1. First let's note that with

$$n(x) = n(0)\,\mathrm{sech}(\alpha x) \tag{1}$$

we have

$$n(x_{\max}) = n(0)\,\mathrm{sech}(\alpha x_{\max}) = n(0)\cos\theta_0 \qquad \Longrightarrow \qquad \cosh(\alpha x_{\max})\cos\theta_0 = 1 \tag{2}$$

We shall show that

$$\alpha x = \sinh^{-1}\left[\sinh(\alpha x_{\max})\sin(\alpha z)\right] \tag{3}$$

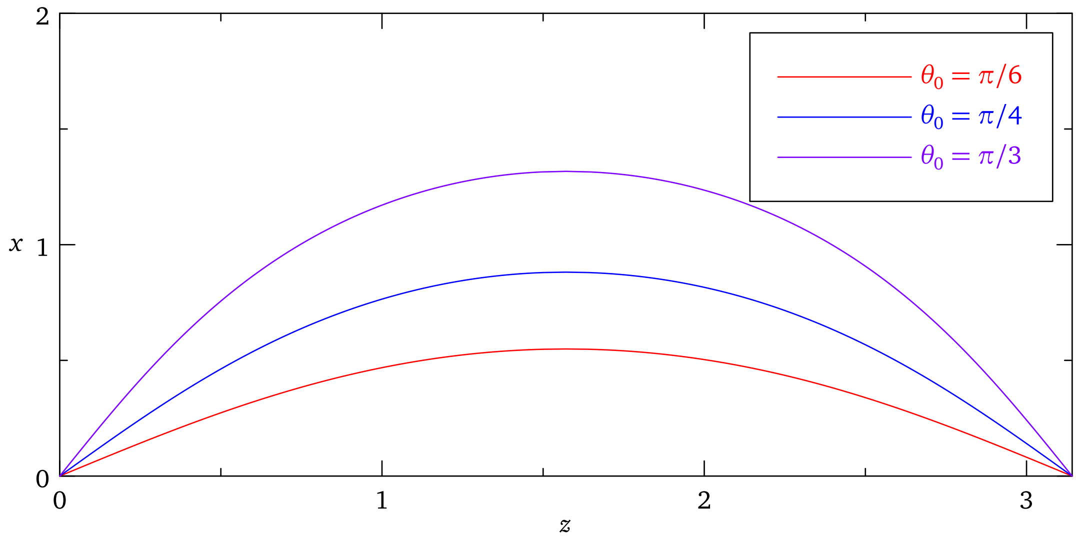
satisfies the Eikonal equation (8.116)

$$\bar{n}^2 \frac{d^2 x}{dz^2} = n(x)\frac{dn(x)}{dx} \qquad \text{or equivalently} \qquad \cos^2\theta_0 \frac{d^2 x}{dz^2} = \mathrm{sech}(\alpha x)\frac{d\,\mathrm{sech}(\alpha x)}{dx} \tag{4}$$

The verification of this result is straightforward with (2) and the following derivatives

$$\frac{d\,\mathrm{sech}\,t}{dt} = -\mathrm{sech}\,t\tanh t \qquad\qquad \frac{d\sinh^{-1} t}{dt} = \frac{1}{\sqrt{1+t^2}} \tag{5}$$

The rays are traced below for $\alpha = 1$.



2. The $z_{\max}$ corresponding to $x_{\max}$ satisfies

$$\sinh(\alpha x_{\max}) = \sinh(\alpha x_{\max})\sin(\alpha z_{\max}) \qquad \Longrightarrow \qquad \alpha z_{\max} = \frac{\pi}{2} \qquad \Longrightarrow \qquad Z = 2z_{\max} = \frac{\pi}{\alpha} \tag{6}$$

which does not depend on the launch angle θ_0.

3. By equation (8.119),

$$\begin{aligned}
L_{\text{opt}} &= 2\int_0^{x_{\max}} \frac{n^2(x)\,dx}{\sqrt{n^2(x)-\bar{n}^2}} \\
&= 2n(0)\int_0^{x_{\max}} \frac{\mathrm{sech}^2(\alpha x)\,dx}{\sqrt{\mathrm{sech}^2(\alpha x)-\cos^2\theta_0}} \\
&= 2n(0)\int_0^{x_{\max}} \frac{1/\cosh(\alpha x)\,dx}{\sqrt{1-\cos^2\theta_0\cosh^2(\alpha x)}} && \text{by (2)} \\
&= 2n(0)\int_0^{x_{\max}} \frac{\cosh(\alpha x_{\max})/\cosh(\alpha x)\,dx}{\sqrt{\cosh^2(\alpha x_{\max})-\cosh^2(\alpha x)}} \\
&= 2n(0)\int_0^{x_{\max}} \frac{\cosh(\alpha x_{\max})/\cosh(\alpha x)\,dx}{\sqrt{\sinh^2(\alpha x_{\max})-\sinh^2(\alpha x)}} \\
&= 2n(0)\frac{\cosh(\alpha x_{\max})}{\sinh(\alpha x_{\max})}\int_0^{x_{\max}} \frac{dx}{\cosh(\alpha x)\cos(\alpha z)}
\end{aligned} \tag{7}$$

Following the hint, let's make a variable change

$$x = \frac{1}{\alpha} \sinh^{-1} \left[\sinh(\alpha x_{\max}) \sin t \right] \qquad \Longrightarrow \qquad dx = \frac{1}{\alpha} \cdot \frac{\sinh(\alpha x_{\max}) \cos t \, dt}{\sqrt{1 + \sinh^2(\alpha x_{\max}) \sin^2 t}} \tag{8}$$

which gives

$$\begin{aligned} L_{\text{opt}} &= 2n(0) \cosh(\alpha x_{\max}) \frac{1}{\alpha} \int_0^{\pi/2} \frac{dt}{1 + \sinh^2(\alpha x_{\max}) \sin^2 t} && \text{by hint} \\ &= 2n(0) \cosh(\alpha x_{\max}) \frac{1}{\alpha} \cdot \frac{\pi}{2 \cosh(\alpha x_{\max})} \\ &= n(0) \frac{\pi}{\alpha} \\ &= n(0) Z \end{aligned} \tag{9}$$

which means rays with different launch angles will travel back to $x = 0$ with the same time interval.