In these notes, we will review the definition of vector spherical harmonics (VSH) and state and prove some of their basic properties.

1. **Definition**

   A set of three vector fields is defined over the 3-space

$$\mathbf{Y}_{lm}(\mathbf{r}) = Y_{lm}(\theta,\phi)\hat{\mathbf{r}} \tag{1}$$
$$\mathbf{\Psi}_{lm}(\mathbf{r}) = r\nabla Y_{lm}(\theta,\phi) \tag{2}$$
$$\mathbf{\Phi}_{lm}(\mathbf{r}) = \mathbf{r}\times\nabla Y_{lm}(\theta,\phi) \tag{3}$$

   where $Y_{lm}(\theta,\phi)$ is the usual (scalar) spherical harmonics. The factor r and $\mathbf{r}$ in (2) and (3) ensure $\mathbf{\Psi}_{lm}$ and $\mathbf{\Phi}_{lm}$ have no r-dependency so all of them are in fact only functions of (θ,ϕ).

2. **Orthogonality**

   Since $\mathbf{Y}_{lm}$ has only $\hat{\mathbf{r}}$ component and Y_{lm} has only θ,ϕ dependency, it is clear from the definition that at $\mathbf{r}$,

$$\mathbf{Y}_{lm}(\mathbf{r})\cdot\mathbf{\Psi}_{lm}(\mathbf{r}) = 0 \qquad \mathbf{Y}_{lm}(\mathbf{r})\cdot\mathbf{\Phi}_{lm}(\mathbf{r}) = 0 \qquad \mathbf{\Psi}_{lm}(\mathbf{r})\cdot\mathbf{\Phi}_{lm}(\mathbf{r}) = 0 \tag{4}$$

   Furthermore, for all l,l' and m,m',

$$\mathbf{Y}_{lm}(\mathbf{r})\cdot\mathbf{\Psi}_{l'm'}(\mathbf{r}) = 0 \qquad \mathbf{Y}_{lm}(\mathbf{r})\cdot\mathbf{\Phi}_{l'm'}(\mathbf{r}) = 0 \tag{5}$$

   They are also orthogonal in Hilbert space:

$$\int \mathbf{Y}_{lm}\cdot\mathbf{Y}^*_{l'm'}d\Omega = \delta_{ll'}\delta_{mm'} \tag{6}$$
$$\int \mathbf{\Psi}_{lm}\cdot\mathbf{\Psi}^*_{l'm'}d\Omega = l(l+1)\delta_{ll'}\delta_{mm'} \tag{7}$$
$$\int \mathbf{\Phi}_{lm}\cdot\mathbf{\Phi}^*_{l'm'}d\Omega = l(l+1)\delta_{ll'}\delta_{mm'} \tag{8}$$
$$\int \mathbf{Y}_{lm}\cdot\mathbf{\Psi}^*_{l'm'}d\Omega = 0 \tag{9}$$
$$\int \mathbf{Y}_{lm}\cdot\mathbf{\Phi}^*_{l'm'}d\Omega = 0 \tag{10}$$
$$\int \mathbf{\Psi}_{lm}\cdot\mathbf{\Phi}^*_{l'm'}d\Omega = 0 \tag{11}$$

   Proof. (6) follows immediately from the orthonormality of spherical harmonics.

   With the vector identity $\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$, we have

$$\int \mathbf{\Psi}_{lm}\cdot\mathbf{\Psi}^*_{l'm'}d\Omega = \int r^2\nabla Y_{lm}\cdot\nabla Y^*_{l'm'}d\Omega = \int r^2\left[\nabla\cdot\left(Y^*_{l'm'}\nabla Y_{lm}\right) - Y^*_{l'm'}\nabla^2 Y_{lm}\right]d\Omega \tag{12}$$

   Denote

$$\mathbf{A}(\theta,\phi) \equiv Y^*_{l'm'}(r\nabla Y_{lm}) \tag{13}$$

   then we see the first term's integral in (12) vanishes because

$$\begin{aligned}\int r\nabla\cdot\mathbf{A}d\Omega &= \int\left[\frac{1}{\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right]d\Omega \\ &= \int_0^{2\pi}d\phi\underbrace{\int_0^{\pi}d\theta\frac{\partial(A_\theta\sin\theta)}{\partial\theta}}_{0} + \int_0^{\pi}d\theta\underbrace{\int_0^{2\pi}d\phi\frac{\partial A_\phi}{\partial\phi}}_{0} = 0\end{aligned} \tag{14}$$

Since the spherical harmonics Y_{lm} satisfies the differential equation

$$r^2\nabla^2 Y_{lm} = -l(l+1)Y_{lm} \tag{15}$$

the second integral of (12) evaluates to $l(l+1)\delta_{ll'}\delta_{mm'}$ due to the orthonormality of spherical harmonics, which completes the proof of (7).

