1. **Prob 11.16**

   (a) Obviously, both sides of the alleged equation

$$J^\alpha - \frac{1}{c^2}\left(U_\beta J^\beta\right)U^\alpha = \frac{\sigma}{c}F^{\alpha\beta}U_\beta \tag{1}$$

   transform contravariantly with respect to the Lorentz transformation. So if they are equal in one inertial frame, the equality shall hold in all inertial frames. It is most convenient to work in the rest frame K' where there U'^α is simplest. The verification of (1) in K' is straightforward by noting that

$$U'^0 = c \qquad U'^i = U'_i = 0 \qquad J'^0 = c\rho' \qquad F^{00} = 0 \qquad F^{i0} = E^i \qquad J'^i = \sigma E^i \tag{2}$$

   (b) If K' is moving at velocity $\mathbf{v}$ with respect to K, the space dimension i for (1) reads

$$\begin{aligned} J^i - \frac{1}{c^2}\left(U_\beta J^\beta\right)U^i &= \frac{\sigma}{c}F^{i\beta}U_\beta \qquad \Longrightarrow \\ J^i - \frac{1}{c^2}\left[\gamma c^2\rho - \gamma\left(\mathbf{v}\cdot\mathbf{J}\right)\right]\gamma v^i &= \frac{\sigma}{c}\left(\gamma c E^i + \gamma F^{ij}v_j\right) \end{aligned} \tag{3}$$

   From (11.137), we see that

$$F^{ij} = \epsilon^{ijk}B_k \tag{4}$$

   where B_k is the k-th covariant component (e.g., $B_3 = -B_z$), which turns (3) into

$$J^i - \gamma^2\rho v^i + \frac{\gamma^2}{c^2}\left(\mathbf{v}\cdot\mathbf{J}\right)v^i = \frac{\gamma\sigma}{c}\left[cE^i + \left(\mathbf{v}\times\mathbf{B}\right)^i\right] \qquad \text{or} \qquad \mathbf{J} - \gamma^2\rho\mathbf{v} + \gamma^2\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{J}\right) = \gamma\sigma\left(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}\right) \tag{5}$$

   Taking $\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\right)$ on both sides gives

$$\begin{aligned} \boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{J}\right) - \gamma^2\beta^2\rho\mathbf{v} + \gamma^2\beta^2\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{J}\right) &= \gamma\sigma\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right) && \Longrightarrow \\ \left(1+\gamma^2\beta^2\right)\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{J}\right) - \gamma^2\beta^2\rho\mathbf{v} &= \gamma\sigma\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right) \qquad \text{note } \gamma^2\beta^2 = \gamma^2 - 1 && \Longrightarrow \\ \gamma^2\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{J}\right) &= \gamma^2\beta^2\rho\mathbf{v} + \gamma\sigma\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right) \end{aligned} \tag{6}$$

   Putting (6) back into (5) yields the desired result

$$\mathbf{J} = \gamma\sigma\left[\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B} - \boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\right] + \rho\mathbf{v} \tag{7}$$

   We see that the time dimension of (1) requires

$$c\rho - \frac{1}{c^2}\left[\gamma c^2\rho - \gamma\left(\mathbf{v}\cdot\mathbf{J}\right)\right]\gamma c = \frac{\gamma\sigma}{c}\mathbf{E}\cdot\mathbf{v} \qquad \text{or} \qquad \left(1-\gamma^2\right)c\rho + \gamma^2\boldsymbol{\beta}\cdot\mathbf{J} = \gamma\sigma\boldsymbol{\beta}\cdot\mathbf{E} \tag{8}$$

   which is consistent with (6).

   (c) Applying the Lorentz transformation to the 4-current gives

$$\begin{bmatrix} c\rho' \\ \mathbf{J}' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\boldsymbol{\beta}^T \\ -\gamma\boldsymbol{\beta} & I + (\gamma-1)\dfrac{\boldsymbol{\beta}\boldsymbol{\beta}^T}{\beta^2} \end{bmatrix} \begin{bmatrix} c\rho \\ \mathbf{J} \end{bmatrix} \tag{9}$$

   The condition that $\rho' = 0$ requires

$$\gamma c\rho - \gamma\boldsymbol{\beta}\cdot\mathbf{J} = 0 \tag{10}$$

   Putting (6) and (10) together, we can solve for $\mathbf{J}$ and ρ and get

$$\rho = \frac{\gamma\sigma\boldsymbol{\beta}\cdot\mathbf{E}}{c} \qquad \mathbf{J} = \gamma\sigma\left(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}\right) \tag{11}$$

