1. **Verification that (5.127) is the solution of (5.126) via Weber–Schafheitlin integral**

   For the *dual integral equations* (5.126)

   $$\int_0^\infty dy\, g(y) J_n(yx) = x^n \qquad\qquad \text{for } 0 \le x < 1 \tag{1}$$

   $$\int_0^\infty dy\, y\, g(y) J_n(yx) = 0 \qquad\qquad \text{for } 1 \le x \tag{2}$$

   It was claimed that (5.127) is a solution

   $$g(y) = \frac{2\Gamma(n+1)}{\sqrt{\pi}\Gamma(n+1/2)} j_n(y) = \frac{\Gamma(n+1)}{\Gamma(n+1/2)}\left(\frac{2}{y}\right)^{1/2} J_{n+1/2}(y) \tag{3}$$

   Let's verify it via the *Weber–Schafheitlin* integral (reference equation 10.22.56 on dlmf.nist.gov)

   for $0 < a < b, \operatorname{Re}(\mu + \nu + 1) > \operatorname{Re}(\lambda) > -1$:

   $$\int_0^\infty \frac{J_\mu(at)J_\nu(bt)}{t^\lambda}dt = \frac{a^\mu \Gamma\left(\frac{\nu}{2}+\frac{\mu}{2}-\frac{\lambda}{2}+\frac{1}{2}\right)}{2^\lambda b^{\mu-\lambda+1}\Gamma\left(\frac{\nu}{2}-\frac{\mu}{2}+\frac{\lambda}{2}+\frac{1}{2}\right)}\mathbf{F}\left(\frac{\mu+\nu-\lambda+1}{2},\frac{\mu-\nu-\lambda+1}{2};\mu+1;\frac{a^2}{b^2}\right) \tag{4}$$

   where $\mathbf{F}$ is the Olver's hypergeometric function

   $$\mathbf{F}(a,b;c;z) = \frac{1}{\Gamma(c)}F(a,b;c;z) = \frac{1}{\Gamma(c)}\left[1 + \frac{ab}{c}z + \frac{1}{2!}\frac{a(a+1)b(b+1)}{c(c+1)}z^2 + \cdots\right] \tag{5}$$

   - To see (1), insert $a = x, b = 1, \mu = n, \nu = n+1/2, \lambda = 1/2$ into (4), we have

   $$\int_0^\infty \frac{J_n(xy)J_{n+1/2}(y)\,dy}{\sqrt{y}} = \frac{x^n\Gamma(n+1/2)}{\sqrt{2}\Gamma(1)}\frac{1}{\Gamma(n+1)}\overbrace{F\left(n+\frac{1}{2},0;n+1;x^2\right)}^{=1} \tag{6}$$

   Thus we see that (3) satisfies (1).
   - For (2), we need $a = 1, b = x, \mu = n+1/2, \nu = n, \lambda = -1/2$, with which the denominator of (4) has $\Gamma(0) = \infty$, hence (4) vanishes, which satisfies (2).

2. **Closed-form formula for (5.129)**

   The text gives the integral representation of the additional potential in equation (5.129)

   $$\Phi^{(1)}(\mathbf{x}) = \frac{2H_0 a^2}{\pi}\overbrace{\int_0^\infty dk\, j_1(ka)\, e^{-k|z|}J_1(k\rho)}^{I}\sin\phi \tag{7}$$

   Now let's calculate the integral I explicitly (We will treat $z > 0$ region only, the $z < 0$ region can be obtained by the reversal of sign in (5.123).)

   With the relation

   $$j_1(x) = -j_0'(x) \tag{8}$$

   I can be written as

   $$\begin{aligned}
   I &= \int_0^\infty dk\left[-j_0'(ka)\right]e^{-kz}J_1(k\rho) \\
   &= \int_0^\infty dk\left[-\frac{dj_0(ka)}{a\,dk}\right]e^{-kz}J_1(k\rho) \\
   &= -\frac{1}{a}j_0(ka)e^{-kz}J_1(k\rho)\Big|_0^\infty + \frac{1}{a}\int_0^\infty dk\, j_0(ka)\frac{d}{dk}\left[e^{-kz}J_1(k\rho)\right]
   \end{aligned} \tag{9}$$

The first term clearly vanishes at both $k = 0$ and $k = \infty$, so

$$
\begin{aligned}
I &= \frac{1}{a} \int_0^\infty dk j_0(ka) \frac{d}{dk} \left[e^{-kz} J_1(k\rho) \right] \\
&= \frac{1}{a} \left[-z \int_0^\infty dk j_0(ka) e^{-kz} J_1(k\rho) + \rho \int_0^\infty dk j_0(ka) e^{-kz} J_1'(k\rho) \right]
\end{aligned}
\tag{10}
$$

With the recurrence relation (see equation 10.6.1 on dlmf.nist.gov)

$$
J_1'(x) = \frac{1}{2} \left[J_0(x) - J_2(x) \right]
\tag{11}
$$

we have

$$
I = \frac{1}{a} \left[-z \int_0^\infty dk j_0(ka) e^{-kz} J_1(k\rho) + \frac{\rho}{2} \int_0^\infty dk j_0(ka) e^{-kz} J_0(k\rho) - \frac{\rho}{2} \int_0^\infty dk j_0(ka) e^{-kz} J_2(k\rho) \right]
\tag{12}
$$

