1. **Plane wave decomposition into spherical waves**

   In Jackson section 10.3, second paragraph, the author briefly outlined the first method to decompose a plane wave into spherical waves. Here we provide a detailed derivation of this method.

   On the one hand, a plane wave $e^{i\mathbf{k}\cdot\mathbf{x}}$ is the solution to the Helmholtz equation

$$\left(\nabla^2+k^2\right)\psi(\mathbf{x})=0 \tag{1}$$

   On the other hand, any solution to (1) can be written as a superposition of spherical waves

$$\psi(\mathbf{x})=\sum_{l=0}^{\infty} j_l(kr)\sum_{m=-l}^{l} A_{lm}Y_{lm}(\theta,\phi) \tag{2}$$

   which is from the general solution (9.92) and the requirement to include origin.

   Without loss of generality, let $\mathbf{k}$ be along the z-axis, then with orthogonality of the spherical harmonics, we have

$$\begin{aligned} j_l(kr)A_{lm} &= \int d\Omega Y^*_{lm}(\theta,\phi)e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \delta_{m0}2\pi\sqrt{\frac{2l+1}{4\pi}}\int_0^{\pi} P_l(\cos\theta)e^{ikr\cos\theta}\sin\theta d\theta \\ &= \delta_{m0}2\pi\sqrt{\frac{2l+1}{4\pi}}\cdot 2i^l j_l(kr) \end{aligned} \tag{3}$$

   where in the last step we have used DLMF 10.54.E2

$$j_n(z)=\frac{(-i)^n}{2}\int_0^{\pi} e^{iz\cos\theta}P_n(\cos\theta)\sin\theta d\theta \tag{4}$$

   From this we have

$$A_{lm}=\delta_{m0}i^l\sqrt{(2l+1)4\pi} \tag{5}$$

   which gives the decomposition of a plane wave into spherical waves

$$e^{i\mathbf{k}\cdot\mathbf{x}}=\sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\theta) \tag{6}$$

2. **Derivation of equation (10.48)**

   Previous notes have established the following relations

$$\mathbf{X}_{lm}=\frac{1}{i\sqrt{l(l+1)}}\mathbf{\Phi}_{lm} \tag{7}$$

$$\nabla\times[h(r)\mathbf{\Phi}_{lm}]=-\frac{l(l+1)}{r}h\mathbf{Y}_{lm}-\left(\frac{dh}{dr}+\frac{h}{r}\right)\mathbf{\Psi}_{lm} \tag{8}$$

   Then the first two equations of (10.48) follow directly from the orthonormality of VSH. For the third equation, note by (8),

$$\begin{aligned} \nabla\times[f_{l'}(r)\mathbf{X}_{l'm'}]^*\cdot\nabla\times[g_l(r)\mathbf{X}_{lm}] = &\left\{\frac{1}{(-i)\sqrt{l'(l'+1)}}\left[-\frac{l'(l'+1)}{r}f^*_{l'}\mathbf{Y}^*_{l'm'}-\left(\frac{df^*_{l'}}{dr}+\frac{f^*_{l'}}{r}\right)\mathbf{\Psi}^*_{l'm'}\right]\right\}\cdot \\ &\left\{\frac{1}{i\sqrt{l(l+1)}}\left[-\frac{l(l+1)}{r}g_l\mathbf{Y}_{lm}-\left(\frac{dg_l}{dr}+\frac{g_l}{r}\right)\mathbf{\Psi}_{lm}\right]\right\} \end{aligned} \tag{9}$$

   Thus again by the orthonormality of VSH, the LHS of the third equation of (10.48) is

$$\frac{1}{k^2}\int \nabla\times[f_{l'}(r)\mathbf{X}_{l'm'}]^*\cdot\nabla\times[g_l(r)\mathbf{X}_{lm}]d\Omega=\delta_{ll'}\delta_{mm'}\left[\frac{l(l+1)}{k^2r^2}f^*_{l'}g_l+\frac{1}{k^2}\left(\frac{df^*_{l'}}{dr}+\frac{f^*_{l'}}{r}\right)\left(\frac{dg_l}{dr}+\frac{g_l}{r}\right)\right] \tag{10}$$

