1. **Prob 6.21**

(a) We can consider the dipole as the limit of a pair of charges $\pm q$ located at $\mathbf{r}_0 \pm \mathbf{l}/2$, with $l \to 0$ while keeping $q\mathbf{l} = \mathbf{p}$. Thus the charge density is

$$\begin{aligned}\rho(\mathbf{x}) &= \lim_{l\to 0}\left\{q\delta\left[\mathbf{x}-\left(\mathbf{r}_0+\frac{\mathbf{l}}{2}\right)\right]-q\delta\left[\mathbf{x}-\left(\mathbf{r}_0-\frac{\mathbf{l}}{2}\right)\right]\right\} \\ &= \lim_{l\to 0} q\left\{\delta\left[(\mathbf{x}-\mathbf{r}_0)-\frac{\mathbf{l}}{2}\right]-\delta\left[(\mathbf{x}-\mathbf{r}_0)+\frac{\mathbf{l}}{2}\right]\right\} \\ &= \lim_{l\to 0} q\mathbf{l}\cdot[-\nabla\delta(\mathbf{x}-\mathbf{r}_0)] = -\mathbf{p}\cdot\nabla\delta(\mathbf{x}-\mathbf{r}_0)\end{aligned} \tag{1}$$

And from $\mathbf{J}(\mathbf{x},t) = \rho(\mathbf{x},t)\mathbf{v}$,

$$\mathbf{J}(\mathbf{x},t) = -\mathbf{v}[\mathbf{p}\cdot\nabla\delta(\mathbf{x}-\mathbf{r}_0)] \tag{2}$$

(b) For the magnetic dipole, applying the equation above (5.58)

$$\mathbf{m} = \frac{1}{2}\sum_i q_i(\mathbf{x}_i\times\mathbf{v}_i) \tag{3}$$

on the dipole $\mathbf{p}$, we have

$$\mathbf{m} = \frac{1}{2}\left[q\left(\mathbf{r}_0+\frac{\mathbf{l}}{2}\right)\times\mathbf{v}-q\left(\mathbf{r}_0-\frac{\mathbf{l}}{2}\right)\times\mathbf{v}\right] = \frac{1}{2}\mathbf{p}\times\mathbf{v} \tag{4}$$

For the electric quadrupole, applying (4.9)

$$Q_{ij} = \int\left(3x'_i x'_j - r'^2\delta_{ij}\right)\rho(\mathbf{x}')\,d^3x' \tag{5}$$

to $\mathbf{p}$, we have

$$\begin{aligned}Q_{ij} &= \int\left(3x'_ix'_j-r'^2\delta_{ij}\right)q\left[\delta\left(\mathbf{x}'-\mathbf{r}_0-\frac{\mathbf{l}}{2}\right)-\delta\left(\mathbf{x}'-\mathbf{r}_0+\frac{\mathbf{l}}{2}\right)\right]d^3x' \\ &= 3q\left[\left(r_{0i}+\frac{l_i}{2}\right)\left(r_{0j}+\frac{l_j}{2}\right)-3\left(r_{0i}-\frac{l_i}{2}\right)\left(r_{0j}-\frac{l_j}{2}\right)\right]-\delta_{ij}q\left(\left|\mathbf{r}_0+\frac{\mathbf{l}}{2}\right|^2-\left|\mathbf{r}_0-\frac{\mathbf{l}}{2}\right|^2\right) \\ &= 3q\left(r_{0i}l_j+r_{0j}l_i\right)-\delta_{ij}q\left[\left(r_0^2+\frac{l^2}{4}+\mathbf{r}_0\cdot\mathbf{l}\right)-\left(r_0^2+\frac{l^2}{4}-\mathbf{r}_0\cdot\mathbf{l}\right)\right] \\ &= 3\left(r_{0i}p_j+r_{0j}p_i\right)-2\delta_{ij}\mathbf{r_0}\cdot\mathbf{p}\end{aligned} \tag{6}$$

(c) By equation (4.10), the contribution to the electric potential from the quadrupole is

$$\Phi^{(2)}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\cdot\frac{1}{2}\sum_{ij}Q_{ij}\frac{x_ix_j}{r^5} \tag{7}$$

