1. Let incident wave's polarization be $\boldsymbol{\epsilon}_\pm = \boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2$, the multipole expansion of the incident wave is given in (10.55)

$$\mathbf{E}_{\text{inc},\pm} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left\{ j_l(k_0 r)\mathbf{X}_{l,\pm1} \pm \frac{1}{k_0}\nabla\times\left[j_l(k_0 r)\mathbf{X}_{l,\pm1}\right]\right\} \tag{1}$$

$$\mathbf{H}_{\text{inc},\pm} = \sqrt{\frac{\epsilon_0}{\mu_0}} \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left\{ -\frac{i}{k_0}\nabla\times\left[j_l(k_0 r)\mathbf{X}_{l,\pm1}\right] \mp i j_l(k_0 r)\mathbf{X}_{l,\pm1}\right\} \tag{2}$$

where $k_0 = \omega\sqrt{\epsilon_0\mu_0}$ is the wave number in free space outside the dielectric sphere.

From (9.122), the multipole expansion of the scattered and internal wave also have similar forms (where we have inserted a convenience factor $i^l\sqrt{4\pi(2l+1)}$)

$$\mathbf{E}_{\text{sc},\pm} = \sum_{l,m} i^l \sqrt{4\pi(2l+1)} \left\{ a_{lm} h_l^{(1)}(k_0 r)\mathbf{X}_{lm} + \frac{i}{k_0} b_{lm}\nabla\times\left[h_l^{(1)}(k_0 r)\mathbf{X}_{lm}\right]\right\} \tag{3}$$

$$\mathbf{H}_{\text{sc},\pm} = \sqrt{\frac{\epsilon_0}{\mu_0}} \sum_{l,m} i^l \sqrt{4\pi(2l+1)} \left\{ -\frac{i}{k_0} a_{lm}\nabla\times\left[h_l^{(1)}(k_0 r)\mathbf{X}_{lm}\right] + b_{lm} h_l^{(1)}(k_0 r)\mathbf{X}_{lm}\right\} \tag{4}$$

$$\mathbf{E}_{\text{int},\pm} = \sum_{l,m} i^l \sqrt{4\pi(2l+1)} \left\{ c_{lm} j_l(nk_0 r)\mathbf{X}_{lm} + \frac{i}{nk_0} d_{lm}\nabla\times\left[j_l(nk_0 r)\mathbf{X}_{lm}\right]\right\} \tag{5}$$

$$\mathbf{H}_{\text{int},\pm} = \sqrt{\frac{\epsilon}{\mu}} \sum_{l,m} i^l \sqrt{4\pi(2l+1)} \left\{ -\frac{i}{nk_0} c_{lm}\nabla\times\left[j_l(nk_0 r)\mathbf{X}_{lm}\right] + d_{lm} j_l(nk_0 r)\mathbf{X}_{lm}\right\} \tag{6}$$

where we have determined the radial function for the scattered wave to be $h_l^{(1)}(k_0 r)$ in anticipation of its asymptotic behavior $e^{ik_0 r}/r$ as $r\to\infty$, as well as the radial function for the internal wave to be $j_l(nk_0 r)$ since the corresponding Helmholtz equation is with respect to the wave number nk_0 inside the media, with $n = \sqrt{\epsilon_r\mu_r}$ being the refractive index of the medium.

Recall that

$$\mathbf{X}_{lm} = \frac{1}{i\sqrt{l(l+1)}}\mathbf{\Phi}_{lm} \qquad \nabla\times\left[f(r)\mathbf{\Phi}_{lm}\right] = -\frac{l(l+1)}{r} f\mathbf{Y}_{lm} - \frac{1}{r}\frac{d(rf)}{dr}\mathbf{\Psi}_{lm} \tag{7}$$

where $\mathbf{\Phi}_{lm}, \mathbf{\Psi}_{lm}$ are transverse and $\mathbf{Y}_{lm}$ is radial, and they are orthogonal functions over the solid angles.

