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Pre-Euclidean geometry and Aeginetan coin design: some further remarks

Gerhard Michael Ambrosi

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Abstract Some ancient Greek coins from the island state of Aegina depict peculiar geometric designs. Hitherto they have been interpreted as anticipations of some Euclidean propositions. But this paper proposes geometrical constructions which establish connections to pre-Euclidean treatments of incommensurability. The earlier Aeginetan coin design from about 500 BC onwards appears as an attempt not only to deal with incommensurability but also to *conceal* it. It might be related to Plato's dialogue *Timaeus*. The newer design from 404 BC onwards *reveals* incommensurability, namely in the context of 'doubling the square'. It thereby covers the same topic but a different geometry as passages in Plato's dialogue *Meno* (385 BC). This coin design incorporates important elements of ancient Greek geometrical analysis of the fifth century BC like the *gnomon*, Hippocrates' squaring of the *lunule* (ca. 430 BC), and a geometrical version of monetary equivalence. Through this venue, the design's conceptual lineage might be traced as far back as Heraclitus' cosmology of about 500 BC.

Keywords Ancient Greece · Geometry · Incommensurability · Euclid · Money · Coins

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...If one would understand the Greek
genius fully, it would be a good plan to
begin with their geometry.
Sir Thomas Heath (1921), vi

1 Introduction

Artmann (1990) has drawn attention to the fact that a number of ancient Greek coins had distinct geometrical motifs. Some of these he related to mathematical literature of that time, thereby giving interesting suggestions about the history of the associated geometrical knowledge. There are, e.g. coins from the Greek island of Melos which show a decomposition of a square into eight triangles, 'exactly the same geometric diagram described by Plato in his *Menon* 82b–85b, where he explains the doubling of a square' (Artmann 1990, p. 44).[1] Since Melos was destroyed by Athens in 416 BC, those coins must have been struck long before Plato's *Menon*, 'our *first* direct, explicit, extended piece of evidence about Greek mathematics; it probably dates from about 385 BC'—as Fowler (1999, p. 7) wrote about Plato's dialogue. But the old coins show that the geometry taught in that dialogue must have been wide-spread knowledge long before Plato put it into his impressive prose.

A similar case was mentioned by Artmann (1990, p. 47) as involving Euclid's *Elements* of about 300 BC. He observed (ibid.) that its proposition II.4 is 'the geometrical version of the binomial theorem $(a + b)^2 = a^2 + 2ab + b^2$'. But there are coins from the island of Aegina from around 400 BC which show 'precisely the diagram of Euclid's Theorem II.4'. A few years later Artmann (1999, p. 63) returned to this point and commented:

> We do not know why the people of Aegina selected this particular design for their money. But in any case the coins indicate the familiarity of Prop. II.4 some hundred years before Euclid's time.

It is clear that we will never know for sure the intentions behind these coins' patterns since there are no documents about them. Nevertheless, it is astonishing that before Artmann's brief remarks there never seems to have been the attempt to claim some sort of meaning for the geometrical Aeginetan coin designs. But as we will see below, if we probe deeper into the coins' geometrical background it is possible to see connections to contemporary geometrical debates. We know from Plato's dialogues that geometry at that time was not just an intellectual pastime but it contained for the contemporaries the laws which even gods had to follow. It therefore should not appear as overstretching claims for the importance of geometrical design if in the following we will eventually make brief allusions to Plato's and to Heraclitus' cosmologies.

For our following arguments, it is interesting that the issue of the geometry behind the Aeginetan coins was recently taken up again in this *Archive*. Aboav (2008, p. 612) praised Artmann's idea to take 'the design on a coin of Aegina as illustrating a prop-

[1] For a picture of the coin see Artmann (1990, Fig. 2). For the corresponding geometrical figure see below Fig. 2b.

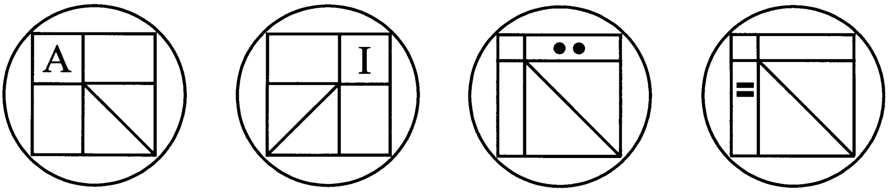
osition of Euclid's *Elements*'. But he disagreed with regard to Artmann's reference to 'Prop. II.4'. Aboav observed that some of the Aeginetan coins which Artmann referred to as depicting anticipated versions of Euclid's proposition II.4 in fact went beyond that proposition. Their geometry allegedly implies not just the decomposition of a square of type $(a + b)^2$ into constituent smaller areas.[2] They also obey a constraint of the type $b^2 = 2a^2$. This means that when the constituent large square which has the algebraic notation b^2 is halved along its diagonal, then we get three areas of size a^2, two of them being triangles. In this case, a large square with four sides, each of its sides of length $(a + b)$, when decomposed into its constituent parts, gives: two areas of size ab and three somewhat smaller areas of size a^2. Squares with this type of decomposition are thus constructed under a special constraint.[3] Aboav (2008, p. 606) labelled them as 'type A' squares and he continued: 'It is a square of this type, rather than the 6-divided square of Euclid II, 4 (Artmann 1999, p. 63), that appears on some coins of Aegina of the fourth century' (Aboav 2008, p. 606).

