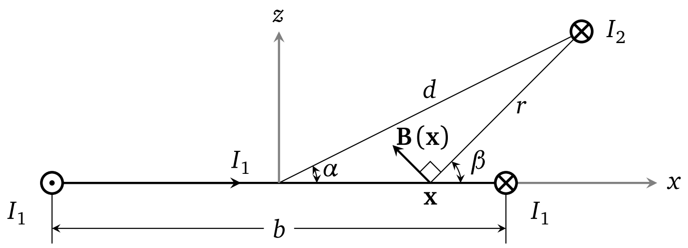
1. As depicted below, the magnitude of the magnetic induction at point $\mathbf{x} = (x, y, 0)$ is

$$B(\mathbf{x}) = \frac{\mu_0 I_2}{2\pi r} \tag{1}$$

hence the flux density through the rectangular loop is its z component

$$B_z(\mathbf{x}) = B(\mathbf{x})\cos\beta = \frac{\mu_0 I_2}{2\pi}\left(\frac{d\cos\alpha - x}{r^2}\right) = \frac{\mu_0 I_2}{2\pi}\left[\frac{d\cos\alpha - x}{(d\cos\alpha - x)^2 + d^2\sin^2\alpha}\right] \tag{2}$$



The total flux is obtained by integrating (2) throughout the rectangular region

$$\begin{aligned}
F_2 &= \int_{-b/2}^{b/2} dx \int_{-a/2}^{a/2} dy \frac{\mu_0 I_2}{2\pi}\left[\frac{d\cos\alpha - x}{(d\cos\alpha - x)^2 + d^2\sin^2\alpha}\right] \\
&= \frac{\mu_0 I_2 a}{4\pi}\int_{-b/2}^{b/2}\frac{2(d\cos\alpha - x)\,dx}{(d\cos\alpha - x)^2 + d^2\sin^2\alpha} \\
&= \frac{\mu_0 I_2 a}{4\pi}\ln\left[(d\cos\alpha - x)^2 + d^2\sin^2\alpha\right]_{x=b/2}^{x=-b/2} \\
&= \frac{\mu_0 I_2 a}{4\pi}\ln\left(\frac{4d^2 + b^2 + 4bd\cos\alpha}{4d^2 + b^2 - 4bd\cos\alpha}\right)
\end{aligned} \tag{3}$$

which gives the interaction magnetic energy

$$W_{12} = I_1 F_2 = \frac{\mu_0 I_1 I_2 a}{4\pi}\ln\left(\frac{4d^2 + b^2 + 4bd\cos\alpha}{4d^2 + b^2 - 4bd\cos\alpha}\right) \tag{4}$$

2. If we write (4) in terms of the wire's Cartesian coordinates, we have

$$W_{12} = \frac{\mu_0 I_1 I_2 a}{4\pi}\ln\left[\frac{4(x^2+z^2) + b^2 + 4bx}{4(x^2+z^2) + b^2 - 4bx}\right] \tag{5}$$

hence

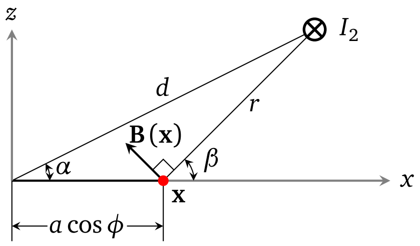
$$F_x = -\frac{\partial W_{12}}{\partial x} = \frac{\mu_0 I_1 I_2 a}{\pi}\left[\frac{2x + b}{4(x^2+z^2) + b^2 + 4bx} - \frac{2x - b}{4(x^2+z^2) + b^2 - 4bx}\right] \tag{6}$$

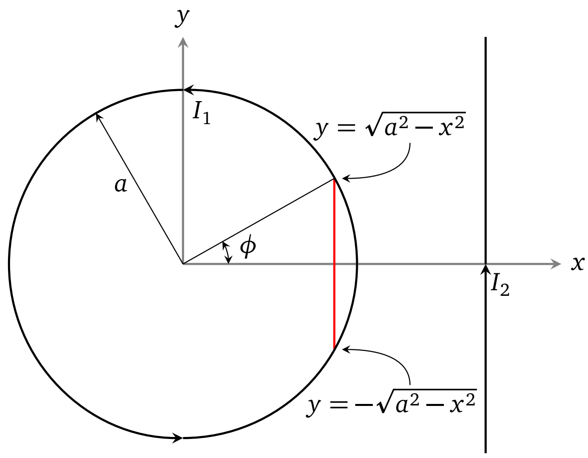
$$F_z = -\frac{\partial W_{12}}{\partial z} = \frac{\mu_0 I_1 I_2 a}{\pi}\left[\frac{2z}{4(x^2+z^2) + b^2 + 4bx} - \frac{2z}{4(x^2+z^2) + b^2 - 4bx}\right] \tag{7}$$

