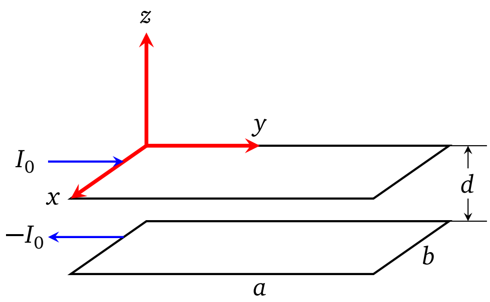
1. **Prob 6.13**



(a) Let the coordinate system be set up as the diagram above. Since we are ignoring the fringing fields, the electromagnetic field between the plates is independent of the x coordinate. Also since $d \ll a, d \ll b$, we are going take the approximation that the fields are independent of the z coordinate too. The whole system is driven by a harmonic input, so the electric and magnetic fields can be written as

$$\mathbf{E}(y,t) = \text{Re}\left[E(y)e^{-i\omega t}\right]\hat{\mathbf{z}} \tag{1}$$
$$\mathbf{B}(y,t) = \text{Re}\left[B(y)e^{-i\omega t}\right]\hat{\mathbf{x}} \tag{2}$$

Similarly, for the upper plate, let the current density and charge density at y be

$$\mathbf{K}(y,t) = \text{Re}\left[K(y)e^{-i\omega t}\right]\hat{\mathbf{y}} \tag{3}$$
$$\sigma(y,t) = \text{Re}\left[\sigma(y)e^{-i\omega t}\right] \tag{4}$$

For the lower plate, $\mathbf{K}$ and σ have an opposite sign.
In the following we will work on the complex amplitudes and ignore their harmonic factor $e^{-i\omega t}$ in favor of the complex Maxwell equation 6.130.
First notice

$$\nabla\cdot\mathbf{D} = \rho \qquad \Longrightarrow \qquad E(y) = -\frac{\sigma(y)}{\epsilon_0} \tag{5}$$

Next notice

$$\nabla\times\mathbf{H} + i\omega\mathbf{D} = \mathbf{J} \qquad \Longrightarrow \qquad B(y) = -\mu_0 K(y) \tag{6}$$

(this can be obtained by applying Stoke's theorem to a rectangular surface parallel to z axis straddling across the upper plate, while noting the displacement current $i\omega\mathbf{D}$ contributes no flux).
Next we have

$$\nabla\times\mathbf{E} - i\omega\mathbf{B} = 0 \qquad \Longrightarrow \qquad \frac{dE(y)}{dy} = i\omega B(y) \qquad \Longrightarrow \qquad \frac{d\sigma(y)}{dy} = \frac{i\omega}{c^2}K(y) \tag{7}$$

Finally, conservation of electric charge at y requires

$$\nabla\cdot\mathbf{K}(y,t) + \frac{\partial\sigma(y,t)}{\partial t} = 0 \qquad \Longrightarrow \qquad \frac{dK(y)}{dy} = i\omega\sigma(y) \tag{8}$$

From (7) and (8), we find the differential equation dictating $\sigma(y)$ and $K(y)$:

$$\frac{d^2\sigma(y)}{dy^2} = -\frac{\omega^2}{c^2}\sigma(y) \qquad \frac{d^2K(y)}{dy^2} = -\frac{\omega^2}{c^2}K(y) \tag{9}$$

With $k = \omega/c$, the general solution can be written as

$$K(y) = Ae^{iky} + Be^{-iky} \qquad \sigma(y) = \frac{A}{c}e^{iky} - \frac{B}{c}e^{-iky} \tag{10}$$

where A, B can be determined by boundary condition $K(y=0) = I_0/b$ and $K(y=a) = 0$, i.e,

$$\frac{I_0}{b} = A + B \qquad 0 = Ae^{ika} + Be^{-ika} \qquad \Longrightarrow$$

$$A = \frac{I_0}{b}\left(\frac{1}{2} + \frac{i}{2}\cot ka\right) \qquad B = \frac{I_0}{b}\left(\frac{1}{2} - \frac{i}{2}\cot ka\right) \tag{11}$$

