1. **Prob 8.7**

(a) From the derivations in section 8.9, we know that

$$u_l(r) = Ar\,j_l(kr) + Brn_l(kr) \qquad \left.\frac{du_l}{dr}\right|_{r=a,b} = 0 \tag{1}$$

Thus the boundary condition requires

$$A\left[j_l(ka) + kaj_l'(ka)\right] + B\left[n_l(ka) + kan_l'(ka)\right] = 0 \tag{2}$$
$$A\left[j_l(kb) + kbj_l'(kb)\right] + B\left[n_l(kb) + kbn_l'(kb)\right] = 0 \tag{3}$$

which gives a trancedental equation for k:

$$\left[j_l(ka) + kaj_l'(ka)\right]\left[n_l(kb) + kbn_l'(kb)\right] = \left[j_l(kb) + kbj_l'(kb)\right]\left[n_l(ka) + kan_l'(ka)\right] \tag{4}$$

(b) For $l = 1$, we can write out the explicit forms of j_l, j_l', n_l, n_l':

$$j_l(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \tag{5}$$
$$j_l'(x) = \left(\frac{x^2\cos x - 2x\sin x}{x^4}\right) - \left(\frac{-x\sin x - \cos x}{x^2}\right) = \frac{2\cos x}{x^2} + \sin x\left(\frac{1}{x} - \frac{2}{x^3}\right) \tag{6}$$
$$n_l(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x} \tag{7}$$
$$n_l'(x) = -\left(\frac{-x^2\sin x - 2x\cos x}{x^4}\right) - \left(\frac{x\cos x - \sin x}{x^2}\right) = \frac{2\sin x}{x^2} - \cos x\left(\frac{1}{x} - \frac{2}{x^3}\right) \tag{8}$$

In anticipation of solving for (4), we also note

$$xj_l'(x) = \frac{2\cos x}{x} + \sin x\left(1 - \frac{2}{x^2}\right) \tag{9}$$
$$xn_l'(x) = \frac{2\sin x}{x} - \cos x\left(1 - \frac{2}{x^2}\right) \tag{10}$$

Denoting $\alpha = ka, \beta = kb$, the LHS of (4) becomes

$$\begin{aligned}
\mathrm{LHS}_{(4)} &= \left[\frac{\sin\alpha}{\alpha^2} - \frac{\cos\alpha}{\alpha} + \frac{2\cos\alpha}{\alpha} + \sin\alpha\left(1 - \frac{2}{\alpha^2}\right)\right]\left[-\frac{\cos\beta}{\beta^2} - \frac{\sin\beta}{\beta} + \frac{2\sin\beta}{\beta} - \cos\beta\left(1 - \frac{2}{\beta^2}\right)\right] \\
&= \left[\frac{\cos\alpha}{\alpha} + \sin\alpha\left(1 - \frac{1}{\alpha^2}\right)\right]\left[\frac{\sin\beta}{\beta} - \cos\beta\left(1 - \frac{1}{\beta^2}\right)\right] \\
&= \frac{\cos\alpha\sin\beta}{\alpha\beta} - \frac{\cos\alpha\cos\beta}{\alpha}\left(1 - \frac{1}{\beta^2}\right) + \frac{\sin\alpha\sin\beta}{\beta}\left(1 - \frac{1}{\alpha^2}\right) - \sin\alpha\cos\beta\left(1 - \frac{1}{\alpha^2}\right)\left(1 - \frac{1}{\beta^2}\right)
\end{aligned} \tag{11}$$

Exchanging $\alpha \longleftrightarrow \beta$ gives the RHS of (4), i.e.,

$$\mathrm{RHS}_{(4)} = \frac{\cos\beta\sin\alpha}{\alpha\beta} - \frac{\cos\beta\cos\alpha}{\beta}\left(1 - \frac{1}{\alpha^2}\right) + \frac{\sin\beta\sin\alpha}{\alpha}\left(1 - \frac{1}{\beta^2}\right) - \sin\beta\cos\alpha\left(1 - \frac{1}{\alpha^2}\right)\left(1 - \frac{1}{\beta^2}\right) \tag{12}$$

Equating (11) and (12) and rearranging terms, we get

$$\begin{aligned}
\frac{\sin(\alpha-\beta)}{\alpha\beta} + \sin(\alpha-\beta)\left(1 - \frac{1}{\alpha^2}\right)\left(1 - \frac{1}{\beta^2}\right) &= \frac{\cos(\alpha-\beta)}{\beta}\left(1 - \frac{1}{\alpha^2}\right) - \frac{\cos(\alpha-\beta)}{\alpha}\left(1 - \frac{1}{\beta^2}\right) && \Longrightarrow \\
\frac{\tan(\alpha-\beta)}{\alpha-\beta} &= \frac{\alpha\beta + 1}{\alpha\beta + (\alpha^2-1)(\beta^2-1)} && \Longrightarrow \\
\frac{\tan kh}{kh} &= \frac{k^2 + \frac{1}{ab}}{k^2 + ab\left(k^2 - \frac{1}{a^2}\right)\left(k^2 - \frac{1}{b^2}\right)}
\end{aligned} \tag{13}$$

(c) With $h = b - a \ll a$, and $k = \omega/c \approx \sqrt{l(l+1)}/a$, up to first order of h/a, the LHS of (13) is approximately unity, which gives an approximate solution of k for (13)

$$k^2 \approx \frac{a^2+b^2}{a^2b^2} \approx \frac{2}{a^2}\left(\frac{1+h/a}{1+2h/a}\right) \approx \frac{2}{a^2}\left(\frac{1}{1+h/a}\right) \qquad \Longrightarrow \qquad k \approx \frac{\sqrt{2}}{a+h/2} \tag{14}$$

agreeing with the statement below (8.105) for $l = 1$.