Define

$$
K_n \equiv \int_0^\infty dk j_0(ka) e^{-kz} J_n(k\rho)
\tag{13}
$$

(12) became

$$
I = \frac{1}{a} \left(-z K_1 + \frac{\rho}{2} K_0 - \frac{\rho}{2} K_2 \right)
\tag{14}
$$

Now let's focus on K_n,

$$
\begin{aligned}
K_n &= \int_0^\infty dk \frac{\sin ka}{ka} e^{-kz} J_n(k\rho) \qquad\qquad (s \equiv z - ia) \\
&= \frac{1}{a} \operatorname{Im} \left[\int_0^\infty e^{-sk} \frac{J_n(k\rho)}{k} dk \right] \\
&= \frac{1}{a} \operatorname{Im} \left[\mathscr{L} \left\{ \frac{J_n(k\rho)}{k} \right\}(s) \right]
\end{aligned}
\tag{15}
$$

(a) **Calculation of K_1, K_2**

For $n = 1, 2$, recall equation 10.6.2 on dlmf.nist.gov

$$
J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)
\tag{16}
$$

and the Laplace transform property

$$
\mathscr{L} \left\{ \frac{df(t)}{dt} \right\}(s) = s \mathscr{L} \{ f(t) \}(s) - f(0^-)
\tag{17}
$$

These yield

$$
\begin{aligned}
\mathscr{L} \left\{ \frac{J_n(k\rho)}{k} \right\}(s) &= \frac{\rho}{n} \mathscr{L} \left\{ J_{n-1}(k\rho) - J_n'(k\rho) \right\}(s) \\
&= \frac{\rho}{n} \left[\mathscr{L} \{ J_{n-1}(k\rho) \}(s) - \mathscr{L} \left\{ \frac{dJ_n(k\rho)}{\rho dk} \right\}(s) \right] \\
&= \frac{\rho}{n} \mathscr{L} \{ J_{n-1}(k\rho) \}(s) - \frac{1}{n} \left[s \mathscr{L} \{ J_n(k\rho) \}(s) - J_n(0^-) \right] \\
&= \frac{\rho}{n} \mathscr{L} \{ J_{n-1}(k\rho) \}(s) - \frac{s}{n} \mathscr{L} \{ J_n(k\rho) \}(s) \qquad \text{for } n = 1, 2
\end{aligned}
\tag{18}
$$

Substituting in (18) with the well known Laplace transforms of $J_n(k\rho)$ (see wikiproof)

$$
\mathscr{L} \{ J_n(k\rho) \}(s) = \frac{\left(\sqrt{s^2 + \rho^2} - s \right)^n}{\rho^n \sqrt{s^2 + \rho^2}}
\tag{19}
$$

gives

$$\mathscr{L}\left\{\frac{J_1(k\rho)}{k}\right\}(s) = \frac{\rho}{\sqrt{s^2+\rho^2}} - s\cdot\frac{\sqrt{s^2+\rho^2}-s}{\rho\sqrt{s^2+\rho^2}} = \frac{\sqrt{s^2+\rho^2}-s}{\rho} \tag{20}$$

$$\mathscr{L}\left\{\frac{J_2(k\rho)}{k}\right\}(s) = \frac{\rho}{2}\cdot\frac{\sqrt{s^2+\rho^2}-s}{\rho\sqrt{s^2+\rho^2}} - \frac{s}{2}\cdot\frac{\left(\sqrt{s^2+\rho^2}-s\right)^2}{\rho^2\sqrt{s^2+\rho^2}}$$

$$= \left(\frac{\sqrt{s^2+\rho^2}-s}{2\rho^2}\right)\left[\frac{\rho^2-s\left(\sqrt{s^2+\rho^2}-s\right)}{\sqrt{s^2+\rho^2}}\right]$$

$$= \frac{\left(\sqrt{s^2+\rho^2}-s\right)^2}{2\rho^2} \tag{21}$$

With (20) and (21) plugged back into (15), we have

$$K_1 = \frac{1}{a}\operatorname{Im}\left[\mathscr{L}\left\{\frac{J_1(k\rho)}{k}\right\}(s)\right] = \frac{1}{a\rho}\left(\operatorname{Im}\sqrt{s^2+\rho^2}-\operatorname{Im}s\right) = \frac{1}{a\rho}\left(\operatorname{Im}\sqrt{s^2+\rho^2}+a\right) \tag{22}$$
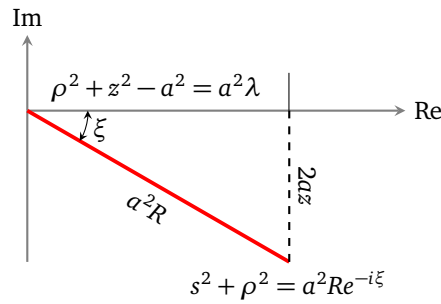
$$K_2 = \frac{1}{a}\operatorname{Im}\left[\mathscr{L}\left\{\frac{J_2(k\rho)}{k}\right\}(s)\right] = \frac{1}{2a\rho^2}\operatorname{Im}\left(\sqrt{s^2+\rho^2}-s\right)^2 \tag{23}$$

