1. We start by examining $\mathbf{E}\times\mathbf{B}$:

$$\begin{aligned}
\mathbf{E}\times\mathbf{B} = \mathbf{E}\times(\nabla\times\mathbf{A}) &= \sum_{ijk}\hat{\mathbf{e}}_k\epsilon_{ijk}E_i\left(\sum_{lmj}\epsilon_{lmj}\frac{\partial A_m}{\partial x_l}\right)\\
&= \sum_{iklm}\hat{\mathbf{e}}_k E_i\frac{\partial A_m}{\partial x_l}\sum_j\epsilon_{ijk}\epsilon_{lmj} = \sum_{iklm}\hat{\mathbf{e}}_k E_i\frac{\partial A_m}{\partial x_l}\left(\delta_{im}\delta_{kl}-\delta_{il}\delta_{km}\right)\\
&= \sum_{ik}\hat{\mathbf{e}}_k E_i\left(\frac{\partial A_i}{\partial x_k}-\frac{\partial A_k}{\partial x_i}\right)\\
&= \sum_i E_i\left(\nabla A_i\right)-(\mathbf{E}\cdot\nabla)\mathbf{A} \qquad (1)
\end{aligned}$$

Thus the integrand becomes

$$\mathbf{x}\times(\mathbf{E}\times\mathbf{B}) = \overbrace{\mathbf{x}\times\left[\sum_i E_i\left(\nabla A_i\right)\right]}^{\mathbf{X}} - \overbrace{\mathbf{x}\times\left[(\mathbf{E}\cdot\nabla)\mathbf{A}\right]}^{\mathbf{Y}} \qquad (2)$$

The first term

$$\mathbf{X} = \sum_{lmk}\hat{\mathbf{e}}_k\epsilon_{lmk}x_l\left(\sum_i E_i\frac{\partial A_i}{\partial x_m}\right) = \sum_i E_i\left(\sum_{lmk}\hat{\mathbf{e}}_k\epsilon_{lmk}x_l\frac{\partial}{\partial x_m}\right)A_i = \sum_i E_i\left(\mathbf{x}\times\nabla\right)A_i \qquad (3)$$

The second term

$$\begin{aligned}
\mathbf{Y} = \sum_{ijk}\hat{\mathbf{e}}_k\epsilon_{ijk}x_i\left(\sum_l E_l\frac{\partial A_j}{\partial x_l}\right) &= \sum_{ijk}\hat{\mathbf{e}}_k\epsilon_{ijk}\left\{\sum_l E_l\left[\frac{\partial\left(x_iA_j\right)}{\partial x_l}-A_j\delta_{il}\right]\right\}\\
&= \sum_l E_l\frac{\partial}{\partial x_l}\sum_{ijk}\hat{\mathbf{e}}_k\epsilon_{ijk}x_iA_j - \sum_{ijk}\hat{\mathbf{e}}_k\epsilon_{ijk}E_iA_j\\
&= \sum_l E_l\frac{\partial\left(\mathbf{x}\times\mathbf{A}\right)}{\partial x_l}-\mathbf{E}\times\mathbf{A}\\
&= (\mathbf{E}\cdot\nabla)(\mathbf{x}\times\mathbf{A})-\mathbf{E}\times\mathbf{A} \qquad (4)
\end{aligned}$$

Thus

$$\int d^3x\,\mathbf{x}\times(\mathbf{E}\times\mathbf{B}) = \int d^3x\left[\mathbf{E}\times\mathbf{A}+\sum_i E_i\left(\mathbf{x}\times\nabla\right)A_i\right] - \int d^3x\,(\mathbf{E}\cdot\nabla)(\mathbf{x}\times\mathbf{A}) \qquad (5)$$

