The three-layer reflection/refraction with general incident angle and media permeability was done in the previous solution for problem 7.2 (see here).

- For perpendicular polarization

$$\text{transmitted amplitude :} \quad E_3 = E_1 \left[\frac{2n_1 \cos\alpha}{(1+r_{23})\, n_1 \cos\alpha + \dfrac{\mu_1}{\mu_2}(1-r_{23})\, n_2 \cos\beta} \right] \left(\frac{2n_2 \cos\beta}{n_2 \cos\beta + \dfrac{\mu_2}{\mu_3} n_3 \cos\gamma} \right) e^{i\psi} \tag{1}$$

$$\text{reflected amplitude :} \quad E_1' = E_1 \left[\frac{(1+r_{23})\, n_1 \cos\alpha - \dfrac{\mu_1}{\mu_2}(1-r_{23})\, n_2 \cos\beta}{(1+r_{23})\, n_1 \cos\alpha + \dfrac{\mu_1}{\mu_2}(1-r_{23})\, n_2 \cos\beta} \right] \tag{2}$$

$$\text{where} \quad r_{23} = \left(\frac{n_2 \cos\beta - \dfrac{\mu_2}{\mu_3} n_3 \cos\gamma}{n_2 \cos\beta + \dfrac{\mu_2}{\mu_3} n_3 \cos\gamma} \right) e^{i2\phi} \tag{3}$$

$$\psi = (k_2 \cos\beta - k_3 \cos\gamma)\, d \qquad \phi = k_2 d \cos\beta = 2\pi \cos\beta \cdot d/\lambda_2 \tag{4}$$

- For parallel polarization

$$\text{transmitted amplitude :} \quad E_3 = E_1 \left[\frac{2n_1 \cos\alpha}{\dfrac{\mu_1}{\mu_2}(1+r_{23})\, n_2 \cos\alpha + (1-r_{23})\, n_1 \cos\beta} \right] \left(\frac{2n_2 \cos\beta}{\dfrac{\mu_2}{\mu_3} n_3 \cos\beta + n_2 \cos\gamma} \right) e^{i\psi} \tag{5}$$

$$\text{reflected amplitude :} \quad E_1' = E_1 \left[\frac{\dfrac{\mu_1}{\mu_2}(1+r_{23})\, n_2 \cos\alpha - (1-r_{23})\, n_1 \cos\beta}{\dfrac{\mu_1}{\mu_2}(1+r_{23})\, n_2 \cos\alpha + (1-r_{23})\, n_1 \cos\beta} \right] \tag{6}$$

$$\text{where} \quad r_{23} = \left(\frac{\dfrac{\mu_2}{\mu_3} n_3 \cos\beta - n_2 \cos\gamma}{\dfrac{\mu_2}{\mu_3} n_3 \cos\beta + n_2 \cos\gamma} \right) e^{i2\phi} \tag{7}$$

In the setting of problem 7.3, we have $n_1 = n_3 = n, n_2 = 1, \mu_i = 1, \alpha = \gamma, \cos\beta = \sqrt{1 - n^2 \sin^2\alpha}$.

1. (a) Perpendicular polarization
Plugging in the special values into (3), we have

$$r_{23} = \left(\frac{\cos\beta - n\cos\alpha}{\cos\beta + n\cos\alpha} \right) e^{i2\phi} \qquad \Longrightarrow \qquad 1 \pm r_{23} = \frac{\cos\beta\left(1 \pm e^{i2\phi}\right) + n\cos\alpha\left(1 \mp e^{i2\phi}\right)}{\cos\beta + n\cos\alpha} \tag{8}$$

thus

$$\begin{aligned}
\frac{E_3}{E_1} &= \left\{ \frac{2n\cos\alpha\,(\cos\beta + n\cos\alpha)}{[\cos\beta\,(1+e^{i2\phi}) + n\cos\alpha\,(1-e^{i2\phi})]\, n\cos\alpha + [\cos\beta\,(1-e^{i2\phi}) + n\cos\alpha\,(1+e^{i2\phi})]\cos\beta} \right\} \times \\
&\quad \left(\frac{2\cos\beta}{\cos\beta + n\cos\alpha} \right) e^{i\psi} \\
&= \left[\frac{4n\cos\alpha\cos\beta}{(n^2\cos^2\alpha + \cos^2\beta)(1-e^{i2\phi}) + 2n\cos\alpha\cos\beta\,(1+e^{i2\phi})} \right] e^{i\psi} \\
&= \left[\frac{4n\cos\alpha\cos\beta}{(n^2\cos^2\alpha + \cos^2\beta)(e^{-i\phi} - e^{i\phi}) + 2n\cos\alpha\cos\beta\,(e^{i\phi} + e^{-i\phi})} \right] e^{i(\psi-\phi)} \qquad \Longrightarrow
\end{aligned}$$

$$T = \frac{|E_3|^2}{|E_1|^2} = \frac{4n^2\cos^2\alpha\cos^2\beta}{4n^2\cos^2\alpha\cos^2\beta + (n^2\cos^2\alpha - \cos^2\beta)^2\sin^2\phi} \tag{9}$$

and

$$\frac{E_1'}{E_1} = \frac{\left(n^2\cos^2\alpha - \cos^2\beta\right)\left(1 - e^{i2\phi}\right)}{(n^2\cos^2\alpha + \cos^2\beta)(1-e^{i2\phi}) + 2n\cos\alpha\cos\beta\,(1+e^{i2\phi})} \qquad \Longrightarrow$$

$$R = \frac{|E_1'|^2}{|E_1|^2} = \frac{\left(n^2\cos^2\alpha - \cos^2\beta\right)^2\sin^2\phi}{4n^2\cos^2\alpha\cos^2\beta + (n^2\cos^2\alpha - \cos^2\beta)^2\sin^2\phi} \tag{10}$$