which gives

$$\begin{aligned} K(y) &= \frac{I_0}{2b}\left(e^{iky} + e^{-iky}\right) + \frac{iI_0}{2b}\cot ka\left(e^{iky} - e^{-iky}\right) \\ &= \frac{I_0}{b}\left(\cos ky - \cot ka \sin ky\right) \\ &= \frac{I_0}{b}\frac{\sin[k(a-y)]}{\sin ka} \end{aligned} \tag{12}$$

$$\begin{aligned} \sigma(y) &= \frac{1}{c}\left[\frac{I_0}{2b}\left(e^{iky} - e^{-iky}\right) + \frac{iI_0}{2b}\cot ka\left(e^{iky} + e^{-iky}\right)\right] \\ &= \frac{iI_0}{cb}\left(\sin ky + \cot ka \cos ky\right) \\ &= \frac{iI_0}{cb}\frac{\cos[k(a-y)]}{\sin ka} \end{aligned} \tag{13}$$

The complex amplitudes can be obtained from (5), (6)

$$E(y) = -\frac{i\mu_0 I_0 c}{b}\frac{\cos[k(a-y)]}{\sin ka} \tag{14}$$

$$B(y) = \frac{-\mu_0 I_0}{b}\frac{\sin[k(a-y)]}{\sin ka} \tag{15}$$

Thus the real fields are

$$\begin{aligned} \mathbf{E}(y,t) &= \mathrm{Re}\left\{-\frac{i\mu_0 I_0 c}{b}\frac{\cos[k(a-y)]}{\sin ka}e^{-i\omega t}\right\}\hat{\mathbf{z}} \qquad \text{if } I_0 \in \mathbb{R} \\ &= -\frac{\mu_0 I_0 c}{b}\frac{\cos[k(a-y)]}{\sin ka}\sin\omega t\,\hat{\mathbf{z}} \end{aligned} \tag{16}$$

$$\begin{aligned} \mathbf{B}(y,t) &= \mathrm{Re}\left\{-\frac{\mu_0 I_0}{b}\frac{\sin[k(a-y)]}{\sin ka}e^{-i\omega t}\right\}\hat{\mathbf{x}} \qquad \text{if } I_0 \in \mathbb{R} \\ &= -\frac{\mu_0 I_0}{b}\frac{\sin[k(a-y)]}{\sin ka}\cos\omega t\,\hat{\mathbf{x}} \end{aligned} \tag{17}$$

(b) From (14) and (15), we get

$$w_m = \frac{1}{4}\mathbf{B}\cdot\mathbf{H}^* = \frac{1}{4}\frac{\mu_0|I_0|^2}{b^2}\frac{\sin^2[k(a-y)]}{\sin^2 ka} \tag{18}$$

$$w_e = \frac{1}{4}\mathbf{E}\cdot\mathbf{D}^* = \frac{1}{4}\frac{\mu_0|I_0|^2}{b^2}\frac{\cos^2[k(a-y)]}{\sin^2 ka} \tag{19}$$

Thus by (6.140), the inductive reactance is

$$X_L = \frac{4\omega}{|I_0|^2}\int w_m d^3x = \frac{\omega\mu_0}{b^2}\int_0^a \frac{\sin^2[k(a-y)]}{\sin^2 ka}(bd)dy \tag{20}$$

If we identify $X_L = \omega L$ as in a series LC circuit, we would have the effective inductance

$$L = \frac{\mu_0 d}{b}\int_0^a \frac{\sin^2[k(a-y)]}{\sin^2 ka}dy \approx \frac{\mu_0 d}{b}\int_0^a \frac{(a-y)^2}{a^2}dy = \frac{\mu_0 ad}{3b} \tag{21}$$

Similarly, the capacitive reactance from (6.140) is

$$X_C = -\frac{4\omega}{|I_0|^2}\int w_e d^3x = -\frac{\omega\mu_0}{b^2}\int_0^a \frac{\cos^2[k(a-y)]}{\sin^2 ka}(bd)\,dy \tag{22}$$

