1. In steady state, let the interior and exterior potential be $\Phi_{\text{int}}$ and $\Phi_{\text{ext}}$ respectively. Usual argument will give their form

$$\Phi_{\text{int}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{1}$$

$$\Phi_{\text{ext}} = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) \tag{2}$$

The potential continuity at boundary requires

$$\sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = \sum_{l=0}^{\infty} B_l a^{-(l+1)} P_l(\cos\theta) \tag{3}$$

which by orthogonality of Legendre polynomials implies

$$B_l = A_l a^{2l+1} \tag{4}$$

The radial components of the interior and exterior field near the boundary are

$$\mathbf{E}_{\text{int}} \cdot \hat{\mathbf{r}} = -\frac{\partial \Phi_{\text{int}}}{\partial r} = -\sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\cos\theta) \tag{5}$$

$$\mathbf{E}_{\text{ext}} \cdot \hat{\mathbf{r}} = -\frac{\partial \Phi_{\text{ext}}}{\partial r} = \sum_{l=0}^{\infty} (l+1) B_l a^{-(l+2)} P_l(\cos\theta) = \sum_{l=0}^{\infty} (l+1) A_l a^{(l-1)} P_l(\cos\theta) \tag{6}$$

Furthermore, in steady state, we have the restriction of continuity of radial current at the boundary

$$\mathbf{J}_{\text{int}} \cdot \hat{\mathbf{r}} = \sigma\left(F\hat{\mathbf{z}} + \mathbf{E}_{\text{int}}\right) \cdot \hat{\mathbf{r}} = \sigma\left[F\cos\theta - \sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\cos\theta)\right] =$$

$$\mathbf{J}_{\text{ext}} \cdot \hat{\mathbf{r}} = \sigma' \mathbf{E}_{\text{ext}} \cdot \hat{\mathbf{r}} = \sigma' \sum_{l=0}^{\infty} (l+1) A_l a^{l-1} P_l(\cos\theta) \qquad \Longrightarrow$$

$$\sigma F \cos\theta = \sum_{l=0}^{\infty} \left[l\sigma + (l+1)\sigma'\right] A_l a^{l-1} P_l(\cos\theta) \tag{7}$$

Again, we apply the orthogonality of Legendre polynomials to conclude that all $A_l$'s vanish except for $l = 1$, and

$$A_1 = \frac{\sigma F}{2\sigma' + \sigma} \tag{8}$$

Going back to (5) and (6), we have

$$\mathbf{E}_{\text{int}} \cdot \hat{\mathbf{r}} = -A_1 \cos\theta \qquad\qquad \mathbf{E}_{\text{ext}} \cdot \hat{\mathbf{r}} = 2A_1 \cos\theta \tag{9}$$

So by Gauss's law, the surface charge density is

$$\eta = \epsilon_0\left(\mathbf{E}_{\text{ext}} \cdot \hat{\mathbf{r}} - \mathbf{E}_{\text{int}} \cdot \hat{\mathbf{r}}\right) = \frac{3\epsilon_0 \sigma F}{2\sigma' + \sigma} \cos\theta \tag{10}$$

and the dipole moment is

$$\mathbf{p} = a^2 \int d\Omega \eta \hat{\mathbf{r}} = a^2 \hat{\mathbf{z}} \int_0^{2\pi} d\phi \int_0^{\pi} \eta a \cos\theta \sin\theta d\theta = 2\pi a^3 \cdot \frac{3\epsilon_0 \sigma F}{2\sigma' + \sigma} \hat{\mathbf{z}} \overbrace{\int_0^{\pi} \sin\theta \cos^2\theta d\theta}^{2/3} = \frac{4\pi\epsilon_0 a^3 \sigma F}{2\sigma' + \sigma} \hat{\mathbf{z}} \tag{11}$$

While (9) gives the radial component of the interior and exterior electric field at the boundary, we can get the angular component of them via

$$\mathbf{E}_{\text{int}} \cdot \hat{\boldsymbol{\theta}} = -\frac{1}{r}\frac{\partial \Phi_{\text{int}}}{\partial \theta} = A_1 \sin\theta \qquad\qquad \mathbf{E}_{\text{ext}} \cdot \hat{\boldsymbol{\theta}} = -\frac{1}{r}\frac{\partial \Phi_{\text{ext}}}{\partial \theta} = A_1 \sin\theta \tag{12}$$

which are equal as expected from $\boldsymbol{\nabla} \times \mathbf{E} = 0$.

