1. **Expansion of Green function from first principles**

Treat $G(\mathbf{x},\mathbf{x}')$ as a function of $\mathbf{x}'$, and expand it into spherical harmonic basis

$$G\left(\mathbf{x},\mathbf{x}'\right)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}R_{lm}\left(\mathbf{x};r'\right)Y_{lm}\left(\theta',\phi'\right) \tag{1}$$

Taking the Laplacian with respect to $\mathbf{x}'$ yields

$$\nabla'^2 G=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{1}{r'^2}\frac{\partial}{\partial r'}\left(r'^2\frac{\partial R_{lm}}{\partial r'}\right)\right]Y_{lm}+R_{lm}\nabla'^2_{\theta',\phi'}Y_{lm} \tag{2}$$

where $\nabla'^2_{\theta',\phi'}$ is the angular components of the Laplacian ∇'^2, for which the spherical harmonics satisfy the differential equation

$$\nabla'^2_{\theta',\phi'}Y_{lm}=-\frac{l(l+1)}{r'^2}Y_{lm} \tag{3}$$

Thus from (2), (3) we have

$$\begin{aligned}\nabla'^2 G=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{1}{r'^2}\frac{\partial}{\partial r'}\left(r'^2\frac{\partial R_{lm}}{\partial r'}\right)-\frac{l(l+1)}{r'^2}R_{lm}\right]Y_{lm}&=-4\pi\delta\left(\mathbf{x}-\mathbf{x}'\right)\\ &=-4\pi\frac{\delta\left(r-r'\right)}{r'^2}\delta\left(\phi-\phi'\right)\delta\left(\cos\theta-\cos\theta'\right)\end{aligned} \tag{4}$$

(It is worth noting that the Green function Laplacian relation has been defined to be with respect to $\mathbf{x}'$ instead of $\mathbf{x}$. In the past notes, since we have been dealing with Dirichlet boundary conditions where $G(\mathbf{x},\mathbf{x}')$ is symmetric in the arguments, we could use ∇^2 and ∇'^2 interchangeably. But for Neumann boundary conditions, Green function is not generally symmetric, we must stick to ∇'^2.)

Since (see (3.56))

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y^*_{lm}(\theta,\phi)Y_{lm}\left(\theta',\phi'\right)=\delta\left(\phi-\phi'\right)\delta\left(\cos\theta-\cos\theta'\right) \tag{5}$$

We write

$$R_{lm}\left(\mathbf{x};r'\right)=h_l\left(r,r'\right)Y^*_{lm}(\theta,\phi) \tag{6}$$

to turn (4) into

$$\sum_{l=0}^{\infty}\left[\frac{1}{r'^2}\frac{\partial}{\partial r'}\left(r'^2\frac{\partial h_l}{\partial r'}\right)-\frac{l(l+1)}{r'^2}h_l\right]\sum_{m=-l}^{l}Y^*_{lm}(\theta,\phi)Y_{lm}\left(\theta',\phi'\right)=-4\pi\frac{\delta\left(r-r'\right)}{r'^2}\delta\left(\phi-\phi'\right)\delta\left(\cos\theta-\cos\theta'\right) \tag{7}$$

Thus we want to find $h_l(r,r')$ satisfying

$$\frac{\partial}{\partial r'}\left(r'^2\frac{\partial h_l}{\partial r'}\right)-l(l+1)h_l=-4\pi\delta\left(r-r'\right) \tag{8}$$

If we scale $h_l(r,r')$ such that

$$g_l\left(r,r'\right)\equiv\frac{2l+1}{4\pi}h_l\left(r,r'\right) \tag{9}$$

then by the addition theorem of spherical harmonics, we can write the Green function as

$$G\left(\mathbf{x},\mathbf{x}'\right)=\sum_{l=0}^{\infty}g_l\left(r,r'\right)\left[\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y^*_{lm}(\theta,\phi)Y_{lm}\left(\theta',\phi'\right)\right]=\sum_{l=0}^{\infty}g_l\left(r,r'\right)P_l(\cos\gamma) \tag{10}$$

which is the hinted form given by the problem statement.

