1. **General solution of arbitrary incident angle**

First let's incorporate Ohm's law into the Maxwell equations in the metal medium. When we consider monochromatic light, every quantity $\mathbf{B}, \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{J}, \rho$ will have a harmonic factor $e^{-i\omega t}$.

From the conservation of charge,

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad \Longrightarrow \quad \sigma \nabla \cdot \mathbf{E} - i\omega\rho = 0 \quad \Longrightarrow \quad \rho = \frac{\sigma}{i\omega} \nabla \cdot \mathbf{E} \tag{1}$$

we can rewrite the two Maxwell equations

$$\nabla \cdot \mathbf{D} - \rho = 0 \quad \Longrightarrow \quad \epsilon \nabla \cdot \mathbf{E} - \frac{\sigma}{i\omega} \nabla \cdot \mathbf{E} = \nabla \cdot \left[\left(\epsilon + \frac{i\sigma}{\omega} \right) \mathbf{E} \right] = 0 \tag{2}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \Longrightarrow \quad \nabla \times \mathbf{H} = (\sigma - i\omega\epsilon)\mathbf{E} = -i\omega \left(\epsilon + \frac{i\sigma}{\omega} \right) \mathbf{E} = \frac{\partial}{\partial t} \left[\left(\epsilon + \frac{i\sigma}{\omega} \right) \mathbf{E} \right] \tag{3}$$

This allows us to replace the real permittivity ϵ with a complex permittivity $\widetilde{\epsilon}(\omega) = \epsilon + i\sigma/\omega$, and treat the media as if it has zero ρ and $\mathbf{J}$ everywhere.

When we consider the field incident onto the metal media, the transmitted field should have a plane wave solution form

$$\mathbf{E}'(\mathbf{x}, t) = \mathbf{E}'_0 e^{i\mathbf{k}' \cdot \mathbf{x} - i\omega t} \tag{4}$$

where $\mathbf{k}' = \mathbf{k}'_R + i\mathbf{k}'_I$ is a complex vector that satisfies

$$\mathbf{k}' \cdot \mathbf{k}' = \omega^2 \mu' \widetilde{\epsilon'} = \omega^2 \mu' \epsilon' + i\omega\mu'\sigma \quad \Longrightarrow$$

$$k_R'^2 - k_I'^2 = \omega^2 \mu' \epsilon' \tag{5}$$

$$\mathbf{k}'_R \cdot \mathbf{k}'_I = \frac{\omega\mu'\sigma}{2} \tag{6}$$

Let the boundary plane between the two media be the $z = 0$ plane. Usual argument of the boundary condition leads to the restriction

$$\mathbf{k} \cdot \mathbf{x} = \mathbf{k}' \cdot \mathbf{x} = \mathbf{k}'_R \cdot \mathbf{x} + i\mathbf{k}'_I \cdot \mathbf{x} \qquad \text{for all } \mathbf{x} \text{ on the } z = 0 \text{ plane} \tag{7}$$

Since the LHS of (7) is real, we must have

$$\mathbf{k} \cdot \mathbf{x} = \mathbf{k}'_R \cdot \mathbf{x} \qquad \mathbf{k}'_I \cdot \mathbf{x} = 0 \qquad \text{for all } \mathbf{x} \text{ on the } z = 0 \text{ plane} \tag{8}$$

An immediate consequence of (6) and (8) is that when $\sigma \neq 0$, $\mathbf{k}'_I$ must be nonzero and along the $\hat{\mathbf{z}}$ direction, which makes the field (4) exponentially attenuate as we go deeper into the metal. But unlike the case where $\sigma = 0$, $\mathbf{k}'_R$ and $\mathbf{k}'_I$ are not orthogonal to each other.