Define

$$\lambda = \frac{z^2+\rho^2-a^2}{a^2} \qquad\qquad R = \sqrt{\lambda^2+\frac{4z^2}{a^2}} \tag{24}$$

then we have

$$\sqrt{\rho^2+s^2} = \left(\rho^2+z^2-a^2-2azi\right)^{1/2} = a\sqrt{R}e^{-i\xi/2} \tag{25}$$

From the diagram above, it's clear that

$$\operatorname{Im}\sqrt{s^2+\rho^2} = -a\sqrt{R}\sin\frac{\xi}{2} \tag{26}$$

$\sin\xi/2$ and $\cos\xi/2$ can be found via

$$1-2\sin^2\frac{\xi}{2} = \cos\xi = \frac{\lambda}{R} \qquad\Longrightarrow\qquad \sin\frac{\xi}{2} = \sqrt{\frac{R-\lambda}{2R}} \qquad \cos\frac{\xi}{2} = \sqrt{\frac{R+\lambda}{2R}} \tag{27}$$

K_1 is obtained by

$$K_1 = \frac{1}{a\rho}\left(-a\sqrt{\frac{R-\lambda}{2}}+a\right) = \frac{1}{\rho}\left(1-\sqrt{\frac{R-\lambda}{2}}\right) \tag{28}$$

For (23),

$$K_2 = \frac{1}{2a\rho^2}\operatorname{Im}\left(\sqrt{s^2+\rho^2}-s\right)^2 = \frac{1}{2a\rho^2}\cdot 2\operatorname{Re}\left(\sqrt{s^2+\rho^2}-s\right)\cdot\operatorname{Im}\left(\sqrt{s^2+\rho^2}-s\right)$$

$$= \frac{1}{a\rho^2}\left(a\sqrt{R}\cos\frac{\xi}{2}-z\right)\left(-a\sqrt{R}\sin\frac{\xi}{2}+a\right) = \frac{1}{a\rho^2}\left(a\sqrt{R}\sqrt{\frac{R+\lambda}{2R}}-z\right)\left(-a\sqrt{R}\sqrt{\frac{R-\lambda}{2R}}+a\right)$$

$$= \frac{1}{a\rho^2}\left(-a^2\frac{\sqrt{R^2-\lambda^2}}{2}-az+az\sqrt{\frac{R-\lambda}{2}}+a^2\sqrt{\frac{R+\lambda}{2}}\right)$$

$$= \frac{1}{\rho^2}\left(-2z+z\sqrt{\frac{R-\lambda}{2}}+a\sqrt{\frac{R+\lambda}{2}}\right) \tag{29}$$

(b) **Calculation of K_0**

With $s = z - ia$, let

$$F(s) = \int_0^\infty e^{-sk} \frac{J_0(k\rho)}{k} dk \tag{30}$$

then by definition $K_0 = \operatorname{Im} F(s)/a$.

Notice

$$F'(s) = -\int_0^\infty e^{-sk} J_0(k\rho)\, dk = -\mathscr{L}\{J_0(k\rho)\}(s) = -\frac{1}{\sqrt{s^2+\rho^2}} \tag{31}$$

Thus $F(s)$ is readily solvable as (reference WolframAlpha)

$$\begin{aligned}
F(s) &= -\tanh^{-1}\left(\frac{s}{\sqrt{\rho^2+s^2}}\right) \\
&= \frac{1}{2}\ln\left(1 - \frac{s}{\sqrt{\rho^2+s^2}}\right) - \frac{1}{2}\ln\left(1 + \frac{s}{\sqrt{\rho^2+s^2}}\right) \\
&= \frac{1}{2}\ln\left(\frac{\sqrt{\rho^2+s^2}-s}{\sqrt{\rho^2+s^2}+s}\right) \\
&= \frac{1}{2}\ln\left[\frac{\left(\sqrt{\rho^2+s^2}-s\right)^2}{\rho^2}\right] \\
&= \ln\left(\frac{\sqrt{\rho^2+s^2}-s}{\rho}\right) \tag{32}
\end{aligned}$$

The imaginary part of $F(s)$ is just the argument of the complex number $\sqrt{\rho^2+s^2}-s$, i.e.,

$$\begin{aligned}
\operatorname{Im}[F(s)] &= \operatorname{Arg}\left(\sqrt{\rho^2+s^2}-s\right) = \operatorname{Arg}\left[a\sqrt{R}\left(\cos\frac{\xi}{2} - i\sin\frac{\xi}{2}\right) - z + ia\right] \\
&= \tan^{-1}\left(\frac{a - a\sqrt{R}\sin\dfrac{\xi}{2}}{a\sqrt{R}\cos\dfrac{\xi}{2} - z}\right) = \tan^{-1}\left(\frac{a - a\sqrt{R}\sqrt{\dfrac{R-\lambda}{2R}}}{a\sqrt{R}\sqrt{\dfrac{R+\lambda}{2R}} - z}\right) \\
&= \tan^{-1}\left(\frac{a - a\sqrt{\dfrac{R-\lambda}{2}}}{a\sqrt{\dfrac{R+\lambda}{2}} - z}\right) = \tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right) \tag{33}
\end{aligned}$$

