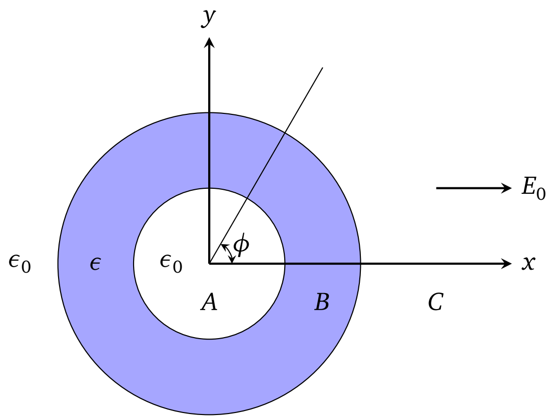

1. This is a 2D problem, the general solution is given in equation (2.71)

$$\Phi(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty}\left(a_n\rho^n \sin n\phi + b_n\rho^n \cos n\phi + c_n\rho^{-n}\sin n\phi + d_n\rho^{-n}\cos n\phi\right) \tag{1}$$

where a_0 is an inconsequential constant which can be set to 0.

For the outer region C, the requirement that $\lim_{\rho\to\infty}\mathbf{\nabla}\Phi = -E_0\hat{\mathbf{x}}$ mandates that $a_n = 0$ for all n, and $b_n = -\delta_{n1}E_0$, therefore

$$\Phi_C(\rho,\phi) = -E_0\rho\cos\phi + \sum_{n=1}^{\infty} c_n\rho^{-n}\sin n\phi + d_n\rho^{-n}\cos n\phi \tag{2}$$

For region A which includes the origin, the potential will have to take form

$$\Phi_A(\rho,\phi) = \sum_{n=1}^{\infty} a_n\rho^n\sin n\phi + b_n\rho^n\cos n\phi \tag{3}$$

and for region B, we cannot eliminate any terms according to asymptotic behavior yet, so

$$\Phi_B(\rho,\phi) = e_0\ln\rho + \sum_{n=1}^{\infty}\left(e_n\rho^n\sin n\phi + f_n\rho^n\cos n\phi + g_n\rho^{-n}\sin n\phi + h_n\rho^{-n}\cos n\phi\right) \tag{4}$$

The B/C boundary satisfies the restrictions

$$\text{tangential } E: \qquad \left.\frac{\partial\Phi_B}{\partial\phi}\right|_{\rho=b} = \left.\frac{\partial\Phi_C}{\partial\phi}\right|_{\rho=b} \tag{5}$$

$$\text{normal } D: \qquad \epsilon\left.\frac{\partial\Phi_B}{\partial\rho}\right|_{\rho=b} = \epsilon_0\left.\frac{\partial\Phi_C}{\partial\rho}\right|_{\rho=b} \tag{6}$$

With the expanded form of Φ_B and Φ_C, (5) turns into

$$\begin{aligned}&\sum_{n=1}^{\infty} n\left(e_n b^n\cos n\phi - f_n b^n\sin n\phi + g_n b^{-n}\cos n\phi - h_n b^{-n}\sin n\phi\right)\\ =&E_0 b\sin\phi + \sum_{n=1}^{\infty} n\left(c_n b^{-n}\cos n\phi - d_n b^{-n}\sin n\phi\right)\end{aligned} \tag{7}$$

Matching coefficients for $\sin n\phi, \cos n\phi$ yields

$$f_1 b + h_1 b^{-1} = -E_0 b + d_1 b^{-1} \tag{8}$$

$$f_n b^n + h_n b^{-n} = d_n b^{-n} \qquad \text{for } n \neq 1 \tag{9}$$

$$e_n b^n + g_n b^{-n} = c_n b^{-n} \qquad \text{for all } n \tag{10}$$

(6) implies

$$\frac{\epsilon}{\epsilon_0}\left[e_0 b^{-1}+\sum_{n=1}^{\infty} n\left(e_n b^{n-1}\sin n\phi+f_n b^{n-1}\cos n\phi-g_n b^{-(n+1)}\sin n\phi-h_n b^{-(n+1)}\cos n\phi\right)\right]$$
$$=-E_0\cos\phi-\sum_{n=1}^{\infty} n\left[c_n b^{-(n+1)}\sin n\phi+d_n b^{-(n+1)}\cos n\phi\right] \tag{11}$$

