TP3 - Statistical learning with extreme

Exercise 2_3

1 - $\blacktriangleright$ $F(a_n x) \underset{n \to +\infty}{\longrightarrow} 1$.

We have : $\exists (a_n)_n > 0$, $F^n(a_n \cdot) \to \Phi_\alpha(\cdot)$,
with $\Phi_\alpha : x \longmapsto e^{-x^{-\alpha}}$, $x > 0$

$\forall x > 0, \forall n \in \mathbb{N}^*$, $F^n(a_n x) = e^{n \log(F(a_n x))}$
$\underset{n \to +\infty}{\longrightarrow} e^{-x^{-\alpha}} \in]0,1[$

As $n \underset{n \to +\infty}{\longrightarrow} +\infty$, we need $\log(F(a_n x)) \underset{n \to +\infty}{\longrightarrow} 0$

ie $F(a_n x) \underset{n \to +\infty}{\longrightarrow} 1$,

otherwise we would have
$e^{n \log(F(a_n x))} \longrightarrow e \in \{0, +\infty\}$ if
$(F(a_n x))_n$ has a limit, and if it does not have a limit then $(e^{n \log(F(a_n x))})$ won't too.

Hence, we must have $\boxed{F(a_n x) \underset{n \to +\infty}{\longrightarrow} 1.}$

▸ $\forall x > 0, f(x) < 1$.

We show it by contradiction. Let's suppose we have $x_0 > 0$ so that $f(x_0) = 1$.
Per monotony of cdf f (f is a cdf), we have:

$$\forall x \geq x_0, \; f(x) = 1.$$

Now, let's show that: $\exists n_0 > 0, \forall n \geq n_0$, $|a_n| > \ell > 0$, $\ell \in \mathbb{R}^+$.

We will use: $\begin{cases} f(a_n x) \xrightarrow[n \to +\infty]{} 1 & (i) \\ f(x) \xrightarrow[x \to 0^+]{} f(0) < 1 & (ii) \end{cases}$

$f(0) < 1$ because otherwise $\forall x \geq 0$ $f(x) = 1$, hence $\forall n \geq 1$, $f(\underbrace{a_n x}_{\geq 0}) = 1$

$$\Rightarrow f^n(a_n x) \xrightarrow[n \to +\infty]{} 1 \neq \phi_\alpha(x)$$

(ii): $\forall \varepsilon > 0, \exists t_0 \geq 0, \forall t \leq t_0$:

$$|f(t) - f(0)| = f(t) - f(0) \leq \varepsilon$$

(i): $\forall \varepsilon > 0, \exists n_0 > 0, \forall n \geq n_0$,

$$1 - f(a_n x) \leq \varepsilon$$

now, let's suppose that $\exists n_1 \geq n_0$, so that $a_{n_1} x \leq t_0$.

Then: $\begin{cases} 1 - f(a_{n_1} x) \leq \varepsilon \\ f(a_{n_1} x) - f(0) \leq \varepsilon \end{cases} \Rightarrow \begin{cases} 1 - f(a_{n_1} x) \leq \varepsilon & (1) \\ -f(a_{n_1} x) \geq -\varepsilon - f(0) & (2) \end{cases}$

$(2) \Rightarrow \quad 1 - f(a_n x) \geq 1 - \varepsilon - f(c)$

if we take $\varepsilon < \underbrace{\frac{1-f(c)}{2}}_{>0}$,

we have:

$$1 - f(a_n x) \geq 1 - \varepsilon - f(c)$$
$$> 1 - f(c) + \frac{f(c)}{2} - \frac{1}{2}$$
$$> \frac{1}{2} - \frac{f(c)}{2}$$
$$> \varepsilon$$

So, by taking the same ε in the definition of (i), we have shown that:

$1 - f(a_n x) > \varepsilon$ which contradicts the definition of $f(a_n x) \xrightarrow[n \to +\infty]{} 1$.

By contradiction, we have shown that:
$\forall n \geq n_0,\ a_n x > t_0$ i.e. $\forall n \geq n_0\ a_n > \underbrace{\frac{t_0}{x}}_{= e} \neq 0$

Now, we supposed that $\forall x \geq x_0,\ f(x) = 1$.

by taking $x = \frac{x_0}{e}$, we have that:

$\forall n \geq n_0, \qquad a_n x \geq e\frac{x_0}{e} > x_0$

i.e. $\forall n \geq n_0,\ f(a_n x) = 1$

i.e. $\forall n \geq n_0,\ f(a_n x) \xrightarrow[n \to +\infty]{} 1 \neq \varphi_\alpha(x)$.

