1. We start by examining $\mathbf{E}\times\mathbf{B}$:

$$\begin{aligned}
\mathbf{E}\times\mathbf{B} = \mathbf{E}\times(\nabla\times\mathbf{A}) &= \sum_{ijk}\hat{\mathbf{e}}_k\epsilon_{ijk}E_i\left(\sum_{lmj}\epsilon_{lmj}\frac{\partial A_m}{\partial x_l}\right)\\
&= \sum_{iklm}\hat{\mathbf{e}}_kE_i\frac{\partial A_m}{\partial x_l}\sum_j\epsilon_{ijk}\epsilon_{lmj} = \sum_{iklm}\hat{\mathbf{e}}_kE_i\frac{\partial A_m}{\partial x_l}(\delta_{im}\delta_{kl}-\delta_{il}\delta_{km})\\
&= \sum_{ik}\hat{\mathbf{e}}_kE_i\left(\frac{\partial A_i}{\partial x_k}-\frac{\partial A_k}{\partial x_i}\right)\\
&= \sum_i E_i(\nabla A_i)-(\mathbf{E}\cdot\nabla)\mathbf{A}
\end{aligned}\tag{1}$$

Thus the integrand becomes

$$\mathbf{x}\times(\mathbf{E}\times\mathbf{B}) = \overbrace{\mathbf{x}\times\left[\sum_i E_i(\nabla A_i)\right]}^{\mathbf{X}}-\overbrace{\mathbf{x}\times[(\mathbf{E}\cdot\nabla)\mathbf{A}]}^{\mathbf{Y}}\tag{2}$$

The first term

$$\mathbf{X} = \sum_{lmk}\hat{\mathbf{e}}_k\epsilon_{lmk}x_l\left(\sum_i E_i\frac{\partial A_i}{\partial x_m}\right) = \sum_i E_i\left(\sum_{lmk}\hat{\mathbf{e}}_k\epsilon_{lmk}x_l\frac{\partial}{\partial x_m}\right)A_i = \sum_i E_i(\mathbf{x}\times\nabla)A_i\tag{3}$$

The second term

$$\begin{aligned}
\mathbf{Y} &= \sum_{ijk}\hat{\mathbf{e}}_k\epsilon_{ijk}x_i\left(\sum_l E_l\frac{\partial A_j}{\partial x_l}\right) = \sum_{ijk}\hat{\mathbf{e}}_k\epsilon_{ijk}\left\{\sum_l E_l\left[\frac{\partial\left(x_iA_j\right)}{\partial x_l}-A_j\delta_{il}\right]\right\}\\
&= \sum_l E_l\frac{\partial}{\partial x_l}\sum_{ijk}\hat{\mathbf{e}}_k\epsilon_{ijk}x_iA_j-\sum_{ijk}\hat{\mathbf{e}}_k\epsilon_{ijk}E_iA_j\\
&= \sum_l E_l\frac{\partial(\mathbf{x}\times\mathbf{A})}{\partial x_l}-\mathbf{E}\times\mathbf{A}\\
&= (\mathbf{E}\cdot\nabla)(\mathbf{x}\times\mathbf{A})-\mathbf{E}\times\mathbf{A}
\end{aligned}\tag{4}$$

Thus

$$\int d^3x\,\mathbf{x}\times(\mathbf{E}\times\mathbf{B}) = \int d^3x\left[\mathbf{E}\times\mathbf{A}+\sum_i E_i(\mathbf{x}\times\nabla)A_i\right]-\int d^3x\,(\mathbf{E}\cdot\nabla)(\mathbf{x}\times\mathbf{A})\tag{5}$$

For the claim to hold, the second integral must vanish.

Indeed, for the free field, $\nabla\cdot\mathbf{E}=0$, we have

$$\begin{aligned}
(\mathbf{E}\cdot\nabla)(\mathbf{x}\times\mathbf{A}) &= (\mathbf{E}\cdot\nabla+\nabla\cdot\mathbf{E})(\mathbf{x}\times\mathbf{A}) \qquad \text{denote } \mathbf{b}=\mathbf{x}\times\mathbf{A}\\
&= \sum_i\left(E_i\frac{\partial}{\partial x_i}+\frac{\partial E_i}{\partial x_i}\right)\sum_j\hat{\mathbf{e}}_jb_j\\
&= \sum_j\hat{\mathbf{e}}_j\left[\sum_i\frac{\partial\left(E_ib_j\right)}{\partial x_i}\right] = \sum_j\hat{\mathbf{e}}_j\nabla\cdot\left(b_j\mathbf{E}\right)
\end{aligned}\tag{6}$$

So the second integral in (5) yields

$$\sum_j\hat{\mathbf{e}}_j\int d^3x\,\nabla\cdot\left(b_j\mathbf{E}\right) = \sum_j\hat{\mathbf{e}}_j\oint_\infty\left(b_j\mathbf{E}\right)\cdot\mathbf{n}\,da = 0\tag{7}$$

where we have used the local distribution assumption so $b_j\mathbf{E}$ vanishes at infinity.

