1. The canonical momentum of the particle is

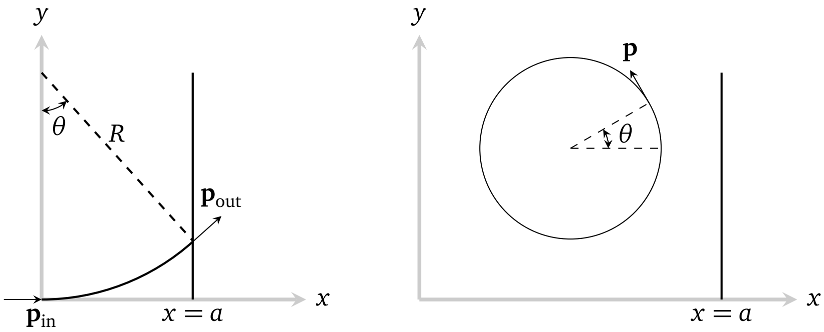
$$\mathbf{G} = \gamma m \mathbf{v} + \frac{q}{c}\mathbf{A} \tag{1}$$

Only knowing $\mathbf{B} = -B\hat{\mathbf{z}}$ is not sufficient to determine $\mathbf{A}$. For example, up to a gauge transformation, we can write

$$\mathbf{A} = -xB\hat{\mathbf{y}} + \nabla\chi \qquad \Longrightarrow \qquad \mathbf{G} = \gamma m \mathbf{v} - \frac{qxB}{c}\hat{\mathbf{y}} + \frac{q}{c}\nabla\chi \tag{2}$$

2. In this part, the radius of the particle's movement inside the magnetic field is

$$R = \frac{cp}{qB} \tag{3}$$



With the help of the diagram on the left, we see that when the particle emerges out of the magnetic field, the components of its mechanical momentum are

$$p_{\text{out},y} = p\sin\theta = \frac{pa}{R} = \frac{qBa}{c} \tag{4}$$
$$p_{\text{out},x} = p\cos\theta \tag{5}$$

With the choice of gauge $\chi = 0$, we see that

$$G_{\text{in},x} = p \qquad G_{\text{in},y} = 0 \qquad G_{\text{in},z} = 0 \tag{6}$$
$$G_{\text{out},x} = p\cos\theta \qquad G_{\text{out},y} = \frac{qBa}{c} - \frac{qBa}{c} = 0 \qquad G_{\text{out},z} = 0 \tag{7}$$

Clearly the y, z component of the canonical momentum are conserved, but not its x component.

This because the Euler-Lagrange equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \tag{8}$$

indicates that for the canonical momentum $G_i = \partial L/\partial \dot{q}_i$ to be conserved, the Lagrangian must not have explicit dependence on the coordinate q_i. Our particular choice of gauge $\chi = 0$ made it so that L has dependence on x, but not y, z, therefore G_y, G_z are conserved, but not G_x.

3. In this part, first let's keep using the zero gauge function $\chi = 0$, which gives

$$\begin{aligned}\mathbf{G}(\theta) &= \mathbf{p}(\theta) - \frac{qxB}{c}\hat{\mathbf{y}} \\ &= \mathbf{p}(\theta) - \frac{qRB}{c}\cos\theta\hat{\mathbf{y}} \\ &= p(-\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}) - p\cos\theta\hat{\mathbf{y}} \\ &= -p\sin\theta\hat{\mathbf{x}}\end{aligned} \tag{9}$$

of which the y component is conserved, but not x.

But if we choose gauge function to be

$$\chi = \frac{xyB}{2} \tag{10}$$

then we have

$$\mathbf{A} = -xB\hat{\mathbf{y}} + \boldsymbol{\nabla}\chi = -xB\hat{\mathbf{y}} + \frac{yB}{2}\hat{\mathbf{x}} + \frac{xB}{2}\hat{\mathbf{y}} = -\frac{B}{2}\left(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}\right) \tag{11}$$

Then the canonical momentum becomes

$$\begin{aligned}
\mathbf{G}(\theta) &= \mathbf{p}(\theta) - \frac{qB}{2c}\left(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}\right) \\
&= \mathbf{p}(\theta) - \frac{qBR}{2c}\left(\cos\theta\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{x}}\right) \\
&= p\left(-\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}\right) - \frac{p}{2}\left(\cos\theta\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{x}}\right) \\
&= \frac{p}{2}\left(-\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}\right) \qquad (12)
\end{aligned}$$

which is not a conserved quantity for either x or y. This is indicated from our choice of χ which makes $\mathbf{A}$, hence L, depend on both x and y.