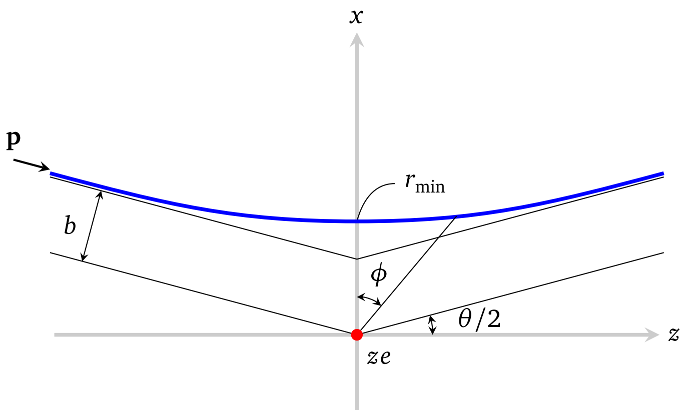
This is the (failed) attempt to derive the relativistic Rutherford scattering formula (Jackson equation 13.1).

We are working in the rest frame of the massive particle with mass $M \to \infty$ and charge ze. The incident electron (mass m, charge $-e$) comes from infinity with initial momentum $\mathbf{p}$, and impact parameter b. The following calculation strongly suggests (although not rigorously and conclusively) that (13.1) is not exact for all scattering angles and all initial velocities (maybe it is an approximation without explicitly stated conditions?).



Let the electron have conserved angular momentum $L = pb$, conserved energy $E = \sqrt{p^2c^2 + m^2c^4}$. Its potential energy at distance r is $V(r) = -ze^2/r$.

With polar coordinates (r, ϕ), conservation of energy and angular momentum require

$$E = \gamma mc^2 + V(r) \quad \Longrightarrow \quad \gamma = \frac{E - V(r)}{mc^2} \tag{1}$$

$$L = \gamma mr^2\dot{\phi} \quad \Longrightarrow \quad \dot{\phi} = \frac{L}{\gamma mr^2} \tag{2}$$

On the other hand, by definition,

$$\frac{1}{\gamma^2} = 1 - \left(\frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right) = 1 - \frac{1}{c^2}\left[\left(\frac{dr}{d\phi}\dot{\phi}\right)^2 + r^2\dot{\phi}^2\right] = 1 - \frac{1}{c^2}\left(\frac{L}{\gamma mr^2}\right)^2\left[\left(\frac{dr}{d\phi}\right)^2 + r^2\right] \tag{3}$$

Combining (3) with (1), we obtain the orbit equation

$$\left(\frac{L}{mr^2}\right)^2\left[\left(\frac{dr}{d\phi}\right)^2 + r^2\right] = \gamma^2c^2\left(1 - \frac{1}{\gamma^2}\right) = \frac{[E - V(r)]^2}{m^2c^2} - c^2 \tag{4}$$

With $u \equiv 1/r$, (4) can be rewritten as

$$\left(\frac{L}{m}\right)^2 u^4\left[\frac{1}{u^4}\left(\frac{du}{d\phi}\right)^2 + \frac{1}{u^2}\right] = \frac{\left(E + ze^2u\right)^2}{m^2c^2} - c^2 \qquad \Longrightarrow$$

$$\left(\frac{du}{d\phi}\right)^2 = \frac{\left(E + ze^2u\right)^2}{c^2L^2} - \frac{m^2c^2}{L^2} - u^2 = \frac{\left(z^2e^4 - c^2L^2\right)u^2 + 2Eze^2u + \overbrace{E^2 - m^2c^4}^{p^2c^2}}{c^2L^2} = Au^2 + Bu + C \tag{5}$$

where

$$A = \frac{z^2e^4 - c^2L^2}{c^2L^2} \qquad B = \frac{2Eze^2}{c^2L^2} \qquad C = \frac{p^2c^2}{c^2L^2} \tag{6}$$

The discriminant is

$$\Delta = B^2 - 4AC = \frac{4\left(p^2c^2 + m^2c^4\right)z^2e^4 - 4\left(z^2e^4 - c^2L^2\right)p^2c^2}{c^4L^4} = \frac{4\left(m^2z^2e^4 + p^2L^2\right)}{L^4} > 0 \tag{7}$$

So we have two real roots of the quadratic equation $Au^2 + Bu + C = 0$

$$u_1 = \frac{-B + \sqrt{\Delta}}{2A} \qquad u_2 = \frac{-B - \sqrt{\Delta}}{2A} \tag{8}$$

Obviously, $C > 0$. The sign of B is the same as z.

