1. **Verification that (5.127) is the solution of (5.126) via Weber–Schafheitlin integral**

For the *dual integral equations* (5.126)

$$\int_0^\infty dy\, g(y) J_n(yx) = x^n \qquad \text{for } 0 \le x < 1 \tag{1}$$

$$\int_0^\infty dy\, y\, g(y) J_n(yx) = 0 \qquad \text{for } 1 \le x \tag{2}$$

It was claimed that (5.127) is a solution

$$g(y) = \frac{2\Gamma(n+1)}{\sqrt{\pi}\Gamma(n+1/2)} j_n(y) = \frac{\Gamma(n+1)}{\Gamma(n+1/2)} \left(\frac{2}{y}\right)^{1/2} J_{n+1/2}(y) \tag{3}$$

Let's verify it via the *Weber–Schafheitlin* integral (reference equation 10.22.56 on dlmf.nist.gov)

for $0 < a < b, \mathrm{Re}(\mu+\nu+1) > \mathrm{Re}(\lambda) > -1$:

$$\int_0^\infty \frac{J_\mu(at) J_\nu(bt)}{t^\lambda} dt = \frac{a^\mu \Gamma\left(\frac{\nu}{2}+\frac{\mu}{2}-\frac{\lambda}{2}+\frac{1}{2}\right)}{2^\lambda b^{\mu-\lambda+1} \Gamma\left(\frac{\nu}{2}-\frac{\mu}{2}+\frac{\lambda}{2}+\frac{1}{2}\right)} F\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\mu-\nu-\lambda+1}{2}; \mu+1; \frac{a^2}{b^2}\right) \tag{4}$$

where F is the hypergeometric function

$$F(a,b;c;z) = 1 + \frac{ab}{c} z + \frac{1}{2!} \frac{a(a+1)b(b+1)}{c(c+1)} z^2 + \cdots \tag{5}$$

- To see (1), insert $a = x, b = 1, \mu = n, \nu = n+1/2, \lambda = 1/2$ into (4), we have

$$\int_0^\infty \frac{J_n(xy) J_{n+1/2}(y)\, dy}{\sqrt{y}} = \frac{x^n \Gamma(n+1/2)}{\sqrt{2}\Gamma(1)} \overbrace{F\left(n+\frac{1}{2}, 0; n+1; x^2\right)}^{=1} \tag{6}$$

  Thus we see that (3) satisfies (1) except for this $\Gamma(n+1)$ factor (**which I think is a mistake on the book**, but fortunately for the $n = 1$ case, it doesn't impact the subsequent discussion).
- For (2), we need $a = 1, b = x, \mu = n+1/2, \nu = n, \lambda = -1/2$, with which the denominator of (4) has $\Gamma(0) = \infty$, hence (4) vanishes, which satisfies (2).

2. **Closed-form formula for (5.129)**

The text gives the integral representation of the additional potential in equation (5.129)

$$\Phi^{(1)}(\mathbf{x}) = \frac{2H_0 a^2}{\pi} \overbrace{\int_0^\infty dk\, j_1(ka) e^{-k|z|} J_1(k\rho)}^{I} \sin\phi \tag{7}$$

Now let's calculate the integral I explicitly (We will treat $z > 0$ region only, the $z < 0$ region can be obtained by the reversal of sign in (5.123).)

With the relation

$$j_1(x) = -j_0'(x) \tag{8}$$

I can be written as

$$\begin{aligned} I &= \int_0^\infty dk \left[-j_0'(ka)\right] e^{-kz} J_1(k\rho) \\ &= \int_0^\infty dk \left[-\frac{dj_0(ka)}{a\,dk}\right] e^{-kz} J_1(k\rho) \\ &= -\frac{1}{a} j_0(ka) e^{-kz} J_1(k\rho)\Big|_0^\infty + \frac{1}{a}\int_0^\infty dk\, j_0(ka) \frac{d}{dk}\left[e^{-kz} J_1(k\rho)\right] \end{aligned} \tag{9}$$

