1. By usual arguments of separation of variable (where we now take k to be real), and the boundary conditions, we can establish the form of the Green function as

$$G\left(\mathbf{x},\mathbf{x}'\right)=\sum_{m=-\infty}^{\infty}e^{im\left(\phi-\phi'\right)}\sum_{n=1}^{\infty}A_{mn}J_m\left(\frac{x_{mn}\rho}{a}\right)\sinh\left(\frac{x_{mn}z_<}{a}\right)\sinh\left[\frac{x_{mn}\left(L-z_>\right)}{a}\right] \tag{1}$$

with the coefficient A_{mn} to be determined.
Integrating the Laplacian of (1) over the infinitesimal range $[z'-\epsilon, z'+\epsilon]$ gives

$$\int_{z'-\epsilon}^{z'+\epsilon}\nabla^2 G dz=\int_{z'-\epsilon}^{z'+\epsilon}\left[\overbrace{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}}^{\text{zero contrib. to integral}}+\frac{\partial^2}{\partial z^2}\right]G dz=\int_{z'-\epsilon}^{z'+\epsilon}-4\pi\frac{\delta\left(\rho-\rho'\right)}{\rho}\delta\left(\phi-\phi'\right)\delta\left(z-z'\right)dz \tag{2}$$

which gives

$$\left.\frac{\partial G}{\partial z}\right|_{z=z'+\epsilon}-\left.\frac{\partial G}{\partial z}\right|_{z=z'-\epsilon}=-4\pi\frac{\delta\left(\rho-\rho'\right)}{\rho}\delta\left(\phi-\phi'\right) \qquad \Longrightarrow$$

$$\sum_{m=-\infty}^{\infty}e^{im\left(\phi-\phi'\right)}\sum_{n=1}^{\infty}A_{mn}J_m\left(\frac{x_{mn}\rho}{a}\right)\left(-\frac{x_{mn}}{a}\right)\sinh\left(\frac{x_{mn}L}{a}\right)=-4\pi\frac{\delta\left(\rho-\rho'\right)}{\rho}\delta\left(\phi-\phi'\right) \tag{3}$$

We already know

$$\sum_{m=-\infty}^{\infty}e^{im\left(\phi-\phi'\right)}=2\pi\delta\left(\phi-\phi'\right) \tag{4}$$

Recall equation (3.96) and (3.97)

$$\text{(3.96)} \qquad f\left(\rho\right)=\sum_{n=1}^{\infty}A_{\nu n}J_\nu\left(\frac{x_{\nu n}\rho}{a}\right)$$

$$\text{(3.97)} \qquad A_{\nu n}=\frac{2}{a^2J_{\nu+1}^2\left(x_{\nu n}\right)}\int_0^a\rho' f\left(\rho'\right)J_\nu\left(\frac{x_{\nu n}\rho'}{a}\right)d\rho'$$

This gives

$$\begin{aligned}f\left(\rho\right)&=\sum_{n=1}^{\infty}\left[\frac{2}{a^2J_{\nu+1}^2\left(x_{\nu n}\right)}\int_0^a\rho' f\left(\rho'\right)J_\nu\left(\frac{x_{\nu n}\rho'}{a}\right)d\rho'\right]J_\nu\left(\frac{x_{\nu n}\rho}{a}\right)\\&=\int_0^a d\rho' f\left(\rho'\right)\underbrace{\left[\rho'\sum_{n=1}^{\infty}\frac{2}{a^2J_{\nu+1}^2\left(x_{\nu n}\right)}J_\nu\left(\frac{x_{\nu n}\rho}{a}\right)J_\nu\left(\frac{x_{\nu n}\rho'}{a}\right)\right]}_{\delta\left(\rho-\rho'\right)}\end{aligned} \tag{5}$$

which gives the closure relation

$$\sum_{n=1}^{\infty}\left[\frac{2}{a^2J_{\nu+1}^2\left(x_{\nu n}\right)}\right]J_\nu\left(\frac{x_{\nu n}\rho}{a}\right)J_\nu\left(\frac{x_{\nu n}\rho'}{a}\right)=\frac{\delta\left(\rho-\rho'\right)}{\rho} \tag{6}$$

