1. **Problem 8.19**

(a) The current density is

$$\mathbf{J} = I_0 \sin\left[\frac{\omega}{c}(h-y)\right]\delta(z)\,\delta(x-X)\left[\Theta(y)-\Theta(h-y)\right]\hat{\mathbf{y}} \tag{1}$$

We use (8.146) to determine the amplitudes of the modes

$$A_\lambda^{(\pm)} = -\frac{Z_\lambda}{2}\int_V \mathbf{J}\cdot\mathbf{E}_\lambda^{(\mp)} d^3x \tag{2}$$

From (8.135)

$$\text{TM}: \qquad E_{ymn} = \frac{2\pi n}{\gamma_{mn} b\sqrt{ab}}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \tag{3}$$

$$\text{TE}: \qquad E_{ymn} = \frac{2\pi m}{\gamma_{mn} a\sqrt{ab}}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \qquad \times\frac{1}{\sqrt{2}} \text{ if } m=0 \text{ or } n=0 \tag{4}$$

where

$$\gamma_{mn}^2 = \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right) \tag{5}$$

Plugging everything into (2) yields

$$\text{TM}: \quad A_{mn}^{(\pm)} = -\frac{Z_{mn}^{\text{TM}} I_0 \pi n}{\gamma_{mn} b\sqrt{ab}}\sin\left(\frac{m\pi X}{a}\right)\overbrace{\int_0^h \sin\left[\frac{\omega}{c}(h-y)\right]\cos\left(\frac{n\pi y}{b}\right)dy}^{I} \tag{6}$$

$$\text{TE}: \quad A_{mn}^{(\pm)} = -\frac{Z_{mn}^{\text{TE}} I_0 \pi m}{\gamma_{mn} a\sqrt{ab}}\sin\left(\frac{m\pi X}{a}\right)\int_0^h \sin\left[\frac{\omega}{c}(h-y)\right]\cos\left(\frac{n\pi y}{b}\right)dy \qquad \times\frac{1}{\sqrt{2}} \text{ if } m=0 \text{ or } n=0 \tag{7}$$

With

$$\sin\left[\frac{\omega}{c}(h-y)\right]\cos\left(\frac{n\pi y}{b}\right) = \frac{1}{2}\left\{\sin\left[\frac{\omega h}{c}-\left(\frac{\omega}{c}-\frac{n\pi}{b}\right)y\right]+\sin\left[\frac{\omega h}{c}-\left(\frac{\omega}{c}+\frac{n\pi}{b}\right)y\right]\right\} \tag{8}$$

the integral in (6) and (7) can be evaluated

$$\begin{aligned} I &= \frac{1}{2}\left\{\left(\frac{1}{\frac{\omega}{c}-\frac{n\pi}{b}}\right)\left[\cos\left(\frac{n\pi h}{b}\right)-\cos\left(\frac{\omega h}{c}\right)\right]+\left(\frac{1}{\frac{\omega}{c}+\frac{n\pi}{b}}\right)\left[\cos\left(\frac{n\pi h}{b}\right)-\cos\left(\frac{\omega h}{c}\right)\right]\right\} \\ &= \left[\frac{\cos\left(\frac{n\pi h}{b}\right)-\cos\left(\frac{\omega h}{c}\right)}{\left(\frac{\omega}{c}\right)^2-\left(\frac{n\pi}{b}\right)^2}\right]\frac{\omega}{c} \end{aligned} \tag{9}$$

As $m \gg 1, n \gg 1$,

$$Z_{mn}^{\text{TM}} = \frac{k_{mn}}{\epsilon\omega} = \frac{1}{\epsilon\omega}\sqrt{\left(\frac{\omega}{c}\right)^2-\gamma_{mn}^2} \qquad \longrightarrow \qquad \frac{i\gamma_{mn}}{\epsilon\omega} \tag{10}$$

$$Z_{mn}^{\text{TE}} = \frac{\mu\omega}{k} = \frac{\mu\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2-\gamma_{mn}^2}} \qquad \longrightarrow \qquad -\frac{i\mu\omega}{\gamma_{mn}} \tag{11}$$

