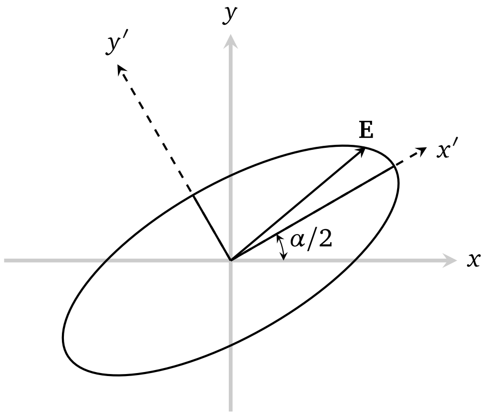
1. **Polarization Ellipse**



Here we shall explicitly prove that for the complex plane wave

$$\mathbf{E}(\mathbf{x},t) = (E_+\boldsymbol{\epsilon}_+ + E_-\boldsymbol{\epsilon}_-)\,e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \tag{1}$$

where

$$\frac{E_-}{E_+} = re^{i\alpha} \tag{2}$$

$$\boldsymbol{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2) \tag{3}$$

the electric field at a fixed point, say $\mathbf{x}=0$, traces out an ellipse as shown in the figure above.
Indeed, the complex electric field $\mathbf{E}$ for $\mathbf{x}=0$ is

$$\begin{aligned}\mathbf{E} &= (E_+\boldsymbol{\epsilon}_+ + E_-\boldsymbol{\epsilon}_-)\,e^{-i\omega t}\\ &= \frac{1}{\sqrt{2}}E_+\left[\boldsymbol{\epsilon}_1\left(1+re^{i\alpha}\right)+i\boldsymbol{\epsilon}_2\left(1-re^{i\alpha}\right)\right]e^{-i\omega t}\end{aligned} \tag{4}$$

As the figure above shows, when we rotate the coordinate system by $\alpha/2$, the coordinates will undergo a transform

$$\begin{bmatrix}x\\y\end{bmatrix} \quad\longrightarrow\quad \begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}\cos\frac{\alpha}{2} & \sin\frac{\alpha}{2}\\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} \tag{5}$$

Identifying $\boldsymbol{\epsilon}_1$ with $\hat{\mathbf{x}}$ and $\boldsymbol{\epsilon}_2$ with $\hat{\mathbf{y}}$, the complex components of the field $\mathbf{E}$ will undergo a transform

$$\begin{aligned}\begin{bmatrix}E_x\\E_y\end{bmatrix} \quad\longrightarrow\quad \begin{bmatrix}E'_x\\E'_y\end{bmatrix} &= \begin{bmatrix}\cos\frac{\alpha}{2} & \sin\frac{\alpha}{2}\\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2}\end{bmatrix}\begin{bmatrix}E_x\\E_y\end{bmatrix} = \frac{1}{\sqrt{2}}E_+e^{-i\omega t}\begin{bmatrix}\cos\frac{\alpha}{2} & \sin\frac{\alpha}{2}\\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2}\end{bmatrix}\begin{bmatrix}1+re^{i\alpha}\\ i(1-re^{i\alpha})\end{bmatrix}\\ &= \frac{1}{\sqrt{2}}E_+e^{-i\omega t}\begin{bmatrix}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)+re^{i\alpha}\left(\cos\frac{\alpha}{2}-i\sin\frac{\alpha}{2}\right)\\ i\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)-ire^{i\alpha}\left(\cos\alpha-i\sin\frac{\alpha}{2}\right)\end{bmatrix}\\ &= \frac{1}{\sqrt{2}}E_+e^{-i\omega t}e^{i\alpha/2}\begin{bmatrix}1+r\\ i(1-r)\end{bmatrix}\end{aligned} \tag{6}$$

It will give more insights to inspect the real parts of (6), which are

$$\operatorname{Re}E'_x = \frac{(1+r)E_+}{\sqrt{2}}\cos\left(\omega t-\frac{\alpha}{2}\right) \tag{7}$$

$$\operatorname{Re}E'_y = \frac{(1-r)E_+}{\sqrt{2}}\sin\left(\omega t-\frac{\alpha}{2}\right) \tag{8}$$

Apparently this traces out an ellipse with semimajor axis $(1+r)E_+/\sqrt{2}$ and semiminor axis $(1-r)E_+/\sqrt{2}$.

