1. For ϵ_r close to 1, the discussion about Born approximation from section 10.2B applies. The differential scattering cross section is given by (10.28)

$$\frac{d\sigma}{d\Omega} = \frac{|\boldsymbol{\epsilon}^* \cdot \mathbf{A}_{\rm sc}|^2}{|\mathbf{D}^{(0)}|^2} \tag{1}$$

where to the first order of $\epsilon_r - 1$,

$$\frac{\boldsymbol{\epsilon}^* \cdot \mathbf{A}_{\rm sc}}{D^{(0)}} = k^2 (\epsilon_r - 1)(\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0)\left[\frac{\sin(qa) - qa\cos(qa)}{q^3}\right] \tag{2}$$

and

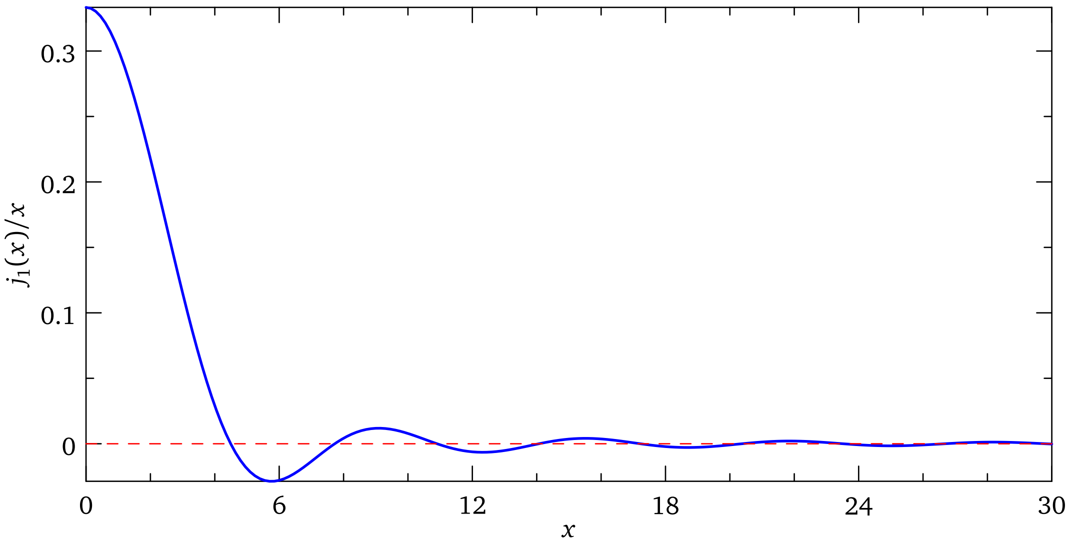
$$\mathbf{q} = k(\mathbf{n}_0 - \mathbf{n}) \tag{3}$$

Recall that

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \tag{4}$$

then (2) can be written as

$$\frac{\boldsymbol{\epsilon}^* \cdot \mathbf{A}_{\rm sc}}{D^{(0)}} = k^2 a^3 (\epsilon_r - 1)(\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0)\left[\frac{j_1(qa)}{qa}\right] \tag{5}$$



From the diagram of $j_1(x)/x$ above, we see that the contribution comes mainly from the range where x is small. Qualitatively, when $ka \gg 1$, we must have a small angle between $\mathbf{n}$ and $\mathbf{n}_0$ for qa to be small, i.e., forward scattering is dominant. The next part has the exact calculation of total scattering cross section.

2. Let θ be the angle between $\mathbf{n}$ and $\mathbf{n}_0$, then $q = 2k\sin(\theta/2)$, thus (1) becomes

$$\frac{d\sigma}{d\Omega} = k^4 a^6 (\epsilon_r - 1)^2 |\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0|^2 \left[\frac{j_1\left(2ka\sin\dfrac{\theta}{2}\right)}{2ka\sin\dfrac{\theta}{2}}\right]^2 \tag{6}$$

Summing over all scattered polarizations turns $|\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0|^2$ into $\left(1+\cos^2\theta\right)/2$ (see (10.6), (10.10)). With $z \equiv 2ka$, (6) can be written as

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= k^4 a^6 (\epsilon_r - 1)^2 \left(\frac{1+\cos^2\theta}{2}\right)\left[\frac{j_1\left(z\sin\dfrac{\theta}{2}\right)}{z\sin\dfrac{\theta}{2}}\right]^2 \\ &= k^4 a^6 (\epsilon_r - 1)^2 \left(1 - 2\sin^2\frac{\theta}{2} + 2\sin^4\frac{\theta}{2}\right)\left[\frac{j_1\left(z\sin\dfrac{\theta}{2}\right)}{z\sin\dfrac{\theta}{2}}\right]^2 \end{aligned} \tag{7}$$

