

1. Let  $y_{vn}$  and  $y_{vm}$  be the  $n$ -th and  $m$ -th root of the eigenequation

$$xJ'_\nu(x) + \lambda J_\nu(x) = 0 \quad (1)$$

We start with the differential equation that  $J_\nu$  satisfies (see Jackson eq (3.77)), i.e.,

$$\begin{aligned} \frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) R &= 0 \\ \frac{1}{x} \frac{d}{dx} \left( x \frac{dR}{dx} \right) + \left(1 - \frac{\nu^2}{x^2}\right) R &= 0 \end{aligned} \quad \Rightarrow \quad (2)$$

Making the variable change  $x = y_{vn}\rho/a$  yields

$$\begin{aligned} \frac{1}{\frac{y_{vn}\rho}{a}} \frac{d}{d\rho} \left[ \frac{y_{vn}\rho}{a} \frac{dJ_\nu\left(\frac{y_{vn}\rho}{a}\right)}{\frac{y_{vn}\rho}{a} d\rho} \right] + \left[ 1 - \frac{\nu^2}{\left(\frac{y_{vn}\rho}{a}\right)^2 \rho^2} \right] J_\nu\left(\frac{y_{vn}\rho}{a}\right) &= 0 \\ \frac{1}{\rho} \frac{d}{d\rho} \left[ \rho \frac{dJ_\nu\left(\frac{y_{vn}\rho}{a}\right)}{d\rho} \right] + \left( \frac{y_{vn}^2}{a^2} - \frac{\nu^2}{\rho^2} \right) J_\nu\left(\frac{y_{vn}\rho}{a}\right) &= 0 \end{aligned} \quad \Rightarrow \quad (3)$$

Multiply both sides of (3) by  $J_\nu(y_{vm}\rho/a)$  and integrate with measure  $\rho d\rho$ , we have

$$\underbrace{\int_0^a \frac{1}{\rho} J_\nu\left(\frac{y_{vm}\rho}{a}\right) \frac{d}{d\rho} \left[ \rho \frac{dJ_\nu\left(\frac{y_{vn}\rho}{a}\right)}{d\rho} \right] \rho d\rho}_{I} + \int_0^a \left( \frac{y_{vn}^2}{a^2} - \frac{\nu^2}{\rho^2} \right) J_\nu\left(\frac{y_{vn}\rho}{a}\right) J_\nu\left(\frac{y_{vm}\rho}{a}\right) \rho d\rho = 0 \quad (4)$$

where

$$I = \underbrace{J_\nu\left(\frac{y_{vm}\rho}{a}\right) \rho \frac{dJ_\nu\left(\frac{y_{vn}\rho}{a}\right)}{d\rho}}_{g(y_{vm}, y_{vn}; \rho)} \bigg|_0^a - \int_0^a \rho \frac{dJ_\nu\left(\frac{y_{vn}\rho}{a}\right)}{d\rho} \frac{dJ_\nu\left(\frac{y_{vm}\rho}{a}\right)}{d\rho} d\rho \quad (5)$$

By the boundary condition at  $\rho = 0$

$$g(y_{vm}, y_{vn}; 0) = 0 \quad (6)$$

(6) enables us to rewrite (4) as

$$g(y_{vm}, y_{vn}; a) - \int_0^a \rho \frac{dJ_\nu\left(\frac{y_{vn}\rho}{a}\right)}{d\rho} \frac{dJ_\nu\left(\frac{y_{vm}\rho}{a}\right)}{d\rho} d\rho + \int_0^a \left( \frac{y_{vn}^2}{a^2} - \frac{\nu^2}{\rho^2} \right) J_\nu\left(\frac{y_{vn}\rho}{a}\right) J_\nu\left(\frac{y_{vm}\rho}{a}\right) \rho d\rho = 0 \quad (7)$$

Apply the exchange  $y_{vm} \leftrightarrow y_{vn}$  to (7) and subtract the resulting equation from (7), we get

$$g(y_{vm}, y_{vn}; a) - g(y_{vn}, y_{vm}; a) + \left( \frac{y_{vn}^2}{a^2} - \frac{y_{vm}^2}{a^2} \right) \int_0^a J_\nu\left(\frac{y_{vn}\rho}{a}\right) J_\nu\left(\frac{y_{vm}\rho}{a}\right) \rho d\rho = 0 \quad (8)$$