If $f_{l'}(r)$ and $g_l(r)$ are linear combinations of spherical Bessel functions, they satisfy

$$\frac{1}{k^2r^2}\frac{d}{dr}\left(r^2\frac{dg_l}{dr}\right)+\left[1-\frac{l(l+1)}{k^2r^2}\right]g_l=0 \qquad \text{similarly for } f_{l'} \tag{11}$$

Thus from the RHS of the third equation of (10.48), we have

$$\begin{aligned} f_{l'}^*g_l+\frac{1}{k^2r^2}\frac{d}{dr}\left[rf_{l'}^*\frac{d(rg_l)}{dr}\right] &= f_{l'}^*g_l+\frac{1}{k^2r^2}\frac{d}{dr}\left(rf_{l'}^*g_l+r^2f_{l'}^*\frac{dg_l}{dr}\right) \\ &= f_{l'}^*g_l+\frac{1}{k^2r^2}\left[f_{l'}^*g_l+r\frac{df_{l'}}{dr}g_l+rf_{l'}^*\frac{dg_l}{dr}+f_{l'}^*\frac{d}{dr}\left(r^2\frac{dg_l}{dr}\right)+r^2\frac{df_{l'}^*}{dr}\frac{dg_l}{dr}\right] \end{aligned} \tag{12}$$

which, after applying (11), is exactly the content in the bracket on the RHS of (10), proving the third equation of (10.48).

3. **The selection of $m=\pm1$ does not depend on circular polarization**

   It is worth emphasizing that the conclusion reached by the end of section 10.3, i.e., that $a(l,m)=b(l,m)=0$ for $m\neq\pm1$, does not depend on the assumption of circular polarization (10.46). In fact, it is a consequence of the incoming plane wave being cylindrically symmetric, i.e., its multipole expansion only has the $m=0$ components (see 10.45).

   To see this more clearly, let's consider any field with only $m=0$ components, with arbitrary polarization direction $\boldsymbol{\epsilon}$,

$$\mathbf{E}(\mathbf{x})=\boldsymbol{\epsilon}\sum_l A_l u_l(kr)Y_{l0}(\theta) \tag{13}$$

   When it is expanded into VSH using (9.122),

$$\mathbf{E}(\mathbf{x})=Z_0\sum_{l,m}\left\{\frac{i}{k}a_E(l,m)\nabla\times[f_l(kr)\mathbf{X}_{lm}]+a_M(l,m)g_l(kr)\mathbf{X}_{lm}\right\} \tag{14}$$

   we can invoke orthonormality of VSH to find

$$Z_0a_M(l,m)g_l(kr)=\int\mathbf{X}_{lm}^*\cdot\mathbf{E}(\mathbf{x})\,d\Omega=\sum_{l'}A_{l'}u_{l'}(kr)\int\mathbf{X}_{lm}^*\cdot\boldsymbol{\epsilon}Y_{l'0}d\Omega \tag{15}$$

   Note that

$$\mathbf{X}_{lm}^*\cdot\boldsymbol{\epsilon}Y_{l'0}\propto\mathbf{L}Y_{lm}^*\cdot\boldsymbol{\epsilon}Y_{l'0}=\left(L_x\epsilon_x+L_y\epsilon_y+L_z\epsilon_z\right)Y_{lm}^*Y_{l'0} \tag{16}$$

   Since L_x, L_y can be written as linear combinations of $L_\pm$, and $L_zY_{lm}\propto m$, it is clear that only the $l=l'$ and $m=\pm1$ (hence $l\geq1$) terms will survive the integration of (16) over all solid angles. The conclusion for $a_E(l,m)$ is similar.

   In section 10.4, the consideration of only $m=\pm1$ terms in (10.57) is justified by the same argument. Note that in the paragraph following (10.57), the comment "*for the restricted class of spherically symmetric problems considered here, only* $m=\pm1$ *occurs*" is inaccurate. Instead of "spherically symmetric", it should be "cylindrically symmetric" (i.e., $\mathbf{E}_{\mathrm{sc}},\mathbf{B}_{\mathrm{sc}}$ only have $m=0$ components, but can have non-zero l components).

