1. First let's understand the density function $\rho(\mathbf{x})$. Recall function $\delta(x)$ represents an infinite impulse at $x=0$, so $\delta'(x)$ represents two infinite pulses at $x=0^-$ and $x=0^+$, with the former positive and latter negative. Therefore, the $\delta'(z)$ in $\rho(\mathbf{x})$ accounted for the dipole along the $-\hat{\mathbf{z}}$ direction, with the remaining factors $\delta(x)\delta(y)\delta(t)$ indicating its point nature in the other space and time dimensions. The form of $J_z(\mathbf{x})$ is the consequence of the charge conservation $\nabla\cdot\mathbf{J}+\partial\rho/\partial t=0$.

   By (6.23), the instantaneous Coulomb potential is

$$\begin{aligned}\Phi(\mathbf{x},t) &= \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ &= \frac{1}{4\pi\epsilon_0}\int\frac{\delta(x')\delta(y')\delta'(z')\delta(t)}{|\mathbf{x}-\mathbf{x}'|}dx'dy'dz' \\ &= \frac{\delta(t)}{4\pi\epsilon_0}\int\frac{\delta'(z')dz'}{\sqrt{x^2+y^2+(z-z')^2}} && \text{recall }\int\delta'(t-a)f(t)dt=-f'(a) \\ &= -\frac{\delta(t)}{4\pi\epsilon_0}\frac{z}{r^3} && \text{where } r=|\mathbf{x}| \qquad (1)\end{aligned}$$

2. By (6.28), the transverse current density is

$$\begin{aligned}\mathbf{J}_t(\mathbf{x},t) &= \frac{1}{4\pi}\nabla\times\left[\nabla\times\int\frac{\mathbf{J}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d^3x'\right] \\ &= -\frac{1}{4\pi}\nabla\times\left[\nabla\times\int\frac{\hat{\mathbf{z}}\delta(x')\delta(y')\delta(z')\delta'(t)}{|\mathbf{x}-\mathbf{x}'|}dx'dy'dz'\right] \\ &= -\frac{\delta'(t)}{4\pi}\nabla\times\left[\nabla\times\left(\frac{\hat{\mathbf{z}}}{r}\right)\right] \\ &= -\frac{\delta'(t)}{4\pi}\left\{\nabla\left[\nabla\cdot\left(\frac{\hat{\mathbf{z}}}{r}\right)\right]-\hat{\mathbf{z}}\nabla^2\left(\frac{1}{r}\right)\right\} \\ &= -\frac{\delta'(t)}{4\pi}\left[\nabla\left(-\frac{z}{r^3}\right)+4\pi\delta(\mathbf{x})\hat{\mathbf{z}}\right] \\ &= -\delta'(t)\left[-\frac{1}{4\pi}\nabla\left(\frac{z}{r^3}\right)+\delta(\mathbf{x})\hat{\mathbf{z}}\right] \qquad (2)\end{aligned}$$

   We could follow the reasoning leading to (4.20) to treat $\nabla\left(z/r^3\right)$, but let's do it more explicitly here. First observe that

$$\nabla\left(\frac{z}{r^3}\right)=z\left(\frac{-3\mathbf{x}}{r^5}\right)+\frac{\hat{\mathbf{z}}}{r^3} \qquad \text{for } \mathbf{x}\neq 0 \qquad (3)$$

   then the function

$$g(\mathbf{x})\equiv\nabla\left(\frac{z}{r^3}\right)-\left(-\frac{3z\mathbf{x}}{r^5}+\frac{\hat{\mathbf{z}}}{r^3}\right) \qquad (4)$$

   is zero whenever $\mathbf{x}\neq 0$. But consider its volume integral for the ball $r<R$:

$$\int_{r<R}g(\mathbf{x})d^3x=\underbrace{\int_{r<R}\nabla\left(\frac{z}{r^3}\right)d^3x}_{I_1}-\underbrace{\int_{r<R}\left(-\frac{3z\mathbf{x}}{r^5}+\frac{\hat{\mathbf{z}}}{r^3}\right)d^3x}_{I_2} \qquad (5)$$

