1. Let R be the rotation that transforms $\hat{\mathbf{z}}$ onto $\mathbf{n}$, the unit normal vector of the loop with spherical angles (θ_0, ϕ_0). R can be achieved by a rotation of θ_0 about y-axis, followed by a rotation of ϕ_0 about the z-axis, i.e., in matrix representation

$$
\begin{aligned}
R = R_z(\phi_0) \cdot R_y(\theta_0) &= \begin{bmatrix} \cos\phi_0 & -\sin\phi_0 & 0 \\ \sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_0 & 0 & \sin\theta_0 \\ 0 & 1 & 0 \\ -\sin\theta_0 & 0 & \cos\theta_0 \end{bmatrix} \\
&= \begin{bmatrix} \cos\phi_0\cos\theta_0 & -\sin\phi_0 & \cos\phi_0\sin\theta_0 \\ \sin\phi_0\cos\theta_0 & \cos\phi_0 & \sin\phi_0\sin\theta_0 \\ -\sin\theta_0 & 0 & \cos\theta_0 \end{bmatrix}
\end{aligned} \tag{1}
$$

Note that the field is a superposition of a "constant" part and a "variable" part

$$
\mathbf{B} = \overbrace{B_0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}^{\mathbf{B}_{\text{const}}} + \overbrace{B_0\beta \begin{bmatrix} y \\ x \\ 0 \end{bmatrix}}^{\mathbf{B}_{\text{var}}} \tag{2}
$$

where the constant part (1st term) has no contribution to the net force on the circular loop due to symmetry.

The points on the loop can be parameterized by an angle $\alpha \in [0, 2\pi]$. I.e., in the rotated coordinate system S' where $\mathbf{n}$ is in the z-direction, a point on the loop, as well as the differential current at this point, are represented by column vectors

$$
\mathbf{x}' = \begin{bmatrix} a\cos\alpha \\ a\sin\alpha \\ 0 \end{bmatrix} \qquad\qquad I d\mathbf{l}' = I a d\alpha \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} \tag{3}
$$

which in the original coordinate system S, have the representations

$$
\mathbf{x} = R\mathbf{x}' = a \begin{bmatrix} \cos\alpha\cos\phi_0\cos\theta_0 - \sin\alpha\sin\phi_0 \\ \cos\alpha\sin\phi_0\cos\theta_0 + \sin\alpha\cos\phi_0 \\ -\cos\alpha\sin\theta_0 \end{bmatrix} \qquad I d\mathbf{l} = IR d\mathbf{l}' = I a d\alpha \begin{bmatrix} -\sin\alpha\cos\phi_0\cos\theta_0 - \cos\alpha\sin\phi_0 \\ -\sin\alpha\sin\phi_0\cos\theta_0 + \cos\alpha\cos\phi_0 \\ \sin\alpha\sin\theta_0 \end{bmatrix} \tag{4}
$$

Thus the variable part of the field at $\mathbf{x}$ is

$$
\mathbf{B}_{\text{var}} = B_0\beta a \begin{bmatrix} \cos\alpha\sin\phi_0\cos\theta_0 + \sin\alpha\cos\phi_0 \\ \cos\alpha\cos\phi_0\cos\theta_0 - \sin\alpha\sin\phi_0 \\ 0 \end{bmatrix} \tag{5}
$$

This gives the net force

$$
\mathbf{F} = \oint_C I d\mathbf{l} \times \mathbf{B}_{\text{var}} = I B_0 \beta a^2 \int_0^{2\pi} \mathbf{f}(\alpha)\, d\alpha \tag{6}
$$

where the components of $\mathbf{f}(\alpha)$ are obtained by taking the cross product of the two column vectors in (4) and (5)

$$
f_x(\alpha) = -\sin\alpha\sin\theta_0 (\cos\alpha\cos\phi_0\cos\theta_0 - \sin\alpha\sin\phi_0) \tag{7}
$$

$$
f_y(\alpha) = \sin\alpha\sin\theta_0 (\cos\alpha\sin\phi_0\cos\theta_0 + \sin\alpha\cos\phi_0) \tag{8}
$$

$$
\begin{aligned}
f_z(\alpha) = &(-\sin\alpha\cos\phi_0\cos\theta_0 - \cos\alpha\sin\phi_0)(\cos\alpha\cos\phi_0\cos\theta_0 - \sin\alpha\sin\phi_0) \\
&-(-\sin\alpha\sin\phi_0\cos\theta_0 + \cos\alpha\cos\phi_0)(\cos\alpha\sin\phi_0\cos\theta_0 + \sin\alpha\cos\phi_0)
\end{aligned} \tag{9}
$$

