1. The two dipoles described in the problem statement can be viewed as the limiting case of the diagram above, where we will eventually take $d \to 0$ while keeping $2qd = p$ constant. With finite d, in spherical coordinates, the charge density and current density are (we use η for charge density since ρ will be used to denote the radial component of cylindrical coordinates later)

$$\eta(\mathbf{x},t) = q\frac{\delta(r-R)}{R^2}\overbrace{[\delta(\phi-\omega t)-\delta(\phi-\omega t-\pi)]}^{f(\phi,t)}\left[\frac{\delta(\theta-\gamma)-\delta(\theta+\gamma-\pi)}{\sin\theta}\right] \tag{1}$$

$$\mathbf{J}(\mathbf{x},t) = \hat{\boldsymbol{\phi}}\omega r\sin\theta\,\eta(\mathbf{x},t) = \hat{\boldsymbol{\phi}}q\omega\frac{r\delta(r-R)}{R^2}\underbrace{[\delta(\phi-\omega t)-\delta(\phi-\omega t-\pi)]}_{f(\phi,t)}[\delta(\theta-\gamma)-\delta(\theta+\gamma-\pi)] \tag{2}$$

where $R = \sqrt{a^2+d^2}$ and $\gamma = \cos^{-1}(d/R)$ is the polar angle of the upper loop.

The Fourier component of the n-th harmonic of $f(\phi,t)$ is

$$f_n(\phi) = \frac{1}{T}\int_0^T \delta(\phi-\omega t)\overbrace{e^{in\omega t}}^{e^{in\phi}/\omega}dt - \frac{1}{T}\int_0^T \delta(\phi-\omega t-\pi)\overbrace{e^{in\omega t}}^{e^{in(\phi-\pi)}/\omega}dt$$

$$= \frac{1}{2\pi}e^{in\phi}[1-(-1)^n] = \begin{cases} \dfrac{1}{\pi}e^{in\phi} & \text{for odd } n \\[2mm] 0 & \text{for even } n \end{cases} \tag{3}$$

which means only odd-numbered harmonics can exist.

The following will assume n to be odd. The n-th harmonic of the charge density and current density are

$$\eta_n(\mathbf{x}) = \frac{q}{\pi}\frac{\delta(r-R)}{R^2}\left[\frac{\delta(\theta-\gamma)-\delta(\theta+\gamma-\pi)}{\sin\theta}\right]e^{in\phi} \tag{4}$$

$$\mathbf{J}_n(\mathbf{x}) = \hat{\boldsymbol{\phi}}\frac{q\omega}{\pi}\frac{r\delta(r-R)}{R^2}[\delta(\theta-\gamma)-\delta(\theta+\gamma-\pi)]e^{in\phi} \tag{5}$$

(a) The electric multipole due to the n-th harmonic is

$$q_{lm}^{(n)} = \int r^l Y_{lm}^*(\theta,\phi)\eta_n(\mathbf{x})\,d^3x$$

$$= \frac{q}{\pi}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\delta_{mn}2\pi\int_0^\infty r^{l+2}\frac{\delta(r-R)}{R^2}dr\int_0^\pi[\delta(\theta-\gamma)-\delta(\theta+\gamma-\pi)]P_l^m(\cos\theta)\,d\theta$$

$$= \delta_{mn}2qR^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\left[\overbrace{P_l^m(\cos\gamma)-P_l^m(-\cos\gamma)}^{\alpha}\right] \tag{6}$$

As $d\to 0$, $R = \sqrt{a^2+d^2} \to a + O(d^2)$, $\cos\gamma = d/R \to d/a + O(d^2)$, and

$$\alpha \to 2\frac{dP_l^m(x)}{dx}\bigg|_{x=0}\cdot\cos\gamma = 2\frac{dP_l^m(x)}{dx}\bigg|_{x=0}\left[\frac{d}{a}+O(d^2)\right] \tag{7}$$