The first term clearly vanishes at both $k=0$ and $k=\infty$, so

$$
\begin{aligned}
I &= \frac{1}{a}\int_0^\infty dk j_0(ka)\frac{d}{dk}\left[e^{-kz}J_1(k\rho)\right] \\
&= \frac{1}{a}\left[-z\int_0^\infty dk j_0(ka)e^{-kz}J_1(k\rho)+\rho\int_0^\infty dk j_0(ka)e^{-kz}J_1'(k\rho)\right] \qquad (10)
\end{aligned}
$$

With the recurrence relation (see equation 10.6.1 on dlmf.nist.gov)

$$
J_1'(x)=\frac{1}{2}\left[J_0(x)-J_2(x)\right] \qquad (11)
$$

we have

$$
I=\frac{1}{a}\left[-z\int_0^\infty dk j_0(ka)e^{-kz}J_1(k\rho)+\frac{\rho}{2}\int_0^\infty dk j_0(ka)e^{-kz}J_0(k\rho)-\frac{\rho}{2}\int_0^\infty dk j_0(ka)e^{-kz}J_2(k\rho)\right] \qquad (12)
$$

Define

$$
K_n\equiv\int_0^\infty dk j_0(ka)e^{-kz}J_n(k\rho) \qquad (13)
$$

(12) became

$$
I=\frac{1}{a}\left(-zK_1+\frac{\rho}{2}K_0-\frac{\rho}{2}K_2\right) \qquad (14)
$$

Now let's focus on K_n,

$$
\begin{aligned}
K_n &= \int_0^\infty dk\frac{\sin ka}{ka}e^{-kz}J_n(k\rho) \qquad\qquad (s\equiv z-ia) \\
&= \frac{1}{a}\,\mathrm{Im}\left[\int_0^\infty e^{-sk}\frac{J_n(k\rho)}{k}dk\right] \\
&= \frac{1}{a}\,\mathrm{Im}\left[\mathscr{L}\left\{\frac{J_n(k\rho)}{k}\right\}(s)\right] \qquad (15)
\end{aligned}
$$

(a) **Calculation of K_1, K_2**
For $n=1,2$, recall equation 10.6.2 on dlmf.nist.gov

$$
J_n'(x)=J_{n-1}(x)-\frac{n}{x}J_n(x) \qquad (16)
$$

and the Laplace transform property

$$
\mathscr{L}\left\{\frac{df(t)}{dt}\right\}(s)=s\mathscr{L}\left\{f(t)\right\}(s)-f\left(0^-\right) \qquad (17)
$$

These yield

$$
\begin{aligned}
\mathscr{L}\left\{\frac{J_n(k\rho)}{k}\right\}(s) &= \frac{\rho}{n}\mathscr{L}\left\{J_{n-1}(k\rho)-J_n'(k\rho)\right\}(s) \\
&= \frac{\rho}{n}\left[\mathscr{L}\left\{J_{n-1}(k\rho)\right\}(s)-\mathscr{L}\left\{\frac{dJ_n(k\rho)}{\rho dk}\right\}(s)\right] \\
&= \frac{\rho}{n}\mathscr{L}\left\{J_{n-1}(k\rho)\right\}(s)-\frac{1}{n}\left[s\mathscr{L}\left\{J_n(k\rho)\right\}(s)-J_n\left(0^-\right)\right] \\
&= \frac{\rho}{n}\mathscr{L}\left\{J_{n-1}(k\rho)\right\}(s)-\frac{s}{n}\mathscr{L}\left\{J_n(k\rho)\right\}(s) \qquad \text{for } n=1,2 \qquad (18)
\end{aligned}
$$

