1. The rotation will not cause the magnetization density to change, so the field inside the sphere is the same as discussed in section 5.10, i.e.,

$$\mathbf{B}_{\text{in}} = \frac{2\mu_0 \mathbf{M}}{3} \tag{1}$$

The Ohm's law for a moving conductor is

$$\mathbf{J} = \sigma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \tag{2}$$

i.e., the contribution of force on the charge now has to add the Lorentz force from the magnetic field **B**.

In steady state where $\mathbf{J} = 0$, we can obtain the electric field

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}_{\text{in}} = -\left(\omega R \sin\theta \hat{\boldsymbol{\phi}}\right) \times \left(\frac{2\mu_0 M}{3}\hat{\mathbf{z}}\right) = -\frac{2\mu_0 \omega M R \sin\theta}{3}\hat{\boldsymbol{\rho}} = -\frac{2\mu_0 \omega M \boldsymbol{\rho}}{3} \tag{3}$$

where $\boldsymbol{\rho}$ is the radial vector in cylindrical coordinates. Thus the electric density is

$$\rho_e = \epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} = \epsilon_0 \frac{1}{\rho} \frac{\partial \left(\rho E_\rho\right)}{\partial \rho} = -\frac{4\mu_0 \epsilon_0 \omega M}{3} = -\frac{m\omega}{\pi c^2 R^3} \tag{4}$$

2. Note here we are essentially dealing with electrostatic problem since there is no current or time dependent magnetic fields. Integrating (3) from $(x, y, z) = (0, 0, R\cos\theta)$ radially outwards (keeping z unchanged) to $(\rho, \theta, z) = (R\sin\theta, \theta, R\cos\theta)$, we have the potential of a point on the surface:

$$\Phi(R, \theta, \phi) = \Phi_0 - \int_0^{R\sin\theta} E d\rho = \Phi_0 + \frac{\mu_0 \omega M R^2 \sin^2\theta}{3} = \Phi_0 + \frac{\mu_0 \omega M R^2}{3} \cdot \frac{2}{3}\left[P_0(\cos\theta) - P_2(\cos\theta)\right] \tag{5}$$

where Φ_0 is an arbitrary constant denonting the electric potential on the part of z axis inside the sphere.

For the exterior of the sphere, the potential should satisfy Laplace equation, which necessarily has the form

$$\Phi_{\text{ext}}(r, \theta, \phi) = A_0 + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \tag{6}$$

Matching (6) with (5) at the boundary $r = R$ gives

$$\begin{aligned} A_0 + \frac{B_0}{R} &= \Phi_0 + \frac{2\mu_0 \omega M R^2}{9} & \frac{B_1}{R^3} &= -\frac{2\mu_0 \omega M R^2}{9} & &\Longrightarrow \\ A_0 &= \Phi_0 + \beta R^2 - \frac{B_0}{R} & B_1 &= -\beta R^5 & &\text{where } \beta \equiv \frac{2\mu_0 \omega M}{9} = \frac{\mu_0 m \omega}{6\pi R^3} \end{aligned} \tag{7}$$

The exterior potential is thus

$$\Phi_{\text{ext}}(r, \theta, \phi) = A_0 + \frac{B_0}{r} + \frac{B_2}{r^3} P_2(\cos\theta) = A_0 + \frac{B_0}{r} - \frac{\beta R^5}{r^3} P_2(\cos\theta) \tag{8}$$

The first term is the gauge constant, the second term is the monopole constribution to the exterior of the sphere, and the third term is the quadrupole contribution.

Since the sphere is electrically neutral, the monopole contribution must vanish, hence $B_0 = 0$.

