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Graphical Choices and Geometrical Thought in the Transmission of Theodosius' *Spherics* from Antiquity to the Renaissance

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Abstract Spherical geometry studies the sphere not simply as a solid object in itself, but chiefly as the spatial context of the elements which interact on it in a complex three-dimensional arrangement. This compels to establish graphical conventions appropriate for rendering on the same plane—the plane of the diagram itself—the spatial arrangement of the objects under consideration. We will investigate such "graphical choices" made in the Theodosius' *Spherics* from antiquity to the Renaissance. Rather than undertaking a minute analysis of every particular element or single variant, we will try to uncover the more general message each author attempted to convey through his particular graphical choices. From this analysis, it emerges that the different kinds of representation are not the result of merely formal requirements but mirror substantial geometrical requirements expressing different ways of interpreting the sphere and testify to different ways of reasoning about the elements that interact on it.

1 Introduction

Spherical geometry allows numerous choices in representing the element under study, since it involves the sphere not simply as a solid object in itself, but chiefly as the spatial context of the arcs, line segments and circles determined on it by the intersection of different inclined plans in a complex three-dimensional spatial arrangement. This allows one to choose the representation and to establish graphical conventions appropriate for rendering on the same plane—the plane of the diagram

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itself—the combined, and sometimes complex, spatial arrangement of the objects under consideration. We will investigate such "graphical choices" made in Theodosius' *Spherics* from antiquity to the Renaissance.

The *Sphaericorum Elementorum libri tres*, composed in the first century BC by Theodosius of Tripoli, contains a logically structured set of 59 propositions divided into three books of increasing interest and difficulty. The first book explains the fundamental properties of circles lying on the sphere. The second book studies the relations between circles which are parallel or tangent or intersect one another and analyses the interactions between them. The third book, whose often long and particularly complex enunciations are nearly all concerned with great circles, considers ratios and proportions between diameters and arcs determined by certain criteria.

Theodosius' work is a fundamental account of spherical geometry: it develops the subject on the basis of the plane and solid geometry of Euclid's *Elements*. The principal reason for its broad diffusion, however, was probably its usefulness as an introduction to astronomy. Being a part of the *Little Astronomy*,[1] the collection of books put together by the Greeks as useful for understanding Ptolemy's *Almagest*, it shared the same process of transmission common to the principal mathematical works of ancient Greece.

The *Spherics* was studied and transmitted in the Greek world for at least ten centuries and has left some manuscript evidence of this transmission.[2] In the ninth century, it was translated into Arabic,[3] entering the emerging Arabo-Islamic culture, which, nourishing an interest in astronomy, translated the Greek *Little Astronomy* and enriched this corpus with new contributions, transforming it into the *Middle Books*, that is to say, the books between Euclid's *Elements* and Ptolemy's *Almagest*, where Theodosius' *Spherics* was still found.[4]

Theodosius' work entered the scientific heritage of the Latin West after the twelfth century exclusively from the Arabic redactions in two different versions: A translation due to Gerard of Cremona, in which the text is largely faithful to the Greek tradition;[5]

[1] This collection is preserved in Greek manuscripts. It included Aristarchus' *The Sizes and Distances of the Sun and Moon*, Autolycus' *On Rising and Settings* and *On the Moving Sphere*, Euclid's *Catoptrics*, *Data*, *Optics*, and *Phenomena*, Hypsicles' *On Ascensions*, Theodosius' *On Geographic Places*, *On Days and Nights*, and *Spherics*. Menelaus' *Sphaericorum libri tres* was also part of this collection, but it is lost in Greek, and it is only known in Arabic and Arabo-Latin versions. For a detailed description of the *Little Astronomy* and its role in Greek culture see Evans (1998, pp. 89–91).

[2] The oldest manuscript known is in the Vaticanus Graecus 204 (tenth century). Heiberg (1927) lists a total of 24 manuscripts containing the Greek text of *Spherics* and made his edition by collating six of them: Vaticanus Graecus 204 (tenth century); Parisinus Graecus 2390 (thirteenth to fourteenth century); Vaticanus Graecus 203 (thirteenth century); Vaticanus Graecus 202 (fourteenth century); Parisinus Graecus 2448 (fourteenth century); Parisinus Graecus 2342 (fourteenth century). Czinczenheim (2000) adds the Vaticanus Graecus 191 (thirteenth century) and the Parisinus Graecus 2364 (fifteenth century).

[3] For the study of diagrams the manuscript Laur.Or.124 has been directly examined.

[4] Lorch (1996, pp. 164–165).

[5] Gerard of Cremona's translation was transmitted in manuscript form and was not printed until the present time. The faithfulness of Gerard's version to the Greek one is based on our analysis of Q.69 sup. in Milan Biblioteca Ambrosiana, but it is confirmed by Czinczenheim (2000, pp. 195, 200): "Par conséquent, à défaut de pouvoir accéder directement aux deux premières traductions arabes, la traduction de Gérard de Crémone représente, après les manuscrits grecs, l'apport le plus précieux à la recherche d'un état ancien du texte.... les traductions arabo-latines permettraient alors de remonter à une branche disparue

and a version usually ascribed to Plato of Tivoli,[6] which presents a heavy reworking of Theodosius' three books. In the scientific renaissance of the sixteenth century no less than six editions were printed.

The many versions of the *Spherics* in existence today allow us to follow the complete tradition of the work and to trace its change from Antiquity to the Renaissance. Throughout this long period the contents had been gradually enriched with the introduction of new properties and results that mark progress in the knowledge of spherical geometry and rendered the demonstrations simpler. The iconographic apparatus also underwent changes and alterations that compel us to try to understand the reasons for these modifications and to clarify the function that the diagrams had among different cultures in various epochs. From our analysis, it emerges that the different styles of representation are not the result of merely formal requirements but reflect substantial geometrical requirements expressing different ways of interpreting the sphere and testify to different ways of reasoning about the elements that interact on it.

In the sixteenth century, printers, guided principally by commercial demand, usually preferred to publish those texts that had broad manuscript circulation to ensure a profit in the immediate future. In Venice in 1518, both Lucantonio Giunta and Ottavianus Scoto's heirs chose to print the version by Plato of Tivoli, which then becomes the *editio princeps* of Theodosius' *Sphaericorum elementorum libri*. These nearly identical printed editions presented a very corrupt and imprecise text: Whole phrases or sentences are omitted or repeated, probably because of copyists' imprecision; incorrect references to the diagram's lettering occur; diagrams are sometimes printed without letters marking their points; some figures are missing entirely; and some diagrams do not refer to the proposition in whose vicinity they are printed.

In order to produce a text free from these blemishes, the Viennese mathematician Johannes Voegelin decided to correct the existing edition[7] using two manuscripts to bring the *Spherics* back to its "original splendour": "Iccirco comparatis duobus vetustis exemplaribus, altero ex Viennensis Gymnasii bibliotheca, altero ex Musaeo Collimitii ...Theodosium ...pristino restitui nitori."[8] He claimed a philological care that, although honestly expressed, remained far from restoring the ancient text of the Greek tradition. At any rate, in 1529 Voegelin published in Vienna a more correct edition of Plato of Tivoli's version than that already printed.[9]

Footnote 5 continued
de la tradition grecque, qui pourrait fournir des leçons authentiques au même titre que les manuscrits grecs conservés." For further informations about Gerard of Cremona's works see Boncompagni (1851b).

6 This attribution was first made by Jean Pena (1558) (see n. 13). His authority was then followed in Boncompagni (1851a, pp. 251–252) and Heiberg (1927, p. VIII). In contrast Lorch (1996) advances the hypothesis that this Latin version is due to Campanus of Novara. This opens the problem of the actual attribution of the changes made to the original text. See n. 29. We will follow tradition and refer to this version as Plato of Tivoli's.

7 The description given by Voegelin precisely describes the real inaccuracies of the *Spherics*' edition of 1518: "... sunt aliquot lacunae, sunt et verba interdum, interdum etiam integrae sententiae praetermissae; schematibus etiam linearibus et multa insunt, et multa desunt, quae lectoris animum perplexum faciunt" (Voegelin 1529, c. sign. A iij).

8 Voegelin (1529, c. sign. A iij recto-verso).

9 Voegelin explicitly declares his original contributions to the text: "Addidi praeterea paucas adnotatiunculas, ut a divinae Matheseos studiosis legi atque intelligi melius possit" (Voegelin 1529, c. sign. A iij verso).

The year 1558 was a turning point in the *Spherics* tradition because in that year the printed versions of the text ended their dependence on the medieval heritage and participated in the Renaissance of mathematics, reflecting its two fundamental aspects, a critical philological approach to manuscript sources and mathematical analysis of the contents of the ancient works. In Paris, Jean Pena, motivated by philological interest, prepared the Greek *editio princeps* of the *Spherics* and restored the ancient work to its original form and language, followed by a faithful Latin translation.[10] The French mathematician considered his own contribution an important "conquest"[11] because, at last, anyone who wanted to compare the medieval Arabo-Latin version of Plato of Tivoli with the Greek text could clearly recognise how much easier and shorter the original was: "... si quis illum Theodosium ex Arabico versum et Venetiis excusum, cum Graeco conferat, incredibile discrimen non modo facilitatis, sed etiam brevitatis inveniet."[12]

In his preface, Pena initiated a concern with the transmission of Theodosius' work. For the first time in the European tradition, he presented the history of the transmission starting from its renown among the Greeks and arriving at the Venetian edition of 1518. Relying on an anonymous work about burning mirrors, Pena ascribed the edition to the Arabo-Latin translator Plato of Tivoli: "... eam versionem annis ab hinc quadraginta Venetiis excuderunt, quam a Platone Tiburtino factam fuisse asseverat author libelli De speculis ustoriis, quisquis ille sit."[13] In this way, it was natural for Pena to assign the origin of all the modifications characterising that version to Arabic mathematicians.[14]

In Messina, in the same year, 1558, Francesco Maurolico interpreted the recovery of the mathematical tradition in a completely different way and published the *Sphaericorum elementorum libri tres ex traditione Maurolyci*. Persuaded of the dignity of mathematical knowledge, Maurolico regretted the silence that had fallen on the ancient works of Euclid, Archimedes, Theodosius, Menelaus, Apollonius, and Serenus, as well as the corruption of their texts.[15] Hence, he carried out a program of

Footnote 9 continued
These contributions were placed at the end of the demonstrations and introduced by his name "Voegelin". Some of them are intended to clarify the mathematical demonstrations, while others pertain to astronomy (see also n. 30). Since none of these additions became a proposition, their number is the same in the two editions of 1518 and 1529.

[10] Before the edition of Jean Pena, the only Latin translation of the Greek text was made by Giorgio Valla who published only a selection of 24 propositions of Theodosius' *Spherics* in the geometrical section of his *De expetendis et fugiendis rebus opus* (Valla 1501).

[11] Jean Pena likens his work on the *Spherics* to the French conquest of Calais made in the same year, 1558 (Pena 1558, c. sign. a ij recto–verso).

