1. Compare equation (3.136) with the desired alternative form, it remains to prove

$$\ln\left(\frac{2}{\sin\theta}\right)-1=\sum_{j=1}^{\infty}\frac{4j+1}{2j(2j+1)}P_{2j}(\cos\theta) \tag{1}$$

Denoting $x=\cos\theta$, we see that (1) is equivalent to

$$\ln\left(\frac{2}{\sqrt{1-x^2}}\right)-1=\sum_{j=1}^{\infty}\frac{4j+1}{2j(2j+1)}P_{2j}(x) \tag{2}$$

For (1) or (2) to converge, we must insist $\theta\in(0,\pi)$ or $x\in(-1,1)$. In other words, $x=\pm1$ must be excluded.
In fact, we will prove a more general claim

$$\ln(1-x)=\ln 2-1-\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}P_n(x) \qquad \text{for } x\in[-1,1) \tag{3}$$

If (3) is true, replacing x with $-x$ in (3) will give

$$\ln(1+x)=\ln 2-1-\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}P_n(-x) \qquad \text{for } x\in(-1,1] \tag{4}$$

then (2) is obtained by adding (3) and (4) for the overlapping range of $(-1,1)$.
The proof of (3) is done in two steps: (a) we will show for arbitrary $x\in[-1,1)$, the derivative of LHS is equal to that of the RHS; and (b) at value $x=-1$, the LHS is equal to the RHS. (a) and (b) together ensures (3) holds for the entire range $[-1,1)$.

(a) Taking the derivative of both sides of (3), we have

$$\mathrm{LHS}'_{(3)}=-\frac{1}{1-x} \qquad \mathrm{RHS}'_{(3)}=-\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}P'_n(x) \tag{5}$$

Thus we wish to prove

$$(1-x)\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}P'_n(x)=1 \tag{6}$$

Now let's recall the well known recurrence relation of Legendre polynomials (reference Wikipedia)

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x) \tag{7}$$

$$\frac{x^2-1}{n}P'_n(x)=xP_n(x)-P_{n-1}(x) \tag{8}$$

By (8), we have

$$(1-x)\frac{P'_n(x)}{n}=\frac{P_{n-1}(x)-xP_n(x)}{1+x} \tag{9}$$

which gives

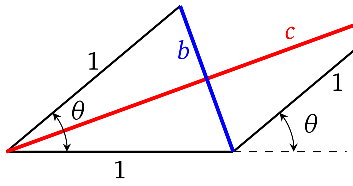
$$\begin{aligned}
\mathrm{LHS}_{(6)}&=\frac{1}{1+x}\sum_{n=1}^{\infty}\frac{2n+1}{n+1}\left[P_{n-1}(x)-xP_n(x)\right] && \text{by (7)}\\
&=\frac{1}{1+x}\sum_{n=1}^{\infty}\frac{2n+1}{n+1}P_{n-1}(x)-\frac{1}{n+1}\left[(n+1)P_{n+1}(x)+nP_{n-1}(x)\right]\\
&=\frac{1}{1+x}\sum_{n=1}^{\infty}\left[P_{n-1}(x)-P_{n+1}(x)\right]\\
&=\frac{1}{1+x}\left[P_0(x)-P_2(x)+P_1(x)-P_3(x)+P_2(x)-P_4(x)+P_3(x)-P_5(x)+\cdots\right]\\
&=\frac{P_0(x)+P_1(x)}{1+x}=1
\end{aligned} \tag{10}$$

(b) This is straightforward: with $x=-1$ inserted to the sum of (3), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}P_n(-1)&=\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}(-1)^n\\&=\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}+\frac{1}{n+1}\right)\\&=-\left(1+\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{1}{3}\right)-\left(\frac{1}{3}+\frac{1}{4}\right)+\cdots=-1\end{aligned}\tag{11}$$

which implies (3) holds for $x=-1$.

2. This is a straightforward application of the expansion (3.38).



$$\frac{1}{2\sin\dfrac{\theta}{2}}=\frac{1}{b}=\sum_{n=0}^{\infty}\frac{1^n}{1^{n+1}}P_n(\cos\theta)\qquad\qquad\frac{1}{2\cos\dfrac{\theta}{2}}=\frac{1}{c}=\sum_{n=0}^{\infty}\frac{1^n}{1^{n+1}}P_n(-\cos\theta)\tag{12}$$

So

$$\frac{1}{2}\left(\frac{1}{\sin\dfrac{\theta}{2}}+\frac{1}{\cos\dfrac{\theta}{2}}\right)=2\sum_{n=0}^{\infty}P_{2n}(\cos\theta)\tag{13}$$

The desired alternative surface charge expression can be obtained by trivially inserting (13) into (3.137).