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The geometry of burning mirrors in Greek antiquity. Analysis, heuristic, projections, lemmatic fragmentation

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Abstract The article analyzes in detail the assumptions and the proofs typical of the research field of the geometry of burning mirrors. It emphasizes the role of two propositions of the Archimedean *Quadratura parabolae*, never brought to bear on this subject, and of a complex system of projections reducing a *sumptōma* of a parabola to some specific linear lemmas. On the grounds of this case-study, the much-debated problem of the heuristic role of analysis is also discussed.

1 Introduction

The research domain of the geometry of burning mirrors has been a long-lived one. Diocles, the main character in the field, was a contemporary of Apollonius; he mentions an earlier contribution by Dositheus, the addressee of most of the surviving treatises of Archimedes; he corrects, in his treatment of spherical burning mirrors, a proposition of the Euclidean *Catoptrics*. Anthemius of Tralles, the latest datable author from whom we have interesting technical contributions, was active in the sixth century of our era. Thanks to a small number of simplifying hypotheses, the study of burning mirrors was given a prominent geometrical connotation: the result is a subgenre mid-way between the writings of pure geometry and the more diffusely argumented treatises of applied mathematics. The surviving studies are short tracts, treating particular problems in the theory of conic sections or, in the case of spherical mirrors, in elementary plane geometry.

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This very circumscribed research subject became the ideal domain for a subtle series of variations on a single theme, offering as side products two pointwise constructions of a parabola burning at a given distance (that is, with given vertex and parameter) and the first attestation of the focus-directrix property. I shall first (Sect. 2) quickly review the "external" evolution of the field, discuss its main hypotheses, and describe the available sources. I shall then offer a detailed analysis of the technical results, setting them in a historical perspective. The main interpretative points of this article are the following:

(1) Showing that there is a very strong link between the proofs of the focal properties of a parabola (Sect. 3) and of a circumference (Sect. 4). I shall also suggest that a spur to carrying out the investigation of the focal properties of a parabola was a concern with showing that the property of the subtangent is characteristic of the curve (in ancient jargon, it is a *sumptōma* of it), namely, a property univocally determining it (Sect. 5).
(2) Making the connection between the Dioclean proof of the focal properties of the parabola and the method of analysis and synthesis explicit (Sect. 3) and discussing, on the basis of this example, the alleged heuristic value of the analysis (Sects. 5, 6). I shall also clarify the reason why Diocles' treatment of the parabola is divided in two parts (proof of its focal properties and construction with given vertex and parameter) and assess the technical and historical import of it (Sect. 6).
(3) Showing that all the attested proofs, a newly discovered text in Arabic probably coming from a Greek source included, can be viewed as successive refinements of one basic idea, and outlining a historical development of the domain of research radically different from those that have been proposed so far (Sects. 4–7). I shall emphasize the role of two propositions of the Archimedean *Quadratura parabolae*, never brought to bear on this subject, and of a complex system of projections reducing a *sumptōma* of a parabola to some specific linear lemmas (Sect. 7).
(4) Finally, identifying and discussing the process of singling out, from Diocles to later authors, a 4-point lemma as the "technical" core of the proof, to be reused and adapted in different geometric configurations (Sects. 6, 7). More generally, I shall use this example as a basis to briefly argue that such a "sensibility to structure", displaying in the identification of the "core" technical lemmas, became a prominent feature of the late approach to mathematical research, witnessed in its most spectacular form by the fragmentation of the *corpus* we observe in Pappus' *Collectio* (Sect. 8).

2 The research domain of the geometry of burning mirrors. Available texts

The study of the visual rays (emanating from the eye and making visual perception possible) was in Greek antiquity kept distinct from the study of light rays, although the related disciplines started from the same hypotheses. This division was a canonical one, and we read an account of it in an extract from Geminus (*Def.* 135.12, in *HOO* IV, 104.9–106.13). The historical record at our disposal, however, attests for a division still unaccomplished in the first treatise dealing, albeit marginally, with burning mirrors: Euclid's *Catoptrica*. In this study, after prop. 28 has emphasized the strategic role of the middle point of the radius of the sphere when objects are seen in concave spherical mirrors, prop. 30 studies, passing from the visual to the solar rays, the burning properties of the same mirrors. Even if it is unsafe to assume that the *Catoptrica* as we read it can be assigned *recta via* to Euclid, Diocles' testimony in the preface of his *On burning mirrors* suggests that the prime mover of the whole tradition was exactly *Catoptrica* 30. In Diocles' treatise, the separation among the research domains is completed, remaining a remarkable feature of all subsequent elaborations.

The sources available in Greek are the short and incomplete tract *On surprising mechanisms* by Anthemius of Tralles, whose manuscript tradition depends entirely on the opening bifolium of the *Vat. gr.* 218 (critical editions in *MGM*, 78–87, and *CG*, 349–59). It contains approximate constructions of an elliptical mirror and of a parabolic mirror burning at a given distance. In the first instance (*MGM*, 78–81), the construction is grounded on the identification of single points on the line, resulting from the equality of the angles that the "broken" straight line drawn from a point of the ellipse to the two foci makes with the tangent to the line through that point (cf. *Con.* III.48). The transition from the approximate solution to the curve is poorly argumented by Anthemius. He shows first that the isolated points on the line he determines in the above way satisfy the property that the sum of their distances from two fixed points A, B is constant, and concludes: 'if we stretch a chord that is fixed around points AB and passes through the origin of the rays that are about to be reflected, the said line will be traced, that will be a part of the so-called ellipse' (*MGM*, 81). It seems thus as if Anthemius thinks that the ellipse is *defined* by this construction, whereas in Apollonius, *Con.* III.52, one finds only the proof of the converse. At about the same period as Anthemius we must assign the anonymous of the *fragmentum mathematicum bobiense*, a very short and incomplete fragment preserved in palimpsest in the ms. *Ambros.* L 99 sup. (*MGM*, 87–90). The latter contains an elegant proof of the focal properties of a parabola and a partial result about spherical mirrors.

The other two sources have been transmitted in Arabic translation only (the Arabic Anthemius does not add anything new; see *CG*, 286–315 and Jones 1987). One of them is Diocles' *On burning mirrors*, in its extant form a compilation of quite disparate results, some of which we read also in a fairly divergent Greek version in Eutocius (*AOO* III, 160.2–174.4). Only five propositions of Diocles' treatise deal in fact with burning properties of mirrors (Toomer 1976, pp. 44–71 = *CG*, 102–116). They are as follows :

Prop. 1: focal properties of parabolic mirrors, followed by the constructions of a paraboloid with exact burning properties and of mirrors burning along a circumference (2 alternative solutions) or, more generally, along a closed path.
Prop. 2 and 3: focal properties of spherical mirrors.
Prop. 4 and 5: pointwise construction of a parabola using the focus-directrix property.

The second source in Arabic is a tract of a certain Dtrūms (*CG*, 155–213), displaying a series of lexical and syntactical features strongly suggesting that it is the translation of a Greek writing (*CG*, 155–157). The first part of this work contains an uninteresting lemma and paraphrases of propositions extracted from Apollonius'

Conica (*Con.* I.1, 3, 4, 6, 11, 20, 33, 35). An anonymous redactor inserted them in place of the first two books of the original treatise, probably containing analogous material. Six propositions follow: props. 10–13 about parabolic mirrors, props. 14–15 about spherical mirrors.

Diocles (Toomer 1976, pp. 34–44 = *CG*, 98–102) precedes his exposition with interesting preliminary remarks. Among other things, he outlines the historical context in which the domain of research developed and points out clearly which were the main assumptions underlying the process of mathematical modelling of burning mirrors. Diocles asserts that the discoverer of the focal properties of the parabola was Dositheus: 'one of those problems, namely the one requiring the construction of a mirror which makes all the rays meet in one point, is the one which was solved practically by Dositheus' (Toomer 1976, p. 34). A problem is raised by the verbal form designating Dositheus' achievement, *amila-hā*, that Rashed translates 'construit' (*CG*, 98). As we see from his translation, Toomer (cf. also 1976, pp. 16 and 140) favours instead a material meaning, and asserts that Dositheus did not propose a geometric proof of the result. If the Greek verb was *kataskeuazein*, then it is not clear whether it had only practical connotations or it was employed metonymically for 'to prove' (cf. for instance Pappus, *Coll.* III.21). A rigorous proof was in fact well within reach for a contemporary of Archimedes: the focal property, as we shall see, is an immediate consequence of two properties that were well known before Apollonius. An 'approximate' or mechanical construction would have raised Diocles' criticisms, who reacts against this in the case of spherical mirrors.