[12] Pena (1558, c. sign. a iiij).

[13] Pena (1558, c. sign. a ij verso-a iiij).

[14] Pena (1558, c. sign. a iiij): "Primo enim Theodosius sex septemve definitionibus tantum contentus fuit. At Arabes alias septem easque fere supervacaneas adiecerunt. Theodosius multitudinem theorematum consulto vitavit, et totum Sphaericum negotium sexaginta propositionibus absolvit. At Arabes hunc numerum triente auxerunt, et pro sexagenis octogenas cumularunt."

[15] F. Maurolico in his letter to Pietro Bembo, sent from Messina in May 1536 says: "Cur silet Euclides praestantissimus? Cur silet Archimedes ac Theodosius? Cur Menelai, Apollonii, Sereni praeclara nusquam

restoring the ancient works so that the knowledge they contained—set free of errors, gaps and imperfections—should shine again.[16] Far from pursuing a philological and faithful reconstruction made by strictly adhering to the manuscript sources, Maurolico was guided by an interest in the enrichment of scientific knowledge. Hence, he integrated the original works with various contributions made throughout the centuries[17] and presented the works' content following a rigour that he regarded as proper to mathematics.

2 Aims and method

This article presents the results of an analysis that focuses attention on the graphical aspect of four versions of Theodosius' *Spherics*: (1) the Greek version, taking the manuscript *Vaticanus Graecus 204* as a characteristic representative of the original tradition; (2) the medieval version of Plato of Tivoli, which, from the graphical point of view, still follows the ancient tradition; (3) the Renaissance version *ex traditione Maurolyci*, which marks an important change in the graphic representation of the spherical elements; and (4) Cristoph Clavius' version of 1586, which adopted Maurolico's iconographic conception, and by somewhat reworking Maurolico's diagrams, started the modern tradition of the diagrams of the *Spherics*.

For each of them, in the following sections, we will faithfully reproduce, by means of the electronic program DRaFT,[18] the more representative diagrams to show their peculiar features and their distinctive traits. In order to make a comparison between the examined versions more evident and immediate, in the Appendix, we present a table of all the diagrams considered in the article.

Rather than undertaking a minute analysis of every particular element or single variant, we will try to uncover the more general message each author attempted to convey through his particular graphical choices. From this analysis, it emerges that the different kinds of representation are not the result of merely formal requirements but mirror substantial geometrical requirements expressing different ways of interpreting the sphere and testify to different ways of reasoning about the elements that interact on it.

From a detailed study of individual diagram–proposition couples, we will attempt to grasp the relation between each proposition's content and the related diagram con-

Footnote 15 continued
adiuntur nomina? ... Et horum si quid circumfertur, tot tantisque scatet mendis, ut vix etiam ab authore ipso emendari possit" Maurolico (2002, III).

[16] Maurolico (2002, III): "Ego ... nisus sum collatis priscis exemplaribus et dictorum authorum et aliorum opera complura emaculare et in suum restituere nitorem ...". For a survey about Maurolico's researches and contributions see Moscheo (1988). See also http://www.dm.unipi.it/pages/maurolic/.

[17] F. Maurolico in a letter to Juan de Vega, sent from Messina in August 1556, says: "... si quid in his a tralatoribus bene traditum aut additum fuisset, id non omitterem. Demum, si quid ego quod ad correctionem aut faciliorem demonstrandi viam faceret excogitassem, id cum venia eruditorum traderem" Maurolico (2002, XII, paragraphs 7–8).

[18] This program enables the redrawing of diagrams while faithfully reproducing the original figure. For more details see http://www.hs.osakafu-u.ac.jp/~ken.saito/diagram/index.html.

firming, with increasing evidence, the strong connection of text and drawing as two elements mutually complementing one another in an inseparable unity: Neither alone is sufficient to understand the work's contents. The diagrams visualise the text, and the text explains the role and properties of the drawn elements according to a complementarity in which the conventions of both the written text and the drawn diagram play a fundamental role.

While the study of each geometrical diagram cannot ignore the content of the related proposition, it is also necessary to situate this particular analysis within the broader context of the whole work: its aims and the scientific domain to which it pertains. Proceeding in this way, it becomes possible to suggest an interpretation of the *Spherics*' diagrams that, while not an exhaustive answer to the issues involved, is presented as a working hypothesis for future research. In fact, the extension of the domain of enquiry and the interaction with current studies[19] will progressively lead to a more complete understanding and to a more conscious comprehension of the role of diagrams in the mathematical works.

3 The diagrams of the *Spherics* in the ancient tradition

In Greek manuscripts of the *Spherics*[20] one can recognise two, clearly distinct, homogeneous kinds of representation, which lead us to consider Theodosius' work as being divided, at least as regards to the graphic aspect, into two parts: The first part covers propositions I.1 to II.16;[21] and the second part covers all the other propositions, from II.17 to III.17.

The common feature of the text of all the propositions of the first part is a constant, explicit reference to the sphere, "in the sphere", as the context where the properties under consideration occur and the demonstrations are developed. Surprisingly, in the diagrams accompanying these same propositions, however, the sphere is not depicted.[22] In a number of cases, the circles produced on the sphere by the intersection of variously inclined planes, are all represented as lying side by side in the plane of the diagram. Both the arcs and linear segments lying on different planes are flattened, and objects lying on the far side of the sphere are turned out, into the plane of the figure, to make them visible on the plane of the diagram.

[19] For a survey of recent results see De Young (2005), Eastwood and Grasshof (2003), Keller (2005), Netz (1999), Saito (2006).

[20] The critical editions in Heiberg (1927) and Czinczenheim (2000) are taken as representatives of the Greek text; while for the study of the diagrams the ancient manuscripts Vat.Gr.204 and Marc.Gr.301 have been directly examined.

[21] Diagrams of III.1 and III.2 can be added to this group: they are two lemmas of solid geometry in which the considered circles do not lie on a sphere.

[22] It should be noted that in Vat.Gr.204, three diagrams—those for propositions I.13, I.14, and I.15—do not agree with this general condition; while in the Hebrew-Arabic manuscript of Florence Biblioteca Laurenziana and in both the Arabo-Latin versions—that of Gerard of Cremona and that commonly attributed to Plato of Tivoli (in manuscripts as well as in the printed editions of 1518, 1529)—their shape is consistent with all the other diagrams of the first part of Theodosius' work, leading one to believe that the latter was their authentic form.

In contrast with this approach, the diagrams of the second part of *Spherics* offer a clear image in perspective view of the sphere as the actual context in which the geometric elements interact. These images invalidate the common hypothesis according to which Greek geometers did not have access to the techniques of three-dimensional representation. They rather compel us to suppose the existence of a precise and consistent iconographic approach according to which the author has clearly distinguished between the two parts of his work through the graphic aspect. This opens the problem of decoding, understanding and bringing out the implicit reasons for this distinction.

Let us consider some general examples, grouping them according to the principal subjects examined in the related propositions.[23]

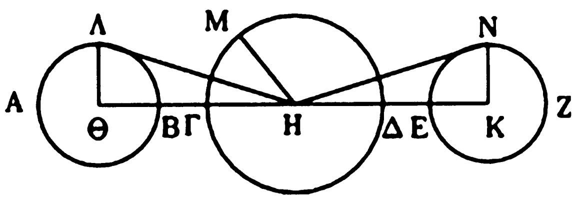
3.1 The first part of the *Spherics*

3.1.1 General properties of circles of the sphere

Proposition I.6

Enunciation: "On a sphere, circles which pass through the sphere's center are the greatest ones; of the others, those which are equally far from the center, are equal, while those, which are farther from the center, are smaller."

Construction: "Let the circles AB, ΓΔ, EZ be on the sphere, of which ΓΔ passes through the sphere's center, while AB, EZ are, in the first case equally far from the sphere's center. I say that ΓΔ is the greatest and AB, EZ are equal. (...) If, in the second case, AB is farther from the sphere's center than EZ; I say that the circle AB is less than EZ."

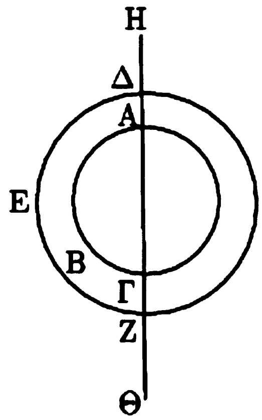


Commentary: In the diagram, the three circles, AB, ΓΔ, and EZ are seen side by side with no attempt at depicting the sphere in which they lie, obscuring any visual evidence that their circumferences are indeed in the same spherical surface. Moreover, the three points Λ, M, and N also lie in the same spherical surface and the line segments joining them to the sphere's centre H are actually radii of the sphere.

[23] In the following, the manuscript Vat.Gr.204 is taken as the representative of the Greek graphical tradition. Its diagrams are faithfully reproduced by means of the electronic program *DRaFT* (see n. 18), only the labelling has been modified by systematically adopting Greek capital letters. The English text of *enunciations* and *constructions* is our translation based on Heiberg's critical edition. The only translations in a modern language of the *Spherics* existing today are the French ones in Ver Eecke (1959) who translates Heiberg Greek text; and Czinczenheim (2000) giving a new critical edition of the Greek text followed by a translation.

This kind of representation, however, clearly shows the plane triangles ΛΘH and NKH: The two planes on which they lie, even if different and differently inclined, are identified with the single plane of the diagram. This makes it easier to visualise their elements on which the proof depends by applying one of Euclid's criteria for congruence to ΛΘH, and NKH[24] to prove that the line segments ΛΘ, and NK are equal. Since these are radii of the circles AB, and EZ, the proposition is demonstrated.

Proposition II.1
Enunciation: "On a sphere, parallel circles have the same poles."
Construction: "For, on a sphere, let circles ABΓ, ΔEZ be parallel. I say circles ABΓ, ΔEZ have the same poles."



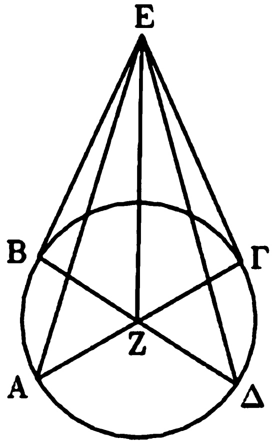
Commentary: The representation clearly shows the parallelism of the two circles ABΓ, and ΔEZ depicted as concentric.
The two points H, and Θ, which are the poles of the parallel circles, are not represented on the spherical surface, but as two general points in the plane of the diagram; and the straight line connecting them is simply superimposed on the two circles without any graphical evidence of the perpendicularity of this line to the planes of the parallel circles, despite the fact that this perpendicularity is a necessary condition for the proof of the proposition.

3.1.2 Properties of the axis of a circle lying on a sphere

Proposition I.7
Enunciation: "If, on a sphere, there is a circle and a straight line is traced, both passing through the sphere's centre and the circle's centre, then this line is perpendicular on the circle."