With (9) substituted into (8), we have the differential equation for g_l:

$$\frac{\partial}{\partial r'}\left(r'^2\frac{\partial g_l}{\partial r'}\right)-l(l+1)g_l=-(2l+1)\delta\left(r-r'\right) \tag{11}$$

which admits general form when $r\neq r'$, with the coefficients as function of r and to be determined:

$$g_l\left(r,r'\right)=\begin{cases}Ar'^l+Br'^{-(l+1)} & \text{for } r'<r\\ Cr'^l+Dr'^{-(l+1)} & \text{for } r'>r\end{cases} \tag{12}$$

Integrating (11) over the infinitesimal range $[r-\epsilon,r+\epsilon]$ gives

$$r'^2\frac{\partial g_l}{\partial r'}\bigg|_{r+\epsilon}-r'^2\frac{\partial g_l}{\partial r'}\bigg|_{r-\epsilon}=-(2l+1) \tag{13}$$

2. **Radial boundary condition of** $g_l\left(r,r'\right)$

   The general Neumann boundary condition is given by

$$\oint_S\frac{\partial G}{\partial n'}da'=-4\pi \tag{14}$$

of which a special case is prescribed as equation (1.45)

$$\frac{\partial G\left(\mathbf{x},\mathbf{x}'\right)}{\partial n'}=-\frac{4\pi}{S} \tag{15}$$

It is a special case since we impose an additional restriction that every point on S has the same normal gradient. We are going to impose (15) to our Green function (10):

$$-\frac{4\pi}{S}=-\frac{1}{a^2+b^2}=\frac{\partial G\left(\mathbf{x},\mathbf{x}'\right)}{\partial n'}=\begin{cases}\sum_{l=0}^{\infty}\frac{\partial g_l\left(r,r'\right)}{\partial r'}\cdot P_l\left(\cos\gamma\right) & \text{for } \mathbf{x}' \text{ on outer shell}\\ \sum_{l=0}^{\infty}-\frac{\partial g_l\left(r,r'\right)}{\partial r'}\cdot P_l\left(\cos\gamma\right) & \text{for } \mathbf{x}' \text{ on inner shell}\end{cases} \tag{16}$$

By orthogonality of Legendre basis, we can conclude

$$\frac{\partial g_l\left(r,r'\right)}{\partial r'}\bigg|_{r'=a\text{ or }b}=\begin{cases}0 & \text{for } l>0\\ \frac{1}{a^2+b^2}\text{ or }-\frac{1}{a^2+b^2} & \text{for } l=0\end{cases} \tag{17}$$

3. **Solution to part (a)**

   To find $g_l\left(r,r'\right)$ for $l>0$, we will apply the following constraints to determine the coefficients in (12):

   (a) Neumann condition (17) on the inner and outer spheres:

$$\frac{\partial g_l\left(r,r'\right)}{\partial r'}\bigg|_{r'=a}=lAa^{l-1}-(l+1)Ba^{-(l+2)}=0 \tag{18}$$

$$\frac{\partial g_l\left(r,r'\right)}{\partial r'}\bigg|_{r'=b}=lCb^{l-1}-(l+1)Db^{-(l+2)}=0 \tag{19}$$

   (b) Continuity at $r'=r$:

$$Ar^l+Br^{-(l+1)}=Cr^l+Dr^{-(l+1)} \tag{20}$$

   (c) Discontinuity of derivative at $r'=r$ (13):

$$lCr^{l+1}-(l+1)Dr^{-l}-lAr^{l+1}+(l+1)Br^{-l}=-(2l+1) \tag{21}$$

From (18), (19):

$$B = \frac{l}{l+1} a^{2l+1} A \tag{22}$$

$$D = \frac{l}{l+1} b^{2l+1} C \tag{23}$$

From (20):

$$D - B = r^{2l+1}(A - C) \tag{24}$$

From (21):

$$\begin{aligned}
l(A-C)\,r^{l+1} + (l+1)(D-B)\,r^{-l} &= 2l+1 && \text{by (24)} && \Longrightarrow \\
(2l+1)(A-C)\,r^{l+1} &= 2l+1 && && \Longrightarrow \\
A - C &= r^{-(l+1)} && \text{by (24)} && \Longrightarrow && (25)\\
D - B &= r^{l} && && && (26)
\end{aligned}$$

Inserting (22), (23), (25) into (26) gives

$$\begin{aligned}
\frac{l}{l+1}\left\{b^{2l+1}\left[A - r^{-(l+1)}\right] - a^{2l+1}A\right\} = r^l & && && \Longrightarrow \\
\left(b^{2l+1} - a^{2l+1}\right)A = \frac{l+1}{l} r^l + \frac{b^{2l+1}}{r^{l+1}} & && && \Longrightarrow \\
A = \frac{1}{b^{2l+1} - a^{2l+1}}\left(\frac{l+1}{l} r^l + \frac{b^{2l+1}}{r^{l+1}}\right) & && \text{by (22)} && \Longrightarrow && (27)\\
B = \frac{1}{b^{2l+1} - a^{2l+1}}\left[a^{2l+1} r^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{r^{l+1}}\right] & && \text{by (26)} && \Longrightarrow && (28)\\
D = \frac{1}{b^{2l+1} - a^{2l+1}}\left[b^{2l+1} r^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{r^{l+1}}\right] & && \text{by (23)} && \Longrightarrow && (29)\\
C = \frac{1}{b^{2l+1} - a^{2l+1}}\left(\frac{l+1}{l} r^l + \frac{a^{2l+1}}{r^{l+1}}\right) & && && && (30)
\end{aligned}$$

