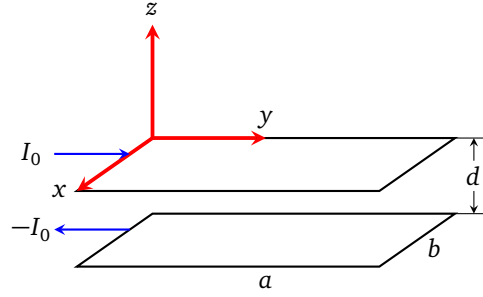


## 1. Prob 6.13



- (a) Let the coordinate system be set up as the diagram above. Since we are ignoring the fringing fields, the electromagnetic field between the plates is independent of the  $x$  coordinate. Also since  $d \ll a, d \ll b$ , we are going to take the approximation that the fields are independent of the  $z$  coordinate too. The whole system is driven by a harmonic input, so the electric and magnetic fields can be written as

$$\mathbf{E}(y, t) = \text{Re} [E(y) e^{-i\omega t}] \hat{\mathbf{z}} \quad (1)$$

$$\mathbf{B}(y, t) = \text{Re} [B(y) e^{-i\omega t}] \hat{\mathbf{x}} \quad (2)$$

Similarly, for the upper plate, let the current density and charge density at  $y$  be

$$\mathbf{K}(y, t) = \text{Re} [K(y) e^{-i\omega t}] \hat{\mathbf{y}} \quad (3)$$

$$\sigma(y, t) = \text{Re} [\sigma(y) e^{-i\omega t}] \quad (4)$$

For the lower plate,  $\mathbf{K}$  and  $\sigma$  have an opposite sign.

In the following we will work on the complex amplitudes and ignore their harmonic factor  $e^{-i\omega t}$  in favor of the complex Maxwell equation 6.130.

First notice

$$\nabla \cdot \mathbf{D} = \rho \quad \Rightarrow \quad E(y) = -\frac{\sigma(y)}{\epsilon_0} \quad (5)$$

Next notice

$$\nabla \times \mathbf{H} + i\omega \mathbf{D} = \mathbf{J} \quad \Rightarrow \quad B(y) = -\mu_0 K(y) \quad (6)$$

(this can be obtained by applying Stoke's theorem to a rectangular surface parallel to  $z$  axis straddling across the upper plate, while noting the displacement current  $i\omega \mathbf{D}$  contributes no flux).

Next we have

$$\nabla \times \mathbf{E} - i\omega \mathbf{B} = 0 \quad \Rightarrow \quad \frac{dE(y)}{dy} = i\omega B(y) \quad \Rightarrow \quad \frac{d\sigma(y)}{dy} = \frac{i\omega}{c^2} K(y) \quad (7)$$

Finally, conservation of electric charge at  $y$  requires

$$\nabla \cdot \mathbf{K}(y, t) + \frac{\partial \sigma(y, t)}{\partial t} = 0 \quad \Rightarrow \quad \frac{dK(y)}{dy} = i\omega \sigma(y) \quad (8)$$

From (7) and (8), we find the differential equation dictating  $\sigma(y)$  and  $K(y)$ :

$$\frac{d^2 \sigma(y)}{dy^2} = -\frac{\omega^2}{c^2} \sigma(y) \quad \frac{d^2 K(y)}{dy^2} = -\frac{\omega^2}{c^2} K(y) \quad (9)$$

With  $k = \omega/c$ , the general solution can be written as

$$K(y) = Ae^{iky} + Be^{-iky} \quad \sigma(y) = \frac{A}{c} e^{iky} - \frac{B}{c} e^{-iky} \quad (10)$$

where  $A, B$  can be determined by boundary condition  $K(y=0) = I_0/b$  and  $K(y=a) = 0$ , i.e.,

$$\begin{aligned} \frac{I_0}{b} &= A + B & 0 &= Ae^{ika} + Be^{-ika} \\ A &= \frac{I_0}{b} \left( \frac{1}{2} + \frac{i}{2} \cot ka \right) & B &= \frac{I_0}{b} \left( \frac{1}{2} - \frac{i}{2} \cot ka \right) \end{aligned} \quad \Rightarrow \quad (11)$$

