1. **Prob 11.11**

   Define

$$f(\lambda) \equiv e^{\lambda(L+\delta L)} e^{-\lambda L} \tag{1}$$

   Since

$$e^{\lambda X} = \sum_{n=0}^{\infty} \frac{(\lambda X)^n}{n!} \qquad \Longrightarrow \qquad \frac{d\left(e^{\lambda X}\right)}{d\lambda} = \sum_{n=0}^{\infty} \frac{n\lambda^{n-1} X^n}{n!} = X \sum_{k=0}^{\infty} \frac{(\lambda X)^k}{k!} = X e^{\lambda X} = e^{\lambda X} X \tag{2}$$

   we have

$$f'(\lambda) = e^{\lambda(L+\delta L)}(L+\delta L)e^{-\lambda L} - e^{\lambda(L+\delta L)} L e^{-\lambda L} = e^{\lambda(L+\delta L)} \delta L e^{-\lambda L} \tag{3}$$

$$f''(\lambda) = e^{\lambda(L+\delta L)}(L+\delta L)\delta L e^{-\lambda L} - e^{\lambda(L+\delta L)} \delta L L e^{-\lambda L} \approx e^{\lambda(L+\delta L)}[L, \delta L] e^{-\lambda L} \qquad \text{1st order in } \delta L \tag{4}$$

   If we denote

$$\left[L^{(k+1)}, \delta L\right] \equiv \left[L, \left[L^{(k)}, \delta L\right]\right] \qquad \text{where} \qquad \left[L^{(0)}, \delta L\right] = \delta L \tag{5}$$

   (3) and (4) show that for $k = 1, 2$, it is true that

$$\frac{d^k f}{d\lambda^k} = e^{\lambda(L+\delta L)} \left[L^{(k-1)}, \delta L\right] e^{-\lambda L} \tag{6}$$

   Then obviously for $k+1$, up to first order in δL,

$$\frac{d^{k+1} f}{d\lambda^{k+1}} = e^{\lambda(L+\delta L)}(L+\delta L)\left[L^{(k-1)}, \delta L\right] e^{-\lambda L} - e^{\lambda(L+\delta L)} \left[L^{(k-1)}, \delta L\right] L e^{-\lambda L} \approx e^{\lambda(L+\delta L)} \left[L^{(k)}, \delta L\right] e^{-\lambda L} \tag{7}$$

   so (6) is true for all $k \geq 1$. Thus at first order in δL, expanding (1) into a Taylor series around 0 gives

$$f(\lambda) \approx e^{\lambda(L+\delta L)} e^{-\lambda L} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \frac{d^n f}{d\lambda^n}\bigg|_{\lambda=0} = I + \lambda \delta L + \frac{\lambda^2}{2!}[L, \delta L] + \frac{\lambda^3}{3!}[L, [L, \delta L]] + \cdots \tag{8}$$

   Taking $\lambda = 1$ gives

$$e^{L+\delta L} e^{-L} \approx I + \delta L + \frac{1}{2!}[L, \delta L] + \frac{1}{3!}[L, [L, \delta L]] + \cdots \tag{9}$$

   Of course for physics problems, we sweep the important convergence issues under the rug.

2. **Prob 11.12**

   (a) Let

$$\beta' \equiv |\boldsymbol{\beta} + \delta\boldsymbol{\beta}| \tag{10}$$

   be a function of $\delta\boldsymbol{\beta}$.
   Then

$$\nabla\beta'\bigg|_{\delta\boldsymbol{\beta}=0} = \frac{\boldsymbol{\beta}}{\beta} \tag{11}$$

   To the first order of $\delta\boldsymbol{\beta}$,

$$\tanh^{-1}\beta' \approx \tanh^{-1}\beta + \left[\left(\frac{1}{1-\beta'^2}\right)\nabla\beta'\right]_{\delta\boldsymbol{\beta}=0} \cdot \delta\boldsymbol{\beta} = \tanh^{-1}\beta + \left(\frac{1}{1-\beta^2}\right)\frac{\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}}{\beta} \tag{12}$$

$$\beta' \approx \beta + \frac{\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}}{\beta} \tag{13}$$

