1. (11.152) can be easily modified to fit the situation described in this problem by $vt \to vt - z$ and $b \to r_\perp$.

$$E_z = -\frac{q\gamma(vt-z)}{\left[r_\perp^2 + \gamma^2(vt-z)^2\right]^{3/2}} \tag{1}$$

$$\mathbf{E}_\perp = \frac{\gamma q \mathbf{r}_\perp}{\left[r_\perp^2 + \gamma^2(vt-z)^2\right]^{3/2}} \tag{2}$$

$$\mathbf{B} = \boldsymbol{\beta} \times \mathbf{E}_\perp \tag{3}$$

Rewriting the above in terms of β gives

$$E_z = -\frac{q\left(1-\beta^2\right)(vt-z)}{\left[\left(1-\beta^2\right)r_\perp^2 + (vt-z)^2\right]^{3/2}} \tag{4}$$

$$\mathbf{E}_\perp = \frac{\left(1-\beta^2\right)q\mathbf{r}_\perp}{\left[\left(1-\beta^2\right)r_\perp^2 + (vt-z)^2\right]^{3/2}} \tag{5}$$

As $\beta \to 1$, we see that $\mathbf{E}_\perp$ vanishes for all z such that $vt - z \neq 0$, but the integral

$$\int_{-\infty}^{\infty} \mathbf{E}_\perp dz = \left(1-\beta^2\right)q\mathbf{r}_\perp \underbrace{\int_{-\infty}^{\infty} \left[\left(1-\beta^2\right)r_\perp^2 + (vt-z)^2\right]^{-3/2} dz}_{\frac{2}{(1-\beta^2)r_\perp^2}} = \frac{2q\mathbf{r}_\perp}{r_\perp^2} \tag{6}$$

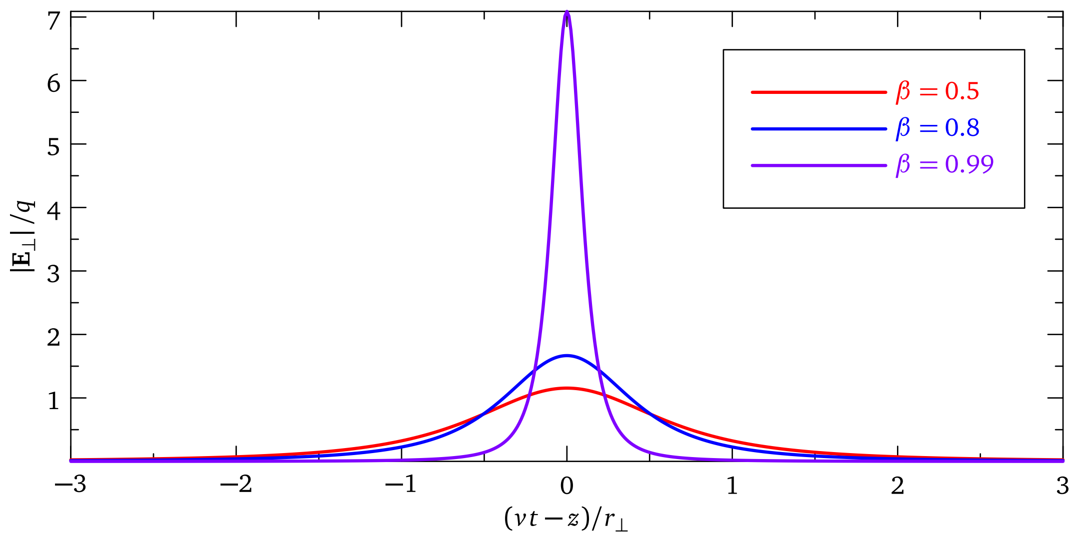
so by definition as $\beta \to 1$, we can write

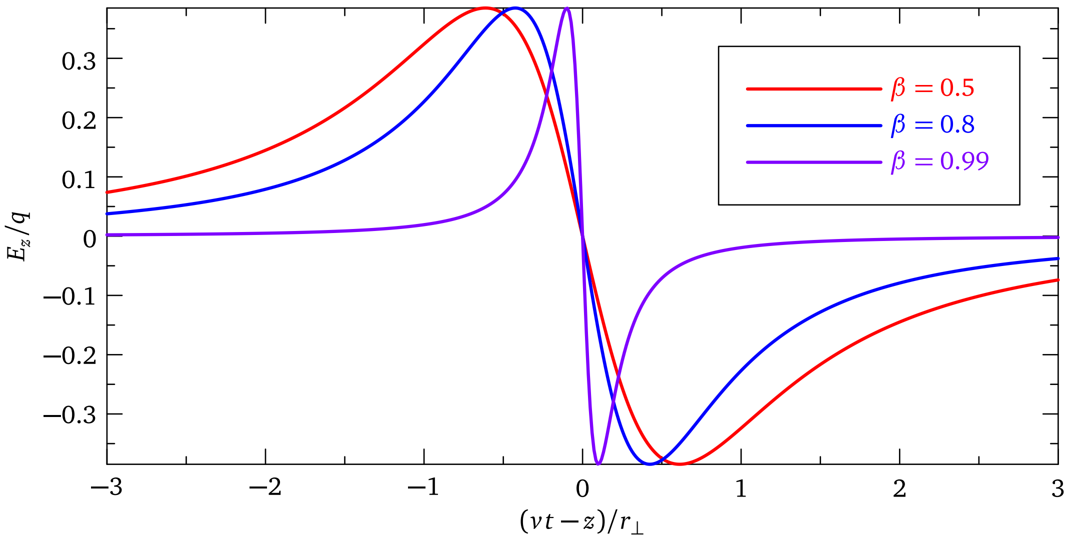
$$\mathbf{E}_\perp = 2q\frac{\mathbf{r}_\perp}{r_\perp^2}\delta(ct-z) \tag{7}$$

