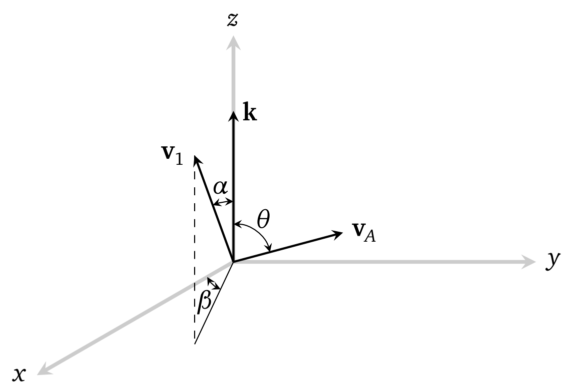

1. Let $\mathbf{k}$ be pointing to the $\hat{\mathbf{z}}$ direction, and let $\mathbf{v}_A$ be in the y-z plane, i.e.,

$$\mathbf{v}_A = v_A(\cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\mathbf{y}}) \tag{1}$$

The most general form of the amplitude $\mathbf{v}_1$ is

$$\mathbf{v}_1 = v_1(\cos\alpha\,\hat{\mathbf{z}} + \sin\alpha\cos\beta\,\hat{\mathbf{x}} + \sin\alpha\sin\beta\,\hat{\mathbf{y}}) \tag{2}$$

Recall equation (7.75)

$$-\omega^2\mathbf{v}_1 + \left(s^2+v_A^2\right)(\mathbf{k}\cdot\mathbf{v}_1)\,\mathbf{k} + (\mathbf{v}_A\cdot\mathbf{k})\left[(\mathbf{v}_A\cdot\mathbf{k})\,\mathbf{v}_1 - (\mathbf{v}_A\cdot\mathbf{v}_1)\,\mathbf{k} - (\mathbf{k}\cdot\mathbf{v}_1)\,\mathbf{v}_A\right] = 0 \tag{3}$$

Inserting (1) and (2) into (3) produces the following component-wise equations

$$\begin{aligned}
\hat{\mathbf{z}}: &\quad -\omega^2\cos\alpha + \left(s^2+v_A^2\right)k^2\cos\alpha + k^2v_A^2\cos^2\theta\cos\alpha \\
&\quad -k^2v_A^2\cos\theta\,(\cos\theta\cos\alpha + \sin\theta\sin\alpha\sin\beta) - k^2v_A^2\cos^2\theta\cos\alpha = 0 &(4)\\
\hat{\mathbf{x}}: &\quad -\omega^2\sin\alpha\cos\beta + k^2v_A^2\cos^2\theta\sin\alpha\cos\beta = 0 &(5)\\
\hat{\mathbf{y}}: &\quad -\omega^2\sin\alpha\sin\beta + k^2v_A^2\cos^2\theta\sin\alpha\sin\beta - k^2v_A^2\cos\theta\cos\alpha\sin\theta = 0 &(6)
\end{aligned}$$

With $u^2 = \omega^2/k^2$, (4)-(6) become

$$\left(-u^2+s^2+v_A^2\sin^2\theta\right)\cos\alpha - v_A^2\cos\theta\sin\theta\sin\alpha\sin\beta = 0 \tag{7}$$

$$\left(-u^2+v_A^2\cos^2\theta\right)\sin\alpha\cos\beta = 0 \tag{8}$$

$$\left(-u^2+v_A^2\cos^2\theta\right)\sin\alpha\sin\beta - v_A^2\cos\theta\sin\theta\cos\alpha = 0 \tag{9}$$

We have the following cases to consider.

(a) If $\sin\alpha\cos\beta \neq 0$, we immediately see from (8) that

$$u^2 = v_A^2\cos^2\theta \tag{10}$$

then (9) requires

$$\cos\theta\sin\theta\cos\alpha = 0 \tag{11}$$

This means

- Either $\cos\alpha = 0$, or
- $\cos\theta\sin\theta = 0$, in which case by (7), we must have $\cos\alpha = 0$ anyway.

To summarize, as long as $\mathbf{v}_1$ has a non-zero $\hat{\mathbf{x}}$ component, it must completely lie within the x-y plane, i.e., it is a transverse wave with phase velocity given by (10).

(b) If $\sin\alpha\cos\beta = 0$, then we must have either $\alpha = 0$ or $\beta = \pi/2$.

i. If $\alpha = 0$, this corresponds to the longitudinal wave $\mathbf{v}_1 \parallel \mathbf{k}$. (9) requires $\cos\theta\sin\theta = 0$, then by (7) we must have

$$u^2 = s^2 + v_A^2\sin^2\theta \tag{12}$$

Thus, depending on whether $\theta = 0$ (i.e., $\mathbf{v}_A \parallel \mathbf{k}$) or $\theta = \pi/2$ (i.e., $\mathbf{v}_A \perp \mathbf{k}$), we have either $u = s$ or $u = \sqrt{s^2+v_A^2}$, which are exactly the two longitudinal wave cases discussed in the text.

ii. If $\beta = \pi/2$

A. If $\alpha = \pi/2$, it's easy to see this is covered by case (a) above.

B. If $\alpha \neq \pi/2$ and $\alpha \neq 0$, this is the most general case which allows us to combine (7) and (9) and cancel the factor $\sin\alpha\cos\alpha\sin\beta$ and obtain

$$\left(-u^2 + s^2 + v_A^2 \sin^2\theta\right)\left(-u^2 + v_A^2\cos^2\theta\right) = \left(v_A^2\cos\theta\sin\theta\right)^2 \tag{13}$$

which yields the solution

$$u_\pm^2 = \frac{1}{2}\left[\left(s^2 + v_A^2\right) \pm \sqrt{\left(s^2 + v_A^2\right)^2 - 4s^2 v_A^2\cos^2\theta}\right] \tag{14}$$

2. The velocity vector's direction was discussed in detail in the cases above.

3. When $v_A \gg s$, up to $O\left(s^2\right)$, (14) can be approximated by

$$u_\pm^2 \approx \frac{1}{2}\left\{\left(s^2 + v_A^2\right) \pm v_A^2\left[1 + \frac{s^2}{v_A^2}\left(1 - 2\cos^2\theta\right)\right]\right\} \qquad \Longrightarrow \qquad u_+^2 \approx v_A^2 + O\left(s^2\right) \qquad u_-^2 \approx O(s^2) \tag{15}$$

Recall the $u_\pm^2$ solutions are the result of case (b).ii.B, with $\beta = \pi/2$ and arbitrary α as long as $\alpha \neq 0, \alpha \neq \pi/2$. But $\beta = \pi/2$ means that $\mathbf{v}_1$ is in the y-z plane, which is the plane of $\mathbf{k}$ and $\mathbf{v}_A$.
Plugging u_+^2 back into (9) and ignoring $O\left(s^2\right)$, we have

$$-\sin\theta^2\sin\alpha - \cos\theta\sin\theta\cos\alpha = 0 \qquad \Longrightarrow \qquad \sin\theta\cos(\theta - \alpha) = 0 \tag{16}$$

this gives a direction of $\mathbf{v}_1$ that is perpendicular to the field $\mathbf{v}_A$.
u_-^2 can be ignored in (9) since it is $O\left(s^2\right)$, this means

$$\cos^2\theta\sin\alpha - \cos\theta\sin\theta\cos\alpha = 0 \qquad \Longrightarrow \qquad \cos\theta\sin(\alpha - \theta) = 0 \tag{17}$$

this gives a direction of $\mathbf{v}_1$ that aligns with the field $\mathbf{v}_A$.