Under the influence of the electric field, the equation of motion of the charge is

$$m\ddot{x} = -\Gamma\dot{x} - kx + eE(x,t) = 0 \qquad \text{where } k = m\omega_0^2 \tag{1}$$

With the assumption that the motion of the charge is small compared to the spatial variation of **E**, we can take the approximation $E(x,t) \approx E(0,t)$, turning (1) into

$$m\ddot{x} + \Gamma\dot{x} + m\omega_0^2 x - eE(0,t) = 0 \tag{2}$$

Taking the Fourier transform of (2) gives

$$\frac{1}{\sqrt{2\pi}}\left[\left(m\frac{d^2}{dt^2} + \Gamma\frac{d}{dt} + m\omega_0^2\right)\int_{-\infty}^{\infty} x(\omega)e^{-i\omega t}d\omega - e\int_{-\infty}^{\infty} E(\omega)e^{-i\omega t}d\omega\right] = 0 \implies$$

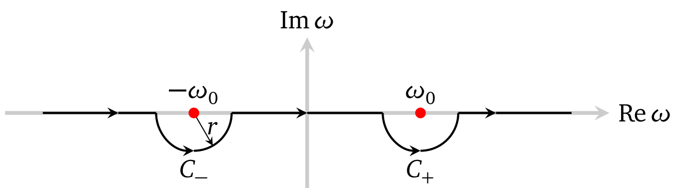
$$\int_{-\infty}^{\infty}\left\{\left[-m\left(\omega^2-\omega_0^2\right) - i\Gamma\omega\right]x(\omega) - eE(\omega)\right\}e^{-i\omega t}d\omega = 0 \tag{3}$$

Orthogonality of the Fourier transform implies that the integrand must vanish for all ω, leading to

$$x(\omega) = \frac{-eE(\omega)}{m\left(\omega^2-\omega_0^2\right) + i\Gamma\omega} \tag{4}$$

The total energy transfer from the field to the charge is given by the integral

$$\begin{aligned}
\Delta E &= \int_{-\infty}^{\infty} eE(0,t)\frac{dx(t)}{dt}dt \\
&= e\int_{-\infty}^{\infty} dt\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega E(\omega)e^{-i\omega t}\right]\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega'\left(-i\omega'\right)x\left(\omega'\right)e^{-i\omega' t}\right] \qquad \omega' \to -\omega' \\
&= e\int_{-\infty}^{\infty} d\omega E(\omega)\int_{-\infty}^{\infty} d\omega'\left(i\omega'\right)x^*\left(\omega'\right)\cdot\overbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} dt e^{-i\left(\omega-\omega'\right)t}}^{\delta\left(\omega-\omega'\right)} \\
&= \frac{e^2}{m}\underbrace{\int_{-\infty}^{\infty}\frac{-|E(\omega)|^2 i\omega}{\left(\omega^2-\omega_0^2\right) - i\Gamma\omega/m}d\omega}_{I}
\end{aligned} \tag{5}$$



Under the limit $\Gamma \to 0$, the integrand has two poles approaching $\pm\omega_0$ from above and the real-axis integral can be decomposed into

$$I = \lim_{r\to 0}\left(\int_{-\infty}^{-\omega_0-r} + \int_{-\omega_0+r}^{\omega_0-r} + \int_{\omega_0+r}^{\infty} + \int_{C_-} + \int_{C_+}\right)$$

where the first three terms give

$$\lim_{r\to 0}\left(\int_{-\infty}^{-\omega_0-r} + \int_{-\omega_0+r}^{\omega_0-r} + \int_{\omega_0}^{\infty}\right) = P.V.\int_{-\infty}^{\infty}\frac{-|E(\omega)|^2 i\omega}{\left(\omega^2-\omega_0^2\right)}d\omega \tag{6}$$

which is a pure imaginary number, having no contribution to ΔE. For the two semi-circular contours, let $z = \pm\omega_0 + re^{i\phi}$, $\phi \in [-\pi, 0]$, then

$$\begin{aligned}
\operatorname{Re} I = \lim_{r\to 0}\left(\int_{C_-} + \int_{C_+}\right) &= \lim_{r\to 0}\left[\int_{-\pi}^{0}\frac{-|E(z)|^2 iz\cdot ire^{i\phi}d\phi}{re^{i\phi}\left(-2\omega_0 + re^{i\phi}\right)} + \int_{-\pi}^{0}\frac{-|E(z)|^2 iz\cdot ire^{i\phi}d\phi}{re^{i\phi}\left(2\omega_0 + re^{i\phi}\right)}\right] \\
&= \frac{\pi}{2}\left[|E(-\omega_0)|^2 + |E(\omega_0)|^2\right] = \pi|E(\omega_0)|^2
\end{aligned} \tag{7}$$

which yields the total energy transfer

$$\Delta E = \frac{\pi e^2|E(\omega_0)|^2}{m} \tag{8}$$