Here we provide two proofs for the mean value theorem.

- **Proof 1:**

  Let the point of interest be the origin for our coordinate system, and let $\mathbf{E}(r,\theta,\phi)$ be the electric field. Let $E_{\text{rad}}(r,\theta,\phi) = \mathbf{E}(r,\theta,\phi)\cdot\mathbf{r}/r$ be the radial component of the electric field. For an arbitrary sphere with radius R centered at the origin, consider a point (R,θ,ϕ) on the sphere. Its potential drop from the origin is given by

$$\Delta\Phi(R,\theta,\phi) = \int_0^R E_{\text{rad}}(r,\theta,\phi)dr \tag{1}$$

  Therefore, the average potential drop of all the points on the sphere at R is then

$$\begin{aligned}\langle\Delta\Phi(r=R)\rangle &= \frac{1}{4\pi R^2}\int_\Omega \Delta\Phi(R,\theta,\phi)R^2 d\Omega \\ &= \frac{1}{4\pi R^2}\int_\Omega\int_0^R E_{\text{rad}}(r,\theta,\phi)drR^2 d\Omega && \text{(exchange integral order)} \\ &= \frac{1}{4\pi}\int_0^R dr\int_\Omega E_{\text{rad}}(r,\theta,\phi)d\Omega \end{aligned} \tag{2}$$

  But by Gauss's theorem, the inner integral is proportional to the total charge inside the sphere of radius r, which by assumption is zero. Hence the average potential drop from the origin on the sphere at R is zero, which means the origin's potential is the average potential of all that of the sphere's surface points.

- **Proof 2:**

  This proof treats the problem as a boundary value problem, where the potentials of all the points on the sphere are taken as given (i.e., Dirichlet boundary condition), and we seek to find the potential at the center $\mathbf{x}$.

  We follow the Green's function method in the text. Define

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x},\mathbf{x}') \tag{3}$$

  where $\nabla'^2 F(\mathbf{x},\mathbf{x}') = 0$ and in the mean time, will also make $G(\mathbf{x},\mathbf{x}') = 0$ for the boundary points. Since our boundary is the sphere with radius $|\mathbf{x}-\mathbf{x}'| = R$, this gives an obvious choice for F:

$$F(\mathbf{x},\mathbf{x}') = -\frac{1}{R} \tag{4}$$

  and hence

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{R} \tag{5}$$

  By equation (1.42)

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x' + \frac{1}{4\pi}\oint_S\left[G(\mathbf{x},\mathbf{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}\right]da' \tag{6}$$

  Recall that we have assumed $\rho(\mathbf{x}') = 0$ everwhere in V, and $G(\mathbf{x},\mathbf{x}') = 0$ everywhere on S. This turns (6) into

$$\begin{aligned}\Phi(\mathbf{x}) &= -\frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}da' && \text{by (5)} \\ &= -\frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\left(-\frac{1}{r^2}\bigg|_{r=R}\right)da' \\ &= \frac{1}{4\pi R^2}\oint_S \Phi(\mathbf{x}')da' \end{aligned} \tag{7}$$

  which is exactly what mean value theorem claims.