These notes are intended to fill the gaps left in Jackson section 3.12.

1. **Direct verification that (3.164) satisfies Poisson equation.**

   (3.164) was derived from the eigenfunction expansion using $\psi_{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}/\sqrt{2\pi}^3$ as complete basis:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{2\pi^2}\int d^3k \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2} \tag{1}$$

   Here we show directly that it satisfies the Poisson equation for $1/|\mathbf{x}-\mathbf{x}'|$.
   Indeed,

$$\begin{aligned}
\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) &= \frac{1}{2\pi^2}\int d^3k \nabla^2\left[\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2}\right] \\
&= \frac{1}{2\pi^2}\int d^3k \left[\frac{-k^2 e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2}\right] \\
&= -\frac{1}{2\pi^2}\int d^3k e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \\
&= -\frac{1}{2\pi^2}\prod_{i=1}^{3}\underbrace{\int_{-\infty}^{\infty} dk_i e^{ik_i(x_i-x_i')}}_{2\pi\delta(x_i-x_i')} = -4\pi\delta(\mathbf{x}-\mathbf{x}')
\end{aligned} \tag{2}$$

2. **Direct proof that (3.164) is the Fourier transform of** $1/|\mathbf{x}-\mathbf{x}'|$.

   By definition of Fourier transform, we have

$$f(\mathbf{x}) \equiv \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \int d^3k \tilde{A}(k)\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{\sqrt{2\pi}^3} \qquad \text{where} \tag{3}$$

$$\tilde{A}(k) = \int d^3x f(\mathbf{x})\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{\sqrt{2\pi}^3} = \int d^3x \frac{1}{|\mathbf{x}-\mathbf{x}'|}\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{\sqrt{2\pi}^3} \tag{4}$$

   Let $\mathbf{q} \equiv \mathbf{x}-\mathbf{x}'$, and let θ be the angle between $\mathbf{k}$ and $\mathbf{q}$, then (4) becomes

$$\begin{aligned}
\tilde{A}(k) &= \frac{1}{\sqrt{2\pi}^3}\int_0^{2\pi} d\phi \int_0^{\infty} q^2 dq \int_0^{\pi} \sin\theta d\theta \left(\frac{1}{q}e^{-ikq\cos\theta}\right) \qquad (y \equiv -\cos\theta) \\
&= \frac{2\pi}{\sqrt{2\pi}^3}\int_0^{\infty} q dq \int_{-1}^{1} dy e^{ikqy} \\
&= \frac{2\pi}{\sqrt{2\pi}^3}\int_0^{\infty} q dq \cdot \frac{1}{ikq}\left(e^{ikq} - e^{-ikq}\right) \\
&= \frac{2\pi}{\sqrt{2\pi}^3 ik}\underbrace{\int_0^{\infty} dq\left(e^{ikq} - e^{-ikq}\right)}_{I}
\end{aligned} \tag{5}$$

   To obtain I, we multiply its integrand with $e^{-\epsilon q}$ where $\epsilon > 0$ and then take $\lim_{\epsilon\to 0}$:

$$\begin{aligned}
\int_0^{\infty} dq e^{-\epsilon q}\left(e^{ikq} - e^{-ikq}\right) &= \left.\frac{e^{(ik-\epsilon)q}}{ik-\epsilon}\right|_0^{\infty} - \left.\frac{e^{-(ik+\epsilon)q}}{-(ik+\epsilon)}\right|_0^{\infty} \\
&= \frac{1}{\epsilon - ik} - \frac{1}{\epsilon + ik} \\
&= \frac{2ik}{\epsilon^2 + k^2}
\end{aligned} \tag{6}$$

   Thus

$$\tilde{A}(k) = \frac{2\pi}{\sqrt{2\pi}^3 ik}\cdot\frac{2ik}{k^2} = \frac{4\pi}{k^2}\frac{1}{\sqrt{2\pi}^3} \tag{7}$$

   which turns (3) into the desired form of (1).

3. **Direct verification that (3.167) satisfies Poisson equation.**

   (3.167) gives the eigenfunction expansion of Green function for the rectangular box:

$$G\left(\mathbf{x},\mathbf{x}'\right)=\frac{32}{\pi abc}\sum_{l,m,n=1}^{\infty}\frac{\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)}{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}} \tag{8}$$

   whose Laplacian is

$$\begin{aligned}\nabla^2 G &= -\frac{32}{\pi abc}\pi^2\sum_{l,m,n=1}^{\infty}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)\\ &= -\frac{32\pi}{abc}\left[\sum_{l=1}^{\infty}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\right]\left[\sum_{m=1}^{\infty}\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\right]\left[\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)\right]\end{aligned} \tag{9}$$

   In the range $[0,a]$ with vanishing boundary values, the functions $U_l(x)=\sqrt{2/a}\sin(l\pi x/a)$ form a complete orthonormal set of functions, hence

$$\sum_{l=1}^{\infty}\frac{2}{a}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)=\delta\left(x-x'\right) \tag{10}$$

   similarly for y and z. This turns the RHS of (9) into $-4\pi\delta\left(\mathbf{x}-\mathbf{x}'\right)$, as expected.