which gives

$$\begin{aligned} K(y) &= \frac{I_0}{2b} (e^{iky} + e^{-iky}) + \frac{iI_0}{2b} \cot ka (e^{iky} - e^{-iky}) \\ &= \frac{I_0}{b} (\cos ky - \cot ka \sin ky) \\ &= \frac{I_0}{b} \frac{\sin[k(a-y)]}{\sin ka} \end{aligned} \quad (12)$$

$$\begin{aligned} \sigma(y) &= \frac{1}{c} \left[ \frac{I_0}{2b} (e^{iky} - e^{-iky}) + \frac{iI_0}{2b} \cot ka (e^{iky} + e^{-iky}) \right] \\ &= \frac{iI_0}{cb} (\sin ky + \cot ka \cos ky) \\ &= \frac{iI_0}{cb} \frac{\cos[k(a-y)]}{\sin ka} \end{aligned} \quad (13)$$

The complex amplitudes can be obtained from (5), (6)

$$E(y) = -\frac{i\mu_0 I_0 c}{b} \frac{\cos[k(a-y)]}{\sin ka} \quad (14)$$

$$B(y) = \frac{-\mu_0 I_0}{b} \frac{\sin[k(a-y)]}{\sin ka} \quad (15)$$

Thus the real fields are

$$\begin{aligned} \mathbf{E}(y, t) &= \text{Re} \left\{ -\frac{i\mu_0 I_0 c}{b} \frac{\cos[k(a-y)]}{\sin ka} e^{-i\omega t} \right\} \hat{\mathbf{z}} & \text{if } I_0 \in \mathbb{R} \\ &= -\frac{\mu_0 I_0 c}{b} \frac{\cos[k(a-y)]}{\sin ka} \sin \omega t \hat{\mathbf{z}} \end{aligned} \quad (16)$$

$$\begin{aligned} \mathbf{B}(y, t) &= \text{Re} \left\{ -\frac{\mu_0 I_0}{b} \frac{\sin[k(a-y)]}{\sin ka} e^{-i\omega t} \right\} \hat{\mathbf{x}} & \text{if } I_0 \in \mathbb{R} \\ &= -\frac{\mu_0 I_0}{b} \frac{\sin[k(a-y)]}{\sin ka} \cos \omega t \hat{\mathbf{x}} \end{aligned} \quad (17)$$

(b) From (14) and (15), we get

$$w_m = \frac{1}{4} \mathbf{B} \cdot \mathbf{H}^* = \frac{1}{4} \frac{\mu_0 |I_0|^2}{b^2} \frac{\sin^2[k(a-y)]}{\sin^2 ka} \quad (18)$$

$$w_e = \frac{1}{4} \mathbf{E} \cdot \mathbf{D}^* = \frac{1}{4} \frac{\mu_0 |I_0|^2}{b^2} \frac{\cos^2[k(a-y)]}{\sin^2 ka} \quad (19)$$

Thus by (6.140), the inductive reactance is

$$X_L = \frac{4\omega}{|I_0|^2} \int w_m d^3x = \frac{\omega\mu_0}{b^2} \int_0^a \frac{\sin^2[k(a-y)]}{\sin^2 ka} (bd) dy \quad (20)$$

If we identify  $X_L = \omega L$  as in a series LC circuit, we would have the effective inductance

$$L = \frac{\mu_0 d}{b} \int_0^a \frac{\sin^2[k(a-y)]}{\sin^2 ka} dy \approx \frac{\mu_0 d}{b} \int_0^a \frac{(a-y)^2}{a^2} dy = \frac{\mu_0 ad}{3b} \quad (21)$$

Similarly, the capacitive reactance from (6.140) is

$$X_C = -\frac{4\omega}{|I_0|^2} \int w_e d^3x = -\frac{\omega\mu_0}{b^2} \int_0^a \frac{\cos^2[k(a-y)]}{\sin^2 ka} (bd) dy \quad (22)$$

