1. **Prob 10.18**

   (a) Since $ka \ll 1$, the fields around the hole can be treated as the static fields. In section 9.5 as well as problem 9.20, we have already shown that the hole is effectively an electric dipole and magnetic dipole (see equation (9.75) and problem 9.20) for the shadow region

$$\mathbf{p}_{\text{eff}} = \frac{4\epsilon_0 a^3}{3}\mathbf{E}_0 \qquad \mathbf{m}_{\text{eff}} = -\frac{8a^3}{3\mu_0}\mathbf{B}_0 \tag{1}$$

   The signs have been chosen to fit the shadow region (see figure 9.4, and figure 5.15).
   Then the scattered (diffracted) field at the radiation zone is given by (10.2)

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0}k^2\frac{e^{ikr}}{r}\left[\left(\hat{\mathbf{k}}\times\mathbf{p}_{\text{eff}}\right)\times\hat{\mathbf{k}} - \hat{\mathbf{k}}\times\mathbf{m}_{\text{eff}}/c\right]\\ &= \frac{e^{ikr}}{3\pi r}k^2a^3\left[2c\hat{\mathbf{k}}\times\mathbf{B}_0 + \hat{\mathbf{k}}\times\left(\mathbf{E}_0\times\hat{\mathbf{k}}\right)\right]\end{aligned} \tag{2}$$

   (b) The angular power distribution on the shadow side is

$$\frac{dP}{d\Omega} = \frac{1}{2Z_0}\operatorname{Re}\left(r^2\mathbf{E}\cdot\mathbf{E}^*\right) = \frac{k^4a^6}{18\pi^2Z_0}\left|2c\hat{\mathbf{k}}\times\mathbf{B}_0 + \hat{\mathbf{k}}\times\left(\mathbf{E}_0\times\hat{\mathbf{k}}\right)\right|^2 \tag{3}$$

   where

$$\begin{aligned}\left|2c\hat{\mathbf{k}}\times\mathbf{B}_0 + \hat{\mathbf{k}}\times\left(\mathbf{E}_0\times\hat{\mathbf{k}}\right)\right|^2 &= \left|2c\hat{\mathbf{k}}\times\mathbf{B}_0 + \left[\mathbf{E}_0 - \left(\hat{\mathbf{k}}\cdot\mathbf{E}_0\right)\hat{\mathbf{k}}\right]\right|^2\\ &= 4c^2\left(\hat{\mathbf{k}}\times\mathbf{B}_0\right)\cdot\left(\hat{\mathbf{k}}\times\mathbf{B}_0\right) + \left|\mathbf{E}_0 - \left(\hat{\mathbf{k}}\cdot\mathbf{E}_0\right)\hat{\mathbf{k}}\right|^2 + 4c\left(\hat{\mathbf{k}}\times\mathbf{B}_0\right)\cdot\left[\mathbf{E}_0 - \left(\hat{\mathbf{k}}\cdot\mathbf{E}_0\right)\hat{\mathbf{k}}\right]\\ &= 4c^2\left[B_0^2 - \left(\hat{\mathbf{k}}\cdot\mathbf{B}_0\right)^2\right] + E_0^2 - \left(\hat{\mathbf{k}}\cdot\mathbf{E}_0\right)^2 + 4c\hat{\mathbf{k}}\cdot\left(\mathbf{B}_0\times\mathbf{E}_0\right)\end{aligned} \tag{4}$$

   Without loss of generality, let $\mathbf{E}_0, \mathbf{B}_0$ be along the $\hat{\mathbf{z}}, \hat{\mathbf{y}}$ direction, and let $\hat{\mathbf{k}}$ be pointing along the (θ,ϕ) direction, then (3) becomes

$$\frac{dP}{d\Omega} = \frac{k^4a^6}{18\pi^2Z_0}\left[4c^2B_0^2\left(1-\sin^2\theta\sin^2\phi\right) + E_0^2\sin^2\theta + 4cB_0E_0\sin\theta\cos\phi\right] \tag{5}$$