2. By (9), the total current coming out of the northern hemisphere is

$$I = a^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta d\theta 2\sigma' A_1 \cos\theta = 2a^2\pi \frac{2\sigma\sigma' F}{2\sigma' + \sigma} \int_0^{\pi/2} \sin\theta \cos\theta d\theta = \frac{2\pi a^2 \sigma\sigma' F}{2\sigma' + \sigma} \tag{13}$$

By Ohm's law, the power dissipation in the exterior region is

$$\begin{aligned}
P_{\text{ext}} &= \frac{1}{\sigma'} \int \mathbf{J}_{\text{ext}}^2 dV = \frac{1}{\sigma'} \int_a^{\infty} r^2 dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \cdot \left(\sigma' \mathbf{E}_{\text{ext}}\right)^2 \\
&= 2\pi\sigma' \int_a^{\infty} r^2 dr \int_0^{\pi} \sin\theta d\theta \left( 2A_1 \frac{a^3}{r^3} \cos\theta \hat{\mathbf{r}} + A_1 \frac{a^3}{r^3} \sin\theta \hat{\boldsymbol{\theta}} \right)^2 \\
&= 2\pi\sigma' a^6 A_1^2 \underbrace{\int_a^{\infty} \frac{dr}{r^4}}_{1/(3a^3)} \overbrace{\int_0^{\pi} \sin\theta d\theta \left(4\cos^2\theta + \sin^2\theta\right)}^{4} \\
&= \frac{8\pi a^3 \sigma' \sigma^2 F^2}{3\left(2\sigma' + \sigma\right)^2}
\end{aligned} \tag{14}$$

Thus

$$R_{\text{ext}} = \frac{P_{\text{ext}}}{I^2} = \frac{2}{3\pi\sigma' a} \qquad V_{\text{ext}} = \frac{P_{\text{ext}}}{I} = \frac{4a\sigma F}{3(2\sigma' + \sigma)} \tag{15}$$

3. The total current is the same for interior and exterior, but the power dissipation for the interior region is

$$\begin{aligned}
P_{\text{int}} &= \frac{1}{\sigma} \int \mathbf{J}_{\text{int}}^2 dV = \frac{1}{\sigma} \int_0^{a} r^2 dr 2\pi \int_0^{\pi} \sin\theta d\theta \left[\sigma \left(F\hat{\mathbf{z}} + \mathbf{E}_{\text{int}}\right)\right]^2 \\
&= 2\pi\sigma \int_0^{a} r^2 dr \int_0^{\pi} \sin\theta d\theta \left[(F - A_1)\cos\theta \hat{\mathbf{r}} - (F - A_1)\sin\theta \hat{\boldsymbol{\theta}}\right]^2 \\
&= 2\pi\sigma \cdot \frac{a^3}{3} \cdot 2\left(F - A_1\right)^2 \\
&= \frac{16\pi a^3 \sigma \sigma'^2 F^2}{3\left(2\sigma' + \sigma\right)^2}
\end{aligned} \tag{16}$$

and therefore

$$R_{\text{int}} = \frac{P_{\text{int}}}{I} = \frac{4}{3\pi\sigma a} \qquad V_{\text{int}} = \frac{8a\sigma' F}{3\left(2\sigma' + \sigma\right)} \tag{17}$$

It follows that the total voltage is

$$V_{\text{total}} = V_{\text{ext}} + V_{\text{int}} = \frac{4aF}{3} \tag{18}$$