Thus the second order electric field is

$$\begin{aligned}4\pi\epsilon_0\cdot\mathbf{E}^{(2)}(\mathbf{x}) &= -\frac{1}{2}\nabla\left(\sum_{ij}Q_{ij}\frac{x_ix_j}{r^5}\right) \\ &= -\frac{1}{2}\sum_\alpha\hat{\mathbf{e}}_\alpha\sum_{ij}Q_{ij}\frac{\partial}{\partial x_\alpha}\left(\frac{x_ix_j}{r^5}\right) \\ &= -\frac{1}{2}\sum_\alpha\hat{\mathbf{e}}_\alpha\sum_{ij}Q_{ij}\left(\frac{\delta_{i\alpha}x_j}{r^5}+\frac{x_i\delta_{\alpha j}}{r^5}-\frac{5x_ix_jx_\alpha}{r^7}\right) \\ &= -\frac{1}{2}\sum_\alpha\hat{\mathbf{e}}_\alpha\left[3\overbrace{\sum_{ij}\left(r_{0i}p_j+r_{0j}p_i\right)\left(\frac{\delta_{i\alpha}x_j}{r^5}+\frac{x_i\delta_{\alpha j}}{r^5}\right)}^{A}-2(\mathbf{r}_0\cdot\mathbf{p})\overbrace{\sum_{ij}\delta_{ij}\left(\frac{\delta_{i\alpha}x_j}{r^5}+\frac{x_i\delta_{\alpha j}}{r^5}\right)}^{B}\right. \\ &\qquad\left.-15\underbrace{\sum_{ij}\left(r_{0i}p_j+r_{0j}p_i\right)\cdot\frac{x_ix_jx_\alpha}{r^7}}_{C}+10(\mathbf{r}_0\cdot\mathbf{p})\underbrace{\sum_{ij}\delta_{ij}\frac{x_ix_jx_\alpha}{r^7}}_{D}\right]\end{aligned} \tag{8}$$

where

$$A = 2\left(r_{0\alpha}\frac{\mathbf{p}\cdot\mathbf{n}}{r^4} + p_\alpha\frac{\mathbf{r}_0\cdot\mathbf{n}}{r^4}\right) \tag{9}$$

$$B = \frac{2x_\alpha}{r^5} \tag{10}$$

$$C = \frac{2x_\alpha(\mathbf{p}\cdot\mathbf{n})(\mathbf{r}_0\cdot\mathbf{n})}{r^5} \tag{11}$$

$$D = \frac{x_\alpha}{r^5} \tag{12}$$

Putting everything back to (8) yields

$$4\pi\epsilon_0\cdot\mathbf{E}^{(2)}(\mathbf{x}) = -\frac{3[\mathbf{r}_0(\mathbf{p}\cdot\mathbf{n}) + \mathbf{p}(\mathbf{r}_0\cdot\mathbf{n})]}{r^4} - \frac{3\mathbf{n}(\mathbf{r}_0\cdot\mathbf{p})}{r^4} + \frac{15\mathbf{n}(\mathbf{p}\cdot\mathbf{n})(\mathbf{r}_0\cdot\mathbf{n})}{r^4} \tag{13}$$

2. **Prob 6.22**

(a) The vector potential due to the moving dipole $\mathbf{p}$ is

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\
&= -\frac{\mu_0\mathbf{v}}{4\pi}\left[\mathbf{p}\cdot\int\frac{\nabla'\delta(\mathbf{x}'-\mathbf{r}_0)}{|\mathbf{x}-\mathbf{x}'|}d^3x'\right] && \text{integration by parts and } \nabla' \leftrightarrow -\nabla \\
&= -\frac{\mu_0\mathbf{v}}{4\pi}\left[\mathbf{p}\cdot\int\delta(\mathbf{x}'-\mathbf{r}_0)\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3x'\right] \\
&= -\frac{\mu_0\mathbf{v}}{4\pi}\left[\mathbf{p}\cdot\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{r}_0|}\right)\right] && \text{let } \mathbf{r}\equiv\mathbf{x}-\mathbf{r}_0 \\
&= \frac{\mu_0}{4\pi}\frac{\mathbf{v}(\mathbf{p}\cdot\mathbf{n})}{r^2}
\end{aligned} \tag{14}$$

The second form

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\left[\frac{1}{2}\frac{(\mathbf{p}\times\mathbf{v})\times\mathbf{r}}{r^3} + \frac{1}{2}\frac{\mathbf{p}(\mathbf{r}\cdot\mathbf{v})+\mathbf{v}(\mathbf{r}\cdot\mathbf{p})}{r^3}\right] \tag{15}$$

follows straightforwardly from the vector identity

$$(\mathbf{b}\times\mathbf{c})\times\mathbf{a} = (\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{a}\cdot\mathbf{c})\mathbf{b} \tag{16}$$

