1. **Prob 5.21**

   (a) Since there is no free current, we can use the scalar potential to describe $\mathbf{H}$: i.e.,

$$\mathbf{H} = -\boldsymbol{\nabla}\Phi_M \tag{1}$$

   Then

$$\int \mathbf{B}\cdot\mathbf{H}d^3x = -\int \mathbf{B}\cdot\boldsymbol{\nabla}\Phi_M d^3x = -\int \boldsymbol{\nabla}\cdot(\Phi_M\mathbf{B})\,d^3x + \overbrace{\int \Phi_M\boldsymbol{\nabla}\cdot\mathbf{B}d^3x}^{=0} = -\oint_{S_\infty}\Phi_M\mathbf{B}\cdot\mathbf{n}da = 0 \tag{2}$$

   where in the last step we have used the fact $\mathbf{B}$ generated by the locally distributed magnetization vanishes at infinity.

   (b) The argument is similar to section 1.11, where we imagine the locally distributed magnetization is brought about by moving the dipoles $\mathbf{M}$ from infinity to its current location one by one, which gives

$$W = -\frac{1}{2}\int \mathbf{M}\cdot\mathbf{B}d^3x = -\frac{\mu_0}{2}\int \mathbf{M}\cdot(\mathbf{M}+\mathbf{H})\,d^3x \tag{3}$$

   Treating the integral $\int \mathbf{M}\cdot\mathbf{M}d^3x$ as an inconsequential constant which is only a property of the medium, we can ignore it and write

$$W = -\frac{\mu_0}{2}\int \mathbf{M}\cdot\mathbf{H}d^3x = -\frac{\mu_0}{2}\int\left(\frac{\mathbf{B}}{\mu_0}-\mathbf{H}\right)\cdot\mathbf{H}d^3x = \frac{\mu_0}{2}\int \mathbf{H}\cdot\mathbf{H}d^3x \tag{4}$$

2. **Prob 5.22**

   Let the bar's length direction be the z direction, hence $\mathbf{M} = M\hat{\mathbf{z}}$. By (5.103), the effective volume current density and the effective surface current density are

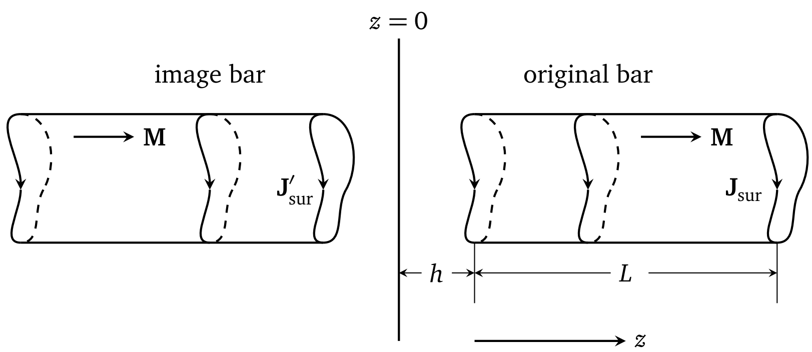
$$\mathbf{J}_{\text{vol}} = \boldsymbol{\nabla}\times M\hat{\mathbf{z}} = 0 \qquad\qquad \mathbf{J}_{\text{sur}} = M\hat{\mathbf{z}}\times\mathbf{n} \tag{5}$$

   On the end caps, $\mathbf{n} = \pm\hat{\mathbf{z}}$, so there is no effective surface current there. On the side though, $\mathbf{n}$ lies parallel to the x-y plane, hence $\mathbf{J}_{\text{sur}}$ is winding around the cylinder parallel to the x-y plane.

   Since the bar is placed against an infinitely permeable flat surface, the effect of the surface on the bar itself can be described by the method of image currents

$$\mathbf{J}'_{\text{sur}}(x,y,-z) = \mathbf{J}_{\text{sur}}(x,y,z) \tag{6}$$

   Effectively, these image currents are equivalent to mirroring the entire bar across the $z=0$ plane, keeping its magnetization $\mathbf{M}$ unchanged.