By the other boundary condition at  $\rho = a$ :

$$\begin{aligned} \frac{d}{d\rho} \ln \left[ J_\nu\left(\frac{y_{vn}\rho}{a}\right) \right] \bigg|_{\rho=a} &= \frac{1}{J_\nu\left(\frac{y_{vn}\rho}{a}\right)} \frac{dJ_\nu\left(\frac{y_{vn}\rho}{a}\right)}{d\rho} \bigg|_{\rho=a} = -\frac{\lambda}{a} \quad \Rightarrow \\ & a \frac{dJ_\nu\left(\frac{y_{vn}\rho}{a}\right)}{d\rho} \bigg|_{\rho=a} = -\lambda J_\nu\left(\frac{y_{vn}\rho}{a}\right) \bigg|_{\rho=a} \quad \Rightarrow \\ g(y_{vm}, y_{vn}; a) &= -\lambda J_\nu(y_{vm}) J_\nu(y_{vn}) \end{aligned} \quad (9)$$

The symmetry between  $y_{vm}$  and  $y_{vn}$  in (9) turns (8) into

$$(y_{vn}^2 - y_{vm}^2) \int_0^a J_\nu\left(\frac{y_{vn}\rho}{a}\right) J_\nu\left(\frac{y_{vm}\rho}{a}\right) \rho d\rho = 0 \quad (10)$$

which shows that for different eigenvalues  $y_{vn} \neq y_{vm}$ ,  $J_\nu(y_{vn}\rho/a)$  and  $J_\nu(y_{vm}\rho/a)$  are orthogonal with respect to the *weighted inner product*

$$\langle f, g \rangle = \int_0^a f(x)g(x)x dx \quad (11)$$

2. First, let's take a closer look at function  $g$  at  $\rho = a$ .

$$g(y_{vm}, y_{vn}; a) = J_\nu(y_{vm}) a \left[ \frac{dJ_\nu\left(\frac{y_{vn}\rho}{a}\right)}{d\rho} \right] \bigg|_{\rho=a} = J_\nu(y_{vm}) y_{vn} J'_\nu(y_{vn}) \quad (12)$$

With this, if in (8) we replace  $y_{vm}$  with a general  $y$  which is not necessarily a root of the eigenequation (1), we have

$$\begin{aligned} \int_0^a J_\nu\left(\frac{y\rho}{a}\right) J_\nu\left(\frac{y_{vn}\rho}{a}\right) \rho d\rho &= \frac{a^2}{y^2 - y_{vn}^2} [g(y, y_{vn}; a) - g(y_{vn}, y; a)] && \text{by (12)} \\ &= \frac{a^2}{y^2 - y_{vn}^2} [J_\nu(y) y_{vn} J'_\nu(y_{vn}) - J_\nu(y_{vn}) y J'_\nu(y)] && y_{vn} \text{ is a root of (1)} \\ &= \frac{a^2}{y^2 - y_{vn}^2} [-J_\nu(y) \lambda J_\nu(y_{vn}) - J_\nu(y_{vn}) y J'_\nu(y)] \\ &= \frac{a^2}{y + y_{vn}} [-J_\nu(y_{vn})] \left[ \frac{\lambda J_\nu(y) + y J'_\nu(y)}{y - y_{vn}} \right] \end{aligned} \quad (13)$$

Now the desired normalization factor

$$N \equiv \int_0^a J_\nu\left(\frac{y_{vn}\rho}{a}\right) J_\nu\left(\frac{y_{vn}\rho}{a}\right) \rho d\rho \quad (14)$$

is just (13) under the limit  $y \rightarrow y_{vn}$ , i.e.,

$$\begin{aligned} N &= \lim_{y \rightarrow y_{vn}} \frac{a^2}{y + y_{vn}} [-J_\nu(y_{vn})] \left[ \frac{\lambda J_\nu(y) + y J'_\nu(y)}{y - y_{vn}} \right] \\ &= -\frac{a^2}{2y_{vn}} J_\nu(y_{vn}) \frac{d[\lambda J_\nu(y) + y J'_\nu(y)]}{dy} \bigg|_{y=y_{vn}} \\ &= -\frac{a^2}{2} J_\nu(y_{vn}) \frac{\lambda J'_\nu(y_{vn}) + y_{vn} J''_\nu(y_{vn}) + J'_\nu(y_{vn})}{y_{vn}} \end{aligned} \quad (15)$$

