1. By Green's theorem (1.35)

$$\int_V \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) d^3x = \oint_S \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right] da \tag{1}$$

Substituting with $\phi = G(\mathbf{x},\mathbf{y})$ and $\psi = G(\mathbf{x}',\mathbf{y})$ yields

$$\int_V \left[G(\mathbf{x},\mathbf{y}) \nabla_{\mathbf{y}}^2 G(\mathbf{x}',\mathbf{y}) - G(\mathbf{x}',\mathbf{y}) \nabla_{\mathbf{y}}^2 G(\mathbf{x},\mathbf{y})\right] d^3y = \oint_S \left[G(\mathbf{x},\mathbf{y}) \frac{\partial G(\mathbf{x}',\mathbf{y})}{\partial n} - G(\mathbf{x}',\mathbf{y}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n}\right] da_{\mathbf{y}} \tag{2}$$

The LHS of (2) is

$$-4\pi \int_V \left[G(\mathbf{x},\mathbf{y}) \delta(\mathbf{y}-\mathbf{x}') - G(\mathbf{x}',\mathbf{y}) \delta(\mathbf{y}-\mathbf{x})\right] d^3y = -4\pi \left[G(\mathbf{x},\mathbf{x}') - G(\mathbf{x}',\mathbf{x})\right] \tag{3}$$

Therefore

$$G(\mathbf{x},\mathbf{x}') - G(\mathbf{x}',\mathbf{x}) = -\frac{1}{4\pi} \oint_S \left[G(\mathbf{x},\mathbf{y}) \frac{\partial G(\mathbf{x}',\mathbf{y})}{\partial n} - G(\mathbf{x}',\mathbf{y}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n}\right] da_{\mathbf{y}} \tag{4}$$

It's clear when G satisfies Dirichlet boundary condition, i.e., $G(\mathbf{x},\mathbf{y}) = G(\mathbf{x}',\mathbf{y}) = 0$ for $\mathbf{y}$ on S, we have $G(\mathbf{x},\mathbf{x}') = G(\mathbf{x}',\mathbf{x})$, i.e., G is symmetric in $\mathbf{x},\mathbf{x}'$.

2. With Neumann boundary condition, where

$$\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n} = \frac{\partial G(\mathbf{x}',\mathbf{y})}{\partial n} = -\frac{4\pi}{S} \qquad \text{for } \mathbf{y} \in S \tag{5}$$

(4) is turned into

$$G(\mathbf{x},\mathbf{x}') - G(\mathbf{x}',\mathbf{x}) = \frac{1}{S} \oint_S \left[G(\mathbf{x},\mathbf{y}) - G(\mathbf{x}',\mathbf{y})\right] da_{\mathbf{y}} \tag{6}$$

which in general does not vanish.

But if we define

$$H(\mathbf{x},\mathbf{x}') \equiv G(\mathbf{x},\mathbf{x}') - \overbrace{\frac{1}{S} \oint_S G(\mathbf{x},\mathbf{y})\, da_{\mathbf{y}}}^{F(\mathbf{x})} \tag{7}$$

we see that by (6)

$$H(\mathbf{x},\mathbf{x}') - H(\mathbf{x}',\mathbf{x}) = \left[G(\mathbf{x},\mathbf{x}') - \frac{1}{S} \oint_S G(\mathbf{x},\mathbf{y})\, da_{\mathbf{y}}\right] - \left[G(\mathbf{x}',\mathbf{x}) - \frac{1}{S} \oint_S G(\mathbf{x}',\mathbf{y})\, da_{\mathbf{y}}\right] = 0 \tag{8}$$

i.e., H is symmetric in $\mathbf{x},\mathbf{x}'$.

3. Recall given the Green function G, the potential satisfies the integral equation (1.42)

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{x}') G(\mathbf{x},\mathbf{x}') d^3x' + \frac{1}{4\pi} \oint_S \left[G(\mathbf{x},\mathbf{x}') \frac{\partial \Phi}{\partial n'} - \Phi(\mathbf{x}') \frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}\right] da' \tag{9}$$

If we were to replace $G(\mathbf{x},\mathbf{x}')$ with $H(\mathbf{x},\mathbf{x}') = G(\mathbf{x},\mathbf{x}') - F(\mathbf{x})$ in (9), we end up with the following extra term on the RHS of (9):

$$-F(\mathbf{x}) \cdot \left[\frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{x}') d^3x' + \frac{1}{4\pi} \oint_S \frac{\partial \Phi}{\partial n'} da'\right] = -\frac{F(\mathbf{x})}{4\pi} \left[\int_V \rho(\mathbf{x}') d^3x + \oint_S \frac{\partial \Phi}{\partial n'} da'\right] \tag{10}$$

which equals zero by Gauss's Theorem.