1. The force can be written as

$$\mathbf{F} = \int_V (\nabla \times \mathbf{M}) \times \mathbf{B}_e d^3x + \oint_S (\mathbf{M} \times \mathbf{n}) \times \mathbf{B}_e da \tag{1}$$

With vector identity (see Jackson inner cover)

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) \tag{2}$$

the first integral in (1) can be written as

$$\begin{aligned}\int_V (\nabla \times \mathbf{M}) \times \mathbf{B}_e d^3x &= -\int_V \mathbf{B}_e \times (\nabla \times \mathbf{M}) d^3x \\ &= \int_V (\mathbf{M} \cdot \nabla)\mathbf{B}_e d^3x + \int_V (\mathbf{B}_e \cdot \nabla)\mathbf{M} d^3x + \int_V \mathbf{M} \times \overbrace{(\nabla \times \mathbf{B}_e)}^{0} d^3x - \int_V \nabla(\mathbf{M} \cdot \mathbf{B}_e) d^3x \\ &= \int_V (\mathbf{M} \cdot \nabla)\mathbf{B}_e d^3x + \int_V (\mathbf{B}_e \cdot \nabla)\mathbf{M} d^3x - \int_V \nabla(\mathbf{M} \cdot \mathbf{B}_e) d^3x \end{aligned} \tag{3}$$

Again with vector identities and vector calculus

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{4}$$

$$\int_V \nabla\psi d^3x = \oint_S \psi \mathbf{n} da \tag{5}$$

the second integral in (1) can be written as

$$\begin{aligned}\oint_S (\mathbf{M} \times \mathbf{n}) \times \mathbf{B}_e da &= \oint_S \mathbf{B}_e \times (\mathbf{n} \times \mathbf{M}) da \\ &= \oint_S (\mathbf{B}_e \cdot \mathbf{M})\mathbf{n} da - \oint_S (\mathbf{B}_e \cdot \mathbf{n})\mathbf{M} da \\ &= \int_V \nabla(\mathbf{B}_e \cdot \mathbf{M}) d^3x - \oint_S (\mathbf{B}_e \cdot \mathbf{n})\mathbf{M} da \end{aligned} \tag{6}$$

With (3) and (6) inserted into (1), we have

$$\mathbf{F} = \int_V (\mathbf{M} \cdot \nabla)\mathbf{B}_e d^3x + \int_V (\mathbf{B}_e \cdot \nabla)\mathbf{M} d^3x - \oint_S (\mathbf{B}_e \cdot \mathbf{n})\mathbf{M} da \tag{7}$$

Now for arbitrary vector field $\mathbf{P}, \mathbf{Q}$, let

$$\mathbf{R} = \int_V (\mathbf{P} \cdot \nabla)\mathbf{Q} d^3x \tag{8}$$

we see (with Einstein summation convention)

$$\begin{aligned} R_j &= \int_V \left(P_i \frac{\partial}{\partial x_i}\right) Q_j d^3x = \int_V P_i \frac{\partial Q_j}{\partial x_i} d^3x \\ &= \int_V \left[\frac{\partial\left(P_i Q_j\right)}{\partial x_i} - Q_j \frac{\partial P_i}{\partial x_i}\right] d^3x \\ &= \int_V \left[\nabla \cdot \left(Q_j \mathbf{P}\right) - Q_j (\nabla \cdot \mathbf{P})\right] d^3x \\ &= \oint_S \left(Q_j \mathbf{P}\right) \cdot \mathbf{n} da - \int_V Q_j (\nabla \cdot \mathbf{P}) d^3x \\ &= \oint_S Q_j (\mathbf{P} \cdot \mathbf{n}) da - \int_V Q_j (\nabla \cdot \mathbf{P}) d^3x \end{aligned} \tag{9}$$