$$F_y = -\frac{\partial W_{12}}{\partial y} = 0 \tag{8}$$

3. When the loop is circular, let's consider the magnetic flux through the differential region $[x, x+dx]$ where $x = a\cos\phi$. The diagram below on the left is looking towards the $y+$ direction, where the one on the right is looking towards the $z-$ direction. We know

$$\begin{aligned}
dF_2\Big|_{x\to x+dx} &= \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy B_z(\mathbf{x})\cos\beta\, dx \\
&= \frac{\mu_0 I_2}{2\pi}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \left(\frac{d\cos\alpha - x}{r^2}\right) dx \\
&= \frac{\mu_0 I_2}{\pi}\cdot\sqrt{a^2 - x^2}\left[\frac{d\cos\alpha - x}{(d\cos\alpha - x)^2 + d^2\sin\alpha^2}\right] dx
\end{aligned} \tag{9}$$





Integrating over $x \in [-a, a]$ gives the total flux

$$\begin{aligned} F_2 &= \int_{-a}^{a} \frac{\mu_0 I_2}{\pi} \left[\frac{d\cos\alpha - x}{(d\cos\alpha - x)^2 + d^2 \sin^2\alpha} \right] \sqrt{a^2-x^2}\,dx \\ &= \frac{\mu_0 I_2 a^2}{\pi d} \int_0^{\pi} \underbrace{\left[\frac{\cos\alpha - \frac{a}{d}\cos\phi}{\left(\cos\alpha - \frac{a}{d}\cos\phi\right)^2 + \sin^2\alpha} \right]}_{A} \sin^2\phi\, d\phi \end{aligned} \tag{10}$$

Define

$$t \equiv \frac{a}{d}\cos\phi \tag{11}$$

then we can write A as

$$A = \frac{\cos\alpha - t}{1 + t^2 - 2t\cos\alpha} \tag{12}$$

On the other hand, observe that

$$\begin{aligned} \frac{1}{t}\operatorname{Re}\left(\frac{1}{1-te^{-i\alpha}} - 1\right) &= \operatorname{Re}\left(\frac{e^{-i\alpha}}{1-te^{-i\alpha}}\right) \\ &= \operatorname{Re}\left(\frac{\cos\alpha - i\sin\alpha}{1 - t\cos\alpha + it\sin\alpha}\right) \\ &= \frac{\cos\alpha\,(1-t\cos\alpha) - (-\sin\alpha)(-t\sin\alpha)}{(1-t\cos\alpha)^2 + t^2\sin^2\alpha} = A \end{aligned} \tag{13}$$

We can thus express A as a sum of series

$$A = \frac{1}{t}\operatorname{Re}\left(\frac{1}{1-te^{-i\alpha}} - 1\right) = \operatorname{Re}\left[\frac{1}{t}\sum_{n=1}^{\infty}\left(te^{-i\alpha}\right)^n\right] = \operatorname{Re}\sum_{n=1}^{\infty}\left(\frac{a}{d}\cos\phi\right)^{n-1} e^{-in\alpha} \tag{14}$$

Let

$$K_n = \int_0^{\pi} \cos^n\phi\, d\phi \tag{15}$$

then

$$\begin{aligned} K_0 = \pi \qquad\qquad K_1 = 0 \qquad\qquad K_{n+1} &= \int_0^{\pi} \cos^n\phi\cos\phi\, d\phi = \cos^n\phi\sin\phi\Big|_0^{\pi} + n\int_0^{\pi} \cos^{n-1}\phi\sin^2\phi\, d\phi \\ &= n\left(K_{n-1} - K_{n+1}\right) \end{aligned} \tag{16}$$

or

$$K_n = \begin{cases} \dfrac{(2k-1)!!}{(2k)!!}\pi & \text{for } n = 2k \\ 0 & \text{for } n = 2k+1 \end{cases} \tag{17}$$

