Here we provide the missing steps leading to the spin equation of motion (11.166). In particular, we should not assume that $d\gamma/d\tau = 0$, which will not be true if the particle is accelerating due to electric field, or gradient of magnetic field.

First, let's show (11.165)

$$S_\lambda \frac{dU^\lambda}{d\tau} = -\gamma \mathbf{S} \cdot \frac{d\mathbf{v}}{d\tau} \tag{1}$$

Indeed, separating the sum on the LSH by time and space components, we have

$$S_\lambda \frac{dU^\lambda}{d\tau} = S_0 \frac{d(\gamma c)}{d\tau} - \mathbf{S} \cdot \frac{d(\gamma \mathbf{v})}{d\tau} = \frac{d\gamma}{d\tau} \overbrace{(cS_0 - \mathbf{S} \cdot \mathbf{v})}^{=0 \text{ by } (11.157)} - \gamma \mathbf{S} \cdot \frac{d\mathbf{v}}{d\tau} = -\gamma \mathbf{S} \cdot \frac{d\mathbf{v}}{d\tau} \tag{2}$$

Recall the general covariant equation of motion (11.162)

$$\frac{dS^\alpha}{d\tau} = \overbrace{\frac{ge}{2mc}\left[F^{\alpha\beta} S_\beta + \frac{1}{c^2} U^\alpha \left(S_\lambda F^{\lambda\mu} U_\mu\right)\right]}^{\equiv F^\alpha} - \frac{1}{c^2} U^\alpha \left(S_\lambda \frac{dU^\lambda}{d\tau}\right) \tag{3}$$

where we have denoted the first term as F^α following Jackson (which is unfortunately conflicting with symbol for the field strength tensor, but it should be clear from the context).

Using (2), the space and time representation of (3) can be written separately as

$$\frac{d\mathbf{S}}{d\tau} = \mathbf{F} + \gamma^2 \boldsymbol{\beta} \left(\mathbf{S} \cdot \frac{d\boldsymbol{\beta}}{d\tau}\right) \tag{4}$$

$$\frac{dS_0}{d\tau} = F_0 + \gamma^2 \left(\mathbf{S} \cdot \frac{d\boldsymbol{\beta}}{d\tau}\right) \tag{5}$$

Taking the proper-time derivative of (11.158) while not assuming $d\gamma/d\tau = 0$ yields

$$\begin{aligned}
\frac{d\mathbf{s}}{d\tau} &= \frac{d\mathbf{S}}{d\tau} - \overbrace{\frac{d}{d\tau}\left(\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta} \cdot \mathbf{S})\boldsymbol{\beta}}^{A} - \left(\frac{\gamma}{\gamma+1}\right)\frac{d(\boldsymbol{\beta} \cdot \mathbf{S})}{d\tau}\boldsymbol{\beta} - \left(\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta} \cdot \mathbf{S})\frac{d\boldsymbol{\beta}}{d\tau} \qquad \text{use (4) and (11.157)} \\
&= \mathbf{F} + \gamma^2 \boldsymbol{\beta}\left(\mathbf{S} \cdot \frac{d\boldsymbol{\beta}}{d\tau}\right) - A - \left(\frac{\gamma}{\gamma+1}\right)\left[F_0 + \gamma^2\left(\mathbf{S} \cdot \frac{d\boldsymbol{\beta}}{d\tau}\right)\right]\boldsymbol{\beta} - \left(\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta} \cdot \mathbf{S})\frac{d\boldsymbol{\beta}}{d\tau} \\
&= \mathbf{F} - \left(\frac{\gamma}{\gamma+1}\right)F_0\boldsymbol{\beta} + \underbrace{\left(\frac{\gamma^2}{\gamma+1}\right)\left(\mathbf{S} \cdot \frac{d\boldsymbol{\beta}}{d\tau}\right)\boldsymbol{\beta}}_{B} - A - \underbrace{\left(\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta} \cdot \mathbf{S})\frac{d\boldsymbol{\beta}}{d\tau}}_{C}
\end{aligned} \tag{6}$$

From (11.157) and (11.159), we see that

$$\boldsymbol{\beta} \cdot \mathbf{S} = \gamma(\boldsymbol{\beta} \cdot \mathbf{s}) \tag{7}$$

Dotting (11.158) with $d\boldsymbol{\beta}/d\tau$ gives

$$\mathbf{S} \cdot \frac{d\boldsymbol{\beta}}{d\tau} = \mathbf{s} \cdot \frac{d\boldsymbol{\beta}}{d\tau} + \left(\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta} \cdot \mathbf{S})\left(\boldsymbol{\beta} \cdot \frac{d\boldsymbol{\beta}}{d\tau}\right) \tag{8}$$

(7) and (8) enable us to write

$$\begin{aligned}
B - A + C &= \left(\frac{\gamma^2}{\gamma+1}\right)\left[\left(\mathbf{s} \cdot \frac{d\boldsymbol{\beta}}{d\tau}\right)\boldsymbol{\beta} - (\boldsymbol{\beta} \cdot \mathbf{s})\frac{d\boldsymbol{\beta}}{d\tau}\right] + \overbrace{\left[\left(\frac{\gamma^2}{\gamma+1}\right)\left(\frac{\gamma}{\gamma+1}\right)\frac{1}{2}\frac{d\beta^2}{d\tau} - \frac{d}{d\tau}\left(\frac{\gamma}{\gamma+1}\right)\right]}^{0}(\boldsymbol{\beta} \cdot \mathbf{S})\boldsymbol{\beta} \\
&= \left(\frac{\gamma^2}{\gamma+1}\right)\left[\mathbf{s} \times \left(\boldsymbol{\beta} \times \frac{d\boldsymbol{\beta}}{d\tau}\right)\right]
\end{aligned} \tag{9}$$

Putting (9) into (6) gives (11.166)

$$\frac{d\mathbf{s}}{d\tau} = \mathbf{F} - \left(\frac{\gamma}{\gamma+1}\right)F_0\boldsymbol{\beta} + \left(\frac{\gamma^2}{\gamma+1}\right)\left[\mathbf{s} \times \left(\boldsymbol{\beta} \times \frac{d\boldsymbol{\beta}}{d\tau}\right)\right] \tag{10}$$