Our goal is to verify that for the potential given by (1.17)

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3 x' \tag{1}$$

its Laplacian is

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \tag{2}$$

I do not really understand why the text chose to divide the whole region into $r \le R$ and $r > R$. In particular, I cannot convince myself of the sentence "*If $\rho(\mathbf{x}')$ is such that (1.17) exists, the contribution to integral (1.30) from the exterior of the sphere will vanish like a^2 as $a \to 0$*".

In fact, we can use the "a-potential" method on $\nabla^2(1/r)$ to prove (1.31) directly, i.e.,

$$\nabla^2 \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) = -4\pi \delta(\mathbf{x}-\mathbf{x}') \tag{3}$$

then it follows that

$$\nabla^2 \Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \nabla^2 \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) d^3 x'$$

$$= \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \left[-4\pi \delta(\mathbf{x}-\mathbf{x}') \right] d^3 x' = -\frac{\rho(\mathbf{x})}{\epsilon_0} \tag{4}$$

To see (3), first notice that for any a,

$$\nabla \left(\frac{1}{\sqrt{r^2+a^2}} \right) = -\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(r^2+a^2)^{3/2}} \qquad \Longrightarrow \tag{5}$$

$$\nabla^2 \left(\frac{1}{\sqrt{r^2+a^2}} \right) = \frac{\partial}{\partial x}\left[\frac{-x}{(r^2+a^2)^{3/2}} \right] + \frac{\partial}{\partial y}\left[\frac{-y}{(r^2+a^2)^{3/2}} \right] + \frac{\partial}{\partial z}\left[\frac{-z}{(r^2+a^2)^{3/2}} \right]$$

$$= \left[\frac{-1}{(r^2+a^2)^{3/2}} + \frac{3x^2}{(r^2+a^2)^{5/2}} \right] + \left[\frac{-1}{(r^2+a^2)^{3/2}} + \frac{3y^2}{(r^2+a^2)^{5/2}} \right] + \left[\frac{-1}{(r^2+a^2)^{3/2}} + \frac{3z^2}{(r^2+a^2)^{5/2}} \right]$$

$$= \frac{-3a^2}{(r^2+a^2)^{5/2}} \tag{6}$$

For $r > 0$, we take $a = 0$, which gives

$$\nabla^2 \left(\frac{1}{r} \right) = 0 \qquad \text{for } r > 0 \tag{7}$$

And the volume integral

$$\int \nabla^2 \left(\frac{1}{\sqrt{r^2+a^2}} \right) d^3 x = -\int d\Omega \int_0^\infty \frac{3a^2}{(r^2+a^2)^{5/2}} r^2 dr \qquad \text{(let } r = a\tan\zeta)$$

$$= -4\pi \int_0^{\pi/2} \frac{3a^2 \left(a^2 \tan^2 \zeta \right) \left(a/\cos^2 \zeta \right) d\zeta}{a^5/\cos^5 \zeta}$$

$$= -4\pi \int_0^{\pi/2} 3\sin^2 \zeta \cos \zeta \, d\zeta = -4\pi \tag{8}$$

By taking the limit of $a \to 0$ and combining (7) and (8), we can conclude (3) by definition of the δ-function.

Even though we didn't use the text's method, it's still worth explaining the expansion of $\rho(\mathbf{x}')$ in the integral

$$-\frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \left[\frac{3a^2}{(r^2+a^2)^{5/2}} \right] d^3 x' = -\frac{1}{\epsilon_0} \int_0^R \frac{3a^2}{(r^2+a^2)^{5/2}} \left[\rho(\mathbf{x}) + \frac{r^2}{6} \nabla^2 \rho + \cdots \right] r^2 dr \tag{9}$$

With the assumption that $\rho(\mathbf{x}')$ is well behaved, we can expand $\rho(\mathbf{x}')$ around $\mathbf{x}$:

$$\rho(\mathbf{x}') = \rho(\mathbf{x}) + \left(\frac{\partial \rho}{\partial x} \Delta x + \frac{\partial \rho}{\partial y} \Delta y + \frac{\partial \rho}{\partial z} \Delta z \right) +$$
$$\frac{1}{2} \left[\frac{\partial^2 \rho}{\partial x^2} (\Delta x)^2 + \frac{\partial^2 \rho}{\partial y^2} (\Delta y)^2 + \frac{\partial^2 \rho}{\partial z^2} (\Delta z)^2 + 2 \frac{\partial^2 \rho}{\partial x \partial y} \Delta x \Delta y + 2 \frac{\partial^2 \rho}{\partial x \partial z} \Delta x \Delta z + 2 \frac{\partial^2 \rho}{\partial y \partial z} \Delta y \Delta z \right] +$$
$$\cdots \tag{10}$$

where all partial derivatives are evaluated at $\mathbf{x}$, hence can be taken out of the integral.

The integral of first order terms will vanish on the LHS of (9), because their null angular integration:

$$\int_\Omega d\Omega \Delta x \propto \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \sin\theta \cos\phi = 0 \tag{11}$$

$$\int_\Omega d\Omega \Delta y \propto \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \sin\theta \sin\phi = 0 \tag{12}$$

$$\int_\Omega d\Omega \Delta z \propto \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \cos\theta = 0 \tag{13}$$

Similarly, the cross terms of the second order will vanish too:

$$\int_\Omega d\Omega \Delta x \Delta y \propto \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \sin^2\theta \cos\phi \sin\phi = 0 \tag{14}$$

$$\int_\Omega d\Omega \Delta x \Delta z \propto \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \sin\theta \cos\phi \cos\theta = 0 \tag{15}$$

$$\int_\Omega d\Omega \Delta y \Delta z \propto \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \sin\theta \sin\phi \cos\theta = 0 \tag{16}$$

The angular integrations of the diagonal second order terms are

$$\int_\Omega d\Omega (\Delta x)^2 = r^2 \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \sin^2\theta \cos^2\phi$$
$$= r^2 \int_0^\pi \sin^3\theta \, d\theta \int_0^{2\pi} \frac{1 + \cos 2\phi}{2} d\phi$$
$$= \pi r^2 \int_{-1}^1 (1 - y^2) dy = \left(2 - \frac{2}{3} \right) \pi r^2 = \frac{4\pi r^2}{3} \tag{17}$$

$$\int_\Omega d\Omega (\Delta y)^2 = r^2 \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \sin^2\theta \sin^2\phi$$
$$= r^2 \int_0^\pi \sin^3\theta \, d\theta \int_0^{2\pi} \frac{1 - \cos 2\phi}{2} d\phi$$
$$= \pi r^2 \int_{-1}^1 (1 - y^2) dy = \left(2 - \frac{2}{3} \right) \pi r^2 = \frac{4\pi r^2}{3} \tag{18}$$

$$\int_\Omega d\Omega (\Delta z)^2 = r^2 \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \cos^2\theta$$
$$= 2\pi r^2 \int_{-1}^1 y^2 dy = \frac{4\pi r^2}{3} \tag{19}$$

which give the RHS of (9).