1. **Prob 12.3**

(a) Without loss of generality, let z-axis be aligned with the direction of $\mathbf{E}_0$, and let x-axis be aligned with the initial velocity $\mathbf{v}_0$. From the equation of motion (12.1), (12.2), we have

$$\frac{d(\gamma m v_z)}{dt} = eE_0 \tag{1}$$

$$\frac{d\left(\gamma mc^2\right)}{dt} = ev_z E_0 \tag{2}$$

Multiplying both sides of (2) by γm, we get

$$\frac{(mc)^2}{2}\frac{d\gamma^2}{dt} = eE_0(\gamma m v_z) \qquad \text{take } d/dt \text{ and use (1)} \qquad \Longrightarrow$$

$$\frac{(mc)^2}{2}\frac{d^2\gamma^2}{dt^2} = (eE_0)^2 \tag{3}$$

This means that as a function of t, γ^2 is quadratic in t:

$$\gamma^2(t) = A + Bt + Ct^2 \tag{4}$$

We can use (3) to find C, and use the value of γ^2 at $t=0$ to find A, and the value at $t=0$ of (2) to find B. The result is

$$\gamma^2(t) = \overbrace{\frac{1}{1-v_0^2/c^2}}^{\gamma_0^2} + \left(\frac{eE_0}{mc}\right)^2 t^2 \tag{5}$$

$v_z(t)$ can be found via (1)

$$v_z(t) = \left(\frac{eE_0}{m}\right)\frac{t}{\gamma} = \frac{\left(\frac{eE_0}{m}\right)t}{\sqrt{\gamma_0^2 + \left(\frac{eE_0}{mc}\right)^2 t^2}} \tag{6}$$

Integrating (6) yields

$$z(t) = \frac{mc^2}{eE_0}\left[\sqrt{\gamma_0^2 + \left(\frac{eE_0}{mc}\right)^2 t^2} - \gamma_0\right] \tag{7}$$

In the x direction, momentum is conserved, so we have

$$v_x(t) = \frac{\gamma_0 m v_0}{\gamma(t) m} = \frac{\gamma_0 v_0}{\sqrt{\gamma_0^2 + \left(\frac{eE_0}{mc}\right)^2 t^2}} \tag{8}$$

hence

$$x(t) = \frac{\gamma_0 m v_0 c}{eE_0}\sinh^{-1}\left(\frac{eE_0 t}{\gamma_0 mc}\right) \tag{9}$$

(b) Inverting (9) gives

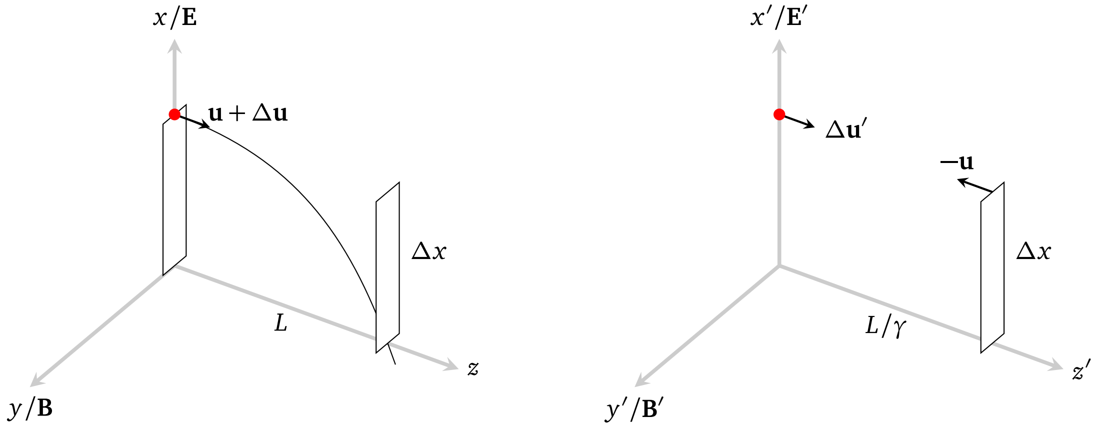
$$t = \frac{\gamma_0 mc}{eE_0}\sinh\left(\frac{eE_0 x}{\gamma_0 m v_0 c}\right) \tag{10}$$

Then by (7),

$$z(x) = \frac{\gamma_0 mc^2}{eE_0}\left[\cosh\left(\frac{eE_0 x}{\gamma_0 m v_0 c}\right) - 1\right] \tag{11}$$

From (9), we can define a characteristic time $T = \gamma_0 mc/eE_0$. When $t \ll T$, we have $x \propto t$ and $z \propto t^2$, the trajectory is approximately parabolic. When $t \gg T$, $x \propto \ln t$ and $z \propto t$, the trajectory is approximately exponential.

