To avoid the terrible abuse of symbol of z and e, we use q to denote the moving charge.

1. The frequency space potentials are given by (13.25)

$$\Phi(\mathbf{k}, \omega) = \frac{2q}{\epsilon(\omega)} \left[\frac{\overbrace{\delta(\omega - \mathbf{k} \cdot \mathbf{v})}^{(1/v)\delta(k_3 - \omega/v)}}{k^2 - \frac{\omega^2 \epsilon(\omega)}{c^2}} \right] = \frac{2q}{v\epsilon(\omega)} \left[\frac{\delta\left(k_3 - \frac{\omega}{v}\right)}{k^2 - \frac{\omega^2 \epsilon(\omega)}{c^2}} \right] \tag{1}$$

$$\mathbf{A}(\mathbf{k}, \omega) = \epsilon(\omega)\boldsymbol{\beta}\Phi(\mathbf{k}, \omega) \tag{2}$$

Applying the Fourier transform $\mathbf{k} \to \mathbf{x}$, we have

$$\Phi(\mathbf{x}, \omega) = \frac{1}{(2\pi)^{3/2}} \int d^3k \, \Phi(\mathbf{k}, \omega) e^{i\mathbf{k} \cdot \mathbf{x}}$$

$$= \frac{2q e^{i\omega z/v}}{(2\pi)^{3/2} v\epsilon(\omega)} \int dk_1 dk_2 \frac{e^{ik_1 x + ik_2 y}}{k_1^2 + k_2^2 + \frac{\omega^2}{v^2} - \frac{\omega^2 \epsilon(\omega)}{c^2}} \qquad \text{choose axis so } y = 0$$

$$= \frac{2q e^{i\omega z/v}}{(2\pi)^{3/2} v\epsilon(\omega)} \int_{-\infty}^{\infty} e^{ik_1\rho} dk_1 \int_{-\infty}^{\infty} \frac{dk_2}{k_2^2 + k_1^2 + \frac{\omega^2}{v^2} - \frac{\omega^2 \epsilon(\omega)}{c^2}} \tag{3}$$

Define

$$\lambda^2 = \frac{\omega^2}{v^2} - \frac{\omega^2 \epsilon(\omega)}{c^2} = \frac{\omega^2}{v^2}\left[1 - \beta^2 \epsilon(\omega)\right] \tag{4}$$

When $\beta^2 \epsilon(\omega) < 1$, we have $\lambda^2 > 0$, the inner integral is elementary,

$$\int_{-\infty}^{\infty} \frac{dk_2}{k_2^2 + k_1^2 + \lambda^2} = \frac{\pi}{\sqrt{k_1^2 + \lambda^2}} \tag{5}$$

This turns (3) into

$$\Phi(\mathbf{x}, \omega) = \frac{2\pi q e^{i\omega z/v}}{(2\pi)^{3/2} v\epsilon(\omega)} \overbrace{\int_{-\infty}^{\infty} \frac{e^{ik_1\rho} dk_1}{\sqrt{k_1^2 + \lambda^2}}}^{2K_0(\lambda\rho)} \qquad \text{see DLMF 10.32.E11}$$

$$= \frac{q e^{i\omega z/v}}{v\epsilon(\omega)} \sqrt{\frac{2}{\pi}} K_0(\lambda\rho) \tag{6}$$

where we have chosen the branch of λ with positive real part to make the asymptotic form of $K_0(\lambda\rho) \propto e^{-\lambda\rho}$ not blow up when $\rho \to \infty$.

2. When ϵ is constant and $\beta^2 \epsilon < 1$, (6) can be written as

$$\Phi(\mathbf{x}, \omega) = \frac{q e^{i\omega z/v}}{v\epsilon} \sqrt{\frac{2}{\pi}} K_0\left(\frac{|\omega|\rho}{\Gamma v}\right) \qquad \text{where } \Gamma = \frac{1}{\sqrt{1 - \beta^2 \epsilon}} \tag{7}$$

Applying Fourier transform, we get

$$\Phi(\mathbf{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(\mathbf{x}, \omega) e^{-i\omega t} d\omega = \frac{1}{\sqrt{2\pi}} \frac{q}{v\epsilon} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} K_0\left(\frac{|\omega|\rho}{\Gamma v}\right) e^{i\omega(z/v - t)} d\omega \tag{8}$$

Here we invoke DLMF 10.43.E20 and DLMF 10.43.E21

$$\int_0^{\infty} \cos(at) K_0(t) dt = \frac{\pi}{2\sqrt{1 + a^2}} \qquad \int_0^{\infty} \sin(at) K_0(t) dt = \frac{\sinh^{-1} a}{\sqrt{1 + a^2}} \tag{9}$$

