1. This is a straightforward application of the relation we have proved back in problem 2.12:

$$\Phi(\rho, \phi) = \frac{1}{2\pi} \int_0^{2\pi} d\phi' \Phi(b, \phi') \left[1 + 2 \sum_{n=1}^{\infty} \frac{\rho^n}{b^n} \cos n(\phi - \phi') \right] \tag{1}$$

Clearly the contribution from the "1" term in the bracket is zero due to the alternating signs. The contribution from n is

$$C_n = \frac{V}{2\pi} \frac{2\rho^n}{b^n} \left(I_1 + I_2 + I_3 + I_4 \right) \tag{2}$$

where

$$I_1 = \int_0^{\pi/2} \cos n \left(\phi' - \phi \right) d\phi' = \frac{1}{n} \left[\sin n \left(\frac{\pi}{2} - \phi \right) - \sin n \left(-\phi \right) \right]$$

$$= \frac{1}{n} \operatorname{Im} \left(e^{in\pi/2 - in\phi} - e^{-in\phi} \right)$$

$$= \frac{1}{n} \operatorname{Im} \left[\left(i^n - 1 \right) e^{-in\phi} \right] \tag{3}$$

$$I_2 = -\frac{1}{n} \operatorname{Im} \left[\left(i^{2n} - i^n \right) e^{-in\phi} \right] \tag{4}$$

$$I_3 = \frac{1}{n} \operatorname{Im} \left[\left(i^{3n} - i^{2n} \right) e^{-in\phi} \right] \tag{5}$$

$$I_4 = -\frac{1}{n} \operatorname{Im} \left[\left(i^{4n} - i^{3n} \right) e^{-in\phi} \right] \tag{6}$$

Therefore

$$I_1 + I_2 + I_3 + I_4 = \frac{1}{n} \operatorname{Im} \left\{ 2e^{-in\phi} \left[i^n - 1 - (-1)^n + (-i)^n \right] \right\} \tag{7}$$

which clearly vanishes unless $n = 4k + 2$, in which case

$$I_1 + I_2 + I_3 + I_4 = \frac{8 \sin n\phi}{n} \qquad \text{when } n = 4k + 2 \tag{8}$$

This gives

$$\Phi(\rho, \phi) = \frac{4V}{\pi} \sum_{k=0}^{\infty} \left(\frac{\rho}{b} \right)^{4k+2} \frac{\sin (4k + 2) \phi}{2k + 1} \tag{9}$$

2. We could sum the series in (9) as the imaginary part of a power series modulated by coefficient $1/(2k + 1)$, which involves $\tan^{-1}$ of complex numbers. I'm not familiar with trigonometry function of complex argument, so I'm going to prove the claim via another route.

Recall in problem 2.12, we proved

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi')\overbrace{\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}}^{A(\phi')}d\phi' \tag{10}$$

With this, we can obtain the interior point's potential via four-part integration:

$$\Phi(\rho,\phi) = \frac{V}{2\pi}\left(\int_0^{\pi/2} A(\phi')d\phi' - \int_{\pi/2}^{\pi} A(\phi')d\phi' + \int_{\pi}^{3\pi/2} A(\phi')d\phi' - \int_{3\pi/2}^{2\pi} A(\phi')d\phi'\right)$$

$$= \frac{V}{2\pi}\left\{\int_0^{\pi/2}\left[A(\phi')+A(\phi'-\pi)\right]d\phi' - \int_{\pi/2}^{\pi}\left[A(\phi')+A(\phi'-\pi)\right]d\phi'\right\} \tag{11}$$

Now define

$$B(\phi') = A(\phi')+A(\phi'-\pi)$$

$$= (b^2-\rho^2)\left[\frac{1}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)} + \frac{1}{b^2+\rho^2+2b\rho\cos(\phi'-\phi)}\right]$$

$$= (b^2-\rho^2)\left[\frac{2(b^2+\rho^2)}{(b^2+\rho^2)^2-4b^2\rho^2\cos^2(\phi'-\phi)}\right] \tag{12}$$

Then the potential

$$\Phi(\rho,\phi) = \frac{V}{2\pi}\left[\int_0^{\pi/2} B(\phi')d\phi' - \int_{\pi/2}^{\pi} B(\phi')d\phi'\right]$$

$$= \frac{V}{2\pi}\int_0^{\pi/2}\left[B(\phi')-B\left(\phi'-\frac{\pi}{2}\right)\right]d\phi'$$

$$= \frac{V}{2\pi}\cdot 2\left(b^4-\rho^4\right)\int_0^{\pi/2}\left[\frac{1}{(b^2+\rho^2)^2-4b^2\rho^2\cos^2(\phi'-\phi)} - \frac{1}{(b^2+\rho^2)^2-4b^2\rho^2\sin^2(\phi'-\phi)}\right]d\phi'$$

$$= \frac{V}{\pi}\left(b^4-\rho^4\right)\int_0^{\pi/2}\frac{4b^2\rho^2\cos 2(\phi'-\phi)d\phi'}{(b^2+\rho^2)^4-4b^2\rho^2(b^2+\rho^2)^2+16b^4\rho^4\cos^2(\phi'-\phi)\sin^2(\phi'-\phi)}$$

$$= \frac{V}{\pi}\left(b^4-\rho^4\right)\int_0^{\pi/2}\frac{4b^2\rho^2\cos 2(\phi'-\phi)d\phi'}{(b^4-\rho^4)^2+4b^4\rho^4\sin^2 2(\phi'-\phi)} \tag{13}$$

Now it's the usual practice of variable change which we used in problem 2.13 – define

$$t \equiv \sin 2\left(\phi'-\phi\right) \tag{14}$$

then

$$\Phi(\rho,\phi) = \frac{V}{\pi}\left(b^4-\rho^4\right)\int_{t_0}^{t_1}\frac{2b^2\rho^2 dt}{(b^4-\rho^4)^2+4b^4\rho^4 t^2} \tag{15}$$

And subsequently, with

$$\tan\xi \equiv \frac{2b^2\rho^2}{b^4-\rho^4}t \tag{16}$$

we get

$$\Phi(\rho,\phi) = \frac{V}{\pi}\left(b^4-\rho^4\right)\int_{\xi_0}^{\xi_1}\frac{2b^2\rho^2\dfrac{b^4-\rho^4}{2b^2\rho^2}\dfrac{1}{\cos^2\xi}d\xi}{(b^4-\rho^4)^2\dfrac{1}{\cos^2\xi}} = \frac{V}{\pi}(\xi_1-\xi_0) \tag{17}$$

From (14) and (16), we have the bounds ξ_0,ξ_1

$$\xi_0 = -\tan^{-1}\left(\frac{2b^2\rho^2}{b^4-\rho^4}\sin 2\phi\right) \qquad \xi_1 = \tan^{-1}\left(\frac{2b^2\rho^2}{b^4-\rho^4}\sin 2\phi\right) \tag{18}$$

which finally yields

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2b^2\rho^2}{b^4-\rho^4}\sin 2\phi\right) \tag{19}$$