1. **Prob 9.22**

(a) In any source-free region, the electric and magnetic field are governed by equation (9.107). Thus the discussion in section 9.7 applies. In particular, the TE modes correspond to the magnetic source case, where the solution is given by (9.116),

$$\mathbf{E}_{lm}^{\mathrm{TE}} = Z_0 g_l(kr)\mathbf{L}Y_{lm}(\theta,\phi) \qquad \mathbf{H}_{lm}^{\mathrm{TE}} = -\frac{i}{kZ_0}\nabla\times\mathbf{E}_{lm}^{\mathrm{TE}} \tag{1}$$

and the TM modes correspond to the electric source case, as given by (9.118),

$$\mathbf{H}_{lm}^{\mathrm{TM}} = f_l(kr)\mathbf{L}Y_{lm}(\theta,\phi) \qquad \mathbf{E}_{lm}^{\mathrm{TM}} = \frac{iZ_0}{k}\nabla\times\mathbf{H}_{lm}^{\mathrm{TM}} \tag{2}$$

where $g_l(kr), f_l(kr)$ have the form

$$(f|g)_l(kr) = A_l^{(1)}h_l^{(1)}(kr) + A_l^{(2)}h_l^{(2)}(kr) \tag{3}$$

Since $h_l^{(1)}(kr), h_l^{(2)}(kr)$ are linear combinations of $j_l(kr), n_l(kr)$, and since for spherical cavity the point $r=0$ is allowed, we know both $f_l(kr)$ and $g_l(kr)$ must be proportional to $j_l(kr)$ because $n_l(kr)$ is divergent at $r=0$. Thus we can rewrite (1) and (2) as

$$\mathbf{E}_{lm}^{\mathrm{TE}} = Z_0 H_0 j_l(kr)\mathbf{L}Y_{lm}(\theta,\phi) \qquad \mathbf{H}_{lm}^{\mathrm{TE}} = -\frac{iH_0}{k}\nabla\times[j_l(kr)\mathbf{L}Y_{lm}(\theta,\phi)] \tag{4}$$

$$\mathbf{H}_{lm}^{\mathrm{TM}} = H_0 j_l(kr)\mathbf{L}Y_{lm}(\theta,\phi) \qquad \mathbf{E}_{lm}^{\mathrm{TM}} = \frac{iZ_0H_0}{k}\nabla\times[j_l(kr)\mathbf{L}Y_{lm}(\theta,\phi)] \tag{5}$$

Before we proceed, let's review the definition and properties of vector spherical harmonics (see previous notes). The vector spherical harmonics are defined as

$$\mathbf{Y}_{lm}(\mathbf{r}) = Y_{lm}(\theta,\phi)\hat{\mathbf{r}} \tag{6}$$

$$\boldsymbol{\Psi}_{lm}(\mathbf{r}) = r\nabla Y_{lm}(\theta,\phi) \tag{7}$$

$$\boldsymbol{\Phi}_{lm}(\mathbf{r}) = \mathbf{r}\times\nabla Y_{lm}(\theta,\phi) = i\mathbf{L}Y_{lm}(\theta,\phi) \tag{8}$$

The curl has the following property

$$\nabla\times[h(r)\boldsymbol{\Phi}_{lm}] = -\frac{l(l+1)}{r}h\mathbf{Y}_{lm} - \left(\frac{dh}{dr}+\frac{1}{r}h\right)\boldsymbol{\Psi}_{lm} = -\frac{l(l+1)}{r}h\mathbf{Y}_{lm} - \frac{d(rh)}{dr}\nabla Y_{lm} \tag{9}$$

Now we can consider the boundary conditions at $r=a$.

- For TE mode, $\mathbf{E}$ is transverse, but since the wall is perfect conductor, the tangent electric field must vanish at the wall, this requires

$$j_l(ka) = 0 \tag{10}$$

  I.e., k must be such that ka is a zero of $j_l(x)$. Also note that applying (9) to $\mathbf{H}_{lm}^{\mathrm{TE}}$ in (4) shows that its radial component is proportional to $j_l(kr)$, thus (10) also enforces the boundary condition that $\mathbf{H}$ be transverse at the wall.
- for TM mode, $\mathbf{H}$ is already transverse, the only requirement is for $\mathbf{E}$ to be normal at the wall, which by (9) requires

$$\left.\frac{d[rj_l(kr)]}{dr}\right|_{r=a} = 0 \tag{11}$$

  I.e., ka must be a zero of $d[xj_l(x)]/dx$.

