1. Since $a \gg b$, we approximate the current distribution as if it is concentrated at the center of the cross section. Then by (5.37)

$$A_z = \frac{\mu_0}{4\pi} \frac{4Ia}{\sqrt{a^2 + r^2 + 2ar\sin\theta}} \left[\frac{\left(2 - k^2\right) K(k) - 2E(k)}{k^2} \right] \tag{1}$$

where

$$r^2 = (a + \rho\cos\phi)^2 + \rho^2 \sin^2\phi \qquad\qquad r\sin\theta = a + \rho\cos\phi \tag{2}$$

$$k^2 = \frac{4ar\sin\theta}{a^2 + r^2 + 2ar\sin\theta} = \frac{4a(a + \rho\cos\phi)}{4a^2 + \rho^2 + 4a\rho\cos\phi} \tag{3}$$

Note we have changed A's subscript from ϕ to z which represents the direction perpendicular to the paper (as opposed to the direction of increasing ϕ in the problem's figure).

Thus

$$1 - k^2 = \frac{\rho^2}{4a^2 + \rho^2 + 4a\rho\cos\phi} \to \frac{\rho^2}{4a^2} \qquad \text{as } \frac{\rho}{a} \to 0 \tag{4}$$

When $k \to 1$, the elliptic integrals can be approximated by (see wolfram, note on that website, $z = k^2$, see their definition page)

$$K(k) \to -\frac{1}{2}\ln\left(1 - k^2\right) + \ln 4 = \ln\left(\frac{8a}{\rho}\right) \tag{5}$$

$$E(k) \to 1 \tag{6}$$

Now (1) can be approximated as

$$A_z \to \frac{\mu_0}{4\pi}\frac{4Ia}{2a}\left[\ln\left(\frac{8a}{\rho}\right) - 2\right] = \frac{\mu_0 I}{2\pi}\left[\ln\left(\frac{8a}{\rho}\right) - 2\right] \qquad \text{as } \frac{\rho}{a} \to 0 \tag{7}$$

2. Using the "straight wire" approximation, for $\rho < b$, the magnetic induction is given by

$$-\frac{\partial A_z}{\partial \rho} = B_\phi(\rho) = \frac{\mu_0 I \rho^2}{2\pi b^2 \rho} = \frac{\mu_0 I \rho}{2\pi b^2} \tag{8}$$

thus

$$A_z = -\frac{\mu_0 I \rho^2}{4\pi b^2} + C \tag{9}$$

C is determined by matching (7) and (9) at $\rho = b$:

$$C = \frac{\mu_0 I}{2\pi}\left[\ln\left(\frac{8a}{b}\right) - 2\right] + \frac{\mu_0 I}{4\pi} \qquad \Longrightarrow \qquad A_z = \frac{\mu_0 I}{4\pi}\left(1 - \frac{\rho^2}{b^2}\right) + \frac{\mu_0 I}{2\pi}\left[\ln\left(\frac{8a}{b}\right) - 2\right] \tag{10}$$

(**It doesn't seem necessary to impose the continuity of radial derivative at $\rho = b$ per the hint.**)

3. From equation (5.149)

$$\begin{aligned} W &= \frac{1}{2}\int \mathbf{A}\cdot\mathbf{J} d^3x \approx \frac{1}{2}\cdot(2\pi a)\int_0^b \rho\, d\rho \int_0^{2\pi} d\phi \left\{ \frac{\mu_0 I}{4\pi}\left(1 - \frac{\rho^2}{b^2}\right) + \frac{\mu_0 I}{2\pi}\left[\ln\left(\frac{8a}{b}\right) - 2\right]\right\} \cdot \frac{I}{\pi b^2} \\ &= \frac{\mu_0 a I^2}{2b^2}\int_0^b \left(1 - \frac{\rho^2}{b^2}\right)\rho\, d\rho + \frac{\mu_0 a I^2}{2}\left[\ln\left(\frac{8a}{b}\right) - 2\right] \\ &= \frac{\mu_0 a I^2}{2}\left[\ln\left(\frac{8a}{b}\right) - \frac{7}{4}\right] \end{aligned} \tag{11}$$

which gives

$$L = \frac{2W}{I^2} = \mu_0 a\left[\ln\left(\frac{8a}{\rho}\right) - \frac{7}{4}\right] \tag{12}$$

If the current only flows on the surface, we will use (7) for the energy integration only, and

$$W = \frac{\mu_0 a I^2}{2}\left[\ln\left(\frac{8a}{b}\right) - 2\right] \qquad \Longrightarrow \qquad L = \mu_0 a\left[\ln\left(\frac{8a}{b}\right) - 2\right] \tag{13}$$