2. The expansion of vector potential in radiation gauge is

$$\mathbf{A}(\mathbf{x},t) = \sum_{\lambda}\int \frac{d^3k}{(2\pi)^3}\left[\boldsymbol{\epsilon}_{\lambda}(\mathbf{k})\,a_{\lambda}(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} + \boldsymbol{\epsilon}^*_{\lambda}(\mathbf{k})\,a^*_{\lambda}(\mathbf{k})\,e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}\right] \tag{8}$$

Let $\mathscr{A}_{\lambda}(\mathbf{k}) = \boldsymbol{\epsilon}_{\lambda}(\mathbf{k})\,a_{\lambda}(\mathbf{k})$ be the amplitude of the constituent plane wave $(\mathbf{k},\lambda)$, hence $\mathscr{A}^*_{\lambda}(\mathbf{k}) = \boldsymbol{\epsilon}^*_{\lambda}(\mathbf{k})\,a^*_{\lambda}(\mathbf{k})$ is the amplitude of the negative frequency plane wave $(-\mathbf{k},\lambda)$.

The corresponding $\mathbf{E}$ field amplitudes, $\mathscr{E}_{\lambda}(\mathbf{k})$ and $\mathscr{E}^*_{\lambda}(\mathbf{k})$ can be obtained by the relation $\mathbf{E} = -\partial\mathbf{A}/\partial t$ implied by the Coulomb gauge (or radiation gauge), i.e.,

$$\mathscr{E}_{\lambda}(\mathbf{k}) = i\omega\mathscr{A}_{\lambda}(\mathbf{k}) = i\omega a_{\lambda}(\mathbf{k})\,\boldsymbol{\epsilon}_{\lambda}(\mathbf{k}) \qquad \mathscr{E}^*_{\lambda}(\mathbf{k}) = -i\omega\mathscr{A}^*_{\lambda}(\mathbf{k}) = -i\omega a^*_{\lambda}(\mathbf{k})\,\boldsymbol{\epsilon}^*_{\lambda}(\mathbf{k}) \tag{9}$$

The spin is thus

$$\begin{aligned}\mathbf{L}_{\text{spin}} =& \frac{1}{\mu_0 c^2}\int d^3x\,\mathbf{E}(\mathbf{x},t)\times\mathbf{A}(\mathbf{x},t)\\ =& \frac{1}{\mu_0 c^2}\int d^3x \sum_{\lambda,\mu}\left\{\int \frac{d^3k}{(2\pi)^3}\left[\mathscr{E}_{\lambda}(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} + \mathscr{E}^*_{\lambda}(\mathbf{k})\,e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}\right]\right\}\times\\ &\left\{\int \frac{d^3k'}{(2\pi)^3}\left[\mathscr{A}_{\mu}(\mathbf{k}')\,e^{i\mathbf{k}'\cdot\mathbf{x}-i\omega t} + \mathscr{A}^*_{\mu}(\mathbf{k}')\,e^{-i\mathbf{k}'\cdot\mathbf{x}+i\omega t}\right]\right\}\end{aligned} \tag{10}$$

For a particular combination of λ,μ, the integral can be written

$$\begin{aligned}I_{\lambda,\mu} = \int d^3x \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\Big[&\mathscr{E}_{\lambda}(\mathbf{k})\times\mathscr{A}_{\mu}(\mathbf{k}')\,e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}e^{-i2\omega t}+\\ &\mathscr{E}_{\lambda}(\mathbf{k})\times\mathscr{A}^*_{\mu}(\mathbf{k}')\,e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}+\\ &\mathscr{E}^*_{\lambda}(\mathbf{k})\times\mathscr{A}_{\mu}(\mathbf{k}')\,e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}+\\ &\mathscr{E}^*_{\lambda}(\mathbf{k})\times\mathscr{A}^*_{\mu}(\mathbf{k}')\,e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}e^{i2\omega t}\Big]\end{aligned} \tag{11}$$

Using $\int d^3x e^{i\mathbf{q}\cdot\mathbf{x}} = (2\pi)^3\,\delta(\mathbf{q})$, the triple integral collapses into a single integral over d^3k. Moreover, the two harmonic terms don't contribute to the time average, leaving

$$\left\langle I_{\lambda,\mu}\right\rangle = \int \frac{d^3k}{(2\pi)^3}\left[\mathscr{E}_{\lambda}(\mathbf{k})\times\mathscr{A}^*_{\mu}(\mathbf{k}) + \mathscr{E}^*_{\lambda}(\mathbf{k})\times\mathscr{A}_{\mu}(\mathbf{k})\right] \tag{12}$$

