1. Let $y_{\nu n}$ and $y_{\nu m}$ be the n-th and m-th root of the eigenequation

$$xJ'_\nu(x)+\lambda J_\nu(x)=0 \tag{1}$$

We start with the differential equation that J_ν satisfies (see Jackson eq (3.77)), i.e.,

$$\begin{aligned}\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}+\left(1-\frac{\nu^2}{x^2}\right)R=0 & \qquad \Longrightarrow \\ \frac{1}{x}\frac{d}{dx}\left(x\frac{dR}{dx}\right)+\left(1-\frac{\nu^2}{x^2}\right)R=0 & \end{aligned} \tag{2}$$

Making the variable change $x=y_{\nu n}\rho/a$ yields

$$\begin{aligned}\frac{1}{\frac{y_{\nu n}\rho}{a}}\frac{d}{\frac{y_{\nu n}}{a}d\rho}\left[\frac{y_{\nu n}\rho}{a}\frac{dJ_\nu\left(\frac{y_{\nu n}\rho}{a}\right)}{\frac{y_{\nu n}}{a}d\rho}\right]+\left[1-\frac{\nu^2}{\left(\frac{y_{\nu n}}{a}\right)^2\rho^2}\right]J_\nu\left(\frac{y_{\nu n}\rho}{a}\right)=0 & \qquad \Longrightarrow \\ \frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dJ\left(\frac{y_{\nu n}\rho}{a}\right)}{d\rho}\right]+\left(\frac{y_{\nu n}^2}{a^2}-\frac{\nu^2}{\rho^2}\right)J_\nu\left(\frac{y_{\nu n}\rho}{a}\right)=0 & \end{aligned} \tag{3}$$

Multiply both sides of (3) by $J_\nu(y_{\nu m}\rho/a)$ and integrate with measure $\rho d\rho$, we have

$$\underbrace{\int_0^a\frac{1}{\rho}J_\nu\left(\frac{y_{\nu m}\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{dJ\left(\frac{y_{\nu n}\rho}{a}\right)}{d\rho}\right]\rho d\rho}_{I}+\int_0^a\left(\frac{y_{\nu n}^2}{a^2}-\frac{\nu^2}{\rho^2}\right)J_\nu\left(\frac{y_{\nu n}\rho}{a}\right)J_\nu\left(\frac{y_{\nu m}\rho}{a}\right)\rho d\rho=0 \tag{4}$$

where

$$I=\underbrace{J_\nu\left(\frac{y_{\nu m}\rho}{a}\right)\rho\frac{dJ\left(\frac{y_{\nu n}\rho}{a}\right)}{d\rho}}_{g(y_{\nu m},y_{\nu n};\rho)}\Bigg|_0^a-\int_0^a\rho\frac{dJ\left(\frac{y_{\nu n}\rho}{a}\right)}{d\rho}\frac{dJ\left(\frac{y_{\nu m}\rho}{a}\right)}{d\rho}d\rho \tag{5}$$

By the boundary condition at $\rho=0$

$$g(y_{\nu m},y_{\nu n};0)=0 \tag{6}$$

(6) enables us to rewrite (4) as

$$g\left(y_{\nu m},y_{\nu n};a\right)-\int_0^a\rho\frac{dJ\left(\frac{y_{\nu n}\rho}{a}\right)}{d\rho}\frac{dJ\left(\frac{y_{\nu m}\rho}{a}\right)}{d\rho}d\rho+\int_0^a\left(\frac{y_{\nu n}^2}{a^2}-\frac{\nu^2}{\rho^2}\right)J_\nu\left(\frac{y_{\nu n}\rho}{a}\right)J_\nu\left(\frac{y_{\nu m}\rho}{a}\right)\rho d\rho=0 \tag{7}$$

Apply the exchange $y_{\nu m}\leftrightarrow y_{\nu n}$ to (7) and subtract the resulting equation from (7), we get

$$g\left(y_{\nu m},y_{\nu n};a\right)-g\left(y_{\nu n},y_{\nu m};a\right)+\left(\frac{y_{\nu n}^2}{a^2}-\frac{y_{\nu m}^2}{a^2}\right)\int_0^a J_\nu\left(\frac{y_{\nu n}\rho}{a}\right)J_\nu\left(\frac{y_{\nu m}\rho}{a}\right)\rho d\rho=0 \tag{8}$$

