1. From previous problems (e.g., 11.22), we know that the center of mass frame moves with velocity

$$\mathbf{u} = \frac{\gamma_1 m_1 \mathbf{v}_1 + \gamma_2 m_2 \mathbf{v}_2}{\gamma_1 m_1 + \gamma_2 m_2}. \tag{1}$$

If we were consistent to require $O(1/c^2)$ accuracy, we would have to calculate each particle's relative velocity to the center of mass frame to that order, and then use (12.82) to get the Darwin Lagrangian which is accurate up to $O\left(1/c^2\right)$. The calculation would have involved equation (11.31) for the CM frame velocity $\mathbf{v}_1', \mathbf{v}_2'$, which is not expressible only by $\mathbf{v}_1 - \mathbf{v}_2$. Also the inverse distance in $q_1 q_2 / r_{12}'$ would have to be kept up to $O\left(1/c^2\right)$ correction after Lorentz transformation into the CM frame. The result is also not expressible by $\mathbf{x}_1 - \mathbf{x}_2$ and $\mathbf{v}_1 - \mathbf{v}_2$ alone. The intention of Jackson is probably just treat the relative velocity and distance in the CM frame as if they are non-relativistic, but this defeats the purpose of keeping $O\left(1/c^2\right)$ accuracy in the Darwin Lagrangian. We will proceed with this (somewhat inconsistent) assumption.

The non-relativistic limit of (1) reads

$$\mathbf{u} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2} \tag{2}$$

giving the relative velocity of each particle to the CM frame

$$\mathbf{v}_1' = \mathbf{v}_1 - \mathbf{u} = \frac{m_2 (\mathbf{v}_1 - \mathbf{v}_2)}{m_1 + m_2} = \frac{m_2 \mathbf{v}}{m_1 + m_2} \qquad \mathbf{v}_2' = \mathbf{v}_2 - \mathbf{u} = -\frac{m_1 \mathbf{v}}{m_1 + m_2} \tag{3}$$

Then the Darwin Lagrangian is given by (12.82)

$$\begin{aligned} \mathbf{L}_{\text{Darwin}} &= \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 + \frac{m_1 v_1'^4}{8c^2} + \frac{m_2 v_2'^4}{8c^2} - \frac{q_1 q_2}{r'} + \frac{1}{2c^2} \frac{q_1 q_2}{r'} \left[\mathbf{v}_1' \cdot \mathbf{v}_2' + \left(\mathbf{v}_1' \cdot \hat{\mathbf{r}}' \right) \left(\mathbf{v}_2' \cdot \hat{\mathbf{r}}' \right) \right] \\ &= \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) v^2 + \frac{1}{8} \left(\frac{m_1 m_2}{m_1 + m_2} \right) \left[\frac{m_1^3 + m_2^3}{(m_1 + m_2)^3} \right] \frac{v^4}{c^2} - \frac{q_1 q_2}{r} - \frac{q_1 q_2}{2c^2 r} \frac{m_1 m_2}{(m_1 + m_2)^2} \left[v^2 + (\mathbf{v} \cdot \hat{\mathbf{r}})^2 \right] \end{aligned} \tag{4}$$

The canonical momentum is given by

$$\mathbf{p} = \nabla_{\mathbf{v}} L = \left(\frac{m_1 m_2}{m_1 + m_2} \right) \mathbf{v} + \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) \left[\frac{m_1^3 + m_2^3}{(m_1 + m_2)^3} \right] \frac{v^2 \mathbf{v}}{c^2} - \frac{q_1 q_2}{c^2 r} \frac{m_1 m_2}{(m_1 + m_2)^2} \left[\mathbf{v} + (\mathbf{v} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} \right] \tag{5}$$

2. The Hamiltonian is

$$\begin{aligned} H &= \mathbf{p} \cdot \mathbf{v} - L \\ &= \mathbf{p} \cdot \mathbf{v} - \left\{ \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) v^2 + \frac{1}{8} \left(\frac{m_1 m_2}{m_1 + m_2} \right) \left[\frac{m_1^3 + m_2^3}{(m_1 + m_2)^3} \right] \frac{v^4}{c^2} - \frac{q_1 q_2}{r} - \frac{q_1 q_2}{2c^2 r} \frac{m_1 m_2}{(m_1 + m_2)^2} \left[v^2 + (\mathbf{v} \cdot \hat{\mathbf{r}})^2 \right] \right\} \\ &= \left[\mathbf{p} \cdot \mathbf{v} - \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) v^2 \right] - \frac{1}{8} \left(\frac{m_1 m_2}{m_1 + m_2} \right) \left[\frac{m_1^3 + m_2^3}{(m_1 + m_2)^3} \right] \frac{v^4}{c^2} + \frac{q_1 q_2}{r} + \frac{q_1 q_2}{2c^2 r} \frac{m_1 m_2}{(m_1 + m_2)^2} \left[v^2 + (\mathbf{v} \cdot \hat{\mathbf{r}})^2 \right] \end{aligned} \tag{6}$$

where the first square bracket is just

$$\mathbf{p} \cdot \mathbf{v} - \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) v^2 = \left(\frac{m_1 + m_2}{2 m_1 m_2} \right) \left\{ p^2 - \left[\mathbf{p} - \left(\frac{m_1 m_2}{m_1 + m_2} \right) \mathbf{v} \right]^2 \right\} \tag{7}$$

From (5), we see that

$$\mathbf{p} = \left(\frac{m_1 m_2}{m_1 + m_2} \right) \mathbf{v} + O\left(\frac{1}{c^2} \right) \tag{8}$$

then from (6) and (7), we can take the 0th order approximation

$$\mathbf{v} \approx \left(\frac{m_1 + m_2}{m_1 m_2} \right) \mathbf{p} \tag{9}$$

to achieve an accuracy of $O\left(1/c^2\right)$ in the Hamiltonian, giving

$$H \approx \frac{p^2}{2} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) - \frac{p^4}{8c^2} \left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) + \frac{q_1 q_2}{r} + \frac{q_1 q_2}{2 m_1 m_2 c^2 r} \left[p^2 + (\mathbf{p} \cdot \hat{\mathbf{r}})^2 \right] \tag{10}$$