1. From

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \oint_C d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{1}$$

and with the obvious choice of coordinate system, for the point $\mathbf{x}$, on the axis, we have

$$\mathbf{x} = (0,0,z) \qquad \mathbf{x}' = \left(a\cos\phi', a\sin\phi', 0\right) \qquad d\mathbf{l}' = \left(-a\sin\phi', a\cos\phi', 0\right) d\phi' \tag{2}$$

Thus

$$\mathbf{B}(0,z) = \frac{\mu_0 I}{4\pi} \cdot 2\pi \frac{a^2}{\sqrt{a^2+z^2}^3} \hat{\mathbf{z}} = \frac{\mu_0 I a^2}{2} \cdot \frac{1}{\sqrt{a^2+z^2}^3} \hat{\mathbf{z}} \tag{3}$$

2. With the two loops and shift of origin, we get

$$\mathbf{B}(0,z) = \frac{\mu_0 I a^2}{2} \left[\frac{1}{\sqrt{a^2+(b/2-z)^2}^3} + \frac{1}{\sqrt{a^2+(b/2+z)^2}^3} \right] \hat{\mathbf{z}} \tag{4}$$

The Taylor expansion

$$(1+x)^{-3/2} = 1 - \frac{3x}{2} + \frac{15x^2}{8} - \frac{35x^3}{16} + \frac{315x^4}{128} + \cdots \tag{5}$$

gives (recall $d^2 = a^2 + b^2/4$)

$$\begin{aligned}
\frac{1}{\sqrt{a^2+(b/2\pm z)^2}^3} &= \frac{1}{d^3}\left(1 \pm \frac{bz}{d^2} + \frac{z^2}{d^2}\right)^{-3/2} \\
&= \frac{1}{d^3}\left[1 - \frac{3}{2}\left(\pm\frac{bz}{d^2} + \frac{z^2}{d^2}\right) + \frac{15}{8}\left(\pm\frac{bz}{d^2} + \frac{z^2}{d^2}\right)^2 \right. \\
&\qquad \left. - \frac{35}{16}\left(\pm\frac{bz}{d^2} + \frac{z^2}{d^2}\right)^3 + \frac{315}{128}\left(\pm\frac{bz}{d^2} + \frac{z^2}{d^2}\right)^4 + \cdots \right] \\
&= \frac{1}{d^3}\left[1 - \frac{3}{2}\left(\pm\frac{bz}{d^2} + \frac{z^2}{d^2}\right) + \frac{15}{8}\left(\frac{b^2z^2}{d^4} \pm 2\frac{bz^3}{d^4} + \frac{z^4}{d^4}\right) \right. \\
&\qquad \left. - \frac{35}{16}\left(\pm\frac{b^3z^3}{d^6} + 3\frac{b^2z^4}{d^6}\right) + \frac{315}{128}\frac{b^4z^4}{d^8} + O\left(z^5\right) \right]
\end{aligned} \tag{6}$$

Notice when we plug (6) into (4), the ± terms cancel and the remaining terms double. This gives

$$\mathbf{B}(0,z) = \hat{\mathbf{z}}\frac{\mu_0 I a^2}{d^3}\left[1 + z^2\left(\frac{15b^2}{8d^4} - \frac{3}{2d^2}\right) + z^4\left(\frac{15}{8d^4} - \frac{105b^2}{16d^6} + \frac{315b^4}{128d^8}\right) + O\left(z^5\right)\right] \tag{7}$$

The claim is proved by noticing

$$\frac{15b^2}{8d^4} - \frac{3}{2d^2} = \frac{15b^2 - 12d^2}{8d^4} = \frac{15b^2 - 12\left(a^2+b^2/4\right)}{8d^4} = \frac{3\left(b^2-a^2\right)}{2d^4} \tag{8}$$

$$\begin{aligned}
\frac{15}{8d^4} - \frac{105b^2}{16d^6} + \frac{315b^4}{128d^8} &= \frac{240d^4 - 840b^2d^2 + 315b^4}{128d^8} \\
&= \frac{240\left(a^4 + a^2b^2/2 + b^4/16\right) - 840\left(b^2a^2 + b^4/4\right) + 315b^4}{128d^8} \\
&= \frac{240a^4 - 720a^2b^2 + 120b^4}{128d^8} = \frac{15\left(b^4 - 6a^2b^2 + 2a^4\right)}{16d^8}
\end{aligned} \tag{9}$$

