1. It's easy to see that the alleged Green function satisfies the Dirichlet boundary condition $G(\phi'=0)=G(\phi'=\beta)=0$. To solve for the Poisson equation

$$\nabla'^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi') \tag{1}$$

we start with the ansatz

$$G(\rho,\phi;\rho',\phi') = \sum_{m=1}^{\infty} g_m(\rho,\rho')\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right) \tag{2}$$

whose Laplacian is

$$\begin{aligned}\nabla'^2 G &= \left[\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right)+\frac{1}{\rho'^2}\frac{\partial}{\partial\phi'^2}\right]G \\ &= \sum_{m=1}^{\infty}\left[\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right)-\frac{1}{\rho'^2}\left(\frac{m\pi}{\beta}\right)^2\right]g_m(\rho,\rho')\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)\end{aligned} \tag{3}$$

In problem 2.24, we have proved

$$\frac{2}{\beta}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right) = \delta(\phi-\phi') \tag{4}$$

So if $g_m(\rho,\rho')$ satisfies

$$\left[\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right)-\frac{1}{\rho'^2}\left(\frac{m\pi}{\beta}\right)^2\right]g_m(\rho,\rho') = \frac{2}{\beta}\cdot\left[-4\pi\frac{\delta(\rho-\rho')}{\rho}\right] \tag{5}$$

we would have the right solution for (1).

The following is almost the same as problem 2.17 except for some constant changes. As per usual, integrate (5) with $\rho' d\rho'$ over the infinitesimal range $[\rho-\epsilon,\rho+\epsilon]$, we obtain the derivative discontinuity condition

$$\rho'\frac{\partial g_m}{\partial\rho'}\bigg|_{\rho+\epsilon} - \rho'\frac{\partial g_m}{\partial\rho'}\bigg|_{\rho-\epsilon} = -\frac{8\pi}{\beta} \tag{6}$$

In the range $\rho'\neq\rho$, the general solution of (5) is

$$g_m(\rho,\rho') = a_m\rho'^{m\pi/\beta} + b_m\rho'^{-m\pi/\beta} \tag{7}$$

(recall in this problem m starts from 1 due to the boundary condition, so we don't have the $\ln\rho$ terms).

Considering the asymptotic behavior for $\rho'\to 0$ and $\rho'\to\infty$, we have

$$g_m(\rho,\rho') = \begin{cases} a_m\rho'^{m\pi/\beta} & \rho'<\rho \\ b_m\rho'^{-m\pi/\beta} & \rho'>\rho\end{cases} \tag{8}$$

Thus

$$\text{by continuity at } \rho'=\rho\ :\qquad a_m\rho^{m\pi/\beta} = b_m\rho^{-m\pi/\beta} \tag{9}$$

$$\text{by derivative discontinuity (5)}\ :\qquad -\left(\frac{m\pi}{\beta}\right)b_m\rho^{-m\pi/\beta} - \left(\frac{m\pi}{\beta}\right)a_m\rho^{m\pi/\beta} = -\frac{8\pi}{\beta} \tag{10}$$

which gives the solution

$$a_m = \frac{4}{m}\rho^{-m\pi/\beta} \qquad\qquad b_m = \frac{4}{m}\rho^{m\pi/\beta} \tag{11}$$

and finally

$$\begin{aligned} g_m(\rho,\rho') &= \begin{cases}\dfrac{4}{m}\rho^{-m\pi/\beta}\rho'^{m\pi/\beta} & \rho'<\rho \\ \dfrac{4}{m}\rho^{m\pi/\beta}\rho'^{-m\pi/\beta} & \rho'>\rho\end{cases} \\ &= \frac{4}{m}\left(\frac{\rho_<}{\rho_>}\right)^{m\pi/\beta}\end{aligned} \tag{12}$$

The full Green function is thus

$$G(\rho,\phi;\rho',\phi') = 4\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^{m\pi/\beta}\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right) \tag{13}$$

2. Using identity

$$\sin\eta\sin\xi = \frac{1}{2}\left[\cos(\eta-\xi)-\cos(\eta+\xi)\right] \tag{14}$$

we can convert (13) into

$$G = 2\sum_{m}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^{m\pi/\beta}\left\{\cos\left[\frac{m\pi(\phi-\phi')}{\beta}\right]-\cos\left[\frac{m\pi(\phi+\phi')}{\beta}\right]\right\} \tag{15}$$

Then by the expansion

$$\ln\left(1+x^2-2x\cos\theta\right) = -2\sum_{m=1}^{\infty}\frac{x^m}{m}\cos m\theta \tag{16}$$

we arrive at

$$\begin{aligned} G &= \ln\left\{\frac{1+\left(\frac{\rho_>}{\rho_>}\right)^{2\pi/\beta}-2\left(\frac{\rho_<}{\rho_>}\right)^{\pi/\beta}\cos\left[\frac{\pi(\phi+\phi')}{\beta}\right]}{1+\left(\frac{\rho_>}{\rho_>}\right)^{2\pi/\beta}-2\left(\frac{\rho_<}{\rho_>}\right)^{\pi/\beta}\cos\left[\frac{\pi(\phi-\phi')}{\beta}\right]}\right\} \\ &= \ln\left\{\frac{\rho^{2\pi/\beta}+\rho'^{2\pi/\beta}-2(\rho\rho')^{\pi/\beta}\cos\left[\frac{\pi(\phi+\phi')}{\beta}\right]}{\rho^{2\pi/\beta}+\rho'^{2\pi/\beta}-2(\rho\rho')^{\pi/\beta}\cos\left[\frac{\pi(\phi-\phi')}{\beta}\right]}\right\} \end{aligned} \tag{17}$$