We have a contradiction.

Hence: $\underline{\forall x > 0,\ f(x) < 1}$.

$\blacktriangleright\ a_n \to +\infty$.

Let's prove it by contradiction.

If non$[a_n \to +\infty]$, then:

$$\exists M > 0,\ \forall n \in \mathbb{N},\ \exists n_0 \in \mathbb{N},$$

$$a_{n_0} < M$$

From this definition, we can construct a $\varphi : \mathbb{N} \to \mathbb{N}$ extractrice such that

$\forall n \in \mathbb{N}, \quad a_{\varphi(n)} < M$.

As such, $\forall x > 0,\ \forall n \geq 0,\ a_{\varphi(n)} x < Mx$

i.e. $\forall n \geq 0,\ f^{n}(a_{\varphi(n)} x) < e^{n \log(\underbrace{f(Mx)}_{<1})}$

because $\forall x > 0,\ f(x) < 1$

$\xrightarrow[n \to +\infty]{} 0$

$\neq \varphi_\alpha(x)$

Hence, by contradiction:

$$\boxed{a_n \xrightarrow[n \to +\infty]{} +\infty}.$$

2 - $\forall x>0$, $F^n(a_n x) \underset{+\infty}{=} e^{-x^{-\alpha}} + o(1)$

$$\Rightarrow e^{n \log F(a_n x)} \underset{+\infty}{=} e^{-x^{-\alpha}} + o(1)$$

$$\Rightarrow n \log F(a_n x) = -x^{-\alpha} + o(1)$$

$$\Rightarrow n \log(1 + F(a_n x) - 1) = -x^{-\alpha} + o(1)$$

$$\Rightarrow + n(F(a_n x) - 1) = + -x^{-\alpha} + o(1)$$

$$\Rightarrow 1 - F(a_n x) = \frac{x^{-\alpha}}{n} + o\left(\frac{1}{n}\right)$$

$$\Rightarrow 1 - F(a_n x) \sim \frac{x^{-\alpha}}{n}$$

We have also:

$$1 - F(a_n) \sim \frac{1}{n}$$

hence:

$$\frac{1 - F(a_n x)}{1 - F(a_n)} \sim \frac{x^{-\alpha}}{n} \frac{n}{1} = x^{-\alpha}$$

~~Let $t>0$, and $n(t) = \min\{n \in \mathbb{N}, a_n \leq t\}$,~~

~~so $n(t) \leq t$.~~

~~As F is non-decreasing,~~

$$\cancel{1 - F(a_{n(t)} x) \geq 1 - F(tx)}$$

Let $t>0$, and $m(t) = \inf\{n \geq 0, a_n \geq t\}$.

We have: $a_{m(t)} \geq t \geq a_{m(t)-1}$.

Hence, as F is non-decreasing:

$$1 - F(a_{m(t)} x) \leq 1 - F(tx) \leq 1 - F(a_{m(t)-1} x)$$

$$1 - F(a_{m(t)}) \leq 1 - F(t) \leq 1 - F(a_{m(t)-1})$$

Hence, $\frac{1 - F(a_{n(t)}x)}{1 - F(a_{n(t)+1}x)} \leq \frac{1 - F(tx)}{1 - F(t)} \leq \frac{1 - F(a_{n(t)+1}x)}{1 - F(a_{n(t)+1}x)}$

As both terms on the side of the inequality converge to $x^{-\alpha}$, we have proved that:
$\left(n(t) \xrightarrow[t \to +\infty]{} +\infty \right)$

$$\boxed{\forall x > 0, \quad \frac{1 - F(tx)}{1 - F(t)} \xrightarrow[t \to +\infty]{} x^{-\alpha} \quad \Leftrightarrow \quad 1 - F \in RV(-\alpha)}$$

3_ for $(a_n)_{n>0}$, we have already shown that:

$$1 - F(a_n) \underset{+\infty}{\sim} \frac{1}{n} + o\left(\frac{1}{n}\right)$$

Let's call $(b_n)_n$, $b_n = \left(\frac{1}{1-F}\right)^{\leftarrow}(n)$ $\forall n \in \mathbb{N}$.

