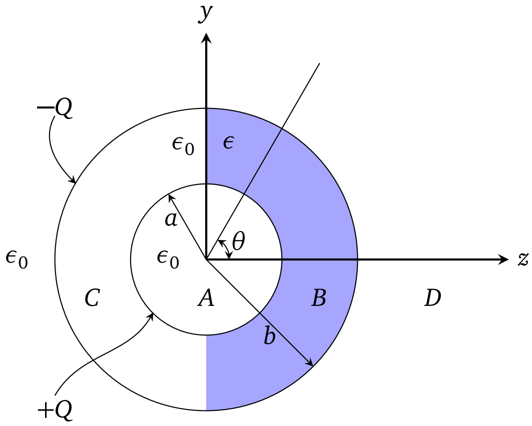


1. Due to the cylindrical symmetry, the potentials in all 4 regions can be written as linear combinations of Legendre polynomials in $\cos\theta$, but considering the limiting behavior of $r \to 0$ and $r \to \infty$, we can write these potentials as the following forms:

$$\Phi_A = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{1}$$

$$\Phi_B = \sum_{l=0}^{\infty} \left[B_l r^l + C_l r^{-(l+1)}\right] P_l(\cos\theta) \tag{2}$$

$$\Phi_C = \sum_{l=0}^{\infty} \left[D_l r^l + E_l r^{-(l+1)}\right] P_l(\cos\theta) \tag{3}$$

$$\Phi_D = \sum_{l=0}^{\infty} F_l r^{-(l+1)} P_l(\cos\theta) \tag{4}$$

Since the two spherical shells are conductors, their surfaces must be at constant potential, denoted $V_a$ and $V_b$ respectively.

Now consider $\Phi_D$ as $r \to b^+$:

$$\Phi_D(b,\theta) = \sum_{l=0}^{\infty} F_l b^{-(l+1)} P_l(\cos\theta) = V_b \qquad \text{for all } 0 \le \theta \le \pi \tag{5}$$

Orthogonality of Legendre polynomials dictates $F_l = 0$ for all $l$ except $F_0$. But since the sphere with sufficiently large radius encloses zero total charge, we know $F_0$ must also vanish by Gauss's law. Therefore we have established

$$\Phi_D = V_b = 0 \tag{6}$$

Similar arguments applied to $\Phi_A$ will lead to the conclusion that

$$\Phi_A = A_0 = V_a \tag{7}$$

with the constant $V_a$ or $A_0$ to be determined. It will not be zero since we expect a constant potential difference between the two spherical shells.

Now consider $\Phi_B$ as $r \to b^-$:

$$\Phi_B(b,\theta) = \sum_{l=0}^{\infty} \left[B_l b^l + C_l b^{-(l+1)}\right] P_l(\cos\theta) = 0 \qquad \text{for } 0 \le \theta \le \frac{\pi}{2} \tag{8}$$

**It would be tempting but wrong to apply the orthogonality of Legendre polynomial to (8) and claim that square bracket vanishes for all $l$.** This is because $\theta$ is ranging only over the half domain $[0, \pi/2]$ instead of $[0, \pi]$. Over the half domain, odd Legendre polynomials are orthogonal to each other, so are even ones, but the odd and even Legendre polynomials are generally not orthogonal to each other.

For later convenience, we rewrite (8) by grouping even and odd orders separately

$$\begin{aligned}0 &= \sum_{k=0}^{\infty}\left[B_{2k}b^{2k}+C_{2k}b^{-(2k+1)}\right]P_{2k}(\cos\theta)\\&+\sum_{k=0}^{\infty}\left[B_{2k+1}b^{2k+1}+C_{2k+1}b^{-(2k+2)}\right]P_{2k+1}(\cos\theta) \qquad \text{for } 0\le\theta\le\frac{\pi}{2}\end{aligned} \tag{9}$$

Similarly, the limit of $\Phi_B$ as $r\to a^+$ yields

$$\begin{aligned}V_a &= \sum_{k=0}^{\infty}\left[B_{2k}a^{2k}+C_{2k}a^{-(2k+1)}\right]P_{2k}(\cos\theta)\\&+\sum_{k=0}^{\infty}\left[B_{2k+1}a^{2k+1}+C_{2k+1}a^{-(2k+2)}\right]P_{2k+1}(\cos\theta) \qquad \text{for } 0\le\theta\le\frac{\pi}{2}\end{aligned} \tag{10}$$

Now consider the region $C$, the corresponding limits with $r\to b^-$ and $r\to a^+$ yield

$$\Phi_C(a,\theta)=\sum_{l=0}^{\infty}\left[D_l a^l+E_l a^{-(l+1)}\right]P_l(\cos\theta)=V_a \qquad \text{for } \frac{\pi}{2}\le\theta\le\pi \tag{11}$$

$$\Phi_C(b,\theta)=\sum_{l=0}^{\infty}\left[D_l b^l+E_l b^{-(l+1)}\right]P_l(\cos\theta)=0 \qquad \text{for } \frac{\pi}{2}\le\theta\le\pi \tag{12}$$