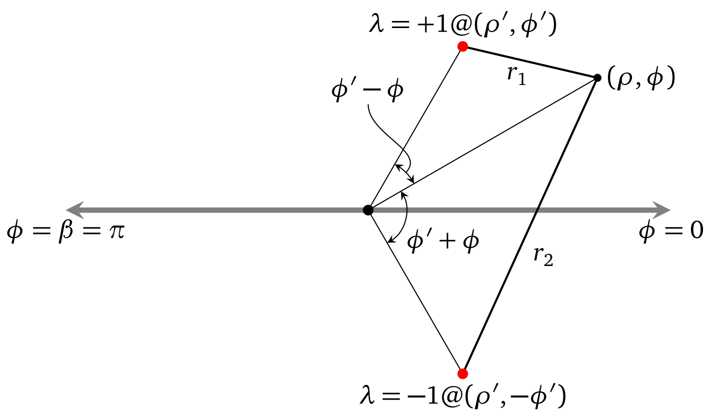
3. For the special case $\beta=\pi$, the Green function is

$$G_\pi = \ln\left[\frac{\rho^2+\rho'^2-2\rho\rho'\cos(\phi+\phi')}{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\right] \tag{18}$$

Recall that in 2d electrostatic fields, the physical significance of the Green function is $4\pi\epsilon_0$ times the potential generated by a unit line charge at position (ρ',ϕ'), so (18) gives a potential

$$\Phi_\pi(\rho,\phi) = \frac{1}{4\pi\epsilon_0}\ln\left[\frac{\rho^2+\rho'^2-2\rho\rho'\cos(\phi+\phi')}{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\right] \tag{19}$$

This agrees exactly with the potential given by the unit line charge at (ρ',ϕ') together with a grounded conducting plane formed by $\phi=0$ and $\phi=\beta=\pi$. Since in this configuration, the field in the upper half space is identical to the potential generated by the original line charge and an oppositely charged image line charge located at $(\rho',-\phi')$.



Clearly, the original and image line charge together generate a potential

$$\Phi(\rho,\phi) = \frac{1}{2\pi\epsilon_0}\ln\left(\frac{r_2}{r_1}\right) = \frac{1}{2\pi\epsilon_0}\ln\left[\frac{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi+\phi')}}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}}\right] \tag{20}$$

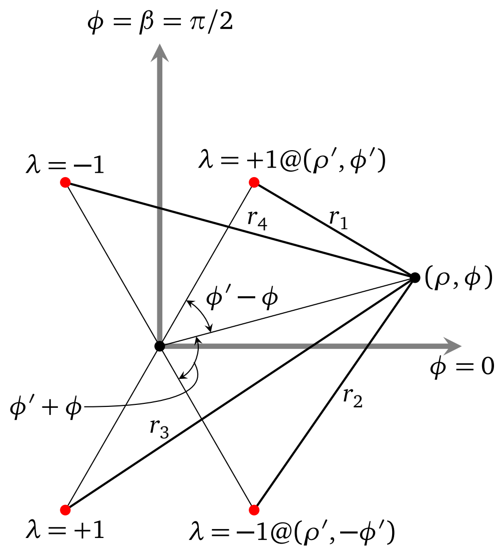
which agrees with (19).

When $\beta=\pi/2$, the Green function by (17) is

$$\begin{aligned} G_{\pi/2} &= \frac{1}{4\pi\epsilon_0}\ln\left[\frac{\rho^4+\rho'^4-2\rho^2\rho'^2\cos 2(\phi+\phi')}{\rho^4+\rho'^4-2\rho^2\rho'^2\cos 2(\phi-\phi')}\right] \\ &= \frac{1}{2\pi\epsilon_0}\ln\left[\frac{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi+\phi')}\sqrt{\rho^2+\rho'^2+2\rho\rho'\cos(\phi+\phi')}}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\sqrt{\rho^2+\rho'^2+2\rho\rho'\cos(\phi-\phi')}}\right] \end{aligned} \tag{21}$$

This is exactly the potential given by the unit line charge at (ρ',ϕ') in the presence of grounded conducting half planes at $\phi=0$ and $\phi=\beta=\pi/2$, which by methods of image charges, is given by

$$\Phi(\rho,\phi) = \frac{1}{2\pi\epsilon_0}\ln\left(\frac{r_2 r_4}{r_1 r_3}\right) \tag{22}$$



By simple geometry illustrated in the diagram, we can clearly see the equivalence between (21) and (22).