If we identify  $X_C = -1/\omega C$ , we would have the effective capacitance

$$C = \frac{1}{\frac{\omega^2 \mu_0 d}{b} \int_0^a \frac{\cos^2[k(a-y)]}{\sin^2 ka} dy} \approx \frac{1}{\frac{\omega^2 \mu_0 d}{b} \int_0^a \frac{1}{k^2 a^2} dy} = \frac{k^2 ab}{\omega^2 \mu_0 d} = \frac{\epsilon_0 ab}{d} \quad (23)$$

## 2. Prob 6.14

- (a) Similar to problem 6.13, based on symmetry arguments, we have the following form of the fields for the space between the plates

$$\mathbf{E}(\rho, t) = \text{Re}[E(\rho)e^{-i\omega t}]\hat{\mathbf{z}} \quad (24)$$

$$\mathbf{B}(\rho, t) = \text{Re}[B(\rho)e^{-i\omega t}]\hat{\boldsymbol{\phi}} \quad (25)$$

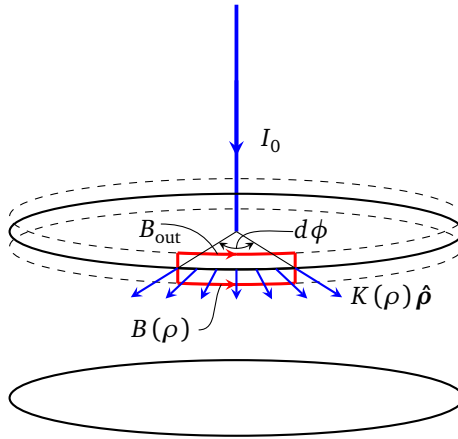
Let the surface current and charge density of the upper plate be

$$\mathbf{K}(\rho, t) = \text{Re}[K(\rho)e^{-i\omega t}]\hat{\boldsymbol{\rho}} \quad (26)$$

$$\sigma(\rho, t) = \text{Re}[\sigma(\rho)e^{-i\omega t}] \quad (27)$$

First,

$$\nabla \cdot \mathbf{D} = \rho \quad \Rightarrow \quad E(\rho) = -\frac{\sigma(\rho)}{\epsilon_0} \quad (28)$$



In order to apply the equation

$$\nabla \times \mathbf{H} + i\omega \mathbf{D} = \mathbf{J} \quad (29)$$

let's consider a "rectangular" loop straddling across the upper plate at radius  $\rho$ , depicted red in the diagram above. The two long sides span an arc of length  $\rho d\phi$ , and the two short sides are infinitesimal.

Applying Stoke's theorem for this patch with (29) yields (note  $\mathbf{D}$  is in the  $\hat{\mathbf{z}}$  direction hence contributes no flux),

$$[B(\rho) - B_{\text{out}}]\rho d\phi = \mu_0 K(\rho)\rho d\phi \quad (30)$$

where  $B_{\text{out}}$  is the magnetic induction at  $\rho$  right above the upper plate, in the  $\hat{\boldsymbol{\phi}}$  direction. In fact  $B_{\text{out}}$  is easy to obtain if we only consider the space above the upper plate, and where the current is along the  $z$  axis, i.e.,

$$B_{\text{out}} = -\frac{\mu_0 I_0}{2\pi\rho} \quad (31)$$

This gives us

$$B(\rho) = \mu_0 K(\rho) + B_{\text{out}} = \mu_0 K(\rho) - \frac{\mu_0 I_0}{2\pi\rho} = \mu_0 \kappa(\rho) \quad \text{where} \quad \kappa(\rho) \equiv K(\rho) - \frac{I_0}{2\pi\rho} \quad (32)$$

Next notice

$$\nabla \times \mathbf{E} - i\omega \mathbf{B} = 0 \quad \Rightarrow \quad -\frac{dE(\rho)}{d\rho} = i\omega B(\rho) \quad \Rightarrow \quad \frac{d\sigma(\rho)}{d\rho} = \frac{i\omega}{c^2} \kappa(\rho) \quad (33)$$

