1. **Prob 11.13**

   (a) In frame K', the charge density $\lambda' = q_0$ gives rise to an electric field along the $\hat{\boldsymbol{\rho}}$ direction,

   $$E'_\rho \cdot 2\pi\rho = 4\pi\lambda' \qquad \Longrightarrow \qquad E'_\rho = \frac{2q_0}{\rho} \tag{1}$$

   There is no current in K' so the magnetic field is zero.
   The electric and magnetic field in the lab frame K are given by the inverse transformation of (11.149), i.e.,

   $$\mathbf{E} = \gamma\left(\mathbf{E}' - \boldsymbol{\beta}\times\mathbf{B}'\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{E}'\right) \tag{2}$$

   $$\mathbf{B} = \gamma\left(\mathbf{B}' + \boldsymbol{\beta}\times\mathbf{E}'\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{B}'\right) \tag{3}$$

   Let $\boldsymbol{\beta}$ be along the $\hat{\mathbf{z}}$ direction, this gives

   $$\mathbf{E} = \gamma E'_\rho \hat{\boldsymbol{\rho}} = \frac{2\gamma q_0}{\rho}\hat{\boldsymbol{\rho}} \tag{4}$$

   $$\mathbf{B} = \gamma\beta E'_\rho \hat{\boldsymbol{\phi}} = \frac{2\gamma\beta q_0}{\rho}\hat{\boldsymbol{\phi}} \tag{5}$$

   (b) Since $(c\lambda, \mathbf{J})$ forms a 4-vector, we can use (11.18) to transform it from K' to K:

   $$c\lambda = \gamma\left(c\lambda' + \beta J'_z\right) = \gamma c q_0 \qquad J_z = \gamma\left(J'_z + \beta c\lambda'\right) = \gamma\beta c q_0 \qquad J_x = J_y = 0 \tag{6}$$

   (c) With the above charge and current density, the K-frame electric and magnetic field are

   $$E_\rho 2\pi\rho = 4\pi\lambda \qquad \Longrightarrow \qquad E_\rho = \frac{2\gamma q_0}{\rho} \tag{7}$$

   $$B_\phi 2\pi\rho = \frac{4\pi J_z}{c} \qquad \Longrightarrow \qquad B_\phi = \frac{2\gamma\beta q_0}{\rho} \tag{8}$$

   agreeing with (4) and (5).

2. **Prob 11.14**

   (a) The matrix representation of $F^{\alpha\beta}, F_{\alpha\beta}, \mathscr{F}^{\alpha\beta}$ are given in (11.137), (11.138) and (11.140), and we can write $\mathscr{F}_{\alpha\beta}$ similarly,

   $$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \qquad F_{\alpha\beta} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

   $$\mathscr{F}^{\alpha\beta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix} \qquad \mathscr{F}_{\alpha\beta} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{bmatrix} \tag{9}$$

   The contractions $F^{\alpha\beta}F_{\alpha\beta}, \mathscr{F}^{\alpha\beta}F_{\alpha\beta}, \mathscr{F}^{\alpha\beta}\mathscr{F}_{\alpha\beta}$ are easily obtained by the summation of position-wise product of the matrix elements, giving

   $$F^{\alpha\beta}F_{\alpha\beta} = 2\left(|\mathbf{B}|^2 - |\mathbf{E}|^2\right) \qquad \mathscr{F}^{\alpha\beta}F_{\alpha\beta} = -4\mathbf{B}\cdot\mathbf{E} \qquad \mathscr{F}^{\alpha\beta}\mathscr{F}_{\alpha\beta} = 2\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right) \tag{10}$$

   To form Lorentz invariant scalar out of rank-2 tensors, the only remaining combination is $F^{\alpha\beta}\mathscr{F}_{\alpha\beta}$, which is the same as $\mathscr{F}^{\alpha\beta}F_{\alpha\beta} = -4\mathbf{B}\cdot\mathbf{E}$.