which requires

$$e_0=0 \tag{12}$$
$$\frac{\epsilon}{\epsilon_0}\left(f_1-h_1 b^{-2}\right)=-E_0-d_1 b^{-2} \tag{13}$$
$$\frac{\epsilon}{\epsilon_0}\left[f_n b^{n-1}-h_n b^{-(n+1)}\right]=-d_n b^{-(n+1)} \qquad \text{for } n\neq 1 \tag{14}$$
$$\frac{\epsilon}{\epsilon_0}\left[e_n b^{n-1}-g_n b^{-(n+1)}\right]=-c_n b^{-(n+1)} \qquad \text{for all } n \tag{15}$$

Similarly for the A/B boundary:

$$\text{tangential } E: \qquad \left.\frac{\partial\Phi_B}{\partial\phi}\right|_{\rho=a}=\left.\frac{\partial\Phi_A}{\partial\phi}\right|_{\rho=a} \tag{16}$$
$$\text{normal } D: \qquad \epsilon\left.\frac{\partial\Phi_B}{\partial\rho}\right|_{\rho=a}=\epsilon_0\left.\frac{\partial\Phi_A}{\partial\rho}\right|_{\rho=a} \tag{17}$$

where (16) turns into

$$\sum_{n=1}^{\infty} n\left(e_n a^n\cos n\phi-f_n a^n\sin n\phi+g_n a^{-n}\cos n\phi-h_n a^{-n}\sin n\phi\right)$$
$$=\sum_{n=1}^{\infty} n\left(a_n a^n\cos n\phi-b_n a^n\sin n\phi\right) \tag{18}$$

which implies for all n,

$$e_n a^n+g_n a^{-n}=a_n a^n \tag{19}$$
$$f_n a^n+h_n a^{-n}=b_n a^n \tag{20}$$

With $e_0=0$ already determined in (12), (17) is rewritten as

$$\frac{\epsilon}{\epsilon_0}\left[\sum_{n=1}^{\infty} n\left(e_n a^{n-1}\sin n\phi+f_n a^{n-1}\cos n\phi-g_n a^{-(n+1)}\sin n\phi-h_n a^{-(n+1)}\cos n\phi\right)\right]$$
$$=\sum_{n=1}^{\infty} n\left(a_n a^{n-1}\sin n\phi+b_n a^{n-1}\cos n\phi\right) \tag{21}$$

which requires for all n,

$$\frac{\epsilon}{\epsilon_0}\left[e_n a^{n-1}-g_n a^{-(n+1)}\right]=a_n a^{n-1} \tag{22}$$
$$\frac{\epsilon}{\epsilon_0}\left[f_n a^{n-1}-h_n a^{-(n+1)}\right]=b_n a^{n-1} \tag{23}$$

Denote $\lambda=\epsilon/\epsilon_0$. Multiply (15) by b and add the result to (10) will produce

$$(1+\lambda)\,e_n b^n+(1-\lambda)\,g_n b^{-n}=0 \qquad \Longrightarrow \qquad g_n=\frac{\lambda+1}{\lambda-1}e_n b^{2n} \tag{24}$$

Multiply (22) by a and subtract the result from (19) will produce

$$(1-\lambda)\,e_n a^n+(1+\lambda)\,g_n a^{-n}=0 \qquad \Longrightarrow \qquad g_n=\frac{\lambda-1}{\lambda+1}e_n a^{2n} \tag{25}$$