1. When $A < 0$, regardless of the sign of B, we have $u_1 < 0 < u_2$. The motion is allowed in the range $u \in [0, u_2]$, i.e., an electron starting from infinity moves all the way to the stationary point at minimum distance $r_{\min} = 1/u_2$, then returns to infinity. This corresponds to the scattering process that works for both repulsive and attractive forces.

2. If $A > 0$, and $B > 0$ (i.e., $z > 0$), we see that $u_2 < u_1 < 0$. By (5), the admissible range for u is $[0, \infty)$. If the electron starts at infinity $u = 0$, it will keep going until $u \to \infty$ ($r = 0$) since the stationary point u_1, u_2 are not in the admissible range. This represents a capture orbit. Qualitatively, the condition $z > 0$ means this is an attractive force for the incident electron, and the condition $A > 0$, or $z^2 e^4 > c^2 p^2 b^2$, indicates the electron is either moving too closely to the axis, or its initial momentum is too small, resulting it eventually being captured by the target particle.

3. When $A > 0$, but $B < 0$, we have $0 < u_2 < u_1$. The motion is allowed in the range $u \in [0, u_2]$, the closest distance is $r_{\min} = 1/u_2$. This corresponds to a scattering orbit in a repulsive force.

For the scattering process as a whole, the change in momentum is equal to the impulse (time integration of the Coulomb force),

$$\Delta \mathbf{p} = \int_{-\infty}^{\infty} \frac{-ze^2}{r^2} (\cos\phi \,\hat{\mathbf{x}} + \sin\phi \,\hat{\mathbf{z}})\, dt \tag{9}$$

By symmetry, the z component vanishes on both sides, leaving the x component equation

$$\begin{aligned}
2p \sin\frac{\theta}{2} &= |\Delta \mathbf{p}| = -ze^2 \int_{-\infty}^{\infty} \frac{\cos\phi}{r^2} dt && \text{by (2)} \\
&= -\frac{mze^2}{L} \int_{-\infty}^{\infty} \gamma \cos\phi \,\dot{\phi}\, dt = -\frac{mze^2}{L} \overbrace{\int_{-\phi_\infty}^{\phi_\infty} \gamma \cos\phi \, d\phi}^{I}
\end{aligned} \tag{10}$$

In non-relativistic limit where $\gamma = 1$, we can obtain the Rutherford scattering formula

$$\begin{aligned}
2p \sin\frac{\theta}{2} &= -\frac{mze^2}{L} \cdot 2 \sin\phi_\infty && \text{note } \phi_\infty = \frac{\pi}{2} - \frac{\theta}{2} && \Longrightarrow \\
2p \sin\frac{\theta}{2} &= -\frac{mze^2}{pb} \cdot 2 \cos\frac{\theta}{2} && && \Longrightarrow \\
b &= -\frac{mze^2}{p^2} \cot\frac{\theta}{2} = -\frac{ze^2}{pv} \cot\frac{\theta}{2} && && \Longrightarrow \\
\frac{d\sigma}{d\Omega} &= \frac{b}{\sin\theta} \left|\frac{db}{d\theta}\right| = \left(\frac{ze^2}{2pv}\right)^2 \frac{1}{\sin^4\theta/2}
\end{aligned} \tag{11}$$

If the particle is moving at relativistic speed, we can rewrite (10) as

$$\begin{aligned}
2p \sin\frac{\theta}{2} &= -\frac{mze^2}{L} \left(\gamma \sin\phi \Big|_{-\phi_\infty}^{\phi_\infty} - \int_{-\phi_\infty}^{\phi_\infty} \frac{d\gamma}{d\phi} \sin\phi \, d\phi \right) \\
&= -\frac{mze^2}{L} \cdot 2\gamma_\infty \sin\phi_\infty + \underbrace{\frac{mze^2}{L} \int_{-\phi_\infty}^{\phi_\infty} \frac{d\gamma}{d\phi} \sin\phi \, d\phi}_{K}
\end{aligned} \tag{12}$$

Now if term K vanishes, we would end up with

$$2p \sin\frac{\theta}{2} = -\frac{2\gamma_\infty mze^2}{pb} \cos\frac{\theta}{2} \qquad \Longrightarrow \qquad b = -\frac{ze^2}{pv} \cot\frac{\theta}{2} \tag{13}$$

which would give (11) as desired.

