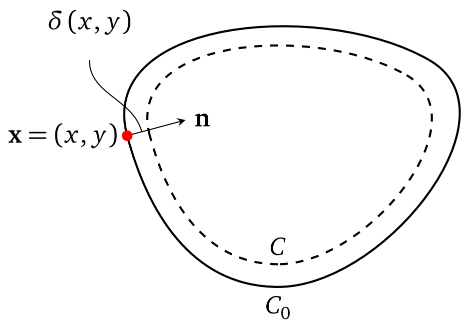
1. **Prob 8.12**



(a) See figure above, let $\mathbf{n}$ be the inward pointing normal vector from the unperturbed contour C_0. Let $\mathbf{x}=(x,y)$ be a point on C_0. For TM mode, we must have

$$\psi_0(\mathbf{x})=0 \tag{1}$$

For the perturbed contour C, since the wall is perfect conductor, the boundary condition requires

$$\psi(\mathbf{x}+\delta\mathbf{n})=0 \tag{2}$$

This can be achieved by letting

$$\psi(\mathbf{x})=\psi_0(\mathbf{x})-\delta\cdot\frac{\partial\psi_0}{\partial n} \tag{3}$$

since expanding $\psi(\mathbf{x}+\delta\mathbf{n})$ around $\mathbf{x}$ to the first order will give

$$\psi(\mathbf{x}+\delta\mathbf{n})\approx\psi(\mathbf{x})+\delta\cdot\frac{\partial\psi}{\partial n}=\psi_0(\mathbf{x})+\delta\left(\frac{\partial\psi}{\partial n}-\frac{\partial\psi_0}{\partial n}\right)\approx\psi_0(x)=0 \tag{4}$$

where in the first approximation, we ignored higher order derivatives, and in the second approximation, we ignored the difference of the first-order normal gradients, which is of second order. Thus ψ satisfies the eigenequation with modified boundary condition

$$\left(\nabla_t^2+\gamma^2\right)\psi=0 \qquad \psi\big|_{C_0}=-\delta\cdot\frac{\partial\psi_0}{\partial n} \tag{5}$$

Following the steps from (8.67) - (8.69) (with one more approximation that the 2D integral of the LHS of Green's identity is to be over the unperturbed cross section S_0), we have

$$\text{TM}: \qquad \gamma_0^2-\gamma^2=-\frac{\oint_{C_0}\delta(x,y)\left|\frac{\partial\psi_0}{\partial n}\right|^2 dl}{\int_{S_0}|\psi_0|^2\,da} \tag{6}$$

For TE mode, the unperturbed boundary condition becomes

$$\frac{\partial\psi_0}{\partial n}(\mathbf{x})=0 \tag{7}$$

then the construction (3) still holds, since

$$\frac{\partial\psi}{\partial n}(\mathbf{x}+\delta\mathbf{n})\approx\frac{\partial\psi}{\partial n}(\mathbf{x})+\delta\cdot\frac{\partial^2\psi}{\partial n^2}=\frac{\partial\psi_0}{\partial n}(\mathbf{x})+\delta\left(\frac{\partial^2\psi}{\partial n^2}-\frac{\partial^2\psi_0}{\partial n^2}\right)\approx 0 \tag{8}$$

which makes ψ satisfy the Neumann boundary condition for C.

Then the same procedure produces

$$\text{TE}: \qquad \gamma_0^2 - \gamma^2 = \frac{\displaystyle\oint_{C_0} \delta(x,y)\,\psi_0^* \frac{\partial^2 \psi_0}{\partial n^2} dl}{\displaystyle\int_{S_0} |\psi|^2\, da} \tag{9}$$

Note our results are the minus of the claim from Jackson, because in the problem statement the normal is said to be "to C". This deviates from the convention used in section 8.6, we stick to the convention here.

