1. **Prob 3.1**

   This is a straightforward application of equation (3.33)

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta) \tag{1}$$

   Boundary condition dictates

$$V_{\text{inner}}(\theta) = \sum_l \left[A_l a^l + B_l a^{-(l+1)}\right] P_l(\cos\theta) \tag{2}$$

$$V_{\text{outer}}(\theta) = \sum_l \left[A_l b^l + B_l b^{-(l+1)}\right] P_l(\cos\theta) \tag{3}$$

   By completeness of $P_l(\cos\theta)$ over the range $[0,\pi]$ (see equation (3.24)),

$$\begin{aligned} A_l a^l + B_l a^{-(l+1)} &= \frac{2l+1}{2}\int_0^{\pi} V_{\text{inner}}(\theta) P_l(\cos\theta)\sin\theta d\theta \\ &= \frac{2l+1}{2}\int_0^{\pi/2} V P_l(\cos\theta)\sin\theta d\theta \\ &= \frac{(2l+1)V}{2}\underbrace{\int_0^1 P_l(x)dx}_{I_l} = \frac{(2l+1)V}{2} I_l \end{aligned} \tag{4}$$

$$\begin{aligned} A_l b^l + B_l b^{-(l+1)} &= \frac{2l+1}{2}\int_0^{\pi} V_{\text{outer}}(\theta) P_l(\cos\theta)\sin\theta d\theta \\ &= \frac{(2l+1)V}{2}\int_{\pi/2}^{\pi} P_l(\cos\theta)\sin\theta d\theta \\ &= \frac{(2l+1)V}{2}\int_{-1}^{0} P_l(x)dx = \frac{(2l+1)V}{2}(-1)^l I_l \end{aligned} \tag{5}$$

   Multiplying (4) by b^l and (5) by a^l and subtract, we have

$$\begin{aligned} B_l b^l a^{-(l+1)} - B^l a^l b^{-(l+1)} &= \frac{(2l+1)VI_l}{2}\left[b^l - (-a)^l\right] \qquad \Longrightarrow \\ B_l &= \frac{(2l+1)VI_l}{2}\cdot\frac{\left[b^l-(-a)^l\right](ab)^{l+1}}{b^{2l+1}-a^{2l+1}} \qquad \left(\text{define } \lambda \equiv \frac{a}{b}\right) \qquad (6) \\ &= \frac{(2l+1)VI_l}{2}\cdot\frac{a^{l+1}\left[1-(-\lambda)^l\right]}{1-\lambda^{2l+1}} \qquad (7) \end{aligned}$$

   Then we can obtain A_l from (4) and (6):

$$\begin{aligned} A_l &= a^{-l}\left[\frac{(2l+1)VI_l}{2} - B_l a^{-(l+1)}\right] \\ &= \frac{(2l+1)VI_l}{2}a^{-l}\left\{1 - \frac{\left[b^l-(-a)^l\right]b^{l+1}}{b^{2l+1}-a^{2l+1}}\right\} \\ &= \frac{(2l+1)VI_l}{2}\cdot\frac{(-1)^l b^{l+1} - a^{l+1}}{b^{2l+1}-a^{2l+1}} \\ &= \frac{(2l+1)VI_l}{2}\cdot\frac{b^{-l}\left[(-1)^l - \lambda^{l+1}\right]}{1-\lambda^{2l+1}} \end{aligned} \tag{8}$$

   Therefore

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\frac{(2l+1)VI_l}{2}\left[\frac{(-1)^l-\lambda^{l+1}}{1-\lambda^{2l+1}}\left(\frac{r}{b}\right)^l + \frac{1-(-\lambda)^l}{1-\lambda^{2l+1}}\left(\frac{a}{r}\right)^{l+1}\right]P_l(\cos\theta) \tag{9}$$

