In these notes, detailed derivation of various equations in Jackson 8.8 is provided. In particular, the Q factor for all TM and TE modes of the cylindrical cavity is calculated in closed forms.

1. **Derivation of Stored Energy U (8.92) for {TM|TE}$_{mnp}$**

   (a) The field amplitudes for the TM$_{mnp}$ mode are

$$E_z = \psi(\rho,\phi)\cos\left(\frac{p\pi z}{d}\right) \qquad \psi(\rho,\phi) = J_m(\gamma_{mn}\rho)\, e^{im\phi} \qquad \gamma_{mn} = \frac{x_{mn}}{R} \tag{1}$$

$$\mathbf{E}_t = -\frac{p\pi}{d\gamma_{mn}^2}\sin\left(\frac{p\pi z}{d}\right)\boldsymbol{\nabla}_t\psi \qquad \mathbf{H}_t = \frac{i\epsilon\omega}{\gamma_{mn}^2}\cos\left(\frac{p\pi z}{d}\right)\hat{\mathbf{z}}\times\boldsymbol{\nabla}_t\psi \tag{2}$$

When $p > 0$, the stored energy can be obtained

$$\begin{aligned}
U =& \int_V \left(\frac{\epsilon|\mathbf{E}|^2}{4} + \frac{\mu|\mathbf{H}|^2}{4}\right) dv \\
=& \frac{\epsilon}{4}\int_0^d dz \int_A \left[|\psi|^2\cos^2\left(\frac{p\pi z}{d}\right) + \left(\frac{p\pi}{d\gamma_{mn}^2}\right)^2 \sin^2\left(\frac{p\pi z}{d}\right)|\boldsymbol{\nabla}_t\psi|^2\right] da + \\
& \frac{\mu}{4}\int_0^d dz \int_A \left(\frac{\epsilon\omega}{\gamma_{mn}^2}\right)^2 \cos^2\left(\frac{p\pi z}{d}\right)|\boldsymbol{\nabla}_t\psi|^2\, da \\
=& \frac{\epsilon}{4}\cdot\frac{d}{2}\int_A \left[|\psi|^2 + \left(\frac{p\pi}{d\gamma_{mn}^2}\right)^2|\boldsymbol{\nabla}_t\psi|^2\right] da + \frac{\mu}{4}\cdot\frac{d}{2}\int_A \left(\frac{\epsilon\omega}{\gamma_{mn}^2}\right)^2 |\boldsymbol{\nabla}_t\psi|^2\, da \\
=& \frac{\epsilon}{4}\cdot\frac{d}{2}\left\{\int_A |\psi|^2\, da + \frac{1}{\gamma_{mn}^2}\int_A \left[\left(\frac{p\pi}{d\gamma_{mn}}\right)^2 + \frac{\mu\epsilon\omega^2}{\gamma_{mn}^2}\right]|\boldsymbol{\nabla}_t\psi|^2\, da\right\} \qquad \text{by (8.78)} \\
=& \frac{\epsilon}{4}\cdot\frac{d}{2}\left\{\int_A |\psi|^2\, da + \frac{1}{\gamma_{mn}^2}\left[1 + 2\left(\frac{p\pi}{d\gamma_{mn}}\right)^2\right]\int_A |\boldsymbol{\nabla}_t\psi|^2\, da\right\}
\end{aligned} \tag{3}$$

Since

$$\boldsymbol{\nabla}_t\psi = \frac{\partial\psi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}} = \gamma_{mn}J_m'(\gamma_{mn}\rho)\, e^{im\phi}\,\hat{\boldsymbol{\rho}} + \frac{1}{\rho}J_m(\gamma_{mn}\rho)(im)\, e^{im\phi}\,\hat{\boldsymbol{\phi}} \tag{4}$$

we have

$$\begin{aligned}
|\boldsymbol{\nabla}_t\psi|^2 &= \gamma_{mn}^2\left[J_m'(\gamma_{mn}\rho)\right]^2 + \frac{m^2}{\rho^2}\left[J_m(\gamma_{mn}\rho)\right]^2 \\
&= \gamma_{mn}^2\left\{\left[J_m'(\gamma_{mn}\rho)\right]^2 + \frac{m^2}{(\gamma_{mn}\rho)^2}\left[J_m(\gamma_{mn}\rho)\right]^2\right\} \qquad \text{define } t \equiv \gamma_{mn}\rho \\
&= \gamma_{mn}^2\left\{\left[J_m'(t)\right]^2 + \frac{m^2}{t^2}\left[J_m(t)\right]^2\right\} \\
&= \gamma_{mn}^2\left\{\left[J_m'(t)\right]^2 + \frac{J_m(t)}{t}\frac{d}{dt}\left[tJ_m'(t)\right] + \left[J_m(t)\right]^2\right\}
\end{aligned} \tag{5}$$

where in the last step, we have invoked the Bessel equation

$$\frac{1}{t}\frac{d}{dt}\left[tJ_m'(t)\right] + \left(1 - \frac{m^2}{t^2}\right)J_m(t) = 0 \tag{6}$$