Substituting in (18) with the well known Laplace transforms of $J_n(k\rho)$ (see wikiproof)

$$
\mathscr{L}\left\{J_n(k\rho)\right\}(s)=\frac{\left(\sqrt{s^2+\rho^2}-s\right)^n}{\rho^n\sqrt{s^2+\rho^2}} \qquad (19)
$$

gives

$$\mathscr{L}\left\{\frac{J_1(k\rho)}{k}\right\}(s) = \frac{\rho}{\sqrt{s^2+\rho^2}} - s\cdot\frac{\sqrt{s^2+\rho^2}-s}{\rho\sqrt{s^2+\rho^2}} = \frac{\sqrt{s^2+\rho^2}-s}{\rho} \tag{20}$$

$$\begin{aligned}\mathscr{L}\left\{\frac{J_2(k\rho)}{k}\right\}(s) &= \frac{\rho}{2}\cdot\frac{\sqrt{s^2+\rho^2}-s}{\rho\sqrt{s^2+\rho^2}} - \frac{s}{2}\cdot\frac{\left(\sqrt{s^2+\rho^2}-s\right)^2}{\rho^2\sqrt{s^2+\rho^2}} \\ &= \left(\frac{\sqrt{s^2+\rho^2}-s}{2\rho^2}\right)\left[\frac{\rho^2 - s\left(\sqrt{s^2+\rho^2}-s\right)}{\sqrt{s^2+\rho^2}}\right] \\ &= \frac{\left(\sqrt{s^2+\rho^2}-s\right)^2}{2\rho^2}\end{aligned} \tag{21}$$

With (20) and (21) plugged back into (15), we have

$$K_1 = \frac{1}{a}\operatorname{Im}\left[\mathscr{L}\left\{\frac{J_1(k\rho)}{k}\right\}(s)\right] = \frac{1}{a\rho}\left(\operatorname{Im}\sqrt{s^2+\rho^2} - \operatorname{Im}s\right) = \frac{1}{a\rho}\left(\operatorname{Im}\sqrt{s^2+\rho^2}+a\right) \tag{22}$$

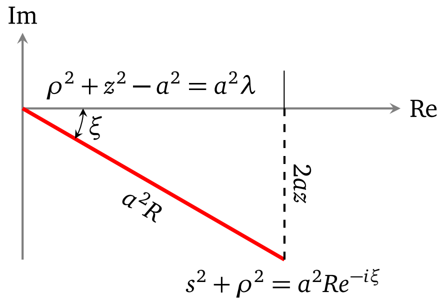
$$K_2 = \frac{1}{a}\operatorname{Im}\left[\mathscr{L}\left\{\frac{J_2(k\rho)}{k}\right\}(s)\right] = \frac{1}{2a\rho^2}\operatorname{Im}\left(\sqrt{s^2+\rho^2}-s\right)^2 \tag{23}$$

Define

$$\lambda = \frac{z^2+\rho^2-a^2}{a^2} \qquad\qquad R = \sqrt{\lambda^2+\frac{4z^2}{a^2}} \tag{24}$$

then we have

$$\sqrt{\rho^2+s^2} = \left(\rho^2+z^2-a^2-2azi\right)^{1/2} = a\sqrt{R}e^{-i\xi/2} \tag{25}$$



From the diagram above, it's clear that

$$\operatorname{Im}\sqrt{s^2+\rho^2} = -a\sqrt{R}\sin\frac{\xi}{2} \tag{26}$$

$\sin\xi/2$ and $\cos\xi/2$ can be found via

$$1-2\sin^2\frac{\xi}{2} = \cos\xi = \frac{\lambda}{R} \qquad \Longrightarrow \qquad \sin\frac{\xi}{2} = \sqrt{\frac{R-\lambda}{2R}} \qquad \cos\frac{\xi}{2} = \sqrt{\frac{R+\lambda}{2R}} \tag{27}$$

K_1 is obtained by

$$K_1 = \frac{1}{a\rho}\left(-a\sqrt{\frac{R-\lambda}{2}}+a\right) = \frac{1}{\rho}\left(1-\sqrt{\frac{R-\lambda}{2}}\right) \tag{28}$$