Comparing (24) with (25) and noting that $a < b$, we conclude the only way for this to hold is

$$e_n = g_n = a_n = c_n = 0 \qquad \text{for all } n \tag{26}$$

Repeating the above procedure for the $n \neq= 1$ case for (9),(15), (20), (23) will produce similar result

$$f_n = h_n = d_n = b_n = 0 \qquad \text{for } n \neq 1 \tag{27}$$

We now solve for the only remaining unknowns b_1, d_1, f_1, h_1.
Multiply (13) by b and add it to (8):

$$(1+\lambda) f_1 b + (1-\lambda) h_1 b^{-1} = -2E_0 b \tag{28}$$

For $n = 1$, multiply (23) by a and subtract it from (20):

$$(1-\lambda) f_1 a + (1+\lambda) h_1 a^{-1} = 0 \tag{29}$$

From (28) and (29) we obtain f_1, h_1 as

$$f_1 = \frac{-2(\lambda+1) E_0 b^2}{(\lambda+1)^2 b^2 - (\lambda-1)^2 a^2} \tag{30}$$

$$h_1 = \frac{-2(\lambda-1) E_0 a^2 b^2}{(\lambda+1)^2 b^2 - (\lambda-1)^2 a^2} \tag{31}$$

which then by (8) and (20) we have

$$d_1 = \frac{E_0 b^2 \left(\lambda^2-1\right)\left(b^2-a^2\right)}{(\lambda+1)^2 b^2 - (\lambda-1)^2 a^2} \tag{32}$$

$$b_1 = \frac{-4\lambda E_0 b^2}{(\lambda+1)^2 b^2 - (\lambda-1)^2 a^2} \tag{33}$$

Finally, the potentials are

$$\Phi_A(\rho,\phi) = b_1 \rho \cos\phi = \frac{-4\epsilon\epsilon_0 E_0 b^2}{(\epsilon+\epsilon_0)^2 b^2 - (\epsilon-\epsilon_0)^2 a^2} \rho \cos\phi \tag{34}$$

$$\Phi_B(\rho,\phi) = \left(f_1\rho + h_1\rho^{-1}\right)\cos\phi = \frac{-2\epsilon_0 E_0 a b^2}{(\epsilon+\epsilon_0)^2 b^2 - (\epsilon-\epsilon_0)^2 a^2}\left[(\epsilon+\epsilon_0)\frac{\rho}{a} + (\epsilon-\epsilon_0)\frac{a}{\rho}\right]\cos\phi \tag{35}$$

$$\Phi_C(\rho,\phi) = \left(-E_0\rho + d_1\rho^{-1}\right)\cos\phi = E_0\left[-\rho + \frac{\left(\epsilon^2-\epsilon_0^2\right)\left(b^2-a^2\right)}{(\epsilon+\epsilon_0)^2 b^2 - (\epsilon-\epsilon_0)^2 a^2}\frac{b^2}{\rho}\right]\cos\phi \tag{36}$$

2. The field lines are attached at the end.
3. When the cylinder becomes solid, we take $a \to 0$, in which case region A no longer exists, and (35)-(36) becomes

$$\Phi_B(\rho,\phi) = \frac{-2\epsilon_0 E_0}{\epsilon+\epsilon_0}\rho\cos\phi \tag{37}$$

$$\Phi_C(\rho,\phi) = E_0\left(-\rho + \frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\frac{b^2}{\rho}\right)\cos\phi \tag{38}$$

For a cylindrical cavity, we can take $b \to \infty$, in which case region C no longer exists, leaving

$$\Phi_A(\rho,\phi) = \frac{-4\epsilon\epsilon_0 E_0}{(\epsilon+\epsilon_0)^2}\rho\cos\phi \tag{39}$$

$$\Phi_B(\rho,\phi) = \frac{-2\epsilon_0 E_0 a}{(\epsilon+\epsilon_0)^2}\left[(\epsilon+\epsilon_0)\frac{\rho}{a} + (\epsilon-\epsilon_0)\frac{a}{\rho}\right]\cos\phi \tag{40}$$

