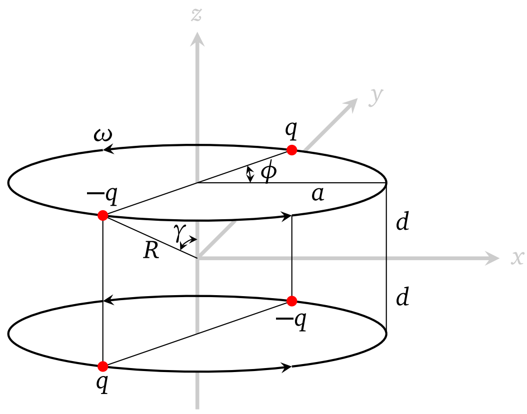

1. The two pairs of dipole described in the problem statement can be viewed as the limiting case of the diagram above, where we will eventually take $d \to 0$ while keeping $2qd = p$ constant. With finite d, in spherical coordinates, the charge density and current density are (we use η for charge density since ρ will be used to denote the radial component of cylindrical coordinates later)

$$\eta(\mathbf{x},t) = q\frac{\delta(r-R)}{R^2}\overbrace{[\delta(\phi-\omega t)-\delta(\phi-\omega t-\pi)]}^{f(\phi,t)}\left[\frac{\delta(\theta-\gamma)-\delta(\theta+\gamma-\pi)}{\sin\theta}\right] \tag{1}$$

$$\mathbf{J}(\mathbf{x},t) = \hat{\boldsymbol{\phi}}\omega r\sin\theta\,\eta(\mathbf{x},t) = \hat{\boldsymbol{\phi}}q\omega\frac{r\delta(r-R)}{R^2}\underbrace{[\delta(\phi-\omega t)-\delta(\phi-\omega t-\pi)]}_{f(\phi,t)}[\delta(\theta-\gamma)-\delta(\theta+\gamma-\pi)] \tag{2}$$

where $R=\sqrt{a^2+d^2}$ and $\gamma=\cos^{-1}(d/R)$ is the polar angle of the upper loop.

The Fourier component of the n-th harmonic of $f(\phi,t)$ is

$$\begin{aligned} f_n(\phi) &= \overbrace{\frac{1}{T}\int_0^T \delta(\phi-\omega t)e^{in\omega t}dt}^{e^{in\phi}/\omega} - \overbrace{\frac{1}{T}\int_0^T \delta(\phi-\omega t-\pi)e^{in\omega t}dt}^{e^{in(\phi-\pi)}/\omega} \\ &= \frac{1}{2\pi}e^{in\phi}\left[1-(-1)^n\right] = \begin{cases} \frac{1}{\pi}e^{in\phi} & \text{for odd } n \\ 0 & \text{for even } n \end{cases} \end{aligned} \tag{3}$$

which means only odd-numbered harmonics can exist.

The following will assume n to be odd. The n-th harmonic of the charge density and current density are

$$\eta_n(\mathbf{x}) = \frac{q}{\pi}\frac{\delta(r-R)}{R^2}\left[\frac{\delta(\theta-\gamma)-\delta(\theta+\gamma-\pi)}{\sin\theta}\right]e^{in\phi} \tag{4}$$

$$\mathbf{J}_n(\mathbf{x}) = \hat{\boldsymbol{\phi}}\frac{q\omega}{\pi}\frac{r\delta(r-R)}{R^2}[\delta(\theta-\gamma)-\delta(\theta+\gamma-\pi)]e^{in\phi} \tag{5}$$

(a) The electric multipole due to the n-th harmonic is

$$\begin{aligned} q_{lm}^{(n)} &= \int r^l Y_{lm}^*(\theta,\phi)\eta_n(\mathbf{x})d^3x \\ &= \frac{q}{\pi}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\delta_{mn}2\pi\int_0^\infty r^{l+2}\frac{\delta(r-R)}{R^2}dr\int_0^\pi[\delta(\theta-\gamma)-\delta(\theta+\gamma-\pi)]P_l^m(\cos\theta)d\theta \\ &= \delta_{mn}2qR^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\left[\overbrace{P_l^m(\cos\gamma)-P_l^m(-\cos\gamma)}^{\alpha}\right] \end{aligned} \tag{6}$$

