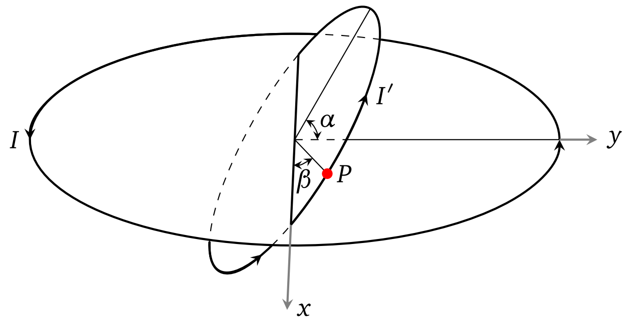

Without loss of generality, let I, I' be the current of the bigger and smaller loop respectively, and let the bigger loop be on the x-y plane, with $z+$ direction chosen that makes I counterclockwise. Let the smaller loop be rotated by α around the x-axis.

Any point P on the smaller loop can be parameterized by an angle β as shown above. In the frame S' where the smaller loop is "upright", point P's coordinate and the differential current at P have coordinates

$$\mathbf{x}' = b\begin{bmatrix}\cos\beta\\ \sin\beta\\ 0\end{bmatrix} \qquad I'd\mathbf{l}' = I'b\begin{bmatrix}-\sin\beta\\ \cos\beta\\ 0\end{bmatrix} \tag{1}$$

In the frame S where the bigger loop is upright, these corresponding representations are obtained by left multiplying the rotation matrix

$$R_x(\alpha) = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{bmatrix} \tag{2}$$

which gives

$$\mathbf{x} = R_x(\alpha)\mathbf{x}' = b\begin{bmatrix}\cos\beta\\ \cos\alpha\sin\beta\\ \sin\alpha\sin\beta\end{bmatrix} \qquad I'd\mathbf{l} = R_x(\alpha)\cdot I'd\mathbf{l}' = I'b\begin{bmatrix}-\sin\beta\\ \cos\alpha\cos\beta\\ \sin\alpha\cos\beta\end{bmatrix} \tag{3}$$

The torque excerted by the magnetic field generated by the outer loop on the inner loop is given by

$$\mathbf{N} = \int \mathbf{x}\times(\mathbf{J}\times\mathbf{B})\,d^3x = \oint_{\text{inner}} \mathbf{x}\times\left(I'd\mathbf{l}\times\mathbf{B}\right) = I'\oint_{\text{inner}}\left[(\mathbf{x}\cdot\mathbf{B})\,d\mathbf{l} - (\mathbf{x}\cdot d\mathbf{l})\,\mathbf{B}\right] \tag{4}$$

The second term vanishes because $\mathbf{x}$ and $d\mathbf{l}$ are orthogonal by virtue of being on a circle centered at origin. In section 5.5, we see that the magnetic field generated by the outer loop has a radial component (equation 5.48)

$$B_r = \frac{\mu_0 I a}{2r}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n n!}\frac{r_<^{2n+1}}{r_>^{2n+2}}P_{2n+1}(\cos\theta) \tag{5}$$

as well as a polar component B_θ. In the first term of integral (4), $\mathbf{x}$ is in the radial direction, so B_θ has no effect for $\mathbf{N}$.

Taking $r_< = |\mathbf{x}| = b, r_> = a$ and inserting (5) into (4), we have

$$\begin{aligned}\mathbf{N} &= \frac{\mu_0 I I' a}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n n!}\frac{b^{2n+1}}{a^{2n+2}}\oint_{\text{inner}}P_{2n+1}(\cos\theta)\,d\mathbf{l}\\ &= \frac{\mu_0 I I' b^2}{2a}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n n!}\left(\frac{b}{a}\right)^{2n}\int_0^{2\pi}\mathbf{f}(\beta)\,d\beta\end{aligned} \tag{6}$$

where the integrand is a vector function in β, given by (3):

$$\mathbf{f}(\beta) = \begin{bmatrix}-P_{2n+1}(\cos\theta)\sin\beta\\ P_{2n+1}(\cos\theta)\cos\alpha\cos\beta\\ P_{2n+1}(\cos\theta)\sin\alpha\cos\beta\end{bmatrix} \tag{7}$$

Recall the polar angle θ can be deduced by the cartesian coordinate in (3), i.e.,

$$\cos\theta = \frac{z}{|\mathbf{x}|} = \sin\alpha\sin\beta \tag{8}$$

Since for any k,

$$\int_0^{2\pi} \sin^k\beta\cos\beta\, d\beta = 0 \tag{9}$$

we know $N_y = N_z = 0$.