   To get the total power through the hole, we integrate (5) over $\theta\in[0,\pi/2]$ and $\phi\in[0,2\pi]$, where the cross term will drop out. The final integral is obtained as

$$\begin{aligned}P_{\text{through}} &= \int_0^{\pi/2}\sin\theta\, d\theta\int_0^{2\pi}d\phi\,\frac{dP}{d\Omega}\\ &= \frac{k^4a^6}{18\pi^2Z_0}\left[4c^2B_0^2\cdot 2\pi\overbrace{\int_0^{\pi/2}\left(1-\frac{\sin^2\theta}{2}\right)\sin\theta\, d\theta}^{2/3} + E_0^2\cdot 2\pi\overbrace{\int_0^{\pi/2}\sin^2\theta\sin\theta\, d\theta}^{2/3}\right]\\ &= \frac{2k^4a^6}{27\pi Z_0}\left(4c^2B_0^2 + E_0^2\right)\end{aligned} \tag{6}$$

2. **Prob 10.19**

   (a) Recall that in the derivation of the static electric and magnetic field over the circular hole (see section 3.13 and 5.13), $\mathbf{E}_0$ and $\mathbf{H}_0$ are the normal electric and tangential magnetic field over the conductor's surface *far away from the hole*, which are also what $\mathbf{E}_0, \mathbf{B}_0$ mean in problem 10.18. With plane wave incident upon the conductor screen with hole, we can take $\mathbf{E}_0$ and $\mathbf{B}_0$ as the *total* field, i.e., incident plus reflected field by an infinite plane of perfect conductor *without* the circular hole. We have the following claim:

   $\mathbf{E}_0$ *is twice the normal component of the incident electric field, and* $\mathbf{B}_0$ *is twice the tangential component of the incident magnetic field.*

   We present two ways to prove this claim.

- By symmetry arguments.
  For infinite planar perfect conductor screen, the total field on the shadow side must vanish. Which means the reflecteds field (due to the surface charge and current on the conductor screen) must cancel out the incident field on the shadow side. Let the screen be in the x-y plane, and let the incident wave come from the $z-$ direction, we must have

$$\begin{array}{ccccccc} E_{\text{inc},z}\Big|_{z=0^-} & = & E_{\text{inc},z}\Big|_{z=0^+} & = & -E_{\text{refl},z}\Big|_{z=0^+} & = & E_{\text{refl},z}\Big|_{z=0^-} \\ E_{\text{inc},x|y}\Big|_{z=0^-} & = & E_{\text{inc},x|y}\Big|_{z=0^+} & = & -E_{\text{refl},x|y}\Big|_{z=0^+} & = & -E_{\text{refl},x|y}\Big|_{z=0^-} \\ B_{\text{inc},z}\Big|_{z=0^-} & = & B_{\text{inc},z}\Big|_{z=0^+} & = & -B_{\text{refl},z}\Big|_{z=0^+} & = & -B_{\text{refl},z}\Big|_{z=0^-} \\ B_{\text{inc},x|y}\Big|_{z=0^-} & = & B_{\text{inc},x|y}\Big|_{z=0^+} & = & -B_{\text{refl},x|y}\Big|_{z=0^+} & = & B_{\text{refl},x|y}\Big|_{z=0^-} \end{array}$$

  where the first column of equality states the continuity of the incident field across $z = 0$, the second is a consequence of the total cancellation of incident and reflected fields on the shadow side, and the third column derives from the parity of the reflected fields about $z = 0$ (see 10.95). From this we can see that on the illuminated side, the total electric field has only normal component, which is twice that of the incident electric field, same for the tangential component of the magnetic field. This proves the claim.
- By explicitly calculating the reflected fields.
  Equation (7.39) and (7.41) gave the exact reflected field for the perpendicular polarization and parallel polarization respectively (see figure 7.6). When the screen is perfect conductor, we can take the medium's index of refraction, n', to be $i\infty$. Then for perpendicular polarization, (7.39) indicates $E'' = -E$, and for parallel polarization, (7.41) indicates $E'' = E$. By referring to figure 7.6, we see that the claim is true.

