1. This is a straightforward application of the Green function obtained in Prob 3.17.

$$G\left(\mathbf{x},\mathbf{x}'\right) = 2\sum_{m=-\infty}^{\infty} e^{im\left(\phi-\phi'\right)}\int_0^\infty dk J_m(k\rho) J_m\left(k\rho'\right)\frac{\sinh(kz_<)\sinh[k(L-z_>)]}{\sinh(kL)} \tag{1}$$

By the Green function method, the interior point's potential is

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G}{\partial n'}da' \tag{2}$$

On the surface S, only points within the disc has non zero potential, which turns (2) into

$$\begin{aligned}\Phi(\rho,z) &= -\frac{1}{4\pi}\int_0^{2\pi} d\phi'\int_0^a V\left(\left.\frac{\partial G}{\partial z'}\right|_{z'=L}\right)\rho' d\rho' \\ &= -\frac{V}{4\pi}\int_0^{2\pi} d\phi'\int_0^a \rho' d\rho'\left[2\sum_{m=-\infty}^{\infty} e^{im\left(\phi-\phi'\right)}\int_0^\infty dk J_m(k\rho) J_m\left(k\rho'\right)\frac{\sinh(kz)(-k)\cosh 0}{\sinh(kL)}\right]\end{aligned} \tag{3}$$

The integration in ϕ' ensures only $m=0$ term survives, thus

$$\begin{aligned}\Phi(\rho,z) &= \frac{V}{4\pi}\cdot 2\pi\int_0^a \rho' d\rho'\cdot 2\int_0^\infty k dk J_0(k\rho) J_0\left(k\rho'\right)\frac{\sinh(kz)}{\sinh(kL)} \\ &= V\int_0^\infty k dk J_0(k\rho)\frac{\sinh(kz)}{\sinh(kL)}\int_0^a \rho' J_0\left(k\rho'\right)d\rho' \\ &= V\int_0^\infty k dk J_0(k\rho)\frac{\sinh(kz)}{\sinh(kL)}\overbrace{\int_0^{ka}\frac{1}{k^2}\left(k\rho'\right)J_0\left(k\rho'\right)d\left(k\rho'\right)}^{(ka)J_1(ka)/k^2} \\ &= V\int_0^\infty dk J_0(k\rho)\frac{\sinh(kz)}{\sinh(kL)} a J_1(ka) \qquad \text{define } \lambda \equiv ka \\ &= V\int_0^\infty d\lambda J_0\left(\frac{\lambda\rho}{a}\right)J_1(\lambda)\frac{\sinh\left(\frac{\lambda z}{a}\right)}{\sinh\left(\frac{\lambda L}{a}\right)}\end{aligned}$$

(4), (5)

where we have used the relation

$$[xJ_1(x)]' = xJ_0(x) \tag{6}$$

to evaluate the inner integral in $d\left(k\rho'\right)$.

2. When $a \to \infty$,

$$J_0\left(\frac{\lambda\rho}{a}\right) \to 1 \qquad\qquad \frac{\sinh\left(\frac{\lambda z}{a}\right)}{\sinh\left(\frac{\lambda L}{a}\right)} \to \frac{z}{L} \tag{7}$$

which makes

$$\Phi(\rho,z) \to V\int_0^\infty d\lambda J_1(\lambda)\cdot\frac{z}{L} = \frac{zV}{L}\int_0^\infty -J_0'(\lambda)d\lambda = \frac{zV}{L} \tag{8}$$

which is expected for a pair of infinite parallel plates, one grounded, the other at potential V.
When a is not infinitely large but meerly "large", we can attempt to expand the integrand into different orders of a^{-1}:

$$J_0\left(\frac{\lambda\rho}{a}\right) = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!j!}\left(\frac{\lambda\rho}{2a}\right)^{2j} = 1 + c_1\left(\frac{\lambda}{a}\right)^2 + c_2\left(\frac{\lambda}{a}\right)^4 + \cdots \tag{9}$$

$$\frac{\sinh\left(\frac{\lambda z}{a}\right)}{\sinh\left(\frac{\lambda L}{a}\right)} = \left[\sum_{k=0}^{\infty}\frac{\left(\frac{\lambda z}{a}\right)^{2k+1}}{(2k+1)!}\right]\left[\sum_{l=0}^{\infty}\frac{\left(\frac{\lambda L}{a}\right)^{2l+1}}{(2l+1)!}\right]^{-1} = \frac{z}{L}\left[1 + d_1\left(\frac{\lambda}{a}\right)^2 + d_2\left(\frac{\lambda}{a}\right)^4 + \cdots\right] \tag{10}$$

The leading order (0-th) of the integral is just (8). But when we go to the higher orders, we encounter problems. For example, for the $2k$-th order correction:

$$\Delta_{2k}\Phi(\rho,z) \propto \frac{1}{a^{2k}}\int_0^\infty J_1(\lambda)\lambda^{2k}d\lambda \tag{11}$$

But recall that (equation (3.91))

$$J_\nu(x) \longrightarrow \sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right) \qquad \text{as } x\to\infty \tag{12}$$

The infinite integral (11) clearly diverges for $k \geq 1$, despite that we will eventually divide this integral by a large amount a^{2k}. This difficulty shows that we cannot arbitrarily switch the order between taking the limit $a\to\infty$ and the infinite integral in λ. Another possible reason for this divergence is that the expansion of $1/\sinh x$ converges only when $|x| < \pi$ (See Wikiproof), so the expansion of (10) is conditional, and is clearly invalid when $\lambda\to\infty$.

The problem also asks for an explicit estimate of the correction, **I cannot figure out this part**.

3. In problem (3.12), we have given the explicit integral representation of the solution

$$\Phi(\rho,\phi,z) = \int_0^\infty \tilde{A}(k)J_0(k\rho)\,e^{-k(L-z)}dk \qquad \text{where} \tag{13}$$

$$\tilde{A}(k) = kV\int_0^a J_0\left(k\rho'\right)\rho' d\rho' \tag{14}$$

(Note here we have a simple change of coordination compared to the 3.12 setup, which necessitates the replacement, in the original 3.12 solution, of e^{-kz} by $e^{-k(L-z)}$.)

Compare this with (4), all we need to show is the asymptotic form

$$\frac{\sinh(kz)}{\sinh(kL)} \to e^{-k(L-z)} \tag{15}$$

as $L\to\infty$ while holding $(L-z)$, a and ρ fixed.

Indeed, if we denote $d = L-z$, then

$$\begin{aligned}\frac{\sinh(kL-kd)}{\sinh(kL)} &= \frac{\sinh(kL)\cosh(kd)-\cosh(kL)\sinh(kd)}{\sinh(kL)}\\ &= \cosh(kd)-\frac{\cosh(kL)}{\sinh(kL)}\sinh(kd)\\ &\to \cosh(kd)-1\cdot\sinh(kd) = e^{-kd} \qquad \text{as } L\to\infty\end{aligned} \tag{16}$$