By parity, in (6), only the integral involving $\cos\left[\omega\left(z/v - t\right)\right]$ survives, giving

$$\Phi\left(\mathbf{x}, t\right) = \frac{q}{\pi v \epsilon} \frac{\Gamma v}{\rho} \frac{\pi}{\sqrt{1 + \left[\dfrac{\Gamma\left(z - vt\right)}{\rho}\right]^2}} = \frac{q}{\epsilon} \frac{\Gamma}{\sqrt{\rho^2 + \Gamma^2\left(z - vt\right)^2}} \tag{10}$$

From (2), when ϵ is constant, the vector potential is simply $\epsilon\boldsymbol{\beta}$ times the scalar potential, i.e.,

$$\mathbf{A}\left(\mathbf{x}, t\right) = \epsilon\boldsymbol{\beta}\,\Phi\left(\mathbf{x}, t\right) \tag{11}$$

The electric field has only $\hat{\mathbf{z}}$ and $\hat{\boldsymbol{\rho}}$ component,

$$
\begin{aligned}
E_z\left(\mathbf{x}, t\right) &= -\frac{\partial\Phi}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t} = -\frac{\partial\Phi}{\partial z} - \frac{\epsilon\beta}{c}\frac{\partial\Phi}{\partial t} \\
&= -\left(1 - \beta^2\epsilon\right)\frac{\partial\Phi}{\partial z} = -\frac{q}{\Gamma^2\epsilon}\left\{\frac{-\Gamma^2\left(z - vt\right)\cdot\Gamma}{\left[\rho^2 + \Gamma^2\left(z - vt\right)^2\right]^{3/2}}\right\} \\
&= \frac{\Gamma q\left(z - vt\right)}{\epsilon\left[\rho^2 + \Gamma^2\left(z - vt\right)^2\right]^{3/2}}
\end{aligned}
\tag{12}
$$

$$E_\rho\left(\mathbf{x}, t\right) = -\frac{\partial\Phi}{\partial\rho} = \frac{\Gamma q\rho}{\epsilon\left[\rho^2 + \Gamma^2\left(z - vt\right)^2\right]^{3/2}} \tag{13}$$

The magnetic field is

$$\mathbf{B}\left(\mathbf{x}, t\right) = \boldsymbol{\nabla}\times\mathbf{A}\left(\mathbf{x}, t\right) = -\frac{\partial A_z}{\partial\rho}\hat{\boldsymbol{\phi}} = -\epsilon\beta\frac{\partial\Phi}{\partial\rho}\hat{\boldsymbol{\phi}} = \epsilon\beta E_\rho\left(\mathbf{x}, t\right)\hat{\boldsymbol{\phi}} = \frac{\beta\Gamma q\rho}{\left[\rho^2 + \Gamma^2\left(z - vt\right)^2\right]^{3/2}}\hat{\boldsymbol{\phi}} \tag{14}$$

These differ from (11.152) by $\gamma \to \Gamma$ and additional factor $1/\epsilon$ for E_z, E_ρ.

3. When $\beta^2\epsilon > 1$, we see that $\lambda^2 < 0$ from (4). Let

$$\mu^2 = -\lambda^2 = \frac{\omega^2}{v^2}\left(\beta^2\epsilon - 1\right) > 0 \tag{15}$$

the integral (3) is now

$$\Phi\left(\mathbf{x}, \omega\right) = \frac{2qe^{i\omega z/v}}{\left(2\pi\right)^{3/2}v\epsilon\left(\omega\right)}\int_{-\infty}^{\infty}e^{ik_1\rho}\,dk_1\underbrace{\int_{-\infty}^{\infty}\frac{dk_2}{k_2^2 + k_1^2 - \mu^2}}_{I} \tag{16}$$