Finally when we let $d \to 0$ while keeping $2qd = p$ constant, the electric multipole moment becomes

$$q_{lm}^{(n)} \to \delta_{mn} 2 (2qd) a^{l-1} \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \left. \frac{dP_l^m(x)}{dx} \right|_{x=0}$$

$$= \delta_{mn} 2p a^{l-1} \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} (l+m) P_{l-1}^m(0) \tag{8}$$

where in the last step we have used the recurrence relation (See 14.10.E5 on DLMF)

$$(1-x^2) \frac{dP_l^m(x)}{dx} = (l+m) P_{l-1}^m(x) - lx P_l^m(x) \tag{9}$$

$P_{l-1}^m(0)$ is non-zero only when $l - m$ is odd and $l \geq |m|$. The lowest of which is $l - 1 = |m| = |n|$. The lowest order electric multipole occurs at $l = 2$ and $m = n = \pm 1$, i.e.,

$$q_{21}^{(1)} = 2pa \sqrt{\frac{5}{4\pi} \cdot \frac{1}{6}} \cdot 3P_1^1(0) = -\sqrt{\frac{15}{2\pi}} pa \qquad q_{2,-1}^{(-1)} = 2pa \sqrt{\frac{5}{4\pi}} \cdot 6 \cdot P_1^{-1}(0) = \sqrt{\frac{15}{2\pi}} pa \tag{10}$$

(b) For the n-th harmonic, the magnetic multipole moments is given by

$$M_{lm}^{(n)} = -\frac{1}{l+1} \int r^l Y_{lm}^*(\theta, \phi) \, \boldsymbol{\nabla} \cdot [\mathbf{x} \times \mathbf{J}_n(\mathbf{x})] d^3 x \tag{11}$$

From (5), we have

$$\mathbf{x} \times \mathbf{J}_n(\mathbf{x}) = -\hat{\boldsymbol{\theta}} \frac{q\omega}{\pi} \frac{r^2 \delta(r-R)}{R^2} e^{in\phi} \overbrace{[\delta(\theta - \gamma) - \delta(\theta + \gamma - \pi)]}^{g(\theta)} \tag{12}$$

$$\boldsymbol{\nabla} \cdot [\mathbf{x} \times \mathbf{J}_n(\mathbf{x})] = -\frac{q\omega}{\pi} \frac{r^2 \delta(r-R)}{R^2} e^{in\phi} \frac{1}{r \sin\theta} \frac{d[g(\theta) \sin\theta]}{d\theta}$$

$$= -\frac{q\omega}{\pi} \frac{r \delta(r-R)}{R^2} e^{in\phi} \left[g'(\theta) + g(\theta) \cot\theta \right] \tag{13}$$

Now (11) becomes

$$M_{lm}^{(n)} = \frac{1}{l+1} \frac{q\omega R^{l+1}}{\pi} \delta_{mn} 2\pi \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} (G+H) \tag{14}$$

where

$$G = \int_0^\pi P_l^m(\cos\theta) g'(\theta) \sin\theta \, d\theta = \int_0^\pi P_l^m(\cos\theta) \left[\delta'(\theta - \gamma) - \delta'(\theta + \gamma - \pi) \right] \sin\theta \, d\theta$$

$$= -\left\{ \left. \frac{d[P_l^m(\cos\theta) \sin\theta]}{d\theta} \right|_{\theta=\gamma} - \left. \frac{d[P_l^m(\cos\theta) \sin\theta]}{d\theta} \right|_{\theta=\pi-\gamma} \right\}$$

$$= -\left\{ -\left. \frac{dP_l^m(x)}{dx} \right|_{x=\cos\gamma} \cdot \sin^2\gamma + P_l^m(\cos\gamma) \cos\gamma + \left. \frac{dP_l^m(x)}{dx} \right|_{x=-\cos\gamma} \cdot \sin^2\gamma - P_l^m(-\cos\gamma)(-\cos\gamma) \right\} \tag{15}$$

$$H = \int_0^\pi P_l^m(\cos\theta) g(\theta) \cot\theta \sin\theta \, d\theta = \int_0^\pi P_l^m(\cos\theta) \cos\theta \left[\delta(\theta - \gamma) - \delta(\theta + \gamma - \pi) \right] d\theta$$

$$= P_l^m(\cos\gamma) \cos\gamma - P_l^m(-\cos\gamma)(-\cos\gamma) \tag{16}$$

After canceling out H in part of G, we have

$$M_{lm}^{(n)} = \delta_{mn} \left(\frac{1}{l+1} \right) 2q\omega R^{l+1} \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \overbrace{\left[\left. \frac{dP_l^m(x)}{dx} \right|_{x=\cos\gamma} - \left. \frac{dP_l^m(x)}{dx} \right|_{x=-\cos\gamma} \right]}^{\beta} \sin^2\gamma \tag{17}$$