2. **Stokes Parameters**

We consider the same complex electric field vector $\mathbf{E}$ expressed in the $\boldsymbol{\epsilon}_1/\boldsymbol{\epsilon}_2$ basis and $\boldsymbol{\epsilon}_+/\boldsymbol{\epsilon}_-$ basis.

$$\mathbf{E} = E_1\boldsymbol{\epsilon}_1 + E_2\boldsymbol{\epsilon}_2 = E_+\boldsymbol{\epsilon}_+ + E_-\boldsymbol{\epsilon}_- \tag{9}$$

Let

$$E_1 = a_1 e^{i\delta_1} \qquad E_2 = a_2 e^{i\delta_2} \tag{10}$$

$$E_+ = a_+ e^{i\delta_+} \qquad E_- = a_- e^{i\delta_-} \tag{11}$$

Then under the $\boldsymbol{\epsilon}_1/\boldsymbol{\epsilon}_2$ and $\boldsymbol{\epsilon}_+/\boldsymbol{\epsilon}_-$ basis, the Stokes parameters are

$\boldsymbol{\epsilon}_1/\boldsymbol{\epsilon}_2$ basis $\qquad\qquad$ $\boldsymbol{\epsilon}_+/\boldsymbol{\epsilon}_-$ basis

$$s_0 = |\boldsymbol{\epsilon}_1\cdot\mathbf{E}|^2 + |\boldsymbol{\epsilon}_2\cdot\mathbf{E}|^2 = a_1^2 + a_2^2 \qquad s_0' = \left|\boldsymbol{\epsilon}_+^*\cdot\mathbf{E}\right| + \left|\boldsymbol{\epsilon}_-^*\cdot\mathbf{E}\right|^2 = a_+^2 + a_-^2 \tag{12}$$

$$s_1 = |\boldsymbol{\epsilon}_1\cdot\mathbf{E}|^2 - |\boldsymbol{\epsilon}_2\cdot\mathbf{E}|^2 = a_1^2 - a_2^2 \qquad s_1' = 2\,\mathrm{Re}\left[\left(\boldsymbol{\epsilon}_+^*\cdot\mathbf{E}\right)^*\left(\boldsymbol{\epsilon}_-^*\cdot\mathbf{E}\right)\right] = 2a_+a_-\cos(\delta_- - \delta_+) \tag{13}$$

$$s_2 = 2\,\mathrm{Re}\left[(\boldsymbol{\epsilon}_1\cdot\mathbf{E})^*(\boldsymbol{\epsilon}_2\cdot\mathbf{E})\right] = 2a_1a_2\cos(\delta_2-\delta_1) \qquad s_2' = 2\,\mathrm{Im}\left[\left(\boldsymbol{\epsilon}_+^*\cdot\mathbf{E}\right)^*\left(\boldsymbol{\epsilon}_-^*\cdot\mathbf{E}\right)\right] = 2a_+a_-\sin(\delta_- - \delta_+) \tag{14}$$

$$s_3 = 2\,\mathrm{Im}\left[(\boldsymbol{\epsilon}_1\cdot\mathbf{E})^*(\boldsymbol{\epsilon}_2\cdot\mathbf{E})\right] = 2a_1a_2\sin(\delta_2-\delta_1) \qquad s_3' = \left|\boldsymbol{\epsilon}_+^*\cdot\mathbf{E}\right| - \left|\boldsymbol{\epsilon}_-^*\cdot\mathbf{E}\right|^2 = a_+^2 - a_-^2 \tag{15}$$