For $|k_1| > \mu$, the double integral can be calculated straightforwardly,

$$
\begin{aligned}
\int_{|k_1|>\mu}e^{ik_1\rho}\,dk_1\int_{-\infty}^{\infty}\frac{dk_2}{k_2^2 + k_1^2 - \mu^2} &= \int_{|k_1|>\mu}e^{ik_1\rho}\,dk_1\cdot\frac{\pi}{\sqrt{k_1^2 - \mu^2}} \\
&= 2\pi\underbrace{\int_{\mu}^{\infty}\frac{\cos\left(k_1\rho\right)dk_1}{\sqrt{k_1^2 - \mu^2}}}_{-Y_0(\mu\rho)\cdot\pi/2} \qquad \text{see DLMF 10.9.E12} \\
&= -\pi^2 Y_0\left(\mu\rho\right)
\end{aligned}
\tag{17}
$$
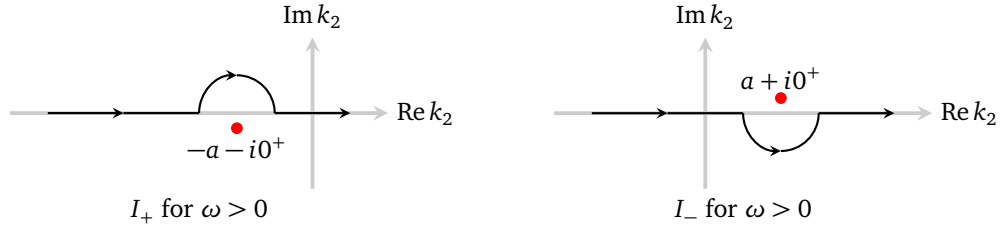
On the other hand, when $|k_1| < \mu$, with

$$a \equiv \sqrt{\mu^2 - k_1^2} = \sqrt{\frac{\omega^2}{v^2}\left(\beta^2\epsilon - 1\right) - k_1^2} \tag{18}$$

the inner integral of (16) becomes

$$I = \int_{-\infty}^{\infty}\frac{dk_2}{\left(k_2 + a\right)\left(k_2 - a\right)} = \frac{1}{2a}\left(\overbrace{\int_{-\infty}^{\infty}\frac{dk_2}{k_2 - a}}^{I_-} - \overbrace{\int_{-\infty}^{\infty}\frac{dk_2}{k_2 + a}}^{I_+}\right) \tag{19}$$

Consider the physical situation where $\mathrm{Im}\,\epsilon\left(\omega\right)$ is infinitesimally positive for positive frequency (absorbing medium),

I_+ for $\omega > 0$ (left diagram, pole at $-a - i0^+$)

I_- for $\omega > 0$ (right diagram, pole at $a + i0^+$)

then the pole of I_+, I_- are indicated by the red dots in the diagram above.

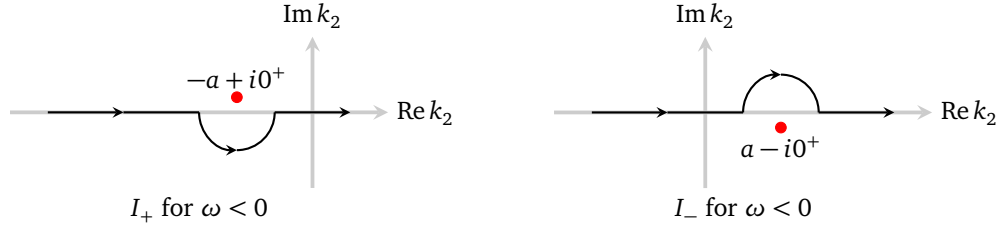To evaluate I_+ with pole at $-a - i0^+$, we choose the contour with infinitesimal semicircle above the singularity

$$I_+ = \lim_{R \to \infty} \lim_{r \to 0} \left(\int_{-R}^{-a-r} \frac{dk_2}{k_2 + a} + \int_{-a+r}^{R} \frac{dk_2}{k_2 + a} + \int_{\text{semicircle}} \frac{dz}{z + a} \right)$$

$$= \lim_{R \to \infty} \lim_{r \to 0} \left[\ln\left(\frac{-r}{-R+a}\right) + \ln\left(\frac{R+a}{r}\right) \right] + \lim_{r \to 0} \int_{\pi}^{0} \frac{i r e^{i\phi} d\phi}{r e^{i\phi}}$$

$$= -i\pi \tag{20}$$

Similarly,

$$I_- = \lim_{R \to \infty} \lim_{r \to 0} \left(\int_{-R}^{a-r} \frac{dk_2}{k_2 - a} + \int_{a+r}^{R} \frac{dk_2}{k_2 - a} + \int_{\text{semicircle}} \frac{dz}{z - a} \right)$$

$$= \lim_{R \to \infty} \lim_{r \to 0} \left[\ln\left(\frac{-r}{-R-a}\right) + \ln\left(\frac{R-a}{r}\right) \right] + \lim_{r \to 0} \int_{-\pi}^{0} \frac{i r e^{i\phi} d\phi}{r e^{i\phi}}$$

$$= i\pi \tag{21}$$

For negative frequencies, the absorption requirement is on the complex conjugate $\epsilon^*(\omega)$, so $\operatorname{Im} \epsilon(\omega) < 0$ for $\omega < 0$, the contours of $I_\pm$ are flipped about the real axis (see diagram below) hence $I_\pm$ gain a minus sign on top of (20), (21).