It is clear that $b_n \xrightarrow[n \to +\infty]{} +\infty$.

We have: $n+1 \geq \left(\frac{1}{1-F}\right)\left(\frac{1}{1-F}\right)^{\leftarrow}(n) \geq n$

$\geq$ by construction

$\leq$ if we assume F continuous (we could have still the result if not, but the upper bound could be higher)

As such: $\frac{1}{n}\left[\left(\frac{1}{1-F}\right)\left(\frac{1}{1-F}\right)^{\leftarrow}(n)\right] \xrightarrow[n \to +\infty]{} 1$

i.e. $\frac{1}{1-F}(b_n) \underset{+\infty}{\sim} n$

Because $1-F \in RV(-\alpha)$,

$x>0$, $\dfrac{1-F(b_n x)}{1-F(b_n)} \longrightarrow x^{-\alpha}$

i.e. $n(1-F(b_n x)) \sim x^{-\alpha}$

$\Rightarrow \quad 1-F(b_n) \underset{+\infty}{\sim} \frac{1}{n} + o\left(\frac{1}{n}\right)$

Hence we have: $F(b_n) \underset{+\infty}{\sim} F(a_n)$.

Now, in our case, let's prove that:

$\boxed{\text{non}[a_n \sim b_n] \Rightarrow \text{non}[U(b_n) \sim U(a_n)]}$, with $U \in RV(-\alpha)$ and $U \searrow$

$\text{non}[a_n \sim b_n] \Leftrightarrow \text{non}\left[\frac{a_n}{b_n} \underset{n \to +\infty}{\longrightarrow} 1\right]$

$\overset{(1)}{\Leftrightarrow} \exists \varepsilon > 0, \forall n_0 \geq 0, \exists n \geq n_0,$

$|a_n - b_n| > \varepsilon b_n$

Now, let's construct an extractrice φ, such that:

- (1) is true for $\varphi(n)$ $\forall n \in \mathbb{N}$
- $a_{\varphi(n)} \geq b_{\varphi(n)}$ for $\varphi(n)$ $\forall n \in \mathbb{N}$, WLOG

We assume that among the $n \in \{n \in \mathbb{N}$, such that (1) is true$\}$, $a_n \geq b_n$ an infinite number of time WLOG.

~~(if $b_n \geq a_n$~~

Then, we have:

$\forall n \in \mathbb{N}$,

$$\frac{U(a\varphi(n))}{U(b\varphi(n))} = \frac{U(\overbrace{a\varphi(n) - b\varphi(n)}^{\geq 0} + b\varphi(n))}{U b\varphi(n)}$$

U is $\searrow$ $\rightarrow$ $$\leq \frac{U(\varepsilon b\varphi(n) + b\varphi(n))}{U b\varphi(n)}$$

$$= \frac{U(b\varphi(n)[\varepsilon+1])}{U b\varphi(n)}$$

$$\xrightarrow[n\to+\infty]{} \frac{1}{(1+\varepsilon)^\alpha} < 1 \quad (\alpha > 0)$$

Therefore, we have that $\lim\limits_{n\to+\infty} \frac{U(a\varphi(n))}{U(b\varphi(n))} < 1$ if the limit exists,

Hence: $\boxed{\text{non}\left[\frac{U(a_n)}{U(b_n)} \xrightarrow[n\to+\infty]{} 1\right]}$

Therefore, by contraposition and because $1-F \in RV(-\alpha)$,

we have that: $\boxed{(1-F)(a_n) \underset{+\infty}{\sim} (1-F)(b_n) \rightarrow a_n \underset{+\infty}{\sim} b_n}$

4_ $x>0$, we suppose that:

$$n(1-F(a_n x)) \underset{n \to +\infty}{\longrightarrow} x^{-\alpha} \neq 0$$

this implies that $F(a_n x) \underset{n \to +\infty}{\longrightarrow} 1$

hence: $\ln(1+1-f(a_n x)) \underset{+\infty}{\sim} 1-f(a_n x)$

$$\ln(1+f(a_n x)-1) \underset{+\infty}{\sim} f(a_n x)-1$$

hence:

$$n\ln(1+f(a_n x)-1) \underset{+\infty}{\sim} -x^{-\alpha}$$

$$\Rightarrow \quad e^{n\ln(f(a_n x))} \underset{+\infty}{\sim} e^{-x^{-\alpha}}$$

$$\Rightarrow \quad \boxed{F^n(a_n x) \underset{+\infty}{\sim} \phi_\alpha(x)}$$

(2-15) holds true.

