In these notes, we justify the validity of method of image when it's applied to the semi-infinite dielectric slab. It needs elaboration since it's not immediately obvious that it's valid to apply the method of image in situations where the boundary is not an equipotential surface.

Let's recall the proof of uniqueness theorem. For a volume V with bounding surface S and charge density $\rho(\mathbf{x})$, let Φ_1 and Φ_2 be two solutions such that

$$\nabla^2\Phi_1 = \nabla^2\Phi_2 = -\rho(\mathbf{x})/\epsilon_0 \tag{1}$$

Let

$$U = \Phi_1 - \Phi_2 \tag{2}$$

which should then satisfy the Laplace equation

$$\nabla^2 U = 0 \tag{3}$$

throughout V.

Jackson equation (1.38) follows from Green's identity

$$\int_V \left(U\nabla^2 U + \nabla U\cdot\nabla U\right)d^3x = \oint_S U\frac{\partial U}{\partial n}da \qquad \Longrightarrow \qquad \int_V \nabla U\cdot\nabla U d^3x = \oint_S U\frac{\partial U}{\partial n}da \tag{4}$$

We can see that when the RHS vanishes

$$\oint_S U\frac{\partial U}{\partial n}da = 0 \tag{5}$$

the two solutions Φ_1 and Φ_2 are identical up to a constant difference. (5) serves as the precondition of uniqueness theorem.

There are two cases in which (5) can be satisfied.

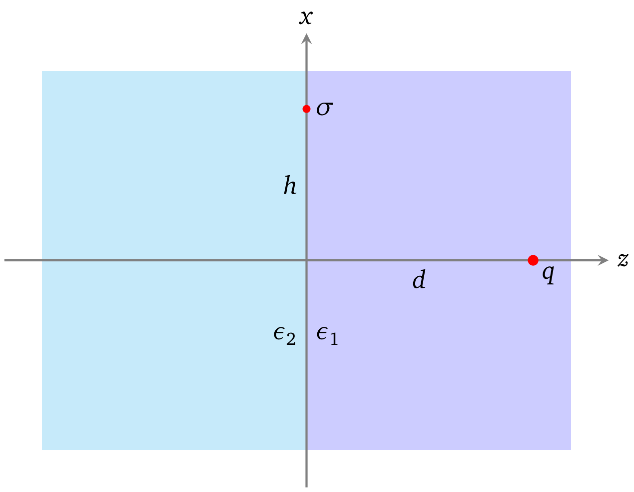
1. U is constant everywhere on the surface S, in which case

$$\oint_S U\frac{\partial U}{\partial n}da = U\oint_S \frac{\partial U}{\partial n}da = U\int_V \nabla^2 U d^3x = 0 \tag{6}$$

   U being constant means S is an equipotential surface for both Φ_1 and Φ_2. This condition has been widely used in classical examples of method of image involving grounded conductors (infinite or spherical).

2. Normal gradient of U vanishes everywhere on surface S, in which case (5) is obviously satisfied. This corresponds to the boundary condition that specifies the normal gradient everywhere on S, so when $\partial\Phi_1/\partial n = \partial\Phi_2/\partial n$ take these boundary values, the normal gradient of U vanishes.

Now coming back to the example from section (4.4). By placing the point charge q at $d\hat{\mathbf{z}}$ near the semi-infinite dielectric slab, we have essentially fixed the normal gradient of potential everywhere on the slab's plane. This can be seen by calculating the induced surface charge density σ at point $\mathbf{x} = h\hat{\mathbf{x}}$.



Let $\mathbf{D}_q(\mathbf{x})$ be the electric displacement at $\mathbf{x}$ due to q, macroscopic law gives

$$\mathbf{D}_q(\mathbf{x}) = \frac{q}{4\pi}\frac{h\hat{\mathbf{x}} - d\hat{\mathbf{z}}}{|h\hat{\mathbf{x}} - d\hat{\mathbf{z}}|^3} \tag{7}$$

Thus the microscopic normal electric field due to q at $\mathbf{x}$ is

$$E_{z,q}(\mathbf{x}) = -\frac{q}{4\pi\epsilon_1}\frac{d}{|h\hat{\mathbf{x}} - d\hat{\mathbf{z}}|^3} \tag{8}$$

Draw a pillbox around $\mathbf{x}$, on the two sides $z = 0^+, 0^-$, the electric field due to σ is $\pm\sigma/2\epsilon_0\hat{\mathbf{z}}$. Then by superposition, the microscopic normal field on the two sides are

$$E_{z,0^+}(\mathbf{x}) = E_{z,q}(\mathbf{x}) + \frac{\sigma}{2\epsilon_0} \qquad\qquad E_{z,0^-}(\mathbf{x}) = E_{z,q}(\mathbf{x}) - \frac{\sigma}{2\epsilon_0} \tag{9}$$

Then normal constraint of the dielectric boundary requires

$$\epsilon_1 E_{z,0^+} = \epsilon_2 E_{z,0^-} \quad\Longrightarrow\quad \epsilon_1\left[E_{z,q}(\mathbf{x}) + \frac{\sigma}{2\epsilon_0}\right] = \epsilon_2\left[E_{z,q}(\mathbf{x}) - \frac{\sigma}{2\epsilon_0}\right] \quad\Longrightarrow\quad \frac{\sigma}{2\epsilon_0} = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}E_{z,q}(\mathbf{x}) \tag{10}$$

which turns (9) into

$$\begin{aligned}
E_{z,0^+}(\mathbf{x}) &= E_{z,q}(\mathbf{x}) + \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}E_{z,q}(\mathbf{x}) \\
&= E_{z,q}(\mathbf{x}) + \frac{q'}{4\pi\epsilon_1}\frac{d}{|h\hat{\mathbf{x}} + d\hat{\mathbf{z}}|^3} && \text{where } q' \equiv -\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}q && (11) \\
E_{z,0^-}(\mathbf{x}) &= \frac{2\epsilon_1}{\epsilon_2 + \epsilon_1}E_{z,q}(\mathbf{x}) \\
&= -\frac{q''}{4\pi\epsilon_2}\frac{d}{|h\hat{\mathbf{x}} - d\hat{\mathbf{z}}|^3} && \text{where } q'' \equiv \frac{2\epsilon_2}{\epsilon_2 + \epsilon_1}q && (12)
\end{aligned}$$

The form of (11) and (12) were written to emphasize the effective normal field generated by the image charges at the boundary, with q' at $-d\hat{\mathbf{z}}$ and q'' at $d\hat{\mathbf{z}}$.

We now have a complete specification of normal field values for both regions' boundaries, and these normal field values are satisfied by the given image charges, we can indeed apply the uniqueness theorem (case 2) and conclude that these image charges generate identical fields for the respective volumes.