$$Y_{l1}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{1}{l(l+1)}} P_l^1(\cos\theta)\, e^{i\phi} \tag{1}$$

$$P_l^1(\cos\theta) = -\sin\theta \frac{dP_l(\cos\theta)}{d\cos\theta} = -\sin\theta P_l'(\cos\theta) \tag{2}$$

$$\mathbf{Y}_{l1} = \sqrt{\frac{2l+1}{4\pi}\frac{1}{l(l+1)}} \left[-\sin\theta P_l'(\cos\theta)\right] e^{i\phi}\,\hat{\mathbf{r}} \tag{3}$$

$$\mathbf{\Psi}_{l1} = r\nabla Y_{l1}(\theta,\phi) = \frac{\partial Y_{l1}}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{\sin\theta}\frac{\partial Y_{l1}}{\partial\phi}\hat{\boldsymbol{\phi}} \tag{4}$$

$$\mathbf{\Phi}_{l1} = \hat{\mathbf{r}}\times\mathbf{\Psi}_{l1} = -\frac{1}{\sin\theta}\frac{\partial Y_{l1}}{\partial\phi}\hat{\boldsymbol{\theta}} + \frac{\partial Y_{l1}}{\partial\theta}\hat{\boldsymbol{\phi}} \tag{5}$$

$$\begin{aligned}\frac{\partial Y_{l1}}{\partial\theta} &= \sqrt{\frac{2l+1}{4\pi}\frac{1}{l(l+1)}}\left[-\cos\theta P_l'(\cos\theta) + \sin^2\theta P_l''(\cos\theta)\right] e^{i\phi} \\ &= \sqrt{\frac{2l+1}{4\pi}\frac{1}{l(l+1)}}\left[\cos\theta P_l'(\cos\theta) - l(l+1)P_l(\cos\theta)\right] e^{i\phi}\end{aligned} \tag{6}$$

$$\frac{1}{\sin\theta}\frac{\partial Y_{l1}}{\partial\phi} = \sqrt{\frac{2l+1}{4\pi}\frac{1}{l(l+1)}}\left[-iP_l'(\cos\theta)\right] e^{i\phi} \tag{7}$$

$$\begin{aligned}\mathbf{F} &= \sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\left\{\alpha z_l(kr)\mathbf{X}_{l1} + \frac{\beta}{k}\nabla\times\left[z_l(kr)\mathbf{X}_{l1}\right]\right\} \\ &= \sum_{l=1}^{\infty} i^{l-1}\sqrt{\frac{4\pi(2l+1)}{l(l+1)}}\left\{\alpha z_l(kr)\mathbf{\Phi}_{l1} + \frac{\beta}{k}\nabla\times\left[z_l(kr)\mathbf{\Phi}_{l1}\right]\right\} \\ &= \sum_{l=1}^{\infty} i^{l-1}\sqrt{\frac{4\pi(2l+1)}{l(l+1)}}\left\{\alpha z_l(kr)\mathbf{\Phi}_{l1} - l(l+1)\beta\frac{z_l(kr)}{kr}\mathbf{Y}_{l1} - \beta\left[z_l'(kr) + \frac{z_l(kr)}{kr}\right]\mathbf{\Psi}_{l1}\right\} \\ &= \sum_{l=1}^{\infty} i^{l-1}\left[\frac{2l+1}{l(l+1)}\right] e^{i\phi}\left(R\hat{\mathbf{r}} + T\hat{\boldsymbol{\theta}} + P\hat{\boldsymbol{\phi}}\right)\end{aligned} \tag{8}$$

$$R = l(l+1)\beta\frac{z_l(kr)}{kr}\sin\theta P_l'(\cos\theta) \tag{9}$$

$$T = i\alpha z_l(kr)P_l'(\cos\theta) - \beta\left[z_l'(kr) + \frac{z_l(kr)}{kr}\right]\left[\cos\theta P_l'(\cos\theta) - l(l+1)P_l(\cos\theta)\right] \tag{10}$$

$$P = \alpha z_l(kr)\left[\cos\theta P_l'(\cos\theta) - l(l+1)P_l(\cos\theta)\right] + i\beta\left[z_l'(kr) + \frac{z_l(kr)}{kr}\right]P_l'(\cos\theta) \tag{11}$$

$$\mathbf{G} = \sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\left\{\alpha z_l(kr)\mathbf{X}_{l,-1} - \frac{\beta}{k}\nabla\times\left[z_l(kr)\mathbf{X}_{l,-1}\right]\right\} \tag{12}$$

