1. For TM mode, the longitudinal electric field $E_{z\lambda}, E_{z\mu}$ satisfy the eigenequation

$$\left(\nabla_t^2 + \gamma_\lambda^2\right) E_{z\lambda} = 0 \qquad \left(\nabla_t^2 + \gamma_\mu^2\right) E_{z\mu} = 0 \tag{1}$$

Thus by Green's Theorem

$$\int_A \left(E_{z\lambda}\nabla_t^2 E_{z\mu} - E_{z\mu}\nabla_t^2 E_{z\lambda}\right) da = \oint_C \left(E_{z\lambda}\nabla_t E_{z\mu} - E_{z\mu}\nabla_t E_{z\lambda}\right)\cdot \mathbf{n} dl \implies$$
$$\left(\gamma_\lambda^2 - \gamma_\mu^2\right)\int_A E_{z\lambda}E_{z\mu} da = \oint_C \left(E_{z\lambda}\frac{\partial E_{z\mu}}{\partial n} - E_{z\mu}\frac{\partial E_{z\lambda}}{\partial n}\right) dl = 0 \tag{2}$$

where we have used the boundary condition $E_{z\lambda}\Big|_S = E_{z\mu}\Big|_S = 0$. Then if the two modes are not degenerate, i.e., $\gamma_\lambda^2 \neq \gamma_\mu^2$, we have the orthogonality condition

$$\int_A E_{z\lambda}E_{z\mu} da = 0 \qquad \text{for } \lambda \neq \mu \tag{3}$$

The orthogonality does not generally hold for degenerate modes, as expected from linear algebra, but they are expected to be orthogonalizable via linear combinations.

For TE, the proof is the same except a different boundary condition is used, i.e., $\partial H_{z\lambda}/\partial n = \partial H_{z\mu}/\partial n = 0$ on S.

2. To see (8.131), recall (8.33)

$$\mathbf{E}_\lambda = \frac{ik_\lambda}{\gamma_\lambda^2}\nabla_t E_{z\lambda} \qquad \mathbf{E}_\mu = \frac{ik_\mu}{\gamma_\mu^2}\nabla_t E_{z\mu} \tag{4}$$

Also by Green's first identity

$$\int_A \left(E_{z\lambda}\nabla_t^2 E_{z\mu} + \nabla_t E_{z\lambda}\cdot\nabla_t E_{z\mu}\right) da = \oint_C E_{z\lambda}\mathbf{n}\cdot\nabla_t E_{z\mu} dl \implies$$
$$-\gamma_\mu^2 \int_A E_{z\lambda}E_{z\mu} da + \int_A \nabla_t E_{z\lambda}\cdot\nabla_t E_{z\mu} da = \oint_C E_{z\lambda}\frac{\partial E_{z\mu}}{\partial n} dl = 0 \implies$$
$$\int_A \mathbf{E}_\lambda\cdot\mathbf{E}_\mu da = -\frac{k_\lambda k_\mu}{\gamma_\lambda^2\gamma_\mu^2}\int_A \nabla_t E_{z\lambda}\cdot\nabla_t E_{z\mu} da = -\frac{k_\lambda k_\mu}{\gamma_\lambda^2}\int_A E_{z\lambda}E_{z\mu} da \propto \delta_{\lambda\mu} \tag{5}$$

This also shows the normalization stated by (8.134)

(8.132), (8.133) follows trivially from (8.31).

For mixed modes though, i.e., when $\mathbf{E}_\lambda^{\text{TM}}$ is the transverse field of the λ-th TM mode and $\mathbf{E}_\mu^{\text{TE}}$ is the transeverse field of the μ-th TE mode, we have

$$\mathbf{E}_\lambda^{\text{TM}} = \frac{ik_\lambda}{\gamma_\lambda^2}\nabla_t E_{z\lambda}^{\text{TM}} \tag{6}$$
$$\mathbf{H}_\mu^{\text{TE}} = \frac{ik_\mu}{\gamma_\mu^2}\nabla_t H_{z\mu}^{\text{TE}} \implies \mathbf{E}_\mu^{\text{TE}} = -Z_\mu\hat{\mathbf{z}}\times\mathbf{H}_\mu^{\text{TE}} = -\frac{ik_\mu Z_\mu}{\gamma_\mu^2}\hat{\mathbf{z}}\times\nabla_t H_{z\mu}^{\text{TE}} \tag{7}$$

Thus

$$\begin{aligned}
\int_A \mathbf{E}_\lambda^{\text{TM}}\cdot\mathbf{E}_\mu^{\text{TE}} da &= \frac{k_\lambda k_\mu Z_\mu}{\gamma_\lambda^2\gamma_\mu^2}\int_A \nabla_t E_{z\lambda}^{\text{TM}}\cdot\left(\hat{\mathbf{z}}\times\nabla_t H_{z\mu}^{\text{TE}}\right) da && \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) \\
&= -\frac{k_\lambda k_\mu Z_\mu}{\gamma_\lambda^2\gamma_\mu^2}\int_A \left(\nabla_t E_{z\lambda}^{\text{TM}}\times\nabla_t H_{z\mu}^{\text{TE}}\right)\cdot\hat{\mathbf{z}} da && \nabla\psi\times\mathbf{a} = \nabla\times(\psi\mathbf{a}) - \psi\nabla\times\mathbf{a} \\
&= -\frac{k_\lambda k_\mu Z_\mu}{\gamma_\lambda^2\gamma_\mu^2}\int_A \left[\nabla_t\times\left(E_{z\lambda}^{\text{TM}}\nabla_t H_{z\mu}^{\text{TE}}\right) - E_{z\lambda}^{\text{TM}}\nabla_t\times\left(\nabla_t H_{z\mu}^{\text{TE}}\right)\right]\cdot\hat{\mathbf{z}} da && \nabla\times\nabla\psi = 0,\ \text{Stokes Theorem} \\
&= -\frac{k_\lambda k_\mu Z_\mu}{\gamma_\lambda^2\gamma_\mu^2}\oint_C E_{z\lambda}^{\text{TM}}\nabla_t H_{z\mu}^{\text{TE}}\cdot d\mathbf{l} && E_{z\lambda}^{\text{TM}}\Big|_S = 0 \\
&= 0
\end{aligned} \tag{8}$$