Now make a variable change $\theta\to\pi-\theta$ for (11) and (12) and recall the parity of Legendre polynomials, we have

$$\begin{aligned}V_a &= \sum_{l=0}^{\infty}\left[D_l a^l+E_l a^{-(l+1)}\right]P_l(-\cos\theta)\\&=\sum_{k=0}^{\infty}\left[D_{2k}a^{2k}+E_{2k}a^{-(2k+1)}\right]P_{2k}(\cos\theta)\\&-\sum_{k=0}^{\infty}\left[D_{2k+1}a^{2k+1}+E_{2k+1}a^{-(2k+2)}\right]P_{2k+1}(\cos\theta) \qquad \text{for } 0\le\theta\le\frac{\pi}{2}\end{aligned} \tag{13}$$

$$\begin{aligned}0 &= \sum_{k=0}^{\infty}\left[D_{2k}b^{2k}+E_{2k}b^{-(2k+1)}\right]P_{2k}(\cos\theta)\\&-\sum_{k=0}^{\infty}\left[D_{2k+1}b^{2k+1}+E_{2k+1}b^{-(2k+2)}\right]P_{2k+1}(\cos\theta) \qquad \text{for } 0\le\theta\le\frac{\pi}{2}\end{aligned} \tag{14}$$

Note (13) and (14) are now over the same half domain $[0,\pi/2]$ just like (9) and (10), but their odd terms now acquire a minus sign.

Potential's continuity along the $B/C$ boundary requires

$$\Phi_B\left(r,\frac{\pi}{2}\right)=\Phi_C\left(r,\frac{\pi}{2}\right) \quad\Longrightarrow\quad \sum_{l=0}^{\infty}\left[B_l r^l+C_l r^{-(l+1)}\right]P_l(0)=\sum_{l=0}^{\infty}\left[D_l r^l+E_l r^{-(l+1)}\right]P_l(0) \tag{15}$$

Since $P_{2k}(0)\neq 0$, matching coefficients of $r$ power terms corresponding to $l=2k$ will give us

$$B_{2k}=D_{2k} \qquad C_{2k}=E_{2k} \qquad \text{for } k\ge 0 \tag{16}$$

However since $P_{2k+1}(0)=0$, (15) does not impose any restrictions on odd-indexed coefficients.

Normal field restriction along the $B/C$ boundary requires

$$\epsilon\left.\frac{\partial\Phi_B}{\partial\theta}\right|_{\theta=\pi/2}=\epsilon_0\left.\frac{\partial\Phi_C}{\partial\theta}\right|_{\theta=\pi/2} \quad\Longrightarrow\quad \frac{\epsilon}{\epsilon_0}\sum_{l=0}^{\infty}\left[B_l r^l+C_l r^{-(l+1)}\right]P_l'(0)=\sum_{l=0}^{\infty}\left[D_l r^l+E_l r^{-(l+1)}\right]P_l'(0) \tag{17}$$

Again, by parity and matching coefficients, we now have the restrictions on the odd-indexed coefficients:

$$\frac{\epsilon}{\epsilon_0}B_{2k+1}=D_{2k+1} \qquad \frac{\epsilon}{\epsilon_0}C_{2k+1}=E_{2k+1} \qquad \text{for } k\ge 0 \tag{18}$$

Subtracting (14) from (9) and invoking (16), (18) gives us

$$0 = \left(1 + \frac{\epsilon}{\epsilon_0}\right) \sum_{k=0}^{\infty} \left[B_{2k+1} b^{2k+1} + C_{2k+1} b^{-(2k+2)}\right] P_{2k+1}(\cos\theta) \qquad \text{for } 0 \le \theta \le \frac{\pi}{2} \tag{19}$$

But this identity must be true for $\theta \in [0, \pi]$ since $P_{2k+1}$ has definite parity. Now we can properly invoke orthogonality of Legendre polynomial over the full range (or equivalently, *purely odd* polynomials over the half range also form an orthogonal basis) and claim

$$B_{2k+1} b^{2k+1} + C_{2k+1} b^{-(2k+2)} = 0 \qquad \text{for } k \ge 0 \tag{20}$$

Similarly, subtracting (13) from (10) will lead to

$$B_{2k+1} a^{2k+1} + C_{2k+1} a^{-(2k+2)} = 0 \qquad \text{for } k \ge 0 \tag{21}$$

Combining (18), (20) and (21), we get

$$B_{2k+1} = C_{2k+1} = D_{2k+1} = E_{2k+1} = 0 \qquad \text{for } k \ge 0 \tag{22}$$

