Let's first write out the charge density function in spherical coordinates:

$$\rho(\mathbf{x},t)=\frac{q}{2\pi r^2}\left[2\delta(r)\,\delta(\cos\theta)-\delta(r-a\cos\omega t)\,\delta(\cos\theta-1)-\delta(r-a\cos\omega t)\,\delta(\cos\theta+1)\right] \tag{1}$$

In particular, it has no ϕ dependency, and the factor $1/2\pi$ is necessary for normalization.

Apparently the current density $\mathbf{J}$ is along the z-axis, so the effective magnetization $\mathscr{M}=\mathbf{x}\times\mathbf{J}/2=0$. Thus by (9.170) and (9.172), only the electric multipole moment q_{lm} will contribute to the radiation.

Let

$$C_{lm}=\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} \tag{2}$$

be the normalization constant for the spherical harmonics. Then by definition, the spherical multipole moment q_{lm} is

$$q_{lm}(t)=\int r^l Y_{lm}^*(\theta,\phi)\,\rho(\mathbf{x},t)\,d^3x=I_1+I_2+I_3 \tag{3}$$

where I_1,I_2,I_3 are the integrals corresponding to the three terms in (1):

$$I_1=\frac{2q}{2\pi}\int_0^\infty r^l\delta(r)\,dr\int_{-1}^1 C_{lm}P_l^m(\cos\theta)\,\delta(\cos\theta)\,d(\cos\theta)\int_0^{2\pi}e^{-im\phi}\,d\phi=2q\sqrt{\frac{1}{4\pi}}\delta_{l0}\delta_{m0} \tag{4}$$

$$\begin{aligned}I_2&=-\frac{q}{2\pi}\int_0^\infty r^l\delta(r-a\cos\omega t)\,dr\int_{-1}^1 C_{lm}P_l^m(\cos\theta)\,\delta(\cos\theta-1)\,d(\cos\theta)\int_0^{2\pi}e^{-im\phi}\,d\phi\\&=-q(a\cos\omega t)^l\sqrt{\frac{2l+1}{4\pi}}P_l^0(1)\,\delta_{m0}\end{aligned} \tag{5}$$

$$I_3=-q(a\cos\omega t)^l\sqrt{\frac{2l+1}{4\pi}}P_l^0(-1)\,\delta_{m0} \tag{6}$$

Obviously I_2+I_3 vanishes for odd l, and when $l=0$, I_2+I_3 corresponds to the monopole $-2q$ which cancels I_1.

In summary

$$q_{lm}(t)=\begin{cases}-2q\sqrt{\dfrac{2l+1}{4\pi}}(a\cos\omega t)^l & \text{for } m=0, l=2,4,6,\cdots\\ 0 & \text{otherwise}\end{cases} \tag{7}$$

For $l=2k$ with $k=1,2,3,\cdots$, we can expand $(a\cos\omega t)^{2k}=\left[a\left(e^{i\omega t}+e^{-i\omega t}\right)/2\right]^{2k}$ to obtain

$$\begin{aligned}q_{2k,0}(t)&=-2q\sqrt{\frac{4k+1}{4\pi}}\left(\frac{a}{2}\right)^{2k}\sum_{n=0}^{2k}\binom{2k}{n}e^{i2(k-n)\omega t}\\&=-2q\sqrt{\frac{4k+1}{4\pi}}\left(\frac{a}{2}\right)^{2k}\left\{\binom{2k}{k}+\sum_{n=0}^{k-1}\binom{2k}{n}e^{i2(k-n)\omega t}+\sum_{n'=k+1}^{2k}\binom{2k}{n'}e^{i2\left(k-n'\right)\omega t}\right\}\qquad\text{let } n'=2k-n\\&=-2q\sqrt{\frac{4k+1}{4\pi}}\left(\frac{a}{2}\right)^{2k}\left\{\binom{2k}{k}+\sum_{n=0}^{k-1}2\binom{2k}{n}\cos\left[2(k-n)\omega t\right]\right\}\end{aligned} \tag{8}$$

We see for every $l=2k$, the multipole $q_{l0}(t)$ has harmonic frequencies $2\omega,4\omega,6\omega,\cdots,2k\omega$. For the lowest order $l=2$ at its only harmonic 2ω, the multipole moment is

$$q_{20}(t)\bigg|_{2\omega}=-qa^2\sqrt{\frac{5}{4\pi}} \tag{9}$$

In general, to calculate the angular power distribution due to a multipole in the radiation zone, we should use (9.151), but in our case, we can take a shortcut. Since q_{20} is the only non-zero component of $l=2$, we can use (4.6) to convert it to the Cartesian representation Q_{ij} using the fact that Q is symmetric and traceless:

$$-2Q_{11}=-2Q_{22}=Q_{33}=2q_{20}\sqrt{\frac{4\pi}{5}}=-2qa^2\qquad\qquad Q_{12}=Q_{13}=Q_{23}=0 \tag{10}$$

This allows us to use (9.51) and (9.52) to get (note $k=2\omega/c$)

$$\frac{dP}{d\Omega}=\frac{c^2Z_0k^6}{512\pi^2}\left(-2qa^2\right)^2\sin^2\theta\cos^2\theta=\frac{Z_0\omega^6q^2a^4}{2\pi^2c^4}\sin^2\theta\cos^2\theta\qquad P=\frac{c^2Z_0k^6}{960\pi}\left(-2qa^2\right)^2=\frac{4Z_0\omega^6q^2a^4}{15\pi c^4} \tag{11}$$