1. Consider the usual separation of variables in cylindrical coordinates, for potential at $z \to \infty$ to vanish, the z-dependency is going to be proportional to $e^{-k|z|}$ for k positive real.

   Given the symmetry in ϕ, the ϕ-dependency $A_m \cos m\phi + B_m \sin m\phi$ must be reduced to a constant, i.e., $m = 0$.

   For radial part, since $\rho = 0$ is in the range, we must only have J_m in the radial solution. In summary, the potential for $z > 0$ will have to have the form

$$\Phi(\rho,\phi,z) = \int_0^\infty \tilde{A}(k) J_0(k\rho) e^{-kz} dk \qquad \text{where by (3.110)} \tag{1}$$

$$\tilde{A}(k) = k \int_0^\infty \Phi(\rho,\phi,0) J_0(k\rho')\rho' d\rho' = kV \int_0^a J_0\left(k\rho'\right)\rho' d\rho' \tag{2}$$

   Plugging (2) into (1), and calculating for the axis point $\rho = 0$:

$$\begin{aligned}\Phi(0,\phi,z) &= \int_0^\infty \left[kV \int_0^a J_0\left(k\rho'\right)\rho' d\rho' \right] J_0(0) e^{-kz} dk \\ &= V \int_0^a \rho' d\rho' \underbrace{\int_0^\infty J_0(k\rho') k e^{-kz} dk}_{I}\end{aligned} \tag{3}$$

   where

$$\begin{aligned} I &= -\frac{d}{dz} \int_0^\infty J_0(k\rho') e^{-kz} dk \\ &= -\frac{d}{dz} \mathscr{L}\{J_0(k\rho')\}(z) \\ &= -\frac{d}{dz} \frac{1}{\sqrt{\rho'^2 + z^2}} = \frac{z}{\sqrt{\rho'^2+z^2}^3}\end{aligned} \tag{4}$$

   Thus

$$\Phi(0,\phi,z) = V \int_0^a \frac{z\rho' d\rho'}{\sqrt{\rho'^2 + z^2}^3} = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right) \tag{5}$$

2. Now we are looking for $\Phi(a,\phi,z)$. From (2)

$$\begin{aligned}\tilde{A}(k) &= kV \int_0^a J_0\left(k\rho'\right)\rho' d\rho' \\ &= kV \int_0^a \sum_{j=0}^\infty \frac{(-1)^j}{j!j!} \left(\frac{k\rho'}{a}\right)^{2j} \rho' d\rho' \\ &= kV \sum_{j=0}^\infty \frac{(-1)^j}{j!j!} \left(\frac{k}{2}\right)^{2j} \int_0^a \rho'^{2j+1} d\rho' \\ &= kV \sum_{j=0}^\infty \frac{(-1)^j}{j!(j+1)!} \frac{k^{2j} a^{2j+2}}{2^{2j+1}} \\ &= kV \cdot \frac{a}{k} J_1(ka) = VaJ_1(ka)\end{aligned} \tag{6}$$

   This gives

$$\Phi(a,\phi,z) = Va \int_0^\infty J_1(ka) J_0(ka) e^{-kz} dk \tag{7}$$

   The integral is non-trivial and can be looked up from the integral table (reference Eduardo Kausel and Mirza M. Irfan Baig, in particular ENS-4.7), the result is

$$\int_0^\infty J_1(ka) J_0(ka) e^{-kz} dk = I_{10}^0 = -\frac{1}{\pi a} \frac{kz}{2\sqrt{a\cdot a}} K(k) - \frac{\operatorname{sgn}(a-a)}{\pi a}\Lambda + \frac{1}{a} H(a-a) \tag{8}$$

where

$$k = \frac{2\sqrt{a \cdot a}}{\sqrt{(a+a)^2 + z^2}} = \frac{2a}{\sqrt{z^2 + 4a^2}} \tag{9}$$

and H is Heaviside step function which takes value of $1/2$ at 0, and K the complete elliptic integral of the first kind. Λ is irrelevant because of the zero factor in front of it.

Plugging these all into (7) gives

$$\Phi(a, \phi, z) = \frac{V}{2}\left[1 - \frac{kz}{\pi a} K(k)\right] \tag{10}$$