1. First we recall problem (6.10) which is a statement about the conservation of angular momentum for the electromagnetic field (here we do not have any particle).

$$\frac{d}{dt}\int_V \mathscr{L}(\mathbf{x})\, d^3x + \int_S \mathbf{n}\cdot \overleftrightarrow{\mathbf{M}}\, da = 0 \tag{1}$$

where $\mathscr{L}$ is the angular momentum density of the field and $\overleftrightarrow{\mathbf{M}} = \overleftrightarrow{\mathbf{T}} \times \mathbf{x}$ is a rank-2 tensor related to the Maxwell stress tensor by

$$M_{il} = \epsilon_{jkl} T_{ij} x_k \qquad \text{and} \tag{2}$$

$$T_{ij} = \epsilon_0 \left[E_i E_j + c^2 B_i B_j - \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E} + c^2 \mathbf{B}\cdot\mathbf{B}\right)\delta_{ij}\right] \qquad \text{see (6.120)} \tag{3}$$

Let's write out the l-th component of the vector $\mathbf{n}\cdot\overleftrightarrow{\mathbf{M}}$:

$$\begin{aligned}
\left(\mathbf{n}\cdot\overleftrightarrow{\mathbf{M}}\right)_l &= n_i M_{il} = \epsilon_{jkl} n_i T_{ij} x_k \\
&= \epsilon_0 \epsilon_{jkl} n_i \left[E_i E_j + c^2 B_i B_j - \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E} + c^2 \mathbf{B}\cdot\mathbf{B}\right)\delta_{ij}\right] x_k \\
&= \epsilon_0 \epsilon_{jkl} \left[(\mathbf{n}\cdot\mathbf{E}) E_j + c^2 (\mathbf{n}\cdot\mathbf{B}) B_j - \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E} + c^2 \mathbf{B}\cdot\mathbf{B}\right) n_j\right] x_k \\
&= \epsilon_0 r \left[(\mathbf{n}\cdot\mathbf{E})(\mathbf{E}\times\mathbf{n})_l + c^2 (\mathbf{n}\cdot\mathbf{B})(\mathbf{B}\times\mathbf{n})_l \right]
\end{aligned} \tag{4}$$

This gives the vector representation

$$\mathbf{n}\cdot\overleftrightarrow{\mathbf{M}} = \epsilon_0 r \left[(\mathbf{n}\cdot\mathbf{E})(\mathbf{E}\times\mathbf{n}) + c^2 (\mathbf{n}\cdot\mathbf{B})(\mathbf{B}\times\mathbf{n}) \right] \tag{5}$$

Taking the volume V to be the whole space, and S the boundary at infinity, we have

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\int \mathscr{L}(\mathbf{x})\, d^3x = -\int_\infty \mathbf{n}\cdot\overleftrightarrow{\mathbf{M}}\, da = \lim_{r\to\infty} \epsilon_0 \int r^3 \left[(\mathbf{n}\cdot\mathbf{E})(\mathbf{n}\times\mathbf{E}) + c^2 (\mathbf{n}\cdot\mathbf{B})(\mathbf{n}\times\mathbf{B}) \right] d\Omega \tag{6}$$

But the LHS of (6) is the *gain* of angular momentum of the field. In this problem, we are interested in the radiated angular momentum at infinity (i.e., loss), which can be obtained through negation of (6):

$$\frac{d\mathbf{L}_{\text{loss}}}{dt} = -\lim_{r\to\infty} \epsilon_0 \int r^3 \left[(\mathbf{n}\cdot\mathbf{E})(\mathbf{n}\times\mathbf{E}) + c^2 (\mathbf{n}\cdot\mathbf{B})(\mathbf{n}\times\mathbf{B}) \right] d\Omega \tag{7}$$

Note (7) is the instantaneous radiated angular momentum at infinity, and holds for any real field $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$. When we deal with complex fields, keep in mind that it is the real part of the fields that are physical.

For the harmonically oscillating fields, we obtain its time average by taking the complex conjugate of one of the factors in the product and multiply the overall result by $1/2$

$$\left\langle \frac{d\mathbf{L}_{\text{loss}}}{dt} \right\rangle = -\lim_{r\to\infty} \frac{\epsilon_0}{2} \int r^3\, \mathrm{Re}\left[(\mathbf{n}\cdot\mathbf{E})(\mathbf{n}\times\mathbf{E}^*) + c^2 (\mathbf{n}\cdot\mathbf{B})(\mathbf{n}\times\mathbf{B}^*) \right] d\Omega \tag{8}$$

Now consider the dipole field given by (9.18)

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left\{ k^2 (\mathbf{n}\times\mathbf{p})\times\mathbf{n}\frac{e^{ikr}}{r} + [3\mathbf{n}(\mathbf{n}\cdot\mathbf{p}) - \mathbf{p}]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{ikr} \right\} \tag{9}$$

$$\mathbf{H}(\mathbf{x}) = \frac{ck^2}{4\pi}\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)\mathbf{n}\times\mathbf{p} \tag{10}$$

we see that the contribution of the magnetic field in (8) vanishes since $\mathbf{H}$ is transverse, and

$$\mathbf{n}\cdot\mathbf{E} = \frac{1}{4\pi\epsilon_0}\cdot 2(\mathbf{n}\cdot\mathbf{p})\left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{ikr} \tag{11}$$

$$\mathbf{n}\times\mathbf{E}^* = \frac{1}{4\pi\epsilon_0}\left\{ k^2 \overbrace{\mathbf{n}\times[(\mathbf{n}\times\mathbf{p}^*)\times\mathbf{n}]}^{\mathbf{n}\times\mathbf{p}^*} \frac{e^{-ikr}}{r} - \mathbf{n}\times\mathbf{p}^*\left(\frac{1}{r^3} + \frac{ik}{r^2}\right) e^{-ikr} \right\} \tag{12}$$