   Using the vector identity

$$\int_V\nabla\psi d^3x=\oint_S\psi\mathbf{n}da \qquad (6)$$

   we obtain

$$\begin{aligned}I_1 &= R^2\int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\phi\frac{z}{R^3}(\cos\theta\hat{\mathbf{z}}+\sin\theta\cos\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\hat{\mathbf{y}}) \\ &= \hat{\mathbf{z}}\cdot 2\pi\int_0^\pi\cos^2\theta\sin\theta d\theta=\frac{4\pi}{3}\hat{\mathbf{z}} \qquad (7)\end{aligned}$$

On the other hand

$$
\begin{aligned}
I_2 &= \int_0^R r^2 dr \int d\Omega \left[-\frac{3zr\left(\cos\theta\hat{\mathbf{z}}+\sin\theta\cos\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\hat{\mathbf{y}}\right)}{r^5}+\frac{\hat{\mathbf{z}}}{r^3}\right] \\
&= 2\pi\cdot\int_0^R r^2 dr\cdot\frac{\hat{\mathbf{z}}}{r^3}\underbrace{\int_0^\pi \sin\theta d\theta\left(1-3\cos^2\theta\right)}_{=0}=0
\end{aligned} \tag{8}
$$

This means

$$
\int_{r<R} g(\mathbf{x})\, d^3x = \frac{4\pi}{3}\hat{\mathbf{z}} \tag{9}
$$

and hence by definition of the δ-function,

$$
g(\mathbf{x}) = \frac{4\pi}{3}\hat{\mathbf{z}}\delta(\mathbf{x}) \qquad \Longrightarrow \qquad \boldsymbol{\nabla}\left(\frac{z}{r^3}\right) = \frac{4\pi}{3}\hat{\mathbf{z}}\delta(\mathbf{x}) - \frac{3z\mathbf{x}}{r^5}+\frac{\hat{\mathbf{z}}}{r^3} \tag{10}
$$

Inserting (10) into (2), we obtain

$$
\mathbf{J}_t(\mathbf{x},t) = -\delta'(t)\left[\frac{2}{3}\hat{\mathbf{z}}\delta(\mathbf{x}) - \frac{\hat{\mathbf{z}}}{4\pi r^3}+\frac{3}{4\pi r^3}\mathbf{n}(\hat{\mathbf{z}}\cdot\mathbf{n})\right] \tag{11}
$$

3. Recall for the general wave function (6.32),

$$
\boldsymbol{\nabla}^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi f(\mathbf{x},t) \tag{12}
$$

whose retarded solution is given by (6.47)

$$
\Psi(\mathbf{x},t) = \int \frac{[f(\mathbf{x}',t')]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{13}
$$

where

$$
\left[f\left(\mathbf{x}',t'\right)\right]_{\text{ret}} = f\left(\mathbf{x}', t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right) \tag{14}
$$

In the Coulomb gauge, the vector potential $\mathbf{A}$ satisfies the wave equation

$$
\boldsymbol{\nabla}^2\mathbf{A} - \frac{1}{c^2}\frac{\partial\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}_t \tag{15}
$$

so it is clear that the vector potential can be written

$$
\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int \frac{[\mathbf{J}_t(\mathbf{x}',t')]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{16}
$$

By following the hint, we write the transverse current density as

$$
\mathbf{J}_t(\mathbf{x},t) = -\delta'(t)\left[\hat{\mathbf{z}}\delta(\mathbf{x}) + \frac{1}{4\pi}\boldsymbol{\nabla}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right] \tag{17}
$$

Plugging (17) into (16) and invoking the definition of retarded function, we have

$$
\begin{aligned}
\mathbf{A}(\mathbf{x},t) &= -\frac{\mu_0}{4\pi}\int \delta'\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\left[\frac{\hat{\mathbf{z}}\delta\left(\mathbf{x}'\right)+\frac{1}{4\pi}\boldsymbol{\nabla}'\frac{\partial}{\partial z'}\left(\frac{1}{r'}\right)}{|\mathbf{x}-\mathbf{x}'|}\right] d^3x' \\
&= -\frac{\mu_0}{4\pi}\left[\delta'\left(t-\frac{r}{c}\right)\frac{\hat{\mathbf{z}}}{r}+\frac{1}{4\pi}\underbrace{\int \delta'\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\frac{\boldsymbol{\nabla}'\frac{\partial}{\partial z'}\left(\frac{1}{r'}\right)}{|\mathbf{x}-\mathbf{x}'|}d^3x'}_{I}\right]
\end{aligned} \tag{18}
$$

