1. **Derivation of differential cross section (10.63)**

It is claimed in (10.63) that for an incident polarization $\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2$, the differential scattering cross section is

$$\frac{d\sigma_{\text{sc}}}{d\Omega} = \frac{\pi}{2k^2}\left|\sum_l \sqrt{2l+1}\left[a_\pm(l)\mathbf{X}_{l,\pm1} \pm i\beta_\pm(l)\,\mathbf{n}\times\mathbf{X}_{l,\pm1}\right]\right|^2 \tag{1}$$

The differential scattering cross section is related to the scattering amplitude via

$$\frac{d\sigma_{\text{sc}}}{d\Omega} = |\mathbf{f}(\mathbf{k},\mathbf{k}_0)|^2 \tag{2}$$

and $\mathbf{f}$ is defined with the far-field approximation $r \to \infty$.

With far-field approximation, we can substitute the asymptotic form of Hankel function

$$h_l^{(1)}(kr) \to (-i)^{l+1}\frac{e^{ikr}}{kr} \tag{3}$$

into (10.57) and obtain (as $r \to \infty$)

$$\mathbf{E}_{\text{sc}} \to \frac{1}{2}\sum_{l=1}^{\infty}(-i)\sqrt{4\pi(2l+1)}\left[\overbrace{\alpha_\pm(l)\frac{e^{ikr}}{kr}\mathbf{X}_{l,\pm1} \pm \frac{\beta_\pm(l)}{k}\nabla\times\left(\frac{e^{ikr}}{kr}\mathbf{X}_{l,\pm1}\right)}^{\mathbf{e}_{l,\pm1}}\right] \tag{4}$$

$$\mathbf{B}_{\text{sc}} \to \frac{1}{2c}\sum_{l=1}^{\infty}(-i)\sqrt{4\pi(2l+1)}\left[\underbrace{\frac{-i\alpha_\pm(l)}{k}\nabla\times\left(\frac{e^{ikr}}{kr}\mathbf{X}_{l,\pm1}\right) \mp i\beta_\pm(l)\frac{e^{ikr}}{kr}\mathbf{X}_{l,\pm1}}_{\mathbf{b}_{l,\pm1}}\right] \tag{5}$$

The differential scattered power per solid angle is thus

$$\frac{dP_{\text{sc}}}{d\Omega} = \frac{1}{2\mu_0}\text{Re}\left[r^2\mathbf{n}\cdot\left(\mathbf{E}_{\text{sc}}\times\mathbf{B}_{\text{sc}}^*\right)\right] = -\frac{1}{2\mu_0}\text{Re}\left[r^2\mathbf{E}_{\text{sc}}\cdot\left(\mathbf{n}\times\mathbf{B}_{\text{sc}}^*\right)\right] \tag{6}$$

With (10.60) we have

$$\nabla\times\left(\frac{e^{ikr}}{kr}\mathbf{X}_{l,m}\right) = \frac{i\sqrt{l(l+1)}}{r}\frac{e^{ikr}}{kr}\mathbf{Y}_{lm} + i\frac{e^{ikr}}{r}\mathbf{n}\times\mathbf{X}_{lm} \tag{7}$$

It is clear that only the transverse components of $\mathbf{E}_{\text{sc}}, \mathbf{B}_{\text{sc}}$ contribute to (6), so we can ignore the $\mathbf{Y}$ components in the above, giving

$$\mathbf{e}_{l,\pm1,\text{trans}} = \frac{e^{ikr}}{kr}\left[\overbrace{\alpha_\pm(l)\mathbf{X}_{l,\pm1} \pm i\beta_\pm(l)\,\mathbf{n}\times\mathbf{X}_{l,\pm1}}^{\mathbf{Z}_{l,\pm1}}\right] \tag{8}$$

$$\begin{aligned}\mathbf{n}\times\mathbf{b}_{l,\pm}^* &= \mathbf{n}\times\left[\frac{i\alpha_\pm^*(l)}{k}\left(-i\frac{e^{-ikr}}{r}\mathbf{n}\times\mathbf{X}_{l,\pm1}^*\right) \pm i\beta_\pm^*(l)\frac{e^{-ikr}}{kr}\mathbf{X}_{l,\pm1}^*\right] \\ &= \frac{e^{-ikr}}{kr}\left[-\alpha_\pm^*(l)\mathbf{X}_{l,\pm1}^* \pm i\beta_\pm^*(l)\,\mathbf{n}\times\mathbf{X}_{l,\pm1}^*\right] = -\frac{e^{-ikr}}{kr}\mathbf{Z}_{l,\pm1}^* \end{aligned}\tag{9}$$

Finally summing all ls in $\mathbf{E}_{\text{sc}}$ and $\mathbf{B}_{\text{sc}}$ and putting them to (6) gives

$$\begin{aligned}\frac{dP_{\text{sc}}}{d\Omega} &= -\frac{1}{2\mu_0}\text{Re}\left\{r^2\left[\frac{1}{2}\sum_l(-i)\sqrt{4\pi(2l+1)}\frac{e^{ikr}}{kr}\mathbf{Z}_{l,\pm1}\right]\cdot\left[\frac{1}{2c}\sum_l i\sqrt{4\pi(2l+1)}\left(-\frac{e^{-ikr}}{kr}\mathbf{Z}_{l,\pm1}^*\right)\right]\right\} \\ &= \frac{1}{\mu_0 c}\cdot\frac{\pi}{2k^2}\left|\sum_l\sqrt{2l+1}\mathbf{Z}_{l,\pm1}\right|^2\end{aligned} \tag{10}$$

(1) is obtained by dividing by the incident flux $1/\mu_0 c$.