If we identify $X_C = -1/\omega C$, we would have the effective capacitance

$$C = \frac{1}{\dfrac{\omega^2\mu_0 d}{b}\displaystyle\int_0^a \frac{\cos^2[k(a-y)]}{\sin^2 ka}dy} \approx \frac{1}{\dfrac{\omega^2\mu_0 d}{b}\displaystyle\int_0^a \frac{1}{k^2a^2}dy} = \frac{k^2ab}{\omega^2\mu_0 d} = \frac{\epsilon_0 ab}{d} \tag{23}$$

2. **Prob 6.14**

(a) Similar to problem 6.13, based on symmetry arguments, we have the following form of the fields for the space between the plates

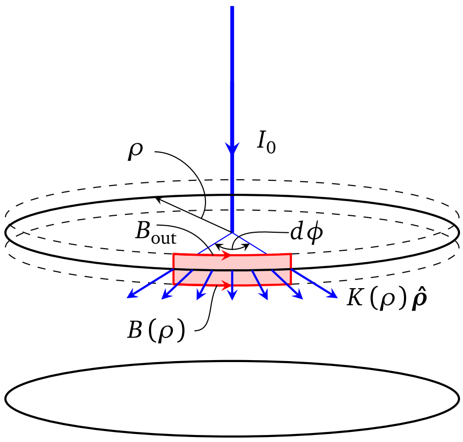
$$\mathbf{E}(\rho,t) = \mathrm{Re}\left[E(\rho)e^{-i\omega t}\right]\hat{\mathbf{z}} \tag{24}$$
$$\mathbf{B}(\rho,t) = \mathrm{Re}\left[B(\rho)e^{-i\omega t}\right]\hat{\boldsymbol{\phi}} \tag{25}$$

Let the surface current and charge density of the upper plate be

$$\mathbf{K}(\rho,t) = \mathrm{Re}\left[K(\rho)e^{-i\omega t}\right]\hat{\boldsymbol{\rho}} \tag{26}$$
$$\sigma(\rho,t) = \mathrm{Re}\left[\sigma(\rho)e^{-i\omega t}\right] \tag{27}$$

First,

$$\nabla\cdot\mathbf{D} = \rho \qquad \Longrightarrow \qquad E(\rho) = -\frac{\sigma(\rho)}{\epsilon_0} \tag{28}$$



In order to apply the equation

$$\nabla\times\mathbf{H} + i\omega\mathbf{D} = \mathbf{J} \tag{29}$$

let's consider a "rectangular" loop straddling across the upper plate at radius ρ, depicted red in the diagram above. The two long sides span an arc of length $\rho d\phi$, and the two short sides are infinitesimal.
Applying Stoke's theorem for this patch with (29) yields (note $\mathbf{D}$ is in the $\hat{\mathbf{z}}$ direction hence contributes no flux),

$$[B(\rho) - B_{\text{out}}]\rho d\phi = \mu_0 K(\rho)\rho d\phi \tag{30}$$

where B_{out} is the magnetic induction at ρ right above the upper plate, in the $\hat{\boldsymbol{\phi}}$ direction. In fact B_{out} is easy to obtain if we only consider the space above the upper plate, and where the current is along the z axis, i.e.,

$$B_{\text{out}} = -\frac{\mu_0 I_0}{2\pi\rho} \tag{31}$$

This gives us

$$B(\rho) = \mu_0 K(\rho) + B_{\text{out}} = \mu_0 K(\rho) - \frac{\mu_0 I_0}{2\pi\rho} = \mu_0\kappa(\rho) \qquad \text{where} \qquad \kappa(\rho) \equiv K(\rho) - \frac{I_0}{2\pi\rho} \tag{32}$$

Next notice

$$\nabla\times\mathbf{E} - i\omega\mathbf{B} = 0 \qquad \Longrightarrow \qquad -\frac{dE(\rho)}{d\rho} = i\omega B(\rho) \qquad \Longrightarrow \qquad \frac{d\sigma(\rho)}{d\rho} = \frac{i\omega}{c^2}\kappa(\rho) \tag{33}$$

