1. (a) **Linearity of transformations in homogeneous spacetime**
Given two inertial frames K and K', let $\mathscr{T}, \mathscr{X}, \mathscr{Y}, \mathscr{Z}$ be the transforms from the unprimed coordinates (t, x, y, z) to the primed coordinates (t', x', y', z'), i.e.,

$$t' = \mathscr{T}(t, x, y, z) \tag{1}$$
$$x' = \mathscr{X}(t, x, y, z) \tag{2}$$
$$y' = \mathscr{Y}(t, x, y, z) \tag{3}$$
$$z' = \mathscr{Z}(t, x, y, z) \tag{4}$$

When we say spacetime is homogeneous, what we mean is that there is no special location or instant in spacetime (a.k.a., translational invariance). Let's consider two events measured in K with the same location but Δt apart in time, $(t_0, \mathbf{x}_0)$ and $(t_0 + \Delta t, \mathbf{x}_0)$. The difference of their time measurements in K' should not depend on the K-frame starting time t_0, i.e.,

$$\mathscr{T}(t_0 + \Delta t, \mathbf{x}_0) - \mathscr{T}(t_0, \mathbf{x}_0) = \mathscr{T}(\Delta t, \mathbf{x}_0) - \mathscr{T}(0, \mathbf{x}_0) \tag{5}$$

where on the RHS, we set the starting point $t_0 = 0$ due to translational invariance.
If $\mathbf{x}_0$ is fixed, (5) can be written more explicitly as a functional equation for the single-argument function $\mathscr{T}_{\mathbf{x}_0}(t)$:

$$\mathscr{T}_{\mathbf{x}_0}(t_0 + \Delta t) - \mathscr{T}_{\mathbf{x}_0}(t_0) = \mathscr{T}_{\mathbf{x}_0}(\Delta t) - \mathscr{T}_{\mathbf{x}_0}(0) \qquad \forall t_0, \Delta t \in \mathbb{R} \tag{6}$$

If we define $f_{\mathbf{x}_0}(t) \equiv \mathscr{T}_{\mathbf{x}_0}(t) - \mathscr{T}_{\mathbf{x}_0}(0)$, then (6) is equivalent to the Cauchy's functional equation

$$f_{\mathbf{x}_0}(t_0 + \Delta t) = f_{\mathbf{x}_0}(t_0) + f_{\mathbf{x}_0}(\Delta t) \qquad \forall t_0, \Delta t \in \mathbb{R} \tag{7}$$

which, under very weak assumptions (e.g., $f_{\mathbf{x}_0}$ is continuous), can be proved to have general solution

$$f_{\mathbf{x}_0}(t) = \alpha_{\mathbf{x}_0} t \tag{8}$$

This restricts the form of $\mathscr{T}_{\mathbf{x}_0}$ to

$$\mathscr{T}_{\mathbf{x}_0}(t) = \alpha_{\mathbf{x}_0} t + \mathscr{T}_{\mathbf{x}_0}(0) \tag{9}$$

But can the slope α really have a location dependence? To see this, let's consider two events described by K-frame coordinates $(t_0, \mathbf{x}_0)$ and $(t_0, \mathbf{y}_0)$. The difference of their time measurements in K' is thus

$$\mathscr{T}_{\mathbf{x}_0}(t_0) - \mathscr{T}_{\mathbf{y}_0}(t_0) = \left(\alpha_{\mathbf{x}_0} - \alpha_{\mathbf{y}_0}\right) t_0 + \left[\mathscr{T}_{\mathbf{x}_0}(0) - \mathscr{T}_{\mathbf{y}_0}(0)\right] \tag{10}$$

With translational invariance, this difference in time measurements in K' should not depend on the K-frame starting time t_0, which implies

$$\alpha_{\mathbf{x}_0} = \alpha_{\mathbf{y}_0} \qquad \forall \mathbf{x}_0, \mathbf{y}_0 \in \mathbb{R}^3 \tag{11}$$

i.e., the slope α is independent of location, thus it can only be a function of the relative motion $\mathbf{v}$ between the two frames.
In summary, from (9), the most general form of the transformation $\mathscr{T}$ is then

$$\mathscr{T}(t, \mathbf{x}) = \alpha(\mathbf{v})\, t + \mathscr{T}(0, \mathbf{x}) \tag{12}$$

It is clear that the above arguments are applicable to any combinations of of $\{\mathscr{T}, \mathscr{X}, \mathscr{Y}, \mathscr{Z}\} \times \{\Delta t, \Delta x, \Delta y, \Delta z\}$ – as long as we assume homogeneity along each of the four spacetime dimensions (plus weak assumptions on the continuity of the transform functions) – we can conclude that the most general form of the transformations must be linear

$$\mathscr{T}(t, x, y, z) = a_0 t + a_1 x + a_2 y + a_3 z \tag{13}$$
$$\mathscr{X}(t, x, y, z) = b_0 t + b_1 x + b_2 y + b_3 z \tag{14}$$
$$\mathscr{Y}(t, x, y, z) = c_0 t + c_1 x + c_2 y + c_3 z \tag{15}$$
$$\mathscr{Z}(t, x, y, z) = d_0 t + d_1 x + d_2 y + d_3 z \tag{16}$$

where all the coefficients can only take dependency on the relative motion $\mathbf{v}$ between the two frames.