2. **Prob 12.4**



Let $\mathbf{E}$ be in the x direction, and let $\mathbf{B}$ be in the y direction, and let the particle move into the device with velocity $(u+\Delta u)\,\hat{\mathbf{z}}$, where $u = cE/B$ is the drift velocity. At the drift velocity, the electric force and the magnetic force balance out. The extra Δu gives rise to a net force of $-e\Delta u B/c\hat{\mathbf{x}}$. With $\Delta u \ll u$, we can treat the z-direction velocity as constant (even though the net force in the $-x$ direction will give the particle some x velocity over time, which will result in an z-acceleration due to magnetic force, the increase in z velocity is of higher order of Δu). The equation of motion in x direction says

$$\frac{dp_x}{dt} = -\frac{e\Delta u B}{c} \qquad \Longrightarrow \qquad \frac{dv_x}{dt} = -\frac{e\Delta u B}{c\gamma m} \tag{12}$$

This is also treating γ as approximately constant which is true given the small Δu. With this, the opening Δx must be no smaller than the x displacement of the particle as it travels the distance L in the z direction, i.e.,

$$\frac{1}{2}\left(\frac{e\Delta u B}{c\gamma m}\right)\left(\frac{L}{u}\right)^2 \leq \Delta x \qquad \Longrightarrow \qquad \Delta u \leq \frac{2c\gamma m u^2 \Delta x}{eBL^2} \tag{13}$$

We can do this calculation in the inertial frame moving at velocity $\mathbf{u}$ (see above figure on the right). In this frame, we have $\mathbf{E}' = 0$, $\mathbf{B}' = \mathbf{B}/\gamma$ by (12.44). The particle's velocity in this frame is given by

$$\Delta u' = \frac{(u+\Delta u)-u}{1-\dfrac{(u+\Delta u)u}{c^2}} \approx \gamma^2 \Delta u \tag{14}$$

The opposing opening is moving towards the particle with velocity $-\mathbf{u}$ from a contracted distance L/γ. Since $\Delta u' \ll u$, the time for them to meet is approximately

$$t' \approx \frac{L}{\gamma u} \tag{15}$$

The equation of motion in the x' direction requires

$$\frac{dp'_x}{dt'} = -\frac{e\Delta u' B'}{c} \qquad \Longrightarrow \qquad \frac{dv'_x}{dt'} \approx -\frac{e\Delta u' B'}{cm} \tag{16}$$

where we have treated the Lorentz factor due to $\Delta u'$ to be approximately unity.

The condition for velocity selection is thus

$$\frac{1}{2}\left(\frac{e\Delta u' B'}{cm}\right)t'^2 \leq \Delta x \qquad \Longrightarrow \qquad \frac{1}{2}\left[\frac{e\left(\gamma^2\Delta u\right)(B/\gamma)}{cm}\right]\left(\frac{L}{\gamma u}\right)^2 \leq \Delta x \tag{17}$$

which gives the same constraint on Δu as (13).

3. **Prob 12.5**

(a) For $\mathbf{E} \perp \mathbf{B}$ and $|\mathbf{E}| < |\mathbf{B}|$, section 12.3 has shown that in a frame K' moving at velocity

$$\mathbf{u} = c\frac{\mathbf{E}\times\mathbf{B}}{B^2} \tag{18}$$

the electric field vanishes, and magnetic field is scaled down by a factor of

$$\gamma = \sqrt{\frac{B^2}{B^2 - E^2}} \tag{19}$$

Translating (12.41) into the coordinate axis given in this problem, i.e., $\boldsymbol{\epsilon}_3 = \hat{\mathbf{y}}, \boldsymbol{\epsilon}_1 = \hat{\mathbf{z}}, \boldsymbol{\epsilon}_2 = \hat{\mathbf{x}}$, we have

$$\mathbf{x}'\left(t'\right) = \mathbf{X}_0' + v_y' t' \hat{\mathbf{y}} + a \sin \omega t' \hat{\mathbf{z}} + a \cos \omega t' \hat{\mathbf{x}} \tag{20}$$

where

$$a = \frac{cp_\perp'}{eB'} = \frac{c\gamma' m\sqrt{v_z'^2 + v_x'^2}}{eB'} \qquad \gamma' = \frac{1}{\sqrt{1 - v'^2/c^2}} \qquad \omega = \frac{eB'}{\gamma' mc} \tag{21}$$

Applying the Lorentz transformation to (20) gives the parametric trajectory in the lab frame

$$z\left(t'\right) = \gamma\left[z'\left(t'\right) + ut'\right] = \gamma\left(z_0' + a \sin \omega t' + ut'\right) \tag{22}$$

$$x\left(t'\right) = x'\left(t'\right) = x_0' + a \cos \omega t' \tag{23}$$

$$y\left(t'\right) = y'\left(t'\right) = y_0' + v_y' t' \tag{24}$$

Note that the parameter for the trajectory is the time in frame K'.