Thus

$$\begin{aligned}
L+\delta L &= -\frac{(\boldsymbol{\beta}+\delta\boldsymbol{\beta})\cdot\mathbf{K}\tanh^{-1}\beta'}{\beta'} \\
&\approx -\frac{(\boldsymbol{\beta}+\delta\boldsymbol{\beta})\cdot\mathbf{K}\left[\tanh^{-1}\beta+\left(\frac{1}{1-\beta^2}\right)\frac{\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}}{\beta}\right]}{\beta}\left(1-\frac{\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}}{\beta^2}\right) \\
&\approx \underbrace{-\frac{\boldsymbol{\beta}\cdot\mathbf{K}\tanh^{-1}\beta}{\beta}}_{L}-\frac{\tanh^{-1}\beta}{\beta}\mathbf{K}\cdot\underbrace{\left[\delta\boldsymbol{\beta}-\frac{\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}\right)}{\beta^2}\right]}_{\delta\boldsymbol{\beta}_\perp}-\left(\frac{1}{1-\beta^2}\right)\mathbf{K}\cdot\underbrace{\frac{\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}\right)}{\beta^2}}_{\delta\boldsymbol{\beta}_\parallel}
\end{aligned} \tag{14}$$

Therefore we can identify

$$\delta L = -\gamma^2\delta\boldsymbol{\beta}_\parallel\cdot\mathbf{K}-\frac{\delta\boldsymbol{\beta}_\perp\cdot\mathbf{K}\tanh^{-1}\beta}{\beta} \tag{15}$$

(b) C_1 and C_2 can be calculated as follows:

$$\begin{aligned}
C_1 = [L,\delta L] &= \left[-\frac{\boldsymbol{\beta}\cdot\mathbf{K}\tanh^{-1}\beta}{\beta}, -\gamma^2\delta\boldsymbol{\beta}_\parallel\cdot\mathbf{K}-\frac{\delta\boldsymbol{\beta}_\perp\cdot\mathbf{K}\tanh^{-1}\beta}{\beta}\right] \\
&= \frac{\gamma^2\tanh^{-1}\beta}{\beta}\left[\beta_i K_i, \delta\beta_{\parallel j}K_j\right]+\left(\frac{\tanh^{-1}\beta}{\beta}\right)^2\left[\beta_i K_i,\delta\beta_{\perp j}K_j\right] \\
&= \frac{\gamma^2\tanh^{-1}\beta}{\beta}\beta_i\delta\beta_{\parallel j}\left[K_i,K_j\right]+\left(\frac{\tanh^{-1}\beta}{\beta}\right)^2\beta_i\delta\beta_{\perp j}\left[K_i,K_j\right] \\
&= \frac{\gamma^2\tanh^{-1}\beta}{\beta}\underbrace{\beta_i\delta\beta_{\parallel j}\left(-\epsilon_{ijk}S_k\right)}_{-\left(\boldsymbol{\beta}\times\delta\boldsymbol{\beta}_\parallel\right)\cdot\mathbf{S}=0}+\left(\frac{\tanh^{-1}\beta}{\beta}\right)^2\underbrace{\beta_i\delta\beta_{\perp j}\left(-\epsilon_{ijk}S_k\right)}_{-\left(\boldsymbol{\beta}\times\delta\boldsymbol{\beta}_\perp\right)\cdot\mathbf{S}} \\
&= -\left(\frac{\tanh^{-1}\beta}{\beta}\right)^2\left(\boldsymbol{\beta}\times\delta\boldsymbol{\beta}_\perp\right)\cdot\mathbf{S}
\end{aligned} \tag{16}$$