Denote $u \equiv z\sin(\theta/2)$, then the total scattering cross section can be obtained as

$$\begin{aligned}
\sigma &= \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi k^4 a^6 (\epsilon_r - 1)^2 \int_0^\pi \left(1 - \frac{2u^2}{z^2} + \frac{2u^4}{z^4}\right)\left[\frac{j_1(u)}{u}\right]^2 \cdot \frac{4u du}{z^2} \\
&= \frac{8\pi k^4 a^6}{z^2}(\epsilon_r - 1)^2 \int_0^z \left(1 - \frac{2u^2}{z^2} + \frac{2u^4}{z^4}\right)\frac{j_1^2}{u} du \\
&= \frac{\pi}{2} k^2 a^4 (\epsilon_r - 1)^2 \cdot \underbrace{\int_0^z \left(\frac{4}{u} - \frac{8u}{z^2} + \frac{8u^3}{z^4}\right) j_1^2 du}_{F(z)}
\end{aligned} \tag{8}$$

To evaluate integral $F(z)$, we first note the following recurrence relations for the spherical Bessel function, (see DLMF 10.51.E2):

$$j_1'(u) = j_0(u) - \frac{2j_1(u)}{u} \tag{9}$$

$$j_0'(u) = -j_1(u) \tag{10}$$

The first part of $F(z)$ is

$$\begin{aligned}
F_1(z) &= 4\int_0^z \frac{j_1}{u} \cdot j_1 du && \text{by (9)} \\
&= 2\int_0^z \left(j_0 - j_1'\right) j_1 du && \text{by (10)} \\
&= 2\left(-\int_0^z j_0 j_0' du - \int_0^z j_1 j_1' du\right) \\
&= -\left(j_0^2 + j_1^2\right)\Big|_0^z \\
&= -\left[\left(\frac{\sin z}{z}\right)^2 - 1 + \left(\frac{\sin z}{z^2} - \frac{\cos z}{z}\right)^2\right] \\
&= -\left(\frac{1}{z^2} - 1 + \frac{\sin^2 z}{z^4} - \frac{\sin 2z}{z^3}\right)
\end{aligned} \tag{11}$$

and the second part is

$$\begin{aligned}
F_2(z) &= -\frac{8}{z^2}\int_0^z u j_1^2 du && \text{by (10)} \\
&= \frac{8}{z^2}\int_0^z u j_1 j_0' du \\
&= \frac{8}{z^2}\left[u j_1 j_0\Big|_0^z - \int_0^z j_0\left(j_1 + u j_1'\right) du\right] && \text{by (9)} \\
&= \frac{8}{z^2}\left[z j_1(z) j_0(z) - \int_0^z j_0(u j_0 - j_1) du\right] && \text{by (10)} \\
&= \frac{8}{z^2}\left[z\left(\frac{\sin z}{z^2} - \frac{\cos z}{z}\right)\frac{\sin z}{z} - \int_0^z u j_0^2 du - \int_0^z j_0 j_0' du\right] \\
&= \frac{8}{z^2}\left[z\left(\frac{\sin z}{z^2} - \frac{\cos z}{z}\right)\frac{\sin z}{z} - \int_0^z u j_0^2 du - \frac{1}{2}\left(\frac{\sin^2 z}{z^2} - 1\right)\right] \\
&= \frac{4\sin^2 z}{z^4} - \frac{4\sin 2z}{z^3} - \frac{8}{z^2}\int_0^z u j_0^2 du + \frac{4}{z^2}
\end{aligned} \tag{12}$$

The third part is a little more involved:

$$\begin{aligned}
F_3(z) &= -\frac{8}{z^4}\left[u^3 j_1 j_0\Big|_0^z - \int_0^z j_0\left(3u^2 j_1 + u^3 j_1'\right) du\right] && \text{by (9)} \\
&= -\frac{8}{z^4}\left[z^3 j_1(z) j_0(z) - \int_0^z j_0\left(u^3 j_0 + u^2 j_1\right) du\right]
\end{aligned} \tag{13}$$

where

$$\int_0^z j_0^2 u^3 du = \int_0^z u \sin^2 u du = \frac{z^2}{4} - \frac{z \sin 2z}{4} + \frac{1}{8} - \frac{\cos 2z}{8} \tag{14}$$

$$\int_0^z u^2 j_0 j_1 du = -\int_0^z u^2 j_0 j_0' du = -\frac{1}{2} j_0^2 u^2 \Big|_0^z + \int_0^z u j_0^2 du = -\frac{\sin^2 z}{2} + \int_0^z u j_0^2 du \tag{15}$$

thus

$$F_3(z) = -\frac{12 \sin^2 z}{z^4} + \frac{2 \sin 2z}{z^3} + \frac{2}{z^2} + \frac{1 - \cos 2z}{z^4} + \frac{8}{z^4} \int_0^z u j_0^2 du \tag{16}$$

Since we can write

$$\int_0^z u j_0^2 du = \int_0^z u \frac{\sin^2 u}{u^2} du = \int_0^z \frac{1 - \cos^2 2u}{2u} du = \frac{1}{2} \int_0^{2z} \frac{1 - \cos t}{t} dt \tag{17}$$

then adding (11), (12) and (16) finally yields

$$F(z) = 1 + \frac{5}{z^2} - \frac{7(1 - \cos 2z)}{2z^4} - \frac{\sin 2z}{z^3} - 4\left(\frac{1}{z^2} - \frac{1}{z^4}\right) \int_0^{2z} \frac{1 - \cos t}{t} dt \tag{18}$$

We see that up to order $O\left(z^0\right)$, $F(z) \approx 1$, and we fall back to part (a)'s result.