Finally conservation of charge requires

$$\nabla \cdot [\hat{\boldsymbol{\rho}} K(\rho)] - i\omega \sigma(\rho) = 0 \quad \Rightarrow \quad \frac{1}{\rho} \frac{d[\rho K(\rho)]}{d\rho} = \frac{1}{\rho} \frac{d[\rho \kappa(\rho)]}{d\rho} = i\omega \sigma(\rho) \quad \Rightarrow \quad \frac{d\kappa}{d\rho} + \frac{\kappa}{\rho} = i\omega \sigma \quad (34)$$

Taking  $d/d\rho$  of (34) and using (33), we get

$$\frac{d^2\kappa}{d\rho^2} + \frac{1}{\rho} \frac{d\kappa}{d\rho} - \frac{\kappa}{\rho^2} = -\frac{\omega^2}{c^2} \kappa \quad \text{or} \quad \frac{d^2\kappa}{d\rho^2} + \frac{1}{\rho} \frac{d\kappa}{d\rho} + \left(k^2 - \frac{1}{\rho^2}\right) \kappa = 0 \quad (35)$$

which is the Bessel equation with  $\nu = 1$ , hence the general solution is

$$\kappa(\rho) = AJ_1(k\rho) + BN_1(k\rho) \quad (36)$$

The boundary condition at  $\rho \rightarrow 0$  requires

$$\lim_{\rho \rightarrow 0} 2\pi\rho K(\rho) = I_0 \quad \text{or} \quad \lim_{\rho \rightarrow 0} \rho\kappa(\rho) = 0 \quad (37)$$

With the asymptotic form of  $J_1(k\rho)$  and  $N_1(k\rho)$  (see Jackson equation 3.89, 3.90)

$$J_1(k\rho) \rightarrow \frac{k\rho}{2} \quad N_1(k\rho) \rightarrow -\frac{1}{\pi} \left( \frac{2}{k\rho} \right) \quad (38)$$

we must have  $B = 0$  in order to satisfy (37).

Boundary condition at  $\rho = a$  requires

$$0 = K(a) = AJ_1(k\rho) + \frac{I_0}{2\pi a} \quad \Rightarrow \quad A = -\frac{I_0}{2\pi a} \frac{1}{J_1(ka)} \quad (39)$$

In summary, we have

$$\kappa(\rho) = -\frac{I_0}{2\pi a} \frac{J_1(k\rho)}{J_1(ka)} \quad (40)$$

By (34),

$$\sigma(\rho) = \frac{1}{i\omega} \left( \frac{d\kappa}{d\rho} + \frac{\kappa}{\rho} \right) \quad (41)$$

Using the recurrence relation for  $J_\nu(z)$  (see equation 10.6.2 on [dlmf.nist.gov](http://dlmf.nist.gov))

$$J'_\nu(z) + \frac{\nu}{z} J_\nu(z) = J_{\nu-1}(z) \quad (42)$$

(41) becomes

$$\sigma(\rho) = \frac{k}{i\omega} AJ_0(k\rho) = \frac{iI_0}{2\pi a c} \frac{J_0(k\rho)}{J_1(ka)} \quad (43)$$

The field amplitudes are

$$B(\rho) = \mu_0 \kappa(\rho) = -\frac{\mu_0 I_0}{2\pi a} \frac{J_1(k\rho)}{J_1(ka)} \quad (44)$$

$$E(\rho) = -\frac{\sigma(\rho)}{\epsilon_0} = -\frac{i\mu_0 I_0 c}{2\pi a} \frac{J_0(k\rho)}{J_1(ka)} \quad (45)$$

(b) Recall Bessel function can be written as series

$$J_\nu(x) = \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{x}{2}\right)^{2j+\nu}}{j! \Gamma(j+\nu+1)} \quad (46)$$

