

## 1. Verification that (5.127) is the solution of (5.126) via Weber–Schafheitlin integral

For the dual integral equations (5.126)

$$\int_0^\infty dy g(y) J_n(yx) = x^n \quad \text{for } 0 \leq x < 1 \quad (1)$$

$$\int_0^\infty dy y g(y) J_n(yx) = 0 \quad \text{for } 1 \leq x \quad (2)$$

It was claimed that (5.127) is a solution

$$g(y) = \frac{2\Gamma(n+1)}{\sqrt{\pi}\Gamma(n+1/2)} j_n(y) = \frac{\Gamma(n+1)}{\Gamma(n+1/2)} \left(\frac{2}{y}\right)^{1/2} J_{n+1/2}(y) \quad (3)$$

Let's verify it via the Weber–Schafheitlin integral (reference [equation 10.22.56 on dlmf.nist.gov](https://dlmf.nist.gov/10.22.56))

for  $0 < a < b, \operatorname{Re}(\mu + \nu + 1) > \operatorname{Re}(\lambda) > -1$  :

$$\int_0^\infty \frac{J_\mu(at) J_\nu(bt)}{t^\lambda} dt = \frac{a^\mu \Gamma\left(\frac{\nu}{2} + \frac{\mu}{2} - \frac{\lambda}{2} + \frac{1}{2}\right)}{2^\lambda b^{\mu-\lambda+1} \Gamma\left(\frac{\nu}{2} - \frac{\mu}{2} + \frac{\lambda}{2} + \frac{1}{2}\right)} F\left(\frac{\mu + \nu - \lambda + 1}{2}, \frac{\mu - \nu - \lambda + 1}{2}; \mu + 1; \frac{a^2}{b^2}\right) \quad (4)$$

where  $F$  is the hypergeometric function

$$F(a, b; c; z) = 1 + \frac{ab}{c}z + \frac{1}{2!} \frac{a(a+1)b(b+1)}{c(c+1)}z^2 + \dots \quad (5)$$

- To see (1), insert  $a = x, b = 1, \mu = n, \nu = n + 1/2, \lambda = 1/2$  into (4), we have

$$\int_0^\infty \frac{J_n(xy) J_{n+1/2}(y) dy}{\sqrt{y}} = \frac{x^n \Gamma(n+1/2)}{\sqrt{2}\Gamma(1)} \overbrace{F\left(n + \frac{1}{2}, 0; n+1; x^2\right)}^{=1} \quad (6)$$

Thus we see that (3) satisfies (1) except for this  $\Gamma(n+1)$  factor (**which I think is a mistake on the book**, but fortunately for the  $n = 1$  case, it doesn't impact the subsequent discussion).

- For (2), we need  $a = 1, b = x, \mu = n + 1/2, \nu = n, \lambda = -1/2$ , with which the denominator of (4) has  $\Gamma(0) = \infty$ , hence (4) vanishes, which satisfies (2).

## 2. Closed-form formula for (5.129)

The text gives the integral representation of the additional potential in equation (5.129)

$$\Phi^{(1)}(\mathbf{x}) = \frac{2H_0 a^2}{\pi} \overbrace{\int_0^\infty dk j_1(ka) e^{-k|z|} J_1(k\rho) \sin \phi}^I \quad (7)$$

Now let's calculate the integral  $I$  explicitly (We will treat  $z > 0$  region only, the  $z < 0$  region can be obtained by the reversal of sign in (5.123).)

With the relation

$$j_1(x) = -j'_0(x) \quad (8)$$

$I$  can be written as

$$\begin{aligned} I &= \int_0^\infty dk [-j'_0(ka)] e^{-kz} J_1(k\rho) \\ &= \int_0^\infty dk \left[ -\frac{dj_0(ka)}{adk} \right] e^{-kz} J_1(k\rho) \\ &= -\frac{1}{a} j_0(ka) e^{-kz} J_1(k\rho) \Big|_0^\infty + \frac{1}{a} \int_0^\infty dk j_0(ka) \frac{d}{dk} [e^{-kz} J_1(k\rho)] \end{aligned} \quad (9)$$

The first term clearly vanishes at both  $k = 0$  and  $k = \infty$ , so

$$\begin{aligned} I &= \frac{1}{a} \int_0^\infty dk j_0(ka) \frac{d}{dk} [e^{-kz} J_1(k\rho)] \\ &= \frac{1}{a} \left[ -z \int_0^\infty dk j_0(ka) e^{-kz} J_1(k\rho) + \rho \int_0^\infty dk j_0(ka) e^{-kz} J_1'(k\rho) \right] \end{aligned} \quad (10)$$

With the recurrence relation (see [equation 10.6.1 on dlmf.nist.gov](#))

$$J_1'(x) = \frac{1}{2} [J_0(x) - J_2(x)] \quad (11)$$

we have

$$I = \frac{1}{a} \left[ -z \int_0^\infty dk j_0(ka) e^{-kz} J_1(k\rho) + \frac{\rho}{2} \int_0^\infty dk j_0(ka) e^{-kz} J_0(k\rho) - \frac{\rho}{2} \int_0^\infty dk j_0(ka) e^{-kz} J_2(k\rho) \right] \quad (12)$$

Define

$$K_n \equiv \int_0^\infty dk j_0(ka) e^{-kz} J_n(k\rho) \quad (13)$$

(12) became

$$I = \frac{1}{a} \left( -zK_1 + \frac{\rho}{2}K_0 - \frac{\rho}{2}K_2 \right) \quad (14)$$

