1. By Prob 3.17 (b), the Green function of a pair of parallel plates is

$$G(\mathbf{x},\mathbf{x}') = 2\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\int_0^{\infty} dk J_m(k\rho) J_m(k\rho')\frac{\sinh(kz_<)\sinh[k(L-z_>)]}{\sinh(kL)} \tag{1}$$

Now with the charge distribution $\sigma(\rho)$ on the disc at $z = z_0$, consider the potential generated on the disc itself:

$$\begin{aligned}\Phi(\rho,z_0) &= \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}') G(\mathbf{x},\mathbf{x}') d^3x' \\ &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\phi' \int_0^R \rho' d\rho' \sigma(\rho')\cdot 2\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\int_0^{\infty} dk J_m(k\rho) J_m(k\rho')\frac{\sinh(kz_0)\sinh[k(L-z_0)]}{\sinh(kL)}\end{aligned} \tag{2}$$

The $\int d\phi'$ integral ensures only $m = 0$ has non-zero contributions, which gives

$$\Phi(\rho,z_0) = \frac{1}{4\pi\epsilon_0}\int_0^R 2\pi\rho' d\rho' \sigma(\rho')\cdot 2\int_0^{\infty} dk J_0(k\rho) J_0(k\rho')\frac{\sinh(kz_0)\sinh[k(L-z_0)]}{\sinh(kL)} \tag{3}$$

Under the limit $L \to \infty$ with $d = L - z_0$ fixed,

$$\begin{aligned}\frac{\sinh(kz_0)\sinh[k(L-z_0)]}{\sinh(kL)} &= \sinh(kd)\frac{\sinh[k(L-d)]}{\sin(kL)} \\ &= \sinh(kd)\frac{\sinh(kL)\cosh(kd)-\cosh(kL)\sinh(kd)}{\sinh(kL)} \\ &= \sinh(kd)\left[\cosh(kd)-\frac{\cosh(kL)}{\sinh(kL)}\sinh(kd)\right] \qquad \text{as } L\to\infty \\ &\to \sinh(kd)[\cosh(kd)-\sinh(kd)] \\ &= \sinh(kd)\cdot e^{-kd} \\ &= \frac{1}{2}\left(1-e^{-2kd}\right)\end{aligned} \tag{4}$$

This turns (3) into

$$\begin{aligned}\Phi(\rho,z_0) &= \frac{1}{2\epsilon_0}\int_0^R \rho' d\rho' \sigma(\rho')\int_0^{\infty} dk J_0(k\rho) J_0(k\rho')\left(1-e^{-2kd}\right) \\ &= \frac{1}{2\epsilon_0}\int_0^{\infty} dk\left(1-e^{-2kd}\right) J_0(k\rho)\int_0^R \sigma(\rho')\rho' J_0(k\rho') d\rho'\end{aligned} \tag{5}$$

Since the disc is a conductor, the potential everywhere on the disc is a constant, denoted $\Phi_0$. For example, if we take $\rho = 0$, we end up with

$$\Phi_0 = \Phi(0,z_0) = \frac{1}{2\epsilon_0}\int_0^{\infty} dk\left(1-e^{-2kd}\right)\int_0^R \sigma(\rho')\rho' J_0(k\rho') d\rho' \tag{6}$$

With the total charge on the disc given by

$$Q = \int_0^R \sigma(\rho) 2\pi\rho d\rho \tag{7}$$

we can have the expression for the capacitance

$$\begin{aligned}\frac{4\pi\epsilon_0}{C} = 4\pi\epsilon_0\cdot\frac{\Phi_0}{Q} &= 4\pi\epsilon_0\frac{\frac{1}{2\epsilon_0}\int_0^{\infty} dk\left(1-e^{-2kd}\right)\int_0^R \sigma(\rho')\rho' J_0(k\rho') d\rho'}{\int_0^R \sigma(\rho) 2\pi\rho d\rho} \\ &= \int_0^{\infty} dk\left(1-e^{-2kd}\right)\frac{\int_0^R \sigma(\rho')\rho' J_0(k\rho') d\rho'}{\int_0^R \sigma(\rho)\rho d\rho}\end{aligned} \tag{8}$$

which seems to differ from the desired form by a square power, but they are indeed equal, because of the restriction that $\sigma(\rho)$ must make the entire disc equipotential. To see this, notice

$$\frac{4\pi\epsilon_0}{C} = 4\pi\epsilon_0 \cdot \frac{\Phi_0}{Q} = 4\pi\epsilon_0 \frac{\Phi_0 Q}{Q^2} \tag{9}$$

Now instead of taking $\Phi_0$ to be the specific value at $\rho = 0$, we take it to be the potential of a general $\rho$ as given by (5), which turns (9) into

$$\begin{aligned}
\frac{4\pi\epsilon_0}{C} &= 4\pi\epsilon_0 \frac{\int_0^R \Phi_0 \sigma(\rho) 2\pi\rho d\rho}{\left[\int_0^R \sigma(\rho) 2\pi\rho d\rho\right]^2} = 4\pi\epsilon_0 \frac{\int_0^R \Phi(\rho, z_0) \sigma(\rho) 2\pi\rho d\rho}{\left[\int_0^R \sigma(\rho) 2\pi\rho d\rho\right]^2} \\
&= 4\pi\epsilon_0 \frac{\frac{1}{2\epsilon_0}\int_0^\infty dk\left(1-e^{-2kd}\right)\left[\int_0^R \sigma(\rho) J_0(k\rho) 2\pi\rho d\rho\right]\left[\int_0^R \sigma(\rho') \rho' J_0(k\rho') d\rho'\right]}{\left[\int_0^R \sigma(\rho) 2\pi\rho d\rho\right]^2} \\
&= \int_0^\infty dk\left(1-e^{-2kd}\right) \frac{\left[\int_0^R \sigma(\rho) J_0(k\rho) \rho d\rho\right]^2}{\left[\int_0^R \sigma(\rho) \rho d\rho\right]^2}
\end{aligned} \tag{10}$$

which is the desired form with the square power.

