1. **Addition of non-collinear velocities**

   Let S be the lab frame, S' be a frame moving with velocity $\mathbf{u}$ relative to S, and S'' be a frame moving with velocity $\mathbf{v}$ relative to S' where $\mathbf{v}$ and $\mathbf{u}$ are not necessarily collinear. Denote $\mathbf{u} \oplus \mathbf{v}$ as the composite velocity (S'' relative to S). Jackson (11.31) gave the parallel and perpendicular components of $\mathbf{u} \oplus \mathbf{v}$ as

$$(\mathbf{u} \oplus \mathbf{v})_{\parallel} = \frac{\mathbf{v}_{\parallel} + \mathbf{u}}{1 + \dfrac{\mathbf{u} \cdot \mathbf{v}}{c^2}} \qquad (\mathbf{u} \oplus \mathbf{v})_{\perp} = \frac{\mathbf{v}_{\perp}}{\gamma_u \left(1 + \dfrac{\mathbf{u} \cdot \mathbf{v}}{c^2}\right)} \tag{1}$$

   With

$$\mathbf{v}_{\parallel} = \frac{(\mathbf{u} \cdot \mathbf{v})\,\mathbf{u}}{u^2} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \frac{(\mathbf{u} \cdot \mathbf{v})\,\mathbf{u}}{u^2} \tag{2}$$

   plugged into (1), we have

$$\begin{aligned}
\mathbf{u} \oplus \mathbf{v} &= \left(\frac{\mathbf{v}_{\parallel} + \mathbf{u}}{1 + \dfrac{\mathbf{u} \cdot \mathbf{v}}{c^2}}\right) + \frac{\mathbf{v}_{\perp}}{\gamma_u \left(1 + \dfrac{\mathbf{u} \cdot \mathbf{v}}{c^2}\right)} \\
&= \left[\frac{\dfrac{(\mathbf{u} \cdot \mathbf{v})\,\mathbf{u}}{u^2} + \mathbf{u}}{1 + \dfrac{\mathbf{u} \cdot \mathbf{v}}{c^2}}\right] + \left[\frac{\mathbf{v} - \dfrac{(\mathbf{u} \cdot \mathbf{v})\,\mathbf{u}}{u^2}}{\gamma_u \left(1 + \dfrac{\mathbf{u} \cdot \mathbf{v}}{c^2}\right)}\right] \\
&= \frac{\left[1 + \left(1 - \dfrac{1}{\gamma_u}\right) \dfrac{(\mathbf{u} \cdot \mathbf{v})}{u^2}\right] \mathbf{u} + \dfrac{\mathbf{v}}{\gamma_u}}{1 + \dfrac{\mathbf{u} \cdot \mathbf{v}}{c^2}}
\end{aligned} \tag{3}$$

   With some straightforward algebra, we can show that

$$|\mathbf{u} \oplus \mathbf{v}| = |\mathbf{v} \oplus \mathbf{u}| \tag{4}$$

   of which the corresponding Lorentz factor is given in (11.34)

$$\gamma_{\mathbf{u} \oplus \mathbf{v}} = \gamma_{\mathbf{v} \oplus \mathbf{u}} = \gamma = \gamma_u \gamma_v \left(1 + \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}\right) \tag{5}$$

   From (3), we see here that when $\mathbf{u}$ and $\mathbf{v}$ are not collinear, $\oplus$ is not commutative, i.e., $\mathbf{u} \oplus \mathbf{v}$ and $\mathbf{v} \oplus \mathbf{u}$ have the same norm but in general have an angle θ between them.

2. **Matrix representation of the boost transformation**

   Let $B(\mathbf{u})$ be the boost transformation by velocity $\mathbf{u}$, the matrix representation (11.98) is simplified to the following block form

$$B(\mathbf{u}) = \begin{bmatrix} \gamma_u & -\gamma_u \dfrac{\mathbf{u}^T}{c} \\ -\gamma_u \dfrac{\mathbf{u}}{c} & I + (\gamma_u - 1)\dfrac{\mathbf{u}\mathbf{u}^T}{u^2} \end{bmatrix} \tag{6}$$

   where in the matrix representation, $\mathbf{u}$ is a column vector.

