To avoid the terrible abuse of symbol of z and e, we use q to denote the moving charge.

1. The frequency space potentials are given by (13.25)

$$\Phi(\mathbf{k},\omega) = \frac{2q}{\epsilon(\omega)}\left[\frac{\overbrace{\delta(\omega-\mathbf{k}\cdot\mathbf{v})}^{(1/v)\delta(k_3-\omega/v)}}{k^2-\dfrac{\omega^2\epsilon(\omega)}{c^2}}\right] = \frac{2q}{v\epsilon(\omega)}\left[\frac{\delta\left(k_3-\dfrac{\omega}{v}\right)}{k^2-\dfrac{\omega^2\epsilon(\omega)}{c^2}}\right] \tag{1}$$

$$\mathbf{A}(\mathbf{k},\omega) = \epsilon(\omega)\boldsymbol{\beta}\Phi(\mathbf{k},\omega) \tag{2}$$

Applying the Fourier transform $\mathbf{k}\to\mathbf{x}$, we have

$$\begin{aligned}\Phi(\mathbf{x},\omega) &= \frac{1}{(2\pi)^{3/2}}\int d^3k\Phi(\mathbf{k},\omega)e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \frac{2qe^{i\omega z/v}}{(2\pi)^{3/2}v\epsilon(\omega)}\int dk_1dk_2\frac{e^{ik_1x+ik_2y}}{k_1^2+k_2^2+\dfrac{\omega^2}{v^2}-\dfrac{\omega^2\epsilon(\omega)}{c^2}} \qquad \text{choose axis so } y=0 \\ &= \frac{2qe^{i\omega z/v}}{(2\pi)^{3/2}v\epsilon(\omega)}\int_{-\infty}^{\infty}e^{ik_1\rho}dk_1\int_{-\infty}^{\infty}\frac{dk_2}{k_2^2+k_1^2+\dfrac{\omega^2}{v^2}-\dfrac{\omega^2\epsilon(\omega)}{c^2}}\end{aligned} \tag{3}$$

Define

$$\lambda^2 = \frac{\omega^2}{v^2}-\frac{\omega^2\epsilon(\omega)}{c^2} = \frac{\omega^2}{v^2}\left[1-\beta^2\epsilon(\omega)\right] \tag{4}$$

When $\beta^2\epsilon(\omega)<1$, we have $\lambda^2>0$, the inner integral is elementary,

$$\int_{-\infty}^{\infty}\frac{dk_2}{k_2^2+k_1^2+\lambda^2} = \frac{\pi}{\sqrt{k_1^2+\lambda^2}} \tag{5}$$

This turns (3) into

$$\begin{aligned}\Phi(\mathbf{x},\omega) &= \frac{2\pi qe^{i\omega z/v}}{(2\pi)^{3/2}v\epsilon(\omega)}\overbrace{\int_{-\infty}^{\infty}\frac{e^{ik_1\rho}dk_1}{\sqrt{k_1^2+\lambda^2}}}^{2K_0(\lambda\rho)} \qquad \text{see DLMF 10.32.E11} \\ &= \frac{qe^{i\omega z/v}}{v\epsilon(\omega)}\sqrt{\frac{2}{\pi}}K_0(\lambda\rho)\end{aligned} \tag{6}$$

where we have chosen the branch of λ with positive real part to make the asymptotic form of $K_0(\lambda\rho)\propto e^{-\lambda\rho}$ not blow up when $\rho\to\infty$.

2. When ϵ is constant and $\beta^2\epsilon<1$, (6) can be written as

$$\Phi(\mathbf{x},\omega) = \frac{qe^{i\omega z/v}}{v\epsilon}\sqrt{\frac{2}{\pi}}K_0\left(\frac{|\omega|\rho}{\Gamma v}\right) \qquad \text{where } \Gamma = \frac{1}{\sqrt{1-\beta^2\epsilon}} \tag{7}$$

Applying Fourier transform, we get

$$\Phi(\mathbf{x},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(\mathbf{x},\omega)e^{-i\omega t}d\omega = \frac{1}{\sqrt{2\pi}}\frac{q}{v\epsilon}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}K_0\left(\frac{|\omega|\rho}{\Gamma v}\right)e^{i\omega(z/v-t)}d\omega \tag{8}$$

Here we invoke DLMF 10.43.E20 and DLMF 10.43.E21

$$\int_0^{\infty}\cos(at)K_0(t)dt = \frac{\pi}{2\sqrt{1+a^2}} \qquad \int_0^{\infty}\sin(at)K_0(t)dt = \frac{\sinh^{-1}a}{\sqrt{1+a^2}} \tag{9}$$

