In these notes, we fill in the details for the example of variational approach on page 46, in particular, we reproduce Figure 1.9 (see associated octave script `pp46_variational_approach.m`).

Define

$$I(k,l)=\int_0^1 \rho^k(1-\rho)^l\,d\rho \tag{1}$$

Expanding the RHS will produce

$$I(k,l)=\sum_{m=0}^{l}\binom{l}{m}(-1)^m\frac{1}{k+m+1} \tag{2}$$

Consider the first trial potential

$$\Psi_1(\rho)=\alpha_1(1-\rho)+\beta_1(1-\rho)^2+\gamma_1(1-\rho)^3 \tag{3}$$

Then (1.63) becomes

$$\frac{1}{2\pi}I[\Psi_1]=\underbrace{\frac{1}{2}\int_0^1\left(\frac{d\Psi_1}{d\rho}\right)^2\rho\,d\rho}_{A}-\overbrace{\int_0^1 g\Psi_1\rho\,d\rho}^{B} \tag{4}$$

where

$$g(\rho)=-5(1-\rho)+10^4\rho^5(1-\rho)^5 \tag{5}$$

The first integral of (4) gives

$$\begin{aligned}
A=\frac{1}{2}\int_0^1\left(\frac{d\Psi_1}{d\rho}\right)^2\rho\,d\rho&=\frac{1}{2}\int_0^1\left[\alpha_1+2\beta_1(1-\rho)+3\gamma_1(1-\rho)^2\right]^2\rho\,d\rho\\
&=\frac{1}{2}\int_0^1\left[9\gamma_1^2(1-\rho)^4+12\beta_1\gamma_1(1-\rho)^3+(6\alpha_1\gamma_1+4\beta_1^2)(1-\rho)^2+4\alpha_1\beta_1(1-\rho)+\alpha_1^2\right]\rho\,d\rho\\
&=\frac{1}{2}\left[9\gamma_1^2I(1,4)+12\beta_1\gamma_1I(1,3)+(6\alpha_1\gamma_1+4\beta_1^2)I(1,2)+4\alpha_1\beta_1I(1,1)+\alpha_1^2I(1,0)\right]
\end{aligned}\tag{6}$$

For the second term,

$$\begin{aligned}
g\Psi_1=&-5\alpha_1(1-\rho)^2-5\beta_1(1-\rho)^3-5\gamma_1(1-\rho)^4+\\
&10^4\alpha_1(1-\rho)^6\rho^5+10^4\beta_1(1-\rho)^7\rho^5+10^4\gamma_1(1-\rho)^8\rho^5
\end{aligned}\tag{7}$$

which gives

$$B=\int_0^1 g\Psi_1\rho\,d\rho=\alpha_1\left[10^4I(6,6)-5I(1,2)\right]+\beta_1\left[10^4I(6,7)-5I(1,3)\right]+\gamma_1\left[10^4I(6,8)-5I(1,4)\right] \tag{8}$$

Now enforce the extremum condition of $I[\Psi_1]$, we get

$$\begin{bmatrix} I(1,0) & 2I(1,1) & 3I(1,2)\\ 2I(1,1) & 4I(1,2) & 6I(1,3)\\ 3I(1,2) & 6I(1,3) & 9I(1,4)\end{bmatrix}\begin{bmatrix}\alpha_1\\ \beta_1\\ \gamma_1\end{bmatrix}=\begin{bmatrix}10^4I(6,6)-5I(1,2)\\ 10^4I(6,7)-5I(1,3)\\ 10^4I(6,8)-5I(1,4)\end{bmatrix} \tag{9}$$

This gives

$$\alpha_1=1.1092\qquad \beta_1=0.5546\qquad \gamma_1=-1.2944 \tag{10}$$

Now for the second trial potential

$$\Psi_2(\rho)=\alpha\rho^2+\beta\rho^3+\gamma\rho^4-(\alpha+\beta+\gamma) \tag{11}$$

the text has already obtained equation (1.73):

$$\frac{1}{2\pi}I[\Psi_2] = \left[\frac{1}{2}\alpha^2 + \frac{6}{5}\alpha\beta + \frac{4}{3}\alpha\gamma + \frac{3}{4}\beta^2 + \frac{12}{7}\beta\gamma + \gamma^2\right] - \left(e_2\alpha + e_3\beta + e_4\gamma\right) \tag{12}$$

where

$$e_n = \int_0^1 g(\rho)(\rho^n - 1)\rho\, d\rho = -5I(n+1,1) + 5I(1,1) + 10^4 I(n+6,5) - 10^4 I(6,5) \tag{13}$$

Enforcing the extremum condition gives

$$\begin{bmatrix} 1 & 6/5 & 4/3 \\ 6/5 & 3/2 & 12/7 \\ 4/3 & 12/7 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} e_2 \\ e_3 \\ e_4 \end{bmatrix} \tag{14}$$

which gives

$$\alpha = 2.9150 \qquad \beta = -7.0306 \qquad \gamma = 3.6422 \tag{15}$$

In order to plot the exact solution, we must solve for $\Psi_E(\rho)$ such that

$$\nabla^2\Psi_E = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Psi_E}{\partial\rho}\right) = -g \tag{16}$$

This can be done by direct integration

$$\Psi_E(\rho) = -\int \frac{1}{\rho}\left(\int \rho g\, d\rho + C_1\right) d\rho + C_2 \tag{17}$$

where C_1 must vanish in order for Ψ_E to be well-behaved at $\rho = 0$. The integration can be calculated exactly since g is a polynomial in ρ, which eventually gives

$$\Psi_E(\rho) = -\left[-\frac{5}{4}\rho^2 + \frac{5}{9}\rho^3 + 10^4\left(\frac{1}{49}\rho^7 - \frac{5}{64}\rho^8 + \frac{10}{81}\rho^9 - \frac{1}{10}\rho^{10} + \frac{5}{121}\rho^{11} - \frac{1}{144}\rho^{12}\right)\right] + C_2 \tag{18}$$

where C_2 takes the bracket's value evaluated at $\rho = 1$ so $\Psi_E(1) = 0$.

The associated octave script reproduces Figure 1.9 exactly as below.