1. From the first Kramers-Kronig relation in (7.120)

$$\mathrm{Re}\left[\epsilon\left(\omega\right)/\epsilon_0\right]=1+\frac{2}{\pi}P\int_0^{\infty}\frac{\omega'\,\mathrm{Im}\left[\epsilon\left(\omega'\right)/\epsilon_0\right]}{\omega'^2-\omega^2}d\omega' \tag{1}$$

With

$$\mathrm{Im}\left[\epsilon\left(\omega'\right)/\epsilon_0\right]=\lambda\left[\theta\left(\omega'-\omega_1\right)-\theta\left(\omega'-\omega_2\right)\right] \tag{2}$$

we have

$$\begin{aligned}\mathrm{Re}\left[\epsilon\left(\omega\right)/\epsilon_0\right]&=1+\frac{2\lambda}{\pi}P\int_{\omega_1}^{\omega_2}\frac{\omega'd\omega'}{\omega'^2-\omega^2} && \text{let } u\equiv\omega'^2-\omega^2\\ &=1+\frac{\lambda}{\pi}P\int_{u_1}^{u_2}\frac{du}{u} && \text{where } u_{1,2}=\omega_{1,2}^2-\omega^2\end{aligned} \tag{3}$$

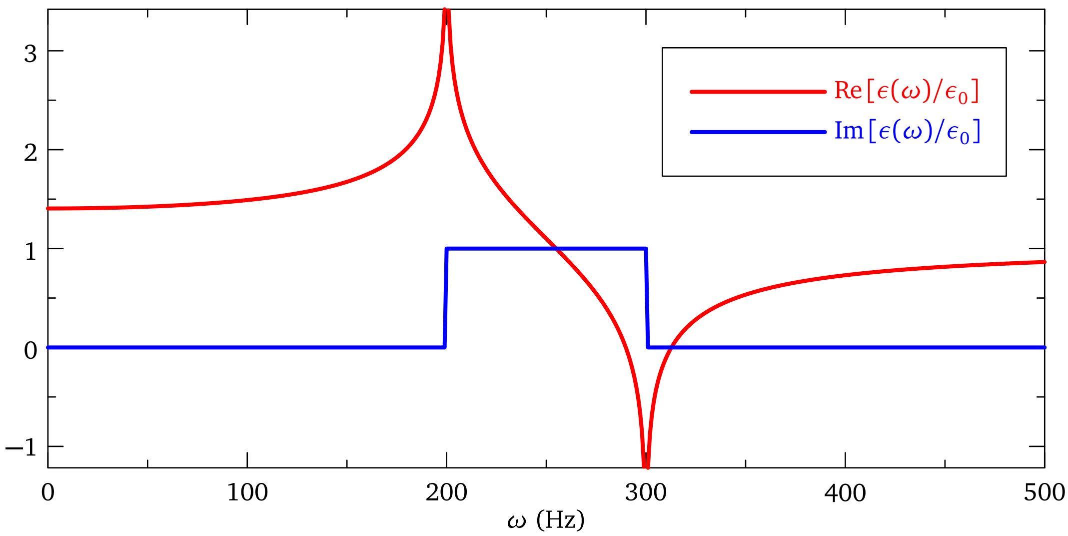
When $\omega<\omega_1$ or $\omega>\omega_2$, the integral is well defined and can be calculated directly,

$$\int_{u_1}^{u_2}\frac{du}{u}=\ln\left(\frac{\omega^2-\omega_2^2}{\omega^2-\omega_1^2}\right) \tag{4}$$

For $\omega_1<\omega<\omega_2$, we can obtain the principal value of the integral via the limiting procedure

$$P\int_{u_1}^{u_2}\frac{du}{u}=\lim_{\delta\to0}\left(\int_{u_1}^{-\delta}\frac{du}{u}+\int_{\delta}^{u_2}\frac{du}{u}\right)=\lim_{\delta\to0}\left[\ln\left(\frac{\delta}{-u_1}\right)+\ln\left(\frac{u_2}{\delta}\right)\right]=\ln\left(\frac{u_2}{-u_1}\right)=\ln\left(\frac{\omega_2^2-\omega^2}{\omega^2-\omega_1^2}\right) \tag{5}$$

Therefore, except for $\omega=\omega_{1,2}$, $\mathrm{Re}\left[\epsilon\left(\omega\right)/\epsilon_0\right]$ is non-singular, the plot below shows the relationship for $\omega_1=200\mathrm{Hz}$, $\omega_2=300\mathrm{Hz}$ and $\lambda=\pi/2$.



2. Let

$$f\left(\omega\right)=\frac{1}{\omega_0^2-\omega^2-i\gamma\omega}\qquad\qquad g(\omega)=\frac{1}{\omega_0^2-\omega^2+i\gamma\omega} \tag{6}$$

then we have

$$\mathrm{Im}\left[\epsilon\left(\omega\right)/\epsilon_0\right]=\frac{\lambda\gamma\omega}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}=\frac{\lambda}{2i}\left(\frac{1}{\omega_0^2-\omega^2-i\gamma\omega}-\frac{1}{\omega_0^2-\omega^2+i\gamma\omega}\right)=\frac{\lambda}{2i}\left[f\left(\omega\right)-g\left(\omega\right)\right] \tag{7}$$

For $f\left(\omega\right)$, its two poles are at $-i\gamma/2\pm\nu_0$ (in the lower halfplane), where $\nu_0^2=\omega_0^2-\gamma^2/4$. For $g\left(\omega\right)$, its two poles are at $i\gamma/2\pm\nu_0$ (in the upper halfplane).