4. **Detailed derivation of alternate form (3.168) of the Green function of a rectangular box.**

   Note the Green function representation (8) is not separable in x,y,z variables due to the coupling of l,m,n in the denominator in the term.

   Recall in section 2.9, we have used separation of variables to solve the rectangular box problem. We will use the similar technique here. Let's assume

$$G\left(\mathbf{x},\mathbf{x}'\right)=A\left(\mathbf{x}'\right)X(x)Y(y)Z(z) \tag{11}$$

   For $\mathbf{x}\neq\mathbf{x}'$,

$$\begin{aligned}\nabla^2 G = A\left(X''YZ+XY''Z+XYZ''\right)=0 \qquad &\Longrightarrow\\ A\left(\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}\right)=0\end{aligned} \tag{12}$$

   Due to the independence of X,Y,Z, we must have constant C such that

$$\frac{Z''}{Z}=C \qquad\qquad \frac{X''}{X}+\frac{Y''}{Y}=-C \tag{13}$$

   Let's focus the case where C is assumed to be positive, then we can see that the solution involves hyperbolic sine/cosine functions in z and sine/cosine functions in x and y. Due to the symmetry between $\mathbf{x}$ and $\mathbf{x}'$, we come up with the following ansatz:

$$G\left(\mathbf{x},\mathbf{x}'\right)=\sum_{l,m=1}^{\infty}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)g_{lm}(z,z') \tag{14}$$

   For this to satisfy the Poinsson equation, we require

$$\begin{aligned}\nabla^2 G &= \sum_{l,m=1}^{\infty}\left\{\left[\frac{d^2}{dz^2}-\left(\frac{l\pi}{a}\right)^2-\left(\frac{m\pi}{b}\right)^2\right]g_{lm}(z)\right\}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\\ &= -4\pi\delta\left(\mathbf{x}-\mathbf{x}'\right)\end{aligned} \tag{15}$$

   With (10), we know that (15) can be satisfied if for all l,m,

$$\left(\frac{d^2}{dz^2}-K_{lm}^2\right)g_{lm}=-\frac{16\pi}{ab}\delta\left(z-z'\right) \qquad \text{where} \qquad K_{lm}=\pi\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}} \tag{16}$$

   Solving (16) is just usual business:

- Boundary condition at $z = 0$ and $z = c$ requires

$$g_{lm}(z) = \begin{cases} A\sinh(K_{lm}z) & z < z' \\ B\sinh[K_{lm}(c-z)] & z > z' \end{cases} \tag{17}$$

- Continuity of g_{lm} at z' requires

$$A\sinh\left(K_{lm}z'\right) = B\sinh\left[K_{lm}(c-z')\right] \tag{18}$$

- Slope discontinuity at $z = z'$ requires

$$\left.\frac{dg_{lm}}{dz}\right|_{z'+\epsilon} - \left.\frac{dg_{lm}}{dz}\right|_{z'-\epsilon} = -\frac{16\pi}{ab} \qquad \Longrightarrow \tag{19}$$

$$-K_{lm}B\cosh\left[K_{lm}\left(c-z'\right)\right] - K_{lm}A\cosh\left(K_{lm}z'\right) = -\frac{16\pi}{ab} \tag{20}$$

Plugging (18) into (20), we obtain

$$A\frac{\sinh(K_{lm}z')\cosh[K_{lm}(c-z')]}{\sinh[K_{lm}(c-z')]} + A\cosh\left(K_{lm}z'\right) = \frac{16\pi}{K_{lm}ab} \qquad \Longrightarrow$$

$$A = \frac{16\pi\sinh[K_{lm}(c-z')]}{K_{lm}ab\sinh(K_{lm}c)} \qquad B = \frac{16\pi\sinh(K_{lm}z')}{K_{lm}ab\sinh(K_{lm}c)} \tag{21}$$

Finally, this gives

$$g_{lm}(z) = \frac{16\pi\sinh[K_{lm}(c-z_>)]\sinh(K_{lm}z_<)}{K_{lm}ab\sinh(K_{lm}c)} \qquad \text{and} \tag{22}$$

$$G\left(\mathbf{x},\mathbf{x}'\right) = \frac{16\pi}{ab}\sum_{l,m=1}^{\infty}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\frac{\sinh[K_{lm}(c-z_>)]\sinh(K_{lm}z_<)}{K_{lm}\sinh(K_{lm}c)} \tag{23}$$

5. **Proof of (3.169) using Fourier transform**

Comparing (8) and (23), we must have (Jackson 3.169)

$$\frac{\sinh[K_{lm}(c-z_>)]\sinh(K_{lm}z_<)}{K_{lm}\sinh(K_{lm}c)} = \frac{2}{c}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)}{K_{lm}^2 + \left(\frac{n\pi}{c}\right)^2} \tag{24}$$

We shall now give a direct proof of this using Fourier transform.