Let r be the angle between $\mathbf{k}'_R$ and $\mathbf{k}'_I$ (i.e., refraction angle). The first equality of (8) gives

$$k \sin i = k'_R \sin r \tag{9}$$

and (6) can be written as

$$k'_R k'_I \cos r = \frac{\omega\mu'\sigma}{2} = \omega^2 \mu\epsilon \cdot \frac{\mu'\epsilon'\sigma}{2\omega\mu\epsilon\epsilon'} = k^2 \left(\frac{n'}{n} \right)^2 \left(\frac{\sigma}{2\omega\epsilon'} \right) \tag{10}$$

Combining (9) and (10) gives

$$\cos^2 r + \sin^2 r = \left(\frac{k^2}{k'_R k'_I} \right)^2 \left(\frac{n'}{n} \right)^4 \left(\frac{\sigma}{2\omega\epsilon'} \right)^2 + \left(\frac{k \sin i}{k'_R} \right)^2 = 1 \quad \Longrightarrow \quad k_R'^2 k_I'^2 - (k \sin i)^2 k_I'^2 - k^4 \left(\frac{n'}{n} \right)^4 \left(\frac{\sigma}{2\omega\epsilon'} \right)^2 = 0 \tag{11}$$

Together with (5),

$$k_R'^2 - k_I'^2 = \omega^2 \mu' \epsilon' = \omega^2 \mu\epsilon \cdot \frac{\mu'\epsilon'}{\mu\epsilon} = k^2 \left(\frac{n'}{n} \right)^2 \tag{12}$$

we finally obtain the quadratic equation for $k_I'^2$:

$$k_I'^4 + k^2\left[\left(\frac{n'}{n}\right)^2 - \sin^2 i\right]k_I'^2 - k^4\left(\frac{n'}{n}\right)^4\left(\frac{\sigma}{2\omega\epsilon'}\right)^2 = 0 \tag{13}$$

which gives

$$k_I'^2 = \frac{k^2}{2}\left\{\left[\sin^2 i - \left(\frac{n'}{n}\right)^2\right] + \sqrt{\left[\left(\frac{n'}{n}\right)^2 - \sin^2 i\right]^2 + \left(\frac{n'}{n}\right)^4\left(\frac{\sigma}{\omega\epsilon'}\right)^2}\right\} \tag{14}$$

$$k_R'^2 = \frac{k^2}{2}\left\{\left[\sin^2 i + \left(\frac{n'}{n}\right)^2\right] + \sqrt{\left[\left(\frac{n'}{n}\right)^2 - \sin^2 i\right]^2 + \left(\frac{n'}{n}\right)^4\left(\frac{\sigma}{\omega\epsilon'}\right)^2}\right\} \tag{15}$$

A few points are noteworthy:

(a) With normal incidence $\sin i = 0$, plugging (14) and (15) into (10) yields $\cos r = 1$, as expected.

(b) when $\sigma = 0$, we see that

$$k_I'^2 = \frac{k^2}{2}\left[\sin^2 i - \left(\frac{n'}{n}\right)^2 + \left|\left(\frac{n'}{n}\right)^2 - \sin^2 i\right|\right] = \begin{cases} 0 & \text{when } \sin i < \frac{n'}{n} \\ k^2\left[\sin^2 i - \left(\frac{n'}{n}\right)^2\right] & \text{when } \sin i \geq \frac{n'}{n} \end{cases} \tag{16}$$

$$k_R'^2 = \frac{k^2}{2}\left[\sin^2 i + \left(\frac{n'}{n}\right)^2 + \left|\left(\frac{n'}{n}\right)^2 - \sin^2 i\right|\right] = \begin{cases} k^2\left(\frac{n'}{n}\right)^2 & \text{when } \sin i < \frac{n'}{n} \\ k^2 \sin^2 i & \text{when } \sin i \geq \frac{n'}{n} \end{cases} \tag{17}$$

which recovers the calculation of non-metal reflection and refraction (including the case of total internal reflection).

