This is very similar to Prob 2.25.

Treating $G(\mathbf{x},\mathbf{x}')$ as a function in $\mathbf{x}$, due to the angular boundary condition, its Fourier decomposition must be of the form

$$G(\mathbf{x},\mathbf{x}') = \sum_{m=1}^{\infty} g_m(\rho,\rho') A_m \sin\left(\frac{m\pi\phi}{\beta}\right) \tag{1}$$

Taking the Laplacian, we have

$$\nabla^2 G = \sum_{m=1}^{\infty} \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{1}{\rho^2}\left(\frac{m\pi}{\beta}\right)^2\right] g_m(\rho,\rho') A_m \sin\left(\frac{m\pi\phi}{\beta}\right) = -4\pi\delta(\phi-\phi')\frac{\delta(\rho-\rho')}{\rho} \tag{2}$$

By Prob 2.24

$$\frac{2}{\beta}\sum_{m=1}^{\infty} \sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right) = \delta(\phi-\phi') \tag{3}$$

we can take

$$A_m = \sin\left(\frac{m\pi\phi'}{\beta}\right) \tag{4}$$

and further require

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{1}{\rho^2}\left(\frac{m\pi}{\beta}\right)^2\right] g_m = -\frac{8\pi}{\beta}\frac{\delta(\rho-\rho')}{\rho} \tag{5}$$

Integrating (5) over the infinitesimal range $[\rho'-\epsilon,\rho'+\epsilon]$ yields

$$\rho\frac{\partial g_m}{\partial\rho}\bigg|_{\rho'+\epsilon} - \rho\frac{\partial g_m}{\partial\rho}\bigg|_{\rho'-\epsilon} = -\frac{8\pi}{\beta} \tag{6}$$

It's easy to see that the general solution of (5) for $\rho \neq \rho'$ is

$$g_m(\rho,\rho') = \begin{cases} a_m\rho^{m\pi/\beta} & \text{for } \rho < \rho' \\ b_m\rho^{m\pi/\beta} + c_m\rho^{-m\pi/\beta} & \text{for } \rho > \rho' \end{cases} \tag{7}$$

Thus

$$\text{by continuity at } \rho=\rho' : \quad b_m\rho'^{m\pi/\beta} + c_m\rho'^{-m\pi/\beta} = a_m\rho'^{m\pi/\beta} \tag{8}$$

$$\text{by derivative discontinuity (6)} : \quad \left(\frac{m\pi}{\beta}\right)b_m\rho'^{m\pi/\beta} - \left(\frac{m\pi}{\beta}\right)c_m\rho'^{-m\pi/\beta} - \left(\frac{m\pi}{\beta}\right)a_m\rho'^{m\pi/\beta} = -\frac{8\pi}{\beta} \implies$$

$$b_m\rho'^{m\pi/\beta} - c_m\rho'^{-m\pi/\beta} - a_m\rho'^{m\pi/\beta} = -\frac{8}{m} \tag{9}$$

$$\text{by boundary condition at } \rho=a : \quad b_m a^{m\pi/\beta} + c_m a^{-m\pi/\beta} = 0 \tag{10}$$

Plugging (8) into (9) gives

$$c_m = \frac{4}{m}\rho'^{m\pi/\beta} \tag{11}$$

Plugging (11) into (10) gives

$$b_m = -\frac{4}{m}\frac{\rho'^{m\pi/\beta}}{a^{2m\pi/\beta}} \tag{12}$$

And finally by (8)

$$a_m = b_m + c_m\rho'^{-2m\pi/\beta} = \frac{4}{m}\left(\frac{1}{\rho'^{m\pi/\beta}} - \frac{\rho'^{m\pi/\beta}}{a^{2m\pi/\beta}}\right) \tag{13}$$

These determine the form of g_m:

$$g_m(\rho,\rho') = \frac{4}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^{m\pi/\beta} - \left(\frac{\rho\rho'}{a^2}\right)^{m\pi/\beta}\right] \tag{14}$$

which gives the desired Green function

$$G(\mathbf{x},\mathbf{x}') = \sum_{m=1}^{\infty}\frac{4}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^{m\pi/\beta} - \left(\frac{\rho\rho'}{a^2}\right)^{m\pi/\beta}\right]\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right) \tag{15}$$