1. Recall Jackson's text before equation (4.25), "*The quadrupole moment of a nuclear state is defined as the value of* $(1/e)Q_{33}$...".
   The tracelessness of Q moment
   $$Q_{11}+Q_{22}+Q_{33}=0 \tag{1}$$
   and the cylindrical symmetry together imply
   $$Q_{11}=Q_{22}=-\frac{Q_{33}}{2} \tag{2}$$
   Also for external field, we have
   $$\nabla\cdot\mathbf{E}=0 \quad\Longrightarrow\quad \frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0 \quad\Longrightarrow\quad \frac{\partial E_x}{\partial x}=\frac{\partial E_y}{\partial y}=-\frac{1}{2}\frac{\partial E_z}{\partial z} \tag{3}$$
   Then the energy of quadrupole interaction is (again due to symmetry, all Q_{ij} where $i\neq j$ vanish)
   $$-\frac{1}{6}\sum_{i,j}Q_{ij}\left(\frac{\partial E_j}{\partial x_i}\right)_0=-\frac{1}{6}\left(1+\frac{1}{4}+\frac{1}{4}\right)Q_{33}\left(\frac{\partial E_z}{\partial z}\right)_0=-\frac{eQ}{4}\left(\frac{\partial E_z}{\partial z}\right)_0 \tag{4}$$

2. The desired quantity is
   $$\frac{\left(\frac{\partial E_z}{\partial z}\right)_0}{\frac{e}{4\pi\epsilon_0 a_0^3}}=-\frac{4W}{eQ\frac{e}{4\pi\epsilon_0 a_0^3}}=-\frac{4(W/h)}{\frac{eQ}{2\pi\hbar}\frac{e}{4\pi\epsilon_0 a_0^3}}=-\frac{32(W/h)\pi^2\hbar\epsilon_0 a_0^3}{e^2 Q}\approx -0.085 \tag{5}$$

3. Assuming the spheroid's rotation axis is the semimajor axis (more justifications at the end). Any point in the volume can be parameterized as
   $$x=bt\sin\theta\cos\phi \tag{6}$$
   $$y=bt\sin\theta\sin\phi \tag{7}$$
   $$z=at\cos\theta \tag{8}$$
   where $t\in[0,1],\theta\in[0,\pi],\phi\in[0,2\pi]$.
   Then
   $$\begin{aligned} dxdydz &= \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial t} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} dtd\theta d\phi \\ &= \begin{vmatrix} b\sin\theta\cos\phi & bt\cos\theta\cos\phi & -bt\sin\theta\sin\phi \\ b\sin\theta\sin\phi & bt\cos\theta\sin\phi & bt\sin\theta\cos\phi \\ a\cos\theta & -at\sin\theta & 0 \end{vmatrix} dtd\theta d\phi \\ &= ab^2t^2\sin\theta dtd\theta d\phi \end{aligned} \tag{9}$$
   The quadrupole moment
   $$\begin{aligned} Q=\frac{1}{e}Q_{33} &= \frac{1}{e}\int_{\text{spheroid}}\rho(\mathbf{x})\left(3z^2-r^2\right)dxdydz \\ &= \frac{1}{e}\frac{Ze\cdot\int_{\text{spheroid}}\left(2z^2-x^2-y^2\right)dxdydz}{\int_{\text{spheroid}}dxdydz} \end{aligned} \tag{10}$$

can be calculated with (9):

$$\int_{\text{spheroid}} \left(2z^2 - x^2 - y^2\right) dx\,dy\,dz = ab^2 \int_0^1 t^2 dt \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi \left(2a^2 t^2 \cos^2\theta - b^2 t^2 \sin^2\theta\right)$$
$$= ab^2 \left(\frac{2\pi}{5}\right)\left(\frac{4}{3}\right)\left(a^2 - b^2\right) \tag{11}$$
$$\int_{\text{spheroid}} dx\,dy\,dz = ab^2 \int_0^1 t^2 dt \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{4\pi}{3} ab^2 \tag{12}$$

This gives

$$Q = \frac{2Z}{5}\left(a^2 - b^2\right) \tag{13}$$

Now

$$\frac{a-b}{R} = \frac{a^2 - b^2}{(a+b)R} = \frac{a^2 - b^2}{2R^2} = \frac{5Q}{4ZR^2} \approx 0.10123 \tag{14}$$

Coming back to the earlier assumption that the spheroid is rotated about its semimajor axis. If it's the opposite (rotation about the semiminor axis), we have a and b switched, in which case Q will be a negative quantity. Our result of $(a-b)/R$ still holds assuming the given Q is the absolute value.