Bringing (3), (4), (6) together, we can determine the coefficient

$$A_{mn}=\frac{4}{a^2J_{m+1}^2\left(x_{mn}\right)}\cdot\frac{a}{x_{mn}\sinh\left(\frac{x_{mn}L}{a}\right)}J_{mn}\left(\frac{x_{mn}\rho'}{a}\right)=\frac{4}{a}\frac{J_m\left(\frac{x_{mn}\rho'}{a}\right)}{J_{m+1}^2\left(x_{mn}\right)x_{mn}\sinh\left(\frac{x_{mn}L}{a}\right)} \tag{7}$$

Therefore the Green function is

$$G\left(\mathbf{x},\mathbf{x}'\right)=\frac{4}{a}\sum_{m=-\infty}^{\infty}e^{im\left(\phi-\phi'\right)}\sum_{n=1}^{\infty}\frac{J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{J_{m+1}^2\left(x_{mn}\right)x_{mn}\sinh\left(\frac{x_{mn}L}{a}\right)}\sinh\left(\frac{x_{mn}z_<}{a}\right)\sinh\left[\frac{x_{mn}\left(L-z_>\right)}{a}\right] \tag{8}$$

and hence the potential generated by the point charge q at (ρ', ϕ', z') is

$$\begin{aligned}\Phi(\rho,\phi,z) &= \frac{1}{4\pi\epsilon_0}\int_V \rho\left(\mathbf{x}'\right) G\left(\mathbf{x},\mathbf{x}'\right) d^3x \\ &= \frac{q}{\pi\epsilon_0 a}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sum_{n=1}^{\infty}\frac{J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{J_{m+1}^2\left(x_{mn}\right)x_{mn}\sinh\left(\frac{x_{mn}L}{a}\right)}\sinh\left(\frac{x_{mn}z_<}{a}\right)\sinh\left[\frac{x_{mn}\left(L-z_>\right)}{a}\right] \end{aligned} \tag{9}$$

2. Now let's take k to be imaginary number, which gives the Green function form

$$G\left(\mathbf{x},\mathbf{x}'\right) = \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)g_{mn}\left(\rho,\rho'\right) \tag{10}$$

Taking the Laplacian of (10), we have

$$\begin{aligned}\nabla^2 G &= \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)-\left[\frac{m^2}{\rho^2}+\left(\frac{n\pi}{L}\right)^2\right]\right\}g_{mn} \\ &= -4\pi\frac{\delta\left(\rho-\rho'\right)}{\rho}\delta\left(\phi-\phi'\right)\delta\left(z-z'\right) \end{aligned} \tag{11}$$

With (4) and

$$\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) = \frac{L}{2}\delta\left(z-z'\right) \tag{12}$$

we are left with

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g_{mn}}{\partial\rho}\right)-\left[\frac{m^2}{\rho^2}+\left(\frac{n\pi}{L}\right)^2\right]g_{mn} = -\frac{4}{L}\frac{\delta\left(\rho-\rho'\right)}{\rho} \tag{13}$$

Now it's clear that g_{mn} is a linear combination of $I_m\left(n\pi\rho/L\right)$ and $K_m\left(n\pi\rho/L\right)$. Considering the range $\rho<\rho'$ includes the origin, we must have

$$g_{mn}\left(\rho,\rho'\right) = \begin{cases} a_m I_m\left(\frac{n\pi\rho}{L}\right) & \text{for } \rho<\rho' \\ b_m I_m\left(\frac{n\pi\rho}{L}\right)+c_m K_m\left(\frac{n\pi\rho}{L}\right) & \text{for } \rho>\rho' \end{cases} \tag{14}$$

Boundary condition at $\rho=a$ and continuity at $\rho=\rho'$ ensure

$$b_m I_m\left(\frac{n\pi a}{L}\right)+c_m K_m\left(\frac{n\pi a}{L}\right) = 0 \tag{15}$$

$$a_m I_m\left(\frac{n\pi\rho'}{L}\right) = b_m I_m\left(\frac{n\pi\rho'}{L}\right)+c_m K_m\left(\frac{n\pi\rho'}{a}\right) \tag{16}$$

Integrating (13) over $[\rho'-\epsilon,\rho'+\epsilon]$ gives

$$\begin{aligned} -\frac{4}{L} &= \left(\rho\frac{\partial g_{mn}}{\partial\rho}\right)\bigg|_{\rho=\rho'+\epsilon}-\left(\rho\frac{\partial g_{mn}}{\partial\rho}\right)\bigg|_{\rho=\rho'-\epsilon} && \Longrightarrow \\ -\frac{4}{L} &= \frac{n\pi\rho'}{L}\left[b_m I_m'\left(\frac{n\pi\rho'}{L}\right)+c_m K_m'\left(\frac{n\pi\rho'}{L}\right)-a_m I_m'\left(\frac{n\pi\rho'}{L}\right)\right] && \Longrightarrow \\ -\frac{4}{n\pi\rho'} &= b_m I_m'\left(\frac{n\pi\rho'}{L}\right)+c_m K_m'\left(\frac{n\pi\rho'}{L}\right)-a_m I_m'\left(\frac{n\pi\rho'}{L}\right) \end{aligned} \tag{17}$$