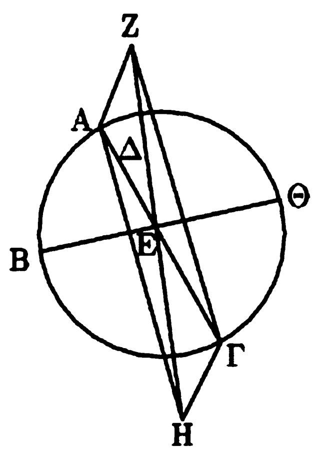
[24] In the plane triangles ΛΘH, and NKH, the angles ΛΘH, and NKH are equal by construction, the sides HΘ, and HK are equal by hypothesis, while the sides ΛH, and HN are equal because they are radii of the sphere.

Construction: "Let ABΓΔ be a circle on a sphere, E the sphere's centre, and Z that of the circle; and let the straight line EZ be traced. I say that EZ is perpendicular to the circle ABΓΔ."



Commentary: The diagram represents the circle ABΓΔ as lying in the plane of the figure, and its diameters AΓ, and BΔ passing through the circle's centre Z.
It is impossible to recognise E as the centre of the sphere by looking at the diagram alone.

Proposition I.8
Enunciation: "If on a sphere there is a circle, the straight line passing through the sphere's centre that is perpendicular to the circle will pass also through the circle's poles."
Construction: "On a sphere let there be the circle ABΓ, and let point Δ be the sphere's centre. Through Δ let there be drawn the straight line perpendicular to ABΓ's plane, intersecting it in the point E; so that E is the centre of the circle ABΓ. Extending the straight line EΔ, it intersects the sphere's surface in points Z and H. I say that Z, H are the poles of the circle ABΓ."



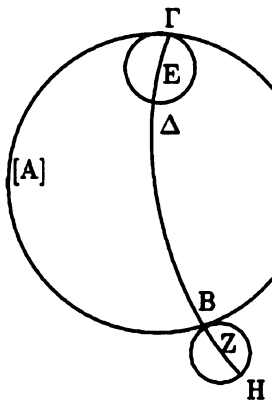
Commentary: As in the previous example, here there is no graphical evidence of the specific function of points Z, Δ, H in the sphere: Δ represents the centre of the sphere, Z and H represent the poles of circle AB which lie on the sphere's surface, which is not represented.

3.1.3 Relations between tangent, parallel, and intersecting circles of the sphere

Proposition II.6

Enunciation: "If on a sphere a great circle is tangent to one of the circles which lies on the sphere, then it is tangent also to another, equal and parallel to it."

Construction: "For, on a sphere, let the great circle ABΓ be tangent to one of the circles, lying on the sphere, ΓΔ in point Γ. I say that the circle ABΓ is also tangent to another circle which is equal and parallel to ΓΔ."



Commentary: In this diagram, through point E, the pole of circle ΓΔ, and the point of tangency Γ, the great circle ΓEΔBZH is drawn,[25] on whose circumference the arc BZ is taken equal to arc ΓE to determine the second pole Z of circle ΓΔ. Thus, Z is also the pole of all the circles parallel to it,[26] of which the one passing through B is tangent to ABΓ at B[27] and is equal to ΓΔ.[28]

The figure depicts all the circles as lying in the plane of the figure so as to show more clearly the property of tangency.

This graphic representation concentrates attention on an auxiliary object, the arc of the circumference ΓEH, and makes it clear that both the points of tangency and the poles of the tangent circles lay on this arc, which is fundamental to the proof of the proposition.

[25] *Theodosius* I.20.

[26] *Theodosius* II.1.

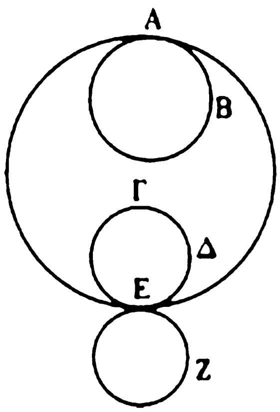
[27] *Theodosius* II.3.

[28] By construction the distance ZB of the pole Z to the circumference BH is equal to the distance EΓ of the pole E to the circumference ΓΔ.

Proposition II.7

Enunciation: "If, on a sphere, there are two equal and parallel circles, then a great circle which is tangent to one of them, is also tangent to the other."

Construction: "Let two equal and parallels circles, AB, ΓΔ, be on a sphere. I say that a great circle which is tangent to AB, is also tangent to ΓΔ."

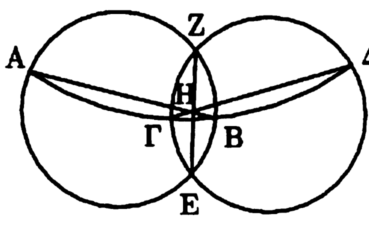


Commentary: The proof uses a *reductio ad absurdum* argument that relies on the impossibility of two equal circles ΓΔE, and EZ both tangent to the great circle AE at the same point E lying on the same sphere.

Proposition II.9

Enunciation: "If, on a sphere, two circles intersect one another, and through their poles is drawn a great circle, then it will cut their arcs in equal parts."

Construction: "For, on a sphere, the circles ZAEB, ZΓEΔ intersect one another in points Z, E, and the circle AΓBΔ is drawn through their poles. I say that AΓBΔ cuts the arcs of ZAEB, ZΓEΔ in equal parts, namely, that the arc ZA is equal to AE, ZB is equal to BE, ZΓ to ΓE, and ZΔ to ΔE."



Commentary: The diagram shows clearly the common section of the two circles, EZ, and their circumferences' arcs whose equality must be shown. The proof points out the relations of perpendicularity and bisection of the linear segments ZE, AB, and ΓΔ determined by the intersection of the planes of the three circles.

Visually, it is not easy to grasp the spatial arrangement of the circles on the sphere.

3.2 A suggested interpretation

The diagrams that we have examined are not anomalous cases. They express general trends to which all the diagrams of the first part of the *Spherics* systematically conform. This kind of drawing makes it difficult to recognise the spherical reality. The correct interpretation of these diagrams is disclosed only by reading the text. In fact, it is the construction given in the text which enables the reader to rebuild every drawn element in its spherical reality, and thus make clear the spatial relationships among them. All of this leads us to interpret this "plane" representation, characterising the first part of the *Spherics*, as a particular choice adopted by the author to graphically concentrate the attention, case by case, on specific elements which have a fundamental role in each proposition, to the detriment of the spherical reality in which these properties take place.

If one considers comprehensively the complete contents of each of these propositions, one notices that they examine general properties pertaining to the domain of pure geometry. One then naturally infers that in this homogeneous technique of representation, Theodosius considered the first part of his *Spherical elements* as an extension of Euclid's *Elements of geometry*,[29] whose graphic tradition—especially that of the stereometric books—is followed by him in the first part of his *Spherics*.

In contrast, starting with the seventeenth proposition of the second book, the diagrams are more consistent with the domain of spherical geometry, which is represented in a realistic way. The sphere becomes fundamental as the context containing all the elements involved in the proposition, which are now drawn in perspective view. The propositions they pertain to analyse specific properties and relations whose chief function is to be an instrument particularly useful in astronomy. It becomes necessary to adopt a more complete graphical representation of the spherical context in its whole by means of diagrams whose content can be easily shifted to the celestial reality. In fact, if one identifies the sphere now actually represented with the celestial sphere, it is easy to recognise the ecliptic, the equator, the solstitial and equinoctial colures, the right and oblique horizon, and parallel and meridian circles, from which one can then compare the relations between particular arcs corresponding to the celestial coordinates, such as declination, right and oblique ascension, and latitude and longitude.

Let us consider some examples of diagrams pertaining to the second part of *Spherics*.

[29] This continuity is confirmed by the version traditionally attributed to Plato of Tivoli, where the three books of Theodosius' *Elements of Spherical Geometry* are numbered the 16th, 17th, and 18th, following the fifteen books of Euclid's *Elements of Geometry*. This occurs not only in manuscripts, but also in both the printed editions of 1518 and 1529, although these volumes do not contain Euclid's *Elements*. In these editions of the *Spherics*, the references to propositions legitimizing Theodosius' demonstrations never specify the title of the reference work—Euclid's *Elements* or Theodosius' *Spherics*. Distinguishing them is clear since books I, VI, and XI are Euclid's, while Theodosius' are numbered XVI, XVII, and XVIII. These citations and the link they seem to reveal between Theodosius' three books and Euclid's traditional fifteen support Lorch (1996)'s attribution of this Latin version of the *Spherics* to Campanus of Novara, who reworked Euclid's *Elements* and may also have joined these two works.

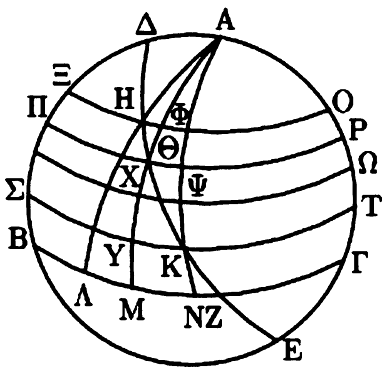
3.3 The second part of the *Spherics*

3.3.1 Relations between arcs employed in astronomy

Proposition III.6

Enunciation: "If the pole of parallel circles lies on the circumference of a great circle, to which two other great circles are perpendicular, one of which is one of the parallels and the other of which is inclined to the parallels, and two equal and consecutive arcs are cut off on the inclined circle on the same side with respect to the greatest of the parallel circles, and the great circles passing both through the parallels' poles and the arcs' extremities, are drawn, then they mark, on the greatest of the parallels, two unequal arcs, of which the one nearer to the original great circle is greater than the farther."

Construction: "For, on the circumference of the great circle ΑΒΓ let Α be the pole of the parallel circles, and the great circle ΑΒΓ be perpendicular to two great circles ΒΖΓ, ΔΖΕ, of which ΒΖΓ is one of the parallels, while ΔΖΕ is inclined to the parallels. On ΔΖΕ let two equal and consecutive arcs, ΚΘ, ΘΗ, be cut off on the same side with respect to the greatest of the parallels ΒΖΓ; and describe the great circles passing through Α and through Η, Θ, Κ respectively, ΑΗΛ, ΑΘΜ, ΑΚΝ. I say that the arc ΛΜ is greater than the arc ΜΝ."



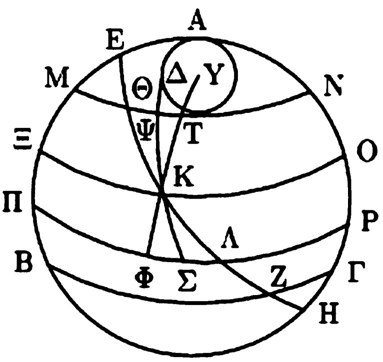
Commentary: Since this proof uses propositions already demonstrated in the *Spherics*, its diagram can be visualised entirely on the surface of the sphere, with no reference to internal objects.

On the celestial sphere, where ΒΖΓ is considered as the equator and ΔΕ as the ecliptic, the proposition shows that the right ascension of equal arcs on the ecliptic increases continuously with the distance from the solsticial point Ζ. For example, Gemini's right ascension is greater than Taurus' right ascension, which is in turn greater than that of Aries.