(b) To see the lowest TE modes, refer to the zeroes x_{ln} of $j_l(x)$ tabulated in the following table

n	$j_0(x)$	$j_1(x)$	$j_2(x)$	$j_3(x)$	$j_4(x)$
1	3.14159	4.49341	5.76346	6.98793	8.18256
2	6.28319	7.72525	9.09501	10.41712	11.70491
3	9.42478	10.90412	12.32294	13.69802	15.03966
4	12.56637	14.06619	15.51460	16.92362	18.30126
5	15.70796	17.22076	18.68904	20.12181	21.52542

Note that $l = 0$ corresponds to the trivial case of a zero field, so the lowest four TE modes correspond to

$$k_{11}^{\mathrm{TE}} a = x_{11} = 4.49341 \qquad \Longrightarrow \qquad \lambda_{11}^{\mathrm{TE}} = \frac{2\pi}{k_{11}^{\mathrm{TE}}} \approx 1.398a$$

$$k_{21}^{\mathrm{TE}} a = x_{21} = 5.76346 \qquad \Longrightarrow \qquad \lambda_{21}^{\mathrm{TE}} = \frac{2\pi}{k_{21}^{\mathrm{TE}}} \approx 1.090a$$

$$k_{31}^{\mathrm{TE}} a = x_{31} = 6.98793 \qquad \Longrightarrow \qquad \lambda_{31}^{\mathrm{TE}} = \frac{2\pi}{k_{31}^{\mathrm{TE}}} \approx 0.8991a$$

$$k_{12}^{\mathrm{TE}} a = x_{12} = 7.72525 \qquad \Longrightarrow \qquad \lambda_{12}^{\mathrm{TE}} = \frac{2\pi}{k_{12}^{\mathrm{TE}}} \approx 0.8133a$$

Similarly for the lowest TM modes, the zeroes x'_{ln} of $d\left[x j_l(x)\right]/dx$ are tabulated in the following table

n	$d[xj_0(x)]/dx$	$d[xj_1(x)]/dx$	$d[xj_2(x)]/dx$	$d[xj_3(x)]/dx$	$d[xj_4(x)]/dx$
1	1.57080	2.74371	3.87024	4.97342	6.06195
2	4.71239	6.11676	7.44309	8.72175	9.96755
3	7.85398	9.31662	10.71301	12.06359	13.38012
4	10.99557	12.48594	13.92052	15.31356	16.67415
5	14.13717	15.64387	17.10274	18.52421	19.91540

giving the lowest four TM modes ($l > 0$)

$$k_{11}^{\mathrm{TM}} a = x'_{11} = 2.74371 \qquad \Longrightarrow \qquad \lambda_{11}^{\mathrm{TM}} = \frac{2\pi}{k_{11}^{\mathrm{TM}}} \approx 2.290a$$

$$k_{21}^{\mathrm{TM}} a = x'_{21} = 3.87024 \qquad \Longrightarrow \qquad \lambda_{21}^{\mathrm{TM}} = \frac{2\pi}{k_{21}^{\mathrm{TM}}} \approx 1.623a$$

$$k_{31}^{\mathrm{TM}} a = x'_{31} = 4.97342 \qquad \Longrightarrow \qquad \lambda_{31}^{\mathrm{TM}} = \frac{2\pi}{k_{31}^{\mathrm{TM}}} \approx 1.263a$$

$$k_{41}^{\mathrm{TM}} a = x'_{41} = 6.06195 \qquad \Longrightarrow \qquad \lambda_{41}^{\mathrm{TM}} = \frac{2\pi}{k_{41}^{\mathrm{TM}}} \approx 1.036a$$

(c) We only explicitly calculate the TE_{11} mode, other $l = 1$ TE modes or TM modes are similar.

$$\begin{aligned}
Y_{11}(\theta,\phi) &= -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} \qquad \Longrightarrow \\
\mathbf{L}Y_{11}(\theta,\phi) &= \frac{1}{i}\mathbf{r}\times\boldsymbol{\nabla}Y_{11}(\theta,\phi) = i\sqrt{\frac{3}{8\pi}}\hat{\mathbf{r}}\times\left(\cos\theta\hat{\boldsymbol{\theta}} + i\hat{\boldsymbol{\phi}}\right)e^{i\phi} \\
&= \sqrt{\frac{3}{8\pi}}\left(i\cos\theta\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\theta}}\right)e^{i\phi} \qquad (12)
\end{aligned}$$

