In these notes, the multipole fields in Jackson section 9.7 and 9.8 will be reformulated in terms of vector spherical harmonics. This allows us to supply derivation details missing from Jackson.

1. **Vector spherical harmonics**

   Vector spherical harmonics are defined as

$$\mathbf{Y}_{lm}(\mathbf{r}) = Y_{lm}(\theta,\phi)\hat{\mathbf{r}} \tag{1}$$
$$\mathbf{\Psi}_{lm}(\mathbf{r}) = r\nabla Y_{lm}(\theta,\phi) \tag{2}$$
$$\mathbf{\Phi}_{lm}(\mathbf{r}) = \mathbf{r}\times\nabla Y_{lm}(\theta,\phi) \tag{3}$$

   where $Y_{lm}(\theta,\phi)$ is the usual (scalar) spherical harmonics. The factor r and $\mathbf{r}$ in (2) and (3) ensure $\mathbf{\Psi}_{lm}$ and $\mathbf{\Phi}_{lm}$ have no r-dependency so all of them are in fact only functions of (θ,ϕ).

2. **Reformulating multipole expansion of electromagnetic fields in VSH**

   In Jackson section 9.7, with the angular momentum operator $\mathbf{L}=\mathbf{r}\times\nabla/i$, we have the following identities

$$\mathbf{L}Y_{lm} = \frac{1}{i}\mathbf{r}\times\nabla Y_{lm} = \frac{1}{i}\mathbf{\Phi}_{lm} \qquad \mathbf{X}_{lm} = \frac{1}{i\sqrt{l(l+1)}}\mathbf{\Phi}_{lm} \tag{4}$$

   The solution (9.116) and (9.118) can be expressed in these symbols

$$\mathbf{E}^{(M)}_{lm} = Z_0 g_l(kr)\mathbf{L}Y_{lm} = \frac{Z_0}{i}g_l(kr)\mathbf{\Phi}_{lm} \tag{5}$$
$$\mathbf{H}^{(M)}_{lm} = -\frac{i}{kZ_0}\nabla\times\mathbf{E}^{(M)}_{lm} = -\frac{1}{k}\nabla\times[g_l(kr)\mathbf{\Phi}_{lm}] \tag{6}$$
$$\mathbf{H}^{(E)}_{lm} = f_l(kr)\mathbf{L}Y_{lm} = \frac{f_l(kr)}{i}\mathbf{\Phi}_{lm} \tag{7}$$
$$\mathbf{E}^{(E)}_{lm} = \frac{iZ_0}{k}\nabla\times\mathbf{H}^{(E)}_{lm} = \frac{Z_0}{k}\nabla\times[f_l(kr)\mathbf{\Phi}_{lm}] \tag{8}$$

   In particular, we recall from the previous notes that

$$\nabla\times[h(r)\mathbf{\Phi}_{lm}] = -\frac{l(l+1)}{r}h\mathbf{Y}_{lm} - \left(\frac{dh}{dr}+\frac{1}{r}h\right)\mathbf{\Psi}_{lm} = -\frac{l(l+1)}{r}h\mathbf{Y}_{lm} - \frac{d(rh)}{dr}\nabla Y_{lm} \tag{9}$$

   Because $\mathbf{\Phi}_{lm}$ is transverse, we see that the magnetic multipole field, (M) is indeed TE, and because of (9), in this mode the magnetic field will generally have a radial component. Vice versa for the electric multipole field, (E), where magnetic field is transverse and electric field generally has a radial component.

   From (9.119) we can also see that

$$\mathbf{r}\times\mathbf{X}_{lm} = \frac{1}{i\sqrt{l(l+1)}}\mathbf{r}\times(\mathbf{r}\times\nabla Y_{lm}) = -\frac{r}{i\sqrt{l(l+1)}}\mathbf{\Psi}_{lm} \tag{10}$$

   so the orthogonality condition (9.120), (9.121) follow from the orthogonality of VSH in Hilbert space.