As $d\to 0$, $R=\sqrt{a^2+d^2}\to a+O(d^2)$, $\cos\gamma = d/R \to d/a+O(d^2)$, and

$$\alpha \to 2\left.\frac{dP_l^m(x)}{dx}\right|_{x=0}\cdot\cos\gamma = 2\left.\frac{dP_l^m(x)}{dx}\right|_{x=0}\left[\frac{d}{a}+O(d^2)\right] \tag{7}$$

Finally when we let $d \to 0$ while keeping $2qd = p$ constant, the electric multipole moment becomes

$$\begin{aligned} q_{lm}^{(n)} &\to \delta_{mn} 2(2qd) a^{l-1} \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \left.\frac{dP_l^m(x)}{dx}\right|_{x=0} \\ &= \delta_{mn} 2pa^{l-1} \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} (l+m) P_{l-1}^m(0) \end{aligned} \tag{8}$$

where in the last step we have used the recurrence relation (See 14.10.E5 on DLMF)

$$\left(1-x^2\right)\frac{dP_l^m(x)}{dx} = (l+m)P_{l-1}^m(x) - lxP_l^m(x) \tag{9}$$

$P_{l-1}^m(0)$ is non-zero only when $l-m$ is odd and $l \geq |m|$. The lowest of which is $l-1 = |m| = |n|$. The lowest order electric multipole occurs at $l=2$ and $m=n=\pm 1$, i.e.,

$$q_{21}^{(1)} = 2pa\sqrt{\frac{5}{4\pi}\cdot\frac{1}{6}}\cdot 3P_1^1(0) = -\sqrt{\frac{15}{2\pi}}pa \qquad q_{2,-1}^{(-1)} = 2pa\sqrt{\frac{5}{4\pi}}\cdot 6\cdot P_1^{-1}(0) = \sqrt{\frac{15}{2\pi}}pa \tag{10}$$

(b) For the n-th harmonic, the magnetic multipole moments is given by

$$M_{lm}^{(n)} = -\frac{1}{l+1}\int r^l Y_{lm}^*(\theta,\phi)\,\boldsymbol{\nabla}\cdot[\mathbf{x}\times\mathbf{J}_n(\mathbf{x})]d^3x \tag{11}$$

From (5), we have

$$\mathbf{x}\times\mathbf{J}_n(\mathbf{x}) = -\hat{\boldsymbol{\theta}}\frac{q\omega}{\pi}\frac{r^2\delta(r-R)}{R^2}e^{in\phi}\overbrace{[\delta(\theta-\gamma)-\delta(\theta+\gamma-\pi)]}^{g(\theta)} \tag{12}$$

$$\begin{aligned} \boldsymbol{\nabla}\cdot[\mathbf{x}\times\mathbf{J}_n(\mathbf{x})] &= -\frac{q\omega}{\pi}\frac{r^2\delta(r-R)}{R^2}e^{in\phi}\frac{1}{r\sin\theta}\frac{d[g(\theta)\sin\theta]}{d\theta} \\ &= -\frac{q\omega}{\pi}\frac{r\delta(r-R)}{R^2}e^{in\phi}\left[g'(\theta)+g(\theta)\cot\theta\right] \end{aligned} \tag{13}$$