Now let's focus on  $K_n$ ,

$$\begin{aligned} K_n &= \int_0^\infty dk \frac{\sin ka}{ka} e^{-kz} J_n(k\rho) & (s \equiv z - ia) \\ &= \frac{1}{a} \operatorname{Im} \left[ \int_0^\infty e^{-sk} \frac{J_n(k\rho)}{k} dk \right] \\ &= \frac{1}{a} \operatorname{Im} \left[ \mathcal{L} \left\{ \frac{J_n(k\rho)}{k} \right\} (s) \right] \end{aligned} \quad (15)$$

(a) **Calculation of  $K_1, K_2$**

For  $n = 1, 2$ , recall [equation 10.6.2 on dlmf.nist.gov](#)

$$J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x) \quad (16)$$

and the Laplace transform property

$$\mathcal{L} \left\{ \frac{df(t)}{dt} \right\} (s) = s \mathcal{L} \{ f(t) \} (s) - f(0^-) \quad (17)$$

These yield

$$\begin{aligned} \mathcal{L} \left\{ \frac{J_n(k\rho)}{k} \right\} (s) &= \frac{\rho}{n} \mathcal{L} \{ J_{n-1}(k\rho) - J_n'(k\rho) \} (s) \\ &= \frac{\rho}{n} \left[ \mathcal{L} \{ J_{n-1}(k\rho) \} (s) - \mathcal{L} \left\{ \frac{dJ_n(k\rho)}{\rho dk} \right\} (s) \right] \\ &= \frac{\rho}{n} \mathcal{L} \{ J_{n-1}(k\rho) \} (s) - \frac{1}{n} [s \mathcal{L} \{ J_n(k\rho) \} (s) - J_n(0^-)] \\ &= \frac{\rho}{n} \mathcal{L} \{ J_{n-1}(k\rho) \} (s) - \frac{s}{n} \mathcal{L} \{ J_n(k\rho) \} (s) & \text{for } n = 1, 2 \end{aligned} \quad (18)$$

Substituting in (18) with the well known Laplace transforms of  $J_n(k\rho)$  ([see wikiproof](#))

$$\mathcal{L} \{ J_n(k\rho) \} (s) = \frac{(\sqrt{s^2 + \rho^2} - s)^n}{\rho^n \sqrt{s^2 + \rho^2}} \quad (19)$$

gives

$$\mathcal{L}\left\{\frac{J_1(k\rho)}{k}\right\}(s) = \frac{\rho}{\sqrt{s^2 + \rho^2}} - s \cdot \frac{\sqrt{s^2 + \rho^2} - s}{\rho \sqrt{s^2 + \rho^2}} = \frac{\sqrt{s^2 + \rho^2} - s}{\rho} \quad (20)$$

$$\begin{aligned} \mathcal{L}\left\{\frac{J_2(k\rho)}{k}\right\}(s) &= \frac{\rho}{2} \cdot \frac{\sqrt{s^2 + \rho^2} - s}{\rho \sqrt{s^2 + \rho^2}} - \frac{s}{2} \cdot \frac{(\sqrt{s^2 + \rho^2} - s)^2}{\rho^2 \sqrt{s^2 + \rho^2}} \\ &= \left(\frac{\sqrt{s^2 + \rho^2} - s}{2\rho^2}\right) \left[\frac{\rho^2 - s(\sqrt{s^2 + \rho^2} - s)}{\sqrt{s^2 + \rho^2}}\right] \\ &= \frac{(\sqrt{s^2 + \rho^2} - s)^2}{2\rho^2} \end{aligned} \quad (21)$$

With (20) and (21) plugged back into (15), we have

$$K_1 = \frac{1}{a} \operatorname{Im} \left[ \mathcal{L} \left\{ \frac{J_1(k\rho)}{k} \right\} (s) \right] = \frac{1}{a\rho} \left( \operatorname{Im} \sqrt{s^2 + \rho^2} - \operatorname{Im} s \right) = \frac{1}{a\rho} \left( \operatorname{Im} \sqrt{s^2 + \rho^2} + a \right) \quad (22)$$

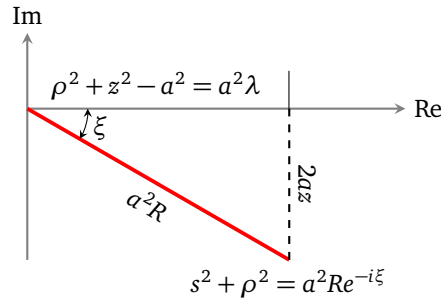
$$K_2 = \frac{1}{a} \operatorname{Im} \left[ \mathcal{L} \left\{ \frac{J_2(k\rho)}{k} \right\} (s) \right] = \frac{1}{2a\rho^2} \operatorname{Im} \left( \sqrt{s^2 + \rho^2} - s \right)^2 \quad (23)$$

Define

$$\lambda = \frac{z^2 + \rho^2 - a^2}{a^2} \quad R = \sqrt{\lambda^2 + \frac{4z^2}{a^2}} \quad (24)$$

then we have

$$\sqrt{\rho^2 + s^2} = (\rho^2 + z^2 - a^2 - 2azi)^{1/2} = a\sqrt{R}e^{-i\xi/2} \quad (25)$$



