1. **Prob 12.11**

   (a) In this configuration, the muon moves with constant speed in a circle, its β and γ are constant. With $\boldsymbol{\beta} \cdot \mathbf{B} = 0$ and $\mathbf{E} = 0$, the equation of motion of its spin is given by (11.170)

$$\frac{d\mathbf{s}}{dt} = \frac{e}{mc}\mathbf{s} \times \left[\left(a + \frac{1}{\gamma}\right)\mathbf{B}\right] = \frac{e}{mc}\left(a + \frac{1}{\gamma}\right)\mathbf{s} \times \mathbf{B} \tag{1}$$

   Let $\mathbf{B}$ be along the z direction, then we immediately see that

$$\frac{ds_z}{dt} = 0 \tag{2}$$

   If the muon is initially polarized longitudinally, we see that the z component of its spin remains zero.
   The differential equation for its x and y components are

$$\frac{ds_x}{dt} = \frac{eB}{mc}\left(a + \frac{1}{\gamma}\right)s_y \tag{3}$$

$$\frac{ds_y}{dt} = -\frac{eB}{mc}\left(a + \frac{1}{\gamma}\right)s_x \tag{4}$$

   Substituting one for the other, we obtain the harmonic oscillator equation

$$\frac{d^2 s_i}{dt^2} = -\left[\frac{eB}{mc}\left(a + \frac{1}{\gamma}\right)\right]^2 s_i \qquad \text{for } i = x, y \tag{5}$$

   This shows that the spin vector oscillates with a frequency

$$\omega_{\text{spin}} = \frac{eB}{mc}\left(a + \frac{1}{\gamma}\right) = \frac{eBa}{mc} + \omega_{\text{orbital}} \tag{6}$$

   where

$$\omega_{\text{orbital}} = \frac{eB}{\gamma mc} \tag{7}$$

   is the orbital frequency of the muon in the magnetic field.
   The difference in frequency explains the precession of the spin.

   (b) The momentum of the particle is

$$p = \frac{eBR}{c} \approx 1275\text{MeV}/c \tag{8}$$

   giving Lorentz factor

$$\gamma = \frac{\sqrt{p^2c^2 + m^2c^4}}{mc^2} \approx 12.1 \tag{9}$$

   The number of precession periods per mean lifetime is

$$N = \frac{\gamma\tau_0}{2\pi/\Omega} \approx 7.12 \tag{10}$$

   (c) From part (a), we have

$$\Omega = \frac{eBa}{mc} = \omega_{\text{orbital}} \times \gamma a \tag{11}$$

   The number of precession periods per orbital rotation is then

$$N = \frac{2\pi/\omega_{\text{orbital}}}{2\pi/\Omega} = \gamma a \tag{12}$$

   - For 300MeV muon, $\gamma \approx 2.8$, $N \approx 3.3 \times 10^{-3}$;
   - For 300MeV electron, $\gamma \approx 587$, $N \approx 0.68$;
   - For 5GeV electron, $\gamma \approx 9.8 \times 10^3$, $N \approx 11.4$.

2. **Prob 12.12**

(a) We start with (11.170)

$$\frac{d\mathbf{s}}{dt}=\frac{e}{mc}\mathbf{s}\times\left[\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)\mathbf{B}-\left(\frac{g}{2}-1\right)\left(\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta}-\left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)\boldsymbol{\beta}\times\mathbf{E}\right] \tag{13}$$

and the identity

$$\frac{d}{dt}\left(\hat{\boldsymbol{\beta}}\cdot\mathbf{s}\right)=\hat{\boldsymbol{\beta}}\cdot\frac{d\mathbf{s}}{dt}+\frac{1}{\beta}\left[\mathbf{s}-\left(\hat{\boldsymbol{\beta}}\cdot\mathbf{s}\right)\hat{\boldsymbol{\beta}}\right]\cdot\frac{d\boldsymbol{\beta}}{dt} \tag{14}$$

where $d\boldsymbol{\beta}/dt$ is given by (11.168)

$$\frac{d\boldsymbol{\beta}}{dt}=\frac{e}{\gamma mc}\left[\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B}-\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\right] \tag{15}$$

Since $\mathbf{s}_\perp=\mathbf{s}-\left(\hat{\boldsymbol{\beta}}\cdot\mathbf{s}\right)\hat{\boldsymbol{\beta}}$, the second term of (14) is just

$$\frac{1}{\beta}\mathbf{s}_\perp\cdot\frac{d\boldsymbol{\beta}}{dt}=\frac{e}{\gamma mc}\mathbf{s}_\perp\cdot\left(\frac{\mathbf{E}}{\beta}+\hat{\boldsymbol{\beta}}\times\mathbf{B}\right) \tag{16}$$

In calculating $\hat{\boldsymbol{\beta}}\cdot d\mathbf{s}/dt$ of (14), note that the term proportional to $\mathbf{s}\times\left[(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta}\right]$ in (13) is normal to $\hat{\boldsymbol{\beta}}$ hence has no contribution, therefore

$$\begin{aligned}
\hat{\boldsymbol{\beta}}\cdot\frac{d\mathbf{s}}{dt}&=\frac{e}{mc}\left\{\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)\hat{\boldsymbol{\beta}}\cdot(\mathbf{s}\times\mathbf{B})-\left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)\hat{\boldsymbol{\beta}}\cdot\left[\mathbf{s}\times(\boldsymbol{\beta}\times\mathbf{E})\right]\right\}\\
&=\frac{e}{mc}\left\{\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)\mathbf{s}\cdot\left(\mathbf{B}\times\hat{\boldsymbol{\beta}}\right)-\left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta}\times\mathbf{E})\cdot\overbrace{\left(\hat{\boldsymbol{\beta}}\times\mathbf{s}\right)}^{\hat{\boldsymbol{\beta}}\times\mathbf{s}_\perp}\right\}\\
&=\frac{e}{mc}\left\{\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)\mathbf{s}_\perp\cdot\left(\mathbf{B}\times\hat{\boldsymbol{\beta}}\right)-\left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)\left(\beta\mathbf{s}_\perp\cdot\mathbf{E}\right)\right\}
\end{aligned} \tag{17}$$

