HW 2 - Convex Optimization

Exercise 1

1 - (P) : $\min_{x \in \mathbb{R}^d} c^T x$, $c \in \mathbb{R}^d$

s.t. $Ax = b$

$x \geq 0$

The lagrangian of this problem is:

$$L : \begin{cases} \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R} \\ (x, \lambda, \nu) \longmapsto c^T x - \lambda^T x + \nu^T (Ax - b) \end{cases}$$

Let's compute $g : \lambda, \nu \longmapsto \inf_{x \in \mathbb{R}^d} L(x, \lambda, \nu)$

$\forall (x, \lambda, \nu) \in (\mathbb{R}^d)^3$, $L(x, \lambda, \nu) = (c^T - \lambda^T + \nu^T A)x - \nu^T b$

We see that if $c^T - \lambda^T + \nu^T A \neq 0$, $\exists (x_n) \in (\mathbb{R}^d)^{\mathbb{N}}$, $L(x_n, \lambda, \nu) \underset{n \to +\infty}{\longrightarrow} -\infty$ for any λ, ν.

hence: $g : (\lambda, \nu) \longmapsto \begin{cases} -\nu^T b & \text{if } c^T - \lambda^T + \nu^T A = 0 \\ -\infty & \text{else} \end{cases}$

The dual is therefore:

$$\max_{\lambda, \nu \in \mathbb{R}^d} g(\lambda, \nu) \quad \text{s.t. } \lambda \geq 0 \ \wedge \ c - \lambda + A^T \nu = 0 \quad \Leftrightarrow \quad \boxed{\max_{\nu \in \mathbb{R}^d} -b^T \nu \quad \text{s.t. } c + A^T \nu \geq 0}$$

2 - (D):

$b \in \mathbb{R}^m$,

$$\max_{y \in \mathbb{R}^m} b^T y \quad \text{s.t.} \quad A^T y \leq c \quad \Leftrightarrow \quad \min_{y \in \mathbb{R}^m} -b^T y \quad \text{s.t.} \quad A^T y \leq c$$

$$L : \begin{vmatrix} \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R} \\ (y, \lambda, \nu) \longmapsto -b^T y + \lambda^T (A^T y - c) + 0 \end{vmatrix}$$

$$g : \lambda \longmapsto \inf_{y \in \mathbb{R}^m} \left(-b^T y + (A\lambda)^T y - \lambda^T c \right)$$

$$\forall \lambda \in \mathbb{R}^m, \quad g(\lambda) = \begin{cases} -\lambda^T c & \text{if } (A\lambda)^T - b^T = 0 \\ -\infty & \text{else} \end{cases}$$

hence, the dual problem is:

$$\boxed{\max_{\lambda \in \mathbb{R}^m} -\lambda^T c \quad \text{s.t.} \quad \lambda \geq 0, \; A\lambda = b \quad \Leftrightarrow \quad \min_{\lambda \in \mathbb{R}^m} c^T \lambda \quad \text{s.t.} \quad \lambda \geq 0, \; A\lambda = b}$$

In fact, (P) is the dual of (D) and the inverse is also true.

3 - (Self-Dual):

$$\min_{x, y} c^T x - b^T y \quad \text{s.t.} \quad Ax = b, \quad x \geq 0, \quad A^T y \leq c$$

$$L: \begin{array}{ccc} \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{n} & \longrightarrow & \mathbb{R} \\ ((x,y),(\lambda_x,\lambda_y),\nu) & \longmapsto & c^T x - b^T y - \lambda_x^T x \\ & & + \lambda_y^T (A^T y - c) \\ & & + \nu^T (b - Ax) \end{array}$$

$\cancel{x, y, \lambda_x, \lambda_y, \nu \in (\mathbb{R}^d \times \mathbb{R}^n)^2 \times \mathbb{R}^d}$,