Multiply (17) by $K_m\left(n\pi\rho'/L\right)$ and (16) by $K_m'\left(n\pi\rho'/L\right)$ and subtract the two products, we obtain

$$\left(b_m-a_m\right)\left[I_m\left(\frac{n\pi\rho'}{L}\right)K_m'\left(\frac{n\pi\rho'}{L}\right)-I_m'\left(\frac{n\pi\rho'}{L}\right)K_m\left(\frac{n\pi\rho'}{L}\right)\right] = \frac{4}{n\pi\rho'}K_m\left(\frac{n\pi\rho'}{L}\right) \tag{18}$$

We recognize the content in the bracket on the LHS as the Wronskian, which equals $-1/x=-L/n\pi\rho'$, this gives us

$$b_m-a_m = -\frac{4}{L}K_m\left(\frac{n\pi\rho'}{L}\right) \tag{19}$$

Plugging (19) into (16) yields

$$c_m = \frac{4}{L} I_m\left(\frac{n\pi\rho'}{L}\right) \tag{20}$$

and then by (15),

$$b_m = -\frac{4}{L}\frac{I_m\left(\frac{n\pi\rho'}{L}\right)K_m\left(\frac{n\pi a}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)} \tag{21}$$

and eventually by (19)

$$a_m = b_m + \frac{4}{L}K_m\left(\frac{n\pi\rho'}{L}\right) = \frac{4}{L}\cdot\frac{\left[I_m\left(\frac{n\pi a}{L}\right)K_m\left(\frac{n\pi\rho'}{L}\right) - I_m\left(\frac{n\pi\rho'}{L}\right)K_m\left(\frac{n\pi a}{L}\right)\right]}{I_m\left(\frac{n\pi a}{L}\right)} \tag{22}$$

Putting all these together, we can write the full $g_{mn}(\rho,\rho')$ as

$$\begin{aligned} g_{mn}(\rho,\rho') &= \begin{cases} \frac{4}{L}\cdot\frac{I_m\left(\frac{n\pi\rho}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)}\left[I_m\left(\frac{n\pi a}{L}\right)K_m\left(\frac{n\pi\rho'}{L}\right) - I_m\left(\frac{n\pi\rho'}{L}\right)K_m\left(\frac{n\pi a}{L}\right)\right] & \text{for } \rho<\rho' \\ \frac{4}{L}\cdot\frac{I_m\left(\frac{n\pi\rho'}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)}\left[I_m\left(\frac{n\pi a}{L}\right)K_m\left(\frac{n\pi\rho}{L}\right) - I_m\left(\frac{n\pi\rho}{L}\right)K_m\left(\frac{n\pi a}{L}\right)\right] & \text{for } \rho>\rho' \end{cases} \\ &= \frac{4}{L}\cdot\frac{I_m\left(\frac{n\pi\rho_<}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)}\left[I_m\left(\frac{n\pi a}{L}\right)K_m\left(\frac{n\pi\rho_>}{L}\right) - I_m\left(\frac{n\pi\rho_>}{L}\right)K_m\left(\frac{n\pi a}{L}\right)\right] \end{aligned} \tag{23}$$

Putting this back into (10) and apply the Green function integral for the point charge gives the potential

$$\begin{aligned} \Phi(\rho,\phi,z) = &\frac{q}{\pi\epsilon_0 L}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)\frac{I_m\left(\frac{n\pi\rho_<}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)} \\ &\times\left[I_m\left(\frac{n\pi a}{L}\right)K_m\left(\frac{n\pi\rho_>}{L}\right) - I_m\left(\frac{n\pi\rho_>}{L}\right)K_m\left(\frac{n\pi a}{L}\right)\right] \end{aligned} \tag{24}$$

