1. **Lemmas**

Before we get into this specific problem, let's mention and prove some results upfront.

First, let's recall the following integral over the $[0,1]$ domain (reference wolfram equation (51), (52)):

$$\int_0^1 P_m(x)P_n(x)dx = \begin{cases} \frac{1}{2n+1} & m=n \\ 0 & m \neq n, m, n \text{ both even or odd} \\ f_{m,n} & m \text{ even}, n \text{ odd} \\ f_{n,m} & m \text{ odd}, n \text{ even} \end{cases} \tag{1}$$

where

$$f_{m,n} = \frac{(-1)^{(m+n+1)/2} m!n!}{2^{m+n-1}(m-n)(m+n+1)\left[\left(\frac{m}{2}\right)!\right]^2 \left[\left(\frac{n-1}{2}\right)!\right]^2} \tag{2}$$

Next, let's prove the following lemmas.

Lemma 1. *The set of even Legendre polynomials, when restricted to the half interval* $[0,1]$*, forms a complete orthogonal set of basis for all functions over domain* $[0,1]$*.*

Proof. Let $g(x)$ be any function over the domain $[0,1]$, we can extend $g(x)$ to a function $h(x)$ over the domain $[-1,1]$ by defining $h(x) = g(-x)$ for $-1 \le x \le 0$. Essentially we mirror $g(x)$ across the y-axis.

By the completeness and orthonormality of the Legendre polynomials over their usual domain $[-1,1]$, we have (see Jackson (3.23),(3.24))

$$h(x) = \sum_{l=0}^{\infty} A_l P_l(x) \qquad \text{where} \tag{3}$$

$$A_l = \frac{2l+1}{2}\int_{-1}^{1} h(x)P_l(x)dx \tag{4}$$

All A_{2l+1}'s must vanish since $h(x)$ is even, but $P_{2l+1}(x)$ is odd. Restricting (3) to the half interval $[0,1]$, we have

$$g(x) = \sum_{l=0}^{\infty} A_{2l} P_{2l}(x) \qquad \text{where} \tag{5}$$

$$A_{2l} = \frac{4l+1}{2}\int_{-1}^{1} h(x)P_{2l}(x)dx = (4l+1)\int_0^1 g(x)P_{2l}(x)dx \tag{6}$$

The fact that $P_{2l}(x)$'s, when restricted to $[0,1]$, are orthogonal, is apparent due to their orthogonality in the full domain $[-1,1]$ and even parity. □

Lemma 2. *The set of odd Legendre polynomials, when restricted to the half interval* $[0,1]$*, forms a complete orthogonal set of basis for all functions over domain* $[0,1]$ *that vanish at* $x=0$*.*

Proof. The proof is similar to Lemma 1. Let $g(x)$ be any function over the domain $[0,1]$ with vanishing value at $x=0$, we can extend $g(x)$ to an odd function $h(x)$ over $[-1,1]$ by defining $h(x) = -g(-x)$ for $-1 \le x \le 0$. Similar arguments will give

$$g(x) = \sum_{l=0}^{\infty} A_{2l+1} P_{2l+1}(x) \qquad \text{where} \tag{7}$$

$$A_{2l+1} = (4l+3)\int_0^1 g(x)P_{2l+1}(x)dx \tag{8}$$

The orthogonality of $P_{2l+1}(x)$'s on $[0,1]$ is ensured by (1). □

Even though only Lemma 1 is used in the following, we present both lemmas here to emphasize the point that over the half interval $[0,1]$, even and odd Legendre polynomials form independent sets of complete orthogonal basis (subject to endpoint requirements such as Lemma 2).

Now comes the solution of Jackson problem 3.3.

