Here we shall give a direct verification that the Green function

$$G_{\pm}\left(\mathbf{x},\mathbf{x}'\right)=\frac{e^{\pm ik\left|\mathbf{x}-\mathbf{x}'\right|}}{\left|\mathbf{x}-\mathbf{x}'\right|} \tag{1}$$

satisfies the inhomogeneous Helmholtz equation with delta source:

$$\left(\boldsymbol{\nabla}^2+k^2\right)G_{\pm}\left(\mathbf{x},\mathbf{x}'\right)=-4\pi\delta\left(\mathbf{x}-\mathbf{x}'\right) \tag{2}$$

Let

$$\mathbf{r}=\mathbf{x}-\mathbf{x}' \tag{3}$$

then (1) can be written as

$$G_{\pm}\left(\mathbf{r}\right)=\frac{e^{\pm ikr}}{r} \tag{4}$$

When $r>0$, with the spherical representation of $\boldsymbol{\nabla}^2$, it's straightforward to verify that

$$\left(\boldsymbol{\nabla}^2+k^2\right)G_{\pm}\left(\mathbf{r}\right)=\frac{1}{r}\frac{d^2}{dr^2}\left(rG_{\pm}\right)+k^2G_{\pm}=0 \qquad \text{for } r>0 \tag{5}$$

Now consider the integration of both sides of (2) over the volume of an infinitesimal ball of radius ϵ. The RHS gives

$$\int_{r<\epsilon}-4\pi\delta\left(\mathbf{r}\right)d^3x=-4\pi \tag{6}$$

and the LHS produces a sum of two integrals

$$\int_{r<\epsilon}\left(\boldsymbol{\nabla}^2+k^2\right)G_{\pm}d^3x=\underbrace{\int_{r<\epsilon}\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}G_{\pm}\right)d^3x}_{I_1}+\underbrace{\int_{r<\epsilon}k^2G_{\pm}d^3x}_{I_2} \tag{7}$$

By divergence theorem,

$$I_1=4\pi\epsilon^2\cdot\left.\frac{\partial G_{\pm}}{\partial r}\right|_{r=\epsilon}=4\pi\epsilon^2\left(\pm ik\frac{e^{\pm ik\epsilon}}{\epsilon}-\frac{e^{\pm ik\epsilon}}{\epsilon^2}\right) \tag{8}$$

and

$$\begin{aligned} I_2&=4\pi\int_0^{\epsilon}k^2G_{\pm}r^2dr=4\pi k^2\int_0^{\epsilon}re^{\pm ikr}dr\\ &=4\pi k^2\left(\left.\frac{1}{\pm ik}re^{\pm ikr}\right|_0^{\epsilon}-\frac{1}{\pm ik}\int_0^{\epsilon}e^{\pm ikr}dr\right)\\ &=4\pi k^2\left[\left.\frac{1}{\pm ik}re^{\pm ikr}\right|_0^{\epsilon}-\left.\left(\frac{1}{\pm ik}\right)^2e^{\pm ikr}\right|_0^{\epsilon}\right] \end{aligned} \tag{9}$$

When we take the limit $\epsilon\to 0$, we see (7) is indeed approaching -4π. Thus by definition of δ function, that is, $\left(\boldsymbol{\nabla}^2+k^2\right)G_{\pm}\left(\mathbf{r}\right)=0$ when $r>0$ but its integration over the infinitesimal ball containing $r=0$ is -4π, (1) is a solution of (2).