From problem 7.6, we have seen that the effective permittivity of a metal medium is

$$\widetilde{\epsilon}(\omega) = \epsilon(\omega) + \frac{i\sigma}{\omega} \tag{1}$$

where $\epsilon(\omega)$ is the "normal" dielectric constant due to the dipoles (i.e., the ϵ_b in equation 7.57).

With the assumption of $\epsilon(\omega) = \widetilde{\epsilon}(\omega) - i\sigma/\omega$ being analytic, we can apply the Kramers-Kronig relation (7.120) to it, i.e.,

$$\operatorname{Re}\left[\frac{\widetilde{\epsilon}(\omega) - i\sigma/\omega}{\epsilon_0}\right] = \operatorname{Re}[\epsilon(\omega)/\epsilon_0] = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \operatorname{Im}[\epsilon(\omega')/\epsilon_0]}{\omega'^2 - \omega^2} d\omega' \tag{2}$$

$$\operatorname{Im}\left[\frac{\widetilde{\epsilon}(\omega) - i\sigma/\omega}{\epsilon_0}\right] = \operatorname{Im}[\epsilon(\omega)/\epsilon_0] = -\frac{2\omega}{\pi} P \int_0^\infty \frac{\operatorname{Re}[\epsilon(\omega')/\epsilon_0 - 1]}{\omega'^2 - \omega^2} d\omega' \tag{3}$$

Since $\operatorname{Re}\widetilde{\epsilon}(\omega) = \operatorname{Re}\epsilon(\omega)$, by (3), we have

$$\operatorname{Im}\widetilde{\epsilon}(\omega) = \frac{\sigma}{\omega} - \frac{2\omega}{\pi} P \int_0^\infty \frac{\operatorname{Re}\widetilde{\epsilon}(\omega') - \epsilon_0}{\omega'^2 - \omega^2} d\omega' \tag{4}$$

as claimed in the problem statement.

But by (2),

$$\begin{aligned}
\operatorname{Re}[\widetilde{\epsilon}(\omega)/\epsilon_0] &= 1 + \frac{2}{\pi} P \left\{ \int_0^\infty \frac{\omega' \operatorname{Im}[\widetilde{\epsilon}(\omega)/\epsilon_0]}{\omega'^2 - \omega^2} d\omega' - \int_0^\infty \frac{\omega' \cdot \dfrac{\sigma}{\epsilon_0 \omega'}}{\omega'^2 - \omega^2} d\omega' \right\} \\
&= 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \operatorname{Im}[\widetilde{\epsilon}(\omega)/\epsilon_0]}{\omega'^2 - \omega^2} d\omega' - \frac{2\sigma}{\pi\epsilon_0} \cdot \frac{1}{2\omega} P \int_0^\infty \left(\frac{1}{\omega' - \omega} - \frac{1}{\omega' + \omega} \right) d\omega'
\end{aligned} \tag{5}$$

But the last term vanishes since

$$\begin{aligned}
P \int_0^\infty \left(\frac{1}{\omega' - \omega} - \frac{1}{\omega' + \omega} \right) d\omega' &= \lim_{R\to\infty} P \left(\int_0^R \frac{d\omega'}{\omega' - \omega} - \int_0^R \frac{d\omega'}{\omega' + \omega} \right) \\
&= \lim_{R\to\infty} \lim_{\delta\to 0} \left(\int_{-\omega}^{-\delta} \frac{du}{u} + \int_\delta^R \frac{du}{u} - \int_\omega^R \frac{du}{u} \right) \\
&= \lim_{R\to\infty} \lim_{\delta\to 0} \left[\ln\left(\frac{\delta}{\omega}\right) + \ln\left(\frac{R}{\delta}\right) - \ln\left(\frac{R}{\omega}\right) \right] = 0
\end{aligned} \tag{6}$$

Thus

$$\operatorname{Re}[\widetilde{\epsilon}(\omega)/\epsilon_0] = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \operatorname{Im}[\widetilde{\epsilon}(\omega)/\epsilon_0]}{\omega'^2 - \omega^2} d\omega' \tag{7}$$