1. By definition, the multipole moment q_{lm} is

$$q_{lm} = \int Y^*_{lm}\left(\theta', \phi'\right) r'^l \rho\left(\mathbf{x}'\right) d^3x' \tag{1}$$

where for case (a), the density $\rho\left(\mathbf{x}'\right)$ from the four point charges is

$$\rho\left(\mathbf{x}'\right) = q \cdot \frac{\delta\left(r'-a\right)\delta\left(\theta'-\pi/2\right)}{r'^2 \sin\theta'}\left[\delta\left(\phi'\right)+\delta\left(\phi'-\frac{\pi}{2}\right)-\delta\left(\phi'-\pi\right)-\delta\left(\phi'-\frac{3\pi}{2}\right)\right] \tag{2}$$

Inserting (2) into (1) yields

$$\begin{aligned} q_{lm} =& q\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int_0^\infty r'^l r'^2 \frac{\delta\left(r'-a'\right)}{r'^2} dr' \int_0^\pi \sin\theta' \frac{\delta\left(\theta'-\pi/2\right)}{\sin\theta'} P_l^m\left(\cos\theta'\right) d\theta' \\ &\times \int_0^{2\pi} e^{-im\phi'}\left[\delta\left(\phi'\right)+\delta\left(\phi'-\frac{\pi}{2}\right)-\delta\left(\phi'-\pi\right)-\delta\left(\phi'-\frac{3\pi}{2}\right)\right] d\phi' \\ =& q\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \cdot a^l P_l^m(0)\left[1+(-i)^m-(-1)^m-i^m\right] \\ =& q\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \cdot a^l P_l^m(0)\left(1-i^m\right)\left[1-(-1)^m\right] \end{aligned} \tag{3}$$

which clearly vanishes unless m is odd.
Recall that the associated Legendre funciton $P_l^m(x)$ has definite parity:

$$P_l^m(-x) = (-1)^{l+m} P_l^m(x) \tag{4}$$

this means for odd m, (3) will vanish unless l is also odd, therefore

$$q_{lm} = \begin{cases} 2q\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} a^l P_l^m(0)\left(1-i^m\right) & \text{for } l, m \text{ odd} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

This gives the first few non-vanishing moments

$$q_{11} = 2q\sqrt{\frac{3}{4\pi\cdot 2}} a(-1)(1-i) = qa\sqrt{\frac{3}{2\pi}}(i-1) \tag{6}$$

$$q_{1,-1} = 2q\sqrt{\frac{3\cdot 2}{4\pi\cdot}} a\left(\frac{1}{2}\right)(1+i) = qa\sqrt{\frac{3}{2\pi}}(1+i) \tag{7}$$

$$q_{33} = 2q\sqrt{\frac{7}{4\pi\cdot 6!}} a^3(-15)(1+i) = -qa^3\sqrt{\frac{35}{16\pi}}(1+i) \tag{8}$$

$$q_{31} = 2q\sqrt{\frac{7}{4\pi\cdot 12}} a^3\left(\frac{3}{2}\right)(1-i) = qa^3\sqrt{\frac{21}{16\pi}}(1-i) \tag{9}$$

$$q_{3,-1} = 2q\sqrt{\frac{7\cdot 12}{4\pi}} a^3\left(-\frac{1}{8}\right)(1+i) = -qa^3\sqrt{\frac{21}{16\pi}}(1+i) \tag{10}$$

$$q_{3,-3} = 2q\sqrt{\frac{7\cdot 6!}{4\pi}} a^3\left(\frac{1}{48}\right)(1+i) = qa^3\sqrt{\frac{35}{16\pi}}(1-i) \tag{11}$$

2. For case (b), the density is

$$\rho\left(\mathbf{x}'\right) = q \cdot \frac{1}{2\pi} \cdot \frac{\delta\left(r'-a\right)}{r'^2} \cdot \frac{\delta\left(\theta'\right)+\delta\left(\theta'-\pi\right)}{\sin\theta'} - 2q\delta\left(\mathbf{x}'\right) \tag{12}$$

(where the $1/2\pi$ factor in the first term is necessary to restore the point charge after integrating over the neighborhood of $z = \pm a$ due to the ϕ independence).

Given the cylindrical symmetry, the integral of (1) will vanish unless $m = 0$, which gives

$$q_{l0} = q\sqrt{\frac{2l+1}{4\pi}}a^l \cdot [P_l(1) + P_l(-1)] - 2q\sqrt{\frac{1}{4\pi}}\delta_{l0}$$
$$= q\sqrt{\frac{2l+1}{4\pi}}a^l\left[1+(-1)^l\right] - \frac{q}{\sqrt{\pi}}\delta_{l0} = \begin{cases} \frac{q}{\sqrt{\pi}}\left[\sqrt{2l+1}a^l - \delta_{l0}\right] & \text{for } l \text{ even} \\ 0 & \text{for } l \text{ odd} \end{cases} \tag{13}$$

The first few non-vanishing moments are

$$q_{20} = qa^2\sqrt{\frac{5}{\pi}} \qquad\qquad q_{40} = qa^4\sqrt{\frac{9}{\pi}} \tag{14}$$

3. For case (b), when we keep only the lowest order term q_{20} (which is the dominant term for $r > a$), the potential can be approximated by equation (4.1)

$$\Phi(\mathbf{x}) \approx \frac{1}{4\pi\epsilon_0} \cdot \frac{4\pi}{5} \cdot qa^2\sqrt{\frac{5}{\pi}}\frac{Y_{20}(\theta,\phi)}{r^3}$$
$$= \frac{qa^2}{5\epsilon_0 r^3}\sqrt{\frac{5}{\pi}}\sqrt{\frac{5}{16\pi}}\left(-1+3\cos^2\theta\right)$$
$$= \frac{qa^2}{4\pi\epsilon_0 r^3}\left(-1+3\cos^2\theta\right) \tag{15}$$

For $r > a$ in the x-y plane, this becomes

$$\Phi(x,y) \approx -\frac{qa^2}{4\pi\epsilon_0 r^3} \tag{16}$$

4. The exact potential in case (b) on the x-y plane is

$$\Phi(x,y) = \frac{2q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{a^2+r^2}} - \frac{1}{r}\right) \tag{17}$$

For $r > a$, this can be approximated by

$$\Phi(x,y) = \frac{q}{2\pi\epsilon_0 r}\left[1 - \frac{1}{2}\left(\frac{a}{r}\right)^2 - 1 + O\left(\frac{1}{r^4}\right)\right] \approx -\frac{qa^2}{4\pi\epsilon_0 r^3} \tag{18}$$

which is well approximated by (17) with an error of order r^{-5}.

The plots of (17) and (18) are shown below