With this, F_2 in (10) can be written as

$$
\begin{aligned}
F_2 &= \frac{\mu_0 I_2 a^2}{\pi d} \cdot \mathrm{Re} \int_0^{\pi} \left[\sum_{n=1}^{\infty} \left(\frac{a}{d}\right)^{n-1} \cos^{n-1}\phi \left(1-\cos^2\phi\right) e^{-in\alpha} d\phi \right] \\
&= \frac{\mu_0 I_2 a^2}{\pi d} \cdot \mathrm{Re} \left[\sum_{n=1}^{\infty} e^{-in\alpha} \left(\frac{a}{d}\right)^{n-1} (K_{n-1} - K_{n+1}) \right] \\
&= \frac{\mu_0 I_2 a^2}{\pi d} \cdot \mathrm{Re} \left[\sum_{n=1}^{\infty} e^{-in\alpha} \left(\frac{a}{d}\right)^{n-1} \frac{K_{n+1}}{n} \right] \\
&= \frac{\mu_0 I_2 a^2}{\pi d} \cdot \mathrm{Re} \left[\sum_{k=0}^{\infty} e^{-i(2k+1)\alpha} \left(\frac{a}{d}\right)^{2k} \frac{(2k-1)!!}{(2k+2)!!} \pi \right] \\
&= \mu_0 I_2 d \cdot \mathrm{Re} \left\{ \left(\frac{a}{d}\right) \sum_{k=0}^{\infty} \left[\left(\frac{a}{d}\right) e^{-i\alpha} \right]^{2k+1} \frac{(2k-1)!!}{(2k+2)!!} \right\}
\end{aligned} \tag{18}
$$

Recall

$$
\sqrt{1-z} = 1 - \sum_{k=1}^{\infty} z^k \frac{(2k-3)!!}{(2k)!!} \tag{19}
$$

then letting

$$
z \equiv \left[\left(\frac{a}{d}\right) e^{-i\alpha} \right]^2 \tag{20}
$$

yields

$$
\frac{1 - \sqrt{1 - \left(\frac{a}{d}\right)^2 e^{-2i\alpha}}}{\left(\frac{a}{d}\right) e^{-i\alpha}} = \sum_{k=1}^{\infty} \left[\left(\frac{a}{d}\right) e^{-i\alpha} \right]^{2k-1} \frac{(2k-3)!!}{(2k)!!} = \sum_{k=0}^{\infty} \left[\left(\frac{a}{d}\right) e^{-i\alpha} \right]^{2k+1} \frac{(2k-1)!!}{(2k+2)!!} \tag{21}
$$

Substituting (21) into (18) gives

$$
F_2 = \mu_0 I_2 d \cdot \mathrm{Re} \left\{ \left(\frac{a}{d}\right) \left[\frac{1 - \sqrt{1 - \left(\frac{a}{d}\right)^2 e^{-2i\alpha}}}{\left(\frac{a}{d}\right) e^{-i\alpha}} \right] \right\} = \mu_0 I_2 d \cdot \mathrm{Re} \left[e^{i\alpha} - \sqrt{e^{2i\alpha} - \left(\frac{a}{d}\right)^2} \right] \tag{22}
$$

and

$$
W_{12} = I_1 F_2 = \mu_0 I_1 I_2 d \cdot \mathrm{Re} \left[e^{i\alpha} - \sqrt{e^{2i\alpha} - \left(\frac{a}{d}\right)^2} \right] \tag{23}
$$

4. When $d \gg a, d \gg b$, from (4)

$$
W_{12} = \frac{\mu_0 I_1 I_2 a}{4\pi} \ln\left(1 + \frac{8bd\cos\alpha}{4d^2 + b^2 - 4bd\cos\alpha}\right) \approx \frac{\mu_0 I_1 I_2 a}{4\pi} \left(\frac{2b\cos\alpha}{d}\right) = I_1 ab \cdot \frac{\mu_0 I_2 \cos\alpha}{2\pi d} = \mathbf{m}_1 \cdot \mathbf{B}_2 \tag{24}
$$

and from (23)

$$
\begin{aligned}
W_{12} &= \mu_0 I_1 I_2 d \cdot \mathrm{Re} \left[e^{i\alpha} - e^{i\alpha} \sqrt{1 - \left(\frac{a}{d} e^{-i\alpha}\right)^2} \right] \\
&\approx \mu_0 I_1 I_2 d \cdot \mathrm{Re} \left(\frac{1}{2} \frac{a^2}{d^2} e^{-i\alpha} \right) = \mu_0 I_1 I_2 \frac{a^2 \cos\alpha}{2d} = I_1 \pi a^2 \cdot \frac{\mu_0 I_2 \cos\alpha}{2\pi d} = \mathbf{m}_1 \cdot \mathbf{B}_2
\end{aligned} \tag{25}
$$