1. Before working on the problem, we have to first show the condition under which the method of image is valid. For electrostatics, the guarantee is the uniqueness theorem. For magnetostatics, it's a variant of the uniqueness theorem. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two vector potentials generated by current distribution $\mathbf{J}$, i.e.,

$$\nabla \times (\nabla \times \mathbf{A}_1) = \nabla \times (\nabla \times \mathbf{A}_2) = \mu_0 \mathbf{J} \tag{1}$$

Let $\mathbf{W} \equiv \mathbf{A}_1 - \mathbf{A}_2$, then

$$\nabla \times (\nabla \times \mathbf{W}) = 0 \tag{2}$$

By Gauss's theorem, for vector field $\mathbf{P}$ and $\mathbf{Q}$

$$\begin{aligned} \oint_S [\mathbf{P} \times (\nabla \times \mathbf{Q})] \cdot \mathbf{n} da &= \int_V \nabla \cdot [\mathbf{P} \times (\nabla \times \mathbf{Q})] d^3x \\ &= \int_V \{(\nabla \times \mathbf{P}) \cdot (\nabla \times \mathbf{Q}) - \mathbf{P} \cdot [\nabla \times (\nabla \times \mathbf{Q})]\} d^3x \end{aligned} \tag{3}$$

where we have used vector identity

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \tag{4}$$

Setting $\mathbf{P} = \mathbf{Q} = \mathbf{W}$ yields

$$\oint_S [\mathbf{W} \times (\nabla \times \mathbf{W})] \cdot \mathbf{n} da = \int_V (\nabla \times \mathbf{W})^2 d^3x \tag{5}$$

With the vector identity

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \tag{6}$$

(5) becomes

$$\int_V (\nabla \times \mathbf{W})^2 d^3x = \oint_S \mathbf{W} \cdot [(\nabla \times \mathbf{W}) \times \mathbf{n}] da \tag{7}$$

which means if

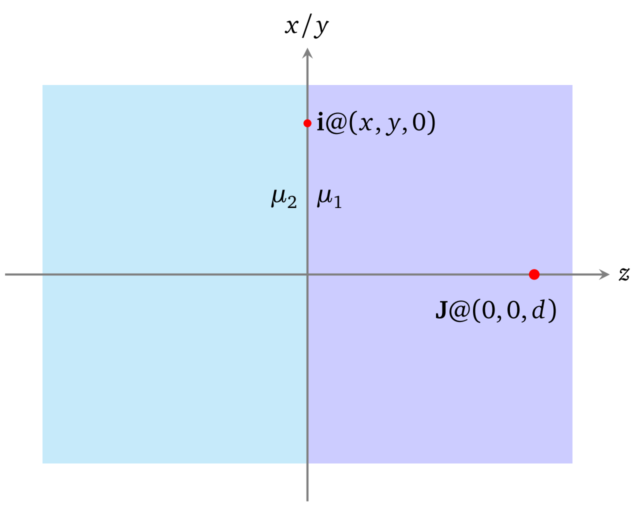
$$(\nabla \times \mathbf{W}) \times \mathbf{n} = (\nabla \times \mathbf{A}_1 - \nabla \times \mathbf{A}_2) \times \mathbf{n} = (\mathbf{B}_1 - \mathbf{B}_2) \times \mathbf{n} = 0 \tag{8}$$

everywhere on the boundary surface, we must end up with

$$\nabla \times \mathbf{W} = \mathbf{B}_1 - \mathbf{B}_2 = 0 \tag{9}$$

everywhere in the volume V, i.e., *the magnetic field solution is unique when the tangential field $\mathbf{B}_{\parallel} = \mathbf{n} \times \mathbf{B}$ is specified on every point of the boundary surface.*

2. Now coming back to problem 5.17. Let's consider the effect on point $\mathbf{x} = (x, y, 0)$ generated by the "point" current $\mathbf{J}$ at $(0, 0, d)$.



Let $\mathbf{H_J}(\mathbf{x})$ be the magnetic field at $\mathbf{x}$ due to $\mathbf{J}$, which can be calculated by the macroscopic law

$$\mathbf{H_J}(\mathbf{x}) = \frac{1}{4\pi}\frac{\mathbf{J}\times(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}-d\hat{\mathbf{z}})}{\sqrt{x^2+y^2+d^2}^3} \tag{10}$$

Thus the tangential component of the microscopic induction flux due to $\mathbf{J}$ at $\mathbf{x}$ is

$$\begin{aligned}
\mathbf{B}_{\|,\mathbf{J}}(\mathbf{x}) &= \hat{\mathbf{z}}\times(\mu_1\mathbf{H_J}) \\
&= \frac{\mu_1}{4\pi}\frac{\hat{\mathbf{z}}\times[\mathbf{J}\times(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}-d\hat{\mathbf{z}})]}{\sqrt{x^2+y^2+d^2}^3} \\
&= \frac{\mu_1}{4\pi}\frac{\hat{\mathbf{z}}\times\left(-J_y x\hat{\mathbf{z}}+J_z x\hat{\mathbf{y}}+J_x y\hat{\mathbf{z}}-J_z y\hat{\mathbf{x}}+J_x d\hat{\mathbf{y}}-J_y d\hat{\mathbf{x}}\right)}{\sqrt{x^2+y^2+d^2}^3} \\
&= \frac{\mu_1}{4\pi}\frac{\left[-(J_z x+J_x d)\hat{\mathbf{x}}-\left(J_z y+J_y d\right)\hat{\mathbf{y}}\right]}{\sqrt{x^2+y^2+d^2}^3}
\end{aligned} \tag{11}$$

Now let $\mathbf{i}$ be the induced surface current density at $\mathbf{x}$. At the microscopic level, $\mathbf{i}$ gives rise to a tangential induction flux $\pm\mu_0\mathbf{i}\times\hat{\mathbf{z}}/2$ on the two sides $0^\pm$.