and by (3) at the same limit,

$$\mathbf{B} = 2q\frac{\hat{\mathbf{v}} \times \mathbf{r}_\perp}{r_\perp^2}\delta(ct-z) \tag{8}$$

The graph of $|\mathbf{E}_\perp|$ and E_z are plotted below. Simple calculations show that $|\mathbf{E}_\perp|$ takes peak value $\gamma q/r_\perp^2$ at $vt - z = 0$, and E_z takes peak value $\pm\sqrt{4/27}q/r_\perp^2$ at $vt - z = \mp r_\perp/\sqrt{2}\gamma$, which under $\gamma \to \infty$, becomes two finite spikes at $0^\pm$ and zero elsewhere. For the practical reasons stated in the paragraph below (11.153), we can ignore the longitudinal component of the field.





2. For charge density

$$\nabla\cdot\mathbf{E} = 2q\delta(ct-z)\nabla_\perp\cdot\left(\frac{\mathbf{r}_\perp}{r_\perp^2}\right) \tag{9}$$

Straightforward calculation shows that the 2D divergence vanishes if $r_\perp \neq 0$, but by Gauss Theorem, integrating it over the disk of radius R gives

$$\int_{r_\perp<R}\nabla_\perp\cdot\left(\frac{\mathbf{r}_\perp}{r_\perp^2}\right)da = \oint_{r_\perp=R}\frac{\mathbf{r}_\perp}{r_\perp^2}\cdot\mathbf{n}dl = 2\pi \tag{10}$$

This enables us to identify $\nabla_\perp\cdot\left(\mathbf{r}_\perp/r_\perp^2\right)$ as $2\pi\delta^{(2)}(\mathbf{r}_\perp)$, hence

$$\nabla\cdot\mathbf{E} = 4\pi q\delta^{(2)}(\mathbf{r}_\perp)\delta(vt-z) \quad\Longrightarrow\quad \rho = q\delta^{(2)}(\mathbf{r}_\perp)\delta(ct-z) \quad\Longrightarrow\quad J^0 = c\rho = qc\delta^{(2)}(\mathbf{r}_\perp)\delta(ct-z) \tag{11}$$

On the other hand, the current density can be obtained through

$$\begin{aligned}
\mathbf{J} &= \frac{c}{4\pi}\left(\nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right)\\
&= \frac{c}{4\pi}\cdot 2q\left\{\nabla\times\left[\frac{\hat{\mathbf{v}}\times\mathbf{r}_\perp}{r_\perp^2}\delta(ct-z)\right] - \frac{1}{c}\frac{\mathbf{r}_\perp}{r_\perp^2}\frac{\partial\delta(ct-z)}{\partial t}\right\}\\
&= \frac{c}{4\pi}\cdot 2q\left\{\underbrace{\nabla\delta(ct-z)\times\left(\frac{\hat{\mathbf{v}}\times\mathbf{r}_\perp}{r_\perp^2}\right)}_{\delta'(ct-z)\mathbf{r}_\perp/r_\perp^2} + \delta(ct-z)\nabla\times\left(\frac{\hat{\mathbf{v}}\times\mathbf{r}_\perp}{r_\perp^2}\right) - \frac{\mathbf{r}_\perp}{r_\perp^2}\delta'(ct-z)\right\}\\
&= \frac{c}{4\pi}\cdot 2q\delta(ct-z)\left\{\hat{\mathbf{v}}\underbrace{\left[\nabla\cdot\left(\frac{\mathbf{r}_\perp}{r_\perp^2}\right)\right]}_{2\pi\delta^{(2)}(\mathbf{r}_\perp)} - \overbrace{(\hat{\mathbf{v}}\cdot\nabla)\frac{\mathbf{r}_\perp}{r_\perp^2}}^{0}\right\}\\
&= qc\hat{\mathbf{v}}\delta^{(2)}(\mathbf{r}_\perp)\delta(ct-z)
\end{aligned} \tag{12}$$

3. For the two given 4-potentials, it's routine check to verify that both of them satisfy

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla A^0 \qquad\text{and}\qquad \mathbf{B} = \nabla\times\mathbf{A} \tag{13}$$

Taking the difference of them yields the gradient of the gauge function

$$\nabla\chi = -2q\delta(ct-z)\ln(\lambda r_\perp)\hat{\mathbf{z}} + 2q\Theta(ct-z)\nabla_\perp\ln(\lambda r_\perp) = \nabla[2q\Theta(ct-z)\ln(\lambda r_\perp)] \tag{14}$$

so we can identify the gauge function

$$\chi = 2q\Theta(ct-z)\ln(\lambda r_\perp) \tag{15}$$