For (23),

$$\begin{aligned}K_2 &= \frac{1}{2a\rho^2}\operatorname{Im}\left(\sqrt{s^2+\rho^2}-s\right)^2 = \frac{1}{2a\rho^2}\cdot 2\operatorname{Re}\left(\sqrt{s^2+\rho^2}-s\right)\cdot\operatorname{Im}\left(\sqrt{s^2+\rho^2}-s\right) \\ &= \frac{1}{a\rho^2}\left(a\sqrt{R}\cos\frac{\xi}{2}-z\right)\left(-a\sqrt{R}\sin\frac{\xi}{2}+a\right) = \frac{1}{a\rho^2}\left(a\sqrt{R}\sqrt{\frac{R+\lambda}{2R}}-z\right)\left(-a\sqrt{R}\sqrt{\frac{R-\lambda}{2R}}+a\right) \\ &= \frac{1}{a\rho^2}\left(-a^2\frac{\sqrt{R^2-\lambda^2}}{2} - az + az\sqrt{\frac{R-\lambda}{2}} + a^2\sqrt{\frac{R+\lambda}{2}}\right) \\ &= \frac{1}{\rho^2}\left(-2z+z\sqrt{\frac{R-\lambda}{2}}+a\sqrt{\frac{R+\lambda}{2}}\right)\end{aligned} \tag{29}$$

(b) **Calculation of K_0**

With $s = z - ia$, let

$$F(s) = \int_0^\infty e^{-sk} \frac{J_0(k\rho)}{k} dk \tag{30}$$

then by definition $K_0 = \operatorname{Im} F(s)/a$.

Notice

$$F'(s) = -\int_0^\infty e^{-sk} J_0(k\rho)\, dk = -\mathscr{L}\{J_0(k\rho)\}(s) = -\frac{1}{\sqrt{s^2+\rho^2}} \tag{31}$$

Thus $F(s)$ is readily solvable as (reference WolframAlpha)

$$\begin{aligned}
F(s) &= -\tanh^{-1}\left(\frac{s}{\sqrt{\rho^2+s^2}}\right) \\
&= \frac{1}{2}\ln\left(1 - \frac{s}{\sqrt{\rho^2+s^2}}\right) - \frac{1}{2}\ln\left(1 + \frac{s}{\sqrt{\rho^2+s^2}}\right) \\
&= \frac{1}{2}\ln\left(\frac{\sqrt{\rho^2+s^2}-s}{\sqrt{\rho^2+s^2}+s}\right) \\
&= \frac{1}{2}\ln\left[\frac{\left(\sqrt{\rho^2+s^2}-s\right)^2}{\rho^2}\right] \\
&= \ln\left(\frac{\sqrt{\rho^2+s^2}-s}{\rho}\right)
\end{aligned} \tag{32}$$

The imaginary part of $F(s)$ is just the argument of the complex number $\sqrt{\rho^2+s^2}-s$, i.e.,

$$\begin{aligned}
\operatorname{Im}[F(s)] &= \operatorname{Arg}\left(\sqrt{\rho^2+s^2}-s\right) = \operatorname{Arg}\left[a\sqrt{R}\left(\cos\frac{\xi}{2} - i\sin\frac{\xi}{2}\right) - z + ia\right] \\
&= \tan^{-1}\left(\frac{a - a\sqrt{R}\sin\frac{\xi}{2}}{a\sqrt{R}\cos\frac{\xi}{2} - z}\right) = \tan^{-1}\left(\frac{a - a\sqrt{R}\sqrt{\frac{R-\lambda}{2R}}}{a\sqrt{R}\sqrt{\frac{R+\lambda}{2R}} - z}\right) \\
&= \tan^{-1}\left(\frac{a - a\sqrt{\frac{R-\lambda}{2}}}{a\sqrt{\frac{R+\lambda}{2}} - z}\right) = \tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right)
\end{aligned} \tag{33}$$