With all odd terms vanishing, adding (9) with (14) and invoking (16), we obtain

$$0 = 2 \sum_{l=0}^{\infty} \left[B_{2k} b^{2k} + C_{2k} b^{-(2k+1)}\right] P_{2k}(\cos\theta) \qquad \text{for } 0 \le \theta \le \frac{\pi}{2} \tag{23}$$

With only purely even polynomials involved, same argument will set all coefficients to zero,

$$B_{2k} b^{2k} + C_{2k} b^{-(2k+1)} = 0 \qquad \text{for } k \ge 0 \tag{24}$$

Similalry, adding (10) and (13) gives

$$V_a = \sum_{k=0}^{\infty} \left[B_{2k} a^{2k} + C_{2k} a^{-(2k+1)}\right] P_{2k}(\cos\theta) \tag{25}$$

from which we obtain

$$B_0 + \frac{C_0}{a} = V_a \qquad B_{2k} a^{2k} + C_{2k} a^{-(2k+1)} = 0 \quad \text{for } k \ge 1 \tag{26}$$

Consider (24) and (26) together, this means

$$B_{2k} = C_{2k} = D_{2k} = E_{2k} = 0 \qquad \text{for } k \ge 1 \tag{27}$$

So far, we can leave $C_0$ as the only unknown and write the potentials as

$$\Phi_A = A_0 = V_a = -\frac{C_0}{b} + \frac{C_0}{a} \qquad \Phi_B(r) = \Phi_C(r) = -\frac{C_0}{b} + \frac{C_0}{r} \qquad \Phi_D = 0 \tag{28}$$

By equation (4.40)

$$\sigma_{\text{out}}(\theta) = \begin{cases} \epsilon \cdot \left.\dfrac{\partial \Phi_B}{\partial r}\right|_{r=b} = -\epsilon \dfrac{C_0}{b^2} & \text{for } 0 \le \theta \le \dfrac{\pi}{2} \\ \epsilon_0 \cdot \left.\dfrac{\partial \Phi_C}{\partial r}\right|_{r=b} = -\epsilon_0 \dfrac{C_0}{b^2} & \text{for } \dfrac{\pi}{2} \le \theta \le \pi \end{cases} \tag{29}$$

Setting the total charge of the outer sphere to $-Q$ gives us

$$-Q = 2\pi b^2 \left[-(\epsilon + \epsilon_0) \frac{C_0}{b^2}\right] \qquad \Longrightarrow \qquad C_0 = \frac{Q}{2\pi(\epsilon + \epsilon_0)} \tag{30}$$

Thus finally

$$\Phi_A = \frac{Q}{2\pi(\epsilon + \epsilon_0)} \left(\frac{1}{a} - \frac{1}{b}\right) \qquad \Phi_B(r) = \Phi_C(r) = \frac{Q}{2\pi(\epsilon + \epsilon_0)} \left(\frac{1}{r} - \frac{1}{b}\right) \qquad \Phi_D = 0 \tag{31}$$

The field in $B$ and $C$ is

$$\mathbf{E}_B = \mathbf{E}_C = \frac{Q}{2\pi(\epsilon + \epsilon_0) r^2} \hat{\mathbf{r}} \tag{32}$$

2. The surface charge density is given by (29)

$$\sigma_{\text{out}} = \begin{cases} \dfrac{-Q\epsilon}{2\pi b^2(\epsilon+\epsilon_0)} & \text{for } 0 \le \theta \le \dfrac{\pi}{2} \\ \dfrac{-Q\epsilon_0}{2\pi b^2(\epsilon+\epsilon_0)} & \text{for } \dfrac{\pi}{2} \le \theta \le \pi \end{cases} \qquad \sigma_{\text{in}} = \begin{cases} \dfrac{Q\epsilon}{2\pi a^2(\epsilon+\epsilon_0)} & \text{for } 0 \le \theta \le \dfrac{\pi}{2} \\ \dfrac{Q\epsilon_0}{2\pi a^2(\epsilon+\epsilon_0)} & \text{for } \dfrac{\pi}{2} \le \theta \le \pi \end{cases} \tag{33}$$

3. From

$$\sigma_{\text{free}} + \sigma_{\text{pol}} = E_r\epsilon_0 \tag{34}$$

we get the polarization charge density of the dielectric at $r = a$:

$$\sigma_{\text{pol}} = E_r\epsilon_0 - \sigma_{\text{free}} = \frac{Q\epsilon_0}{2\pi a^2(\epsilon+\epsilon_0)} - \frac{Q\epsilon}{2\pi a^2(\epsilon+\epsilon_0)} = \frac{Q(\epsilon_0-\epsilon)}{2\pi a^2(\epsilon+\epsilon_0)} \tag{35}$$