3. This is an application of the discussion in section (3.12). Define

$$\psi_{mkn}(\rho,\phi,z) = A_{mkn}\cdot e^{im\phi}\sin\left(\frac{k\pi z}{L}\right)J_m\left(\frac{x_{mn}\rho}{a}\right) \tag{25}$$

where A_{mkn} is the normalization constant to be determined.
Let's calculate the Laplacian of ψ_{mkn}:

$$\begin{aligned} \nabla^2\psi_{mkn} &= \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]\psi_{mkn} \\ &= A_{mkn}e^{im\phi}\sin\left(\frac{k\pi z}{L}\right)\left\{\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) - \left[\frac{m^2}{\rho^2} + \left(\frac{k\pi}{L}\right)^2\right]\right\}J_m\left(\frac{x_{mn}\rho}{a}\right) \end{aligned} \tag{26}$$

Recall that $J_m(x_{mn}\rho/a)$, being the Bessel function of the first kind, satisfies the Bessel equation (see equation 3.77)

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dJ_m\left(\frac{x_{mn}\rho}{a}\right)}{d\rho}\right] + \left[\left(\frac{x_{mn}}{a}\right)^2 - \frac{m^2}{\rho^2}\right]J_m\left(\frac{x_{mn}\rho}{a}\right) = 0 \tag{27}$$

With (27) plugged into (26), we obtain

$$\nabla^2\psi_{mkn} = A_{mkn}e^{im\phi}\sin\left(\frac{k\pi z}{L}\right)\left[-\left(\frac{k\pi}{L}\right)^2 - \left(\frac{x_{mn}}{a}\right)^2\right]J_m\left(\frac{x_{mn}\rho}{a}\right) = -\left[\left(\frac{k\pi}{L}\right)^2 + \left(\frac{x_{mn}}{a}\right)^2\right]\psi_{mkn} \tag{28}$$

Clearly ψ_{mkn} is the eigenfunction (in the sense of equation (3.154)) with eigenvalue

$$\lambda_{mkn} = \left(\frac{k\pi}{L}\right)^2 + \left(\frac{x_{mn}}{a}\right)^2 \tag{29}$$

For the normalization constant, we require

$$\int_V \psi^*_{mkn}(\mathbf{x})\psi_{mkn}(\mathbf{x})\,d^3x = A^2_{mkn}\int_0^{2\pi} d\phi \overbrace{\int_0^a J_m^2\left(\frac{x_{mn}\rho}{a}\right)\rho\,d\rho}^{a^2 J^2_{m+1}(x_{mn})/2} \underbrace{\int_0^L \sin^2\left(\frac{k\pi z}{L}\right)dz}_{L/2} = 1 \qquad \Longrightarrow$$

$$A^2_{mkn} = \frac{2}{\pi L a^2 J^2_{m+1}(x_{mn})} \tag{30}$$

By (3.160)

$$\begin{aligned} G\left(\mathbf{x},\mathbf{x}'\right) &= 4\pi \sum_{m=-\infty}^{\infty}\sum_{k,n=1}^{\infty} \frac{\psi^*_{mkn}(\mathbf{x}')\psi_{mkn}(\mathbf{x})}{\lambda_{mkn}} \\ &= \frac{8}{La^2}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sum_{k=1}^{\infty}\sin\left(\frac{k\pi z}{L}\right)\sin\left(\frac{k\pi z'}{L}\right)\sum_{n=1}^{\infty}\frac{J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{\left[\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2\right]J^2_{m+1}(x_{mn})} \end{aligned} \tag{31}$$

Then the potential generated by the point charge is simply

$$\Phi(\rho,\phi,z) = \frac{2q}{\pi\epsilon_0 La^2}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sum_{k=1}^{\infty}\sin\left(\frac{k\pi z}{L}\right)\sin\left(\frac{k\pi z'}{L}\right)\sum_{n=1}^{\infty}\frac{J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{\left[\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2\right]J^2_{m+1}(x_{mn})} \tag{32}$$

The relationship between (32) and (9) is explained in the Fourier representation equation (3.169) (which I had proved in details in my other notes). Translated with symbols in this problem:

$$\frac{\sinh\left(\frac{x_{mn}z_<}{a}\right)\sinh\left[\frac{x_{mn}(L-z_>)}{a}\right]}{\left(\frac{x_{mn}}{a}\right)\sinh\left(\frac{x_{mn}L}{a}\right)} = \frac{2}{L}\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\pi z}{L}\right)\sin\left(\frac{k\pi z'}{L}\right)}{\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2} \tag{33}$$

Plugging (33) into (9) will produce (32) exactly.