Recall the recurrence relation of Legendre polynomials (reference Wikipedia):

$$P_l(x) = \frac{1}{2l+1}\frac{d}{dx}\left[P_{l+1}(x) - P_{l-1}(x)\right] \tag{10}$$

which implies

$$I_l = \int_0^1 P_l(x)dx = \frac{1}{2l+1}\left[P_{l+1}(x) - P_{l-1}(x)\right]\Big|_0^1 = \frac{1}{2l+1}\left[P_{l-1}(0) - P_{l+1}(0)\right] \tag{11}$$

For even l except 0, we see that both $P_{l+1}(0)$ and $P_{l-1}(0)$ vanish due to their odd parity, so the only even l term survived in (9) is $l = 0$, which enables us to simplify (9) as

$$\Phi(r,\theta) = \frac{V}{2} + \frac{V}{2}\sum_{l\text{ odd}}\left[P_{l-1}(0) - P_{l+1}(0)\right]\left[\frac{(-1)^l - \lambda^{l+1}}{1-\lambda^{2l+1}}\left(\frac{r}{b}\right)^l + \frac{1-(-\lambda)^l}{1-\lambda^{2l+1}}\left(\frac{a}{r}\right)^{l+1}\right]P_l(\cos\theta) \tag{12}$$

In the limit $a \to 0$, $\lambda \to 0$, the first few terms are

$$\begin{aligned}
\Phi(r,\theta) =& \frac{V}{2} - \frac{V}{2}\left[P_0(0) - P_2(0)\right]\left(\frac{r}{b}\right)P_1(\cos\theta) - \\
& \frac{V}{2}\left[P_2(0) - P_4(0)\right]\left(\frac{r}{b}\right)^3 P_3(\cos\theta) - \frac{V}{2}\left[P_4(0) - P_6(0)\right]\left(\frac{r}{b}\right)^5 P_5(\cos\theta) - \cdots \\
=& \frac{V}{2} - \frac{V}{2}\left[\frac{3}{2}\left(\frac{r}{b}\right)P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{b}\right)^3 P_3(\cos\theta) + \frac{11}{16}\left(\frac{r}{b}\right)^5 P_5(\cos\theta) - \cdots\right]
\end{aligned} \tag{13}$$

which agrees with (3.36) considering the zero-potential difference of the two problem statements.

2. **Prob 3.2**

   (a) Let's consider the potential of a point on the z-axis inside the sphere, which is

$$\begin{aligned}
\Phi(r<R, \theta = 0 \text{ or } \pi) &= \int_0^{2\pi} d\phi' \int_\alpha^\pi \sin\theta' d\theta' \frac{R^2\sigma(\theta',\phi')}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2+R^2-2Rr\cos\theta'}} \\
&= \frac{Q}{8\pi\epsilon_0}\int_\alpha^\pi \sum_{l=0}^\infty \frac{r^l}{R^{l+1}} P_l(\cos\theta')\sin\theta' d\theta' \\
&= \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^\infty \frac{r^l}{R^{l+1}}\int_{-1}^{\cos\alpha} P_l(x)dx \qquad \text{by (10)} \\
&= \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^\infty \frac{r^l}{R^{l+1}}\cdot\frac{1}{2l+1}\left[P_{l+1}(x) - P_{l-1}(x)\right]\Big|_{-1}^{\cos\alpha} \\
&= \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^\infty \frac{r^l}{R^{l+1}}\cdot\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]
\end{aligned} \tag{14}$$

   where in the last step, we have used the fact that $P_{l+1}(x)$ and $P_{l-1}(x)$ are of the same parity, thus take the same value at $x = -1$. Also note that for $l = 0$, using the convention $P_{-1}(\cos\alpha) = -1$ will be consistent with the result directly obtained from $\int_{-1}^{\cos\alpha} P_0(x)dx$.

   Knowing the potential of points on the z axis, it's a simple matter to multiply each l term with $P_l(\cos\theta)$ to obtain the off-axis points that are inside the sphere (see uniqueness argument on page 102), thus for $r < R$,

$$\Phi(r,\theta) = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^\infty \frac{r^l}{R^{l+1}}\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]P_l(\cos\theta) \tag{15}$$

   For points outside, we just need to replace r^l/R^{l+1} with R^l/r^{l+1}.