Adding (16) and (17) gives (11.171)

$$\begin{aligned}
\frac{d}{dt}\left(\hat{\boldsymbol{\beta}}\cdot\mathbf{s}\right)&=-\frac{e}{mc}\mathbf{s}_\perp\cdot\left[\left(\frac{g}{2}-1\right)\hat{\boldsymbol{\beta}}\times\mathbf{B}+\left(\frac{g\beta}{2}-\frac{\gamma\beta}{\gamma+1}-\frac{1}{\gamma\beta}\right)\mathbf{E}\right]\\
&=-\frac{e}{mc}\mathbf{s}_\perp\cdot\left[\left(\frac{g}{2}-1\right)\hat{\boldsymbol{\beta}}\times\mathbf{B}+\left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\mathbf{E}\right]
\end{aligned} \tag{18}$$

(b) (18) can be written in terms of $|\mathbf{s}|$ and θ,

$$\frac{d\left(|\mathbf{s}|\cos\theta\right)}{dt}=-\frac{e}{mc}\mathbf{s}_\perp\cdot\left[\left(\frac{g}{2}-1\right)\hat{\boldsymbol{\beta}}\times\mathbf{B}+\left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\mathbf{E}\right] \tag{19}$$

From 11.167 and 11.155 we see that $d\mathbf{s}/dt$ is perpendicular to $\mathbf{s}$, so $|\mathbf{s}|$ is constant, which means

$$\frac{d\left(|\mathbf{s}|\cos\theta\right)}{dt}=-|\mathbf{s}|\sin\theta\frac{d\theta}{dt}=-|\mathbf{s}_\perp|\frac{d\theta}{dt} \tag{20}$$

Dividing both sides of (19) by $-|\mathbf{s}_\perp|$ gives

$$\frac{d\theta}{dt}=\frac{e}{mc}\hat{\mathbf{n}}\cdot\left[\left(\frac{g}{2}-1\right)\hat{\boldsymbol{\beta}}\times\mathbf{B}+\left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\mathbf{E}\right] \tag{21}$$

(c) For particle to be undeflected by the $\mathbf{E}\times\mathbf{B}$ selector, it must satisfy $\boldsymbol{\beta}=\mathbf{E}\times\mathbf{B}/B^2$. This means $\left\{\hat{\mathbf{E}},\hat{\mathbf{B}},\hat{\boldsymbol{\beta}}\right\}$ is a set of right-handed orthogonal unit vectors. But since $\left(\hat{\mathbf{n}}\times\hat{\boldsymbol{\beta}}\right)\cdot\mathbf{B}=B$, $\left\{\hat{\mathbf{n}},\hat{\boldsymbol{\beta}},\hat{\mathbf{B}}\right\}$ is also such a set. Therefore $\hat{\mathbf{n}}$ must be antiparallel to $\mathbf{E}$, i.e., $\hat{\mathbf{n}}\cdot\mathbf{E}=-|\mathbf{E}|=-\beta B$. This turns (21) to

$$\frac{d\theta}{dt}=\frac{eB}{mc}\left[\left(\frac{g}{2}-1\right)-\left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\beta\right]=\frac{eB}{mc}\cdot\frac{g}{2\gamma^2}=\frac{g}{2\gamma}\left(\frac{eB}{\gamma mc}\right) \tag{22}$$

(d) Given $L^\alpha=\left(\gamma\beta,\gamma\hat{\boldsymbol{\beta}}\right),N^\alpha=(0,\hat{\mathbf{n}})$, let's evaluate the scalar $\left(gL_\alpha/2-U_\alpha/v\right)F^{\alpha\beta}N_\beta$ in time and space components,

For $\alpha=0$
$$\left(\frac{g}{2}L_0-\frac{U_0}{v}\right)F^{0\beta}N_\beta=\left(\frac{g}{2}\gamma\beta-\frac{\gamma c}{v}\right)\hat{\mathbf{n}}\cdot\mathbf{E}=\gamma\left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\hat{\mathbf{n}}\cdot\mathbf{E} \tag{23}$$

For $\alpha=i$
$$\left(\frac{g}{2}L_i-\frac{U_i}{v}\right)F^{i\beta}N_\beta=-\left(\frac{g}{2}\gamma\hat{\boldsymbol{\beta}}-\frac{\gamma\mathbf{v}}{v}\right)\cdot(\hat{\mathbf{n}}\times\mathbf{B})=\gamma\left(\frac{g}{2}-1\right)\hat{\mathbf{n}}\cdot\left(\hat{\boldsymbol{\beta}}\times\mathbf{B}\right) \tag{24}$$

Comparing the sum of (23) and (24) with (21) gives the covariant form

$$\frac{e}{mc}\left(\frac{g}{2}L_\alpha-\frac{U_\alpha}{v}\right)F^{\alpha\beta}N_\beta=\gamma\frac{d\theta}{dt}=\frac{d\theta}{d\tau} \tag{25}$$