After a description of the parabolic mirror, Diocles expounds properties and use of spherical burning mirrors. One finds a clear reference to *Catoptr.* 30, and then the following statement (Toomer 1976, p. 38):

> an ingenious method has been found for a burning-mirror to burn without being turned to face the sun; instead it is fixed in one and the same position, and indicates the hours of the day without a gnomon. It does this by burning a trace to which the rays are reflected: the reflecting produces a trace for the position of the hour which is sought.

It has long been supposed that the mirror indicating the hours without a gnomon should be a parabolic one, and the first of Diocles' constructions of a mirror burning along a circumference has been regarded as directed to this end (Toomer 1976, pp. 143–144; Hogendijk 1985). However, the most likely hypothesis is that it is a spherical mirror, since any such surface has an axis of symmetry at every point: therefore, it always presents to the sun a portion capable of good burning performances (Sesiano 1988).

The proof of the focal properties of a paraboloid requires that the rays coming from the sun are parallel. Diocles, following a long-standing tradition, formulates this condition as the request that the earth be considered as a point. He discusses the issue at length in his preface, probably because this assumption, which is absent in the *Catoptrica*, was the break-through idea that allowed one to set up a soluble model. Diocles expressly asserts that this hypothesis is strictly applicable only to the model developed in the treatise, and that the falsity of an unqualified statement does not prevent one to consider it as acceptable in a restricted context.

A further, implicit, hypothesis has been assumed already in the *Catoptrica*: it is enough to study a suitable section of a spherical mirror in order to determine its properties; the mirror is then obtained by rotating the section. In the spherical case, in fact, an incident and a reflected ray always lie in one and the same plane (*El.* XI.2), which is univocally determined by the further constraint that the mirror image be located along the line joining the object with the centre of the sphere (*Catoptr.* def. 5); this plane identifies a great circle on the surface of the sphere as a section. In the case of a paraboloid, one supposes by analogy that the plane of reflection passes through its axis. Therefore, one needs only to study the focal properties of a parabola, the mirror being again obtained by rotation. All of this entails that any burning mirror must have an axis.

3 Parabolic burning mirrors

A parabola was obtained, before the Apollonian reform, by cutting a right-angled cone with a plane perpendicular to one of its sides. It is not difficult to show that the resulting line is univocally determined by the following «principal» *sumptōma* (see Fig. 1): if from a point A of a parabola of vertex V a straight line AH is dropped perpendicular to the axis, then

$$s(AH) = r(VH, a),$$

where $s(AH)$ is the square described on straight line AH, $r(VH, a)$ the rectangle contained between straight lines VH and a (a notation used henceforth), a is a given segment, called "upright side" (cf. *Con.* I.11, where the validity of the condition is extended to the "ordinates" referred to any diameter). The *sumptōma* was often used in the "ratio form", where the upright side is "factored out": if A' is another point on the parabola and straight line $A'H'$ is dropped perpendicular to the axis, then (cf. *Con.* I.20)

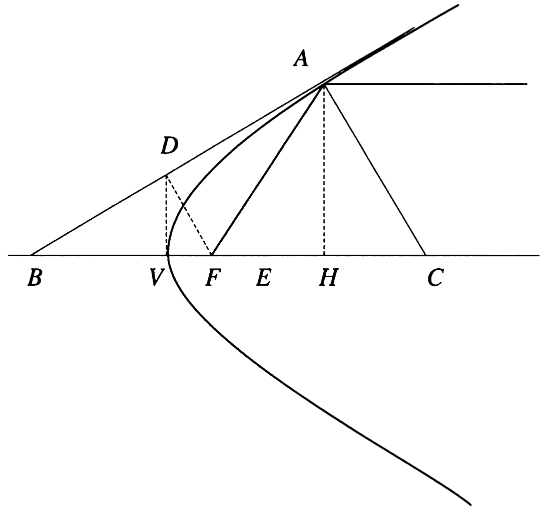
$$VH : VH' :: s\,(AH) : s\,(A'H')\,.$$

If from A we draw the tangent AB and the perpendicular to it AC as far as their intersections with the axis, then a triangle ABC is produced which is right-angled at A and let me call it the *fundamental triangle* associated to A. Of the two abscissas determined by the perpendicular AH, BH is called *subtangent*, HC *subnormal*. One shows that

(1) the subtangent is bisected by the vertex of the parabola, $BV = VH$,
(2) the subnormal is constant and equal to half the upright side, $HC = a/2$.

These two basic properties were known at least as early as Archimedes (cf. *Quadr.* 2 and *Con.* I.35; *Fluit.* II.4 and *Con.* V.13, 27, respectively). In enunciating the subtangent property, he asserts that it is an elementary result and does not bother to prove it. It is important to keep in mind that the subtangent property holds with respect to any diameter, the perpendicular to the axis being replaced by the ordinate dropped from A

Fig. 1



(this is the so-called "oblique conjugation"), while the subnormal property obviously holds only with respect to the axis ("orthogonal conjugation").

I now propose an analysis of prop. 1 of Diocles' *On burning mirrors*, clarifying which properties of reflection of light rays on a line do depend on the nature of the line and which do not (the crucial assumptions are in italics). As a by-product, the reason behind a surprising feature of Diocles' proof will become clear, namely, the fact that he *assumes* that the reflected rays concentrate on a single point at a well-defined distance from the vertex, and *proves* that reflection occurs at equal angles.

Let us suppose that *the line has an axis BC* and that a light ray parallel to the axis meets the line at A (Fig. 1 again). Let ABC be the fundamental triangle associated with A, and AF the reflected ray. *If reflection occurs at equal angles*, then the normal AC bisects the angle between incident and reflected ray; moreover, *the incident ray being parallel to the axis*, each of the two halves of this angle is equal to angle ACF; finally, the incident and the reflected rays make equal angles also with the tangent AB, any of these angles being equal to angle ABF: therefore the triangles ABF and AFC are isosceles on bases AB and AC, respectively. Since AF is common, it results that $BF = FC$. Conversely, if this equality holds, then reflection occurs at equal angles (all deductive steps can be inverted). Now, let a perpendicular AH be dropped from A to the axis of the line. *If the line is a parabola*, then two further constraints are imposed: HC is constant and the vertex of the parabola is the middle point of BH. Accordingly, let us mark on the axis a segment VE equal to half the upright side of the parabola: by the property of the subnormal one has $VE = a/2 = HC$. Let us now turn back to the crucial equality $BF = FC$: it holds if

$$VF = FE \tag{1}$$

$$BV = EC \tag{2}$$

Of these equalities, (1) determines where to place the intersection of the reflected ray with the axis (namely, at a distance from the vertex equal to 1/4 of the upright side), the bright idea being to realize that (2) depends only on the system tangent–normal attached to a point of the parabola, independently of any considerations about reflected or incident rays. To see this, it is enough to observe that, on the one hand, the property of the subtangent gives $BV = VH$, while, on the other hand, VE is such that $VE = HC$; adding EH to both sides of this equality gives $VH = EC$; from this and from $BV = VH$, (2) immediately follows by transitivity. One has only to reassemble the several pieces of the proof in a consequent way.

As said above, that an analysis of this kind underlay Diocles' proof of his *theorem* is suggested by the fact that he proves the equality of the angles and assumes reflection on a single point of the axis of a parabola. Of course, this presupposes that the parabola was already identified as the line enjoying the exact focal properties. How could this guess be made? We shall return on this in Sect. 5. After Diocles' proof is known, it is reasonable to restore the "most natural" logical order, namely, the one starting from the equality of angles to prove that, in the case of a parabola, the reflected rays concentrate on a single point on its axis. This is done by Dtrūms in his prop. 10. His proof can most naturally be viewed as singling out what really is crucial in Diocles': the fact that ABF is an isosceles triangle. Dtrūms' elegant argument amounts to an application of *El.* II.8 to the four points $HFVB$, namely, in this order: projection of the point of incidence on the axis/focus/vertex/ intersection with the axis of the tangent through the point of incidence. In fact (Fig. 1), since $BF = FA$, by *El.* I.47 one has that

$$s(BF) = s(AH) + s(FH),$$

but, by *El.* II.8,

$$s(BF) = s(FH) + 4r(HV, VF),$$

since $BF = HV + VF$: therefore,

$$s(AH) = 4r(HV, VF),$$

that is, the principal *sumptōma*, referred to the axis, of a parabola with vertex V and upright side $4VF$. This move allows Dtrūms to bypass the property of the subnormal and eliminate the most contrived part of Diocles' proof. As we shall see in Sect. 6, *El.* II.8 applied to the same configuration of points yielded the same result features in Diocles' prop. 5: this shows that Dtrūms' approach is a refinement of Diocles' proof, and hence depends on it.