(b) For TM_{11}

$$\psi_0(x,y) = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) \tag{10}$$

The contour integral on the numerator of (6) only involves the two vertical walls, on each of which

$$\left|\frac{\partial \psi_0}{\partial n}\right|_{\text{ver}} = \left(\frac{\pi}{a}\right)\sin\left(\frac{\pi y}{b}\right) \tag{11}$$

Also

$$\delta(x,y)\Big|_{\text{ver}} = \frac{\delta y}{b} \tag{12}$$

Then by (6) we have

$$\gamma_0^2 - \gamma^2 = -\frac{2\displaystyle\int_0^b \left(\frac{\pi}{a}\right)^2 \sin^2\left(\frac{\pi y}{b}\right)\left(\frac{\delta y}{b}\right) dy}{\displaystyle\int_0^b \sin^2\left(\frac{\pi y}{b}\right) dy \int_0^a \sin^2\left(\frac{\pi x}{a}\right) dx} = -\frac{\dfrac{\pi^2 \delta b}{2a^2}}{\dfrac{ab}{4}} = -\frac{2\pi^2\delta}{a^3} \tag{13}$$

For TE_{10},

$$\psi_0(x,y) = \cos\left(\frac{\pi x}{a}\right) \tag{14}$$

On the $x = 0$ vertical wall, $\mathbf{n}$ points to the $+\hat{\mathbf{x}}$ direction, and on the $x = a$ vertical wall, $\mathbf{n}$ points to the $-\hat{\mathbf{x}}$ direction, hence

$$\psi_0\Big|_{\text{ver},x=0} = 1 \qquad \psi_0\Big|_{\text{ver},x=a} = -1 \qquad \left.\frac{\partial^2\psi_0}{\partial n^2}\right|_{\text{ver},x=0} = -\left(\frac{\pi}{a}\right)^2 \qquad \left.\frac{\partial^2\psi_0}{\partial n^2}\right|_{\text{ver},x=a} = \left(\frac{\pi}{a}\right)^2 \tag{15}$$

which by (9) gives

$$\gamma_0^2 - \gamma^2 = \frac{2\displaystyle\int_0^b -\left(\frac{\pi}{a}\right)^2 \frac{\delta y}{b} dy}{\displaystyle\int_0^b dy \int_0^a \cos^2\left(\frac{\pi x}{a}\right) dx} = -\frac{\dfrac{\pi^2\delta b}{a^2}}{\dfrac{ab}{2}} = -\frac{2\pi^2\delta}{a^3} \tag{16}$$

2. **Prob 8.13**

(a) Let the k-th perturbed solution ψ_k be a perturbation from the linear combination

$$\psi_k \approx \sum_i a_i \psi_0^{(i)} \tag{17}$$

For the finite conductivity case, ψ_k satisfies the equation and boundary condition

$$\left(\boldsymbol{\nabla}_t^2 + \gamma_k^2\right)\psi_k = 0 \qquad\qquad \psi_k\big|_S = f\sum_i a_i \frac{\partial \psi_0^{(i)}}{\partial n} \tag{18}$$

From the Green's identity

$$\int_A \left(\phi\,\boldsymbol{\nabla}_t^2\psi - \psi\,\boldsymbol{\nabla}_t^2\phi\right)da = \oint_C \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right)dl \tag{19}$$

we can identify ψ with ψ_k and ϕ with $\psi_0^{(j)*}$. Then (19) becomes

$$\left(\gamma_0^2 - \gamma_k^2\right)\int_A \psi_0^{(j)*}\psi_k da = \oint_C \psi_k \frac{\partial \psi_0^{(j)*}}{\partial n} dl \tag{20}$$

We can approximate ψ_k in the LHS integrand with (17), and with orthogonality of $\psi_0^{(i)}$s, we have

$$\text{LHS}_{(20)} = \left(\gamma_0^2 - \gamma_k^2\right)a_j\int_A \left|\psi_0^{(j)}\right|^2 da = \left(\gamma_0^2 - \gamma_k^2\right)N_j a_j \tag{21}$$

And the RHS of (20) is

$$\text{RHS}_{(20)} = f\sum_i a_i \oint_C \frac{\partial \psi_0^{(i)}}{\partial n}\frac{\partial \psi_0^{(j)*}}{\partial n} dl = \sum_i a_i \Delta_{ji} \tag{22}$$

Equating (21) with (22) gives the linear system

$$\sum_i \left[\left(\gamma_k^2 - \gamma_0^2\right)N_j\delta_{ji} + \Delta_{ji}\right]a_i = 0 \qquad\qquad \text{for } j = 1, 2, \cdots, N \tag{23}$$

γ_k^2 can be obtained from the eigenvalue of the matrix in (23), and a_is correspond to the components of the k-th eigenvector.