As usual, when $d \to 0$, $\sin^2\gamma \to 1 - O(d^2)$ and

$$\beta \to 2 \left. \frac{d^2 P_l^m(x)}{dx^2} \right|_{x=0} \cdot \cos\gamma = 2 \left. \frac{d^2 P_l^m(x)}{dx^2} \right|_{x=0} \left[\frac{d}{a} + O(d^2) \right] \tag{18}$$

which gives

$$M_{lm}^{(n)} \rightarrow \delta_{mn} \left(\frac{2}{l+1}\right)(2qd)\,\omega a^l \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\left.\frac{d^2 P_l^m(x)}{dx^2}\right|_{x=0}$$

$$= \delta_{mn}\left(\frac{2}{l+1}\right)p\omega a^l \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\left[m^2 - l(l+1)\right]P_l^m(0) \tag{19}$$

where we have used the differential equation for the associated Legendre function

$$\left(1-x^2\right)\frac{d^2 P_l^m(x)}{dx^2} - 2x\frac{dP_l^m(x)}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m(x) = 0 \tag{20}$$

Parity of $P_l^m(x)$ requires $l-m$ to be even for $M_{lm}^{(n)}$ to be non-zero. The $n = \pm 1$ harmonics have $l = 1, m = \pm 1$ as the lowest non-zero magnetic multipole moment, i.e.,

$$M_{1,\pm 1}^{(1)} = \pm\sqrt{\frac{3}{8\pi}}\,pa\omega \tag{21}$$

2. From problem 6.21, for a dipole $\mathbf{p}$ moving along the trajectory $\mathbf{r}(t)$, the charge density and current density are

$$\eta(\mathbf{x}, t) = -(\mathbf{p}\cdot\nabla)\,\delta\left[\mathbf{x} - \mathbf{r}(t)\right] \tag{22}$$

$$\mathbf{J}(\mathbf{x}, t) = -\frac{d\mathbf{r}}{dt}(\mathbf{p}\cdot\nabla)\,\delta\left[\mathbf{x} - \mathbf{r}(t)\right] \tag{23}$$

Let subscript $+/-$ represent dipole whose initial position is at $(\pm a, 0, 0)$ respectively, then in cylindrical coordinates

$$(\mathbf{p}_+\cdot\nabla)\,\delta\left[\mathbf{x} - \mathbf{r}_+(t)\right] = p\frac{\partial}{\partial z}\left[\frac{\delta(\rho - a)}{a}\delta(z)\,\delta(\phi - \omega t)\right] = p\frac{\delta(\rho - a)}{a}\delta'(z)\,\delta(\phi - \omega t) \tag{24}$$

$$(\mathbf{p}_-\cdot\nabla)\,\delta\left[\mathbf{x} - \mathbf{r}_-(t)\right] = -p\frac{\partial}{\partial z}\left[\frac{\delta(\rho - a)}{a}\delta(z)\,\delta(\phi - \omega t - \pi)\right] = -p\frac{\delta(\rho - a)}{a}\delta'(z)\,\delta(\phi - \omega t - \pi) \tag{25}$$

With both dipoles considered, the charge density and current density are

$$\eta(\mathbf{x}, t) = -p\frac{\delta(\rho - a)}{a}\delta'(z)\left[\delta(\phi - \omega t) - \delta(\phi - \omega t - \pi)\right] \tag{26}$$

$$\mathbf{J}(\mathbf{x}, t) = \hat{\boldsymbol{\phi}}\,\omega\rho\cdot\eta(\mathbf{x}, t) = -\hat{\boldsymbol{\phi}}\,\omega p\frac{\rho\,\delta(\rho - a)}{a}\delta'(z)\left[\delta(\phi - \omega t) - \delta(\phi - \omega t - \pi)\right] \tag{27}$$