Proposition III.7

Enunciation: "If, on a sphere, a great circle is tangent to one of the circles lying on the sphere, and another great circle, inclined to parallels, is tangent to another circle that is parallel to the first and greater than it, being the tangent point on the first great circle considered; and on the inclined great circle two equal and consecutive arcs are cut off, on the same side with respect to the greatest of parallels; the parallel circles, passing through the extremities of these arcs, then cut off on the first great circle considered two unequal arcs, of which the one nearer to the greatest parallel is greater than the farer one."

Construction: "For, on a sphere, let the great circle ABΓ be tangent to one of the circles lying on the sphere AΔ at point A; let another great circle EZH, inclined to parallel circles, be tangent to a circle which is greater than AΔ, having the tangent points E, H on ABΓ; and let BZΓ be the greatest of parallels. Then, on the inclined circle EZH, let two equal and consecutive arcs ΛK, KΘ be cut off on the same side with respect to the greatest parallel BZΓ; and through the points Λ, K, Θ let the parallel circles ΠΛP, ΞKO, MΘN be described. I say that the arc ΠΞ is greater than the arc ΞM."



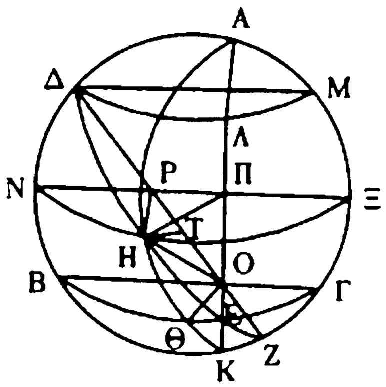
Commentary: Since this proof uses propositions already demonstrated in the *Spherics*, its diagram can be visualised entirely on the surface of the sphere, with no reference to internal objects.

On the celestial sphere, where ABΓ is the oblique horizon, Y the visible pole, and EZH the ecliptic, then the ortive amplitude of equal arcs taken on the ecliptic diminishes with the distance from the solstitial point Z. For example, the ortive amplitude of Gemini is smaller than that of Taurus, which is in its turn smaller than that of Aries.

Proposition III.11

Enunciation: "If the pole of parallel circles lies on the circumference of a great circle to which two other great circles are perpendicular, one of which is one of the parallels and the other of which is inclined to the parallels, and another great circle described through the poles of the parallels intersects the inclined one between the greatest of the parallels and the parallel circle which is tangent to the inclined one; then the ratio between the diameter of the sphere and the diameter of the circle which is tangent to the inclined one is greater than the ratio between the arcs determined respectively on the greatest parallel and the great inclined circle by the first great circle and the one described through the poles."

Construction: "For, on the circumference of the great circle ABΓ is placed the pole A of parallel circles, and the circle ABΓ is perpendicular to two great circles BEΓ and ΔEZ, of which BEΓ is the greatest of parallels and ΔEZ is inclined on them; another great circle AHK described through the parallels' poles intersects ΔEZ between BEΓ and the circle which is tangent to ΔEZ, and let ΔΛM be the circle which is tangent to ΔEZ. I say that the ratio between the diameter of the sphere and the diameter of the circle ΔΛM is greater than the ratio between the arc BΘ and the arc ΔH."



Commentary: The diagram shows both the elements on the sphere's surface, whose interactions produce the arcs considered in the thesis, as well as the plane sections inside the sphere producing secondary auxiliary lines marking the plane triangles, which allow one to prove the thesis using Euclid's plane theorems.

On the celestial sphere, BEΓ being the equator, ΔEZ the ecliptic and ΔM a tropic circle, this theorem concludes that the ratio between the right ascension and the corresponding arc of the ecliptic is smaller than the ratio between the sphere's diameter and the tropic circle's diameter.

This drawing reveals that Greek geometers could just as well have adopted this same graphic technique in the diagrams of the first part of the *Spherics*, and offers a further argument that the graphic choices are motivated by reasons linked to the distinction between the ambits of interest. It is worthy of note that the text of the *Spherics* always expresses itself in purely geometrical terms without any explicit reference to astronomy.[30] It is only the diagrams which translate and reveal different conceptual exigencies such as that the work's content pertains to two different domains of mathematical knowledge, geometry and astronomy.

[30] Some remarks containing the transposition of Theodosius' geometric properties on the circles of the celestial sphere appear for the first time in Voegelin (1529). Voegelin himself says in his note to I.17: "Accommoda schema hoc sphaerae armillari et nihil te remorabitur". These contributions are placed at the end of the demonstrations and introduced by his name "Voegelin", see also n. 9. Only in the 17th century, did Pierre Herigone, in his edition of the *Spherics*, introduce as "Corollaries" the astronomical transpositions of Theodosius' theorems, being the first author to follow Voegelin's suggestion. See Herigone (1644).

4 The early printed diffusion of *Spherics*: Plato of Tivoli's version

In the course of our analysis, particular attention must be paid to the Arabo-Latin medieval version usually ascribed to Plato of Tivoli,[31] for it was the basis of the early printed diffusion of the *Spherics* with the three editions in the sixteenth century. These editions transmitted Theodosius' work with additions that enriched and widened its contents in a substantial manner: The definitions were doubled; the number of propositions increased from 59 to 79; and the work's structure was modified, with some propositions originally in the third book being transferred to the second. These textual changes do not correspond to any similar attitude towards the graphical content, which was maintained unchanged, preserving the previous tradition to which the new diagrams, relating to the new propositions, also conformed, as the following examples will show.[32]

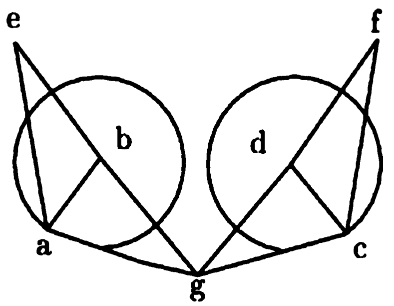
4.1 The first part of the *Spherics*

4.1.1 General properties of circles of the sphere

Proposition I.33

Enunciation: "The circles described on a sphere which have straight lines joining their poles to their circumferences equal are equal."

Construction: "Let two circles *ab*, *cd*, whose poles are *e* and *f*, be described on a sphere whose center is *g*, and extend the straight lines from their poles to their circumferences *ea*, *fc*, which are equal. I say that circles *ab*, *cd* are equal."



Commentary: This style of representation is clearly similar to those found in the diagrams of the Greek tradition that are adopted to illustrate the general properties of circles on the sphere. As in I.6, here the two circles *ab*, *cd* are seen side by side with no attempt at depicting the sphere in which they lie, obscuring any visual evidence that a single spherical surface holds both the poles *e*, *f* and the circumferences of

[31] See n. 6.

[32] For the study of this Latin version the manuscripts Q.112.sup and C.241.inf in Milan, Biblioteca Ambrosiana, and the printed editions of Lucantonio Giunta and Ottaviano Scoto's heirs have been directly examined. The diagrams of the printed edition published in Venice by Lucatonio Giunta in 1518 are faithfully reproduced here by means of the electronic program *DRaFT* (see n. 18). This edition uses both capital and lower case Latin letters on the diagrams (sometimes also in the same diagram); we always use lower case letters.

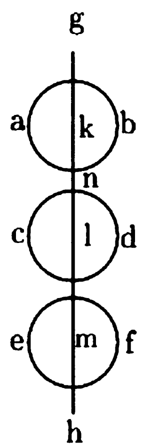
the circles, in particular the points a and c. Moreover, the four lines ag, eg, fg and cg joining these points to the sphere's centre, g—whose role is also unrecognizable in such a drawing—are radii of the sphere.

The plane triangles *aeg*, *gfc*, however, are clearly shown, and the two planes on which they lie, even if different and differently inclined, are identified with the plane of the diagram: the same plane where we also see the two circles *ab*, *cd*, even if they are orthogonal to them.

Proposition II.3

Enunciation: "On a sphere there are exactly two equal and parallel circles."

Construction: "Otherwise, let three equal and parallel circles *ab*, *cd*, *ef* be described on a sphere: by II.1 they will have the same poles g, h, and by I.12 the line joining g and h will be perpendicular to their planes and pass through their centres, which are points k, l, m, and through the sphere's centre which is point n. Since the straight lines *nk*, *nl* and *nm*, by definition, determine the distance of each circle from the sphere's centre, it will follow, by I.6, that these three lines are equal, but it is impossible. So, on a sphere there are exactly two equal and parallel circles."



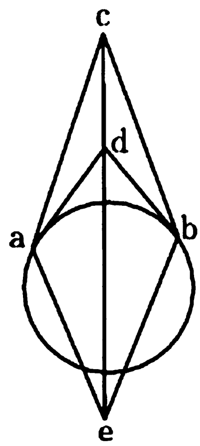
Commentary: The diagram places side by side in the plane of the drawing all three circles supposed equal and parallel on the same sphere, and the straight line joining the two common poles g, h. This line is drawn through them without any graphical evidence that it is orthogonal to the circles' planes. In this representation one can easily recognise the same graphic choice adopted by the ancient tradition for II.1. The sphere is not depicted, and it is impossible to graphically recognise the point n's role as the centre of the sphere.

4.1.2 Properties of the axis of a circle lying on the sphere

Proposition I.13

Enunciation: "Every straight line joining one of the poles of a circle to the sphere's centre, if prolonged, will necessarily also pass through the second pole of the circle."

Construction: "For, let the circle *ab*, whose pole is *c*, be described on a sphere, whose centre is *d*, and draw the line *cd* which, being prolonged, intersects the sphere's surface in point *e*. I say that point *e* is the second pole of the circle *ab*."



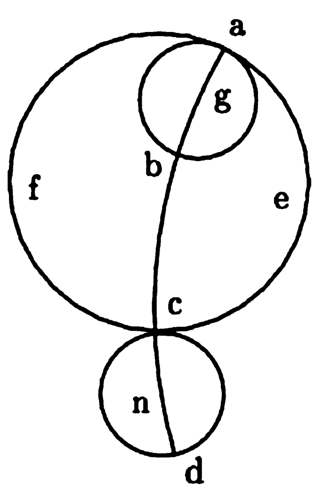
Commentary: As in the ancient tradition, here there is no graphical evidence of the specific functions of points *c*, *e*, *d* on the sphere: The point *d* represents the sphere's centre; and *c* and *e* represent the poles of circle *ab* lying on the sphere's surface, which is not depicted at all.

4.1.3 Relations between tangent, parallel, and intersecting circles of the sphere

Proposition II.9

Enunciation: "Every pair of parallel circles on the sphere to both of which a <single> great circle is tangent are equal."

Construction: "On a sphere, let *ab*, *cd* be two parallel circles to both of which the great circle *ef* is tangent in points *a*, *c*. I say that circles *ab*, *cd* are equal."



Commentary: In this representation, one can easily recognise the same graphic choice adopted in the ancient tradition.