   The Lorentz transformation Λ from S to S'' is thus given by the two successive boosts, $B(\mathbf{u})$ followed by $B(\mathbf{v})$, i.e.

$$\begin{aligned}
\Lambda = B(\mathbf{v})B(\mathbf{u}) &= \begin{bmatrix} \gamma_v & -\gamma_v \dfrac{\mathbf{v}^T}{c} \\ -\gamma_v \dfrac{\mathbf{v}}{c} & I + (\gamma_v - 1)\dfrac{\mathbf{v}\mathbf{v}^T}{v^2} \end{bmatrix} \begin{bmatrix} \gamma_u & -\gamma_u \dfrac{\mathbf{u}^T}{c} \\ -\gamma_u \dfrac{\mathbf{u}}{c} & I + (\gamma_u - 1)\dfrac{\mathbf{u}\mathbf{u}^T}{u^2} \end{bmatrix} \\
&= \begin{bmatrix} \gamma_u \gamma_v \left(1 + \dfrac{\mathbf{v}^T \mathbf{u}}{c^2}\right) & -\gamma_u \gamma_v \dfrac{\mathbf{u}^T}{c} - \gamma_v \dfrac{\mathbf{v}^T}{c} - \gamma_v (\gamma_u - 1)\dfrac{\mathbf{v}^T \mathbf{u}\mathbf{u}^T}{c u^2} \\ -\gamma_u \gamma_v \dfrac{\mathbf{v}}{c} - \gamma_u \dfrac{\mathbf{u}}{c} - \gamma_u (\gamma_v - 1)\dfrac{\mathbf{v}\mathbf{v}^T \mathbf{u}}{c v^2} & \gamma_u \gamma_v \dfrac{\mathbf{v}\mathbf{u}^T}{c^2} + \left[I + (\gamma_v - 1)\dfrac{\mathbf{v}\mathbf{v}^T}{v^2}\right]\left[I + (\gamma_u - 1)\dfrac{\mathbf{u}\mathbf{u}^T}{u^2}\right] \end{bmatrix}
\end{aligned} \tag{7}$$

which can be re-expressed using (3) and (5) as

$$\Lambda = B(\mathbf{v})B(\mathbf{u}) = \begin{bmatrix} \gamma & -\gamma\frac{(\mathbf{u}\oplus\mathbf{v})^T}{c} \\ -\gamma\frac{(\mathbf{v}\oplus\mathbf{u})}{c} & M \end{bmatrix} = \begin{bmatrix} \gamma & -\mathbf{a}^T \\ -\mathbf{b} & M \end{bmatrix} \tag{8}$$

where

$$\mathbf{a} = \frac{\gamma}{c}(\mathbf{u}\oplus\mathbf{v}) \qquad \mathbf{b} = \frac{\gamma}{c}(\mathbf{v}\oplus\mathbf{u}) \qquad M = \gamma_u\gamma_v\frac{\mathbf{v}\mathbf{u}^T}{c^2} + \left[I + (\gamma_v - 1)\frac{\mathbf{v}\mathbf{v}^T}{v^2}\right]\left[I + (\gamma_u - 1)\frac{\mathbf{u}\mathbf{u}^T}{u^2}\right] \tag{9}$$

Clearly

$$\Lambda^{-1} = [B(\mathbf{v})B(\mathbf{u})]^{-1} = B(-\mathbf{u})B(-\mathbf{v}) = \begin{bmatrix} \gamma & \mathbf{b}^T \\ \mathbf{a} & M^T \end{bmatrix} \tag{10}$$

We can obtain some useful relations by requiring $\Lambda\Lambda^{-1} = I$:

$$I = \Lambda\Lambda^{-1} = \begin{bmatrix} \gamma & -\mathbf{a}^T \\ -\mathbf{b} & M \end{bmatrix}\begin{bmatrix} \gamma & \mathbf{b}^T \\ \mathbf{a} & M^T \end{bmatrix} = \begin{bmatrix} \gamma^2 - \mathbf{a}^T\mathbf{a} & \gamma\mathbf{b}^T - \mathbf{a}^T M^T \\ -\gamma\mathbf{b} + M\mathbf{a} & -\mathbf{b}\mathbf{b}^T + MM^T \end{bmatrix} \tag{11}$$

or,

$$\gamma^2 - \mathbf{a}^T\mathbf{a} = 1 \qquad M\mathbf{a} = \gamma\mathbf{b} \qquad MM^T = I + \mathbf{b}\mathbf{b}^T \tag{12}$$

