1. **Prob 12.9**

(a) The magnetic induction generated by dipole $\mathbf{M}$ is

$$\mathbf{B}(\mathbf{x}) = \frac{1}{r^3}\left[3(\mathbf{M}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{M}\right] \tag{1}$$

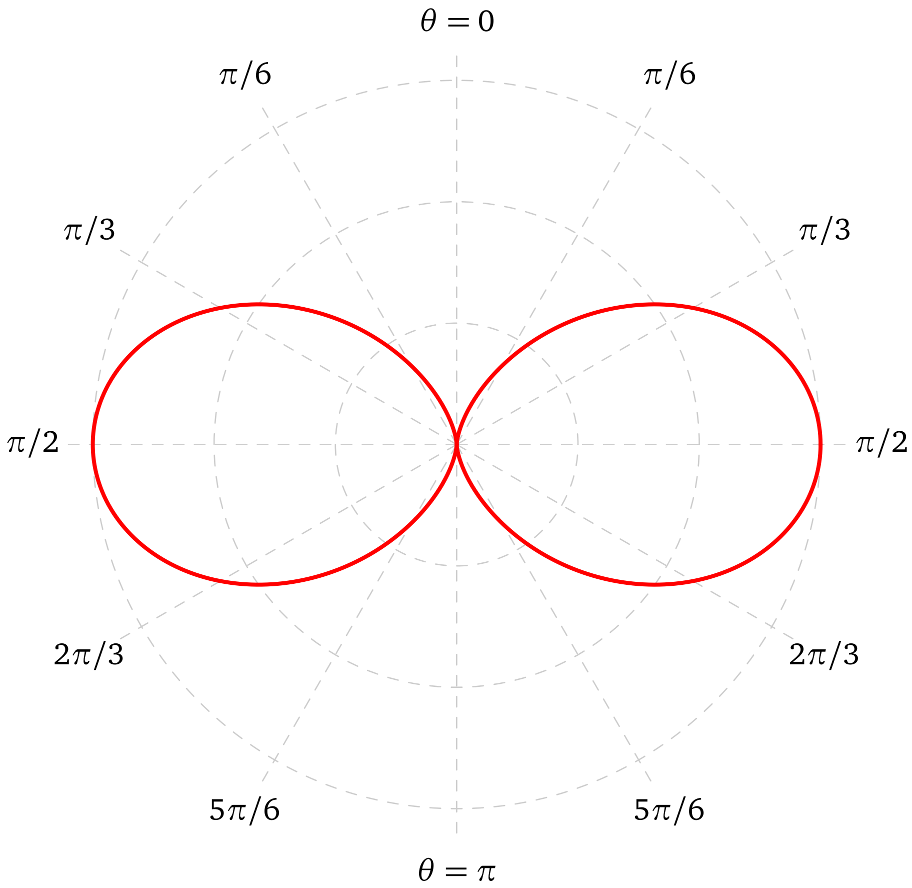
Let $\mathbf{M}$ be along the z-axis, and with $\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$, we can write the above as

$$\mathbf{B}(\mathbf{x}) = \frac{1}{r^3}\left[3M\cos\theta\,\hat{\mathbf{r}} - M\left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\right)\right] = \frac{1}{r^3}\left(2M\cos\theta\,\hat{\mathbf{r}} + M\sin\theta\,\hat{\boldsymbol{\theta}}\right) \tag{2}$$

The tangent of the field line must be proportional to the ratio of the corresponding components of $\mathbf{B}$, i.e.

$$\frac{dr}{r\,d\theta} = \frac{B_r}{B_\theta} = \frac{2\cos\theta}{\sin\theta} \quad\Longrightarrow\quad \frac{dr}{r} = 2\frac{d(\sin\theta)}{\sin\theta} \quad\Longrightarrow\quad r \propto \sin^2\theta \tag{3}$$

Below is a plot of a field line.



(b) At $\theta = \pi/2$, the radius of curvature is

$$R_C = \left.\frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}\right|_{r=R,\theta=\pi/2} = \frac{R}{3} \tag{4}$$

The gradient drift velocity can be obtained via (12.61) with $v_\parallel = 0$,

$$\mathbf{v}_C = \frac{1}{\omega_B R_C}\cdot\frac{v_\perp^2}{2}\left(\frac{\mathbf{R}_C\times\mathbf{B}_0}{R_C B_0}\right) = \frac{3}{2}\frac{v_\perp^2}{\omega_B R} = \frac{3}{2}\frac{\omega_B a^2}{R}\hat{\boldsymbol{\phi}} \tag{5}$$