(b) If $|\mathbf{E}| > |\mathbf{B}|$, we transform into frame K' that moves with velocity

$$\mathbf{u} = c\frac{\mathbf{E}\times\mathbf{B}}{E^2} \tag{25}$$

In this frame, there is only electric field of strength $\mathbf{E}' = \mathbf{E}/\gamma$, and the magnetic field is zero. We can use the result of problem 12.3 by properly choosing the zero time at which the velocity along x direction is zero and velocities along y and z are v_{y0}' and v_{z0}'. In this case

$$x'\left(t'\right) = \frac{mc^2}{eE'}\left[\sqrt{\gamma_0'^2 + \left(\frac{eE'}{mc}\right)^2 t'^2} - \gamma_0'\right] \tag{26}$$

$$y'\left(t'\right) = \frac{\gamma_0' m v_{y0}' c}{eE'} \sinh^{-1}\left(\frac{eE' t'}{\gamma_0' mc}\right) \tag{27}$$

$$z'\left(t'\right) = \frac{\gamma_0' m v_{z0}' c}{eE'} \sinh^{-1}\left(\frac{eE' t'}{\gamma_0' mc}\right) \tag{28}$$

where

$$\gamma_0' = \frac{1}{\sqrt{1 - v_0'^2/c^2}} \qquad v_0' = \sqrt{v_{y0}'^2 + v_{z0}'^2} \tag{29}$$

Applying Lorentz transformation back to K gives the parametric trajectory

$$x\left(t'\right) = x'\left(t'\right) = \frac{mc^2}{eE'}\left[\sqrt{\gamma_0'^2 + \left(\frac{eE'}{mc}\right)^2 t'^2} - \gamma_0'\right] \tag{30}$$

$$y\left(t'\right) = y'\left(t'\right) = \frac{\gamma_0' m v_{y0}' c}{eE'} \sinh^{-1}\left(\frac{eE' t'}{\gamma_0' mc}\right) \tag{31}$$

$$z\left(t'\right) = \gamma\left[z'\left(t'\right) + ut'\right] = \gamma\left[\frac{\gamma_0' m v_{z0}' c}{eE'} \sinh^{-1}\left(\frac{eE' t'}{\gamma_0' mc}\right) + ut'\right] \tag{32}$$

4. **Prob 12.6**

(a) From problem 11.15, we see that when $\mathbf{E}$ and $\mathbf{B}$ are not already parallel, we can always change into a frame K' where $\mathbf{E}'$ and $\mathbf{B}'$ are parallel. The relative velocity of K' to the original frame is along the direction of $\mathbf{E}\times\mathbf{B}$. Once we solve the equation of motion in K', applying the Lorentz transformation back to K will give the trajectory in K. In part (b) we will derive such solution in K'.

(b) Let $\mathbf{E},\mathbf{B}$ be parallel and along the z direction. The equation of motion is

$$\frac{d\left(\gamma m\mathbf{u}\right)}{dt} = e\left(\mathbf{E}+\frac{\mathbf{u}\times\mathbf{B}}{c}\right) \tag{33}$$

$$\frac{d\left(\gamma mc^2\right)}{dt} = e\mathbf{u}\cdot\mathbf{E} \tag{34}$$

Their component forms are

$$\frac{d\left(\gamma mu_z\right)}{dt} = eE \qquad \frac{d\left(\gamma mc^2\right)}{dt} = eu_zE \tag{35}$$

$$\frac{d\left(\gamma mu_x\right)}{dt} = \left(\frac{eB}{c}\right)u_y \qquad \frac{d\left(\gamma mu_y\right)}{dt} = -\left(\frac{eB}{c}\right)u_x \tag{36}$$

We can properly choose the starting time at which $u_z(0)=0$. The solution of (35) is already obtained in problem 12.3. Rewriting (5) and (6) gives

$$\gamma^2(t) = \gamma_0^2+\lambda^2t^2 \qquad \text{where} \qquad \lambda=\frac{eE}{mc} \qquad \gamma_0=\frac{1}{\sqrt{1-u_{\perp 0}^2/c^2}} \tag{37}$$

$$u_z(t) = \frac{\lambda ct}{\gamma} \tag{38}$$

From the definition of γ, we have

$$\frac{1}{\gamma^2} = 1-\left(\frac{u_z^2+u_\perp^2}{c^2}\right) \qquad \Longrightarrow \qquad u_\perp^2 = \frac{c^2\left(\gamma_0^2-1\right)}{\gamma^2} \tag{39}$$

