1. From (3.185)

$$\Phi^{(1)}(\rho,z) = \frac{(E_0-E_1)\,a}{\pi}\left[\sqrt{\frac{R-\lambda}{2}} - \frac{|z|}{a}\tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right)\right] \qquad \text{where}$$

$$\lambda = \frac{z^2+\rho^2-a^2}{a^2} \qquad R = \sqrt{\lambda^2 + \frac{4z^2}{a^2}} \tag{1}$$

we have

$$\text{top surface:} \qquad \Delta\sigma_+(\rho) = -\epsilon_0 \left.\frac{\partial\Phi^{(1)}}{\partial z}\right|_{z=0^+} \tag{2}$$

$$\text{bottom surface:} \qquad \Delta\sigma_-(\rho) = -\epsilon_0 \left.\left(-\frac{\partial\Phi^{(1)}}{\partial z}\right)\right|_{z=0^-} \tag{3}$$

For $z \to 0^+$,

$$\frac{\partial\Phi^{(1)}}{\partial z} = \frac{(E_0-E_1)\,a}{\pi}\left[\overbrace{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{R-\lambda}{2}}}\cdot\frac{\partial(R-\lambda)}{\partial z}}^{A} - \overbrace{\frac{1}{a}\tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right)}^{B} - \overbrace{\frac{z}{a}\frac{\partial}{\partial z}\tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right)}^{C}\right] \tag{4}$$

Clearly, $C=0$ as $z\to 0$, and the evaluation of B at $z=0$ straightforward

$$B\Big|_{z=0} = \frac{1}{a}\tan^{-1}\frac{a}{\sqrt{\rho^2-a^2}} = \frac{1}{a}\sin^{-1}\left(\frac{a}{\rho}\right) \tag{5}$$

The evaluation of A is tricky since $R\to\lambda$ as $z\to 0$, so it's evaluated with a limiting procedure:

$$R-\lambda = \lambda\sqrt{1+\frac{4z^2}{\lambda^2a^2}} - \lambda \longrightarrow \frac{2z^2}{\lambda a^2} \qquad \text{as } z\to 0 \tag{6}$$

$$\frac{\partial(R-\lambda)}{\partial z} = \frac{1}{2}\frac{2\lambda\frac{\partial\lambda}{\partial z}+\frac{8z}{a^2}}{R} - \frac{\partial\lambda}{\partial z} \qquad R\to\lambda \text{ as } z\to 0$$

$$\longrightarrow \frac{4z}{a^2\lambda} \tag{7}$$

Therefore as $z\to 0^+$:

$$A \longrightarrow \frac{1}{4}\cdot\frac{a\sqrt{\lambda}}{z}\cdot\frac{4z}{a^2\lambda} \longrightarrow \frac{1}{a}\frac{a}{\sqrt{\rho^2-a^2}} \tag{8}$$

Inserting (5) and (8) into (4) and (2) gives

$$\Delta\sigma_+(\rho) = -\epsilon_0\left.\frac{\partial\Phi^{(1)}}{\partial z}\right|_{z=0^+} = -\epsilon_0\frac{E_0-E_1}{\pi}\left[\frac{a}{\sqrt{\rho^2-a^2}} - \sin^{-1}\left(\frac{a}{\rho}\right)\right] \tag{9}$$

For the bottom surface, in (4), B will flip its sign because of $|z|$ in (1), which cancels the sign in (3). The sign in A is also flipped when taking the square root of (6), which again cancels the sign in (3). In summary, the bottom surface density is the same as the top surface density, i.e.,

$$\Delta\sigma_+(\rho) = \Delta\sigma_-(\rho) = \Delta\sigma(\rho) = -\epsilon_0\frac{E_0-E_1}{\pi}\left[\frac{a}{\sqrt{\rho^2-a^2}} - \sin^{-1}\left(\frac{a}{\rho}\right)\right] \tag{10}$$

2. With part (a)'s result

$$\int_a^R \rho\,d\rho\,(\sigma_+ + \sigma_-) = -\epsilon_0\int_a^R (E_0-E_1)\,\rho\,d\rho + 2\int_a^R \Delta\sigma(\rho)\,\rho\,d\rho \tag{11}$$

Therefore, to prove

$$\lim_{R\to\infty}\left[2\pi\int_a^R d\rho\rho\left(\sigma_+ + \sigma_-\right) + 2\pi\epsilon_0\int_0^R d\rho\rho\left(E_0 - E_1\right)\right] = 0 \tag{12}$$

It's equivalent to prove

$$\lim_{R\to\infty}\left[4\pi\int_a^R \Delta\sigma\left(\rho\right)\rho d\rho + 2\pi\epsilon_0\int_0^a (E_0 - E_1)\rho d\rho\right] = 0 \tag{13}$$

Indeed, by (10)

$$4\pi\int_a^R \Delta\sigma\left(\rho\right)\rho d\rho = -4\epsilon_0(E_0 - E_1)\overbrace{\int_a^R\left[\frac{a}{\sqrt{\rho^2 - a^2}} - \sin^{-1}\left(\frac{a}{\rho}\right)\right]\rho d\rho}^{I} \tag{14}$$

where

$$\begin{aligned}
I &= \int_a^R \frac{a}{\sqrt{\rho^2 - a^2}}\rho d\rho - \left[\frac{\rho^2}{2}\sin^{-1}\left(\frac{a}{\rho}\right)\bigg|_a^R - \int_a^R \frac{\rho^2}{2}\frac{1}{\sqrt{1 - \frac{a^2}{\rho^2}}}\left(-\frac{a}{\rho^2}\right)d\rho\right] \\
&= \frac{1}{2}\int_a^R \frac{a}{\sqrt{\rho^2 - a^2}}\rho d\rho - \frac{R^2}{2}\sin^{-1}\left(\frac{a}{R}\right) + \frac{a^2}{2}\frac{\pi}{2} \\
&= \frac{a}{2}\sqrt{R^2 - a^2} - \frac{R^2}{2}\sin^{-1}\left(\frac{a}{R}\right) + \frac{a^2\pi}{4} \\
&= \frac{aR}{2}\left[1 - \frac{a^2}{2R^2} + O\left(\frac{1}{R^4}\right)\right] - \frac{R^2}{2}\left[\frac{a}{R} + \frac{1}{6}\left(\frac{a}{R}\right)^3 + O\left(\frac{1}{R^5}\right)\right] + \frac{a^2\pi}{4}
\end{aligned} \tag{15}$$

Clearly, as $R\to\infty$, $I\to a^2\pi/4$. Together with (14), this proves (13). The physical meaning of (12) is of course that the grounded plane with circular hole has zero net charges.