Then by (4) and (5)

$$\begin{aligned}
\mathbf{E}_{11}^{\mathrm{TE}} &= Z_0 H_0 j_1\left(k_{11}^{\mathrm{TE}} r\right)\sqrt{\frac{3}{8\pi}}\left(i\cos\theta\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\theta}}\right)e^{i\phi} \\
\mathbf{H}_{11}^{\mathrm{TE}} &= -\frac{H_0}{k_{11}^{\mathrm{TE}}}\left\{-\frac{2}{r}j_1\left(k_{11}^{\mathrm{TE}} r\right)Y_{11}(\theta,\phi)\hat{\mathbf{r}} - \frac{d\left[r j_1\left(k_{11}^{\mathrm{TE}} r\right)\right]}{dr}\boldsymbol{\nabla}Y_{11}(\theta,\phi)\right\} \\
&= -\sqrt{\frac{3}{8\pi}}\frac{H_0}{k_{11}^{\mathrm{TE}}}\left\{\frac{2j_1\left(k_{11}^{\mathrm{TE}} r\right)}{r}\sin\theta\hat{\mathbf{r}} + \frac{1}{r}\frac{d\left[r j_1\left(k_{11}^{\mathrm{TE}} r\right)\right]}{dr}\left(\cos\theta\hat{\boldsymbol{\theta}} + i\hat{\boldsymbol{\phi}}\right)\right\}e^{i\phi} \qquad (13)
\end{aligned}$$

2. **Prob 9.23**

(a) TE mode

To calculate the Q factor, it is most convenient to express the fields (4) in terms of VSH (6)–(8):

$$\mathbf{E}_{lm}^{\mathrm{TE}} = -iZ_0H_0 j_l(kr)\mathbf{\Phi}_{lm} \qquad \mathbf{H}_{lm}^{\mathrm{TE}} = \frac{H_0}{k}\left\{\frac{l(l+1)}{r}j_l(kr)\mathbf{Y}_{lm} + \frac{1}{r}\frac{d[rj_l(kr)]}{dr}\mathbf{\Psi}_{lm}\right\} \tag{14}$$

The stored energy in TE mode is

$$U_{lm}^{\mathrm{TE}} = \int\left(\frac{\epsilon_0}{4}\left|\mathbf{E}_{lm}^{\mathrm{TE}}\right|^2 + \frac{\mu_0}{4}\left|\mathbf{H}_{lm}^{\mathrm{TE}}\right|^2\right)d^3x \tag{15}$$

By the orthonormality of the vector spherical harmonics

$$\int \mathbf{Y}_{lm}\cdot\mathbf{Y}_{l'm'}^{*}d\Omega = \delta_{ll'}\delta_{mm'} \qquad \int \mathbf{\Psi}_{lm}\cdot\mathbf{\Psi}_{l'm'}^{*}d\Omega = l(l+1)\delta_{ll'}\delta_{mm'} \qquad \int \mathbf{\Phi}_{lm}\cdot\mathbf{\Phi}_{l'm'}^{*}d\Omega = l(l+1)\delta_{ll'}\delta_{mm'} \tag{16}$$

$$\int \mathbf{Y}_{lm}\cdot\mathbf{\Psi}_{l'm'}^{*}d\Omega = 0 \qquad \int \mathbf{Y}_{lm}\cdot\mathbf{\Phi}_{l'm'}^{*}d\Omega = 0 \qquad \int \mathbf{\Psi}_{lm}\cdot\mathbf{\Phi}_{l'm'}^{*}d\Omega = 0 \tag{17}$$

no cross term in (15) survives the integration over solid angle, reducing the stored energy to a radial integral

$$U_{lm}^{\mathrm{TE}} = \frac{l(l+1)\mu_0H_0^2}{4}\overbrace{\int_0^a\left\{j_l^2 + l(l+1)\frac{j_l^2}{k^2r^2} + \frac{1}{k^2r^2}\left[\frac{d(rj_l)}{dr}\right]^2\right\}r^2dr}^{I} \tag{18}$$