Note from (36), we have

$$u_x\frac{d\left(\gamma mu_x\right)}{dt}+u_y\frac{d\left(\gamma mu_y\right)}{dt} = 0 \qquad \Longrightarrow \qquad \frac{d\gamma}{dt}u_\perp^2+\frac{\gamma}{2}\frac{du_\perp^2}{dt}=0 \qquad \Longrightarrow \qquad \frac{du_\perp^2}{u_\perp^2}=-2\frac{d\gamma}{\gamma} \tag{40}$$

which is consistent with (39).
Let ϕ be the angle made between $\mathbf{u}_\perp$ and the x-axis, i.e.,

$$u_x = u_\perp\cos\phi \qquad u_y = u_\perp\sin\phi \tag{41}$$

Using the fact that $\gamma u_\perp$ is a constant (by (39)), we see that (36) implies

$$\left.\begin{aligned}\frac{d\left(\gamma mu_\perp\cos\phi\right)}{dt} &= \left(\frac{eB}{c}\right)u_\perp\sin\phi\\ \frac{d\left(\gamma mu_\perp\sin\phi\right)}{dt} &= -\left(\frac{eB}{c}\right)u_\perp\cos\phi\end{aligned}\right\} \Longrightarrow \qquad \frac{d\phi}{dt}=-\frac{\mu}{\gamma} \qquad \text{where} \qquad \mu=\frac{eB}{mc} \tag{42}$$

Plugging (37) into (42) gives

$$\frac{d\phi}{dt} = -\frac{\mu}{\sqrt{\gamma_0^2+\lambda^2t^2}} = -\frac{1}{\sqrt{\left(\frac{\gamma_0}{\mu}\right)^2+\left(\frac{\lambda}{\mu}\right)^2t^2}} \tag{43}$$

the solution of which is

$$\phi(t) = -\left(\frac{\mu}{\lambda}\right)\sinh^{-1}\left(\frac{\lambda t}{\gamma_0}\right) = -\left(\frac{B}{E}\right)\sinh^{-1}\left(\frac{eEt}{\gamma_0 mc}\right) \tag{44}$$

assuming the x axis is aligned with the direction of $\mathbf{u}_\perp(0)$.

Inverting (44) yields

$$t(\phi) = -\frac{\gamma_0 mc}{eE} \sinh\left[\left(\frac{E}{B}\right)\phi\right] \tag{45}$$

If we define

$$R = \frac{mc^2}{eB} \qquad\qquad \rho = \frac{E}{B} \tag{46}$$

(45) can be written in the desired form

$$ct = -\frac{R}{\rho}\gamma_0 \sinh(\rho\phi) \tag{47}$$

In these variables, the z-displacement (7) can be written as

$$z = \frac{R}{\rho}\gamma_0 [\cosh(\rho\phi) - 1] \tag{48}$$

Note these are the same as the form in the problem after some translation in z and reversing the rotation direction ϕ.

We can obtain the x displacement by the integration

$$\begin{aligned}
x &= \int u_x dt = \int u_\perp \cos\phi\, dt && \text{use (39)} \\
&= \int c\sqrt{\gamma_0^2 - 1}\cos\phi \frac{dt}{\gamma} && \text{by (42) } \frac{dt}{\gamma} = -\frac{d\phi}{\mu} \\
&= -\frac{c}{\mu}\sqrt{\gamma_0^2 - 1}\sin\phi \\
&= -R\sqrt{\gamma_0^2 - 1}\sin\phi
\end{aligned} \tag{49}$$

and similarly for the y displacement

$$y = \int u_y dt = R\sqrt{\gamma_0^2 - 1}\cos\phi \tag{50}$$

after conveniently choosing the origin in the x-y plane to eliminate the integration constant.

The A in the problem statement is just $\sqrt{\gamma_0^2 - 1}$, which cannot be arbitrary but has to be determined by the initial transverse velocity $\mathbf{u}_\perp(0)$.

The statement that ϕ is c/R times the proper time can be seen from (42).

It is rather surprising that despite persistent acceleration by the electric field, the particle's projected trajectory in the x-y plane remains a perfect circle. As t increases, by (44), the angular displacement grows approximately via $\phi \sim \ln t$.