Now (11) becomes

$$M_{lm}^{(n)} = \frac{1}{l+1}\frac{q\omega R^{l+1}}{\pi}\delta_{mn}2\pi\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}(G+H) \tag{14}$$

where

$$\begin{aligned} G &= \int_0^\pi P_l^m(\cos\theta)g'(\theta)\sin\theta d\theta = \int_0^\pi P_l^m(\cos\theta)\left[\delta'(\theta-\gamma)-\delta'(\theta+\gamma-\pi)\right]\sin\theta d\theta \\ &= -\left\{\left.\frac{d\left[P_l^m(\cos\theta)\sin\theta\right]}{d\theta}\right|_{\theta=\gamma} - \left.\frac{d\left[P_l^m(\cos\theta)\sin\theta\right]}{d\theta}\right|_{\theta=\pi-\gamma}\right\} \\ &= -\left\{-\left.\frac{dP_l^m(x)}{dx}\right|_{x=\cos\gamma}\cdot\sin^2\gamma + P_l^m(\cos\gamma)\cos\gamma + \left.\frac{dP_l^m(x)}{dx}\right|_{x=-\cos\gamma}\cdot\sin^2\gamma - P_l^m(-\cos\gamma)(-\cos\gamma)\right\} \end{aligned} \tag{15}$$

$$\begin{aligned} H &= \int_0^\pi P_l^m(\cos\theta)g(\theta)\cot\theta\sin\theta d\theta = \int_0^\pi P_l^m(\cos\theta)\cos\theta[\delta(\theta-\gamma)-\delta(\theta+\gamma-\pi)]d\theta \\ &= P_l^m(\cos\gamma)\cos\gamma - P_l^m(-\cos\gamma)(-\cos\gamma) \end{aligned} \tag{16}$$

After canceling out H in part of G, we have

$$M_{lm}^{(n)} = \delta_{mn}\left(\frac{1}{l+1}\right)2q\omega R^{l+1}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\left[\overbrace{\left.\frac{dP_l^m(x)}{dx}\right|_{x=\cos\gamma} - \left.\frac{dP_l^m(x)}{dx}\right|_{x=-\cos\gamma}}^{\beta}\right]\sin^2\gamma \tag{17}$$

As usual, when $d\to 0$, $\sin^2\gamma \to 1 - O\left(d^2\right)$ and

$$\beta \to 2\left.\frac{d^2P_l^m(x)}{dx^2}\right|_{x=0}\cdot\cos\gamma = 2\left.\frac{d^2P_l^m(x)}{dx^2}\right|_{x=0}\left[\frac{d}{a}+O\left(d^2\right)\right] \tag{18}$$

which gives

$$
\begin{aligned}
M_{lm}^{(n)} &\to \delta_{mn}\left(\frac{2}{l+1}\right)(2qd)\,\omega a^{l}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\frac{d^{2}P_{l}^{m}(x)}{dx^{2}}\bigg|_{x=0} \\
&= \delta_{mn}\left(\frac{2}{l+1}\right)p\omega a^{l}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\left[m^{2}-l(l+1)\right]P_{l}^{m}(0)
\end{aligned}
\tag{19}
$$

where we have used the differential equation for the associated Legendre function

$$
\left(1-x^{2}\right)\frac{d^{2}P_{l}^{m}(x)}{dx^{2}}-2x\frac{dP_{l}^{m}(x)}{dx}+\left[l(l+1)-\frac{m^{2}}{1-x^{2}}\right]P_{l}^{m}(x)=0 \tag{20}
$$

Parity of $P_{l}^{m}(x)$ requires $l-m$ to be even for $M_{lm}^{(n)}$ to be non-zero. The $n=\pm1$ harmonics have $l=1, m=\pm1$ as the lowest non-zero magnetic multipole moment, i.e.,

$$
M_{1,\pm1}^{(1)}=\pm\sqrt{\frac{3}{8\pi}}pa\omega \tag{21}
$$

2. From problem 6.21, for a dipole $\mathbf{p}$ moving along the trajectory $\mathbf{r}(t)$, the charge density and current density are

$$
\begin{aligned}
\eta(\mathbf{x},t) &= -(\mathbf{p}\cdot\boldsymbol{\nabla})\,\delta\left[\mathbf{x}-\mathbf{r}(t)\right] && (22)\\
\mathbf{J}(\mathbf{x},t) &= -\frac{d\mathbf{r}}{dt}(\mathbf{p}\cdot\boldsymbol{\nabla})\,\delta\left[\mathbf{x}-\mathbf{r}(t)\right] && (23)
\end{aligned}
$$

