\mathbf{x})\, e^{-in\omega_0 t}\right] \tag{7}$$

If we take $\rho(\mathbf{x},t)$ to be the rotating source (1), the n-th harmonic (4) can be written

$$\rho_n(\mathbf{x})=\frac{1}{T}\int_0^T \rho_0(r,\theta,\phi-\omega_0 t')\, e^{in\omega_0 t'}dt' \tag{8}$$

whose contribution to q_{lm} is

$$\begin{aligned}
q^n_{lm} &= \int Y^*_{lm}(\theta',\phi')\, r'^l \rho_n(\mathbf{x}')\, d^3x'\\
&= \int Y^*_{lm}(\theta',\phi')\, r'^l\left[\frac{1}{T}\int_0^T \rho_0(r',\theta',\phi'-\omega_0 t')\, e^{in\omega_0 t'}dt'\right] d^3x' \qquad \text{variable change } \phi'\to\phi'+\omega_0 t'\\
&= \int Y^*_{lm}(\theta',\phi'+\omega_0 t')\, r'^l\left[\frac{1}{T}\int_0^T \rho_0(r',\theta',\phi')\, e^{in\omega_0 t'}dt'\right] d^3x'\\
&= \overbrace{\frac{1}{T}\int_0^T e^{i(n-m)\omega_0 t'}dt'}^{\delta_{mn}}\int Y^*_{lm}(\theta',\phi')\, r'^l \rho_0(r',\theta',\phi')\, d^3x = \delta_{mn} q_{lm}(0)
\end{aligned} \tag{9}$$

The interpretation of this connection is that the lm-th multipole moment q_{lm} has contribution only from the m-th harmonic $\rho_m(\mathbf{x})$ of the source distribution.

3. For the rotating single charge, we can express the time-dependent charge density as

$$\rho(\mathbf{x}, t) = \frac{q}{R^2}\delta(r-R)\,\delta(\cos\theta)\,\delta(\phi-\omega_0 t) \tag{10}$$

Thus by (2)

$$\begin{aligned}
q_{lm}(t) &= e^{-im\omega_0 t}\int Y_{lm}^*\left(\theta',\phi'\right) r'^l \rho_0\left(\mathbf{x}'\right) d^3x' \\
&= e^{-im\omega_0 t}\frac{q}{R^2}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\overbrace{\int_0^\infty r'^{l+2}\delta\left(r'-R\right)dr'}^{R^{l+2}}\underbrace{\int_0^\pi \delta\left(\cos\theta'\right)P_l^m\left(\cos\theta'\right)\sin\theta' d\theta'}_{P_l^m(0)}\overbrace{\int_0^{2\pi}\delta\left(\phi'\right)e^{-im\phi'}d\phi'}^{1} \\
&= e^{-im\omega_0 t}qR^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0) \qquad (11)
\end{aligned}$$

For $l=0,1$,

$$q_{00}(t) = q\sqrt{\frac{1}{4\pi}} \tag{12}$$

$$q_{11}(t) = -e^{-i\omega_0 t}qR\sqrt{\frac{3}{8\pi}} \qquad q_{10}(t) = 0 \qquad q_{1,-1}(t) = e^{i\omega_0 t}qR\sqrt{\frac{3}{8\pi}} \tag{13}$$

Method in part (b) would have given the same result due to the more general result (9).
From (11) we see that higher moments exist if $P_l^m(0) \neq 0$. By the parity of associated Legendre function

$$P_l^m(-x) = (-1)^{l+m}P_l^m(x) \tag{14}$$

the higher moments exist if $l+m$ is even. The radiation frequency is $\pm m\omega_0$ for any even $m+l$.