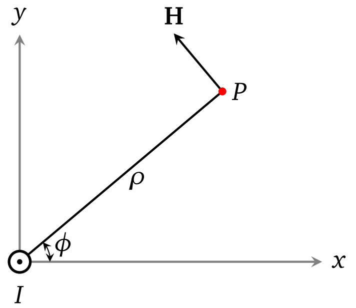


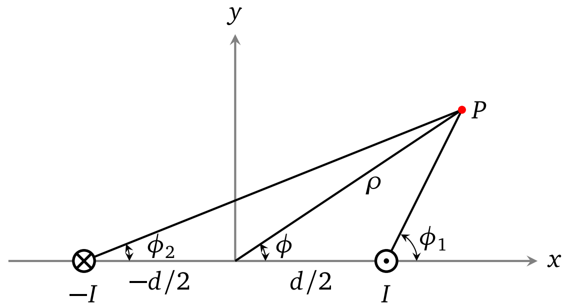


1. From the figure on the left, when we have a current coming out the paper in the $z+$ direction, the $\mathbf{H}$ field at point $(\rho, \phi)$ is

$$\mathbf{H} = \frac{I}{2\pi\rho}\hat{\boldsymbol{\phi}} \tag{1}$$

Since for $\rho \neq 0$,

$$\boldsymbol{\nabla} \times \mathbf{H} = \frac{1}{\rho}\frac{\partial\left(\rho H_\phi\right)}{\partial\rho}\hat{\mathbf{z}} = 0 \tag{2}$$

we can write $\mathbf{H}$ as the gradient of a scalar potential field

$$\mathbf{H} = -\boldsymbol{\nabla}\Phi_M \tag{3}$$

Since in polar coordinates, the gradient of a scalar field $f$ is

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \tag{4}$$

we can simply take

$$\Phi_M = -\frac{I\phi}{2\pi} \tag{5}$$

to satisfy (1) and (3).

This is an unusual scalar field since when $\phi$ goes one revolution, the value of $\Phi_M$ does not go back, i.e., $\Phi_M$ is a multi-valued function of $\phi$. But this is not an issue as long as the gradient $\boldsymbol{\nabla}\Phi_M = -\mathbf{H}$ is physical.

Now refer to the figure on the right, when we place current $\pm I$ at $x = \pm d/2$, the scalar potential at $P$ is

$$\begin{aligned}
\Phi_M(\rho,\phi) &= -\frac{I}{2\pi}(\phi_1 - \phi_2) = -\frac{I}{2\pi}\left[\tan^{-1}\left(\frac{y}{x-d/2}\right) - \tan^{-1}\left(\frac{y}{x+d/2}\right)\right] \\
&= -\frac{I}{2\pi}\tan^{-1}\left[\frac{\left(\frac{y}{x-d/2}\right) - \left(\frac{y}{x+d/2}\right)}{1+\left(\frac{y}{x-d/2}\right)\left(\frac{y}{x+d/2}\right)}\right] \\
&= -\frac{I}{2\pi}\tan^{-1}\left(\frac{dy}{x^2+y^2-d^2/4}\right) \\
&= -\frac{I}{2\pi}\tan^{-1}\left(\frac{d\rho\sin\phi}{\rho^2-d^2/4}\right) \\
&= -\frac{I}{2\pi}\tan^{-1}\left[\frac{\left(\frac{d}{\rho}\right)\sin\phi}{1-\frac{1}{4}\left(\frac{d}{\rho}\right)^2}\right]
\end{aligned} \tag{6}$$

When $\theta \to 0$, $\tan^{-1}\theta \to \theta$, therefore as $d/\rho \to 0$,

$$\Phi_M(\rho,\phi) \to -\frac{Id\sin\phi}{2\pi\rho} \tag{7}$$

2. By linear superposition and the general solution to the 2D Laplace equation (2.71), we can write the scalar potentials of the three regions as

$$\Phi_{\text{in}} = -\frac{Id\sin\phi}{2\pi\rho} + \sum_{n=1}^{\infty}\left(a_n\rho^n\cos n\phi + b_n\rho^n\sin n\phi\right) \tag{8}$$

$$\Phi_{\text{ring}} = c_0\ln\rho + \sum_{n=1}^{\infty}\left(c_n\rho^n\cos n\phi + d_n\rho^n\sin n\phi + e_n\rho^{-n}\cos n\phi + f_n\rho^{-n}\sin n\phi\right) \tag{9}$$

$$\Phi_{\text{out}} = \sum_{n=1}^{\infty}\left(g_n\rho^{-n}\cos n\phi + h_n\rho^{-n}\sin n\phi\right) \tag{10}$$