$$= \sum_{l=1}^{\infty} -i^{l-1}\sqrt{\frac{4\pi(2l+1)}{l(l+1)}}\left\{\alpha z_l(kr)\mathbf{\Phi}_{l1}^* + l(l+1)\beta\frac{z_l(kr)}{kr}\mathbf{Y}_{l1}^* + \beta\left[z_l'(kr) + \frac{z_l(kr)}{kr}\right]\mathbf{\Psi}_{l1}^*\right\} \tag{13}$$

$$= \sum_{l=1}^{\infty} -i^{l-1}\left[\frac{2l+1}{l(l+1)}\right] e^{-i\phi}\left(R'\hat{\mathbf{r}} + T'\hat{\boldsymbol{\theta}} + P'\hat{\boldsymbol{\phi}}\right) \tag{14}$$

$$R' = -R \tag{15}$$

$$T' = -i\alpha z_l(kr)P_l'(\cos\theta) + \beta\left[z_l'(kr) + \frac{z_l(kr)}{kr}\right]\left[\cos\theta P_l'(\cos\theta) - l(l+1)P_l(\cos\theta)\right] \tag{16}$$

$$P' = \alpha z_l(kr)\left[\cos\theta P_l'(\cos\theta) - l(l+1)P_l(\cos\theta)\right] + i\beta\left[z_l'(kr) + \frac{z_l(kr)}{kr}\right]P_l'(\cos\theta) \tag{17}$$

$$\tag{18}$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_l(x) = \left(\frac{2l-1}{l}\right) x P_{l-1}(x) - \left(\frac{l-1}{l}\right) P_{l-2}(x) \tag{19}$$

$$P_0'(x) = 0 \qquad P_1'(x) = 1 \qquad P_l'(x) = l P_{l-1}(x) + x P_{l-1}'(x) \tag{20}$$

$$j_0(x) = \frac{\sin x}{x} \qquad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad z_l(x) = (2l-1)\frac{z_{l-1}(x)}{x} - z_{l-2}(x) \tag{21}$$

$$n_0(x) = -\frac{\cos x}{x} \qquad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x} \qquad z_l'(x) = z_{l-1}(x) - (l+1)\frac{z_l(x)}{x} \tag{22}$$

$$j_l'(0) = \left.\frac{j_l(x)}{x}\right|_{x=0} = \begin{cases} \frac{1}{3} & \text{for } l = 1 \\ 0 & \text{for } l > 1 \end{cases} \tag{23}$$

$$J = j_l(ka) \qquad H = h_l^{(1)}(ka) \qquad N = j_l(nka) \tag{24}$$

$$J' = ka j_l'(ka) + j_l(ka) \qquad H' = ka h_l^{(1)\prime}(ka) + h_l^{(1)}(ka) \qquad N' = nka j_l'(nka) + j_l(nka) \tag{25}$$

$$\text{incident:} \qquad \alpha = 1 \qquad \beta = 1 \tag{26}$$

$$\text{scattered:} \qquad \alpha = \frac{JN' - \mu_r J'N}{\mu_r H'N - HN'} \qquad \beta = \frac{n^2 J'N - \mu_r JN'}{\mu_r HN' - n^2 H'N} \tag{27}$$

$$\text{internal:} \qquad \alpha = \frac{\mu_r (JH' - J'H)}{\mu_r H'N - HN'} \qquad \beta = \frac{n\mu_r (J'H - JH')}{\mu_r HN' - n^2 H'N} \qquad k \to nk \tag{28}$$

$$\mathbf{E}_{\text{inc}} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left\{ j_l(kr)\mathbf{X}_{l1} + \frac{1}{k}\boldsymbol{\nabla} \times [j_l(kr)\mathbf{X}_{l1}] \right\} \tag{29}$$

$$\mathbf{E}_{\text{sc}} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left\{ a_{l1} h_l^{(1)}(kr)\mathbf{X}_{l1} + \frac{i}{k} b_{l1}\boldsymbol{\nabla} \times \left[h_l^{(1)}(kr)\mathbf{X}_{l1}\right] \right\} \tag{30}$$

$$\mathbf{E}_{\text{int}} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left\{ c_{l1} j_l(nkr)\mathbf{X}_{l1} + \frac{i}{nk} d_{l1}\boldsymbol{\nabla} \times [j_l(nkr)\mathbf{X}_{l1}] \right\} \tag{31}$$