1. From problem 9.7, we have

$$\frac{dP(t)}{d\Omega}=\frac{Z_0}{16\pi^2c^2}\left|\left[\mathbf{n}\times\ddot{\mathbf{p}}\left(t'\right)\right]\times\mathbf{n}\right|^2=\frac{Z_0}{16\pi^2c^2}\left|\mathbf{n}\times\ddot{\mathbf{p}}\left(t'\right)\right|^2=\frac{Z_0}{16\pi^2c^2}\left|\ddot{\mathbf{p}}\left(t'\right)\right|^2\sin^2\theta \qquad \text{for } t'=t-\frac{r}{c} \tag{1}$$

which gives the *instantaneous* total power

$$P(t)=\frac{Z_0}{16\pi^2c^2}\left|\ddot{\mathbf{p}}\left(t'\right)\right|^2\overbrace{\int\sin^2\theta\, d\Omega}^{8\pi/3}=\frac{1}{6\pi\epsilon_0c^3}\left|\ddot{\mathbf{p}}\left(t'\right)\right|^2 \tag{2}$$

In order to derive the *instantaneous* rate of radiation of angular momentum, we need to go back to problem 9.8 which relates this quantity to the instantaneous real fields (for details, see the solution to problem 9.8)

$$\left.\frac{d\mathbf{L}_{\text{loss}}(t)}{dt}\right|_{>r}=-\epsilon_0\int r^3\left\{\left[\mathbf{n}\cdot\mathbf{E}(\mathbf{x},t)\right]\left[\mathbf{n}\times\mathbf{E}(\mathbf{x},t)\right]+c^2\left[\mathbf{n}\cdot\mathbf{B}(\mathbf{x},t)\right]\left[\mathbf{n}\times\mathbf{B}(\mathbf{x},t)\right]\right\}d\Omega \qquad \text{for } \mathbf{E},\mathbf{B} \text{ real} \tag{3}$$

where $|_{>r}$ indicates our interests in the radiated angular momentum outside of sphere with radius r.

However when $\mathbf{E},\mathbf{B}$ are in complex forms, standard routine requires us to change one of the factors into its complex conjugate followed by taking the overall real part, i.e.,

$$\left.\frac{d\mathbf{L}_{\text{loss}}(t)}{dt}\right|_{>r}=-\epsilon_0\,\text{Re}\int r^3\left\{\left[\mathbf{n}\cdot\mathbf{E}(\mathbf{x},t)\right]\left[\mathbf{n}\times\mathbf{E}^*(\mathbf{x},t)\right]+c^2\left[\mathbf{n}\cdot\mathbf{B}(\mathbf{x},t)\right]\left[\mathbf{n}\times\mathbf{B}^*(\mathbf{x},t)\right]\right\}d\Omega \quad \text{for } \mathbf{E},\mathbf{B} \text{ complex} \tag{4}$$

Let the Fourier decomposition of $\mathbf{E}(\mathbf{x},t)$ be

$$\mathbf{E}(\mathbf{x},t)=\int\mathbf{E}(\mathbf{x},\omega)e^{-i\omega t}d\omega \tag{5}$$

where $\mathbf{E}(\mathbf{x},\omega)$ is given by (9.18)

$$\mathbf{E}(\mathbf{x},\omega)=\frac{1}{4\pi\epsilon_0}\left\{k^2\left[\mathbf{n}\times\mathbf{p}(\omega)\right]\times\mathbf{n}\frac{e^{ikr}}{r}+\left\{3\mathbf{n}\left[\mathbf{n}\cdot\mathbf{p}(\omega)\right]-\mathbf{p}(\omega)\right\}\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\} \tag{6}$$

with $\mathbf{p}(\omega)$ the Fourier transform of $\mathbf{p}(t)$.

We could do the similar to the magnetic field but (9.18) indicates that for all frequencies, $\mathbf{B}(\mathbf{x},\omega)$ is transverse, hence $\mathbf{B}(\mathbf{x},t)$ is transverse, rendering its effect on (4) to be nil.