2. **Prob 8.8**

(a) At Schumann resonance, $u_l(r) = \text{const}$, so the fields are (setting unit to be 1 since it will not take effect due to the definition of Q, see below)

$$\mathbf{B} = \frac{1}{r} P_l^1(\cos\theta)\,\hat{\boldsymbol{\phi}} \qquad \mathbf{E} = -\frac{ic^2}{\omega}\frac{l(l+1)}{r^2} P_l(\cos\theta)\,\hat{\boldsymbol{\theta}} \tag{15}$$

and to the first order of h/a, the frequency is

$$\omega_l \approx \sqrt{l(l+1)}\left(\frac{c}{a+h/2}\right) \tag{16}$$

We go back to the definition to calculate Q, i.e.,

$$Q = \omega_l \frac{U}{P_{\text{loss}}} \tag{17}$$

where

$$U = \int_V \left(\frac{|\mathbf{B}|^2}{4\mu_0} + \frac{\epsilon_0 |E|^2}{4}\right) dv \tag{18}$$

$$P_{\text{loss}} = \frac{\mu_i \omega \delta_i}{4}\int_{R=b} |\mathbf{n}\times\mathbf{H}|^2\, da + \frac{\mu_e \omega \delta_e}{4}\int_{R=a} |\mathbf{n}\times\mathbf{H}|^2\, da \tag{19}$$

Noting the orthonormality for $P_l(x)$ and $P_l^m(x)$,

$$\int_{-1}^{1} [P_l(x)]^2\, dx = \frac{2}{2l+1} \qquad \int_{-1}^{1} \left[P_l^m(x)\right] dx = \frac{(l+m)!}{(l-m)!}\frac{2}{2l+1} \tag{20}$$

we can calculate the stored energy U,

$$\begin{aligned} U &= 2\pi \int_0^{\pi} \sin\theta\, d\theta \int_a^b r^2 dr \left\{\frac{1}{4\mu_0}\cdot\frac{1}{r^2}\left[P_l^1(\cos\theta)\right]^2 + \frac{\epsilon_0 c^4}{4\omega^2}\cdot\frac{[l(l+1)]^2}{r^4}[P_l(\cos\theta)]^2\right\} \\ &= \frac{2\pi(b-a)}{4\mu_0}\cdot\frac{2l(l+1)}{2l+1} + \frac{2\pi\epsilon_0 c^4}{4\omega^2}\left(\frac{1}{a}-\frac{1}{b}\right)[l(l+1)]^2\cdot\frac{2}{2l+1} \\ &= \frac{\pi h l(l+1)}{2l+1}\left[\frac{1}{\mu_0} + \frac{\epsilon_0 c^4}{\omega^2}\frac{l(l+1)}{ab}\right] \end{aligned} \tag{21}$$

At this point, because of the h factor, we can treat the square bracket to the $O\left(h^0\right)$ order, with which $\omega \approx \sqrt{l(l+1)}c/a$ and $b \approx a$, this gives

$$U \approx \frac{2\pi h l(l+1)}{(2l+1)\mu_0} \tag{22}$$

For P_{loss}, notice at $R = a$ or $R = b$,

$$\int_R |\mathbf{n}\times\mathbf{H}|^2\, da = 2\pi R^2 \int_0^{\pi} \sin\theta\, d\theta \cdot \frac{1}{\mu_0^2 R^2}\left[P_l^1(\cos\theta)\right]^2 = \frac{4\pi l(l+1)}{(2l+1)\mu_0^2} \tag{23}$$

With the approximation $\mu_e \approx \mu_i \approx \mu_0$, we have

$$P_{\text{loss}} \approx \frac{\omega(\delta_i+\delta_e)}{4}\cdot\frac{4\pi l(l+1)}{(2l+1)\mu_0} \tag{24}$$

Putting (22) and (24) into (17) yields

$$Q = \frac{2h}{\delta_e + \delta_i} \tag{25}$$

(b) With $\nu = 10.6\text{Hz}$, we have $\omega = 2\pi\nu = 66.57\text{Hz}$, then $\delta_e = \sqrt{2/\mu_e\omega\sigma_e} \approx 488.31\text{m}$, $\delta_i = \sqrt{2/\mu_i\omega\sigma_i} \approx 48831\text{m}$. Then with $h = 10^5\text{m}$, the Q-factor is $Q = 2h/(\delta_e+\delta_i) \approx 4.05$.

(c) The center assumption of the approximation is treating ionosphere as an excellent conductor. The rough measure of how good a conductor is can be seen from equation (7.57), where a perfect conductor should have infinite imaginary part. Thus we calculate

$$\frac{\sigma}{\epsilon_0\omega} \approx 1.7\times 10^4 \tag{26}$$

to measure how good of a conductor the ionosphere is for the frequency ω, which is a very good approximation for this extremely low frequency.