From (1) and (2) we have

$$\lambda J'_\nu(y_{vn}) = -\frac{\lambda^2 J_\nu(y_{vn})}{y_{vn}} \quad (16)$$

$$y_{vn} J''_\nu(y_{vn}) + J'_\nu(y_{vn}) = -y_{vn} \left( 1 - \frac{\nu^2}{y_{vn}^2} \right) J_\nu(y_{vn}) \quad (17)$$

This turns (15) into

$$N = \frac{a^2}{2} \left( 1 + \frac{\lambda^2 - \nu^2}{y_{vn}^2} \right) J_\nu^2(y_{vn}) \quad (18)$$

This is one of the equivalent forms at the end of the problem.

With the normalization factor  $N$  decided (subsequently subscripted with  $vn$  to emphasize the dependency on  $\nu$  and  $n$ ), we can properly claim that the functions

$$U_{vn}(\rho) = \frac{1}{\sqrt{N_{vn}}} J_\nu\left(\frac{y_{vn}\rho}{a}\right) \quad (19)$$

form an orthonormal set of basis over the range  $[0, a]$  with respect to the weighted inner product (11).

Now if we further assume that this set is complete, for any function  $f(\rho)$  satisfying the boundary condition, we can expand it into

$$f(\rho) = \sum_{n=1}^{\infty} B_n U_{\nu n}(\rho) = \sum_{n=1}^{\infty} B_n \frac{1}{\sqrt{N_{\nu n}}} J_{\nu} \left( \frac{y_{\nu n} \rho}{a} \right) \quad \text{where} \quad (20)$$

$$B_n = \int_0^a f(\rho') U_{\nu n}(\rho') \rho' d\rho' = \int_0^a f(\rho') \frac{1}{\sqrt{N_{\nu n}}} J_{\nu} \left( \frac{y_{\nu n} \rho'}{a} \right) \rho' d\rho' \quad (21)$$

Thus if we define  $A_n = B_n / \sqrt{N_{\nu n}}$ , we have

$$f(\rho) = \sum_{n=1}^{\infty} A_n J_{\nu} \left( \frac{y_{\nu n} \rho}{a} \right) \quad \text{where} \quad (22)$$

$$\begin{aligned} A_n &= \frac{1}{N_{\nu n}} \int_0^a f(\rho') J_{\nu} \left( \frac{y_{\nu n} \rho'}{a} \right) \rho' d\rho' \\ &= \frac{2}{a^2} \left[ \left( 1 + \frac{\lambda^2 - \nu^2}{y_{\nu n}^2} \right) J_{\nu}^2(y_{\nu n}) \right]^{-1} \int_0^a f(\rho') J_{\nu} \left( \frac{y_{\nu n} \rho'}{a} \right) \rho' d\rho' \end{aligned} \quad (23)$$

### 3. Proof of the alternative forms of normalization constant.

(a)

$$\left[ \left( 1 - \frac{\nu^2}{y_{\nu n}^2} \right) J_{\nu}^2(y_{\nu n}) + J_{\nu}'^2(y_{\nu n}) \right] = \left( 1 + \frac{\lambda^2 - \nu^2}{y_{\nu n}^2} \right) J_{\nu}^2(y_{\nu n}) \quad (24)$$

This follows trivially from (1).

(b)

$$\left( 1 + \frac{\lambda^2 - \nu^2}{y_{\nu n}^2} \right) J_{\nu}^2(y_{\nu n}) = \left( 1 + \frac{y_{\nu n}^2 - \nu^2}{\lambda^2} \right) J_{\nu}'^2(y_{\nu n}) \quad (25)$$

This also follows trivially from (1).

(c)

$$\left[ \left( 1 - \frac{\nu^2}{y_{\nu n}^2} \right) J_{\nu}^2(y_{\nu n}) + J_{\nu}'^2(y_{\nu n}) \right] = J_{\nu}^2(y_{\nu n}) - J_{\nu-1}(y_{\nu n}) J_{\nu+1}(y_{\nu n}) \quad (26)$$

This can be proved by using the recurrence relation (reference [Wolfram](#))

$$\frac{\nu}{x} J_{\nu}(x) = \frac{J_{\nu+1}(x) + J_{\nu-1}(x)}{2} \quad (27)$$

$$J_{\nu}'(x) = -\frac{J_{\nu+1}(x) - J_{\nu-1}(x)}{2} \quad (28)$$