(b) **Implications of isotropic spacetime**

If K' is moving with relative velocity $\mathbf{v} = v\hat{\mathbf{x}}$ as seen from K, we want to use the homogeneity and isotropy of spacetime to simplify the coefficients in (13)-(16). Since $\mathbf{v}$ is along the x direction, in frame K, all points with the same t and x coordinates are indistinguishable due to homogeneity. Consequently, their contributions to (13) and (14) must also be indistinguishable, requiring

$$a_2 = a_3 = 0 \qquad\qquad b_2 = b_3 = 0 \tag{17}$$

Now consider two events in K frame (t_0, x_0, y_0, z_0) and $(t_0 + \Delta t, x_0, y_0, z_0)$. By (15), the difference of their y' measurements in K' is $c_0 \Delta t$. Because of isotropy, there is no preferred positive or negative direction in the y' axis in frame K', hence there is no reason for $c_0 \Delta t$ to be either positive or negative, we must have

$$c_0 = 0 \tag{18}$$

The same argument with Δt replaced by Δx gives

$$c_1 = 0 \tag{19}$$

From the indistinguishability of (t_0, x_0, y_0, z_0) and $(t_0, x_0, y_0, z_0 + \Delta z)$ in K, we can also conclude in (15) that

$$c_3 = 0 \tag{20}$$

Similar argument applied to (16) gives

$$d_0 = d_1 = d_2 = 0 \tag{21}$$

At this point, (13)-(16) are simplified to

$$t' = a_0 t + a_1 x \tag{22}$$
$$x' = b_0 t + b_1 x \tag{23}$$
$$y' = c_2 y \tag{24}$$
$$z' = d_3 z \tag{25}$$

By reversing the roles of K and K', we can also conclude that the scaling factor c_2 and d_3 must satisfy $|c_2| = |d_3| = 1$. Since as $v \to 0$, $\mathscr{Y}, \mathscr{Z}$ will become identity transforms, we must set $c_2 = d_3 = 1$, further simplifying (22)-(25) to

$$t' = a_0 t + a_1 x \tag{26}$$
$$x' = b_0 t + b_1 x \tag{27}$$
$$y' = y \tag{28}$$
$$z' = z \tag{29}$$

To determine these remaining coefficients (which are functions of v), let's consider the origin point of frame K' of which $x' = 0$, but as seen from K, $x = vt$. Thus (27) requires for all t that

$$0 = b_0 t + b_1 v t \qquad \Longrightarrow \qquad b_0 = -v b_1 \tag{30}$$

consequently, (27) is turned into

$$x' = b_1(v)(x - vt) \tag{31}$$

We now prove that $b_1(v)$ is an even function of v. To see this, consider the event E_1 described by $x = x_0, t = 0$ in K. Then in K', this event has $x' = b_1(v) x_0$. Let K'' be a frame like K' except it moves with velocity $-v$ relative to K. Consider the event E_2 described by $x = -x_0, t = 0$ in K, (31) implies that $x'' = b_1(-v)(-x_0)$. With isotropy of spacetime, the two situations described above are symmetric – i.e. the distance of E_1's location to the origin of K', as measured in K', should be the same as the distance of E_2's location to the origin of K'' as measured in K'', except at the opposite direction of the x-axis. This implies $x' = -x''$ or $b_1(v) = b_1(-v)$ which allows us to write

$$x' = f\left(v^2\right)(x - vt) \tag{32}$$

To determine a_0, a_1, we can rewrite (26) equivalently as

$$t' = a_0(v)\, t \overbrace{- v \widetilde{a}_1(v)}^{a_1(v)} x \tag{33}$$

We can prove that both $a_0(v)$ and $\widetilde{a}_1(v)$ are even functions of v, as follows:

- Consider an event described by $x=0, t=t_0$ in K. The time measurement of this event in K' is $t'=a_0(v)\,t_0$, and for K'' it is $t''=a_0(-v)\,t_0$. Isotropy requires these two time measurements to be identical, therefore $a_0(v)=a_0(-v)$.
- Similarly, consider event E_1 described by $x=x_0, t=0$ in K, then its time measurement in K' is $t'=-v\widetilde{a}_1(v)\,x_0$. For event E_2 described by $x=-x_0, t=0$ in K, its time measurement in K'' is $t''=+v\widetilde{a}_1(-v)(-x_0)$. Again, these two situations are isotropic, hence $t'=t''$, or $a_1(v)=a_1(-v)$.

We can now write (33) as

$$t'=g\left(v^2\right)t-vh\left(v^2\right)x \tag{34}$$

In summary, from the homogeneity and isotropy of spacetime, for the given setup of K and K', the most general form of transformations between the two frames is

$$x'=f\left(v^2\right)(x-vt) \qquad t'=g\left(v^2\right)t-vh\left(v^2\right)x \qquad y'=y \qquad z'=z \tag{35}$$

$$x=f\left(v^2\right)\left(x'+vt'\right) \qquad t=g\left(v^2\right)t'+vh\left(v^2\right)x' \qquad y=y' \qquad z=z' \tag{36}$$

2. Plugging (36) into (35) gives

$$x'=f\left(fx'+vft'\right)-vf(gt'+vhx')=\left(f^2-v^2fh\right)x'+vf(f-g)\,t' \tag{37}$$

Since x' and t' are independent, we must have

$$f=g \qquad f^2-v^2fh=1 \tag{38}$$

3. Let the position of this physical entity at $t'=0$ be at the origin of K', then

$$x'=u't' \tag{39}$$

Plugging (35) into the above gives the $x\sim t$ relation in K (recall $f=g$):

$$f(x-vt)=u'(gt-vhx) \qquad \Longrightarrow \qquad \frac{x}{t}=\frac{u'+v}{1+u'vh/f} \tag{40}$$

If by postulate $2'$, there is a universal limiting speed C, then the combined velocity of $u'=C$ and v must also be C, i.e.,

$$C=\frac{C+v}{1+Cvh/f} \qquad \Longrightarrow \qquad h=\frac{f}{C^2} \tag{41}$$

By (38), we recover the Lorentz transformation

$$f=\frac{1}{\sqrt{1-\dfrac{v^2}{C^2}}} \tag{42}$$