For the distortion case, following problem 8.12, we can change the boundary condition in (18) as

$$\text{TM}: \qquad \psi_k\big|_S = -\delta(x,y)\sum_i a_i \frac{\partial \psi_0^{(i)}}{\partial n} \tag{24}$$

$$\text{TE}: \qquad \left.\frac{\partial \psi_k}{\partial n}\right|_S = -\delta(x,y)\sum_i a_i \frac{\partial^2 \psi_0^{(i)}}{\partial n^2} \tag{25}$$

Then the RHS of (20) will become

$$\text{TM}: \qquad \text{RHS}_{(20)} = -\sum_i a_i \oint_C \delta(x,y)\frac{\partial \psi_0^{(i)}}{\partial n}\frac{\partial \psi_0^{(j)*}}{\partial n} dl \tag{26}$$

$$\text{TE}: \qquad \text{RHS}_{(20)} = \sum_i a_i \oint_C \delta(x,y)\psi_0^{(j)*}\frac{\partial^2 \psi_0^{(i)}}{\partial n^2} dl \tag{27}$$

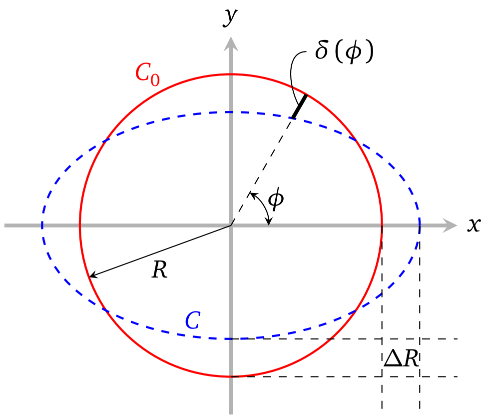
Thus by defining (**again, we have a minus sign compared to the claim from Jackson, see explanations above for part (a) of problem 8.12**)

$$\text{TM}: \qquad \Delta_{ji} = -\oint_C \delta(x,y)\frac{\partial \psi_0^{(i)}}{\partial n}\frac{\partial \psi_0^{(j)*}}{\partial n} dl \tag{28}$$

$$\text{TE}: \qquad \Delta_{ji} = \oint_C \delta(x,y)\psi_0^{(j)*}\frac{\partial^2 \psi_0^{(i)}}{\partial n^2} dl \tag{29}$$

the system of linear equations (23) still holds its form.

(b) The distortion of the circular cross section is depicted below.



We wish to find the displacement function $\delta(\phi)$, accurate to the first order of $\Delta R/R$. Given that $\delta(0) = -\Delta R$, and $\delta(\pi/2) = \Delta R$ (again, normal vector $\mathbf{n}$ is pointing inward), an obvious guess would be

$$\delta(\phi) = -\Delta R \cos 2\phi \tag{30}$$

To verify it, we test it against the ellipse up to $O(\Delta R/R)$.

$$\begin{aligned}\frac{x^2}{(R+\Delta R)^2} + \frac{y^2}{(R-\Delta R)^2} =& \frac{(R\cos\phi + \Delta R\cos 2\phi\cos\phi)^2}{R^2\left(1+\frac{\Delta R}{R}\right)^2} + \frac{(R\sin\phi + \Delta R\cos 2\phi\sin\phi)^2}{R^2\left(1-\frac{\Delta R}{R}\right)^2} \\ =& \left[\cos^2\phi + 2\left(\frac{\Delta R}{R}\right)\cos 2\phi\cos^2\phi\right]\left(1-\frac{2\Delta R}{R}\right) + \\ & \left[\sin^2\phi + 2\left(\frac{\Delta R}{R}\right)\cos 2\phi\sin^2\phi\right]\left(1+\frac{2\Delta R}{R}\right) + O\left[\left(\frac{\Delta R}{R}\right)^2\right] \\ =& 1 + O\left[\left(\frac{\Delta R}{R}\right)^2\right]\end{aligned} \tag{31}$$