(b) Parallel polarization

In this case

$$r_{23} = \left(\frac{n\cos\beta - \cos\alpha}{n\cos\beta + \cos\alpha}\right)e^{i2\phi} \qquad \Longrightarrow \qquad 1 \pm r_{23} = \frac{n\cos\beta\left(1 \pm e^{i2\phi}\right) + \cos\alpha\left(1 \mp e^{i2\phi}\right)}{n\cos\beta + \cos\alpha} \tag{11}$$

which gives

$$\begin{aligned} \frac{E_3}{E_1} &= \left[\frac{4n\cos\alpha\cos\beta}{(\cos^2\alpha + n^2\cos^2\beta)(1-e^{i2\phi}) + 2n\cos\alpha\cos\beta\,(1+e^{i2\phi})}\right]e^{i\psi} \qquad \Longrightarrow \\ T = \frac{|E_3|^2}{|E_1|^2} &= \frac{4n^2\cos^2\alpha\cos^2\beta}{4n^2\cos^2\alpha\cos^2\beta + (n^2\cos^2\beta - \cos^2\alpha)^2\sin^2\phi} \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{E_1'}{E_1} &= \frac{\left(\cos^2\alpha - n^2\cos^2\beta\right)\left(1-e^{i2\phi}\right)}{(\cos^2\alpha + n^2\cos^2\beta)(1-e^{i2\phi}) + 2n\cos\alpha\cos\beta\,(1+e^{i2\phi})} \qquad \Longrightarrow \\ R = \frac{\left|E_1'\right|^2}{|E_1|^2} &= \frac{\left(n^2\cos^2\beta - \cos^2\alpha\right)^2\sin^2\phi}{4n^2\cos^2\alpha\cos^2\beta + (n^2\cos^2\beta - \cos^2\alpha)^2\sin^2\phi} \end{aligned} \tag{13}$$

Note in both perpendicular and parallel cases, we have $T + R = 1$, as expected.

2. When incident angle α is greater than the critical angle, we should still use the formal expression (1) and (5) to get the complex amplitude of E_3, except that $\cos\beta$ is now a purely imaginary number, so we should replace $\cos\beta$ with $i\cos\eta$, where $\cos\eta = \sqrt{n^2\sin^2\alpha - 1}$. In particular, the "phases" ϕ and ψ are now

$$\phi = ik_2\cos\eta d \equiv i\xi \qquad\qquad \psi = ik_2\cos\eta d - k_3\cos\alpha d = i\xi - k_3\cos\alpha d \tag{14}$$

Then the transmission amplitude in the perpendicular polarization case becomes

$$\frac{E_3}{E_1} = \left[\frac{i4n\cos\alpha\cos\eta}{(n^2\cos^2\alpha - \cos^2\eta)(e^{\xi} - e^{-\xi}) + i2n\cos\alpha\cos\eta\,(e^{\xi} + e^{-\xi})}\right]e^{-ik_3\cos\alpha d} \tag{15}$$

which gives

$$T = \frac{|E_3|^2}{|E_1|^2} = \frac{4n^2\cos^2\alpha\cos^2\eta}{4n^2\cos^2\alpha\cos^2\eta + (n^2\cos^2\alpha + \cos^2\eta)^2\sinh^2\xi} \tag{16}$$

For reflection,

$$\frac{E_1'}{E_1} = \frac{\left(n^2\cos^2\alpha + \cos^2\eta\right)\left(1 - e^{-2\xi}\right)}{(n^2\cos^2\alpha - \cos^2\eta)(1 - e^{-2\xi}) + i2n\cos\alpha\cos\eta\,(1 + e^{-2\xi})} \tag{17}$$

hence

$$R = \frac{\left|E_1'\right|^2}{|E_1|^2} = \frac{\left(n^2\cos^2\alpha + \cos^2\eta\right)^2\sinh^2\xi}{4n^2\cos^2\alpha\cos^2\eta + (n^2\cos^2\alpha + \cos^2\eta)^2\sinh^2\xi} \tag{18}$$

