1. We start with the differential equation that J_ν satisfies (see Jackson eq (3.77)), i.e.,

$$\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}+\left(1-\frac{\nu^2}{x^2}\right)R=0 \qquad\Longrightarrow$$
$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dR}{dx}\right)+\left(1-\frac{\nu^2}{x^2}\right)R=0 \tag{1}$$

Making the variable change $x=k\rho$ yields

$$\frac{1}{k\rho}\frac{d}{kd\rho}\left[k\rho\frac{dJ_\nu(k\rho)}{kd\rho}\right]+\left(1-\frac{\nu^2}{k^2\rho^2}\right)J_\nu(k\rho)=0 \qquad\Longrightarrow$$
$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dJ_\nu(k\rho)}{d\rho}\right]+\left(k^2-\frac{\nu^2}{\rho^2}\right)J_\nu(k\rho)=0 \tag{2}$$

Multiply both sides of (2) by $J_\nu(k'\rho)$ and integrate with measure $\rho d\rho$, we have

$$\underbrace{\int_0^a\frac{1}{\rho}J_\nu\left(k'\rho\right)\frac{d}{d\rho}\left[\rho\frac{dJ_\nu(k\rho)}{d\rho}\right]\rho d\rho}_{I}+\int_0^a\left(k^2-\frac{\nu^2}{\rho^2}\right)J_\nu(k\rho)J_\nu\left(k'\rho\right)\rho d\rho=0 \tag{3}$$

where

$$I=\underbrace{J_\nu\left(k'\rho\right)\rho\frac{dJ_\nu(k\rho)}{d\rho}}_{g(k',k;\rho)}\Bigg|_0^a-\int_0^a\rho\frac{dJ_\nu(k\rho)}{d\rho}\frac{dJ_\nu(k'\rho)}{d\rho}d\rho \tag{4}$$

Exchange $k'\longleftrightarrow k$ in (3) and subtract the resulting equation from (3), we end up with

$$g\left(k',k;a\right)-g\left(k',k;0\right)-g\left(k,k';a\right)+g\left(k,k';0\right)+\left(k^2-k'^2\right)\int_0^a J_\nu(k\rho)J_\nu\left(k'\rho\right)\rho d\rho=0 \tag{5}$$

Now by the series expansion of J_ν function (for non-integer ν)

$$g\left(k',k;\rho\right)=\left[\sum_{l=0}^\infty\frac{(-1)^l}{l!\Gamma(l+\nu+1)}\left(\frac{k'\rho}{2}\right)^{2l+\nu}\right]\rho\frac{d}{d\rho}\left[\sum_{m=0}^\infty\frac{(-1)^m}{m!\Gamma(m+\nu+1)}\left(\frac{k\rho}{2}\right)^{2m+\nu}\right]$$
$$=\sum_{l,m=0}^\infty\frac{(-1)^{l+m}}{l!m!\Gamma(l+\nu+1)\Gamma(m+\nu+1)}\left(\frac{k'}{2}\right)^{2l+\nu}\left(\frac{k}{2}\right)^{2m+\nu}(2m+\nu)\rho^{2l+2m+2\nu} \tag{6}$$

Exchanging $k\longleftrightarrow k'$ gives

$$g\left(k,k';\rho\right)=\sum_{l,m=0}^\infty\frac{(-1)^{l+m}}{l!m!\Gamma(l+\nu+1)\Gamma(m+\nu+1)}\left(\frac{k}{2}\right)^{2l+\nu}\left(\frac{k'}{2}\right)^{2m+\nu}(2m+\nu)\rho^{2l+2m+2\nu} \tag{7}$$