Similarly, from

$$I = \Lambda^{-1}\Lambda = \begin{bmatrix} \gamma & \mathbf{b}^T \\ \mathbf{a} & M^T \end{bmatrix}\begin{bmatrix} \gamma & -\mathbf{a}^T \\ -\mathbf{b} & M \end{bmatrix} = \begin{bmatrix} \gamma^2 - \mathbf{b}^T\mathbf{b} & -\gamma\mathbf{a}^T + \mathbf{b}^T M \\ \gamma\mathbf{a} - M^T\mathbf{b} & -\mathbf{a}\mathbf{a}^T + M^T M \end{bmatrix} \tag{13}$$

we must have

$$\gamma^2 - \mathbf{b}^T\mathbf{b} = 1 \qquad \mathbf{b}^T M = \gamma\mathbf{a}^T \qquad M^T M = I + \mathbf{a}\mathbf{a}^T \tag{14}$$

3. **Transformation of 4-velocity between frames, hint of rotation**

In section 11.4 of Jackson, it has been established that $\begin{bmatrix} \gamma_v c \\ \gamma_v \mathbf{v} \end{bmatrix}$ is a 4-vector, i.e., its transformation between inertial frames is given by the Lorentz transformation.

Seen from S, S'' moves with 4-velocity $\begin{bmatrix} \gamma c \\ \gamma(\mathbf{u}\oplus\mathbf{v}) \end{bmatrix}$. Transformation of this 4-velocity into the S' frame will require

$$B(\mathbf{u})\begin{bmatrix} \gamma c \\ \gamma(\mathbf{u}\oplus\mathbf{v}) \end{bmatrix} = \begin{bmatrix} \gamma_v c \\ \gamma_v \mathbf{v} \end{bmatrix} \tag{15}$$

where the LHS describes the result of applying the Lorentz transformation between S and S' to the 4-velocity, and the RHS is by definition this same 4-velocity as seen from S'. Similarly applying the $B(\mathbf{v})$ to (15) will result in

$$B(\mathbf{v})B(\mathbf{u})\begin{bmatrix} \gamma c \\ \gamma(\mathbf{u}\oplus\mathbf{v}) \end{bmatrix} = B(\mathbf{v})\begin{bmatrix} \gamma_v c \\ \gamma_v \mathbf{v} \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix} \tag{16}$$

This is nothing extraordinary since it is a statement that S'' is at rest relative to S''. But if we continue to left-apply $B^{-1}(\mathbf{v}\oplus\mathbf{u})$ to the above, we will have

$$B^{-1}(\mathbf{v}\oplus\mathbf{u})B(\mathbf{v})B(\mathbf{u})\begin{bmatrix} \gamma c \\ \gamma(\mathbf{u}\oplus\mathbf{v}) \end{bmatrix} = B^{-1}(\mathbf{v}\oplus\mathbf{u})\begin{bmatrix} c \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma c \\ \gamma(\mathbf{v}\oplus\mathbf{u}) \end{bmatrix} \tag{17}$$

where the last step is interpreted as the transformation of the zero 4-velocity in S'' to a frame (called S_R), moving with relative velocity $-(\mathbf{v}\oplus\mathbf{u}) = (-\mathbf{v})\oplus(-\mathbf{u})$ to S''. Now *this* is an extraordinary result because $(-\mathbf{v})\oplus(-\mathbf{u})$ happens to be the velocity of S relative to S''. What it means is if we have a 4-velocity $\begin{bmatrix} \gamma c \\ \gamma(\mathbf{u}\oplus\mathbf{v}) \end{bmatrix}$ in S, by transforming it successively via a boost $B(\mathbf{u})$ into S', then a boost $B(\mathbf{v})$ into S'', then finally by a boost $B[(-\mathbf{v})\oplus(-\mathbf{u})]$ into S_R, we end up with a *rotated* 4-velocity $\begin{bmatrix} \gamma c \\ \gamma(\mathbf{v}\oplus\mathbf{u}) \end{bmatrix}$ in S_R. Apparently S_R is not the same as S, despite they both having the same relative velocity to S''. This is our hint of a rotation.

Conversely, similar arguments will lead to the relation

$$B(\mathbf{v})B(\mathbf{u})B^{-1}(\mathbf{u}\oplus\mathbf{v})\begin{bmatrix}\gamma c\\-\gamma(\mathbf{u}\oplus\mathbf{v})\end{bmatrix}=B(\mathbf{v})B(\mathbf{u})\begin{bmatrix}c\\0\end{bmatrix}=B(\mathbf{v})\begin{bmatrix}\gamma_u c\\-\gamma_u\mathbf{u}\end{bmatrix}=\begin{bmatrix}\gamma c\\-\gamma(\mathbf{v}\oplus\mathbf{u})\end{bmatrix}\tag{18}$$

A comparison between (17) and (18) should justify our

speculation that $$B^{-1}(\mathbf{v}\oplus\mathbf{u})B(\mathbf{v})B(\mathbf{u})=R(\theta)=B(\mathbf{v})B(\mathbf{u})B^{-1}(\mathbf{u}\oplus\mathbf{v})\tag{19}$$

where $R(\theta)$ is a rotation transformation (by angle θ) that brings $\mathbf{u}\oplus\mathbf{v}$ into $\mathbf{v}\oplus\mathbf{u}$.

