Note: the word "exact" in both problem 9.16 and 9.17 should be interpreted as "exact after taking the far-zone approximation but without taking the long wavelength approximation."

1. **Prob 9.16**

(a) Let the antenna be along the z direction. The current density is thus

$$\mathbf{J}(\mathbf{x}) = \hat{\mathbf{z}} I_0 \sin(kz)\,\delta(x)\,\delta(y) \qquad \text{where } k = \frac{2\pi}{d} \text{ and } |z| \le \frac{d}{2} \tag{1}$$

By (9.8), the vector potential is

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \mathbf{J}(\mathbf{x}')\,e^{-ik\mathbf{n}\cdot\mathbf{x}'}\,d^3x' = \hat{\mathbf{z}}\frac{I_0\mu_0}{4\pi}\frac{e^{ikr}}{r}\underbrace{\int_{-d/2}^{d/2}\sin(kz')\,e^{-ik\cos\theta z'}\,dz'}_{I} \tag{2}$$

where

$$\begin{aligned}
I &= -\frac{1}{k}\cos(kz')\,e^{-ikz'\cos\theta}\Big|_{-d/2}^{d/2} - \int_{-d/2}^{d/2}\left[-\frac{1}{k}\cos(kz')(-ik\cos\theta)\,e^{-ikz'\cos\theta}\right]dz' \\
&= \frac{1}{k}\left(e^{-i\pi\cos\theta} - e^{i\pi\cos\theta}\right) - i\cos\theta\int_{-d/2}^{d/2}\cos(kz')\,e^{-ikz'\cos\theta}\,dz' \\
&= -\frac{2i}{k}\sin(\pi\cos\theta) - i\cos\theta\left\{\underbrace{\frac{1}{k}\sin(kz')\,e^{-ikz'\cos\theta}\Big|_{-d/2}^{d/2}}_{0} - \int_{-d/2}^{d/2}\left[\frac{1}{k}\sin(kz')(-ik\cos\theta)\,e^{-ikz'\cos\theta}\right]dz'\right\} \\
&= -\frac{2i}{k}\sin(\pi\cos\theta) + \cos^2\theta\cdot I
\end{aligned} \tag{3}$$

Therefore

$$I = -\frac{2i\sin(\pi\cos\theta)}{k\sin^2\theta} \tag{4}$$

and

$$\mathbf{A}(\mathbf{x}) = -\hat{\mathbf{z}}\frac{iI_0\mu_0}{2\pi k}\frac{e^{ikr}}{r}\frac{\sin(\pi\cos\theta)}{\sin^2\theta} \tag{5}$$

The magnetic field is given by

$$\begin{aligned}
\mathbf{H}(\mathbf{x}) &= \frac{ik}{\mu_0}\mathbf{n}\times\mathbf{A}(\mathbf{x}) \\
&= \frac{I_0}{2\pi}\frac{e^{ikr}}{r}\frac{\sin(\pi\cos\theta)}{\sin^2\theta}(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}})\times\hat{\mathbf{z}} \\
&= -\frac{I_0}{2\pi}\frac{e^{ikr}}{r}\frac{\sin(\pi\cos\theta)}{\sin\theta}\hat{\boldsymbol{\phi}}
\end{aligned} \tag{6}$$

