Let the rotation be about the z-axis. At (θ', ϕ'), the differential charge is

$$dq\left(\theta', \phi'\right) = \sigma a^2 \sin\theta' d\theta' d\phi' \tag{1}$$

and moves with velocity

$$\mathbf{v}\left(\theta', \phi'\right) = \omega(a\sin\theta')\hat{\boldsymbol{\phi}} \tag{2}$$

Thus the differential current at (θ', ϕ') is

$$d\mathbf{I}\left(\theta', \phi'\right) = dq\mathbf{v} = \sigma\omega a^3 \sin^2\theta' d\theta' d\phi' \hat{\boldsymbol{\phi}} = \sigma\omega a^3 \sin^2\theta' d\theta' d\phi' \left(-\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}}\right) \tag{3}$$

Define

$$d\widetilde{I}\left(\theta', \phi'\right) = \sigma\omega a^3 \sin^2\theta' e^{i\phi'} d\theta' d\phi' \tag{4}$$

thus we have

$$dI_x = -\operatorname{Im} d\widetilde{I} \qquad dI_y = \operatorname{Re} d\widetilde{I} \qquad dI_z = 0 \tag{5}$$

Similarly, with the complex potential

$$\widetilde{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\oint_{\text{sphere}} \frac{d\widetilde{I}}{|\mathbf{x}-\mathbf{x}'|} \tag{6}$$

we can obtain the vector potential as

$$A_x = -\operatorname{Im}\widetilde{A} \qquad A_y = \operatorname{Re}\widetilde{A} \qquad A_z = 0 \tag{7}$$

From equation (3.70), we can expand (6) as

$$\begin{aligned}
\widetilde{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\cdot\left(\sigma\omega a^3\right)\int_0^{2\pi} e^{i\phi'}d\phi' \int_0^{\pi} \frac{\sin^2\theta' d\theta'}{|\mathbf{x}-\mathbf{x}'|} \\
&= \frac{\mu_0\sigma\omega a^3}{4\pi}\int_0^{2\pi} e^{i\phi'}d\phi' \int_0^{\pi} \sin^2\theta' d\theta' \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{4\pi}{2l+1}\right)\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*\left(\theta', \phi'\right) Y_{lm}(\theta, \phi) \\
&= \frac{\mu_0\sigma\omega a^3}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{4\pi}{2l+1}\right)\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]\frac{r_<^l}{r_>^{l+1}} e^{im\phi}P_l^m(\cos\theta)\underbrace{\int_0^{2\pi} e^{i(1-m)\phi'}d\phi'}_{2\pi\delta_{m1}}\int_0^{\pi}\sin^2\theta' P_l^m\left(\cos\theta'\right)d\theta' \\
&= \frac{\mu_0\sigma\omega a^3}{2}\sum_{l=1}^{\infty}\frac{e^{i\phi}P_l^1(\cos\theta)}{l(l+1)}\frac{r_<^l}{r_>^{l+1}}\int_0^{\pi}\sin^2\theta' P_l^1\left(\cos\theta'\right)d\theta'
\end{aligned} \tag{8}$$

Applying the orthogonality of associated Legendre function to the integral produces

$$\int_0^{\pi}\sin^2\theta' P_l^1\left(\cos\theta'\right)d\theta' = \int_{-1}^{1} -P_1^1(y)P_l^1(y)\,dy = -\frac{4}{3}\delta_{l1} \tag{9}$$

Hence

$$\widetilde{A}(\mathbf{x}) = \frac{\mu_0\sigma\omega a^3}{2}\cdot\frac{2}{3}e^{i\phi}\sin\theta\frac{r_<}{r_>^2} = \frac{\mu_0\sigma\omega a^3}{3}e^{i\phi}\sin\theta\frac{r_<}{r_>^2} \tag{10}$$

Applying (7) will give

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0\sigma\omega a^3}{3}\sin\theta\frac{r_<}{r_>^2}\hat{\boldsymbol{\phi}} = \begin{cases} \dfrac{\mu_0\sigma\omega a r\sin\theta}{3}\hat{\boldsymbol{\phi}} & \text{for } r < a \\ \dfrac{\mu_0\sigma\omega a^4\sin\theta}{3r^2}\hat{\boldsymbol{\phi}} & \text{for } r \geq a \end{cases} \tag{11}$$

Then the field is given by $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, which is

$$\mathbf{B}(\mathbf{x}) = \begin{cases} \dfrac{2\mu_0\sigma\omega a}{3}\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right) = \dfrac{2\mu_0\sigma\omega a}{3}\hat{\mathbf{z}} & \text{for } r < a \\ \dfrac{\mu_0\sigma\omega a^4}{3r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right) & \text{for } r \geq a \end{cases} \tag{12}$$