Finally, the *anonymus bobiensis* adopts the same deductive approach as Diocles, while still perceiving that triangle AFC is irrelevant (Fig. 1 again). Its proof that AFB is isosceles is independent of Dtrūms'. The idea is to show that the straight line joining the focus with the middle point D of AB—point D being determined by the property of the subtangent as the intersection of AB with the tangent through the vertex—is perpendicular to AB itself. This is an easy consequence of the characteristic property of the parabola and of the property of the subtangent. For from $BH = 2BV$ and by similar triangles, one has $AH = 2DV$. From this and

$$s(AH) = 4r(HV, VF)$$

it results

$$4s(DV) = 4r(HV, VF),$$

that is,

$$s(DV) = r(HV, VF) = r(BV, VF).$$

It follows that, if DV is perpendicular to BC, triangle BDF is right-angled at D.

Before passing to the spherical mirrors, I briefly discuss two points.

First, some more details about *El.* II.8. It is a four-point lemma whose basic configuration is a line AB cut at a chance point C; one makes $BD = CB$ on AB produced: it results that

$$s(AD) = s(AC) + 4r(AB, BC):$$

the greater square can be obtained from the lesser one by application of a "double gnomon". In adapting this lemma to the geometry of burning mirrors, two cases may occur, depending on whether projection H falls between the focus F and the vertex V or not (Fig. 1 represents the second alternative). Once the segments have been identified allowing the application of the lemma, one has in both cases that $s(FB) = s(FH) + 4r(HV, VF)$, as we have seen. A moment's reflection shows that the equality $BV = VH$ (namely, the property of the subtangent) proves crucial in allowing the application of the lemma.

Second, and most importantly, I explain why all of the above proofs hold only with reference to the axis, namely, in orthogonal conjugation. This issue will prove crucial in the subsequent discussion of the focus–directrix property.

(1) *Diocles*. He employs the property of the subnormal, which holds only if the ordinate AH is orthogonal to the diameter, that is, only if the diameter of reference coincides with the axis of the parabola.
(2) *Dtrūms*. He applies *El.* I.47 in order to have $s(BF) = s(AH) + s(FH)$, and this of course holds only if triangle AHF is right-angled at H, that is, again, only if the ordinate AH is orthogonal to the diameter.
(3) The *anonymus bobiensis*. The relation $s(DV) = r(BV, VF)$ entails that triangle BDF is right-angled at D only if DV is perpendicular to BC. Since DV is the tangent through the vertex, it is perpendicular to the diameter identifying that vertex if and only if the ordinate AH is (*Con.* I.17, and cf. Archimedes, *Quadr.* 1).

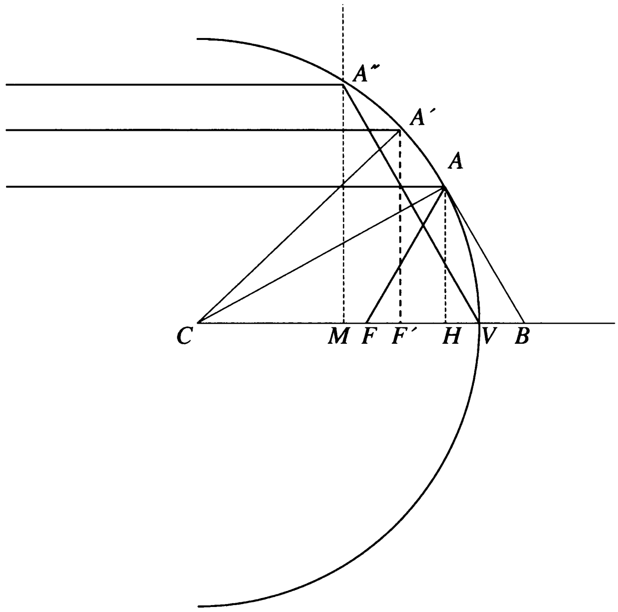


Fig. 2

4 Spherical burning mirrors

Before answering the question left open in the preceding section, let us turn to the spherical burning mirrors. Let a semicircumference be given, C be its centre and CV its axis (Fig. 2). Let us assume as usual that the incident rays are all parallel to the axis. In *Catoptr*. 30, it is shown that the rays coming from a single point on CV produced meet, after reflection, on the segment CV only. Therefore, C is the limiting point of incidence for these rays, this limit being attained only by the reflection of a ray along CV itself. This fact remains true even if the point from which the incident rays emanate 'translates at infinity', in such a way as to make the rays parallel. Diocles' first result (prop. 2) consists in refining the determination of the limiting point in the hypothesis that the rays are parallel: this is the middle point M of radius CV (cf. *Catoptr.* 28). For, let us suppose that a ray parallel to CV and incident at A be reflected as AF. Let CA be joined, and let the fundamental triangle ABC be considered. We have seen in the preceding section that the equality $AF = FC$ does not depend on the nature of the line. But $AF > FV$ (*El.* III.7): therefore, $FC > FV$, and point F lies between M and V. After that, Diocles claims, but does not prove, that the nearer to V a ray falls, the nearer to M will it meet CV, when reflected.

The preceding proof implicitly assumes that the rays meet CV after one single reflection. Dtrūms deals with the issue of multiple reflections; further properties of the reflected rays are investigated by Diocles and the *anonymus bobiensis*. These results can be summarized as follows. Between the semicircumference and its axis CV, let us adapt in succession half of the side of a regular hexagon, of a square and of an equilateral triangle inscribed in the circle that completes the semicircumference. Let them be in this order: AH, $A'F'$, $A''M$ (recall that M is the middle point of CV). Let us now consider the point of intersection with CV of rays parallel to it and falling on arc $A''V$. It is easy to show that the ray falling on the mirror at A'' is reflected at V. As a consequence, only those rays falling on arc $A''V$ intersect the radius CV after one single reflection. Those falling on the complementary arc still intersect CV, but after multiple reflections. In prop. 14(a), Dtrūms shows that in this case, the intersection with radius CV will take place on segment $F'V$ only: for otherwise, the rays intersecting MF' would form, at their last reflection, acute angles with the radius of the semicircumference drawn from C to the point of incidence. This is impossible for rays intersecting CV after multiple reflections.

Let us now return to the case of a single reflection and consider the ray falling at A': since $A'F'$ is half the side of an inscribed square and since reflection occurs at equal angles, $A'F'$ itself is the reflected ray. Therefore, the rays falling on arc $A''A'$ will arrive on segment $F'V$ (Dtrūms, prop. 14(b)), and those falling on arc $A'V$ will arrive on segment MF' (Dtrūms, prop. 14(b), and the same is proven in the Bobbio fragment). But why mark point A? Because Diocles shows in his prop. 3 that the rays falling on arc AV arrive, after one reflection, on a segment MF such that $FV > 5MF$ (and therefore $MF < 1/12\ CV$): a spherical burning mirror of small opening angle concentrates the solar rays well enough. Dtrūms' prop. 15 has no geometrical content that is independent of prop. 14.

The relationships among the several versions are clear: both the *fragmentum bobiense* and Dtrūms' results are refinements of Diocles'. In particular, Diocles seems not to have caught the importance of the ray reflecting perpendicular to the axis of the mirror. On the other hand, it is not entirely clear if he appreciated the limiting role of the arc subtended by the side of an equilateral triangle: the introduction of such an arc in prop. 3 seems motivated by the aim of providing a simple construction of the arc with which the proof will deal, namely, AV (Diocles never mentions the inscribed polygons). The relationships between Dtrūms and the remains of the *fragmentum bobiense* are even clearer than this; the proof of Dtrūms' prop. 14(b) is identical with the one in the fragment. The coincidences between the two versions are so strict as to make one suspect that one of them is a compilation of the other or, more likely, that they draw from a common source. In this case, Dtrūms would have performed a more selective abridgement, by eliminating the originality claims made by the *anonymus bobiensis*.

5 Guessing the focal property of the parabola

Let us now return to the question asked in Sect. 2: how to guess that the parabola is the line enjoying the exact focal properties. The problem seems desperate in a modern perspective only, since one has to establish a criterion singling out the parabola among infinitely many curves. Yet, this argument neglects the fact that Greek geometry knew only of a limited number of 'established' varieties of line: a preliminary, and reasonable, strategy is to see if any of them have the preferred property.

Even when not granting this, authors such as Dositheus or Diocles had a guide in their search for exact burning mirrors: the approximate focal properties of the circumference. The 'fundamental triangle', quite a natural object to single out in the case of a circumference since the normal there coincides with the radius, does not depend on the nature of the curve, and hence it can be transferred identically to the analysis of reflection on any line. In the case of the parabola, this triangle acquires a special relevance, after one realizes that two well-known properties of the parabola, namely, the properties of the subtangent and of the subnormal, impose further constraints on two of its elements: the reflected ray (that is, the median relative to the hypotenuse), and the perpendicular from the point of incidence to the axis. It is really a very short step from here to conceiving an analysis such as the one leading to Diocles' proof. In assessing my proposal, one must also bear in mind the role of tradition, which operates as a motive force in all fields of Greek mathematics: it was only natural to try to refine the partial investigations about spherical mirrors made in *Catoptr.* 30, having as goals both a satisfactory analysis of spherical aberration and finding a surface with exact focal properties.

