In these notes, we fill the details omitted in Jackson section 3.13 *Mixed Boundary Conditions; Conducting Plane with a Circular Hole.*

1. **Proof of equation (3.175)**

It was claimed (3.175) that

$$B_l = \frac{1}{l!}\left(-\frac{d}{d|z|}\right)^l \int_0^\infty dk e^{-k|z|} J_0(k\rho) = \frac{1}{l!}\left(-\frac{d}{d|z|}\right)^l \left(\frac{1}{\sqrt{\rho^2 + z^2}}\right) \tag{1}$$

For this, we need to prove the integral identity

$$\int_0^\infty dk e^{-k|z|} J_0(k\rho) = \frac{1}{\sqrt{\rho^2 + z^2}} \tag{2}$$

To see this, insert the expansion of $J_0(k\rho)$ into the LHS

$$\begin{aligned}
\text{LHS}_{(2)} &= \int_0^\infty dk e^{-k|z|} \sum_{j=0}^\infty \frac{(-1)^j}{j!j!}\left(\frac{k\rho}{2}\right)^{2j} \\
&= \sum_{j=0}^\infty \frac{(-1)^j}{j!j!}\left(\frac{\rho}{2}\right)^{2j} \int_0^\infty dk e^{-k|z|} k^{2j} \\
&= \sum_{j=0}^\infty \frac{(-1)^j}{j!j!}\left(\frac{\rho}{2}\right)^{2j} \cdot \left[\frac{(2j)!}{|z|^{2j+1}}\right] \\
&= \frac{1}{|z|}\sum_{j=0}^\infty \frac{1}{j!}\frac{(-1)^j}{2^j}\left(\frac{\rho}{|z|}\right)^{2j}\frac{(2j)!}{2^j j!} \\
&= \frac{1}{|z|}\sum_{j=0}^\infty \frac{1}{j!}\left[\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2j-1}{2}\right)\right]\left(\frac{\rho}{|z|}\right)^{2j}
\end{aligned} \tag{3}$$

where in the second line, we have used the integral

$$\int_0^\infty e^{-ax} x^n dx = \frac{n!}{a^{n+1}} \tag{4}$$

Then (3) can be readily recognized as the Taylor series for

$$\frac{1}{|z|}\frac{1}{\sqrt{1+\left(\frac{\rho}{|z|}\right)^2}} = \frac{1}{\sqrt{\rho^2 + z^2}} \tag{5}$$

2. **Proof of equation (3.176)**

Next, we have a claim that

$$B_l = \frac{1}{l!}\left(-\frac{d}{d|z|}\right)^l \left(\frac{1}{\sqrt{\rho^2+z^2}}\right) = \frac{P_l(|\cos\theta|)}{r^{l+1}} \qquad \text{where } \cos\theta = \frac{z}{r}, r = \sqrt{\rho^2 + z^2} \tag{6}$$

We can prove this by induction:

- $l = 0$, trivially true;
- $l = 1$:

$$-\frac{d}{d|z|}\frac{1}{\sqrt{\rho^2+z^2}} = \frac{|z|}{r^3} = \frac{P_1(|\cos\theta|)}{r^2} \tag{7}$$

- By induction, if

$$\frac{1}{(l-1)!}\left(-\frac{d}{d|z|}\right)^{l-1}\left(\frac{1}{r}\right) = \frac{P_{l-1}(|\cos\theta|)}{r^l} \tag{8}$$

Then

$$
\begin{aligned}
B_l &= \frac{1}{l}\left(-\frac{d}{d|z|}\right)B_{l-1} \\
&= -\frac{1}{l}\frac{d}{d|z|}\left[\frac{P_{l-1}(|z|/r)}{r^l}\right] \\
&= -\frac{1}{l}\left[\frac{1}{r^l}\frac{dP_{l-1}(|z|/r)}{d|z|} + P_{l-1}(|z|/r)\frac{(-l)}{r^{l+1}}\frac{dr}{d|z|}\right] \\
&= -\frac{1}{l}\left[\frac{1}{r^l}P'_{l-1}(|z|/r)\frac{d(|z|/r)}{d|z|} - lP_{l-1}(|z|/r)\frac{1}{r^{l+1}}\frac{|z|}{r}\right] \\
&= -\frac{1}{l}\left[\frac{P'_{l-1}(|z|/r)}{r^l}\left(\frac{1}{r}-\frac{|z|}{r^2}\cdot\frac{|z|}{r}\right) - \frac{lP_{l-1}(|z|/r)|z|/r}{r^{l+1}}\right] \\
&= -\frac{1}{r^{l+1}}\left[\frac{1}{l}P'_{l-1}(|z|/r)\left(1-\frac{|z|^2}{r^2}\right) - P_{l-1}(|z|/r)|z|/r\right] \\
&= \frac{P_l(|z|/r)}{r^{l+1}}
\end{aligned}
\tag{9}
$$

where in the last step, we have used the recurrence relation of Legendre polynomials (reference: equation 14.10.4 on dlmf.nist.gov)

$$
\frac{1}{l}P'_{l-1}(x)\left(1-x^2\right) - P_{l-1}(x)x = -P_l(x)
\tag{10}
$$

3. **Verification that (3.180) is the solution of (3.179) via Weber–Schafheitlin integral**

