1. **Prob 11.8**

   By (11.29), the frequency of the light as measured from the moving liquid satisfies

$$\omega' = \gamma(\omega \mp vk) \tag{1}$$

where k is the wave number of light in the liquid measured from the lab frame, i.e.,

$$k = \frac{n(\omega)\,\omega}{c} \tag{2}$$

and the $\mp$ sign is taken when the liquid is moving parallel or antiparallel with the light's direction of propagation. Up to first order of v, (1) yields

$$\omega' = \gamma\omega\left[1 \mp \frac{n(\omega)\,v}{c}\right] \approx \omega\left(1+\frac{v^2}{2c^2}\right)\left[1 \mp \frac{n(\omega)\,v}{c}\right] \approx \omega \mp \overbrace{\frac{n(\omega)\,v\omega}{c}}^{\Delta\omega} \tag{3}$$

With velocity addition law, the speed of light as measured in the lab frame is

$$c_{\text{lab}} = \frac{\pm v + u'}{1 \pm \dfrac{vu'}{c^2}} = \frac{\pm v + \dfrac{c}{n(\omega')}}{1 \pm \dfrac{v}{n(\omega')\,c}} \approx \left[\frac{c}{n(\omega')} \pm v\right]\left[1 \mp \frac{v}{n(\omega')\,c}\right] \approx \frac{c}{n(\omega')} \pm v \mp \frac{v}{n^2(\omega')} \tag{4}$$

Expanding $n(\omega'), n^2(\omega')$ around ω and up to first order of v, the above gives

$$\begin{aligned} c_{\text{lab}} &\approx c\left[\frac{1}{n(\omega)} - \frac{1}{n^2(\omega)}\frac{dn}{d\omega}\bigg|_{\omega}\Delta\omega\right] \pm v \mp \frac{v}{n^2(\omega)} \\ &= \frac{c}{n(\omega)} \pm v\left[1 - \frac{1}{n^2(\omega)} + \frac{\omega}{n(\omega)}\frac{dn}{d\omega}\bigg|_{\omega}\right] \end{aligned} \tag{5}$$

2. **Prob 11.9**

   (a) From the given transforms

$$x'^{\alpha} = \left(g^{\alpha\beta} + \epsilon^{\alpha\beta}\right)x_{\beta} \tag{6}$$
$$x^{\alpha} = \left(g^{\alpha\beta} + \epsilon'^{\alpha\beta}\right)x'_{\beta} \tag{7}$$

and the relation between contravariant and covariant vector

$$x_{\beta} = g_{\beta\gamma}x^{\gamma} \tag{8}$$

we have (plugging (8) into (6))

$$\begin{aligned} x'^{\alpha} &= \left(g^{\alpha\beta} + \epsilon^{\alpha\beta}\right)g_{\beta\gamma}x^{\gamma} && \text{expand } x^{\gamma} \text{ using (7)} \\ &= \left(g^{\alpha\beta} + \epsilon^{\alpha\beta}\right)g_{\beta\gamma}\left(g^{\gamma\delta} + \epsilon'^{\gamma\delta}\right)x'_{\delta} \\ &= \left(g^{\alpha\beta} + \epsilon^{\alpha\beta}\right)\left(\delta_{\beta}{}^{\delta} + g_{\beta\gamma}\epsilon'^{\gamma\delta}\right)x'_{\delta} && \text{ignore } O\left(\epsilon^2\right) \text{ for infinitesimal } \epsilon \\ &= g^{\alpha\beta}\delta_{\beta}{}^{\delta}x'_{\delta} + \epsilon^{\alpha\beta}\delta_{\beta}{}^{\delta}x'_{\delta} + g^{\alpha\beta}g_{\beta\gamma}\epsilon'^{\gamma\delta}x'_{\delta} \\ &= \underbrace{g^{\alpha\beta}x'_{\beta}}_{x'^{\alpha}} + \underbrace{\epsilon^{\alpha\beta}x'_{\beta} + \epsilon'^{\alpha\delta}x'_{\delta}}_{(\epsilon^{\alpha\beta}+\epsilon'^{\alpha\beta})x'_{\beta}} \end{aligned} \tag{9}$$