   For the region $z > 0$, the magnetic field is the superposition of the field generated by the original bar and the field generated by the image bar, i.e.,

$$\mathbf{B} = \mathbf{B}_{\text{img}} + \mathbf{B}_{\text{orig}} \tag{7}$$

hence the total energy is

$$W = -\frac{1}{2}\int_V \mathbf{M}\cdot\mathbf{B}d^3x = -\frac{1}{2}\int_V \mathbf{M}\cdot\mathbf{B}_{\text{orig}}d^3x - \frac{1}{2}\int_V \mathbf{M}\cdot\mathbf{B}_{\text{img}}d^3x \tag{8}$$

The first term is a constant property of the bar itself, and thus is independent of the distance h from the bar's bottom to the $z=0$ plane. By the principle of virtual work,

$$\mathbf{F} = -\frac{\partial W}{\partial h}\hat{\mathbf{z}}\bigg|_{h=0} = \frac{1}{2}\frac{\partial}{\partial h}\int_V \mathbf{M}\cdot\mathbf{B}_{\text{img}}d^3x\bigg|_{h=0} \tag{9}$$

We now take the approximation that $\mathbf{B}_{\text{img}}$ is constant on all points of the cross section at z, and that $\mathbf{B}_{\text{img}}$ is pointing to the z direction (which is true if the cross section is a circle and the observation point is on the axis). With this approxmation we have

$$F_z \approx \frac{1}{2}\frac{d}{dh}\int_h^{h+L} MB_{\text{img}}(z)Adz\bigg|_{h=0} = \frac{MA}{2}\left[\frac{d}{dh}\int_h^{h+L} B_{\text{img}}(z)\,dz\right]_{h=0} \tag{10}$$

It's tempting to use the hint that L is large to claim

$$\frac{d}{dh}\int_h^{h+L} B_{\text{img}}(z)\,dz \approx \frac{d}{dh}\int_h^{\infty} B_{\text{img}}(z)\,dz = -B_{\text{img}}(h)$$

we will see below that this is wrong, because the integral $\int_h^\infty B_{\text{img}}(z)\,dz$ diverges. So the correct way is to keep L finite and evaluate (10), then take the limit $L\to\infty$.

We will use the result from problem 5.19, which gives

$$B_{\text{img}}(z) = -\frac{\mu_0 M}{2}\left[\frac{z+h}{\sqrt{a^2+(z+h)^2}} - \frac{z+h+L}{\sqrt{a^2+(z+h+L)^2}}\right] \tag{11}$$

hence

$$\begin{aligned}
\int_h^{h+L} B_{\text{img}}(z)\,dz &= -\frac{\mu_0 M}{2}\left[\sqrt{a^2+(z+h)^2} - \sqrt{a^2+(z+h+L)^2}\right]_{z=h}^{z=h+L} \\
&= -\frac{\mu_0 M}{2}\left[2\sqrt{a^2+(2h+L)^2} - \sqrt{a^2+(2h+2L)^2} - \sqrt{a^2+(2h)^2}\right] \qquad \Longrightarrow \\
\frac{d}{dh}\int_h^{h+L} B_{\text{img}}(z)\,dz\bigg|_{h=0} &= -\frac{\mu_0 M}{2}\left[\frac{4(2h+L)}{\sqrt{a^2+(2h+L)^2}} - \frac{2(2h+2L)}{\sqrt{a^2+(2h+2L)^2}} - \frac{4h}{\sqrt{a^2+(2h)^2}}\right]_{h=0} \\
&= -\frac{\mu_0 M}{2}\left[\frac{4L}{\sqrt{a^2+L^2}} - \frac{4L}{\sqrt{a^2+4L^2}}\right] \\
&\to -\mu_0 M \qquad \text{as} \qquad L\to\infty
\end{aligned} \tag{12}$$

Plugging (12) back to (10) gives

$$F_z = -\frac{\mu_0 A M^2}{2} \tag{13}$$

On the other hand, if we approximate (11) with $L\to\infty$ and use the tempted way, we end up with

$$\frac{d}{dh}\int_h^{\infty} B_{\text{img}}(z)\,dz\bigg|_{h=0} = -B_{\text{img}}(h=0) \approx = -\frac{\mu_0 M}{2}$$

which is a factor of 2 off compared to the correct result (12).