Let subscript +/— represent dipole whose initial position is at $(\pm a,0,0)$ respectively, then in cylindrical coordinates

$$
\begin{aligned}
(\mathbf{p}_{+}\cdot\boldsymbol{\nabla})\,\delta\left[\mathbf{x}-\mathbf{r}_{+}(t)\right] &= p\frac{\partial}{\partial z}\left[\frac{\delta(\rho-a)}{a}\delta(z)\,\delta(\phi-\omega t)\right]=p\frac{\delta(\rho-a)}{a}\delta'(z)\,\delta(\phi-\omega t) && (24)\\
(\mathbf{p}_{-}\cdot\boldsymbol{\nabla})\,\delta\left[\mathbf{x}-\mathbf{r}_{-}(t)\right] &= -p\frac{\partial}{\partial z}\left[\frac{\delta(\rho-a)}{a}\delta(z)\,\delta(\phi-\omega t-\pi)\right]=-p\frac{\delta(\rho-a)}{a}\delta'(z)\,\delta(\phi-\omega t-\pi) && (25)
\end{aligned}
$$

With both dipoles considered, the charge density and current density are

$$
\begin{aligned}
\eta(\mathbf{x},t) &= -p\frac{\delta(\rho-a)}{a}\delta'(z)\left[\delta(\phi-\omega t)-\delta(\phi-\omega t-\pi)\right] && (26)\\
\mathbf{J}(\mathbf{x},t) &= \hat{\boldsymbol{\phi}}\omega\rho\cdot\eta(\mathbf{x},t)=-\hat{\boldsymbol{\phi}}\omega p\frac{\rho\,\delta(\rho-a)}{a}\delta'(z)\left[\delta(\phi-\omega t)-\delta(\phi-\omega t-\pi)\right] && (27)
\end{aligned}
$$

The Fourier transform (3) gives us the n-th harmonic of the charge density and current density

$$
\eta_{n}(\mathbf{x})=-\frac{p}{\pi}\frac{\delta(\rho-a)}{a}\delta'(z)\,e^{in\phi} \qquad \mathbf{J}_{n}(\mathbf{x})=-\hat{\boldsymbol{\phi}}\frac{\omega p}{\pi}\frac{\rho\,\delta(\rho-a)}{a}\delta'(z)\,e^{in\phi} \qquad \text{for odd } n \tag{28}
$$

The n-th harmonic's contribution to the vector potential in the radiation zone ($kr \gg 1$) is given by (9.8)

$$
\mathbf{A}_{n}(\mathbf{x})=\frac{\mu_{0}}{4\pi}\frac{e^{ik_{n}r}}{r}\int \mathbf{J}_{n}\left(\mathbf{x}'\right)e^{-ik_{n}\mathbf{n}\cdot\mathbf{x}'}d^{3}x' \qquad k_{n}=\frac{n\omega}{c} \tag{29}
$$

Let the unit vector from the origin to the observation point be $\mathbf{n}=\sin\theta\cos\phi\,\hat{\mathbf{x}}+\sin\theta\sin\phi\,\hat{\mathbf{y}}+\cos\theta\,\hat{\mathbf{z}}$, then

$$
\mathbf{n}\cdot\mathbf{x}'=\rho'\cos\phi'\sin\theta\cos\phi+\rho'\sin\phi'\sin\theta\sin\phi+z'\cos\theta=\rho'\sin\theta\cos\left(\phi'-\phi\right)+z'\cos\theta \tag{30}
$$