For the claim to hold, the second integral must vanish.
Indeed, for the free field, $\nabla\cdot\mathbf{E}=0$, we have

$$\begin{aligned}
(\mathbf{E}\cdot\nabla)(\mathbf{x}\times\mathbf{A}) &= (\mathbf{E}\cdot\nabla+\nabla\cdot\mathbf{E})(\mathbf{x}\times\mathbf{A}) \qquad \text{denote } \mathbf{b}=\mathbf{x}\times\mathbf{A}\\
&= \sum_i\left(E_i\frac{\partial}{\partial x_i}+\frac{\partial E_i}{\partial x_i}\right)\sum_j\hat{\mathbf{e}}_j b_j\\
&= \sum_j\hat{\mathbf{e}}_j\left[\sum_i\frac{\partial\left(E_ib_j\right)}{\partial x_i}\right] = \sum_j\hat{\mathbf{e}}_j\nabla\cdot\left(b_j\mathbf{E}\right) \qquad (6)
\end{aligned}$$

So the second integral in (5) yields

$$\sum_j\hat{\mathbf{e}}_j\int d^3x\,\nabla\cdot\left(b_j\mathbf{E}\right) = \sum_j\hat{\mathbf{e}}_j\oint_\infty\left(b_j\mathbf{E}\right)\cdot\mathbf{n}\,da = 0 \qquad (7)$$

where we have used the local distribution assumption so $b_j\mathbf{E}$ vanishes at infinity.

2. The expansion of vector potential in radiation gauge is

$$\mathbf{A}(\mathbf{x},t)=\sum_{\lambda}\int \frac{d^3k}{(2\pi)^3}\left[\boldsymbol{\epsilon}_{\lambda}(\mathbf{k})\,a_{\lambda}(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}+\boldsymbol{\epsilon}^*_{\lambda}(\mathbf{k})\,a^*_{\lambda}(\mathbf{k})\,e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}\right] \qquad \text{where } \omega=c\,|\mathbf{k}| \tag{8}$$

Let $\mathscr{A}_{\lambda}(\mathbf{k})=\boldsymbol{\epsilon}_{\lambda}(\mathbf{k})\,a_{\lambda}(\mathbf{k})$ be the amplitude of the constituent plane wave $(\mathbf{k},\lambda)$, hence $\mathscr{A}^*_{\lambda}(\mathbf{k})=\boldsymbol{\epsilon}^*_{\lambda}(\mathbf{k})\,a^*_{\lambda}(\mathbf{k})$ is the amplitude of the negative frequency plane wave $(-\mathbf{k},\lambda)$.

The corresponding $\mathbf{E}$ field amplitudes, $\mathscr{E}_{\lambda}(\mathbf{k})$ and $\mathscr{E}^*_{\lambda}(\mathbf{k})$ can be obtained by the relation $\mathbf{E}=-\partial\mathbf{A}/\partial t$ implied by the Coulomb gauge (or radiation gauge), i.e.,

$$\mathscr{E}_{\lambda}(\mathbf{k})=i\omega\mathscr{A}_{\lambda}(\mathbf{k})=i\omega a_{\lambda}(\mathbf{k})\,\boldsymbol{\epsilon}_{\lambda}(\mathbf{k}) \qquad \mathscr{E}^*_{\lambda}(\mathbf{k})=-i\omega\mathscr{A}^*_{\lambda}(\mathbf{k})=-i\omega a^*_{\lambda}(\mathbf{k})\,\boldsymbol{\epsilon}^*_{\lambda}(\mathbf{k}) \tag{9}$$

The spin is thus

$$\begin{aligned}
\mathbf{L}_{\text{spin}} &=\frac{1}{\mu_0c^2}\int d^3x\,\mathbf{E}(\mathbf{x},t)\times\mathbf{A}(\mathbf{x},t)\\
&=\frac{1}{\mu_0c^2}\int d^3x\sum_{\lambda,\mu}\left\{\int\frac{d^3k}{(2\pi)^3}\left[\mathscr{E}_{\lambda}(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}+\mathscr{E}^*_{\lambda}(\mathbf{k})\,e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}\right]\right\}\times\\
&\qquad\left\{\int\frac{d^3k'}{(2\pi)^3}\left[\mathscr{A}_{\mu}\left(\mathbf{k}'\right)e^{i\mathbf{k}'\cdot\mathbf{x}-i\omega' t}+\mathscr{A}^*_{\mu}\left(\mathbf{k}'\right)e^{-i\mathbf{k}'\cdot\mathbf{x}+i\omega' t}\right]\right\}
\end{aligned} \tag{10}$$