From the diagram above, it's clear that

$$\operatorname{Im} \sqrt{s^2 + \rho^2} = -a\sqrt{R} \sin \frac{\xi}{2} \quad (26)$$

$\sin \xi/2$  and  $\cos \xi/2$  can be found via

$$1 - 2\sin^2 \frac{\xi}{2} = \cos \xi = \frac{\lambda}{R} \quad \Rightarrow \quad \sin \frac{\xi}{2} = \sqrt{\frac{R - \lambda}{2R}} \quad \cos \frac{\xi}{2} = \sqrt{\frac{R + \lambda}{2R}} \quad (27)$$

$K_1$  is obtained by

$$K_1 = \frac{1}{a\rho} \left( -a\sqrt{\frac{R - \lambda}{2}} + a \right) = \frac{1}{\rho} \left( 1 - \sqrt{\frac{R - \lambda}{2}} \right) \quad (28)$$

For (23),

$$\begin{aligned} K_2 &= \frac{1}{2a\rho^2} \operatorname{Im} \left( \sqrt{s^2 + \rho^2} - s \right)^2 = \frac{1}{2a\rho^2} \cdot 2 \operatorname{Re} \left( \sqrt{s^2 + \rho^2} - s \right) \cdot \operatorname{Im} \left( \sqrt{s^2 + \rho^2} - s \right) \\ &= \frac{1}{a\rho^2} \left( a\sqrt{R} \cos \frac{\xi}{2} - z \right) \left( -a\sqrt{R} \sin \frac{\xi}{2} + a \right) = \frac{1}{a\rho^2} \left( a\sqrt{R} \sqrt{\frac{R + \lambda}{2R}} - z \right) \left( -a\sqrt{R} \sqrt{\frac{R - \lambda}{2R}} + a \right) \\ &= \frac{1}{a\rho^2} \left( -a^2 \frac{\sqrt{R^2 - \lambda^2}}{2} - az + az \sqrt{\frac{R - \lambda}{2}} + a^2 \sqrt{\frac{R + \lambda}{2}} \right) \\ &= \frac{1}{\rho^2} \left( -2z + z \sqrt{\frac{R - \lambda}{2}} + a \sqrt{\frac{R + \lambda}{2}} \right) \end{aligned} \quad (29)$$

(b) **Calculation of  $K_0$**

With  $s = z - ia$ , let

$$F(s) = \int_0^\infty e^{-sk} \frac{J_0(k\rho)}{k} dk \quad (30)$$

then by definition  $K_0 = \text{Im } F(s)/a$ .

Notice

$$F'(s) = - \int_0^\infty e^{-sk} J_0(k\rho) dk = -\mathcal{L}\{J_0(k\rho)\}(s) = -\frac{1}{\sqrt{s^2 + \rho^2}} \quad (31)$$

Thus  $F(s)$  is readily solvable as (reference [WolframAlpha](#))

$$\begin{aligned} F(s) &= -\tanh^{-1}\left(\frac{s}{\sqrt{\rho^2 + s^2}}\right) \\ &= \frac{1}{2} \ln\left(1 - \frac{s}{\sqrt{\rho^2 + s^2}}\right) - \frac{1}{2} \ln\left(1 + \frac{s}{\sqrt{\rho^2 + s^2}}\right) \\ &= \frac{1}{2} \ln\left(\frac{\sqrt{\rho^2 + s^2} - s}{\sqrt{\rho^2 + s^2} + s}\right) \\ &= \frac{1}{2} \ln\left[\frac{(\sqrt{\rho^2 + s^2} - s)^2}{\rho^2}\right] \\ &= \ln\left(\frac{\sqrt{\rho^2 + s^2} - s}{\rho}\right) \end{aligned} \quad (32)$$

The imaginary part of  $F(s)$  is just the argument of the complex number  $\sqrt{\rho^2 + s^2} - s$ , i.e.,

$$\begin{aligned} \text{Im}[F(s)] &= \text{Arg}\left(\sqrt{\rho^2 + s^2} - s\right) = \text{Arg}\left[a\sqrt{R}\left(\cos\frac{\xi}{2} - i\sin\frac{\xi}{2}\right) - z + ia\right] \\ &= \tan^{-1}\left(\frac{a - a\sqrt{R}\sin\frac{\xi}{2}}{a\sqrt{R}\cos\frac{\xi}{2} - z}\right) = \tan^{-1}\left(\frac{a - a\sqrt{R}\sqrt{\frac{R-\lambda}{2R}}}{a\sqrt{R}\sqrt{\frac{R+\lambda}{2R}} - z}\right) \\ &= \tan^{-1}\left(\frac{a - a\sqrt{\frac{R-\lambda}{2}}}{a\sqrt{\frac{R+\lambda}{2}} - z}\right) = \tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right) \end{aligned} \quad (33)$$