As mentioned earlier, in the range $[0, c]$, with vanishing boundary conditions, the functions $U_n(z) = \sqrt{2/c}\sin(n\pi z/c)$ form a complete orthonormal set of basis. Let

$$f(z) = \frac{\sinh[K_{lm}(c-z_>)]\sinh(K_{lm}z_<)}{K_{lm}\sinh(K_{lm}c)} \tag{25}$$

then by Fourier transform

$$f(z) = \sum_{n=1}^{\infty}A_n\sqrt{\frac{2}{c}}\sin\left(\frac{n\pi z}{c}\right) \qquad \text{where} \tag{26}$$

$$A_n = \int_0^c f(z)\sqrt{\frac{2}{c}}\sin\left(\frac{n\pi z}{c}\right)dz \tag{27}$$

Let's focus on the integral

$$\begin{aligned} I &= \int_0^c f(z)\sin\left(\frac{n\pi z}{c}\right)dz \\ &= \int_0^{z'}\frac{\sinh[K_{lm}(c-z')]\sinh(K_{lm}z)}{K_{lm}\sinh(K_{lm}c)}\sin\left(\frac{n\pi z}{c}\right)dz + \int_{z'}^{c}\frac{\sinh[K_{lm}(c-z)]\sinh(K_{lm}z')}{K_{lm}\sinh(K_{lm}c)}\sin\left(\frac{n\pi z}{c}\right)dz \\ &= \frac{\sinh[K_{lm}(c-z')]}{K_{lm}\sinh(K_{lm}c)}\underbrace{\int_0^{z'}\sinh(K_{lm}z)\sin\left(\frac{n\pi z}{c}\right)dz}_{A} + \frac{\sinh(K_{lm}z')}{K_{lm}\sinh(K_{lm}c)}\underbrace{\int_{z'}^{c}\sinh[K_{lm}(c-z)]\sin\left(\frac{n\pi z}{c}\right)dz}_{B} \end{aligned} \tag{28}$$

Using the well known integration formula (see Wikipedia)

$$\int \sinh(ax+b)\sin(cx+d)\,dx = \frac{a}{a^2+c^2}\cosh(ax+b)\sin(cx+d) - \frac{c}{a^2+c^2}\sinh(ax+b)\cos(cx+d) + C \tag{29}$$

we have

$$\begin{aligned} A &= \left.\frac{K_{lm}\cosh(K_{lm}z)\sin\left(\frac{n\pi z}{c}\right) - \left(\frac{n\pi}{c}\right)\sinh(K_{lm}z)\cos\left(\frac{n\pi z}{c}\right)}{K_{lm}^2+\left(\frac{n\pi}{c}\right)^2}\right|_0^{z'} \\ &= \frac{K_{lm}\cosh\left(K_{lm}z'\right)\sin\left(\frac{n\pi z'}{c}\right) - \left(\frac{n\pi}{c}\right)\sinh\left(K_{lm}z'\right)\cos\left(\frac{n\pi z'}{c}\right)}{K_{lm}^2+\left(\frac{n\pi}{c}\right)^2} \end{aligned} \tag{30}$$

$$\begin{aligned} B &= \left.\frac{-K_{lm}\cosh[K_{lm}(c-z)]\sin\left(\frac{n\pi z}{c}\right) - \left(\frac{n\pi}{c}\right)\sinh[K_{lm}(c-z)]\cos\left(\frac{n\pi z}{c}\right)}{K_{lm}^2+\left(\frac{n\pi}{c}\right)^2}\right|_{z'}^{c} \\ &= \frac{K_{lm}\cosh\left[K_{lm}\left(c-z'\right)\right]\sin\left(\frac{n\pi z'}{c}\right) + \left(\frac{n\pi}{c}\right)\sinh\left[K_{lm}\left(c-z'\right)\right]\cos\left(\frac{n\pi z'}{c}\right)}{K_{lm}^2+\left(\frac{n\pi}{c}\right)^2} \end{aligned} \tag{31}$$

Plugging these back into (28) yields

$$\begin{aligned} I &= \frac{K_{lm}\sin\left(\frac{n\pi z'}{c}\right)\left\{\overbrace{\sinh\left[K_{lm}\left(c-z'\right)\right]\cosh\left(K_{lm}z'\right) + \sinh\left(K_{lm}z'\right)\cosh\left[K_{lm}\left(c-z'\right)\right]}^{\sinh(K_{lm}c)}\right\}}{K_{lm}\sinh(K_{lm}c)\left[K_{lm}^2+\left(\frac{n\pi}{c}\right)^2\right]} \\ &= \frac{\sin\left(\frac{n\pi z'}{c}\right)}{K_{lm}^2+\left(\frac{n\pi}{c}\right)^2} \end{aligned} \tag{32}$$

(24) is recovered by plugging (32) back into (27) and (26).