Par la seconde partie, on montre cette fois que $\underline{\tilde{a_n} \sim a_n \Rightarrow (1-F)(a_n) \sim (1-F)(\tilde{a_n})}$

$\tilde{a_n} \sim a_n \Rightarrow \forall \varepsilon>0, \exists N>0, \forall n \geq N$

$$-\varepsilon \leq \frac{\tilde{a_n}-a_n}{a_n} \leq \varepsilon$$

$$\frac{(1-F)(\tilde{a_n})}{(1-F)(a_n)} = \frac{(1-F)(\tilde{a_n}+a_n-a_n)}{(1-F)(a_n)} = \frac{(1-F)(a_n[1+\frac{\tilde{a_n}-a_n}{a_n}])}{(1-F)(a_n)}$$

$1-F$ est décroissante, donc:

$$\frac{(1-F)(a_n(1+\varepsilon))}{(1-F)(a_n)} \leq \frac{(1-F)(a_n[1+\frac{\tilde{a_n}-a_n}{a_n}])}{(1-F)(a_n)} \leq \frac{(1-F)(a_n(1-\varepsilon))}{(1-F)(a_n)}$$

De plus, comme (a_n) vérifie (2.15), alors $1-F \in RV(-\alpha)$, donc pour ε fixé, quand on fait tendre n vers l'∞, on a

$$(i) : \frac{1}{(1+\varepsilon)^{\alpha}} \leq \lim_{n\to+\infty} \frac{(1-F)(\tilde{a}_n)}{(1-F)(a_n)} \leq \frac{1}{(1-\varepsilon)^{\alpha}}$$

donc, lorsque $\varepsilon \to 0$ on aura bien

$$\frac{(1-F)(\tilde{a}_n)}{(1-F)(a_n)} \xrightarrow[n\to+\infty]{} 1 \quad \text{ie} \quad \boxed{(1-F)(\tilde{a}_n) \sim (1-F)(a_n)}$$

note : on a écrit dans (i) $\lim\limits_{n\to+\infty} \frac{(1-F)(\tilde{a}_n)}{(1-F)(a_n)}$, mais elle ne pouvait pas exister. La limite $\varepsilon \to 0$ explique son existence, donc nous nous sommes autorisés cet abus de notation.

Enfin :

$$(1-F)(a_n) \sim (1-F)(\tilde{a}_n)$$

$$\Rightarrow \quad n(1-F(a_n)) \sim (1-F)(\tilde{a}_n)n$$

~~ie $n(1-F)(\tilde{a}_n) \longrightarrow x^{-\alpha}$~~

$n>0$, $x>0$

$$\frac{1-F(a_n x)}{1-F(\tilde{a}_n x)} = \frac{1-F(a_n x)}{1-F(a_n)} \cdot \frac{1-F(a_n)}{1-F(\tilde{a}_n x)}$$

$$= \underbrace{\frac{1-F(a_n x)}{1-F(a_n)}}_{\to \phi_\alpha(x)} \cdot \underbrace{\frac{1-F(\tilde{a}_n)}{1-F(\tilde{a}_n x)}}_{\to \phi_\alpha^{-1}(x)} \cdot \frac{1-F(a_n)}{1-F(\tilde{a}_n)}$$

$$\sim \frac{1-F(a_n)}{1-F(\tilde{a}_n)} \sim 1$$

$$\Rightarrow \forall x>0, \quad 1-F(a_n x) \sim 1-F(\tilde{a}_n x)$$

ie $\boxed{n\left[1-F(a_n^\nu x)\right] \sim n\left[1-F(a_n x)\right] \sim x^{-\lambda}}$

Exercise 2.7

$n \in \mathbb{N}^*$
$m_n = \frac{1}{n} \sum_{i \geq 1} \delta\left(\frac{i}{n}\right)$, m lebesgue measure

Let's show that: $m_n \xrightarrow{v} m$

$\forall f$ continuous ~~and~~ with compact support $K \subset]0,+\infty[$
We can write $[a,b] = K$, for $0 < a < b < +\infty$.