Finally, with K_0, K_1, K_2 inserted to (14) and (7), we get

$$\begin{aligned}
\Phi^{(1)}(\mathbf{x}) &= \frac{2H_0 a}{\pi}\left(-zK_1 + \frac{\rho}{2}K_0 - \frac{\rho}{2}K_2\right)\sin\phi \\
&= \frac{2H_0 a}{\pi}\left[-\frac{z}{\rho}\left(1 - \sqrt{\frac{R-\lambda}{2}}\right) + \frac{\rho}{2a}\tan^{-1}\sqrt{\frac{2}{R+\lambda}} - \frac{1}{2\rho}\left(-2z + z\sqrt{\frac{R-\lambda}{2}} + a\sqrt{\frac{R+\lambda}{2}}\right)\right]\sin\phi \\
&= \frac{2H_0 a}{\pi}\left(\overbrace{\frac{z}{2\rho}\sqrt{\frac{R-\lambda}{2}}}^{A} + \underbrace{\frac{\rho}{2a}\tan^{-1}\sqrt{\frac{2}{R+\lambda}}}_{B} - \overbrace{\frac{a}{2\rho}\sqrt{\frac{R+\lambda}{2}}}^{C}\right)\sin\phi \tag{34}
\end{aligned}$$

3. Asymptotic behavior with large r, equation (5.129)

Let the observation point $\mathbf{x} = (\rho, \phi, z)$ be equivalently expressed in spherical coordinate (r, θ, ϕ), then

$$z = r\cos\theta \qquad\qquad \rho = r\sin\theta \tag{35}$$

The benefit of using this form is so that we can write (34) in increasing negative powers of r.

By definition,

$$\lambda = \left(\frac{r}{a}\right)^2 - 1 \tag{36}$$

$$R = \sqrt{\left[\left(\frac{r}{a}\right)^2 - 1\right]^2 + \frac{4z^2}{a^2}} = \sqrt{\left(\frac{r}{a}\right)^4 - 2\left(\frac{r}{a}\right)^2 + 1 + \frac{4z^2}{a^2}} = \left(\frac{r}{a}\right)^2 \sqrt{1 - 2\left(\frac{a}{r}\right)^2 + \left(\frac{a}{r}\right)^4 + \frac{4a^2 \cos^2\theta}{r^2}} \tag{37}$$

Using the Taylor expansion

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots \tag{38}$$

we have

$$
\begin{aligned}
R &= \left(\frac{r}{a}\right)^2 \left\{1 + \frac{1}{2}\left[-2\left(\frac{a}{r}\right)^2 + \left(\frac{a}{r}\right)^4 + \frac{4a^2\cos^2\theta}{r^2}\right] - \frac{1}{8}\left[4\left(\frac{a}{r}\right)^4 - \frac{16a^4\cos^2\theta}{r^4} + \frac{16a^4\cos^4\theta}{r^4}\right] + O\left(\frac{1}{r^6}\right)\right\} \\
&= \left(\frac{r}{a}\right)^2 - 1 + \frac{1}{2}\left(\frac{a}{r}\right)^2 + 2\cos^2\theta - \frac{1}{2}\left(\frac{a}{r}\right)^2 + \frac{2a^2\cos^2\theta}{r^2} - \frac{2a^2\cos^4\theta}{r^2} + O\left(\frac{1}{r^4}\right) \\
&= \left(\frac{r}{a}\right)^2 - 1 + 2\cos^2\theta + \frac{2a^2\cos^2\theta\sin^2\theta}{r^2} + O\left(\frac{1}{r^4}\right)
\end{aligned}
\tag{39}
$$

hence

$$\frac{R-\lambda}{2} = \cos^2\theta + \frac{a^2\cos^2\theta\sin^2\theta}{r^2} + O\left(\frac{1}{r^4}\right) \qquad\Longrightarrow$$

$$\sqrt{\frac{R-\lambda}{2}} = \cos\theta\left[1 + \frac{a^2\sin^2\theta}{r^2} + O\left(\frac{1}{r^4}\right)\right]^{1/2} = \cos\theta\left[1 + \frac{1}{2}\frac{a^2\sin^2\theta}{r^2} + O\left(\frac{1}{r^4}\right)\right] \tag{40}$$

The dominating orders of A in (34) are

$$A = \frac{\cos\theta}{2\sin\theta}\cdot\cos\theta\left[1 + \frac{1}{2}\frac{a^2\sin^2\theta}{r^2} + O\left(\frac{1}{r^4}\right)\right] = \frac{\cos^2\theta}{2\sin\theta} + \frac{1}{4}\frac{a^2\sin\theta\cos^2\theta}{r^2} + O\left(\frac{1}{r^4}\right) \tag{41}$$

Notice

$$\frac{2}{R+\lambda} = \frac{a^2}{z^2}\left(\frac{R-\lambda}{2}\right) \tag{42}$$

gives

$$\tan^{-1}\sqrt{\frac{2}{R+\lambda}} = \tan^{-1}\left(\frac{a}{z}\sqrt{\frac{R-\lambda}{2}}\right) = \tan^{-1}\left(\frac{a}{r\cos\theta}\sqrt{\frac{R-\lambda}{2}}\right) = \tan^{-1}\left[\frac{a}{r} + \frac{1}{2}\frac{a^3\sin^2\theta}{r^3} + O\left(\frac{1}{r^5}\right)\right] \tag{43}$$