$x, y, \lambda_x, \lambda_y, \nu \in \mathrm{dom}(L)$,

$$L((x,y),(\lambda_x,\lambda_y),\nu) = \nu^T b - \lambda_y^T c + (c^T - \lambda_x^T - \nu^T A)x$$
$$+ (-b^T + \lambda_y^T A^T) y$$

hence, we have:

$$g: (\lambda_x, \lambda_y, \nu) \longmapsto \begin{cases} \nu^T b - \lambda_y^T c & \text{if } c - \lambda_x - A^T \nu = 0 \text{ and } b - \lambda_y A \lambda_y = 0 \\ -\infty & \text{else} \end{cases}$$

The dual is:

$$\max_{\lambda_x, \lambda_y, \nu} \quad \nu^T b - \lambda_y^T c$$

$$\text{s.t.} \quad \lambda_x \geq 0, \ \lambda_y \geq 0$$
$$A \lambda_y = b$$
$$c - A^T \nu = \lambda_x$$

$$\Leftrightarrow \boxed{\begin{array}{ll} \max_{\lambda_y \in \mathbb{R}^d, \nu \in \mathbb{R}^n} & b^T \nu - c^T \lambda_y \\ \text{s.t.} & \lambda_y \geq 0 \\ & c \geq A^T \nu \\ & A \lambda_y = b \end{array}}$$

$$\Leftrightarrow \min_{x \in \mathbb{R}^d, y \in \mathbb{R}^n} c^T x - b^T y \quad \text{s.t.} \quad \begin{array}{l} x \geq 0 \\ c \geq A^T y \\ Ax = b \end{array}$$

4- Let x^*, y^* be optimal solution of (P) and (D).

Because we assumed that the problem (Self-Dual) is feasible, then it is also the case for (P) and (D).

We have:

- $\forall x \in \mathbb{R}^d$, $Ax = b$, $x \geq 0$, $c^T x \geq c^T x^*$
- $\forall y \in \mathbb{R}^n$, $A^T y \leq c$, $b^T y \leq b^T y^*$

Hence:

$\forall x, y$ feasible,

$$c^T x - b^T y \geq c^T x^* - b^T y^*,$$

hence $[x^*, y^*]$ is optimal for the (Self-Dual).

We proved that any optimal x^*, y^* of (P) and (D) is optimal for the (Self-Dual). We just need to add that for a given $[x^*, y^*]$ solution of (Self-Dual), x^* is also optimal for (P) and y^* for (D) because if we fix x or y in (Self-Dual) we have (P) and (D).

Hence, for a given optimal $[x^*, y^*]$ of (Self-Dual), we could obtained it by solving (P) and (D)

- Because (P) is a convex problem, feasible and bounded, we can seperate two possibilities:
  - if A is such that $Ax = 0 \Rightarrow x = 0$ and $b = 0$, then the optimal solution for (P) is $x = 0$ and optimal value 0, same for (D) since $b^T y = 0$ so we have the optimal value of the (Self-Dual) 0

- if not, then by Slater's constraint, we have strong duality.

Let x^*, y^* be optimals for (P) and (D), (x^*, y^*) is optimal for the (self-dual),

$$c^T x^* - b^T y^* = 0 .$$

Hence: the optimal value is 0.

Exercise 2

1) Conjugate of $x \mapsto \|x\|_1$.

$$f : \left| \begin{array}{l} \mathbb{R}^d \longrightarrow \mathbb{R} \\ y \longmapsto \sup_{x \in \mathbb{R}^d} (y^T x - \|x\|_1) \end{array} \right.$$

- $y \in \mathbb{R}^d$, $x \in \mathbb{R}^d$, we have:

$$y^T x - \|x\|_1 \underset{\text{if } x_i \neq 0}{=} \sum_{i=1}^{d} |x_i| \left(\frac{x_i}{|x_i|} y_i - 1 \right)$$