To calculate $g\left(k,k';0\right)-g\left(k',k;0\right)$ in (5), we need to subtract (6) from (7) with $\rho\to 0$, which we hope to vanish. The leading power $\rho^{2\nu}$ in this subtraction vanishes since its coefficient vanishes given $l=m=0$. But the next significant power $\rho^{2\nu+2}$ in this subtraction has contribution from both $l=0,m=1$ and $l=1,m=0$, whose coefficient is proportional to

$$\left[\overbrace{\underbrace{\left(\frac{k}{2}\right)^\nu\left(\frac{k'}{2}\right)^{\nu+2}(\nu+2)}_{l=0,m=1}+\underbrace{\left(\frac{k}{2}\right)^{\nu+2}\left(\frac{k'}{2}\right)^\nu\nu}_{l=1,m=0}}^{\text{from(7)}}\right]-\left[\overbrace{\underbrace{\left(\frac{k'}{2}\right)^\nu\left(\frac{k}{2}\right)^{\nu+2}(\nu+2)}_{l=0,m=1}+\underbrace{\left(\frac{k'}{2}\right)^{\nu+2}\left(\frac{k}{2}\right)^\nu\nu}_{l=1,m=0}}^{\text{from(6)}}\right]$$
$$\propto$$
$$(kk')^\nu\left(k'^2-k^2\right) \tag{8}$$

which does not vanish. Thus for this leading power $\rho^{2\nu+2}$ to converge as $\rho \to 0$, we require Re $\nu > -1$. Now with this condition imposed, (5) is simplified as

$$g\left(k',k;a\right)-g\left(k,k';a\right)+\left(k^2-k'^2\right)\int_0^a J_\nu(k\rho)J_\nu\left(k'\rho\right)\rho d\rho = 0 \qquad \Longrightarrow$$

$$\int_0^a J_\nu(k\rho)J_\nu\left(k'\rho\right)\rho d\rho = \frac{g\left(k',k;a\right)-g\left(k,k';a\right)}{k'^2-k^2} \tag{9}$$

By definition of g,

$$g\left(k',k;a\right)=J_\nu\left(k'a\right)kaJ'_\nu(ka) \qquad\qquad g\left(k,k';a\right)=J_\nu(ka)k'aJ'_\nu\left(k'a\right) \tag{10}$$

With the equation (3.91), as $x\to\infty$,

$$J_\nu(x)\to\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right) \tag{11}$$

$$J'_\nu(x)\to\sqrt{\frac{2}{\pi}}\left[-\frac{1}{2\sqrt{x}^3}\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)-\frac{1}{\sqrt{x}}\sin\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right] \tag{12}$$

we have

$$g\left(k',k;a\right)\to\sqrt{\frac{2}{\pi k'a}}\cos\overbrace{\left(k'a-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}^{\xi'}ka\sqrt{\frac{2}{\pi}}\left[-\frac{1}{2\sqrt{ka}^3}\cos\overbrace{\left(ka-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}^{\xi}-\frac{1}{\sqrt{ka}}\sin\left(ka-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right]$$

$$=\frac{2}{\pi}\left(-\frac{1}{2}\frac{1}{\sqrt{kk'}a}\cos\xi'\cos\xi-\sqrt{\frac{k}{k'}}\sin\xi\cos\xi'\right) \tag{13}$$

$$g\left(k,k';a\right)\to\frac{2}{\pi}\left(-\frac{1}{2}\frac{1}{\sqrt{kk'}a}\cos\xi'\cos\xi-\sqrt{\frac{k'}{k}}\sin\xi'\cos\xi\right) \tag{14}$$

hence

$$\begin{aligned}\int_0^a J_\nu(k\rho)J_\nu\left(k'\rho\right)\rho d\rho &= \frac{1}{k'^2-k^2}\cdot\frac{2}{\pi}\left(\sqrt{\frac{k'}{k}}\sin\xi'\cos\xi-\sqrt{\frac{k}{k'}}\sin\xi\cos\xi'\right)\\ &=\frac{1}{k'^2-k^2}\frac{2}{\pi}\left\{\sqrt{\frac{k'}{k}}\frac{1}{2}\left[\sin\left(\xi'+\xi\right)+\sin\left(\xi'-\xi\right)\right]-\sqrt{\frac{k}{k'}}\frac{1}{2}\left[\sin\left(\xi+\xi'\right)+\sin\left(\xi-\xi'\right)\right]\right\}\\ &=\frac{1}{k'^2-k^2}\frac{1}{\pi}\left[\sin\left(\xi'+\xi\right)\left(\sqrt{\frac{k'}{k}}-\sqrt{\frac{k}{k'}}\right)+\sin\left(\xi'-\xi\right)\left(\sqrt{\frac{k'}{k}}+\sqrt{\frac{k}{k'}}\right)\right]\\ &=\frac{1}{\pi}\frac{1}{\sqrt{kk'}}\left[\frac{\sin\left(\xi'+\xi\right)}{k'+k}+\frac{\sin\left(\xi'-\xi\right)}{k'-k}\right]\end{aligned} \tag{15}$$