Thus

$$J_1(k\rho) \approx \frac{k\rho}{2} - \frac{(k\rho)^3}{16} \quad (47)$$

$$J_0(k\rho) \approx 1 - \frac{(k\rho)^2}{4} \quad (48)$$

and

$$\begin{aligned}
\frac{J_1(k\rho)}{J_1(ka)} &\approx \left( \frac{k\rho}{2} - \frac{k^3\rho^3}{16} \right) \left[ \left( \frac{ka}{2} \right) \left( 1 - \frac{k^2a^2}{8} \right) \right]^{-1} \\
&\approx \left( \frac{\rho}{a} - \frac{k^2\rho^2}{8a} \right) \left( 1 + \frac{k^2a^2}{8} \right) \\
&\approx \left( \frac{\rho}{a} \right) \left( 1 - \frac{k^2\rho^2}{8} \right) \left( 1 + \frac{k^2a^2}{8} \right) \\
&\approx \left( \frac{\rho}{a} \right) \left( 1 - \frac{k^2\rho^2}{8} + \frac{k^2a^2}{8} \right)
\end{aligned} \tag{49}$$

So

$$\begin{aligned}
\int_V w_m d^3x &= \frac{1}{4} \cdot 2\pi d \int_0^a \rho d\rho BH^* \\
&= \frac{1}{4} \cdot 2\pi d \cdot \frac{\mu_0 |I_0|^2}{(2\pi a)^2} \int_0^a \rho d\rho \left[ \frac{J_1(k\rho)}{J_1(ka)} \right]^2 \\
&\approx \frac{\mu_0 |I_0|^2 d}{8\pi a^2} \int_0^a \rho d\rho \left[ \left( \frac{\rho^2}{a^2} \right) \left( 1 - \frac{k^2\rho^2}{4} + \frac{k^2a^2}{4} \right) \right] \\
&= \frac{\mu_0 |I_0|^2 d}{8\pi a^2} \left( \frac{a^2}{4} - \frac{k^2a^4}{24} + \frac{k^2a^4}{16} \right) \\
&= \frac{\mu_0 |I_0|^2 d}{32\pi} \left( 1 + \frac{\omega^2 a^2}{12c^2} \right)
\end{aligned} \tag{50}$$

Also

$$\frac{J_0(k\rho)}{J_1(ka)} \approx \frac{2}{ka} + O(k^0) \tag{51}$$

hence

$$\begin{aligned}
\int_V w_e d^3x &= \frac{1}{4} \cdot 2\pi d \int_0^a \rho d\rho ED^* \\
&= \frac{1}{4} \cdot 2\pi d \cdot \frac{\mu_0^2 |I_0|^2 c^2 \epsilon_0}{(2\pi a)^2} \int_0^a \rho d\rho \left[ \frac{J_0(k\rho)}{J_1(ka)} \right]^2 \\
&\approx \frac{|I_0|^2 d}{8\pi a^2 c^2 \epsilon_0} \int_0^a \rho d\rho \cdot \frac{4}{k^2 a^2} \\
&= \frac{|I_0|^2 d}{4\pi \epsilon_0 \omega^2 a^2}
\end{aligned} \tag{52}$$

(c) We can get the effective inductance and capacitance via (6.140):

$$X_L = \omega L = \frac{4\omega}{|I_0|^2} \int w_m d^3x \quad \Rightarrow \quad L \approx \frac{\mu_0 d}{8\pi} \tag{53}$$

$$X_C = -\frac{1}{\omega C} = -\frac{4\omega}{|I_0|^2} \int w_e d^3x \quad \Rightarrow \quad C \approx \frac{\pi \epsilon_0 a^2}{d} \tag{54}$$

Resonance frequency is

$$\omega_{\text{res}} = \frac{1}{\sqrt{LC}} \approx \frac{2\sqrt{2}c}{a} \approx \frac{2.828c}{a} \tag{55}$$

while the first root of  $J_0(ka)$  happens at

$$ka = \frac{\omega a}{c} = 2.405 \quad \Rightarrow \quad \omega = \frac{2.405c}{a} \tag{56}$$