1. As usual, given the boundary condition, the Green function $G(\mathbf{x},\mathbf{x}')$, viewed as a function of $\mathbf{x}$, can be written in separate variable form as

$$G(\mathbf{x},\mathbf{x}') = \sum_{m=-\infty}^{\infty} e^{im\phi} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{L}\right) g_{mn}(\rho,\rho') \tag{1}$$

where $g_{mn}(\rho,\rho')$, viewed as a function of ρ, must have the form (imposed by the convergence at $\rho = 0$ and $\rho \to \infty$)

$$g_{mn}(\rho,\rho') = \begin{cases} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right) & \text{for } \rho < \rho' \\ B_{mn} K_m\left(\frac{n\pi\rho}{L}\right) & \text{for } \rho > \rho' \end{cases} \tag{2}$$

At $\rho = \rho'$, continuity requires

$$A_{mn} I_m\left(\frac{n\pi\rho'}{L}\right) = B_{mn} K_m\left(\frac{n\pi\rho'}{L}\right) \tag{3}$$

So we can write g_{mn} in the general form

$$g_{mn}(\rho,\rho') = C_{mn} \overbrace{I_m\left(\frac{n\pi\rho_<}{L}\right) K_m\left(\frac{n\pi\rho_>}{L}\right)}^{h_{mn}(\rho,\rho')} \tag{4}$$

Taking the Laplacian of (1) gives

$$\begin{aligned}\nabla^2 G &= \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} e^{im\phi} \sin\left(\frac{n\pi z}{L}\right) \left\{ \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho \frac{\partial g_{mn}}{\partial\rho}\right) - \left[\left(\frac{n\pi}{L}\right)^2 + \frac{m^2}{\rho^2}\right] g_{mn} \right\} \\ &= -4\pi \frac{\delta(\rho-\rho')}{\rho} \delta(\phi-\phi') \delta(z-z')\end{aligned} \tag{5}$$

Since

$$\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} = 2\pi\delta(\phi-\phi') \tag{6}$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) = \frac{L}{2}\delta(z-z') \tag{7}$$

We will have an ansatz for C_{mn} in the form of

$$C_{mn} = D_{mn} e^{-im\phi'} \sin\left(\frac{n\pi z'}{L}\right) \tag{8}$$

which turns (5) into the restriction

$$D_{mn} \left\{ \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial h_{mn}}{\partial\rho}\right) - \left[\left(\frac{n\pi}{L}\right)^2 + \frac{m^2}{\rho^2}\right] h_{mn} \right\} = -\frac{4}{L}\frac{\delta(\rho-\rho')}{\rho} \tag{9}$$

Multiplying (9) with ρ and then integrating from $\rho' - \epsilon$ to $\rho' + \epsilon$ produces

$$\begin{aligned} D_{mn}\left(\rho \frac{\partial h_{mn}}{\partial\rho}\bigg|_{\rho=\rho'+\epsilon} - \rho\frac{\partial h_{mn}}{\partial\rho}\bigg|_{\rho=\rho'-\epsilon}\right) &= -\frac{4}{L} \qquad \Longrightarrow \\ D_{mn}\rho' \left\{ \frac{n\pi}{L}\left[I_m\left(\frac{n\pi\rho'}{L}\right) K'_m\left(\frac{n\pi\rho'}{L}\right) - I'_m\left(\frac{n\pi\rho'}{L}\right) K_m\left(\frac{n\pi\rho'}{L}\right)\right]\right\} &= -\frac{4}{L}\end{aligned} \tag{10}$$

We recognize the content inside the bracket as the Wronskian (see equation (3.147))

$$W[I_m(x), K_m(x)] = -\frac{1}{x} \tag{11}$$

which yields

$$D_{mn} = \frac{4}{L} \qquad \Longrightarrow \qquad G(\mathbf{x},\mathbf{x}') = \frac{4}{L} \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi\rho_<}{L}\right) K_m\left(\frac{n\pi\rho_>}{L}\right) \tag{12}$$

2. When we take k to be real and consider the boundary condition, we can express the Green function, as a function of $\mathbf{x}$, as

$$G\left(\mathbf{x},\mathbf{x}'\right)=\sum_{m=-\infty}^{\infty}e^{im\phi}\int_0^{\infty}dk g_{mk}\left(\rho,\rho'\right)h_{mk}\left(z,z'\right) \tag{13}$$

where

$$h_{mk}\left(z,z'\right)=\begin{cases}A_{mk}\sinh\left(kz\right) & \text{for } z<z'\\ B_{mk}\sinh\left[k\left(L-z\right)\right] & \text{for } z>z'\end{cases} \tag{14}$$