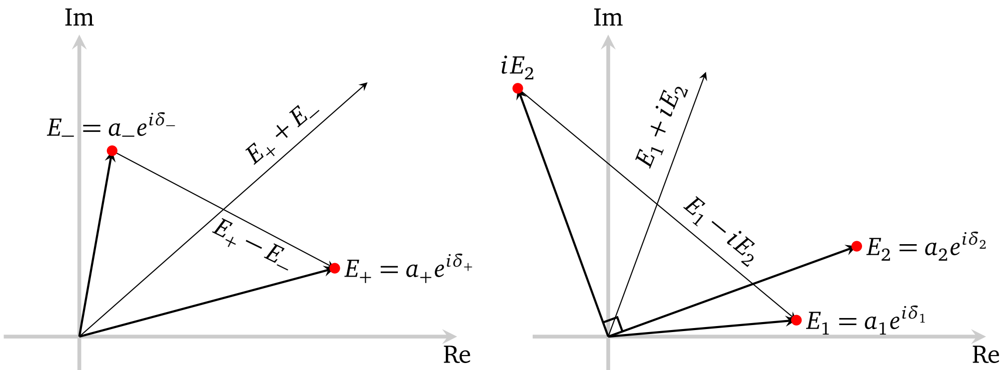
These identities are straightforward to prove given the relations (9)-(11), but their equivalence in two bases is less obvious.

In fact, by (3), we have

$$E_1\boldsymbol{\epsilon}_1 + E_2\boldsymbol{\epsilon}_2 = \frac{E_+}{\sqrt{2}}(\boldsymbol{\epsilon}_1 + i\boldsymbol{\epsilon}_2) + \frac{E_-}{\sqrt{2}}(\boldsymbol{\epsilon}_1 - i\boldsymbol{\epsilon}_2) \implies$$

$$E_1 = \frac{1}{\sqrt{2}}(E_+ + E_-) \qquad E_2 = \frac{i}{\sqrt{2}}(E_+ - E_-) \implies \tag{16}$$

$$E_+ = \frac{1}{\sqrt{2}}(E_1 - iE_2) \qquad E_- = \frac{1}{\sqrt{2}}(E_1 + iE_2) \tag{17}$$



Referring to the diagram above on the left, we see that

$$2a_1^2 = |E_+ + E_-|^2 = a_+^2 + a_-^2 + 2a_+a_-\cos(\delta_- - \delta_+) \tag{18}$$

$$2a_2^2 = |E_+ - E_-|^2 = a_+^2 + a_-^2 - 2a_+a_-\cos(\delta_- - \delta_+) \tag{19}$$

Similarly, referring to the diagram above on the right, we have

$$2a_+^2 = |E_1 - iE_2|^2 = a_1^2 + a_2^2 + 2a_1a_2\sin(\delta_2 - \delta_1) \tag{20}$$

$$2a_-^2 = |E_1 + iE_2|^2 = a_1^2 + a_2^2 - 2a_1a_2\sin(\delta_2 - \delta_1) \tag{21}$$

Adding (18) to (19) will give $s_0 = s_0'$. Subtracting (19) from (18) will give $s_1 = s_1'$. Subtracting (21) from (20) gives $s_3 = s_3'$.

Lastly, from (17), we know

$$E_+ - iE_- = \frac{1-i}{\sqrt{2}}(E_1 + E_2) \tag{22}$$

Using the two diagrams above (with $E_1 \longleftrightarrow E_+$ and $E_2 \longleftrightarrow E_-$), we have

$$a_+^2 + a_-^2 + 2a_+a_-\sin(\delta_- - \delta_+) = |E_+ - iE_-|^2 = |E_1 + E_2|^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\delta_2 - \delta_1) \implies$$

$$s_2 = s_2' \tag{23}$$