1. **Prob 7.14**

(a) From the Maxwell equation

$$\nabla\times\mathbf{E}-i\omega\mathbf{B}=0 \qquad \nabla\times\mathbf{H}+i\omega\mathbf{D}=0 \tag{1}$$

we have the equation for $\mathbf{E}$:

$$\nabla\times(\nabla\times\mathbf{E})-\omega^2\mu_0\epsilon(z)\mathbf{E}=0 \quad\Longrightarrow\quad \nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}-\omega^2\mu_0\epsilon(z)\mathbf{E}=0 \tag{2}$$

If we assume the electric field has the form

$$\mathbf{E}(\mathbf{x},t)=\left[\hat{\mathbf{x}}E_x(z)+\hat{\mathbf{y}}E_y(z)+\hat{\mathbf{z}}E_z(z)\right]e^{ikx-i\omega t} \tag{3}$$

the equation $\nabla\cdot\mathbf{D}=0$ requires

$$0=\nabla\cdot[\epsilon(z)\mathbf{E}]=\frac{d\epsilon(z)}{dz}E_z e^{ikx}+\epsilon(z)\nabla\cdot\mathbf{E} \quad\Longrightarrow\quad \nabla\cdot\mathbf{E}=-\frac{1}{\epsilon}\frac{d\epsilon}{dz}E_z e^{ikx} \tag{4}$$

Plugging (4) into (2), we have

$$\nabla\left(\frac{1}{\epsilon}\frac{d\epsilon}{dz}E_z e^{ikx}\right)+\nabla^2\mathbf{E}+\omega^2\mu_0\epsilon\mathbf{E}=0 \tag{5}$$

or by components:

$$\hat{\mathbf{x}}: \qquad ik\cdot\frac{1}{\epsilon}\frac{d\epsilon}{dz}E_z+\frac{d^2E_x}{dz^2}-k^2E_x+\omega^2\mu_0\epsilon E_x=0 \tag{6}$$

$$\hat{\mathbf{y}}: \qquad \frac{d^2E_y}{dz^2}-k^2E_y+\omega^2\mu_0\epsilon E_y=0 \tag{7}$$

$$\hat{\mathbf{z}}: \qquad \frac{d}{dz}\left(\frac{1}{\epsilon}\frac{d\epsilon}{dz}E_z\right)+\frac{d^2E_z}{dz^2}-k^2E_z+\omega^2\mu_0\epsilon E_z=0 \tag{8}$$

There are a few comments here. First, notice $\nabla\cdot\mathbf{D}=0$ also requires

$$\nabla\cdot\left\{\epsilon(z)\left[\hat{\mathbf{x}}E_x(z)+\hat{\mathbf{y}}E_y(z)+\hat{\mathbf{z}}E_z(z)\right]e^{ikx}\right\}=0 \implies ik\epsilon E_x+\frac{d}{dz}(\epsilon E_z)=0 \implies E_x=\frac{i}{k}\cdot\frac{1}{\epsilon}\frac{d}{dz}(\epsilon E_z) \tag{9}$$

Plugging (9) into (6), we end up with

$$\begin{aligned} ik\cdot\frac{1}{\epsilon}\frac{d\epsilon}{dz}E_z+\frac{i}{k}\cdot\frac{d^2}{dz^2}\left[\frac{1}{\epsilon}\frac{d}{dz}(\epsilon E_z)\right]-ik\cdot\frac{1}{\epsilon}\frac{d}{dz}(\epsilon E_z)+\frac{i}{k}\cdot\omega^2\mu_0\frac{d}{dz}(\epsilon E_z)=0 &\quad\Longrightarrow \\ \omega^2\mu_0\frac{d}{dz}(\epsilon E_z)-k^2\frac{dE_z}{dz}+\frac{d^2}{dz^2}\left[\frac{1}{\epsilon}\frac{d}{dz}(\epsilon E_z)\right]=0 & \end{aligned} \tag{10}$$

which is the z-derivative of equation (8). This verifies that the assumed solution form (3) is consistent. Next, the horizontal polarization equation (7) for E_y can be written

$$\frac{d^2F(z)}{dz^2}+q^2(z)F(z)=0 \qquad \text{where } F(z)=E_y(z) \qquad q^2(z)=\omega^2\mu_0\epsilon(z)-k^2 \tag{11}$$