With the elementary results

$$
\int_0^{2\pi} \sin\alpha\cos\alpha d\alpha = 0 \qquad \text{and} \qquad \int_0^{2\pi} \sin^2\alpha d\alpha = \int_0^{2\pi} \cos^2\alpha d\alpha = \pi \tag{10}
$$

we eventually get

$$\mathbf{F} = IB_0\beta\pi a^2 \begin{bmatrix} \sin\theta_0 \sin\phi_0 \\ \sin\theta_0 \cos\phi_0 \\ 0 \end{bmatrix} \tag{11}$$

On the other hand, with the approximation (5.69), we get

$$\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}) = \boldsymbol{\nabla}\left(I\pi a^2 \begin{bmatrix} \sin\theta_0\cos\phi_0 \\ \sin\theta_0\sin\phi_0 \\ \cos\theta_0 \end{bmatrix} \cdot \begin{bmatrix} B_0 + B_0\beta y \\ B_0 + B_0\beta x \\ 0 \end{bmatrix} \right) = IB_0\beta\pi a^2 \begin{bmatrix} \sin\theta_0\sin\phi_0 \\ \sin\theta_0\cos\phi_0 \\ 0 \end{bmatrix} \tag{12}$$

It is the same as the exact solution becase our given field $\mathbf{B}$'s Taylor expansion has only up to the 1st derivative in equation (5.65), therefore the force in (5.66) only has one contribution from the dipole term, which gives rise to the lowest order formula $\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B})$, in this case, an exact formula.

2. The lowest order term of the torque is given by equation (5.71)

$$\mathbf{N} = \mathbf{m}\times\mathbf{B}(0) = I\pi a^2 \begin{bmatrix} \sin\theta_0\cos\phi_0 \\ \sin\theta_0\sin\phi_0 \\ \cos\theta_0 \end{bmatrix} \times \begin{bmatrix} B_0 \\ B_0 \\ 0 \end{bmatrix} = I\pi a^2 B_0 \begin{bmatrix} -\cos\theta_0 \\ \cos\theta_0 \\ \sin\theta_0\cos\phi_0 - \sin\theta_0\sin\phi_0 \end{bmatrix} \tag{13}$$

To see the contribution from the higher orders, we go back to the expansion of $\mathbf{B}$

$$B_k(\mathbf{x}) = B_k(0) + \mathbf{x}\cdot\boldsymbol{\nabla}B_k(0) + \cdots \tag{14}$$

where for the given $\mathbf{B}$ of the problem, we only have up to the second term, which is the 1st order field $\mathbf{B}_{\text{var}}$ above. Thus the contribution to torque from $\mathbf{B}_{\text{var}}$ is

$$\mathbf{N}_{\text{var}} = \int \mathbf{x}\times(\mathbf{J}\times\mathbf{B}_{\text{var}})\, d^3x \tag{15}$$

where the integrand

$$\mathbf{x}\times(\mathbf{J}\times\mathbf{B}_{\text{var}}) \propto \begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \left(\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} \times \begin{bmatrix} y \\ x \\ 0 \end{bmatrix} \right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} -xJ_z \\ yJ_z \\ xJ_x - yJ_y \end{bmatrix} = \begin{bmatrix} xyJ_x - y^2J_y - yzJ_z \\ -xzJ_z - x^2J_x + xyJ_y \\ 2xyJ_z \end{bmatrix} \tag{16}$$

Since the current density is distributed along the circular loop, the integration in (15) is actually a line integral along the tilted circle centered at the origin. We can see that the integral of each term (for example xyJ_y) in the final column of (16) will be zero. This is because for a pair of antipodal points $\mathbf{x}$ and $-\mathbf{x}$ on the circle, their current components have opposite signs (e.g., $J_y(\mathbf{x}) = -J_y(-\mathbf{x})$), but their coordinate products have the same value (e.g., $xy = (-x)(-y)$), thus their contributions to the line integral cancel each other.

The vanishing contribution from $\mathbf{B}_{\text{var}}$ seems to be particular to this highly symmetric setup. If the loop's shape is not perfect circle, or the center is not at the origin, the symmetry argument above will not be applicable, so we don't expect the integral of (15) to vanish in general.

For a concrete counter example, consider the current flowing in a square wire as shown below. The integration of the z-component of (16) (i.e., $2xyJ_z$) along the wire is apparently non-zero, which means the 1st order torque $\mathbf{N}_{\text{var}}$ at least has a non-zero z-component.