For the *dual integral equations* (3.179)

$$
\int_0^\infty dy\, y\, g(y) J_n(yx) = x^n \qquad\qquad \text{for } 0 \le x < 1
\tag{11}
$$

$$
\int_0^\infty dy\, g(y) J_n(yx) = 0 \qquad\qquad \text{for } 1 \le x
\tag{12}
$$

It was claimed that (3.180) is a solution

$$
g(y) = \frac{\Gamma(n+1)}{\sqrt{\pi}\Gamma(n+3/2)} j_{n+1}(y) = \frac{\Gamma(n+1)}{\Gamma(n+3/2)}\frac{J_{n+3/2}(y)}{\sqrt{2y}}
\tag{13}
$$

Let's verify it via the *Weber–Schafheitlin* integral (reference equation 10.22.56 on dlmf.nist.gov)

for $0 < a < b, \mathrm{Re}\,(\mu + \nu + 1) > \mathrm{Re}(\lambda) > -1:$

$$
\int_0^\infty \frac{J_\mu(at)J_\nu(bt)}{t^\lambda}dt = \frac{a^\mu\Gamma\left(\frac{\nu}{2}+\frac{\mu}{2}-\frac{\lambda}{2}+\frac{1}{2}\right)}{2^\lambda b^{\mu-\lambda+1}\Gamma\left(\frac{\nu}{2}-\frac{\mu}{2}+\frac{\lambda}{2}+\frac{1}{2}\right)}F\left(\frac{\mu+\nu-\lambda+1}{2},\frac{\mu-\nu-\lambda+1}{2};\mu+1;\frac{a^2}{b^2}\right)
\tag{14}
$$

where F is the hypergeometric function

$$
F(a,b;c;z) = 1 + \frac{ab}{c}z + \frac{1}{2!}\frac{a(a+1)b(b+1)}{c(c+1)}z^2 + \cdots
\tag{15}
$$

- To see (11), insert $a = x, b = 1, \mu = n, \nu = n+3/2, \lambda = -1/2$ into (14), we have

$$
\int_0^\infty \sqrt{y}J_n(xy)J_{n+3/2}(y)dy = \frac{\sqrt{2}x^n\Gamma(n+3/2)}{\Gamma(1)}\overbrace{F\left(n+\frac{3}{2},0,n+1;\frac{a^2}{b^2}\right)}^{=1}
\tag{16}
$$

Thus we see that (13) satisfies (11) except for this $\Gamma(n+1)$ factor (**which I think is a mistake on the book**, but fortunately for the $n = 0$ case, it doesn't impact the subsequent discussion).

- For (12), we need $a = 1, b = x, \mu = n+3/2, \nu = n, \lambda = 1/2$, with which the denominator of (14) has $\Gamma(0) = \infty$, hence (14) vanishes, which satisfies (12).

4. **Steps leading from (3.184) to (3.185)**

We are now going to calculate the integration (3.184):

$$\Phi^{(1)}(\rho,z) = \frac{(E_0 - E_1)a^2}{\pi} \underbrace{\int_0^\infty dk\, j_1(ka)e^{-k|z|}J_0(k\rho)}_{I} \tag{17}$$

that leads to the claimed result

$$\Phi^{(1)}(\rho,z) = \frac{(E_0 - E_1)a}{\pi}\left[\sqrt{\frac{R-\lambda}{2}} - \frac{|z|}{a}\tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right)\right] \qquad \text{where}$$

$$\lambda = \frac{z^2 + \rho^2 - a^2}{a^2} \qquad R = \sqrt{\lambda^2 + \frac{4z^2}{a^2}} \tag{18}$$