By the other boundary condition at $\rho=a$:

$$\begin{aligned}\frac{d}{d\rho}\ln\left[J_\nu\left(\frac{y_{\nu n}\rho}{a}\right)\right]\Bigg|_{\rho=a}=\frac{1}{J_\nu\left(\frac{y_{\nu n}\rho}{a}\right)}\frac{dJ\left(\frac{y_{\nu n}\rho}{a}\right)}{d\rho}\Bigg|_{\rho=a} &=-\frac{\lambda}{a} \qquad \Longrightarrow \\ a\frac{dJ\left(\frac{y_{\nu n}\rho}{a}\right)}{d\rho}\Bigg|_{\rho=a} &=-\lambda J_\nu\left(\frac{y_{\nu n}\rho}{a}\right)\Bigg|_{\rho=a} \qquad \Longrightarrow \\ g\left(y_{\nu m},y_{\nu n};a\right) &=-\lambda J_\nu\left(y_{\nu m}\right)J_\nu\left(y_{\nu n}\right)\end{aligned} \tag{9}$$

The symmetry between $y_{\nu m}$ and $y_{\nu n}$ in (9) turns (8) into

$$\left(y_{\nu n}^2 - y_{\nu m}^2\right) \int_0^a J_\nu\left(\frac{y_{\nu n}\rho}{a}\right) J_\nu\left(\frac{y_{\nu m}\rho}{a}\right) \rho d\rho = 0 \tag{10}$$

which shows that for different eigenvalues $y_{\nu n} \neq y_{\nu m}$, $J_\nu(y_{\nu n}\rho/a)$ and $J_\nu(y_{\nu m}\rho/a)$ are orthogonal with respect to the *weighted inner product*

$$\langle f, g \rangle = \int_0^a f(x)g(x)x dx \tag{11}$$

2. First, let's take a closer look at function g at $\rho = a$.

$$g(y_{\nu m}, y_{\nu n}; a) = J_\nu(y_{\nu m}) a \left[\frac{dJ_\nu\left(\frac{y_{\nu n}\rho}{a}\right)}{d\rho}\right]\Bigg|_{\rho=a} = J_\nu(y_{\nu m}) y_{\nu n} J_\nu'(y_{\nu n}) \tag{12}$$

With this, if in (8) we replace $y_{\nu m}$ with a general y which is not necessarily a root of the eigenequation (1), we have

$$\begin{aligned}
\int_0^a J_\nu\left(\frac{y\rho}{a}\right) J_\nu\left(\frac{y_{\nu n}\rho}{a}\right) \rho d\rho &= \frac{a^2}{y^2 - y_{\nu n}^2}\left[g(y, y_{\nu n}; a) - g(y_{\nu n}, y; a)\right] && \text{by (12)} \\
&= \frac{a^2}{y^2 - y_{\nu n}^2}\left[J_\nu(y) y_{\nu n} J_\nu'(y_{\nu n}) - J_\nu(y_{\nu n}) y J_\nu'(y)\right] && y_{\nu n} \text{ is a root of (1)} \\
&= \frac{a^2}{y^2 - y_{\nu n}^2}\left[-J_\nu(y)\lambda J_\nu(y_{\nu n}) - J_\nu(y_{\nu n}) y J_\nu'(y)\right] \\
&= \frac{a^2}{y + y_{\nu n}}\left[-J_\nu(y_{\nu n})\right]\left[\frac{\lambda J_\nu(y) + y J_\nu'(y)}{y - y_{\nu n}}\right]
\end{aligned} \tag{13}$$

Now the desired normalization factor

$$N \equiv \int_0^a J_\nu\left(\frac{y_{\nu n}\rho}{a}\right) J_\nu\left(\frac{y_{\nu n}\rho}{a}\right) \rho d\rho \tag{14}$$

is just (13) under the limit $y \to y_{\nu n}$, i.e.,

$$\begin{aligned}
N &= \lim_{y \to y_{\nu n}} \frac{a^2}{y + y_{\nu n}}\left[-J_\nu(y_{\nu n})\right]\left[\frac{\lambda J_\nu(y) + y J_\nu'(y)}{y - y_{\nu n}}\right] \\
&= -\frac{a^2}{2y_{\nu n}} J_\nu(y_{\nu n}) \frac{d\left[\lambda J_\nu(y) + y J_\nu'(y)\right]}{dy}\Bigg|_{y=y_{\nu n}} \\
&= -\frac{a^2}{2} J_\nu(y_{\nu n}) \frac{\lambda J_\nu'(y_{\nu n}) + y_{\nu n} J_\nu''(y_{\nu n}) + J_\nu'(y_{\nu n})}{y_{\nu n}}
\end{aligned} \tag{15}$$