Note that at $\theta=\pi/2$, $\mathbf{B}$ is along the direction $\hat{\boldsymbol{\theta}}$, and $\mathbf{R}_C$ is pointing from the center of curvature to the charge, so $\mathbf{v}_C$ is in the direction of $\hat{\boldsymbol{\phi}}$. Since $\mathbf{M}$ points to the south, $\boldsymbol{\phi}$ points to the west. The drift angular frequency is given by v_C/R, or in terms of time dependency of east longitude, we have

$$\phi(t)=\phi_0-\frac{3}{2}\left(\frac{a}{R}\right)^2\omega_B(t-t_0) \tag{6}$$

(c) Let $\xi=\pi/2-\theta$ be the (south) latitude angle (recall that $\mathbf{M}$ points to the south). Then (2) can be rewritten in terms of ξ

$$\mathbf{B}(\mathbf{x})=\frac{M}{r^3}\left(2\sin\xi\hat{\mathbf{r}}+\cos\xi\hat{\boldsymbol{\theta}}\right) \tag{7}$$

For the force line at $r=R$, we can substitude $r=R\sin^2\theta=R\cos^2\xi$ into (7) and get the magnitude of $\mathbf{B}$ as

$$B=\frac{M}{R^3}\frac{\sqrt{\cos^2\xi+4\sin^2\xi}}{\cos^6\xi}\approx B_0\left(1+3\xi^2\right)^{1/2}\left(1-\frac{\xi^2}{2}\right)^{-6}\approx B_0\left(1+\frac{9\xi^2}{2}\right) \tag{8}$$

We can apply the adiabatic invariance discussed in section 12.5 to the particle and invoke equation (12.72)

$$v_{\parallel}^2(\xi)=v_0^2-v_{\perp 0}^2\frac{B(\xi)}{B_0}\approx v_0^2-v_{\perp 0}^2\left(1+\frac{9\xi^2}{2}\right)=v_{\parallel}^2(0)-\frac{9}{2}(\omega_B a)^2\xi^2 \tag{9}$$

As is interpreted in the text, this is equivalent to a harmonic potential

$$V(z)=\frac{1}{2}m\cdot\frac{9}{2}\left(\frac{\omega_B a}{R}\right)^2 z^2 \tag{10}$$

giving rise to an angular frequency of

$$\Omega=\frac{3}{\sqrt{2}}\omega_B\left(\frac{a}{R}\right) \tag{11}$$

The change in longitude per cycle of oscillation in latitude is then

$$\Delta\phi=\frac{2\pi}{\Omega}\cdot\frac{3}{2}\left(\frac{a}{R}\right)^2\omega_B=\sqrt{2}\pi\left(\frac{a}{R}\right) \tag{12}$$

(d) For $M=8.1\times10^{25}$gauss-cm^3, at $R=3\times10^9$cm, we have

$$B_0=\frac{M}{R^3}=3\times10^{-7}\text{Tesla} \tag{13}$$

For an electron with kinetic energy of 10MeV,

$$(\gamma-1)mc^2=10\text{MeV}\qquad\Longrightarrow\qquad\gamma=1+\frac{10\text{MeV}}{0.511\text{MeV}}\approx20.6 \tag{14}$$

Then (in SI units)

$$\omega_B=\frac{eB_0}{\gamma m}=2.57\times10^3\text{rad/s} \tag{15}$$

giving the gyration radius

$$a\approx\frac{c}{\omega_B}\approx1.17\times10^5\text{m} \tag{16}$$

Furthermore, from (11) and (5), we have

$$T_\theta=\frac{2\pi}{\Omega}\approx0.3\text{s}\qquad\qquad T_\phi=\frac{2\pi}{\frac{3}{2}\left(\frac{a}{R}\right)^2\omega_B}\approx107\text{s} \tag{17}$$

Repeating the calculation for kinetic energy of 10KeV or $\gamma\approx1.02$, we have

$$\omega_B\approx5.19\times10^4\text{rad/s}\qquad a\approx1.13\times10^3\text{m}\qquad T_\theta\approx1.52\text{s}\qquad T_\phi\approx5.7\times10^4\text{s} \tag{18}$$

2. **Prob 12.10**

The adiabatic invariance derivation allows us to plug the exact form of (8) into (9), giving

$$v_{\parallel}^2(\xi) = v_0^2 - v_{\perp 0}^2 \frac{\sqrt{1+3\sin^2\xi}}{\cos^6\xi} \implies$$
$$v_{\parallel}^2(\xi) = v_{\parallel}^2(0) - v_{\perp 0}^2\left(\frac{\sqrt{1+3\sin^2\xi}}{\cos^6\xi} - 1\right) \implies$$
$$\frac{v_{\parallel}^2(\xi)}{v_{\perp 0}^2} = \tan^2\alpha - \left(\frac{\sqrt{1+3\sin^2\xi}}{\cos^6\xi} - 1\right) \tag{19}$$