hence

$$\text{TM}: \qquad A_{mn}^{(\pm)} \qquad \longrightarrow \qquad \text{const}\times\frac{1}{n}\left[\cos\left(\frac{\omega h}{c}\right)-\cos\left(\frac{n\pi h}{b}\right)\right]\sin\left(\frac{m\pi X}{a}\right) \tag{12}$$

$$\text{TE}: \qquad A_{mn}^{(\pm)} \qquad \longrightarrow \qquad \text{const}\times\frac{m}{n^2}\left(\frac{1}{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\right)\left[\cos\left(\frac{\omega h}{c}\right)-\cos\left(\frac{n\pi h}{b}\right)\right]\sin\left(\frac{m\pi X}{a}\right) \tag{13}$$

(b) For the lowest propagating mode TE_{10}, we have (don't forget the factor of $1/\sqrt{2}$ for $n=0$)

$$\begin{aligned}
A_{10}^{\pm} &= -\frac{1}{\sqrt{2}}\frac{Z_{10}^{\mathrm{TE}} I_0 \pi}{\pi\sqrt{ab}} \sin\left(\frac{\pi X}{a}\right)\cdot\left[\frac{1-\cos\left(\frac{\omega h}{c}\right)}{\left(\frac{\omega}{c}\right)^2}\right]\frac{\omega}{c} \\
&= -\sqrt{2}\frac{Z_{10}^{\mathrm{TE}} I_0}{\sqrt{ab}} \sin\left(\frac{\pi X}{a}\right)\cdot\sin^2\left(\frac{\omega h}{2c}\right)\cdot\frac{c}{\omega}
\end{aligned} \tag{14}$$

Then by (8.133) with proper normalization and that $Z_{10}^{\mathrm{TE}} = \mu\omega/k$,

$$P^{(\pm)} = \frac{\left[A_{10}^{(\pm)}\right]^2}{2Z_{10}^{\mathrm{TE}}} = \frac{\mu c^2 I_0^2}{\omega k a b}\sin^2\left(\frac{\pi X}{a}\right)\sin^4\left(\frac{\omega h}{2c}\right) \tag{15}$$

(c) If the guide has an end at $-L$, the wave will be reflected. We can refer back to problem 7.4 and see that the reflection of normal incident waves will introduce a phase change of π. To maximize the propagating wave, we can make $2kL=\pi$ so the reflected wave will constructively interfere with the forward propagating wave, in which case the power is quadrupled. The radiation resistance is thus calculated as

$$R = \frac{4P^{(+)}}{\frac{1}{2}I^2\big|_{y=0}} = \frac{4\frac{\mu c^2 I_0^2}{\omega k a b}\sin^2\left(\frac{\pi X}{a}\right)\sin^4\left(\frac{\omega h}{2c}\right)}{\frac{1}{2}I_0^2\sin^2\left(\frac{\omega h}{c}\right)} = \frac{2\mu c^2}{\omega k a b}\sin^2\left(\frac{\pi X}{a}\right)\tan^2\left(\frac{\omega h}{2c}\right) \tag{16}$$

2. **Problem 8.20**

(a) Let the current go counter clockwise in the diagram. Then by (8.146) and (8.135) the TM mode amplitudes are

$$\begin{aligned}
A_{mn}^{(\pm)} =& -\frac{Z_{mn}^{\mathrm{TM}}}{2}\int_{-\pi/2}^{\pi/2} R\,d\phi\left(J_x E_{xmn} + J_y E_{ymn}\right) \\
=& -\frac{Z_{mn}^{\mathrm{TM}} R}{2}\int_{-\pi/2}^{\pi/2} I_0(-\sin\phi)\cdot\frac{2\pi m}{\gamma_{mn} a\sqrt{ab}}\cos\left(\frac{m\pi R\cos\phi}{a}\right)\sin\left[\frac{n\pi(h+R\sin\phi)}{b}\right]d\phi \\
& -\frac{Z_{mn}^{\mathrm{TM}} R}{2}\int_{-\pi/2}^{\pi/2} I_0\cos\phi\cdot\frac{2\pi n}{\gamma_{mn} b\sqrt{ab}}\sin\left(\frac{m\pi R\cos\phi}{a}\right)\cos\left[\frac{n\pi(h+R\sin\phi)}{b}\right]d\phi \\
=& \frac{Z_{mn}^{\mathrm{TM}}\pi I_0 R}{\gamma_{mn}\sqrt{ab}}\left(I_x - I_y\right)
\end{aligned} \tag{17}$$

