1. **Prob 3.19**

   (a) Recall the *Green's reciprocation theorem* from Prob 1.12

$$\int_V \rho\Phi' d^3x + \int_S \sigma\Phi' da = \int_V \rho'\Phi d^3x + \int_S \sigma'\Phi da \tag{1}$$

We take the "primed" configuration to be one where both plates are grounded ($\Phi' = 0$ for $z = 0, L$), with a point charge at z_0, and the "unprimed" configuration to be one where the upper plate has the center disc held at potential V (i.e., $\Phi = V$ for $\rho < a$), but there is no point charge between the plates. (1) is then translated into

$$0 = q\Phi(z_0, 0) + V\int_{\text{disc}} \sigma' da \qquad \Longrightarrow \qquad Q_L(a) = -\frac{q}{V}\Phi(z_0, 0) \tag{2}$$

   (b) By Prob 3.17 (b), the Green function of a pair of parallel plates is

$$G(\mathbf{x}, \mathbf{x}') = 2\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\int_0^\infty dk J_m(k\rho) J_m(k\rho')\frac{\sinh(kz_<)\sinh[k(L-z_>)]}{\sinh(kL)} \tag{3}$$

With the point charge at z_0, the interior potential is given by equation (1.44)

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') d^3x' \\
&= \frac{q}{4\pi\epsilon_0}\cdot 2\sum_{m=-\infty}^{\infty} e^{im\phi}\int_0^\infty dk J_m(k\rho) J_m(0)\frac{\sinh(kz_<)\sinh[k(L-z_>)]}{\sinh(kL)} \qquad (J_m(0) = \delta_{m0}) \\
&= \frac{q}{2\pi\epsilon_0}\int_0^\infty dk J_0(k\rho)\frac{\sinh(kz_<)\sinh[k(L-z_>)]}{\sinh(kL)}
\end{aligned} \tag{4}$$

Thus the surface point charge at $z = L$ is (note for interior point, the surface normal is in the $-z$ direction)

$$\begin{aligned}
\sigma(\mathbf{x}) &= -\epsilon_0\left(-\frac{\partial\Phi}{\partial z}\right)\bigg|_{z=L} \\
&= \frac{q}{2\pi}\int_0^\infty dk J_0(k\rho)\frac{\sinh(kz_0)(-k)\cosh 0}{\sinh(kL)} \\
&= -\frac{q}{2\pi}\int_0^\infty dk\frac{\sinh(kz_0)}{\sinh(kL)} k J_0(k\rho)
\end{aligned} \tag{5}$$

Note if we integrate (5) for the disc region $\rho < a$, we end up with (2) since Prob 3.18 (a) gives the integral form of $\Phi(z_0, 0)$.
The mentioned reference *Gradshteyn, Ryzhik* formula 6.666 says for $|\text{Re}\,\alpha| < \pi, \text{Re}\,\nu > -1$,

$$\int_0^\infty x^{\nu+1}\sinh(\alpha x)\,\text{cosech}(\pi x) J_\nu(\beta x)\,dx = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1} n^{\nu+1}\sin(n\alpha) K_\nu(n\beta) \tag{6}$$

To apply (6) on (5), we need to make the following identifications

$$\nu = 0 \qquad \pi x = kL \text{ or } k = \frac{\pi}{L}x \qquad \alpha = \frac{k}{x}z_0 = \frac{\pi z_0}{L} \qquad \beta = \frac{k}{x}\rho = \frac{\pi\rho}{L} \tag{7}$$

With these, (5) is turned into

$$\begin{aligned}
\sigma(\rho) &= -\frac{q}{2\pi}\int_0^\infty \left(\frac{\pi}{L}\right) dx\frac{\sinh(\alpha x)}{\sinh(\pi x)}\left(\frac{\pi}{L}\right) x J_0(\beta x) \\
&= -\frac{q}{2\pi}\left(\frac{\pi}{L}\right)^2\cdot\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1} n\sin\left(\frac{n\pi z_0}{L}\right) K_0\left(\frac{n\pi\rho}{L}\right) \\
&= \frac{q}{L^2}\sum_{n=1}^{\infty}(-1)^n n\sin\left(\frac{n\pi z_0}{L}\right) K_0\left(\frac{n\pi\rho}{L}\right)
\end{aligned} \tag{8}$$

(c) From (5), we have

$$\begin{aligned}
\sigma(0) &= -\frac{q}{2\pi}\int_0^\infty \frac{e^{kz_0}-e^{-kz_0}}{e^{kL}-e^{-kL}}kdk \\
&= -\frac{q}{2\pi}\int_0^\infty \left[e^{k(z_0-L)}-e^{-k(z_0+L)}\right]\left(\frac{1}{1-e^{-2kL}}\right)kdk \\
&= -\frac{q}{2\pi}\int_0^\infty \left[e^{k(z_0-L)}-e^{-k(z_0+L)}\right]\left[\sum_{n=0}^{\infty}\left(e^{-2kL}\right)^n\right]kdk \\
&= -\frac{q}{2\pi}\sum_{n=0}^{\infty}\left\{\int_0^\infty e^{-k[-z_0+(2n+1)L]}kdk - \int_0^\infty e^{-k[z_0+(2n+1)L]}kdk\right\} \\
&= -\frac{q}{2\pi}\sum_{n=0}^{\infty}\left\{\frac{1}{[(2n+1)L-z_0]^2}-\frac{1}{[(2n+1)L+z_0]^2}\right\} \\
&= -\frac{q}{2\pi L^2}\sum_{n>0,\mathrm{odd}}\left[\frac{1}{\left(n-\frac{z_0}{L}\right)^2}-\frac{1}{\left(n+\frac{z_0}{L}\right)^2}\right]
\end{aligned} \tag{9}$$

