1. **Prob 5.35**

   (a) Recall in problem 5.13, we have calculated the magnetic field due to a uniformly charged sphere that rotates around its axis.

$$\mathbf{A}(\mathbf{x}) = \frac{\mu\eta\omega a^3}{3}\sin\theta\frac{r_<}{r_>^2}\hat{\boldsymbol{\phi}} \tag{1}$$

   where η is the surface charge density and ω is the angular velocity.
   We have also derived that inside the sphere, the magnetic field is uniform

$$\mathbf{B}_0 = \frac{2\mu\eta\omega a}{3}\hat{\mathbf{z}} \tag{2}$$

   Thus the vector potential can be expressed in B_0:

$$\mathbf{A}(\mathbf{x}) = \frac{B_0 a^2}{2}\sin\theta\frac{r_<}{r_>^2}\hat{\boldsymbol{\phi}} \tag{3}$$

   With this configuration, the charge density is

$$\rho(\mathbf{x}) = \delta(r-a)\eta \tag{4}$$

   hence the current density is

$$\mathbf{J}(\mathbf{x}) = \rho\mathbf{v}(\mathbf{x}) = \delta(r-a)\eta\omega a\sin\theta\hat{\boldsymbol{\phi}} = \frac{3B_0}{2\mu}\delta(r-a)\sin\theta\hat{\boldsymbol{\phi}} \tag{5}$$

   (b) Consider a general diffusion equation

$$\nabla^2 f(\mathbf{x},t) = D\frac{\partial f(\mathbf{x},t)}{\partial t} \qquad\qquad D>0 \tag{6}$$

   We can attempt to write the solution in separate variable form,

$$f(\mathbf{x},t) = T(t)R(r)\Theta(\theta)\Phi(\phi) \tag{7}$$

   (6) now becomes

$$\begin{aligned}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} = D\frac{\partial f}{\partial t} \qquad \Longrightarrow \\ \frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = \frac{D}{T}\frac{dT}{dt}\end{aligned} \tag{8}$$

   The two sides are functions of two disjoint sets of independent variables, hence they must both equal to a constant, say $-\lambda^2$. It must be a negative number for $T(t)$ to diminish at $t\to\infty$. This gives

$$T(t) \propto e^{-\lambda^2 t/D} \tag{9}$$

   The remaining equation becomes

$$\sin^2\theta\left[\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \lambda^2 r^2\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \tag{10}$$

   Now again, both sides have to be equal to a constant, denoted m^2. Here m has to be integer since Φ is a single-valued function of space. This means $\Phi(\phi)$ must be a linear combination of $\cos m\phi$ and $\sin m\phi$.
   Working on the LHS of (10) further,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \lambda^2 r^2 = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{m^2}{\sin^2\theta} \tag{11}$$

Once again, both sides of (12) must equal to a constant, denoted $l(l+1)$. This gives rise to

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0 \tag{12}$$

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left[\lambda^2-\frac{l(l+1)}{r^2}\right]R=0 \tag{13}$$

The two linearly independent solutions of (12) are $P_l^m(\cos\theta)$ and $Q_l^m(\cos\theta)$ - associated Legendre functions of the first and second kind. The two linearly independent solutions of (13) are $j_l(\lambda r)$ and $y_l(\lambda r)$ - spherical Bessel functions of the first and second kind.

For the physical problem at hand, we have to discard non-integer l values since they cause $\theta=0$ to diverge. Similarly y_l and Q_l^m will be discarded because of their divergence at $r=0$ and $\cos\theta=1$.