Now (4) becomes

$$\begin{aligned}\left.\frac{d\mathbf{L}_{\text{loss}}(t)}{dt}\right|_{>r}&=-\epsilon_0\,\text{Re}\int r^3\left[\mathbf{n}\cdot\mathbf{E}(\mathbf{x},t)\right]\left[\mathbf{n}\times\mathbf{E}^*(\mathbf{x},t)\right]d\Omega\\&=-\epsilon_0\,\text{Re}\int r^3d\Omega\int d\omega e^{-i\omega t}\int d\omega' e^{i\omega' t}\left[\mathbf{n}\cdot\mathbf{E}(\mathbf{x},\omega)\right]\left[\mathbf{n}\times\mathbf{E}^*\left(\mathbf{x},\omega'\right)\right]\end{aligned} \tag{7}$$

Using (6), we see that

$$\mathbf{n}\cdot\mathbf{E}(\mathbf{x},\omega)=\frac{1}{4\pi\epsilon_0}\cdot 2\left[\mathbf{n}\cdot\mathbf{p}(\omega)\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr} \tag{8}$$

$$\mathbf{n}\times\mathbf{E}^*\left(\mathbf{x},\omega'\right)=\frac{1}{4\pi\epsilon_0}\left\{k'^2\overbrace{\mathbf{n}\times\left\{\left[\mathbf{n}\times\mathbf{p}^*\left(\omega'\right)\right]\times\mathbf{n}\right\}}^{\mathbf{n}\times\mathbf{p}^*\left(\omega'\right)}\frac{e^{-ik'r}}{r}-\mathbf{n}\times\mathbf{p}^*\left(\omega'\right)\left(\frac{1}{r^3}+\frac{ik'}{r^2}\right)e^{-ik'r}\right\} \tag{9}$$

and

$$\begin{aligned}\left[\mathbf{n}\cdot\mathbf{E}(\mathbf{x},\omega)\right]\left[\mathbf{n}\times\mathbf{E}^*\left(\mathbf{x},\omega'\right)\right]&=\left\{-\frac{ikk'^2}{8\pi^2\epsilon_0^2r^3}\left[\mathbf{n}\cdot\mathbf{p}(\omega)\right]\left[\mathbf{n}\times\mathbf{p}^*\left(\omega'\right)\right]+O\left(\frac{1}{r^4}\right)\right\}e^{i\left(k-k'\right)r} \qquad \text{(for large } r\text{)}\\&\approx\frac{1}{8\pi^2\epsilon_0^2c^3r^3}\left[-i\omega e^{i\omega r/c}\mathbf{n}\cdot\mathbf{p}(\omega)\right]\left[\omega'^2e^{-i\omega' r/c}\mathbf{n}\times\mathbf{p}^*\left(\omega'\right)\right]\end{aligned} \tag{10}$$

With $t' = t - r/c$, putting (10) back to (7) yields

$$\left.\frac{d\mathbf{L}_{\text{loss}}(t)}{dt}\right|_{>r} \approx -\frac{1}{8\pi^2\epsilon_0 c^3}\,\mathrm{Re}\int d\Omega \overbrace{\int(-i\omega)e^{-i\omega t'}\mathbf{n}\cdot\mathbf{p}(\omega)\,d\omega}^{\mathbf{n}\cdot\dot{\mathbf{p}}(t')}\underbrace{\int \omega'^2 e^{i\omega' t'}\mathbf{n}\times\mathbf{p}^*(\omega')\,d\omega'}_{-\mathbf{n}\times\ddot{\mathbf{p}}^*(t')=-\mathbf{n}\times\ddot{\mathbf{p}}(t')}$$

$$= \frac{1}{8\pi^2\epsilon_0 c^3}\int\left[\mathbf{n}\cdot\dot{\mathbf{p}}(t')\right]\left[\mathbf{n}\times\ddot{\mathbf{p}}(t')\right]d\Omega$$