(b) Using the vector identity

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a} \tag{17}$$

we obtain the symmetric magnetic field

$$\begin{aligned}
\mathbf{B}_{\text{sym}} &= \frac{\mu_0}{8\pi}\nabla\times\left[\frac{\mathbf{p}(\mathbf{r}\cdot\mathbf{v})+\mathbf{v}(\mathbf{r}\cdot\mathbf{p})}{r^3}\right] \\
&= \frac{\mu_0}{8\pi}\left[\nabla\left(\frac{\mathbf{r}\cdot\mathbf{v}}{r^3}\right)\times\mathbf{p} + \nabla\left(\frac{\mathbf{r}\cdot\mathbf{p}}{r^3}\right)\times\mathbf{v}\right]
\end{aligned} \tag{18}$$

Note that

$$\nabla\left(\frac{\mathbf{r}\cdot\mathbf{v}}{r^3}\right) = \sum_i\hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}\left(\sum_j\frac{r_jv_j}{r^3}\right) = \sum_{ij}\hat{\mathbf{e}}_iv_j\left(\frac{\delta_{ij}}{r^3} - \frac{3r_ir_j}{r^5}\right) = \frac{\mathbf{v}}{r^3} - \frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{v})}{r^3} \tag{19}$$

and similarly

$$\nabla\left(\frac{\mathbf{r}\cdot\mathbf{p}}{r^3}\right) = \frac{\mathbf{p}}{r^3} - \frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})}{r^3} \tag{20}$$

Then it is clear

$$\mathbf{B}_{\text{sym}} = -\frac{3\mu_0}{8\pi r^3}\mathbf{n}\times[\mathbf{p}(\mathbf{n}\cdot\mathbf{v}) + \mathbf{v}(\mathbf{n}\cdot\mathbf{p})] \tag{21}$$

(c) Let's compute the curl of one term in (21), after which the full curl can be obtained using the $\mathbf{p} \longleftrightarrow \mathbf{v}$ symmetry. With (17),

$$\nabla \times \left[(\mathbf{n} \times \mathbf{p}) \left(\frac{\mathbf{n} \cdot \mathbf{v}}{r^3} \right) \right] = \nabla \left(\frac{\mathbf{n} \cdot \mathbf{v}}{r^3} \right) \times (\mathbf{n} \times \mathbf{p}) + \left(\frac{\mathbf{n} \cdot \mathbf{v}}{r^3} \right) \nabla \times (\mathbf{n} \times \mathbf{p}) \tag{22}$$

With the identity

$$\nabla (\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) \tag{23}$$

we have

$$\begin{aligned}
\nabla \left(\frac{\mathbf{n} \cdot \mathbf{v}}{r^3} \right) &= (\mathbf{v} \cdot \nabla) \left(\frac{\mathbf{n}}{r^3} \right) + \mathbf{v} \times \overbrace{\left[\nabla \times \left(\frac{\mathbf{n}}{r^3} \right) \right]}^{=0} \\
&= \sum_i \left(v_i \frac{\partial}{\partial x_i} \right) \left(\sum_j \hat{\mathbf{e}}_j \frac{r_j}{r^4} \right) \\
&= \sum_{ij} v_i \hat{\mathbf{e}}_j \left(\frac{\delta_{ij}}{r^4} - \frac{4 r_i r_j}{r^6} \right) \\
&= \frac{\mathbf{v}}{r^4} - \frac{4 \mathbf{n} (\mathbf{n} \cdot \mathbf{v})}{r^4}
\end{aligned} \tag{24}$$

Thus the first term of the RHS of (22) becomes

$$\begin{aligned}
\nabla \left(\frac{\mathbf{n} \cdot \mathbf{v}}{r^3} \right) \times (\mathbf{n} \times \mathbf{p}) &= \frac{\mathbf{v} \times (\mathbf{n} \times \mathbf{p})}{r^4} - \frac{4 \mathbf{n} \cdot \mathbf{v}}{r^4} [\mathbf{n} \times (\mathbf{n} \times \mathbf{p})] \\
&= \frac{[(\mathbf{v} \cdot \mathbf{p}) \mathbf{n} - (\mathbf{v} \cdot \mathbf{n}) \mathbf{p}]}{r^4} - \frac{4 \mathbf{n} \cdot \mathbf{v}}{r^4} [(\mathbf{n} \cdot \mathbf{p}) \mathbf{n} - \mathbf{p}] \\
&= \frac{1}{r^4} [(\mathbf{v} \cdot \mathbf{p}) \mathbf{n} + 3 (\mathbf{v} \cdot \mathbf{n}) \mathbf{p} - 4 (\mathbf{n} \cdot \mathbf{v}) (\mathbf{n} \cdot \mathbf{p}) \mathbf{n}]
\end{aligned} \tag{25}$$