Assembling R_j's back into $\mathbf{R}$,

$$\int_V (\mathbf{P}\cdot\boldsymbol{\nabla})\,\mathbf{Q}d^3x = \mathbf{R} = \oint_S \mathbf{Q}\,(\mathbf{P}\cdot\mathbf{n})\,da - \int_V \mathbf{Q}\,(\boldsymbol{\nabla}\cdot\mathbf{P})\,d^3x \tag{10}$$

Applying (10) to (7) gives

$$\begin{aligned}\mathbf{F} &= \left[\oint_S \mathbf{B}_e\,(\mathbf{M}\cdot\mathbf{n})\,da - \int_V \mathbf{B}_e\,(\boldsymbol{\nabla}\cdot\mathbf{M})\,d^3x\right] + \left[\oint_S \mathbf{M}\,(\mathbf{B}_e\cdot\mathbf{n})\,da - \int_V \mathbf{M}\overbrace{(\boldsymbol{\nabla}\cdot\mathbf{B}_e)}^{0}d^3x\right] - \oint_S (\mathbf{B}_e\cdot\mathbf{n})\,\mathbf{M}da \\ &= \oint_S \mathbf{B}_e\,(\mathbf{M}\cdot\mathbf{n})\,da - \int_V \mathbf{B}_e\,(\boldsymbol{\nabla}\cdot\mathbf{M})\,d^3x \end{aligned} \tag{11}$$

as desired.

2. This is a brute force calculation. The magnetization is uniform

$$\mathbf{M}(r,\theta,\phi) = M\begin{bmatrix}\sin\theta_0\cos\phi_0\\ \sin\theta_0\sin\phi_0\\ \cos\theta_0\end{bmatrix} \tag{12}$$

so the second term in (11) vanishes. The force only has contribution from the surface integral, which is

$$\mathbf{F} = \oint_S \mathbf{B}_e\,(\mathbf{M}\cdot\mathbf{n})\,da \tag{13}$$

where

$$\mathbf{B}_e\,(\theta,\phi) = B_0\begin{bmatrix}1+R\beta\sin\theta\sin\phi\\ 1+R\beta\sin\theta\cos\phi\\ 0\end{bmatrix} \tag{14}$$

$$\begin{aligned}\mathbf{M}\cdot\mathbf{n} &= M\begin{bmatrix}\sin\theta_0\cos\phi_0\\ \sin\theta_0\sin\phi_0\\ \cos\theta_0\end{bmatrix}\cdot\begin{bmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{bmatrix}\\ &= M\,(\sin\theta_0\cos\phi_0\sin\theta\cos\phi + \sin\theta_0\sin\phi_0\sin\theta\sin\phi + \cos\theta_0\cos\theta)\end{aligned} \tag{15}$$

Thus

$$\mathbf{F} = B_0MR^3\int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta\,d\theta\,\mathbf{f}(\theta,\phi) \tag{16}$$

where

$$f_1\,(\theta,\phi) = \beta\sin\theta\sin\phi\,(\sin\theta_0\cos\phi_0\sin\theta\cos\phi + \sin\theta_0\sin\phi_0\sin\theta\sin\phi + \cos\theta_0\cos\theta) \tag{17}$$
$$f_2\,(\theta,\phi) = \beta\sin\theta\cos\phi\,(\sin\theta_0\cos\phi_0\sin\theta\cos\phi + \sin\theta_0\sin\phi_0\sin\theta\sin\phi + \cos\theta_0\cos\theta) \tag{18}$$
$$f_3\,(\theta,\phi) = 0 \tag{19}$$

where we have ignored the "1" in (14) since it has no net contribution to the force.

While performing the integral, notice the terms with $\sin\phi, \cos\phi, \sin\phi\cos\phi$ will all vanish after the $d\phi$ integral, so the result is actually easy to obtain, which is

$$\mathbf{F} = \beta B_0M\left(\frac{4\pi R^3}{3}\right)(\sin\theta_0\sin\phi_0\hat{\mathbf{x}} + \sin\theta_0\cos\phi_0\hat{\mathbf{y}}) \tag{20}$$