Similar to problem 5.14, boundary conditions can be used to conclude that coefficients of all homogeneous terms vanish, i.e.,

$$c_0 = 0 \qquad a_n = c_n = e_n = g_n = 0 \text{ for } n \geq 1 \qquad b_n = d_n = f_n = h_n = 0 \text{ for } n \geq 2 \tag{11}$$

The remaining unknowns are $b_1, d_1, f_1, h_1$, which can be solved using the boundary conditions

$$\left.\frac{\partial\Phi_{\text{in}}}{\partial\phi}\right|_{\rho=a} = \left.\frac{\partial\Phi_{\text{ring}}}{\partial\phi}\right|_{\rho=a} \implies -\frac{Id}{2\pi} + b_1a^2 = d_1a^2 + f_1 \tag{12}$$

$$\left.\frac{\partial\Phi_{\text{out}}}{\partial\phi}\right|_{\rho=b} = \left.\frac{\partial\Phi_{\text{ring}}}{\partial\phi}\right|_{\rho=b} \implies h_1 = d_1b^2 + f_1 \tag{13}$$

$$\left.\frac{\partial\Phi_{\text{in}}}{\partial\rho}\right|_{\rho=a} = \mu_r\left.\frac{\partial\Phi_{\text{ring}}}{\partial\rho}\right|_{\rho=a} \implies \frac{Id}{2\pi} + b_1a^2 = \mu_r\left(d_1a^2 - f_1\right) \tag{14}$$

$$\left.\frac{\partial\Phi_{\text{out}}}{\partial\rho}\right|_{\rho=b} = \left.\frac{\partial\Phi_{\text{ring}}}{\partial\rho}\right|_{\rho=b} \implies -h_1 = \mu_r\left(d_1b^2 - f_1\right) \tag{15}$$

Eventually, we obtain

$$b_1 = \frac{Id}{2\pi a^2}\left[\frac{\left(\mu_r^2-1\right)\left(b^2-a^2\right)}{(\mu_r+1)^2b^2-(\mu_r-1)^2a^2}\right] \tag{16}$$

$$d_1 = \frac{-Id}{\pi}\left[\frac{(\mu_r-1)}{(\mu_r+1)^2b^2-(\mu_r-1)^2a^2}\right] \tag{17}$$

$$f_1 = \frac{-Id}{\pi}\left[\frac{(\mu_r+1)b^2}{(\mu_r+1)^2b^2-(\mu_r-1)^2a^2}\right] \tag{18}$$

$$h_1 = \frac{-2Id}{\pi}\left[\frac{\mu_r b^2}{(\mu_r+1)^2b^2-(\mu_r-1)^2a^2}\right] \tag{19}$$

or, in other words, the potentials are

$$\Phi_{\text{in}} = \frac{-Id\sin\phi}{2\pi}\left[\frac{1}{\rho} - \frac{\rho}{a^2}\cdot\frac{\left(\mu_r^2-1\right)\left(b^2-a^2\right)}{(\mu_r+1)^2b^2-(\mu_r-1)^2a^2}\right] \tag{20}$$

$$\Phi_{\text{ring}} = \frac{-Id\sin\phi}{\pi}\left[\frac{(\mu_r-1)\rho + (\mu_r+1)\dfrac{b^2}{\rho}}{(\mu_r+1)^2b^2-(\mu_r-1)^2a^2}\right] \tag{21}$$

$$\Phi_{\text{out}} = \frac{-2Id\sin\phi}{\pi\rho}\left[\frac{\mu_r b^2}{(\mu_r+1)^2b^2-(\mu_r-1)^2a^2}\right] \tag{22}$$

The "shielding factor"

$$F = \frac{4\mu_r b^2}{(\mu_r+1)^2b^2-(\mu_r-1)^2a^2} \tag{23}$$

of this problem is the same as that of 5.14. But in these two problems, external fields are placed in different parts.

3. For the given numerical values, the shielding factor is

$$F \approx 0.046 \tag{24}$$

Below is the visualization of the field lines using $\mu_r = 100, a/b = 0.9$. The few visible stray lines come from $\phi = 90°, 90° \pm 0.25°, 90° \pm 0.5°, 90° \pm 0.75°$. The code is available here.