   We can also express the general solution (9.122) in terms of VHS,

$$\mathbf{H} = \sum_{l,m}\left\{\frac{a_E(l,m)f_l(kr)}{i\sqrt{l(l+1)}}\mathbf{\Phi}_{lm} - \frac{a_M(l,m)}{k\sqrt{l(l+1)}}\nabla\times[g_l(kr)\mathbf{\Phi}_{lm}]\right\} \tag{11}$$
$$\mathbf{E} = Z_0\sum_{l,m}\left\{\frac{a_E(l,m)}{k\sqrt{l(l+1)}}\nabla\times[f_l(kr)\mathbf{\Phi}_{lm}] + \frac{a_M(l,m)g_l(kr)}{i\sqrt{l(l+1)}}\mathbf{\Phi}_{lm}\right\} \tag{12}$$

   Then (9.123) can be obtained from (9) and the fact that $\mathbf{\Phi}_{lm}$ is transverse, as well as the orthonormality of spherical harmonics.

3. **Angular momentum of both electric and magnetic multipoles, (9.145)**

   Before we prove (9.145), we must clarify the dimensions of a_E, a_M, f_l and g_l. From the definition of f_l and g_l in (9.116), (9.118), we see that both f_l and g_l have the same dimension as $\mathbf{H}$, hence both a_E and a_M are dimensionless. With the correct dimension, in the radiation zone as $r\to\infty$, by asymptotic form (9.89) we have

$$\begin{Bmatrix} f_l(kr) \\ g_l(kr) \end{Bmatrix} \longrightarrow H_0(-i)^{l+1}\frac{e^{ikr}}{kr} \tag{13}$$

Strictly speaking (9.145) is missing H_0^2 on the RHS, but in Jackson's context, H_0 should be understood to be 1. Now we prove (9.145). If we include both electric and magnetic multipoles, the triple cross product in (9.137) is

$$\begin{aligned}\mathbf{r}\times(\mathbf{E}\times\mathbf{H}^*) &= \mathbf{r}\times\left\{\left[\mathbf{E}^{(M)}+\mathbf{E}^{(E)}\right]\times\left[\mathbf{H}^{(M)}+\mathbf{H}^{(E)}\right]^*\right\} && \mathbf{E}^{(M)},\mathbf{H}^{(E)}\text{ are transverse}\\ &= \left[\mathbf{r}\cdot\mathbf{H}^{(M)*}\right]\left[\mathbf{E}^{(M)}+\mathbf{E}^{(E)}\right]-\left[\mathbf{r}\cdot\mathbf{E}^{(E)}\right]\left[\mathbf{H}^{(M)}+\mathbf{H}^{(E)}\right]^* && (14)\end{aligned}$$

Taking the conjugate of the minus sign term will produce the same real part, i.e.,

$$\mathrm{Re}\left[\mathbf{r}\times(\mathbf{E}\times\mathbf{H}^*)\right]=\mathrm{Re}\left\{\left[\mathbf{r}\cdot\mathbf{H}^{(M)*}\right]\mathbf{E}^{(M)}+\left[\mathbf{r}\cdot\mathbf{H}^{(M)*}\right]\mathbf{E}^{(E)}-\left[\mathbf{r}\cdot\mathbf{E}^{(E)*}\right]\mathbf{H}^{(M)}-\left[\mathbf{r}\cdot\mathbf{E}^{(E)*}\right]\mathbf{H}^{(E)}\right\} \tag{15}$$

where the fields have the following explicit forms by (11)

$$\mathbf{H}^{(E)}=\sum_{l,m}\frac{a_E(l,m)f_l(kr)}{i\sqrt{l(l+1)}}\mathbf{\Phi}_{lm} \qquad \mathbf{E}^{(E)}=\frac{Z_0}{k}\sum_{l,m}\frac{a_E(l,m)}{\sqrt{l(l+1)}}\nabla\times\left[f_l(kr)\mathbf{\Phi}_{lm}\right] \tag{16}$$

$$\mathbf{E}^{(M)}=Z_0\sum_{l,m}\frac{a_M(l,m)g_l(kr)}{i\sqrt{l(l+1)}}\mathbf{\Phi}_{lm} \qquad \mathbf{H}^{(M)}=-\frac{1}{k}\sum_{l,m}\frac{a_M(l,m)}{\sqrt{l(l+1)}}\nabla\times\left[g_l(kr)\mathbf{\Phi}_{lm}\right] \tag{17}$$