Finally, with K_0, K_1, K_2 inserted to (14) and (7), we get

$$\begin{aligned}
\Phi^{(1)}(\mathbf{x}) &= \frac{2H_0 a}{\pi}\left(-zK_1 + \frac{\rho}{2}K_0 - \frac{\rho}{2}K_2\right)\sin\phi \\
&= \frac{2H_0 a}{\pi}\left[-\frac{z}{\rho}\left(1 - \sqrt{\frac{R-\lambda}{2}}\right) + \frac{\rho}{2a}\tan^{-1}\sqrt{\frac{2}{R+\lambda}} - \frac{1}{2\rho}\left(-2z + z\sqrt{\frac{R-\lambda}{2}} + a\sqrt{\frac{R+\lambda}{2}}\right)\right]\sin\phi \\
&= \frac{2H_0 a}{\pi}\left(\overbrace{\frac{z}{2\rho}\sqrt{\frac{R-\lambda}{2}}}^{A} + \underbrace{\frac{\rho}{2a}\tan^{-1}\sqrt{\frac{2}{R+\lambda}}}_{B} - \overbrace{\frac{a}{2\rho}\sqrt{\frac{R+\lambda}{2}}}^{C}\right)\sin\phi
\end{aligned} \tag{34}$$

3. **Asymptotic behavior with large r, equation (5.129)**

Let the observation point $\mathbf{x} = (\rho, \phi, z)$ be equivalently expressed in spherical coordinate (r, θ, ϕ), then

$$z = r\cos\theta \qquad \rho = r\sin\theta \tag{35}$$

The benefit of using this form is so that we can write (34) in increasing negative powers of r.

By definition,

$$\lambda = \left(\frac{r}{a}\right)^2 - 1 \tag{36}$$

$$R = \sqrt{\left[\left(\frac{r}{a}\right)^2 - 1\right]^2 + \frac{4z^2}{a^2}} = \sqrt{\left(\frac{r}{a}\right)^4 - 2\left(\frac{r}{a}\right)^2 + 1 + \frac{4z^2}{a^2}} = \left(\frac{r}{a}\right)^2 \sqrt{1 - 2\left(\frac{a}{r}\right)^2 + \left(\frac{a}{r}\right)^4 + \frac{4a^2\cos^2\theta}{r^2}} \tag{37}$$

Using the Taylor expansion

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots \tag{38}$$

we have

$$\begin{aligned} R &= \left(\frac{r}{a}\right)^2 \left\{1 + \frac{1}{2}\left[-2\left(\frac{a}{r}\right)^2 + \left(\frac{a}{r}\right)^4 + \frac{4a^2\cos^2\theta}{r^2}\right] - \frac{1}{8}\left[4\left(\frac{a}{r}\right)^4 - \frac{16a^4\cos^2\theta}{r^4} + \frac{16a^4\cos^4\theta}{r^4}\right] + O\left(\frac{1}{r^6}\right)\right\} \\ &= \left(\frac{r}{a}\right)^2 - 1 + \frac{1}{2}\left(\frac{a}{r}\right)^2 + 2\cos^2\theta - \frac{1}{2}\left(\frac{a}{r}\right)^2 + \frac{2a^2\cos^2\theta}{r^2} - \frac{2a^2\cos^4\theta}{r^2} + O\left(\frac{1}{r^4}\right) \\ &= \left(\frac{r}{a}\right)^2 - 1 + 2\cos^2\theta + \frac{2a^2\cos^2\theta\sin^2\theta}{r^2} + O\left(\frac{1}{r^4}\right) \end{aligned} \tag{39}$$

hence

$$\begin{aligned} \frac{R-\lambda}{2} &= \cos^2\theta + \frac{a^2\cos^2\theta\sin^2\theta}{r^2} + O\left(\frac{1}{r^4}\right) \qquad \Longrightarrow \\ \sqrt{\frac{R-\lambda}{2}} &= \cos\theta\left[1 + \frac{a^2\sin^2\theta}{r^2} + O\left(\frac{1}{r^4}\right)\right]^{1/2} = \cos\theta\left[1 + \frac{1}{2}\frac{a^2\sin^2\theta}{r^2} + O\left(\frac{1}{r^4}\right)\right] \end{aligned} \tag{40}$$