At this point, we have fully established the complex wave vector $\mathbf{k}'$, but for the subsequent calculations, we shall write it in component forms (assuming the plane of incidence is the x-z plane)

$$\mathbf{k}' = \mathbf{k}_R' + i\mathbf{k}_I' = \hat{\mathbf{x}}k_R'\sin r + \hat{\mathbf{z}}\left(k_R'\cos r + ik_I'\right) \equiv \hat{\mathbf{x}}k_x' + \hat{\mathbf{z}}k_z' \qquad \text{where} \tag{18}$$

$$k_x' = k_R'\sin r = k\sin i \tag{19}$$

$$k_z' = k_R'\cos r + ik_I' \tag{20}$$

Note that

$$k_x'^2 + k_z'^2 = k_R'^2 - k_I'^2 + 2k_R'k_I'\cos r = \omega^2\mu'\widetilde{\epsilon'} \tag{21}$$

With the help of figure 7.6, we can also write $\mathbf{k}$ and $\mathbf{k}''$ in component forms

$$\mathbf{k} = \hat{\mathbf{x}}k\sin i + \hat{\mathbf{z}}k\cos i \tag{22}$$

$$\mathbf{k}'' = \hat{\mathbf{x}}k\sin i - \hat{\mathbf{z}}k\cos i \tag{23}$$

After replacing $\epsilon' \rightarrow \widetilde{\epsilon'}$ in Jackson (7.37), the general boundary conditions are

$$\left[\epsilon\left(\mathbf{E}_0 + \mathbf{E}_0''\right) - \widetilde{\epsilon'}\mathbf{E}_0'\right]\cdot\hat{\mathbf{z}} = 0 \tag{24}$$

$$\left(\mathbf{k}\times\mathbf{E}_0 + \mathbf{k}''\times\mathbf{E}_0'' - \mathbf{k}'\times\mathbf{E}_0'\right)\cdot\hat{\mathbf{z}} = 0 \tag{25}$$

$$\left(\mathbf{E}_0 + \mathbf{E}_0'' - \mathbf{E}_0'\right)\times\hat{\mathbf{z}} = 0 \tag{26}$$

$$\left[\frac{1}{\mu}\left(\mathbf{k}\times\mathbf{E}_0 + \mathbf{k}''\times\mathbf{E}_0''\right) - \frac{1}{\mu'}\left(\mathbf{k}'\times\mathbf{E}_0'\right)\right]\times\hat{\mathbf{z}} = 0 \tag{27}$$

As in the text, we treat two polarization modes differently.

(a) **$\mathbf{E}_0$ is perpendicular to the plane of incidence**
In this case all of $\mathbf{E}_0, \mathbf{E}_0'', \mathbf{E}_0'$ are along the $\hat{\mathbf{y}}$ direction. (26) implies

$$E_0 + E_0'' - E_0' = 0 \tag{28}$$

and (24) is useless given the assumption of polarization. Substituting $\mathbf{k}, \mathbf{k}'', \mathbf{k}'$ for their component forms in (25) gives a redundant restriction that multiplies the LHS of (28) by $\sin i$.
Finally (27) requires

$$\frac{k\cos i}{\mu}\left(E_0 - E_0''\right) - \frac{k_z'}{\mu'}E_0' = 0 \tag{29}$$

Combining (28) and (29) gives

$$E_0'' = E_0\left(\frac{\cos i - \dfrac{\mu}{\mu'}\cdot\dfrac{k_z'}{k}}{\cos i + \dfrac{\mu}{\mu'}\cdot\dfrac{k_z'}{k}}\right) \qquad E_0' = E_0\left(\frac{2\cos i}{\cos i + \dfrac{\mu}{\mu'}\cdot\dfrac{k_z'}{k}}\right) \tag{30}$$

We see that this can be obtained from Jackson (7.39) by the substitution

$$\cos r \longrightarrow \frac{n}{n'}\cdot\frac{k_z'}{k} \tag{31}$$

which is identically true when $\mathbf{k}'$ is a real vector.