Let $E_i, B_i = E_i/c$ be the amplitude of the incident electric and magnetic field, then using the above claim, we can set E_0, B_0 in problem 10.18 as (again, please see figure 7.6)

$$\begin{array}{lll} E_0 = 2E_i \sin\alpha & B_0 = 2B_i = \dfrac{2E_i}{c} & \text{for parallel polarization} \\ E_0 = 0 & B_0 = 2B_i\cos\alpha = \dfrac{2E_i\cos\alpha}{c} & \text{for perpendicular polarization} \end{array}$$

Plugging these into (5) yields

$$\begin{aligned} \frac{dP}{d\Omega}\bigg|_{\parallel} &= \frac{k^4a^6}{18\pi^2 Z_0}\left[4c^2\cdot\frac{4E_i^2}{c^2}\left(1-\sin^2\theta\sin^2\phi\right) + 4E_i^2\sin^2\alpha\sin^2\theta + 4c\cdot\frac{2E_i}{c}\cdot 2E_i\sin\alpha\sin\theta\cos\phi\right] \\ &= \frac{2k^4a^6E_i^2}{9\pi^2 Z_0}\left[4(1-\sin^2\theta\sin^2\phi) + \sin^2\alpha\sin^2\theta + 4\sin\alpha\sin\theta\cos\phi\right] \qquad (7) \\ \frac{dP}{d\Omega}\bigg|_{\perp} &= \frac{k^4a^6}{18\pi^2 Z_0}\left[4c^2\cdot\frac{4E_i^2\cos^2\alpha}{c^2}\left(1-\sin^2\theta\sin^2\phi\right)\right] \\ &= \frac{8k^4a^6E_i^2\cos^2\alpha}{9\pi^2 Z_0}\left(1-\sin^2\theta\cos^2\phi\right) \qquad (8) \end{aligned}$$

With

$$P_i = \frac{E_i^2}{2Z_0}\pi a^2\cos\alpha \qquad (9)$$

these can also be expressed as

$$\begin{aligned} \frac{dP}{d\Omega}\bigg|_{\parallel} &= P_i\cdot\frac{4(ka)^4}{9\pi^3}\left[\frac{4\left(1-\sin^2\theta\sin^2\phi\right)+\sin^2\alpha\sin^2\theta + 4\sin\alpha\sin\theta\cos\phi}{\cos\alpha}\right] \qquad (10) \\ \frac{dP}{d\Omega}\bigg|_{\perp} &= P_i\cdot\frac{16(ka)^4}{9\pi^3}\cos\alpha\left(1-\sin^2\theta\cos^2\phi\right) \qquad (11) \end{aligned}$$

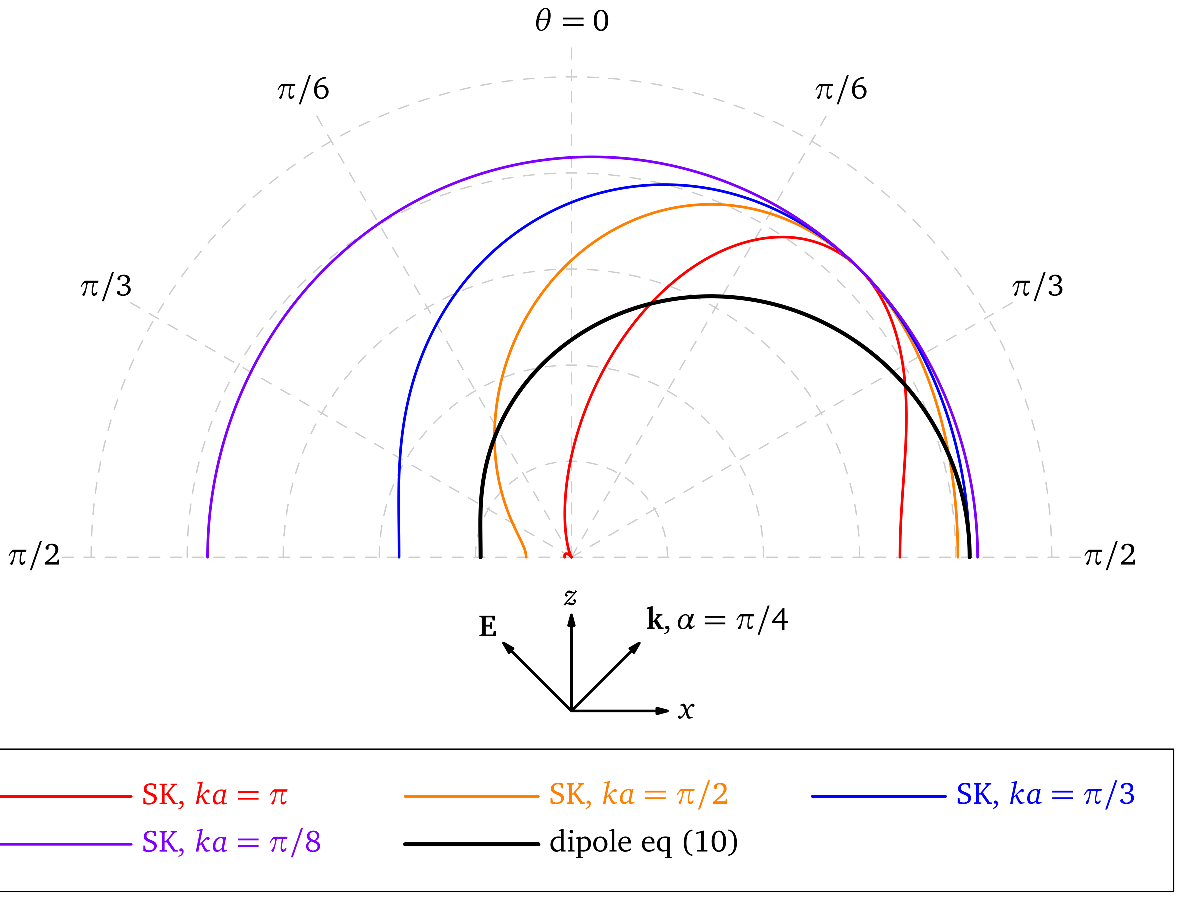
Comparing the above to (10.114) (parallel polarization), and results of problem 10.12 (perpendicular polarization) which are computed from the vector Smythe-Kirchhoff approximation,