4. **Derivation of total scattering and absorption cross sections (10.61)**

   For a general field solution of the form

$$\mathbf{E}(\mathbf{x})=\sum_{l=1}^{\infty}i^l\sqrt{4\pi(2l+1)}\left\{\overbrace{u_{l,\pm1}(kr)\mathbf{X}_{l,\pm1}\pm\frac{1}{k}\nabla\times\left[w_{l,\pm1}(kr)\mathbf{X}_{l,\pm1}\right]}^{\mathbf{e}_{l,\pm1}}\right\} \tag{17}$$

$$\mathbf{B}(\mathbf{x})=\frac{1}{c}\sum_{l=1}^{\infty}i^l\sqrt{4\pi(2l+1)}\left\{\underbrace{-\frac{i}{k}\nabla\times\left[u_{l,\pm1}(kr)\mathbf{X}_{l,\pm1}\right]\mp iw_{l,\pm1}(r)\mathbf{X}_{l,\pm1}}_{\mathbf{b}_{l,\pm1}}\right\} \tag{18}$$

   Let's calculate the quantity

$$P=\frac{a^2}{2\mu_0}\mathrm{Re}\int_{r=a}\mathbf{E}\cdot(\mathbf{n}\times\mathbf{B}^*)\,d\Omega \tag{19}$$

Expanding $\mathbf{b}_{l,\pm1}$ by (8), and noting that the $\mathbf{Y}$ component is radial, we have

$$\begin{aligned}\mathbf{n}\times\mathbf{b}^*_{l,\pm1} &= \mathbf{n}\times\left\{\frac{i}{k}\cdot\frac{1}{i\sqrt{l(l+1)}}\left[\frac{du^*_{l,\pm1}(kr)}{dr}+\frac{u^*_{l,\pm1}(kr)}{r}\right]\boldsymbol{\Psi}^*_{l,\pm1}\pm iw^*_{l,\pm1}(kr)\frac{1}{(-i)\sqrt{l(l+1)}}\boldsymbol{\Phi}^*_{l,\pm1}\right\}\\ &= \frac{1}{\sqrt{l(l+1)}}\left\{\left[\frac{du^*_{l,\pm1}(kr)}{d(kr)}+\frac{u^*_{l,\pm1}(kr)}{kr}\right]\boldsymbol{\Phi}^*_{l,\pm1}\pm w^*_{l,\pm1}(kr)\boldsymbol{\Psi}^*_{l,\pm1}\right\}\end{aligned}\tag{20}$$

where we have used the VSH relations

$$\mathbf{n}\times\boldsymbol{\Psi}_{lm}=\boldsymbol{\Phi}_{lm}\qquad\qquad \mathbf{n}\times\boldsymbol{\Phi}_{lm}=-\boldsymbol{\Psi}_{lm}\tag{21}$$

Orthogonality guarantees that only the transverse components of $\mathbf{e}_{l,\pm1}$ will engage with $\mathbf{n}\times\mathbf{b}^*_{l,\pm1}$ in the solid angle integration, and by (8)

$$\mathbf{e}_{l,\pm1,\text{trans}}=\frac{1}{i\sqrt{l(l+1)}}\left\{u_{l,\pm1}(kr)\boldsymbol{\Phi}_{l,\pm1}\mp\left[\frac{dw_{l,\pm1}(kr)}{d(kr)}+\frac{w_{l,\pm1}(kr)}{kr}\right]\boldsymbol{\Psi}_{l,\pm1}\right\}\tag{22}$$

producing

$$\begin{aligned}\mathrm{Re}\int_{r=a}\mathbf{e}_{l,\pm1}\cdot\left(\mathbf{n}\times\mathbf{b}^*_{l,\pm1}\right)d\Omega &= \mathrm{Re}\left[\frac{1}{i}\left(u_{l,\pm1}u^{*\prime}_{l,\pm1}+\frac{u_{l,\pm1}u^*_{l,\pm1}}{ka}-w'_{l,\pm1}w^*_{l,\pm1}-\frac{w_{l,\pm1}w^*_{l,\pm1}}{ka}\right)\right]\\ &= \mathrm{Im}\left(u_{l,\pm1}u^{*\prime}_{l,\pm1}-w'_{l,\pm1}w^*_{l,\pm1}\right)\end{aligned}\tag{23}$$

where u,w and their derivatives are evaluated at ka. This gives

$$P=\frac{a^2}{2\mu_0 c}\sum_{l=1}^{\infty}4\pi(2l+1)\,\mathrm{Im}\left(u_{l,\pm1}u^{*\prime}_{l,\pm1}-w'_{l,\pm1}w^*_{l,\pm1}\right)\tag{24}$$

