With the given field

$$E_z = J_m(\gamma r) e^{im\phi} e^{i\beta z - i\omega t} \qquad H_z = 0 \tag{1}$$

$$E_\phi = -\frac{m\beta}{\gamma^2}\frac{E_z}{r} \qquad H_r = -\frac{k}{Z_0\beta}E_\phi \tag{2}$$

$$E_r = \frac{i\beta}{\gamma^2}\frac{\partial E_z}{\partial r} \qquad H_\phi = \frac{k}{Z_0\beta}E_r \tag{3}$$

The time averaged angular momentum density is

$$\mathbf{m} = \frac{1}{2c^2}\,\mathrm{Re}\left[\mathbf{x}\times(\mathbf{E}\times\mathbf{H}^*)\right] = \frac{1}{2c^2}\,\mathrm{Re}\left[(r\hat{\mathbf{r}} + z\hat{\mathbf{z}})\times(\mathbf{E}\times\mathbf{H}^*)\right] \tag{4}$$

so the z component of which is

$$\begin{aligned} m_z &= \frac{1}{2c^2}\,\mathrm{Re}\left[r\left(\mathbf{E}\times\mathbf{H}^*\right)_\phi\right] = \frac{1}{2c^2}\,\mathrm{Re}\left[r\left(E_z H_r^* - E_r H_z^*\right)\right] \\ &= \frac{1}{2c^2}\,\mathrm{Re}\left[r E_z\left(-\frac{k}{Z_0\beta}\right)\left(-\frac{m\beta}{\gamma^2}\right)\frac{E_z^*}{r}\right] \\ &= \frac{mk}{2c^2 Z_0\gamma^2}\left|E_z\right|^2 \qquad \text{let } t = \gamma r \\ &= \frac{mk}{2c^2 Z_0 \gamma^2} J_m^2(t) \end{aligned} \tag{5}$$

Its integration over the cross section at a given z is

$$\int m_z da = 2\pi\cdot\frac{mk}{2c^2 Z_0\gamma^4}\overbrace{\int_0^{\gamma R} J_m^2(t)\,t\,dt}^{I} = 2\pi\cdot\frac{mk}{2c^2 Z_0\gamma^4} I \tag{6}$$

On the other hand, the time averaged energy density is

$$\begin{aligned} u &= \frac{\epsilon_0}{4}\left(\mathbf{E}\cdot\mathbf{E}^* + Z_0^2\mathbf{H}\cdot\mathbf{H}^*\right) \\ &= \frac{\epsilon_0}{4}\left[\left|E_z\right|^2 + \left(\frac{m\beta}{\gamma^2}\right)^2\frac{\left|E_z\right|^2}{r^2} + \left(\frac{\beta}{\gamma^2}\right)^2\left|\frac{\partial E_z}{\partial r}\right|^2 + \left(\frac{k}{\beta}\right)^2\left(\frac{m\beta}{\gamma^2}\right)^2\frac{\left|E_z\right|^2}{r^2} + \left(\frac{k}{\beta}\right)^2\left(\frac{\beta}{\gamma^2}\right)^2\left|\frac{\partial E_z}{\partial r}\right|^2\right] \\ &= \frac{\epsilon_0}{4}\left[\left|E_z\right|^2 + \frac{m^2\left(\beta^2 + k^2\right)}{\gamma^4}\frac{\left|E_z\right|^2}{r^2} + \frac{\left(\beta^2+k^2\right)}{\gamma^4}\left|\frac{\partial E_z}{\partial r}\right|^2\right] \\ &= \frac{\epsilon_0}{4}\left\{J_m^2(t) + \frac{\left(\beta^2+k^2\right)}{\gamma^2}\left[\frac{m^2}{t^2}J_m^2(t) + J_m'^2(t)\right]\right\} \end{aligned} \tag{7}$$

whose integration over the cross section at a given z is

$$\int u da = 2\pi\cdot\frac{\epsilon_0}{4}\left\{\frac{I}{\gamma^2} + \frac{\left(\beta^2+k^2\right)}{\gamma^4}\overbrace{\int_0^{\gamma R}\left[\frac{m^2}{t^2}J_m^2(t) + J_m'^2(t)\right]t\,dt}^{I'}\right\} \tag{8}$$

With the Bessel equation

$$\frac{1}{t}\frac{d}{dt}\left[tJ_m'(t)\right] + \left(1 - \frac{m^2}{t^2}\right)J_m(t) = 0 \tag{9}$$

we have

$$\begin{aligned} I' &= \int_0^{\gamma R}\left\{J_m(t)\frac{d}{dt}\left[tJ_m'(t)\right] + J_m'^2(t)\,t + J_m^2(t)\,t\right\}dt \\ &= \int_0^{\gamma R}\frac{d}{dt}\left[tJ_m(t)J_m'(t)\right]dt + I \qquad \text{note } J_m(\gamma R) = 0 \\ &= I \end{aligned} \tag{10}$$

Then (8) becomes

$$\begin{aligned}\int uda &= 2\pi\cdot\frac{\epsilon_0}{4}\frac{1}{\gamma^2}\left[1+\frac{\left(\beta^2+k^2\right)}{\gamma^2}\right]I \qquad\qquad \text{recall } k^2=\beta^2+\gamma^2\\ &= 2\pi\cdot\frac{\epsilon_0}{2\gamma^2}\left(1+\frac{\beta^2}{\gamma^2}\right)I \end{aligned} \tag{11}$$

The ratio of the z component of the angular momentum to the energy is thus

$$\frac{\int m_z da}{\int uda}=\frac{\dfrac{mk}{c^2Z_0\gamma^4}}{\dfrac{\epsilon_0}{\gamma^2}\left(1+\dfrac{\beta^2}{\gamma^2}\right)}=\frac{mk}{(\beta^2+\gamma^2)c}=\frac{mk}{k(kc)}=\frac{m\hbar}{\hbar\omega} \tag{12}$$