1. The Proca field equation is given in (12.92)
$$\partial^\beta F_{\beta\alpha} + \mu^2 A_\alpha = \frac{4\pi}{c} J_\alpha \tag{1}$$
Taking ∂^α on both sides yields
$$\partial^\alpha \partial^\beta F_{\beta\alpha} + \mu^2 \partial^\alpha A_\alpha = \frac{4\pi}{c} \partial^\alpha J_\alpha \qquad \Longrightarrow \qquad \partial_\alpha A^\alpha = 0 \tag{2}$$
I.e., it is necessary for the Proca field to satisfy the Lorenz gauge condition.
With the spatial component of the Proca field given as plane wave
$$\mathbf{A} = \boldsymbol{\epsilon}_0 e^{ikz - i\omega t} \tag{3}$$
Lorenz gauge condition requires
$$\frac{\partial A^0}{\partial t} + ik\hat{\mathbf{z}} \cdot \boldsymbol{\epsilon}_0 e^{ikz - i\omega t} = 0 \tag{4}$$
If A^0 has harmonic time dependence $e^{-i\omega t}$, then the above gives the constraint
$$A^0 = \frac{k}{\omega} \hat{\mathbf{z}} \cdot \boldsymbol{\epsilon}_0 e^{ikz - i\omega t} \tag{5}$$
With the Lorenz gauge condition, (1) is equivalent to
$$\Box A_\alpha + \mu^2 A_\alpha = \frac{4\pi}{c} J_\alpha \tag{6}$$
For a source-free field, this gives the constraint
$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 + \mu^2\right) A_\alpha = 0 \qquad \Longrightarrow \qquad \omega^2 = k^2 + \mu^2 \tag{7}$$
With these, the field strengths can be calculated
$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = ik\hat{\mathbf{z}} \times \boldsymbol{\epsilon}_0 e^{ikz - i\omega t} \tag{8}$$
$$\mathbf{E} = -\boldsymbol{\nabla} A^0 - \frac{\partial \mathbf{A}}{\partial t} = i\left[-\frac{k^2}{\omega} \hat{\mathbf{z}} (\hat{\mathbf{z}} \cdot \boldsymbol{\epsilon}_0) + \omega \boldsymbol{\epsilon}_0\right] e^{ikz - i\omega t} \tag{9}$$
From problem 12.16, we have obtained the symmetric stress tensor
$$\Theta^{00} = \frac{1}{8\pi}\left[\mathbf{E}^2 + \mathbf{B}^2 + \mu^2 \left(A^0 A^0 + \mathbf{A} \cdot \mathbf{A}\right)\right] \tag{10}$$
$$\Theta^{i0} = \frac{1}{4\pi}\left[(\mathbf{E} \times \mathbf{B})_i + \mu^2 A^i A^0\right] \tag{11}$$
Using the general rule of time average of product of complex quantities,
$$\langle XY \rangle = \frac{1}{2} \operatorname{Re}(XY^*) \tag{12}$$
we obtain
$$\begin{aligned} \left\langle \Theta^{00} \right\rangle &= \frac{1}{16\pi}\left\{\left|-\frac{k^2}{\omega} \hat{\mathbf{z}} (\hat{\mathbf{z}} \cdot \boldsymbol{\epsilon}_0) + \omega \boldsymbol{\epsilon}_0\right|^2 + k^2 \left|\hat{\mathbf{z}} \times \boldsymbol{\epsilon}_0\right|^2 + \mu^2 \left[\left(\frac{k}{\omega} \hat{\mathbf{z}} \cdot \boldsymbol{\epsilon}_0\right)^2 + 1\right]\right\} \qquad \text{by (7)} \\ &= \frac{\omega^2 - k^2 (\hat{\mathbf{z}} \cdot \boldsymbol{\epsilon}_0)^2}{8\pi} \end{aligned} \tag{13}$$
$$\begin{aligned} \left\langle \Theta^{30} \right\rangle &= \frac{1}{8\pi}\left\{\omega k\left[1 - (\hat{\mathbf{z}} \cdot \boldsymbol{\epsilon}_0)^2\right] + \mu^2 (\hat{\mathbf{z}} \cdot \boldsymbol{\epsilon}_0) \cdot \frac{k}{\omega} (\hat{\mathbf{z}} \cdot \boldsymbol{\epsilon}_0)\right\} \\ &= \left(\frac{k}{\omega}\right)\left[\frac{\omega^2 - k^2 (\hat{\mathbf{z}} \cdot \boldsymbol{\epsilon}_0)^2}{8\pi}\right] \end{aligned} \tag{14}$$
It is then straightforward to see that

- For transverse wave $\hat{\mathbf{z}} \cdot \boldsymbol{\epsilon}_0 = 0$, $F_t = \left\langle \Theta^{30} \right\rangle = \omega k / 8\pi$.
- For longitudinal wave $\hat{\mathbf{z}} \cdot \boldsymbol{\epsilon}_0 = 1$, $F_l = \left\langle \Theta^{30} \right\rangle = \mu^2 k / 8\pi\omega = (\mu/\omega)^2 F_t$.
- For arbitrary polarization, $\left\langle \Theta^{30} \right\rangle / \left\langle \Theta^{00} \right\rangle = k/\omega$.