3. To calculate field for small ρ, we will use the result from problem 5.4:

$$B_z(\rho,z) \approx B_z(0,z) - \frac{\rho^2}{4}\left[\frac{\partial^2 B_z(0,z)}{\partial z^2}\right] \tag{10}$$

$$B_\rho(\rho,z) \approx -\frac{\rho}{2}\left[\frac{\partial B_z(0,z)}{\partial z}\right] \tag{11}$$

Observe that

$$\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{a^2+(b/2\pm z)^2}^3}\right) = \frac{\left(-\frac{3}{2}\right)(\pm 2)(b/2\pm z)}{\sqrt{a^2+(b/2\pm z)^2}^5} = \frac{\mp 3(b/2\pm z)}{\sqrt{a^2+(b/2\pm z)^2}^5} \tag{12}$$

Using Taylor expansion

$$(1+x)^{-5/2} = 1 - \frac{5x}{2} + \frac{35x^2}{8} + \cdots \tag{13}$$

we get (12) up to the z^2 order:

$$\begin{aligned}
\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{a^2+(b/2\pm z)^2}^3}\right) &= \mp 3(b/2\pm z)\cdot\frac{1}{d^5}\left[1-\frac{5}{2}\left(\pm\frac{bz}{d^2}+\frac{z^2}{d^2}\right)+\frac{35}{8}\frac{b^2z^2}{d^4}+O\left(z^3\right)\right] \\
&\approx \frac{1}{d^5}\left(\mp\frac{3b}{2}-3z\right)\left[1\mp\frac{5bz}{2d^2}+z^2\left(\frac{35b^2}{8d^4}-\frac{5}{2d^2}\right)\right] \\
&\approx \frac{1}{d^5}\left\{\mp\frac{3b}{2}+z\left(\frac{15b^2}{4d^2}-3\right)\pm z^2\left[\frac{15b}{2d^2}-\frac{3b}{2}\left(\frac{35b^2}{8d^4}-\frac{5}{2d^2}\right)\right]\right\}
\end{aligned} \tag{14}$$

When (14) is inserted into (11), the $\pm$ terms will cancel, which leaves

$$\begin{aligned}
B_\rho(\rho,z) &= \left(-\frac{\rho}{2}\right)\cdot\left(\frac{\mu_0 I a^2}{2}\right)\frac{2}{d^5}\left(\frac{15b^2}{4d^2}-3\right)z+O\left(z^3\right) \\
&\approx -\rho z\cdot\left(\frac{\mu_0 I a^2}{2}\right)\cdot\left(\frac{15b^2-12d^2}{4d^7}\right) \\
&= -\rho z\underbrace{\left(\frac{\mu_0 I a^2}{2}\right)\left[\frac{3\left(b^2-a^2\right)}{d^7}\right]}_{\equiv\sigma_2}
\end{aligned} \tag{15}$$

From (12), we have

$$\begin{aligned}
\frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{a^2+(b/2\pm z)^2}^3}\right) &= \frac{-3}{\sqrt{a^2+(b/2\pm z)^2}^5} \mp \frac{3(b/2\pm z)\left(-\frac{5}{2}\right)(\pm 2)(b/2\pm z)}{\sqrt{a^2+(b/2\pm z)^2}^7} \\
&= \frac{-3\left[a^2+(b/2\pm z)^2\right]+15(b/2\pm z)^2}{\sqrt{a^2+(b/2\pm z)^2}^7} \\
&= \frac{12(b/2\pm z)^2-3a^2}{\sqrt{a^2+(b/2\pm z)^2}^7} \\
&= \frac{3\left(b^2-a^2\right)}{d^7}+O(z)
\end{aligned} \tag{16}$$

