$$\begin{aligned}
\Phi_{(b)}(\mathbf{x}) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\int d\Omega' Y_{lm}^{*}\left(\theta',\phi'\right)V\left(\theta',\phi'\right)\right]\left(\frac{r}{a}\right)^{l} Y_{lm}(\theta,\phi) \\
&= \sum_{l=0}^{\infty}\left(\frac{r}{a}\right)^{l}\int d\Omega' V\left(\theta',\phi'\right)\sum_{m=-l}^{l} Y_{lm}^{*}\left(\theta',\phi'\right)Y_{lm}(\theta,\phi) \qquad \text{by addition theorem} \\
&= \sum_{l=0}^{\infty}\left(\frac{r}{a}\right)^{l}\int d\Omega' V\left(\theta',\phi'\right)\frac{2l+1}{4\pi}P_l(\cos\gamma) \\
&= \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\left(\frac{r}{a}\right)^{l}\int d\Omega' V\left(\theta',\phi'\right)P_l(\cos\gamma)
\end{aligned} \tag{1}$$

On the other hand

$$\Phi_{(a)}(\mathbf{x}) = \frac{a\left(a^2-r^2\right)}{4\pi}\int \frac{d\Omega' V(\theta',\phi')}{(r^2+a^2-2ar\cos\gamma)^{3/2}} \tag{2}$$

Compare (1) and (2), it's sufficient to prove

$$\sum_{l=0}^{\infty}(2l+1)\left(\frac{r}{a}\right)^{l}P_l(\cos\gamma) = \frac{a\left(a^2-r^2\right)}{(r^2+a^2-2ar\cos\gamma)^{3/2}} \tag{3}$$

Indeed, define

$$t \equiv \frac{r}{a} \qquad x \equiv \cos\gamma \qquad g(t,x) \equiv \frac{1}{\sqrt{1+t^2-2tx}} \tag{4}$$

then it's easy to see

$$\begin{aligned}
\text{RHS}_{(3)} &= \frac{a^3\left(1-t^2\right)}{a^3(1+t^2-2tx)^{3/2}} \\
&= \frac{\left(1+t^2-2tx\right)+2t(x-t)}{(1+t^2-2tx)^{3/2}} \\
&= \frac{1}{\sqrt{1+t^2-2tx}} + 2t\cdot\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{1+t^2-2tx}}\right) \\
&= \left(1+2t\frac{\partial}{\partial t}\right)g(t,x)
\end{aligned} \tag{5}$$

But $g(t,x)$, being the generating function of the Legendre polynomials, can be expanded as

$$g(t,x) = \sum_{l=0}^{\infty}P_l(x)t^l \tag{6}$$

Therefore

$$\left(1+2t\frac{\partial}{\partial t}\right)g(t,x) = \sum_{l=0}^{\infty}\left[P_l(x)t^l + 2tP_l(x)lt^{l-1}\right] = \sum_{l=0}^{\infty}(2l+1)P_l(x)t^l = \text{LHS}_{(3)} \tag{7}$$