   (b) Since both $|\mathbf{B}|^2 - |\mathbf{E}|^2$ and $\mathbf{B}\cdot\mathbf{E}$ are Lorentz invariant, if $\mathbf{E} = 0$ in some reference frame, we must have

   $$|\mathbf{B}|^2 \geq |\mathbf{E}|^2 \qquad \mathbf{B}\cdot\mathbf{E} = 0 \tag{11}$$

   in all reference frames.

(c) We can similarly form $G^{\alpha\beta}, G_{\alpha\beta}, \mathscr{G}^{\alpha\beta}, \mathscr{G}_{\alpha\beta}$ using $\mathbf{D}, \mathbf{H}$ like (9), we would have $\{G^{\alpha\beta}, \mathscr{G}^{\alpha\beta}\} \times \{G_{\alpha\beta}, \mathscr{G}_{\alpha\beta}\}$ combinations like (10)

$$G^{\alpha\beta} G_{\alpha\beta} = 2\left(|\mathbf{H}|^2 - |\mathbf{D}|^2\right) \qquad \mathscr{G}^{\alpha\beta} G_{\alpha\beta} = G^{\alpha\beta}\mathscr{G}_{\alpha\beta} = -4\mathbf{H}\cdot\mathbf{D} \qquad \mathscr{G}^{\alpha\beta}\mathscr{G}_{\alpha\beta} = 2\left(|\mathbf{D}|^2 - |\mathbf{H}|^2\right) \tag{12}$$

as well as additional F/G combinations

$$\begin{aligned}
F^{\alpha\beta} G_{\alpha\beta} &= G^{\alpha\beta} F_{\alpha\beta} = 2\left(\mathbf{B}\cdot\mathbf{H} - \mathbf{E}\cdot\mathbf{D}\right) \\
\mathscr{F}^{\alpha\beta} G_{\alpha\beta} &= G^{\alpha\beta}\mathscr{F}_{\alpha\beta} = F^{\alpha\beta}\mathscr{G}_{\alpha\beta} = \mathscr{G}^{\alpha\beta} F_{\alpha\beta} = -2\left(\mathbf{B}\cdot\mathbf{D} + \mathbf{E}\cdot\mathbf{H}\right) \\
\mathscr{F}^{\alpha\beta}\mathscr{G}_{\alpha\beta} &= \mathscr{G}^{\alpha\beta}\mathscr{F}_{\alpha\beta} = 2\left(\mathbf{E}\cdot\mathbf{D} - \mathbf{B}\cdot\mathbf{H}\right)
\end{aligned} \tag{13}$$

3. **Prob 11.15**

If in frame K', $\mathbf{E}'$ is parallel to $\mathbf{B}'$, then in that frame, $\mathbf{E}'\times\mathbf{B}' = 0$, i.e., both the energy flux and field momentum density are zero. Although $\mathbf{E}\times\mathbf{B}$ is not part of a 4-vector (it is part of the rank-2 energy stress tensor), we can use the momentum analogy to inspire us to guess the direction of the relative velocity $\mathbf{v}$ of K' with respect to K. The guess is for $\mathbf{v}$ to be along the direction of $\mathbf{E}\times\mathbf{B}$, in which case, the γ^2 term of (11.149) will vanish.

In general, let

$$\mathbf{E} = E_0\hat{\mathbf{x}} \qquad\qquad \mathbf{B} = \alpha E_0\left(\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}\right) \tag{14}$$

then with the trial $\boldsymbol{\beta} = \beta\hat{\mathbf{z}}$, $\mathbf{E}$ and $\mathbf{B}$ transform via

$$\mathbf{E}' = \gamma\left(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}\right) = \gamma\left(E_0\hat{\mathbf{x}} + \alpha\beta E_0\cos\theta\,\hat{\mathbf{y}} - \alpha\beta E_0\sin\theta\,\hat{\mathbf{x}}\right) \tag{15}$$

$$\mathbf{B}' = \gamma\left(\mathbf{B} - \boldsymbol{\beta}\times\mathbf{E}\right) = \gamma\left(\alpha E_0\cos\theta\,\hat{\mathbf{x}} + \alpha E_0\sin\theta\,\hat{\mathbf{y}} - \beta E_0\hat{\mathbf{y}}\right) \tag{16}$$