Using the curl formula for $\mathbf{\Phi}_{lm}$ (9), we have

$$\mathbf{r}\cdot\mathbf{H}^{(M)}=-\frac{1}{k}\sum_{l,m}\frac{a_M(l,m)}{\sqrt{l(l+1)}}\left[-l(l+1)\right]g_l(kr)Y_{lm}=\frac{1}{k}\sum_{l,m}a_M(l,m)g_l(kr)\mathbf{L}\cdot\mathbf{X}_{lm} \tag{18}$$

and similarly

$$\mathbf{r}\cdot\mathbf{E}^{(E)}=-\frac{Z_0}{k}\sum_{l,m}a_E(l,m)f_l(kr)\mathbf{L}\cdot\mathbf{X}_{lm} \tag{19}$$

Then we obtain

$$\begin{aligned}\left[\mathbf{r}\cdot\mathbf{H}^{(M)*}\right]\mathbf{E}^{(M)} &= \frac{1}{k}\left[\sum_{l,m}a_M^*(l,m)g_l^*(kr)(\mathbf{L}\cdot\mathbf{X}_{lm})^*\right]\left[Z_0\sum_{l,m}a_M(l,m)g_l(kr)\mathbf{X}_{lm}\right]\\ &= \frac{Z_0}{k}\sum_{l,l',m,m'}a_M^*\left(l',m'\right)a_M(l,m)g_{l'}^*(kr)g_l(kr)(\mathbf{L}\cdot\mathbf{X}_{l'm'})^*\mathbf{X}_{lm} && (20)\\ \left[\mathbf{r}\cdot\mathbf{E}^{(E)*}\right]\mathbf{H}^{(E)} &= -\frac{Z_0}{k}\sum_{l,l',m,m'}a_E^*\left(l',m'\right)a_E(l,m)f_{l'}^*(kr)f_l(kr)(\mathbf{L}\cdot\mathbf{X}_{l'm'})^*\mathbf{X}_{lm} && (21)\end{aligned}$$

It is then clear that by putting (20) and (21) into (15) and taking the radiation zone approximation (13) will produce the first part of (9.145),

$$\left.\frac{d\mathbf{M}}{dr}\right|_{1\text{st}}=\frac{\mu_0H_0^2}{2\omega k^2}\mathrm{Re}\sum_{l,l',m,m'}\left[a_E^*\left(l',m'\right)a_E(l,m)+a_M^*\left(l',m'\right)a_M(l,m)\right]\int(\mathbf{L}\cdot\mathbf{X}_{l'm'})^*\mathbf{X}_{lm}d\Omega \tag{22}$$

To calculate the E/M interference terms in (15), we shall separate the radial component from transverse components of the curls in (16) and (17). By (9),

$$E_{\text{rad}}^{(E)}=\frac{Z_0}{k}\sum_{l,m}\frac{a_E(l,m)}{\sqrt{l(l+1)}}\left[-\frac{l(l+1)}{r}\right]f_l(kr)Y_{lm}=-\frac{Z_0}{kr}\sum_{l,m}a_E(l,m)f_l(kr)\sqrt{l(l+1)}Y_{lm} \tag{23}$$

$$H_{\text{rad}}^{(M)}=-\frac{1}{k}\sum_{l,m}\frac{a_M(l,m)}{\sqrt{l(l+1)}}\left[-\frac{l(l+1)}{r}\right]g_l(kr)Y_{lm}=\frac{1}{kr}\sum_{lm}a_M(l,m)g_l(kr)\sqrt{l(l+1)}Y_{lm} \tag{24}$$

which gives

$$\left[\mathbf{r}\cdot\mathbf{H}^{(M)*}\right]E_{\text{rad}}^{(E)}=-\frac{Z_0}{k^2r}\sum_{l,l',m,m'}a_M^*\left(l',m'\right)g_{l'}^*(kr)a_E(l,m)f_l(kr)\sqrt{l(l+1)}(\mathbf{L}\cdot\mathbf{X}_{l'm'})^*Y_{lm} \tag{25}$$

$$\left[\mathbf{r}\cdot\mathbf{E}^{(E)*}\right]H_{\text{rad}}^{(M)}=-\frac{Z_0}{k^2r}\sum_{l,l',m,m'}a_E^*\left(l',m'\right)f_{l'}^*(kr)a_M(l,m)g_l(kr)\sqrt{l(l+1)}(\mathbf{L}\cdot\mathbf{X}_{l'm'})^*Y_{lm} \tag{26}$$

Since $\mathbf{L}\cdot\mathbf{X}_{l'm'} = \mathbf{L}\cdot\mathbf{L}Y_{l'm'}/\sqrt{l'(l'+1)} = \sqrt{l'(l'+1)}Y_{l'm'}$, we see that (25) and (26) are complex conjugate of each other so their contributions to the real part (15) will cancel. Note this is exact without radiation zone approximation and is true at every point.