Overall, we obtain the general form of solution

$$f(\mathbf{x},t)=\int_0^\infty d\lambda e^{-\lambda^2 t/D}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}j_l(\lambda r)P_l^m(\cos\theta)\left[a_{lm}(\lambda)\cos m\phi+b_{lm}(\lambda)\sin m\phi\right] \tag{14}$$

Coming back to the diffusion of vector potential (5.160)

$$\nabla^2\mathbf{A}(\mathbf{x},t)=\mu\sigma\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t} \tag{15}$$

which, in Cartesian coordinates, is equivalent to 3 scalar diffusion equations

$$\nabla^2 A_i(\mathbf{x},t)=\mu\sigma\frac{\partial A_i(\mathbf{x},t)}{\partial t}\qquad\qquad i=x,y,z \tag{16}$$

The initial condition is given by (3), i.e., at $t=0$,

$$A_x=\frac{B_0a^2}{2}\frac{r_<}{r_>^2}\sin\theta(-\sin\phi) \tag{17}$$

$$A_y=\frac{B_0a^2}{2}\frac{r_<}{r_>^2}\sin\theta(\cos\phi) \tag{18}$$

$$A_z=0 \tag{19}$$

Matching (17) against (14) at $t=0$ and using the appropriate orthogonality relations, we can conclude that only $b_{1,\pm1}(\lambda)$ can be non-zero. Similarly for (18), only $a_{1,\pm1}(\lambda)$ can be non-zero. In either case, we have the integral representation of the radial function

$$A(r)\equiv\frac{B_0a^2}{2}\frac{r_<}{r_>^2}=\int_0^\infty d\lambda j_1(\lambda r)\widetilde{A}(\lambda) \tag{20}$$

By (3.113),

$$\begin{aligned}\widetilde{A}(\lambda)&=\frac{2\lambda^2}{\pi}\int_0^\infty r^2A(r)j_1(\lambda r)dr\\&=\frac{B_0a^2\lambda^2}{\pi}\int_0^\infty r^2\left(\frac{r_<}{r_>^2}\right)j_1(\lambda r)dr\\&=\frac{B_0a^2\lambda^2}{\pi}\left[\underbrace{\int_0^a\frac{r^3}{a^2}j_1(\lambda r)dr}_{I_1}+\underbrace{\int_a^\infty aj_1(\lambda r)dr}_{I_2}\right]\end{aligned} \tag{21}$$

Recall the recurrence relation of spherical Bessel functions

$$j_l'(x)=j_{l-1}(x)-\frac{l+1}{x}j_l(x) \tag{22}$$

This yields a closed-form result for the integral

$$\int_0^{x_0}x^{l+1}j_{l-1}(x)dx=\int_0^{x_0}x^{l+1}\left[j_l'(x)+\frac{l+1}{x}j_l(x)\right]dx=x_0^{l+1}j_l(x_0) \tag{23}$$

Thus

$$I_1 = \frac{1}{\lambda^4 a^2}\int_0^{\lambda a} (\lambda r)^3 j_1(\lambda r)\, d(\lambda r) = \frac{1}{\lambda^4 a^2}(\lambda a)^3 j_2(\lambda a) = \frac{a}{\lambda} j_2(\lambda a) \tag{24}$$

$$I_2 = \frac{a}{\lambda}\int_{\lambda a}^{\infty} \left[-j_0'(\lambda r)\right] d(\lambda r) = \frac{a}{\lambda} j_0(\lambda a) \tag{25}$$

Together with the recurrence relation

$$j_{l-1}(x) + j_{l+1}(x) = \frac{2l+1}{x} j_l(x) \tag{26}$$

we can establish

$$\widetilde{A}(\lambda) = \frac{B_0 a^2 \lambda^2}{\pi}\cdot\frac{a}{\lambda}\left[j_0(\lambda a) + j_2(\lambda a)\right] = \frac{B_0 a^2\lambda^2}{\pi}\frac{a}{\lambda}\left[\frac{3}{\lambda a} j_1(\lambda a)\right] = \frac{3B_0 a^2}{\pi} j_1(\lambda a) \tag{27}$$