$$\begin{aligned}
C_2 = [L,C_1] &= \left(\frac{\tanh^{-1}\beta}{\beta}\right)^3\left[\boldsymbol{\beta}\cdot\mathbf{K},\left(\boldsymbol{\beta}\times\delta\boldsymbol{\beta}_\perp\right)\cdot\mathbf{S}\right] \\
&= \left(\frac{\tanh^{-1}\beta}{\beta}\right)^3\beta_i\epsilon_{lmk}\beta_l\delta\beta_{\perp m}\left[K_i,S_k\right] \\
&= \left(\frac{\tanh^{-1}\beta}{\beta}\right)^3\beta_i\epsilon_{lmk}\beta_l\delta\beta_{\perp m}\left(-\epsilon_{kin}K_n\right) \\
&= \left(\frac{\tanh^{-1}\beta}{\beta}\right)^3\left(\delta_{im}\delta_{ln}-\delta_{il}\delta_{mn}\right)\beta_i\beta_l\delta\beta_{\perp m}K_n \\
&= \left(\frac{\tanh^{-1}\beta}{\beta}\right)^3\left[\overbrace{\left(\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}_\perp\right)}^{0}\left(\boldsymbol{\beta}\cdot\mathbf{K}\right)-\beta^2\delta\boldsymbol{\beta}\cdot\mathbf{K}\right] \\
&= \left(\tanh^{-1}\beta\right)^2\delta L_\perp
\end{aligned} \tag{17}$$

From (16), we already see that $\left[L,\delta L_\parallel\right]=0$ and $[L,\delta L_\perp]=C_1$, then it is easy to obtain

$$C_3 = [L,C_2] = \left(\tanh^{-1}\beta\right)^2[L,\delta L_\perp] = \left(\tanh^{-1}\beta\right)^2 C_1 \tag{18}$$

$$C_4 = [L,C_3] = \left(\tanh^{-1}\beta\right)^2[L,C_1] = \left(\tanh^{-1}\beta\right)^4\delta L_\perp \tag{19}$$

(c) Using the results from problem 11.11, to the first order in δL, we have

$$
\begin{aligned}
A =& I + \delta L_{\parallel} + \delta L_{\perp} + \frac{C_1}{2!} + \frac{C_2}{3!} + \frac{C_3}{4!} + \frac{C_4}{5!} + \cdots \\
=& I - \gamma^2 \delta \boldsymbol{\beta}_{\parallel} \cdot \mathbf{K} - \frac{\delta \boldsymbol{\beta}_{\perp} \cdot \mathbf{K}}{\beta} \left[\tanh^{-1}\beta + \frac{\left(\tanh^{-1}\beta\right)^3}{3!} + \frac{\left(\tanh^{-1}\beta\right)^5}{5!} + \cdots \right] \\
& - \frac{(\boldsymbol{\beta} \times \delta \boldsymbol{\beta}_{\perp}) \cdot \mathbf{S}}{\beta^2} \left[\frac{\left(\tanh^{-1}\beta\right)^2}{2!} + \frac{\left(\tanh^{-1}\beta\right)^4}{4!} + \cdots \right] \\
=& I - \gamma^2 \delta \boldsymbol{\beta}_{\parallel} \cdot \mathbf{K} - \frac{\delta \boldsymbol{\beta}_{\perp} \cdot \mathbf{K}}{\beta} \underbrace{\sinh\left(\tanh^{-1}\beta\right)}_{\gamma\beta} - \frac{(\boldsymbol{\beta} \times \delta \boldsymbol{\beta}_{\perp}) \cdot \mathbf{S}}{\beta^2} \underbrace{\left[\cosh\left(\tanh^{-1}\beta\right) - 1\right]}_{\gamma - 1} \\
=& I - \gamma^2 \delta \boldsymbol{\beta}_{\parallel} \cdot \mathbf{K} - \gamma \delta \boldsymbol{\beta}_{\perp} \cdot \mathbf{K} - \frac{\gamma^2}{\gamma + 1} (\boldsymbol{\beta} \times \delta \boldsymbol{\beta}_{\perp}) \cdot \mathbf{S}
\end{aligned}
\tag{20}
$$