But it is clear that K does not vanish, since both $d\gamma/d\phi$ and $\sin\phi$ are odd in ϕ. *This is a strong hint that (11) does not hold in relativistic regime.*

We can also try to calculate integral I directly. By (1), the integral I becomes

$$I = \int_{-\phi_\infty}^{\phi_\infty} \left(\frac{E + ze^2 u}{mc^2} \right) \cos\phi \, d\phi = \frac{2E}{mc^2} \sin\phi_\infty + \frac{ze^2}{mc^2} \overbrace{\int_{-\phi_\infty}^{\phi_\infty} u \cos\phi \, d\phi}^{J} \tag{14}$$

Differentiating (5) on both sides yields

$$2u'u'' = 2Auu' + Bu' \qquad \Longrightarrow \qquad u'' - Au = \frac{B}{2} \tag{15}$$

Then integrating with $\cos\phi\, d\phi$ gives

$$\begin{aligned}
B\sin\phi_\infty = \int_{-\phi_\infty}^{\phi_\infty} \frac{B}{2}\cos\phi\, d\phi &= \int_{-\phi_\infty}^{\phi_\infty} \left(u'' - Au\right)\cos\phi\, d\phi && \text{integrate by parts} \\
&= u'\cos\phi\Big|_{-\phi_\infty}^{\phi_\infty} + \int_{-\phi_\infty}^{\phi_\infty} u'\sin\phi\, d\phi - AJ && \text{integrate by parts again} \\
&= u'\cos\phi\Big|_{-\phi_\infty}^{\phi_\infty} + \underbrace{u\sin\phi\Big|_{-\phi_\infty}^{\phi_\infty}}_{0} - (A+1)J && (16)
\end{aligned}$$

By the sign convention indicated by the diagram, we have $u' = du/d\phi > 0$ for $\phi < 0$ (left half of diagram, as ϕ increases, u increases), and $u' < 0$ for $\phi > 0$ (right half of diagram, as ϕ increases, u decreases). With the correct sign, (5) gives

$$u'(-\phi_\infty) = \sqrt{C} = -u'(\phi_\infty) \qquad \Longrightarrow \qquad u'\cos\phi\Big|_{-\phi_\infty}^{\phi_\infty} = -2\sqrt{C}\cos\phi_\infty \tag{17}$$

hence by (16)

$$J = -\frac{2\sqrt{C}\cos\phi_\infty}{A+1} - \frac{B\sin\phi_\infty}{A+1} \tag{18}$$

This looks promising except that after putting (18) into (14) to obtain I, and then putting I back to (10), one would yield the useless identity

$$2p\sin\frac{\theta}{2} = 2p\sin\frac{\theta}{2}$$

This is not a surprise at all since in a central field, energy conservation implies impulse-momentum relation, so this way of calculating integral I (by using orbit equation (5) which is derived using energy conservation) yields no new relationship between b and θ.

Let's solve for the exact deflection angle for the two scattering cases.

Note that after completing the square and separating the variable, for the range $\phi \in [-\phi_\infty, 0]$ where $du/d\phi > 0$, we can rewrite (5) as

$$\frac{du}{\sqrt{A\left[\left(u+\frac{B}{2A}\right)^2-\frac{\Delta}{4A^2}\right]}} = d\phi \tag{19}$$

1. Scattering mode where $A < 0$.

   From (19), we have

$$\begin{aligned}
\frac{du}{\sqrt{\left(\frac{\sqrt{\Delta}}{-2A}\right)^2-\left(u+\frac{B}{2A}\right)^2}} &= \sqrt{-A}\cdot d\phi && \text{let } u+\frac{B}{2A} = \left(\frac{\sqrt{\Delta}}{-2A}\right)\sin\xi && \Longrightarrow \\
\frac{\left(\frac{\sqrt{\Delta}}{-2A}\right)\cos\xi\, d\xi}{\left(\frac{\sqrt{\Delta}}{-2A}\right)\cos\xi} &= \sqrt{-A}\cdot d\phi && \text{integrate both sides} && \Longrightarrow \\
\frac{\pi}{2}+\sin^{-1}\left(\frac{B}{\sqrt{\Delta}}\right) &= \sqrt{-A}\left(\frac{\pi}{2}-\frac{\theta}{2}\right) && && (20)
\end{aligned}$$

   where the integration of $d\xi$ is taken in the range of ξ corresponding to $u \in [0, u_2]$.

