1. Let $\psi(x,y)e^{\pm ikz}$ be $E_z(H_z)$ for TM(TE) mode. The governing differential equation is given by (8.34)

$$\left(\nabla_t^2+\gamma^2\right)\psi=0 \tag{1}$$

Given the cylindrical symmetry, we write ψ in separate variable form

$$\psi(\rho,\phi)=R(\rho)\Phi(\phi) \tag{2}$$

Then (1) becomes

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)\Phi+\frac{R}{\rho^2}\frac{d^2\Phi}{d\phi^2}+\gamma^2 R\Phi=0 \tag{3}$$

Multiplying by ρ^2/ψ and rearranging the terms, we get

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\gamma^2\rho^2=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \tag{4}$$

The two sides depend on independent variables ρ,ϕ, so they must both be equal to a constant, denoted m^2, i.e.,

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2 \tag{5}$$

$$\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}+\left(\gamma^2-\frac{m^2}{\rho^2}\right)R=0 \tag{6}$$

From (5), we conclude m must be an integer for ϕ to be a single-valued function of space, hence

$$\Phi(\phi)=Ae^{im\phi} \tag{7}$$

From (6), the radial solution is a linear combination of $J_m(\gamma\rho)$ and $N_m(\gamma\rho)$. We discard the $N_m(\gamma\rho)$ contribution for its divergence at $\rho=0$. Therefore, the solution form of ψ is

$$\psi(\rho,\phi)=\psi_0 J_m(\gamma\rho)e^{im\phi} \tag{8}$$

For TE/TM mode, boundary conditions require

$$\text{TM}: \quad E_z(R,\phi)=E_0J_m(\gamma R)e^{im\phi}=0 \tag{9}$$

$$\text{TE}: \quad \left.\frac{\partial H_z(\rho,\phi)}{\partial\rho}\right|_{\rho=R}=H_0\gamma J_m'(\gamma R)e^{im\phi}=0 \tag{10}$$

Evidently, for TM mode, γ can take values x_{mk}/R where x_{mk} is the k-th zero of $J_m(x)$. For nontrivial TE mode, γ can take values x'_{mk}/R where x'_{mk} is the k-th root of $J'_m(x)$.

k	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

Referring to the zeros table of $J_m(x)$ above (see WolframAlpha), we see that the lowest γ values for the TM mode are

$$\gamma_{01}=2.4048/R \qquad \gamma_{11}=3.8317/R \qquad \gamma_{21}=5.1356/R \qquad \gamma_{02}=5.5201/R \qquad \gamma_{31}=6.3802/R \tag{11}$$

So the dominant frequency and the next four cutoff frequencies are

$$\omega_{\text{dom}}=\omega_{01}=\frac{2.4048}{\sqrt{\mu\epsilon}R} \quad \omega_{11}=1.5934\omega_{\text{dom}} \quad \omega_{21}=2.1356\omega_{\text{dom}} \quad \omega_{02}=2.2955\omega_{\text{dom}} \quad \omega_{31}=2.6531\omega_{\text{dom}} \tag{12}$$

k	$J_0'(x)$	$J_1'(x)$	$J_2'(x)$	$J_3'(x)$	$J_4'(x)$	$J_5'(x)$
1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

Similarly from the zeros table of $J_m'(x)$ above, the first few γ's are

$$\gamma_{11}' = 1.8412/R \qquad \gamma_{21}' = 3.0542/R \qquad \gamma_{01}' = 3.8317/R \qquad \gamma_{31}' = 4.2012/R \qquad \gamma_{41}' = 5.3175/R \tag{13}$$

So the dominant frequency and the next four cutoff frequencies are

$$\omega_{\text{dom}}' = \omega_{11}' = \frac{1.8412}{\sqrt{\mu\epsilon}R} \quad \omega_{21}' = 1.6588\omega_{\text{dom}}' \quad \omega_{01}' = 2.0811\omega_{\text{dom}}' \quad \omega_{31}' = 2.2818\omega_{\text{dom}}' \quad \omega_{41}' = 2.8881\omega_{\text{dom}}' \tag{14}$$