Finally conservation of charge requires

$$\nabla\cdot[\hat{\boldsymbol{\rho}}K(\rho)] - i\omega\sigma(\rho) = 0 \quad \Longrightarrow \quad \frac{1}{\rho}\frac{d[\rho K(\rho)]}{d\rho} = \frac{1}{\rho}\frac{d[\rho\kappa(\rho)]}{d\rho} = i\omega\sigma(\rho) \quad \Longrightarrow \quad \frac{d\kappa}{d\rho} + \frac{\kappa}{\rho} = i\omega\sigma \tag{34}$$

Taking $d/d\rho$ of (34) and using (33), we get

$$\frac{d^2\kappa}{d\rho^2}+\frac{1}{\rho}\frac{d\kappa}{d\rho}-\frac{\kappa}{\rho^2}=-\frac{\omega^2}{c^2}\kappa \qquad \text{or} \qquad \frac{d^2\kappa}{d\rho^2}+\frac{1}{\rho}\frac{d\kappa}{d\rho}+\left(k^2-\frac{1}{\rho^2}\right)\kappa=0 \tag{35}$$

which is the Bessel equation with $\nu=1$, hence the general solution is

$$\kappa(\rho)=AJ_1(k\rho)+BN_1(k\rho) \tag{36}$$

The boundary condition at $\rho\to 0$ requires

$$\lim_{\rho\to 0}2\pi\rho K(\rho)=I_0 \qquad \text{or} \qquad \lim_{\rho\to 0}\rho\kappa(\rho)=0 \tag{37}$$

With the asymptotic form of $J_1(k\rho)$ and $N_1(k\rho)$ (see Jackson equation 3.89, 3.90)

$$J_1(k\rho)\to\frac{k\rho}{2} \qquad N_1(k\rho)\to-\frac{1}{\pi}\left(\frac{2}{k\rho}\right) \tag{38}$$

we must have $B=0$ in order to satisfy (37).
Boundary condition at $\rho=a$ requires

$$0=K(a)=AJ_1(k\rho)+\frac{I_0}{2\pi a} \qquad \Longrightarrow \qquad A=-\frac{I_0}{2\pi a}\frac{1}{J_1(ka)} \tag{39}$$

In summary, we have

$$\kappa(\rho)=-\frac{I_0}{2\pi a}\frac{J_1(k\rho)}{J_1(ka)} \tag{40}$$

By (34),

$$\sigma(\rho)=\frac{1}{i\omega}\left(\frac{d\kappa}{d\rho}+\frac{\kappa}{\rho}\right) \tag{41}$$

Using the recurrence relation for $J_\nu(z)$(see equation 10.6.2 on dlmf.nist.gov)

$$J'_\nu(z)+\frac{\nu}{z}J_\nu(z)=J_{\nu-1}(z) \tag{42}$$

(41) becomes

$$\sigma(\rho)=\frac{k}{i\omega}AJ_0(k\rho)=\frac{iI_0}{2\pi ac}\frac{J_0(k\rho)}{J_1(ka)} \tag{43}$$

The field amplitudes are

$$B(\rho)=\mu_0\kappa(\rho)=-\frac{\mu_0 I_0}{2\pi a}\frac{J_1(k\rho)}{J_1(ka)} \tag{44}$$

$$E(\rho)=-\frac{\sigma(\rho)}{\epsilon_0}=-\frac{i\mu_0 I_0 c}{2\pi a}\frac{J_0(k\rho)}{J_1(ka)} \tag{45}$$

(b) Recall Bessel function can be written as series

$$J_\nu(x)=\sum_{j=0}^{\infty}\frac{(-1)^j\left(\frac{x}{2}\right)^{2j+\nu}}{j!\Gamma(j+\nu+1)} \tag{46}$$