Thus for inversion to hold, we must require

$$\epsilon^{\alpha\beta} = -\epsilon'^{\alpha\beta} \tag{10}$$

(b) Writing the squared distance of the prime frame in terms of the unprimed frame, we get

$$\begin{aligned}
x'^{\alpha}x'_{\alpha} &= \left[\left(g^{\alpha\beta}+\epsilon^{\alpha\beta}\right)x_{\beta}\right]\left(g_{\alpha\delta}x'^{\delta}\right)\\
&= \left[\left(g^{\alpha\beta}+\epsilon^{\alpha\beta}\right)g_{\beta\gamma}x^{\gamma}\right]\left[g_{\alpha\delta}\left(g^{\delta\mu}+\epsilon^{\delta\mu}\right)x_{\mu}\right]\\
&= \left(\delta^{\alpha}{}_{\gamma}+\epsilon^{\alpha\beta}g_{\beta\gamma}\right)\left(\delta_{\alpha}{}^{\mu}+g_{\alpha\delta}\epsilon^{\delta\mu}\right)x^{\gamma}x_{\mu} && \text{ignore } O\left(\epsilon^{2}\right) \text{ for infinitesimal } \epsilon\\
&= x^{\alpha}x_{\alpha}+\epsilon^{\alpha\beta}\underbrace{g_{\beta\gamma}x^{\gamma}}_{x_{\beta}}x_{\alpha}+\epsilon^{\delta\mu}\underbrace{g_{\alpha\delta}x^{\alpha}}_{x_{\delta}}x_{\mu}\\
&= x^{\alpha}x_{\alpha}+2\epsilon^{\alpha\beta}x_{\alpha}x_{\beta} && (11)
\end{aligned}$$

If norm is preserved, the second term must vanish for any covariant vector x. Keep in mind that we are using repeated-index convention, the condition that the second term vanishes is equivalent to requiring ϵ being antisymmetric, i.e.,

$$\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} \tag{12}$$

This can be seen by, for example, setting $x=(0,1,0,0)$, the second term is $2\epsilon^{11}$, so all of the diagonal components of ϵ are clearly zero. Subsequently by setting, e.g., $x=(0,1,0,1)$, the second term yields $2\left(\epsilon^{13}+\epsilon^{31}+\epsilon^{11}+\epsilon^{33}\right)$, so we must have $\epsilon^{13}=-\epsilon^{31}$, and so on. Thus, condition (12) is necessary for the norm to be preserved.

(c) (6) can be rewritten as transform between contravariant vectors

$$x'^{\alpha} = \left(g^{\alpha\beta}+\epsilon^{\alpha\beta}\right)g_{\beta\gamma}x^{\gamma} = \left(\delta^{\alpha}{}_{\gamma}+\epsilon^{\alpha\beta}g_{\beta\gamma}\right)x^{\gamma} \tag{13}$$

Since $\epsilon^{\alpha\beta}$ is antisymmetric, we can write its matrix representation as

$$\epsilon = \begin{bmatrix} 0 & \zeta_1 & \zeta_2 & \zeta_3 \\ -\zeta_1 & 0 & -\omega_3 & \omega_2 \\ -\zeta_2 & \omega_3 & 0 & -\omega_1 \\ -\zeta_3 & -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{14}$$

with infinitesimal $\boldsymbol{\zeta}$ and $\boldsymbol{\omega}$, the matrix representation of the transform in (13) can be explicitly written out

$$\delta^{\alpha}{}_{\gamma}+\epsilon^{\alpha\beta}g_{\beta\gamma} = \begin{bmatrix} 1 & -\zeta_1 & -\zeta_2 & -\zeta_3 \\ -\zeta_1 & 1 & \omega_3 & -\omega_2 \\ -\zeta_2 & -\omega_3 & 1 & \omega_1 \\ -\zeta_3 & \omega_2 & -\omega_1 & 1 \end{bmatrix} = I - \boldsymbol{\zeta}\cdot\mathbf{K} - \boldsymbol{\omega}\cdot\mathbf{S} = I + L \approx e^{L} \tag{15}$$