2. **Solution to part (a)**

Following the hint, let the charge density at ρ be

$$\sigma(\rho) = \frac{A}{\sqrt{R^2-\rho^2}} \tag{9}$$

We can determine A by requiring the center of the disc to have potential V:

$$\begin{aligned}
V &= \int_0^R \frac{\sigma(\rho)2\pi\rho d\rho}{4\pi\epsilon_0\rho} = \int_0^R \frac{A}{2\epsilon_0}\frac{d\rho}{\sqrt{R^2-\rho^2}} && (\text{let } \rho = R\sin\xi) \\
&= \frac{A}{2\epsilon_0}\int_0^{\pi/2}\frac{R\cos\xi d\xi}{R\cos\xi} = \frac{A\pi}{4\epsilon_0} && \Longrightarrow \qquad (10) \\
A &= \frac{4\epsilon_0 V}{\pi} && (11)
\end{aligned}$$

Now consider a field point on z-axis with distance r from the origin. View the disc as infinite number of rings occupying the radial range $[\rho,\rho+d\rho]$. Following the derivation for Figure 3.4, we have the differential potential contribution from the ring to the field point as

$$\begin{aligned}
d\Phi(z=r) &= \frac{\sigma(\rho)2\pi\rho d\rho}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l\left(\cos\frac{\pi}{2}\right) \\
&= \frac{2V}{\pi}\frac{\rho d\rho}{\sqrt{R^2-\rho^2}}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(0) \\
&= \frac{2V}{\pi}\frac{\rho d\rho}{\sqrt{R^2-\rho^2}}\sum_{l=0}^{\infty}\frac{r_<^{2l}}{r_>^{2l+1}}P_{2l}(0) && (12)
\end{aligned}$$

where $r_<, r_>$ are the smaller and bigger of ρ and r (see Figure 3.4), and where in the last step, all odd terms dropped out because of Legendre polynomial's odd parity.

In this sub problem, we assume $r > R \geq \rho$, so integrating (12) will give

$$\begin{aligned}
\Phi(z=r) &= \int_0^R \frac{2V}{\pi}\frac{\rho d\rho}{\sqrt{R^2-\rho^2}}\sum_{l=0}^{\infty}\frac{\rho^{2l}}{r^{2l+1}}P_{2l}(0) \\
&= \frac{2V}{\pi}\sum_{l=0}^{\infty}P_{2l}(0)\cdot\frac{1}{r^{2l+1}}\int_0^R\frac{\rho^{2l+1}d\rho}{\sqrt{R^2-\rho^2}} \\
&= \frac{2V}{\pi}\sum_{l=0}^{\infty}P_{2l}(0)\cdot\frac{1}{r^{2l+1}}\int_0^{\pi/2}\frac{R^{2l+1}\sin^{2l+1}\xi R\cos\xi d\xi}{R\cos\xi} \\
&= \frac{2V}{\pi}\sum_{l=0}^{\infty}P_{2l}(0)\cdot\frac{R^{2l+1}}{r^{2l+1}}\underbrace{\int_0^{\pi/2}\sin^{2l+1}\xi d\xi}_{I_{2l+1}} && (13)
\end{aligned}$$

I_{2l+1} can be obtained by reduction:

$$\begin{aligned}
I_{2l+1} &= \int_0^{\pi/2}\sin^{2l}\xi\sin\xi d\xi \\
&= (-\cos\xi)\sin^{2l}\xi\Big|_0^{\pi/2} - \int_0^{\pi/2}(-\cos\xi)\cdot 2l\sin^{2l-1}\xi\cos\xi d\xi \\
&= 2l\,(I_{2l-1} - I_{2l+1}) && \Longrightarrow \\
I_{2l+1} &= \left(\frac{2l}{2l+1}\right)I_{2l-1} = \left(\frac{2l}{2l+1}\right)\left(\frac{2l-2}{2l-1}\right)I_{2l-3} = \cdots \\
&= \left(\frac{2l}{2l+1}\right)\left(\frac{2l-2}{2l-1}\right)\cdots\left(\frac{2}{3}\right)I_1 = \frac{(2l)!!}{(2l+1)!!} && (14)
\end{aligned}$$