Thus

$$J_1(k\rho)\approx\frac{k\rho}{2}-\frac{(k\rho)^3}{16} \tag{47}$$

$$J_0(k\rho)\approx 1-\frac{(k\rho)^2}{4} \tag{48}$$

and

$$
\begin{aligned}
\frac{J_1(k\rho)}{J_1(ka)} &\approx \left(\frac{k\rho}{2}-\frac{k^3\rho^3}{16}\right)\left[\left(\frac{ka}{2}\right)\left(1-\frac{k^2a^2}{8}\right)\right]^{-1} \\
&\approx \left(\frac{\rho}{a}-\frac{k^2\rho^3}{8a}\right)\left(1+\frac{k^2a^2}{8}\right) \\
&\approx \left(\frac{\rho}{a}\right)\left(1-\frac{k^2\rho^2}{8}\right)\left(1+\frac{k^2a^2}{8}\right) \\
&\approx \left(\frac{\rho}{a}\right)\left(1-\frac{k^2\rho^2}{8}+\frac{k^2a^2}{8}\right)
\end{aligned} \tag{49}
$$

So

$$
\begin{aligned}
\int_V w_m d^3x &= \frac{1}{4}\cdot 2\pi d\int_0^a \rho d\rho BH^* \\
&= \frac{1}{4}\cdot 2\pi d\cdot\frac{\mu_0|I_0|^2}{(2\pi a)^2}\int_0^a \rho d\rho\left[\frac{J_1(k\rho)}{J_1(ka)}\right]^2 \\
&\approx \frac{\mu_0|I_0|^2 d}{8\pi a^2}\int_0^a \rho d\rho\left[\left(\frac{\rho^2}{a^2}\right)\left(1-\frac{k^2\rho^2}{4}+\frac{k^2a^2}{4}\right)\right] \\
&= \frac{\mu_0|I_0|^2 d}{8\pi a^2}\left(\frac{a^2}{4}-\frac{k^2a^4}{24}+\frac{k^2a^4}{16}\right) \\
&= \frac{\mu_0|I_0|^2 d}{32\pi}\left(1+\frac{\omega^2a^2}{12c^2}\right)
\end{aligned} \tag{50}
$$

Also

$$
\frac{J_0(k\rho)}{J_1(ka)} \approx \frac{2}{ka}+O\left(k^0\right) \tag{51}
$$

hence

$$
\begin{aligned}
\int_V w_e d^3x &= \frac{1}{4}\cdot 2\pi d\int_0^a \rho d\rho ED^* \\
&= \frac{1}{4}\cdot 2\pi d\cdot\frac{\mu_0^2|I_0|^2c^2\epsilon_0}{(2\pi a)^2}\int_0^a \rho d\rho\left[\frac{J_0(k\rho)}{J_1(ka)}\right]^2 \\
&\approx \frac{|I_0|^2 d}{8\pi a^2c^2\epsilon_0}\int_0^a \rho d\rho\cdot\frac{4}{k^2a^2} \\
&= \frac{|I_0|^2 d}{4\pi\epsilon_0\omega^2a^2}
\end{aligned} \tag{52}
$$

(c) We can get the effective inductance and capacitance via (6.140):

$$
X_L = \omega L = \frac{4\omega}{|I_0|^2}\int w_m d^3x \quad\Longrightarrow\quad L \approx \frac{\mu_0 d}{8\pi} \tag{53}
$$

$$
X_C = -\frac{1}{\omega C} = -\frac{4\omega}{|I_0|^2}\int w_e d^3x \quad\Longrightarrow\quad C \approx \frac{\pi\epsilon_0a^2}{d} \tag{54}
$$

Resonance frequency is

$$
\omega_{\text{res}} = \frac{1}{\sqrt{LC}} \approx= \frac{2\sqrt{2}c}{a} \approx \frac{2.828c}{a} \tag{55}
$$

while the first root of $J_0(ka)$ happens at

$$
ka = \frac{\omega a}{c} = 2.405 \quad\Longrightarrow\quad \omega = \frac{2.405c}{a} \tag{56}
$$