The investigations in the domain of burning mirrors were thus very likely done first on the sphere, and the analysis of this problem suggested the main technical tools needed to find an exact solution. This is a first phase that can rightly be termed as 'heuristic', and on a global level: the aim is to single out a *corpus* of notions and tools developed during the solution of a problem, and to export it to attack another problem, still unsolved and hopefully connected with the solved problem. For instance, recall that the limiting point on the axis of the semicircumference is placed at 1/4 of the diameter, and suppose that the conjecture was made that the line with exact focal properties is the parabola. Why not suppose that the focus is located away from the vertex at exactly 1/4 of the upright side? Of course, the parallelism is by and large imperfect, since the diameter of a circumference cannot be said to 'correspond' in any simple mathematical manner to the radius of a circle, but it remains true that diameter and upright side are the only dimensional parameters attached to a circle and to a parabola. Is it not by conjecturing that mathematical research proceeds?

After the mathematical tools have been identified, a local analysis comes into play. The argument expounded in Sect. 2 justifying the discovery of the focal property might well be the 'analysis' that is not made explicit by Diocles (an analysis of a *theorem*, one must stress). It is clear that the argument in Sect. 2 is neither an exact inverse of the attested synthesis nor an approximate construction, just as the analysis that is proposed in the next section leading to the focus–directrix property of the parabola will not be an inverted synthesis or an approximate construction. There is much more freedom in the heuristic phase than the one we are used to thinking. In this perspective, all the analyses reconstructed in the literature are marred by a degree of arbitrariness, ably concealed by the fact of being an exact inversion of an attested synthesis, making them nothing but rhetorical exercises pointing more to the cleverness of their redactors than to some historically sound substrate. The reconstructed analyses, in fact, oscillate between the two extreme poles of overdetermination (they must be the inverse of known syntheses) and underdetermination, when approximate constructions (having a sense only after an exact solution has been found) or logically inconsistent guessings are allowed to serve as analyses. On the contrary, a heuristic analysis can at the same time be non-approximate and non-formalized: the one that we shall see leading to the focus-directrix property is a perfect example. An *attested* analysis, however, precisely because it is written and thereby is in a strong sense 'formalized', need not be identical with the heuristic analysis, nor, a fortiori, an attested synthesis needs to be identical with the inverse of the heuristic analysis.

6 The pointwise construction of the parabola: Diocles

Constructing a mirror burning at a given distance, that is, a parabola with given vertex and focal distance, is a problem, not a theorem like the one establishing the focal properties of a parabola, and therefore requires a different approach. In order to solve it, Diocles, Dtrūms and Anthemius adopted the following deductive progression: they singled out first a particular *sumptōma* allowing them to set up a pointwise construction of a line with exact focal properties; they subsequently showed, in a separate proposition, that the points determined in this way lie on a parabola, identified as the line enjoying the 'principal' *sumptōma* that we have seen at the beginning of Sect. 2. This is nothing but a variant of the canonical strategy for solving a *locus* theorem, where the constraint in the enunciation identifying the *locus* is reduced to a *sumptōma* of a known curve, and eventually, for instance, in the case of conic sections, to a 'principal' *sumptōma* in one of its forms.

Diocles' particular *sumptōma* is nothing but the property of equidistance of any point on the parabola from the focus and from a given line, nowadays called 'directrix'; Dtrūms performs a more complex construction, details of which we shall see in the next section; Anthemius elaborates a non-rigorous version of Diocles' construction.

Let us thus turn to prop. 4 of *On burning mirrors* (Fig. 3). Fixing the focal distance VF of a parabola amounts to assuming some parameters of the curve as given: the focus, the vertex, and the axis are given in position, the upright side (4 times VF) or, alternatively, the 'breadth' FK of the parabola is given in magnitude. To see the last point, recall that the principal *sumptōma* of a parabola with upright side $4VF$, applied to the point K orthogonal ordinate of which falls on the focus, gives

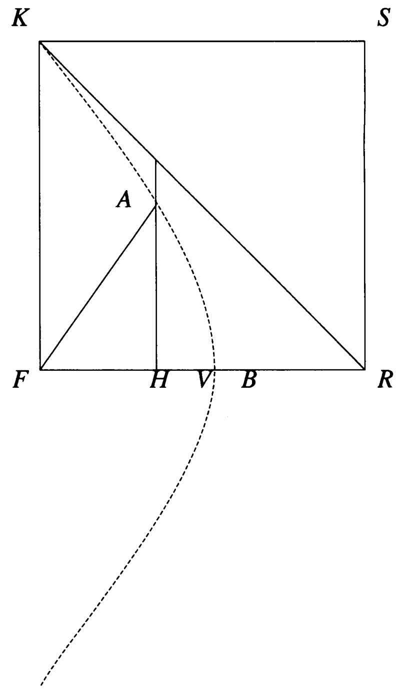
$$s(FK) = r(VF, 4VF).$$

fROM this, one immediately obtains $FK = 2VF$. Doubling VF beyond V as far as R, one has $FR = FK$. Diocles completes the square $KFRS$ and draws diagonal KR (KR is tangent to the parabola at K; cf. *Con.* I.33).

Diocles goes on to determine points on the curve other than V and K. To this end, he proposes a simple construction of the points that have the same distance from the focus F and from the straight line RS. Given in fact a point H on FV, a point B is taken on VR such that $HV = VB$ (and therefore $FB = HR$ since V is the midpoint of FR). A circle is then traced with centre F and radius FB: the circle intersects the perpendicular erected from H at A. The proposition ends with the claim that all points A determined in this way lie on a parabola (namely, the one with upright side $4VF$ and vertex V).

How could Diocles have conceived this construction? Did he know that the property of equidistance of any point from the focus and from a given line is a *sumptōma* of a parabola? Complex and ingenious answers have been proposed; as we shall see in

Fig. 3



a moment, Knorr even ventured to call into play the extreme resource of 'plain good luck' (1983), and proposed accordingly that the approximate construction compiled by Anthemius was the source of Diocles' exact solution. As a matter of fact, things are far simpler than that: the fact that the parabola enjoys the focus-directrix property is actually *immediate*; moreover, as the property is stated as an equality, it automatically follows that it is a *sumptōma*. To prove this, let us notice first that, in order to construct a mirror burning at a given distance, the problem of identifying the line concentrating the rays at one single point must be supposed as solved. We may thus assume that the proof of prop. 1 is known (Fig. 4).

Recall the pivotal equality, $AF = FB$. From point H, let a segment $HR = BF$ be cut off on the axis, on the same side as V. It is enough to show that the position of R is univocally determined by F and V, which are given *if the assigned curve is a parabola*. But from

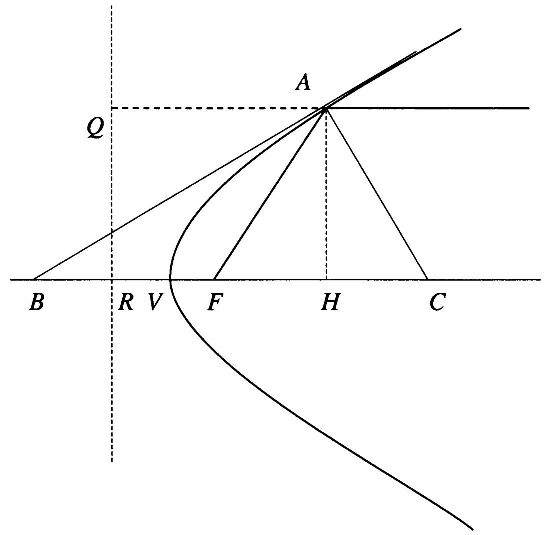
$$HR = BF$$

let $HV = BV$ (by the property of the subtangent) be subtracted: one obtains

$$VF = VR.$$

But V and F are given: therefore, R is also given. End of proof. In other terms, starting from whatever point H which is the projection on the axis of a parabola of a point A on the line, and cutting off from the axis a segment HR equal to the distance of A from

Fig. 4



the focus, the endpoint R of the segment is always the same. As HA is perpendicular to the axis, point A is equidistant from the focus and from a straight line through R perpendicular to the axis.