- if $\|y\|_\infty > 1$, ie $\exists i_0 \leq d$, $|y_{i_0}| > 1$, then with

$$x_n = (0, \ldots, \underset{i_0}{n \frac{y_{i_0}}{|y_{i_0}|}}, \ldots, 0) , n \geq 1$$

we have:

$$y^T x_n - \|x_n\|_1 = \sum_{i=1}^{d} \left(|x_i| \left(\frac{x_i}{|x_i|} y_i - 1 \right) \right) \delta_{i_0 i}$$

$$= n \left(\frac{y_{i_0}^2}{|y_{i_0}|} - 1 \right) \geq n$$

Therefore we have: $f(y) = +\infty$

- if $\|y\|_\infty \leq 1$, then

$$y^T x - \|x\|_1 = \sum_{i=1}^{d} |x_i| \left(\frac{x_i}{|x_i|} y_i - 1 \right)$$

and $\forall i \leq d$, $\frac{x_i}{|x_i|} y_i - 1 \leq 0$ because

$$\frac{x_i}{|x_i|} y_i - 1 = \begin{cases} |y_i| - 1 \leq 0 \\ -|y_i| - 1 \leq -1 \end{cases}$$

hence, for y so that $\|y\|_\infty \leq 1$ we have that

$\forall x \in \mathbb{R}^d$, $y^T x - \|x\|_1 = \sum_{i=1}^{d} y_i x_i - |x_i|$

$$= \sum_{i=1}^{d} \underbrace{|x_i|}_{\geq 0} \underbrace{\left(\frac{x_i}{|x_i|} y_i - 1\right)}_{\leq 0}$$

if $x_i \neq 0$,
otherwise we have 0 in the sum

≤ 0

with equality for $x = 0$.

Hence:

$$\boxed{f^* : y \longmapsto \begin{cases} 0 & \text{if } \|y\|_\infty \leq 1 \\ +\infty & \text{else} \end{cases}}$$

2 _

(RLS) : $\min_x \|Ax - b\|_2^2 + \|x\|_1$

$\iff$ (P) : $\min_{x,y} \|y\|_2^2 + \|x\|_1$

s.t. $y = Ax - b$

We compute the dual:

$\mathcal{L}(x, y, \nu) = \|y\|_2^2 + \|x\|_1 + \nu^T (Ax - b - y)$

$g(\nu) = \inf_{x,y} \mathcal{L}(x, y, \nu)$

$= \inf_{x,y} \|y\|_2^2 + \|x\|_1 + \nu^T A x - \nu^T y - \nu^T b$

$= \inf_y (\|y\|_2^2 - \nu^T y) + \inf_x (\|x\|_1 + \nu^T A x) - \nu^T b$

$$\inf_x (\|x\|_1 + \nu^T A x) = - \sup_x (-\|x\|_1 + (-\nu^T A x))$$
$$= -f(-\nu^T A) = \begin{cases} 0 & \text{if } \|\nu^T A\|_\infty \leq 1 \\ -\infty & \text{else} \end{cases}$$

Let's compute $\inf_{y} (\|y\|_2^2 - \nu^T y)$.

Consider $g : \begin{array}{l} \mathbb{R}^p \longrightarrow \mathbb{R} \\ y \longmapsto \|y\|_2^2 - \nu^T y \end{array}$

g is convex, differentiable and $\forall y \in \mathbb{R}^p$,
$$g(y) = \|y\|_2^2 - \nu^T y \underset{C.S}{\geq} \|y\|_2 (\|y\|_2 - \|\nu\|_2) \xrightarrow[\|y\| \to +\infty]{} +\infty$$

Hence, g has a minimum value obtained for y^* so that $\nabla g(y^*) = 0$

$$\nabla g : y \longmapsto 2y - \nu$$
$$\nabla g(y^*) = 0 \iff y^* = \frac{\nu}{2}$$