The condition $\mathbf{E}'\times\mathbf{B}' = 0$ requires

$$\left(1 - \alpha\beta\sin\theta\right)\left(\alpha\sin\theta - \beta\right) = \alpha^2\beta\cos^2\theta \qquad \text{or} \qquad \alpha\sin\theta - \left(1+\alpha^2\right)\beta + \alpha\sin\theta\,\beta^2 = 0 \tag{17}$$

for which the solution

$$\beta = \frac{1 + \alpha^2 \pm \sqrt{\left(1+\alpha^2\right)^2 - 4\alpha^2\sin^2\theta}}{2\alpha\sin\theta} \tag{18}$$

exists for all θ and α. But since $\beta < 1$, we can only take the "$-$" sign.

Let's verify the Lorentz invariants of problem 11.14

$$\begin{aligned}
\left|\mathbf{B}'\right|^2 - \left|\mathbf{E}'\right|^2 &= \gamma^2 E_0^2\left[\left(\alpha\cos\theta\right)^2 + \left(\alpha\sin\theta - \beta\right)^2 - \left(1 - \alpha\beta\sin\theta\right)^2 - \left(\alpha\beta\cos\theta\right)^2\right] \\
&= \gamma^2 E_0^2\left[\left(1-\alpha^2\right)\beta^2 + \alpha^2 - 1\right] = \left(\alpha^2 - 1\right)E_0^2 = |\mathbf{B}|^2 - |\mathbf{E}|^2
\end{aligned} \tag{19}$$

$$\begin{aligned}
\mathbf{B}'\cdot\mathbf{E}' &= \gamma^2 E_0^2\left[\left(\alpha\cos\theta\right)\left(1 - \alpha\beta\sin\theta\right) + \left(\alpha\sin\theta - \beta\right)\left(\alpha\beta\cos\theta\right)\right] \\
&= \gamma^2\alpha E_0^2\cos\theta\left(1 - \beta^2\right) = \alpha E_0^2\cos\theta = \mathbf{B}\cdot\mathbf{E}
\end{aligned} \tag{20}$$

Substituting $\alpha = 2$ for this problem, we have

$$\beta = \frac{5 - \sqrt{25 - 16\sin^2\theta}}{4\sin\theta} \tag{21}$$

- when $\theta \to 0$, $\beta \to 2\sin\theta/5$, i.e., when $\mathbf{B}, \mathbf{E}$ are already almost parallel, one does not need to boost very fast to make them perfectly parallel.
- when $\theta \to \pi/2$, $\beta \to 1/2$, in which case $\mathbf{E}' \to 0$, which is trivially parallel to $\mathbf{B}'$ while still maintaining the invariance of $\mathbf{B}\cdot\mathbf{E}$.

It is important to question the uniqueness of K'. Could there be another frame K'', moving relative to K with velocity $\mathbf{u}$, in which $\mathbf{E}''$ and $\mathbf{B}''$ are also parallel? Let's assume K'' and $\mathbf{u}$ do exist. The transformation between K and K', as well as between K and K'' are given by the two boosts $B(\mathbf{v})$ and $B(\mathbf{u})$ respectively,

$$K' \quad \xleftarrow{B(\mathbf{v})} \quad K \quad \xrightarrow{B(\mathbf{u})} \quad K''$$