Finally, with  $K_0, K_1, K_2$  inserted to (14) and (7), we get

$$\begin{aligned} \Phi^{(1)}(\mathbf{x}) &= \frac{2H_0 a}{\pi} \left(-zK_1 + \frac{\rho}{2}K_0 - \frac{\rho}{2}K_2\right) \sin\phi \\ &= \frac{2H_0 a}{\pi} \left[-\frac{z}{\rho} \left(1 - \sqrt{\frac{R-\lambda}{2}}\right) + \frac{\rho}{2a} \tan^{-1}\sqrt{\frac{2}{R+\lambda}} - \frac{1}{2\rho} \left(-2z + z\sqrt{\frac{R-\lambda}{2}} + a\sqrt{\frac{R+\lambda}{2}}\right)\right] \sin\phi \\ &= \frac{2H_0 a}{\pi} \left(\overbrace{\frac{z}{2\rho} \sqrt{\frac{R-\lambda}{2}}}^A + \underbrace{\frac{\rho}{2a} \tan^{-1}\sqrt{\frac{2}{R+\lambda}}}_B - \overbrace{\frac{a}{2\rho} \sqrt{\frac{R+\lambda}{2}}}^C\right) \sin\phi \end{aligned} \quad (34)$$

### 3. Asymptotic behavior with large $r$ , equation (5.129)

Let the observation point  $\mathbf{x} = (\rho, \phi, z)$  be equivalently expressed in spherical coordinate  $(r, \theta, \phi)$ , then

$$z = r \cos \theta \quad \rho = r \sin \theta \quad (35)$$

The benefit of using this form is so that we can write (34) in increasing negative powers of  $r$ .

By definition,

$$\lambda = \left(\frac{r}{a}\right)^2 - 1 \quad (36)$$

$$R = \sqrt{\left[\left(\frac{r}{a}\right)^2 - 1\right]^2 + \frac{4z^2}{a^2}} = \sqrt{\left(\frac{r}{a}\right)^4 - 2\left(\frac{r}{a}\right)^2 + 1 + \frac{4z^2}{a^2}} = \left(\frac{r}{a}\right)^2 \sqrt{1 - 2\left(\frac{a}{r}\right)^2 + \left(\frac{a}{r}\right)^4 + \frac{4a^2 \cos^2 \theta}{r^2}} \quad (37)$$

Using the Taylor expansion

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots \quad (38)$$

we have

$$\begin{aligned} R &= \left(\frac{r}{a}\right)^2 \left\{ 1 + \frac{1}{2} \left[ -2\left(\frac{a}{r}\right)^2 + \left(\frac{a}{r}\right)^4 + \frac{4a^2 \cos^2 \theta}{r^2} \right] - \frac{1}{8} \left[ 4\left(\frac{a}{r}\right)^4 - \frac{16a^4 \cos^2 \theta}{r^4} + \frac{16a^4 \cos^4 \theta}{r^4} \right] + O\left(\frac{1}{r^6}\right) \right\} \\ &= \left(\frac{r}{a}\right)^2 - 1 + \frac{1}{2} \left(\frac{a}{r}\right)^2 + 2 \cos^2 \theta - \frac{1}{2} \left(\frac{a}{r}\right)^2 + \frac{2a^2 \cos^2 \theta}{r^2} - \frac{2a^2 \cos^4 \theta}{r^2} + O\left(\frac{1}{r^4}\right) \\ &= \left(\frac{r}{a}\right)^2 - 1 + 2 \cos^2 \theta + \frac{2a^2 \cos^2 \theta \sin^2 \theta}{r^2} + O\left(\frac{1}{r^4}\right) \end{aligned} \quad (39)$$

hence

$$\begin{aligned} \frac{R-\lambda}{2} &= \cos^2 \theta + \frac{a^2 \cos^2 \theta \sin^2 \theta}{r^2} + O\left(\frac{1}{r^4}\right) \implies \\ \sqrt{\frac{R-\lambda}{2}} &= \cos \theta \left[ 1 + \frac{a^2 \sin^2 \theta}{r^2} + O\left(\frac{1}{r^4}\right) \right]^{1/2} = \cos \theta \left[ 1 + \frac{1}{2} \frac{a^2 \sin^2 \theta}{r^2} + O\left(\frac{1}{r^4}\right) \right] \end{aligned} \quad (40)$$

The dominating orders of  $A$  in (34) are

$$A = \frac{\cos \theta}{2 \sin \theta} \cdot \cos \theta \left[ 1 + \frac{1}{2} \frac{a^2 \sin^2 \theta}{r^2} + O\left(\frac{1}{r^4}\right) \right] = \frac{\cos^2 \theta}{2 \sin \theta} + \frac{1}{4} \frac{a^2 \sin \theta \cos^2 \theta}{r^2} + O\left(\frac{1}{r^4}\right) \quad (41)$$

Notice

$$\frac{2}{R+\lambda} = \frac{a^2}{z^2} \left( \frac{R-\lambda}{2} \right) \quad (42)$$

gives

$$\tan^{-1} \sqrt{\frac{2}{R+\lambda}} = \tan^{-1} \left( \frac{a}{z} \sqrt{\frac{R-\lambda}{2}} \right) = \tan^{-1} \left( \frac{a}{r \cos \theta} \sqrt{\frac{R-\lambda}{2}} \right) = \tan^{-1} \left[ \frac{a}{r} + \frac{1}{2} \frac{a^3 \sin^2 \theta}{r^3} + O\left(\frac{1}{r^5}\right) \right] \quad (43)$$