Integrating (5) over A, we have

$$\begin{aligned}
\int_A |\boldsymbol{\nabla}_t\psi|^2\, da &= 2\pi\int_0^R |\boldsymbol{\nabla}_t\psi|^2\rho\, d\rho \\
&= 2\pi\left\{\int_0^{x_{mn}}\left[J_m'(t)\right]^2 t\, dt + \int_0^{x_{mn}} J_m(t)\frac{d}{dt}\left[tJ_m'(t)\right] dt + \int_0^{x_{mn}}\left[J_m(t)\right]^2 t\, dt\right\}
\end{aligned} \tag{7}$$

After integrating by parts for the second integral, and noticing x_{mn} is a zero of $J_m(t)$, we see that it exactly cancels the first integral, leaving

$$\int_A |\nabla_t \psi|^2 \, da = 2\pi \int_0^{x_{mn}} [J_m(t)]^2 \, t dt = \gamma_{mn}^2 \int_A [J_m(\gamma_{mn}\rho)]^2 \, da = \gamma_{mn}^2 \int_A |\psi|^2 \, da \tag{8}$$

Plugging (8) back to (3) finally gives the time-averaged stored energy

$$U = \frac{\epsilon d}{4} \left[1 + \left(\frac{p\pi}{d\gamma_{mn}} \right)^2 \right] \int_A |\psi|^2 \, da \tag{9}$$

When $p = 0$, as we go from the second line to the third line in (3), the dz integral should result in d instead of $d/2$, which will eventually produce (9) with d replaced by $2d$.

(b) For TE_{mnp} mode, the field amplitudes are

$$H_z = \psi(\rho, \phi) \sin\left(\frac{p\pi z}{d} \right) \qquad \psi(\rho, \phi) = J_m(\gamma_{mn}\rho) e^{im\rho} \qquad \gamma_{mn} = \frac{x'_{mn}}{R} \tag{10}$$

$$\mathbf{E}_t = -\frac{i\omega\mu}{\gamma_{mn}^2} \sin\left(\frac{p\pi z}{d} \right) \hat{\mathbf{z}} \times \nabla_t \psi \qquad \mathbf{H}_t = \frac{p\pi}{d\gamma_{mn}^2} \cos\left(\frac{p\pi z}{d} \right) \nabla_t \psi \tag{11}$$

Then it is easy to see the stored energy expression for U in (3) is the same except for the $\epsilon \leftrightarrow \mu$ replacement. Everything afterwards follows until (7) where the first two integrals still cancel, but this time for a different reason that x'_{mn} is a zero of $J'_m(t)$. Eventually

$$U = \frac{\mu d}{4} \left[1 + \left(\frac{p\pi}{d\gamma_{mn}} \right)^2 \right] \int_A |\psi|^2 \, da \tag{12}$$

2. **Derivation of P_{loss} and Q factor (8.94), (8.95) for TM_{mnp}**

From (8.93)

$$P_{\text{loss}} = \frac{1}{2\sigma\delta} \left(\overbrace{\oint_C dl \int_0^d dz \, |\mathbf{n} \times \mathbf{H}|^2_{\text{sides}}}^{I_1} + 2 \overbrace{\int_A da \, |\mathbf{n} \times \mathbf{H}|^2_{\text{ends}}}^{I_2} \right) \tag{13}$$

On the sides,

$$\mathbf{n} \times \mathbf{H} = \frac{i\epsilon\omega}{\gamma_{mn}^2} \cos\left(\frac{p\pi z}{d} \right) \mathbf{n} \times (\hat{\mathbf{z}} \times \nabla_t \psi) = \frac{i\epsilon\omega}{\gamma_{mn}^2} \cos\left(\frac{p\pi z}{d} \right) \frac{\partial \psi}{\partial n} \hat{\mathbf{z}} \tag{14}$$