Denoting

$$g\left(\mathbf{x}'\right)=\frac{\delta'\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x}-\mathbf{x}'|} \tag{19}$$

we perform integration by parts twice to obtain I:

$$\begin{aligned}
I &= \int g\left(\mathbf{x}'\right)\nabla'\frac{\partial}{\partial z'}\left(\frac{1}{r'}\right)d^3x' \\
&= \underbrace{\int \nabla'\left[g\left(\mathbf{x}'\right)\frac{\partial}{\partial z'}\left(\frac{1}{r'}\right)\right]d^3x'}_{=0\text{ by (6) at }\infty}-\int \frac{\partial}{\partial z'}\left(\frac{1}{r'}\right)\nabla'\left[g\left(\mathbf{x}'\right)\right]d^3x' \\
&= \underbrace{-\left\{\iint \frac{1}{r'}\nabla'\left[g\left(\mathbf{x}'\right)\right]dxdy\right\}\Big|_{z=-\infty}^{z=\infty}}_{=0\text{ due to boundary condition at }\infty}+\int \frac{1}{r'}\frac{\partial}{\partial z'}\nabla'\left[g\left(\mathbf{x}'\right)\right]d^3x' \qquad \text{apply } \nabla'\leftrightarrow-\nabla, \frac{\partial}{\partial z'}\leftrightarrow-\frac{\partial}{\partial z} \text{ by (19)} \\
&= \nabla\frac{\partial}{\partial z}\int \frac{1}{r'}\frac{\delta'\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\
&= \frac{\partial}{\partial t}\nabla\frac{\partial}{\partial z}\underbrace{\int \frac{1}{r'}\frac{\delta\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x}-\mathbf{x}'|}d^3x'}_{I'}
\end{aligned} \tag{20}$$

Recall the delta function property

$$\int \delta\left[f\left(\mathbf{x}'\right)\right]g\left(\mathbf{x}'\right)d^3x'=\int_{f^{-1}(0)}\frac{g\left(\mathbf{x}'\right)}{|\nabla' f|}d\sigma\left(\mathbf{x}'\right) \tag{21}$$

where $d\sigma(\mathbf{x}')$ is the differential area on the surface $f^{-1}(0)=\{\mathbf{x}'|f(\mathbf{x}')=0\}$.
For our case (20),

$$f\left(\mathbf{x}'\right)=t-\frac{|\mathbf{x}-\mathbf{x}'|}{c} \tag{22}$$

When $t>0$, the surface $f^{-1}(0)$ consists of all points $\mathbf{x}'$ at distance ct from $\mathbf{x}$. For any $\mathbf{x}'_0\in f^{-1}(0)$,

$$\left|\left(\nabla' f\right)\left(\mathbf{x}'_0\right)\right|=\frac{1}{c} \tag{23}$$

This turns the integral in (20) into

$$\begin{aligned}
I' = \int \frac{1}{r'}\frac{\delta\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x}-\mathbf{x}'|}d^3x' &= c\int_{f^{-1}(0)}\frac{1}{r'}\frac{1}{|\mathbf{x}-\mathbf{x}'|}d\sigma\left(\mathbf{x}'\right) \\
&= c\cdot\frac{1}{ct}\int_{f^{-1}(0)}\frac{1}{r'}d\sigma\left(\mathbf{x}'\right)
\end{aligned} \tag{24}$$