Recall the δ function representation (reference Wolfram)

$$\delta(x)=\lim_{\epsilon\to 0}\frac{1}{\pi x}\sin\left(\frac{x}{\epsilon}\right) \tag{16}$$

makes the second term of (15)

$$\frac{1}{\sqrt{kk'}}\frac{1}{\pi\left(k'-k\right)}\sin\left[\frac{\left(k'-k\right)}{1/a}\right] \qquad \longrightarrow \qquad \frac{\delta\left(k'-k\right)}{k} \qquad \text{as } a\to\infty \tag{17}$$

Now as $a\to\infty$, the first term

$$\frac{1}{\pi}\frac{1}{\sqrt{kk'}}\frac{\sin\left[\left(k'+k\right)a-\left(\nu+\frac{1}{2}\right)\pi\right]}{k'+k}=\frac{1}{\pi}\frac{1}{\sqrt{kk'}}\left[\frac{k'+k-\frac{1}{a}\left(\nu+\frac{1}{2}\right)\pi}{k'+k}\right]\frac{\sin\left[\dfrac{k'+k-\frac{1}{a}\left(\nu+\frac{1}{2}\right)\pi}{1/a}\right]}{k'+k-\frac{1}{a}\left(\nu+\frac{1}{2}\right)\pi} \tag{18}$$

will have an asymptotic form that's proportional to $\delta(k'+k)$ which can be taken as zero since k', k are positive. In summary

$$\int_0^\infty J_\nu(k\rho) J_\nu(k'\rho)\rho d\rho = \lim_{a\to\infty}\int_0^a J_\nu(k\rho) J_\nu(k'\rho)\rho d\rho = \frac{\delta(k'-k)}{k} \tag{19}$$

and by a trivial symbolic exchange $\rho \longleftrightarrow k$,

$$\int_0^\infty J_\nu(k\rho) J_\nu(k\rho') k dk = \frac{\delta(\rho'-\rho)}{\rho} \tag{20}$$

2. If we treat

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{21}$$

as a function in $\mathbf{x}$, which satisfies Laplace equation when $\mathbf{x} \neq \mathbf{x}'$, we can expand it into the basis function in separate variables ρ, ϕ, z. Considering the boundary condition at $\rho = 0$, and $z = \pm\infty$, we have the following form

$$G(\mathbf{x}-\mathbf{x}') = \begin{cases} \sum_{m=-\infty}^{\infty} e^{im\phi}\int_0^\infty A_{km} J_m(k\rho) e^{-kz} dk & \text{for } z > z' \\ \sum_{m=-\infty}^{\infty} e^{im\phi}\int_0^\infty B_{km} J_m(k\rho) e^{kz} dk & \text{for } z < z' \end{cases} \tag{22}$$

If we insist G to be continuous at $z = z'$ (not at the same time $\phi = \phi'$ and $\rho = \rho'$ in which case we know G has singularity), we will have

$$A_{km} e^{-kz'} = B_{km} e^{kz'} \equiv C_{km} \tag{23}$$

Then (22) can be written more uniformly as

$$G(\mathbf{x}-\mathbf{x}') = \sum_{m=-\infty}^{\infty} e^{im\phi}\int_0^\infty C_{km} J_m(k\rho) e^{-k|z-z'|} dk \tag{24}$$