The dominating orders of A in (34) are

$$A = \frac{\cos\theta}{2\sin\theta} \cdot \cos\theta\left[1 + \frac{1}{2}\frac{a^2\sin^2\theta}{r^2} + O\left(\frac{1}{r^4}\right)\right] = \frac{\cos^2\theta}{2\sin\theta} + \frac{1}{4}\frac{a^2\sin\theta\cos^2\theta}{r^2} + O\left(\frac{1}{r^4}\right) \tag{41}$$

Notice

$$\frac{2}{R+\lambda} = \frac{a^2}{z^2}\left(\frac{R-\lambda}{2}\right) \tag{42}$$

gives

$$\tan^{-1}\sqrt{\frac{2}{R+\lambda}} = \tan^{-1}\left(\frac{a}{z}\sqrt{\frac{R-\lambda}{2}}\right) = \tan^{-1}\left(\frac{a}{r\cos\theta}\sqrt{\frac{R-\lambda}{2}}\right) = \tan^{-1}\left[\frac{a}{r} + \frac{1}{2}\frac{a^3\sin^2\theta}{r^3} + O\left(\frac{1}{r^5}\right)\right] \tag{43}$$

Using the Taylor expansion

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots \tag{44}$$

we can write

$$\tan^{-1}\sqrt{\frac{2}{R+\lambda}} = \frac{a}{r} + \frac{1}{2}\frac{a^3\sin^2\theta}{r^3} - \frac{1}{3}\left(\frac{a}{r}\right)^3 + O\left(\frac{1}{r^5}\right) \tag{45}$$

which gives the B term in (34)

$$B = \frac{r\sin\theta}{2a}\left[\frac{a}{r} + \frac{1}{2}\frac{a^3\sin^2\theta}{r^3} - \frac{1}{3}\left(\frac{a}{r}\right)^3 + O\left(\frac{1}{r^5}\right)\right] = \frac{\sin\theta}{2} + \frac{1}{4}\frac{a^2\sin^3\theta}{r^2} - \frac{1}{6}\frac{a^2\sin\theta}{r^2} + O\left(\frac{1}{r^4}\right) \tag{46}$$

Lastly, from (39)

$$\begin{aligned} \frac{R+\lambda}{2} &= \left(\frac{r}{a}\right)^2 - 1 + \cos^2\theta + O\left(\frac{1}{r^2}\right) \qquad \Longrightarrow \\ \sqrt{\frac{R+\lambda}{2}} &= \left(\frac{r}{a}\right)\left[1 - \sin^2\theta\left(\frac{a}{r}\right)^2 + O\left(\frac{1}{r^4}\right)\right]^{1/2} = \frac{r}{a} - \frac{1}{2}\frac{a\sin^2\theta}{r} + O\left(\frac{1}{r^3}\right) \end{aligned} \tag{47}$$

which gives

$$C = \frac{a}{2r\sin\theta}\left[\frac{r}{a} - \frac{1}{2}\frac{a\sin^2\theta}{r} + O\left(\frac{1}{r^3}\right)\right] = \frac{1}{2\sin\theta} - \frac{1}{4}\frac{a^2\sin\theta}{r^2} + O\left(\frac{1}{r^4}\right) \tag{48}$$

Insert A, B, C back into (34) and ignore the $O(r^{-4})$ terms,

$$A+B-C \approx \frac{\cos^2\theta}{2\sin\theta} + \frac{a^2\sin\theta\cos^2\theta}{4r^2} + \frac{\sin\theta}{2} + \frac{a^2\sin^3\theta}{4r^2} - \frac{a^2\sin\theta}{6r^2} - \frac{1}{2\sin\theta} + \frac{a^2\sin\theta}{4r^2} \approx \frac{a^2\sin\theta}{3r^2} \implies$$
$$\Phi^{(1)}(\mathbf{x}) \approx \frac{2H_0a^3}{3\pi}\cdot\frac{r\sin\theta\sin\phi}{r^3} = \frac{2H_0a^3}{3\pi}\frac{y}{r^3} \tag{49}$$

