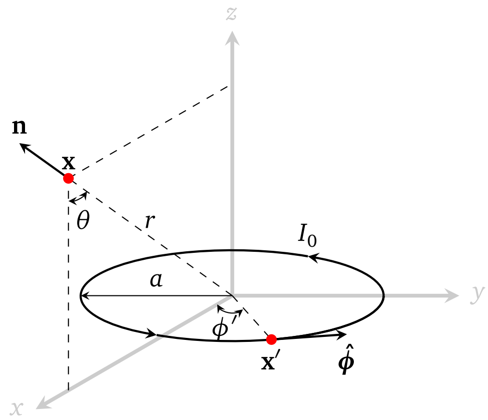

1. For the radiation zone $kr \gg 1$, we can use (9.8) to find the vector potential

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \mathbf{J}\left(\mathbf{x}'\right)e^{-ik\mathbf{n}\cdot\mathbf{x}'}d^3x' \tag{1}$$

We choose the coordinate frame so that the observation point $\mathbf{x}$ is at $(r\sin\theta, 0, r\cos\theta)$, hence $\mathbf{n} = \sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}$ and a point on the loop has the form $\mathbf{x}' = a\cos\phi'\hat{\mathbf{x}} + a\sin\phi'\hat{\mathbf{y}}$.

The integral in (1) becomes

$$\begin{aligned}\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int_0^{2\pi} d\phi'\hat{\boldsymbol{\phi}}I_0 e^{-ika\sin\theta\cos\phi'} \\ &= \frac{\mu_0 I_0}{4\pi}\frac{e^{ikr}}{r}\int_0^{2\pi} e^{-ika\sin\theta\cos\phi'}\left(-\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}}\right)d\phi' \end{aligned}\tag{2}$$

The $\hat{\mathbf{x}}$ component of the integral is obviously zero, the $\hat{\mathbf{y}}$ component can be obtained by recalling DLMF 10.9.E2

$$J_n(z) = \frac{i^{-n}}{\pi}\int_0^{\pi} e^{iz\cos\alpha}\cos(n\alpha)\,d\alpha \tag{3}$$

and letting $n = 1, z = -ka\sin\theta$, which gives

$$\mathbf{A}(\mathbf{x}) = \hat{\mathbf{y}}\frac{\mu_0 I_0}{4\pi}\frac{e^{ikr}}{r}\frac{2\pi}{i^{-1}}J_1(-ka\sin\theta) = -\hat{\mathbf{y}}\frac{i\mu_0 I_0}{2}\frac{e^{ikr}}{r}J_1(ka\sin\theta) \tag{4}$$

The radiation zone fields can be obtained from the vector potential by

$$\mathbf{H} = \frac{ik}{\mu_0}\mathbf{n}\times\mathbf{A} = \frac{kI_0}{2}\frac{e^{ikr}}{r}J_1(ka\sin\theta)\,\mathbf{n}\times\hat{\mathbf{y}} = \frac{kI_0}{2}\frac{e^{ikr}}{r}J_1(ka\sin\theta)(\sin\theta\hat{\mathbf{z}} - \cos\theta\hat{\mathbf{x}}) \tag{5}$$

$$\mathbf{E} = Z_0\mathbf{H}\times\mathbf{n} = \frac{Z_0 kI_0}{2}\frac{e^{ikr}}{r}J_1(ka\sin\theta)(\sin\theta\hat{\mathbf{z}} - \cos\theta\hat{\mathbf{x}})\times\mathbf{n} = \frac{Z_0 kI_0}{2}\frac{e^{ikr}}{r}J_1(ka\sin\theta)\hat{\mathbf{y}} \tag{6}$$

and the power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{1}{2}\mathrm{Re}\left[r^2\mathbf{n}\cdot(\mathbf{E}\times\mathbf{H}^*)\right] = \frac{Z_0 k^2 I_0^2}{4}\left[J_1(ka\sin\theta)\right]^2 \tag{7}$$

2. By (9.170), (9.172), the only feasible multipole moments are M_{lm}

$$M_{lm} = -\frac{1}{l+1}\int r^l Y_{lm}^*(\theta,\phi)\,\nabla\cdot(\mathbf{x}\times\mathbf{J})d^3x \tag{8}$$

where $\mathbf{J}$ is most conveniently expressed in spherical coordinates (see 5.33)

$$\mathbf{J}(r,\theta,\phi) = I_0\frac{\delta(r-a)}{a}\sin\theta\,\delta(\cos\theta)\,\hat{\boldsymbol{\phi}} = I_0\frac{\delta(r-a)}{a}\delta\left(\theta - \frac{\pi}{2}\right)\hat{\boldsymbol{\phi}} \tag{9}$$

thus

$$
\begin{aligned}
\mathbf{x}\times\mathbf{J} &= -rI_0\frac{\delta(r-a)}{a}\delta\left(\theta-\frac{\pi}{2}\right)\hat{\boldsymbol{\theta}} && \Longrightarrow \\
\nabla\cdot(\mathbf{x}\times\mathbf{J}) &= -rI_0\frac{\delta(r-a)}{a}\cdot\frac{1}{r\sin\theta}\frac{d}{d\theta}\left[\delta\left(\theta-\frac{\pi}{2}\right)\sin\theta\right] \\
&= -I_0\frac{\delta(r-a)}{a}\left[\delta'\left(\theta-\frac{\pi}{2}\right)+\delta\left(\theta-\frac{\pi}{2}\right)\cot\theta\right] \qquad (10)
\end{aligned}
$$

giving

$$
\begin{aligned}
M_{lm} &= \frac{I_0}{l+1}\int_0^\infty r^l\frac{\delta(r-a)}{a}r^2dr\int Y_{lm}^*(\theta,\phi)\left[\delta'\left(\theta-\frac{\pi}{2}\right)+\overbrace{\delta\left(\theta-\frac{\pi}{2}\right)\cot\theta}^{\text{no effect on integral}}\right]d\Omega \\
&= \frac{I_0a^{l+1}}{l+1}\delta_{m0}\cdot 2\pi\sqrt{\frac{2l+1}{4\pi}}\int_0^\pi P_l(\cos\theta)\,\delta'\left(\theta-\frac{\pi}{2}\right)\sin\theta\,d\theta \\
&= \delta_{m0}\frac{2\pi I_0a^{l+1}}{l+1}\sqrt{\frac{2l+1}{4\pi}}\left\{-\frac{d\left[P_l(\cos\theta)\sin\theta\right]}{d\theta}\right\}_{\theta=\pi/2} \\
&= \delta_{m0}\frac{2\pi I_0a^{l+1}}{l+1}\sqrt{\frac{2l+1}{4\pi}}P_l'(0) \qquad (11)
\end{aligned}
$$

The only non-vanishing multipole moments are M_{l0} where l is odd, the lowest of which is the dipole $l=1$,

$$
M_{10}=\sqrt{\frac{3}{4\pi}}\cdot I_0\pi a^2 \qquad (12)
$$

where $\mathbf{m}=\hat{\mathbf{z}}I_0\pi a^2$ is the dipole moment of the current loop in Cartesian tensor form, and the factor $\sqrt{3/4\pi}$ is required by the spherical tensor normalization (see 4.5).