

When the wires are perfect conductors, both the electric charge and the current stay at the surface. When they carry equal but opposite currents, the current gave rise to the surface charge distribution as well. Therefore the current density and charge density are proportional to each other. We can express this proportionality as

$$\frac{I}{q} = \frac{j(\rho, \phi)}{\sigma(\rho, \phi)} \quad (1)$$

where I is the total current, q is the total charge per length, j is the surface current density, and σ is the surface charge density.

The electric scalar potential and the magnetic vector potential at any point in the space around the wire can be written as

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon} \left[\int_{\text{wire-1}} \frac{\sigma_1(\rho', \phi')}{|\mathbf{x} - \mathbf{x}'|} d^3x' + \int_{\text{wire-2}} \frac{\sigma_2(\rho', \phi')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \right] \quad (2)$$

$$\mathbf{A}(\mathbf{x}) = \frac{\mu}{4\pi} \left[\int_{\text{wire-1}} \frac{j_1(\rho', \phi')}{|\mathbf{x} - \mathbf{x}'|} d^3x' + \int_{\text{wire-2}} \frac{j_2(\rho', \phi')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \right] \hat{\mathbf{z}} \quad (3)$$

We can see the two potentials are proportional too:

$$\frac{\mathbf{A}}{\Phi} = \mu\epsilon \frac{I}{q} \hat{\mathbf{z}} \quad (4)$$

The fields in the space around the wire are

$$\mathbf{E} = -\nabla\Phi = -\frac{\partial\Phi}{\partial\rho}\hat{\rho} - \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\phi} \quad (5)$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{\rho}\frac{\partial A_z}{\partial\phi}\hat{\rho} - \frac{\partial A_z}{\partial\rho}\hat{\phi} \quad (6)$$

and inside the wire's volume, the fields are zero.

The electric and magnetic energy density in the space around the wire are

$$\mathcal{E}_E = \frac{1}{2}\epsilon\mathbf{E}^2 = \frac{\epsilon}{2} \left[\left(\frac{\partial\Phi}{\partial\rho} \right)^2 + \left(\frac{1}{\rho} \frac{\partial\Phi}{\partial\phi} \right)^2 \right] \quad (7)$$

$$\mathcal{E}_B = \frac{1}{2}\frac{\mathbf{B}^2}{\mu} = \frac{1}{2\mu} \left[\left(\frac{\partial A_z}{\partial\phi} \right)^2 + \left(\frac{1}{\rho} \frac{\partial A_z}{\partial\rho} \right)^2 \right] \quad (8)$$

Let the per-length inductance and capacitance be L and C , then the electric and magnetic energy in the range $[0, z_0]$ can be expressed by

$$\frac{1}{2} \frac{(qz_0)^2}{Cz_0} = \int_0^{z_0} dz \int \mathcal{E}_E dx dy \quad \frac{1}{2} (Lz_0) I^2 = \int_0^{z_0} dz \int \mathcal{E}_B dx dy \quad (9)$$

where the $dx dy$ integral is done in the x - y plane excluding the wire's cross sections, since inside the cross-section, energy density is zero.

Finally taking the ratio gives

$$\frac{\frac{1}{2} \frac{(qz_0)^2}{Cz_0}}{\frac{1}{2} (Lz_0) I^2} = \frac{\mathcal{E}_E}{\mathcal{E}_B} = \mu\epsilon \left(\frac{\Phi}{A} \right)^2 = \frac{q^2}{I^2} \frac{1}{\mu\epsilon} \quad \Rightarrow \quad LC = \mu\epsilon \quad (10)$$