1. Without loss of generality, let the observation point be $\mathbf{x} = (\rho, \phi, z = 0)$. The vector potential at $\mathbf{x}$ due to the current distribution is

$$\mathbf{A}(\mathbf{x}) = \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \int \frac{K(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{1}$$

Write the inverse distance using Bessel series (see problem 3.16(b)),

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{m=-\infty}^{\infty} \int_0^\infty dk e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') e^{-k|z'|} \tag{2}$$

we have

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} dz' \int_0^{2\pi} R d\phi' \cdot \frac{I\cos\phi'}{2R} \sum_{m=-\infty}^{\infty} \int_0^\infty dk e^{im(\phi-\phi')} J_m(k\rho) J_m(kR) e^{-k|z'|} \\
&= \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \frac{I}{2} \int_{-\infty}^{\infty} dz' \sum_{m=-\infty}^{\infty} e^{im\phi} \underbrace{\int_0^{2\pi} \cos\phi' e^{-im\phi'} d\phi'}_{\delta_{m1}\pi + \delta_{m,-1}\pi} \int_0^\infty dk J_m(k\rho) J_m(kR) e^{-k|z'|} \\
&= \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \frac{I}{2} 2\pi\cos\phi \int_{-\infty}^{\infty} dz' \int_0^\infty dk J_1(k\rho) J_1(kR) e^{-k|z'|} \\
&= \hat{\mathbf{z}} \frac{\mu_0 I\cos\phi}{4} \int_0^\infty dk J_1(k\rho) J_1(kR) \cdot 2\int_0^\infty e^{-kz'} dz' \\
&= \hat{\mathbf{z}} \frac{\mu_0 I\cos\phi}{4} \underbrace{\int_0^\infty dk J_1(k\rho) J_1(kR) \frac{2}{k}}_{X}
\end{aligned} \tag{3}$$

Using equation (10.22.58) on dlmf.nist.gov

$$\int_0^\infty \frac{J_\nu(at) J_\nu(bt)}{t^\lambda} dt = \frac{(ab)^\nu \Gamma(\nu - \lambda/2 + 1/2)}{2^\lambda (a^2+b^2)^{\nu-\lambda/2+1/2} \Gamma(\lambda/2+1/2)} F\left[\frac{2\nu+1-\lambda}{4}, \frac{2\nu+3-\lambda}{4}; \nu+1; \frac{4a^2b^2}{(a^2+b^2)^2}\right] \tag{4}$$

we have

$$\begin{aligned}
X &= 2 \cdot \frac{\rho R \Gamma(1)}{2(\rho^2+R^2)\Gamma(1)} F\left[\frac{1}{2}, 1; 2; \frac{4\rho^2R^2}{(\rho^2+R^2)^2}\right] \qquad \text{define } u \equiv \frac{2\rho R}{\rho^2+R^2} \\
&= \frac{u}{2} F\left(\frac{1}{2}, 1; 2; u^2\right)
\end{aligned} \tag{5}$$

The hypergeometric function can be written as (see 15.2.1 on dlmf.nist.gov)

$$F\left(\frac{1}{2}, 1; 2; u^2\right) = \frac{\Gamma(2)}{\Gamma(1/2)\Gamma(1)} \sum_{n=0}^{\infty} \frac{\Gamma(1/2+n)\Gamma(1+n)}{\Gamma(2+n) n!} u^{2n} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n (n+1)!} u^{2n} \tag{6}$$

Recall

$$\sqrt{1-z} = 1 - \sum_{n=0}^{\infty} z^{n+1} \frac{(2n-1)!!}{(2n+2)!!} \tag{7}$$

then

$$F\left(\frac{1}{2}, 1; 2; u^2\right) = \frac{2}{u^2} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^{n+1}(n+1)!} (u^2)^{n+1} = \frac{2}{u^2}\left(1 - \sqrt{1-u^2}\right) \tag{8}$$

which gives

$$X = \frac{u}{2} F\left(\frac{1}{2}, 1; 2; u^2\right) = \frac{1}{u} - \sqrt{\frac{1}{u^2} - 1} = \frac{\rho^2+R^2}{2\rho R} - \frac{|\rho^2 - R^2|}{2\rho R} = \begin{cases} \frac{\rho}{R} & \text{for } \rho < R \\ \frac{R}{\rho} & \text{for } \rho > R \end{cases} \tag{9}$$

Then it's clear for $\rho < R$, (3) becomes

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I \rho \cos\phi}{4R}\hat{\mathbf{z}} = \frac{\mu_0 I x}{4R}\hat{\mathbf{z}} \qquad \Longrightarrow \qquad \mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A} = -\frac{\mu_0 I}{4R}\hat{\mathbf{y}} \tag{10}$$

However for $\rho > R$,

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I R \cos\phi}{4\rho}\hat{\mathbf{z}} = \frac{\mu_0 I R x}{4\rho^2}\hat{\mathbf{z}} = \frac{\mu_0}{4}\frac{\overbrace{(-IR\hat{\mathbf{y}})}^{\mathbf{m}} \times (x\hat{\mathbf{x}} + y\hat{\mathbf{y}})}{\rho^2} = \frac{\mu_0}{4}\frac{\mathbf{m} \times \boldsymbol{\rho}}{|\boldsymbol{\rho}|^2} \tag{11}$$

which is manifestly the 2D analog of the 3D dipole vector potential given in equation (5.55)

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{x}}{|\mathbf{x}|^3} \tag{12}$$

The magnetic induction is most conveniently expressed in polar coordinates

$$\mathbf{B}(\mathbf{x}) = \frac{1}{\rho}\frac{\partial A_z}{\partial \phi}\hat{\boldsymbol{\rho}} - \frac{\partial A_z}{\partial \rho}\hat{\boldsymbol{\phi}} = -\frac{\mu_0 I R}{4\rho^2}\left(\sin\phi\,\hat{\boldsymbol{\rho}} - \cos\phi\,\hat{\boldsymbol{\phi}}\right) \tag{13}$$

2. Per unit length in z, the energy is

$$\begin{aligned} W &= \int_{\text{in}} \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{4R}\right)^2 d^3x + \int_{\text{out}} \frac{1}{2\mu_0}\left(\frac{\mu_0 I R}{4\rho^2}\right)^2 d^3x \\ &= \frac{\mu_0 I^2}{32R^2}\cdot \pi R^2 + \int_R^{\infty} 2\pi\rho\, d\rho \cdot \frac{\mu_0 I^2 R^2}{32\rho^4} \\ &= \frac{\mu_0 \pi I^2}{32} + \frac{\mu_0 \pi I^2}{32} \end{aligned} \tag{14}$$

And the energy inside and outside are equal.

3. The total current in the right hemisphere $\phi \in [-\pi/2, \pi/2]$ is

$$\int_{-\pi/2}^{\pi/2} K(\phi) R\, d\phi = \int_{-\pi/2}^{\pi/2} \frac{I}{2}\cos\phi\, d\phi = I \tag{15}$$

and the total current in the left hemisphere is $-I$, so the setup is a circuit carrying current I, whose self inductance is

$$L = \frac{2W}{I^2} = \frac{\pi\mu_0}{8} \tag{16}$$