To find the quadrupole moment tensor components, recall the multipole expansion of potential in spherical tensor forms, equation (4.1)

$$\Phi = \frac{1}{4\pi\epsilon_0} \sum_{l,m} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \tag{9}$$

Matching (9) with (8) for $l = 2$ gives

$$\frac{1}{4\pi\epsilon_0} \sum_{m=-2}^{2} \frac{4\pi}{5} q_{2m} \frac{Y_{2m}(\theta, \phi)}{r^3} = -\frac{\beta R^5}{r^3} P_2(\cos\theta) \tag{10}$$

This allows us to find q_{2m} as the inner product

$$\begin{aligned}
\frac{1}{4\pi\epsilon_0}\frac{4\pi}{5}q_{2m} &= -\beta R^5 \int d\Omega Y^*_{2m}(\theta,\phi) P_2(\cos\theta) \\
&= -\beta R^5 \int_0^{2\pi} d\phi e^{-im\phi} \int_0^{\pi} \sin\theta d\theta \sqrt{\frac{5}{4\pi}} P_l^m(\cos\theta) P_2(\cos\theta) \\
&= -\beta R^5 \cdot 2\pi\delta_{m0}\sqrt{\frac{5}{4\pi}} \underbrace{\int_0^{\pi} \sin\theta d\theta P_2(\cos\theta) P_2(\cos\theta)}_{2/5} \qquad \Longrightarrow
\end{aligned}$$

$$q_{2m} = -\sqrt{\frac{5}{4\pi}} \cdot 4\pi\epsilon_0 \beta R^5 \delta_{m0} \tag{11}$$

The traceless Cartesian quadrupole moments can be obtained from (4.6), i.e.,

$$Q_{33} = 2\sqrt{\frac{4\pi}{5}} q_{20} = -8\pi\epsilon_0\beta R^5 = -8\pi\epsilon_0 R^5 \cdot \frac{\mu_0 m\omega}{6\pi R^3} = -\frac{4\omega m R^2}{3c^2} \tag{12}$$

$$Q_{11} = Q_{22} = -\frac{Q_{33}}{2} \tag{13}$$

3. We established $B_0 = 0$ above using the arguments of vanishing monopole contribution, but we can also determine that explicitly by calculating the surface charge density $\sigma(\theta)$.

   From (8), the radial component of the field just outside of the sphere is:

$$E_{R^+} = -\left.\frac{\partial\Phi_{\text{ext}}}{\partial r}\right|_{r=R} = \frac{B_0}{R^2} - 3\beta R P_2(\cos\theta) \tag{14}$$

   From (3), the radial component of the field just inside the sphere is

$$E_{R^-} = -|E|\sin\theta = -3\beta R \sin^2\theta = -2\beta R[1 - P_2(\cos\theta)] \tag{15}$$

   The surface density is thus

$$\sigma = \epsilon_0(E_{R^+} - E_{R^-}) = \frac{\epsilon_0 B_0}{R^2} + \epsilon_0\beta R[2 - 5P_2(\cos\theta)] \tag{16}$$

   Zero net charge requires

$$\begin{aligned}
R^2 \int d\Omega\sigma(\theta) + \frac{4\pi R^3}{3}\rho_e &= 0 \qquad \Longrightarrow \\
4\pi\epsilon_0 B_0 + 8\pi\epsilon_0\beta R^3 + \frac{4\pi R^3}{3}\left(-\frac{4\mu_0\epsilon_0\omega M}{3}\right) &= 0 \qquad \Longrightarrow \\
B_0 &= 0 \qquad (17)
\end{aligned}$$

   Agreeing with the vanishing monopole argument, of course.

   (16) gives

$$\sigma(\theta) = 2\epsilon_0\beta R\left[1 - \frac{5}{2}P_2(\cos\theta)\right] = \frac{m\omega}{3\pi c^2 R^2}\left[1 - \frac{5}{2}P_2(\cos\theta)\right] \tag{18}$$

4. The potential drop from the equator to the pole can be obtained from (5)

$$V = \Phi\left(R, \theta = \frac{\pi}{2}\right) - \Phi(R, \theta = 0) = \left.\frac{\mu_0\omega M R^2 \sin^2\theta}{3}\right|_0^{\pi/2} = \frac{\mu_0\omega m}{4\pi R} \tag{19}$$

   which is the same as the line integral of electric field from the equator to the pole via any path.