The power per unit solid angle is

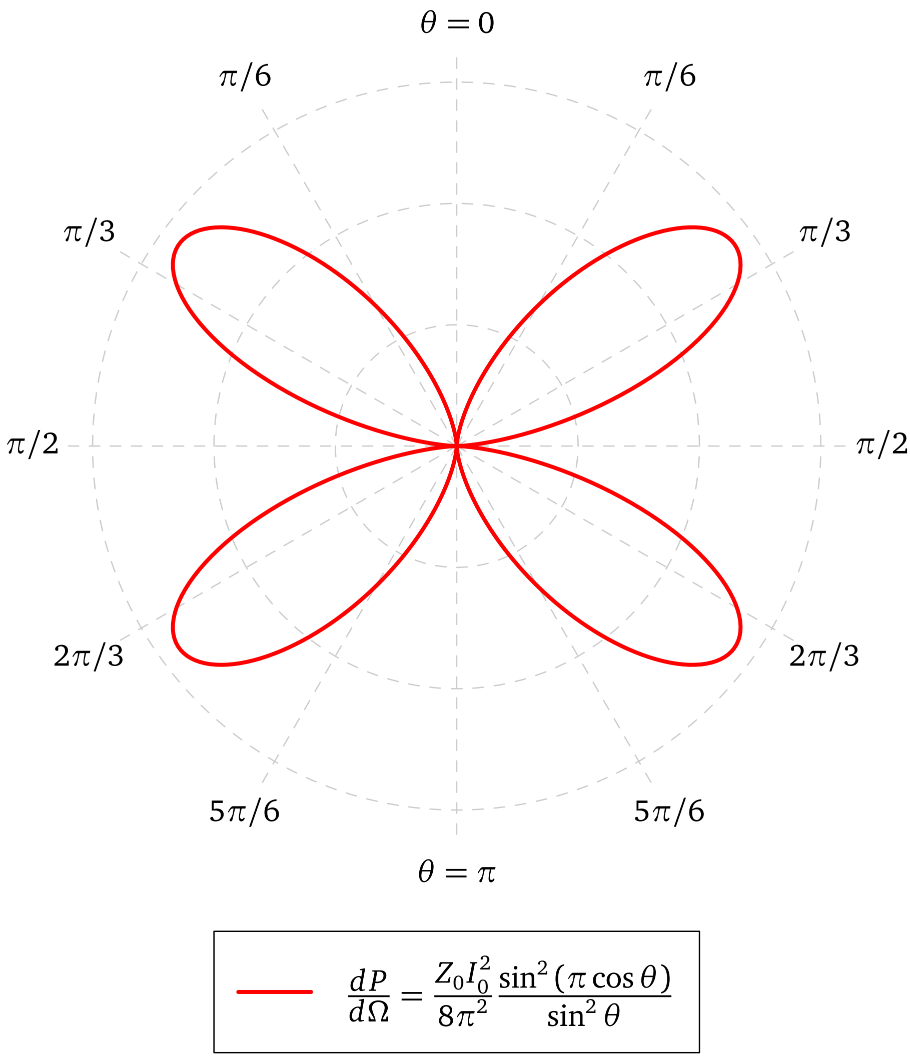
$$\frac{dP}{d\Omega} = \frac{Z_0}{2}r^2\mathbf{H}\cdot\mathbf{H}^* = \frac{Z_0 I_0^2}{8\pi^2}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta} \tag{7}$$

of which the polar plot is shown below.
The total power radiated is

$$P = \frac{Z_0 I_0^2}{8\pi^2}\cdot 2\pi\int_{-1}^{1}\frac{\sin^2(\pi\cos\theta)}{1-\cos^2\theta}\,d(\cos\theta) \tag{8}$$

The integral can be evaluated numerically. The result is 1.5572, which gives the radiation resistance

$$R_{\text{rad}} = \frac{2P}{I_0^2} \approx \frac{Z_0}{2\pi}\times 1.5572 = 0.248 Z_0 \approx 93.36\Omega \tag{9}$$



2. **Prob 9.17**

   (a) With the current density given by (1) (assumed to have harmonic time dependency $e^{-i\omega t}$), we can find the charge density via charge conservation:

$$\frac{\partial \rho(\mathbf{x}, t)}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J}(\mathbf{x}, t) = 0 \implies \rho(\mathbf{x}) = \frac{1}{i\omega} \boldsymbol{\nabla} \cdot \mathbf{J}(\mathbf{x}) = \frac{k I_0}{i\omega} \cos(kz)\,\delta(x)\,\delta(y) = \frac{I_0}{ic} \cos(kz)\,\delta(x)\,\delta(y) \quad (10)$$

When calculating multipoles, it is most convenient to express this in spherical coordinates,

$$\rho(\mathbf{x}) = \frac{I_0}{ic} \frac{1}{2\pi} \frac{\cos(kr\cos\theta)}{r^2} \left[\frac{\delta(\theta) + \delta(\theta - \pi)}{\sin\theta} \right] \qquad \text{where } r \le \frac{d}{2} \quad (11)$$

