In these notes, we fill the details omitted in Jackson section 3.13 *Mixed Boundary Conditions; Conducting Plane with a Circular Hole*.

1. **Proof of equation (3.175)**

   It was claimed (3.175) that

$$B_l = \frac{1}{l!}\left(-\frac{d}{d|z|}\right)^l \int_0^\infty dk e^{-k|z|} J_0(k\rho) = \frac{1}{l!}\left(-\frac{d}{d|z|}\right)^l \left(\frac{1}{\sqrt{\rho^2+z^2}}\right) \tag{1}$$

   For this, we need to prove the integral identity

$$\int_0^\infty dk e^{-k|z|} J_0(k\rho) = \frac{1}{\sqrt{\rho^2+z^2}} \tag{2}$$

   To see this, insert the expansion of $J_0(k\rho)$ into the LHS

$$\begin{aligned}
\mathrm{LHS}_{(2)} &= \int_0^\infty dk e^{-k|z|} \sum_{j=0}^\infty \frac{(-1)^j}{j!j!}\left(\frac{k\rho}{2}\right)^{2j} \\
&= \sum_{j=0}^\infty \frac{(-1)^j}{j!j!}\left(\frac{\rho}{2}\right)^{2j} \int_0^\infty dk e^{-k|z|} k^{2j} \\
&= \sum_{j=0}^\infty \frac{(-1)^j}{j!j!}\left(\frac{\rho}{2}\right)^{2j} \cdot \left[\frac{(2j)!}{|z|^{2j+1}}\right] \\
&= \frac{1}{|z|}\sum_{j=0}^\infty \frac{1}{j!}\frac{(-1)^j}{2^j}\left(\frac{\rho}{|z|}\right)^{2j} \frac{(2j)!}{2^j j!} \\
&= \frac{1}{|z|}\sum_{j=0}^\infty \frac{1}{j!}\left[\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2j-1}{2}\right)\right]\left(\frac{\rho}{|z|}\right)^{2j}
\end{aligned} \tag{3}$$

   where in the second line, we have used the integral

$$\int_0^\infty e^{-ax} x^n dx = \frac{n!}{a^{n+1}} \tag{4}$$

   Then (3) can be readily recognized as the Taylor series for

$$\frac{1}{|z|}\frac{1}{\sqrt{1+\left(\frac{\rho}{|z|}\right)^2}} = \frac{1}{\sqrt{\rho^2+z^2}} \tag{5}$$

2. **Proof of equation (3.176)**

   Next, we have a claim that

$$B_l = \frac{1}{l!}\left(-\frac{d}{d|z|}\right)^l \left(\frac{1}{\sqrt{\rho^2+z^2}}\right) = \frac{P_l(|\cos\theta|)}{r^{l+1}} \qquad \text{where } \cos\theta = \frac{z}{r}, r = \sqrt{\rho^2+z^2} \tag{6}$$

   We can prove this by induction:

   - $l = 0$, trivially true;
   - $l = 1$:

$$-\frac{d}{d|z|}\frac{1}{\sqrt{\rho^2+z^2}} = \frac{|z|}{r^3} = \frac{P_1(|\cos\theta|)}{r^2} \tag{7}$$

   - By induction, if

$$\frac{1}{(l-1)!}\left(-\frac{d}{d|z|}\right)^{l-1}\left(\frac{1}{r}\right) = \frac{P_{l-1}(|\cos\theta|)}{r^l} \tag{8}$$

Then

$$
\begin{aligned}
B_l &= \frac{1}{l}\left(-\frac{d}{d|z|}\right)B_{l-1} \\
&= -\frac{1}{l}\frac{d}{d|z|}\left[\frac{P_{l-1}(|z|/r)}{r^l}\right] \\
&= -\frac{1}{l}\left[\frac{1}{r^l}\frac{dP_{l-1}(|z|/r)}{d|z|} + P_{l-1}(|z|/r)\frac{(-l)}{r^{l+1}}\frac{dr}{d|z|}\right] \\
&= -\frac{1}{l}\left[\frac{1}{r^l}P'_{l-1}(|z|/r)\frac{d(|z|/r)}{d|z|} - lP_{l-1}(|z|/r)\frac{1}{r^{l+1}}\frac{|z|}{r}\right] \\
&= -\frac{1}{l}\left[\frac{P'_{l-1}(|z|/r)}{r^l}\left(\frac{1}{r}-\frac{|z|}{r^2}\cdot\frac{|z|}{r}\right) - \frac{lP_{l-1}(|z|/r)|z|/r}{r^{l+1}}\right] \\
&= -\frac{1}{r^{l+1}}\left[\frac{1}{l}P'_{l-1}(|z|/r)\left(1-\frac{|z|^2}{r^2}\right) - P_{l-1}(|z|/r)|z|/r\right] \\
&= \frac{P_l(|z|/r)}{r^{l+1}}
\end{aligned}
\tag{9}
$$

where in the last step, we have used the recurrence relation of Legendre polynomials (reference: equation 14.10.4 on dlmf.nist.gov)

$$
\frac{1}{l}P'_{l-1}(x)\left(1-x^2\right) - P_{l-1}(x)x = -P_l(x) \tag{10}
$$

3. **Verification that (3.180) is the solution of (3.179) via Weber–Schafheitlin integral**