Using the Taylor expansion

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots \tag{44}$$

we can write

$$\tan^{-1}\sqrt{\frac{2}{R+\lambda}} = \frac{a}{r} + \frac{1}{2}\frac{a^3\sin^2\theta}{r^3} - \frac{1}{3}\left(\frac{a}{r}\right)^3 + O\left(\frac{1}{r^5}\right) \tag{45}$$

which gives the B term in (34)

$$B = \frac{r\sin\theta}{2a}\left[\frac{a}{r} + \frac{1}{2}\frac{a^3\sin^2\theta}{r^3} - \frac{1}{3}\left(\frac{a}{r}\right)^3 + O\left(\frac{1}{r^5}\right)\right] = \frac{\sin\theta}{2} + \frac{1}{4}\frac{a^2\sin^3\theta}{r^2} - \frac{1}{6}\frac{a^2\sin\theta}{r^2} + O\left(\frac{1}{r^4}\right) \tag{46}$$

Lastly, from (39)

$$\frac{R+\lambda}{2} = \left(\frac{r}{a}\right)^2 - 1 + \cos^2\theta + O\left(\frac{1}{r^2}\right) \qquad\Longrightarrow$$

$$\sqrt{\frac{R+\lambda}{2}} = \left(\frac{r}{a}\right)\left[1 - \sin^2\theta\left(\frac{a}{r}\right)^2 + O\left(\frac{1}{r^4}\right)\right]^{1/2} = \frac{r}{a} - \frac{1}{2}\frac{a\sin^2\theta}{r} + O\left(\frac{1}{r^3}\right) \tag{47}$$

which gives

$$C = \frac{a}{2r\sin\theta}\left[\frac{r}{a} - \frac{1}{2}\frac{a\sin^2\theta}{r} + O\left(\frac{1}{r^3}\right)\right] = \frac{1}{2\sin\theta} - \frac{1}{4}\frac{a^2\sin\theta}{r^2} + O\left(\frac{1}{r^4}\right) \tag{48}$$

Insert A, B, C back into (34) and ignore the $O(r^{-4})$ terms,

$$A + B - C \approx \frac{\cos^2\theta}{2\sin\theta} + \frac{a^2\sin\theta\cos^2\theta}{4r^2} + \frac{\sin\theta}{2} + \frac{a^2\sin^3\theta}{4r^2} - \frac{a^2\sin\theta}{6r^2} - \frac{1}{2\sin\theta} + \frac{a^2\sin\theta}{4r^2} \approx \frac{a^2\sin\theta}{3r^2} \quad\Longrightarrow$$

$$\Phi^{(1)}(\mathbf{x}) \approx \frac{2H_0 a^3}{3\pi} \cdot \frac{r\sin\theta\sin\phi}{r^3} = \frac{2H_0 a^3}{3\pi}\frac{y}{r^3} \tag{49}$$

4. **Problem 5.24, tangential field on the surface $z = 0^+$**

   At $z = 0^+$,

   $$\lambda = R = \frac{\rho^2}{a^2} - 1 \tag{50}$$

   By (34),

   $$\Phi^{(1)}(\rho, \phi, z = 0) = \frac{2H_0 a}{\pi}\left(\frac{\rho}{2a}\tan^{-1}\frac{a}{\sqrt{\rho^2 - a^2}} - \frac{a}{2\rho}\cdot\frac{\sqrt{\rho^2 - a^2}}{a}\right)\sin\phi$$

   $$= \frac{2H_0 a}{\pi}\left[\frac{\rho}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) - \frac{\sqrt{\rho^2 - a^2}}{2\rho}\right]\sin\phi \tag{51}$$

   The derivatives are

   $$\frac{\partial\Phi^{(1)}}{\partial\rho} = \frac{2H_0 a}{\pi}\left[\frac{1}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) + \frac{\rho}{2a}\frac{1}{\sqrt{1 - \left(\frac{a}{\rho}\right)^2}}\left(-\frac{a}{\rho^2}\right) + \frac{\sqrt{\rho^2 - a^2}}{2\rho^2} - \frac{1}{2\rho}\frac{\rho}{\sqrt{\rho^2 - a^2}}\right]\sin\phi$$

   $$= \frac{2H_0 a}{\pi}\left[\frac{1}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) - \frac{1}{2\sqrt{\rho^2 - a^2}} + \frac{\sqrt{\rho^2 - a^2}}{2\rho^2} - \frac{1}{2\sqrt{\rho^2 - a^2}}\right]\sin\phi$$

   $$= \frac{2H_0 a}{\pi}\left[\frac{1}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) - \frac{1}{\sqrt{\rho^2 - a^2}} + \frac{\sqrt{\rho^2 - a^2}}{2\rho^2}\right]\sin\phi \tag{52}$$

   $$\frac{1}{\rho}\frac{\partial\Phi^{(1)}}{\partial\phi} = \frac{2H_0 a}{\pi}\left[\frac{1}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) - \frac{\sqrt{\rho^2 - a^2}}{2\rho^2}\right]\cos\phi \tag{53}$$