4. **Problem 5.24, tangential field on the surface $z = 0^+$**
At $z = 0^+$,

$$\lambda = R = \frac{\rho^2}{a^2} - 1 \tag{50}$$

By (34),

$$\begin{aligned}\Phi^{(1)}(\rho,\phi,z=0) &= \frac{2H_0a}{\pi}\left(\frac{\rho}{2a}\tan^{-1}\frac{a}{\sqrt{\rho^2-a^2}} - \frac{a}{2\rho}\cdot\frac{\sqrt{\rho^2-a^2}}{a}\right)\sin\phi \\ &= \frac{2H_0a}{\pi}\left[\frac{\rho}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) - \frac{\sqrt{\rho^2-a^2}}{2\rho}\right]\sin\phi\end{aligned} \tag{51}$$

The derivatives are

$$\begin{aligned}\frac{\partial\Phi^{(1)}}{\partial\rho} &= \frac{2H_0a}{\pi}\left[\frac{1}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) + \frac{\rho}{2a}\frac{1}{\sqrt{1-\left(\frac{a}{\rho}\right)^2}}\left(-\frac{a}{\rho^2}\right) + \frac{\sqrt{\rho^2-a^2}}{2\rho^2} - \frac{1}{2\rho}\frac{\rho}{\sqrt{\rho^2-a^2}}\right]\sin\phi \\ &= \frac{2H_0a}{\pi}\left[\frac{1}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) - \frac{1}{2\sqrt{\rho^2-a^2}} + \frac{\sqrt{\rho^2-a^2}}{2\rho^2} - \frac{1}{2\sqrt{\rho^2-a^2}}\right]\sin\phi \\ &= \frac{2H_0a}{\pi}\left[\frac{1}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) - \frac{1}{\sqrt{\rho^2-a^2}} + \frac{\sqrt{\rho^2-a^2}}{2\rho^2}\right]\sin\phi\end{aligned} \tag{52}$$

$$\frac{1}{\rho}\frac{\partial\Phi^{(1)}}{\partial\phi} = \frac{2H_0a}{\pi}\left[\frac{1}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) - \frac{\sqrt{\rho^2-a^2}}{2\rho^2}\right]\cos\phi \tag{53}$$

The field in polar coordinates are

$$H_\rho^{(1)} = -\frac{\partial\Phi^{(1)}}{\partial\rho} \qquad H_\phi^{(1)} = -\frac{1}{\rho}\frac{\partial\Phi^{(1)}}{\partial\phi} \tag{54}$$

and in Cartesian coordinates are

$$H_x^{(1)} = H_\rho^{(1)}\cos\phi - H_\phi^{(1)}\sin\phi = \frac{2H_0a}{\pi}\left(\frac{1}{\sqrt{\rho^2-a^2}} - \frac{\sqrt{\rho^2-a^2}}{\rho^2}\right)\sin\phi\cos\phi = \frac{2H_0a^3}{\pi}\frac{xy}{\rho^4\sqrt{\rho^2-a^2}} \tag{55}$$

$$\begin{aligned}H_y^{(1)} &= H_\rho^{(1)}\sin\phi + H_\phi^{(1)}\cos\phi \\ &= -\frac{2H_0a}{\pi}\left\{\left[\frac{1}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) - \frac{1}{\sqrt{\rho^2-a^2}} + \frac{\sqrt{\rho^2-a^2}}{2\rho^2}\right]\sin^2\phi + \left[\frac{1}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) - \frac{\sqrt{\rho^2-a^2}}{2\rho^2}\right]\cos^2\phi\right\} \\ &= -\frac{2H_0a}{\pi}\left[\frac{1}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) - \left(\frac{1}{\sqrt{\rho^2-a^2}} - \frac{\sqrt{\rho^2-a^2}}{\rho^2}\right)\sin^2\phi - \frac{\sqrt{\rho^2-a^2}}{2\rho^2}\right] \\ &= \frac{H_0}{\pi}\left[\frac{a\sqrt{\rho^2-a^2}}{\rho^2} - \sin^{-1}\left(\frac{a}{\rho}\right)\right] + \frac{2H_0a^3}{\pi}\frac{y^2}{\rho^4\sqrt{\rho^2-a^2}}\end{aligned} \tag{56}$$