5 Francesco Maurolico's graphic innovation

Throughout the course of the transmission of the *Spherics*, in Greek, Arabic and Latin, Theodosius' diagrams remained essentially the same[33] until 1558, when Francesco Maurolico published his *Theodosii sphaericorum elementorum libri tres ex traditione Maurolyci,* [34] which inaugurated a more advanced phase of the Renaissance recovery of ancient mathematics. The Sicilian mathematician took as his source Plato of Tivoli's Arabo-Latin medieval version, of which he retained all the propositions, adopting the enunciations verbatim. Nevertheless, he introduced substantial changes to the work, adding further propositions, whose number increased from 79 to 84, reworking the demonstrations according to the requirements of modernity and renewal, which also involved the iconographic apparatus.

Maurolico's graphic choice is not a mere formality, nor is it simply motivated by Renaissance texts on perspective. Instead, it mirrors his own attitude toward objects in the sphere, which leads him to prefer to work directly on the sphere's surface. This inclination is attested by Maurolico's original proofs in which arguments concerning arcs are preferred to the arguments concerning plane and rectilinear sections inside the sphere that had prevailed throughout the earlier tradition.

One case may be taken to illustrate the difference between the traditional approach to spherical geometry and Maurolico's. It is problem II.15 in the Greek version, which becomes II.19 *ex traditione Maurolyci*. The two diagrams are very different and mirror the different procedures, whose principal steps can be outlined as follows using faithful reproductions of the figures.

Enunciation: "To construct on a sphere a great circle passing through a given point on the sphere's surface and tangent to a given circle."

Commentary: In order to make the comparison easier, Greek and Latin labels have been changed so as to mark the same points on the different diagrams by means of the same Latin capital letters.[35]

On the sphere's surface, let AB be the given circumference on the sphere, and C the given point. The pole of AB is taken as D.[36]

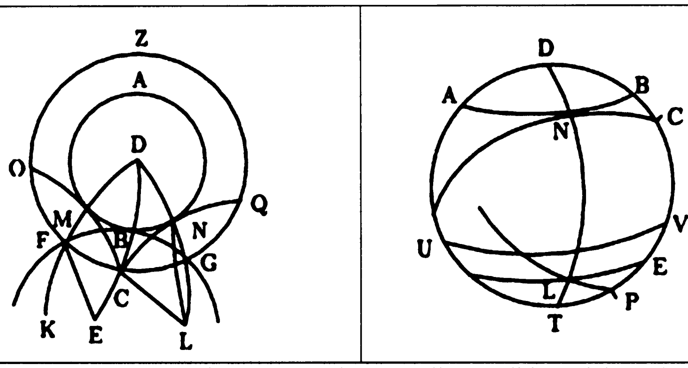
The problem is solved by locating the pole of the required tangent circle. The required pole is a point of the sphere's surface satisfying two conditions: (1) its

[33] We refer to the general stability of the graphic aspects in the overall transmission of the *Spherics*, ignoring minor, individual variants, such as a rotation of 90 degrees.

[34] Maurolico's contribution to spherical geometry is known only through this edition of 1558. No manuscript autograph sources have been found. For further information concerning the editorial vicissitudes of this volume, see Moscheo (1998, pp. 100–132, 309–316); for a bibliographical description of the whole volume's contents, see Moscheo (1998, pp. 283–308).

[35] The diagram of Vat.Gr.204 is reproduced and employed here to illustrate the solution of the traditional version. Both the diagram and the solution remain the same for the entire tradition in all the versions which we have examined; the first meaningful change was made by Maurolico in 1558.

[36] *Theodosius* I.21, corresponding to *Maurolico* I.32.



II. 15 in Vat. Gr. 204 (left), and II.19 in Maurolico's edition (right). The labelling has been modified in order to indicate the same points of the two diagrams with the same letters.

distance from the given point C is a quadrant of a great circumference—or, as the Greek text says, is an arc of a great circle subtended by the side of a square inscribed in it—; and (2) its distance from the given circle AB is a quadrant of a great circumference—or, as the Greek text says, is an arc of a great circle subtended by the side of a square inscribed in it.

The traditional—both Greek and Arabo-Latin—version draws the circle FCG with D as a pole and line DC as a distance.

Then, both the traditional version and Maurolico construct the great circle DCE through points D and C.[37] On this great circle, the traditional version cuts off the arc BE subtended by the side of a square inscribed in it, while Maurolico cuts off the arc BE equal to a quadrant of the circumference ABC: then he draws the circle EL parallel to AB passing through E; while the traditional version draws the circle FBG with E as a pole and line EB as a distance.

FBG is a great circle[38] tangent to circle AB at B[39] and cutting the circumference FCG at F and G. Through points F and G the traditional version considers, respectively, the great circles DFK and DGL also passing through D. On their circumferences arcs FK, GL are cut off equal to CE.

Maurolico, on the other hand, cuts off the arc CP, on the great circumference ABC, equal to a quadrant and then draws the circle PL with the given point C as a pole and arc CEP as a distance. Great circle PL intersects the circumference of EL in two points, one of which is L, whose distance from C is a quadrant of a great circumference by construction. In this way, Maurolico has found a point satisfying the first condition for the pole of the tangent great circle[40]: The great circle with L as a pole passes through C.

[37] *Theodosius* I.20, corresponding to *Maurolico* I.31.

[38] *Theodosius* I.17: by construction, the segment EB, from the pole of circle FBG to its circumference, is equal to the side of a square inscribed in a great circle.

[39] *Theodosius* II.3: circles AB and FBG intersect the great circle DB passing through their poles at the same point on its circumference, B.

[40] *Maurolico* I.21.

The traditional version is more complex and has more steps than Maurolico's. It demonstrates that: (1) arc FC is equal to CG[41]; (2) arcs FK, CE, GL are equal by construction; and (3) the three circles passing through the pole D of FCG are orthogonal to FCG.[42] This means that all the conditions to apply II.12 are verified,[43] and it is possible to conclude that the rectilinear segment FE is equal to the rectilinear segment CL. However, since FE is equal to the side of a square inscribed in a great circle,[44] also CL is equal to the side of a square inscribed in a great circle: L satisfies the first condition, and the great circle having L as a pole passes through C.

Let us now consider the second condition. Maurolico does this more simply by considering the great circle through L and D.[45] On the circumference LD, the parallel circles AB and LE cut off an arc NL equal to arc BE,[46] which is a quadrant by construction. This means that L satisfies the second condition, and the great circle CN having L as a pole will be tangent to AB at N. The traditional version draws the rectilinear segment NL, proving that it is the side of a square inscribed in a great circle as follows. Arcs DF, DC, and DG are equal, since D is a pole of circle FCG,[47] while arcs DM, DB, and DN are equal, since D is a pole of circle AB,[48] and hence their differences, MF, BC, and NG are equal. By construction, arcs FK, CE, and GL are equal; hence, the sums MK, BE, and NL are equal. By construction, arc BE is subtended by the side of a square inscribed in a great circle; then, arc NL is subtended by the side of a square inscribed in a great circle, too. That means that the rectilinear segment NL is also the side of a square inscribed in a great circle. Therefore, L satisfies the second condition, and it will be the pole of the great circle CN tangent to AB in N.

The traditional version proves in the same manner that the point K also satisfies the conditions, and hence, on the same sphere there are exactly two great circles both passing through the same given point C and tangent to the given circle AB. Maurolico states this in a corollary to the proposition.

6 Maurolico's graphic realism

Maurolico substantially modified those figures which in the previous tradition were given a plane schematic representation. He reinterpreted them according to the new perspective. The sphere itself was given priority, it became the actual context in which

41 *Theodosius* II.9: circle FBG intersects FCG in points F and G, and DBC is the great circle through their poles, so DBC divides arcs FBG, FCG in halves.

42 *Theodosius* I.15.

43 *Theodosius* II.12: CE is orthogonal to FC, and LG is orthogonal to CG, arc FC is equal to arc CG and arc CE is equal to arc GL.

44 *Theodosius* I.16.

45 *Maurolico* I.31.

46 *Maurolico* II.15.

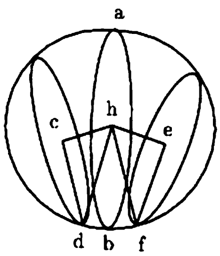
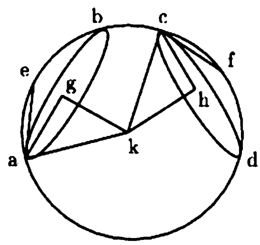
47 *Theodosius* definition 5 of the first book.

48 *Theodosius* definition 5 of the first book.

the geometric elements interact. The diagrams are drawn three-dimensionally and express the content of the propositions by showing the objects involved in correct proportions and as a whole. His aim is to represent the relations and properties, geometrically proved, in a realistic way.

The following examples reproduce some diagrams *ex traditione Maurolyci* to show how Maurolico transformed the traditional ones.[49] We have chosen diagrams which correspond to the propositions already examined above. For this reason the enunciations are not repeated in the following table.

6.1 The first part of the *Spherics*

<table>
<tr><td colspan="3">1. General properties of circles of the sphere</td></tr>
<tr><td>Proposition I.6

</td><td>Proposition I.34

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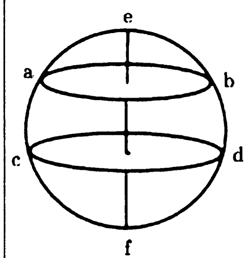
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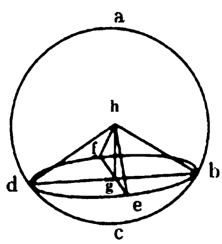
<tr><td colspan="3">2. Properties of the axis of a circle lying on the sphere</td></tr>
<tr><td>Proposition I.8

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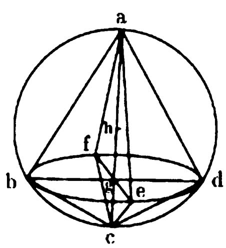
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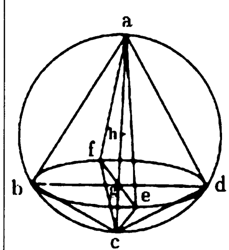
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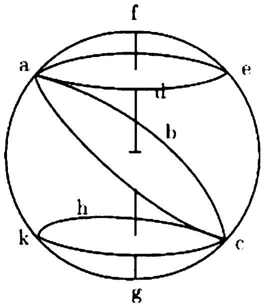
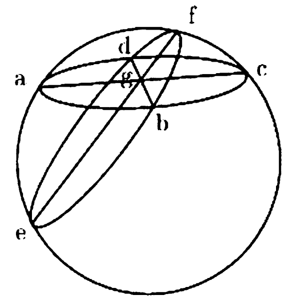
</table>

[49] The diagrams from the printed edition of *Theodosii Sphaericorum elementorum libri tres ex traditione Maurolyci* published in Messina in 1558 are faithfully reproduced by means of the electronic program *DRaFT* (see n.18).