For a particular combination of λ,μ, the integral can be written

$$\begin{aligned}
I_{\lambda,\mu}=\int d^3x\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\Big[&\mathscr{E}_{\lambda}(\mathbf{k})\times\mathscr{A}_{\mu}\left(\mathbf{k}'\right)e^{i\left(\mathbf{k}+\mathbf{k}'\right)\cdot\mathbf{x}}e^{-i\left(\omega+\omega'\right)t}+\\
&\mathscr{E}_{\lambda}(\mathbf{k})\times\mathscr{A}^*_{\mu}\left(\mathbf{k}'\right)e^{i\left(\mathbf{k}-\mathbf{k}'\right)\cdot\mathbf{x}}e^{-i\left(\omega-\omega'\right)t}+\\
&\mathscr{E}^*_{\lambda}(\mathbf{k})\times\mathscr{A}_{\mu}\left(\mathbf{k}'\right)e^{-i\left(\mathbf{k}-\mathbf{k}'\right)\cdot\mathbf{x}}e^{i\left(\omega-\omega'\right)t}+\\
&\mathscr{E}^*_{\lambda}(\mathbf{k})\times\mathscr{A}^*_{\mu}\left(\mathbf{k}'\right)e^{-i\left(\mathbf{k}+\mathbf{k}'\right)\cdot\mathbf{x}}e^{i\left(\omega+\omega'\right)t}\Big]
\end{aligned} \tag{11}$$

Using $\int d^3x e^{i\mathbf{q}\cdot\mathbf{x}}=(2\pi)^3\,\delta(\mathbf{q})$, the triple integral collapses into a single integral over d^3k. The first and last term will be left with factor $e^{\mp i2\omega t}$, whose time average vanishs. The middle two terms will have no time dependency. Thus,

$$\left\langle I_{\lambda,\mu}\right\rangle=\int\frac{d^3k}{(2\pi)^3}\left[\mathscr{E}_{\lambda}(\mathbf{k})\times\mathscr{A}^*_{\mu}(\mathbf{k})+\mathscr{E}^*_{\lambda}(\mathbf{k})\times\mathscr{A}_{\mu}(\mathbf{k})\right] \tag{12}$$

Since

$$\boldsymbol{\epsilon}_{\lambda}(\mathbf{k})\times\boldsymbol{\epsilon}^*_{\mu}(\mathbf{k})=\begin{cases}-i\lambda\hat{\mathbf{k}} & \text{for } \mu=\lambda\\ 0 & \text{for } \mu\neq\lambda\end{cases} \tag{13}$$

we have

$$\mathscr{E}_{\lambda}(\mathbf{k})\times\mathscr{A}^*_{\mu}(\mathbf{k})=i\omega a_{\lambda}(\mathbf{k})\,a^*_{\mu}(\mathbf{k})\left[\boldsymbol{\epsilon}_{\lambda}(\mathbf{k})\times\boldsymbol{\epsilon}^*_{\mu}(\mathbf{k})\right]=\begin{cases}\lambda\omega\,|a_{\lambda}(\mathbf{k})|^2\,\hat{\mathbf{k}}=\lambda c\mathbf{k}\,|a_{\lambda}(\mathbf{k})|^2 & \text{for } \mu=\lambda\\ 0 & \text{for } \mu\neq\lambda\end{cases} \tag{14}$$

Finally summing all $I_{\lambda,\mu}$'s in (10) and taking the time average, we obtain

$$\left\langle\mathbf{L}_{\text{spin}}\right\rangle=\frac{2}{\mu_0c}\int\frac{d^3k}{(2\pi)^3}\mathbf{k}\left[|a_+(\mathbf{k})|^2-|a_-(\mathbf{k})|^2\right] \tag{15}$$

The energy of the field is

$$U=\frac{\epsilon_0}{2}\int d^3x\left[\mathbf{E}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)+c^2\mathbf{B}(\mathbf{x},t)\cdot\mathbf{B}(\mathbf{x},t)\right] \tag{16}$$

Expanding the field, we have

$$\int d^3x \mathbf{E}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t) = \sum_{\lambda,\mu} E_{\lambda,\mu} \qquad\qquad \int d^3x \mathbf{B}(\mathbf{x},t)\cdot\mathbf{B}(\mathbf{x},t) = \sum_{\lambda,\mu} B_{\lambda,\mu} \tag{17}$$

where $E_{\lambda,\mu}, B_{\lambda,\mu}$ have similar forms as (11) with the appropriate amplitude and substitution of cross product by dot product.