Finally putting the Cartesian components back into the vector potential form of (14) gives

$$\begin{aligned}
\mathbf{A}(\mathbf{x},t) &= (-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}})\sin\theta \int_0^{\infty} d\lambda\, e^{-\lambda^2 t/\mu\sigma}\widetilde{A}(\lambda)\, j_1(\lambda r) \\
&= \hat{\boldsymbol{\phi}}\frac{3B_0 a^2}{\pi}\sin\theta\int_0^{\infty} d\lambda\, e^{-\lambda^2 t/\mu\sigma} j_1(\lambda a)\, j_1(\lambda r) \qquad \text{define } k \equiv \lambda a,\ \nu \equiv \frac{1}{\mu\sigma a^2} \\
&= \hat{\boldsymbol{\phi}}\sin\theta\cdot\underbrace{\frac{3B_0 a}{\pi}\int_0^{\infty} dk\, e^{-\nu k^2 t} j_1(k)\, j_1\left(\frac{kr}{a}\right)}_{A_\phi(r,t)}
\end{aligned} \tag{28}$$

Consider the magnetic induction

$$\begin{aligned}
\mathbf{B} = \nabla\times\mathbf{A} &= \frac{1}{r\sin\theta}\frac{\partial\left[A_\phi(r,t)\sin^2\theta\right]}{\partial\theta}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left[rA_\phi(r,t)\sin\theta\right]}{\partial r}\hat{\boldsymbol{\theta}} \\
&= \frac{2A_\phi(r,t)}{r}\cos\theta\,\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left[rA_\phi(r,t)\right]}{\partial r}\sin\theta\,\hat{\boldsymbol{\theta}} \\
&= \frac{2A_\phi}{r}\cos\theta\left(\cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\boldsymbol{\rho}}\right) - \frac{1}{r}\frac{\partial\left(rA_\phi\right)}{\partial r}\sin\theta\left(-\sin\theta\,\hat{\mathbf{z}} + \cos\theta\,\hat{\boldsymbol{\rho}}\right) \\
&= \left[\frac{2A_\phi}{r}\cos^2\theta + \frac{1}{r}\frac{\partial\left(rA_\phi\right)}{\partial r}\sin^2\theta\right]\hat{\mathbf{z}} + \left[\frac{2A_\phi}{r} - \frac{1}{r}\frac{\partial\left(rA_\phi\right)}{\partial r}\right]\sin\theta\cos\theta\,\hat{\boldsymbol{\rho}}
\end{aligned} \tag{29}$$

Since as $r \to 0$:

$$\frac{2j_1\left(\frac{kr}{a}\right)}{r} = \frac{2}{r}\cdot\sqrt{\frac{\pi a}{2kr}} J_{3/2}\left(\frac{kr}{a}\right) = \sqrt{\frac{2\pi a}{k}}\, r^{-3/2}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{kr}{2a}\right)^{2n+3/2}}{n!\,\Gamma\left(n+\frac{5}{2}\right)} \quad\longrightarrow\quad \frac{\sqrt{\pi}}{\Gamma(5/2)}\left(\frac{k}{2a}\right) \tag{30}$$

$$\begin{aligned}
\frac{1}{r}\frac{d\left[r j_1\left(\frac{kr}{a}\right)\right]}{dr} &= \frac{1}{r}\frac{d}{dr}\left[r\sqrt{\frac{\pi a}{2kr}} J_{3/2}\left(\frac{kr}{a}\right)\right] \\
&= \sqrt{\frac{\pi a}{2k}}\frac{1}{r}\frac{d}{dr}\left[\sum_{n=0}^{\infty} r^{1/2}\frac{(-1)^n\left(\frac{kr}{2a}\right)^{2n+3/2}}{n!\,\Gamma\left(n+\frac{5}{2}\right)}\right] \quad\longrightarrow\quad \frac{\sqrt{\pi}}{\Gamma(5/2)}\left(\frac{k}{2a}\right)
\end{aligned} \tag{31}$$

This gives the induction at origin

$$\begin{aligned}
\mathbf{B}(\mathbf{0},t) &= \hat{\mathbf{z}}\frac{3B_0 a}{\pi}\frac{\sqrt{\pi}}{\Gamma(5/2)}\int_0^{\infty} dk\, e^{-\nu k^2 t} j_1(k)\left(\frac{k}{2a}\right) \\
&= \hat{\mathbf{z}}\frac{3B_0 a}{\pi}\frac{\sqrt{\pi}}{3\sqrt{\pi}/4}\frac{1}{2a}\int_0^{\infty} dk\, e^{-\nu k^2 t} j_1(k)\, k \\
&= \hat{\mathbf{z}}\frac{2B_0}{\pi}\int_0^{\infty} dk\, e^{-\nu k^2 t}\left(\frac{\sin k}{k} - \cos k\right)
\end{aligned} \tag{32}$$