Lastly, the vertical polarization equation (8) for E_z can be written

$$\frac{d^2F(z)}{dz^2}+q^2(z)F(z)=0 \quad \text{where } F(z)=\sqrt{\frac{\epsilon}{\epsilon_0}}E_z(z) \qquad q^2(z)=\omega^2\mu_0\epsilon(z)+\frac{1}{2\epsilon}\frac{d^2\epsilon}{dz^2}-\frac{3}{4\epsilon^2}\left(\frac{d\epsilon}{dz}\right)^2-k^2 \tag{12}$$

(b) The advantage of writing differential equation of $F(z)$ in the form

$$\frac{d^2F(z)}{dz^2}+q^2(z)F(z)=0 \tag{13}$$

is it allows us to follow the standard WKB treatment, i.e., for region where $q^2(z) > 0$, the solution has the form

$$F(z) = \frac{1}{[q^2(z)]^{1/4}} \left\{ A \exp\left[i \int^z \sqrt{q^2(t)} dt \right] + B \exp\left[-i \int^z \sqrt{q^2(t)} dt \right] \right\} \tag{14}$$

and for region where $q^2(z) < 0$,

$$F(z) = \frac{1}{[-q^2(z)]^{1/4}} \left\{ C \exp\left[\int^z \sqrt{-q^2(t)} dt \right] + D \exp\left[\int^z \sqrt{-q^2(t)} dt \right] \right\} \tag{15}$$

For horizontal polarization with $k = 0$, by (11) and (7.59) we have

$$q^2(z) = \omega^2 \mu_0 \epsilon(z) = \omega^2 \mu_0 \epsilon_0 \left[1 - \frac{\omega_p^2(z)}{\omega^2} \right] = \mu_0 \epsilon_0 \left[\omega^2 - \omega_p^2(z) \right] \tag{16}$$

Similarly for vertical polarization, if we only keep the zeroth derivative of $\epsilon(z)$, we can also use (16) to approximate $q^2(z)$.
Note by (7.60)

$$\omega_p^2(z) \propto N(z) \tag{17}$$

and if $N(z)$ follows the curve shown in Figure 7.11, we can see that there is a critical height $z = h_0$ below which $\omega_p(z)$ is smaller than ω and consequently $q^2(z) > 0$. In this range, the field given by (14) is oscillating with varying z. But when the height is over h_0, where $q^2(z)$ becomes negative, the field $F(z)$ will be exponentially growing or decaying with z. Exponentially growing solution will be discarded for being unphysical, so the field will actually be exponentially decaying with $z > h_0$, explaning the total reflection.

(c) At height z, the index of refraction is

$$n(z) = \sqrt{\frac{\epsilon(z)}{\epsilon_0}} = \sqrt{1 - \frac{\omega_p^2(z)}{\omega^2}} \tag{18}$$

then the group velocity of the pulse is given by (7.88)

$$v_g(z) = \frac{c}{n(\omega) + \omega(dn/d\omega)} = c\sqrt{1 - \frac{\omega_p^2(z)}{\omega^2}} \tag{19}$$

Let h_0 be the height at which $q^2(z)$ becomes zero, i.e., h_0 is the root of

$$\omega^2 = \omega_p^2(z) = \frac{N(z) e^2}{m\epsilon_0} \tag{20}$$

then the time for the pulse to travel with group velocity from $z = 0$ to $z = h_0$ is

$$\frac{T}{2} = \int_0^{h_0} \frac{dz}{v_g(z)} = \int_0^{h_0} \frac{dz}{c} \left[1 - \frac{\omega_p^2(z)}{\omega^2} \right]^{-1/2} \tag{21}$$

So the effective height of the ionosphere is

$$h(\omega) = \frac{cT}{2} = \int_0^{h_0} dz \left[1 - \frac{\omega_p^2(z)}{\omega^2} \right] \tag{22}$$

2. **Prob 7.15**

(a) The group velocity derived in (19) is applicable here, so we can directly use (21) to answer this question, i.e.,

$$t(\omega) = \int_0^R \frac{dz}{v_g(z)} = \int_0^R \frac{dz}{c} \left[1 - \frac{\omega_p^2(z)}{\omega^2} \right]^{-1/2} \approx \int_0^R \frac{dz}{c} \left[1 + \frac{\omega_p^2(z)}{2\omega^2} \right] = \frac{R}{c} + \frac{1}{c} \int_0^R \frac{n(z) e^2}{2m\epsilon_0 \omega^2} dz \tag{23}$$

where the approximation takes place due to the assumption $\omega_p \ll \omega$. So

$$ct(\omega) = R + \frac{e^2}{2m\epsilon_0 \omega^2} \int_0^R n(z)\, dz \tag{24}$$