I_+ for $\omega < 0$ (left diagram, pole at $-a + i0^+$)

I_- for $\omega < 0$ (right diagram, pole at $a - i0^+$)

In summary, the inner integral of (16), considering both positive and negative frequencies, is

$$I = \frac{1}{2a}(I_- - I_+) = \operatorname{sgn}(\omega) \frac{i\pi}{\sqrt{\mu^2 - k_1^2}} \tag{22}$$

which gives the remaining part of the double integral

$$\int_{|k_1| < \mu} e^{ik_1 \rho} dk_1 \int_{-\infty}^{\infty} \frac{dk_2}{k_2^2 + k_1^2 - \mu^2} = \operatorname{sgn}(\omega) 2\pi i \overbrace{\int_{0}^{\mu} \frac{\cos(k_1 \rho) dk_1}{\sqrt{\mu^2 - k_1^2}}}^{J_0(\mu\rho) \cdot \pi/2} \qquad \text{see DLMF 10.9.E4}$$

$$= \operatorname{sgn}(\omega) \pi^2 i J_0(\mu\rho) \tag{23}$$

Putting (17) and (23) back to (16) gives the desired result

$$\Phi(\mathbf{x}, \omega) = \frac{q e^{i\omega z/v}}{v \epsilon(\omega)} \sqrt{\frac{\pi}{2}} \left[-Y_0(\mu\rho) + \operatorname{sgn}(\omega) i J_0(\mu\rho) \right] \tag{24}$$

Finally, when $\epsilon(\omega)$ is independent of frequency, applying Fourier transform $\omega \to t$ yields

$$\Phi(\mathbf{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(\mathbf{x}, \omega) e^{-i\omega t} d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \frac{q}{v\epsilon} \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} e^{i\omega(z/v - t)} \left[-Y_0\left(\frac{|\omega|\rho}{v} \sqrt{\beta^2 \epsilon - 1}\right) + \operatorname{sgn}(\omega) i J_0\left(\frac{|\omega|\rho}{v} \sqrt{\beta^2 \epsilon - 1}\right) \right] d\omega \tag{25}$$

By parity,

$$\int_{-\infty}^{\infty} e^{i\omega(z/v-t)} Y_0\left(\frac{|\omega|\rho}{v}\sqrt{\beta^2\epsilon-1}\right)d\omega = 2\int_0^{\infty} \cos\left[\omega\left(z/v-t\right)\right] Y_0\left(\frac{\omega\rho}{v}\sqrt{\beta^2\epsilon-1}\right)d\omega \tag{26}$$

$$\int_{-\infty}^{\infty} e^{i\omega(z/v-t)}\operatorname{sgn}(\omega)\,iJ_0\left(\frac{|\omega|\rho}{v}\sqrt{\beta^2\epsilon-1}\right)d\omega = -2\int_0^{\infty} \sin\left[\omega\left(z/v-t\right)\right] J_0\left(\frac{\omega\rho}{v}\sqrt{\beta^2\epsilon-1}\right)d\omega \tag{27}$$

The latter integrals are given in 6.671.12 and 6.671.7 of *I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products Eighth Edition*

$$\int_0^{\infty} Y_0(ax)\cos(bx)\,dx = \begin{cases} -\dfrac{1}{\sqrt{b^2-a^2}} & \text{for } b > a > 0 \\ 0 & \text{for } a > b > 0 \end{cases} \tag{28}$$

$$\int_0^{\infty} J_0(ax)\sin(bx)\,dx = \begin{cases} \dfrac{1}{\sqrt{b^2-a^2}} & \text{for } b > a > 0 \\ 0 & \text{for } a > b > 0 \end{cases} \tag{29}$$

When $z > vt$, the sum of the two integrals in (25) vanishes, regardless of the relative size of a and b. But for $z < vt$, the sum enhances the result for the $b > a > 0$ case. In summary, the full scalar potential is

$$\Phi(\mathbf{x},t) = \begin{cases} \dfrac{2q}{\epsilon\sqrt{(z-vt)^2-\rho^2\left(\beta^2\epsilon-1\right)}} & \text{for } z < vt \text{ and } \rho < \dfrac{vt-z}{\sqrt{\beta^2\epsilon-1}} = \dfrac{vt-z}{\tan\theta_c} \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

Multiplying (30) by $\boldsymbol{\beta}/\epsilon$, as indicated by (2), will give us the same vector potential as (13.51).

The non-zero potential region is the interior of the shockwave lightcone, as depicted below.