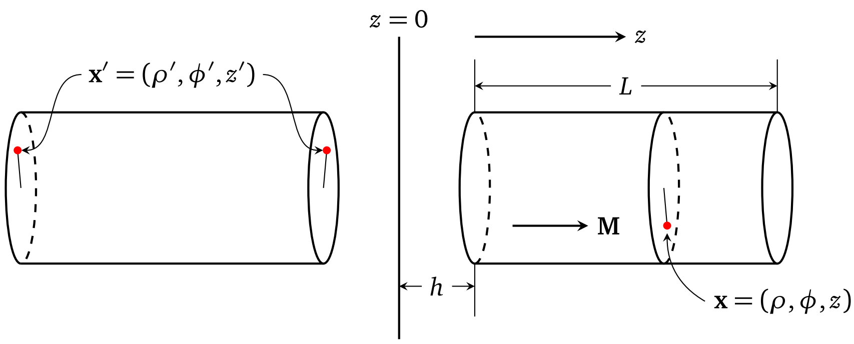
3. **Prob 5.23**

Similar to the argument in problem 5.22, the force on the original bar can be considered as excerted from the image bar depicted below. The effect of the image bar is most conveniently calculated from the "magnetic charge" point of view, where the effective volume magnetic charge density and effective surface charge density are

$$\rho = \nabla\cdot\mathbf{M} = 0 \qquad\qquad \sigma = \mathbf{M}\cdot\mathbf{n} \tag{14}$$

The only non-zero charge density are located on the end caps:

$$\sigma_{z=-h} = M \qquad\qquad \sigma_{z=-(h+L)} = -M \tag{15}$$



At the observation point $\mathbf{x}=(\rho,\phi,z)$, the differential scalar potential contribution from the image charge at $\mathbf{x}'=(\rho',\phi',z')$ is

$$d\Phi_M\left(\mathbf{x},\mathbf{x}'\right)=\frac{\mu_0 M}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho' d\rho' d\phi' \tag{16}$$

(Here we include the μ_0 factor so $-\boldsymbol{\nabla}\Phi_M$ gives the $\mathbf{B}_{\text{img}}$ field).
We can expand the inverse distance as (recall problem 3.16(b))

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\sum_{m=-\infty}^{\infty}\int_0^\infty dk e^{im(\phi-\phi')}J_m(k\rho)J_m\left(k\rho'\right)e^{-k|z-z'|} \tag{17}$$

The total scalar potential at $\mathbf{x}$ is thus (16) integrated over all $\mathbf{x}'$ on both end caps.

$$\Phi_M(\mathbf{x})=\frac{\mu_0 M}{4\pi}\int_0^{2\pi}d\phi'\int_0^a\rho' d\rho'\left\{\sum_{m=-\infty}^{\infty}\int_0^\infty dk e^{im(\phi-\phi')}J_m(k\rho)J_m\left(k\rho'\right)\left[e^{-k(z+h)}-e^{-k(z+h+L)}\right]\right\} \tag{18}$$

The integration in ϕ' ensures only $m=0$ has contribution, which is typical for cylindrically symmetric problems.

$$\Phi_M(\mathbf{x})=\frac{\mu_0 M}{2}\int_0^\infty dk J_0(k\rho)\left[e^{-k(z+h)}-e^{-k(z+h+L)}\right]\int_0^a J_0\left(k\rho'\right)\rho' d\rho' \tag{19}$$

By 10.22.1 on dlmf.nist.gov,

$$\int_0^a J_0\left(k\rho'\right)\rho' d\rho'=\frac{a}{k}J_1(ka) \tag{20}$$

hence

$$\Phi_M(\mathbf{x})=\frac{\mu_0 M a}{2}\int_0^\infty dk\frac{J_0(k\rho)J_1(ka)}{k}\left[e^{-k(z+h)}-e^{-k(z+h+L)}\right] \tag{21}$$

which gives the z-component of the induction at $\mathbf{x}$

$$B_z(\mathbf{x})=-\frac{\partial\Phi_M}{\partial z}=\frac{\mu_0 M a}{2}\int_0^\infty dk J_0(k\rho)J_1(ka)\left[e^{-k(z+h)}-e^{-k(z+h+L)}\right] \tag{22}$$

