The goal of this document is to provide detailed explanation for Jackson section 6.8, in particular the steps leading to 6.126 for a harmonic field with time-varying amplitude.

1. **About linearity of the media**

   The linearity of the media in response to the electric field is a linear relationship in the frequency domain

$$\mathbf{D}(\mathbf{x},\omega)=\epsilon(\omega)\mathbf{E}(\mathbf{x},\omega) \tag{1}$$

   By the convolution theorem of Fourier transform, this implies a convolution relationship in the time domain

$$\mathbf{D}(\mathbf{x},t)=\int \epsilon\left(t-t'\right)\mathbf{E}\left(\mathbf{x},t'\right)dt' \tag{2}$$

   which signifies the non-instantaneous response of the material's polarization to the external field.

2. **Complex field notation**

   For the remaining discussion, we try to obtain 6.126 for a field

$$\mathbf{E}(t)=\mathbf{A}(t)\cos(\omega_0 t+\alpha) \tag{3}$$

   where the amplitude $\mathbf{A}(t)$ changes slowly compared to ω_0. Rewriting (3) in equivalent form

$$\begin{aligned}\mathbf{E}(t)&=\mathbf{A}_r(t)\cos\omega_0 t+\mathbf{A}_i(t)\sin\omega_0 t\\ &=\mathrm{Re}\left[\underbrace{(\mathbf{A}_r+i\mathbf{A}_i)}_{\equiv\widetilde{\mathbf{A}}(t)}(\cos\omega_0 t-i\sin\omega_0 t)\right]\\ &=\mathrm{Re}\left[\widetilde{\mathbf{A}}(t)e^{-i\omega_0 t}\right]\end{aligned} \tag{4}$$

   we see that the field is the real part of complex harmonic field with time-varying, complex amplitude. This form is more friendly for Fourier transform manipulations.

3. **Derivation of 6.126**

   Let the Fourier transform of the time-varying complex amplitude $\widetilde{\mathbf{A}}(t)$ be

$$\widetilde{\mathbf{A}}(t)=\int \widetilde{\mathbf{A}}(\nu)e^{-i\nu t}d\nu \tag{5}$$

   Then the field $\mathbf{E}(t)$ can be written as

$$\begin{aligned}\mathbf{E}(t)&=\mathrm{Re}\int \widetilde{\mathbf{A}}(\nu)e^{-i(\omega_0+\nu)t}d\nu && \text{let } \omega=\omega_0+\nu\\ &=\mathrm{Re}\int \widetilde{\mathbf{A}}(\omega-\omega_0)e^{-i\omega t}d\omega\end{aligned} \tag{6}$$

   which is the Fourier transform of the whole field $\widetilde{\mathbf{A}}(t)e^{-i\omega_0 t}$ (i.e., not just the amplitude).

   With the linear response assumption, we have

$$\begin{aligned}\mathbf{D}(t)&=\mathrm{Re}\int \epsilon(\omega)\widetilde{\mathbf{A}}(\omega-\omega_0)e^{-i\omega t}d\omega\\ &=\mathrm{Re}\int \epsilon(\omega_0+\nu)\widetilde{\mathbf{A}}(\nu)e^{-i(\omega_0+\nu)t}d\nu\end{aligned} \tag{7}$$

   whose time derivative is

$$\frac{\partial\mathbf{D}(t)}{\partial t}=\mathrm{Re}\int -i(\omega_0+\nu)\epsilon(\omega_0+\nu)\widetilde{\mathbf{A}}(\nu)e^{-i(\omega_0+\nu)t}d\nu \tag{8}$$

Since the amplitude is changing slowly compared to ω_0, i.e., $\widetilde{\mathbf{A}}(\nu)$ has support in the region $\nu \ll \omega_0$, we can expand the function $\omega\epsilon(\omega)$ around ω_0 to the first order and get

$$\begin{aligned}
\frac{\partial \mathbf{D}(t)}{\partial t} &\approx \mathrm{Re}\int -i\left[\omega_0\epsilon(\omega_0)+\nu\left.\frac{d(\omega\epsilon)}{d\omega}\right|_{\omega_0}\right]\widetilde{\mathbf{A}}(\nu)e^{-i(\omega_0+\nu)t}d\nu \\
&= \mathrm{Re}\left[-i\omega_0\epsilon(\omega_0)\widetilde{\mathbf{A}}(t)e^{-i\omega_0 t}\right]+\mathrm{Re}\left[\left.\frac{d(\omega\epsilon)}{d\omega}\right|_{\omega_0}\cdot\frac{\partial\widetilde{\mathbf{A}}(t)}{\partial t}e^{-i\omega_0 t}\right]
\end{aligned} \tag{9}$$