Now by (9) and (10), the transverse components of $\mathbf{E}^{(E)}$ and $\mathbf{H}^{(M)}$ are

$$\mathbf{E}^{(E)}_{\text{trans}} = -\frac{Z_0}{k}\sum_{l,m}\frac{a_E(l,m)}{\sqrt{l(l+1)}}\frac{d[rf_l(kr)]}{dr}\nabla Y_{lm} = \frac{iZ_0}{kr}\sum_{l,m}a_E(l,m)\frac{d[rf_l(kr)]}{dr}\hat{\mathbf{r}}\times\mathbf{X}_{lm} \tag{27}$$

$$\mathbf{H}^{(M)}_{\text{trans}} = \frac{1}{k}\sum_{l,m}\frac{a_M(l,m)}{\sqrt{l(l+1)}}\frac{d[rg_l(kr)]}{dr}\nabla Y_{lm} = -\frac{i}{kr}\sum_{l,m}a_M(l,m)\frac{d[rg_l(kr)]}{dr}\hat{\mathbf{r}}\times\mathbf{X}_{lm} \tag{28}$$

so

$$\left[\mathbf{r}\cdot\mathbf{H}^{(M)*}\right]\mathbf{E}^{(E)}_{\text{trans}} = \frac{iZ_0}{k^2r}\sum_{l,l',m,m'}a_M^*\left(l',m'\right)a_E(l,m)\,g_{l'}^*(kr)\frac{d[rf_l(kr)]}{dr}\left(\mathbf{L}\cdot\mathbf{X}_{l'm'}\right)^*\hat{\mathbf{r}}\times\mathbf{X}_{lm} \tag{29}$$

$$\left[\mathbf{r}\cdot\mathbf{E}^{(E)*}\right]\mathbf{H}^{(M)}_{\text{trans}} = \frac{iZ_0}{k^2r}\sum_{l,l',m,m'}a_E^*\left(l',m'\right)a_M(l,m)\,f_{l'}^*(kr)\frac{d[rg_l(kr)]}{dr}\left(\mathbf{L}\cdot\mathbf{X}_{l'm'}\right)^*\hat{\mathbf{r}}\times\mathbf{X}_{lm} \tag{30}$$

In the radiation zone

$$\left\{\begin{matrix} g_{l'}^*(kr)\dfrac{d[rf_l(kr)]}{dr} \\ f_{l'}^*(kr)\dfrac{d[rg_l(kr)]}{dr}\end{matrix}\right\} \longrightarrow H_0^2 i^{l'+1}\frac{e^{-ikr}}{kr}\cdot(-i)^{l+1}\,ie^{ikr} = i^{l'-l+1}\frac{H_0^2}{kr} \tag{31}$$

thus

$$\begin{aligned}&\left[\mathbf{r}\cdot\mathbf{H}^{(M)*}\right]\mathbf{E}^{(E)}_{\text{trans}} - \left[\mathbf{r}\cdot\mathbf{E}^{(E)*}\right]\mathbf{H}^{(M)}_{\text{trans}} \longrightarrow \\ &\frac{Z_0H_0^2}{k^3r^2}\sum_{l,l',m,m'}i^{l'-l}\left[a_E^*\left(l',m'\right)a_M(l,m) - a_M^*\left(l',m'\right)a_E(l,m)\right]\left(\mathbf{L}\cdot\mathbf{X}_{l'm'}\right)^*\hat{\mathbf{r}}\times\mathbf{X}_{lm}\end{aligned} \tag{32}$$

This finally gives the interference part of (9.145)

$$\left.\frac{d\mathbf{M}}{dr}\right|_{\text{2nd}} = \frac{\mu_0H_0^2}{2\omega k^2}\,\text{Re}\sum_{l,l',m,m'}i^{l'-l}\left[a_E^*\left(l'm'\right)a_M(l,m) - a_M^*\left(l',m'\right)a_E(l,m)\right]\int\left(\mathbf{L}\cdot\mathbf{X}_{l'm'}\right)^*\hat{\mathbf{r}}\times\mathbf{X}_{lm}d\Omega \tag{33}$$