Recall $j_l(kr)$ satisfies the spherical Bessel equation

$$\frac{1}{k^2r^2}\frac{d}{dr}\left(r^2\frac{dj_l}{dr}\right) + \left[1 - \frac{l(l+1)}{k^2r^2}\right]j_l = 0 \tag{19}$$

So the integral I becomes

$$I = \int_0^a 2j_l^2r^2dr + \int_0^a\frac{1}{k^2r^2}\overbrace{\left\{j_l\frac{d}{dr}\left(r^2\frac{dj_l}{dr}\right) + \left[\frac{d(rj_l)}{dr}\right]^2\right\}}^{K}r^2dr \tag{20}$$

Mariculously, we see that K is a total derivative

$$K = j_l\frac{d}{dr}\left(r^2\frac{dj_l}{dr}\right) + \left[\frac{d(rj_l)}{dr}\right]^2 = \frac{d}{dr}\left[rj_l\frac{d(rj_l)}{dr}\right] \tag{21}$$

which renders the second integral in (25) zero becaue $j_l(ka) = 0$ for the TE modes. In a passing note, K's vanishing contribution means the stored energy is evenly distributed between the electric and magnetic fields. In summary, the stored energy in the TE mode is

$$\begin{aligned} U_{lm}^{\mathrm{TE}} &= \frac{l(l+1)\mu_0H_0^2}{2}\int_0^a j_l^2(kr)r^2dr && \text{let } t = \frac{r}{a} \\ &= \frac{l(l+1)\mu_0H_0^2}{2}\cdot a^3\int_0^1 j_l^2(x_{ln}t)t^2dt && \text{by orthonormality of } j_l(x) \\ &= \frac{l(l+1)\mu_0H_0^2a^3}{4}j_{l+1}^2(x_{ln}) \end{aligned} \tag{22}$$

On the other hand, the power loss is the following integral over the wall

$$\begin{aligned} P_{\mathrm{loss}}^{\mathrm{TE}} &= \frac{1}{2\sigma\delta}\int\left|\hat{\mathbf{r}}\times\mathbf{H}_{lm}^{\mathrm{TE}}\right|^2da \\ &= \frac{1}{2\sigma\delta}\frac{H_0^2}{k^2}\left[\left.\frac{d(rj_l)}{dr}\right|_{r=a}\right]^2\int\left|\hat{\mathbf{r}}\times\mathbf{\Psi}_{lm}\right|^2d\Omega && \text{note } \hat{\mathbf{r}}\times\mathbf{\Psi}_{lm} = \mathbf{\Phi}_{lm} \text{ and use } j_l(ka) = 0 \\ &= \frac{l(l+1)}{2\sigma\delta}\frac{H_0^2}{k^2}\left[a\left.\frac{dj_l(kr)}{dr}\right|_{r=a}\right]^2 \\ &= \frac{l(l+1)}{2\sigma\delta}H_0^2a^2\left[j_l'(x_{ln})\right]^2 && \text{use recursion } j_l'(x) = -j_{l+1}(x) + \frac{l}{x}j_l(x) \\ &= \frac{l(l+1)}{2\sigma\delta}H_0^2a^2j_{l+1}^2(x_{ln}) \end{aligned} \tag{23}$$

Then by definition, the Q factor for all TE modes is

$$Q_{lm}^{\mathrm{TE}} = \omega_0 \frac{U_{lm}^{\mathrm{TE}}}{P_{\mathrm{loss}}^{\mathrm{TE}}} = \frac{\omega_0 \mu_0 \sigma \delta a}{2} = \frac{a}{\delta} \tag{24}$$

with skin depth relation (8.8).

(b) TM mode

The fields (5) expressed in VSH are

$$\mathbf{H}_{lm}^{\mathrm{TM}} = -iH_0 j_l(kr) \mathbf{\Phi}_{lm} \qquad \mathbf{E}_{lm}^{\mathrm{TM}} = -\frac{Z_0 H_0}{k}\left\{\frac{l(l+1)}{r} j_l(kr)\mathbf{Y}_{lm} + \frac{1}{r}\frac{d[r j_l(kr)]}{dr}\mathbf{\Psi}_{lm}\right\} \tag{25}$$

Following the same steps (15) – (18), we get the stored energy in TM mode

$$U_{lm}^{\mathrm{TM}} = \frac{l(l+1)\mu_0 H_0^2}{4}\int_0^a \left\{ j_l^2 + l(l+1)\frac{j_l^2}{k^2 r^2} + \frac{1}{k^2 r^2}\left[\frac{d(r j_l)}{dr}\right]^2\right\} r^2 dr \tag{26}$$