To calculate $P_{2l}(0)$, note the recurrence relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{15}$$

gives (for $l \geq 1$)

$$\begin{aligned}
(2l)P_{2l}(0) &= 0 - (2l-1)P_{2l-2}(0) \qquad\qquad \Longrightarrow \\
P_{2l}(0) &= \left(-\frac{2l-1}{2l}\right)P_{2l-2}(0) = \left(-\frac{2l-1}{2l}\right)\left(-\frac{2l-3}{2l-2}\right)P_{2l-4}(0) = \cdots \\
&= \left(-\frac{2l-1}{2l}\right)\left(-\frac{2l-3}{2l-2}\right)\cdots\left(-\frac{1}{2}\right)P_0(0) = (-1)^l\frac{(2l-1)!!}{(2l)!!}
\end{aligned} \tag{16}$$

If we adopt the convention $0!! = (-1)!! = 1$, we can write for all $l \geq 0$:

$$P_{2l}(0) = (-1)^l\frac{(2l-1)!!}{(2l)!!} \tag{17}$$

Inserting (14) and (16) back into (13), we have

$$\Phi(z=r) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l+1} \tag{18}$$

Having obtained the z-axis potential in this form, we can apply the uniqueness argument on page 102 (in particular the B_l term in (3.37)) to get the off-axis field points' $(r > R)$ potential by simply multiplying $P_{2l}(\cos\theta)$ to each term, i.e.,

$$\Phi(r > R, \theta) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l+1}P_{2l}(\cos\theta) \tag{19}$$

Note here it doesn't matter whether we have used $z > 0$ or $z < 0$ for on-axis field calculation, since only the even terms enter, the factor $(-1)^{2l}$ factor for the $z < 0$ case would not have made any difference.

3. **Solution to part (b)**

It is much more involved to treat the $r < R$ range, the following is inspired by the solution provided by Stanislav Boldyrev, Physics Department, UW-Madison.

When $r < R$, we must do the integration of (12) in two segments, i.e.,

$$\begin{aligned}
\Phi(z=r<R) &= \int_0^R \frac{2V}{\pi}\frac{\rho\,d\rho}{\sqrt{R^2-\rho^2}}\sum_{l=0}^{\infty}\frac{r_<^{2l}}{r_>^{2l+1}}P_{2l}(0) \\
&= \frac{2V}{\pi}\sum_{l=0}^{\infty}P_{2l}(0)\left[\underbrace{\int_0^r\frac{\rho\,d\rho}{\sqrt{R^2-\rho^2}}\frac{\rho^{2l}}{r^{2l+1}}}_{X_l} + \underbrace{\int_r^R\frac{\rho\,d\rho}{\sqrt{R^2-\rho^2}}\frac{r^{2l}}{\rho^{2l+1}}}_{Y_l}\right]
\end{aligned} \tag{20}$$

We already know that the radial solution of the Laplace equation must be of the form

$$R(r) = \sum_{n=0}^{\infty}A_n r^n + B_n r^{-(n+1)} \tag{21}$$

For our range $r < R$ to include the origin, all B_n's must vanish, that is, if we carry out the two integrations in (20), we shall eventually end up with the series sum

$$\Phi(z=r<R) = \sum_{n=0}^{\infty}A_n r^n \tag{22}$$

Now we will prove a crucial lemma

Lemma 3. *(20)'s power expansion series (22) only contains $n = 0$ and odd n terms.*

Proof. In fact, the $n = 0$ term is easy to determine, it's the limit of (20) as $r \to 0$, in which case $X_l \to 0$ for all l and and $Y_l \to 0$ unless $l = 0$. Thus

$$A_0 = \lim_{r\to 0} \Phi(z = r < R) = \frac{2V}{\pi} P_0(0) \int_0^R \frac{d\rho}{\sqrt{R^2-\rho^2}} = V \tag{23}$$

