

Here we give a detailed derivation of field-strength transformation under boost (11.149).

The matrix representation of the field-strength tensor $F^{a\beta}$ is given by (11.137)

$$F = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \quad (1)$$

which can be written in block form as

$$F = \begin{bmatrix} 0 & -\mathbf{E}^T \\ \mathbf{E} & \mathbf{B} \cdot \mathbf{S} \end{bmatrix} \quad (2)$$

where S_i 's are 3×3 matrices corresponding to the the generators of rotations in the Lorentz group, i.e., the lower-right 3×3 block of (11.91).

To simplify the proof, we take note of the following relations

$$S_i \mathbf{u} = \hat{\mathbf{e}}_i \times \mathbf{u} \quad (3)$$

$$\mathbf{a}\mathbf{b}^T - \mathbf{b}\mathbf{a}^T = -(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{S} \quad (4)$$

As is shown in (11.147), under Lorentz transformation A , the field-strength tensor transforms as

$$F' = AFA^T \quad (5)$$

where if we only have boost, A can be written in block form (see 11.98)

$$A = \begin{bmatrix} \gamma & -\gamma \boldsymbol{\beta}^T \\ -\gamma \boldsymbol{\beta} & \underbrace{I + \frac{(\gamma-1)}{\beta^2} \boldsymbol{\beta} \boldsymbol{\beta}^T}_M \end{bmatrix} = A^T \quad (6)$$

In these block forms, (5) becomes

$$\begin{aligned} F' &= AFA^T = \begin{bmatrix} \gamma & -\gamma \boldsymbol{\beta}^T \\ -\gamma \boldsymbol{\beta} & M \end{bmatrix} \begin{bmatrix} 0 & -\mathbf{E}^T \\ \mathbf{E} & \mathbf{B} \cdot \mathbf{S} \end{bmatrix} \begin{bmatrix} \gamma & -\gamma \boldsymbol{\beta}^T \\ -\gamma \boldsymbol{\beta} & M \end{bmatrix} \\ &= \begin{bmatrix} -\gamma \boldsymbol{\beta}^T \mathbf{E} & -\gamma \mathbf{E}^T - \gamma \boldsymbol{\beta}^T (\mathbf{B} \cdot \mathbf{S}) \\ M \mathbf{E} & \gamma \boldsymbol{\beta} \mathbf{E}^T + M (\mathbf{B} \cdot \mathbf{S}) \end{bmatrix} \begin{bmatrix} \gamma & -\gamma \boldsymbol{\beta}^T \\ -\gamma \boldsymbol{\beta} & M \end{bmatrix} \\ &= \begin{bmatrix} -\gamma^2 \boldsymbol{\beta}^T \mathbf{E} + \gamma^2 \mathbf{E}^T \boldsymbol{\beta} + \gamma^2 \boldsymbol{\beta}^T (\mathbf{B} \cdot \mathbf{S}) \boldsymbol{\beta} & \gamma^2 \boldsymbol{\beta}^T \mathbf{E} \boldsymbol{\beta}^T - \gamma \mathbf{E}^T M - \gamma \boldsymbol{\beta}^T (\mathbf{B} \cdot \mathbf{S}) M \\ \gamma M \mathbf{E} - \gamma^2 \boldsymbol{\beta} \mathbf{E}^T \boldsymbol{\beta} - \gamma M (\mathbf{B} \cdot \mathbf{S}) \boldsymbol{\beta} & -\gamma M \mathbf{E} \boldsymbol{\beta}^T + \gamma \boldsymbol{\beta} \mathbf{E}^T M + M (\mathbf{B} \cdot \mathbf{S}) M \end{bmatrix} \end{aligned} \quad (7)$$

From (3), we can see $F'^{00} = 0$ as expected. From the fact that $\mathbf{B} \cdot \mathbf{S}$ is antisymmetric, we see that the off-diagonal blocks are antisymmetric too, with the column block $[1 : 3, 0]$ representing the transformed electric field, i.e.,

$$\begin{aligned} \mathbf{E}' &= F'^{[1:3,0]} = \gamma M \mathbf{E} - \gamma^2 \boldsymbol{\beta} \mathbf{E}^T \boldsymbol{\beta} - \gamma M (\mathbf{B} \cdot \mathbf{S}) \boldsymbol{\beta} \\ &= \gamma \left[I + \frac{(\gamma-1)}{\beta^2} \boldsymbol{\beta} \boldsymbol{\beta}^T \right] \mathbf{E} - \gamma^2 \boldsymbol{\beta} \mathbf{E}^T \boldsymbol{\beta} - \gamma \left[I + \frac{(\gamma-1)}{\beta^2} \boldsymbol{\beta} \boldsymbol{\beta}^T \right] (\mathbf{B} \cdot \mathbf{S}) \boldsymbol{\beta} \quad \text{using (3)} \\ &= \gamma \mathbf{E} + \gamma \left[\frac{(\gamma-1)}{\beta^2} - \gamma \right] \boldsymbol{\beta} (\boldsymbol{\beta} \cdot \mathbf{E}) - \gamma \overbrace{(\mathbf{B} \cdot \mathbf{S}) \boldsymbol{\beta}}^{\mathbf{B} \times \boldsymbol{\beta}} \\ &= \gamma (\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma+1} \boldsymbol{\beta} (\boldsymbol{\beta} \cdot \mathbf{E}) \end{aligned} \quad (8)$$

