Integrating the differential magnetic induction

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{1}$$

around the loops gives the field at point $\mathbf{x}$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \oint_C d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{2}$$

By the sign convention of the solid angle Ω as stated in the problem,

$$\Omega = \int_S \frac{-\mathbf{n}' \cdot (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} da' \tag{3}$$

The desired field form is thus

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \boldsymbol{\nabla}\Omega = -\frac{\mu_0 I}{4\pi} \boldsymbol{\nabla} \int_S \frac{\mathbf{n}' \cdot (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} da' \tag{4}$$

Comparing(4) with (2), we see that it remains to prove

$$\oint_C d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\boldsymbol{\nabla} \int_S \frac{\mathbf{n}' \cdot (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} da' \tag{5}$$

Define

$$\mathbf{v}(\mathbf{x},\mathbf{x}') \equiv \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{6}$$

We readily recognize

$$\mathbf{v}(\mathbf{x},\mathbf{x}') = -\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = \boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{7}$$

which turns the LHS of (5) into

$$\text{LHS}_{(5)} = \oint_C d\mathbf{l}' \times \mathbf{v} \tag{8}$$

whose k-th component is just

$$\text{LHS}_{(5)}\Big|_k = \epsilon_{ijk} \oint_C dl_i' \cdot v_j \tag{9}$$

where here, as well as in the following, we use the Einstein's summation convention.

Application of the vector identity (see front cover of Jackson)

$$\oint_C \psi d\mathbf{l} = \int_S \mathbf{n} \times \boldsymbol{\nabla}\psi da \tag{10}$$

to (9) gives (remember the gradient must be taken with respect to $\mathbf{x}'$)

$$\begin{aligned}
\text{LHS}_{(5)}\Big|_k &= \epsilon_{ijk} \left(\int_S \mathbf{n}' \times \boldsymbol{\nabla}' v_j da' \right)_i \\
&= \int_S \epsilon_{ijk} \left(\mathbf{n}' \times \boldsymbol{\nabla}' v_j\right)_i da' \\
&= \int_S \epsilon_{ijk}\epsilon_{lmi} \left(n_l' \frac{\partial v_j}{\partial x_m'} \right) da'
\end{aligned} \tag{11}$$

On the other hand, the k-th component of the RHS of (5) is

$$\mathrm{RHS}_{(5)}\Big|_k = -\frac{\partial}{\partial x_k}\int_S \mathbf{n}'\cdot\mathbf{v}\,da' = -\int_S n_i'\frac{\partial v_i}{\partial x_k}da' \qquad \left(\text{use } \frac{\partial v_i}{\partial x_k} = -\frac{\partial v_i}{\partial x_k'}\right)$$
$$= \int_S n_i'\frac{\partial v_i}{\partial x_k'}da' \tag{12}$$

Then comparing (11) and (12), we see it's sufficient to prove

$$\epsilon_{ijk}\epsilon_{lmi}\left(n_l'\frac{\partial v_j}{\partial x_m'}\right) = n_i'\frac{\partial v_i}{\partial x_k'} \tag{13}$$

With the identity

$$\epsilon_{ijk}\epsilon_{lmi} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{14}$$

the LHS of (13) becomes

$$\mathrm{LHS}_{(13)} = \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right)\left(n_l'\frac{\partial v_j}{\partial x_m'}\right) = n_j'\frac{\partial v_j}{\partial x_k'} - n_k'\frac{\partial v_j}{\partial x_j'} \tag{15}$$

whose first term (after Einstein summation) is equal to the RHS of (13), and whose second term vanishes since

$$n_k'\frac{\partial v_j}{\partial x_j'} = n_k'\boldsymbol{\nabla}'^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = 0 \tag{16}$$

when $\mathbf{x} \neq \mathbf{x}'$.