(a) For the scattered fields given by (10.57),

$$\mathbf{E}_{\text{sc}}=\frac{1}{2}\sum_{l=1}^{\infty}i^l\sqrt{4\pi(2l+1)}\left\{\alpha_\pm(l)h_l^{(1)}(kr)\mathbf{X}_{l,\pm1}\pm\frac{\beta_\pm(l)}{k}\nabla\times\left[h_l^{(1)}(kr)\mathbf{X}_{l,\pm1}\right]\right\}\tag{25}$$

$$\mathbf{B}_{\text{sc}}=\frac{1}{2c}\sum_{l=1}^{\infty}i^l\sqrt{4\pi(2l+1)}\left\{\frac{-i\alpha_\pm(l)}{k}\nabla\times\left[h_l^{(1)}(kr)\mathbf{X}_{l,\pm1}\right]\mp i\beta_\pm(l)h_l^{(1)}(kr)\mathbf{X}_{l,\pm1}\right\}\tag{26}$$

We identify

$$u_{l,\pm1}=\frac{\alpha_\pm(l)}{2}h_l^{(1)}\qquad\qquad w_{l,\pm1}=\frac{\beta_\pm(l)}{2}h_l^{(1)}\tag{27}$$

giving

$$\begin{aligned}\mathrm{Im}\left(u_{l,\pm1}u^{*\prime}_{l,\pm1}-w'_{l,\pm1}w^*_{l,\pm1}\right) &= \frac{1}{4}\mathrm{Im}\left\{|\alpha_\pm(l)|^2h_l^{(1)}h_l^{(2)\prime}-|\beta_\pm(l)|^2h_l^{(1)\prime}h_l^{(2)}\right\}\\ &= \frac{1}{4}\mathrm{Im}\left\{\left[|\alpha_\pm(l)|^2+|\beta_\pm(l)|^2\right]h_l^{(1)}h_l^{(2)\prime}-\overbrace{|\beta_\pm(l)|^2\left[h_l^{(1)}h_l^{(2)}\right]'}^{\text{real}}\right\}\\ &= \frac{1}{4}\mathrm{Im}\left\{\left[|\alpha_\pm(l)|^2+|\beta_\pm(l)|^2\right](j_l+in_l)(j'_l-in'_l)\right\}\\ &= -\frac{1}{4}\left[|\alpha_\pm(l)|^2+|\beta_\pm(l)|^2\right]W(j_l,n_l) \qquad \text{by (9.91)}\\ &= -\left[\frac{|\alpha_\pm(l)|^2+|\beta_\pm(l)|^2}{4k^2a^2}\right]\end{aligned}\tag{28}$$

The total scattered power (10.58) is the negative of (24)

$$\begin{aligned}P_{\text{sc}} &= -\frac{a^2}{2\mu_0 c}\cdot\sum_{l=1}^{\infty}4\pi(2l+1)\left\{-\left[\frac{|\alpha_\pm(l)|^2+|\beta_\pm(l)|^2}{4k^2a^2}\right]\right\}\\ &= \frac{1}{\mu_0 c}\cdot\frac{\pi}{2k^2}\sum_{l=1}^{\infty}(2l+1)\left[|\alpha_\pm(l)|^2+|\beta_\pm(l)|^2\right]\end{aligned}\tag{29}$$

Finally, dividing this by the incident flux $1/\mu_0 c$ gives the total scattering cross section

$$\sigma_{\text{sc}} = \frac{\pi}{2k^2}\sum_{l=1}^{\infty}(2l+1)\left[|\alpha_\pm(l)|^2 + |\beta_\pm(l)|^2\right] \tag{30}$$

(Noate, the $\pm$ is understood to be a summation of both signs, with $\alpha_\pm, \beta_\pm$ to be determined by the polarization of the incident wave, as well as boundary conditions)