where

$$I_x = \int_{-\pi/2}^{\pi/2}\frac{m}{a}\sin\phi\cos\left(\frac{m\pi R\cos\phi}{a}\right)\sin\left[\frac{n\pi(h+R\sin\phi)}{b}\right]d\phi \tag{18}$$

$$I_y = \int_{-\pi/2}^{\pi/2}\frac{n}{b}\cos\phi\sin\left(\frac{m\pi R\cos\phi}{a}\right)\cos\left[\frac{n\pi(h+R\sin\phi)}{b}\right]d\phi \tag{19}$$

With

$$\sin\left[\frac{n\pi(h+R\sin\phi)}{b}\right] = \sin\left(\frac{n\pi h}{b}\right)\cos\left(\frac{n\pi R\sin\phi}{b}\right) + \cos\left(\frac{n\pi h}{b}\right)\sin\left(\frac{n\pi R\sin\phi}{b}\right) \tag{20}$$

$$\cos\left[\frac{n\pi(h+R\sin\phi)}{b}\right] = \cos\left(\frac{n\pi h}{b}\right)\cos\left(\frac{n\pi R\sin\phi}{b}\right) - \sin\left(\frac{n\pi h}{b}\right)\sin\left(\frac{n\pi R\sin\phi}{b}\right) \tag{21}$$

and taking advantage of the parity of the integrand in (18) and (19), we have

$$I_x = \cos\left(\frac{n\pi h}{b}\right)\int_{-\pi/2}^{\pi/2}\frac{m}{a}\sin\phi\cos\left(\frac{m\pi R\cos\phi}{a}\right)\overbrace{\sin\left(\frac{n\pi R\sin\phi}{b}\right)}^{u(\phi)}d\phi \tag{22}$$

$$I_y = \cos\left(\frac{n\pi h}{b}\right)\int_{-\pi/2}^{\pi/2}\frac{n}{b}\cos\phi\underbrace{\sin\left(\frac{m\pi R\cos\phi}{a}\right)}_{v(\phi)}\cos\left(\frac{n\pi R\sin\phi}{b}\right)d\phi \tag{23}$$

Also notice that

$$\frac{m}{a}\sin\phi\cos\left(\frac{m\pi R\cos\phi}{a}\right)=-\frac{1}{\pi R}\frac{d}{d\phi}\left[\sin\left(\frac{m\pi R\cos\phi}{a}\right)\right]=-\frac{1}{\pi R}\frac{dv(\phi)}{d\phi} \tag{24}$$

$$\frac{n}{b}\cos\phi\cos\left(\frac{n\pi R\sin\phi}{b}\right)=\frac{1}{\pi R}\frac{d}{d\phi}\left[\sin\left(\frac{n\pi R\sin\phi}{b}\right)\right]=\frac{1}{\pi R}\frac{du(\phi)}{d\phi} \tag{25}$$

This allows us to conveniently obtain

$$\begin{aligned}I_x-I_y&=-\frac{1}{\pi R}\cos\left(\frac{n\pi h}{b}\right)\int_{-\pi/2}^{\pi/2}\left[\frac{dv(\phi)}{d\phi}u(\phi)+\frac{du(\phi)}{d\phi}v(\phi)\right]d\phi\\&=-\frac{1}{\pi R}\cos\left(\frac{n\pi h}{b}\right)u(\phi)v(\phi)\Big|_{-\pi/2}^{\pi/2}=0\end{aligned} \tag{26}$$

Going back to (17), we see that none of the TM modes survive.