Now expand X_l into r's power series:

$$\begin{aligned} X_l &= \int_0^r \frac{\rho^{2l+1} d\rho}{r^{2l+1}\sqrt{R^2-\rho^2}} \\ &= \frac{R^{2l+1}}{r^{2l+1}} \int_0^r \left(\frac{\rho}{R}\right)^{2l+1} \frac{1}{R} \frac{1}{\sqrt{1-\left(\frac{\rho}{R}\right)^2}} d\rho \end{aligned} \tag{24}$$

Define $y = \rho/R, y_0 = r/R$, and with the Taylor expansion

$$\frac{1}{\sqrt{1-y^2}} = \sum_{k=0}^{\infty} c_k y^{2k} \qquad\qquad c_k = \frac{(2k-1)!!}{(2k)!!} \tag{25}$$

(24) is turned into

$$\begin{aligned} X_l &= \frac{1}{y_0^{2l+1}} \int_0^{y_0} y^{2l+1} \sum_{k=0}^{\infty} c_k y^{2k} dy \\ &= \sum_{k=0}^{\infty} c_k \frac{1}{y_0^{2l+1}} \frac{y_0^{2l+2k+2}}{2l+2k+2} = \sum_{k=0}^{\infty} c_k \frac{y_0^{2k+1}}{2l+2k+2} \end{aligned} \tag{26}$$

which clearly only contains r's odd power terms.

However if we do the same for Y_l,

$$\begin{aligned} Y_l &= \int_r^R \frac{r^{2l} d\rho}{\rho^{2l}\sqrt{R^2-\rho^2}} \\ &= r^{2l} \int_{y_0}^1 \frac{dy}{R^{2l} y^{2l}\sqrt{1-y^2}} \\ &= y_0^{2l} \int_{y_0}^1 \sum_{k=0}^{\infty} y^{-2l} c_k y^{2k} dy \\ &= y_0^{2l} \left(\sum_{k=0}^{\infty} c_k \frac{y^{2k-2l+1}}{2k-2l+1}\right)\Bigg|_{y_0}^1 \\ &= \sum_{k=0}^{\infty} \frac{c_k y_0^{2l}}{2k-2l+1} - \sum_{k=0}^{\infty} \frac{c_k y_0^{2k+1}}{2k-2l+1} \end{aligned} \tag{27}$$

we see it seems to contain even powered terms in the first sum. In fact, we must at least have y_0^0 term to account for A_0. But unfortunately, it's not obvious that the first sum of (27) will vanish for all but $l = 0$ (see the reference above for a direct proof, using complex contour integrals), so let's resort to another way to find the Y_l integral.

Letting $\rho = R\sin\xi$, and $\xi_0 = \sin^{-1}(r/R)$, we can calculate Y_l as

$$Y_l = \int_{\xi_0}^{\pi/2} \frac{r^{2l} R\cos\xi d\xi}{R^{2l}\sin^{2l}\xi R\cos\xi} = \sin^{2l}\xi_0 \underbrace{\int_{\xi_0}^{\pi/2} \frac{d\xi}{\sin^{2l}\xi}}_{J_{2l}} \tag{28}$$

For $l > 0$, we can calculate J_{2l} using the same reduction trick:

$$\begin{aligned} J_{2l} &= \int_{\xi_0}^{\pi/2} \left(\frac{d\xi}{\sin^2\xi}\right)\left(\frac{1}{\sin^{2l-2}\xi}\right) \\ &= -\cot\xi\left(\frac{1}{\sin^{2l-2}\xi}\right)\Bigg|_{\xi_0}^{\pi/2} + \int_{\xi_0}^{\pi/2} \cot\xi\,[-(2l-2)]\left(\frac{1}{\sin^{2l-1}\xi}\right)\cos\xi d\xi \\ &= \frac{\cos\xi_0}{\sin^{2l-1}\xi_0} - (2l-2)(J_{2l} - J_{2l-2}) \end{aligned} \tag{29}$$