(b) **$\mathbf{E}_0$ is parallel to the plane of incidence**
In this case $\mathbf{E}_0$ and $\mathbf{E}_0'$ can be written in component forms

$$\mathbf{E}_0 = -\hat{\mathbf{x}}E_0\cos i + \hat{\mathbf{z}}E_0\sin i \tag{32}$$
$$\mathbf{E}_0'' = \hat{\mathbf{x}}E_0''\cos i + \hat{\mathbf{z}}E_0''\sin i \tag{33}$$

If we write

$$\mathbf{E}_0' = \hat{\mathbf{x}}E_{0x}' + \hat{\mathbf{z}}E_{0z}' \tag{34}$$

transverse wave condition requires

$$\mathbf{k}'\cdot\mathbf{E}_0' = 0 \qquad \Longrightarrow \qquad k_x'E_{0x}' + k_z'E_{0z}' = 0 \tag{35}$$

The boundary condition (25) yields nothing and (26) requires

$$\cos i\left(E_0 - E_0''\right) + E_{0x}' = 0 \tag{36}$$

With

$$\mathbf{k}\times\mathbf{E}_0 = (\hat{\mathbf{x}}k\sin i + \hat{\mathbf{z}}k\cos i)\times(-\hat{\mathbf{x}}E_0\cos i + \hat{\mathbf{z}}E_0\sin i) = -\hat{\mathbf{y}}kE_0 \tag{37}$$
$$\mathbf{k}''\times\mathbf{E}_0'' = (\hat{\mathbf{x}}k\sin i - \hat{\mathbf{z}}k\cos i)\times\left(\hat{\mathbf{x}}E_0''\cos i + \hat{\mathbf{z}}E_0''\sin i\right) = -\hat{\mathbf{y}}kE_0'' \tag{38}$$
$$\begin{aligned}\mathbf{k}'\times\mathbf{E}_0' &= \left(\hat{\mathbf{x}}k_x' + \hat{\mathbf{z}}k_z'\right)\times\left(\hat{\mathbf{x}}E_{0x}' + \hat{\mathbf{z}}E_{0z}'\right) = -\hat{\mathbf{y}}\left(k_x'E_{0z}' - k_z'E_{0x}'\right)\\ &= -\hat{\mathbf{y}}\left(k_x' + \frac{k_z'^2}{k_x'}\right)E_{0z}' = -\hat{\mathbf{y}}\left(\frac{\omega^2\mu'\widetilde{\epsilon}'}{k_x'}\right)E_{0z}' = -\hat{\mathbf{y}}\left(\frac{k}{\sin i}\right)\left(\frac{\mu'\widetilde{\epsilon}'}{\mu\epsilon}\right)E_{0z}'\end{aligned} \tag{39}$$

(27) turns into

$$\left(E_0 + E_0''\right) - \left(\frac{\widetilde{\epsilon}'}{\epsilon}\right)\left(\frac{1}{\sin i}\right)E_{0z}' = 0 \tag{40}$$

which duplicates (24).
Putting (35), (36), and (40) together yields

$$E_{0z}' = E_0\left(\frac{2\cos i\sin i}{\dfrac{\widetilde{\epsilon}'}{\epsilon}\cos i + \dfrac{k_z'}{k}}\right) \qquad E_{0x}' = -E_0\left(\frac{2\cos i\cdot\dfrac{k_z'}{k}}{\dfrac{\widetilde{\epsilon}'}{\epsilon}\cos i + \dfrac{k_z'}{k}}\right) \qquad E_0'' = E_0\left(\frac{\dfrac{\widetilde{\epsilon}'}{\epsilon}\cos i - \dfrac{k_z'}{k}}{\dfrac{\widetilde{\epsilon}'}{\epsilon}\cos i + \dfrac{k_z'}{k}}\right) \tag{41}$$

In Jackson (7.41), note if we write $\mathbf{E}'_0$ in component forms, i.e., $\mathbf{E}'_0 = E'_0(-\hat{\mathbf{x}}\cos r + \hat{\mathbf{z}}\sin r)$ and then do the following substitution (which are identically true when $\mathbf{k}'$ is real)

$$\cos r \longrightarrow \frac{n}{n'}\cdot\frac{k'_z}{k} \qquad\qquad \sin r \longrightarrow \frac{n}{n'}\cdot\sin i \qquad\qquad n'^2 \longrightarrow c^2\mu'\widetilde{\epsilon'} \tag{42}$$

we will end up with (41).