The Fourier transform (3) gives us the n-th harmonic of the charge density and current density

$$\eta_n(\mathbf{x}) = -\frac{p}{\pi}\frac{\delta(\rho - a)}{a}\delta'(z)\,e^{in\phi} \qquad \mathbf{J}_n(\mathbf{x}) = -\hat{\boldsymbol{\phi}}\frac{\omega p}{\pi}\frac{\rho\,\delta(\rho - a)}{a}\delta'(z)\,e^{in\phi} \qquad \text{for odd } n \tag{28}$$

The n-th harmonic's contribution to the vector potential in the radiation zone $(kr \gg 1)$ is given by (9.8)

$$\mathbf{A}_n(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ik_n r}}{r}\int \mathbf{J}_n(\mathbf{x}')\,e^{-ik_n\mathbf{n}\cdot\mathbf{x}'}\,d^3x' \qquad k_n = \frac{n\omega}{c} \tag{29}$$

Let the unit vector from the origin to the observation point be $\mathbf{n} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$, then

$$\mathbf{n}\cdot\mathbf{x}' = \rho'\cos\phi'\sin\theta\cos\phi + \rho'\sin\phi'\sin\theta\sin\phi + z'\cos\theta = \rho'\sin\theta\cos(\phi' - \phi) + z'\cos\theta \tag{30}$$

hence

$$\mathbf{A}_n(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ik_n r}}{r}\frac{\omega p}{\pi}\int\left(-\hat{\boldsymbol{\phi}}\right)\frac{\rho'\delta(\rho' - a)}{a}\delta'(z')\,e^{in\phi'}e^{-ik_n\rho'\sin\theta\cos(\phi' - \phi)}e^{-ik_n z'\cos\theta}\,d^3x'$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ik_n r}}{r}\frac{\omega pa}{\pi}\int_0^{2\pi}\left(\sin\phi'\,\hat{\mathbf{x}} - \cos\phi'\,\hat{\mathbf{y}}\right)e^{in\phi'}e^{-ik_n a\sin\theta\cos(\phi' - \phi)}\,d\phi'\overbrace{\int_{-\infty}^{\infty}\delta'(z')\,e^{-ik_n z'\cos\theta}\,dz}^{ik_n\cos\theta}$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ik_n r}}{r}\frac{\omega pa}{\pi}ik_n\cos\theta\,(X\hat{\mathbf{x}} + Y\hat{\mathbf{y}}) \tag{31}$$

where

$$X = \int_0^{2\pi} \sin\phi' e^{in\phi'} e^{-ik_n a \sin\theta \cos(\phi'-\phi)} d\phi' \qquad Y = -\int_0^{2\pi} \cos\phi' e^{in\phi'} e^{-ik_n a \sin\theta \cos(\phi'-\phi)} d\phi' \qquad (32)$$

Define $\xi \equiv \phi' - \phi$ and $\beta \equiv k_n a \sin\theta$ then

$$X = e^{in\phi} \int_0^{2\pi} \sin(\xi+\phi) e^{in\xi} e^{-i\beta\cos\xi} d\xi$$

$$= e^{in\phi} \int_0^{2\pi} (\sin\xi\cos\phi + \cos\xi\sin\phi)(\cos n\xi + i\sin n\xi) e^{-i\beta\cos\xi} d\xi \qquad \text{terms odd in } \xi \text{ can be dropped}$$

$$= e^{in\phi} \int_0^{2\pi} (\sin\phi\cos\xi\cos n\xi + i\cos\phi\sin\xi\sin n\xi) e^{-i\beta\cos\xi} d\xi$$

$$= e^{in\phi} \int_0^{2\pi} \left\{ \frac{-ie^{i\phi}}{2} \cos[(n+1)\xi] + \frac{ie^{-i\phi}}{2} \cos[(n-1)\xi] \right\} e^{-i\beta\cos\xi} d\xi$$

$$= \frac{i}{2} \left\{ e^{i(n-1)\phi} \int_0^{2\pi} \cos[(n-1)\xi] e^{-i\beta\cos\xi} d\xi - e^{i(n+1)\phi} \int_0^{2\pi} \cos[(n+1)\xi] e^{-i\beta\cos\xi} d\xi \right\}$$