This implies

$$\begin{aligned}
J_{2l} &= \left(\frac{1}{2l-1}\right)\left(\frac{\cos\xi_0}{\sin^{2l-1}\xi_0}\right) + \left(\frac{2l-2}{2l-1}\right)J_{2l-2} \\
&= \left(\frac{1}{2l-1}\right)\left(\frac{\cos\xi_0}{\sin^{2l-1}\xi_0}\right) + \left(\frac{2l-2}{2l-1}\right)\left(\frac{1}{2l-3}\right)\left(\frac{\cos\xi_0}{\sin^{2l-3}\xi_0}\right) + \left(\frac{2l-2}{2l-1}\right)\left(\frac{2l-4}{2l-3}\right)J_{2l-4} \\
&= \underbrace{\left(\frac{1}{2l-1}\right)}_{d_l}\left(\frac{\cos\xi_0}{\sin^{2l-1}\xi_0}\right) + \underbrace{\left(\frac{2l-2}{2l-1}\right)\left(\frac{1}{2l-3}\right)}_{d_{l-1}}\left(\frac{\cos\xi_0}{\sin^{2l-3}\xi_0}\right) + \cdots + \underbrace{\left[\frac{(2l-2)(2l-4)\cdots 4}{(2l-1)(2l-3)\cdots 5\cdot 3}\right]}_{d_2}\left(\frac{\cos\xi_0}{\sin^{3}\xi_0}\right) + \\
&\quad \underbrace{\left[\frac{(2l-2)(2l-4)\cdots 2}{(2l-1)(2l-3)\cdots 3}\right]}_{d_1}J_2 \qquad \text{where} \qquad J_2 = \frac{\cos\xi_0}{\sin\xi_0}
\end{aligned} \tag{30}$$

Going back to (28), we see that for $l > 0$, Y_l has the form

$$Y_l = \sin^{2l}\xi_0 \sum_{k=1}^{l} d_k \left(\frac{\cos\xi_0}{\sin^{2k-1}\xi_0}\right) = \sum_{k=1}^{l} d_k \cos\xi_0 \sin^{2l-2k+1}\xi_0 \tag{31}$$

If we further expand $\cos\xi_0 = \left(1-\sin^2\xi_0\right)^{1/2}$, it's clear that for any $l > 0$, (31) has only odd powers of $\sin\xi_0 = r/R$. Furthermore, by (28), $Y_0 = \pi/2 - \xi_0$, which only contains the 0-power constant of $\pi/2$ and the expansion of odd-powered terms of r.

In summary, we have established that the first sum of (27) is equal to $\delta_{l0}\cdot\pi/2$, and with (26), (27) inserted back into (20), we finally have

$$\begin{aligned}
\Phi(z = r < R) &= \frac{2V}{\pi}\sum_{l=0}^{\infty} P_{2l}(0)\left[\frac{\pi}{2}\delta_{l0} + \sum_{k=0}^{\infty} c_k \left(\frac{1}{2k+2l+2} - \frac{1}{2k-2l+1}\right)\left(\frac{r}{R}\right)^{2k+1}\right] \\
&= V + \frac{2V}{\pi}\sum_{l=0}^{\infty} P_{2l}(0)\sum_{k=0}^{\infty}\frac{-(4l+1)c_k}{(2k+2l+2)(2k-2l+1)}\left(\frac{r}{R}\right)^{2k+1} \qquad \text{by (17), (25)} \\
&= V + \frac{2V}{\pi}\sum_{k=0}^{\infty}\left[\sum_{l=0}^{\infty}\underbrace{(-1)^{l+1}\frac{(2l-1)!!}{(2l)!!}\frac{(4l+1)(2k-1)!!}{(2k)!!(2k+2l+2)(2k-2l+1)}}_{\lambda_{lk}}\right]\left(\frac{r}{R}\right)^{2k+1}
\end{aligned} \tag{32}$$