If this were true, we would have

$$B(\mathbf{v})B(\mathbf{u})=R(\theta)B(\mathbf{u}\oplus\mathbf{v})=B(\mathbf{v}\oplus\mathbf{u})R(\theta)\tag{20}$$

i.e., two successive boosts are equivalent to a boost of a composite velocity followed (or preceded) by a rotation. This is the essence of the Thomas (Wigner) rotation.

4. **Rigorous proof of (19), the Thomas rotation**

To see the first half of (19), let's examine the matrix representation of $B^{-1}(\mathbf{v}\oplus\mathbf{u})B(\mathbf{v})B(\mathbf{u})$. First notice by replacing $\mathbf{u}$ with $-(\mathbf{v}\oplus\mathbf{u})$ in (6), we have

$$B^{-1}(\mathbf{v}\oplus\mathbf{u})=\begin{bmatrix}\gamma & \gamma\dfrac{(\mathbf{v}\oplus\mathbf{u})^T}{c}\\ \gamma\dfrac{(\mathbf{v}\oplus\mathbf{u})}{c} & I+(\gamma-1)\dfrac{(\mathbf{v}\oplus\mathbf{u})(\mathbf{v}\oplus\mathbf{u})^T}{|\mathbf{v}\oplus\mathbf{u}|^2}\end{bmatrix}=\begin{bmatrix}\gamma & \mathbf{b}^T\\ \mathbf{b} & I+(\gamma-1)\dfrac{\mathbf{b}\mathbf{b}^T}{\mathbf{b}^T\mathbf{b}}\end{bmatrix}\tag{21}$$

Multiplying (8) yields

$$\begin{aligned}B^{-1}(\mathbf{v}\oplus\mathbf{u})B(\mathbf{v})B(\mathbf{u})&=\begin{bmatrix}\gamma & \mathbf{b}^T\\ \mathbf{b} & I+(\gamma-1)\dfrac{\mathbf{b}\mathbf{b}^T}{\mathbf{b}^T\mathbf{b}}\end{bmatrix}\begin{bmatrix}\gamma & -\mathbf{a}^T\\ -\mathbf{b} & M\end{bmatrix}\\
&=\begin{bmatrix}\gamma^2-\mathbf{b}^T\mathbf{b} & -\gamma\mathbf{a}^T+\mathbf{b}^TM\\ \gamma\mathbf{b}-\mathbf{b}-(\gamma-1)\dfrac{\mathbf{b}\mathbf{b}^T\mathbf{b}}{\mathbf{b}^T\mathbf{b}} & -\mathbf{b}\mathbf{a}^T+M+(\gamma-1)\dfrac{\mathbf{b}\mathbf{b}^TM}{\mathbf{b}^T\mathbf{b}}\end{bmatrix} \qquad \text{use (14)}\\
&=\begin{bmatrix}1 & 0\\ 0 & M-\dfrac{\mathbf{b}\mathbf{a}^T}{\gamma+1}\end{bmatrix}\end{aligned}\tag{22}$$

Similarly, for the second half of (19)

$$\begin{aligned}B(\mathbf{v})B(\mathbf{u})B^{-1}(\mathbf{u}\oplus\mathbf{v})&=\begin{bmatrix}\gamma & -\mathbf{a}^T\\ -\mathbf{b} & M\end{bmatrix}\begin{bmatrix}\gamma & \mathbf{a}^T\\ \mathbf{a} & I+(\gamma-1)\dfrac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}\end{bmatrix}\\
&=\begin{bmatrix}\gamma^2-\mathbf{a}^T\mathbf{a} & \gamma\mathbf{a}^T-\mathbf{a}^T-(\gamma-1)\dfrac{\mathbf{a}^T\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}\\ -\gamma\mathbf{b}+M\mathbf{a} & -\mathbf{b}\mathbf{a}^T+M+(\gamma-1)\dfrac{M\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}\end{bmatrix} \qquad \text{use (12)}\\
&=\begin{bmatrix}1 & 0\\ 0 & M-\dfrac{\mathbf{b}\mathbf{a}^T}{\gamma+1}\end{bmatrix}\end{aligned}\tag{23}$$