<table>
<tr><td colspan="2">3. Relations between tangent, parallel, and intersecting circles of the sphere</td></tr>
<tr><td>Proposition II.7

</td><td>Proposition II.12

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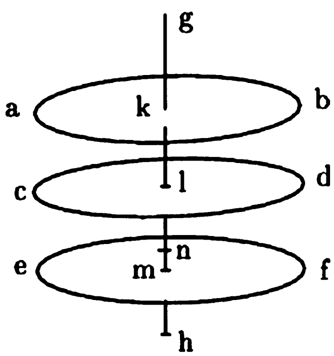
6.2 *Reductio ad absurdum*

Even diagrams belonging to propositions whose demonstrations use *reductio ad absurdum* arguments maintain a coherence with the realism of the representation generally adopted by Maurolico. These proofs assume situations which cannot actually occur on the sphere, so that they are depicted by means of diagrams where the circles, supposed in particular unlikely conditions, remain indefinitely suspended in the plane of the figure without any reference to a sphere. The following is a selection of the more meaningful examples.

Proposition II.3
Enunciation: "On a sphere there are exactly two equal and parallel circles."
Construction: "Otherwise, if possible, three equal and parallel circles *ab*, *cd*, *ef* are described on a sphere …"



Commentary: This proposition corresponds to II.3 in Plato of Tivoli's version, and it is not found in the Greek tradition.

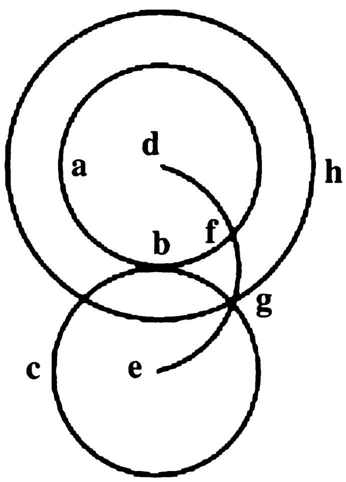
Maurolico maintains the same lettering as his mediaeval source, but he modifies the diagram's shape, using perspective techniques to represent the three circles lying on three different parallel planes and showing the perpendicular position of their axis by means of a realistic representation that depicts only its visible parts and interrupts it in the parts covered by the circles' surfaces.

The demonstration, using a *reductio ad absurdum* argument, proves, as in the mediaeval source, the impossibility of three equal and parallel circles existing on the same sphere. The diagram represents them as suspended in space without locating them on a sphere.

Proposition II.5

Enunciation: "On a sphere, every great circle passing through the poles of a pair of circles that intersect one another will pass also through their point of tangency."

Construction: "On a sphere, let circles *ab*, *bc* be tangent in point *b*; and a great circle pass through their poles *d*, *e*. I say that the great circle *de* passes through the point *b*."



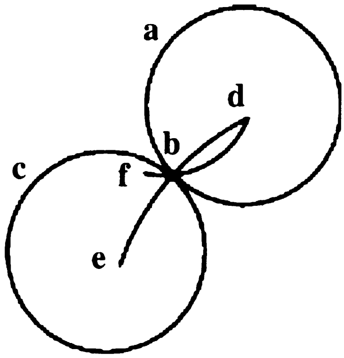
Commentary: This proposition corresponds to II.5 in Plato of Tivoli's version and to II.4 in Vat.Gr.204.

Maurolico repeats the same figure and the same demonstration, which used a *reductio ad absurdum* assumption already in the ancient tradition.

Proposition II.6

Enunciation: "On a sphere two tangent circles are given, the great circle passing through the poles of one of them and through the tangent point will also pass through the poles of the second circle."

Construction: "On a sphere, let circles *ab*, *bc*, be tangent at point *b*; and let a great circle pass through the pole *d* <of *ab*> and the tangent point *b*. I say that the great circle *db* will also pass through the pole *e* <of *bc*>."



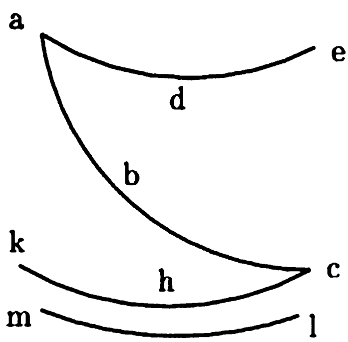
Commentary: This proposition corresponds to II.6 in Plato of Tivoli's version and to II.5 in Vat.Gr.204.

Maurolico uses the same demonstration, which, already in the ancient tradition, used *reductio ad absurdum* assumptions.

Proposition II.8

Enunciation: "If on a sphere there are two equal and parallel circles, every great circle which is tangent to one of them is also tangent to the other."

Construction: "On a sphere, let there be two equal and parallel circles *ade*, *chk*, and let a great circle *abc* be tangent to *ade* in point *a*. I say that the same circle *abc* is tangent to *chk*, too."



Commentary: This proposition corresponds to II.8 in Plato of Tivoli's version and to II.7 in Vat.Gr.204.

Maurolico definitively modified the traditional figure. The contrast between this diagram and the realistic approach of II.7 in Maurolico, seen above, in which circles are physically represented as partially covering the axis *fg*, reveals the author's desire to make graphically evident the impossibility of the "absurd" hypotheses actually occurring on a sphere. In fact, the proof uses a *reductio ad absurdum* argument that relies on the impossibility of three equal and parallel circles *ade, chk, lm* existing on the same sphere. Consequently, the sphere is not drawn.

6.3 *Consequentia mirabilis*

Propositions I.1, I.16, I.17, and I.33 seem to contradict Maurolico's graphic approach as presented thus far. The style of the diagrams is the same as those in the demonstrations that use *reductio ad absurdum* assumptions, in which the explicit negation of the thesis leads immediately and evidently to the absurdity of the assumed condition. As a matter of fact, in I.1, I.16, I.17, and I.33 the negation of the thesis is also implicitly assumed, but, this time, the reasoning leads instead to a proof of the thesis in a direct way, without any contradiction occurring:

> It is the most surprising thing to be discovered since the origins of the world, namely to prove something from its negation by means of a demonstration which does not lead to an impossible condition, and so the demonstration could not be done without appealing exactly to the negation of the conclusion.[50]

And, as Clavius will say in the *Scholium* to proposition I.12 in his version of Theodosius' *Spherics*:[51]

> Here, you indeed see a surprising kind of demonstration. In fact, from the fact that G is said not to be the sphere's centre, it is proved, by an affirmative demonstration, that G is the sphere's centre.[52]

In Maurolico's *Spherics*, these diagrams represent the circles and the straight lines without placing them on the sphere; however, in their demonstrations no "absurd" hypothesis is explicitly evoked. Hence, the simple addition of a circle would be enough to bring these diagrams into line with the rest of Maurolico's figures.

Actually, these cases show that Maurolico's iconographic approach is more refined and goes beyond the simple formal distinction between real and impossible conditions. They reveal that the Sicilian mathematician pays particular attention to demonstrations and, more precisely, to the distinction between different methods of proof.

For these four propositions, in which the reasoning is carried out following on the *consequentia mirabilis*,[53] he adopts a technique of representation similar to the one adopted for the *reductio ad absurdum*. The negation of the thesis, which remains implicit at the textual level, becomes evident exclusively through the image joined to the proposition. Therefore, the fact that the sphere is not portrayed in the diagram is not simple carelessness. It represents an important clue, because it reveals Maurolico's

[50] Cardanus (1570, p. 231) *Propositio 201, Scholium*: "Et est res admirabilior quae inventa sit ab orbe condito, scilicet ostendere aliquid ex suo opposito, demonstratione non ducente ad impossibilem et ita, ut non possit demonstrari ea demonstratione nisi per illud suppositum quod est contrarium conclusioni, velut si quis demonstraret quod Socrates est albus quia est niger, et non posset demonstrare aliter, et ideo est longe maius Chrysippeo Syllogismo."

[51] It corresponds to Maurolico's I.17, quoted below.

[52] Clavius (1586, p. 17): "Hic vides mirabilem sane argumentandi modum. Nam ex eo, quod G, dicitur non esse centrum sphaerae, demonstratum est demonstratione affirmativa, G esse centrum sphaerae."

[53] Both Cardanus and Clavius, in their praise of this kind of demonstration, define it as "surprising", in Latin "admirabilis" or "mirabilis". For this reason, today we refer to it as to the *consequentia mirabilis*. Maurolico never refers to it in an explicit way, but the diagrams reveal his awareness of the application of this specific method. For further details see Freguglia (1999).

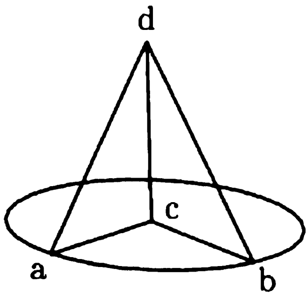
mastery of logical-deductive discourse, understanding and consciously applying the *consequentia mirabilis*.

This attention to logical-deductive discourse, which we see in these latest diagrams, testifies again to Maurolico's sensitivity to the problems discussed at this time and in particular to the debate *de certitudine mathematicarum*, which attempted to articulate the methods which must be followed to produce valid geometrical reasoning.

Proposition I.1

Enunciation: "When a plane surface intersects a spherical surface, the intersection is a circle."

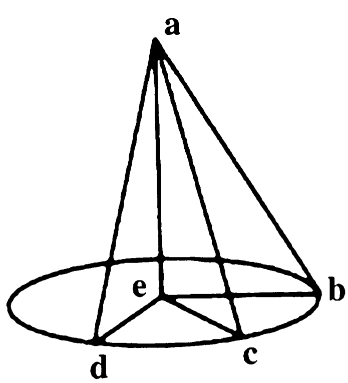
Construction: "A plane surface intersects a sphere, and let the line *ab* be their intersection. I say that *ab* is the circumference of a circle."



Proposition I.33

Enunciation: "If from a point on a sphere's surface, more than two equal straight lines are drawn to the circumference of a circle lying on the same sphere, then the point is a pole of the circumference."

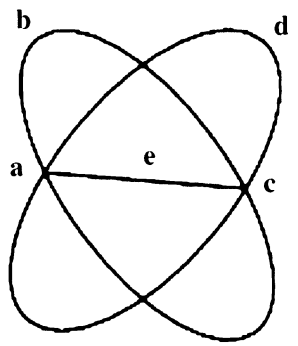
Construction: "From the point *a* on the sphere's surface to the circumference *bcd* of a circle are drawn three straight lines *ba*, *ac*, *ad* which are equal. I say that the point *a* is a pole of the circle *bcd*."



Proposition I.16

Enunciation: "All the great circles on a sphere divide themselves into equal parts."

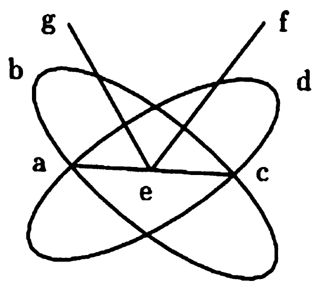
Construction: "Let two great circles *abc*, *adc* be described on a sphere whose centre is e. I say that each of them divides the other into equal parts."



