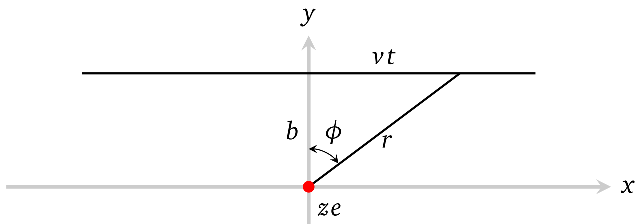
1. In the rest frame of the heavy particle, the electron is coming in with impact parameter b and velocity v. The change in momentum is given by the impulse

$$\Delta\mathbf{p} = \int_{-\infty}^{\infty} -\nabla V(r)\,dt = \int_{-\infty}^{\infty} -\nabla\left[\frac{ze^2 e^{-k_D r}}{r}\right]dt = ze^2\int_{-\infty}^{\infty}\frac{e^{-k_D r}}{r^2}(1+k_D r)\hat{\mathbf{r}}\,dt \tag{1}$$

Transforming back to the lab frame, we can obtain the energy loss

$$\Delta E = \frac{|\Delta\mathbf{p}|^2}{2m} \tag{2}$$



As shown in the diagram above, the longitudinal component (x) of the integral in (1) vanishes, giving

$$\Delta E = \frac{z^2e^4}{2m}I^2 \qquad \text{where} \qquad I = \int_{-\infty}^{\infty}\frac{e^{-k_D r}}{r^2}(1+k_D r)\cos\phi\,dt \tag{3}$$

For a fast velocity, the trajectory is approximated as a straight line (small deflection angle), so we have

$$r = \sqrt{b^2+v^2t^2} = b\sqrt{1+u^2} \qquad \text{where} \qquad u \equiv \frac{vt}{b} \qquad \cos\phi = \frac{b}{r} \tag{4}$$

The integral I becomes

$$\begin{aligned} I &= \int_{-\infty}^{\infty}\frac{e^{-k_D b\sqrt{1+u^2}}}{bv(1+u^2)^{3/2}}\left(1+k_D b\sqrt{1+u^2}\right)du \qquad \text{let } u=\sinh w \\ &= \frac{1}{bv}\int_{-\infty}^{\infty}\frac{e^{-k_D b\cosh w}}{\cosh^3 w}(1+k_D b\cosh w)\cosh w\,dw \\ &= \frac{1}{bv}\int_{-\infty}^{\infty}\frac{e^{-k_D b\cosh w}}{\cosh^2 w}(1+k_D b\cosh w)\,dw \end{aligned} \tag{5}$$

Let $\kappa \equiv k_D b$, then

$$\frac{d(bvI)}{d\kappa} = \int_{-\infty}^{\infty} e^{-\kappa\cosh w}\left[\frac{-\cosh w(1+\kappa\cosh w)+\cosh w}{\cosh^2 w}\right]dw = -\kappa\int_{-\infty}^{\infty}e^{-\kappa\cosh w}dw \tag{6}$$

Recall the integral representation of $K_\nu(\kappa)$, DLMF 10.32.E9

$$K_\nu(\kappa) = \int_0^{\infty} e^{-\kappa\cosh t}\cosh(\nu t)\,dt \tag{7}$$

and the recurrence relation DLMF 10.29.E2

$$(-1)^\nu K_\nu'(\kappa) = (-1)^{\nu-1}K_{\nu-1}(\kappa) - \frac{\nu}{\kappa}(-1)^\nu K_\nu(\kappa) \tag{8}$$

we see that

$$\frac{d(bvI)}{d\kappa} = -2\kappa K_0(\kappa) = 2\left[\kappa K_1'(\kappa)+K_1(\kappa)\right] = 2\frac{d[\kappa K_1(\kappa)]}{d\kappa} \qquad \Longrightarrow \qquad I = \frac{2}{bv}[\kappa K_1(\kappa)]+C \tag{9}$$

With DLMF 10.25.E3

$$K_\nu(\kappa) \to \sqrt{\frac{\pi}{2\kappa}}e^{-\kappa} \qquad \text{as } \kappa\to\infty \tag{10}$$

matching the behavior of I as $\kappa\to\infty$ establishes $C=0$. Putting I back to ΔE yields

$$\Delta E = \frac{2z^2e^4}{mv^2}k_D^2 K_1^2(k_D b) \tag{11}$$

2. The energy loss per unit distance can be obtained by the weighted integral over all possible impact parameters (see Jackson (13.7))

$$\frac{dE}{dx} = 2\pi NZ \int_{b_{\min}}^{\infty} \Delta E(b)\, b\, db = \frac{4\pi NZz^2e^4}{mv^2} \int_{b_{\min}}^{\infty} k_D^2 K_1^2(k_D b)\, b\, db = \frac{4\pi NZz^2e^4}{mv^2} \int_{\kappa_{\min}}^{\infty} K_1^2(\kappa)\, \kappa\, d\kappa \tag{12}$$

The integral can be looked up from integration tables (e.g., see 5.54.2 from [1])

$$\int \kappa K_1^2(\kappa)\, d\kappa = \underbrace{\frac{\kappa^2}{2}\left[K_1^2(\kappa) - K_0(\kappa) K_2(\kappa)\right]}_{F(\kappa)} + C \tag{13}$$

From the asymptotic form (10), we see that $F(\infty) = 0$. In small argument $\kappa \to 0$, we have (see Jackson (3.103))

$$K_1(\kappa) \to \frac{1}{\kappa} \qquad K_0(\kappa) \to -\left[\ln\left(\frac{\kappa}{2}\right) + \gamma\right] \qquad K_2(\kappa) \to \frac{2}{\kappa^2} \tag{14}$$

thus

$$\begin{aligned} F(\kappa) &\to \frac{\kappa^2}{2}\left[\frac{1}{\kappa^2} + \left[\ln\left(\frac{\kappa}{2}\right) + \gamma\right] \cdot \frac{2}{\kappa^2}\right] \\ &= \frac{1}{2} + \ln\left(\frac{\kappa}{2}\right) + \gamma \\ &= \ln\left(\frac{\kappa e^{\gamma+1/2}}{2}\right) \\ &\approx \ln(1.47\kappa) \end{aligned} \tag{15}$$

The energy loss per unit distance is then

$$\begin{aligned} \frac{dE}{dx} &\approx \frac{4\pi NZz^2e^4}{mv^2} \ln\left(\frac{1}{1.47 k_D b_{\min}}\right) \qquad \text{by (7.60) in Gaussian units } \omega_p^2 \equiv \frac{4\pi NZe^2}{m} \\ &= \frac{z^2e^2}{v^2}\omega_p^2 \ln\left(\frac{1}{1.47 k_D b_{\min}}\right) \end{aligned} \tag{16}$$

[1] **I. S. Gradshteyn and I. M. Ryzhik**, *Table of Integrals,Series and Products Eighth Edition.*