

Our goal is to verify that for the potential given by (1.17)

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \quad (1)$$

its Laplacian is

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \quad (2)$$

I do not really understand why the text chose to divide the whole region into $r \leq R$ and $r > R$. In particular, I cannot convince myself of the sentence "If $\rho(\mathbf{x}')$ is such that (1.17) exists, the contribution to integral (1.30) from the exterior of the sphere will vanish like a^2 as $a \rightarrow 0$ ".

In fact, we can use the "a-potential" method on $\nabla^2(1/r)$ to prove (1.31) directly, i.e.,

$$\nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -4\pi\delta(\mathbf{x} - \mathbf{x}') \quad (3)$$

then it follows that

$$\begin{aligned} \nabla^2 \Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) d^3x' \\ &= \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') [-4\pi\delta(\mathbf{x} - \mathbf{x}')] d^3x' = -\frac{\rho(\mathbf{x})}{\epsilon_0} \end{aligned} \quad (4)$$

To see (3), first notice that for any a ,

$$\nabla \left(\frac{1}{\sqrt{r^2 + a^2}} \right) = -\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(r^2 + a^2)^{3/2}} \implies \quad (5)$$

$$\begin{aligned} \nabla^2 \left(\frac{1}{\sqrt{r^2 + a^2}} \right) &= \frac{\partial}{\partial x} \left[\frac{-x}{(r^2 + a^2)^{3/2}} \right] + \frac{\partial}{\partial y} \left[\frac{-y}{(r^2 + a^2)^{3/2}} \right] + \frac{\partial}{\partial z} \left[\frac{-z}{(r^2 + a^2)^{3/2}} \right] \\ &= \left[\frac{-1}{(r^2 + a^2)^{3/2}} + \frac{3x^2}{(r^2 + a^2)^{5/2}} \right] + \left[\frac{-1}{(r^2 + a^2)^{3/2}} + \frac{3y^2}{(r^2 + a^2)^{5/2}} \right] + \left[\frac{-1}{(r^2 + a^2)^{3/2}} + \frac{3z^2}{(r^2 + a^2)^{5/2}} \right] \\ &= \frac{-3a^2}{(r^2 + a^2)^{5/2}} \end{aligned} \quad (6)$$

For $r > 0$, we take $a = 0$, which gives

$$\nabla^2 \left(\frac{1}{r} \right) = 0 \quad \text{for } r > 0 \quad (7)$$

And the volume integral

$$\begin{aligned} \int \nabla^2 \left(\frac{1}{\sqrt{r^2 + a^2}} \right) d^3x &= - \int d\Omega \int_0^\infty \frac{3a^2}{(r^2 + a^2)^{5/2}} r^2 dr \quad (\text{let } r = a \tan \zeta) \\ &= -4\pi \int_0^{\pi/2} \frac{3a^2 (a^2 \tan^2 \zeta) (a / \cos^2 \zeta) d\zeta}{a^5 / \cos^5 \zeta} \\ &= -4\pi \int_0^{\pi/2} 3 \sin^2 \zeta \cos \zeta d\zeta = -4\pi \end{aligned} \quad (8)$$

By taking the limit of $a \rightarrow 0$ and combining (7) and (8), we can conclude (3) by definition of the δ -function.

Even though we didn't use the text's method, it's still worth explaining the expansion of $\rho(\mathbf{x}')$ in the integral

$$-\frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \left[\frac{3a^2}{(r^2 + a^2)^{5/2}} \right] d^3x' = -\frac{1}{\epsilon_0} \int_0^R \frac{3a^2}{(r^2 + a^2)^{5/2}} \left[\rho(\mathbf{x}) + \frac{r^2}{6} \nabla^2 \rho + \dots \right] r^2 dr \quad (9)$$

With the assumption that $\rho(\mathbf{x}')$ is well behaved, we can expand $\rho(\mathbf{x}')$ around \mathbf{x} :

$$\begin{aligned} \rho(\mathbf{x}') = & \rho(\mathbf{x}) + \left(\frac{\partial \rho}{\partial x} \Delta x + \frac{\partial \rho}{\partial y} \Delta y + \frac{\partial \rho}{\partial z} \Delta z \right) + \\ & \frac{1}{2} \left[\frac{\partial^2 \rho}{\partial x^2} (\Delta x)^2 + \frac{\partial^2 \rho}{\partial y^2} (\Delta y)^2 + \frac{\partial^2 \rho}{\partial z^2} (\Delta z)^2 + 2 \frac{\partial^2 \rho}{\partial x \partial y} \Delta x \Delta y + 2 \frac{\partial^2 \rho}{\partial x \partial z} \Delta x \Delta z + 2 \frac{\partial^2 \rho}{\partial y \partial z} \Delta y \Delta z \right] + \\ & \dots \end{aligned} \quad (10)$$

where all partial derivatives are evaluated at \mathbf{x} , hence can be taken out of the integral.

The integral of first order terms will vanish on the LHS of (9), because their null angular integration:

$$\int_{\Omega} d\Omega \Delta x \propto \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \sin \theta \cos \phi = 0 \quad (11)$$

$$\int_{\Omega} d\Omega \Delta y \propto \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \sin \theta \sin \phi = 0 \quad (12)$$

$$\int_{\Omega} d\Omega \Delta z \propto \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \cos \theta = 0 \quad (13)$$

Similarly, the cross terms of the second order will vanish too:

$$\int_{\Omega} d\Omega \Delta x \Delta y \propto \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \sin^2 \theta \cos \phi \sin \phi = 0 \quad (14)$$

$$\int_{\Omega} d\Omega \Delta x \Delta z \propto \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \sin \theta \cos \phi \cos \theta = 0 \quad (15)$$

$$\int_{\Omega} d\Omega \Delta y \Delta z \propto \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \sin \theta \sin \phi \cos \theta = 0 \quad (16)$$

The angular integrations of the diagonal second order terms are

$$\begin{aligned} \int_{\Omega} d\Omega (\Delta x)^2 &= r^2 \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \sin^2 \theta \cos^2 \phi \\ &= r^2 \int_0^{\pi} \sin^3 \theta d\theta \int_0^{2\pi} \frac{1 + \cos 2\phi}{2} d\phi \\ &= \pi r^2 \int_{-1}^1 (1 - y^2) dy = \left(2 - \frac{2}{3} \right) \pi r^2 = \frac{4\pi r^2}{3} \end{aligned} \quad (17)$$

$$\begin{aligned} \int_{\Omega} d\Omega (\Delta y)^2 &= r^2 \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \sin^2 \theta \sin^2 \phi \\ &= r^2 \int_0^{\pi} \sin^3 \theta d\theta \int_0^{2\pi} \frac{1 - \cos 2\phi}{2} d\phi \\ &= \pi r^2 \int_{-1}^1 (1 - y^2) dy = \left(2 - \frac{2}{3} \right) \pi r^2 = \frac{4\pi r^2}{3} \end{aligned} \quad (18)$$

$$\begin{aligned} \int_{\Omega} d\Omega (\Delta z)^2 &= r^2 \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \cos^2 \theta \\ &= 2\pi r^2 \int_{-1}^1 y^2 dy = \frac{4\pi r^2}{3} \end{aligned} \quad (19)$$

which gives the RHS of (9).