2. For the TM mode mk

$$E_z(\rho,\phi) = E_0 J_m\left(x_{mk}\frac{\rho}{R}\right)e^{im\phi} \tag{15}$$

Thus by (8.59)

$$\begin{aligned} -\frac{dP}{dz} &= \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mk}}\right)^2 \oint_C \frac{1}{\mu^2\omega_{mk}^2}\left|\frac{\partial E_z}{\partial\rho}\right|^2 dl \\ &= \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mk}}\right)^2 \cdot 2\pi R\left(\frac{1}{\mu^2\omega_{mk}^2}\right)\cdot|E_0|^2\left(\frac{x_{mk}}{R}\right)^2\left[J_m'(x_{mk})\right]^2 \end{aligned} \tag{16}$$

On the other hand, by (8.51)

$$\begin{aligned} P &= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_{mk}}\right)^2\sqrt{1-\frac{\omega_{mk}^2}{\omega^2}}\int_A |E_z|^2\, da \\ &= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_{mk}}\right)^2\sqrt{1-\frac{\omega_{mk}^2}{\omega^2}}\cdot 2\pi|E_0|^2\int_0^R\left[J_m\left(x_{mk}\frac{\rho}{R}\right)\right]^2\rho\, d\rho \qquad \text{let } t\equiv\rho/R \\ &= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_{mk}}\right)^2\sqrt{1-\frac{\omega_{mk}^2}{\omega^2}}\cdot 2\pi|E_0|^2R^2\int_0^1\left[J_m(x_{mk}t)\right]^2 t\, dt \\ &= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_{mk}}\right)^2\sqrt{1-\frac{\omega_{mk}^2}{\omega^2}}\cdot 2\pi|E_0|^2R^2\cdot\frac{1}{2}\left[J_m'(x_{mk})\right]^2 \end{aligned} \tag{17}$$

where in the last step, we have used the orthonormality of Bessel functions (see 10.22.37 on dlmf.nist.gov)

$$\int_0^1 J_\nu(x_{\nu l}t)J_\nu(x_{\nu m}t)\,t\,dt = \frac{1}{2}\left[J_\nu'(x_{\nu l})\right]^2\delta_{lm} \qquad \text{for } x_{\nu l}, x_{\nu m} \text{ zeros of } J_\nu(x) \tag{18}$$

Then we can obtain the attenuation constant

$$\begin{aligned} \beta_{mk} &= \frac{-dP/dz}{2P} = \sqrt{\frac{\mu}{\epsilon}}\left(\frac{x_{mk}^2}{\sigma\delta\mu^2\omega_{mk}^2R^3}\right)\left(\sqrt{1-\frac{\omega_{mk}^2}{\omega^2}}\right)^{-1} \qquad \text{recall } \frac{x_{mk}^2}{R^2\omega_{mk}^2} = \frac{\gamma_{mk}^2}{\omega_{mk}^2} = \mu\epsilon \\ &= \sqrt{\frac{\epsilon}{\mu}}\left(\frac{1}{\sigma\delta R}\right)\left(\sqrt{1-\frac{\omega_{mk}^2}{\omega^2}}\right)^{-1} \qquad \text{recall } \delta = \delta_{mk}\sqrt{\frac{\omega_{mk}}{\omega}} \\ &= \sqrt{\frac{\epsilon}{\mu}}\left(\frac{1}{\sigma\delta_{mk}R}\right)\sqrt{\frac{\omega}{\omega_{mk}}}\left(\sqrt{1-\frac{\omega_{mk}^2}{\omega^2}}\right)^{-1} \end{aligned} \tag{19}$$

For the TE mode mk

$$H_z(\rho,\phi) = H_0 J_m\left(x'_{mk}\frac{\rho}{R}\right)e^{im\phi} \tag{20}$$