hence

$$
\begin{aligned}
\mathbf{A}_{n}(\mathbf{x}) &= \frac{\mu_{0}}{4\pi}\frac{e^{ik_{n}r}}{r}\frac{\omega p}{\pi}\int\left(-\hat{\boldsymbol{\phi}}\right)\frac{\rho'\delta(\rho'-a)}{a}\delta'\left(z'\right)e^{in\phi'}e^{-ik_{n}\rho'\sin\theta\cos\left(\phi'-\phi\right)}e^{-ik_{n}z'\cos\theta}d^{3}x' \\
&= \frac{\mu_{0}}{4\pi}\frac{e^{ik_{n}r}}{r}\frac{\omega pa}{\pi}\int_{0}^{2\pi}\left(\sin\phi'\hat{\mathbf{x}}-\cos\phi'\hat{\mathbf{y}}\right)e^{in\phi'}e^{-ik_{n}a\sin\theta\cos\left(\phi'-\phi\right)}d\phi'\overbrace{\int_{-\infty}^{\infty}\delta'\left(z'\right)e^{-ik_{n}z'\cos\theta}dz}^{ik_{n}\cos\theta} \\
&= \frac{\mu_{0}}{4\pi}\frac{e^{ik_{n}r}}{r}\frac{\omega pa}{\pi}ik_{n}\cos\theta\,(X\hat{\mathbf{x}}+Y\hat{\mathbf{y}})
\end{aligned}
\tag{31}
$$

where

$$X=\int_0^{2\pi}\sin\phi' e^{in\phi'}e^{-ik_n a\sin\theta\cos\left(\phi'-\phi\right)}d\phi' \qquad\qquad Y=-\int_0^{2\pi}\cos\phi' e^{in\phi'}e^{-ik_n a\sin\theta\cos\left(\phi'-\phi\right)}d\phi' \tag{32}$$

Define $\xi\equiv\phi'-\phi$ and $\beta\equiv k_n a\sin\theta$ then

$$\begin{aligned}
X&=e^{in\phi}\int_0^{2\pi}\sin(\xi+\phi)\,e^{in\xi}e^{-i\beta\cos\xi}d\xi\\
&=e^{in\phi}\int_0^{2\pi}(\sin\xi\cos\phi+\cos\xi\sin\phi)(\cos n\xi+i\sin n\xi)\,e^{-i\beta\cos\xi}d\xi \qquad \text{terms odd in } \xi \text{ can be dropped}\\
&=e^{in\phi}\int_0^{2\pi}(\sin\phi\cos\xi\cos n\xi+i\cos\phi\sin\xi\sin n\xi)\,e^{-i\beta\cos\xi}d\xi\\
&=e^{in\phi}\int_0^{2\pi}\left\{\frac{-ie^{i\phi}}{2}\cos[(n+1)\xi]+\frac{ie^{-i\phi}}{2}\cos[(n-1)\xi]\right\}e^{-i\beta\cos\xi}d\xi\\
&=\frac{i}{2}\left\{e^{i(n-1)\phi}\int_0^{2\pi}\cos[(n-1)\xi]\,e^{-i\beta\cos\xi}d\xi-e^{i(n+1)\phi}\int_0^{2\pi}\cos[(n+1)\xi]\,e^{-i\beta\cos\xi}d\xi\right\}\\
&=\pi i\left[\underbrace{i^{n-1}e^{i(n-1)\phi}J_{n-1}(\beta)}_{K_{n-1}}-\underbrace{i^{n+1}e^{i(n+1)\phi}J_{n+1}(\beta)}_{K_{n+1}}\right]
\end{aligned}\tag{33}$$

where we have used DLMF 10.9.E2

$$J_n(z)=\frac{i^{-n}}{\pi}\int_0^{\pi}e^{iz\cos\alpha}\cos(n\alpha)\,d\alpha \tag{34}$$

and the fact that n is odd.