For any y, the integral below has a closed form:

$$g(y) \equiv \int_0^\infty dk e^{-\nu k^2 t} e^{iky} = \int_0^\infty dk e^{-y^2/4\nu t} \exp\left\{-\nu t \left[k^2 - \frac{iky}{\nu t} + \left(\frac{iy}{2\nu t}\right)^2\right]\right\}$$
$$= e^{-y^2/4\nu t} \int_0^\infty dk \exp\left[-\nu t \left(k - \frac{iy}{2\nu t}\right)^2\right] = e^{-y^2/4\nu t} \frac{1}{2}\sqrt{\frac{\pi}{\nu t}} \tag{33}$$

where we have used the Gaussian integral

$$\int_{-\infty}^\infty e^{-p(x+c)^2} dx = \sqrt{\frac{\pi}{p}} \qquad \text{for } p, c \in \mathbb{C}, \operatorname{Re} p > 0 \tag{34}$$

The two integrals in (32) can be evaluated using (33):

$$\int_0^\infty dk e^{-\nu k^2 t} \frac{\sin k}{k} = \int_0^\infty dk e^{-\nu k^2 t} \int_0^1 \cos ky\, dy = \int_0^1 dy \operatorname{Re}[g(y)]$$
$$= \frac{1}{2}\sqrt{\frac{\pi}{\nu t}} \int_0^1 dy e^{-y^2/4\nu t} \qquad u \equiv \frac{y}{2\sqrt{\nu t}}$$
$$= \frac{1}{2}\sqrt{\frac{\pi}{\nu t}} 2\sqrt{\nu t} \int_0^{\frac{1}{2\sqrt{\nu t}}} e^{-u^2} du$$
$$= \frac{1}{2}\sqrt{\frac{\pi}{\nu t}} 2\sqrt{\nu t} \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{1}{2\sqrt{\nu t}}\right) = \frac{\pi}{2} \operatorname{erf}\left(\frac{1}{2\sqrt{\nu t}}\right) \tag{35}$$

$$\int_0^\infty dk e^{-\nu k^2 t} \cos k = \operatorname{Re}[g(1)] = \frac{e^{-1/4\nu t}}{2}\sqrt{\frac{\pi}{\nu t}} \tag{36}$$

Back to (32):

$$\mathbf{B}(\mathbf{0}, t) = \hat{\mathbf{z}} B_0 \left[\operatorname{erf}\left(\frac{1}{2\sqrt{\nu t}}\right) - \frac{1}{\sqrt{\pi \nu t}} e^{-1/4\nu t}\right] \tag{37}$$

(c) The text has stated that the current density $\mathbf{J}$ is subject to the same diffusion equation (the proof is easy). With the similar arguments that lead to (20), we can establish the integral representation for radial component of $\mathbf{J}$:

$$J(r) = \delta(r-a) \frac{3B_0}{2\mu} = \int_0^\infty d\lambda j_1(\lambda r) \tilde{J}(\lambda) \tag{38}$$

and

$$\tilde{J}(\lambda) = \frac{2\lambda^2}{\pi} \int_0^\infty r^2 J(r) j_1(\lambda r) dr = \frac{2\lambda^2 a^2}{\pi} \frac{3B_0}{2\mu} j_1(\lambda a) \tag{39}$$

so the diffused current dentsity is

$$\mathbf{J}(\mathbf{x}, t) = \hat{\boldsymbol{\phi}} \sin\theta \int_0^\infty d\lambda e^{-\lambda^2 t/\mu\sigma} \tilde{J}(\lambda) j_1(\lambda r)$$
$$= \hat{\boldsymbol{\phi}} \sin\theta \int_0^\infty d\lambda e^{-\lambda^2 t/\mu\sigma} \frac{3B_0 \lambda^2 a^2}{\pi\mu} j_1(\lambda a) j_1(\lambda r)$$
$$= \hat{\boldsymbol{\phi}} \sin\theta \frac{3B_0}{\pi\mu a} \int_0^\infty dk e^{-\nu k^2 t} k^2 j_1(k) j_1\left(\frac{kr}{a}\right) \tag{40}$$