For $p > 0$, with the definition of ξ_{mn} from (8.62),

$$I_1 = \left(\frac{\epsilon\omega}{\gamma_{mn}^2} \right)^2 \frac{d}{2} \oint_C \left| \frac{\partial \psi}{\partial n} \right|^2 dl = \left(\frac{\epsilon\omega}{\gamma_{mn}^2} \right)^2 \frac{d}{2} \cdot \omega_{mn}^2 \xi_{mn} \mu\epsilon \frac{C}{A} \int_A |\psi|^2 \, da = \left(\xi_{mn} \frac{Cd}{2A} \right) \left(\frac{\epsilon^2\omega^2}{\gamma_{mn}^2} \right) \int_A |\psi|^2 \, da \tag{15}$$

On the end caps,

$$\mathbf{n} \times \mathbf{H} = \frac{i\epsilon\omega}{\gamma_{mn}^2} \hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times \nabla_t \psi) = -\frac{i\epsilon\omega}{\gamma_{mn}^2} \nabla_t \psi \tag{16}$$

Then by (8),

$$I_2 = \left(\frac{\epsilon\omega}{\gamma_{mn}^2} \right)^2 \int_A |\nabla_t \psi|^2 \, da = \frac{\epsilon^2\omega^2}{\gamma_{mn}^2} \int_A |\psi|^2 \, da \tag{17}$$

Thus we can readily obtain (8.94) since

$$\begin{aligned}
P_{\text{loss}} = \frac{1}{2\sigma\delta}(I_1 + 2I_2) &= \frac{1}{\sigma\delta} \left(\frac{\epsilon^2\omega^2}{\gamma_{mn}^2} \right) \left(1 + \xi_{mn} \frac{Cd}{4A} \right) \int_A |\psi|^2 \, da \\
&= \frac{\epsilon}{\sigma\delta\mu} \left(\frac{\epsilon\mu\omega^2}{\gamma_{mn}^2} \right) \left(1 + \xi_{mn} \frac{Cd}{4A} \right) \int_A |\psi|^2 \, da \qquad \text{by (8.78)} \\
&= \frac{\epsilon}{\sigma\delta\mu} \left[1 + \left(\frac{p\pi}{d\gamma_{mn}} \right)^2 \right] \left(1 + \xi_{mn} \frac{Cd}{4A} \right) \int_A |\psi|^2 \, da
\end{aligned} \tag{18}$$

Then with the definition of Q and δ, we get (8.95),

$$Q = \frac{\omega_{mn} U}{P_{\text{loss}}} = \frac{\mu}{\mu_c}\frac{d}{\delta}\frac{1}{2\left(1+\xi_{mn}\dfrac{Cd}{4A}\right)} \tag{19}$$

When $p = 0$, the dz integral in I_1 will double, as will U in (9). The Q factor in (19) must have d replaced by $2d$ (or equivalently as Jackson states, Q has to be multiplied by 2 and ξ_{mn} replaced by $2\xi_{mn}$).

In fact, ξ_{mn} can be calculated explicitly for cylindrical cavity. By definition (8.62)

$$\xi_{mn} = \frac{1}{\omega_{mn}^2 \mu\epsilon}\frac{A}{C}\frac{\displaystyle\oint_C \left|\frac{\partial\psi}{\partial n}\right|^2 dl}{\displaystyle\int_A |\psi|^2\, da} = \frac{A}{C}\frac{1}{\gamma_{mn}^2}\frac{2\pi R\gamma_{mn}^2\left[J_m'(\gamma_{mn}R)\right]^2}{2\pi\displaystyle\int_0^R \left[J_m(\gamma_{mn}\rho)\right]^2 \rho\, d\rho} = \frac{\pi R^2}{2\pi R}\frac{1}{\gamma_{mn}^2}\frac{2\pi R\gamma_{mn}^2\left[J_m'(\gamma_{mn}R)\right]^2}{\pi R^2\left[J_m'(\gamma_{mn}R)\right]^2} = 1 \tag{20}$$

where we have used the orthonormality of Bessel functions (see 10.22.37 on dlmf.nist.gov)

$$\int_0^1 J_\nu(x_{\nu l}t)J_\nu(x_{\nu m}t)\,t\,dt = \frac{1}{2}\left[J_\nu'(x_{\nu l})\right]^2\delta_{lm} \qquad \text{for } x_{\nu l}, x_{\nu m} \text{ zeros of } J_\nu(x) \tag{21}$$

In summary, for any TM_{mnp}, the formula for the Q factor is

$$Q_{mnp}^{\text{TM}} = \begin{cases} \dfrac{\mu}{\mu_c}\dfrac{d}{\delta}\left(\dfrac{1}{2+d/R}\right) & \text{for } p > 0 \\[2ex] \dfrac{\mu}{\mu_c}\dfrac{d}{\delta}\left(\dfrac{1}{1+d/R}\right) & \text{for } p = 0 \end{cases} \tag{22}$$

Curiously, it has no dependency on m or n.