   The deflection angle θ is given by

$$\theta = \pi\left(1-\frac{1}{\sqrt{-A}}\right)-\frac{2}{\sqrt{-A}}\sin^{-1}\left(\frac{B}{\sqrt{\Delta}}\right) \tag{21}$$

   It is understood that the value of $\sin^{-1}(B/\Delta)$ will be properly chosen so θ is in the range $[0, \pi]$.

2. Scattering mode where $A > 0, B < 0$.

   Similarly, (19) implies

$$\begin{aligned}
\frac{du}{\sqrt{\left(u+\frac{B}{2A}\right)^2-\frac{\Delta}{4A^2}}} &= \sqrt{A}d\phi && \text{let } u+\frac{B}{2A} = -\left(\frac{\sqrt{\Delta}}{2A}\right)\cosh\xi && \Longrightarrow \\
\frac{-\left(\frac{\sqrt{\Delta}}{2A}\right)\sinh\xi\, d\xi}{\left(\frac{\sqrt{\Delta}}{2A}\right)\sinh\xi} &= \sqrt{A}d\phi && \text{integrate both sides} && \Longrightarrow \\
\cosh^{-1}\left(\frac{-B}{\sqrt{\Delta}}\right) &= \sqrt{A}\left(\frac{\pi}{2}-\frac{\theta}{2}\right) && && (22)
\end{aligned}$$

   We have chosen the minus sign in the definition of ξ so we have a real ξ at $u_2 = (-B-\sqrt{\Delta})/2A$. Also in the last step, we have chosen the positve branch of $\cosh^{-1}$ to satisfy the range requirement of θ.

   In this scattering mode, the deflection angle is

$$\theta = \pi - \frac{2}{\sqrt{A}}\cosh^{-1}\left(\frac{-B}{\sqrt{\Delta}}\right) \tag{23}$$

In order to see the condition under which the exact relativistic deflection angle (21) or (23) may be approximated by the non-relativistic Rutherford scattering formula (Jackson 13.1), let's first find the non-relativistic orbit equation. In non-relativistic regime, the conserved energy is $E = p^2/2m$, then from energy conservation, we have

$$\frac{1}{2}mv^2-\frac{ze^2}{r}=\frac{p^2}{2m} \quad\Longrightarrow\quad \frac{1}{2}m\left(r'^2+r^2\right)\dot{\phi}^2-\frac{ze^2}{r}=\frac{p^2}{2m} \quad\Longrightarrow\quad \frac{1}{2}m\left(r'^2+r^2\right)\left(\frac{L}{mr^2}\right)^2-\frac{ze^2}{r}=\frac{p^2}{2m} \tag{24}$$

Changing to $u = 1/r$ produces

$$\left(\frac{du}{d\phi}\right)^2 = -u^2 + \frac{2mze^2}{L^2}u + \frac{1}{b^2} \tag{25}$$

Comparing this with (5) and (6), we see the following changes in quadratic coefficients A, B, C:

relativistic orbit equation:	$A = \frac{z^2e^4}{c^2L^2} - 1$	$B = \frac{2\gamma_\infty mze^2}{L^2}$	$C = \frac{1}{b^2}$
non-relativistic orbit equation:	$A = -1$	$B = \frac{2mze^2}{L^2}$	$C = \frac{1}{b^2}$

Two important points are worth noting:

- When $cL = cpb \gg \left|ze^2\right|$, i.e., high initial momentum p, or large impact parameter b, or both, we have $A_{\text{rel}} \to -1$. The relativistic orbit is the same as the non-relativistic orbit except for the $m \to \gamma_\infty m$ substitution in B.
- The relativistic scattering mode with $A > 0, B < 0$ has no corresponding non-relativistic counterpart, since the latter requires $A = -1$. This corresponds to the repulsive Coulomb field with small initial angular momentum $L = pb < \left|ze^2\right|/c$ and it cannot be approximated by any non-relativistic scattering via limiting procedure.

With $A \to -1$, (21) implies

$$\sin\frac{\theta}{2} = -\frac{B}{\sqrt{\Delta}} \qquad \Longrightarrow \qquad \tan\frac{\theta}{2} = \frac{-B/\sqrt{\Delta}}{\sqrt{1 - B^2/\Delta}} = -\frac{B}{\sqrt{4C}} = -\frac{ze^2}{pbv} \tag{26}$$

which restores the Rutherford scattering formula (see (11)). But the condition $A \to -1$ implies $pb \to \infty$, which only produces $\theta \to 0$, i.e., small deflection angles.