Going back to (12), we obtain $g_l(r, r')$ for $l > 0$:

$$\begin{aligned}
g_l\left(r, r'\right) &= \begin{cases}
\dfrac{1}{b^{2l+1} - a^{2l+1}}\left[\dfrac{l+1}{l}\left(rr'\right)^l + b^{2l+1}\dfrac{r'^l}{r^{l+1}} + a^{2l+1}\dfrac{r^l}{r'^{l+1}} + \dfrac{l}{l+1}\dfrac{(ab)^{2l+1}}{(rr')^{l+1}}\right] & \text{for } r' < r \\
\dfrac{1}{b^{2l+1} - a^{2l+1}}\left[\dfrac{l+1}{l}\left(rr'\right)^l + a^{2l+1}\dfrac{r'^l}{r^{l+1}} + b^{2l+1}\dfrac{r^l}{r'^{l+1}} + \dfrac{l}{l+1}\dfrac{(ab)^{2l+1}}{(rr')^{l+1}}\right] & \text{for } r' > r
\end{cases} \\
&= \frac{r_<^l}{r_>^{l+1}} + \frac{1}{b^{2l+1} - a^{2l+1}}\left[\frac{l+1}{l}(rr')^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{(rr')^{l+1}} + a^{2l+1}\left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}}\right)\right]
\end{aligned} \tag{31}$$

Take a note of the symmetry in r, r', we will discuss in more details in the last section.

4. **Solution to part (b)**

   Let's apply the same constraints for $g_0(r, r')$ in form (12):

   (a) Neumann condition (17):

$$\left.\frac{\partial g_l(r, r')}{\partial r'}\right|_{r'=a} = -\frac{B}{a^2} = \frac{1}{a^2 + b^2} \qquad \Longrightarrow \qquad B = -\frac{a^2}{a^2 + b^2} \tag{32}$$

$$\left.\frac{\partial g_l(r, r')}{\partial r'}\right|_{r'=b} = -\frac{D}{b^2} = -\frac{1}{a^2 + b^2} \qquad \Longrightarrow \qquad D = \frac{b^2}{a^2 + b^2} \tag{33}$$

   (b) Continuity at $r' = r$:

$$A + \frac{B}{r} = C + \frac{D}{r} \tag{34}$$

(c) Discontinuity of derivative at $r' = r$ (13):

$$-D + B = -1 \tag{35}$$

We see that (35) is redundant given (32), (33), so we have an underconstrained system. We can arbitrarily set

$$C = f(r) \qquad \text{and thus} \qquad A = f(r) + \frac{1}{r} \tag{36}$$

In summary

$$\begin{aligned} g_0(r, r) &= \begin{cases} f(r) + \dfrac{1}{r} - \left(\dfrac{a^2}{a^2 + b^2}\right)\dfrac{1}{r'} & \text{for } r' < r \\ f(r) + \left(\dfrac{b^2}{a^2 + b^2}\right)\dfrac{1}{r'} & \text{for } r' > r \end{cases} \\ &= \frac{1}{r_>} - \left(\frac{a^2}{a^2 + b^2}\right)\frac{1}{r'} + f(r) \end{aligned} \tag{37}$$

The fact that $f(r)$ will not affect the potential calculation is proved in a similar way to Prob 1.14, where $f(r)$ can be taken out of the integral in d^3x' and da', and use Gauss's theorem to show its contribution to $\Phi(\mathbf{x})$ is zero.

5. **Symmetry of Neumann Green function $G(\mathbf{x}, \mathbf{x}')$ in $\mathbf{x}, \mathbf{x}'$**

In general, Neumann Green functions are not necessarily symmetric in their arguments $\mathbf{x}, \mathbf{x}'$, but in problem (1.14) we have proved that

$$H\left(\mathbf{x}, \mathbf{x}'\right) = G\left(\mathbf{x}, \mathbf{x}'\right) - F(\mathbf{x}) \qquad \text{where} \qquad F(\mathbf{x}) = \frac{1}{S}\oint_S G\left(\mathbf{x}, \mathbf{x}'\right) da' \tag{38}$$

is symmetric in $\mathbf{x}, \mathbf{x}'$, i.e., $H(\mathbf{x}, \mathbf{x}') = H(\mathbf{x}', \mathbf{x})$.

