

1. The  $i$ -th component of the force on the charge is

$$F_i = \int \rho(\mathbf{x}) E_i(\mathbf{x}) d^3x \quad (1)$$

Expanding  $E_i(\mathbf{x})$  around origin gives

$$E_i(\mathbf{x}) = E_i(0) + \mathbf{x} \cdot (\nabla E_i)_0 + \frac{1}{2} \sum_{j,k} x_j x_k \left( \frac{\partial^2 E_i}{\partial x_j \partial x_k} \right)_0 + \dots \quad (2)$$

Inserting (2) into (1) gives the contributions of the  $i$ -th component of the force from monopole, dipole, quadrupole etc:

- contribution from monopole:

$$F_i^{(0)} = \int \rho(\mathbf{x}) E_i(0) d^3x = q E_i(0) \quad (3)$$

- contribution from dipole:

$$\begin{aligned} F_i^{(1)} &= \int \rho(\mathbf{x}) \mathbf{x} \cdot (\nabla E_i)_0 d^3x = \mathbf{p} \cdot (\nabla E_i)_0 = \sum_j p_j \left( \frac{\partial E_i}{\partial x_j} \right)_0 \\ &= \sum_j p_j \left( -\frac{\partial^2 \Phi}{\partial x_j \partial x_i} \right)_0 = \sum_j p_j \left( \frac{\partial E_j}{\partial x_i} \right)_0 = \frac{\partial}{\partial x_i} [\mathbf{p} \cdot \mathbf{E}(\mathbf{x})]_0 \end{aligned} \quad (4)$$

- contribution from quadrupole:

$$\begin{aligned} F_0^{(2)} &= \frac{1}{2} \sum_{j,k} \int \rho(\mathbf{x}) x_j x_k \left( \frac{\partial^2 E_i}{\partial x_j \partial x_k} \right)_0 d^3x \\ &= \frac{1}{6} \sum_{j,k} \int \rho(\mathbf{x}) (3x_j x_k - r^2 \delta_{jk}) \left( \frac{\partial^2 E_i}{\partial x_j \partial x_k} \right)_0 d^3x + \frac{1}{6} \sum_j \int \rho(\mathbf{x}) r^2 \left( \frac{\partial^2 E_i}{\partial x_j^2} \right)_0 d^3x \\ &= \frac{1}{6} \sum_{j,k} Q_{jk} \left( \frac{\partial^2 E_i}{\partial x_j \partial x_k} \right)_0 + \frac{1}{6} \int \rho(\mathbf{x}) r^2 \underbrace{\left[ \sum_j \left( \frac{\partial^2 E_i}{\partial x_j^2} \right)_0 \right]}_{-(\partial/\partial x_i)(\nabla^2 \Phi)_0=0} d^3x \\ &= \frac{1}{6} \sum_{j,k} Q_{jk} \left( -\frac{\partial^3 \Phi}{\partial x_i \partial x_j \partial x_k} \right)_0 = \frac{\partial}{\partial x_i} \left[ \frac{1}{6} \sum_{j,k} Q_{jk} \left( \frac{\partial E_j}{\partial x_k} \right)_0 \right] \end{aligned} \quad (5)$$

Putting this  $i$ -th component form back into vector form, we finally get

$$\mathbf{F} = q\mathbf{E}(0) + \{\nabla[\mathbf{p} \cdot \mathbf{E}(\mathbf{x})]\}_0 + \left\{ \nabla \left[ \frac{1}{6} \sum_{j,k} Q_{jk} \frac{\partial E_j}{\partial x_k} \right] \right\}_0 + \dots \quad (6)$$

The relation to equation (4.24) embodies the usual relation that the force is the negative gradient of potential energy.

2. The  $i$ -th component of the total torque is

$$\begin{aligned} N_i &= \sum_{j,k} \epsilon_{ijk} \int \rho(\mathbf{x}) x_j E_k(\mathbf{x}) d^3x \\ &= \sum_{j,k} \epsilon_{ijk} \left[ \int \rho(\mathbf{x}) x_j E_k(0) d^3x + \int \rho(\mathbf{x}) x_j \mathbf{x} \cdot (\nabla E_k)_0 d^3x + \dots \right] \\ &= \underbrace{\sum_{j,k} \epsilon_{ijk} p_j E_k(0)}_{N_i^{(1)}} + \underbrace{\sum_{j,k} \epsilon_{ijk} \int \rho(\mathbf{x}) x_j \sum_l x_l \left( \frac{\partial E_k}{\partial x_l} \right)_0 d^3x}_{N_i^{(2)}} + \dots \end{aligned} \quad (7)$$

where  $N_i^{(1)}$  is readily recognized as

$$N_i^{(1)} = [\mathbf{p} \times \mathbf{E}(0)]_i \quad (8)$$

Furthermore,

$$\begin{aligned} N_i^{(2)} &= \sum_{j,k,l} \epsilon_{ijk} \int \rho(\mathbf{x}) x_j x_l \left( \frac{\partial E_k}{\partial x_l} \right)_0 d^3x \\ &= \frac{1}{3} \sum_{j,k,l} \epsilon_{ijk} \int \rho(\mathbf{x}) (3x_j x_l - r^2 \delta_{jl}) \left( \frac{\partial E_k}{\partial x_l} \right)_0 d^3x + \underbrace{\frac{1}{3} \sum_{j,k} \epsilon_{ijk} \int \rho(\mathbf{x}) r^2 \left( \frac{\partial E_k}{\partial x_j} \right)_0 d^3x}_{\propto (\nabla \times \mathbf{E})(0)_i = 0} \\ &= \frac{1}{3} \sum_{j,k,l} \epsilon_{ijk} Q_{jl} \left( \frac{\partial E_k}{\partial x_l} \right)_0 \quad \left( \text{use } \frac{\partial E_k}{\partial x_l} = \frac{\partial E_l}{\partial x_k} \right) \\ &= \frac{1}{3} \sum_{j,k} \epsilon_{ijk} \frac{\partial}{\partial x_k} \left[ \sum_l Q_{jl} E_l(\mathbf{x}) \right]_0 \end{aligned} \quad (9)$$