Similar argument leads to the conclusion that in (20) K's contribution vanishes (but this time we are making use of the TM mode's boundary condition $d[x j_l(x)]/dx = 0$), so the stored energy becomes

$$U_{lm}^{\mathrm{TM}} = \frac{l(l+1)\mu_0 H_0^2}{2}\int_0^a j_l^2(kr) r^2 dr = \frac{l(l+1)\mu_0 H_0^2}{2}\cdot a^3 \int_0^1 j_l^2\left(x'_{ln} t\right) t^2 dt \tag{27}$$

Here we invoke 10.22.38 of https://dlmf.nist.gov which was also proved in problem 3.11:

$$\int_0^1 J_\nu(\alpha_l t) J_\nu(\alpha_m t)\, t\, dt = \left(\frac{a^2}{b^2} + \alpha_l^2 - \nu^2\right)\frac{[J_\nu(\alpha_l)]^2}{2\alpha_l^2}\delta_{lm} \quad \text{for } \alpha_l, \alpha_m \text{ positive zeros of } aJ_\nu(x) + bxJ'_\nu(x) \tag{28}$$

With the relation

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+1/2}(x) \tag{29}$$

the TM bounary condition indicates

$$0 = \left.\frac{d[x j_l(x)]}{dx}\right|_{x = x'_{ln}} = \sqrt{\frac{\pi}{2}}\left[\frac{1}{2}\frac{1}{\sqrt{x'_{ln}}} J_{l+1/2}\left(x'_{ln}\right) + \sqrt{x'_{ln}} J'_{l+1/2}\left(x'_{ln}\right)\right] \tag{30}$$

i.e., a ratio of $a/b = 1/2$ in (28).

Applying (28) to the integral in (27) gives

$$\begin{aligned}\int_0^1 j_l^2\left(x'_{ln} t\right) t^2 dt &= \frac{\pi}{2x'_{ln}}\int_0^1 J_{l+1/2}^2\left(x'_{ln} t\right) t\, dt = \frac{\pi}{2x'_{ln}}\left[\frac{1}{4} + x'^2_{ln} - \left(l + \frac{1}{2}\right)^2\right]\frac{J_{l+1/2}^2\left(x'_{ln}\right)}{2x'^2_{ln}} \\ &= \frac{1}{2}\left[1 - \frac{l(l+1)}{x'^2_{ln}}\right] j_l^2\left(x'_{ln}\right)\end{aligned} \tag{31}$$

so the stored energy is

$$U_{lm}^{\mathrm{TM}} = \frac{l(l+1)\mu_0 H_0^2 a^3}{4}\left[1 - \frac{l(l+1)}{x'^2_{ln}}\right] j_l^2\left(x'_{ln}\right) \tag{32}$$

On the other hand, the power loss is

$$\begin{aligned} P_{\mathrm{loss}}^{\mathrm{TM}} &= \frac{1}{2\sigma\delta}\int \left|\hat{\mathbf{r}} \times \mathbf{H}_{lm}^{\mathrm{TM}}\right|^2 da \\ &= \frac{1}{2\sigma\delta} H_0^2 \int j_l^2(kr)\left|\hat{\mathbf{r}} \times (\mathbf{r} \times \boldsymbol{\nabla} Y_{lm})\right|^2 da \\ &= \frac{1}{2\sigma\delta} H_0^2 a^2 j_l^2(ka)\int \left|\mathbf{\Psi}_{lm}\right|^2 d\Omega \\ &= \frac{l(l+1)}{2\sigma\delta} H_0^2 a^2 j_l^2\left(x'_{ln}\right)\end{aligned} \tag{33}$$

Finally we get the Q factor for TM modes

$$Q_{lm}^{\mathrm{TM}} = \omega_0 \frac{U_{lm}^{\mathrm{TM}}}{P_{\mathrm{loss}}^{\mathrm{TM}}} = \frac{\omega_0 \mu_0 \sigma \delta a}{2}\left[1 - \frac{l(l+1)}{x'^2_{ln}}\right] = \frac{a}{\delta}\left[1 - \frac{l(l+1)}{x'^2_{ln}}\right] \tag{34}$$