Similarly,

$$\begin{aligned}
Y&=-e^{in\phi}\int_0^{2\pi}\cos(\xi+\phi)\,e^{in\xi}e^{-i\beta\cos\xi}d\xi\\
&=-e^{in\phi}\int_0^{2\pi}(\cos\xi\cos\phi-\sin\xi\sin\phi)(\cos n\xi+i\sin n\xi)\,e^{-i\beta\cos\xi}d\xi\\
&=-e^{in\phi}\int_0^{2\pi}(\cos\phi\cos\xi\cos n\xi-i\sin\phi\sin\xi\sin n\xi)\,e^{-i\beta\cos\xi}d\xi\\
&=-e^{in\phi}\int_0^{2\pi}\left\{\frac{e^{i\phi}}{2}\cos[(n+1)\xi]+\frac{e^{-i\phi}}{2}\cos[(n-1)\xi]\right\}e^{-i\beta\cos\xi}d\xi\\
&=-\frac{1}{2}\left\{e^{i(n-1)\phi}\int_0^{2\pi}\cos[(n-1)\xi]\,e^{-i\beta\cos\xi}d\xi+e^{i(n+1)\phi}\int_0^{2\pi}\cos[(n+1)\xi]\,e^{-i\beta\cos\xi}d\xi\right\}\\
&=-\pi\left[i^{n-1}e^{i(n-1)\phi}J_{n-1}(\beta)+i^{n+1}e^{i(n+1)\phi}J_{n+1}(\beta)\right]\\
&=-\pi\,(K_{n-1}+K_{n+1})
\end{aligned}\tag{35}$$

Putting X and Y back into (31) gives

$$\begin{aligned}
\mathbf{A}_n(\mathbf{x})&=\frac{\mu_0}{4\pi}\frac{e^{ik_n r}}{r}\frac{\omega pa}{\pi}ik_n\cos\theta\,\pi i\left[(K_{n-1}-K_{n+1})\hat{\mathbf{x}}+i(K_{n-1}+K_{n+1})\hat{\mathbf{y}}\right] \qquad \text{recall } \omega=\frac{ck_n}{n}\\
&=-\frac{\mu_0}{4\pi}\frac{e^{ik_n r}}{r}\frac{cpak_n^2}{n}\cos\theta\left[(K_{n-1}-K_{n+1})\hat{\mathbf{x}}+i(K_{n-1}+K_{n+1})\hat{\mathbf{y}}\right]
\end{aligned}\tag{36}$$

The radiation zone magnetic field due to the n-th harmonic is thus

$$\begin{aligned}
\mathbf{H}_n(\mathbf{x}) &= \frac{ik_n}{\mu_0}\mathbf{n}\times\mathbf{A}_n(\mathbf{x}) \\
&= -\frac{ik_n}{\mu_0}\frac{\mu_0}{4\pi}\frac{e^{ik_nr}}{r}\frac{cpak_n^2}{n}\cos\theta\,(\sin\theta\cos\phi\,\hat{\mathbf{x}}+\sin\theta\sin\phi\,\hat{\mathbf{y}}+\cos\theta\,\hat{\mathbf{z}})\times[(K_{n-1}-K_{n+1})\,\hat{\mathbf{x}}+i\,(K_{n-1}+K_{n+1})\,\hat{\mathbf{y}}] \\
&= -i\frac{cpa}{4\pi}\frac{e^{ik_nr}}{r}\frac{k_n^3}{n}\cos\theta\Big\{-i\cos\theta\,(K_{n-1}+K_{n+1})\,\hat{\mathbf{x}}+\cos\theta\,(K_{n-1}-K_{n+1})\,\hat{\mathbf{y}} \\
&\qquad +i\sin\theta\,[\cos\phi\,(K_{n-1}+K_{n+1})+i\sin\phi\,(K_{n-1}-K_{n+1})]\,\hat{\mathbf{z}}\Big\} \\
&= -\frac{cpa}{4\pi}\frac{e^{ik_nr}}{r}\frac{k_n^3}{n}\cos\theta\Big\{\cos\theta\,(K_{n-1}+K_{n+1})\,\hat{\mathbf{x}}+i\cos\theta\,(K_{n-1}-K_{n+1})\,\hat{\mathbf{y}} \\
&\qquad -\sin\theta\,[\cos\phi\,(K_{n-1}+K_{n+1})+i\sin\phi\,(K_{n-1}-K_{n+1})]\,\hat{\mathbf{z}}\Big\}
\end{aligned} \tag{37}$$