Similarly for current density

$$\mathbf{J}(\mathbf{x}) = \hat{\mathbf{z}} \frac{I_0}{2\pi} \frac{\sin(kr\cos\theta)}{r^2} \left[\frac{\delta(\theta) + \delta(\theta - \pi)}{\sin\theta} \right] \qquad \text{where } r \le \frac{d}{2} \quad (12)$$

Then the electric multipole moment is

$$\begin{aligned} q_{lm} &= \int r^l Y_{lm}^*(\theta, \phi) \rho(\mathbf{x})\, d^3x = \frac{I_0}{ic} \frac{1}{2\pi} \int r^l Y_{lm}^*(\theta, \phi) \frac{\cos(kr\cos\theta)}{r^2} \left[\frac{\delta(\theta) + \delta(\theta - \pi)}{\sin\theta} \right] r^2 dr \sin\theta\, d\theta\, d\phi \\ &= \frac{I_0}{ic} \frac{1}{2\pi} \delta_{m0} 2\pi \sqrt{\frac{2l+1}{4\pi}} \int_0^{d/2} r^l dr \int_0^{\pi} P_l(\cos\theta) \cos(kr\cos\theta) [\delta(\theta) + \delta(\theta - \pi)]\, d\theta \\ &= \delta_{m0} \frac{I_0}{ic} \sqrt{\frac{2l+1}{4\pi}} \left[1 + (-1)^l\right] \int_0^{d/2} r^l \cos(kr)\, dr \end{aligned} \quad (13)$$

It is clear that only even l contributes. The integral for an arbitrary even value of l can be calculated recursively from $l=0$ with increments of 2 (or see Wikipedia). The monopole, dipole and quadrupole moments are

$$q_{00}=0 \qquad q_{10}=0 \qquad q_{20}=\frac{I_0}{ic}\sqrt{\frac{5}{\pi}}\frac{d}{k^2} \tag{14}$$

Note the quadrupole moment is expressed in spherical tensor form. To convert it to Cartesian form, we can use (4.6) and get

$$Q_{33}=2\sqrt{\frac{4\pi}{5}}q_{20}=\frac{4I_0d}{ick^2}=\frac{8\pi I_0}{ick^3} \qquad Q_{11}=Q_{22}=-\frac{Q_{33}}{2} \tag{15}$$

For magnetic multipoles

$$M_{lm}=-\frac{1}{l+1}\int r^l Y_{lm}^*(\theta,\phi)\boldsymbol{\nabla}\cdot[\mathbf{x}\times\mathbf{J}(\mathbf{x})]d^3x \tag{16}$$

Note that $\mathbf{x}\times\hat{\mathbf{z}}$ points to the $-\hat{\boldsymbol{\phi}}$ direction, but $\mathbf{J}(\mathbf{x})$ has no ϕ dependency, thus the divergence vanishes, rendering all megnetic multipole moments zero.

(Note the definition of the multipole moment does not need the long wavelength approximation at all, thus the calculation in this part is exact.)

(b) Let's go back to (9.167) and (9.168) for the exact electric and magnetc multipole coefficients $a_E(l,m)$ and $a_M(l,m)$. Clearly $a_M(l,m)=0$, and

$$a_E(l,m)=\frac{k^2}{i\sqrt{l(l+1)}}\int Y_{lm}^*(\theta,\phi)\left\{c\rho(\mathbf{x})\frac{\partial[rj_l(kr)]}{\partial r}+ik[\mathbf{x}\cdot\mathbf{J}(\mathbf{x})]j_l(kr)\right\}d^3x \tag{17}$$

