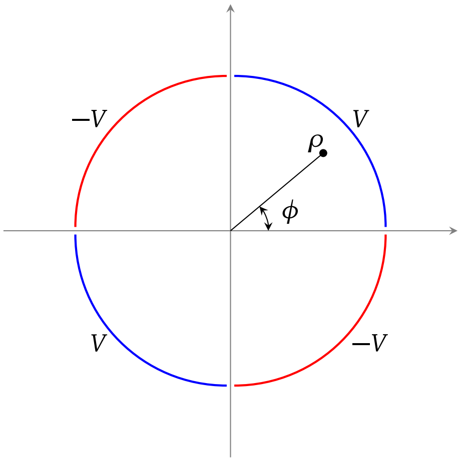

1. This is a straightforward application of the relation we have proved back in problem 2.12:

$$\Phi(\rho,\phi)=\frac{1}{2\pi}\int_0^{2\pi} d\phi'\Phi(b,\phi')\left[1+2\sum_{n=1}^{\infty}\frac{\rho^n}{b^n}\cos n(\phi-\phi')\right] \tag{1}$$

Clearly the contribution from the "1" term in the bracket is zero due to the alternating signs. The contribution from n is

$$C_n=\frac{V}{2\pi}\frac{2\rho^n}{b^n}\left(I_1+I_2+I_3+I_4\right) \tag{2}$$

where

$$\begin{aligned} I_1 &= \int_0^{\pi/2}\cos n\left(\phi'-\phi\right)d\phi' = \frac{1}{n}\left[\sin n\left(\frac{\pi}{2}-\phi\right)-\sin n(-\phi)\right] \\ &= \frac{1}{n}\,\mathrm{Im}\left(e^{in\pi/2-in\phi}-e^{-in\phi}\right) \\ &= \frac{1}{n}\,\mathrm{Im}\left[\left(i^n-1\right)e^{-in\phi}\right] && (3) \\ I_2 &= -\frac{1}{n}\,\mathrm{Im}\left[\left(i^{2n}-i^n\right)e^{-in\phi}\right] && (4) \\ I_3 &= \frac{1}{n}\,\mathrm{Im}\left[\left(i^{3n}-i^{2n}\right)e^{-in\phi}\right] && (5) \\ I_4 &= -\frac{1}{n}\,\mathrm{Im}\left[\left(i^{4n}-i^{3n}\right)e^{-in\phi}\right] && (6) \end{aligned}$$

Therefore

$$I_1+I_2+I_3+I_4=\frac{1}{n}\,\mathrm{Im}\left\{2e^{-in\phi}\left[i^n-1-(-1)^n+(-i)^n\right]\right\} \tag{7}$$

which clearly vanishes unless $n=4k+2$, in which case

$$I_1+I_2+I_3+I_4=\frac{8\sin n\phi}{n} \qquad \text{when } n=4k+2 \tag{8}$$

This gives

$$\Phi(\rho,\phi)=\frac{4V}{\pi}\sum_{k=0}^{\infty}\left(\frac{\rho}{b}\right)^{4k+2}\frac{\sin(4k+2)\phi}{2k+1} \tag{9}$$

2. We could sum the series in (9) as the imaginary part of a power series modulated by coefficient $1/(2k+1)$, which involves $\tan^{-1}$ of complex numbers. I'm not familiar with trigonometry function of complex argument, so I'm going to prove the claim via another route.

Recall in problem 2.12, we proved

$$\Phi(\rho,\phi)=\frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\overbrace{\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}}^{A(\phi')}d\phi' \tag{10}$$

With this, we can obtain the interior point's potential via four-part integration:

$$\begin{aligned}\Phi(\rho,\phi)&=\frac{V}{2\pi}\left[\int_0^{\pi/2}A(\phi')d\phi'-\int_{\pi/2}^{\pi}A(\phi')d\phi'+\int_{\pi}^{3\pi/2}A(\phi')d\phi'-\int_{3\pi/2}^{2\pi}A(\phi')d\phi'\right]\\&=\frac{V}{2\pi}\left\{\int_0^{\pi/2}\left[A(\phi')+A(\phi'-\pi)\right]d\phi'-\int_{\pi/2}^{\pi}\left[A(\phi')+A(\phi'-\pi)\right]d\phi'\right\}\end{aligned} \tag{11}$$