Using the Taylor expansion

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots \quad (44)$$

we can write

$$\tan^{-1} \sqrt{\frac{2}{R+\lambda}} = \frac{a}{r} + \frac{1}{2} \frac{a^3 \sin^2 \theta}{r^3} - \frac{1}{3} \left(\frac{a}{r}\right)^3 + O\left(\frac{1}{r^5}\right) \quad (45)$$

which gives the  $B$  term in (34)

$$B = \frac{r \sin \theta}{2a} \left[ \frac{a}{r} + \frac{1}{2} \frac{a^3 \sin^2 \theta}{r^3} - \frac{1}{3} \left(\frac{a}{r}\right)^3 + O\left(\frac{1}{r^5}\right) \right] = \frac{\sin \theta}{2} + \frac{1}{4} \frac{a^2 \sin^3 \theta}{r^2} - \frac{1}{6} \frac{a^2 \sin \theta}{r^2} + O\left(\frac{1}{r^4}\right) \quad (46)$$

Lastly, from (39)

$$\begin{aligned} \frac{R+\lambda}{2} &= \left(\frac{r}{a}\right)^2 - 1 + \cos^2 \theta + O\left(\frac{1}{r^2}\right) \implies \\ \sqrt{\frac{R+\lambda}{2}} &= \left(\frac{r}{a}\right) \left[ 1 - \sin^2 \theta \left(\frac{a}{r}\right)^2 + O\left(\frac{1}{r^4}\right) \right]^{1/2} = \frac{r}{a} - \frac{1}{2} \frac{a \sin^2 \theta}{r} + O\left(\frac{1}{r^3}\right) \end{aligned} \quad (47)$$

which gives

$$C = \frac{a}{2r \sin \theta} \left[ \frac{r}{a} - \frac{1}{2} \frac{a \sin^2 \theta}{r} + O\left(\frac{1}{r^3}\right) \right] = \frac{1}{2 \sin \theta} - \frac{1}{4} \frac{a^2 \sin \theta}{r^2} + O\left(\frac{1}{r^4}\right) \quad (48)$$

Insert  $A, B, C$  back into (34) and ignore the  $O(r^{-4})$  terms,

$$\begin{aligned} A + B - C &\approx \frac{\cos^2 \theta}{2 \sin \theta} + \frac{a^2 \sin \theta \cos^2 \theta}{4r^2} + \frac{\sin \theta}{2} + \frac{a^2 \sin^3 \theta}{4r^2} - \frac{a^2 \sin \theta}{6r^2} - \frac{1}{2 \sin \theta} + \frac{a^2 \sin \theta}{4r^2} \approx \frac{a^2 \sin \theta}{3r^2} \implies \\ \Phi^{(1)}(\mathbf{x}) &\approx \frac{2H_0 a^3}{3\pi} \cdot \frac{r \sin \theta \sin \phi}{r^3} = \frac{2H_0 a^3}{3\pi} \frac{y}{r^3} \end{aligned} \quad (49)$$

#### 4. Problem 5.24, tangential field on the surface $z = 0^+$

At  $z = 0^+$ ,

$$\lambda = R = \frac{\rho^2}{a^2} - 1 \quad (50)$$

By (34),

$$\begin{aligned} \Phi^{(1)}(\rho, \phi, z = 0) &= \frac{2H_0 a}{\pi} \left( \frac{\rho}{2a} \tan^{-1} \frac{a}{\sqrt{\rho^2 - a^2}} - \frac{a}{2\rho} \cdot \frac{\sqrt{\rho^2 - a^2}}{a} \right) \sin \phi \\ &= \frac{2H_0 a}{\pi} \left[ \frac{\rho}{2a} \sin^{-1} \left( \frac{a}{\rho} \right) - \frac{\sqrt{\rho^2 - a^2}}{2\rho} \right] \sin \phi \end{aligned} \quad (51)$$

The derivatives are

$$\begin{aligned} \frac{\partial \Phi^{(1)}}{\partial \rho} &= \frac{2H_0 a}{\pi} \left[ \frac{1}{2a} \sin^{-1} \left( \frac{a}{\rho} \right) + \frac{\rho}{2a} \frac{1}{\sqrt{1 - \left( \frac{a}{\rho} \right)^2}} \left( -\frac{a}{\rho^2} \right) + \frac{\sqrt{\rho^2 - a^2}}{2\rho^2} - \frac{1}{2\rho} \frac{\rho}{\sqrt{\rho^2 - a^2}} \right] \sin \phi \\ &= \frac{2H_0 a}{\pi} \left[ \frac{1}{2a} \sin^{-1} \left( \frac{a}{\rho} \right) - \frac{1}{2\sqrt{\rho^2 - a^2}} + \frac{\sqrt{\rho^2 - a^2}}{2\rho^2} - \frac{1}{2\sqrt{\rho^2 - a^2}} \right] \sin \phi \\ &= \frac{2H_0 a}{\pi} \left[ \frac{1}{2a} \sin^{-1} \left( \frac{a}{\rho} \right) - \frac{1}{\sqrt{\rho^2 - a^2}} + \frac{\sqrt{\rho^2 - a^2}}{2\rho^2} \right] \sin \phi \end{aligned} \quad (52)$$

$$\frac{1}{\rho} \frac{\partial \Phi^{(1)}}{\partial \phi} = \frac{2H_0 a}{\pi} \left[ \frac{1}{2a} \sin^{-1} \left( \frac{a}{\rho} \right) - \frac{\sqrt{\rho^2 - a^2}}{2\rho^2} \right] \cos \phi \quad (53)$$

