1. For steady-state currents, the Proca equation reduces to the vector equation

$$\nabla^2\mathbf{A}-\mu^2\mathbf{A}=-\frac{4\pi}{c}\mathbf{J}=-\frac{4\pi}{c}\cdot c\nabla\times[\mathbf{m}f(x)]=-4\pi\nabla f\times\mathbf{m} \tag{1}$$

Recall in section 6.4, we have established that for Helmholtz equation with delta source (6.36)

$$\left(\nabla^2+k^2\right)G_k\left(\mathbf{x},\mathbf{x}'\right)=-4\pi\delta\left(\mathbf{x}-\mathbf{x}'\right) \tag{2}$$

the solution is Green's function

$$G_k\left(\mathbf{x},\mathbf{x}'\right)=\frac{e^{\pm ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \tag{3}$$

Then it follows that for arbitrary source funciton $g(\mathbf{x})$, the solution to the equation

$$\left(\nabla^2+k^2\right)\Psi(\mathbf{x})=-4\pi g(\mathbf{x}) \tag{4}$$

is the convolution

$$\Psi(\mathbf{x})=\int G_k\left(\mathbf{x},\mathbf{x}'\right)g\left(\mathbf{x}'\right)d^3x' \tag{5}$$

For the Proca equation, we can identify $k=i\mu$ (or $-i\mu$) and choose the positive (or negative) sign in (3) to ensure convergence in the exponential, then we have

$$\begin{aligned}\mathbf{A}(\mathbf{x})&=\int\left[\nabla'f\left(\mathbf{x}'\right)\times\mathbf{m}\right]\left(\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\right)d^3x'\\&=\left[\int\nabla'f\left(\mathbf{x}'\right)\left(\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\right)d^3x'\right]\times\mathbf{m} && \text{integration by parts}\\&=\left[-\int f\left(\mathbf{x}'\right)\nabla'\left(\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\right)d^3x'\right]\times\mathbf{m} && \nabla'\leftrightarrow-\nabla\\&=-\mathbf{m}\times\nabla\int f\left(\mathbf{x}'\right)\left(\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\right)d^3x'\end{aligned} \tag{6}$$

2. For $f(\mathbf{x})=\delta(\mathbf{x})$, (6) is simplified to

$$\mathbf{A}(\mathbf{x})=-\mathbf{m}\times\nabla\left(\frac{e^{-\mu r}}{r}\right)=\nabla\times\left[\mathbf{m}\left(\frac{e^{-\mu r}}{r}\right)\right] \tag{7}$$

giving

$$\begin{aligned}\mathbf{B}(\mathbf{x})=\nabla\times\mathbf{A}(\mathbf{x})&=\nabla\left\{\nabla\cdot\left[\mathbf{m}\left(\frac{e^{-\mu r}}{r}\right)\right]\right\}-\nabla^2\left[\mathbf{m}\left(\frac{e^{-\mu r}}{r}\right)\right]\\&=\nabla\left[\mathbf{m}\cdot\underbrace{\nabla\left(\frac{e^{-\mu r}}{r}\right)}_{-(\mu r+1)e^{-\mu r}/r^2\hat{\mathbf{r}}}\right]-\mathbf{m}\underbrace{\nabla^2\left(\frac{e^{-\mu r}}{r}\right)}_{\mu^2e^{-\mu r}/r}\\&=-\nabla(\mathbf{m}\cdot\mathbf{r})\left[\frac{(\mu r+1)e^{-\mu r}}{r^3}\right]-(\mathbf{m}\cdot\mathbf{r})\nabla\left[\frac{(\mu r+1)e^{-\mu r}}{r^3}\right]-\mu^2\mathbf{m}\frac{e^{-\mu r}}{r}\\&=\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}\right]\left(1+\mu r+\frac{\mu^2r^2}{3}\right)\frac{e^{-\mu r}}{r^3}-\frac{2\mu^2\mathbf{m}}{3}\frac{e^{-\mu r}}{r}\end{aligned} \tag{8}$$

3. At the equator, the ratio of the "external field" to the dipole field gives

$$\frac{-\dfrac{2\mu^2\mathbf{m}}{3}\dfrac{e^{-\mu R}}{R}}{-\mathbf{m}\left(1+\mu R+\dfrac{\mu^2R^2}{3}\right)\dfrac{e^{-\mu R}}{R^3}}\le 4\times10^{-3}\qquad\Longrightarrow\qquad -0.074\le\mu R\le0.08 \tag{9}$$

hence

$$m_{\text{photon}}\le\frac{\mu\hbar}{c}\approx4.4\times10^{-51}\text{kg} \tag{10}$$