1. Let $\mathbf{R} = (X, Y, Z)$ be the translation vector from old origin to the new origin. Thus a point at coordinate $\mathbf{x}$ in the old system will have coordinate $\mathbf{x}' = \mathbf{x} - \mathbf{R}$ in the new system. Also, let $\rho'(\mathbf{x}')$ be the charge density function in the new coordinate system, clearly the following holds

$$\rho'\left(\mathbf{x}'\right) = \rho\left(\mathbf{x}' + \mathbf{R}\right) = \rho(\mathbf{x}) \tag{1}$$

Let l be the lowest order non-vanishing multipole moment in the old system. Consider the order-l moment in the new system

$$\begin{aligned} M'^{(l)}_{\alpha\beta\gamma} &= \int \rho'\left(\mathbf{x}'\right) x'^{\alpha} y'^{\beta} z'^{\gamma} d^3x' \\ &= \int \rho(\mathbf{x})(x - X)^{\alpha}(y - Y)^{\beta}(z - Z)^{\gamma} d^3x \\ &= \int \rho(\mathbf{x}) x^{\alpha} y^{\beta} z^{\gamma} d^3x + LM \\ &= M^{(l)}_{\alpha\beta\gamma} + LM \end{aligned} \tag{2}$$

where LM represents a linear combination of lower-order moments measured in the old system, all of which vanish by assumption. Thus

$$M'^{(l)}_{\alpha\beta\gamma} = M^{(l)}_{\alpha\beta\gamma} \tag{3}$$

I.e., the lowest-order non-vanishing moment is invariant under translation. The proof above also shows in general, non-lowest-order moments are not translation invariant.

We have proved this for the Cartesian moments, but from the discussion of reducibility of $M^{(l)}_{\alpha\beta\gamma}$ in prob 4.3, we know that q_{lm} is necessarily the traceless linear combination of $M^{(l)}_{\alpha\beta\gamma}$, so the translation invariance conclusion applies to the spherical moments q_{lm} as well.

2. In the untranslated system,

$$q = \int \rho(\mathbf{x}) d^3x \tag{4}$$

$$\mathbf{p} = \int \rho(\mathbf{x}) \mathbf{x} d^3x \tag{5}$$

$$Q_{ij} = \int \rho(\mathbf{x}) x_i x_j d^3x \tag{6}$$

But in the translated system,

$$q' = \int \rho'\left(\mathbf{x}'\right) d^3x' = \int \rho(x) d^3x = q \tag{7}$$

$$\begin{aligned} \mathbf{p}' &= \int \rho'\left(\mathbf{x}'\right) \mathbf{x}' d^3x' \\ &= \int \rho(\mathbf{x})(\mathbf{x} - \mathbf{R}) d^3x = \mathbf{p} - q\mathbf{R} \end{aligned} \tag{8}$$

$$\begin{aligned} Q'_{ij} &= \int \rho'\left(\mathbf{x}'\right) x'_i x'_j d^3x' \\ &= \int \rho(\mathbf{x})(x_i - R_i)\left(x_j - R_j\right) d^3x \\ &= Q_{ij} - \left(R_i p_j + R_j p_i\right) + qR_iR_j \end{aligned} \tag{9}$$

3. By (8), if $q \neq 0$, letting $\mathbf{R} = \mathbf{p}/q$ can make $\mathbf{p}' = 0$.

However, for general Q_{ij} without traceless requirement, we have 6 independent choices for all the Q_{ij}s (by symmetry requirement $Q_{ij} = Q_{ji}$), but (9) has 3 unknowns in $\mathbf{R}$, so in general we cannot find $\mathbf{R}$ to satisfy $Q'_{ij} = 0$. Even if we impose an additional requirement that Q_{ij} be traceless (upon which (9) will have to be modified), we have 5 free choices for Q_{ij} and 3 unknowns in $\mathbf{R}$, a solution is not guaranteed to exist in general.