Proposition I.17

Enunciation: "All the circles on a sphere dividing themselves into equal parts are great circles."

Construction: "On a sphere let the circles *abc*, *adc* divide themselves in semicircles. I say that they are two great circles."



7 Concluding remarks

In every mathematical work the diagrams and the text are the end result of some process of selection, made by the author of each redaction, of those elements that are essential to each proposition. However, this is the result of a previous elaboration during which the mathematician makes properties already proved and new elements converge in a complex interaction gradually leading to the determination of every new enunciation. This prior elaboration—at least in the specific case of spherical geometry—cannot be purely mental or verbal: it also requires the support of graphic elements on which to consolidate the spatial disposition of the geometric objects under consideration. During this prior elaboration each diagram can be considered as a working instrument reflecting the author's choice, so that it then assumes personal value in support of his geometrical thought. The study of images could surely find useful material in authors' working autograph drafts, by means of which one could follow the course of the geometrical reasoning.

The tradition of Theodosius' work, both manuscript and printed, consists only of versions that are in their definitive form. Nevertheless, even in this final form, not only each diagram but the entire iconographic apparatus exhibits the choices of the author, while reflecting the practices of an epoch: the ancient one represented by Theodosius; that of the Renaissance represented by Maurolico and then Clavius, whose version, followed in successive editions, began the modern tradition.

In each version of the *Spherics*, the diagrams retain an inseparable and deep connection with the text, and they express the theoretical commitments of the mathematician who produced them. By means of the iconographic apparatus the author gives prominence to specific information, committing to the diagrams important conceptual messages that can not always be understood directly from the text alone.

If the distinguishing feature of Greek mathematical texts has always been acknowledged to be rigour and concision—avoiding redundant hypothesis and arguments, or superfluous definitions, postulates and enunciations—it seems that now we must recognise that the same formal precision and attention was also extended to the graphical aspect, revealing an even closer connection between these two parts of the mathematical expression than had hitherto been recognised.

It seems meaningful to note that the context "on the sphere" is expressed in the text, not only when the sphere is not depicted in diagrams, but also when just a generically spherical context is represented: "on the sphere" is missing from the text only when it is graphically drawn in a canonical configuration identified by a family of parallel circles and a great circle passing through their poles, perpendicular to them.[54] In this case, the sphere was consistently drawn in a realistic manner that made it redundant to mention the sphere itself in the text.

It is therefore evident that the ancient tradition paid particular attention to the iconographic apparatus, which also shows the different approach that Greek mathematicians took with regard to the two different domains—the geometrical and the astronomical—by means of two different ways of representing the sphere. The analysis of purely geometrical properties and relations is usually represented on the plane without any indication of spatiality, thereby making spherical geometry conform to Euclidean plane geometry by virtue of the individual elements' belonging to the plane sections which generate them and on which they lie. In contrast, the analysis of the relation between celestial magnitudes and coordinates is represented on the sphere, which thence becomes the actual context in which the astronomical phenomena under consideration occur.

In the Renaissance, Francesco Maurolico, far from the classical sensibility, modified the diagrams of *Sphaericorum elementorum libri*. He eliminates the previous distinction between geometrical and astronomical domains introduced by the Greeks, and prefers to give to the graphical aspect the role of bearing different messages that integrate and complete the text. The Sicilian mathematician reinterprets his medieval source, developing a version whose title *ex traditione Maurolyci* emphasises his departure from the tradition. His new and individual manner of interpreting the spherical

[54] See, for example, *supra* III.6 of the Greek tradition.

elements certainly concerns the text itself, but it is also reflected, in an evident and substantial way, in the representation of the diagrams supporting the text. In order to make as clear and evident as possible the interactions between the geometrical objects in true perspective, he sacrifices the representation of individual specific properties in favour of a holistic view in which he brings into the foreground the sphere, to which the entire work is devoted. Thus one can recognise in Maurolico the propensity to visualise all the elements involved in the proposition representing them in the way they actually interact.

It is precisely because of this realism that, in the edition of the *Sphaericorum elementorum libri ex traditione Maurolyci,* we find two homogeneous groups of diagrams whose distinguishing character remains the presence or the absence of the sphere. Nevertheless, this distinguishing element is adopted by Maurolico to express the possibility or impossibility that the conditions expressed in the hypotheses could realise themselves on the spherical context. So, while the Greeks, by means of the graphical aspect, rigourously distinguished the domains, geometry and astronomy, to which the examined properties pertain, in Maurolico's work this distinction changed into a differentiation between the different demonstrative methods implicit in the text.

In 1586, the edition of the *Spherics* realised by Clavius was published in Rome. Working in an epoch where the recovery of ancient mathematics was complete, he could consider didactical problems, which he then addressed in textbooks. This project included the edition of the *Spherics*. Clavius' principal aim was to make the work's contents clear and comprehensible through both the text and diagrams. For this reason, he states that he preferred the diagrams introduced by Maurolico to those of the ancient tradition, because they "make understanding the spherical elements much more easy."[55] He introduces some changes to his source's images: all the parts involved in the proposition are depicted by perspective on the sphere, even when they cannot really occur in the spherical context. In Clavius' version most of the diagrams pertaining to the technique of *reductio ad absurdum* and all those pertaining to the *consequentia mirabilis* modify those of Maurolico, adding a circle which represents the sphere.[56] In this way, Clavius made a conscious decision to leave awareness of the different demonstrative methods employed to the text and the reader's competence.[57] In subsequent editions of the

[55] Clavius (1586, p. 3): "Figuras quoque, quae in graeco exemplari extant, plerunque neglegimus, quod illae, quas Maurolicus pinxit, commodiores sint, et ad intelligendas res sphaericas multo faciliores." For informations about the personal acquaintance of Maurolico and Clavius see Moscheo (1998, pp. 214–221).

[56] See, for example, Clavius' propositions I.1; I.11; I.12; *Scholium* I following I.21; *Scholium* following II.2.

[57] For Clavius' attitude toward explicitly commenting on demonstrative methods, one should note his *Scholium* to I.1 of Euclid's *Elements*. This *scholium* presents all the syllogisms involved in a single demonstration, giving an example of the so-called *demonstratio explicita*. He concludes the example by saying that the structure of every proposition, not only those of Euclid but of all other mathematicians, is the same, but this "resolutio" is not explicitly given by mathematicians who prove the assumption in a briefer and easier way: "Ut autem videas, plures demonstrationes in una propositione contineri, placuit primam hanc propositionem resoluere in prima sua principia, initio facto ab ultimo syllogismo demonstrativo.... Non aliter resolui poterunt omnes aliae propositiones, non solum Euclidis, verum etiam caeterorum mathematicorum. Negligunt tamen Mathematici resolutionem istam in suis demonstrationibus, eo quod brevius, ac facilius

Spherics, Clavius' diagrams were adopted, beginning the more modern graphical tradition.

The complex transmission of the *Spherics* shows that, as a matter of fact, there was no evolution or progressive refinement and improvement of the graphical expression. Rather, one recognises a number of very different and distinct manners of conceiving of the function of the diagram, which, while overlapping one another, each time removed earlier criteria and imposed new graphic choices.

Appendix

Table of diagrams examined

Since a complete edition was not the aim of this study, the following table intends to give a comprehensive overview of the diagrams examined in this article. In order to show similarities and differences in the whole tradition, they are faithfully reproduced[58] from the three representative versions examined: (1) the manuscript *Vaticanus Graecus 204*; (2) the printed edition of the medieval Arabo-Latin version of Plato of Tivoli, published by Lucantonio Giunta in 1518; and (3) the Renaissance version of Francesco Maurolico printed in 1558.

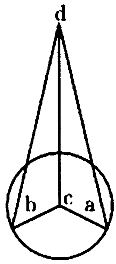
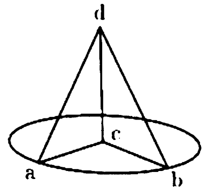
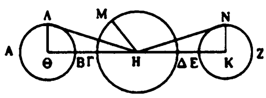
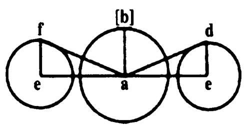
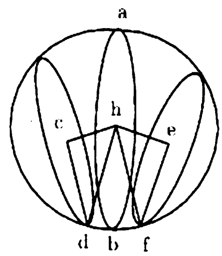
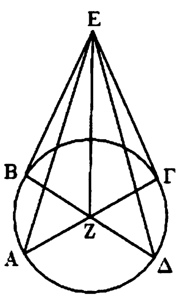
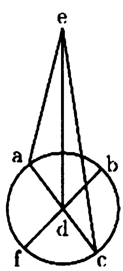
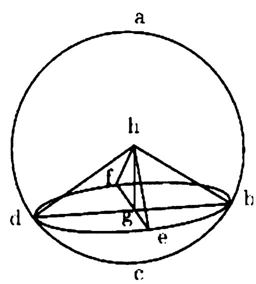
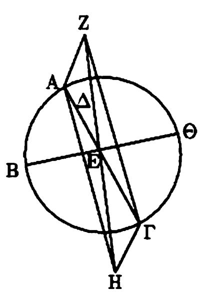
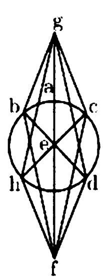
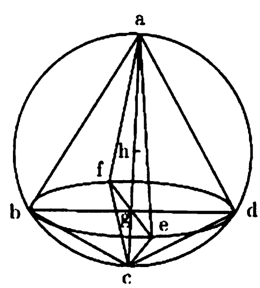
Comment on diagram II.18 in Plato of Tivoli's version. In this diagram, the arc *abc* of circle *abc* is not drawn; the circle *edh* lacks the arc *dh* which allows one to mark an important point for the construction, namely, *h*, which is the pole of the great circle *fbg*; and the linear segments *ak*, *dk*, *cl*, and *gh* are not drawn.

Comment on diagram III.4 in Plato of Tivoli's version. This diagram contains some inaccuracies. The three great circles which must pass through *h*, *m*, and *q*, respectively, do not pass through them. On the circumference of the circle *ebd*, the point *z* should instead be *i*, being the intersection of the circumference *ebd* with the circumference of the great circle through *z* (the pole of parallels) and *m*. The arc *pti* should be *pqt*, where *q* is the extremity of one of the two equal and consecutive arcs marked on *abc*, and *t* (not *i*) is the extremity of the parallel circle through *q*. There are two points *v* on the parallel circle which should pass through a point *x* which is not indicated.

Comment on diagram III.12 in Plato of Tivoli's version. This diagram too contains some inaccuracies. The point *p* should be *d*. The linear segment *ch* is the intersection of the planes of circles *nhs*, *phz*, and so in the diagram the point *c* should be placed exactly on the section of the two linear segments *ns*, *pz*, while it is marked as a point of the diameter *pz* which does not lie on *ns*. Similarly, the point *h* is the intersection of three circumferences, *nhs*, *phz*, and *ahtk*, while the diagram represents three different points: the intersection of the circumferences *ns* and *pz*, *atk* and *pz*, *atk* and *ns*.

