Here we provide the detailed explanation for equation (7.90) and (7.91). It is worth noting that in (7.90), the abbreviation "c.c." means "complex conjugate". Written in full, (7.90) should read

$$u(x,t) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} A(k)\, e^{ikx-i\omega(k)t} dk + \int_{-\infty}^{\infty} A^*(k)\, e^{-ikx+i\omega(k)t} dk\right] \tag{1}$$

By construction, $u(x,t)$ is a real function. At $t=0$,

$$u(x,0) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} A(k)\, e^{ikx} dk + \int_{-\infty}^{\infty} A^*(k)\, e^{-ikx} dk\right] \tag{2}$$

$$\frac{\partial u}{\partial t}(x,0) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty} A(k)[-i\omega(k)]\, e^{ikx} dk + \int_{-\infty}^{\infty} A^*(k)[i\omega(k)]\, e^{-ikx} dk\right\} \tag{3}$$

Integrating both sides of (2) and (3) with $e^{-ik'x}dx$, and recalling $\int_{-\infty}^{\infty} e^{ipx} dx = 2\pi\delta(p)$, we get

$$\begin{aligned}
\int_{-\infty}^{\infty} u(x,0)\, e^{-ik'x} dx &= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} dk A(k) \int_{-\infty}^{\infty} e^{i(k-k')x} dx + \int_{-\infty}^{\infty} dk A^*(k) \int_{-\infty}^{\infty} e^{-i(k+k')x} dx\right] \\
&= \frac{\sqrt{2\pi}}{2}\left[\int_{-\infty}^{\infty} dk A(k)\, \delta(k-k') + \int_{-\infty}^{\infty} dk A^*(k)\, \delta(k+k')\right] \\
&= \frac{\sqrt{2\pi}}{2}\left[A(k') + A^*(-k')\right]
\end{aligned} \tag{4}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x,0)\, e^{-ik'x} dx &= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\Bigg\{\int_{-\infty}^{\infty} dk A(k)[-i\omega(k)] \int_{-\infty}^{\infty} e^{i(k-k')x} dx + \\
&\qquad \int_{-\infty}^{\infty} dk A^*(k)[i\omega(k)] \int_{-\infty}^{\infty} e^{-i(k+k')x} dx\Bigg\} \\
&= \frac{\sqrt{2\pi}}{2}\left\{\int_{-\infty}^{\infty} dk A(k)[-i\omega(k)]\, \delta(k-k') + \int_{-\infty}^{\infty} dk A^*(k)[i\omega(k)]\, \delta(k+k')\right\} \\
&= \frac{\sqrt{2\pi}}{2}\left\{[-i\omega(k')]A(k') + [i\omega(-k')]A^*(-k')\right\} \qquad \text{if we assume } \omega(k') = \omega(-k') \\
&= \frac{\sqrt{2\pi}}{2}[-i\omega(k')]\left[A(k') - A^*(-k')\right]
\end{aligned} \tag{5}$$

Recombining (4) and (5) while relabeling $k' \to k$, we have

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[u(x,0) + \frac{i}{\omega(k)}\frac{\partial u}{\partial t}(x,0)\right] e^{-ikx} dx \tag{6}$$

$$A^*(-k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[u(x,0) - \frac{i}{\omega(k)}\frac{\partial u}{\partial t}(x,0)\right] e^{-ikx} dx \tag{7}$$

We see that (7) is consistent with (6) when $\omega(k) = \omega(-k)$.