Now we make $\mathbf{x}'=\mathbf{x}-\mathbf{R}$, where $\mathbf{R}$ ranges over the surface points on a sphere centered at $\mathbf{x}$ with radius ct, which gives

$$\begin{aligned}
I' &= c\cdot\frac{1}{ct}(ct)^2\int d\Omega'\frac{1}{|\mathbf{x}-\mathbf{R}|} \\
&= c^2t\int_0^\pi \sin\theta' d\theta'\int_0^{2\pi}d\phi'\cdot 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*\left(\theta',\phi'\right)Y_{lm}(\theta,\phi) \\
&= c^2t\cdot 4\pi\sum_{l=0}^{\infty}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\sum_{m=-l}^{l}Y_{lm}(\theta,\phi)\int_0^\pi \sin\theta' d\theta'\int_0^{2\pi}d\phi'\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m\left(\cos\theta'\right)e^{-im\phi'} \\
&= c^2t\cdot 4\pi\sum_{l=0}^{\infty}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{l0}(\theta,\phi)\cdot 2\pi\int_0^\pi \sin\theta' d\theta'\sqrt{\frac{2l+1}{4\pi}}P_l\left(\cos\theta'\right) \\
&= c^2t\cdot 4\pi\frac{1}{r_>}
\end{aligned} \tag{25}$$

where $r_>, r_<$ are the greater and smaller between $r = |\mathbf{x}|$ and ct. Note I' can be expressed in terms of the Heaviside step function Θ,

$$I' = 4\pi c^2 t \frac{1}{r_>} = 4\pi c^2 t \left[\frac{\Theta(r-ct)}{r} + \frac{\Theta(ct-r)}{ct} \right] = 4\pi c \left[\frac{ct}{r}\Theta(r-ct) + \Theta(ct-r) \right] \tag{26}$$

(26) is almost the right result for I', except for the case where $f^{-1}(0)$ is empty, in which case $I' = 0$. But the only situation where $f^{-1}(0)$ is empty is for $t < 0$, so the full form of I' is obtained by multiplying (26) with $\Theta(t)$, i.e.,

$$I' = 4\pi c \left[\frac{ct}{r}\Theta(r-ct) + \Theta(ct-r) \right] \Theta(t) \tag{27}$$

Plugging this back to (20) yields

$$\begin{aligned} I &= 4\pi c \nabla \frac{\partial}{\partial z}\frac{\partial}{\partial t}\left[\frac{ct}{r}\Theta(r-ct)\Theta(t) + \Theta(ct-r)\Theta(t) \right] \\ &= 4\pi c \nabla \frac{\partial}{\partial z}\left[\frac{c}{r}\Theta(r-ct)\Theta(t) + \frac{ct}{r}(-c)\delta(r-ct)\Theta(t) + \frac{ct}{r}\Theta(r-ct)\delta(t) + \right. \\ &\qquad \left. c\delta(ct-r)\Theta(t) + \Theta(ct-r)\delta(t) \right] \\ &= 4\pi c^2 \nabla \frac{\partial}{\partial z}\left[\frac{\Theta(r-ct)\Theta(t)}{r} \right] \end{aligned} \tag{28}$$

where we see in the bracket, the second term and the fourth term cancel each other, while the third and fifth term are both zero.

Going back to (18), the vector potential is

$$\mathbf{A}(\mathbf{x}, t) = -\frac{\mu_0}{4\pi}\delta'\left(t - \frac{r}{c}\right)\frac{\hat{\mathbf{z}}}{r} - \frac{\mu_0}{4\pi}c^2 \nabla \frac{\partial}{\partial z}\left[\frac{\Theta(r-ct)\Theta(t)}{r} \right] \tag{29}$$

and

$$\begin{aligned} -\frac{\partial \mathbf{A}}{\partial t} &= \frac{\mu_0}{4\pi}\delta''\left(t - \frac{r}{c}\right)\frac{\hat{\mathbf{z}}}{r} + \frac{1}{4\pi\epsilon_0}\nabla \frac{\partial}{\partial z}\left[\frac{-c\delta(r-ct)\Theta(t) + \Theta(r-ct)\delta(t)}{r} \right] \\ &= \frac{\mu_0}{4\pi}\delta''\left(t - \frac{r}{c}\right)\frac{\hat{\mathbf{z}}}{r} - \frac{c}{4\pi\epsilon_0}\nabla \frac{\partial}{\partial z}\left[\frac{\delta(r-ct)}{r} \right] + \frac{1}{4\pi\epsilon_0}\nabla \frac{\partial}{\partial z}\left[\frac{\delta(t)}{r} \right] \end{aligned} \tag{30}$$