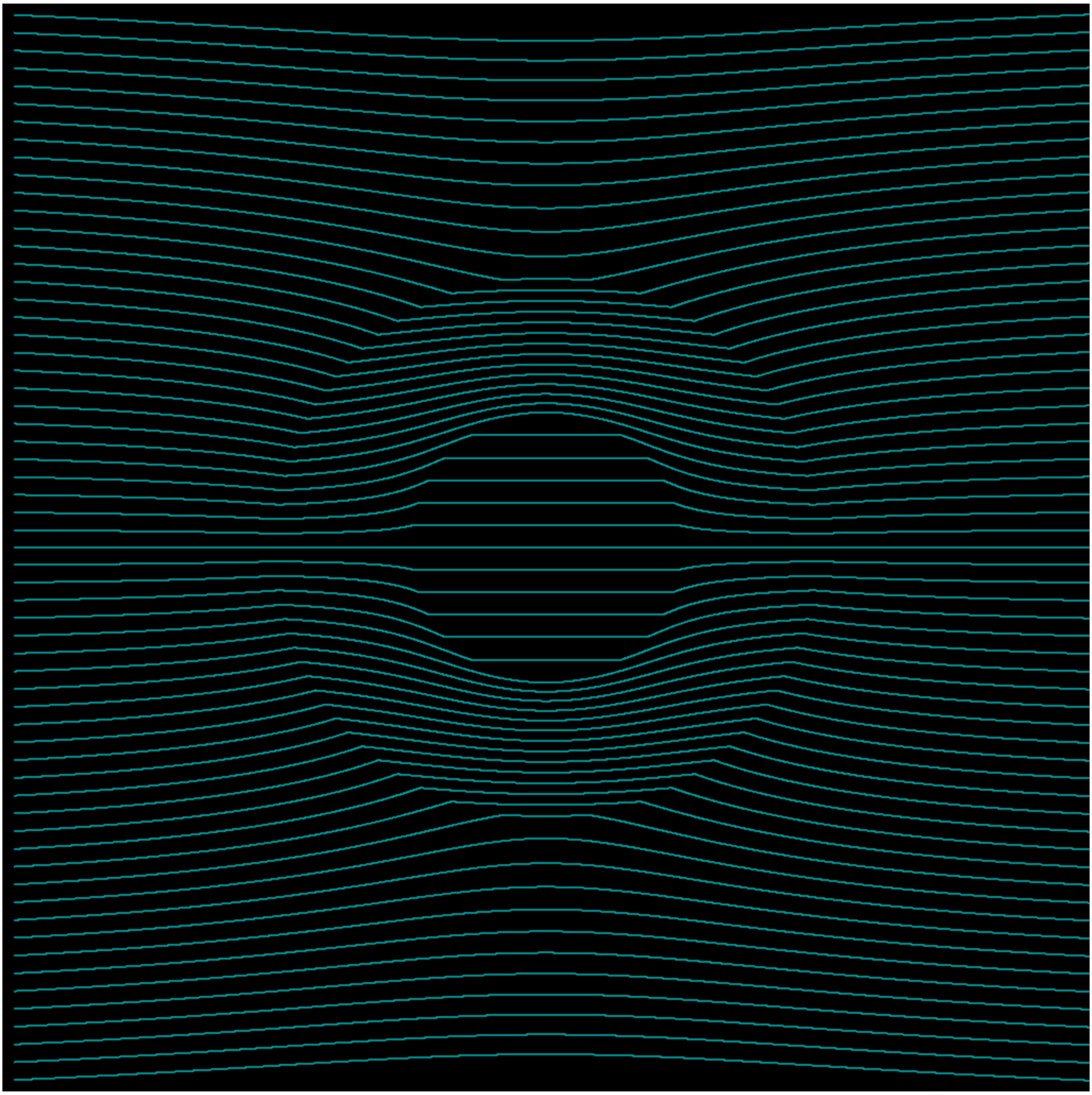

Figure 1: Field lines with $b = 2a > 0$