First, the relation

$$j_1(x) = -j_0'(x) \tag{19}$$

will turn the integral I in (17) into

$$
\begin{aligned}
I &= \int_0^\infty dk\left[-j_0'(ka)\right]e^{-k|z|}J_0(k\rho) \\
&= \int_0^\infty dk\left[-\frac{dj_0(ka)}{adk}\right]e^{-k|z|}J_0(k\rho) \\
&= -\frac{1}{a}j_0(ka)e^{-k|z|}J_0(k\rho)\Big|_0^\infty + \frac{1}{a}\int_0^\infty dk\,j_0(ka)\frac{d}{dk}\left[e^{-k|z|}J_0(k\rho)\right] \\
&= \frac{1}{a}\left[1 + \int_0^\infty j_0(ka)(-|z|)e^{-k|z|}J_0(k\rho)dk + \int_0^\infty j_0(ka)e^{-k|z|}J_0'(k\rho)\rho\, dk\right] \\
&= \frac{1}{a}\left[1 - |z|\underbrace{\int_0^\infty j_0(ka)e^{-k|z|}J_0(k\rho)dk}_{K_0} - \rho\underbrace{\int_0^\infty j_0(ka)e^{-k|z|}J_1(k\rho)dk}_{K_1}\right]
\end{aligned}
\tag{20}
$$

where in the last step, we have used

$$J_0'(x) = -J_1(x) \tag{21}$$

(a) **Calculation of K_1**

Since $j_0(ka) = \sin ka/(ka)$, then

$$
\begin{aligned}
K_1 &= \int_0^\infty \frac{\sin ka}{ka}e^{-k|z|}J_1(k\rho)dk \\
&= \frac{1}{a}\operatorname{Im}\left[\int_0^\infty e^{ika}e^{-k|z|}\frac{J_1(k\rho)}{k}dk\right] \qquad\qquad \text{define } s \equiv |z| - ia \\
&= \frac{1}{a}\operatorname{Im}\left[\int_0^\infty e^{-sk}\frac{J_1(k\rho)}{k}dk\right]
\end{aligned}
\tag{22}
$$

The content of the square bracket is the Laplace transform of $J_1(k\rho)/k$, denoted $\mathscr{L}\{J_1(k\rho)/k\}(s)$. But before calculating it, we have to take a detour.

It is clear that the proof that established (2) can be used to produce

$$\mathscr{L}\{J_0(k\rho)\}(s) = \frac{1}{\sqrt{\rho^2 + s^2}} \tag{23}$$

With (21) and the Laplace transform property

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\}(s) = s\mathscr{L}\{f(t)\}(s) - f(0^-) \tag{24}$$

we have

$$\mathscr{L}\{J_1(k\rho)\}(s) = \mathscr{L}\left\{-J_0'(k\rho)\right\}(s) = \mathscr{L}\left\{-\frac{dJ_0(k\rho)}{\rho\,dk}\right\}(s) = -\frac{1}{\rho}\left(\frac{s}{\sqrt{s^2+\rho^2}}-1\right) = \frac{1}{\rho}\left(1-\frac{s}{\sqrt{\rho^2+s^2}}\right) \quad (25)$$

Note the recurrence relation of Bessel functions (reference equation 10.6.2 on dlmf.nist.gov)

$$J_\nu'(x) = J_{\nu-1}(x) - \frac{\nu}{x}J_\nu(x) \quad (26)$$
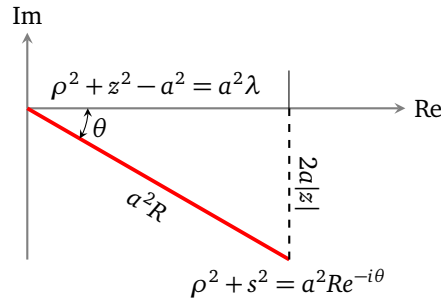
Apply the Laplace transform to (26) with $\nu = 1$,

$$\mathscr{L}\left\{\frac{J_1(k\rho)}{k\rho}\right\}(s) = \mathscr{L}\{J_0(k\rho)\}(s) - \mathscr{L}\{J_1'(k\rho)\}(s)$$

$$= \mathscr{L}\{J_0(k\rho)\}(s) - \mathscr{L}\left\{\frac{dJ_1(k\rho)}{\rho\,dk}\right\}(s) \qquad \text{by (23),(24),(25)}$$

$$= \frac{1}{\sqrt{\rho^2+s^2}} - \frac{1}{\rho}\left[\frac{s}{\rho}\left(1-\frac{s}{\sqrt{\rho^2+s^2}}\right)\right]$$

$$= \frac{\rho^2 - s\sqrt{\rho^2+s^2}+s^2}{\rho^2\sqrt{\rho^2+s^2}} = \frac{1}{\rho^2}\left(\sqrt{\rho^2+s^2}-s\right) \qquad \Longrightarrow$$

$$\mathscr{L}\left\{\frac{J_1(k\rho)}{k}\right\}(s) = \frac{1}{\rho}\left(\sqrt{\rho^2+s^2}-s\right) \quad (27)$$