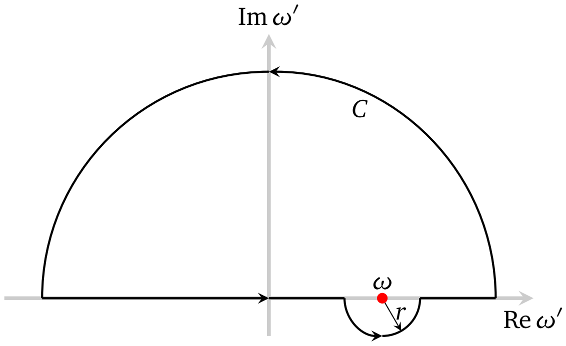
Then by (7.119),

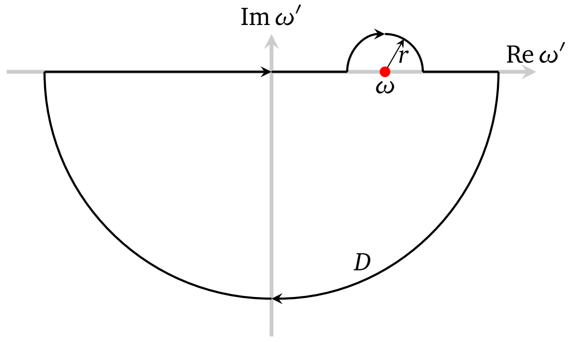
$$\operatorname{Re}\left[\epsilon(\omega)/\epsilon_0\right] = 1 + \frac{\lambda}{2\pi i} P \int_{-\infty}^{\infty} \left[\frac{f(\omega') - g(\omega')}{\omega' - \omega}\right] d\omega' \tag{8}$$

To evaluate the principal value

$$F = P \int_{-\infty}^{\infty} \frac{f(\omega')}{\omega' - \omega} d\omega' \tag{9}$$

consider the integral along the contour C depicted in the figure on the left below





$$\begin{aligned}\lim_{r\to 0} \oint_C \frac{f(\omega')}{\omega' - \omega} d\omega' = \lim_{r\to 0} \Bigg[& \int_{\text{arc}@\infty} \frac{f(\omega')}{\omega' - \omega} d\omega + \\ & \int_{-\infty}^{\omega - r} \frac{f(\omega')}{\omega' - \omega} d\omega' + \int_{\omega + r}^{\infty} \frac{f(\omega')}{\omega' - \omega} d\omega' + \\ & \int_{-\pi}^{0} \frac{f\left(\omega + re^{i\phi}\right)}{re^{i\phi}} ire^{i\phi} d\phi \Bigg]\end{aligned} \tag{10}$$

The first term vanishes due to the behavior of $f(\omega')$ at ∞, the next two terms produce F by definition of the principal value, and the last term gives $i\pi f(\omega)$. Sicne ω is the only pole within this contour, by the residue theorem, we have

$$F + i\pi f(\omega) = 2\pi i f(\omega) \qquad \Longrightarrow \qquad F = i\pi f(\omega) \tag{11}$$

Similarly, for

$$G = P \int_{-\infty}^{\infty} \frac{g(\omega')}{\omega' - \omega} d\omega' \tag{12}$$

consider the integral along the contour D depicted on the right,

$$\begin{aligned}\lim_{r\to 0} \oint_D \frac{g(\omega')}{\omega' - \omega} d\omega' = \lim_{r\to 0} \Bigg[& \int_{\text{arc}@\infty} \frac{g(\omega')}{\omega' - \omega} d\omega + \\ & \int_{-\infty}^{\omega - r} \frac{g(\omega')}{\omega' - \omega} d\omega' + \int_{\omega + r}^{\infty} \frac{g(\omega')}{\omega' - \omega} d\omega' + \\ & \int_{\pi}^{0} \frac{g\left(\omega + re^{i\phi}\right)}{re^{i\phi}} ire^{i\phi} d\phi \Bigg]\end{aligned} \tag{13}$$

from which we obtain

$$G - i\pi g(\omega) = -2\pi i g(\omega) \qquad \Longrightarrow \qquad G = -i\pi g(\omega) \tag{14}$$

Plugging (11) and (14) back to (8), we have

$$\mathrm{Re}[\epsilon(\omega)/\epsilon_0] = 1 + \frac{\lambda}{2}[f(\omega) + g(\omega)] = 1 + \frac{\lambda\left(\omega_0^2 - \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2} \tag{15}$$

Below is the plot for $\omega_0 = 200\text{Hz}, \gamma = 15\text{Hz}, \lambda = 10^4$.