For the *dual integral equations* (3.179)

$$
\int_0^\infty dy\, y\, g(y) J_n(yx) = x^n \qquad \text{for } 0 \le x < 1 \tag{11}
$$

$$
\int_0^\infty dy\, g(y) J_n(yx) = 0 \qquad \text{for } 1 \le x \tag{12}
$$

It was claimed that (3.180) is a solution

$$
g(y) = \frac{\Gamma(n+1)}{\sqrt{\pi}\Gamma(n+3/2)} j_{n+1}(y) = \frac{\Gamma(n+1)}{\Gamma(n+3/2)}\frac{J_{n+3/2}(y)}{\sqrt{2y}} \tag{13}
$$

Let's verify it via the *Weber–Schafheitlin* integral (reference equation 10.22.56 on dlmf.nist.gov)

$$
\text{for } 0 < a < b, \mathrm{Re}\,(\mu+\nu+1) > \mathrm{Re}(\lambda) > -1:
$$

$$
\int_0^\infty \frac{J_\mu(at)J_\nu(bt)}{t^\lambda}dt = \frac{a^\mu \Gamma\left(\frac{\nu}{2}+\frac{\mu}{2}-\frac{\lambda}{2}+\frac{1}{2}\right)}{2^\lambda b^{\mu-\lambda+1}\Gamma\left(\frac{\nu}{2}-\frac{\mu}{2}+\frac{\lambda}{2}+\frac{1}{2}\right)} F\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\mu-\nu-\lambda+1}{2}; \mu+1; \frac{a^2}{b^2}\right) \tag{14}
$$

where $F$ is the hypergeometric function

$$
F(a,b;c;z) = 1 + \frac{ab}{c}z + \frac{1}{2!}\frac{a(a+1)b(b+1)}{c(c+1)}z^2 + \cdots \tag{15}
$$

- To see (11), insert $a = x, b = 1, \mu = n, \nu = n+3/2, \lambda = -1/2$ into(14), we have

$$
\int_0^\infty \sqrt{y} J_n(xy) J_{n+3/2}(y) dy = \frac{\sqrt{2}x^n \Gamma(n+3/2)}{\Gamma(1)} \overbrace{F\left(n+\frac{3}{2}, 0, n+1; \frac{a^2}{b^2}\right)}^{=1} \tag{16}
$$

  Thus we see that (13) satisfies (11) except for this $\Gamma(n+1)$ factor (**which I think is a mistake on the book**, but fortunately for the $n=0$ case, it doesn't impact the subsequent discussion).

- For (12), we need $a = 1, b = x, \mu = n+3/2, \nu = n, \lambda = 1/2$, with which the denominator of (14) has $\Gamma(0) = \infty$, hence (14) vanishes, which satisfies (12).

4. **Steps leading from (3.184) to (3.185)**

We are now going to calculate the integration (3.184):

$$\Phi^{(1)}(\rho,z)=\frac{(E_0-E_1)\,a^2}{\pi}\underbrace{\int_0^\infty dk j_1(ka)e^{-k|z|}J_0(k\rho)}_{I} \tag{17}$$

that leads to the claimed result

$$\Phi^{(1)}(\rho,z)=\frac{(E_0-E_1)\,a}{\pi}\left[\sqrt{\frac{R-\lambda}{2}}-\frac{|z|}{a}\tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right)\right] \qquad \text{where}$$

$$\lambda=\frac{z^2+\rho^2-a^2}{a^2} \qquad R=\sqrt{\lambda^2+\frac{4z^2}{a^2}} \tag{18}$$