$$\left.\frac{dP}{d\Omega}\right|_{\parallel,\mathrm{SK}} = P_i \frac{(ka)^2}{4\pi} \cos\alpha \left(\cos^2\theta + \cos^2\phi \sin^2\theta\right) \left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2 \tag{12}$$

$$\left.\frac{dP}{d\Omega}\right|_{\perp,\mathrm{SK}} = P_i \frac{(ka)^2}{4\pi} \left(\frac{\cos^2\theta + \sin^2\phi \sin^2\theta}{\cos\alpha}\right) \left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2 \tag{13}$$

we see that when $ka \ll 1$, results from the vector Smythe-Kirchhoff approximation are $O(k^2a^2)$ larger.

(b) The E plane plot for the angular distribution of power is shown below. Each line has its own scale so they all fit in one diagram (i.e., the relative intensity between lines is not meaningful here, but the shape is accurate).



(c) The transmission coefficients can be obtained by integrating (10) and (11) over $\theta \in [0, \pi/2]$ and $\phi \in [0, 2\pi]$,

$$\begin{aligned} T_\parallel &= \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{2\pi} d\phi \frac{4(ka)^4}{9\pi^3} \left[\frac{4\left(1-\sin^2\theta\sin^2\phi\right) + \sin^2\alpha\sin^2\theta + 4\sin\alpha\sin\theta\cos\phi}{\cos\alpha}\right] \\ &= \frac{4(ka)^4}{9\pi^3}\frac{1}{\cos\alpha}\cdot 2\pi \left[4\overbrace{\int_0^{\pi/2}\left(1-\frac{\sin^2\theta}{2}\right)\sin\theta\, d\theta}^{2/3} + \sin^2\alpha \overbrace{\int_0^{\pi/2} \sin^2\theta\sin\theta\, d\theta}^{2/3}\right] \\ &= \frac{16(ka)^4}{27\pi^2}\left(\frac{4+\sin^2\alpha}{\cos\alpha}\right) \end{aligned} \tag{14}$$

$$\begin{aligned} T_\perp &= \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{2\pi} d\phi \frac{16(ka)^4}{9\pi^3} \cos\alpha\left(1-\sin^2\theta\cos^2\phi\right) \\ &= \frac{16(ka)^4}{9\pi^3}\cos\alpha\cdot 2\pi\cdot\frac{2}{3} \\ &= \frac{64(ka)^4}{27\pi^2}\cos\alpha \end{aligned} \tag{15}$$