In summary, the Rutherford scattering formula, Jackson (13.1)

$$\frac{d\sigma}{d\Omega} = \left(\frac{ze^2}{2pv}\right)^2 \frac{1}{\sin^4\theta/2}$$

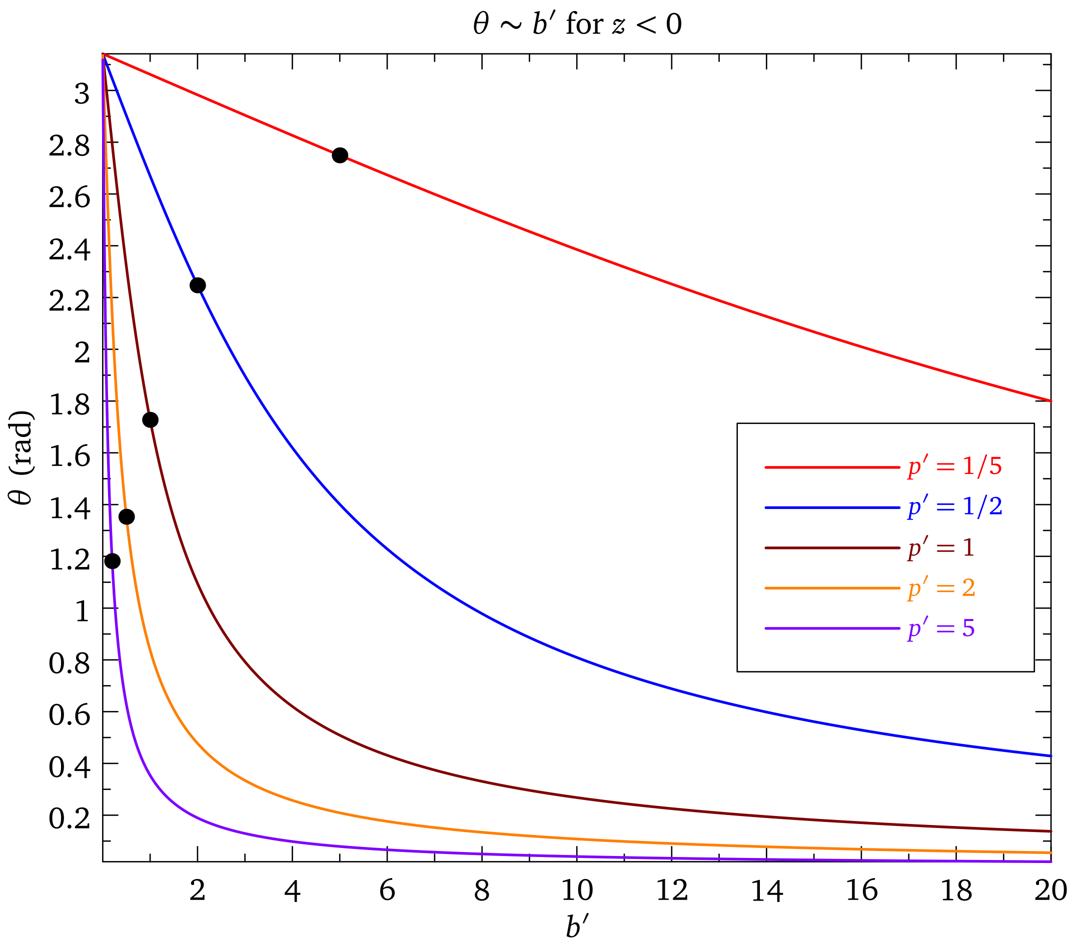
is correct only under non-relativistic limit. But under the *additional* assumption that $cL \gg \left|ze^2\right|$ (i.e., $A \to -1$), it is also valid if we write $p = \gamma_\infty mv$ to account for the relativistic velocity at infinity, but this additional assumption is consistent only with small deflection angles. This additional assumption should have been stated in Jackson before (13.1) is used in relativistic settings. Furthermore, it is inconsistent to use the relativistic Rutherford scattering formula for scenarios where $\theta \to \pi$ (e.g., paragraph above equation 13.4) since the prerequisite to Rutherford is $A \to -1$ which does not allow large θ.