In anticipation of the multiplication by r^3 and eventually taking the limit $r \to \infty$ in (8), we keep $(\mathbf{n}\cdot\mathbf{E})(\mathbf{n}\times\mathbf{E}^*)$ up to $O\left(1/r^3\right)$:

$$(\mathbf{n}\cdot\mathbf{E})(\mathbf{n}\times\mathbf{E}^*) = -\frac{ik^3}{8\pi^2\epsilon_0^2 r^3}(\mathbf{n}\cdot\mathbf{p})(\mathbf{n}\times\mathbf{p}^*) + O\left(\frac{1}{r^4}\right) \tag{13}$$

which turns (8) into

$$\begin{aligned}\left\langle \frac{d\mathbf{L}_{\text{loss}}}{dt}\right\rangle &= \frac{k^3}{16\pi^2\epsilon_0}\operatorname{Re}\left[i\int(\mathbf{n}\cdot\mathbf{p})(\mathbf{n}\times\mathbf{p}^*)\,d\Omega\right]\\ &= -\frac{k^3}{16\pi^2\epsilon_0}\operatorname{Im}\int(\mathbf{n}\cdot\mathbf{p})(\mathbf{n}\times\mathbf{p}^*)\,d\Omega\\ &= -\frac{k^3}{16\pi^2\epsilon_0}\operatorname{Im}\int n_l p_l \hat{\mathbf{e}}_k\epsilon_{ijk}n_i p_j^*\,d\Omega\\ &= -\frac{k^3}{16\pi^2\epsilon_0}\operatorname{Im}\left(\hat{\mathbf{e}}_k\epsilon_{ijk}p_l p_j^*\overbrace{\int n_l n_i\,d\Omega}^{\delta_{li}\cdot 4\pi/3}\right)\\ &= \frac{k^3}{12\pi\epsilon_0}\operatorname{Im}(\mathbf{p}^*\times\mathbf{p})\end{aligned} \tag{14}$$

2. From (9.24), the total power of radiation is

$$P = \frac{c^2 Z_0 k^4}{12\pi}|\mathbf{p}|^2 \tag{15}$$

thus

$$\frac{\langle d\mathbf{L}_{\text{loss}}/dt\rangle}{P} = \frac{1}{\omega}\frac{\operatorname{Im}(\mathbf{p}^*\times\mathbf{p})}{|\mathbf{p}|^2} \tag{16}$$

Our expectation is that this ratio has a connection to Quantum Mechanics, where the numerator is in units of $m\hbar$ and the denominator is $\hbar\omega$. Recall (4.5) for mapping $\mathbf{p}$ from Cartesian tensor components to the spherical tensor components

$$q_{1,\pm1} = \mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}\left(p_x \mp i p_y\right) \qquad\qquad q_{10} = \frac{1}{2}\sqrt{\frac{3}{\pi}}p_z \tag{17}$$

- To have only $m=-1$ component: we set $\mathbf{p} = p_0(1,-i,0)$, then $\operatorname{Im}(\mathbf{p}^*\times\mathbf{p})/|\mathbf{p}|^2 = -\hat{\mathbf{z}}$.
- To have only $m=0$ component, we set $\mathbf{p} = p_0(0,0,1)$, then $\operatorname{Im}(\mathbf{p}^*\times\mathbf{p})/|\mathbf{p}|^2 = 0$.
- To have only $m=1$ component, we set $\mathbf{p} = p_0(1,i,0)$, then $\operatorname{Im}(\mathbf{p}^*\times\mathbf{p})/|\mathbf{p}|^2 = \hat{\mathbf{z}}$.

which aligns with our expectation.

3. For a charge q moving in a circle of radius a with angular velocity ω, the dipole moment is

$$\mathbf{p} = qa(\cos\omega t, \sin\omega t, 0) = \operatorname{Re}\left[qae^{-i\omega t}(1,i,0)\right] \qquad \Longrightarrow \qquad \operatorname{Im}(\mathbf{p}^*\times\mathbf{p}) = 2q^2a^2\hat{\mathbf{z}} \tag{18}$$

Applying (14) gives

$$\left\langle\frac{d\mathbf{L}_{\text{loss}}}{dt}\right\rangle = \frac{k^3q^2a^2}{6\pi\epsilon_0}\hat{\mathbf{z}} \tag{19}$$

For charge oscillating along the z axis,

$$\mathbf{p} = qa(0,0,\cos\omega t) = \operatorname{Re}\left[qae^{-i\omega t}(0,0,1)\right] \qquad \Longrightarrow \qquad \operatorname{Im}(\mathbf{p}^*\times\mathbf{p}) = 0 \tag{20}$$

therefore it does not radiate angular momentum.

4. As shown in section 9.3, the magnetic dipole fields can be obtained from electric dipole with the following substitution

$$\mathbf{p}\to\frac{\mathbf{m}}{c} \qquad\qquad \mathbf{E}\to Z_0\mathbf{H} \qquad\qquad \mathbf{H}\to-\frac{\mathbf{E}}{Z_0} \tag{21}$$

So the equivalent of (14) for magnetic dipole is

$$\left\langle\frac{d\mathbf{L}_{\text{loss}}}{dt}\right\rangle = \frac{\mu_0 k^3}{12\pi}\operatorname{Im}(\mathbf{m}^*\times\mathbf{m}) \tag{22}$$