1. Let $z+$ direction be pointing from the center of the sphere to the charge q. Let Φ_{int} and Φ_{ext} be the interior and exterior potential of the sphere. Since there are no charge in the interior region, Φ_{int} must be solution to the Laplace equation. Moreover, given the cylindrical symmetry and the fact that the point $r = 0$ is in the interior region, Φ_{int} must have the form

$$\Phi_{\text{int}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{1}$$

For the exterior, by linear superposition, we can write Φ_{ext} as the sum of point charge potential from q, plus a solution of Laplace equation that doesn't involve singularity, i.e.,

$$\begin{aligned}\Phi_{\text{ext}} &= \left[\sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)\right] + \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{x} - d\hat{\mathbf{z}}|} \\ &= \left[\sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)\right] + \left[\frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)\right] \\ &= \sum_{l=0}^{\infty}\left[B_l r^{-(l+1)} + \frac{q}{4\pi\epsilon_0}\frac{r_<^l}{r_>^{l+1}}\right] P_l(\cos\theta) \end{aligned} \tag{2}$$

where $r_<$ and $r_>$ represent the smaller and greater between r and d.

Now we impose the boundary conditions

$$\text{tangential } E: \qquad \left.\frac{\partial\Phi_{\text{int}}}{\partial\theta}\right|_{r=a} = \left.\frac{\partial\Phi_{\text{ext}}}{\partial\theta}\right|_{r=a} \tag{3}$$

$$\text{normal } D: \qquad \epsilon\left.\frac{\partial\Phi_{\text{int}}}{\partial r}\right|_{r=a} = \epsilon_0\left.\frac{\partial\Phi_{\text{ext}}}{\partial r}\right|_{r=a} \tag{4}$$

(3) indicates

$$\sum_{l=0}^{\infty} A_l a^l P_l'(\cos\theta)(-\sin\theta) = \sum_{l=0}^{\infty}\left[B_l a^{-(l+1)} + \frac{q}{4\pi\epsilon_0}\frac{a^l}{d^{l+1}}\right] P_l'(\cos\theta)(-\sin\theta) \tag{5}$$

which, by the orthogonality of $P_l^m(x)$ (here $m = 1$), requires for all l:

$$A_l a^l = B_l a^{-(l+1)} + \frac{q}{4\pi\epsilon_0}\frac{a^l}{d^{l+1}} \tag{6}$$

(4) indicates (letting $\lambda \equiv \epsilon/\epsilon_0$)

$$\lambda\sum_{l=0}^{\infty} A_l l a^{l-1} P_l(\cos\theta) = \sum_{l=0}^{\infty}\left[-B_l(l+1)a^{-(l+2)} + \frac{q}{4\pi\epsilon_0}\frac{l a^{l-1}}{d^{l+1}}\right] P_l(\cos\theta) \tag{7}$$

which requires for all l:

$$\lambda l A_l a^{l-1} = -B_l(l+1)a^{-(l+2)} + \frac{q}{4\pi\epsilon_0}\frac{l a^{l-1}}{d^{l+1}} \tag{8}$$

From (6) and (8), we have

$$A_l = \frac{q}{4\pi\epsilon_0}\frac{2l+1}{(\lambda+1)l+1}\frac{1}{d^{l+1}} \tag{9}$$

$$B_l = -\frac{q}{4\pi\epsilon_0}\frac{(\lambda-1)l}{(\lambda+1)l+1}\frac{a^{2l+1}}{d^{l+1}} \tag{10}$$

Inserting (9) and (10) back into (1) and (2) yields

$$\Phi_{\text{int}} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{2l+1}{\left(\frac{\epsilon}{\epsilon_0}+1\right)l+1}\frac{r^l}{d^{l+1}} P_l(\cos\theta) \tag{11}$$

$$\Phi_{\text{ext}} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{r_<^l}{r_>^{l+1}} - \frac{(\epsilon-\epsilon_0)l}{(\epsilon+\epsilon_0)l+\epsilon_0}\frac{a^{2l+1}}{d^{l+1}}\frac{1}{r^{l+1}}\right] P_l(\cos\theta) \tag{12}$$

2. Near the origin, (11) is dominated by the low-l terms:

$$\Phi_{\text{int}} \approx \frac{q}{4\pi\epsilon_0}\left[\frac{1}{d}+\left(\frac{3}{2+\epsilon/\epsilon_0}\right)\frac{z}{d^2}\right] \tag{13}$$

hence

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi_{\text{int}} \approx -\frac{q}{4\pi\epsilon_0 d^2}\left(\frac{3}{2+\epsilon/\epsilon_0}\right)\hat{\mathbf{z}} \tag{14}$$

3. When $\epsilon \to \infty$, it's clear that

$$A_l \to \begin{cases} \dfrac{q}{4\pi\epsilon_0 d} & \text{for } l=0 \\ 0 & \text{for } l>0 \end{cases} \qquad \Longrightarrow \qquad \Phi_{\text{int}} \to \frac{q}{4\pi\epsilon_0 d} = \frac{1}{4\pi\epsilon_0}\left[\frac{0-(-qa/d)}{a}\right] \tag{15}$$

which agrees with the interior potential of the uncharged insulated conducting sphere in the presence of point charge (see page 61 with $Q=0, q'=-qa/d$).

Similarly, as $\epsilon \to \infty$,

$$B_l \to \begin{cases} 0 & \text{for } l=0 \\ -\dfrac{q}{4\pi\epsilon_0}\dfrac{a^{2l+1}}{d^{l+1}} & \text{for } l>0 \end{cases} \tag{16}$$

which, by (2) gives

$$\begin{aligned}
\Phi_{\text{ext}} &= \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{x}-d\hat{\mathbf{z}}|} - \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{a^{2l+1}}{d^{l+1}r^{l+1}}P_l(\cos\theta) + \frac{q}{4\pi\epsilon_0}\frac{a}{dr} \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{x}-d\hat{\mathbf{z}}|} + \frac{1}{4\pi\epsilon_0}\left(-\frac{qa}{d}\right)\sum_{l=0}^{\infty}\frac{\left(\frac{a^2}{d}\right)^l}{r^{l+1}}P_l(\cos\theta) + \frac{1}{4\pi\epsilon_0}\frac{qa/d}{r} \qquad \text{(denote } q'=-qa/d\text{)} \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{x}-d\hat{\mathbf{z}}|} + \frac{q'}{4\pi\epsilon_0}\frac{1}{\left|\mathbf{x}-\left(\frac{a^2}{d}\right)\hat{\mathbf{z}}\right|} + \frac{0-q'}{4\pi\epsilon_0 r}
\end{aligned} \tag{17}$$

which again agrees with the the example on page 61 (equation 2.8) as the sphere becomes conducting.