From (1) and (2) we have

$$\lambda J_\nu'(y_{\nu n}) = -\frac{\lambda^2 J_\nu(y_{\nu n})}{y_{\nu n}} \tag{16}$$

$$y_{\nu n} J_\nu''(y_{\nu n}) + J_\nu'(y_{\nu n}) = -y_{\nu n}\left(1 - \frac{\nu^2}{y_{\nu n}^2}\right) J_{\nu n}(y_{\nu n}) \tag{17}$$

This turns (15) into

$$N = \frac{a^2}{2}\left(1 + \frac{\lambda^2 - \nu^2}{y_{\nu n}^2}\right) J_\nu^2(y_{\nu n}) \tag{18}$$

This is one of the equivalent forms at the end of the problem.

With the normalization factor N decided (subsequently subscripted with νn to emphasize the dependency on ν and n), we can properly claim that the functions

$$U_{\nu n}(\rho) = \frac{1}{\sqrt{N_{\nu n}}} J_\nu\left(\frac{y_{\nu n}\rho}{a}\right) \tag{19}$$

form an orthonormal set of basis over the range $[0, a]$ with respect to the weighted inner product (11).

Now *if we further assume that this set is complete*, for any function $f(\rho)$ satisfying the boundary condition, we can expand it into

$$f(\rho) = \sum_{n=1}^{\infty} B_n U_{\nu n}(\rho) = \sum_{n=1}^{\infty} B_n \frac{1}{\sqrt{N_{\nu n}}} J_\nu\left(\frac{y_{\nu n}\rho}{a}\right) \quad \text{where} \tag{20}$$

$$B_n = \int_0^a f(\rho')U_{\nu n}(\rho')\rho' d\rho' = \int_0^a f(\rho')\frac{1}{\sqrt{N_{\nu n}}} J_\nu\left(\frac{y_{\nu n}\rho'}{a}\right)\rho' d\rho' \tag{21}$$

Thus if we define $A_n = B_n/\sqrt{N_{\nu n}}$, we have

$$f(\rho) = \sum_{n=1}^{\infty} A_n J_\nu\left(\frac{y_{\nu n}\rho}{a}\right) \quad \text{where} \tag{22}$$

$$\begin{aligned} A_n &= \frac{1}{N_{\nu n}} \int_0^a f(\rho')J_\nu\left(\frac{y_{\nu n}\rho'}{a}\right)\rho' d\rho' \\ &= \frac{2}{a^2}\left[\left(1 + \frac{\lambda^2 - \nu^2}{y_{\nu n}^2}\right)J_\nu^2(y_{\nu n})\right]^{-1} \int_0^a f(\rho')J_\nu\left(\frac{y_{\nu n}\rho'}{a}\right)\rho' d\rho' \end{aligned} \tag{23}$$

3. Proof of the alternative forms of normalization constant.

   (a)

$$\left[\left(1 - \frac{\nu^2}{y_{\nu n}^2}\right)J_\nu^2(y_{\nu n}) + J_\nu'^2(y_{\nu n})\right] = \left(1 + \frac{\lambda^2 - \nu^2}{y_{\nu n}^2}\right)J_\nu^2(y_{\nu n}) \tag{24}$$

   This follows trivially from (1).

   (b)

$$\left(1 + \frac{\lambda^2 - \nu^2}{y_{\nu n}^2}\right)J_\nu^2(y_{\nu n}) = \left(1 + \frac{y_{\nu n}^2 - \nu^2}{\lambda^2}\right)J_\nu'^2(y_{\nu n}) \tag{25}$$

   This also follows trivially from (1).

   (c)

$$\left[\left(1 - \frac{\nu^2}{y_{\nu n}^2}\right)J_\nu^2(y_{\nu n}) + J_\nu'^2(y_{\nu n})\right] = J_\nu^2(y_{\nu n}) - J_{\nu-1}(y_{\nu n})J_{\nu+1}(y_{\nu n}) \tag{26}$$

   This can be proved by using the recurrence relation (reference Wolfram)

$$\frac{\nu}{x}J_\nu(x) = \frac{J_{\nu+1}(x) + J_{\nu-1}(x)}{2} \tag{27}$$

$$J_\nu'(x) = -\frac{J_{\nu+1}(x) - J_{\nu-1}(x)}{2} \tag{28}$$