$$= \frac{1}{8\pi^2\epsilon_0 c^3}\int n_l\dot{p}_l(t')\hat{\mathbf{e}}_k\epsilon_{ijk}n_i\ddot{p}_j(t')\,d\Omega$$

$$= \frac{1}{8\pi^2\epsilon_0 c^3}\hat{\mathbf{e}}_k\epsilon_{ijk}\dot{p}_l(t')\ddot{p}_j(t')\overbrace{\int n_l n_i\,d\Omega}^{\delta_{il}\cdot 4\pi/3}$$

$$= \frac{1}{6\pi\epsilon_0 c^3}\dot{\mathbf{p}}(t')\times\ddot{\mathbf{p}}(t') \tag{11}$$

Note this result is the instantaneous rate of radiated angular momentum through the sphere of radius r and the only approximation used is r being large so we can drop $O\left(r^{-4}\right)$ in (10).

2. Since $\mathbf{x}, r$ are already used for the observation point, to avoid confusion, let's use $\mathbf{x}', r'$ for the charged particle. Its equation of motion is

$$m\ddot{\mathbf{x}}' = -\nabla' V = -\frac{dV}{dr'}\mathbf{n}' \qquad\Longrightarrow\qquad \ddot{\mathbf{p}} = q\ddot{\mathbf{x}}' = -\frac{q}{m}\frac{dV}{dr'}\mathbf{n}' \tag{12}$$

When the charged particle can be approximated as a point dipole, by (2) and (11) we have

$$P(t) = \frac{q^2}{6\pi\epsilon_0 m^2 c^3}\left(\frac{dV}{dr'}\right)^2 = \frac{\tau}{m}\left(\frac{dV}{dr'}\right)^2 \tag{13}$$

$$\frac{d\mathbf{L}_{\text{em}}}{dt} = \frac{1}{6\pi\epsilon_0 c^3}\left(q\dot{\mathbf{x}}'\right)\times\left(-\frac{q}{m}\frac{dV}{dr'}\mathbf{n}'\right) = \frac{1}{6\pi\epsilon_0 c^3}\frac{q^2}{m^2}\left(\frac{1}{r'}\frac{dV}{dr'}\right)\overbrace{m\mathbf{x}'\times\dot{\mathbf{x}}'}^{\mathbf{L}} = \frac{\tau}{m}\left(\frac{1}{r'}\frac{dV}{dr'}\right)\mathbf{L} \tag{14}$$

where the evaluations of the above are at the retarded time $t' = t - r/c$.

3. The inverse time ratio is

$$\lambda = \frac{d\mathbf{L}_{\text{em}}/dt}{\mathbf{L}} = \frac{e^2}{6\pi\epsilon_0 m^2 c^3}\left(\frac{1}{r'}\frac{dV}{dr'}\right) \tag{15}$$

With the Coulomb potential in the hydrogen atom

$$V(r') = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r'} \tag{16}$$

we have

$$\lambda = \frac{e^2}{6\pi\epsilon_0 m^2 c^3}\left(\frac{1}{r'}\frac{e^2}{4\pi\epsilon_0 r'^2}\right) = \frac{e^4}{24\pi^2\epsilon_0^2 m^2 c^3 r'^3} \approx \frac{e^4}{24\pi^2\epsilon_0^2 m^2 c^3 a_0^3} \tag{17}$$

where we take $r' \approx a_0 = \hbar/mc\alpha$. Furthermore, with $\alpha = e^2/4\pi\epsilon_0\hbar c$, we get

$$\lambda \approx \frac{2}{3}\frac{\alpha^4 c}{a_0} \tag{18}$$

4. As pointed out in problem 9.6, the result of problem 9.8 (harmonically oscillating dipole) and 9.9 (dipole with arbitrary time dependency) can be related by the substitution

$$-i\omega \longleftrightarrow \frac{\partial}{\partial t} \qquad\text{and}\qquad \mathbf{p}e^{ikr-i\omega t} \longleftrightarrow \mathbf{p}_{\text{ret}}(t') \tag{19}$$