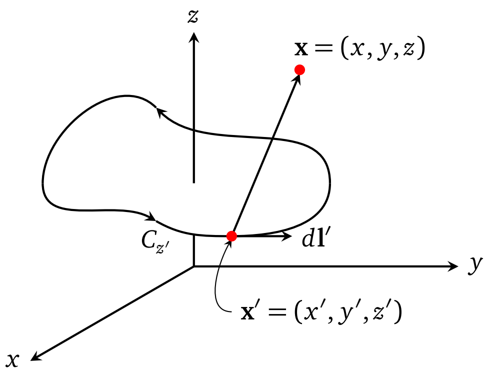

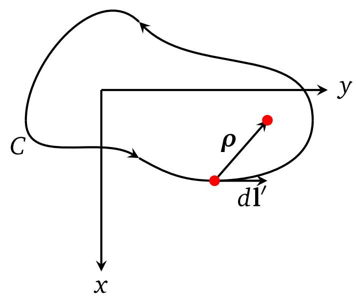

1. Consider a thin slice of cross section at $z' \to z' + dz'$, let $C_{z'}$ denote its boundary loop which lies in the $z = z'$ plane. Its contribution to the field at $\mathbf{x} = (x, y, z)$ is

$$d\mathbf{B}_{z'} = \frac{\mu_0 N I dz'}{4\pi} \oint_{C_{z'}} d\mathbf{l}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \tag{1}$$

Integrating (1) with $z' : -\infty \to \infty$ gives the field

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 N I}{4\pi} \int_{-\infty}^{\infty} dz' \oint_{C_{z'}} d\mathbf{l}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \tag{2}$$

whose k-th component is

$$B_k = \frac{\mu_0 N I}{4\pi} \int_{-\infty}^{\infty} dz' \epsilon_{ijk} \oint_{C_{z'}} dl'_i \frac{x_j - x'_j}{|\mathbf{x} - \mathbf{x}'|^3} \tag{3}$$

Since the current flows parallel to the x-y plane, all loops $C_{z'}$ are actually planer loops, whose dl'_3 component is zero, thus we can exchange the integration order to obtain

$$B_k = \frac{\mu_0 N I}{4\pi} \epsilon_{ijk} \oint_C dl'_i \int_{-\infty}^{\infty} dz' \frac{x_j - x'_j}{|\mathbf{x} - \mathbf{x}'|^3} \tag{4}$$

where C is taken to be in the $z = 0$ plane without loss of generality. Furthermore, the inner integral vanishes for $j = 3$ due to its odd parity. This reduces (4) to

$$B_1 = B_2 = 0 \tag{5}$$

$$B_3 = \frac{\mu_0 N I}{4\pi} \epsilon_{ij3} \oint_C dl'_i \left(x_j - x'_j\right) \overbrace{\int_{-\infty}^{\infty} \frac{dz'}{|\mathbf{x} - \mathbf{x}'|^3}}^{I} \tag{6}$$

where

$$I = \int_{-\infty}^{\infty} \frac{dz'}{\sqrt{\underbrace{\left(x - x'\right)^2 + \left(y - y'\right)^2}_{\equiv \rho^2} + (z - z')^2}^{\,3}} \qquad \left(\text{let } z' - z \equiv \rho \tan \xi\right)$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\rho / \cos^2 \xi \, d\xi}{\rho^3 / \cos^3 \xi} = \frac{2}{\rho^2} \tag{7}$$

Now (6) becomes

$$B_3 = \frac{\mu_0 N I}{2\pi} \left(\oint_C d\mathbf{l}' \times \frac{\boldsymbol{\rho}}{\rho^2} \right)_3 \qquad \text{where } \boldsymbol{\rho} = (x - x', y - y', 0) \tag{8}$$

From the 2D figure above on the right, it's clear that $|d\mathbf{l}' \times \boldsymbol{\rho}|$ is, as viewed from the eye at $(x, y, 0)$, the area sweeped by the differential line segment $d\mathbf{l}'$, hence $|d\mathbf{l}' \times \boldsymbol{\rho}|/\rho^2$ is the sweeped angle. Therefore the integral in (8) gives the total sweeped angle as the eye traces the loop for one round. So for $\mathbf{x}$ enclosed in the solenoid, this integral produces 2π, but vanishes if $\mathbf{x}$ is outside.

In summary

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \begin{cases} NI\hat{\mathbf{z}} & \text{for inside points} \\ 0 & \text{for outside points} \end{cases} \tag{9}$$

2. With a "realistic" solenoid, a single wire is arranged in a helix shape where it slowly stacks up in the z-direction. Consider a plane parallel to the x-y plane, the coil's wire intersects with this plane at one point, which means the current through this plane (hence in the z-direction) is I. If we integrate the field along a circle with radius b outside of the solenoid, we will have

$$B_\phi 2\pi b = \mu_0 I \qquad \Longrightarrow \qquad B_\phi = \frac{\mu_0 I}{2\pi b} \tag{10}$$

I.e., we have a tangential field in the $\hat{\boldsymbol{\phi}}$ direction outside of the solenoid, this is the same field generated by the current I flowing in the z direction.