1. **Prob 10.2**

We can treat the elliptic polarization

$$\epsilon = \frac{1}{\sqrt{1+r^2}}\left(\epsilon_+ + re^{i\alpha}\epsilon_-\right) \tag{1}$$

as the superposition of two circular polarizations, whose respective coefficients $\alpha_\pm(1), \beta_\pm(1)$ were given by the unnumbered equation above (10.71):

$$\alpha_\pm(1) = -\frac{2i}{3}(ka)^3 \qquad \beta_\pm(1) = \frac{4i}{3}(ka)^3 \tag{2}$$

In the long-wavelength limit, only $l=1$ contribution is worth keeping.

The differential scattering cross section can be obtained from (10.63) via the coherent sum of two circular polarizations, with the corresponding weight given in the incident elliptic polarization expression (1),

$$\begin{aligned}\frac{d\sigma_{sc}}{d\Omega} &= \frac{3\pi}{2k^2}\left|\frac{1}{\sqrt{1+r^2}}\left[\alpha_+(1)\mathbf{X}_{1,1} + i\beta_+(1)\mathbf{n}\times\mathbf{X}_{1,1}\right] + \frac{re^{i\alpha}}{\sqrt{1+r^2}}\left[\alpha_-(1)\mathbf{X}_{1,-1} - i\beta_-(1)\mathbf{n}\times\mathbf{X}_{1,-1}\right]\right|^2 \\ &= \frac{3\pi}{2k^2}\cdot k^6a^6\cdot\left(\frac{1}{1+r^2}\right)\left|-\frac{2i}{3}\mathbf{X}_{1,1} - \frac{4}{3}\mathbf{n}\times\mathbf{X}_{1,1} + re^{i\alpha}\left(-\frac{2i}{3}\mathbf{X}_{1,-1} + \frac{4}{3}\mathbf{n}\times\mathbf{X}_{1,-1}\right)\right|^2\end{aligned} \tag{3}$$

Referring to the explicit expressions of VSH ($l=1$) on Wikipedia, we have

$$\mathbf{X}_{1,1} = \frac{1}{i\sqrt{2}}\mathbf{\Phi}_{11} = \sqrt{\frac{3}{16\pi}}e^{i\phi}\left(\hat{\boldsymbol\theta} + i\cos\theta\hat{\boldsymbol\phi}\right) \tag{4}$$

$$\mathbf{X}_{1,-1} = -\mathbf{X}^*_{1,1} = -\sqrt{\frac{3}{16\pi}}e^{-i\phi}\left(\hat{\boldsymbol\theta} - i\cos\theta\hat{\boldsymbol\phi}\right) \tag{5}$$

$$\mathbf{n}\times\mathbf{X}_{1,1} = \frac{1}{i\sqrt{2}}\mathbf{n}\times\mathbf{\Phi}_{11} = -\frac{1}{i\sqrt{2}}\mathbf{\Psi}_{11} = \sqrt{\frac{3}{16\pi}}e^{i\phi}\left(-i\cos\theta\hat{\boldsymbol\theta} + \hat{\boldsymbol\phi}\right) \tag{6}$$

$$\mathbf{n}\times\mathbf{X}_{1,-1} = -\mathbf{n}\times\mathbf{X}^*_{1,1} = -\sqrt{\frac{3}{16\pi}}e^{-i\phi}\left(i\cos\theta\hat{\boldsymbol\theta} + \hat{\boldsymbol\phi}\right) \tag{7}$$

Collecting the coefficients of the $\hat{\boldsymbol\theta}, \hat{\boldsymbol\phi}$ components within the absolute value sign in (3), we get

$$\begin{aligned}\hat{\boldsymbol\theta}\text{ coefficient} &= \sqrt{\frac{3}{16\pi}}\left[-\frac{2i}{3}e^{i\phi} + \frac{4i}{3}\cos\theta e^{i\phi} + \frac{2i}{3}re^{i(\alpha-\phi)} - \frac{4i}{3}r\cos\theta e^{i(\alpha-\phi)}\right] \\ &= \sqrt{\frac{3}{16\pi}}\frac{2i}{3}e^{i\phi}\left[-1 + 2\cos\theta + re^{i(\alpha-2\phi)} - 2r\cos\theta e^{i(\alpha-2\phi)}\right] \\ &= \sqrt{\frac{3}{16\pi}}\frac{2i}{3}e^{i\phi}\left\{(2\cos\theta - 1)\left[1 - r\cos(\alpha-2\phi) - ir\sin(\alpha-2\phi)\right]\right\}\end{aligned} \tag{8}$$