With

$$\psi_0^{(\pm)} = J_1(\gamma_0\rho)e^{\pm i\phi} \qquad \gamma_0 R \approx 1.841 \qquad J_1'(\gamma_0 R) = 0 \tag{32}$$

we have

$$\left.\frac{\partial^2\psi_0^{(\pm)}}{\partial n^2}\right|_{C_0} = \gamma_0^2 J_1''(\gamma_0 R)e^{\pm i\phi} \tag{33}$$

Then by (29) it is clear that

$$\Delta_{++} = \Delta_{--} = 0 \tag{34}$$

$$\begin{aligned}K \equiv \Delta_{+-} = \Delta_{-+} &= -\gamma_0^2 J_1(\gamma_0 R)J_1''(\gamma_0 R)R\Delta R\int_0^{2\pi}\cos 2\phi\, e^{\pm i2\phi}\,d\phi \\ &= -\pi\gamma_0^2 J_1(\gamma_0 R)J_1''(\gamma_0 R)R\Delta R\end{aligned} \tag{35}$$

Also by invoking 10.22.38 of https://dlmf.nist.gov which was also proved in problem 3.11:

$$\int_0^1 J_\nu(\alpha_l t)J_\nu(\alpha_m t)\,t\,dt = \left(\frac{a^2}{b^2} + \alpha_l^2 - \nu^2\right)\frac{[J_\nu(\alpha_l)]^2}{2\alpha_l^2}\delta_{lm} \quad \text{for } \alpha_l, \alpha_m \text{ positive zeros of } aJ_\nu(x) + bxJ_\nu'(x) \tag{36}$$

we know

$$\begin{aligned}N \equiv N_\pm = \int_A \left|\psi_0^{(\pm)}\right|^2 da &= 2\pi\int_0^R [J_1(\gamma_0\rho)]^2\rho\,d\rho \\ &= 2\pi R^2\int_0^1\left[J_1\left(\gamma_0 Rt\right)\right]^2 t\,dt \\ &= \pi R^2[J_1(\gamma_0 R)]^2\left[1 - \frac{1}{(\gamma_0 R)^2}\right]\end{aligned} \tag{37}$$

Putting these back to (23), we obtain a system of linear equations

$$\begin{bmatrix} \left(\gamma^2-\gamma_0^2\right)N & K \\ K & \left(\gamma^2-\gamma_0^2\right)N \end{bmatrix}\begin{bmatrix} a_+ \\ a_- \end{bmatrix}=0 \tag{38}$$

which yields the eigenvalues

$$\gamma_\pm^2=\gamma_0^2\pm\frac{K}{N}=\gamma_0^2\left(1\pm\lambda\frac{\Delta R}{R}\right) \tag{39}$$

where

$$\lambda=\frac{1}{\gamma_0^2}\cdot\frac{R}{\Delta R}\cdot\left(\frac{K}{N}\right)=-\frac{J_1''(\gamma_0 R)(\gamma_0 R)^2}{J_1(\gamma_0 R)\left[(\gamma_0 R)^2-1\right]}=1 \tag{40}$$

Where the last step is a result of recognizing the Bessel equation

$$x^2 J_1''(x)+xJ_1'(x)+\left(x^2-1\right)=0 \tag{41}$$

with $x=\gamma_0 R$ being the zero of $J_1'(x)$.
By plugging the value of $\gamma_\pm^2$ back into (38), we can solve for the eigenvectors,

$$\begin{bmatrix} a_+ \\ a_- \end{bmatrix}=\begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} \tag{42}$$

with $\pm$ sign matching the eigenvalue $\gamma_\mp^2$.
Then the two modes that are lifted out of the degeneracy are

$$\psi_\pm(\rho,\phi)\propto J_1(\gamma_0\rho)\times\begin{cases} \sin\phi & \text{for } \gamma_+^2=\gamma_0^2\left(1+\dfrac{\Delta R}{R}\right) \\ \cos\phi & \text{for } \gamma_-^2=\gamma_0^2\left(1-\dfrac{\Delta R}{R}\right) \end{cases} \tag{43}$$