Then the Lorentz transformation from K' to K'' is just $\Lambda \equiv B(\mathbf{u})B(-\mathbf{v})$. Due to Thomas (Wigner) rotation, Λ is not a pure boost in general, but it can be decomposed into a boost and a rotation in either order

$$\Lambda = B(\mathbf{u})B(-\mathbf{v}) = R(\theta)B\left[(-\mathbf{v})\oplus\mathbf{u}\right] = B\left[\mathbf{u}\oplus(-\mathbf{v})\right]R(\theta) \tag{22}$$

where $\oplus$ is the non-commutative addition of velocities (see my notes on Thomas Rotation for the derivation of (22)). We take the former order $\Lambda = R(\theta) B(\mathbf{w})$ with $\mathbf{w} = (-\mathbf{v}) \oplus \mathbf{u}$.

$$K' \xrightarrow{B(\mathbf{w})} \Sigma \xrightarrow{R(\theta)} K''$$

This decomposition allows us to define a reference frame Σ as the transformation of K' via $B(\mathbf{w})$, and K'' is subsequently obtained from Σ by a rotation $R(\theta)$ without boost.

From (11.147), we see that the matrix representation of the field-strength tensor F transforms from K' to K'' as

$$F'' = \Lambda F' \Lambda^T = R(\theta) F_\Sigma [R(\theta)]^T \qquad \text{where} \qquad F_\Sigma = B(\mathbf{w}) F' [B(\mathbf{w})]^T \tag{23}$$

We can write F_Σ in block form (see (11.137))

$$F_\Sigma = \begin{bmatrix} 0 & -\mathbf{E}_\Sigma^T \\ \mathbf{E}_\Sigma & \mathbf{B}_\Sigma \cdot \mathbf{S} \end{bmatrix} \tag{24}$$

where S_i's are the 3×3 matrices corresponding to the generators of rotations in the Lorentz group, i.e., the lower-right 3×3 block of (11.91).

With

$$R(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & r(\theta) \end{bmatrix} \tag{25}$$

where $r(\theta) \in SO(3)$ is the 3×3 rotation matrix, we can use the orthogonality of r to verify that

$$F'' = R(\theta) F_\Sigma [R(\theta)]^T = \begin{bmatrix} 0 & -\mathbf{E}_\Sigma^T r^T \\ r\mathbf{E}_\Sigma & (r\mathbf{B}_\Sigma) \cdot \mathbf{S} \end{bmatrix} \tag{26}$$

as expected from subjecting the 3-vectors $\mathbf{E}_\Sigma, \mathbf{B}_\Sigma$ to a rotation r. If $\mathbf{E}'' = r\mathbf{E}_\Sigma$ were to be parallel to $\mathbf{B}'' = r\mathbf{B}_\Sigma$ in K'', $\mathbf{E}_\Sigma$ and $\mathbf{B}_\Sigma$ must be parallel to each other in Σ in the first place.

Recall (23) that $\mathbf{E}_\Sigma, \mathbf{B}_\Sigma$ can be obtained from the mutually parallel $\mathbf{E}', \mathbf{B}'$ by a boost $B(\mathbf{w})$. In components parallel and perpendicular to $\mathbf{w}$, the transformation reads

$$\mathbf{E}_{\Sigma\|} = \mathbf{E}'_\| \qquad \mathbf{E}_{\Sigma\perp} = \gamma_\mathbf{w}\left(\mathbf{E}'_\perp + \boldsymbol{\beta}_\mathbf{w} \times \mathbf{B}'\right) \tag{27}$$

$$\mathbf{B}_{\Sigma\|} = \mathbf{B}'_\| \qquad \mathbf{B}_{\Sigma\perp} = \gamma_\mathbf{w}\left(\mathbf{B}'_\perp - \boldsymbol{\beta}_\mathbf{w} \times \mathbf{E}'\right) \tag{28}$$