Since

$$\nabla^2 G = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial\phi^2} + \frac{\partial^2 G}{\partial z^2} = -4\pi\delta(\mathbf{x}-\mathbf{x}') = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')\delta(z-z') \tag{25}$$

Integrating (25) across the infinitesimal range $[z'-\epsilon, z'+\epsilon]$ should give

$$\int_{z'-\epsilon}^{z'+\epsilon}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial\phi^2} + \frac{\partial^2 G}{\partial z^2}\right] dz = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi') \tag{26}$$

The first two terms in the bracket, being continuous in z, will produce zero after the integral, but the third term will produce the following equation

$$\left.\frac{\partial G}{\partial z}\right|_{z'+\epsilon} - \left.\frac{\partial G}{\partial z}\right|_{z'-\epsilon} = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi') \quad \Longrightarrow \tag{27}$$

$$\sum_{m=-\infty}^{\infty} e^{im\phi}\int_0^\infty C_{km} J_m(k\rho)(-2k) dk = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi') \quad \Longrightarrow$$

$$\sum_{m=-\infty}^{\infty} e^{im\phi}\int_0^\infty C_{km} J_m(k\rho) k dk = 2\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi') \tag{28}$$

In light of (20), as well as the Fourier completeness (equation (3.139))

$$\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} = 2\pi\delta(\phi-\phi') \tag{29}$$

we can set

$$C_{km} = e^{-im\phi'} J_m(k\rho') \tag{30}$$

Therefore

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{m=-\infty}^{\infty} \int_0^\infty e^{im\left(\phi-\phi'\right)} J_m(k\rho) J_m\left(k\rho'\right) e^{-k|z-z'|} dk \tag{31}$$

Note we didn't violate the Re $\nu > -1$ condition in (1) since that condition is necessary only for non-integer νs.

3. (a)

$$\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^\infty e^{-k|z|} J_0(k\rho)\, dk \tag{32}$$

Proof. This is easily obtained by setting $\mathbf{x}' = 0$ in (31) and use the fact that $J_m(k\rho')$ vanishes for all ms but $m = 0$. □

(b)

$$J_0\left(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi}\right) = \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho) J_m\left(k\rho'\right) \tag{33}$$

Proof. We will prove a slightly more general result

$$\overbrace{J_0\left[k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\right]}^{L} = \overbrace{\sum_{m=-\infty}^{\infty} e^{im\left(\phi-\phi'\right)} J_m(k\rho) J_m\left(k\rho'\right)}^{R} \tag{34}$$

Choose arbitrary z, z' such that $z \neq z'$. Then

$$\overbrace{\int_0^\infty e^{-k|z-z'|} R\, dk = \frac{1}{|\mathbf{x}-\mathbf{x}'|}}^{\text{by (31)}} = \underbrace{\frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2}} = \int_0^\infty e^{-k|z-z'|} L\, dk}_{\text{by (32)}} \tag{35}$$

□

This implies

$$\int_0^\infty e^{-k|z-z'|} (L-R)\, dk = 0 \tag{36}$$

But since $e^{-k|z-z'|}$ is positive everywhere in the integration range, we must have $L = R$ everywhere for $k \geq 0$.

(c)

$$e^{ik\rho\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(k\rho) \tag{37}$$

Proof. This is easily proved by recalling the generating function of $J_m(x)$:

$$g(x,t) = \exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{m=-\infty}^{\infty} J_m(x)\, t^m \tag{38}$$

Setting $t = ie^{i\phi}$ and $x = k\rho$ gives the result. □

4.

$$J_m(x) = \frac{1}{2\pi i^m} \int_0^{2\pi} e^{ix\cos\phi - im\phi}\, d\phi \tag{39}$$

Proof. This is just interpreting $i^m J_m(k\rho)$ in (37) as the Fourier expansion coefficients for the function $f(\phi) = e^{ik\rho\cos\phi}$. I.e., by (2.41), (2.42)

$$\sqrt{2\pi}\, i^m J_m(k\rho) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{ik\rho\cos\phi} e^{-im\phi}\, d\phi \tag{40}$$

□