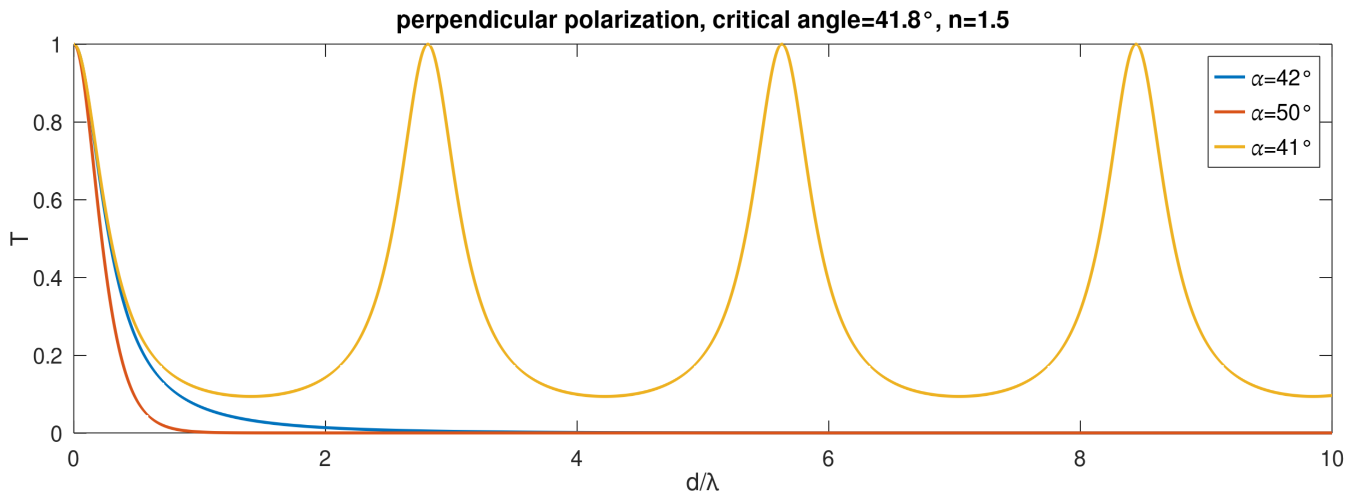
still satisfying $T + R = 1$.

For parallel polarization case, we can obtain the following similarly

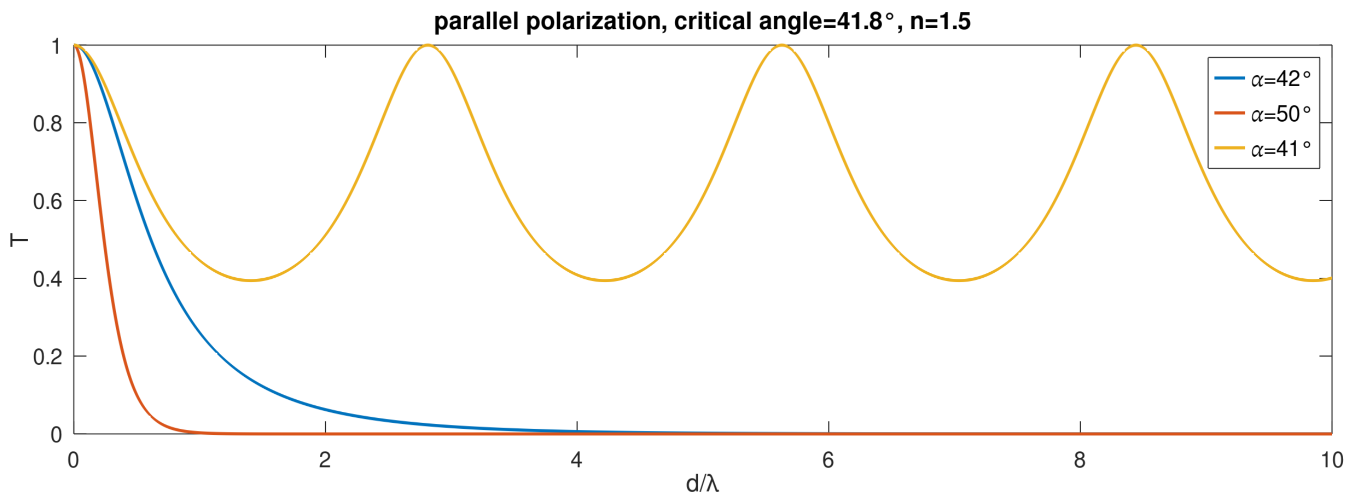
$$T = \frac{4n^2\cos^2\alpha\cos^2\eta}{4n^2\cos^2\alpha\cos^2\eta + (\cos^2\alpha + n^2\cos^2\eta)^2\sinh^2\xi} \tag{19}$$

$$R = \frac{\left(\cos^2\alpha + n^2\cos^2\eta\right)^2\sinh^2\xi}{4n^2\cos^2\alpha\cos^2\eta + (\cos^2\alpha + n^2\cos^2\eta)^2\sinh^2\xi} \tag{20}$$

The diagram below shows the transmission coefficient T for different incident angles with $n = 1.5$ for which the critical angle is $41.8°$.



perpendicular polarization, critical angle=41.8°, n=1.5
T
d/λ
α=42°
α=50°
α=41°



parallel polarization, critical angle=41.8°, n=1.5
T
d/λ
α=42°
α=50°
α=41°