3. **Derivation of geometric factor (8.97) for TE_{111}**

Note that from the TE fields (10), (11),

$$\mathbf{H}_t = \frac{p\pi}{d\gamma_{mn}^2}\cos\left(\frac{p\pi z}{d}\right)\boldsymbol{\nabla}_t\psi = \frac{p\pi}{d\gamma_{mn}^2}\cos\left(\frac{p\pi z}{d}\right)\left[\gamma_{mn}J_m'(\gamma_{mn}\rho)\,e^{im\phi}\,\hat{\boldsymbol{\rho}} + \frac{im}{\rho}J_m(\gamma_{mn}\rho)\,e^{im\phi}\,\hat{\boldsymbol{\phi}}\right] \tag{23}$$

Thus on the side, where $\rho = R$ and $\mathbf{n} = \hat{\boldsymbol{\rho}}$,

$$\mathbf{n}\times\mathbf{H} = \hat{\boldsymbol{\rho}}\times(\mathbf{H}_t + H_z\hat{\mathbf{z}}) = \frac{p\pi}{d\gamma_{mn}^2}\frac{im}{R}J_m(\gamma_{mn}R)\cos\left(\frac{p\pi z}{d}\right)e^{im\phi}\,\hat{\mathbf{z}} - J_m(\gamma_{mn}R)\,e^{im\phi}\sin\left(\frac{p\pi z}{d}\right)\hat{\boldsymbol{\phi}} \tag{24}$$

Thus the first integral in (13) is

$$\begin{aligned} I_1 &= \oint_C dl\int_0^d dz\,|\mathbf{n}\times\mathbf{H}|^2 = 2\pi R\left\{\frac{d}{2}\left(\frac{p\pi}{d\gamma_{mn}^2}\right)^2\left(\frac{m}{R}\right)^2\left[J_m(\gamma_{mn}R)\right]^2 + \frac{d}{2}\left[J_m(\gamma_{mn}R)\right]^2\right\} \\ &= \pi R d\left[\left(\frac{p\pi}{d\gamma_{mn}^2}\right)^2\left(\frac{m}{R}\right)^2 + 1\right]\left[J_m(\gamma_{mn}R)\right]^2 \end{aligned} \tag{25}$$

On the end caps

$$\mathbf{n}\times\mathbf{H} = \hat{\mathbf{z}}\times(\mathbf{H}_t + H_z\hat{\mathbf{z}}) = \frac{p\pi}{d\gamma_{mn}^2}\left[\gamma_{mn}J_m'(\gamma_{mn}\rho)\,e^{im\phi}\,\hat{\boldsymbol{\phi}} - \frac{im}{\rho}J_m(\gamma_{mn}\rho)\,e^{im\phi}\,\hat{\boldsymbol{\rho}}\right] \tag{26}$$

which gives

$$\begin{aligned} I_2 &= \int_A |\mathbf{n}\times\mathbf{H}|^2\, da = \left(\frac{p\pi}{d\gamma_{mn}^2}\right)^2\int_A\left\{\gamma_{mn}^2\left[J_m'(\gamma_{mn}\rho)\right]^2 + \frac{m^2}{\rho^2}\left[J_m(\gamma_{mn}\rho)\right]^2\right\}da \qquad \text{by (5), (8)} \\ &= \left(\frac{p\pi}{d\gamma_{mn}^2}\right)^2\gamma_{mn}^2\int_A\left[J_m(\gamma_{mn}\rho)\right]^2 da \end{aligned} \tag{27}$$

Then we get the Q factor

$$Q = \frac{\omega_{mn} U}{P_{\text{loss}}} = \frac{\omega_{mn} \cdot \frac{\mu d}{4}\left[1 + \left(\frac{p\pi}{d\gamma_{mn}}\right)^2\right] \int_A [J_m(\gamma_{mn}\rho)]^2\, da}{\frac{1}{2\sigma\delta}\left\{\pi R d\left[\left(\frac{p\pi}{d\gamma_{mn}^2}\right)^2 \left(\frac{m}{R}\right)^2 + 1\right][J_m(\gamma_{mn} R)]^2 + 2\left(\frac{p\pi}{d\gamma_{mn}^2}\right)^2 \gamma_{mn}^2 \int_A [J_m(\gamma_{mn}\rho)]^2\, da\right\}} \tag{28}$$