From (1), we see that

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left[\frac{\delta(t)}{r} \right] \tag{31}$$

Thus when we calculate the electric field

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi \tag{32}$$

the two instantaneous terms involving $\delta(t)$ cancel each other and finally gives

$$\begin{aligned} \mathbf{E} &= \frac{\mu_0}{4\pi}\delta''\left(t - \frac{r}{c}\right)\frac{\hat{\mathbf{z}}}{r} - \frac{c}{4\pi\epsilon_0}\nabla \frac{\partial}{\partial z}\left[\frac{\delta(r-ct)}{r} \right] \\ &= \frac{c}{4\pi\epsilon_0}\delta''(ct-r)\frac{\hat{\mathbf{z}}}{r} - \frac{c}{4\pi\epsilon_0}\nabla \frac{\partial}{\partial z}\left[\frac{\delta(ct-r)}{r} \right] \end{aligned} \tag{33}$$

Expanding the spatial gradient term, we get

$$\begin{aligned} \nabla \frac{\partial}{\partial z}\left[\frac{\delta(ct-r)}{r} \right] &= \nabla \left[-\frac{\delta'(ct-r)z}{r^2} - \frac{\delta(ct-r)z}{r^3} \right] \\ &= \nabla \left[-r\delta'(ct-r) \right]\frac{z}{r^3} - r\delta'(ct-r)\nabla\left(\frac{z}{r^3}\right) - \nabla\left[\delta(ct-r)\right]\frac{z}{r^3} - \delta(ct-r)\nabla\left(\frac{z}{r^3}\right) \\ &= \frac{z}{r^3}\left[-\mathbf{n}\delta'(ct-r) + r\delta''(ct-r)\mathbf{n} + \delta'(ct-r)\mathbf{n} \right] - \left[r\delta'(ct-r) + \delta(ct-r) \right]\nabla\left(\frac{z}{r^3}\right) \\ &= \delta''(ct-r)\frac{z\mathbf{n}}{r^2} - \left[r\delta'(ct-r) + \delta(ct-r) \right]\nabla\left(\frac{z}{r^3}\right) \end{aligned} \tag{34}$$

Using (10) and ignoring the $\delta(\mathbf{x})$ term (since we are considering the field off from $\mathbf{x}=0$), we have

$$\nabla\frac{\partial}{\partial z}\left[\frac{\delta(ct-r)}{r}\right]=\delta''(ct-r)\frac{z\mathbf{n}}{r^2}-\left[r\delta'(ct-r)+\delta(ct-r)\right]\left(\frac{\hat{\mathbf{z}}}{r^3}-\frac{3z\mathbf{n}}{r^4}\right)\tag{35}$$

Combining with (33), we see for the $\mathbf{n}=\hat{\mathbf{x}},\mathbf{n}=\hat{\mathbf{y}}$ direction,

$$\mathbf{E}_{\mathbf{n}=\hat{\mathbf{x}}|\hat{\mathbf{y}}}=\frac{c}{4\pi\epsilon_0 r}\left[-\delta''(ct-r)-\frac{3\delta'(ct-r)}{r}-\frac{3\delta(ct-r)}{r^2}\right]\cos\theta\,\mathbf{n}\tag{36}$$

and lastly, for the $\mathbf{n}=\hat{\mathbf{z}}$ direction,

$$\mathbf{E}_{\mathbf{n}=\hat{\mathbf{z}}}=\frac{c}{4\pi\epsilon_0 r}\left\{\sin^2\theta\,\delta''(ct-r)+\left(1-3\cos^3\theta\right)\left[\frac{\delta'(ct-r)}{r}+\frac{\delta(ct-r)}{r^2}\right]\right\}\hat{\mathbf{z}}\tag{37}$$

Note (36) and (37) agree with the claim in Jackson since δ,δ'' are even in its arguments, while δ' is odd.