2. The "efficiency" is defined relative to the scattering of massless electromagnetic field. In both cases, the electric field **E** induces oscillation of the electron, which in turn generates outgoing waves. A massless field is always transverse, so the amplitude of the electric field is equal to ω (assuming the incident wave of **A** has unit amplitude), which is the same for a transverse Proca field (see (9)). However for longitudinal Proca field, the amplitude of the electric field is $\omega - k^2/\omega$ (again, see (9)), which is reduced from transverse amplitude by a factor of μ^2/ω^2. Therefore

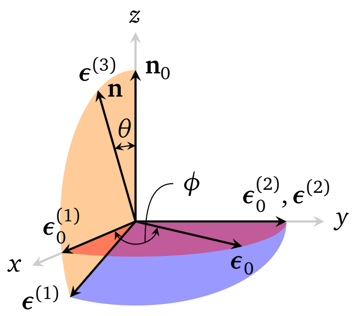
$$E_0 = \begin{cases} 1 & \text{for transverse incident Proca field} \\ \left(\frac{\mu}{\omega}\right)^2 & \text{for longitudinal incident Proca field} \end{cases} \tag{15}$$

The differential cross section is thus defined by the usual way with this additional efficiency factor

$$\frac{d\sigma}{d\Omega} = r_0^2 E_0 \left|\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0\right|^2 \frac{F_{\text{out}}}{F_{\text{in}}} \tag{16}$$

3. When the incident wave is transverse, its polarization $\boldsymbol{\epsilon}_0$ is in the x-y plane (see diagram below), which can be parameterized as

$$\boldsymbol{\epsilon}_0 = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \tag{17}$$



The three orthogonal outgoing polarizations are

$$\boldsymbol{\epsilon}^{(1)} = \cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{z}} \qquad \boldsymbol{\epsilon}^{(2)} = \hat{\mathbf{y}} \qquad \boldsymbol{\epsilon}^{(3)} = \sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}} \tag{18}$$

The differential cross section, averaged over all incident polarizations, and summed over all outgoing polarizations, is given by

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left\langle \frac{d\sigma_1}{d\Omega} + \frac{d\sigma_2}{d\Omega} + \frac{d\sigma_3}{d\Omega} \right\rangle_{\text{avg over }\phi} \tag{19}$$

where by (16),

$$\frac{d\sigma_1}{d\Omega} = r_0^2 \cdot 1 \cdot \left|\boldsymbol{\epsilon}^{(1)*} \cdot \boldsymbol{\epsilon}_0\right|^2 \frac{F_t}{F_t} = r_0^2 \cos^2\phi \cos^2\theta \tag{20}$$

$$\frac{d\sigma_2}{d\Omega} = r_0^2 \cdot 1 \cdot \left|\boldsymbol{\epsilon}^{(2)*} \cdot \boldsymbol{\epsilon}_0\right|^2 \frac{F_t}{F_t} = r_0^2 \sin^2\phi \tag{21}$$

$$\frac{d\sigma_3}{d\Omega} = r_0^2 \cdot 1 \cdot \left|\boldsymbol{\epsilon}^{(3)*} \cdot \boldsymbol{\epsilon}_0\right|^2 \frac{F_l}{F_t} = r_0^2 \left(\frac{\mu}{\omega}\right)^2 \cos^2\phi \sin^2\theta \tag{22}$$

Here we used $E_0 = 1$ for the transverse incident wave.

Taking the average in ϕ gives

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle_t = \frac{r_0^2}{2}\left[1 + \cos^2\theta + \left(\frac{\mu}{\omega}\right)^2 \sin^2\theta\right] \tag{23}$$

4. If the incident wave is longitudinal, we take

$$\boldsymbol{\epsilon}_0 = \hat{\mathbf{z}} \tag{24}$$

The calculation of the differential cross section is similar to the transverse case, except with $E_0 = (\mu/\omega)^2$, giving

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle_l = \left(\frac{\mu}{\omega}\right)^2 r_0^2 \left[\sin^2\theta + \left(\frac{\mu}{\omega}\right)^2 \cos^2\theta\right] \tag{25}$$