This magnificent result can be explained by considering the path integral along the loop formed by the semicircle followed by a straight line on the wall that close the loop,

$$\oint_C \mathbf{E}_{mn}^{\mathrm{TM}}\cdot d\mathbf{l}\propto\int_V \mathbf{J}\cdot\mathbf{E}_{mn}^{\mathrm{TM}}d^3x\propto A_{mn} \tag{27}$$

With constant current on the semicircle, and by Stoke's theorem, the LHS of (27) is proportional to the z-direction flux of B_z which vanishes for the TM mode.

(b) We can repeat the above steps for the TE mode,

$$\begin{aligned}A_{mn}^{(\pm)}&=-\frac{Z_{mn}^{\mathrm{TE}}R}{2}\int_{-\pi/2}^{\pi/2}I_0(-\sin\phi)\cdot\left(\frac{-2\pi n}{\gamma_{mn}b\sqrt{ab}}\right)\cos\left(\frac{m\pi R\cos\phi}{a}\right)\sin\left[\frac{n\pi(h+R\sin\phi)}{b}\right]d\phi\\&\quad-\frac{Z_{mn}^{\mathrm{TE}}R}{2}\int_{-\pi/2}^{\pi/2}I_0\cos\phi\cdot\frac{2\pi m}{\gamma_{mn}a\sqrt{ab}}\sin\left(\frac{m\pi R\cos\phi}{a}\right)\cos\left[\frac{n\pi(h+R\sin\phi)}{b}\right]d\phi\\&=-\frac{Z_{mn}^{\mathrm{TE}}\pi I_0R}{\gamma_{mn}\sqrt{ab}}\left(I_x+I_y\right)\end{aligned} \tag{28}$$

where

$$\begin{aligned}I_x&=\int_{-\pi/2}^{\pi/2}\frac{n}{b}\sin\phi\cos\left(\frac{m\pi R\cos\phi}{a}\right)\sin\left[\frac{n\pi(h+R\sin\phi)}{b}\right]d\phi \qquad\text{by (20) and parity}\\&=\cos\left(\frac{n\pi h}{b}\right)\int_{-\pi/2}^{\pi}\frac{n}{b}\sin\phi\cos\left(\frac{m\pi R\cos\phi}{a}\right)\sin\left(\frac{n\pi R\sin\phi}{b}\right)d\phi\end{aligned} \tag{29}$$

$$\begin{aligned}I_y&=\int_{-\pi/2}^{\pi/2}\frac{m}{a}\cos\phi\sin\left(\frac{m\pi R\cos\phi}{a}\right)\cos\left[\frac{n\pi(h+R\sin\phi)}{b}\right]d\phi \qquad\text{by (21) and parity}\\&=\cos\left(\frac{n\pi h}{b}\right)\int_{-\pi/2}^{\pi/2}\frac{m}{a}\cos\phi\sin\left(\frac{m\pi R\cos\phi}{a}\right)\cos\left(\frac{n\pi R\sin\phi}{b}\right)d\phi\end{aligned} \tag{30}$$

Unlike the TM modes, I_x+I_y does not conveniently vanish, but we can calculate the lowest mode $m=1, n=0$,

$$I_x=0 \tag{31}$$

$$\begin{aligned}I_y&=\int_{-\pi/2}^{\pi/2}\frac{1}{a}\cos\phi\sin\left(\frac{\pi R\cos\phi}{a}\right)d\phi \qquad\text{for } R\ll a\\&\approx\frac{1}{a}\int_{-\pi/2}^{\pi/2}\frac{\pi R}{a}\cos^2\phi\, d\phi=\frac{\pi^2R}{2a^2}\end{aligned} \tag{32}$$

With the extra factor of $1/\sqrt{2}$, the amplitude for TE_{10} is

$$A_{10}^{(\pm)}=-\frac{Z_{10}^{\mathrm{TE}}\pi^2I_0R^2}{2\sqrt{2}a\sqrt{ab}} \tag{33}$$

(c) The power for TE_{10} is

$$P^{(\pm)}=\frac{\left[A_{10}^{(\pm)}\right]^2}{2Z_{10}^{\mathrm{TE}}}=\frac{I_0^2}{16}Z_{10}^{\mathrm{TE}}\frac{\pi^4R^4}{a^3b} \tag{34}$$