Integrating $d\phi$ yields $\delta_{m0}2\pi$, and integrating in $d\theta$ selects $\theta=0$ and $\theta=\pi$ contributions. Thus we can write

$$\begin{aligned}a_E(l,m)=&\frac{k^2}{i\sqrt{l(l+1)}}\delta_{m0}2\pi\sqrt{\frac{2l+1}{4\pi}}\frac{I_0}{2\pi}\left[1+(-1)^l\right]\times\\&\left\{\int_0^{d/2}(-i)\frac{\cos(kr)}{r^2}\frac{\partial[rj_l(kr)]}{\partial r}r^2dr+\int_0^{d/2}ik\frac{r\sin(kr)}{r^2}j_l(kr)r^2dr\right\}\end{aligned} \tag{18}$$

The content in the brace is just

$$-i\int_0^{d/2}\frac{\partial[\cos(kr)rj_l(kr)]}{\partial r}dr=\frac{idj_l(\pi)}{2} \tag{19}$$

Thus

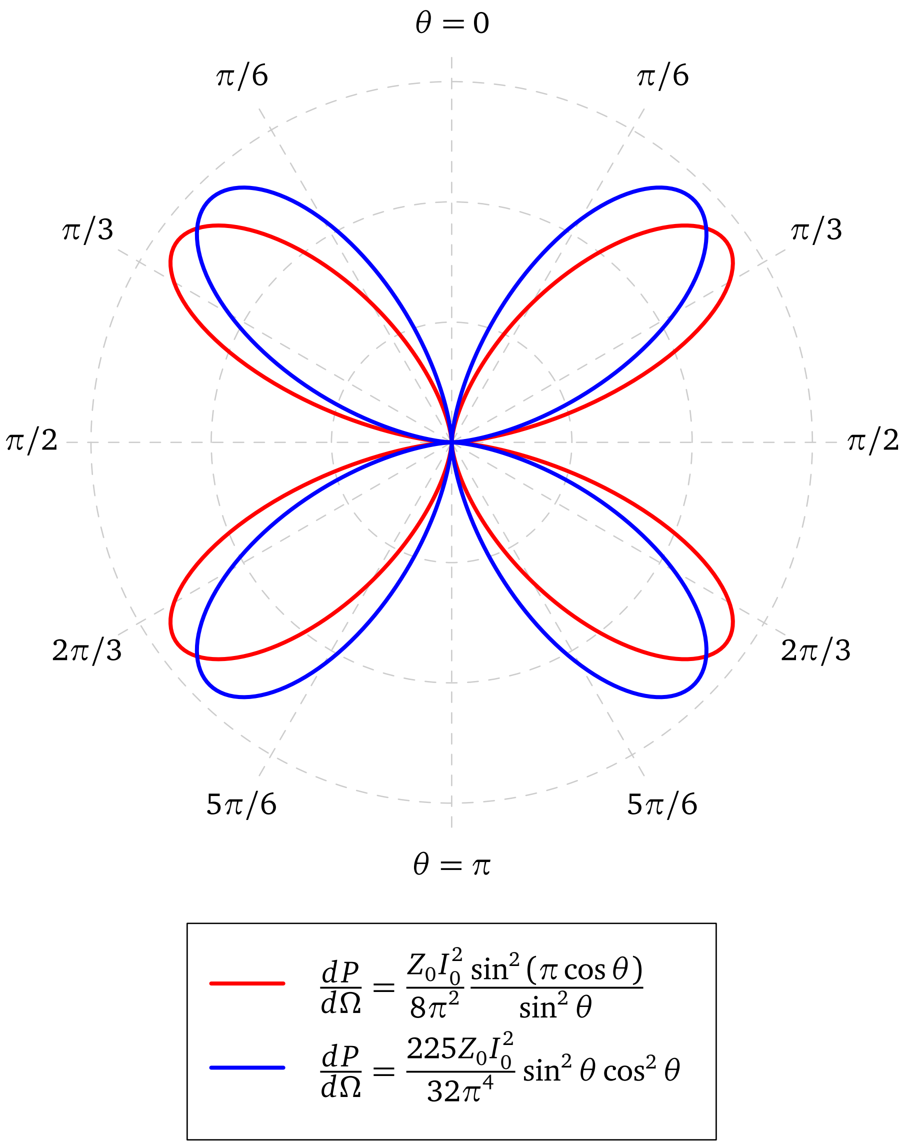
$$\begin{aligned}a_E(l,m)&=\frac{k^2}{i\sqrt{l(l+1)}}\delta_{m0}2\pi\sqrt{\frac{2l+1}{4\pi}}\frac{I_0}{2\pi}\left[1+(-1)^l\right]\cdot\frac{idj_l(\pi)}{2}\\&=\delta_{m0}\frac{\pi I_0k}{\sqrt{l(l+1)}}\sqrt{\frac{2l+1}{4\pi}}\left[1+(-1)^l\right]j_l(\pi)\end{aligned} \tag{20}$$

