Since there is no current, we can use the scalar potential method. This is a 2D problem, so the general solution of (2.71) applies. For the three regions, write the general solution of potentials as

$$\Phi_{\text{out}}(\rho,\phi) = -H_0\rho\cos\phi + \sum_{n=1}^{\infty}\left(a_n\rho^{-n}\cos n\phi + b_n\rho^{-n}\sin n\phi\right) \tag{1}$$

$$\Phi_{\text{ring}}(\rho,\phi) = c_0\ln\rho + \sum_{n=1}^{\infty}\left(c_n\rho^{-n}\cos n\phi + d_n\rho^{-n}\sin n\phi + e_n\rho^{n}\cos n\phi + f_n\rho^{n}\sin n\phi\right) \tag{2}$$

$$\Phi_{\text{in}}(\rho,\phi) = \sum_{n=1}^{\infty}\left(g_n\rho^{n}\cos n\phi + h_n\rho^{n}\sin n\phi\right) \tag{3}$$

The tangential $\mathbf{H}$ boundary condition at $\rho = b$

$$\left.\frac{\partial\Phi_{\text{out}}}{\partial\phi}\right|_{\rho=b} = \left.\frac{\partial\Phi_{\text{ring}}}{\partial\phi}\right|_{\rho=b} \tag{4}$$

requires

$$\begin{aligned} &H_0 b\sin\phi + \sum_{n=1}^{\infty}\left(-na_n b^{-n}\sin n\phi + nb_n b^{-n}\cos n\phi\right) \\ =&\sum_{n=1}^{\infty}\left(-nc_n b^{-n}\sin n\phi + nd_n b^{-n}\cos n\phi - ne_n b^{n}\sin n\phi + nf_n b^{n}\cos n\phi\right) \end{aligned} \tag{5}$$

which gives

$$H_0 b^2 - a_1 = -c_1 - e_1 b^2 \tag{6}$$

$$a_n = c_n + e_n b^{2n} \qquad \text{for } n \geq 2 \tag{7}$$

$$b_n = d_n + f_n b^{2n} \qquad \text{for } n \geq 1 \tag{8}$$

The normal $\mathbf{H}$ boundary condition at $\rho = b$

$$\left.\frac{\partial\Phi_{\text{out}}}{\partial\rho}\right|_{\rho=b} = \mu_r\left.\frac{\partial\Phi_{\text{ring}}}{\partial\rho}\right|_{\rho=b} \tag{9}$$

requires

$$\begin{aligned} &-H_0\cos\phi + \sum_{n=1}^{\infty}\left[-na_n b^{-(n+1)}\cos n\phi - nb_n b^{-(n+1)}\sin n\phi\right] \\ =&\mu_r\left\{c_0 b^{-1} + \sum_{n=1}^{\infty}\left[-nc_n b^{-(n+1)}\cos n\phi - nd_n b^{-(n+1)}\sin n\phi + ne_n b^{n-1}\cos n\phi + nf_n b^{n-1}\sin n\phi\right]\right\} \end{aligned} \tag{10}$$

which gives

$$c_0 = 0 \tag{11}$$

$$H_0 b^2 + a_1 = \mu_r\left(c_1 - e_1 b^2\right) \tag{12}$$

$$a_n = \mu_r\left(c_n - e_n b^{2n}\right) \qquad \text{for } n \geq 2 \tag{13}$$

$$b_n = \mu_r\left(d_n - f_n b^{2n}\right) \qquad \text{for } n \geq 1 \tag{14}$$

The tangential $\mathbf{H}$ boundary condition at $\rho = a$

$$\left.\frac{\partial\Phi_{\text{in}}}{\partial\phi}\right|_{\rho=a} = \left.\frac{\partial\Phi_{\text{ring}}}{\partial\phi}\right|_{\rho=a} \tag{15}$$

requires

$$\begin{aligned} &\sum_{n=1}^{\infty}\left(-ng_n a^{n}\sin n\phi + nh_n a^{n}\cos n\phi\right) \\ =&\sum_{n=1}^{\infty}\left(-nc_n a^{-n}\sin n\phi + nd_n a^{-n}\cos n\phi - ne_n a^{n}\sin n\phi + nf_n a^{n}\cos n\phi\right) \end{aligned} \tag{16}$$

which gives

$$g_n a^{2n} = c_n + e_n a^{2n} \qquad \text{for } n \geq 1 \tag{17}$$
$$h_n a^{2n} = d_n + f_n a^{2n} \qquad \text{for } n \geq 1 \tag{18}$$