Note (37) is obtained with only the radiation zone approximation $kr \gg 1$, in particular it does not assume $ka \ll 1$ yet. If we now invoke the nonrelativistic assumption $|k_na| \ll 1$, therefore $|\beta| = |k_na\sin\theta| \ll 1$, then we will have

$$|K_{n-1}| \gg |K_{n+1}| \ \text{ for } n = 1,3,5,\cdots \qquad \text{and} \qquad |K_{n+1}| \gg |K_{n-1}| \ \text{ for } n = -1,-3,-5,\cdots \tag{38}$$

Now consider the dominating harmonics $n = \pm 1$, where only $K_0 = 1$ is retained in (37). With $k \equiv k_1 = \omega/c = -k_{-1}$, we have

$$\mathbf{H}_1(\mathbf{x},t) = -\frac{cpa}{4\pi}\frac{e^{ikr}}{r}k^3\cos\theta\left[\cos\theta\,(\hat{\mathbf{x}}+i\hat{\mathbf{y}})-\sin\theta e^{i\phi}\hat{\mathbf{z}}\right]e^{-i\omega t} \tag{39}$$

$$\mathbf{H}_{-1}(\mathbf{x},t) = -\frac{cpa}{4\pi}\frac{e^{-ikr}}{r}k^3\cos\theta\left[\cos\theta\,(\hat{\mathbf{x}}-i\hat{\mathbf{y}})-\sin\theta e^{-i\phi}\hat{\mathbf{z}}\right]e^{i\omega t} \tag{40}$$

$\mathbf{H}_1$ and $\mathbf{H}_{-1}$ are complex conjugate of each other, therefore the combined physical field is twice the real part of each. Up to a global phase factor, it's justifiable to write the complex magnetic field for the base frequency ω as

$$\mathbf{H}(\mathbf{x}) = \frac{cpa}{2\pi}\frac{e^{ikr}}{r}k^3\cos\theta\left[\cos\theta\,(\hat{\mathbf{x}}+i\hat{\mathbf{y}})-\sin\theta e^{i\phi}\hat{\mathbf{z}}\right] \tag{41}$$

3. In the radiation zone, the angular distribution of time-averaged radiated power is

$$\begin{aligned}
\left\langle\frac{dP}{d\Omega}\right\rangle &= \frac{1}{2}\mathrm{Re}\left[r^2\mathbf{n}\cdot(\mathbf{E}\times\mathbf{H}^*)\right] \qquad\qquad \mathbf{E} = Z_0\mathbf{n}\times\mathbf{H} \\
&= \frac{Z_0}{2}r^2\mathbf{H}\cdot\mathbf{H}^* \\
&= \frac{Z_0}{2}\left(\frac{cpa}{2\pi}k^3\right)^2\cos^2\theta\left(1+\cos^2\theta\right)
\end{aligned} \tag{42}$$

Integrating over all solid angles gives the total time-averaged radiated power

$$P = \frac{Z_0}{2}\left(\frac{cpa}{2\pi}k^3\right)^2 2\pi\int_0^\pi\left(\cos^4\theta+\cos^2\theta\right)\sin\theta\,d\theta = \frac{4}{15\pi\epsilon_0}ck^6p^2a^2 \tag{43}$$