$\forall n \geq 1$,
$$\int_0^{+\infty} f(x)\, m_n(x) = \int_0^{+\infty} f(x) \frac{1}{n} \sum_{i=0}^{\infty} \delta\left(\frac{i}{n}\right)(dx)$$

Let $N > n$,
$$\int_0^{+\infty} f(x) \frac{1}{n} \sum_{i=0}^{N} \delta\left(\frac{i}{n}\right) dx$$
$$= \sum_{i=0}^{N} \int_{\mathbb{R}_+} \frac{1}{n} f(x)\, \delta\left(\frac{i}{n}\right) dx$$
$$= \frac{1}{n} \sum_{i=0}^{N} f\left(\frac{i}{n}\right)$$

$\left(\sum_{i=0}^{N} f\left(\frac{i}{n}\right)\right)_N$ converges when $N \to +\infty$, because $\mathcal{S}(f) \in [a,b]$, hence $\exists n_0 > 0$, $\forall i \geq n_0$, $f\left(\frac{i}{n}\right) = 0$.

Hence: $\int_{\mathbb{R}_+} f(x) \frac{1}{n} \sum_{i=0}^{\infty} \delta\left(\frac{i}{n}\right)(dx) = \frac{1}{n} \sum_{i=0}^{\infty} \int_{\mathbb{R}_+} f(x) \delta\left(\frac{i}{n}\right)(dx)$
$$= \frac{1}{n} \sum_{i=0}^{\infty} f\left(\frac{i}{n}\right)$$
$$f \text{ compact on } [a,b] \Rightarrow \frac{1}{n} \sum_{i=0}^{m} f\left(b \frac{i}{n}\right)$$

We recognize a Rieman sum:

$$= \frac{1}{n} \sum_{i=0}^{bn} f\left(\frac{i}{n}\right) \xrightarrow[n \to +\infty]{} \int_a^b f(x) \lambda(dx)$$

λ lebesgue measure

$$\cancel{= \frac{1}{n} \sum_{i=0}^{n} f\left(a + i \frac{b-a}{n}\right)}$$

because $t_0 = \lfloor \frac{na}{b-a} \rfloor < t_1 \dots < t_n = b$

form a subdivision of $[a,b]$

<u>Hence:</u> $\boxed{m_n \xrightarrow{v} m}$

Exercise 2.4: Compacts of $(0, +\infty]$

Let $a \in]0, +\infty[$. We want to show that $[a, +\infty]$ is compact in $E = (0, +\infty]$.

Let $(U_i)_{i \in I}$ be a family of open sets such that $[a, +\infty] \subset (U_i)_{i \in I}$. For all $x \in E$ such that $x > a$, we have that $[x, +\infty] \subset (U_i)_{i \in I}$.

Let us define S as the set of $x \in E$ such that $[x, +\infty]$ is covered by a finite number of U_i sets. We know there exist i^* such that $+\infty \in U_{i^*}$ so that $+\infty \in S$ and $S \neq \emptyset$.

S is lower bounded by a and non-empty, hence there exist x^* such that $x^* = \inf_{x \in S} x$. If $x^* > a$ then there exists j^* such that $x^* \in U_{j^*}$ and $\epsilon > 0$ such that $[x^* - \epsilon, x^* + \epsilon] \subset [a, +\epsilon]$ and $[x^* - \epsilon, x^* + \epsilon] \subset U_{j^*}$.

As $x^* + \epsilon > x^* = \inf_{x \in S} x$, there exist $x' \in [x^*, x^* + \epsilon]$ such that $[x', +\infty]$ is covered by a finite number of U_i sets. Hence, $[x', +\infty] \subset \cup_i U_i$ and $[x' - \epsilon, +\infty] \subset \cup_i U_i \cup U_{j^*}$ so that $[x' - \epsilon, +\infty]$ admits a finite cover which contradicts the definition of x^*. Therefore, $x^* \leq a$.

As $x^* = \inf_{x \in S} x \leq a$ we can find a finite cover I' such that $[a, +\infty] \subset \cup_{i \in I'} U_i$. Therefore, $[a, +\infty]$ is compact in E.