   (b) For the field, we have

$$\mathbf{E} = -\frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\boldsymbol{\theta}} \tag{16}$$

   Note at the origin, only the $l = 1$ term in (15) survives the differentiation, which gives

$$\begin{aligned}
\mathbf{E} &= -\frac{Q}{8\pi\epsilon_0}\frac{1}{R^2}\frac{1}{3}\left[P_2(\cos\alpha) - P_0(\cos\alpha)\right]\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right) \\
&= -\frac{Q}{24\pi\epsilon_0 R^2}\left(\frac{3\cos^2\alpha}{2} - \frac{1}{2} - 1\right)\hat{\mathbf{z}} \\
&= \frac{Q\sin^2\alpha}{16\pi\epsilon_0 R^2}\hat{\mathbf{z}}
\end{aligned} \tag{17}$$

(c) For the limit of $\alpha \to 0$, notice

$$\begin{aligned}
P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha) &\approx P_{l+1}\left(1-\frac{\alpha^2}{2}\right) - P_{l-1}\left(1-\frac{\alpha^2}{2}\right) \\
&\approx \left[P_{l+1}(1) - P'_{l+1}(1)\left(\frac{\alpha^2}{2}\right)\right] - \left[P_{l-1}(1) - P'_{l-1}(1)\left(\frac{\alpha^2}{2}\right)\right] \\
&= -\left(\frac{\alpha^2}{2}\right)\left[P'_{l+1}(1) - P'_{l-1}(1)\right] \qquad \text{by (10)} \\
&= -\left(\frac{\alpha^2}{2}\right)(2l+1)\,P_l(1) = -\frac{\alpha^2}{2}\,(2l+1) \qquad (18)
\end{aligned}$$

which turns on-axis potential (14) into

$$\begin{aligned}
\Phi(r<R, \theta = 0 \text{ or } \pi) &\approx -\frac{Q}{8\pi\epsilon_0 R}\left(\frac{\alpha^2}{2}\right)\sum_{l=0}^{\infty}\left(\frac{r}{R}\right)^l \\
&= -\frac{Q}{8\pi\epsilon_0 R}\left(\frac{\alpha^2}{2}\right)\frac{1}{1-\dfrac{r}{R}} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{\left(\dfrac{Q}{4\pi R^2}\right)\left(\pi R^2\alpha^2\right)}{R-r} \qquad (19)
\end{aligned}$$

which is exactly the on-axis potential generated by the negatively charged disc of radius $R\alpha$ at the north pole. For the $\alpha \to \pi$ limit, we do similar things as (18) except with $\cos\alpha \approx -1 + \alpha^2/2$, we end up with

$$\begin{aligned}
P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha) &\approx \left(\frac{\alpha^2}{2}\right)\left[P'_{l+1}(-1) - P'_{l-1}(-1)\right] \\
&= \left(\frac{\alpha^2}{2}\right)(2l+1)\,P_l(-1) = (-1)^l\,\frac{\alpha^2}{2}\,(2l+1) \qquad (20)
\end{aligned}$$

which turns (14) into

$$\begin{aligned}
\Phi(r<R, \theta = 0 \text{ or } \pi) &\approx \frac{Q}{8\pi\epsilon_0 R}\left(\frac{\alpha^2}{2}\right)\sum_{l=0}^{\infty}\left(-\frac{r}{R}\right)^l \\
&= \frac{Q}{8\pi\epsilon_0 R}\left(\frac{\alpha^2}{2}\right)\frac{1}{1+\dfrac{r}{R}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{\left(\dfrac{Q}{4\pi R^2}\right)\left(\pi R^2\alpha^2\right)}{R+r} \qquad (21)
\end{aligned}$$

which is recognized as the potential generated by the now-tiny disc of radius $R\alpha$ at the south pole. Similar interpretation can be applied to the field equation (17) when $\alpha \to 0$ or $\alpha \to \pi$.