□

The coefficient λ_{lk} may look intimidating, but in fact it can be simplified greatly. To see this, we invoke Lemma 1 and expand function $g(x) = P_{2k+1}(x)$ in the basis of $P_{2l}(x)$ using (5) and (6):

$$\begin{aligned}
P_{2k+1}(x) &= \sum_{l=0}^{\infty}\left[(4l+1)\int_0^1 P_{2k+1}(x)P_{2l}(x)dx\right]P_{2l}(x) \qquad \text{by (1)} \\
&= \sum_{l=0}^{\infty}\left[(4l+1)\frac{(-1)^{k+l+1}(2l)!(2k+1)!}{2^{2(k+l)}(2l-2k-1)(2k+2l+2)(l!)^2(k!)^2}\right]P_{2l}(x) \\
&= \sum_{l=0}^{\infty}\left[(4l+1)\cdot\frac{(-1)^{k+l}}{(2k-2l+1)(2k+2l+2)}\cdot\frac{(2l)!}{2^l l! 2^l l!}\cdot\frac{(2k+1)!}{2^k k! 2^k k!}\right]P_{2l}(x) \\
&= \sum_{l=0}^{\infty}\left[\frac{(4l+1)(-1)^{k+l}(2l-1)!!(2k+1)!!}{(2k-2l+1)(2k+2l+2)(2l)!!(2k)!!}\right]P_{2l}(x) \qquad \text{factor out } \lambda_{lk} \\
&= (-1)^{k+1}(2k+1)\sum_{l=0}^{\infty}\lambda_{lk}P_{2l}(x)
\end{aligned} \tag{33}$$

Putting $x = 1$ will yield

$$\sum_{l=0}^{\infty}\lambda_{lk} = \frac{(-1)^{k+1}}{2k+1} \tag{34}$$

This turns the "on-axis" potential for $r < R$ (32) into

$$\Phi(z = r < R) = V + \frac{2V}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k+1}\left(\frac{r}{R}\right)^{2k+1} \tag{35}$$

With the odd powers in r, the "off-axis" field point's potential must multiply the k-th term by

$$\begin{aligned} &P_{2k+1}(\cos\theta) && \text{for } 0 \le \theta \le \frac{\pi}{2} \\ &(-1)^{2k+1}P_{2k+1}(\cos\theta) = -P_{2k+1}(\cos\theta) && \text{for } \frac{\pi}{2} \le \theta \le \pi \end{aligned}$$

The full potential with $r < R$ is thus

$$\Phi(r < R, \theta) = \begin{cases} V + \dfrac{2V}{\pi}\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k+1}\left(\frac{r}{R}\right)^{2k+1}P_{2k+1}(\cos\theta) & \text{for } 0 \le \theta \le \dfrac{\pi}{2} \\ V - \dfrac{2V}{\pi}\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k+1}\left(\frac{r}{R}\right)^{2k+1}P_{2k+1}(\cos\theta) & \text{for } \dfrac{\pi}{2} \le \theta \le \pi \end{cases} \tag{36}$$

4. **Potential's continuity check at $r = R$**

   Let's first verify the potential's continuity at $r = R$ for a few special angles.

   (a) $\theta = 0$:

   By (19) $$\Phi(R^+, \theta = 0) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1} = \frac{2V}{\pi}\cdot\frac{\pi}{4} = \frac{V}{2}$$

   By (36) $$\Phi(R^-, \theta = 0) = V + \frac{2V}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k+1} = V - \frac{2V}{\pi}\cdot\frac{\pi}{4} = \frac{V}{2}$$

   (b) $\theta = \pi$: exactly the same as $\theta = 0$ as can be seen by a simple sign flip of $P_{2k+1}(-1) = -1$.

   (c) $\theta = \pi/2$:

   By (19) $$\Phi\left(R^+, \theta = \frac{\pi}{2}\right) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}P_{2l}(0) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(2l-1)!!}{(2l)!!(2l+1)} = \frac{2V}{\pi}\sin^{-1}1 = V$$

   By (36) $$\Phi\left(R^-, \theta = \frac{\pi}{2}\right) = V \pm \frac{2V}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k+1}\cdot 0 = V$$

   which is not only consistent between inner and outer solution, but also consistent with the boundary condition at the edge of the disc.