With this, we can calculate the total magnetic energy at $t > 0$:

$$W_m(t) = \frac{1}{2} \int \mathbf{J}(\mathbf{x}, t) \cdot \mathbf{A}(\mathbf{x}, t) d^3x$$
$$= \frac{1}{2} \int_0^\infty r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \left(\frac{3B_0 a}{\pi}\right)\left(\frac{3B_0}{\pi\mu a}\right) \sin^2\theta \times$$
$$\left[\int_0^\infty dk_1 e^{-\nu k_1^2 t} j_1(k_1) j_1\left(\frac{k_1 r}{a}\right)\right]\left[\int_0^\infty dk_2 e^{-\nu k_2^2 t} k_2^2 j_1(k_2) j_1\left(\frac{k_2 r}{a}\right)\right] \tag{41}$$

With the orthogonality of spherical Bessel functions

$$\int_0^\infty x^2 j_\alpha(ux)\, j_\alpha(vx)\, dx = \frac{\pi}{2u^2}\delta(u-v) \tag{42}$$

the dr integral is reduced to

$$\int_0^\infty r^2 j_1\left(\frac{k_1 r}{a}\right) j_1\left(\frac{k_2 r}{a}\right) dr = \frac{\pi a^2}{2k_1^2}\delta\left(\frac{k_1-k_2}{a}\right) = \frac{\pi a^3}{2k_1^2}\delta(k_1-k_2) \tag{43}$$

with which we can continue the rest of the integrals in (41):

$$W_m(t) = \frac{1}{2}\cdot 2\pi\left(\frac{\pi a^3}{2}\right)\left(\frac{9B_0^2}{\pi^2\mu}\right)\overbrace{\int_0^\pi \sin^3\theta\, d\theta}^{4/3}\int_0^\infty e^{-2\nu k^2 t} j_1^2(k)\, dk = \frac{6B_0^2 a^3}{\mu}\int_0^\infty e^{-2\nu k^2 t} j_1^2(k)\, dk \tag{44}$$

To see the decay of energy when $\nu t \to \infty$, we refer to the integral (6.633) from *I.S. Gradshteyn and I.M. Ryzhik: Table of Integrals, Series and Products*

$$\begin{aligned}\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_\mu(bx) J_\xi(cx)\, dx = &\frac{b^\mu c^\xi \alpha^{-(\mu+\xi+\lambda+2)/2}}{2^{\xi+\mu+1}\Gamma(\xi+1)}\times \\ &\sum_{m=0}^\infty \frac{\Gamma\left(m+\frac{\xi}{2}+\frac{\mu}{2}+\frac{\lambda}{2}+1\right)}{m!\,\Gamma(m+\mu+1)}\left(-\frac{b^2}{4\alpha}\right)^m F\left(-m, -\mu-m; \xi+1; \frac{c^2}{b^2}\right) \\ &\text{for } \operatorname{Re}\alpha > 0, \operatorname{Re}(\mu+\xi+\lambda) > -2, b > 0, c > 0\end{aligned} \tag{45}$$

Setting $\alpha = 2\nu t$, $b = c = 1$, $\lambda = -2$, $\mu = \xi = 3/2$, we can see when $\nu t \to \infty$, the integral in (44) is dominated by the $m = 0$ order, which yields

$$W_m(t) \approx \frac{6B_0^2 a^3}{\mu}\frac{\pi}{2}\cdot\frac{(2\nu t)^{-3/2}}{2^4\Gamma(5/2)}\cdot\frac{\Gamma(3/2)}{\Gamma(5/2)} = \frac{6B_0^2 a^3}{\mu}\frac{\pi}{2}\frac{1}{16\cdot 2\sqrt{2(\nu t)}^3\cdot 3\sqrt{\pi}/4}\frac{\sqrt{\pi}/2}{3\sqrt{\pi}/4} = \frac{\sqrt{2\pi}B_0^2 a^3}{24\mu\sqrt{\nu t}^3} \tag{46}$$