Footnote 57 continued
sine ea demonstrent id quod proponitur, ut perspicuum esse potest ex superiori demonstratione" (Clavius 1612, T. I, p. 28). As regards the *consequentia mirabilis*, Clavius describes this method in a *Scholium* to IX.12 of Euclid's *Elements* and to I.12 of Theodosius' *Spherics*.

58 The same editorial criteria, already described in the course of the article (see notes 18, 23, 32, 49) have also been applied in the following appendix.

Vat.Gr. 204	Plato of Tivoli	Maurolico
I.1 Figure omitted	I.1 	I.1 
I.6 	I.6 	I.6 
I.7 	I.7 	I.8 
I.8 	I.8 	I.9 

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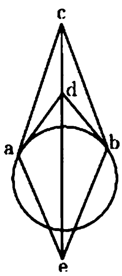
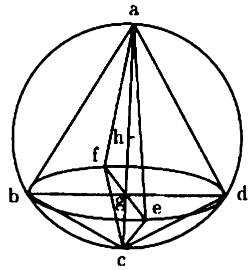
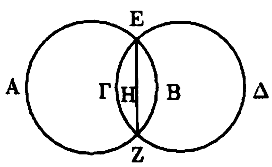
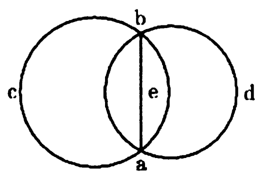
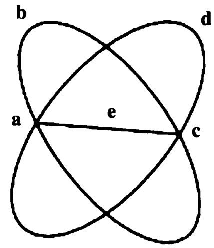
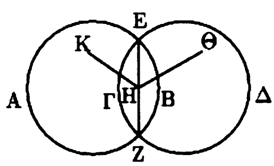
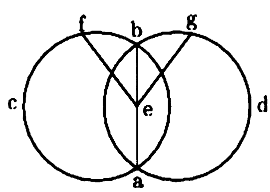
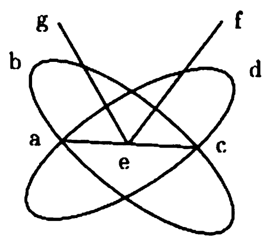
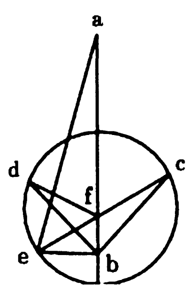
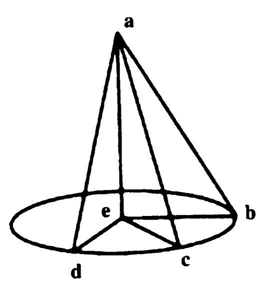
Lacking proposition	I.13 	I.15 
I.11 	I.15 	I.16 
I.12 	I.16 	I.17 
Lacking proposition	I.32 	I.33 

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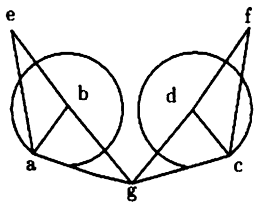
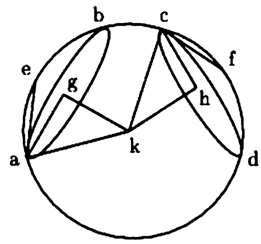
</td><td>I.34
</td></tr>
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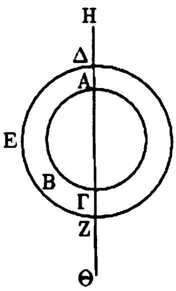
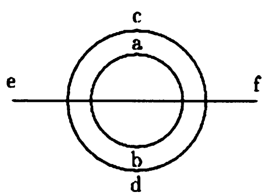
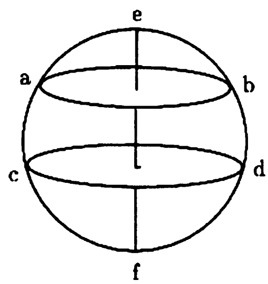
</td><td>II.1
</td><td>II.1
</td></tr>
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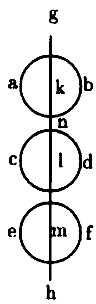
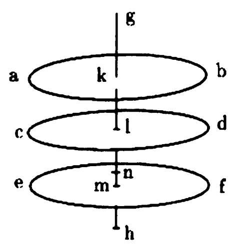
</td><td>II.3
</td></tr>
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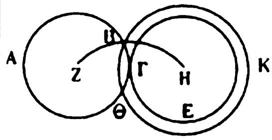
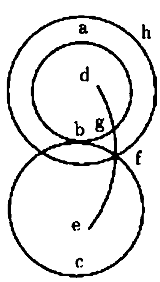
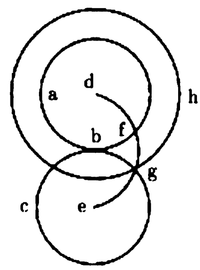
</td><td>II.5
</td><td>II.5
</td></tr>
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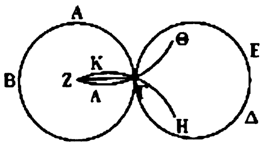
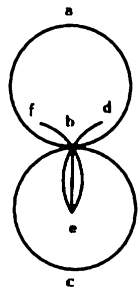
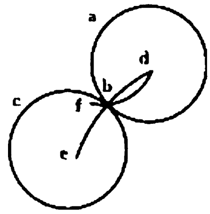
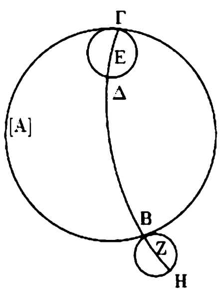
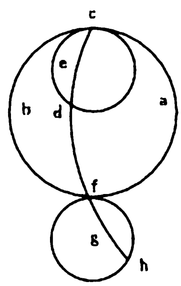
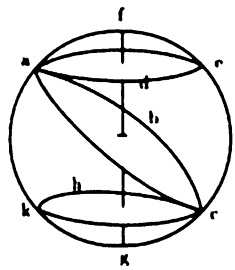
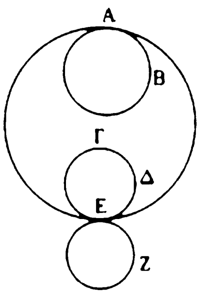
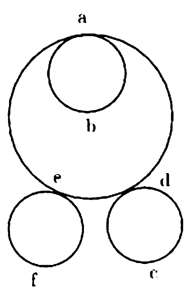
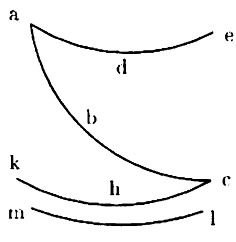
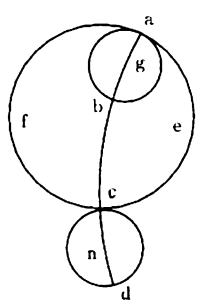
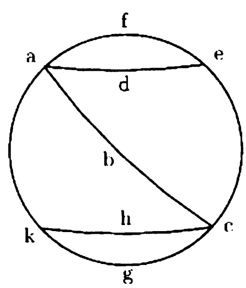
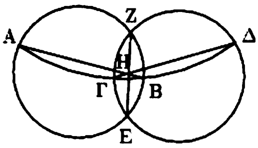
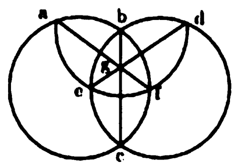
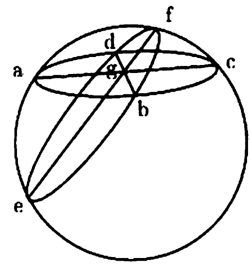
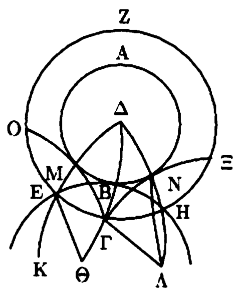
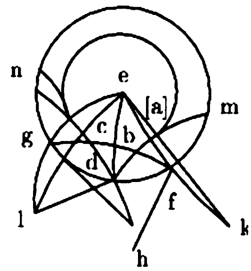
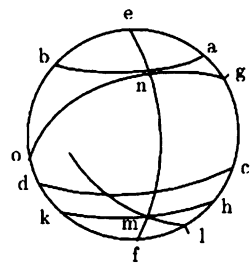
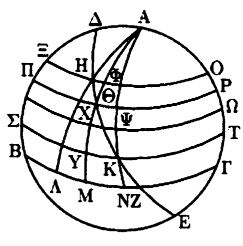
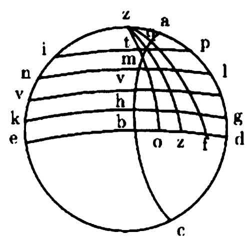
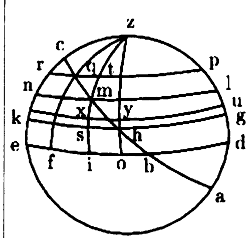
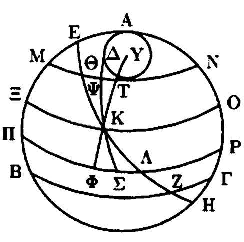
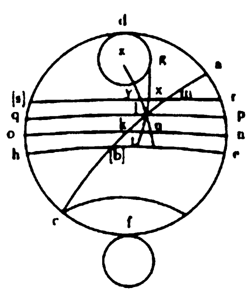
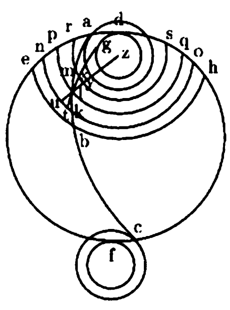
Vat.Gr. 204	Plato of Tivoli	Maurolico
II.5 	II.6 	II.6 
II.6 	II.7 	II.7 
II.7 	II.8 	II.8 
Lacking proposition	II.9 	II.9 

Vat.Gr. 204	Plato of Tivoli	Maurolico
II.9 	II.12 	II.12 
II.15 	II.18 	II.19 
III.6 	III.4 	III.4 
III.7 	III.5 	III.5 

Vat.Gr. 204	Plato of Tivoli	Maurolico
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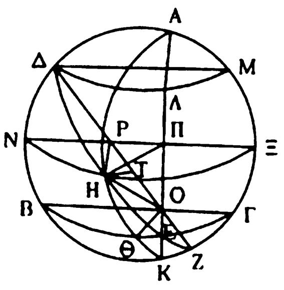	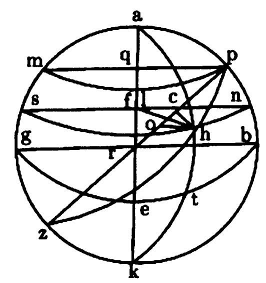	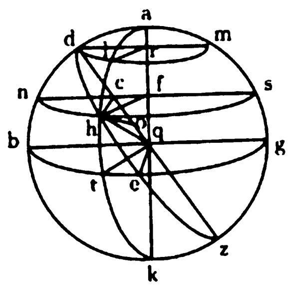

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