The energy due to the image charge on the original bar is thus obtained by the integration

$$\begin{aligned}
W&=-\frac{1}{2}\int_V \mathbf{M}\cdot\mathbf{B}_{\text{img}}(\mathbf{x})\,d^3x\\
&=-\frac{1}{2}\int_0^{2\pi}d\phi\int_0^a\rho d\rho\int_h^{h+L}dz\cdot\frac{\mu_0 M^2 a}{2}\int_0^\infty dk J_0(k\rho)J_1(ka)\left[e^{-k(z+h)}-e^{-k(z+h+L)}\right]\\
&=-\frac{1}{2}\int_0^{2\pi}d\phi\int_0^a\rho d\rho\cdot\frac{\mu_0 M^2 a}{2}\int_0^\infty dk J_0(k\rho)J_1(ka)\int_h^{h+L}\left[e^{-k(z+h)}-e^{-k(z+h+L)}\right]dz\\
&=-\frac{1}{2}\int_0^{2\pi}d\phi\int_0^a\rho d\rho\cdot\frac{\mu_0 M^2 a}{2}\int_0^\infty dk J_0(k\rho)J_1(ka)\left(-\frac{1}{k}\right)\left[e^{-k(2h+L)}-e^{-k(2h)}-e^{-k(2h+2L)}+e^{-k(2h+L)}\right]
\end{aligned} \tag{23}$$

Compare (23) with (21), we can see the mutual energy can also be interpreted as the effective magnetic charges, on the end caps of the original bar, coupling with the scalar potential Φ_M with the usual 1/2 factor.

By principle of virtual work

$$F_z = -\frac{dW}{dh}\bigg|_{h=0} = -\frac{\mu_0 \pi M^2 a}{2}\int_0^a \rho d\rho \int_0^\infty dk J_0(k\rho) J_1(ka)\frac{1}{k}\left(-2ke^{-kL}+2k+2ke^{-2kL}-2ke^{-kL}\right)$$
$$= -\mu_0 \pi M^2 a \int_0^\infty dk J_1(ka)\left(1+e^{-2kL}-2e^{-kL}\right)\int_0^a \rho d\rho J_0(k\rho)$$
$$= -\mu_0 \pi M^2 a^2 \int_0^\infty dk \frac{J_1^2(ka)}{k}\left(1+e^{-2kL}-2e^{-kL}\right) \tag{24}$$

The integral can be looked up from the table (reference Eduardo Kausel and Mirza M. Irfan Baig, in particular ENS-4.9)

$$I_{11}^{-1}(a,a,s) = \int_0^\infty dk \frac{J_1^2(ka)}{k}e^{-sk} = \frac{s}{\pi a}\left[\frac{1}{\kappa}\mathbf{E}(\kappa) - \frac{\kappa}{2a^2}\left(2a^2+\frac{s^2}{2}\right)\mathbf{K}(\kappa)\right]+\frac{1}{2} \quad \text{where} \quad \kappa = \frac{2a}{\sqrt{4a^2+s^2}}$$
$$= \frac{s}{\pi a}\left[\frac{\mathbf{E}(\kappa)-\mathbf{K}(\kappa)}{\kappa}\right]+\frac{1}{2} \tag{25}$$

which gives the results

$$\int_0^\infty dk \frac{J_1^2(ka)}{k} = I_{11}^{-1}(a,a,0) = \frac{1}{2} \tag{26}$$
$$\int_0^\infty dk \frac{J_1^2(ka)}{k}e^{-2kL} = I_{11}^{-1}(a,a,2L) = \frac{2L}{\pi a}\left[\frac{\mathbf{E}(\kappa_1)-\mathbf{K}(\kappa_1)}{\kappa_1}\right]+\frac{1}{2} \qquad \kappa_1 = \frac{a}{\sqrt{a^2+L^2}} \tag{27}$$
$$\int_0^\infty dk \frac{J_1^2(ka)}{k}e^{-kL} = I_{11}^{-1}(a,a,L) = \frac{L}{\pi a}\left[\frac{\mathbf{E}(\kappa_2)-\mathbf{K}(\kappa_2)}{\kappa_2}\right]+\frac{1}{2} \qquad \kappa_2 = \frac{2a}{\sqrt{4a^2+L^2}} \tag{28}$$

Plugging (26)-(28) back into (24), we obtain

$$F_z = -2\mu_0 M^2 a L\left[\frac{\mathbf{E}(\kappa_1)-\mathbf{K}(\kappa_1)}{\kappa_1} - \frac{\mathbf{E}(\kappa_2)-\mathbf{K}(\kappa_2)}{\kappa_2}\right] \tag{29}$$

When $L \to \infty$, only (26) contributes in (24), which gives

$$\lim_{L\to\infty} F_z = -\frac{\mu_0 M^2 \left(\pi a^2\right)}{2} \tag{30}$$

which agrees with the result from problem 5.22.