Note

$$\nabla_t H_z(\rho,\phi) = \frac{\partial H_z}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial H_z}{\partial\phi}\hat{\boldsymbol{\phi}} \qquad \Longrightarrow \qquad |\hat{\boldsymbol{\rho}}\times\nabla_t H_z|^2 = \frac{m^2|H_0|^2}{\rho^2}\left[J_m\left(x'_{mk}\frac{\rho}{R}\right)\right]^2 \tag{21}$$

Plugging this into the lower line of (8.59) yields

$$\begin{aligned}
-\frac{dP}{dz} &= \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega'_{mk}}\right)^2\oint_C\left[\frac{1}{\mu\epsilon\omega'^2_{mk}}\left(1-\frac{\omega'^2_{mk}}{\omega^2}\right)|\hat{\boldsymbol{\rho}}\times\nabla_t H_z|^2 + \frac{\omega'^2_{mk}}{\omega^2}|H_z|^2\right]dl \\
&= \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega'_{mk}}\right)^2\left[\frac{m^2}{\mu\epsilon\omega'^2_{mk}R^2}\left(1-\frac{\omega'^2_{mk}}{\omega^2}\right)+\frac{\omega'^2_{mk}}{\omega^2}\right]|H_0|^2\cdot 2\pi R\left[J_m\left(x'_{mk}\right)\right]^2 \\
&= \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega'_{mk}}\right)^2\left[\frac{m^2}{x'^2_{mk}}\left(1-\frac{\omega'^2_{mk}}{\omega^2}\right)+\frac{\omega'^2_{mk}}{\omega^2}\right]|H_0|^2\cdot 2\pi R\left[J_m\left(x'_{mk}\right)\right]^2
\end{aligned} \tag{22}$$

On the other hand by (8.51)

$$P = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega}{\omega'_{mk}}\right)^2\sqrt{1-\frac{\omega'^2_{mk}}{\omega^2}}\cdot 2\pi|H_0|^2\int_0^R\left[J_m\left(x'_{mk}\frac{\rho}{R}\right)\right]^2\rho\,d\rho \tag{23}$$

Here we invoke 10.22.38 of https://dlmf.nist.gov which was also proved in problem 3.11:

$$\int_0^1 J_\nu(\alpha_l t)J_\nu(\alpha_m t)\,t\,dt = \left(\frac{a^2}{b^2}+\alpha_l^2-\nu^2\right)\frac{[J_\nu(\alpha_l)]^2}{2\alpha_l^2}\delta_{lm} \qquad \text{for } \alpha_l,\alpha_m \text{ positive zeros of } aJ_\nu(x)+bxJ'_\nu(x) \tag{24}$$

(23) becomes

$$P = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega}{\omega'_{mk}}\right)^2\sqrt{1-\frac{\omega'^2_{mk}}{\omega^2}}\cdot 2\pi|H_0|^2R^2\cdot\frac{1}{2}\left(1-\frac{m^2}{x'^2_{mk}}\right)\left[J_m\left(x'_{mk}\right)\right]^2 \tag{25}$$

Thus the attenuation constant can be obtained

$$\begin{aligned}
\beta'_{mk} = \frac{-dP/dz}{2P} &= \sqrt{\frac{\epsilon}{\mu}}\left(\frac{1}{\sigma\delta R}\right)\left[\frac{m^2+\left(\frac{\omega'^2_{mk}}{\omega^2}\right)\left(x'^2_{mk}-m^2\right)}{x'^2_{mk}-m^2}\right]\left(\sqrt{1-\frac{\omega'^2_{mk}}{\omega^2}}\right)^{-1} \\
&= \sqrt{\frac{\epsilon}{\mu}}\left(\frac{1}{\sigma\delta'_{mk}R}\right)\left(\frac{m^2}{x'^2_{mk}-m^2}+\frac{\omega'^2_{mk}}{\omega^2}\right)\sqrt{\frac{\omega}{\omega'_{mk}}}\left(\sqrt{1-\frac{\omega'^2_{mk}}{\omega^2}}\right)^{-1}
\end{aligned} \tag{26}$$

The the $\beta_\lambda \sim \omega$ dependency is plotted below for the lowest TM and TE mode.