We have $\forall y \in \mathbb{R}^p$, $g(y) \geq g(y^*) = \frac{1}{4}\|\nu\|_2^2 - \frac{1}{2}\|\nu\|_2^2$
$$= -\frac{1}{4}\|\nu\|_2^2$$

finally, we have:
$$g(\nu) = \begin{cases} -\nu^T b - \frac{1}{4}\|\nu\|_2^2 & \text{if } \|A^T \nu\|_\infty \leq 1 \\ -\infty & \text{else} \end{cases}$$

Hence, the dual problem is:
$$\max_{\nu \in \mathbb{R}^n} \; -\nu^T b - \frac{1}{4}\|\nu\|_2^2$$
$$\text{s.t. } \|A^T \nu\|_\infty \leq 1$$

$$\iff \boxed{- \min_{\nu} \; \nu^T b + \frac{1}{4}\|\nu\|_2^2 \quad \text{s.t. } \|A^T \nu\|_\infty \leq 1}$$

Exercise 3

$$(\text{Seq } 1) \iff \min_{w} \frac{1}{n\tau} \sum_{i=1}^{m} \ell(w, x_i, y_i) + \frac{1}{2}\|w\|_2^2$$

$$\iff \min_{w} \frac{1}{n\tau} \sum_{i=1}^{m} \max(0, 1 - y_i(w^T x_i)) + \frac{1}{2}\|w\|_2^2$$

$$\iff \min_{w,z} \frac{1}{n\tau} \sum_{i=1}^{m} z + \frac{1}{2}\|w\|_2^2$$

s.t. $z_i = \max(0, 1 - y_i(w^T x_i))$

$z \geq 0$

$$(\text{Seq } 2) \iff \min_{w,z} \frac{1}{n\tau} \mathbb{1}^T z + \frac{1}{2}\|w\|_2^2$$

s.t. $z_i \geq 1 - y_i(w^T x_i)$

$z \geq 0$

~~$(\text{Seq } 1) \iff \min_{w,z} \frac{1}{n\tau} \mathbb{1}^T z + \frac{1}{2}\|w\|_2^2$~~

~~$z_i = 1 - y_i(w^T x_i)$~~

~~$z \geq 0$~~

We have $(\text{Seq } 1) \Rightarrow (\text{Seq } 2)$

If $[w^*, z^*]$ is a solution of (Seq 1), then it is also a solution of (Seq 2) since the objective functions are the same and that w^*, z^* respect the constraints of (Seq 2).

Let's name the objective function:

$$g : w, z \mapsto \frac{1}{n\tau} \mathbb{1}^T z + \frac{1}{2}\|w\|_2^2$$

Let $[w^*, z^*]$ be optimals for (Seq 2)

Let's prove that $\forall i$, $\xi^*_i \geq 0$ ~~and~~ or $\xi^*_i = 1 - y_i(w^T x_i)$,

$$\xi^*_i = \max(0, 1 - y_i(w^T x_i))$$

$i_0 \leq n$, if $\xi^*_{i_0} = 0$ we have the result.

if not, we have $\xi^*_{i_0} \geq 1 - y_i(w^T x_i)$.

if $\xi^*_{i_0} > 1 - y_i(w^T x_i)$, then $\xi^*_{i_0}$ cannot be optimal since ~~by ex~~:

$$g(w^*, \xi^*) > g(w^*, \xi') \quad \text{with}$$

$$\xi' = (\xi^*_1, \xi^*_2, \dots, \underbrace{1 - y_{i_0}(w^T x_{i_0})}_{i_0^{th} \text{ position}}, \dots, \xi^*_n)$$

Therefore, we have a contradiction (proof by contradiction).
Then: $\forall i \leq n$, $\xi_i^* = \max(0, 1 - y_i(w^T x_i))$,
so $[w^*, \xi^*]$ is optimal for (Eq. 1).