2. **Prob 11.17**

(a) Both sides of the alleged equation

$$F^{\alpha\beta} = \frac{q}{c}\frac{\left(X^\alpha U^\beta - X^\beta U^\alpha\right)}{\left[\frac{1}{c^2}\left(U_\mu X^\mu\right)^2 - X_\mu X^\mu\right]^{3/2}} \tag{12}$$

are anti-symmetric contravariant tensor of rank 2, so they must transform contravariantly under Lorentz transformation.
In frame K,

$$X^\alpha = \begin{bmatrix} ct - ct \\ \mathbf{x}_p - \mathbf{x}_q \end{bmatrix} = \begin{bmatrix} 0 \\ b\hat{\mathbf{e}}_2 - vt\hat{\mathbf{e}}_1 \end{bmatrix} \qquad U^\alpha = \begin{bmatrix} \gamma c \\ \gamma v \hat{\mathbf{e}}_1 \end{bmatrix} \tag{13}$$

so

$$U_\mu X^\mu = \gamma v^2 t \qquad X_\mu X^\mu = -\left(b^2 + v^2 t^2\right) \qquad \Longrightarrow \qquad \frac{1}{c^2}\left(U_\mu X^\mu\right)^2 - X_\mu X^\mu = b^2 + \gamma^2 v^2 t^2 \tag{14}$$

giving various components of the RHS of (12)

$$\mathrm{RHS}^{10} = \frac{q}{c}\frac{\left(X^1 U^0 - X^0 U^1\right)}{(b^2 + \gamma^2 v^2 t)^{3/2}} = \frac{q}{c}\frac{(-\gamma c v t)}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} = -\frac{q\gamma v t}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \tag{15}$$

$$\mathrm{RHS}^{20} = \frac{q}{c}\frac{\left(X^2 U^0 - X^0 U^2\right)}{(b^2 + \gamma^2 v^2 t)^{3/2}} = \frac{q}{c}\frac{b\gamma c}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} = \frac{q\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \tag{16}$$

$$\mathrm{RHS}^{21} = \frac{q}{c}\frac{\left(X^2 U^1 - X^1 U^2\right)}{(b^2 + \gamma^2 v^2 t)^{3/2}} = F^{21} = \frac{q}{c}\frac{\gamma b v}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} = \beta E^2 \tag{17}$$

and all other components are zero. These agree with the corresponding components of F in K as shown in (11.152). This means the tensorial equation (12) holds in K.

(b) We will prove that the equation

$$F^{\alpha\beta} = \frac{q}{c}\frac{\left(Y^\alpha U^\beta - Y^\beta U^\alpha\right)}{\left(-Y_\mu Y^\mu\right)^{3/2}} \tag{18}$$

holds in K'.
Indeed,

$$Y'^\alpha = \begin{bmatrix} ct' - ct' \\ \mathbf{x}'_p - \mathbf{x}'_q \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{r}' \end{bmatrix} \qquad U'^\alpha = \begin{bmatrix} c \\ 0 \end{bmatrix} \tag{19}$$

where $\mathbf{r}'$ is the position of the observation point in the rest frame of the charge. It's easy to see that

$$Y'_\mu Y'^\mu = -\left|\mathbf{r}'\right|^2 \tag{20}$$

Then we have

$$(\alpha,\beta) = (i,0): \qquad Y'^\alpha U'^\beta - Y'^\beta U'^\alpha = Y'^i U'^0 - Y'^0 U'^i = c r'^i \qquad \Longrightarrow \qquad \mathrm{RHS}^{i0} = \frac{q r'^i}{|\mathbf{r}'|^3}$$

$$(\alpha,\beta) = (i,j): \qquad Y'^\alpha U'^\beta - Y'^\beta U'^\alpha = Y'^i U'^j - X'^j U'^i = 0 \qquad \Longrightarrow \qquad \mathrm{RHS}^{ij} = 0$$

which are exactly the static electric and magnetic (null) field due to the at-rest charge in K', so (18) holds in K'.

(c) Let the retarded time be $t - \Delta t$, by definition, it must satisfy

$$b^2 + [v(t - \Delta t)]^2 = (c\Delta t)^2 \tag{21}$$

Let $R = c\Delta t$, then (22) becomes

$$\left(1 - \beta^2\right)R^2 + 2\beta v t R - \left(b^2 + v^2 t^2\right) = 0 \tag{22}$$

For

$$Z^\mu = \begin{bmatrix} R \\ b\hat{\mathbf{y}} - \beta\,(ct - R)\,\hat{\mathbf{x}} \end{bmatrix} \tag{23}$$

with (22), we have

$$\frac{1}{c} U_\mu Z^\mu = \frac{1}{c}\left[\gamma c R + \beta\gamma v\,(ct - R)\right] = \gamma R\left(1-\beta^2\right) + \beta\gamma v t = \sqrt{b^2 + \gamma^2 v^2 t^2} \tag{24}$$

Then the routine check for the components shows that the antisymmetric tensor

$$\frac{q}{c}\frac{\left(Z^\alpha U^\beta - Z^\beta U^\alpha\right)}{\left(\frac{1}{c} U_\mu Z^\mu\right)^{3/2}} \tag{25}$$

is equivallent to the F tensor in K.

It is worth mentioning that the tensor form (18) and (25) are simplified versions of (12), thanks to the specially constructed Y'^μ and Z^μ, so that $U'_\mu Y'^\mu = 0$ and $Z_\mu Z^\mu = 0$.