1. From problem 5.30, we know if there were no iron around the current, the vector potential generated by the current is given by

$$\mathbf{A}(\mathbf{x}) = \begin{cases} \dfrac{\mu_0 N I \rho \cos\phi}{4R}\hat{\mathbf{z}} & \text{for } \rho < R \\ \dfrac{\mu_0 N I R \cos\phi}{4\rho}\hat{\mathbf{z}} & \text{for } \rho > R \end{cases} \tag{1}$$

It's easy to verify that in region where $\rho \neq R$, $\mathbf{A}$ satisfies the vector Laplace equation

$$\nabla^2 \mathbf{A} = 0 \tag{2}$$

With the presence of the iron, there are induced currents in the iron, which will contribute a vector potential $\mathbf{A}'(\mathbf{x})$ in the region $\rho < R'$, which must satisfy Laplace equation as well

$$\nabla^2 \mathbf{A}' = 0 \tag{3}$$

Due to the 2D nature of this problem, $\mathbf{A}'(\mathbf{x}) = A'(\rho,\phi)\hat{\mathbf{z}}$, where $A'(\rho,\phi)$ has the general form (2.71)

$$A'(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty}\left(a_n\rho^n \cos n\phi + b_n \rho^n \sin n\phi + c_n\rho^{-n}\cos n\phi + d_n\rho^{-n}\sin n\phi\right) \tag{4}$$

We are interested in the region $\rho < R'$, whose admissible form is reduced to

$$A'(\rho,\phi) = a_0 + \sum_{n=1}^{\infty}\left(a_1\rho^n \cos n\phi + b_n\rho^n \sin n\phi\right) \tag{5}$$

The boundary condition at $\rho = R'$ requires the net tangential field to vanish, i.e.,

$$\begin{aligned} 0 = B_\phi = -\left.\frac{\partial (A + A')}{\partial\rho}\right|_{\rho=R'} & \qquad \Longrightarrow \\ 0 = \sum_{n=1}^{\infty}\left(a_n n\rho^{n-1}\cos n\phi + b_n n \rho^{n-1}\sin n\phi\right) - \frac{\mu_0 N I R\cos\phi}{4R'^2} & \qquad \text{for all } \phi \in [0, 2\pi] \end{aligned} \tag{6}$$

By orthogonality, we know

$$a_1 = \frac{\mu_0 N I R}{4R'^2} \tag{7}$$

and all the other coefficients vanish (a_0 is chosen to be zero as it's inconsequential when we calculate $\mathbf{B}$), i.e.,

$$A'(\rho,\phi) = \frac{\mu_0 N I R \rho\cos\phi}{4R'^2} \tag{8}$$

Combining with (1), the net potential for $\rho < R'$ is

$$\mathbf{A}_{\text{net}}(\mathbf{x}) = \begin{cases} \dfrac{\mu_0 N I \rho\cos\phi}{4R}\left(1 + \dfrac{R^2}{R'^2}\right)\hat{\mathbf{z}} & \text{for } \rho < R \\ \dfrac{\mu_0 N I R\cos\phi}{4R'}\left(\dfrac{R'}{\rho} + \dfrac{\rho}{R'}\right)\hat{\mathbf{z}} & \text{for } R < \rho < R' \end{cases} \tag{9}$$

Therefore for $\rho < R$,

$$\mathbf{B}_{\text{net}}(\mathbf{x}) = \nabla \times \mathbf{A}_{\text{net}} = -\frac{\mu_0 N I}{4R}\left(1 + \frac{R^2}{R'^2}\right)\hat{\mathbf{y}} \tag{10}$$

and for $\rho > R$,

$$\begin{aligned} \mathbf{B}_{\text{net}}(\mathbf{x}) = \nabla \times \mathbf{A}_{\text{net}} &= \frac{1}{\rho}\frac{\partial A_{\text{net},z}}{\partial\phi}\hat{\boldsymbol{\rho}} - \frac{\partial A_{\text{net},z}}{\partial\rho}\hat{\boldsymbol{\phi}} \\ &= \frac{\mu_0 N I R}{4R'}\left[\left(\frac{R'}{\rho^2} + \frac{1}{R'}\right)(-\sin\phi)\hat{\boldsymbol{\rho}} - \left(-\frac{R'}{\rho^2} + \frac{1}{R'}\right)\cos\phi\,\hat{\boldsymbol{\phi}}\right] \end{aligned} \tag{11}$$

2. Due to the infinite permeability of the region $\rho > R'$, there must be no energy distribution there. So the energy (per length in z) is distributed as the following

$$W_{\rho<R} = \pi R^2 \cdot \frac{1}{2\mu_0}\left[\frac{\mu_0 NI}{4R}\left(1+\frac{R^2}{R'^2}\right)\right]^2 = \frac{\mu_0 \pi N^2 I^2}{32}\left(1+\frac{R^2}{R'^2}\right)^2 \tag{12}$$

$$\begin{aligned} W_{R<\rho<R'} &= \int_R^{R'} \rho\, d\rho \int_0^{2\pi} d\phi \frac{1}{2\mu_0}\left(\frac{\mu_0 NIR}{4R'}\right)^2 \left[\left(\frac{R'}{\rho^2}+\frac{1}{R'}\right)^2 \sin^2\phi + \left(\frac{1}{R'}-\frac{R'}{\rho^2}\right)^2 \cos^2\phi\right] \\ &= \frac{\mu_0 N^2 I^2 R^2}{32R'^2}\cdot 2\pi \int_R^{R'} \left(\frac{R'^2}{\rho^3}+\frac{\rho}{R'^2}\right) d\rho \\ &= \frac{\mu_0 N^2 I^2 R^2}{32R'^2}\cdot 2\pi \left[\frac{R'^2}{2}\left(\frac{1}{R^2}-\frac{1}{R'^2}\right)+\frac{1}{2R'^2}\left(R'^2-R^2\right)\right] \\ &= \frac{\mu_0 \pi N^2 I^2}{32}\left(1-\frac{R^4}{R'^4}\right) \end{aligned} \tag{13}$$

3. From (12) and (13),

$$W = \frac{\mu_0 \pi N^2 I^2}{16}\left(1+\frac{R^2}{R'^2}\right) \qquad \Longrightarrow \qquad L = \frac{2W}{I^2} = \frac{\mu_0 \pi N^2}{8}\left(1+\frac{R^2}{R'^2}\right) \tag{14}$$