2. $\exists (a_n)_n \to \infty$ *and* $\exists\ \nu \in \mathrm{M}(E)$ *such that*

$$n\mathbb{P}\left(a_n^{-1} X \in \cdot\right) \xrightarrow{v} \nu$$

where ν *is a homogeneous measure with index* $-\alpha$, i.e. $\nu(tA) = t^{-\alpha}\nu(A)$, *for* $t > 0$ *and* $A \subset E$, *measurable.*

3. *There exist* H *a probability measure on the sphere* S_{d-1}, *a constant* $c > 0$, *and a sequence* $a_n \to \infty$ *such that letting* $R = \|X\|$ *and* $W = X/R$,

$$n\mathbb{P}\left((R/a_n, W) \in \cdot\right) \xrightarrow{v} c\nu_\alpha \otimes H$$

in $\mathrm{M}((0,\infty] \times S_{d-1})$, *where* $\nu_\alpha[x,\infty] = x^{-\alpha}, x > 0$.

Exercise 3.13 (A simple question):
Considering a random pair $X = (X_1, X_2)$ (ex: asset returns/ river flows / temperatures at two locations) such that X_1 and X_2 are likely to take extreme values simultaneously. The general goal is to quantify the strength of their association. To be more precise, the question is: how to estimate the probability that the second components is very large, given that the first component is very large?

In this exercise this vague question is rephrased in such a way that we may anwser easily. Assume that X satisfies (3.2) for some limit G. In order to be able to apply the framework of (standard) multivariate regular variation, let $V = (V_1, V_2)$ with $V_j = 1/(1 - F_j(X_j))$ where F_j is the marginal cdf of X_j.

1. Download the dataset 'data.txt' from the website
   (e.g. through the command `read.table("data.txt")` in R).

2. write a function `ranktransformer` which transforms the dataset $(X_i, i \le n)$ into $\hat{V}_i, i \le n)$ where $\hat{V}_{i,j}$ is obtained as and empirical version of $V_{i,j}$, *i.e.* using an empirical estimate of F_j instead of F_j. In order to avoid division by zero, it is recommended to use

$$\hat{F}_j(x) = \frac{1}{n+1} \sum_{i \le n} \mathbb{1}\{X_{i,j} \le x\}$$

3. Prove (theoretically) that as $t \to \infty$, and for any fixed $\lambda > 0$, the quantity

$$p_t(\lambda) = \mathbb{P}\left(V_2 > \lambda t \mid V_1 > t\right)$$

   converges to some limit $p_\infty(\lambda)$. Express $p(\lambda)$ as a function of the limit measure μ in Equation (3.14). You may of course use the results from the course.

   In the sequel the goal is to estimate $p_\infty(\lambda)$ for some fixed values of λ, say $\lambda = 1$ or $\lambda = 2$.

4. Propose an empirical estimator $\hat{p}_k(\lambda)$ of $p_\infty(\lambda)$ based on its subasymptotic version $p_t(\lambda)$ and the data $X_i, i \le n$, where t is taken as the k^{th} order statistic (with $k \ll n$) of the marginal dataset $\hat{V}_{i,1}, i \le n$

5. Implement the estimator as a function taking as arguments `(Data, krange, Lambda )` where `krange` is a vector of values of k for which the estimator must be computed. The function should return a vector of probabilities of same length as `krange`.

6. Let $\lambda = 2$. Based on plots of the quantities $\hat{p}_k(\lambda)$ as a function of k, propose a reasonable range of values k in order to estimate $p_\infty(\lambda)$. Give a final estimate and a an order of magnitude of its variability (as k varies within your 'reasonable' range).