The boundary condition at the interface requires normal $\mathbf{B}, \mathbf{D}$ to be continuous, and tangential $\mathbf{E}, \mathbf{H}$ to be continuous. Let's consider tangential requirement for now. At $r = R$, due to the orthonality of VSH, the coefficients for the tangential component $\mathbf{\Phi}_{lm}$ and $\mathbf{\Psi}_{lm}$ must match between $\mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{sc}}/\mathbf{E}_{\text{int}}$ as well as between $\mathbf{H}_{\text{inc}} + \mathbf{H}_{\text{sc}}/\mathbf{H}_{\text{int}}$.

In other words,

$$\mathbf{\Phi}_{lm} \text{ for } \mathbf{E}_{\tan}: \qquad \delta_{m,\pm1} j_l(k_0 R) + a_{lm} h_l^{(1)}(k_0 R) = c_{lm} j_l(nk_0 R) \tag{8}$$

$$\mathbf{\Psi}_{lm} \text{ for } \mathbf{E}_{\tan}: \qquad \pm\delta_{m,\pm1} \left.\frac{d\left[r j_l(k_0 r)\right]}{dr}\right|_{r=R} + i b_{lm} \left.\frac{d\left[r h_l^{(1)}(k_0 r)\right]}{dr}\right|_{r=R} = \frac{i d_{lm}}{n} \left.\frac{d\left[r j_l(nk_0 r)\right]}{dr}\right|_{r=R} \tag{9}$$

$$\mathbf{\Phi}_{lm} \text{ for } \mathbf{H}_{\tan}: \qquad \sqrt{\frac{\epsilon_0}{\mu_0}}\left[\mp i\delta_{m,\pm1} j_l(k_0 R) + b_{lm} h_l^{(1)}(k_0 R)\right] = \sqrt{\frac{\epsilon}{\mu}} d_{lm} j_l(nk_0 R) \tag{10}$$

$$\mathbf{\Psi}_{lm} \text{ for } \mathbf{H}_{\tan}: \qquad \sqrt{\frac{\epsilon_0}{\mu_0}}\left\{ \delta_{m,\pm1} \left.\frac{d\left[r j_l(k_0 r)\right]}{dr}\right|_{r=R} + a_{lm} \left.\frac{d\left[r h_l^{(1)}(k_0 r)\right]}{dr}\right|_{r=R}\right\} = \sqrt{\frac{\epsilon}{\mu}}\frac{c_{lm}}{n} \left.\frac{d\left[r j_l(nk_0 r)\right]}{dr}\right|_{r=R} \tag{11}$$

For $m \neq \pm1$, the unknowns $a_{lm}, b_{lm}, c_{lm}, d_{lm}$ satisfy a homogeneous system of equations, which has only trivial solution for general values of ϵ, μ, R. We see the conclusion, that the scattered and internal wave only have m component matching that of the incident wave, is a consequence of the boundary condition about tangential $\mathbf{E}, \mathbf{H}$ continuity.

With the simplifying constants

$$J_l \equiv j_l(k_0 R) \qquad H_l \equiv h_l^{(1)}(k_0 R) \qquad N_l \equiv j_l(nk_0 R) \tag{12}$$

$$J_l' \equiv \left.\frac{d\left[k_0 r j_l(k_0 r)\right]}{d(k_0 r)}\right|_{r=R} \qquad H_l' \equiv \left.\frac{d\left[k_0 r h_l^{(1)}(k_0 r)\right]}{d(k_0 r)}\right|_{r=R} \qquad N_l' \equiv \left.\frac{d\left[nk_0 r j_l(nk_0 r)\right]}{d(nk_0 r)}\right|_{r=R} \tag{13}$$

