In the text, the Green function for the sphere is obtained from the method of image charge, where the Green function is given in equation (2.16)

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - aF(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a}{x'\left|\mathbf{x}-\frac{a^2}{x'^2}\mathbf{x}'\right|} \tag{1}$$

It is clear that the method of image charge will guarantee G to vanish on the boundary S, but it's not immediately obvious that $F(\mathbf{x},\mathbf{x}')$ will have a zero Laplacian with respect to $\mathbf{x}' \in V$.

Here we give a explicit (tedious) proof that it is indeed the case, i.e.,

$$\nabla'^2 F(\mathbf{x},\mathbf{x}') = \nabla'^2 \left(\frac{1}{x'\left|\mathbf{x}-\frac{a^2}{x'^2}\mathbf{x}'\right|} \right) = 0 \tag{2}$$

To see this, first write F in spherical coordinates, where θ represents the angle between $\mathbf{x}$ and $\mathbf{x}'$, and $p = |\mathbf{x}|, q = |\mathbf{x}'|$.

$$F(\mathbf{x},\mathbf{x}') = \left(q\sqrt{p^2 + \frac{a^4}{q^2} - \frac{2pa^2\cos\theta}{q}} \right)^{-1} = \left(p^2q^2 + a^4 - 2pqa^2\cos\theta\right)^{-1/2} \tag{3}$$

In spherical coordinates, the Laplacian is

$$\nabla'^2 F = \overbrace{\frac{1}{q^2}\frac{\partial}{\partial q}\left(q^2\frac{\partial F}{\partial q}\right)}^{A} + \underbrace{\frac{1}{q^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right)}_{B} + \overbrace{\frac{1}{q^2\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2}}^{C} \tag{4}$$

where $C = 0$ since F does not have dependence on ϕ.

Now come the tedious calculations.

$$\begin{aligned}
\frac{\partial F}{\partial q} &= -\frac{1}{2}\left(2p^2q - 2pa^2\cos\theta\right)\left(p^2q^2 + a^4 - 2pqa^2\cos\theta\right)^{-3/2} && \Longrightarrow \\
q^2\frac{\partial F}{\partial q} &= \left(pq^2a^2\cos\theta - p^2q^3\right)\left(p^2q^2 + a^4 - 2pqa^2\cos\theta\right)^{-3/2} && \Longrightarrow \\
q^2A = \frac{\partial}{\partial q}\left(q^2\frac{\partial F}{\partial q}\right) &= \left(2pqa^2\cos\theta - 3p^2q^2\right)\left(p^2q^2 + a^4 - 2pqa^2\cos\theta\right)^{-3/2} + \\
&\quad \left(pq^2a^2\cos\theta - p^2q^3\right)\left[-\frac{3}{2}\left(2p^2q - 2pa^2\cos\theta\right)\left(p^2q^2 + a^4 - 2pqa^2\cos\theta\right)^{-5/2}\right] \\
&= \frac{U}{(p^2q^2 + a^4 - 2pqa^2\cos\theta)^{5/2}}
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
U =& \left[\left(2pqa^2\cos\theta - 3p^2q^2\right)\left(p^2q^2 + a^4 - 2pqa^2\cos\theta\right) + \left(pq^2a^2\cos\theta - p^2q^3\right)\left(3pa^2\cos\theta - 3p^2q\right)\right] \\
=& 2p^3q^3a^2\cos\theta + 2pqa^6\cos\theta - 4p^2q^2a^4\cos^2\theta - 3p^4q^4 - 3p^2q^2a^4 + 6p^3q^3a^2\cos\theta + \\
& 3p^2q^2a^4\cos^2\theta - 3p^3q^3a^2\cos\theta - 3p^3q^3a^2\cos\theta + 3p^4q^4 \\
=& -p^2q^2a^4\cos^2\theta + 2pqa^6\cos\theta + 2p^3q^3a^2\cos\theta - 3p^2q^2a^4
\end{aligned} \tag{6}$$

For the B term:

$$\begin{aligned}
\frac{\partial F}{\partial\theta} &= -\frac{1}{2}\left(2pqa^2\sin\theta\right)\left(p^2q^2 + a^4 - 2pqa^2\cos\theta\right)^{-3/2} && \Longrightarrow \\
\sin\theta\frac{\partial F}{\partial\theta} &= \left(-pqa^2\sin^2\theta\right)\left(p^2q^2 + a^4 - 2pqa^2\cos\theta\right)^{-3/2} && \Longrightarrow \\
\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) &= \left(-2pqa^2\sin\theta\cos\theta\right)\left(p^2q^2 + a^4 - 2pqa^2\cos\theta\right)^{-3/2} + \\
&\quad \left(-pqa^2\sin^2\theta\right)\left[-\frac{3}{2}\left(2pqa^2\sin\theta\right)\left(p^2q^2 + a^4 - 2pqa^2\cos\theta\right)^{-5/2}\right] && \Longrightarrow \\
q^2B = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) &= \frac{V}{(p^2q^2 + a^4 - 2pqa^2\cos\theta)^{5/2}}
\end{aligned} \tag{7}$$

where

$$
\begin{aligned}
V =& \left(-2pqa^2 \cos\theta\right)\left(p^2q^2 + a^4 - 2pqa^2\cos\theta\right) + \left(pqa^2 \sin\theta\right)\left(3pqa^2 \sin\theta\right) \\
=& -2p^3q^3a^2\cos\theta - 2pqa^6\cos\theta + 4p^2q^2a^4\cos^2\theta + 3p^2q^2a^4 - 3p^2q^2a^4\cos^2\theta \\
=& p^2q^2a^4\cos^2\theta - 2pqa^6\cos\theta - 2p^3q^3a^2\cos\theta + 3p^2q^2a^4 \qquad (8)
\end{aligned}
$$

Notice in (6) and (8), $U + V = 0$, and $F(\mathbf{x}, \mathbf{x}')$ is non-singular for all $\mathbf{x}, \mathbf{x}' \in V$, this proves (2), and hence (1) is indeed the Green function for the sphere.