This shows that the solutions we obtained above, (30) and (41), are the complex analytic continuation of Jackson (7.39) and (7.41), as ϵ' changes into complex value $\widetilde{\epsilon'} = \epsilon + i\sigma/\omega$.

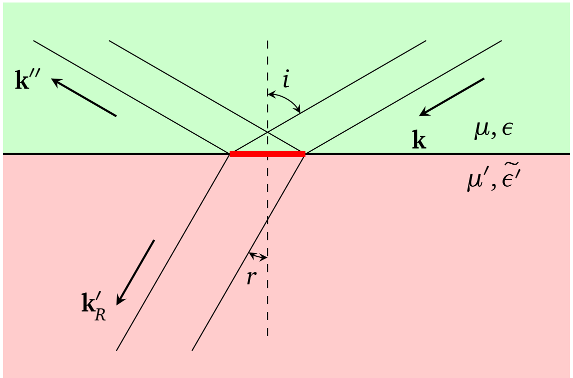
2. **Verification of energy conservation**

Since $\mathbf{k}'$ is complex, the energy conservation with arbitrary incident angle needs more elaboration. When we deal with complex $\mathbf{E}$ and $\mathbf{H}$ field, the time-averaged energy flux is given by

$$\mathbf{S} = \mathrm{Re}\left(\frac{1}{2}\mathbf{E}_0 \times \mathbf{H}_0^*\right) \tag{43}$$

For a general complex wave vector $\mathbf{k}$, this is

$$\mathbf{S} = \mathrm{Re}\left[\frac{1}{2}\mathbf{E}_0 \times \left(\frac{\mathbf{k}\times\mathbf{E}_0}{\mu\omega}\right)^*\right] = \frac{1}{2\mu\omega}\cdot\mathrm{Re}\left[\mathbf{E}_0 \times \left(\mathbf{k}^* \times \mathbf{E}_0^*\right)\right] = \frac{1}{2\mu\omega}\,|\mathbf{E}_0|^2\,\mathrm{Re}\,\mathbf{k}^* \tag{44}$$



Consider a patch on the metal boundary with area A, the local energy conservation requires

$$\overbrace{\frac{1}{2\mu\omega}\,|E_0|^2\,k\cdot A\cos i}^{\text{incident energy}} = \overbrace{\frac{1}{2\mu\omega}\,\left|E''_0\right|^2 k\cdot A\cos i}^{\text{reflected energy}} + \overbrace{\frac{1}{2\mu'\omega}\,\left|E'_0\right|^2 k'_R\cdot A\cos r}^{\text{transmitted energy}} \tag{45}$$

or equivalently

$$\frac{k\cos i}{\mu} = \frac{\left|E''_0\right|^2}{|E_0|^2}\cdot\frac{k\cos i}{\mu} + \frac{\left|E'_0\right|^2}{|E_0|^2}\cdot\frac{k'_R\cos r}{\mu'} \tag{46}$$

Explicit verification of (46) using (30) and (41) is extremely tedious, here we use an indirect method.

For the perpendicular polarization, multiplying (29) with the complex conjugate of (28) yields

$$\frac{k\cos i}{\mu}\left(|E_0|^2 - \left|E''_0\right|^2 + \overbrace{E_0E''^*_0 - E''_0E^*_0}^{\text{purely imaginary}}\right) = \frac{k'_z}{\mu'}\left|E'_0\right|^2 \tag{47}$$

(46) is obtained by taking the real part of (47).