the inhomogeneous system of equations for $m = \pm 1$ can be written as

$$J_l + a_{l,\pm 1} H_l = c_{l,\pm 1} N_l \tag{14}$$

$$\pm J_l' + i b_{l,\pm 1} H_l' = \frac{i d_{l,\pm 1}}{n} N_l' \tag{15}$$

$$\mp i J_l + b_{l,\pm 1} H_l = \frac{n \mu_0}{\mu} d_{l,\pm 1} N_l \tag{16}$$

$$J_l' + a_{l,\pm 1} H_l' = \frac{\mu_0}{\mu} c_{l,\pm 1} N_l' \tag{17}$$

for which the solutions are

$$a_{l,\pm 1} = -\frac{\mu_0 J_l N_l' - \mu J_l' N_l}{\mu_0 H_l N_l' - \mu H_l' N_l} = -\frac{J_l N_l' - \mu_r J_l' N_l}{H_l N_l' - \mu_r H_l' N_l} \tag{18}$$

$$b_{l,\pm 1} = \pm \frac{i\left(\mu J_l N_l' - n^2 \mu_0 J_l' N_l\right)}{\mu H_l N_l' - n^2 \mu_0 H_l' N_l} = \pm \frac{i\left(J_l N_l' - \epsilon_r J_l' N_l\right)}{H_l N_l' - \epsilon_r H_l' N_l} \tag{19}$$

$$c_{l,\pm 1} = -\frac{\mu\left(J_l H_l' - J_l' H_l\right)}{\mu_0 H_l N_l' - \mu H_l' N_l} = -\frac{\mu_r\left(J_l H_l' - J_l' H_l\right)}{H_l N_l' - \mu_r H_l' N_l} \tag{20}$$

$$d_{l,\pm 1} = \pm \frac{i n \mu\left(J_l H_l' - J_l' H_l\right)}{\mu H_l N_l' - n^2 \mu_0 H_l' N_l} = \pm \frac{i\sqrt{\epsilon_r \mu_r}\left(J_l H_l' - J_l' H_l\right)}{H_l N_l' - \epsilon_r H_l' N_l} \tag{21}$$

With this solution, we find that the normal $\mathbf{B}, \mathbf{D}$ continuity constraints, i.e.,

$$\mathbf{Y}_{lm} \text{ for } \mathbf{D}_{\text{norm}}: \qquad \epsilon_0\left(\pm \delta_{m,\pm 1} J_l + i b_{lm} H_l\right) = \epsilon \frac{i d_{lm}}{n} N_l \tag{22}$$

$$\mathbf{Y}_{lm} \text{ for } \mathbf{B}_{\text{norm}}: \qquad \mu_0 \sqrt{\frac{\epsilon_0}{\mu_0}}\left(\delta_{m,\pm 1} J_l + a_{lm} H_l\right) = \mu \sqrt{\frac{\epsilon}{\mu}} \frac{c_{lm}}{n} N_l \tag{23}$$

are automatically satisfied (e.g., for $m = \pm 1$, (22), (23) are equivalent to (16), (14) respectively). The seemingly redundant normal constraints should be no surprise at all, since the general form of multipole expansion (3) – (6) was derived using the divergence equations.

To analyze the phase shift, we can rewite the scattered wave in form of (10.57)

$$\mathbf{E}_{\text{sc},\pm} = \frac{1}{2} \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left\{ \alpha_{l,\pm 1} h_l^{(1)}(k_0 r) \mathbf{X}_{l,\pm 1} \pm \frac{\beta_{l,\pm 1}}{k_0} \boldsymbol{\nabla} \times \left[h_l^{(1)}(k_0 r) \mathbf{X}_{l,\pm 1} \right] \right\} \tag{24}$$

Comparing (24) with (3) yields

$$\alpha_{l,\pm 1} = 2 a_{l,\pm 1} = -\frac{2\left(J_l N_l' - \mu_r J_l' N_l\right)}{H_l N_l' - \mu_r H_l' N_l} \qquad \beta_{l,\pm 1} = \pm 2 i b_{l,\pm 1} = -\frac{2\left(J_l N_l' - \epsilon_r J_l' N_l\right)}{H_l N_l' - \epsilon_r H_l' N_l} \tag{25}$$

With the relation

$$2 J_l = H_l + H_l^* \qquad 2 J_l' = H_l' + H_l'^* \tag{26}$$

we see that both of

$$\alpha_{l,\pm 1} + 1 = -\frac{H_l^* N_l' - \mu_r H_l'^* N_l}{H_l N_l' - \mu_r H_l' N_l} \qquad \beta_{l,\pm 1} + 1 = -\frac{H_l^* N_l' - \epsilon_r H_l'^* N_l}{H_l N_l' - \epsilon_r H_l' N_l} \tag{27}$$

have modulus unity (a complex number divided by its complex conjugate).