First, the relation

$$j_1(x)=-j_0'(x) \tag{19}$$

will turn the integral $I$ in (17) into

$$\begin{aligned}
I&=\int_0^\infty dk\left[-j_0'(ka)\right]e^{-k|z|}J_0(k\rho)\\
&=\int_0^\infty dk\left[-\frac{dj_0(ka)}{adk}\right]e^{-k|z|}J_0(k\rho)\\
&=-\frac{1}{a}j_0(ka)e^{-k|z|}J_0(k\rho)\Big|_0^\infty+\frac{1}{a}\int_0^\infty dk j_0(ka)\frac{d}{dk}\left[e^{-k|z|}J_0(k\rho)\right]\\
&=\frac{1}{a}\left[1+\int_0^\infty j_0(ka)(-|z|)e^{-k|z|}J_0(k\rho)dk+\int_0^\infty j_0(ka)e^{-k|z|}J_0'(k\rho)\rho dk\right]\\
&=\frac{1}{a}\left[1-|z|\underbrace{\int_0^\infty j_0(ka)e^{-k|z|}J_0(k\rho)dk}_{K_0}-\rho\underbrace{\int_0^\infty j_0(ka)e^{-k|z|}J_1(k\rho)dk}_{K_1}\right]
\end{aligned} \tag{20}$$

where in the last step, we have used

$$J_0'(x)=-J_1(x) \tag{21}$$

(a) **Calculation of $K_1$**

Since $j_0(ka)=\sin ka/(ka)$, then

$$\begin{aligned}
K_1&=\int_0^\infty \frac{\sin ka}{ka}e^{-k|z|}J_1(k\rho)dk\\
&=\frac{1}{a}\,\mathrm{Im}\left[\int_0^\infty e^{ika}e^{-k|z|}\frac{J_1(k\rho)}{k}dk\right] \qquad \text{define } s\equiv|z|-ia\\
&=\frac{1}{a}\,\mathrm{Im}\left[\int_0^\infty e^{-sk}\frac{J_1(k\rho)}{k}dk\right]
\end{aligned} \tag{22}$$

The content of the square bracket is the Laplace transform of $J_1(k\rho)/k$, denoted $\mathscr{L}\{J_1(k\rho)/\rho\}(s)$. But before calculating it, we have to take a detour.

It is clear that the proof that established (2) can be used to produce

$$\mathscr{L}\{J_0(k\rho)\}(s)=\frac{1}{\sqrt{\rho^2+s^2}} \tag{23}$$

With (21) and the Laplace transform property

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\}(s)=s\mathscr{L}\{f(t)\}(s)-f(0^-) \tag{24}$$

we have

$$\mathscr{L}\left\{J_1(k\rho)\right\}(s)=\mathscr{L}\left\{-J_0'(k\rho)\right\}(s)=\mathscr{L}\left\{-\frac{dJ_0(k\rho)}{\rho dk}\right\}(s)=-\frac{1}{\rho}\left(\frac{s}{\sqrt{s^2+\rho^2}}-1\right)=\frac{1}{\rho}\left(1-\frac{s}{\sqrt{\rho^2+s^2}}\right) \tag{25}$$

Note the recurrence relation of Bessel functions (reference equation 10.6.2 on dlmf.nist.gov)

$$J_\nu'(x)=J_{\nu-1}(x)-\frac{\nu}{x}J_\nu(x) \tag{26}$$

Apply the Laplace transform to (26) with $\nu=1$,

$$\begin{aligned}
\mathscr{L}\left\{\frac{J_1(k\rho)}{k\rho}\right\}(s)&=\mathscr{L}\left\{J_0(k\rho)\right\}(s)-\mathscr{L}\left\{J_1'(k\rho)\right\}(s)\\
&=\mathscr{L}\left\{J_0(k\rho)\right\}(s)-\mathscr{L}\left\{\frac{dJ_1(k\rho)}{\rho dk}\right\}(s) && \text{by (23),(24),(25)}\\
&=\frac{1}{\sqrt{\rho^2+s^2}}-\frac{1}{\rho}\left[\frac{s}{\rho}\left(1-\frac{s}{\sqrt{\rho^2+s^2}}\right)\right]\\
&=\frac{\rho^2-s\sqrt{\rho^2+s^2}+s^2}{\rho^2\sqrt{\rho^2+s^2}}=\frac{1}{\rho^2}\left(\sqrt{\rho^2+s^2}-s\right) && \Longrightarrow
\end{aligned}$$

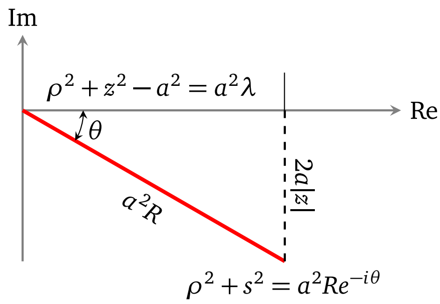
$$\mathscr{L}\left\{\frac{J_1(k\rho)}{k}\right\}(s)=\frac{1}{\rho}\left(\sqrt{\rho^2+s^2}-s\right) \tag{27}$$