By parity, in (6), only the integral involving $\cos\left[\omega\left(z/v-t\right)\right]$ survives, giving

$$\Phi\left(\mathbf{x},t\right)=\frac{q}{\pi v\epsilon}\frac{\Gamma v}{\rho}\frac{\pi}{\sqrt{1+\left[\frac{\Gamma\left(z-vt\right)}{\rho}\right]^{2}}}=\frac{q}{\epsilon}\frac{\Gamma}{\sqrt{\rho^{2}+\Gamma^{2}\left(z-vt\right)^{2}}}\tag{10}$$

From (2), when ϵ is constant, the vector potential is simply $\epsilon\boldsymbol{\beta}$ times the scalar potential, i.e.,

$$\mathbf{A}\left(\mathbf{x},t\right)=\epsilon\boldsymbol{\beta}\Phi\left(\mathbf{x},t\right)\tag{11}$$

The electric field has only $\hat{\mathbf{z}}$ and $\hat{\boldsymbol{\rho}}$ component,

$$\begin{aligned}E_{z}\left(\mathbf{x},t\right)&=-\frac{\partial\Phi}{\partial z}-\frac{1}{c}\frac{\partial A_{z}}{\partial t}=-\frac{\partial\Phi}{\partial z}-\frac{\epsilon\beta}{c}\frac{\partial\Phi}{\partial t}\\&=-\left(1-\beta^{2}\epsilon\right)\frac{\partial\Phi}{\partial z}=-\frac{q}{\Gamma^{2}\epsilon}\left\{\frac{-\Gamma^{2}\left(z-vt\right)\cdot\Gamma}{\left[\rho^{2}+\Gamma^{2}\left(z-vt\right)^{2}\right]^{3/2}}\right\}\\&=\frac{\Gamma q\left(z-vt\right)}{\epsilon\left[\rho^{2}+\Gamma^{2}\left(z-vt\right)^{2}\right]^{3/2}}\end{aligned}\tag{12}$$

$$E_{\rho}\left(\mathbf{x},t\right)=-\frac{\partial\Phi}{\partial\rho}=\frac{\Gamma q\rho}{\epsilon\left[\rho^{2}+\Gamma^{2}\left(z-vt\right)^{2}\right]^{3/2}}\tag{13}$$

The magnetic field is

$$\mathbf{B}\left(\mathbf{x},t\right)=\boldsymbol{\nabla}\times\mathbf{A}\left(\mathbf{x},t\right)=-\frac{\partial A_{z}}{\partial\rho}\hat{\boldsymbol{\phi}}=-\epsilon\beta\frac{\partial\Phi}{\partial\rho}\hat{\boldsymbol{\phi}}=\epsilon\beta E_{\rho}\left(\mathbf{x},t\right)\hat{\boldsymbol{\phi}}=\frac{\beta\Gamma q\rho}{\left[\rho^{2}+\Gamma^{2}\left(z-vt\right)^{2}\right]^{3/2}}\tag{14}$$

These differ from (11.152) by $\gamma\to\Gamma$ and additional factor $1/\epsilon$ for E_{z},E_{ρ}.

3. When $\beta^{2}\epsilon>1$, we see that $\lambda^{2}<0$ from (4). Let

$$\mu^{2}=-\lambda^{2}=\frac{\omega^{2}}{v^{2}}\left(\beta^{2}\epsilon-1\right)>0\tag{15}$$

the integral (3) is now

$$\Phi\left(\mathbf{x},\omega\right)=\frac{2qe^{i\omega z/v}}{\left(2\pi\right)^{3/2}v\epsilon\left(\omega\right)}\int_{-\infty}^{\infty}e^{ik_{1}\rho}dk_{1}\underbrace{\int_{-\infty}^{\infty}\frac{dk_{2}}{k_{2}^{2}+k_{1}^{2}-\mu^{2}}}_{I}\tag{16}$$

For $|k_{1}|>\mu$, the double integral can be calculated straightforwardly,

$$\begin{aligned}\int_{|k_{1}|>\mu}e^{ik_{1}\rho}dk_{1}\int_{-\infty}^{\infty}\frac{dk_{2}}{k_{2}^{2}+k_{1}^{2}-\mu^{2}}&=\int_{|k_{1}|>\mu}e^{ik_{1}\rho}dk_{1}\cdot\frac{\pi}{\sqrt{k_{1}^{2}-\mu^{2}}}\\&=2\pi\underbrace{\int_{\mu}^{\infty}\frac{\cos\left(k_{1}\rho\right)dk_{1}}{\sqrt{k_{1}^{2}-\mu^{2}}}}_{-Y_{0}\left(\mu\rho\right)\cdot\pi/2}\qquad\text{see DLMF 10.9.E12}\\&=-\pi^{2}Y_{0}\left(\mu\rho\right)\end{aligned}\tag{17}$$