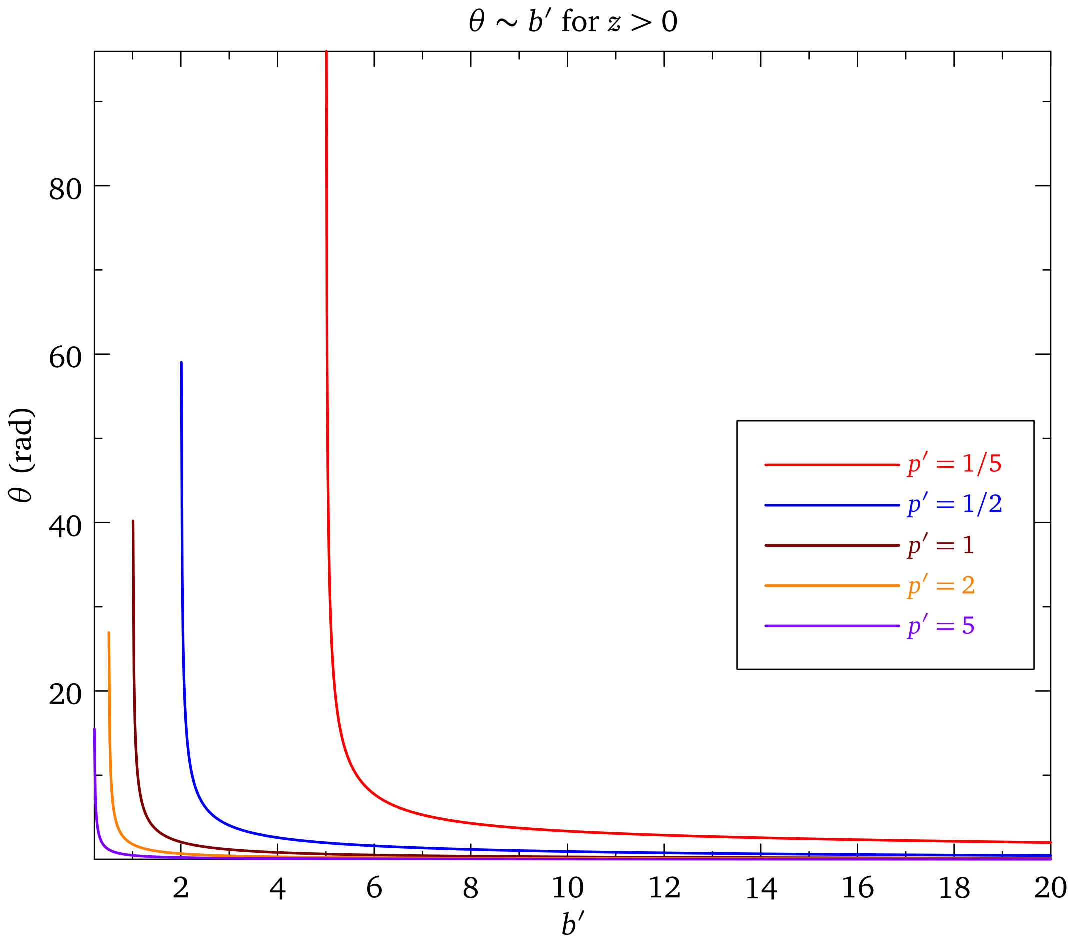
If we define the characteristic length $b_0 = \left|ze^2\right|/mc^2$ and characteristic momentum $p_0 = mc$, we can rewrite A and $B/\sqrt{\Delta}$ in (21) and (23) as functions of dimensionless variables $b' = b/b_0$ and $p' = p/p_0$

$$A = \left(\frac{1}{b'p'}\right)^2 - 1 \qquad \frac{B}{\sqrt{\Delta}} = \operatorname{sgn}(z)\sqrt{\frac{1 + p'^2}{1 + p'^4 b'^2}} \tag{27}$$

The first graph below shows the $\theta \sim b'$ relationship given by (21) and (23) for negative z (repulsive force) and a series of fixed values of p'. The black marker on each line is the point where $A = 0$, to the left of which is governed by (23) (i.e., $A > 0$), and to the right of which is governed by (21) (i.e., $A < 0$).

The second graph is for positive z (attractive force). In this graph, we don't regulate θ to be in the range $[0, \pi]$ so it shows steep slope for b' near the critical impact parameter (below which the particle will be captured). Near the critical impact parameter $b' = 1/p'$, the deflection angle changes quickly with small change in b', manifested as rapid oscillation of deflection angle θ once it is mapped back to $[0, \pi]$. This behavior resembles the divergence of deflection angle near critical impact parameter in the context of general relativity.


$\theta \sim b'$ for $z < 0$
θ (rad)
b'
$p' = 1/5$
$p' = 1/2$
$p' = 1$
$p' = 2$
$p' = 5$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