In these notes, we supply a few derivation details for the density effect formulae.

1. **Fermi's energy loss formula (13.36)**

First let's derive (13.36)

$$\left(\frac{dE}{dx}\right)_{b>a} = \frac{2}{\pi}\frac{(ze)^2}{v^2}\,\mathrm{Re}\int_0^\infty i\omega\lambda^* a K_1(\lambda^* a)K_0(\lambda a)\left[\frac{1}{\epsilon(\omega)}-\beta^2\right]d\omega \tag{1}$$

where

$$\lambda^2 = \left(\frac{\omega}{v}\right)^2\left[1-\beta^2\epsilon(\omega)\right] \tag{2}$$

Recall (13.34) and (13.35)

$$\Delta E(b) = \frac{1}{2\pi N}\,\mathrm{Re}\int_0^\infty -i\omega\epsilon(\omega)\left|\mathbf{E}(\omega)\right|^2 d\omega \tag{3}$$

$$\left(\frac{dE}{dx}\right)_{b>a} = 2\pi N\int_a^\infty \Delta E(b)\, b\,db \tag{4}$$

Combining them yields

$$\begin{aligned}\left(\frac{dE}{dx}\right)_{b>a} &= \mathrm{Re}\int_0^\infty -i\omega\epsilon(\omega)\,d\omega\int_a^\infty \left|\mathbf{E}(\omega)\right|^2 b\,db\\ &= \mathrm{Re}\int_0^\infty i\omega\epsilon^*(\omega)\,d\omega\int_a^\infty \left|\mathbf{E}(\omega)\right|^2 b\,db\end{aligned} \tag{5}$$

The two orthogonal components of $\mathbf{E}(\omega)$ are given in (13.32) and (13.33)

$$E_1(\omega) = -\frac{ize\omega}{v^2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{\epsilon(\omega)}-\beta^2\right]K_0(\lambda b) \tag{6}$$

$$E_2(\omega) = \frac{ze}{v}\sqrt{\frac{2}{\pi}}\frac{\lambda}{\epsilon(\omega)}K_1(\lambda b) \tag{7}$$

thus

$$\begin{aligned}\left|\mathbf{E}(\omega)\right|^2 =& E_1(\omega)E_1^*(\omega)+E_2(\omega)E_2^*(\omega)\\ =&\left(\frac{ze\omega}{v^2}\right)^2\left(\frac{2}{\pi}\right)\left(\frac{1}{\epsilon}-\beta^2\right)\left(\frac{1}{\epsilon^*}-\beta^2\right)K_0(\lambda b)K_0^*(\lambda b)+\\ &\left(\frac{ze}{v}\right)^2\left(\frac{2}{\pi}\right)\frac{\lambda\lambda^*}{\epsilon\epsilon^*}K_1(\lambda b)K_1^*(\lambda b) && K_\nu^*(z)=K_\nu(z^*)\\ =&\left(\frac{ze}{v}\right)^2\left(\frac{2}{\pi}\right)\left[\left(\frac{\omega}{v}\right)^2\left(\frac{1}{\epsilon}-\beta^2\right)\left(\frac{1}{\epsilon^*}-\beta^2\right)K_0(\lambda b)K_0(\lambda^* b)+\frac{\lambda\lambda^*}{\epsilon\epsilon^*}K_1(\lambda b)K_1(\lambda^* b)\right] && \text{use conjugate of (2)}\\ =&\left(\frac{ze}{v}\right)^2\left(\frac{2}{\pi}\right)\frac{\lambda^*}{\epsilon\epsilon^*}\left[\lambda^*\left(1-\beta^2\epsilon\right)K_0(\lambda b)K_0(\lambda^* b)+\lambda K_1(\lambda b)K_1(\lambda^* b)\right]\end{aligned} \tag{8}$$

Putting (8) into (5) gives

$$\begin{aligned}\left(\frac{dE}{dx}\right)_{b>a} =& \left(\frac{ze}{v}\right)^2\left(\frac{2}{\pi}\right)\mathrm{Re}\int_0^\infty \frac{i\omega\lambda^*}{\epsilon}d\omega\times\\ &\left[\lambda^*\left(1-\beta^2\epsilon\right)\underbrace{\int_a^\infty K_0(\lambda b)K_0(\lambda^* b)\,b\,db}_{I_0}+\lambda\overbrace{\int_a^\infty K_1(\lambda b)K_1(\lambda^* b)\,b\,db}^{I_1}\right]\end{aligned} \tag{9}$$