<u>In conclusion, the two problems are equivalent.</u>

2 _

$$L : (w, \xi, (\lambda_i)_{i \leq n}, \pi) \longmapsto \frac{1}{n\tau} \mathbb{1}^T \xi + \frac{1}{2}\|w\|_2^2 + \sum_{i=1}^{n} \lambda_i (1 - y_i(w^T x_i) - \xi_i) - \pi^T \xi$$

$$g : (\lambda_i)_{i \leq n}, \pi \longmapsto \inf_{w, \xi} L(w, \xi, (\lambda_i), \pi)$$

$\forall (\lambda_i)_{i \leq n}, \pi$,

$$g((\lambda_i), \pi) = \inf_{\xi} \left(\frac{1}{n\tau} \mathbb{1}^T \xi - \pi^T \xi - \lambda^T \xi \right) + \lambda^T \mathbb{1} + \inf_{w} \left(\frac{1}{2}\|w\|_2^2 - \sum_{i=1}^{n} \lambda_i y_i (w^T x_i) \right)$$

$\cdot \cancel{\inf_{\mathfrak{z}} \left(\frac{1}{n\tau} \mathfrak{z}^T (-1 \right.}$

$$\cdot \inf_{\mathfrak{z}} \left(\mathfrak{z}^T \left(\frac{1}{n\tau} - \pi - y\lambda \right) \right) = \begin{cases} 0 & \text{if } \frac{1}{n\tau} - \pi - \lambda = 0 \\ -\infty & \text{else} \end{cases}$$

Let's consider $g : w \longmapsto \frac{1}{2}\|w\|_2^2 - \sum_{i=1}^{m} \lambda_i y_i (w^T x_i)$

g is differentiable

$\forall w$, $\nabla g(w) = w - \sum_{i=1}^{m} \lambda_i y_i x_i$

$$\nabla g(w) = 0 \iff w = \sum_{i=1}^{m} \lambda_i y_i x_i$$

As g is convexe and infinite at infinity, we have: $\forall w, \quad g(w) \geq g(w^*), w^* = \sum_{i=1}^{m} \lambda_i y_i x_i$

Hence,

$$\inf_{w} \left(\frac{1}{2}\|w\|_2^2 - \sum_{i=1}^{m} \lambda_i y_i (w^T x_i) \right)$$

$$= \frac{1}{2} \left\| \sum_{i=1}^{m} \lambda_i y_i x_i \right\|_2^2 - \sum_{i=1}^{m} \lambda_i y_i \sum_{j=1}^{m} \lambda_j y_j x_j^T x_i$$

$$= \frac{1}{2} \left\| \sum_{i=1}^{m} \lambda_i y_i x_i \right\|_2^2 - \sum_{i=1}^{m} \sum_{j=1}^{m} (\lambda_i y_i x_i)^T (\lambda_j y_j x_j)$$

$$= \frac{1}{2} \left\| \sum_{i=1}^{m} \lambda_i y_i x_i \right\|_2^2 - \left\| \sum_{i=1}^{m} \lambda_i y_i x_i \right\|_2^2$$

$$= -\frac{1}{2} \left\| \sum_{i=1}^{m} \lambda_i y_i x_i \right\|_2^2$$

hence, we have:

$(\lambda_i), \pi,$

$$g((\lambda_i), \pi) = \begin{cases} \lambda^T \mathbb{1} - \frac{1}{2} \| \sum_{i=1}^{n} \lambda_i y_i x_i \|_2^2 \\ \text{if} \quad \frac{\mathbb{1}}{n\tau} - \pi - \lambda = 0 \\ -\infty \quad \text{else} \end{cases}$$

The dual is:

$$\cancel{\max \quad g(\lambda, \pi) = \lambda^T \mathbb{1} - \frac{1}{2} \sum_{i=1}}$$

$$\begin{aligned} \max \quad & \lambda^T \mathbb{1} - \frac{1}{2} \| \sum_{i=1}^{n} \lambda_i y_i x_i \|_2^2 \\ \text{s.t.} \quad & \lambda \geq 0 \\ & \pi \geq 0 \\ & \frac{\mathbb{1}}{n\tau} - \pi - \lambda = 0 \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} \max \quad & \lambda^T \mathbb{1} - \frac{1}{2} \| \sum_{i=1}^{n} \lambda_i y_i x_i \|_2^2 \\ \text{s.t.} \quad & \lambda \geq 0 \\ & \lambda \leq \frac{1}{n\tau} \mathbb{1} \end{aligned}}$$