The characterization of the parabola by means of the focus–directrix property allows one to set up a pointwise construction of the line using only plane methods. Diocles does this in prop. 4, as we have just seen. In prop. 5, he shows that, by 'inverting' and formalizing the argument just expounded, the curve resulting from the pointwise construction is a parabola. In fact (going back to Fig. 3), the arrangment of the points $FHVB$ is suitable for an application of *El.* II.8, and this theorem provides the crucial equality entailing that any point identified by its equidistance from a point F and a straight line RS satisfies the *sumptōma* of a parabola with axis along VF, vertex V and upright side $4VF$. For let us consider one such point A, of which projection on FR is H. Let us cut off, on the opposite side from V, a segment $VB = VH$ (and therefore, $FB = HR$). By *El.* II.8 one has that

$$s(FB) = s(FH) + 4r(HV, VF).$$

But FAH is a right-angled triangle, so that

$$s(AF) = s(FH) + s(AH).$$

Therefore, if we stipulate $AF = FB$ (= HR), as Diocles does at the end of his construction, then we have

$$s(AH) = 4r(HV, VF) = r(HV, 4VF),$$

and point A is on the parabola with upright side $4VF$ and vertex V.

The problem of the heuristic behind Diocles' construction was put forward with force by Knorr (1983), in the only sustained attempt at assessing the relationships among the several authors involved in researches about burning mirrors. It is worthwhile to discuss his reconstruction in detail. (Knorr did not know Dtrūms' treatise; the analysis offered in *CG*, *passim*, simply states, repeatedly and without really arguing the case, the mutual independency of all contributions involved, a thesis that is refuted throughout this article.)

Knorr (ibid., p. 59) starts by pointing out two deductive drawbacks in the series of prop. 1, 4 and 5:

(a) It is nowhere stated nor proved that the parabola enjoys the focus–directrix property;
(b) 'The curve as constructed in [prop. 4] does *not* satisfy the property of the burning mirror established in Proposition 1', since this proposition 'defines the parabola in terms of the ordinate-abscissa property, not the focus-directrix property'.

According to him, these supposed lacunae can be amended by supposing that props 4–5 are the relics of the analysis, which would have led Diocles to the proof of his prop. 1. As a matter of fact, Diocles' reduction strategy is exactly the canonical one for solving a *locus* problem, as we have seen at the beginning of this section. Apparently, Knorr did not appreciate this point, as from his perspective, the characterization of the curve with exact focal properties in terms of the property of equidistance from a given point and a given straight line must logically and temporally precede the proof of prop. 1: only afterwards, the same property of equidistance shall be proven to identify a parabola. After the curve has been identified, it becomes relatively easy to set up a direct proof such as the one that we read in prop. 1. According to Knorr, Diocles would have kept the superfluous props. 4 and 5 in his treatise as 'vestiges of the preliminary analytic investigation' (ibid., p. 60).

Knorr puts much emphasis on the heuristic character of the analysis because he thinks that the technical difficulties arising in trying to reduce the construction of the curve with exact focal properties to the one identified by the equidistance of its points from a given point and a given straight line could not be overcome by a Greek geometer. In his words (ibid., italics in the original): 'One thus requires a way of reducing the construction of the ray-focusing curve to the curve defined by the focus-directrix property. Now, if the problem is framed in this manner, it would seem to be beyond the capacities of Greek geometry. For the focal property specifies the curve through the orientation of the *tangent* at each point; [...] But to work out the curve from its tangent entails, in effect, finding the solution of a differential equation. Lacking any method comparable to the differential triangle, the Greeks were at great disadvantage as far as the *finding* of solutions to such problems is concerned. On the other hand, once *a*solution was found, by whatever method, even a fortuitous one, its proof could be provided in rigorous fashion [...]'. Knorr even wonders (ibid., italics mine): 'Why, then has Diocles injected consideration of this *ostensibly irrelevant* property as if it were somehow the essential characteristic of the burning curve?'.

In order to get out of the *impasse*, Knorr submits that what we read in Anthemius was a relic, even if relating to the ellipse, of the preliminary heuristic leading to Diocles' construction (ibid., pp. 60–63); however, he is forced by his own argument to suppose that this approximate construction was arrived at by pure chance: 'Anthemius has been helped along to no small degree by plain good luck' (ibid., p. 62).

We see here a historian surrendering to his own prejudices: whenever a rigorous analysis is (regarded as) not available, it is replaced by a non-rigorous argument that is ascribed to an author who is early and shadowy enough, in order not to destroy his reputation. Dositheus is such a character, to whom an 'approximate construction' can be very fittingly attributed on the basis of what Diocles says in his prefatory epistle (see Sect. 2). In Knorr's view, Anthemius' construction should be ascribed to Dositheus. This construction is grounded on the simple remark that the tangent to the line at the point of incidence of a ray bisects the angle between the incident ray produced and the reflected ray. The construction starts from the outer edge of the mirror, which is supposed to be given, as well as the point where burning must occur. Using these assumptions and the focus–directrix property, Anthemius builds his composite mirror; the Greek text ends abruptly before the proposition is completed. In analogy with what we read in the case of an elliptical mirror, Knorr surmises that the missing portion of Anthemius' tract should have contained a 'proof' of the fact that a curve having the focus–directrix property is a parabola. The Arabic translation and the version revised by 'Uṭārid (*CG*, 286–315) show that this was not the case: Anthemius was only interested in transcribing arguments with an explicit practical import. Knorr further identifies the 'missing proof' of Anthemius' work with what is contained in the Bobbio fragment (ibid., pp. 63–70). In doing this, he inserts himself in a tradition of scholars (among whom Heiberg 1883, pp. 127–129 features) who entertain the illusion that history of Greek mathematics is a game of complete information.

Knorr' argument does not stand up because of two unwarranted, and demonstrably false, assumptions. The first is that an analysis revealing the crucial role of the focus–directrix property might be conducted only by explicitly using the tangent. The second, of an ideological nature, is that a property so important (to us) as the focus-directrix could not be reduced to the status of mere functionality to a pointwise determination of a parabola, as indisputably happens in Diocles' approach and as Knorr himself has to recognize (1983, p. 59): 'the role of the [given] line is so submerged in the construction that one strains to view Diocles as working toward the solution of a problem of locus as such'.

The first assumption, whose anachronistic nature is revealed by the long quotation above, is falsified by the very simple analysis proposed at the beginning of this section, which, in my opinion, has the additional virtue of being a direct consequence of the focal property established in prop. 1, and is in fact an immediate development of the argument set out in that proof. Yet, Knorr had at his disposal an analysis not depending on prop. 1: it was enough to 'invert' props. 4–5. For, once a point B is given on the axis of a parabola having vertex V and focal distance FV given (Fig. 3 again), it is enough to mark point H such that $HV = VB$, draw the perpendicular HA from the axis and take its intersection with the circle of centre F and radius FB, to have the point A the tangent through which meets the axis at B. Very likely, Knorr regarded the use of *El.* II.8 as too contrived, since it assigns no *explicit* role to the tangent.

This suggests that the whole story can be read in a different perspective: the focal properties of the parabola were obtained as a by-product of an attempt at proving that the property of the subtangent is a *sumptōma* of the curve. About the relevance of this property to the eyes of the ancient geometers it is not necessary to spend so many words. Recall in fact that the property of the subtangent was well known to Archimedes, that the attested proof in *Con.* I.33 and 35 can be transferred identically to orthogonal conjugation, and that precisely *El.* II.8 provides a crucial result in the final step of the proof of *Con.* I.33, a reference that both Eutocius (it is almost certain that the postponed explicative closing the theorem at *AGE* I, 100.21–22, must be ascribed to him) and Heiberg (*AGE* I, 101 *in int.*) have misinterpreted as a reference to *El.* II.5. As a consequence, the focal property proves crucial to employ the subtangent as a method to construct a parabola with given vertex and parameter (cf. *Con.* I.52, that, however, does not lead in an immediate way to a pointwise construction) and makes it possible to solve by reduction a particular case of the problem of tracing a conic section after some of its tangents are known (it is a particular case since one of the tangents must pass through the vertex; an alternative method follows from *Con.* III.41).

As for the second assumption, the historiographical debate about the first appearance of the focus–directrix property is biased by the relevance that this characterization has assumed in the modern characterization of conic sections. Of this property, there is no trace in Apollonius' *Conica* as they are now extant (recall that it holds in reference to the axis only, and therefore it is not a *sumptōma* that can be referred to any diameter). Yet, to mention just one consideration that could have suggested including it, had it been known, use of the focal property of the parabola would have greatly simplified the expression of the new parameter when the reference diameter is changed in *Con.* I.49—it is simply four times the distance of the vertex of the new diameter from the focus.