Here we invoke 10.22.38 of https://dlmf.nist.gov which was also proved in problem 3.11:

$$\int_0^1 J_\nu(\alpha_l t) J_\nu(\alpha_m t)\, t\, dt = \left(\frac{a^2}{b^2} + \alpha_l^2 - \nu^2\right) \frac{[J_\nu(\alpha_l)]^2}{2\alpha_l^2} \delta_{lm} \quad \text{for } \alpha_l, \alpha_m \text{ positive zeros of } aJ_\nu(x) + bxJ_\nu'(x) \tag{29}$$

For TE mode, $\gamma_{mn} = x'_{mn}/R$, for x'_{mn} the n-th zero of $J_m(x)$.

$$\int_A [J_m(\gamma_{mn}\rho)]^2\, da = 2\pi \int_0^R [J_m(\gamma_{mn}\rho)]^2 \rho\, d\rho = \pi R^2 \left(1 - \frac{m^2}{x'^2_{mn}}\right)[J_m(x'_{mn})]^2 \tag{30}$$

This simplifies (28) into

$$\begin{aligned}
Q &= \frac{\omega_{mn} \cdot \frac{\mu d}{4}\left[1 + \left(\frac{p\pi}{d\gamma_{mn}}\right)^2\right] \pi R^2 \left(1 - \frac{m^2}{x'^2_{mn}}\right)}{\frac{1}{2\sigma\delta}\left\{\pi R d\left[\left(\frac{p\pi}{d\gamma_{mn}^2}\right)^2 \left(\frac{m}{R}\right)^2 + 1\right] + 2\left(\frac{p\pi}{d\gamma_{mn}}\right)^2 \pi R^2 \left(1 - \frac{m^2}{x'^2_{mn}}\right)\right\}} \\
&= \left(\frac{\omega_{mn}\mu d\sigma\delta}{2}\right) \cdot \overbrace{\left\{\frac{\left[1 + \left(\frac{p\pi}{d\gamma_{mn}}\right)^2\right] \pi R^2 \left(1 - \frac{m^2}{x'^2_{mn}}\right)}{\pi R d\left[\left(\frac{p\pi}{d\gamma_{mn}^2}\right)^2 \left(\frac{m}{R}\right)^2 + 1\right] + 2\left(\frac{p\pi}{d\gamma_{mn}}\right)^2 \pi R^2 \left(1 - \frac{m^2}{x'^2_{mn}}\right)}\right\}}^{X} = \frac{\mu}{\mu_c}\frac{d}{\delta} \cdot X
\end{aligned} \tag{31}$$

Equating this with the alternate form involving geometric factor G (8.96)

$$Q = \frac{\mu}{\mu_c}\frac{V}{S\delta} G = \frac{\mu}{\mu_c}\frac{\pi R^2 d}{2\pi R(d+R)\delta} G = \frac{\mu}{\mu_c}\frac{d}{\delta} \cdot \frac{R}{2(d+R)} G \tag{32}$$

we get the expression for G,

$$G = 2\left(1 + \frac{d}{R}\right) X = \left(1 + \frac{d}{R}\right)\left[\frac{1 + \left(\frac{x'_{mn}}{p\pi}\right)^2 \frac{d^2}{R^2}}{1 + \frac{1}{2}\left(\frac{m^2}{x'^2_{mn} - m^2}\right)\frac{d}{R} + \frac{1}{2}\left(\frac{x'^2_{mn}}{x'^2_{mn} - m^2}\right)\left(\frac{x'_{mn}}{p\pi}\right)^2 \frac{d^3}{R^3}}\right] \tag{33}$$

Plugging in the numbers for TE_{111}, with $x'_{11} = 1.841$, we restore (8.97)

$$G = \left(1 + \frac{d}{R}\right)\left(\frac{1 + 0.343\frac{d^2}{R^2}}{1 + 0.209\frac{d}{R} + 0.244\frac{d^3}{R^3}}\right) \tag{34}$$

Then the closed form formula for Q_{mnp} for the TE mode is

$$Q^{\text{TE}}_{mnp} = \frac{\mu}{\mu_c}\frac{d}{\delta}\left[\frac{1 + \left(\frac{x'_{mn}}{p\pi}\right)^2 \frac{d^2}{R^2}}{2 + \left(\frac{m^2}{x'^2_{mn} - m^2}\right)\frac{d}{R} + \left(\frac{x'^2_{mn}}{x'^2_{mn} - m^2}\right)\left(\frac{x'_{mn}}{p\pi}\right)^2 \frac{d^3}{R^3}}\right] \tag{35}$$

Unlike the TM mode, the TE mode Q factor has explicit dependency on m and n.