Since

$$\boldsymbol{\epsilon}_{\lambda}(\mathbf{k})\times\boldsymbol{\epsilon}^*_{\mu}(\mathbf{k}) = \begin{cases}-i\lambda\hat{\mathbf{k}} & \text{for } \mu=\lambda\\ 0 & \text{for } \mu\neq\lambda\end{cases} \tag{13}$$

we have

$$\mathscr{E}_{\lambda}(\mathbf{k})\times\mathscr{A}^*_{\mu}(\mathbf{k}) = i\omega a_{\lambda}(\mathbf{k})\,a^*_{\mu}(\mathbf{k})\left[\boldsymbol{\epsilon}_{\lambda}(\mathbf{k})\times\boldsymbol{\epsilon}^*_{\mu}(\mathbf{k})\right] = \begin{cases}\lambda\omega\,|a_{\lambda}(\mathbf{k})|^2\,\hat{\mathbf{k}} = \lambda c\mathbf{k}\,|a_{\lambda}(\mathbf{k})|^2 & \text{for } \mu=\lambda\\ 0 & \text{for } \mu\neq\lambda\end{cases} \tag{14}$$

Finally summing all $I_{\lambda,\mu}$'s in (10) and taking the time average, we obtain

$$\left\langle\mathbf{L}_{\text{spin}}\right\rangle = \frac{2}{\mu_0 c}\int \frac{d^3k}{(2\pi)^3}\mathbf{k}\left[|a_+(\mathbf{k})|^2 - |a_-(\mathbf{k})|^2\right] \tag{15}$$

The energy of the field is

$$U = \frac{\epsilon_0}{2}\int d^3x\left[\mathbf{E}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t) + c^2\mathbf{B}(\mathbf{x},t)\cdot\mathbf{B}(\mathbf{x},t)\right] \tag{16}$$

Expanding the field, we have

$$\int d^3x \mathbf{E}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t) = \sum_{\lambda,\mu} E_{\lambda,\mu} \qquad \int d^3x \mathbf{B}(\mathbf{x},t)\cdot\mathbf{B}(\mathbf{x},t) = \sum_{\lambda,\mu} B_{\lambda,\mu} \tag{17}$$

where $E_{\lambda,\mu}, B_{\lambda,\mu}$ have similar forms as (11) with the appropriate amplitude and substitution of cross product by dot product.

We shall end up with

$$\left\langle E_{\lambda,\mu}\right\rangle = \int \frac{d^3k}{(2\pi)^3}\left[\mathscr{E}_\lambda(\mathbf{k})\cdot\mathscr{E}^*_\mu(\mathbf{k}) + \mathscr{E}^*_\lambda(\mathbf{k})\cdot\mathscr{E}_\mu(\mathbf{k})\right] \tag{18}$$

$$\left\langle B_{\lambda,\mu}\right\rangle = \int \frac{d^3k}{(2\pi)^3}\left[\mathscr{B}_\lambda(\mathbf{k})\cdot\mathscr{B}^*_\mu(\mathbf{k}) + \mathscr{B}^*_\lambda(\mathbf{k})\cdot\mathscr{B}_\mu(\mathbf{k})\right] \tag{19}$$

With

$$\boldsymbol{\epsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\epsilon}^*_\mu(\mathbf{k}) = \delta_{\lambda\mu} \tag{20}$$

we know

$$\mathscr{E}_\lambda(\mathbf{k})\cdot\mathscr{E}^*_\mu(\mathbf{k}) = \omega^2 a_\lambda(\mathbf{k})a^*_\mu(\mathbf{k})\left[\boldsymbol{\epsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\epsilon}^*_\mu(\mathbf{k})\right] = \begin{cases}\omega^2\left|a_\lambda(\mathbf{k})\right|^2 & \text{for } \mu=\lambda\\ 0 & \text{for } \mu\neq\lambda\end{cases} \tag{21}$$

Also from $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, we have

$$\mathscr{B}_\lambda(\mathbf{k}) = i a_\lambda(\mathbf{k})\left[\mathbf{k}\times\boldsymbol{\epsilon}_\lambda(\mathbf{k})\right] \tag{22}$$

which gives

$$\begin{aligned}\mathscr{B}_\lambda(\mathbf{k})\cdot\mathscr{B}^*_\mu(\mathbf{k}) &= a_\lambda(\mathbf{k})a^*_\mu(\mathbf{k})\left\{\left[\mathbf{k}\times\boldsymbol{\epsilon}_\lambda(\mathbf{k})\right]\cdot\left[\mathbf{k}\times\boldsymbol{\epsilon}^*_\mu(\mathbf{k})\right]\right\}\\ &= k^2 a_\lambda(\mathbf{k})a^*_\mu(\mathbf{k})\left[\boldsymbol{\epsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\epsilon}^*_\mu(\mathbf{k})\right]\\ &= \begin{cases}k^2\left|a_\lambda(\mathbf{k})\right|^2 & \text{for } \mu=\lambda\\ 0 & \text{for } \mu\neq\lambda\end{cases}\end{aligned} \tag{23}$$

Putting everything together in (16), we have

$$\langle U\rangle = \frac{2}{\mu_0}\int \frac{d^3k}{(2\pi)^3}k^2\left[\left|a_+(\mathbf{k})\right|^2 + \left|a_-(\mathbf{k})\right|^2\right] \tag{24}$$

Clearly, $a_\pm(\mathbf{k})$ can be associated with photons with positive or negative helicity. (15) shows the net spin of a collection of photons, some with positive helicity, some negative. (24) gives their total energy.