Rearranging the terms into parallel and perpendicular components, we have

$$\mathbf{E}' = \gamma (\mathbf{E}_{\parallel} + \mathbf{E}_{\perp}) + \gamma \boldsymbol{\beta} \times \mathbf{B} - (\gamma-1) \underbrace{\frac{\boldsymbol{\beta} (\boldsymbol{\beta} \cdot \mathbf{E})}{\beta^2}}_{\mathbf{E}_{\parallel}} = \mathbf{E}_{\parallel} + \gamma (\mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B}) \implies \mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \quad \mathbf{E}'_{\perp} = \gamma (\mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B}) \quad (9)$$

The lower-right 3×3 block of (7) gives us

$$\begin{aligned}
\mathbf{B}' \cdot \mathbf{S} &= F^{[1:3,1:3]} = -\gamma M \mathbf{E} \boldsymbol{\beta}^T + \gamma \boldsymbol{\beta} \mathbf{E}^T M + M (\mathbf{B} \cdot \mathbf{S}) M \\
&= -\gamma \left[I + \frac{(\gamma-1)}{\beta^2} \boldsymbol{\beta} \boldsymbol{\beta}^T \right] \mathbf{E} \boldsymbol{\beta}^T + \gamma \boldsymbol{\beta} \mathbf{E}^T \left[I + \frac{(\gamma-1)}{\beta^2} \boldsymbol{\beta} \boldsymbol{\beta}^T \right] \\
&\quad + \left[I + \frac{(\gamma-1)}{\beta^2} \boldsymbol{\beta} \boldsymbol{\beta}^T \right] (\mathbf{B} \cdot \mathbf{S}) \left[I + \frac{(\gamma-1)}{\beta^2} \boldsymbol{\beta} \boldsymbol{\beta}^T \right] \quad \text{using (3)} \\
&= -\gamma (\mathbf{E} \boldsymbol{\beta}^T - \boldsymbol{\beta} \mathbf{E}^T) + \mathbf{B} \cdot \mathbf{S} + \frac{(\gamma-1)}{\beta^2} \left\{ \boldsymbol{\beta} [(\mathbf{B} \cdot \mathbf{S})^T \boldsymbol{\beta}]^T + (\mathbf{B} \cdot \mathbf{S}) \boldsymbol{\beta} \boldsymbol{\beta}^T \right\} \quad \text{using (4), } \mathbf{B} \cdot \mathbf{S} \text{ antisymmetric} \\
&= -\gamma (\boldsymbol{\beta} \times \mathbf{E}) \cdot \mathbf{S} + \mathbf{B} \cdot \mathbf{S} + \frac{(\gamma-1)}{\beta^2} [-\boldsymbol{\beta} (\mathbf{B} \times \boldsymbol{\beta})^T + (\mathbf{B} \times \boldsymbol{\beta}) \boldsymbol{\beta}^T] \\
&= -\gamma (\boldsymbol{\beta} \times \mathbf{E}) \cdot \mathbf{S} + \mathbf{B} \cdot \mathbf{S} + \frac{(\gamma-1)}{\beta^2} [-(\mathbf{B} \times \boldsymbol{\beta}) \times \boldsymbol{\beta}] \cdot \mathbf{S} \\
&= -\gamma (\boldsymbol{\beta} \times \mathbf{E}) \cdot \mathbf{S} + \mathbf{B} \cdot \mathbf{S} + \frac{(\gamma-1)}{\beta^2} [\beta^2 \mathbf{B} - \boldsymbol{\beta} (\boldsymbol{\beta} \cdot \mathbf{B})] \cdot \mathbf{S} \\
&= \left[\gamma (\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma+1} \boldsymbol{\beta} (\boldsymbol{\beta} \cdot \mathbf{B}) \right] \cdot \mathbf{S} \quad (10)
\end{aligned}$$

allowing us to identify

$$\mathbf{B}' = \gamma (\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma+1} \boldsymbol{\beta} (\boldsymbol{\beta} \cdot \mathbf{B}) \quad (11)$$

of which the parallel and perpendicular components are

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \quad \mathbf{B}'_{\perp} = \gamma (\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E}) \quad (12)$$