The integral I_0, I_1 can be looked up from 2.2.4(a), 2.2.4(b) of [1]

$$I_0 = \left.\frac{\lambda^* b K_0(\lambda b)K_1(\lambda^* b)-\lambda b K_1(\lambda b)K_0(\lambda^* b)}{\lambda^2-\lambda^{*2}}\right|_{b=a}^{\infty} = \frac{\lambda a K_1(\lambda a)K_0(\lambda^* a)-\lambda^* a K_0(\lambda a)K_1(\lambda^* a)}{\lambda^2-\lambda^{*2}} \tag{10}$$

$$I_1 = \left.\frac{\lambda^* b K_1(\lambda b)K_0(\lambda^* b)-\lambda b K_0(\lambda b)K_1(\lambda^* b)}{\lambda^2-\lambda^{*2}}\right|_{b=a}^{\infty} = \frac{\lambda a K_0(\lambda a)K_1(\lambda^* a)-\lambda^* a K_1(\lambda a)K_0(\lambda^* a)}{\lambda^2-\lambda^{*2}} \tag{11}$$

[1] **WernerRosenheinrich2025**, *Tables of Some Indefinite Integrals of Bessel Functions of Integer Order, 2025.*

where the choice of root λ in the fourth quadrant (top of pp634) ensures that $K_\nu(\lambda b)$ exponentially decays as $b \to \infty$. Denote $A \equiv K_1(\lambda^* a) K_0(\lambda a)$, then I_0, I_1 can be simplified as

$$I_0 = \frac{a(\lambda A^* - \lambda^* A)}{\lambda^2 - \lambda^{*2}} \qquad I_1 = \frac{a(\lambda A - \lambda^* A^*)}{\lambda^2 - \lambda^{*2}} \tag{12}$$

hence the square bracket in (9) becomes (see diagram above)

$$\begin{aligned}[\ldots] &= \left(\frac{a}{\lambda^2 - \lambda^{*2}}\right)\left[\lambda^*\left(1-\beta^2\epsilon\right)(\lambda A^* - \lambda^* A) + \lambda(\lambda A - \lambda^* A^*)\right] \\ &= \left(\frac{a}{\lambda^2 - \lambda^{*2}}\right)\left[\left(\lambda^2 - \lambda^{*2}\right)\left(1-\beta^2\epsilon\right)A + \lambda\beta^2\epsilon(\lambda A - \lambda^* A^*)\right] \\ &= a\left(1-\beta^2\epsilon\right)A + a\beta^2\lambda\epsilon\left(\frac{\lambda A - \lambda^* A^*}{\lambda^2 - \lambda^{*2}}\right)\end{aligned} \tag{13}$$

Putting (13) into (9) yields (13.36)

$$\begin{aligned}\left(\frac{dE}{dx}\right)_{b>a} &= \left(\frac{ze}{v}\right)^2\left(\frac{2}{\pi}\right)\mathrm{Re}\left\{\int_0^\infty i\omega\lambda^* a\left(\frac{1}{\epsilon} - \beta^2\right)A\,d\omega + \overbrace{\int_0^\infty i\omega a\beta^2\left[\lambda\lambda^*\left(\frac{\lambda A - \lambda^* A^*}{\lambda^2 - \lambda^{*2}}\right)\right]d\omega}^{\text{purely imaginary}}\right\} \\ &= \left(\frac{ze}{v}\right)^2\left(\frac{2}{\pi}\right)\mathrm{Re}\int_0^\infty i\omega\lambda^* a\left(\frac{1}{\epsilon} - \beta^2\right)K_1(\lambda^* a)K_0(\lambda a)\,d\omega\end{aligned} \tag{14}$$

2. **Recovering nonrelativistic form (13.9)**

   It was claimed in the paragraph after (13.38) that in nonrelativistic limit, Fermi's formula (13.36) will have form (13.9). Let's derive that.