It is reasonable to speculate that there are some relationships between (8) and (9), both in series forms. However for $\rho = 0$, (8) manifestly blows up because $K_0(x)$ is divergent at $x = 0$. **I didn't dig deeper into the relationship between the two series sums of (8) and (9), and how $K_0(0)$'s divergence is to be thought of. In fact, the sum (9) does not even seem to be obviously converging.**

2. **Prob 3.20**

(a) This is a straightforward application of the Green function in Prob 3.17 (a), where

$$G\left(\mathbf{x},\mathbf{x}'\right) = \frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)I_m\left(\frac{n\pi\rho_<}{L}\right)K_m\left(\frac{n\pi\rho_>}{L}\right) \tag{10}$$

With the point charge q at z_0, the interior point's potential is given by

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int_V \rho\left(\mathbf{x}'\right)G\left(\mathbf{x},\mathbf{x}'\right)d^3x' \\
&= \frac{q}{4\pi\epsilon_0}\cdot\frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im\phi}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z_0}{L}\right)I_m(0)K_m\left(\frac{n\pi\rho}{L}\right) \qquad (I_m(0)=\delta_{m0}) \\
&= \frac{q}{\pi\epsilon_0 L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right)
\end{aligned} \tag{11}$$

(b) The surface charge at $z = 0$ is

$$\begin{aligned}
\sigma_0(\rho) = -\epsilon_0\left.\frac{\partial\Phi}{\partial z}\right|_{z=0} &= -\frac{q}{\pi L}\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)\cos 0\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right) \\
&= -\frac{q}{L^2}\sum_{n=1}^{\infty} n\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right)
\end{aligned} \tag{12}$$

Similarly, the surface charge at $z = L$ is

$$\begin{aligned}
\sigma_L(\rho) = -\epsilon_0\left.\left(-\frac{\partial\Phi}{\partial z}\right)\right|_{z=L} &= \frac{q}{\pi L}\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)\cos(n\pi)\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right) \\
&= \frac{q}{L^2}\sum_{n=1}^{\infty}(-1)^n n\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right)
\end{aligned} \tag{13}$$

which agrees with (8).

(c) The total charge at plane $z = L$ is given by

$$
\begin{aligned}
Q_L &= \int_0^\infty \sigma_L(\rho) 2\pi\rho d\rho \\
&= \frac{2\pi q}{L^2} \sum_{n=1}^\infty (-1)^n n \sin\left(\frac{n\pi z_0}{L}\right) \int_0^\infty \rho d\rho K_0\left(\frac{n\pi\rho}{L}\right)
\end{aligned} \tag{14}
$$

The integral can be evaluated using equation (10.43.19) from nist.gov

$$
\int_0^\infty t^{\mu-1} K_\nu(t) dt = 2^{\mu-2} \Gamma\left(\frac{\mu-\nu}{2}\right) \Gamma\left(\frac{\mu+\nu}{2}\right) \tag{15}
$$

where in our case $\mu = 2$, $\nu = 0$. This turns (14) into

$$
\begin{aligned}
Q_L &= \frac{2\pi q}{L^2} \sum_{n=1}^\infty (-1)^n n \sin\left(\frac{n\pi z_0}{L}\right) \left(\frac{L}{n\pi}\right)^2 \underbrace{\int_0^\infty \left(\frac{n\pi\rho}{L}\right) d\left(\frac{n\pi\rho}{L}\right) K_0\left(\frac{n\pi\rho}{L}\right)}_{1} \\
&= \frac{2q}{\pi} \sum_{n=1}^\infty \frac{(-1)^n}{n} \sin\left(\frac{n\pi z_0}{L}\right) \\
&= \frac{2q}{\pi} \operatorname{Im}\left[\sum_{n=1}^\infty \frac{(-1)^n \left(e^{i\pi z_0/L}\right)^n}{n}\right] \\
&= \frac{2q}{\pi} \operatorname{Im}\left[-\ln\left(1 + e^{i\pi z_0/L}\right)\right] \\
&= -\frac{2q}{\pi} \operatorname{Im}\left[\ln\left(1 + e^{i\pi z_0/L}\right)\right]
\end{aligned} \tag{16}
$$

Note that

$$
\operatorname{Im}\left[\ln\left(re^{i\theta}\right)\right] = \theta = \operatorname{Arg}\left(re^{i\theta}\right) \tag{17}
$$

Thus we have

$$
Q_L = -\frac{2q}{\pi} \cdot \operatorname{Arg}\left(1 + e^{i\pi z_0/L}\right) = -\frac{2q}{\pi} \cdot \frac{1}{2} \frac{\pi z_0}{L} = -\frac{q z_0}{L} \tag{18}
$$

which agrees exactly with the result obtained in Prob 1.13 using reciprocation theorem.