Similar argument as part 1 can be applied to get

$$h_{mk}\left(z,z'\right)=C_{mk}\sinh\left(kz_<\right)\sinh\left[k\left(L-z_>\right)\right] \tag{15}$$

In the following, we will absorb the coefficient C_{mk} into g_{mk} and treat it as 1 in h_{mk}. For g_{mk}'s form, by the boundary condition, all we can say at this point is

$$g_{mk}\left(\rho,\rho'\right)=\begin{cases}D_{mk}J_m\left(k\rho\right) & \text{for } \rho<\rho'\\ E_{mk}J_m\left(k\rho\right)+F_{mk}N_m\left(k\rho\right) & \text{for } \rho>\rho'\end{cases} \tag{16}$$

It's hard to establish the restrictions among D_{mk},E_{mk},F_{mk} by the continuity requirement alone (although by symmetry argument, we can guess that g_{mk} is a product of $J_m\left(k\rho\right)$ and $J_m\left(k\rho'\right)$). So let's take another route.

In fact, if we write the Laplacian of (13) in cylindrical coordinates and integrate it across the infinitesimal range $[z'-\epsilon,z'+\epsilon]$, we get

$$\int_{z'-\epsilon}^{z'+\epsilon}\nabla^2 G dz=\int_{z'-\epsilon}^{z'+\epsilon}\left[\overbrace{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 G}{\partial\phi^2}}^{\text{zero contrib. for integral}}+\frac{\partial^2 G}{\partial z^2}\right]dz=\int_{z'-\epsilon}^{z'+\epsilon}-4\pi\frac{\delta\left(\rho-\rho'\right)}{\rho}\delta\left(\phi-\phi'\right)\delta\left(z-z'\right)dz \implies$$

$$\left.\frac{\partial G}{\partial z}\right|_{z=z'+\epsilon}-\left.\frac{\partial G}{\partial z}\right|_{z=z'-\epsilon}=-4\pi\frac{\delta\left(\rho-\rho'\right)}{\rho}\delta\left(\phi-\phi'\right) \tag{17}$$

Since

$$\left.\frac{\partial G}{\partial z}\right|_{z=z'+\epsilon}=\sum_{m=-\infty}^{\infty}e^{im\phi}\int_0^{\infty}dk g_{mk}\left(\rho,\rho'\right)\sinh\left(kz'\right)(-k)\cosh\left[k\left(L-z'\right)\right] \tag{18}$$

$$\left.\frac{\partial G}{\partial z}\right|_{z=z'-\epsilon}=\sum_{m=-\infty}^{\infty}e^{im\phi}\int_0^{\infty}dk g_{mk}\left(\rho,\rho'\right)k\cosh\left(kz'\right)\sinh\left[k\left(L-z'\right)\right] \tag{19}$$

(17) is equivalent to

$$\sum_{m=-\infty}^{\infty}e^{im\phi}\int_0^{\infty}dk g_{mk}\left(\rho,\rho'\right)k\sinh\left(kL\right)=4\pi\frac{\delta\left(\rho-\rho'\right)}{\rho}\delta\left(\phi-\phi'\right) \tag{20}$$

The separation of variable procedure ensures g_{mk} satisfies the Bessel differential equation in the argument $k\rho$. Recall from Prob 3.16

$$\int_0^k kJ_\nu\left(k\rho\right)J_\nu\left(k\rho'\right)dk=\frac{\delta\left(\rho,\rho'\right)}{\rho} \tag{21}$$

as well as (6), we can take g_{mk} to be

$$g_{mk}\left(\rho,\rho'\right)=\frac{2}{\sinh\left(kL\right)}J_m\left(k\rho\right)J_m\left(k\rho'\right)e^{-im\phi'} \tag{22}$$

to satisfy (20).

In summary

$$G\left(\mathbf{x},\mathbf{x}'\right)=2\sum_{m=-\infty}^{\infty}e^{im\left(\phi-\phi'\right)}\int_0^{\infty}dk J_m\left(k\rho\right)J_m\left(k\rho'\right)\frac{\sinh\left(kz_<\right)\sinh\left[k\left(L-z_>\right)\right]}{\sinh\left(kL\right)} \tag{23}$$