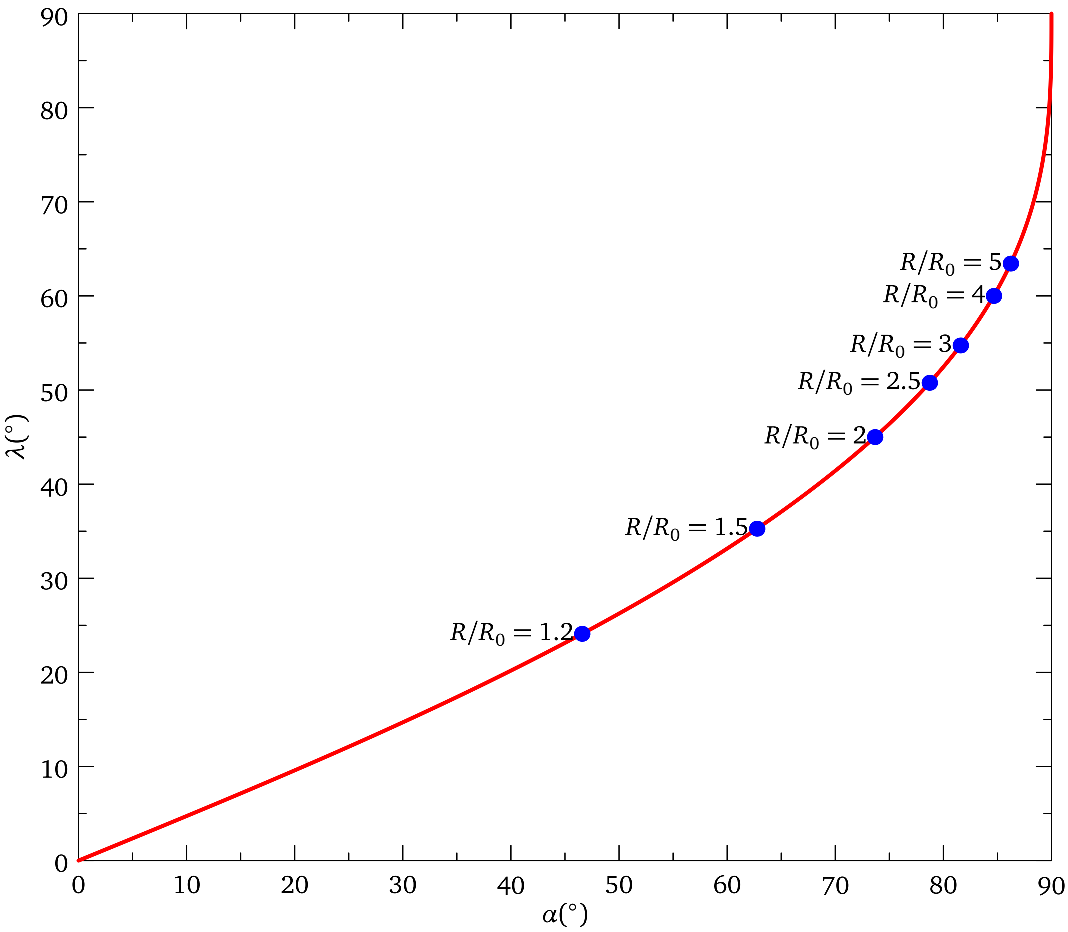
The maximum ξ is achieved at $\xi = \lambda$ where the LHS is zero, or

$$\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda} = \sec^2\alpha \tag{20}$$

For the plot, it is easier to calculate the inverse function

$$\alpha = \cos^{-1}\left[\frac{\cos^3\lambda}{\left(1+3\sin^2\lambda\right)^{1/4}}\right] \tag{21}$$

and then transpose the graph.



From the field line equation in the previous problem, for the particle to hit earth, we must have

$$\cos^2\lambda = \frac{R_0}{R} \implies \lambda = \cos^{-1}\sqrt{\frac{R_0}{R}} \tag{22}$$

The numerical values are recorded in the table below and marked in the graph above.

R/R_0	1.2	1.5	2	2.5	3	4	5
α	46.58°	62.76°	73.67°	78.72°	81.59°	84.66°	86.22°