We shall end up with

$$E_{\lambda,\mu} = \int \frac{d^3k}{(2\pi)^3}\left[\mathscr{E}_\lambda(\mathbf{k})\cdot\mathscr{E}_\mu(-\mathbf{k})e^{-i2\omega t} + \mathscr{E}_\lambda(\mathbf{k})\cdot\mathscr{E}^*_\mu(\mathbf{k}) + \text{c.c.}\right] \tag{18}$$

$$B_{\lambda,\mu} = \int \frac{d^3k}{(2\pi)^3}\left[\mathscr{B}_\lambda(\mathbf{k})\cdot\mathscr{B}_\mu(-\mathbf{k})e^{-i2\omega t} + \mathscr{B}_\lambda(\mathbf{k})\cdot\mathscr{B}^*_\mu(\mathbf{k}) + \text{c.c.}\right] \tag{19}$$

Also from $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, we have

$$\mathscr{B}_\lambda(\mathbf{k}) = i a_\lambda(\mathbf{k})\left[\mathbf{k}\times\boldsymbol{\epsilon}_\lambda(\mathbf{k})\right] \tag{20}$$

Now we explicitly calculate the dot products in (18) and (19):

$$\mathscr{E}_\lambda(\mathbf{k})\cdot\mathscr{E}_\mu(-\mathbf{k}) = -\omega^2 a_\lambda(\mathbf{k})a_\mu(-\mathbf{k})\left[\boldsymbol{\epsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\epsilon}_\mu(-\mathbf{k})\right] \tag{21}$$

$$\mathscr{B}_\lambda(\mathbf{k})\cdot\mathscr{B}_\mu(-\mathbf{k}) = k^2 a_\lambda(\mathbf{k})a_\mu(-\mathbf{k})\left[\boldsymbol{\epsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\epsilon}_\mu(-\mathbf{k})\right] \tag{22}$$

$$\mathscr{E}_\lambda(\mathbf{k})\cdot\mathscr{E}^*_\mu(\mathbf{k}) = \omega^2 a_\lambda(\mathbf{k})a^*_\mu(\mathbf{k})\left[\boldsymbol{\epsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\epsilon}^*_\mu(\mathbf{k})\right] = \delta_{\lambda\mu}\omega^2|a_\lambda(\mathbf{k})|^2 \tag{23}$$

$$\mathscr{B}_\lambda(\mathbf{k})\cdot\mathscr{B}^*_\mu(\mathbf{k}) = k^2 a_\lambda(\mathbf{k})a^*_\mu(\mathbf{k})\left[\boldsymbol{\epsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\epsilon}^*_\mu(\mathbf{k})\right] = \delta_{\lambda\mu}k^2|a_\lambda(\mathbf{k})|^2 \tag{24}$$

We see that when we combine them in (16), the $e^{\mp i2\omega t}$ terms conveniently cancel, and the total energy is

$$\begin{aligned} U &= \frac{\epsilon_0}{2}\int \frac{d^3k}{(2\pi)^3}\cdot 4c^2k^2\left[|a_+(\mathbf{k})|^2 + |a_-(\mathbf{k})|^2\right] \\ &= \frac{2}{\mu_0}\int \frac{d^3k}{(2\pi)^3}k^2\left[|a_+(\mathbf{k})|^2 + |a_-(\mathbf{k})|^2\right] \end{aligned} \tag{25}$$

which does not depend on time (unlike spin which must take the time average to remove the time dependency).

Clearly, $a_\pm(\mathbf{k})$ can be associated with photons with positive or negative helicity. (15) shows the net spin of a collection of photons, some with positive helicity, some negative. (25) gives their total energy.

However it seems quite curious that the total energy is exact at any time, but the spin result has to depend on the time average.