Our goal is to verify that for the potential given by (1.17)

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{1}$$

its Laplacian is

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \tag{2}$$

I do not really understand why the text chose to divide the whole region into $r \le R$ and $r > R$. In particular, I cannot convince myself of the sentence "*If $\rho(\mathbf{x}')$ is such that (1.17) exists, the contribution to integral (1.30) from the exterior of the sphere will vanish like a^2 as $a \to 0$*".

In fact, we can use the "a-potential" method on $\nabla^2(1/r)$ to prove (1.31) directly, i.e.,

$$\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{3}$$

then it follows that

$$\begin{aligned}\nabla^2\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3x' \\ &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')\big[-4\pi\delta(\mathbf{x}-\mathbf{x}')\big]d^3x' = -\frac{\rho(\mathbf{x})}{\epsilon_0}\end{aligned} \tag{4}$$

To see (3), first notice that for any a,

$$\nabla\left(\frac{1}{\sqrt{r^2+a^2}}\right) = -\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(r^2+a^2)^{3/2}} \qquad \Longrightarrow \tag{5}$$

$$\begin{aligned}\nabla^2\left(\frac{1}{\sqrt{r^2+a^2}}\right) &= \frac{\partial}{\partial x}\left[\frac{-x}{(r^2+a^2)^{3/2}}\right]+\frac{\partial}{\partial y}\left[\frac{-y}{(r^2+a^2)^{3/2}}\right]+\frac{\partial}{\partial z}\left[\frac{-z}{(r^2+a^2)^{3/2}}\right] \\ &= \left[\frac{-1}{(r^2+a^2)^{3/2}}+\frac{3x^2}{(r^2+a^2)^{5/2}}\right]+\left[\frac{-1}{(r^2+a^2)^{3/2}}+\frac{3y^2}{(r^2+a^2)^{5/2}}\right]+\left[\frac{-1}{(r^2+a^2)^{3/2}}+\frac{3z^2}{(r^2+a^2)^{5/2}}\right] \\ &= \frac{-3a^2}{(r^2+a^2)^{5/2}}\end{aligned} \tag{6}$$

For $r > 0$, we take $a = 0$, which gives

$$\nabla^2\left(\frac{1}{r}\right) = 0 \qquad \text{for } r > 0 \tag{7}$$

And the volume integral

$$\begin{aligned}\int \nabla^2\left(\frac{1}{\sqrt{r^2+a^2}}\right)d^3x &= -\int d\Omega\int_0^\infty \frac{3a^2}{(r^2+a^2)^{5/2}}r^2dr \qquad\qquad (\text{let } r = a\tan\zeta) \\ &= -4\pi\int_0^{\pi/2}\frac{3a^2\left(a^2\tan^2\zeta\right)\left(a/\cos^2\zeta\right)d\zeta}{a^5/\cos^5\zeta} \\ &= -4\pi\int_0^{\pi/2}3\sin^2\zeta\cos\zeta d\zeta = -4\pi\end{aligned} \tag{8}$$

By taking the limit of $a \to 0$ and combining (7) and (8), we can conclude (3) by definition of the δ-function.

Even though we didn't use the text's method, it's still worth explaining the expansion of $\rho(\mathbf{x}')$ in the integral

$$-\frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')\left[\frac{3a^2}{(r^2+a^2)^{5/2}}\right]d^3x' = -\frac{1}{\epsilon_0}\int_0^R \frac{3a^2}{(r^2+a^2)^{5/2}}\left[\rho(\mathbf{x})+\frac{r^2}{6}\nabla^2\rho+\cdots\right]r^2dr \tag{9}$$

With the assumption that $\rho(\mathbf{x}')$ is well behaved, we can expand $\rho(\mathbf{x}')$ around $\mathbf{x}$:

$$\begin{aligned}\rho(\mathbf{x}') =& \rho(\mathbf{x}) + \left(\frac{\partial \rho}{\partial x}\Delta x + \frac{\partial \rho}{\partial y}\Delta y + \frac{\partial \rho}{\partial z}\Delta z\right) + \\ & \frac{1}{2}\left[\frac{\partial^2 \rho}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 \rho}{\partial y^2}(\Delta y)^2 + \frac{\partial^2 \rho}{\partial z^2}(\Delta z)^2 + 2\frac{\partial^2 \rho}{\partial x \partial y}\Delta x \Delta y + 2\frac{\partial^2 \rho}{\partial x \partial z}\Delta x \Delta z + 2\frac{\partial^2 \rho}{\partial y \partial z}\Delta y \Delta z\right] + \\ & \cdots \end{aligned} \tag{10}$$

where all partial derivatives are evaluated at $\mathbf{x}$, hence can be taken out of the integral.

The integral of first order terms will vanish on the LHS of (9), because their null angular integration:

$$\int_\Omega d\Omega \Delta x \propto \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \sin\theta \cos\phi = 0 \tag{11}$$

$$\int_\Omega d\Omega \Delta y \propto \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \sin\theta \sin\phi = 0 \tag{12}$$

$$\int_\Omega d\Omega \Delta z \propto \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \cos\theta = 0 \tag{13}$$

Similarly, the cross terms of the second order will vanish too:

$$\int_\Omega d\Omega \Delta x \Delta y \propto \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \sin^2\theta \cos\phi \sin\phi = 0 \tag{14}$$

$$\int_\Omega d\Omega \Delta x \Delta z \propto \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \sin\theta \cos\phi \cos\theta = 0 \tag{15}$$

$$\int_\Omega d\Omega \Delta y \Delta z \propto \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \sin\theta \sin\phi \cos\theta = 0 \tag{16}$$

The angular integrations of the diagonal second order terms are

$$\begin{aligned}\int_\Omega d\Omega (\Delta x)^2 &= r^2 \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \sin^2\theta \cos^2\phi \\ &= r^2 \int_0^\pi \sin^3\theta d\theta \int_0^{2\pi} \frac{1+\cos 2\phi}{2} d\phi \\ &= \pi r^2 \int_{-1}^{1} (1-y^2) dy = \left(2 - \frac{2}{3}\right)\pi r^2 = \frac{4\pi r^2}{3} \end{aligned} \tag{17}$$

$$\begin{aligned}\int_\Omega d\Omega (\Delta y)^2 &= r^2 \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \sin^2\theta \sin^2\phi \\ &= r^2 \int_0^\pi \sin^3\theta d\theta \int_0^{2\pi} \frac{1-\cos 2\phi}{2} d\phi \\ &= \pi r^2 \int_{-1}^{1} (1-y^2) dy = \left(2 - \frac{2}{3}\right)\pi r^2 = \frac{4\pi r^2}{3} \end{aligned} \tag{18}$$

$$\begin{aligned}\int_\Omega d\Omega (\Delta z)^2 &= r^2 \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \cos^2\theta \\ &= 2\pi r^2 \int_{-1}^{1} y^2 dy = \frac{4\pi r^2}{3} \end{aligned} \tag{19}$$

which give the RHS of (9).