(d) For the vector potential (28), we can also make use of (45), for which we set $\alpha = \nu t$, $b = r/a$, $c = 1$, $\mu = \xi = 3/2$, $\lambda = -2$. To the leading order $m = 0$,

$$A_\phi(r,t) \approx \frac{3B_0 a}{\pi}\frac{\pi}{2}\sqrt{\frac{a}{r}}\frac{\left(\frac{r}{a}\right)^{3/2}(\nu t)^{-3/2}}{2^4\Gamma(5/2)}\frac{\Gamma(3/2)}{\Gamma(5/2)} = \frac{3B_0 a}{2}\frac{r}{a}\frac{1}{\sqrt{\nu t}^3 16\cdot 3\sqrt{\pi}/4}\frac{\sqrt{\pi}/2}{3\sqrt{\pi}/4} = \frac{B_0 r}{12\sqrt{\pi}\sqrt{\nu t}^3} \tag{47}$$

With this approximation plugged into (29), we see this gives a constant field

$$\mathbf{B} \approx \frac{B_0}{6\sqrt{\pi}\sqrt{\nu t}^3}\hat{\mathbf{z}} \tag{48}$$

But from the series form of (45), we know that the approximation (47) is good only when $b^2/4\alpha = r^2/4a^2\nu t$ is small, or when $r \ll a\sqrt{\nu t}$. The distance $R = a\sqrt{\nu t}$ represents how far the diffusion has reached. So when $r \ll R$, it's well within the diffused area, which has now gone into the steady state, i.e., constant fields. For $r \gg R$, the diffusion has not reached that far yet, so there the field still looks like a dipole (I didn't find a quantitative way of showing the $r \gg R$ case corresponds to the original dipole field).

2. **Prob 5.36**

   (a) In fact, we have already solved the time-dependent electric field by subjecting the diffused current density expression (40) to Ohm's law. Note (40) is a result of the relations listed in (5.159), from which we derived the diffusion equation for all of $\mathbf{B}, \mathbf{J}, \mathbf{E}, \mathbf{A}$. We can arrive at the solution for $\mathbf{E}$ alternatively by applying

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \tag{49}$$

   on (28), which yields

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \hat{\boldsymbol{\phi}} \sin\theta \cdot \frac{3B_0 a}{\pi} \int_0^\infty dk e^{-\nu k^2 t} \left(\nu k^2\right) j_1(k) j_1\left(\frac{kr}{a}\right) \tag{50}$$

   which is exactly $1/\sigma$ of (40). Certainly, when $t = 0$, applying the orthogonality of spherical Bessel functions to (50) gives $1/\sigma$ of the initial surface current distribution.

   (b) The total power dissipated in the resistive medium is

$$\begin{aligned}
P(t) =& \int \mathbf{J} \cdot \mathbf{E} d^3x = \int \frac{\mathbf{J}^2}{\sigma} d^3x \\
=& \frac{1}{\sigma} \int_0^\infty r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \left(\sin\theta \frac{3B_0}{\pi\mu a}\right)^2 \left[\int_0^\infty dk e^{-\nu k^2 t} k^2 j_1(k) j_1\left(\frac{kr}{a}\right)\right]^2 \qquad \text{use (42)} \\
=& \frac{1}{\sigma} \cdot 2\pi \left(\frac{\pi a^3}{2}\right) \left(\frac{3B_0}{\pi\mu a}\right)^2 \cdot \frac{4}{3} \int_0^\infty e^{-2\nu k^2 t} k^2 j_1^2(k) dk \\
=& \frac{12 B_0 a^3 \nu}{\mu} \int_0^\infty e^{-2\nu k^2 t} k^2 j_1^2(k) dk \\
=& -\frac{\partial W_m(t)}{\partial t}
\end{aligned} \tag{51}$$