Letting

$$\alpha_{l,\pm 1} + 1 = e^{i 2 \delta_{l,\pm 1}} \qquad \beta_{l,\pm 1} + 1 = e^{i 2 \delta_{l,\pm 1}'} \tag{28}$$

we find the phase shifts

$$\delta_{l,\pm 1} = \tan^{-1}\left(\frac{J_l N_l' - \mu_r J_l' N_l}{Y_l N_l' - \mu_r Y_l' N_l}\right) \qquad \delta_{l,\pm 1}' = \tan^{-1}\left(\frac{J_l N_l' - \epsilon_r J_l' N_l}{Y_l N_l' - \epsilon_r Y_l' N_l}\right) \tag{29}$$

where Y_l, Y_l' are the imaginary part of H_l, H_l' respectively.

2. Recall the differential and total scattering cross section (10.63), (10.61)

$$\frac{d\sigma_{\text{sc},\pm}}{d\Omega} = \frac{\pi}{2k_0^2}\left|\sum_{l=1}^{\infty}\sqrt{2l+1}\left(\alpha_{l,\pm1}\mathbf{X}_{l,\pm1} \pm i\beta_{l,\pm1}\mathbf{n}\times\mathbf{X}_{l,\pm1}\right)\right|^2 \tag{30}$$

$$\sigma_{\text{sc},\pm} = \frac{\pi}{2k_0^2}\sum_{l=1}^{\infty}(2l+1)\left(\left|\alpha_{l,\pm1}\right|^2 + \left|\beta_{l,\pm1}\right|^2\right) \tag{31}$$

For $k_0R \ll 1$, we expect $l=1$ terms to dominate. Let's verify this by using the asymptotic form of spherical Bessel functions with small argument (9.88)

$$j_l(x) = \frac{x^l}{(2l+1)!!} + O\left(x^{l+2}\right) \qquad\qquad y_l(x) = -\frac{(2l-1)!!}{x^{l+1}} + O\left(\frac{1}{x^{l-1}}\right) \tag{32}$$

With $\rho \equiv k_0R$, up to the leading order, the boundary constants are

$$J_l = j_l(\rho) = \frac{\rho^l}{(2l+1)!!} + O\left(\rho^{l+2}\right) \qquad J_l' = j_l(\rho) + \rho j_l'(\rho) = \frac{(l+1)\rho^l}{(2l+1)!!} + O\left(\rho^{l+2}\right) \tag{33}$$

$$Y_l = y_l(\rho) = -\frac{(2l-1)!!}{\rho^{l+1}} + O\left(\frac{1}{\rho^{l-1}}\right) \qquad Y_l' = y_l(\rho) + \rho y_l'(\rho) = \frac{l(2l-1)!!}{\rho^{l+1}} + O\left(\frac{1}{\rho^{l-1}}\right) \tag{34}$$

$$N_l = j_l(n\rho) = \frac{(n\rho)^l}{(2l+1)!!} + O\left(\rho^{l+2}\right) \qquad N_l' = j_l(n\rho) + n\rho j_l'(n\rho) = \frac{(l+1)(n\rho)^l}{(2l+1)!!} + O\left(\rho^{l+2}\right) \tag{35}$$