(8) is implied by (7) after invoking the vector identity

$$\mathbf{\Phi}_{lm}\cdot\mathbf{\Phi}^*_{l'm'} = (\mathbf{r}\times\nabla Y_{lm})\cdot\left(\mathbf{r}\times\nabla Y^*_{l'm'}\right) = r^2\nabla Y_{lm}\cdot\nabla Y^*_{l'm'} - \overbrace{\left(\mathbf{r}\cdot\nabla Y^*_{l'm'}\right)}^{0}\overbrace{\left(\mathbf{r}\cdot\nabla Y_{lm}\right)}^{0} = \mathbf{\Psi}_{lm}\cdot\mathbf{\Psi}^*_{l'm'} \tag{16}$$

(9) and (10) are trivial since $\mathbf{Y}_{lm}$ points radially and $\mathbf{\Psi}^*_{l'm'}, \mathbf{\Phi}^*_{l'm'}$ are transverse.

To see (11), note that

$$\mathbf{\Psi}_{lm}\cdot\mathbf{\Phi}^*_{l'm'} = r\nabla Y_{lm}\cdot\left(\mathbf{r}\times\nabla Y^*_{l'm'}\right) = \mathbf{r}\cdot\left[\nabla Y^*_{l'm'}\times(r\nabla Y_{lm})\right] = \mathbf{r}\cdot\left[\nabla\times\left(Y^*_{l'm'}r\nabla Y_{lm}\right) - Y^*_{l'm'}\nabla\times(r\nabla Y_{lm})\right] \tag{17}$$

The second term in the bracket has no $\hat{\mathbf{r}}$ component since

$$\nabla\times(r\nabla Y_{lm}) = \nabla r\times\nabla Y_{lm} + r\,\overbrace{\nabla\times(\nabla Y_{lm})}^{0} = \hat{\mathbf{r}}\times\nabla Y_{lm} \tag{18}$$

The first term is just $\nabla\times\mathbf{A}(\theta,\phi)$ with $\mathbf{A}$ defined in (13). This gives

$$\begin{aligned}\int \mathbf{\Psi}_{lm}\cdot\mathbf{\Phi}^*_{l'm'}d\Omega &= \int r\cdot\frac{1}{r\sin\theta}\left[\frac{\partial\left(A_\phi\sin\theta\right)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]d\Omega \\ &= \int_0^{2\pi}d\phi\underbrace{\int_0^{\pi}d\theta\frac{\partial\left(A_\phi\sin\theta\right)}{\partial\theta}}_{0} - \int_0^{\pi}d\theta\underbrace{\int_0^{2\pi}d\phi\frac{\partial A_\theta}{\partial\phi}}_{0} = 0\end{aligned} \tag{19}$$

□

3. **Divergence**

We have the following divergence relations

$$\nabla\cdot[f(r)\mathbf{Y}_{lm}] = \left(\frac{df}{dr} + \frac{2}{r}f\right)Y_{lm} \tag{20}$$

$$\nabla\cdot[f(r)\mathbf{\Psi}_{lm}] = -\frac{l(l+1)}{r}fY_{lm} \tag{21}$$

$$\nabla\cdot[f(r)\mathbf{\Phi}_{lm}] = 0 \tag{22}$$

Proof. Indeed,

$$\nabla\cdot[f(r)\mathbf{Y}_{lm}] = \nabla f\cdot\mathbf{Y}_{lm} + f\nabla\cdot\mathbf{Y}_{lm} = \frac{df}{dr}Y_{lm} + f\nabla\cdot\mathbf{Y}_{lm} = \frac{df}{dr}Y_{lm} + f\frac{1}{r^2}\frac{\partial\left(r^2Y_{lm}\right)}{\partial r} = \left(\frac{df}{dr} + \frac{2}{r}f\right)Y_{lm} \tag{23}$$

$$\nabla\cdot[f(r)\mathbf{\Psi}_{lm}] = \nabla\cdot[f(r)r\nabla Y_{lm}] = \underbrace{\frac{d(rf)}{dr}\hat{\mathbf{r}}\cdot\nabla Y_{lm}}_{0} + rf\nabla^2 Y_{lm} = -\frac{l(l+1)}{r}fY_{lm} \tag{24}$$

$$\begin{aligned}\nabla\cdot[f(r)\mathbf{\Phi}_{lm}] &= \nabla\cdot[f\mathbf{r}\times\nabla Y_{lm}] = \underbrace{\frac{df}{dr}\hat{\mathbf{r}}\cdot(\mathbf{r}\times\nabla Y_{lm})}_{0} + f\nabla\cdot(\mathbf{r}\times\nabla Y_{lm}) \\ &= f\left[\nabla Y_{lm}\cdot(\nabla\times\mathbf{r}) - \mathbf{r}\cdot(\nabla\times\nabla Y_{lm})\right] = 0\end{aligned} \tag{25}$$