$$= \pi i \left[\underbrace{i^{n-1} e^{i(n-1)\phi} J_{n-1}(\beta)}_{K_{n-1}} - \underbrace{i^{n+1} e^{i(n+1)\phi} J_{n+1}(\beta)}_{K_{n+1}} \right] \qquad (33)$$

where we have used DLMF 10.9.E2

$$J_n(z) = \frac{i^{-n}}{\pi} \int_0^{\pi} e^{iz\cos\alpha} \cos(n\alpha) d\alpha \qquad (34)$$

and the fact that n is odd.
Similarly,

$$Y = -e^{in\phi} \int_0^{2\pi} \cos(\xi+\phi) e^{in\xi} e^{-i\beta\cos\xi} d\xi$$

$$= -e^{in\phi} \int_0^{2\pi} (\cos\xi\cos\phi - \sin\xi\sin\phi)(\cos n\xi + i\sin n\xi) e^{-i\beta\cos\xi} d\xi$$

$$= -e^{in\phi} \int_0^{2\pi} (\cos\phi\cos\xi\cos n\xi - i\sin\phi\sin\xi\sin n\xi) e^{-i\beta\cos\xi} d\xi$$

$$= -e^{in\phi} \int_0^{2\pi} \left\{ \frac{e^{i\phi}}{2} \cos[(n+1)\xi] + \frac{e^{-i\phi}}{2} \cos[(n-1)\xi] \right\} e^{-i\beta\cos\xi} d\xi$$

$$= -\frac{1}{2} \left\{ e^{i(n-1)\phi} \int_0^{2\pi} \cos[(n-1)\xi] e^{-i\beta\cos\xi} d\xi + e^{i(n+1)\phi} \int_0^{2\pi} \cos[(n+1)\xi] e^{-i\beta\cos\xi} d\xi \right\}$$

$$= -\pi \left[i^{n-1} e^{i(n-1)\phi} J_{n-1}(\beta) + i^{n+1} e^{i(n+1)\phi} J_{n+1}(\beta) \right]$$

$$= -\pi (K_{n-1} + K_{n+1}) \qquad (35)$$

Putting X and Y back into (31) gives

$$\mathbf{A}_n(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{e^{ik_n r}}{r} \frac{\omega p a}{\pi} i k_n \cos\theta \, \pi i \left[(K_{n-1} - K_{n+1})\hat{\mathbf{x}} + i(K_{n-1} + K_{n+1})\hat{\mathbf{y}} \right] \qquad \text{recall } \omega = \frac{ck_n}{n}$$

$$= -\frac{\mu_0}{4\pi} \frac{e^{ik_n r}}{r} \frac{cpak_n^2}{n} \cos\theta \left[(K_{n-1} - K_{n+1})\hat{\mathbf{x}} + i(K_{n-1} + K_{n+1})\hat{\mathbf{y}} \right] \qquad (36)$$

The radiation zone magnetic field due to the n-th harmonic is thus

$$
\begin{aligned}
\mathbf{H}_n(\mathbf{x}) &= \frac{ik_n}{\mu_0}\mathbf{n}\times\mathbf{A}_n(\mathbf{x})\\[6pt]
&= -\frac{ik_n}{\mu_0}\frac{\mu_0}{4\pi}\frac{e^{ik_nr}}{r}\frac{cpak_n^2}{n}\cos\theta\,(\sin\theta\cos\phi\,\hat{\mathbf{x}}+\sin\theta\sin\phi\,\hat{\mathbf{y}}+\cos\theta\,\hat{\mathbf{z}})\times[(K_{n-1}-K_{n+1})\hat{\mathbf{x}}+i\,(K_{n-1}+K_{n+1})\hat{\mathbf{y}}]\\[6pt]
&= -i\frac{cpa}{4\pi}\frac{e^{ik_nr}}{r}\frac{k_n^3}{n}\cos\theta\Big\{-i\cos\theta\,(K_{n-1}+K_{n+1})\hat{\mathbf{x}}+\cos\theta\,(K_{n-1}-K_{n+1})\hat{\mathbf{y}}\\[6pt]
&\qquad\qquad +i\sin\theta\,[\cos\phi\,(K_{n-1}+K_{n+1})+i\sin\phi\,(K_{n-1}-K_{n+1})]\hat{\mathbf{z}}\Big\}\\[6pt]
&= -\frac{cpa}{4\pi}\frac{e^{ik_nr}}{r}\frac{k_n^3}{n}\cos\theta\Big\{\cos\theta\,(K_{n-1}+K_{n+1})\hat{\mathbf{x}}+i\cos\theta\,(K_{n-1}-K_{n+1})\hat{\mathbf{y}}\\[6pt]
&\qquad\qquad -\sin\theta\,[\cos\phi\,(K_{n-1}+K_{n+1})+i\sin\phi\,(K_{n-1}-K_{n+1})]\hat{\mathbf{z}}\Big\}
\end{aligned}
\tag{37}
$$

