Recall the spherical Green function is (equation (3.125))

$$G\left(\mathbf{x},\mathbf{x}'\right)=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^{*}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left(r_<^l-\frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right) \tag{1}$$

And by (1.44),

$$\Phi(\mathbf{x})=-\frac{1}{4\pi}\oint_S\Phi\left(\mathbf{x}'\right)\frac{\partial G}{\partial n'}da' \tag{2}$$

By the problem statement, we only need to do surface integral for the inner upper hemisphere and outer lower hemisphere.

For the inner sphere (remember n' direction is outwards from the volume of interest)

$$\begin{aligned}\frac{\partial G}{\partial n'}&=-\left.\frac{\partial G}{\partial r'}\right|_{r'=a} \qquad\qquad \text{where } r_<=r', r_>=r\\&=-4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^{*}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left[la^{l-1}+(l+1)\frac{a^{2l+1}}{a^{l+2}}\right]\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)\\&=-4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^{*}(\theta',\phi')Y_{lm}(\theta,\phi)}{1-\left(\frac{a}{b}\right)^{2l+1}}a^{l-1}\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)\end{aligned} \tag{3}$$

And for the outer sphere,

$$\begin{aligned}\frac{\partial G}{\partial n'}&=\left.\frac{\partial G}{\partial r'}\right|_{r'=b} \qquad\qquad \text{where } r_<=r, r_>=r'\\&=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^{*}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right)\left[-(l+1)\frac{1}{b^{l+2}}-l\frac{1}{b^{l+2}}\right]\\&=-4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^{*}(\theta',\phi')Y_{lm}(\theta,\phi)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right)\frac{1}{b^{l+2}}\end{aligned} \tag{4}$$

The potential contribution from the inner hemisphere is

$$\begin{aligned}\Phi_{\text{inner}}(\mathbf{x})&=-\frac{1}{4\pi}\int_0^{2\pi}d\phi'\int_0^{\pi/2}\sin\theta' d\theta' Va^2\left(-\left.\frac{\partial G}{\partial r'}\right|_{r'=a}\right)\\&=V\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}(\theta,\phi)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left[\left(\frac{a}{r}\right)^{l+1}-\frac{a^{l+1}r^l}{b^{2l+1}}\right]\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\overbrace{\int_0^{2\pi}e^{-im\phi'}d\phi'}^{2\pi\delta_{m0}}\int_0^{\pi/2}\sin\theta' P_l^m\left(\cos\theta'\right)d\theta'\\&=V\sum_{l=0}^{\infty}\frac{Y_{l0}(\theta,\phi)}{1-\lambda^{2l+1}}\left[\left(\frac{a}{r}\right)^{l+1}-\frac{a^{l+1}r^l}{b^{2l+1}}\right]\sqrt{\frac{2l+1}{4\pi}}2\pi\underbrace{\int_0^1 P_l(x)dx}_{I_l} \qquad \text{define } \lambda\equiv\frac{a}{b}\\&=\frac{V}{2}\sum_{l=0}^{\infty}\frac{(2l+1)P_l(\cos\theta)}{1-\lambda^{2l+1}}I_l\left[\left(\frac{a}{r}\right)^{l+1}-\frac{a^{l+1}r^l}{b^{2l+1}}\right]\\&=\frac{V}{2}\sum_{l=0}^{\infty}\frac{(2l+1)P_l(\cos\theta)}{1-\lambda^{2l+1}}I_l\left[\left(\frac{a}{r}\right)^{l+1}-\lambda^{l+1}\left(\frac{r}{b}\right)^l\right]\end{aligned} \tag{5}$$

and the contribution from the outer hemisphere is

$$
\begin{aligned}
\Phi_{\text{outer}}(\mathbf{x}) &= -\frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_{\pi/2}^{\pi} \sin\theta' d\theta' V b^2 \left(\left.\frac{\partial G}{\partial r'}\right|_{r'=b} \right) \\
&= V \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Y_{lm}(\theta,\phi)}{1-\left(\frac{a}{b}\right)^{2l+1}} \left[\left(\frac{r}{b}\right)^l - \frac{a^{2l+1}}{b^l r^{l+1}}\right] \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \int_0^{2\pi} e^{-im\phi'} d\phi' \int_{\pi/2}^{\pi} \sin\theta' P_l^m\left(\cos\theta'\right) d\theta' \\
&= V \sum_{l=0}^{\infty} \frac{Y_{l0}(\theta,\phi)}{1-\lambda^{2l+1}} \left[\left(\frac{r}{b}\right)^l - \frac{a^{2l+1}}{b^l r^{l+1}}\right] \sqrt{\frac{2l+1}{4\pi}} 2\pi \underbrace{\int_{-1}^{0} P_l(x) dx}_{(-1)^l I_l} \\
&= \frac{V}{2} \sum_{l=0}^{\infty} \frac{(2l+1)P_l(\cos\theta)}{1-\lambda^{2l+1}} I_l (-1)^l \left[\left(\frac{r}{b}\right)^l - \frac{a^{2l+1}}{b^l r^{l+1}}\right] \\
&= \frac{V}{2} \sum_{l=0}^{\infty} \frac{(2l+1)P_l(\cos\theta)}{1-\lambda^{2l+1}} I_l \left[(-1)^l \left(\frac{r}{b}\right)^l - (-\lambda)^l \left(\frac{a}{r}\right)^{l+1}\right] \qquad (6)
\end{aligned}
$$

Now adding (5) and (6) will recover exactly equation (9) from my earlier solution for Prob 3.1, which proves the point of this problem.