Similarly, for the parallel polarization case, multiplying (40) with the complex conjugate of (36) gives

$$\cos i\left(|E_0|^2 - \left|E''_0\right|^2 + \overbrace{E''_0E^*_0 - E_0E''^*_0}^{\text{purely imaginary}}\right) = -\left(\frac{\widetilde{\epsilon'}}{\epsilon}\right)\left(\frac{1}{\sin i}\right)E'^*_{0x}E'_{0z} = \left(\frac{\widetilde{\epsilon'}}{\epsilon}\right)\left(\frac{1}{\sin i}\right)\left|E'_{0z}\right|^2\left(\frac{k'^*_z}{k'_x}\right) \tag{48}$$

Since

$$\left|E'_0\right|^2 = \left|E'_{0x}\right|^2 + \left|E'_{0z}\right|^2 \qquad\qquad \Longrightarrow \qquad\qquad \left|E'_{0z}\right|^2 = \left|E'_0\right|^2\left(\frac{k'^2_x}{k'^2_x + \left|k'_z\right|^2}\right) \tag{49}$$

Inserting (49) into (48) and taking the real part, we get

$$\cos i\left(|E_0|^2-\left|E_0''\right|^2\right)=\frac{\left|E_0'\right|^2}{\epsilon}\left(\frac{1}{\sin i}\right)\left(\frac{k_x'}{k_x'^2+\left|k_z'\right|^2}\right)\operatorname{Re}\left(\widetilde{\epsilon}'k_z'^*\right) \tag{50}$$

Multiplying both sides with $\omega^2\epsilon=k^2/\mu$ and invoking (21),

$$\begin{aligned}
\frac{k^2\cos i}{\mu}\left(|E_0|^2-\left|E_0''\right|^2\right)&=\frac{\left|E_0'\right|^2}{\mu'}\left(\frac{1}{\sin i}\right)\left(\frac{k\sin i}{k_x'^2+\left|k_z'\right|^2}\right)\operatorname{Re}\left[\left(k_x'^2+k_z'^2\right)k_z'^*\right] \qquad \Longrightarrow\\
\frac{k\cos i}{\mu}\left(|E_0|^2-\left|E_0''\right|^2\right)&=\frac{\left|E_0'\right|^2}{\mu'}\left\{\frac{\operatorname{Re}\left[\left(k_x'^2+k_R'^2\cos^2 r-k_I'^2+i\cdot 2k_R'k_I'\cos r\right)\left(k_R'\cos r-ik_I'\right)\right]}{k_x'^2+k_R'^2\cos^2 r+k_I'^2}\right\}\\
&=\left|E_0'\right|^2\cdot\frac{k_R'\cos r}{\mu'}
\end{aligned} \tag{51}$$

3. **Part (b) of 7.4**

For part (b) of the problem, we take the special case with $i=r=0$ (i.e., normal incidence) and $\mu/\mu'=1$. The limiting case $\sigma\to 0$ was discussed after (14) and (15). For very large σ, k_I' is significant by (14), which gives a small skin depth δ (recall the attenuation factor is $e^{-k_I'z}=e^{-z/\delta}$):

$$k_I'=\frac{1}{\delta} \tag{52}$$

hence by (12)

$$k_R'=\left[k_I'^2+k^2\left(\frac{n'}{n}\right)^2\right]^{1/2}=\frac{1}{\delta}\left[1+k^2\delta^2\left(\frac{n'}{n}\right)^2\right]^{1/2} \tag{53}$$

The reflection coefficient R can be computed from (30) (which can be shown to be the same if we compute from (41), since for normal incidence the two polarization cases coincide):

$$R=\left|\frac{1-\frac{k_R'}{k}-i\cdot\frac{k_I'}{k}}{1+\frac{k_R'}{k}+i\cdot\frac{k_I'}{k}}\right|^2=1-4\cdot\overbrace{\frac{\frac{k_R'}{k}}{\left(1+\frac{k_R'}{k}\right)^2+\left(\frac{k_I'}{k}\right)^2}}^{T} \tag{54}$$

Multiplying $k^2\delta^2$ to T's denominator and numerator yields

$$T=\frac{4k_R'\cdot k\delta^2}{\left(\delta+k_R'\delta\right)^2+1}=\frac{4k\delta\left[1+O\left(\delta^2\right)\right]}{2+O\left(\delta^2\right)}=2k\delta\left[1+O\left(\delta^2\right)\right]\approx 2k\delta=2\frac{\omega}{c}\delta \tag{55}$$