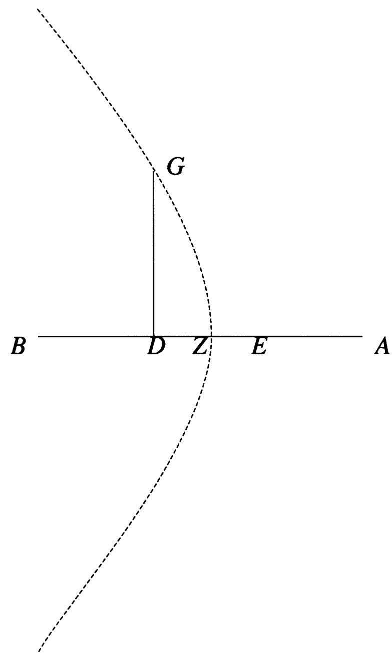
Toomer (1976, pp.16–17) surmised that Diocles himself was the discoverer of the property in the case of the parabola, its validity being extended after him to the other conic sections. The few lemmas Pappus offers, at the very end of book VII of his *Collectio*, for the Euclidean *Loci on a surface* are alleged to support this reconstruction. Pappus (partly) proves in VII.313–318, by reduction to the principal *sumptōma* of a conic section used before Apollonius, the following *locus* theorem:

> Let a straight line *AB* be <given> in position, and *G* given in the same plane, and let be *DG* drawn across, a perpendicular *DE*, and let the ratio of *GD* to *DE*G <be given>. I say that *D* touches a section of a cone, and if the ratio be equal to equal a parabola, if less to greater an ellipse, if greater to less a hyperbola.

It is the only attestation in the ancient *corpus* of the fact that the focus–directrix property is a *sumptōma* of the several conic sections (the formulation as a *locus* bears this out exactly). The really interesting point in Pappus' proof is, from our perspective, the following. He first reduces the just quoted *locus* to another one, solved in VII.314–317 (the reduction is immediate). The enunciation of this *locus* reads thus (Fig. 5):

> Given two <points> *AB* and *GD* at right angles, let the ratio of the square of *AD* to the squares of *GD DB* <be given>. I say that *G* touches a section of a cone, whether the ratio is equal to equal a parabola or greater to less or less to greater.

Fig. 5



Therefore, the *sumptōma* here assumed is nothing but a generalized focus–directrix property modified by squaring the ratio and applying *El.* I.47 to the distance from the focus (the same *sumptōma* identifies a hyperbola, one of the two *loci* whose intersection solves the problem of trisection of an angle, at *Coll.* IV.68). It is important to stress that this is a *generalized* formulation of the focus–directrix property: the ratio of the two distances is given in square. Even if the reduction to the 'unsquared' form is an immediate consequence of *El.* I.47, it obviously works one way only. Accordingly, the generalized formulation holds in any conjugation, while, as we have repeatedly seen, the 'unsquared' one—as well as the focal property that is strictly linked with it—holds in orthogonal conjugation only (this is one more reason 'explaining' as to why the focus–directrix property was not 'seen' by the Greek geometers).

Let us see how the *locus* is solved, considering only the case of the parabola (*Coll.* VII.315). In order to reduce the condition

$$s(AD) = s(GD) + s(DB)$$

to the principal *sumptōma*

$$s(GD) = r(4BZ, DZ),$$

one has 'only' to make the rectangle $r(4BZ,DZ)$ appear in place of $s(AD) - s(DB)$. This can be done in two ways. One possibility is to follow Diocles: bisect AB at Z and mark E such that $DZ=ZE$; and apply *El.* II.8 to the four points $BDZE$. As $AD=BE$, it immediately results that

$$s(AD) - s(DB) = s(BE) - s(DB) = 4r(BZ, DZ).$$

Pappus chooses another route. He marks E such that $BD = DE$ and applies *El.* II.6 to the four points $BDEA$. Thus he has

$$s(AD) - s(DB) = r(BA, AE),$$

which in turn is equal, by marking Z such that $BZ = ZA$ and by observing that

$$AE = AB - BE = 2BZ - 2BD = 2DZ,$$

to $4r(BZ,DZ)$.

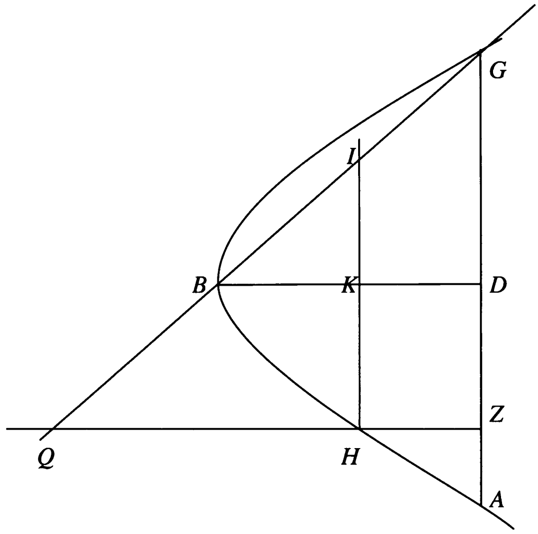
One wonders what proposition of the lost Euclidean treatise might have required Pappus' lemma. Zeuthen (1886, pp. 210–215 and 367–371) observes that, if Pappus had to make such a lemma explicit, then the focus–directrix property should have been taken for granted in the *Loci on a surface*. Zeuthen's remark can be completed by the conjecture that the most "natural" place where the property was established, of course in the above form of a *sumptōma* formulated as a constraint identifying a *locus*, is in Aristaeus' *Solid loci* (an inference that is another product of the illusion that history of Greek mathematics is a game of complete information). As a matter of fact, what Pappus proves is which conic section must be associated with which ratio, his formulation implying that the connection between the (generalized) focus–directrix property and conic sections has to be taken for granted. In sum, if one is not willing to venture in empty conjecturing, then a minimal hypothesis is that the property had been employed in the *Loci on a surface* as an intermediate step in the analysis of some *locus*, but that it was simply considered as a *sumptōma* of a conic section.

7 The pointwise construction of the parabola: Dtrūms and Archimedes

The pointwise construction of a parabola proposed by Dtrūms in his props. 12–13 follows Diocles' scheme but does not resort to the focus-directrix property. As a matter of fact, Dtrūms solves a different problem from that of constructing a parabola having vertex and focal distance given. What is given are three particular points: one is the vertex, and the other two are the endpoints of the same ordinate; it is required to draw the parabola passing through them. As is the case with Diocles, the particular position of the points makes the three tangents through them given in position as well, and from this, it is easy to determine the parabola pointwise using, for instance, the proportions derived by Apollonius in *Con.* III.41. Dtrūms does not follow this route: his procedure has as its model the Archimedean proofs in *Quadr.* 4 and 5. Some peculiarities of Dtrūms' proofs can in fact be explained only as a reworking of analogous features of those of Archimedes' proofs, as we are going to see. Let us read first *Quadr.* 4 (*AOO* II, 268.5–270.3, and see Fig. 6):

> Let it be a segment ABG contained by a straight line and a section of a right-angled cone [*scil.* a parabola], and from the middle of AG a <straight line>

Fig. 6



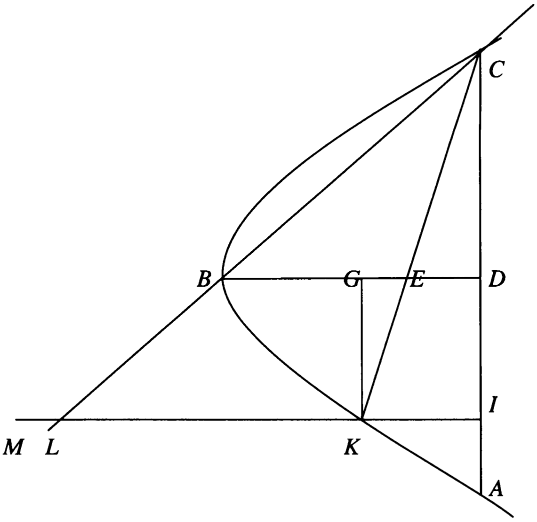
BD be drawn parallel to the diameter—or let it itself be a diameter—and let the joining straight line *BG* be produced.
Then, if another <straight line> *ZQ* be drawn parallel to *BD* cutting the straight line through BG, *ZQ* to *QH* will have the same ratio as *DA* to *DZ*.
Let a <straight line> *KH* be drawn through *H* parallel to *AG*: therefore it is as *BD* to *BK* in length, so *DG* to *KH* in power—for this has been proven. Therefore, it will be as *BG* to *BI* in length, so *BG* to *BQ* in power—for *DZ KH* are equal—therefore, lines *BG BQ BI* are in proportion, so that *BG* to *BQ* has the same ratio as *GQ* to *QI*: therefore, it is as *GD* to *DZ*, so *QZ* to *QH*. And equal to *DG* is *DA*: it is thus manifest that *DA* to *DZ* has the same ratio as *ZQ* to *QH*.