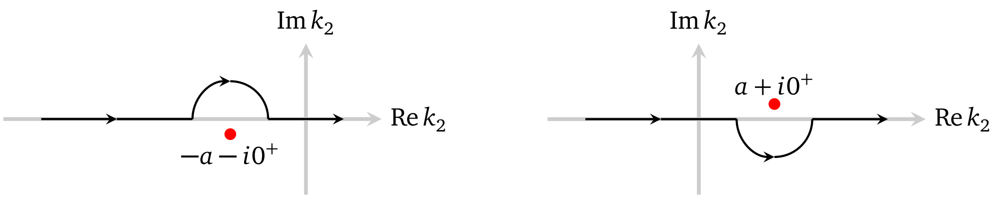
On the other hand, when $|k_{1}|<\mu$, with

$$a\equiv\sqrt{\mu^{2}-k_{1}^{2}}=\sqrt{\frac{\omega^{2}}{v^{2}}\left(\beta^{2}\epsilon-1\right)-k_{1}^{2}}\tag{18}$$

the inner integral of (16) becomes

$$I=\int_{-\infty}^{\infty}\frac{dk_{2}}{\left(k_{2}+a\right)\left(k_{2}-a\right)}=\frac{1}{2a}\left(\overbrace{\int_{-\infty}^{\infty}\frac{dk_{2}}{k_{2}-a}}^{I_{-}}-\overbrace{\int_{-\infty}^{\infty}\frac{dk_{2}}{k_{2}+a}}^{I_{+}}\right)\tag{19}$$

Consider the physical situation where $\operatorname{Im}\epsilon\left(\omega\right)$ is infinitesimally positive for positive frequency (absorbing medium),



I_+ for $\omega>0$ $\qquad$ I_- for $\omega>0$

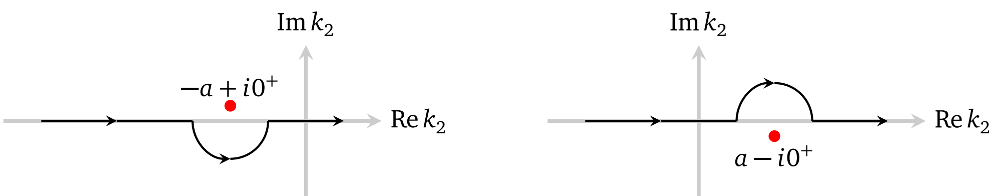
then the pole of I_+, I_- are indicated by the red dots in the diagram above.
To evaluate I_+ with pole at $-a-i0^+$, we choose the contour with infinitesimal semicircle above the singularity

$$\begin{aligned}
I_+ &= \lim_{R\to\infty}\lim_{r\to 0}\left(\int_{-R}^{-a-r}\frac{dk_2}{k_2+a}+\int_{-a+r}^{R}\frac{dk_2}{k_2+a}+\int_{\text{semicircle}}\frac{dz}{z+a}\right)\\
&= \lim_{R\to\infty}\lim_{r\to 0}\left[\ln\left(\frac{-r}{-R+a}\right)+\ln\left(\frac{R+a}{r}\right)\right]+\lim_{r\to 0}\int_{\pi}^{0}\frac{ire^{i\phi}d\phi}{re^{i\phi}}\\
&= -i\pi
\end{aligned}\tag{20}$$

Similarly,

$$\begin{aligned}
I_- &= \lim_{R\to\infty}\lim_{r\to 0}\left(\int_{-R}^{a-r}\frac{dk_2}{k_2-a}+\int_{a+r}^{R}\frac{dk_2}{k_2-a}+\int_{\text{semicircle}}\frac{dz}{z-a}\right)\\
&= \lim_{R\to\infty}\lim_{r\to 0}\left[\ln\left(\frac{-r}{-R-a}\right)+\ln\left(\frac{R-a}{r}\right)\right]+\lim_{r\to 0}\int_{-\pi}^{0}\frac{ire^{i\phi}d\phi}{re^{i\phi}}\\
&= i\pi
\end{aligned}\tag{21}$$

For negative frequencies, the absorption requirement is on the complex conjugate $\epsilon^*(\omega)$, so $\operatorname{Im}\epsilon(\omega)<0$ for $\omega<0$, the contours of $I_\pm$ are flipped about the real axis (see diagram below) hence $I_\pm$ gain a minus sign on top of (20), (21).



