1. **Prob 12.1**

   (a) Given the Lagrangian

$$L(x^\alpha, U^\alpha, \tau) = -\frac{m}{2}U_\alpha U^\alpha - \frac{q}{c}U_\alpha A^\alpha = -\frac{m}{2}g_{\alpha\beta}U^\beta U^\alpha - \frac{q}{c}U^\alpha A_\alpha \tag{1}$$

   The Euler-Lagrange equation states

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0 \tag{2}$$

   in which

$$\begin{aligned}\frac{\partial L}{\partial U^\mu} &= -\frac{m}{2}\left(g_{\alpha\mu}U^\alpha + g_{\mu\beta}U^\beta\right) - \frac{q}{c}A_\mu = -mU_\mu - \frac{q}{c}A_\mu \qquad \Longrightarrow \\ \frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}\right) &= -\frac{dp_\mu}{d\tau} - \frac{q}{c}\frac{dx^\alpha}{d\tau}\frac{\partial A_\mu}{\partial x^\alpha} = -\frac{dp_\mu}{d\tau} - \frac{q}{c}U^\alpha\partial_\alpha A_\mu \end{aligned}\tag{3}$$

   and

$$\frac{\partial L}{\partial x^\mu} = -\frac{q}{c}U^\alpha \partial_\mu A_\alpha \tag{4}$$

   Then (2) is equivalent to

$$\frac{dp_\mu}{d\tau} = \frac{q}{c}U^\alpha\left(\partial_\mu A_\alpha - \partial_\alpha A_\mu\right) \qquad \Longleftrightarrow \qquad \frac{dp^\mu}{d\tau} = \frac{q}{c}U_\alpha\left(\partial^\mu A^\alpha - \partial^\alpha A^\mu\right) = \frac{q}{c}F^{\mu\alpha}U_\alpha \tag{5}$$

   the latter form of which is the Lorentz force law (see (11.144)).

   As a side note, since this Lagrangian is parameterized by proper time τ, itself is Lorentz invariant. This is in contrast to section 12.1A, where L is parameterized by t, therefore γL is Lorentz invariant.

   (b) Using (12.33), the canonical momenta is

$$P^\alpha = -\frac{\partial L}{\partial U_\alpha} = mU^\alpha + \frac{q}{c}A^\alpha \tag{6}$$

   and the Hamiltonian is

$$\begin{aligned} H &= P_\alpha U^\alpha + L \\ &= \left(mU_\alpha + \frac{q}{c}A_\alpha\right)U^\alpha - \frac{m}{2}U_\alpha U^\alpha - \frac{q}{c}U_\alpha A^\alpha \\ &= \frac{m}{2}U_\alpha U^\alpha \qquad (7) \\ &= \frac{m}{2}\left(P_\alpha - \frac{q}{c}A_\alpha\right)\left(P^\alpha - \frac{q}{c}A^\alpha\right) \qquad (8) \end{aligned}$$

   (7) indicates that the Hamiltonian is a Lorentz invariant with value $mc^2/2$.

2. **Prob 12.2**

   (a) Let L be the original Lagrangian, and let

$$L' = L + \frac{d}{dt}F(\mathbf{x}, t) \tag{9}$$

   be another Lagrangian differing from L by a total time derivative of function $F(\mathbf{x}, t)$. Here it is possible that $\mathbf{x}$ depends on the time t.

   For any given path between two spacetime events $(\mathbf{x}_1, t_1), (\mathbf{x}_2, t_2)$, we can calculate the actions due to L and L'

$$A = \int L dt \tag{10}$$

$$A' = \int L' dt = \int L dt + \int \frac{d}{dt}F(\mathbf{x}, t)\, dt = A + F(\mathbf{x}_2, t_2) - F(\mathbf{x}_1, t_1) \tag{11}$$

   Since the difference is constant, it will not chane the path that extremizes the action. Therefore the equations of motion are unchanged.

(b) Let $\Lambda(\mathbf{x}, t)$ be the guage function giving rise to the gauge transformation

$$A^{\alpha} \rightarrow A^{\alpha} + \partial^{\alpha}\Lambda \tag{12}$$

This can be more explicitly written in "1+3" notation as

$$\Phi \rightarrow \Phi + \frac{1}{c}\frac{\partial}{\partial t}\Lambda \qquad\qquad \mathbf{A} \rightarrow \mathbf{A} - \boldsymbol{\nabla}\Lambda \tag{13}$$

From (12.12) we see that this will change the Lagrangian by

$$L \rightarrow L - \frac{e}{c}\left(\frac{\partial \Lambda}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\Lambda\right) \tag{14}$$

Although the function $\Lambda(\mathbf{x}, t)$ itself is defined with respect to independent $\mathbf{x}$ and t, as L is integrated along a path connecting two spacetime events $(\mathbf{x}_1, t_1), (\mathbf{x}_2, t_2)$, the arguments $(\mathbf{x}, t)$ of Λ in (6) are constrained by the path, i.e. $\Lambda = \Lambda(\mathbf{x}(t), t)$. This implies that the quantity

$$\frac{\partial \Lambda}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\Lambda = \frac{d}{dt}\Lambda \tag{15}$$

is a total time derivative. By part (a), the Lagrangians are equivalent.