The normal **H** boundary condition at $\rho = a$

$$\left.\frac{\partial \Phi_{\text{in}}}{\partial \rho}\right|_{\rho=a} = \mu_r \left.\frac{\partial \Phi_{\text{ring}}}{\partial \rho}\right|_{\rho=a} \tag{19}$$

requires

$$\begin{aligned}&\sum_{n=1}^{\infty} \left(n g_n a^{n-1} \cos n\phi + n h_n a^{n-1} \sin n\phi\right) \\ =&\mu_r \sum_{n=1}^{\infty} \left[-n c_n a^{-(n+1)} \cos n\phi - n d_n a^{-(n+1)} \sin n\phi + n e_n a^{n-1} \cos n\phi + n f_n a^{n-1} \sin n\phi\right]\end{aligned} \tag{20}$$

which gives

$$g_n a^{2n} = \mu_r \left(-c_n + e_n a^{2n}\right) \qquad \text{for } n \geq 1 \tag{21}$$
$$h_n a^{2n} = \mu_r \left(-d_n + f_n a^{2n}\right) \qquad \text{for } n \geq 1 \tag{22}$$

From (7) and (13), we have

$$(\mu_r - 1) c_n = (\mu_r + 1)\, e_n b^{2n} \qquad \text{for } n \geq 2 \tag{23}$$

From (17) and (21), we have

$$(\mu_r + 1)\, c_n = (\mu_r - 1)\, e_n a^{2n} \qquad \text{for } n \geq 1 \tag{24}$$

Since $\mu_r > 0$ and $b > a$, for $n \geq 2$, the only possibility for (23) and (24) to hold is to have

$$c_n = e_n = 0 = a_n = g_n \qquad \text{for } n \geq 2 \tag{25}$$

Similar arguments lead to

$$d_n = f_n = 0 = b_n = h_n \qquad \text{for } n \geq 1 \tag{26}$$

Add (6) to (12) and use (24), we have

$$2H_0 b^2 = (\mu_r - 1)\, c_1 - (\mu_r + 1)\, e_1 b^2 = \frac{(\mu_r - 1)^2}{\mu_r + 1} e_1 a^2 - (\mu_r + 1)\, e_1 b^2 \qquad \Longrightarrow$$

$$e_1 = \frac{-2H_0 b^2 (\mu_r + 1)}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2} \qquad \Longrightarrow \tag{27}$$

$$c_1 = \frac{-2H_0 a^2 b^2 (\mu_r - 1)}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2} \qquad \Longrightarrow \tag{28}$$

$$g_1 = c_1 a^{-2} + e_1 = \frac{-4H_0 b^2 \mu_r}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2} \tag{29}$$

Finally, subtracting (6) from (12) yields

$$a_1 = \frac{1}{2}\left[(\mu_r + 1)\, c_1 - (\mu_r - 1)\, e_1 b^2\right] = \frac{H_0 b^2 \left(b^2 - a^2\right)\left(\mu_r^2 - 1\right)}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2} \tag{30}$$

In summary,

$$\Phi_{\text{out}} = -H_0 \cos\phi \left[\rho - \frac{1}{\rho} \cdot \frac{b^2 \left(b^2 - a^2\right)\left(\mu_r^2 - 1\right)}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2}\right] \tag{31}$$

$$\Phi_{\text{ring}} = \left[\frac{-2H_0 b^2 \cos\phi}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2}\right]\left[(\mu_r + 1)\,\rho + \frac{a^2 (\mu_r - 1)}{\rho}\right] \tag{32}$$

$$\Phi_{\text{in}} = \frac{-4H_0 b^2 \mu_r \rho \cos\phi}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2} \tag{33}$$

For the region $\rho < a$, the flux density is constant:

$$\mathbf{B}_{\text{in}} = -\mu_0 \nabla \Phi_{\text{in}} = \frac{4B_0 b^2 \mu_r \hat{\mathbf{x}}}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2} \qquad \Longrightarrow \qquad \frac{|\mathbf{B}_{\text{in}}|}{B_0} = \frac{4\mu_r}{(\mu_r + 1)^2 - (\mu_r - 1)^2 \left(\frac{a}{b}\right)^2} \tag{34}$$