Equation (3.108) was given without proof

$$\int_0^\infty xJ_m(kx)J_m\left(k'x\right)dx = \frac{1}{k}\delta(k'-k) \tag{1}$$

It seems the rigorous proof of this is non-trivial (See *Revisiting the orthogonality of Bessel functions of the first kind on an infinite interval* by J. Ponce de Leon 2015). Here we are going to show why this relation is important and how it conveniently makes Hankel transform and inverse Hankel transform symmetric.

Recall the discussion of orthonormality of $a_n\sqrt{x}J_\nu(x_{\nu n}\rho/a)$ where $x_{\nu n}$ is the n-th root of $J_\nu(x)$. Now with continuous parameter k, we would still hope to have a complete orthonormal set of functions

$$U_{\nu k}(x) = a_k\sqrt{x}J_\nu(kx) \tag{2}$$

This would allow any function $A(x)$ to have expansion

$$A(x) = \int_0^\infty \tilde{A}(k)\cdot a_k\sqrt{x}J_\nu(kx)dk \tag{3}$$

where the coefficient is given by the inner product

$$\tilde{A}(k) = \int_0^\infty A\left(x'\right)\cdot a_k\sqrt{x'}J_\nu\left(kx'\right)dx' \tag{4}$$

Plugging (4) into (3) gives

$$\begin{aligned} A(x) &= \int_0^\infty \left[\int_0^\infty A\left(x'\right)a_k\sqrt{x'}J_\nu\left(kx'\right)dx'\right]a_k\sqrt{x}J_\nu(kx)\,dk \\ &= \int_0^\infty A\left(x'\right)dx'\left[\int_0^\infty a_k^2\sqrt{xx'}J_\nu(kx)J_\nu\left(kx'\right)dk\right] \end{aligned} \tag{5}$$

The content in the square bracket must be identified with $\delta(x'-x)$ so (5) gives the familiar selection property of the δ function.

This would require

$$\int_0^\infty a_k^2 J_\nu(kx)J_\nu\left(kx'\right)dk = \frac{\delta(x'-x)}{x} \tag{6}$$

Now compare (3) and (4), the role of x and k are almost symmetric other than the difference between a_k v.s. $\sqrt{x}$. If it so happens that $a_k = \sqrt{k}$, this symmetry will be established, in which case (6) will become

$$\int_0^\infty kJ_\nu(kx)J_\nu\left(kx'\right)dk = \frac{\delta(x'-x)}{x} \tag{7}$$

and by symmetry between $k \longleftrightarrow x$,

$$\int_0^\infty xJ_\nu(kx)J_\nu\left(k'x\right)dx = \frac{\delta(k'-k)}{k} \tag{8}$$

which is basically the same as (1).

Note all these were just wishful thinking, which if true, would turn (3) and (4) into perfect symmetric Hankel transform and its inverse transform

$$\tilde{A}(k) = \int_0^\infty A(x)\sqrt{kx}J_\nu(kx)dx \tag{9}$$

$$A(x) = \int_0^\infty \tilde{A}(k)\sqrt{kx}J_\nu(kx)dk \tag{10}$$

These are slightly different than the form given in equation (3.110), but if we let

$$\tilde{A}'(k) = \sqrt{k}\tilde{A}(k) \qquad\qquad A'(x) = \frac{A(x)}{\sqrt{x}} \tag{11}$$

we end up with the same form as (3.110)

$$\tilde{A}'(k) = k\int_0^\infty xA'(x)J_\nu(kx)dx \tag{12}$$

$$A'(x) = \int_0^\infty \tilde{A}'(k)J_\nu(kx)\,dk \tag{13}$$

Finally, the wishful thinking must be proved independently (which the reference above provides), and we must also qualify ν with the condition that $\mathrm{Re}(\nu) > -1$ to make $J_\nu(kx)$ a complete orthogonal set (note technically, it is $\sqrt{kx}J_\nu(kx)$ that is the complete orthonormal set, but knowing we can always transform the given $A(x)$ with (11) first, we can speak in loose terms by calling $J_\nu(kx)$ the complete set, in which case any subsequent integrations must be done with the xdx measure).