From (25), we know that

$$\left|\alpha_{l,\pm1}\right|^2 = \frac{4\left(J_lN_l' - \mu_rJ_l'N_l\right)^2}{\left(J_lN_l' - \mu_rJ_l'N_l\right)^2 + \left(Y_lN_l' - \mu_rY_l'N_l\right)^2} \tag{36}$$

where by (33) – (35)

$$J_lN_l' - \mu_rJ_l'N_l = (1-\mu_r)\frac{(l+1)n^l\rho^{2l}}{[(2l+1)!!]^2} + O\left(\rho^{2l+2}\right) \tag{37}$$

$$Y_lN_l' - \mu_rY_l'N_l = -\frac{(\mu_rl+l+1)n^l}{(2l+1)\rho} + O(\rho) \tag{38}$$

giving

$$\left|\alpha_{l,\pm1}\right|^2 = 4\left\{\left(\frac{1-\mu_r}{\mu_rl+l+1}\right)\frac{l+1}{[(2l+1)!!][(2l-1)!!]}\rho^{2l+1} + O\left(\rho^{2l+3}\right)\right\}^2 \qquad \text{similarly with } \mu_r \to \epsilon_r \tag{39}$$

$$\left|\beta_{l,\pm1}\right|^2 = 4\left\{\left(\frac{1-\epsilon_r}{\epsilon_rl+l+1}\right)\frac{l+1}{[(2l+1)!!][(2l-1)!!]}\rho^{2l+1} + O\left(\rho^{2l+3}\right)\right\}^2 \tag{40}$$

Indeed, $\alpha_{l,\pm1}, \beta_{l,\pm1}$ are dominated by the ρ^{2l+1} order, unless $\mu_r = 1$ or $\epsilon_r = 1$, in which case the leading term is of the next higher order.
In this problem, $\mu_r = 1$ so the only meaningful term is

$$\left|\beta_{1,\pm1}\right|^2 \approx 4\left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)^2 \cdot \frac{4}{9}(k_0R)^6 \tag{41}$$

which turns (31) into

$$\sigma_{\text{sc},\pm} \approx \frac{\pi}{2k_0^2}\cdot 3\cdot 4\left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)^2 \cdot \frac{4}{9}k_0^6R^6 = \frac{8\pi}{3}\left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)^2 k_0^4R^6 \tag{42}$$

For differential scattering cross section (30), refer to table 9.1, we have

$$\frac{d\sigma_{\text{sc},\pm}}{d\Omega} \approx \frac{\pi}{2k_0^2}\cdot 3\cdot 4\left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)^2 \cdot \frac{4}{9}k_0^6R^6 \cdot \overbrace{\frac{3}{16\pi}\left(1+\cos^2\theta\right)}^{\left|\mathbf{n}\times\mathbf{X}_{1,\pm1}\right|^2} = \left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)^2 k_0^4R^6\left(\frac{1+\cos^2\theta}{2}\right) \tag{43}$$

matching equation (10.11) and (10.10) from section 10.1.

3. As $\epsilon_r \to \infty$, the limiting form of (27) can be seen by writing it explicitly while noticing $n = \sqrt{\epsilon_r \mu_r}$,

$$\alpha_{l,\pm 1} + 1 = -\frac{h_l^{(2)}(\rho)\left[j_l(n\rho) + n\rho j_l'(n\rho)\right] - \mu_r\left[h_l^{(2)}(\rho) + \rho h_l^{(2)\prime}(\rho)\right] j_l(n\rho)}{h_l^{(1)}(\rho)\left[j_l(n\rho) + n\rho j_l'(n\rho)\right] - \mu_r\left[h_l^{(1)}(\rho) + \rho h_l^{(1)\prime}(\rho)\right] j_l(n\rho)} \quad \to \quad -\frac{H_l^*}{H_l} \tag{44}$$

$$\beta_{l,\pm 1} + 1 = -\frac{h_l^{(2)}(\rho)\left[j_l(n\rho) + n\rho j_l'(n\rho)\right] - \epsilon_r\left[h_l^{(2)}(\rho) + \rho h_l^{(2)\prime}(\rho)\right] j_l(n\rho)}{h_l^{(1)}(\rho)\left[j_l(n\rho) + n\rho j_l'(n\rho)\right] - \epsilon_r\left[h_l^{(1)}(\rho) + \rho h_l^{(1)\prime}(\rho)\right] j_l(n\rho)} \quad \to \quad -\frac{H_l'^*}{H_l'} \tag{45}$$

agreeing with (10.66) for perfect conductor $Z_s = 0$.