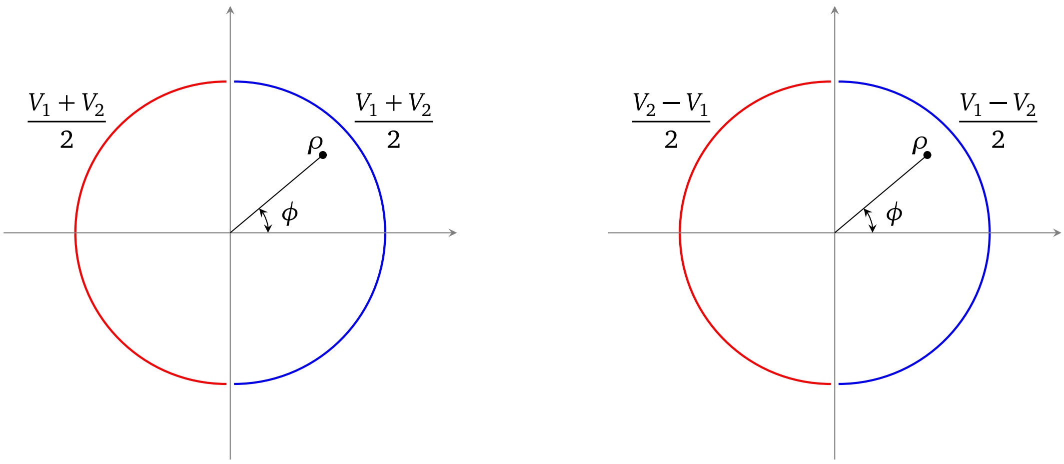

1. By linear superposition, the final potential distribution inside the cylinder is the sum of the two configurations

   (a) Both halves are at the average potential $(V_1+V_2)/2$;

   (b) The two halves are at potential $(V_1-V_2)/2$ and $(V_2-V_1)/2$ respectively.

   We have proved in (2.12) the following relations

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} d\phi'\Phi(b,\phi')\left[1+2\sum_{n=1}^{\infty}\frac{\rho^n}{b^n}\cos n(\phi-\phi')\right] \tag{1}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi' \tag{2}$$

   For configuration (a), simple argument using Gauss's theorem will show that any interior point (ρ,ϕ) should have the same potential as the cylinder. Which is also clear from (1) where integration $\int_0^{2\pi}\cos n(\phi-\phi')\,d\phi'$ will vanish for all $n \geq 1$.

   For contribution from (b), we have to use (2) to do the two-part integration.

$$\begin{aligned}
\Phi_{(b)}(\rho,\phi) =& \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{V_1-V_2}{2}\right)\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi' - \\
& \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}\left(\frac{V_1-V_2}{2}\right)\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi' \qquad (\text{let } \phi'\to\phi'-\pi \text{ in 2nd integral}) \\
=& \frac{1}{2\pi}\left(\frac{V_1-V_2}{2}\right)\int_{-\pi/2}^{\pi/2}\left[\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)} - \frac{b^2-\rho^2}{b^2+\rho^2+2b\rho\cos(\phi'-\phi)}\right]d\phi' \\
=& \frac{1}{2\pi}\frac{V_1-V_2}{2}\int_{-\pi/2}^{\pi/2}\frac{4\left(b^2-\rho^2\right)b\rho\cos(\phi'-\phi)}{(b^2+\rho^2)^2-4b^2\rho^2\cos^2(\phi'-\phi)}d\phi'
\end{aligned} \tag{3}$$

   With the change of variable $t=\sin(\phi'-\phi)$, the integrand becomes

$$\frac{4\left(b^2-\rho^2\right)b\rho\,dt}{(b^2+\rho^2)^2-4b^2\rho^2(1-t^2)} = \frac{4\left(b^2-\rho^2\right)b\rho\,dt}{(b^2-\rho^2)^2+4b^2\rho^2t^2} = \frac{b^2-\rho^2}{b\rho}\frac{dt}{\left(\dfrac{b^2-\rho^2}{2b\rho}\right)^2+t^2} \tag{4}$$

   With one more change of variable $t=\left(b^2-\rho^2\right)/(2b\rho)\tan\xi$, (4) becomes

$$\frac{b^2-\rho^2}{b\rho}\frac{\dfrac{b^2-\rho^2}{2b\rho}\dfrac{1}{\cos^2\xi}d\xi}{\left(\dfrac{b^2-\rho^2}{2b\rho}\right)^2\dfrac{1}{\cos^2\xi}} = 2d\xi \tag{5}$$

The bounds of ξ are achieved with

$$\begin{aligned}
\xi_{\text{lower}} &= \tan^{-1}\left[\frac{2b\rho}{b^2-\rho^2}\sin\left(-\frac{\pi}{2}-\phi\right)\right] = -\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right) && (6)\\
\xi_{\text{upper}} &= \tan^{-1}\left[\frac{2b\rho}{b^2-\rho^2}\sin\left(\frac{\pi}{2}-\phi\right)\right] = \tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right) && (7)
\end{aligned}$$

Plugging all these into (3) yields

$$\Phi_{(b)}(\rho,\phi) = \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right) \tag{8}$$

Summing with the configuration (a) finally gives the full interior potential formula

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right) \tag{9}$$

2. The surface charge density can be calculated as

$$\begin{aligned}
\sigma = -\epsilon_0 \frac{\partial\Phi}{\partial\rho}\bigg|_{\rho=b} &= -\epsilon_0\frac{V_1-V_2}{\pi}\frac{1}{1+\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)^2}\frac{\left(b^2-\rho^2\right)2b+2\rho(2b\rho)}{\left(b^2-\rho^2\right)^2}\cos\phi\Bigg|_{\rho=b}\\
&= -\epsilon_0\frac{V_1-V_2}{\pi}\frac{\left(b^2+\rho^2\right)2b\cos\phi}{(b^2-\rho^2)^2+4b^2\rho^2\cos^2\phi}\Bigg|_{\rho=b}\\
&= -\epsilon_0\frac{V_1-V_2}{\pi b\cos\phi} && (10)
\end{aligned}$$