Form (10) gives rise to

$$\begin{aligned} H\left(\mathbf{x}, \mathbf{x}'\right) &= G\left(\mathbf{x}, \mathbf{x}'\right) - \frac{1}{S}\oint_S \sum_{l=0}^{\infty} g_l\left(r, r'\right) P_l(\cos\gamma)\, da' \\ &= G\left(\mathbf{x}, \mathbf{x}'\right) - \frac{1}{4\pi(a^2 + b^2)} \sum_{l=0}^{\infty} \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \left[g_l(r, a)\, a^2 + g_l(r, b)\, b^2\right] P_l(\cos\gamma) \end{aligned} \tag{39}$$

Using the addition theorem, we can calculate the angular integral

$$\begin{aligned} \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' P_l(\cos\gamma) &= \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*(\theta, \phi) \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \underbrace{\sqrt{\frac{2l+1}{4\pi}} e^{im\phi'} P_l^m\left(\cos\theta'\right)}_{Y_{lm}(\theta', \phi')} \\ &= \sqrt{\frac{4\pi}{2l+1}} \sum_{m=-l}^{l} Y_{lm}^*(\theta, \phi)\, 2\pi\delta_{m0} \int_0^{\pi} \sin\theta' d\theta' P_l^m\left(\cos\theta'\right) \\ &= \sqrt{\frac{4\pi}{2l+1}} Y_{l0}^*(\theta, \phi)\, 2\pi \int_{-1}^{1} dx P_l(x) \\ &= \sqrt{\frac{4\pi}{2l+1}} Y_{l0}^*(\theta, \phi)\, 4\pi\delta_{l0} = 4\pi\delta_{l0} \end{aligned} \tag{40}$$

which turns (39) into

$$\begin{aligned} H\left(\mathbf{x}, \mathbf{x}'\right) &= G\left(\mathbf{x}, \mathbf{x}'\right) - \underbrace{\frac{g_0(r, a)\, a^2 + g_0(r, b)\, b^2}{a^2 + b^2}}_{\equiv \overline{g_0}(r)} \\ &= g_0\left(r, r'\right) - \overline{g_0}(r) + \sum_{l=1}^{\infty} g_l\left(r, r'\right) P_l(\cos\gamma) \end{aligned} \tag{41}$$

Then the symmetry $H(\mathbf{x}, \mathbf{x}') = H(\mathbf{x}', \mathbf{x})$ implies

$$g_0\left(r, r'\right) - \overline{g_0}(r) + \sum_{l=1}^{\infty} g_l\left(r, r'\right) P_l(\cos\gamma) = g_0\left(r', r\right) - \overline{g_0}\left(r'\right) + \sum_{l=1}^{\infty} g_l\left(r', r\right) P_l\left(\cos\gamma'\right) \tag{42}$$

where

$$\cos\gamma = \sin\theta\sin\theta'\cos\left(\phi-\phi'\right)+\cos\theta\cos\theta' \tag{43}$$

and $\cos\gamma'$ is obtained from $\cos\gamma$ by the exchange $\theta \longleftrightarrow \theta', \phi \longleftrightarrow \phi'$, which gives the same value as $\cos\gamma$.
Applying the orthogonality of Legendre basis to (42) yields

$$g_0\left(r,r'\right)-\overline{g_0}\left(r\right)=g_0\left(r',r\right)-\overline{g_0}\left(r'\right) \tag{44}$$
$$g_l\left(r,r'\right)=g_l\left(r',r\right) \qquad \text{for } l>0 \tag{45}$$

This shows explicitly that $g_l\left(r,r'\right)$ is symmetric for $l>0$, and we can choose $f\left(r\right)=-\overline{g_0}\left(r\right)$ to make $g_0\left(r,r'\right)$ symmetric. In particular, for (37), we end up with

$$f\left(r\right)=-\overline{g_0}\left(r\right)=-\left(\frac{a^2}{a^2+b^2}\right)\frac{1}{r}+C \tag{46}$$

Ignoring the inconsequential constant C, we end up with a symmetric g_0:

$$g_0\left(r,r'\right)=\left(\frac{b^2}{a^2+b^2}\right)\frac{1}{r_>}-\left(\frac{a^2}{a^2+b^2}\right)\frac{1}{r_<} \tag{47}$$