Exercise 4

Let's compute the dual of $\sup_{a \in P} a^T x$

$$\sup_{a \in P} a^T x \iff - \inf_{C^T a \leq d} -a^T x \iff -(P)$$

The dual of (P) is:

$x, \lambda,$

$$\mathcal{L}(x, \lambda) = -a^T x + \lambda^T (C^T a - d)$$

$$g : \lambda \longmapsto \inf_{a} a^T(-x + C\lambda) - \lambda^T d$$

$\forall \lambda,$

$$g(\lambda) = \begin{cases} -\lambda^T d & \text{if } C\lambda = x \\ -\infty & \text{else} \end{cases}$$

hence, the dual is:

$$\boxed{\begin{array}{ll} \max & g(\lambda) = -\lambda^T d \\ \text{s.t} & \\ & \lambda \geq 0 \\ & C\lambda = x \end{array}}$$

Moreover, $\sup_{a \in P} a^T x$ is a linear optimization problem. It is convex, and assuming it is feasible we have the strong duality property.

Hence, for z^* optimal of the dual, we can write:

$$\sup_{a \in P} a^T x \iff -(-d^T z^*)$$

with $z^* \geq 0$

$Cz^* = x$

Hence, we have:

$$\boxed{\begin{array}{ll} \min c^T x & \iff \min_x c^T x \\ \text{s.t. } \sup_{a \in P} a^T x \leq b & \text{s.t. } d^T z \leq b \\ & z \geq 0 \\ & Cz = x \end{array}}$$

Exercise 5

1_ Let's consider:

(P) :

$$\min c^T x$$
$$\text{s.t. } Ax \leq b$$
$$x_i(1-x_i) = 0, \; i = 1, \dots, n$$

The dual is:

$$L: (x, \lambda, \nu) \longmapsto c^T x + \lambda^T (Ax - b) + \sum_{i=1}^{m} \nu_i x_i (1 - x_i)$$

λ, ν,

$$\inf_x L(x, \lambda, \nu) = -\lambda^T b + \inf_x \left\{ c^T x + \lambda^T A x + \sum_{i=1}^{m} \nu_i x_i (1 - x_i) \right.$$

Let's define $g: x \longmapsto c^T x + \lambda^T A x + \sum_{i=1}^{m} \nu_i x_i (1 - x_i)$

$\forall x,$

$$g(x) = \sum_{i=1}^{m} c_i x_i + [A^T\lambda]_i x_i + \nu_i x_i - \nu_i x_i^2$$

$$= \sum_{i=1}^{m} x_i(c_i + [A^T\lambda]_i + \nu_i) - \nu_i x_i^2$$

$$= \sum_{i=1}^{m} \alpha_i x_i + \beta_i x_i^2 = \sum_{i=1}^{m} P_i(x_i)$$

g is the sum of n polynomes of degree 2, applied in distinct values of $x = (x_1, \dots, x_n)$. Hence, minimizing g is minimizing P_i for $i = 1, \dots, m$.

First, let's notice that if $\beta_i \leq 0$, then $\inf_x g(x) = -\infty$. Therefore in the following we assume that $\beta_i > 0 \ \forall i \leq n$.