When (16) is inserted back into (10), we see the $\rho^2 z^0$ term has coefficient

$$-\frac{1}{4}\cdot 2\cdot\left[\frac{3\left(b^2-a^2\right)}{d^7}\right]\cdot\left(\frac{\mu_0 I a^2}{2}\right) = -\frac{\sigma_2}{2} \tag{17}$$

The $z^2\rho^0$ and $z^0\rho^0$ terms of (10) both come from $B_z(0,z)$, which are available from (7) and (8)

$$\rho^0 z^0: \qquad \frac{\mu_0 I a^2}{d^3} = \sigma_0 \tag{18}$$

$$z^2 \rho^0: \qquad \left(\frac{\mu_0 I a^2}{2}\right)\cdot\left[\frac{3\left(b^2-a^2\right)}{d^7}\right] = \sigma_2 \tag{19}$$

4. For large $|z|$, we will expand (4) alternatively:

$$\begin{aligned}\frac{1}{\sqrt{a^2+(b/2\pm z)^2}^3} &= \frac{1}{|z|^3}\left(1\pm\frac{b}{z}+\frac{d^2}{z^2}\right)^{-3/2}\\ &= \frac{1}{|z|^3}\left[1-\frac{3}{2}\left(\pm\frac{b}{z}+\frac{d^2}{z^2}\right)+\frac{15}{8}\left(\pm\frac{b}{z}+\frac{d^2}{z^2}\right)^2\right.\\ &\qquad\left.-\frac{35}{16}\left(\pm\frac{b}{z}+\frac{d^2}{z^2}\right)^3+\frac{315}{128}\left(\pm\frac{b}{z}+\frac{d^2}{z^2}\right)^4+\cdots\right]\\ &= \frac{1}{|z|^3}\left[1-\frac{3}{2}\left(\pm\frac{b}{z}+\frac{d^2}{z^2}\right)+\frac{15}{8}\left(\frac{b^2}{z^2}\pm 2\frac{bd^2}{z^3}+\frac{d^4}{z^4}\right)\right.\\ &\qquad\left.-\frac{35}{16}\left(\pm\frac{b^3}{z^3}+3\frac{b^2d^2}{z^4}\right)+\frac{315}{128}\frac{b^4}{z^4}+O\left(\frac{1}{z^5}\right)\right]\end{aligned}\tag{20}$$

Similarly collecting the non-canceling terms in (4) gives

$$\begin{aligned}B_z(0,z) &\approx \frac{\mu_0 I a^2}{|z|^3}\left[1+z^{-2}\left(\frac{15b^2}{8}-\frac{3d^2}{2}\right)+z^{-4}\left(\frac{15d^4}{8}-\frac{105b^2d^2}{16}+\frac{315b^4}{128}\right)\right]\\ &= \frac{\mu_0 I a^2}{|z|^3}\left[1+\frac{3\left(b^2-a^2\right)}{2z^2}+\frac{15\left(b^4-6a^2b^2+2a^4\right)}{16z^4}\right]\end{aligned}\tag{21}$$

which is exactly result (b), i.e., equation (7)-(9) with the substitution $d\to|z|$.

5. When $a=b$, the ratio of deviation from field at origin in (9) is

$$\epsilon = \frac{15\left(-3b^4z^4\right)}{16\left(d^2\right)^4} = \frac{-45b^4z^4}{16\cdot\left(\frac{5b^2}{4}\right)^4} = \left(\frac{z}{b}\right)^4\cdot\left[\frac{-45}{16\left(\frac{5}{4}\right)^4}\right]\approx -1.152\left(\frac{z}{b}\right)^4\tag{22}$$

Thus

$$\begin{aligned}&\text{for } |\epsilon|\le 10^{-4} && \Longrightarrow && \frac{z}{b}\le 0.096\\ &\text{for } |\epsilon|\le 10^{-2} && \Longrightarrow && \frac{z}{b}\le 0.305\end{aligned}\tag{23}$$