In order to get $K_1$ in (22), it remains to extract the imaginary part of (27). Apparently

$$\operatorname{Im}\left[\frac{1}{\rho}\left(\sqrt{\rho^2+s^2}-s\right)\right]=\frac{1}{\rho}\left(\operatorname{Im}\sqrt{\rho^2+s^2}-\operatorname{Im}s\right)=\frac{1}{\rho}\left(\operatorname{Im}\sqrt{\rho^2+s^2}+a\right) \tag{28}$$

Now using the diagram below, we obtain

$$\sqrt{\rho^2+s^2}=\left(\rho^2+z^2-a^2-2a|z|i\right)^{1/2}=a\sqrt{R}e^{-i\theta} \tag{29}$$




hence

$$\operatorname{Im}\sqrt{\rho^2+s^2}=-a\sqrt{R}\sin\frac{\theta}{2} \tag{30}$$

$\sin\theta/2$ can be found via

$$1-2\sin^2\frac{\theta}{2}=\cos\theta=\frac{\lambda}{R}\qquad\Longrightarrow\qquad\sin\frac{\theta}{2}=\sqrt{\frac{R-\lambda}{2R}} \tag{31}$$

Inserting (30), (31) into (28) gives

$$\operatorname{Im}\mathscr{L}\left\{\frac{J_1(k\rho)}{k}\right\}(s)=\frac{a}{\rho}\left(1-\sqrt{\frac{R-\lambda}{2}}\right) \tag{32}$$

which reduces (20) into

$$I=\frac{1}{a}\left(\sqrt{\frac{R-\lambda}{2}}-|z|K_0\right) \tag{33}$$

Compare with the desired result (18), it remains to show

$$K_0 = \int_0^\infty \frac{\sin ka}{ka} e^{-k|z|} J_0(k\rho) dk = \frac{1}{a} \tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right) \qquad \text{or}$$

$$\int_0^\infty \frac{\sin ka}{k} e^{-k|z|} J_0(k\rho) dk_{F(|z|)} = \tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right) \tag{34}$$

(b) **Calculation of $K_0$**

Similar to the $K_1$ case, define

$$F(s) \equiv \int_0^\infty \frac{e^{-sk}}{k} J_0(k\rho) dk, \qquad s = |z| - ia \tag{35}$$

we just need to show

$$\mathrm{Im}[F(s)] = \tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right) \tag{36}$$

Notice by (23)

$$F'(s) = -\int_0^\infty e^{-sk} J_0(k\rho) dk = -\frac{1}{\sqrt{\rho^2+s^2}} \tag{37}$$

Thus $F(s)$ is readily solvable as (reference WolframAlpha)

$$\begin{aligned} F(s) &= -\tanh^{-1}\left(\frac{s}{\sqrt{\rho^2+s^2}}\right) \\ &= \frac{1}{2}\ln\left(1 - \frac{s}{\sqrt{\rho^2+s^2}}\right) - \frac{1}{2}\left(1 + \frac{s}{\sqrt{\rho^2+s^2}}\right) \\ &= \frac{1}{2}\ln\left(\frac{\sqrt{\rho^2+s^2}-s}{\sqrt{\rho^2+s^2}+s}\right) \\ &= \frac{1}{2}\ln\left[\frac{\left(\sqrt{\rho^2+s^2}-s\right)^2}{\rho^2}\right] \\ &= \ln\left(\frac{\sqrt{\rho^2+s^2}-s}{\rho}\right) \end{aligned} \tag{38}$$

The imaginary part of $F(s)$ is just the argument of the complex number $\sqrt{\rho^2+s^2}-s$, i.e.,

$$\begin{aligned} \mathrm{Im}[F(s)] &= \mathrm{Arg}\left(\sqrt{\rho^2+s^2}-s\right) \qquad \text{(see diagram above)} \\ &= \mathrm{Arg}\left[a\sqrt{R}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) - |z| + ia\right] \\ &= \tan^{-1}\left(\frac{a - a\sqrt{R}\sin\frac{\theta}{2}}{a\sqrt{R}\cos\frac{\theta}{2} - |z|}\right) \end{aligned} \tag{39}$$

From (31), we have

$$\cos\frac{\theta}{2} = \sqrt{\frac{R+\lambda}{2R}} \tag{40}$$

hence

$$\mathrm{Im}[F(s)] = \tan^{-1}\left(\frac{a - a\sqrt{R}\sqrt{\frac{R-\lambda}{2R}}}{a\sqrt{R}\sqrt{\frac{R+\lambda}{2R}} - |z|}\right) = \tan^{-1}\left(\frac{a - a\sqrt{\frac{R-\lambda}{2}}}{a\sqrt{\frac{R+\lambda}{2}} - |z|}\right) = \tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right) \tag{41}$$