In *Quadr.* 5 the result is extended, with a simple proof that is omitted here, to show that

$$AZ : ZG :: ZQ : QL$$

holds, where L is the intersection of *ZQ* and of the tangent through G. Archimedes needs this new result as it is the central one in the mechanical method of quadrature of a segment of a parabola. On the other hand, while appearing in Dtrūms' prop. 12, the result is completely useless in Dtrūms' approach, as he applies prop. 12 in prop. 13 without any need to introduce the tangent. A hasty redaction of prop. 12 is suggested by other elements too—the enunciation states two results: one referring to the proportion obtaining with the point on the tangent; and the other to the proportion obtaining with the point on the secant through the vertex. In the conclusion itself, the author observes that the introduction of the tangent is without effect; before this, he sets out a convoluted argument that can be replaced by a single application of *convertendo*. These steps are completely useless as a support to the subsequent construction, which can be effected using the proportion (identical with the Archimedean one established

Fig. 7



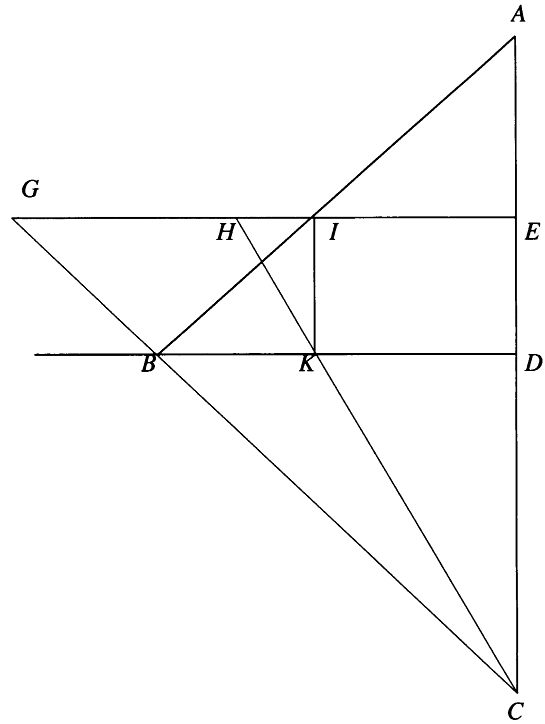
at the end of *Quadr.* 4 and that I shall call (*)) to which these steps are initially applied. Let us read then the relevant portion of the text. As I am unable to directly translate from Arabic, I transcribe the French version (*CG*, 200–201, and see Fig. 7):

> Soit la section parabolique *ABC* te *BD* son diamètre; soit la droite ordonnée *AC*; [...] soit la droite *MKI* parallèle au diamètre, qui traverse la section au point *K*. [...]
>
> Joignons la droite *CB* et prolongeons-la jusqu'au point *L*; menons du point *K* une droite parallèle à la droite *CA*, soit *KG*; joignons également la droite *KC* qui coupe la droite *BD* au point E. On a dans la section le rapport de *DB* à *BG* égal au rapport du carré de *AD* – c'est-à-dire le carré de *CD* – au carré de *KG* et le rapport du carré de *CD* au carré de *KG* est égal au rapport du carré de *DE* au carré de *EG*: le rapport de *DB* à *BG* est donc égal au rapport du carré de *DE* au carré de *EG*. Et en raison de ce que nous avons démontré à ce sujet, on a le produit de *DB* par *BG* égal au carré de *BE* et le rapport de *DB* à BG – qui est égal au rapport du carré de *AD* au carré de *DI* – est égal au rapport du carré de *DB* au carré de *BE*; mais ceci est égal au rapport du carré de *IL* au carré de *LK*: on a donc également en longueur le rapport de *IL* à *LK* égal au rapport de *DA* à *DI*.

The proof gives a simple method for determining a parabola passing through three points with the special properties listed above: *A* and *C* are the endpoints of the same ordinate, and *B* is the vertex of the diameter conjugated to that ordinate. The construction of the problem and the beginning of its synthesis may well be identical with Dtrūms' (*CG*, 201–202, and see Fig. 8):

> Posons la droite *AC* diamètre d'un cercle, qui est le bord du miroir ardent que nous voulons construire [...] cherchons à trouver la parabole que nous traçons sur les points *A*, *B* et *C* et sur l'axe *BD*. Joignons la droite *AB* et menons du

Fig. 8



point E – qui est entre A et D – une droite parallèle à BD, soit la droite EH. Joignons la droite CK et prolongeons-la jusqu'à qu'elle coupe la droite EIH au point H; alors le point H est sur la parabole que nous traçons sur les points A, B et C. [...] Menons la droite EH, qu'elle coupe la droite CB au point G. Si on construit la parabole, le rapport de GE à EH est égal au rapport de BD à DK et le rapport de BD à DK est égal au rapport de BD à EI; et ceci est égal au rapport de DA à AE; la parabole qui passe par les points A, B et C passe donc par le point H.

The synthesis of the problem, namely, to show that H lies on a given parabola, can be completed by inverting either the passages of Dtrūms' prop. 11 or those of the Archimedean proof: as we have read, Dtrūms omits this part of the synthesis and satisfies himself with observing that proportion (*) holds, the rest being obvious since all passages of Archimedes' proof can be inverted. As a consequence, point H on ZQ satisfying (*) satisfies also the principal *sumptōma* in ratio form of the parabola of given vertex and passing through two given points lying at the extremes of the same ordinate. Our discussion will focus henceforth on a comparison between the proofs of Dtrūms' prop. 12 and of *Quadr*. 4; accordingly, prop. 13 of Dtrūms will be left out.

In order to better understand the relationships between the two propositions, it is useful to set in parallel the deductive steps in a schematic manner, as follows:

Archimedes: Enunciation of the principal *sumptōma* of the parabola in the ratio form; identification of four segments, two of which are parallel to the ordinate, and two to the diameter; projection *on the secant through the vertex* of the four segments, pairwise independently, in such a way as to make the projection of two segments coincide: the three resulting projections A, B, and C satisfy a proportion of the kind

$$\mathrm{A} : \mathrm{B} :: s(\mathrm{A}) : s(\mathrm{C}) :$$

therefore, the three projections satisfy a continuous proportion with middle term C; application of *El.* V.12 or of V.19; the four resulting segments are projected back, pairwise independently, a pair on the ordinate and the other pair on the parallel to the diameter through the arbitrary point: one obtains the final proportion (*).

Dtrūms: Enunciation of the principal *sumptōma* of the parabola in the ratio form; identification of four segments, two of which are parallel to the ordinate, and two to the diameter; projection *on the axis* of the four segments, pairwise independently, using as an auxiliary line, the line joining the given point and the arbitrary point (two of these segments already lie on the axis); application of a technical lemma proven in prop. 11 ('Si on partage une droite en parties telles que le rapport du carré de l'une des partie au carré de la partie intermédiaire soit égal au rapport de la droite toute entière à la partie qui reste, alors le produit formé de a droite toute entière par la troisième partie que nous avons mentionnée est égal au carré de la droite composée de la partie intermédiaire et de la partie qui reste'): therefore three suitable combinations A, C, and B of the projections satisfy a continuous proportion with middle term C: therefore three projections satisfy a proportion of the kind

$$\mathrm{A} : \mathrm{B} :: s(\mathrm{A}) : s(\mathrm{C});$$

the first ratio gives, via the principal *sumptōma* in ratio form, a ratio between the squares on the ordinates; A and C in the second ratio are independently projected back from the axis on the parallel to the diameter through the arbitrary point: one obtains the same final proportion (*) as Archimedes'.

There are two essential differences between the two proofs:

(1) Archimedes projects on the secant through the vertex and the given point; Dtrūms projects on the axis. The key proportion A : B :: s(A) : s(C) is different in the two authors, since it is strictly correlated with the choice of the projections: as a matter of fact, it is quite difficult to obtain Archimedes' projection from Dtrūms', and vice versa. In both cases, the two proportions are the natural ones after the projection is chosen.
(2) The most surprising feature of Dtrūms' approach, however, is that he singles out as prop. 11 a four-point technical lemma, whose role is analogous to *El.* II.8 in Diocles' construction, and this allows him to determine by plane methods the single points of a parabola by means of a *sumptōma* which is hardly perspicuous. It would have been much more immediate to offer a pointwise construction of the parabola on the grounds of the first part of the Archimedean proof (see Fig. 6 again): after one knows that BG, BQ, and BI are in continuous proportion, it is enough to notice that B, G, and Q are given. Therefore I is also given, and the construction of a point on a parabola can be effected by slightly modifying the one in Dtrūms.