   If we take the approximation that the second term of (13.38)

$$\epsilon(\omega) = 1 + \frac{4\pi N e^2}{m}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\Gamma_j} \tag{15}$$

   is small, we have

$$\mathrm{Im}\left[\frac{1}{\epsilon(\omega)}\right] \approx \mathrm{Im}\left(1 - \frac{4\pi N e^2}{m}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\Gamma_j}\right) \approx \frac{4\pi N e^2}{m}\,\mathrm{Im}\sum_j\left(\frac{f_j}{\omega^2 - \omega_j^2 + i\omega\Gamma_j}\right) \tag{16}$$

   Putting $\lambda \approx \omega/v$ (i.e., real) into (13.36) gives

$$\begin{aligned}\left(\frac{dE}{dx}\right)_{b>a} &= \frac{2}{\pi}\frac{(ze)^2}{v^2}\int_0^\infty -\omega\lambda a K_1(\lambda a)K_0(\lambda a)\,\mathrm{Im}\left[\frac{1}{\epsilon(\omega)}\right]d\omega \\ &= \frac{2}{\pi}\frac{(ze)^2}{v^2}\cdot\frac{4\pi N e^2}{m}\sum_j f_j\,\mathrm{Im}\int_0^\infty \frac{F(\omega)\,d\omega}{\omega^2 - \omega_j^2 + i\omega\Gamma_j}\end{aligned} \tag{17}$$

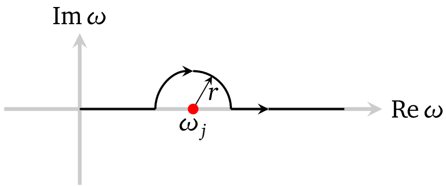
   where

$$\begin{aligned}F(\omega) &= -\omega\lambda a K_1(\lambda a)K_0(\lambda a) && \text{small argument approximation of } K_{0,1}(z) \\ &\approx -\omega(\lambda a)\left(\frac{1}{\lambda a}\right)\ln\left(\frac{2e^{-\gamma_E}}{\lambda a}\right) && \gamma_E \approx 0.5772 \\ &\approx -\omega\ln\left(\frac{1.123v}{\omega a}\right)\end{aligned} \tag{18}$$

   To evaluate the imaginary part of the integral in (17), notice that for the j-th frequency, the poles are at

$$\omega_\pm = \frac{-i\Gamma_j \pm \sqrt{-\Gamma_j^2 + 4\omega_j^2}}{2} \approx \pm\omega_j - \frac{i\Gamma_j}{2} \tag{19}$$

   which in the limit $\Gamma_j \to 0^+$, approach $\pm\omega_j$ from below.



Thus the integral in (17) becomes

$$\int_0^\infty = \lim_{r\to 0}\left(\int_0^{\omega_j - r} + \int_{\omega_j + r}^\infty + \int_{\text{semi-circle}}\right) \tag{20}$$

The sum of the first two terms is the principal value integral which is real and has no contribution to the imaginary part in (17). The integral over the semi-circle is

$$\lim_{r\to 0}\int_{\text{semi-circle}} = \int_\pi^0 \frac{F\left(\omega_j\right) i r e^{i\phi} d\phi}{2\omega_j r e^{i\phi}} = -\frac{i\pi}{2}\frac{F\left(\omega_j\right)}{\omega_j} \tag{21}$$

Substituting $F(\omega)$ using (18) gives

$$\mathrm{Im}\int_0^\infty \frac{F(\omega)}{\omega^2 - \omega_j^2 + i\omega\Gamma_j} d\omega \approx \frac{\pi}{2}\ln\left(\frac{1.123v}{\omega_j a}\right) \tag{22}$$

Putting everything back to (17), we obtain the desired form (13.9)

$$\begin{aligned}
\left(\frac{dE}{dx}\right)_{b>a} &\approx \frac{2}{\pi}\frac{(ze)^2}{v^2}\frac{4\pi N e^2}{m}\frac{\pi}{2}\sum_j f_j \ln\left(\frac{1.123v}{\omega_j a}\right) \\
&= \frac{4\pi N Z z^2 e^4}{m v^2}\frac{1}{Z}\sum_j\left[f_j \ln\left(\frac{1.123v}{a}\right) - f_j \ln\omega_j\right] && \text{use (13.11), (7.52)} \\
&= \frac{4\pi N Z z^2 e^4}{m v^2}\ln\left(\frac{1.123v}{a\langle\omega\rangle}\right)
\end{aligned} \tag{23}$$

with

$$B_c = \frac{1.123v}{v\langle\omega\rangle} \tag{24}$$