Now define

$$\begin{aligned}B(\phi')&=A(\phi')+A(\phi'-\pi)\\&=\left(b^2-\rho^2\right)\left[\frac{1}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}+\frac{1}{b^2+\rho^2+2b\rho\cos(\phi'-\phi)}\right]\\&=\left(b^2-\rho^2\right)\left[\frac{2\left(b^2+\rho^2\right)}{(b^2+\rho^2)^2-4b^2\rho^2\cos^2(\phi'-\phi)}\right]\end{aligned} \tag{12}$$

Then the potential

$$\begin{aligned}\Phi(\rho,\phi)&=\frac{V}{2\pi}\left[\int_0^{\pi/2}B(\phi')d\phi'-\int_{\pi/2}^{\pi}B(\phi')d\phi'\right]\\&=\frac{V}{2\pi}\int_0^{\pi/2}\left[B(\phi')-B\left(\phi'-\frac{\pi}{2}\right)\right]d\phi'\\&=\frac{V}{2\pi}\cdot 2\left(b^4-\rho^4\right)\int_0^{\pi/2}\left[\frac{1}{(b^2+\rho^2)^2-4b^2\rho^2\cos^2(\phi'-\phi)}-\frac{1}{(b^2+\rho^2)^2-4b^2\rho^2\sin^2(\phi'-\phi)}\right]d\phi'\\&=\frac{V}{\pi}\left(b^4-\rho^4\right)\int_0^{\pi/2}\frac{4b^2\rho^2\cos 2(\phi'-\phi)\,d\phi'}{(b^2+\rho^2)^4-4b^2\rho^2(b^2+\rho^2)^2+16b^4\rho^4\cos^2(\phi'-\phi)\sin^2(\phi'-\phi)}\\&=\frac{V}{\pi}\left(b^4-\rho^4\right)\int_0^{\pi/2}\frac{4b^2\rho^2\cos 2(\phi'-\phi)\,d\phi'}{(b^4-\rho^4)^2+4b^4\rho^4\sin^2 2(\phi'-\phi)}\end{aligned} \tag{13}$$

Now it's the usual practice of variable change which we used in problem 2.13 – define

$$t\equiv\sin 2\left(\phi'-\phi\right) \tag{14}$$

then

$$\Phi(\rho,\phi)=\frac{V}{\pi}\left(b^4-\rho^4\right)\int_{t_0}^{t_1}\frac{2b^2\rho^2dt}{(b^4-\rho^4)^2+4b^4\rho^4t^2} \tag{15}$$

And subsequently, with

$$\tan\xi\equiv\frac{2b^2\rho^2}{b^4-\rho^4}t \tag{16}$$

we get

$$\Phi(\rho,\phi)=\frac{V}{\pi}\left(b^4-\rho^4\right)\int_{\xi_0}^{\xi_1}\frac{2b^2\rho^2\dfrac{b^4-\rho^4}{2b^2\rho^2}\dfrac{1}{\cos^2\xi}d\xi}{(b^4-\rho^4)^2\dfrac{1}{\cos^2\xi}}=\frac{V}{\pi}(\xi_1-\xi_0) \tag{17}$$

From (14) and (16), we have the bounds ξ_0,ξ_1

$$\xi_0=-\tan^{-1}\left(\frac{2b^2\rho^2}{b^4-\rho^4}\sin 2\phi\right)\qquad \xi_1=\tan^{-1}\left(\frac{2b^2\rho^2}{b^4-\rho^4}\sin 2\phi\right) \tag{18}$$

which finally yields

$$\Phi(\rho,\phi)=\frac{2V}{\pi}\tan^{-1}\left(\frac{2b^2\rho^2}{b^4-\rho^4}\sin 2\phi\right) \tag{19}$$