   The field in polar coordinates are

   $$H_\rho^{(1)} = -\frac{\partial\Phi^{(1)}}{\partial\rho} \qquad\qquad H_\phi^{(1)} = -\frac{1}{\rho}\frac{\partial\Phi^{(1)}}{\partial\phi} \tag{54}$$

   and in Cartesian coordinates are

   $$H_x^{(1)} = H_\rho^{(1)}\cos\phi - H_\phi^{(1)}\sin\phi = \frac{2H_0 a}{\pi}\left(\frac{1}{\sqrt{\rho^2 - a^2}} - \frac{\sqrt{\rho^2 - a^2}}{\rho^2}\right)\sin\phi\cos\phi = \frac{2H_0 a^3}{\pi}\frac{xy}{\rho^4\sqrt{\rho^2 - a^2}} \tag{55}$$

   $$H_y^{(1)} = H_\rho^{(1)}\sin\phi + H_\phi^{(1)}\cos\phi$$

   $$= -\frac{2H_0 a}{\pi}\left\{\left[\frac{1}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) - \frac{1}{\sqrt{\rho^2 - a^2}} + \frac{\sqrt{\rho^2 - a^2}}{2\rho^2}\right]\sin^2\phi + \left[\frac{1}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) - \frac{\sqrt{\rho^2 - a^2}}{2\rho^2}\right]\cos^2\phi\right\}$$

   $$= -\frac{2H_0 a}{\pi}\left[\frac{1}{2a}\sin^{-1}\left(\frac{a}{\rho}\right) - \left(\frac{1}{\sqrt{\rho^2 - a^2}} - \frac{\sqrt{\rho^2 - a^2}}{\rho^2}\right)\sin^2\phi - \frac{\sqrt{\rho^2 - a^2}}{2\rho^2}\right]$$

   $$= \frac{H_0}{\pi}\left[\frac{a\sqrt{\rho^2 - a^2}}{\rho^2} - \sin^{-1}\left(\frac{a}{\rho}\right)\right] + \frac{2H_0 a^3}{\pi}\frac{y^2}{\rho^4\sqrt{\rho^2 - a^2}} \tag{56}$$

5. **Field in the hole, i.e., equation (5.132)**

When $\rho < a$ and $z \to 0^+$

$$\lambda \to \frac{\rho^2}{a^2} - 1 \qquad\qquad R \to 1 - \frac{\rho^2}{a^2} \qquad\qquad \Longrightarrow$$

$$\frac{R-\lambda}{2} \to 1 - \frac{\rho^2}{a^2} \qquad \text{hence and by (42)} \qquad \frac{2}{R+\lambda} \to \frac{a^2}{z^2}\left(1 - \frac{\rho^2}{a^2}\right) \tag{57}$$

Also

$$\frac{\partial A}{\partial z} = \frac{1}{2\rho}\sqrt{\frac{R-\lambda}{2}} + \frac{z}{2\rho}\frac{1}{2}\frac{1}{\sqrt{\frac{R-\lambda}{2}}}\frac{1}{2}\left(\frac{\partial R}{\partial z} - \frac{\partial \lambda}{\partial z}\right) \to \frac{1}{2\rho}\sqrt{1 - \frac{\rho^2}{a^2}} \tag{58}$$

$$\frac{\partial B}{\partial z} = \frac{\rho}{2a}\left(\frac{1}{1 + \frac{2}{R+\lambda}}\right)\left(-\frac{1}{2}\right)\sqrt{\frac{2}{R+\lambda}}^{\,3}\frac{1}{2}\left(\frac{\partial R}{\partial z} + \frac{\partial \lambda}{\partial z}\right)$$

$$= \frac{\rho}{2a}\left(\frac{R+\lambda}{R+\lambda+2}\right)\left(-\frac{1}{4}\right)\sqrt{\frac{2}{R+\lambda}}^{\,3}\left(\frac{\lambda}{R}\cdot\frac{2z}{a^2} + \frac{4z}{a^2 R} + \frac{2z}{a^2}\right)$$

$$= \frac{\rho}{2a}\left(\frac{R+\lambda}{R+\lambda+2}\right)\left(-\frac{1}{4}\right)\sqrt{\frac{2}{R+\lambda}}^{\,3}\frac{2z}{a^2 R}(\lambda + 2 + R)$$

$$= -\frac{\rho}{2a}\sqrt{\frac{2}{R+\lambda}}\cdot\frac{z}{a^2 R}$$

$$\to -\frac{\rho}{2a^3}\cdot\frac{a}{z}\sqrt{1 - \frac{\rho^2}{a^2}}\frac{z}{\left(1 - \frac{\rho^2}{a^2}\right)} = -\frac{\rho}{2a^2}\frac{1}{\sqrt{1 - \frac{\rho^2}{a^2}}} \tag{59}$$

$$-\frac{\partial C}{\partial z} = -\frac{a}{2\rho}\frac{1}{2}\frac{1}{\sqrt{\frac{R+\lambda}{2}}}\frac{1}{2}\left(\frac{\partial R}{\partial z} + \frac{\partial \lambda}{\partial z}\right)$$

$$= -\frac{a}{8\rho}\sqrt{\frac{2}{R+\lambda}}\frac{2z}{a^2 R}(\lambda + 2 + R)$$

$$\to -\frac{a}{8\rho}\cdot\frac{a}{z}\sqrt{1 - \frac{\rho^2}{a^2}}\cdot\frac{2z}{a^2\left(1 - \frac{\rho^2}{a^2}\right)}\cdot 2 = -\frac{1}{2\rho}\frac{1}{\sqrt{1 - \frac{\rho^2}{a^2}}} \tag{60}$$

