1. Let the charge's location at $t=0$ be $\mathbf{x}=0$, then the charge density can be written

$$\rho(\mathbf{x},t)=Ze\delta(\mathbf{x}-\mathbf{v}t) \tag{1}$$

The Fourier transform in both space and time gives

$$\begin{aligned}\rho(\mathbf{q},\omega)&=\frac{1}{(2\pi)^4}\int dte^{i\omega t}\int d^3qe^{-i\mathbf{q}\cdot\mathbf{x}}\rho(\mathbf{x},t)\\ &=\frac{Ze}{(2\pi)^4}\int dt\int\delta(\mathbf{x}-\mathbf{v}t)e^{-i\mathbf{q}\cdot\mathbf{x}+i\omega t}d^3x\\ &=\frac{Ze}{(2\pi)^4}\int dte^{-i\mathbf{q}\cdot\mathbf{v}t+i\omega t}\\ &=\frac{Ze}{(2\pi)^4}\cdot 2\pi\delta(\omega-\mathbf{q}\cdot\mathbf{v})\\ &=\frac{Ze}{(2\pi)^3}\delta(\omega-\mathbf{q}\cdot\mathbf{v})\end{aligned} \tag{2}$$

2. In general, if $f(\mathbf{q},\omega)$ is the Fourier transform of a scalar function $f(\mathbf{x},t)$, i.e.,

$$f(\mathbf{x},t)=\int d\omega\int d^3qf(\mathbf{q},\omega)e^{i\mathbf{q}\cdot\mathbf{x}-i\omega t} \tag{3}$$

the gradient of $f(\mathbf{x},t)$ is given by

$$\nabla f(\mathbf{x},t)=\int d\omega\int d^3q(i\mathbf{q})f(\mathbf{q},\omega)e^{i\mathbf{q}\cdot\mathbf{x}-i\omega t} \tag{4}$$

which means

$$\nabla f(\mathbf{x},t)\qquad\longleftrightarrow\qquad i\mathbf{q}f(\mathbf{q},\omega) \tag{5}$$

is a Fourier pair.

Similarly

$$\nabla\cdot\mathbf{g}(\mathbf{x},t)\qquad\longleftrightarrow\qquad i\mathbf{q}\cdot\mathbf{g}(\mathbf{q},\omega) \tag{6}$$

is a Fourier pair too.

Then applying the Fourier transform to the relations $\nabla\cdot\mathbf{D}(\mathbf{x},t)=\rho(\mathbf{x},t)$ and $\mathbf{E}(\mathbf{x},t)=-\nabla\Phi(\mathbf{x},t)$ yields

$$\begin{aligned}i\mathbf{q}\cdot\mathbf{D}(\mathbf{q},\omega)=\rho(\mathbf{q},\omega)\quad&\Longrightarrow\quad i\mathbf{q}\cdot[\epsilon(\mathbf{q},\omega)\mathbf{E}(\mathbf{q},\omega)]=\rho(\mathbf{q},\omega)\quad\Longrightarrow\\ i\mathbf{q}\cdot[\epsilon(\mathbf{q},\omega)(-i\mathbf{q})\Phi(\mathbf{q},\omega)]=\rho(\mathbf{q},\omega)\quad&\Longrightarrow\quad\Phi(\mathbf{q},\omega)=\frac{\rho(\mathbf{q},\omega)}{q^2\epsilon(\mathbf{q},\omega)}\end{aligned} \tag{7}$$

3. Writing

$$\mathbf{J}(\mathbf{x},t)=\int d\omega\int d^3q\mathbf{J}(\mathbf{q},\omega)e^{i\mathbf{q}\cdot\mathbf{x}-i\omega t}\qquad\mathbf{E}(\mathbf{x},t)=\int d\omega'\int d^3q'\mathbf{E}(\mathbf{q}',\omega')e^{i\mathbf{q}'\cdot\mathbf{x}-i\omega' t} \tag{8}$$

enables us to obtain

$$
\begin{aligned}
\frac{dW}{dt} &= \int \mathbf{J}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)\,d^3x = \int d\omega \int d^3q \int d\omega' \int d^3q' \mathbf{J}(\mathbf{q},\omega)\cdot\mathbf{E}(\mathbf{q}',\omega')\,e^{-i(\omega+\omega')t} \overbrace{\int e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}} d^3x}^{(2\pi)^3\delta(\mathbf{q}+\mathbf{q}')} \\
&= (2\pi)^3 \int d\omega \int d^3q \int d\omega' \mathbf{J}(\mathbf{q},\omega)\cdot\mathbf{E}(-\mathbf{q},\omega')\,e^{-i(\omega+\omega')t} \\
&= (2\pi)^3 \int d\omega \int d^3q \int d\omega' \sigma(\mathbf{q},\omega)\mathbf{E}(\mathbf{q},\omega)\cdot\mathbf{E}(-\mathbf{q},\omega')\,e^{-i(\omega+\omega')t} \\
&= (2\pi)^3 \int d\omega \int d^3q \int d\omega' \sigma(\mathbf{q},\omega)\left[\frac{-i\mathbf{q}\rho(\mathbf{q},\omega)}{q^2\epsilon(\mathbf{q},\omega)}\right]\cdot\left[\frac{i\mathbf{q}\rho(-\mathbf{q},\omega')}{q^2\epsilon(-\mathbf{q},\omega')}\right] e^{-i(\omega+\omega')t} \\
&= \frac{Z^2e^2}{(2\pi)^3} \int d\omega \int d^3q \int d\omega' \left[\frac{\sigma(\mathbf{q},\omega)}{q^2\epsilon(\mathbf{q},\omega)\epsilon(-\mathbf{q},\omega')}\right]\delta(\omega-\mathbf{q}\cdot\mathbf{v})\,\delta(\omega'+\mathbf{q}\cdot\mathbf{v})\,e^{-i(\omega+\omega')t} \\
&= \frac{Z^2e^2}{(2\pi)^3} \int \frac{d^3q}{q^2} \int d\omega \left[\frac{\sigma(\mathbf{q},\omega)}{\epsilon(\mathbf{q},\omega)\epsilon(-\mathbf{q},-\omega)}\right]\delta(\omega-\mathbf{q}\cdot\mathbf{v})
\end{aligned}
\tag{9}
$$