I_+ for $\omega<0$ $\qquad$ I_- for $\omega<0$

In summary, the inner integral of (16), considering both positive and negative frequencies, is

$$I=\frac{1}{2a}(I_- - I_+)=\operatorname{sgn}(\omega)\frac{i\pi}{\sqrt{\mu^2-k_1^2}}\tag{22}$$

which gives the remaining part of the double integral

$$\begin{aligned}
\int_{|k_1|<\mu}e^{ik_1\rho}dk_1\int_{-\infty}^{\infty}\frac{dk_2}{k_2^2+k_1^2-\mu^2} &= \operatorname{sgn}(\omega)2\pi i\overbrace{\int_0^{\mu}\frac{\cos(k_1\rho)dk_1}{\sqrt{\mu^2-k_1^2}}}^{J_0(\mu\rho)\cdot\pi/2} \qquad \text{see DLMF 10.9.E4}\\
&= \operatorname{sgn}(\omega)\pi^2 iJ_0(\mu\rho)
\end{aligned}\tag{23}$$

Putting (17) and (23) back to (16) gives the desired result

$$\Phi(\mathbf{x},\omega)=\frac{qe^{i\omega z/v}}{v\epsilon(\omega)}\sqrt{\frac{\pi}{2}}\left[-Y_0(\mu\rho)+\operatorname{sgn}(\omega)iJ_0(\mu\rho)\right]\tag{24}$$

Finally, when $\epsilon(\omega)$ is independent of frequency, applying Fourier transform $\omega\to t$ yields

$$\begin{aligned}
\Phi(\mathbf{x},t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(\mathbf{x},\omega)e^{-i\omega t}d\omega\\
&= \frac{1}{\sqrt{2\pi}}\frac{q}{v\epsilon}\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty}e^{i\omega(z/v-t)}\left[-Y_0\left(\frac{|\omega|\rho}{v}\sqrt{\beta^2\epsilon-1}\right)+\operatorname{sgn}(\omega)iJ_0\left(\frac{|\omega|\rho}{v}\sqrt{\beta^2\epsilon-1}\right)\right]d\omega
\end{aligned}\tag{25}$$

By parity,

$$\int_{-\infty}^{\infty} e^{i\omega(z/v-t)} Y_0\left(\frac{|\omega|\rho}{v}\sqrt{\beta^2\epsilon-1}\right)d\omega = 2\int_0^{\infty} \cos\left[\omega\left(z/v-t\right)\right] Y_0\left(\frac{\omega\rho}{v}\sqrt{\beta^2\epsilon-1}\right)d\omega \tag{26}$$

$$\int_{-\infty}^{\infty} e^{i\omega(z/v-t)}\,\mathrm{sgn}\left(\omega\right) iJ_0\left(\frac{|\omega|\rho}{v}\sqrt{\beta^2\epsilon-1}\right)d\omega = -2\int_0^{\infty} \sin\left[\omega\left(z/v-t\right)\right] J_0\left(\frac{\omega\rho}{v}\sqrt{\beta^2\epsilon-1}\right)d\omega \tag{27}$$

The latter integrals are given in 6.671.12 and 6.671.7 of *I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals,Series and Products Eighth Edition*

$$\int_0^{\infty} Y_0\left(ax\right)\cos\left(bx\right)dx = \begin{cases} -\dfrac{1}{\sqrt{b^2-a^2}} & \text{for } b > a > 0 \\ 0 & \text{for } a > b > 0 \end{cases} \tag{28}$$

$$\int_0^{\infty} J_0\left(ax\right)\sin\left(bx\right)dx = \begin{cases} \dfrac{1}{\sqrt{b^2-a^2}} & \text{for } b > a > 0 \\ 0 & \text{for } a > b > 0 \end{cases} \tag{29}$$

When $z > vt$, the sum of the two integrals in (25) vanishes, regardless of the relative size of a and b. But for $z < vt$, the sum enhances the result for the $b > a > 0$ case. In summary, the full scalar potential is

$$\Phi\left(\mathbf{x}, t\right) = \begin{cases} \dfrac{2q}{\epsilon\sqrt{\left(z-vt\right)^2-\rho^2\left(\beta^2\epsilon-1\right)}} & \text{for } z < vt \text{ and } \rho < \dfrac{vt-z}{\sqrt{\beta^2\epsilon-1}} = \dfrac{vt-x}{\tan\theta_c} \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

Multiplying (30) by $\boldsymbol{\beta}/\epsilon$, as indicated by (2), will give us the same vector potential as (13.51).

The non-zero potential region is the interior of the shockwave lightcone, as depicted below.