To avoid confusion with the original $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ directions, we take $\boldsymbol{\beta}_\mathbf{w}$ to be along the $\boldsymbol{\epsilon}_1$ direction, and take $\mathbf{E}'$ ($\mathbf{B}'$) to be along the direction $\cos\xi\boldsymbol{\epsilon}_1 + \sin\xi\boldsymbol{\epsilon}_2$, then (27) and (28) can be written as

$$\mathbf{E}_{\Sigma\|} = E'\cos\xi\boldsymbol{\epsilon}_1 \qquad \mathbf{E}_{\Sigma\perp} = \gamma_\mathbf{w}\left(E'\sin\xi\boldsymbol{\epsilon}_2 + \beta_\mathbf{w} B'\sin\xi\boldsymbol{\epsilon}_3\right) \tag{29}$$

$$\mathbf{B}_{\Sigma\|} = B'\cos\xi\boldsymbol{\epsilon}_1 \qquad \mathbf{B}_{\Sigma\perp} = \gamma_\mathbf{w}\left(B'\sin\xi\boldsymbol{\epsilon}_2 - \beta_\mathbf{w} E'\sin\xi\boldsymbol{\epsilon}_3\right) \tag{30}$$

The requirement $\mathbf{E}_\Sigma \parallel \mathbf{B}_\Sigma$, or $\mathbf{E}_\Sigma \times \mathbf{B}_\Sigma = 0$, mandates that

$$0 = (\mathbf{E}_\Sigma \times \mathbf{B}_\Sigma)_1 = E_{\Sigma2}B_{\Sigma3} - E_{\Sigma3}B_{\Sigma2} = -\gamma_\mathbf{w}^2\beta_\mathbf{w}\left(E'^2 + B'^2\right)\sin^2\xi \tag{31}$$

$$0 = (\mathbf{E}_\Sigma \times \mathbf{B}_\Sigma)_2 = E_{\Sigma3}B_{\Sigma1} - E_{\Sigma1}B_{\Sigma3} = \gamma_\mathbf{w}\beta_\mathbf{w}\left(E'^2 + B'^2\right)\sin\xi\cos\xi \tag{32}$$

(The $\boldsymbol{\epsilon}_3$-component is trivially zero). For (31) and (32) to hold, either there is no relative movement ($\beta_\mathbf{w} = 0$) or the relative movement $\mathbf{w}$ is along the common direction of $\mathbf{E}'$ and $\mathbf{B}'$ ($\sin\xi = 0$).

Of course the condition $\beta_\mathbf{w} = 0$ corresponds to the known solution $K'' = K'$. To see what $\sin\xi = 0$ entails, let's come back to the original coordinate axis, where $\mathbf{v}$ points to the $\hat{\mathbf{z}}$ direction. The addition of velocity formula (11.31) gives the z-component and the transverse component of $\mathbf{w} = (-\mathbf{v}) \oplus \mathbf{u}$

$$w_z = \frac{u_z - v}{1 - \dfrac{u_z v}{c^2}} \qquad w_t = \frac{u_t}{\gamma_v\left(1 - \dfrac{u_z v}{c^2}\right)} \tag{33}$$

Let $\hat{\boldsymbol{\rho}}$ be the unit transverse direction (in x-y plane) of $\mathbf{E}'$ and $\mathbf{B}'$ (see (15) and (16)), for $\mathbf{w}$ to be parallel to $\hat{\boldsymbol{\rho}}$ (i.e., $\sin\xi = 0$), we must have

$$\mathbf{u} = v\mathbf{z} + u_t\hat{\boldsymbol{\rho}} = c\beta\hat{\mathbf{z}} + u_t\hat{\boldsymbol{\rho}} \tag{34}$$

where β is the "$-$" sign solution of (18), and u_t can be arbitrary as long as $|\mathbf{u}|^2 = u_z^2 + u_t^2 < c^2$.

In summary, any reference frame K'' moving relative to K with velocity $\mathbf{u}$ will observe parallel electric and magnetic fields. K' is a just special case where $u_t = 0$.