where in the last step we have used the two $\delta$ functions to select $\omega = -\omega' = \mathbf{q}\cdot\mathbf{v}$.
Similar to the derivation of (7.105), we can write

$$\mathbf{D}(\mathbf{x},t) = \epsilon_0\left[\mathbf{E}(\mathbf{x},t) + \int d\tau \int d^3r\, G(\mathbf{r},\tau)\,\mathbf{E}(\mathbf{x}-\mathbf{r},t-\tau)\right] \quad \text{where} \tag{10}$$

$$G(\mathbf{r},\tau) = \int d\omega \int d^3q\, [\epsilon(\mathbf{q},\omega)/\epsilon_0 - 1]\, e^{i\mathbf{q}\cdot\mathbf{r}-i\omega\tau} \tag{11}$$

And then the same argument in 7.10.C regarding the reality of $G(\mathbf{r},\tau)$ applies to yield the relation

$$\epsilon(-\mathbf{q},-\omega) = \epsilon^*(\mathbf{q}^*,\omega^*) \tag{12}$$

for generally complex $\mathbf{q},\omega$.
Also

$$\sigma(\mathbf{q},\omega) = i\omega\,[\epsilon_0 - \epsilon(\mathbf{q},\omega)] \tag{13}$$

indicates

$$\sigma(\mathbf{q},\omega) = \omega\,\mathrm{Im}\,[\epsilon(\mathbf{q},\omega)] \tag{14}$$

which turns the bracket of (9) into

$$\frac{\sigma(\mathbf{q},\omega)}{\epsilon(\mathbf{q},\omega)\epsilon(-\mathbf{q},-\omega)} = \frac{\omega\,\mathrm{Im}\,[\epsilon(\mathbf{q},\omega)]}{\epsilon(\mathbf{q},\omega)\epsilon^*(\mathbf{q},\omega)} = -\omega\,\mathrm{Im}\left[\frac{1}{\epsilon(\mathbf{q},\omega)}\right] \tag{15}$$

Plugging this back to (9) yields

$$
\begin{aligned}
-\frac{dW}{dt} &= \frac{Z^2e^2}{(2\pi)^3}\int \frac{d^3q}{q^2}\int_{-\infty}^{\infty} \underbrace{d\omega\,\omega\,\mathrm{Im}\left[\frac{1}{\epsilon(\mathbf{q},\omega)}\right]\delta(\omega-\mathbf{q}\cdot\mathbf{v})}_{h(\mathbf{q},\omega)} \\
&= \frac{Z^2e^2}{(2\pi)^3}\int \frac{d^3q}{q^2}\left[\int_0^\infty d\omega h(\mathbf{q},\omega) + \int_{-\infty}^0 d\omega h(\mathbf{q},\omega)\right] && \text{relabel } \omega' = -\omega \text{ in the 2nd integral} \\
&= \frac{Z^2e^2}{(2\pi)^3}\left[\int \frac{d^3q}{q^2}\int_0^\infty d\omega h(\mathbf{q},\omega) + \int \frac{d^3q}{q^2}\int_0^\infty d\omega' h(\mathbf{q},-\omega')\right] && \text{relabel } \mathbf{q}' = -\mathbf{q} \text{ in the 2nd integral} \\
&= \frac{Z^2e^2}{(2\pi)^3}\left[\int \frac{d^3q}{q^2}\int_0^\infty d\omega h(\mathbf{q},\omega) + \int \frac{d^3q'}{q'^2}\int_0^\infty d\omega' h(-\mathbf{q}',-\omega')\right] && \text{note } h(\mathbf{q},\omega) = h(-\mathbf{q},-\omega) \\
&= \frac{Z^2e^2}{4\pi^3}\int \frac{d^3q}{q^2}\int_0^\infty d\omega\omega\,\mathrm{Im}\left[\frac{1}{\epsilon(\mathbf{q},\omega)}\right]\delta(\omega-\mathbf{q}\cdot\mathbf{v})
\end{aligned}
\tag{16}
$$