(b) For the absorbed power, by definition (10.59)

$$P_{\text{abs}} = \frac{a^2}{2\mu_0}\operatorname{Re}\int_{r=a}\mathbf{E}\cdot(\mathbf{n}\times\mathbf{B}^*)\,d\Omega \tag{31}$$

where $\mathbf{E}, \mathbf{B}$ are the total fields, i.e., incident plus scattered fields, just outside the sphere.
Given the multipole expansion of the incident plane wave (10.55),

$$\mathbf{E}_{\text{inc}}(\mathbf{x}) = \sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\left\{j_l(kr)\mathbf{X}_{l,\pm 1} \pm \frac{1}{k}\boldsymbol{\nabla}\times\left[j_l(kr)\mathbf{X}_{l,\pm 1}\right]\right\} \tag{32}$$

$$\mathbf{B}_{\text{inc}}(\mathbf{x}) = \frac{1}{c}\sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\left\{-\frac{i}{k}\boldsymbol{\nabla}\times\left[j_l(kr)\mathbf{X}_{l,\pm 1}\right] \mp i j_l(r)\mathbf{X}_{l,\pm 1}\right\} \tag{33}$$

Adding (25), (26) to (32), (33) gives the total fields in the general form (17), (18) with

$$u_{l,\pm 1} = j_l + \frac{\alpha_\pm(l)}{2}h_l^{(1)} = \left[1 + \frac{\alpha_\pm(l)}{2}\right]j_l + i\frac{\alpha_\pm(l)}{2}n_l \tag{34}$$

$$w_{l,\pm 1} = j_l + \frac{\beta_\pm(l)}{2}h_l^{(1)} = \left[1 + \frac{\beta_\pm(l)}{2}\right]j_l + i\frac{\beta_\pm(l)}{2}n_l \tag{35}$$

Then

$$\begin{aligned}
\operatorname{Im}\left(u_{l,\pm 1}u_{l,\pm 1}^{*\prime}\right) &= \operatorname{Im}\left\{\left[\left(1+\frac{\alpha_\pm}{2}\right)j_l + i\frac{\alpha_\pm}{2}n_l\right]\left[\left(1+\frac{\alpha_\pm^*}{2}\right)j_l' - i\frac{\alpha_\pm^*}{2}n_l'\right]\right\} && j_l j_l', n_l n_l' \text{ terms are real} \\
&= \operatorname{Im} i\left[\overbrace{\frac{\alpha_\pm}{2}\left(1+\frac{\alpha_\pm^*}{2}\right)}^{\lambda} j_l' n_l - \overbrace{\frac{\alpha_\pm^*}{2}\left(1+\frac{\alpha_\pm}{2}\right)}^{\lambda^*} j_l n_l'\right] && j_l n_l' = j_l' n_l + \frac{1}{k^2a^2} \text{ by (9.91)} \\
&= \operatorname{Im} i\left[\overbrace{(\lambda-\lambda^*)\,j_l' n_l}^{\text{imaginary}} - \frac{\lambda^*}{k^2a^2}\right] \\
&= -\frac{\operatorname{Re}\lambda}{k^2a^2} = -\left(\frac{|\alpha_\pm|^2 + 2\operatorname{Re}\alpha_\pm}{4k^2a^2}\right) = \frac{1-|\alpha_\pm+1|^2}{4k^2a^2}
\end{aligned} \tag{36}$$

Similarly, we have

$$-\operatorname{Im}\left(w_{l,\pm 1}' w_{l,\pm 1}^*\right) = \operatorname{Im}\left(w_{l,\pm 1}w_{l,\pm 1}^{*\prime}\right) = \frac{1-|\beta_\pm+1|^2}{4k^2a^2} \tag{37}$$

giving

$$\begin{aligned}
P_{\text{abs}} &= \frac{a^2}{2\mu_0 c}\sum_{l=1}^{\infty}4\pi(2l+1)\left[\frac{1-|\alpha_\pm(l)+1|^2}{4k^2a^2} + \frac{1-|\beta_\pm(l)+1|^2}{4k^2a^2}\right] \\
&= \frac{1}{\mu_0 c}\cdot\frac{\pi}{2k^2}\sum_{l=1}^{\infty}(2l+1)\left[2 - |\alpha_\pm(l)+1|^2 - |\beta_\pm(l)+1|^2\right]
\end{aligned} \tag{38}$$

and

$$\sigma_{\text{abs}} = \frac{\pi}{2k^2}\sum_{l=1}^{\infty}(2l+1)\left[2 - |\alpha_\pm(l)+1|^2 - |\beta_\pm(l)+1|^2\right] \tag{39}$$