Also using the identity

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} (\nabla \cdot \mathbf{b}) - \mathbf{b} (\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b} \tag{26}$$

we have

$$\begin{aligned}
\nabla \times (\mathbf{n} \times \mathbf{p}) &= -\mathbf{p} (\nabla \cdot \mathbf{n}) + (\mathbf{p} \cdot \nabla) \mathbf{n} \\
&= -\mathbf{p} \sum_i \frac{\partial}{\partial x_i} \frac{r_i}{r} + \sum_i p_i \frac{\partial}{\partial x_i} \left(\sum_j \hat{\mathbf{e}}_j \frac{r_j}{r} \right) \\
&= -\mathbf{p} \sum_i \left(\frac{1}{r} - \frac{r_i^2}{r^3} \right) + \sum_{ij} p_i \hat{\mathbf{e}}_j \left(\frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3} \right) \\
&= -\frac{\mathbf{p}}{r} - \frac{(\mathbf{p} \cdot \mathbf{n}) \mathbf{n}}{r}
\end{aligned} \tag{27}$$

so the second term of the RHS of (22) becomes

$$\left(\frac{\mathbf{n} \cdot \mathbf{v}}{r^3} \right) \nabla \times (\mathbf{n} \times \mathbf{p}) = \frac{1}{r^4} [-(\mathbf{n} \cdot \mathbf{v}) \mathbf{p} - (\mathbf{n} \cdot \mathbf{v}) (\mathbf{p} \cdot \mathbf{n}) \mathbf{n}] \tag{28}$$

Adding (25) and (28) turns (22) into

$$\nabla \times \left[(\mathbf{n} \times \mathbf{p}) \left(\frac{\mathbf{n} \cdot \mathbf{v}}{r^3} \right) \right] = \frac{1}{r^4} [(\mathbf{v} \cdot \mathbf{p}) \mathbf{n} + 2 (\mathbf{v} \cdot \mathbf{n}) \mathbf{p} - 5 (\mathbf{n} \cdot \mathbf{v}) (\mathbf{n} \cdot \mathbf{p}) \mathbf{n}] \tag{29}$$

The second term of (21) can be obtained by $\mathbf{v} \longleftrightarrow \mathbf{p}$ in (29), which finally gives the curl of $\mathbf{B}_{\text{sym}}$,

$$\nabla \times \mathbf{B}_{\text{sym}} = \frac{3\mu_0}{4\pi r^4} [5 (\mathbf{n} \cdot \mathbf{v}) (\mathbf{n} \cdot \mathbf{p}) \mathbf{n} - (\mathbf{v} \cdot \mathbf{p}) \mathbf{n} - (\mathbf{v} \cdot \mathbf{n}) \mathbf{p} - (\mathbf{p} \cdot \mathbf{n}) \mathbf{v}] \tag{30}$$

Comparing (30) with (13), we see that they embodied the Maxwell equation $\nabla \times \mathbf{H}_{\text{sym}} = \partial \mathbf{D}^{(2)} / \partial t$.

(d) The total magnetic field computed from the curl of (14) is

$$\mathbf{B} = \frac{\mu_0}{4\pi} \nabla \times \left[\frac{\mathbf{v} (\mathbf{p} \cdot \mathbf{n})}{r^2} \right] = \frac{\mu_0}{4\pi} \underbrace{\nabla \left(\mathbf{p} \cdot \frac{\mathbf{n}}{r^2} \right)}_{\text{see (20)}} \times \mathbf{v} = \frac{\mu_0}{4\pi} \mathbf{v} \times \frac{[3 \mathbf{n} (\mathbf{n} \cdot \mathbf{p}) - \mathbf{p}]}{r^3} = \frac{\mathbf{v}}{c^2} \times \mathbf{E}_{\text{dipole}} \tag{31}$$