In order to get K_1 in (22), it remains to extract the imaginary part of (27). Apparently

$$\mathrm{Im}\left[\frac{1}{\rho}\left(\sqrt{\rho^2+s^2}-s\right)\right] = \frac{1}{\rho}\left(\mathrm{Im}\sqrt{\rho^2+s^2}-\mathrm{Im}\,s\right) = \frac{1}{\rho}\left(\mathrm{Im}\sqrt{\rho^2+s^2}+a\right) \quad (28)$$

Now using the diagram below, we obtain

$$\sqrt{\rho^2+s^2} = \left(\rho^2+z^2-a^2-2a|z|i\right)^{1/2} = a\sqrt{R}e^{-i\theta/2} \quad (29)$$

hence

$$\mathrm{Im}\sqrt{\rho^2+s^2} = -a\sqrt{R}\sin\frac{\theta}{2} \quad (30)$$

$\sin\theta/2$ can be found via

$$1 - 2\sin^2\frac{\theta}{2} = \cos\theta = \frac{\lambda}{R} \quad \Longrightarrow \quad \sin\frac{\theta}{2} = \sqrt{\frac{R-\lambda}{2R}} \quad (31)$$

Inserting (30), (31) into (28) gives

$$\mathrm{Im}\,\mathscr{L}\left\{\frac{J_1(k\rho)}{k}\right\}(s) = \frac{a}{\rho}\left(1-\sqrt{\frac{R-\lambda}{2}}\right) \quad (32)$$

which reduces (20) into

$$I = \frac{1}{a}\left(\sqrt{\frac{R-\lambda}{2}} - |z|K_0\right) \quad (33)$$

Compare with the desired result (18), it remains to show

$$K_0 = \int_0^\infty \frac{\sin ka}{ka} e^{-k|z|} J_0(k\rho)dk = \frac{1}{a}\tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right) \qquad\qquad \text{or}$$

$$\int_0^\infty \frac{\sin ka}{k} e^{-k|z|} J_0(k\rho)dk_{F(|z|)} = \tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right) \qquad\qquad (34)$$

(b) **Calculation of K_0**

Similar to the K_1 case, define

$$F(s) \equiv \int_0^\infty \frac{e^{-sk}}{k} J_0(k\rho)dk, \qquad\qquad s = |z| - ia \qquad\qquad (35)$$

we just need to show

$$\text{Im}[F(s)] = \tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right) \qquad\qquad (36)$$

Notice by (23)

$$F'(s) = -\int_0^\infty e^{-sk} J_0(k\rho)dk = -\frac{1}{\sqrt{\rho^2 + s^2}} \qquad\qquad (37)$$

Thus $F(s)$ is readily solvable as (reference WolframAlpha)

$$\begin{aligned} F(s) &= -\tanh^{-1}\left(\frac{s}{\sqrt{\rho^2 + s^2}}\right) \\ &= \frac{1}{2}\ln\left(1 - \frac{s}{\sqrt{\rho^2 + s^2}}\right) - \frac{1}{2}\ln\left(1 + \frac{s}{\sqrt{\rho^2 + s^2}}\right) \\ &= \frac{1}{2}\ln\left(\frac{\sqrt{\rho^2 + s^2} - s}{\sqrt{\rho^2 + s^2} + s}\right) \\ &= \frac{1}{2}\ln\left[\frac{\left(\sqrt{\rho^2 + s^2} - s\right)^2}{\rho^2}\right] \\ &= \ln\left(\frac{\sqrt{\rho^2 + s^2} - s}{\rho}\right) \end{aligned} \qquad\qquad (38)$$

The imaginary part of $F(s)$ is just the argument of the complex number $\sqrt{\rho^2 + s^2} - s$, i.e.,

$$\begin{aligned} \text{Im}[F(s)] &= \text{Arg}\left(\sqrt{\rho^2 + s^2} - s\right) \qquad\qquad \text{(see diagram above)} \\ &= \text{Arg}\left[a\sqrt{R}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) - |z| + ia\right] \\ &= \tan^{-1}\left(\frac{a - a\sqrt{R}\sin\frac{\theta}{2}}{a\sqrt{R}\cos\frac{\theta}{2} - |z|}\right) \end{aligned} \qquad\qquad (39)$$

From (31), we have

$$\cos\frac{\theta}{2} = \sqrt{\frac{R+\lambda}{2R}} \qquad\qquad (40)$$

hence

$$\text{Im}[F(s)] = \tan^{-1}\left(\frac{a - a\sqrt{R}\sqrt{\frac{R-\lambda}{2R}}}{a\sqrt{R}\sqrt{\frac{R+\lambda}{2R}} - |z|}\right) = \tan^{-1}\left(\frac{a - a\sqrt{\frac{R-\lambda}{2}}}{a\sqrt{\frac{R+\lambda}{2}} - |z|}\right) = \tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right) \qquad (41)$$