$$\begin{aligned}\hat{\boldsymbol\phi}\text{ coefficient} &= \sqrt{\frac{3}{16\pi}}\left[\frac{2}{3}\cos\theta e^{i\phi} - \frac{4}{3}e^{i\phi} + \frac{2}{3}r\cos\theta e^{i(\alpha-\phi)} - \frac{4}{3}re^{i(\alpha-\phi)}\right] \\ &= \sqrt{\frac{3}{16\pi}}\frac{2}{3}e^{i\phi}\left[\cos\theta - 2 + r\cos\theta e^{i(\alpha-2\phi)} - 2re^{i(\alpha-2\phi)}\right] \\ &= \sqrt{\frac{3}{16\pi}}\frac{2}{3}e^{i\phi}\left\{(\cos\theta - 2)\left[1 + r\cos(\alpha-2\phi) + ir\sin(\alpha-2\phi)\right]\right\}\end{aligned} \tag{9}$$

Putting these back to (3) gives the desired result

$$\begin{aligned}\frac{d\sigma_{sc}}{d\Omega} =& \frac{3\pi}{2k^2}k^6a^6\cdot\left(\frac{1}{1+r^2}\right)\frac{3}{16\pi}\frac{4}{9}\Big\{(2\cos\theta-1)^2\left[1+r^2-2r\cos(\alpha-2\phi)\right] + \\ &\qquad (\cos\theta-2)^2\left[1+r^2+2r\cos(\alpha-2\phi)\right]\Big\} \\ =& k^4a^6\left[\frac{5}{8}\left(1+\cos^2\theta\right) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(\alpha-2\phi)\right]\end{aligned} \tag{10}$$

2. **Prob 10.3**

(a) With $\omega R/c \ll 1$, the size of the sphere is much smaller than the wave length, hence the external field can be considered as constant for the spherical region. This problem is already solved in section 5.12, in particular by equation (5.117), (5.121). The exterior scalar magnetic potential is

$$\Phi_M = -H_0 r\cos\theta + \left(\frac{\mu_c-\mu_0}{\mu_c+2\mu_0}\right) H_0 \frac{R^3}{r^2}\cos\theta \tag{11}$$

Then the magnetic field just outside of the sphere is

$$\mathbf{H} = -\boldsymbol{\nabla}\Phi_M\bigg|_{r=R} = H_0\left\{\hat{\mathbf{r}}\cos\theta\left[1+2\left(\frac{\mu_c-\mu_0}{\mu_c+2\mu_0}\right)\right] - \hat{\boldsymbol{\theta}}\sin\theta\left[1-\left(\frac{\mu_c-\mu_0}{\mu_c+2\mu_0}\right)\right]\right\} \tag{12}$$

It is worth noting that we have not used the assumption of "perfect conductor" here. If the sphere is perfect conductor, the normal component shall vanish, which is satisfied only when $\mu_c = 0$.

(b) Under the assumption of "good conductor", the discussion in section 8.1 applies, in particular equation (8.8), (8.12)

$$\frac{dP_{\text{abs}}(\theta,\phi)}{da} = \frac{\mu_c\omega\delta}{4}\left|\mathbf{H}_{\|}\right|^2 \qquad\qquad \delta = \sqrt{\frac{2}{\mu_c\omega\sigma}} \tag{13}$$

Thus the total absorbed power per unit area is

$$\begin{aligned}
P_{\text{abs}} &= R^2\int \frac{\mu_c\omega}{4}\sqrt{\frac{2}{\mu_c\omega\sigma}}\,|H_\theta|^2\,d\Omega \\
&= \frac{R^2}{4}\sqrt{\frac{2\mu_c\omega}{\sigma}}\left(1-\frac{\mu_c-\mu_0}{\mu_c+2\mu_0}\right)^2 H_0^2 \overbrace{\int_0^{2\pi} d\phi\int_0^{\pi}\sin^2\theta\sin\theta\, d\theta}^{8\pi/3} \\
&= \frac{R^2}{4}\sqrt{\frac{2\mu_c\omega}{\sigma}}\left(\frac{3\mu_0}{\mu_c+2\mu_0}\right)^2 H_0^2\frac{8\pi}{3} \\
&= 6\pi R^2 H_0^2\sqrt{\frac{2\mu_c\omega}{\sigma}}\left(\frac{\mu_0}{\mu_c+2\mu_0}\right)^2
\end{aligned} \tag{14}$$

Dividing this by the incident energy flux $c\mu_0 H_0^2/2$ gives the total absorption cross section

$$\sigma_{\text{abs}} = \frac{12\pi R^2}{c\mu_0}\sqrt{\frac{2\mu_c\omega}{\sigma}}\left(\frac{\mu_0}{\mu_c+2\mu_0}\right)^2 = 12\pi R^2\sqrt{\frac{2\mu_c\epsilon_0\omega}{\mu_0\sigma}}\left(\frac{\mu_0}{\mu_c+2\mu_0}\right)^2 \tag{15}$$