The field in polar coordinates are

$$H_\rho^{(1)} = -\frac{\partial \Phi^{(1)}}{\partial \rho} \quad H_\phi^{(1)} = -\frac{1}{\rho} \frac{\partial \Phi^{(1)}}{\partial \phi} \quad (54)$$

and in Cartesian coordinates are

$$H_x^{(1)} = H_\rho^{(1)} \cos \phi - H_\phi^{(1)} \sin \phi = \frac{2H_0 a}{\pi} \left( \frac{1}{\sqrt{\rho^2 - a^2}} - \frac{\sqrt{\rho^2 - a^2}}{\rho^2} \right) \sin \phi \cos \phi = \frac{2H_0 a^3}{\pi} \frac{xy}{\rho^4 \sqrt{\rho^2 - a^2}} \quad (55)$$

$$\begin{aligned} H_y^{(1)} &= H_\rho^{(1)} \sin \phi + H_\phi^{(1)} \cos \phi \\ &= -\frac{2H_0 a}{\pi} \left\{ \left[ \frac{1}{2a} \sin^{-1} \left( \frac{a}{\rho} \right) - \frac{1}{\sqrt{\rho^2 - a^2}} + \frac{\sqrt{\rho^2 - a^2}}{2\rho^2} \right] \sin^2 \phi + \left[ \frac{1}{2a} \sin^{-1} \left( \frac{a}{\rho} \right) - \frac{\sqrt{\rho^2 - a^2}}{2\rho^2} \right] \cos^2 \phi \right\} \\ &= -\frac{2H_0 a}{\pi} \left[ \frac{1}{2a} \sin^{-1} \left( \frac{a}{\rho} \right) - \left( \frac{1}{\sqrt{\rho^2 - a^2}} - \frac{\sqrt{\rho^2 - a^2}}{\rho^2} \right) \sin^2 \phi - \frac{\sqrt{\rho^2 - a^2}}{2\rho^2} \right] \\ &= \frac{H_0}{\pi} \left[ \frac{a \sqrt{\rho^2 - a^2}}{\rho^2} - \sin^{-1} \left( \frac{a}{\rho} \right) \right] + \frac{2H_0 a^3}{\pi} \frac{y^2}{\rho^4 \sqrt{\rho^2 - a^2}} \end{aligned} \quad (56)$$

### 5. Field in the hole, i.e., equation (5.132)

When  $\rho < a$  and  $z \rightarrow 0^+$

$$\begin{aligned} \lambda &\rightarrow \frac{\rho^2}{a^2} - 1 & R &\rightarrow 1 - \frac{\rho^2}{a^2} & \Rightarrow \\ \frac{R-\lambda}{2} &\rightarrow 1 - \frac{\rho^2}{a^2} & \text{hence and by (42)} & \frac{2}{R+\lambda} &\rightarrow \frac{a^2}{z^2} \left(1 - \frac{\rho^2}{a^2}\right) \end{aligned} \quad (57)$$

Also

$$\frac{\partial A}{\partial z} = \frac{1}{2\rho} \sqrt{\frac{R-\lambda}{2}} + \frac{z}{2\rho} \frac{1}{2} \frac{1}{\sqrt{\frac{R-\lambda}{2}}} \frac{1}{2} \left( \frac{\partial R}{\partial z} - \frac{\partial \lambda}{\partial z} \right) \rightarrow \frac{1}{2\rho} \sqrt{1 - \frac{\rho^2}{a^2}} \quad (58)$$

$$\begin{aligned} \frac{\partial B}{\partial z} &= \frac{\rho}{2a} \left( \frac{1}{1 + \frac{2}{R+\lambda}} \right) \left( -\frac{1}{2} \right) \sqrt{\frac{2}{R+\lambda}} \frac{1}{2} \left( \frac{\partial R}{\partial z} + \frac{\partial \lambda}{\partial z} \right) \\ &= \frac{\rho}{2a} \left( \frac{R+\lambda}{R+\lambda+2} \right) \left( -\frac{1}{4} \right) \sqrt{\frac{2}{R+\lambda}} \left( \frac{\lambda}{R} \cdot \frac{2z}{a^2} + \frac{4z}{a^2 R} + \frac{2z}{a^2} \right) \\ &= \frac{\rho}{2a} \left( \frac{R+\lambda}{R+\lambda+2} \right) \left( -\frac{1}{4} \right) \sqrt{\frac{2}{R+\lambda}} \frac{2z}{a^2 R} (\lambda + 2 + R) \\ &= -\frac{\rho}{2a} \sqrt{\frac{2}{R+\lambda}} \cdot \frac{z}{a^2 R} \\ &\rightarrow -\frac{\rho}{2a^3} \cdot \frac{a}{z} \sqrt{1 - \frac{\rho^2}{a^2}} \frac{z}{\left(1 - \frac{\rho^2}{a^2}\right)} = -\frac{\rho}{2a^2} \frac{1}{\sqrt{1 - \frac{\rho^2}{a^2}}} \end{aligned} \quad (59)$$

$$\begin{aligned} -\frac{\partial C}{\partial z} &= -\frac{a}{2\rho} \frac{1}{2} \frac{1}{\sqrt{\frac{R+\lambda}{2}}} \frac{1}{2} \left( \frac{\partial R}{\partial z} + \frac{\partial \lambda}{\partial z} \right) \\ &= -\frac{a}{8\rho} \sqrt{\frac{2}{R+\lambda}} \frac{2z}{a^2 R} (\lambda + 2 + R) \\ &\rightarrow -\frac{a}{8\rho} \cdot \frac{a}{z} \sqrt{1 - \frac{\rho^2}{a^2}} \cdot \frac{2z}{a^2 \left(1 - \frac{\rho^2}{a^2}\right)} \cdot 2 = -\frac{1}{2\rho} \frac{1}{\sqrt{1 - \frac{\rho^2}{a^2}}} \end{aligned} \quad (60)$$