Note (20) is as exact as (9.165), in particular we have not used long wavelength approximation $kd\ll 1$ which would have allowed us to drop the integral contribution from $ik[\mathbf{x}\cdot\mathbf{J}(\mathbf{x})]j_l(kr)$ in (17) and would eventually give rise to the approximated expression of $a_E(l,m)$ in (9.169).

When we consider only the electric quadrupole moment (the lowest non-zero multipole moment), the exact power angular distribution is given by (9.151)

$$\begin{aligned}\frac{dP(2,0)}{d\Omega}&=\frac{Z_0}{2k^2}|a_E(2,0)|^2|\mathbf{X}_{20}|^2 && \text{by table 9.1}\\&=\frac{Z_0}{2k^2}\frac{\pi^2I_0^2k^2}{2\cdot3}\frac{5}{4\pi}\cdot2^2[j_2(\pi)]^2\cdot\frac{15}{8\pi}\sin^2\theta\cos^2\theta && j_2(\pi)=\frac{3}{\pi^2}\\&=\frac{225Z_0I_0^2}{32\pi^4}\sin^2\theta\cos^2\theta\end{aligned} \tag{21}$$

The plot below is depicting (21) and (7) together.



However if we use the long wavelength approximation, the approximated $a_E(2,0)$ as given in (9.169) would yield a power angular distribution solely generated by the quadrupole moment (15), which, by (9.51)

$$\left.\frac{dP}{d\Omega}\right|_{kd\ll 1} = \frac{c^2 Z_0 k^6}{512\pi^2}|Q_{33}|^2\sin^2\theta\cos^2\theta = \frac{Z_0 I_0^2}{8}\sin^2\theta\cos^2\theta \tag{22}$$

(c) The exact total power radiated due to the quadrupole moment is given by (9.154)

$$P(2,0) = \frac{Z_0}{2k^2}|a_E(2,0)|^2 = \frac{Z_0}{2k^2}\frac{\pi^2 I_0^2 k^2}{2\cdot 3}\frac{5}{4\pi}\cdot 2^2\frac{9}{\pi^4} = \frac{15 Z_0 I_0^2}{4\pi^3} \tag{23}$$

so the radiation resistance is

$$R_{\text{rad}} = \frac{2P}{I_0^2} = \frac{15 Z_0}{2\pi^3} \approx 0.242 Z_0 \approx 91.12\Omega \tag{24}$$

In contrast, the total power radiated calculated using the long wavelength approximation (22) yields

$$\left.P(2,0)\right|_{kd\ll 1} = \frac{c^2 Z_0 k^6}{960\pi}|Q_{33}|^2 = \frac{\pi Z_0 I_0^2}{15} \qquad \Longrightarrow \qquad \left.R_{\text{rad}}\right|_{kd\ll 1} = \frac{2P}{I_0^2} = \frac{2\pi Z_0}{15} \approx 157.79\Omega \tag{25}$$

The total power radiation calculated using long wavelength approximation (25) is significantly larger than the exact value (8). This seems a paradox since (9.155) indicates that the total power is an incoherent sum of power radiated by all the multipoles, so any pure multipole's contribution is necessarily smaller than the sum. But the long wavelength approximation is very poor in the first place with $kd = 2\pi$, in which we are dropping a significant term in $a_E(l,m)$ in (9.167). There is no paradox if we consider the exact power radiation due to the quadrupole moment (23). Out of the total exact radiation resistance of 93.36Ω, 91.12Ω is due to the quadrupole moment with the remaining 2.24Ω due to the higher moments.