The x component of the torque is

$$N_x = \frac{\mu_0 I I' b^2}{2a} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)!!}{2^n n!} \left(\frac{b}{a}\right)^{2n} \overbrace{\int_0^{2\pi} P_{2n+1}(\sin\alpha\sin\beta)\sin\beta\, d\beta}^{I} \tag{10}$$

Here we make reference to the *Addition Theorem of Associated Legendre Functions* (see equation 14.18.2 on nist.gov), which states that for $\theta_1, \theta_2, \theta_1+\theta_2 \in [0,\pi)$ and ϕ real,

$$P_l(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi) = \sum_{m=-l}^{l} (-1)^m P_l^{-m}(\cos\theta_1) P_l^m(\cos\theta_2)\cos(m\phi) \tag{11}$$

Assigning $\theta_1 = \pi/2, \theta_2 = \alpha$ in (11) gives

$$P_l(\sin\alpha\cos\phi) = \sum_{m=-l}^{l} (-1)^m P_l^{-m}(0) P_l^m(\cos\alpha)\cos(m\phi) \tag{12}$$

With variable change $\beta = \pi/2 - \phi$, integral I in (10) becomes

$$\begin{aligned}
I &= \int_{-3\pi/2}^{\pi/2} P_{2n+1}(\sin\alpha\cos\phi)\cos\phi\, d\phi \\
&= \sum_{m=-(2n+1)}^{2n+1} (-1)^m P_{2n+1}^{-m}(0) P_{2n+1}^m(\cos\alpha) \overbrace{\int_{-3\pi/2}^{\pi/2} \cos(m\phi)\cos\phi\, d\phi}^{\pi\delta_{m1}+\pi\delta_{m,-1}} \\
&= -\pi\left[P_{2n+1}^{-1}(0)P_{2n+1}^1(\cos\alpha) + P_{2n+1}^1(0)P_{2n+1}^{-1}(\cos\alpha)\right] && \text{(by 3.51)} \\
&= \left[\frac{2\pi}{(2n+1)(2n+2)}\right]\cdot P_{2n+1}^1(0)P_{2n+1}^1(\cos\alpha) && \text{(by 5.45)} \\
&= \left[\frac{\pi}{(2n+1)(n+1)}\right]\left[\frac{(-1)^{n+1}\Gamma(n+3/2)}{\Gamma(n+1)\Gamma(3/2)}\right] P_{2n+1}^1(\cos\alpha) && (13)
\end{aligned}$$

Plugging (13) back into (10) yields

$$\begin{aligned}
N_x &= \frac{\mu_0 I I' \pi b^2}{2a} \sum_{n=0}^{\infty} \left[\frac{(2n-1)!!}{2^n n!(n+1)}\right]\left[\frac{\Gamma(n+3/2)}{\Gamma(n+1)\Gamma(3/2)}\right]\left(\frac{b}{a}\right)^{2n} P_{2n+1}^1(\cos\alpha) \\
&= \frac{\mu_0 I I' \pi b^2}{2a} \sum_{n=0}^{\infty} \left(\frac{n+1}{2n+1}\right)\left[\frac{(2n+1)!!}{2^n(n+1)!}\right]\left[\frac{\Gamma(n+3/2)}{\Gamma(n+2)\Gamma(3/2)}\right]\left(\frac{b}{a}\right)^{2n} P_{2n+1}^1(\cos\alpha) \\
&= \frac{\mu_0 I I' \pi b^2}{2a} \sum_{n=0}^{\infty} \left(\frac{n+1}{2n+1}\right)\left[\frac{\Gamma(n+3/2)}{\Gamma(n+2)\Gamma(3/2)}\right]^2\left(\frac{b}{a}\right)^{2n} P_{2n+1}^1(\cos\alpha) && (14)
\end{aligned}$$

Note that $P_1^1(\cos\alpha) = -\sin\alpha$, thus the leading order of (14) is negative for $\alpha \in [0,\pi]$, which agrees with the 0th order approxmiation obtained from $\mathbf{N} = \mathbf{m}\times\mathbf{B}(0)$.