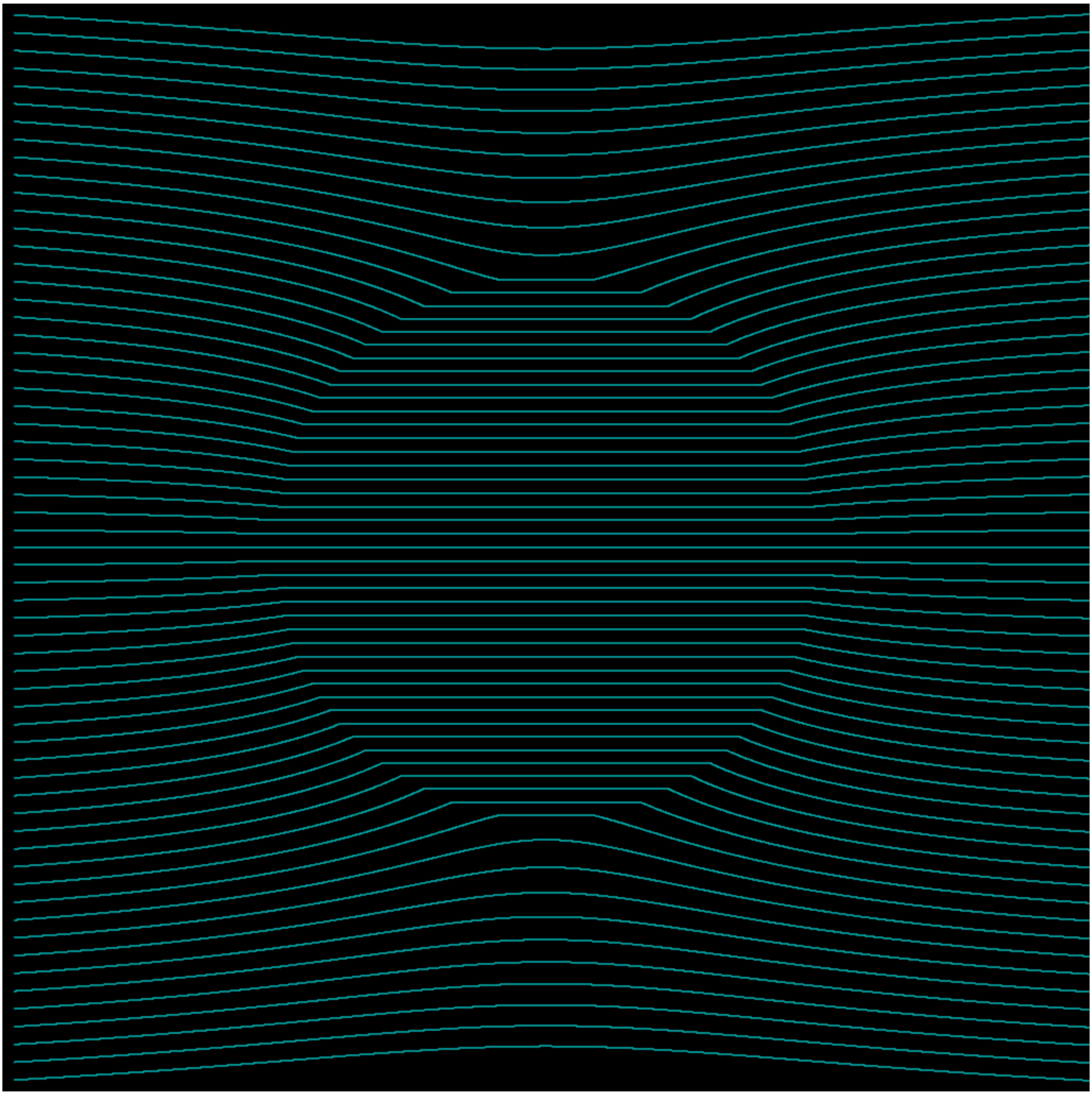

Figure 2: Field lines with $a = 0, b > 0$, i.e., solid cylinder

