

With the given field

$$E_z = J_m(\gamma r) e^{im\phi} e^{i\beta z - i\omega t} \quad H_z = 0 \quad (1)$$

$$E_\phi = -\frac{m\beta}{\gamma^2} \frac{E_z}{r} \quad H_r = -\frac{k}{Z_0\beta} E_\phi \quad (2)$$

$$E_r = \frac{i\beta}{\gamma^2} \frac{\partial E_z}{\partial r} \quad H_\phi = \frac{k}{Z_0\beta} E_r \quad (3)$$

The time averaged angular momentum density is

$$\mathbf{m} = \frac{1}{2c^2} \text{Re}[\mathbf{x} \times (\mathbf{E} \times \mathbf{H}^*)] = \frac{1}{2c^2} \text{Re}[(r\hat{\mathbf{r}} + z\hat{\mathbf{z}}) \times (\mathbf{E} \times \mathbf{H}^*)] \quad (4)$$

so the z component of which is

$$\begin{aligned} m_z &= \frac{1}{2c^2} \text{Re}[r(\mathbf{E} \times \mathbf{H}^*)_\phi] = \frac{1}{2c^2} \text{Re}[r(E_z H_r^* - E_r H_z^*)] \\ &= \frac{1}{2c^2} \text{Re}\left[r E_z \left(-\frac{k}{Z_0\beta}\right) \left(-\frac{m\beta}{\gamma^2}\right) \frac{E_z^*}{r}\right] \\ &= \frac{mk}{2c^2 Z_0 \gamma^2} |E_z|^2 \quad \text{let } t = \gamma r \\ &= \frac{mk}{2c^2 Z_0 \gamma^2} J_m^2(t) \end{aligned} \quad (5)$$

Its integration over the cross section at a given z is

$$\int m_z da = 2\pi \cdot \frac{mk}{2c^2 Z_0 \gamma^4} \overbrace{\int_0^{\gamma R} J_m^2(t) t dt}^I = 2\pi \cdot \frac{mk}{2c^2 Z_0 \gamma^4} I \quad (6)$$

On the other hand, the time averaged energy density is

$$\begin{aligned} u &= \frac{\epsilon_0}{4} (\mathbf{E} \cdot \mathbf{E}^* + Z_0^2 \mathbf{H} \cdot \mathbf{H}^*) \\ &= \frac{\epsilon_0}{4} \left[|E_z|^2 + \left(\frac{m\beta}{\gamma^2}\right)^2 \frac{|E_z|^2}{r^2} + \left(\frac{\beta}{\gamma^2}\right)^2 \left|\frac{\partial E_z}{\partial r}\right|^2 + \left(\frac{k}{\beta}\right)^2 \left(\frac{m\beta}{\gamma^2}\right)^2 \frac{|E_z|^2}{r^2} + \left(\frac{k}{\beta}\right)^2 \left(\frac{\beta}{\gamma^2}\right)^2 \left|\frac{\partial E_z}{\partial r}\right|^2 \right] \\ &= \frac{\epsilon_0}{4} \left[|E_z|^2 + \frac{m^2(\beta^2 + k^2)}{\gamma^4} \frac{|E_z|^2}{r^2} + \frac{(\beta^2 + k^2)}{\gamma^4} \left|\frac{\partial E_z}{\partial r}\right|^2 \right] \\ &= \frac{\epsilon_0}{4} \left\{ J_m^2(t) + \frac{(\beta^2 + k^2)}{\gamma^2} \left[\frac{m^2}{t^2} J_m^2(t) + J_m'^2(t) \right] \right\} \end{aligned} \quad (7)$$

whose integration over the cross section at a given z is

$$\int u da = 2\pi \cdot \frac{\epsilon_0}{4} \left\{ \frac{I}{\gamma^2} + \frac{(\beta^2 + k^2)}{\gamma^4} \overbrace{\int_0^{\gamma R} \left[\frac{m^2}{t^2} J_m^2(t) + J_m'^2(t) \right] t dt}^{I'} \right\} \quad (8)$$

With the Bessel equation

$$\frac{1}{t} \frac{d}{dt} [t J_m'(t)] + \left(1 - \frac{m^2}{t^2}\right) J_m(t) = 0 \quad (9)$$

we have

$$\begin{aligned} I' &= \int_0^{\gamma R} \left\{ J_m(t) \frac{d}{dt} [t J_m'(t)] + J_m'^2(t) t + J_m^2(t) t \right\} dt \\ &= \int_0^{\gamma R} \frac{d}{dt} [t J_m(t) J_m'(t)] dt + I \quad \text{note } J_m(\gamma R) = 0 \\ &= I \end{aligned} \quad (10)$$

Then (8) becomes

$$\begin{aligned}
 \int u da &= 2\pi \cdot \frac{\epsilon_0}{4} \frac{1}{\gamma^2} \left[1 + \frac{(\beta^2 + k^2)}{\gamma^2} \right] I && \text{recall } k^2 = \beta^2 + \gamma^2 \\
 &= 2\pi \cdot \frac{\epsilon_0}{2\gamma^2} \left(1 + \frac{\beta^2}{\gamma^2} \right) I
 \end{aligned} \tag{11}$$

The ratio of the z component of the angular momentum to the energy is thus

$$\frac{\int m_z da}{\int u da} = \frac{\frac{mk}{c^2 Z_0 \gamma^4}}{\frac{\epsilon_0}{\gamma^2} \left(1 + \frac{\beta^2}{\gamma^2} \right)} = \frac{mk}{(\beta^2 + \gamma^2) c} \tag{12}$$