Combining (58)-(60) gives

$$H_z\left(\rho, \phi, z = 0^+\right) = -\lim_{z\to 0^+}\frac{\partial \Phi^{(1)}}{\partial z} = -\lim_{z\to 0^+}\frac{2H_0 a}{\pi}\left(\frac{\partial A}{\partial z} + \frac{\partial B}{\partial z} - \frac{\partial C}{\partial z}\right)\sin\phi$$

$$= -\frac{2H_0 a}{\pi}\left(\frac{1}{2a\rho}\sqrt{a^2 - \rho^2} - \frac{\rho}{2a}\frac{1}{\sqrt{a^2 - \rho^2}} - \frac{a}{2\rho}\frac{1}{\sqrt{a^2 - \rho^2}}\right)\sin\phi$$

$$= -\frac{2H_0 a}{\pi}\frac{1}{2a\rho\sqrt{a^2 - \rho^2}}\left[(a^2 - \rho^2) - \rho^2 - a^2\right]\sin\phi$$

$$= \frac{2H_0}{\pi}\frac{\rho}{\sqrt{a^2 - \rho^2}}\sin\phi \tag{61}$$

For the tangential fields, first observe

$$H_\phi^{(1)}\left(\rho, \phi, z = 0^+\right) = -\lim_{z\to 0^+}\frac{1}{\rho}\frac{\partial \Phi^{(1)}}{\partial \phi}$$

$$= -\lim_{z\to 0^+}\frac{2H_0 a}{\pi}\left[\frac{z}{2\rho^2}\sqrt{\frac{R-\lambda}{2}} + \frac{1}{2a}\tan^{-1}\sqrt{\frac{2}{R+\lambda}} - \frac{a}{2\rho^2}\sqrt{\frac{R+\lambda}{2}}\right]\cos\phi$$

$$= -\frac{2H_0 a}{\pi}\cdot\frac{1}{2a}\frac{\pi}{2}\cos\phi = -\frac{H_0}{2}\cos\phi \tag{62}$$

The derivatives in the $\hat{\boldsymbol{\rho}}$ direction in the $z \to 0^+$ limits are

$$\frac{\partial A}{\partial \rho} = -\frac{z}{2\rho^2}\sqrt{\frac{R-\lambda}{2}} + \frac{z}{2\rho}\frac{1}{2}\frac{1}{\sqrt{\frac{R-\lambda}{2}}}\frac{1}{2}\left(\frac{\partial R}{\partial \rho} - \frac{\partial \lambda}{\partial \rho}\right) \to 0 \tag{63}$$

$$\frac{\partial B}{\partial \rho} = \frac{1}{2a}\tan^{-1}\sqrt{\frac{2}{R+\lambda}} + \frac{\rho}{2a}\left(\frac{1}{1+\frac{2}{R+\lambda}}\right)\left(-\frac{1}{2}\right)\sqrt{\frac{2}{R+\lambda}}^{-3}\frac{1}{2}\left(\frac{\partial R}{\partial \rho} + \frac{\partial \lambda}{\partial \rho}\right)$$

$$\to \frac{1}{2a}\frac{\pi}{2} + \underbrace{\frac{\rho}{2a}\left(\frac{R+\lambda}{R+\lambda+2}\right)\left(-\frac{1}{4}\right)\sqrt{\frac{2}{R+\lambda}}^{-3}\frac{2\rho}{a^2 R}(\lambda+R)}_{\propto \sqrt{R+\lambda} \text{ hence } \to 0} = \frac{1}{2a}\frac{\pi}{2} \tag{64}$$

$$-\frac{\partial C}{\partial \rho} = \frac{a}{2\rho^2}\sqrt{\frac{R+\lambda}{2}} - \frac{a}{2\rho}\frac{1}{2}\frac{1}{\sqrt{\frac{R+\lambda}{2}}}\frac{1}{2}\left(\frac{\partial R}{\partial \rho} + \frac{\partial \lambda}{\partial \rho}\right) \to 0 \tag{65}$$

Therefore

$$H_\rho^{(1)}\left(\rho, \phi, z = 0^+\right) = -\lim_{z\to 0^+}\frac{\partial \Phi^{(1)}}{\partial \rho} = -\lim_{z\to 0^+}\frac{2H_0 a}{\pi}\left(\frac{\partial A}{\partial \rho} + \frac{\partial B}{\partial \rho} - \frac{\partial C}{\partial \rho}\right)\sin\phi = -\frac{H_0}{2}\sin\phi \tag{66}$$

Writing (62) and (66) in Cartesian coordinates gives

$$H_x^{(1)}\left(x, y, 0^+\right) = H_\rho^{(1)}\cos\phi - H_\phi^{(1)}\sin\phi = 0 \qquad H_y^{(1)}\left(x, y, 0^+\right) = H_\rho^{(1)}\sin\phi + H_\phi^{(1)}\cos\phi = -\frac{H_0}{2} \tag{67}$$

Remember for $z > 0$, $\mathbf{H}^{(1)}$ needs to combine with $H_0\hat{\mathbf{y}}$ to give the whole field, i.e.,

$$\mathbf{H}_{\text{tan}} = \frac{\mathbf{H}_0}{2} \tag{68}$$

It's worth noting that the net field $\mathbf{H}$ is continuous from $0^+ \to 0^-$ across the opening, since for $z < 0$, $\mathbf{H}^{(1)}$ is the net field, and $H_x^{(1)}, H_y^{(1)}$ are odd in z but H_z is even in z.