**(8) and (10) are mathematically equivalent, so I'm not sure why the problem chose (10). Maybe because of its quadratic form, which could be more useful to apply the variational principle later on.**

2. When we take $\sigma(\rho)$ as constant, (10) is turned into

$$\begin{aligned}
\frac{4\pi\epsilon_0}{C} &= \int_0^\infty dk\left(1-e^{-2kd}\right) \frac{\left[\int_0^R J_0(k\rho) \rho d\rho\right]^2}{\frac{R^4}{4}} \qquad \text{use } xJ_0(x) = [xJ_1(x)]' \\
&= \frac{4}{R^4}\int_0^\infty dk\left(1-e^{-2kd}\right)\left[\frac{kRJ_1(kR)}{k^2}\right]^2 \\
&= \frac{4}{R^2}\int_0^\infty dk\left(1-e^{-2kd}\right)\frac{J_1^2(kR)}{k^2}
\end{aligned} \tag{11}$$

When $d \ll R$, (11) is dominated by the leading order $1 - e^{-2kd} \approx 2kd$,

$$\frac{4\pi\epsilon_0}{C} \approx \frac{4}{R^2}\int_0^\infty dk \cdot 2kd \cdot \frac{J_1^2(kR)}{k^2} = \frac{8d}{R^2}\int_0^\infty \frac{J_1^2(kR)}{k} dk = \frac{4d}{R^2} \qquad \Longrightarrow \qquad C \approx \frac{\pi\epsilon_0 R^2}{d} \tag{12}$$

For a pair of parallel plates with $d \ll R$ and uniform charge density, we have

$$E = \frac{\sigma}{\epsilon_0} \qquad \Longrightarrow \qquad \Delta\Phi = \frac{\sigma d}{\epsilon_0} \qquad \Longrightarrow \qquad C = \frac{Q}{\Delta\Phi} = \frac{\pi R^2 \sigma}{\sigma d/\epsilon_0} = \frac{\pi R^2 \epsilon_0}{d} \tag{13}$$

which agrees with (12).

When $d$ is big, we approximate $1 - e^{-2kd} \approx 1$, so that

$$\frac{4\pi\epsilon_0}{C} \approx \frac{4}{R^2}\int_0^\infty dk \frac{J_1^2(kR)}{k^2} = \frac{4}{R^2} \cdot R\frac{4}{3\pi} = \frac{16}{3\pi R} \tag{14}$$

The ratio of it to the exact result is

$$\frac{16/(3\pi R)}{\pi/(2R)} = \frac{32}{3\pi^2} \approx 1.082 \tag{15}$$

3. I cannot complete this part, but I'd like to record a few thoughts while they are still fresh. They may or may not be useful down the road.

   - Regarding the "variational principle": my understanding of expressing the inverse of capacitance in the form of (10) is to make the point that it is a functional of $\sigma(\rho)$. But for variational principle to be applicable, we must first establish the fact that $4\pi\epsilon_0/C$ achieves minimum value when $\sigma$ is the real charge distribution that gives rise to the equipotential on the conducting disc. The physical meaning is clear, since when the insulated conductor has a total charge $Q$, its self energy ($\propto Q^2/C$) is inversely proportional to $C$, so the variational principle is really just stating the energy minimization. There is an article *A variational principle and its application to estimating the electrical capacitance of a perfect conductor* that has proved similar claim. Although the equation (1.4) from that article

$$C^{-1} = \min_{v\in L^2(S)} \frac{\displaystyle\int_S\int_S \frac{v(t)v(s)dsdt}{4\pi r_{st}}}{\left|\displaystyle\int_S v(t)dt\right|^2} \tag{16}$$

     manifestly differs from (10), its physical meaning was clear: the numerator is proportional to the self energy, analogous to the $\Phi_0 Q$ in (9), and its denominator is just $Q^2$.

   - The better trial can be taken to be

$$\sigma(\rho) = \alpha + \frac{\beta}{\sqrt{R^2-\rho^2}} \tag{17}$$

     We are supposed to plug this into (10) and find the minimizing $\alpha, \beta$. The $\alpha$ part of $\rho$-integrals was figured out in part (b), for the $\beta$ part, the integral formula 10.22.13 on nist.gov seems useful:

$$\int_0^{\pi/2} J_{2\nu}(2z\cos\theta)\cos(2\mu\theta)\,d\theta = \frac{\pi}{2}J_{\nu+\mu}(z)J_{\nu-\mu}(z) \qquad \text{for Re } \nu > -\frac{1}{2} \tag{18}$$

     which enables us to calculate (with $\rho = R\cos\theta$)

$$\begin{aligned}\int_0^R \frac{\rho J_0(k\rho)}{\sqrt{R^2-\rho^2}}d\rho &= \int_{\pi/2}^0 \frac{R\cos\theta J_0(kR\cos\theta)}{R\sin\theta}(-R\sin\theta\, d\theta)\\ &= \int_0^{\pi/2} R\cos\theta J_0(kR\cos\theta)\,d\theta = \frac{\pi R}{2}J_{1/2}\left(\frac{kR}{2}\right)J_{-1/2}\left(\frac{kR}{2}\right)\end{aligned} \tag{19}$$

     But after the $\rho$-integrals are worked out, the $k$-integral seems prohibitively difficult. This is where I get stuck.