Both (22) and (23) are Lorentz transformation without boost (or relative motion), it is essentially a 3×3 matrix. If we define the 3×3 matrix

$$R\equiv M-\frac{\mathbf{b}\mathbf{a}^T}{\gamma+1}\tag{24}$$

we can see that

$$RR^T=\left(M-\frac{\mathbf{b}\mathbf{a}^T}{\gamma+1}\right)\left(M^T-\frac{\mathbf{a}\mathbf{b}^T}{\gamma+1}\right)=MM^T-\frac{\mathbf{b}\mathbf{a}^TM^T}{\gamma+1}-\frac{M\mathbf{a}\mathbf{b}^T}{\gamma+1}+\frac{\mathbf{b}\mathbf{a}^T\mathbf{a}\mathbf{b}^T}{(\gamma+1)^2}=I\tag{25}$$

Taking the determinant of (23) and using the fact that boost matrices have determinant +1, we see $\det R = +1$. This means that $R \in SO(3)$ thus it is indeed a rotation matrix.

Also expectedly,

$$R\mathbf{a} = M\mathbf{a} - \frac{\mathbf{b}\mathbf{a}^T\mathbf{a}}{\gamma+1} = \gamma\mathbf{b} - (\gamma-1)\,\mathbf{b} = \mathbf{b} \tag{26}$$

that is, it rotates $\mathbf{u} \oplus \mathbf{v}$ into $\mathbf{v} \oplus \mathbf{u}$.

5. **Parameters of Thomas rotation**

Since R rotates $\mathbf{u} \oplus \mathbf{v}$ into $\mathbf{v} \oplus \mathbf{u}$, both of which are in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$, it is clear that the axis of rotation is proportional to $\mathbf{u} \times \mathbf{v}$, which vanishes if $\mathbf{u}$ and $\mathbf{v}$ are collinear.

We can find the rotation angle θ by the well known relation

$$\operatorname{tr} R = 1 + 2\cos\theta \tag{27}$$

From (24) and (9), we have

$$\begin{aligned} \operatorname{tr} R &= \operatorname{tr} M - \frac{\operatorname{tr}\left(\mathbf{b}\mathbf{a}^T\right)}{\gamma+1} \\ &= \gamma_u\gamma_v \frac{\operatorname{tr}\left(\mathbf{v}\mathbf{u}^T\right)}{c^2} + \operatorname{tr} I + (\gamma_u - 1)\,\frac{\operatorname{tr}\left(\mathbf{u}\mathbf{u}^T\right)}{u^2} + (\gamma_v - 1)\,\frac{\operatorname{tr}\left(\mathbf{v}\mathbf{v}^T\right)}{v^2} + (\gamma_u - 1)\,(\gamma_v - 1)\,\frac{\operatorname{tr}\left(\mathbf{v}\mathbf{v}^{\mathbf{T}}\mathbf{u}\mathbf{u}^{\mathbf{T}}\right)}{u^2v^2} - \frac{\operatorname{tr}\left(\mathbf{b}\mathbf{a}^T\right)}{\gamma+1} \end{aligned} \tag{28}$$

Note that for any column vectors $\mathbf{m}$ and $\mathbf{n}$, $\operatorname{tr}\left(\mathbf{m}\mathbf{n}^T\right) = \mathbf{m}\cdot\mathbf{n}$, as well as $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|^2\cos\theta = \left(\gamma^2 - 1\right)\cos\theta$, (28) becomes

$$1 + 2\cos\theta = \gamma_u\gamma_v\frac{\mathbf{u}\cdot\mathbf{v}}{c^2} + 3 + (\gamma_u - 1) + (\gamma_v - 1) + (\gamma_u - 1)\,(\gamma_v - 1)\,\frac{(\mathbf{u}\cdot\mathbf{v})^2}{u^2v^2} - (\gamma - 1)\cos\theta \tag{29}$$

Writing u^2, v^2 in terms of γ_u, γ_v, and replacing $\gamma_u\gamma_v\,(\mathbf{u}\cdot\mathbf{v})\,/c^2$ with $\gamma - \gamma_u\gamma_v$, we can eventually write $\cos\theta$ in a neater form

$$\cos\theta = \frac{(\gamma + \gamma_u + \gamma_v + 1)^2}{(\gamma+1)\,(\gamma_u + 1)\,(\gamma_v + 1)} - 1 \tag{30}$$