The field line visualization is programmed here. The following are notes for the algorithm, specifically the derivatives of A, B, C terms in (34).

For $z > 0$:

$$\frac{\partial \lambda}{\partial z} = \frac{2z}{a^2}$$

$$\frac{\partial R}{\partial z} = \frac{\lambda}{R} \frac{\partial \lambda}{\partial z} + \frac{4z/a^2}{R}$$

$$\frac{\partial A}{\partial z} = \frac{1}{2\rho} \sqrt{\frac{R-\lambda}{2}} + \frac{z}{2\rho} \frac{1}{2} \frac{1}{\sqrt{\frac{R-\lambda}{2}}} \frac{1}{2}\left(\frac{\partial R}{\partial z} - \frac{\partial \lambda}{\partial z}\right)$$

$$\frac{\partial A}{\partial \rho} = -\frac{z}{2\rho^2} \sqrt{\frac{R-\lambda}{2}} + \frac{z}{2\rho} \frac{1}{2} \frac{1}{\sqrt{\frac{R-\lambda}{2}}} \frac{1}{2}\left(\frac{\partial R}{\partial \rho} - \frac{\partial \lambda}{\partial \rho}\right)$$

$$\frac{\partial B}{\partial z} = \frac{\rho}{2a} \left(\frac{1}{1 + \frac{2}{R+\lambda}}\right)\left(-\frac{1}{2}\right) \sqrt{\frac{2}{R+\lambda}}^{\,-3} \frac{1}{2}\left(\frac{\partial R}{\partial z} + \frac{\partial \lambda}{\partial z}\right)$$

$$\frac{\partial B}{\partial \rho} = \frac{1}{2a} \tan^{-1}\sqrt{\frac{2}{R+\lambda}} + \frac{\rho}{2a}\left(\frac{1}{1 + \frac{2}{R+\lambda}}\right)\left(-\frac{1}{2}\right)\sqrt{\frac{2}{R+\lambda}}^{\,-3}\frac{1}{2}\left(\frac{\partial R}{\partial \rho} + \frac{\partial \lambda}{\partial \rho}\right)$$

$$-\frac{\partial C}{\partial z} = -\frac{a}{2\rho}\frac{1}{2}\frac{1}{\sqrt{\frac{R+\lambda}{2}}}\frac{1}{2}\left(\frac{\partial R}{\partial z} + \frac{\partial \lambda}{\partial z}\right)$$

$$-\frac{\partial C}{\partial \rho} = \frac{a}{2\rho^2}\sqrt{\frac{R+\lambda}{2}} - \frac{a}{2\rho}\frac{1}{2}\frac{1}{\sqrt{\frac{R+\lambda}{2}}}\frac{1}{2}\left(\frac{\partial R}{\partial \rho} + \frac{\partial \lambda}{\partial \rho}\right)$$

$$H_z^{(1)} = -\frac{\partial \Phi^{(1)}}{\partial z} = -\frac{2H_0 a}{\pi}\left(\frac{\partial A}{\partial z} + \frac{\partial B}{\partial z} - \frac{\partial C}{\partial z}\right)\sin\phi$$

$$H_\rho^{(1)} = -\frac{\partial \Phi^{(1)}}{\partial \rho} = -\frac{2H_0 a}{\pi}\left(\frac{\partial A}{\partial \rho} + \frac{\partial B}{\partial \rho} - \frac{\partial C}{\partial \rho}\right)\sin\phi$$

$$H_\phi^{(1)} = -\frac{1}{\rho}\frac{\partial \Phi^{(1)}}{\partial \phi} = -\frac{2H_0 a}{\pi}\left[\frac{z}{2\rho^2}\sqrt{\frac{R-\lambda}{2}} + \frac{1}{2a}\tan^{-1}\sqrt{\frac{2}{R+\lambda}} - \frac{a}{2\rho^2}\sqrt{\frac{R+\lambda}{2}}\right]\cos\phi$$

$$H_x^{(1)} = H_\rho^{(1)}\cos\phi - H_\phi^{(1)}\sin\phi$$

$$H_y^{(1)} = H_\rho^{(1)}\sin\phi + H_\phi^{(1)}\cos\phi$$

$$\frac{\partial \lambda}{\partial \rho} = \frac{2\rho}{a^2}$$

$$\frac{\partial R}{\partial \rho} = \frac{\lambda}{R}\frac{\partial \lambda}{\partial \rho}$$

Finally, after adding in the external field:

$$H_x = H_x^{(1)} \qquad\qquad H_y = H_0 + H_y^{(1)} \qquad\qquad H_z = H_z^{(1)}$$

and for $-z < 0$, by parity,

$$H_x(x, y, -z) = -H_x(x, y, z)$$
$$H_y(x, y, -z) = -H_y(x, y, z)$$
$$H_z(x, y, -z) = H_z(x, y, z)$$