Combining (58)-(60) gives

$$\begin{aligned} H_z(\rho, \phi, z=0^+) &= -\lim_{z \rightarrow 0^+} \frac{\partial \Phi^{(1)}}{\partial z} = -\lim_{z \rightarrow 0^+} \frac{2H_0 a}{\pi} \left( \frac{\partial A}{\partial z} + \frac{\partial B}{\partial z} - \frac{\partial C}{\partial z} \right) \sin \phi \\ &= -\frac{2H_0 a}{\pi} \left( \frac{1}{2a\rho} \sqrt{a^2 - \rho^2} - \frac{\rho}{2a} \frac{1}{\sqrt{a^2 - \rho^2}} - \frac{a}{2\rho} \frac{1}{\sqrt{a^2 - \rho^2}} \right) \sin \phi \\ &= -\frac{2H_0 a}{\pi} \frac{1}{2a\rho \sqrt{a^2 - \rho^2}} [(a^2 - \rho^2) - \rho^2 - a^2] \sin \phi \\ &= \frac{2H_0}{\pi} \frac{\rho}{\sqrt{a^2 - \rho^2}} \sin \phi \end{aligned} \quad (61)$$

For the tangential fields, first observe

$$\begin{aligned} H_\phi^{(1)}(\rho, \phi, z=0^+) &= -\lim_{z \rightarrow 0^+} \frac{1}{\rho} \frac{\partial \Phi^{(1)}}{\partial \phi} \\ &= -\lim_{z \rightarrow 0^+} \frac{2H_0 a}{\pi} \left[ \frac{z}{2\rho^2} \sqrt{\frac{R-\lambda}{2}} + \frac{1}{2a} \tan^{-1} \sqrt{\frac{2}{R+\lambda}} - \frac{a}{2\rho^2} \sqrt{\frac{R+\lambda}{2}} \right] \cos \phi \\ &= -\frac{2H_0 a}{\pi} \cdot \frac{1}{2a} \frac{\pi}{2} \cos \phi = -\frac{H_0}{2} \cos \phi \end{aligned} \quad (62)$$

The derivatives in the  $\hat{\rho}$  direction in the  $z \rightarrow 0^+$  limits are

$$\frac{\partial A}{\partial \rho} = -\frac{z}{2\rho^2} \sqrt{\frac{R-\lambda}{2}} + \frac{z}{2\rho} \frac{1}{2} \frac{1}{\sqrt{\frac{R-\lambda}{2}}} \frac{1}{2} \left( \frac{\partial R}{\partial \rho} - \frac{\partial \lambda}{\partial \rho} \right) \rightarrow 0 \quad (63)$$

$$\begin{aligned} \frac{\partial B}{\partial \rho} &= \frac{1}{2a} \tan^{-1} \sqrt{\frac{2}{R+\lambda}} + \frac{\rho}{2a} \left( \frac{1}{1 + \frac{2}{R+\lambda}} \right) \left( -\frac{1}{2} \right) \sqrt{\frac{2}{R+\lambda}}^3 \frac{1}{2} \left( \frac{\partial R}{\partial \rho} + \frac{\partial \lambda}{\partial \rho} \right) \\ &\rightarrow \frac{1}{2a} \frac{\pi}{2} + \underbrace{\frac{\rho}{2a} \left( \frac{R+\lambda}{R+\lambda+2} \right) \left( -\frac{1}{4} \right) \sqrt{\frac{2}{R+\lambda}}^3 \frac{2\rho}{a^2 R} (\lambda+R)}_{\propto \sqrt{R+\lambda} \text{ hence } \rightarrow 0} = \frac{1}{2a} \frac{\pi}{2} \end{aligned} \quad (64)$$

$$-\frac{\partial C}{\partial \rho} = \frac{a}{2\rho^2} \sqrt{\frac{R+\lambda}{2}} - \frac{a}{2\rho} \frac{1}{2} \frac{1}{\sqrt{\frac{R+\lambda}{2}}} \frac{1}{2} \left( \frac{\partial R}{\partial \rho} + \frac{\partial \lambda}{\partial \rho} \right) \rightarrow 0 \quad (65)$$

Therefore

$$H_\rho^{(1)}(\rho, \phi, z=0^+) = -\lim_{z \rightarrow 0^+} \frac{\partial \Phi^{(1)}}{\partial \rho} = -\lim_{z \rightarrow 0^+} \frac{2H_0 a}{\pi} \left( \frac{\partial A}{\partial \rho} + \frac{\partial B}{\partial \rho} - \frac{\partial C}{\partial \rho} \right) \sin \phi = -\frac{H_0}{2} \sin \phi \quad (66)$$

Writing (62) and (66) in Cartesian coordinates gives

$$H_x^{(1)}(x, y, 0^+) = H_\rho^{(1)} \cos \phi - H_\phi^{(1)} \sin \phi = 0 \quad H_y^{(1)}(x, y, 0^+) = H_\rho^{(1)} \sin \phi + H_\phi^{(1)} \cos \phi = -\frac{H_0}{2} \quad (67)$$

Remember for  $z > 0$ ,  $\mathbf{H}^{(1)}$  needs to combine with  $H_0 \hat{\mathbf{y}}$  to give the whole field, i.e.,

$$\mathbf{H}_{\text{tan}} = \frac{\mathbf{H}_0}{2} \quad (68)$$

It's worth noting that the net field  $\mathbf{H}$  is continuous from  $0^+ \rightarrow 0^-$  across the opening, since for  $z < 0$ ,  $\mathbf{H}^{(1)}$  is the net field, and  $H_x^{(1)}, H_y^{(1)}$  are odd in  $z$  but  $H_z$  is even in  $z$ .