(b) For a linearly polarized field

$$\mathbf{E}(\mathbf{x},t) = \mathbf{e}_1 E_0 e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} \tag{25}$$

Its decomposition into the positive and negative helicity fields is through the relations

$$\mathbf{e}_\pm = \frac{\mathbf{e}_1 \pm i\mathbf{e}_2}{\sqrt{2}} \qquad \mathbf{e}_1 = \frac{\mathbf{e}_+ + \mathbf{e}_-}{\sqrt{2}} \qquad \mathbf{e}_2 = \frac{\mathbf{e}_+ - \mathbf{e}_-}{\sqrt{2}i} \tag{26}$$

(Here we use $\mathbf{e}_{\pm,1,2}$, rather than $\boldsymbol{\epsilon}_{\pm,1,2}$ to represent the polarization direction since ϵ is later used as dielectric constant.) I.e.,

$$\mathbf{E}(\mathbf{x},t) = \left(\frac{\mathbf{e}_+ + \mathbf{e}_-}{\sqrt{2}}\right) E_0 e^{i\mathbf{k}\cdot\mathbf{x} - \omega t} \tag{27}$$

If along the journey, the two circularly polarized fields had accumulated different phase factor θ_+, θ_-, at the end of the journey, the field's amplitude will be

$$\begin{aligned}\frac{E_0}{\sqrt{2}}\left(\mathbf{e}_+ e^{i\theta_+} + \mathbf{e}_- e^{i\theta_-}\right) &= E_0\left[\mathbf{e}_1\left(\frac{e^{i\theta_+} + e^{i\theta_-}}{2}\right) + i\mathbf{e}_2\left(\frac{e^{i\theta_+} - e^{i\theta_-}}{2}\right)\right] \\ &= E_0 e^{i(\theta_+ + \theta_-)/2}\left[\mathbf{e}_1 \cos\left(\frac{\theta_+ - \theta_-}{2}\right) - \mathbf{e}_2 \sin\left(\frac{\theta_+ - \theta_-}{2}\right)\right] \end{aligned} \tag{28}$$

Ignoring the global phase factor $e^{i(\theta_+ + \theta_-)/2}$, we see that the result is a linearly polarized field rotated by an angle $-(\theta_+ - \theta_-)/2$.

For the positive helicity and negative helicity field in a magnetic field parallel to the direction of propagation, Jackson (7.67) gives its dielectric constant (note Jackson used ϵ_-, ϵ_+ to represent the positive and negative helicity respectively, to avoid the confusion, we reverse that order here to keep it consistent with the definition in (26)),

$$\frac{\epsilon_\pm(z)}{\epsilon_0} = 1 - \frac{\omega_p^2(z)}{\omega\left[\omega \mp \omega_B(z)\right]} \tag{29}$$

This gives the index of refraction

$$n_\pm(z) = \sqrt{\frac{\epsilon_\pm(z)}{\epsilon_0}} = \sqrt{1 - \frac{\omega_p^2(z)}{\omega\left[\omega \mp \omega_B(z)\right]}} \approx 1 - \frac{1}{2}\frac{\omega_p^2(z)}{\omega^2}\left[1 \pm \frac{\omega_B(z)}{\omega}\right] \tag{30}$$

The accumulated phase for the two helicities are

$$\theta_\pm = \int_0^R \frac{n_\pm(z)\,\omega}{c} dz \tag{31}$$

which gives the rotation angle of the polarization

$$-\frac{(\theta_+ - \theta_-)}{2} = \frac{1}{2}\int_0^R \frac{\omega_p^2(z)\,\omega_B(z)}{\omega^2 c} dz = \frac{e^3}{2m^2\epsilon_0\omega^2 c}\int_0^R n(z) B_\parallel(z)\,dz \tag{32}$$

My result has the opposite sign compared to the claim of the problem. I think this is a result of Jackson using the reversed sign for $\epsilon_\mp$ in equation (7.67).

(c) (Not sure). If we can measure the energy transfer time $t(\omega)$ for a wave packet from the pulsar to us, then we can use (24) to evaluate the integral, hence the mean electron density. If we can have independent measure of the pulsar's polarization, then measuring the received light's polarization will tell us the mean magnetic field according to (32).