Note (37) is obtained with only the radiation zone approximation $kr\gg1$, in particular it does not assume $ka\ll1$ yet. If we now invoke the nonrelativistic assumption $|k_na|\ll1$, therefore $|\beta|=|k_na\sin\theta|\ll1$, then we will have

$$
|K_{n-1}|\gg|K_{n+1}| \text{ for } n=1,3,5,\cdots \qquad\text{and}\qquad |K_{n+1}|\gg|K_{n-1}| \text{ for } n=-1,-3,-5,\cdots
\tag{38}
$$

Now consider the dominating harmonics $n=\pm1$, where only $K_0=1$ is retained in (37). With $k\equiv k_1=\omega/c=-k_{-1}$, we have

$$
\mathbf{H}_1(\mathbf{x},t) = -\frac{cpa}{4\pi}\frac{e^{ikr}}{r}k^3\cos\theta\left[\cos\theta\,(\hat{\mathbf{x}}+i\hat{\mathbf{y}})-\sin\theta e^{i\phi}\hat{\mathbf{z}}\right]e^{-i\omega t}
\tag{39}
$$

$$
\mathbf{H}_{-1}(\mathbf{x},t) = -\frac{cpa}{4\pi}\frac{e^{-ikr}}{r}k^3\cos\theta\left[\cos\theta\,(\hat{\mathbf{x}}-i\hat{\mathbf{y}})-\sin\theta e^{-i\phi}\hat{\mathbf{z}}\right]e^{i\omega t}
\tag{40}
$$

$\mathbf{H}_1$ and $\mathbf{H}_{-1}$ are complex conjugate of each other, therefore the combined physical field is twice the real part of each. Up to a global phase factor, it's justifiable to write the complex magnetic field for the base frequency ω as

$$
\mathbf{H}(\mathbf{x}) = \frac{cpa}{2\pi}\frac{e^{ikr}}{r}k^3\cos\theta\left[\cos\theta\,(\hat{\mathbf{x}}+i\hat{\mathbf{y}})-\sin\theta e^{i\phi}\hat{\mathbf{z}}\right]
\tag{41}
$$

3. In the radiation zone, the angular distribution of time-averaged radiated power is

$$
\begin{aligned}
\left\langle\frac{dP}{d\Omega}\right\rangle &= \frac{1}{2}\mathrm{Re}\left[r^2\mathbf{n}\cdot(\mathbf{E}\times\mathbf{H}^*)\right] \qquad\qquad\qquad \mathbf{E}=Z_0\mathbf{n}\times\mathbf{H}\\[6pt]
&= \frac{Z_0}{2}r^2\mathbf{H}\cdot\mathbf{H}^*\\[6pt]
&= \frac{Z_0}{2}\left(\frac{cpa}{2\pi}k^3\right)^2\cos^2\theta\left(1+\cos^2\theta\right)
\end{aligned}
\tag{42}
$$

Integrating over all solid angles gives the total time-averaged radiated power

$$
P = \frac{Z_0}{2}\left(\frac{cpa}{2\pi}k^3\right)^2 2\pi\int_0^\pi\left(\cos^4\theta+\cos^2\theta\right)\sin\theta\,d\theta = \frac{4}{15\pi\epsilon_0}ck^6p^2a^2
\tag{43}
$$