The field line visualization is programmed [here](#). The following are notes for the algorithm, specifically the derivatives of  $A, B, C$  terms in (34).

For  $z > 0$ :

$$\begin{aligned}
\frac{\partial \lambda}{\partial z} &= \frac{2z}{a^2} & \frac{\partial \lambda}{\partial \rho} &= \frac{2\rho}{a^2} \\
\frac{\partial R}{\partial z} &= \frac{\lambda}{R} \frac{\partial \lambda}{\partial z} + \frac{4z/a^2}{R} & \frac{\partial R}{\partial \rho} &= \frac{\lambda}{R} \frac{\partial \lambda}{\partial \rho} \\
\frac{\partial A}{\partial z} &= \frac{1}{2\rho} \sqrt{\frac{R-\lambda}{2}} + \frac{z}{2\rho} \frac{1}{2} \frac{1}{\sqrt{\frac{R-\lambda}{2}}} \frac{1}{2} \left( \frac{\partial R}{\partial z} - \frac{\partial \lambda}{\partial z} \right) \\
\frac{\partial A}{\partial \rho} &= -\frac{z}{2\rho^2} \sqrt{\frac{R-\lambda}{2}} + \frac{z}{2\rho} \frac{1}{2} \frac{1}{\sqrt{\frac{R-\lambda}{2}}} \frac{1}{2} \left( \frac{\partial R}{\partial \rho} - \frac{\partial \lambda}{\partial \rho} \right) \\
\frac{\partial B}{\partial z} &= \frac{\rho}{2a} \left( \frac{1}{1 + \frac{2}{R+\lambda}} \right) \left( -\frac{1}{2} \right) \sqrt{\frac{2}{R+\lambda}} \frac{1}{2} \left( \frac{\partial R}{\partial z} + \frac{\partial \lambda}{\partial z} \right) \\
\frac{\partial B}{\partial \rho} &= \frac{1}{2a} \tan^{-1} \sqrt{\frac{2}{R+\lambda}} + \frac{\rho}{2a} \left( \frac{1}{1 + \frac{2}{R+\lambda}} \right) \left( -\frac{1}{2} \right) \sqrt{\frac{2}{R+\lambda}} \frac{1}{2} \left( \frac{\partial R}{\partial \rho} + \frac{\partial \lambda}{\partial \rho} \right) \\
-\frac{\partial C}{\partial z} &= -\frac{a}{2\rho} \frac{1}{2} \frac{1}{\sqrt{\frac{R+\lambda}{2}}} \frac{1}{2} \left( \frac{\partial R}{\partial z} + \frac{\partial \lambda}{\partial z} \right) \\
-\frac{\partial C}{\partial \rho} &= \frac{a}{2\rho^2} \sqrt{\frac{R+\lambda}{2}} - \frac{a}{2\rho} \frac{1}{2} \frac{1}{\sqrt{\frac{R+\lambda}{2}}} \frac{1}{2} \left( \frac{\partial R}{\partial \rho} + \frac{\partial \lambda}{\partial \rho} \right) \\
H_z^{(1)} &= -\frac{\partial \Phi^{(1)}}{\partial z} = -\frac{2H_0 a}{\pi} \left( \frac{\partial A}{\partial z} + \frac{\partial B}{\partial z} - \frac{\partial C}{\partial z} \right) \sin \phi \\
H_\rho^{(1)} &= -\frac{\partial \Phi^{(1)}}{\partial \rho} = -\frac{2H_0 a}{\pi} \left( \frac{\partial A}{\partial \rho} + \frac{\partial B}{\partial \rho} - \frac{\partial C}{\partial \rho} \right) \sin \phi \\
H_\phi^{(1)} &= -\frac{1}{\rho} \frac{\partial \Phi^{(1)}}{\partial \phi} = -\frac{2H_0 a}{\pi} \left[ \frac{z}{2\rho^2} \sqrt{\frac{R-\lambda}{2}} + \frac{1}{2a} \tan^{-1} \sqrt{\frac{2}{R+\lambda}} - \frac{a}{2\rho^2} \sqrt{\frac{R+\lambda}{2}} \right] \cos \phi \\
H_x^{(1)} &= H_\rho^{(1)} \cos \phi - H_\phi^{(1)} \sin \phi \\
H_y^{(1)} &= H_\rho^{(1)} \sin \phi + H_\phi^{(1)} \cos \phi
\end{aligned}$$

Finally, after adding in the external field:

$$H_x = H_x^{(1)} \quad H_y = H_0 + H_y^{(1)} \quad H_z = H_z^{(1)}$$

and for  $-z < 0$ , by parity,

$$\begin{aligned}
H_x(x, y, -z) &= -H_x(x, y, z) \\
H_y(x, y, -z) &= -H_y(x, y, z) \\
H_z(x, y, -z) &= H_z(x, y, z)
\end{aligned}$$