Our goal is to calculate the time average of the quantity $\mathbf{E}(t)\cdot\partial\mathbf{D}(t)/\partial t$ over a period $2\pi/\omega_0$, where $\mathbf{E}(t)$ is given by (4) and $\partial\mathbf{D}(t)/\partial t$ is given by (9).

In general, for any complex $\widetilde{x}(t), \widetilde{y}(t)$ whose time variances are slow in the period $2\pi/\omega_0$, we have

$$\begin{aligned}
\left\langle \mathrm{Re}\left[\widetilde{x}(t)e^{-i\omega_0 t}\right]\cdot\mathrm{Re}\left[\widetilde{y}(t)e^{-i\omega_0 t}\right]\right\rangle &= \langle[x_r(t)\cos\omega_0 t + x_i(t)\sin\omega_0 t][y_r(t)\cos\omega_0 t+y_i(t)\sin\omega_0 t]\rangle \\
&= \frac{1}{2}[x_r(t)y_r(t)+x_i(t)y_i(t)] \\
&= \frac{1}{2}\mathrm{Re}[\widetilde{x}^*(t)\widetilde{y}(t)]
\end{aligned} \tag{10}$$

Here we have treated $x_r(t)$ etc., as constants over the period for which we take the average.

Using (10), we have

$$\left\langle \mathbf{E}(t)\cdot\frac{\partial\mathbf{D}(t)}{\partial t}\right\rangle = \frac{1}{2}\mathrm{Re}\left\{\widetilde{\mathbf{A}}^*(t)\cdot\left[-i\omega_0\epsilon(\omega_0)\widetilde{\mathbf{A}}(t)\right]\right\}+\frac{1}{2}\mathrm{Re}\left\{\widetilde{\mathbf{A}}^*(t)\cdot\left[\left.\frac{d(\omega\epsilon)}{d\omega}\right|_{\omega_0}\frac{\partial\widetilde{\mathbf{A}}(t)}{\partial t}\right]\right\} \tag{11}$$

Note that

$$\widetilde{\mathbf{A}}^*(t)\cdot\widetilde{\mathbf{A}}(t) = \mathbf{A}_r^2+\mathbf{A}_i^2 = 2\left\langle|\mathbf{E}(t)|^2\right\rangle \qquad \text{and} \qquad \widetilde{\mathbf{A}}^*(t)\cdot\frac{\partial\widetilde{\mathbf{A}}(t)}{\partial t} = \frac{\partial}{\partial t}\left\langle|\mathbf{E}(t)|^2\right\rangle \tag{12}$$

are both real, (11) becomes

$$\begin{aligned}
\left\langle \mathbf{E}(t)\cdot\frac{\partial\mathbf{D}(t)}{\partial t}\right\rangle &= \mathrm{Re}\left[-i\omega_0\epsilon(\omega_0)\left\langle|\mathbf{E}(t)|^2\right\rangle\right]+\frac{1}{2}\mathrm{Re}\left[\left.\frac{d(\omega\epsilon)}{d\omega}\right|_{\omega_0}\frac{\partial}{\partial t}\left\langle|\mathbf{E}(t)|^2\right\rangle\right] \\
&= \omega_0\,\mathrm{Im}\,\epsilon(\omega_0)\left\langle|\mathbf{E}(t)|^2\right\rangle+\frac{1}{2}\mathrm{Re}\left[\left.\frac{d(\omega\epsilon)}{d\omega}\right|_{\omega_0}\right]\frac{\partial}{\partial t}\left\langle|\mathbf{E}(t)|^2\right\rangle
\end{aligned} \tag{13}$$

We recover 6.126 by putting in the magnetic counterpart.

The comments after 6.127 is a very good summary of the local conservation of electromagnetic energy. I could not paraphrase better than what's already stated there.