The current density vector can be written in Cartesian basis as

$$\mathbf{J} = J(r,\theta)\,\hat{\boldsymbol{\phi}} = J(r,\theta)(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}) \tag{1}$$

Thus if we define the *complex* current density $\widetilde{J}(\mathbf{x}) \equiv J(r,\theta)\,e^{-i\phi}$, we end up with $J_x = \mathrm{Im}\widetilde{J}$ and $J_y = \mathrm{Re}\widetilde{J}$.

Correspondingly, the vector potential

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{2}$$

can be written as $A_x = \mathrm{Im}\widetilde{A}$ and $A_y = \mathrm{Re}\widetilde{A}$, where

$$\widetilde{A} \equiv \frac{\mu_0}{4\pi}\int \frac{\widetilde{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' = \frac{\mu_0}{4\pi}\int \frac{J(r',\theta')\,e^{-i\phi'}}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{3}$$

With equation (3.70)

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*\left(\theta',\phi'\right)Y_{lm}(\theta,\phi) \tag{4}$$

(3) is turned into

$$\widetilde{A} = \mu_0\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\int\frac{r_<^l}{r_>^{l+1}}J\left(r',\theta'\right)e^{-i\phi'}Y_{lm}^*\left(\theta',\phi'\right)Y_{lm}(\theta,\phi)\,d^3x' \tag{5}$$

Due to the $e^{-i\phi'}$ factor, only $m = -1$ will contribute in the sum, hence

$$\begin{aligned}\widetilde{A} &= \mu_0\sum_{l=1}^{\infty}\frac{1}{2l+1}\int\frac{r_<^l}{r_>^{l+1}}J\left(r',\theta'\right)\left[\frac{2l+1}{4\pi}\frac{(l-1)!}{(l+1)!}\right]P_l^1\left(\cos\theta'\right)P_l^1(\cos\theta)\,e^{-i\phi}\,d^3x' \\ &= \frac{\mu_0}{4\pi}e^{-i\phi}\sum_{l=1}^{\infty}P_l^1(\cos\theta)\overbrace{\left[\frac{1}{l(l+1)}\int\frac{r_<^l}{r_>^{l+1}}J\left(r',\theta'\right)P_l^1\left(\cos\theta'\right)d^3x'\right]}^{X}\end{aligned} \tag{6}$$

where we take a note that the phase factor $e^{-i\phi}$ is obtained from the relation $Y_{l,-m} = (-1)^m Y_{lm}^*$.

For the "interior" where $r_< = r$ and $r_> = r'$, X of (6) becomes

$$X = -r^l\overbrace{\left[-\frac{1}{l(l+1)}\int r'^{-(l+1)}J\left(r',\theta'\right)P_l^1\left(\cos\theta'\right)d^3x'\right]}^{\equiv m_l} \tag{7}$$

Similarly, for exterior where $r_< = r'$ and $r_> = r$, we have

$$X = -r^{-(l+1)}\overbrace{\left[-\frac{1}{l(l+1)}\int r'^{l}J\left(r',\theta'\right)P_l^1\left(\cos\theta'\right)d^3x'\right]}^{\equiv \mu_l} \tag{8}$$

Thus the complex potential

$$\widetilde{A} = \begin{cases} -\dfrac{\mu_0}{4\pi}e^{-i\phi}\displaystyle\sum_{l=1}^{\infty}P_l^1(\cos\theta)\,r^l m_l & \text{for interior} \\ -\dfrac{\mu_0}{4\pi}e^{-i\phi}\displaystyle\sum_{l=1}^{\infty}P_l^1(\cos\theta)\,r^{-(l+1)}\mu_l & \text{for exterior}\end{cases} \tag{9}$$

gives the azimuthal component of the vector potential

$$A_\phi = A_y\cos\phi - A_x\sin\phi = \cos\phi\cdot\mathrm{Re}\widetilde{A} - \sin\phi\cdot\mathrm{Im}\widetilde{A} = \begin{cases} -\dfrac{\mu_0}{4\pi}\displaystyle\sum_{l=1}^{\infty}P_l^1(\cos\theta)\,r^l m_l & \text{for interior} \\ -\dfrac{\mu_0}{4\pi}\displaystyle\sum_{l=1}^{\infty}P_l^1(\cos\theta)\,r^{-(l+1)}\mu_l & \text{for exterior}\end{cases} \tag{10}$$

The minus sign in m_l and μ_l is to cancel the conventional minus sign from P_l^1 (see eq 3.49). Also note l starts from 1 (we dropped $l = 0$ contribution when we selected $m = -1$), so there is no contribution of magnetic monopole, as expected.