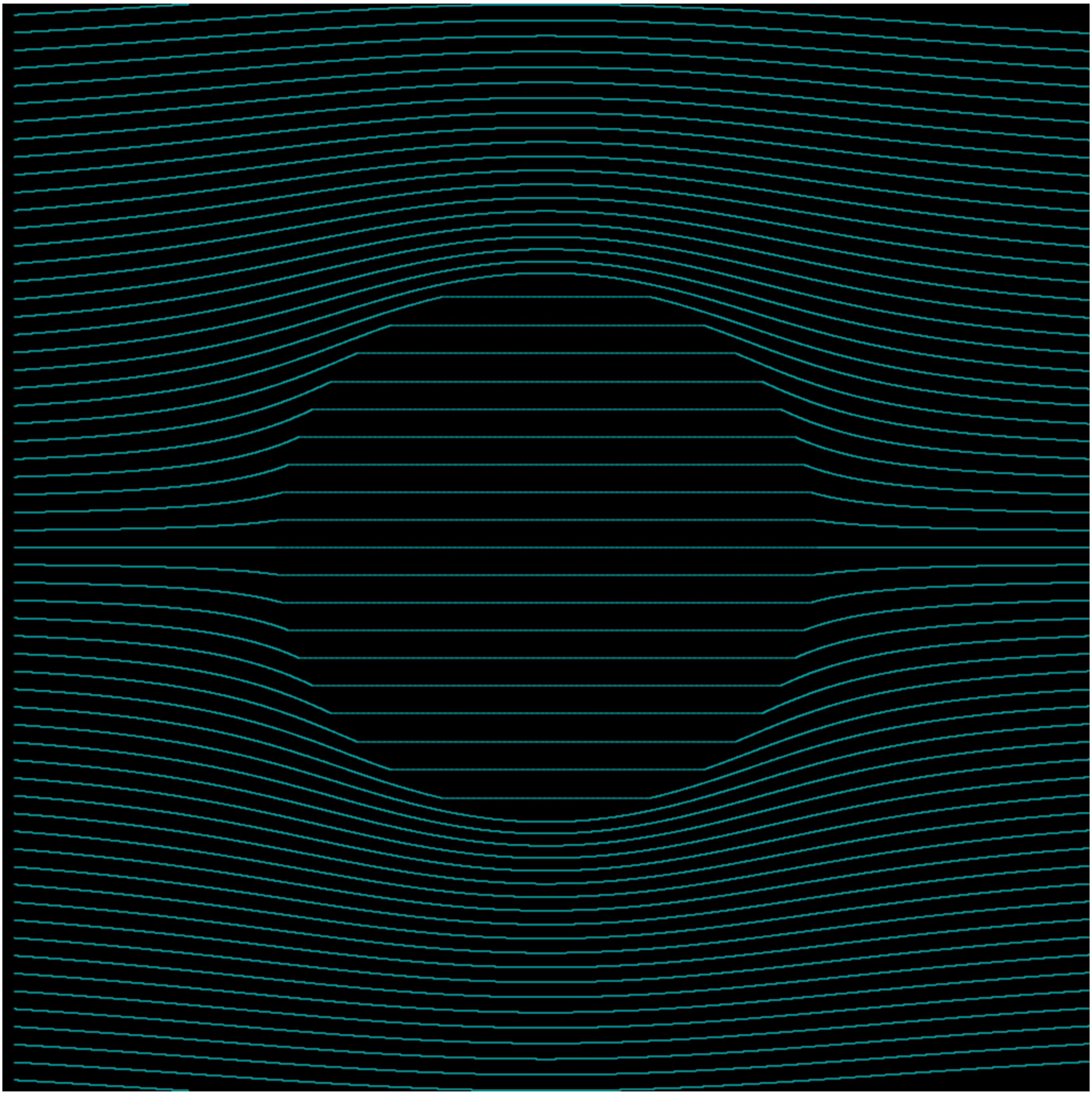

Figure 3: Field lines with $a > 0, b \to \infty$, i.e., cavity