   Now let's show the continuity for arbitrary angle $\theta \in [0, \pi/2]$.
   By (19), at (R^+, θ),

$$\Phi(R^+, \theta) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}P_{2l}(\cos\theta) \tag{37}$$

   and by (36), at (R^-, θ),

$$\Phi(R^-, \theta) = V + \frac{2V}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k+1}P_{2k+1}(\cos\theta) \tag{38}$$

   Continuity of potential at (R, θ) thus requires

$$1 + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k+1}P_{2k+1}(x) = \frac{2}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}P_{2l}(x) \qquad \text{for } 0 \le x = \cos\theta \le 1 \tag{39}$$

   In order to prove (39), let $f(x)$ denote its LHS, and by (33),

$$f(x) = 1 + \frac{2}{\pi}\sum_{k=0}^{\infty}\left[\sum_{n=0}^{\infty}\lambda_{nk}P_{2n}(x)\right] \tag{40}$$

If we express $f(x)$ as expansion with basis $P_{2l}(x)$ we have

$$\begin{aligned}
f(x) &= \sum_{l=0}^{\infty} A_{2l} P_{2l}(x) \qquad \text{where} \\
A_{2l} &= (4l+1)\int_0^1 f(x) P_{2l}(x)dx \\
&= \left[(4l+1)\int_0^1 P_{2l}(x)dx\right] + \left\{(4l+1)\frac{2}{\pi}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{nk}\left[\int_0^1 P_{2n}(x)P_{2l}(x)dx\right]\right\} \\
&= (4l+1)\,\delta_{l0} + (4l+1)\frac{2}{\pi}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{nk}\delta_{ln}\frac{1}{4l+1} \\
&= \delta_{l0} + \frac{2}{\pi}\sum_{k=0}^{\infty}\lambda_{lk}
\end{aligned} \tag{41}$$

The goal is thus to prove

$$\delta_{l0} + \frac{2}{\pi}\sum_{k=0}^{\infty}\lambda_{lk} = \frac{2}{\pi}\frac{(-1)^l}{(2l+1)} \tag{42}$$

Going back to the definition of λ_{lk} in (32), we see that

$$\sum_{k=0}^{\infty}\lambda_{lk} = P_{2l}(0)\sum_{k=0}^{\infty}c_k\left(\frac{1}{2k+2l+2} - \frac{1}{2k-2l+1}\right) \tag{43}$$

Using the expansion (25), we can convert the sums in (43) into their integral representation which can be calculated exactly

$$\sum_{k=0}^{\infty}\frac{c_k}{2k+2l+2} = \int_0^1 \frac{\rho^{2l+1}d\rho}{\sqrt{1-\rho^2}} = I_{2l+1} = \frac{(2l)!!}{(2l+1)!!} \qquad \text{by equation (14)} \tag{44}$$

$$\sum_{k=0}^{\infty}\frac{c_k}{2k-2l+1} = \int_0^1 \frac{d\rho}{\rho^{2l}\sqrt{1-\rho^2}} = \delta_{l0}\frac{\pi}{2} \qquad \text{by equations (27)-(31), under the limit } \xi_0 \to 0 \tag{45}$$

Together with (17), we have

$$\sum_{k=0}^{\infty}\lambda_{lk} = (-1)^l\frac{(2l-1)!!}{(2l)!!}\left[\frac{(2l)!!}{(2l+1)!!} - \delta_{l0}\frac{\pi}{2}\right] \tag{46}$$

which proves (42), hence proves the continuity at $r = R$ everywhere in the northern hemisphere. Continuity in the southern hemisphere follows trivially with the usual sign flip.

5. **Solution to part (c)**

$$C = \frac{Q}{V} = \frac{1}{V}\int_0^R \sigma(\rho)2\pi\rho\, d\rho = \frac{1}{V}\int_0^R \frac{4\epsilon_0 V}{\pi}\frac{2\pi\rho\, d\rho}{\sqrt{R^2-\rho^2}} = 8\epsilon_0 R \tag{47}$$