The microscopic tangential induction fluxes on the two sides of 0 can be obtained by superposition

$$\mathbf{B}_{\|,0^+}(\mathbf{x}) = \mathbf{B}_{\|,\mathbf{J}}(\mathbf{x}) + \frac{\mu_0\mathbf{i}\times\hat{\mathbf{z}}}{2} \qquad \mathbf{B}_{\|,0^-}(\mathbf{x}) = \mathbf{B}_{\|,\mathbf{J}}(\mathbf{x}) - \frac{\mu_0\mathbf{i}\times\hat{\mathbf{z}}}{2} \tag{12}$$

and they must satisfy the boundary constraint

$$\frac{1}{\mu_1}\mathbf{B}_{\|,0^+} = \frac{1}{\mu_2}\mathbf{B}_{\|,0^-} \implies \mu_2\left(\mathbf{B}_{\|,\mathbf{J}}+\frac{\mu_0\mathbf{i}\times\hat{\mathbf{z}}}{2}\right) = \mu_1\left(\mathbf{B}_{\|,\mathbf{J}}-\frac{\mu_0\mathbf{i}\times\hat{\mathbf{z}}}{2}\right) \implies \frac{\mu_0\mathbf{i}\times\hat{\mathbf{z}}}{2} = -\left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)\mathbf{B}_{\|,\mathbf{J}}(\mathbf{x}) \tag{13}$$

which turns (12) into

$$\mathbf{B}_{\|,0^+}(\mathbf{x}) = \mathbf{B}_{\|,\mathbf{J}}(\mathbf{x}) - \left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)\mathbf{B}_{\|,\mathbf{J}}(\mathbf{x}) \tag{14}$$

$$\mathbf{B}_{\|,0^-}(\mathbf{x}) = \left(\frac{2\mu_2}{\mu_2+\mu_1}\right)\mathbf{B}_{\|,\mathbf{J}}(\mathbf{x}) \tag{15}$$

The additional term in (14) can be reformulated as

$$\begin{aligned}
-\left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)\mathbf{B}_{\|,\mathbf{J}}(\mathbf{x}) = \frac{\mu_1}{4\pi}\frac{1}{\sqrt{x^2+y^2+d^2}^3}\Bigg\{ &-\left[-\left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)J_z x+\left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)J_x(-d)\right]\hat{\mathbf{x}} \\
&-\left[-\left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)J_z y+\left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)J_y(-d)\right]\hat{\mathbf{y}}\Bigg\}
\end{aligned} \tag{16}$$

Referring to (11), we can interpret (16) as

$$\frac{\mu_1}{4\pi}\frac{\hat{\mathbf{z}}\times[\mathbf{J}'\times(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+d\hat{\mathbf{z}})]}{\sqrt{x^2+y^2+(-d)^2}^3} \tag{17}$$

where $\mathbf{J}'$ is the image "point" current located at $(0,0,-d)$, with components

$$J'_x = \left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)J_x \qquad J'_y = \left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)J_y \qquad J'_z = -\left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)J_z \tag{18}$$

Similarly, the RHS of (15) can be reformulated as

$$\begin{aligned}
\left(\frac{2\mu_2}{\mu_2+\mu_1}\right)\mathbf{B}_{\|,\mathbf{J}}(\mathbf{x}) = \frac{\mu_1}{4\pi}\frac{1}{\sqrt{x^2+y^2+d^2}^3}\Bigg\{ &-\left[\left(\frac{2\mu_2}{\mu_2+\mu_1}\right)J_z x+\left(\frac{2\mu_2}{\mu_2+\mu_1}\right)J_x d\right]\hat{\mathbf{x}} \\
&-\left[\left(\frac{2\mu_2}{\mu_2+\mu_1}\right)J_z y+\left(\frac{2\mu_2}{\mu_2+\mu_1}\right)J_y d\right]\hat{\mathbf{y}}\Bigg\}
\end{aligned} \tag{19}$$

interpreted as

$$\frac{\mu_1}{4\pi}\frac{\hat{\mathbf{z}}\times[\mathbf{J}''\times(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}-d\hat{\mathbf{z}})]}{\sqrt{x^2+y^2+d^2}^3} \tag{20}$$

with image point current $\mathbf{J}'' = 2\mu_2\mathbf{J}/(\mu_2+\mu_1)$ at $(0,0,d)$ in a medium of permeability μ_1.

What we have proved is that the image currents generate the specified tangential induction flux $\mathbf{n}\times\mathbf{B}$ everywhere on the corresponding region's boundary surface. Applying the uniqueness theorem proved above, we conclude that these image current distributions will give the desired field in the corresponding region's volume.

It is worth noting that although we used the "point" current above, the method of image developed above applies for the whole current distribution due to superposition.