In fact, the lemma proved by Dtrūms in his prop. 11 must have constituted a well known technical tool, since we find it in Pappus', *Coll.* VII.89–90, 220–221, and 277, who offers five proofs, only slightly different one from another, whereas Dtrūms proceeds by *reductio*. The first two proofs of Pappus are among the lemmas referring to Apollonius' *Sectio determinata*, whose connection with the determination of conic sections starting from given points was established long ago (Zeuthen 1886, pp. 195–202). The third and fourth occurrences are in the part dealing with Euclid's *Porisms*. The fifth proof is a lemma to *Con.* V. The transmitted Arabic text of book V does not make it clear to which proposition Pappus' lemma refers, but some of the surrounding lemmas pertain to material related to the group V.51–55. Add to this the resonance (only a resonance, since the four points do not lie on the same straight line) with the fundamental lemma in the construction of the *locus* known as "Apollonian circles" (cf. *Coll.* VII.185 and 223), and Zeuthen's remark (1886, pp. 59–62) that the proportion derived in *Quadr.* 4 shows that the parabola satisfied certain particular cases of the four-line *locus*.

8 Lemmatic fragmentation

The useless complications in Dtrūms' proof, even if he might well be considerably later than Archimedes, cannot be explained by simply invoking his low technical level or accidents of transmission: the copy is too conformal to the Archimedean model and, at the same time, the complications are inexplicable if viewed only in the context of adapting the original proof to the contingent exigencies of constructing a parabola passing through three points. As we have seen, in fact, *Quadr.* 4 offers itself, so to speak, as an effective synthesis of this problem, dictating the form of the construction. I submit that the attested modifications of the Archimedean original had a strategic goal, and it is of no importance whether this has happened with Dtrūms or he found some paradigmatic examples in the tradition. The goal was to give prominence to either of the distinctive features of the construction (projection and technical lemma), even if the resulting construction gets considerably more involved.

Yet, it is not easy to disentangle the complex relationships between the identification of the lemma and the choice of the projection. It may be that the choice of the projection induced the necessity of elaborating a series of steps that eventually coalesced in the lemma. In favour of this hypothesis, one may argue that the axis of a parabola is connected in an invariant way to the curve, whereas the secant introduced by Archimedes is not. On the other hand, the choice of the projection might have been dictated by the need to give prominence to a lemma that has manifold applications. As a matter of fact, the presence of the lemma in the tradition might have suggested modifying Archimedes' projection, which presents itself as a most "natural" one: let us recall in fact that the position of the diameter can be made to depend on that of the (points on the) ordinate, and that this projection is already formulated in oblique conjugation. The combination of two contravariant canonizing elements—a standard projection that can be exported to any conjugation and a lemma to be used in other contexts—produces a proof that has many "links" with a larger class of problems but that surely is not the simplest possible one. All in all, my impression is that, at least when Dtrūms redacted his version, extracting the lemma was the primary aim, as I shall briefly argue in what follows.

Dtrūms' whole approach displays a feeling for what we nowadays would call 'structure': the linear lemmas are the real core of a proof, beyond the contingent geometrical embellishments; and the fact that they can be applied in disparate configurations testifies to their belonging to an order of mathematical reflection that somehow extracts the 'essential geometrical content'—sequences of points on a straight line—from a particular geometrical configuration. The linear lemmas dictated the directive lines of mathematical research: as basic demonstrative tools, they imposed their own presence in the proofs, thereby conditioning the kind of available results. Pappus' *Collectio* is the place where this tendency comes to its extreme and results in a disintegration of the *corpus*.

I have surveyed the n-point lemmas presented in the ancient mathematical *corpus*; the following prospect lists them, indexed by the number of points involved and by the presence or absence among the assumptions of any relation among figures constructed on the segments cut off by the points (the index 'g' means that the lemma is formulated in the language of the givens; * marks the lemmas stating an inequality; '*in*' precedes occurrences of linear lemmas assumed in a Pappian proof, but not proved by him).

	relation not assumed	*relation assumed*
3 pts	*El.* II.2–4, 7–8, X.33/34, 59/60, *Coll.* VII.192g	XIII.1–6, XIV.11, *Coll.* V.78*–9*, 83, VII.305g
4 pts	*El.* II.5–6, 9–10, X.34/35, 41/42*, *Coll.* V.49-50, VII.54*–5*, *in* 142–3, 186, 191	*Coll.* VII.89–90, 216–21, 267–9, 277
5 pts	*Coll.* VII.56*, 70–1, 79, 111–2, *in* 189, 228, 239, 249–50, 263–6, 307*	*Coll.* VII.60–3, 68–9, 72, 77–8, 80, 83–5,86–8, 91–8, 117, 189–90, 270–1, 308
6 pts	Eutocius *in Con. II.23*	*Coll.* VII.99–110
n pts	*El.* II.1, *Coll.* III.14	

One might even suggest that the whole book X of the *Elements* should be included in the list. I add just a few remarks.

(1) Eutocius' lemmas to Apollonius' *Conica* coincide with those in Pappus, the only exception being a lemma to *Con.* II.23.
(2) The lemmas in *Coll.* VII. 217, 219 and 228 prove three results each; VII.95–7 four results.
(3) VII.69, 78, 84–5, 87–8, 92, 94 are alternative proofs of the immediately preceding lemmas.
(4) Pappus' lemmas for Apollonius' *Sectio determinata* are contained in *Coll.* VII.68–119. It is no surprise, then, to find so many linear lemmas (39 in all, most of which five-point lemmas) in this chapter range of the *Collectio*. Other chapter ranges in *Coll.* VII, with the number of associated linear lemmas, are: *De sectione rationis* et *spatii*: 43–67 (9); *Neuseis*: 120–157 (2); *Tactiones* I: 158–184 (0); *Loci plani* II: 185–192 (6); *Porismata* I (7): 193–232; and *Conica*: 233–311 (16).
(5) All the occurrences of three-point lemmas in the second column refer to a segment cut in extreme and mean ratio.

(6) Among the lemmas to the *Neuseis* are included some interesting n-point-and-2-line theorems that will not be discussed here (*Coll.* VII.120–5).

It may well be that such a feeling for structure is a consequence of cultural factors such as the phenomenon of canonization of literary products and the prevalence of a rhetorical curriculum, entailing a renewed attention to the building blocks of any chain of argument. What is more, the lemmas may become a central research tool if one is obsessed by the ideology of the 'discovery' and by reconstructing the heuristic methods of the 'ancients'. In this perspective, there is relevance in the fact that the identification of the parabola via the focus–directrix *sumptōma* split into three branches, depending on the technical lemma employed (add to them the *sumptōma* of the parabola, established by Pappus, *Coll.* IV.79, in orthogonal conjugation but valid with reference to any diameter, where the "core" lemma is *El.* II.5):

(1) *El.* II.8 (used also in *Data* 86 and in Diophantus, *Pol. num.* 1–3, *Metrica* I.26, but elsewhere not to be found in the ancient *corpus*), applied by Diocles to produce a pointwise construction, even if, as we have seen at the end of Sect. 6, it may yield a proof that the (generalized) focus-directrix property is a *sumptōma* of the parabola.
(2) Dtrūms' lemma, applied to produce a pointwise construction.
(3) *El.* II.6, applied by Pappus in the context of a *locus* theorem, showing that the (generalized) focus-directrix property really is a *sumptōma*.

One should not, however, regard the interest in singling out lemmas as a typical attitude of late antiquity: book II of the *Elements* and Apollonius' three *Sectiones* (a warning: these are problems, not theorems) are there to show that the opposite is true, but this is another story.

9 Sources and their sigla

The abbreviations of the titles of the writings of mathematical authors are self-explanatory, as for instance *El.* = *Elements*, *Con.* = *Conica*, *Coll.* = *Collectio*; propositions are referred to by book and number, as for instance "*El.* III.15". Pappus' *Collectio* is cited by book and chapter. Other sources are cited according to the following sigla:

AGE: *Apollonii Pergaei quae graece exstant, cum commentariis antiquis*, edidit et latine interpretatus est J. L. Heiberg. 2 vol. Leipzig, B.G. Teubner 1891–1893.
AOO: *Archimedis opera omnia, cum commentariis Eutocii*, iterum edidit J. L. Heiberg. 3 vol. Leipzig, B.G. Teubner 1910–1915.
CG: *Les catoptriciens grecs I. Les miroirs ardents*. Textes établis, traduits et commentés par R. Rashed. Paris, Les Belles Lettres 2000.
HOO: *Heronis Alexandrini opera quae supersunt omnia*, recensuerunt G. Schmidt (vol. I), L. Nix et W. Schmidt (vol. II), H. Schoene (vol. III), J. L. Heiberg (vol. IV et V). 5 vol. Leipzig, B.G. Teubner 1899–1914.
MGM: *Mathematici Graeci Minores*, edidit J. L. Heiberg. Det Kongelige Danske Videnskabernes Selskabs, Historisk-filologiske Meddelelser XIII,3. København, Bianco Lunos Bogtrykkeri 1927.

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Knorr, W.R. 1983. The geometry of burning-mirrors in antiquity. *Isis* 74: 53–73.
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Toomer, G.J., ed. 1976. *Diocles, On burning mirrors*. Berlin: Springer.
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