□

4. **Curl**

The following are true

$$\nabla\times[f(r)\mathbf{Y}_{lm}] = -\frac{1}{r}f\mathbf{\Phi}_{lm} \tag{26}$$

$$\nabla\times[f(r)\mathbf{\Psi}_{lm}] = \left(\frac{df}{dr} + \frac{1}{r}f\right)\mathbf{\Phi}_{lm} \tag{27}$$

$$\nabla\times[f(r)\mathbf{\Phi}_{lm}] = -\frac{l(l+1)}{r}f\mathbf{Y}_{lm} - \left(\frac{df}{dr} + \frac{1}{r}f\right)\mathbf{\Psi}_{lm} \tag{28}$$

Proof. (26) and (27) are straightforward,

$$\nabla \times [f(r)\mathbf{Y}_{lm}] = \overbrace{\frac{df}{dr}\hat{\mathbf{r}} \times \mathbf{Y}_{lm}}^{0} + f\,\nabla \times \mathbf{Y}_{lm} = f\left(\nabla Y_{lm} \times \hat{\mathbf{r}} + Y_{lm}\overbrace{\nabla \times \hat{\mathbf{r}}}^{0}\right) = -\frac{1}{r} f\,\mathbf{r} \times \nabla Y_{lm} = -\frac{1}{r} f\,\mathbf{\Phi}_{lm} \tag{29}$$

$$\begin{aligned}\nabla \times [f(r)\mathbf{\Psi}_{lm}] &= \nabla \times [f(r)\, r\nabla Y_{lm}] = \nabla(rf) \times \nabla Y_{lm} + (rf)\overbrace{\nabla \times \nabla Y_{lm}}^{0} \\ &= \left(f + r\frac{df}{dr}\right)\hat{\mathbf{r}} \times \nabla Y_{lm} = \left(\frac{df}{dr} + \frac{1}{r}f\right)\mathbf{\Phi}_{lm}\end{aligned} \tag{30}$$

To see (28), first note

$$\nabla \times [f(r)\mathbf{\Phi}_{lm}] = \overbrace{\frac{df}{dr}\hat{\mathbf{r}} \times (\mathbf{r} \times \nabla Y_{lm})}^{S} + f\,\overbrace{\nabla \times (\mathbf{r} \times \nabla Y_{lm})}^{T} \tag{31}$$

It is clear that

$$S = \frac{df}{dr}(-r)\nabla Y_{lm} = -\frac{df}{dr}\mathbf{\Psi}_{lm} \tag{32}$$

Expanding T using identity $\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$, we get

$$T = \overbrace{\mathbf{r}\nabla^2 Y_{lm}}^{T_1} - \overbrace{\nabla Y_{lm}(\nabla \cdot \mathbf{r})}^{T_2} + \overbrace{(\nabla Y_{lm} \cdot \nabla)\mathbf{r}}^{T_3} - \overbrace{(\mathbf{r} \cdot \nabla)\nabla Y_{lm}}^{T_4} \tag{33}$$

where

$$T_1 = -\frac{l(l+1)}{r}\hat{\mathbf{r}} Y_{lm} = -\frac{l(l+1)}{r}\mathbf{Y}_{lm} \tag{34}$$

$$T_2 = 3\nabla Y_{lm} \tag{35}$$

$$T_3 = \left[(\nabla Y_{lm})_x \frac{\partial}{\partial x} + (\nabla Y_{lm})_y \frac{\partial}{\partial y} + (\nabla Y_{lm})_z \frac{\partial}{\partial z}\right](x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) = \nabla Y_{lm} \tag{36}$$

$$T_4 = \left(r\frac{\partial}{\partial r}\right)\left[\frac{1}{r}\left(\frac{\partial Y_{lm}}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial \phi}\hat{\boldsymbol{\phi}}\right)\right] = -\frac{1}{r}\left(\frac{\partial Y_{lm}}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial \phi}\hat{\boldsymbol{\phi}}\right) = -\nabla Y_{lm} \tag{37}$$

which gives

$$T = -\frac{l(l+1)}{r}\mathbf{Y}_{lm} - \nabla Y_{lm} = -\frac{l(l+1)}{r}\mathbf{Y}_{lm} - \frac{1}{r}\mathbf{\Psi}_{lm} \tag{38}$$

Putting (38) and (32) back into (31) yields the desired identity (28). □