2. **Asymptotic form of $\alpha_\pm(l)$ (10.69), (10.70)**

Recall the exact form of $\alpha_\pm(l)$ (10.66)

$$\alpha_\pm(l)+1=-\left\{\frac{h_l^{(2)}-i\left(\frac{Z_s}{Z_0}\right)\frac{1}{x}\frac{d\left[xh_l^{(2)}\right]}{dx}}{h_l^{(1)}-i\left(\frac{Z_s}{Z_0}\right)\frac{1}{x}\frac{d\left[xh_l^{(1)}\right]}{dx}}\right\} \tag{11}$$

(a) For "small argument" approximation (9.88)

$$j_l(x)\approx\frac{x^l}{(2l+1)!!} \qquad\qquad n_l(x)\approx-\frac{(2l-1)!!}{x^{l+1}} \tag{12}$$

we have

$$h_l^{(1,2)}\approx\frac{x^l}{(2l+1)!!}\mp i\frac{(2l-1)!!}{x^{l+1}}=\frac{x^{2l+1}\mp i(2l+1)[(2l-1)!!]^2}{(2l+1)!!x^{l+1}} \tag{13}$$

$$\begin{aligned}i\left(\frac{Z_s}{Z_0}\right)\frac{1}{x}\frac{d\left[xh_l^{(1,2)}\right]}{dx}&\approx i\left(\frac{Z_s}{Z_0}\right)\left[\frac{(l+1)x^{l-1}}{(2l+1)!!}\pm i\frac{l(2l-1)!!}{x^{l+2}}\right]\\&=i\left(\frac{Z_s}{Z_0}\right)\left[\frac{(l+1)x^{2l+1}\pm il(2l+1)[(2l-1)!!]^2}{(2l+1)!!x^{l+2}}\right]\end{aligned} \tag{14}$$

Subtracting (14) from (13) yields

$$h_l^{(1,2)}-i\left(\frac{Z_s}{Z_0}\right)\frac{1}{x}\frac{d\left[xh_l^{(1,2)}\right]}{dx}\approx\frac{x^{2l+1}\left[x-i\left(\frac{Z_s}{Z_0}\right)(l+1)\right]\mp i(2l+1)[(2l-1)!!]^2\left[x+i\left(\frac{Z_s}{Z_0}\right)l\right]}{(2l+1)!!x^{l+2}} \tag{15}$$

which, via (11), gives

$$\begin{aligned}\alpha_\pm(l)&\approx-\left\{\frac{x^{2l+1}\left[x-i\left(\frac{Z_s}{Z_0}\right)(l+1)\right]+i(2l+1)[(2l-1)!!]^2\left[x+i\left(\frac{Z_s}{Z_0}\right)l\right]}{x^{2l+1}\left[x-i\left(\frac{Z_s}{Z_0}\right)(l+1)\right]-i(2l+1)[(2l-1)!!]^2\left[x+i\left(\frac{Z_s}{Z_0}\right)l\right]}+1\right\}\\
&=\frac{-2x^{2l+1}\left[x-i\left(\frac{Z_s}{Z_0}\right)(l+1)\right]}{x^{2l+1}\left[x-i\left(\frac{Z_s}{Z_0}\right)(l+1)\right]-i(2l+1)[(2l-1)!!]^2\left[x+i\left(\frac{Z_s}{Z_0}\right)l\right]}\\
&=\frac{-2ix^{2l+1}}{(2l+1)[(2l-1)!!]^2}\left\{\frac{x-i\left(\frac{Z_s}{Z_0}\right)(l+1)}{\underbrace{\frac{ix^{2l+1}}{(2l+1)[(2l-1)!!]^2}\left[x-i\left(\frac{Z_s}{Z_0}\right)(l+1)\right]}_{\text{negligible for small }x}+\left[x+i\left(\frac{Z_s}{Z_0}\right)l\right]}\right\}\\
&\approx\frac{-2ix^{2l+1}}{(2l+1)[(2l-1)!!]^2}\left[\frac{x-i\left(\frac{Z_s}{Z_0}\right)(l+1)}{x+i\left(\frac{Z_s}{Z_0}\right)l}\right]\end{aligned} \tag{16}$$

(b) For "large argument" approximation (9.89)

$$h_l^{(1,2)}(x)\approx(\mp i)^{l+1}\frac{e^{\pm ix}}{x} \qquad\qquad i\left(\frac{Z_s}{Z_0}\right)\frac{1}{x}\frac{d\left[xh_l^{(1,2)}\right]}{dx}\approx i\left(\frac{Z_s}{Z_0}\right)(\mp i)^l\frac{e^{\pm ix}}{x} \tag{17}$$

giving

$$\alpha_\pm(l)\approx-\left\{\frac{\frac{e^{-ix}}{x}\left[i^{l+1}-i\left(\frac{Z_s}{Z_0}\right)i^l\right]}{\frac{e^{ix}}{x}\left[(-i)^{l+1}-i\left(\frac{Z_s}{Z_0}\right)(-i)^l\right]}\right\}-1=-e^{-i2x}(-1)^{k+1}\left(\frac{1-\frac{Z_s}{Z_0}}{1+\frac{Z_s}{Z_0}}\right)-1 \tag{18}$$