$\forall i$, P_i is minimal in $x_i^* = -\frac{\alpha_i}{2\beta_i}$,

$$P_i(x_i^*) = -\frac{\alpha_i^2}{2\beta_i}$$

Hence: $$\inf_x g(x) = \sum_{i=1}^{m} -\frac{\alpha_i^2}{4\beta_i}$$

now, $\forall i \leq n$ $\alpha_i = c_i + [A^T\lambda]_i + \nu_i$, $\beta_i = -\nu_i$ so

$$\inf_x g(x) = \sum_{i=1}^{m} -\frac{(c_i + [A^T\lambda]_i + \nu_i)^2}{-4\nu_i}$$

Hence the dual is:

$$\max_{\lambda,\nu} \; -\lambda^T b + \sum_{i=1}^{m} -\frac{(c_i + [A^T\lambda]_i + \nu_i)^2}{4\nu_i}$$

$$\text{s.t.} \quad \lambda \geq 0$$
$$\nu > 0$$

As the objective function is continuous, we have:

$$\sup_{\substack{\lambda \geq 0 \\ \nu > 0}} -\lambda^T b + \sum_{i=1}^{m} -\frac{(c_i + [A^T\lambda]_i - \nu_i)^2}{4\nu_i}$$

$$= \sup_{\lambda \geq 0} \left(-\lambda^T b + \sup_{\nu > 0} \sum_{i=1}^{m} -\frac{(c_i + [A^T\lambda]_i - \nu_i)^2}{4\nu_i} \right)$$

$$\boxed{= \sup_{\lambda \geq 0} \left(-\lambda^T b + \sum_{i=1}^{m} \min(0, c_i + [A^T\lambda]_i) \right)}$$

which is convexe
using the indication

d) (2)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & \left. \begin{aligned} x_i \leq 1 \\ -x_i \leq 0 \end{aligned} \right\} i = 1, \dots, n \end{aligned}$$

$$\mathcal{L} : x, \lambda, \pi_1, \pi_2 \longmapsto c^T x + \lambda^T (Ax - b) + \pi_1^T (x - \mathbb{1}) - \pi_2^T x$$

$$\inf_x \mathcal{L}(x, \lambda, \pi_1, \pi_2) = \begin{cases} -b\lambda^T - \pi_1^T \mathbb{1} & \text{if } c + A^T\lambda - \pi_2 + \pi_1 = 0 \\ -\infty & \end{cases}$$

the dual is:

$$\begin{aligned} \sup_{\lambda, \pi_1, \pi_2} \quad & -b\lambda^T - \pi_1 \mathbb{1}^T \\ \text{s.t.} \quad & \pi_1 \geq 0 \\ & \pi_2 \geq 0 \\ & \lambda \geq 0 \\ & c + A^T\lambda - \pi_2 + \pi_1 = 0 \end{aligned}$$

$$\iff \sup_{\lambda, \pi_2} \quad -b\lambda^T + \mathbb{1}^T[c + A^T\lambda - \pi_2]$$

$$\lambda \geq 0$$
$$c + A^T\lambda - \pi_2 \leq 0$$
$$\pi_2 \geq 0$$

We have: for λ, $\sup_{\pi_2} \quad \mathbb{1}^T[c + A^T\lambda - \pi_2] \; : (P')$

s.t. $\pi_2 \geq 0$

$\pi_2 \geq c + A^T\lambda$

π_2^{λ} optimal of (P') is obtained for π_2 being the lowest possible in each coordinate.

for $i \leq m$,

if $[c + A^T\lambda]_i < 0$, then $[\pi_2]_i = 0$

if $[c + A^T\lambda]_i \geq 0$, then $[\pi_2]_i = [c + A^T\lambda]_i$

Hence: $\mathbb{1}^T[c + A^T\lambda - \pi_2^*] = \sum_{i=1}^{m} \min(0, [c + A^T\lambda]_i)$

Hence, $\boxed{(2) \iff \sup_{\lambda} \; -\lambda^T b + \sum_{i=1}^{m} \min(0, [c + A^T\lambda]_i) \quad \lambda \geq 0}$

The dual of the two problems are equivalent. The problems primals are linear, if they are feasible then by strong duality the lower bounds of the original problem obtained using the duals are the same.