Here now Aboav's interesting contributions set in (i) a geometrical construction for 'type A' squares as just defined, (ii) a construction of what he called 'type B' squares with five equal constituent areas, (iii) a demonstration that results reached under point (ii) can be related not primarily to Euclid's *Elements* but rather to his book *On divisions [of figures]*. This book is lost but it was convincingly reconstructed by Archibald (1915). Furthermore, (iv) Aboav stresses that a number of Aeginetan coins of the fifth century BC were based on type B squares as constructed under the just mentioned step (ii) and he suggested that with this design the Aeginetans might have wanted to commemorate Pythagoras' death in Metapontum (which Aboav put at ca. 490 BC).

The following relates to the work just mentioned in that alternative constructions will be offered for Aboav's squares. One of our alternatives can be seen as a variant solution for the problem of doubling the area of the unit square. Thus, it fits quite well with our just mentioned knowledge that the ancient coins from Melos anticipated the mathematics discussed in Plato's *Menon*. There might also be some anticipation of Plato's dialogue *Timaeus*, namely by the Aeginetan coin design with skewed transversals. The message of some Aeginetan coins with perpendicular transversals seems not to go beyond just confirming Artmann's impression that their design anticipated important content of Euclid's *Elements*. We will briefly comment on this aspect also in the following. In the end, we offer a synthesis and extension of some recent contributions to the discussion of the geometry of ancient Aeginetan coin design.

[2] We are aware of the objection that the ancient Greek geometry is misrepresented when the claim is made that it is a 'geometrical algebra' (Unguru 1975/76). We use the algebraic notation here only as a brief reference to former contributions to this debate, not as a characterization of the substance of Greek geometry.

[3] For Aboav's geometrical construction see Aboav (2008, p. 605, Fig. 2). An alternative construction is offered below in Fig. 2a where the 'ab'-rectangles are drawn as shaded rectangles.



(a) cf. Kroll (2002), Nr.576 **(b)** cf. Head (1911), Pl.XXV, 1, 3, 4, 13 **(c)** cf. Kroll (2002), Nr.572, 573, 574 **(d)** cf. Head (1911), Pl.XXIV, 13, 14

Fig. 1 'Type A' squares and a generalization (Aegina, fifth and fourth century BC

2 A typology of perpendicular Aeginetan coin design

According to Artmann (1999, p. 162) 'the coinage of Aegina was the leading currency (like the US \$) around the Aegean sea'.[4] He refers this to the time 580–480 BC. But it seems that even at as late a time as the one when Aristotle's *Metaphysics* was written—in the fourth century BC—Aegina was proverbial for monetary importance.[5] Head (1911, p. 395) attests that the design of the Aeginetan coinage exhibits 'a uniformity which characterizes it as *de facto* an international, and not a mere local, currency'. We can infer from these observations that Aegina's coin design was an important element of ancient Greek public life in general and not just for the Aeginetans themselves. It therefore might be well that we understand a bit more about the workings of the Greek mind if we understand more about the designs and the messages of the Aeginetan coins.

Figure 1 gives a schematic overview of Aeginetan coins with perpendicular transversals. This class of coin designs can be divided into two subgroups. Figure 1a, b depict schemes based on 'type A' squares as defined by Aboav. They are so constructed that they obey the '$3a^2$'-constraint just mentioned. The squares of Fig. 1c, d obviously violate that constraint in that the upper small square is much smaller than a triangle contained in the larger square. In this sense, they represent a second subgroup. One can imagine a third subgroup where the constituent squares are all equal so that the triangles in the lower square have only half the area of the upper square, but this type of coins is extremely rare.

This suggests to enquire about the empirical relevance of these classes. Aboav (2008, p. 605) observes in the context of the Aeginetan perpendicular coin design that on the real coins typically 'the area of this little square to that of one of the triangles

[4] See also von Reden (2010, p. 72) who refers to 'Aiginetan currency domination' until ca. 480 BC and writes (ibid.) 'Aiginetan coins travelled earlier and further beyond the Aegean than others. The prominence of the Aiginetan weight standard is usually attributed to Aigina's dominant role in trade during the late
Footnote 4 continued
archaic period …however, the Aiginetan standard created a degree of monetary consolidation which in turn increased its role in trade'.

[5] In the attempt to clarify the term 'necessary', Aristotle, *Metaphysics* IV, 4, wrote that there is one sense of 'necessary' like the taking of medicine when one is ill or like having to sail to Aegina in order to get 'necessaries'—*chremata* in the original, a term which can be translated as 'coins' or as 'money' (see Ross 1928, 1015a25 for the latter).

is roughly as $4{:}4\frac{1}{2}$' so that the area of the smaller square to that of the larger square typically is 4:9 in his way of counting or simply 1:2.25. If the triangles were of a size equal to that of the small square, the ratio between the upper square and the lower one should be exactly 1:2. Even by Aboav's own casual empiricism, the 'type A' square is an ideal type and there are many extant Aeginetan coins with perpendicular transversals which are not even remotely near to that ratio. The cases of the latter ones are covered schematically by Fig. 1c or d.

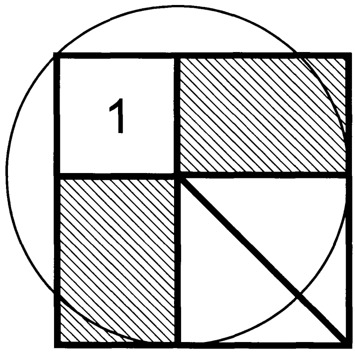
All Aeginetan coins with perpendicular transversals have the geometric pattern of Euclid's proposition II.4 as mentioned above (Artmann 1999, 1990). Thus, they all contain two '*ab*'-type rectangles in the sense of Artmann (1990, p. 47). In the geometrical terminology of Euclid and his precursors, these rectangles are 'complements' and proposition I.43 of Euclid's *Elements* makes a special point of proving that these rectangular complements 'about' the diagonal—i.e. below and above the diagonal—are of equal area (see Artmann 1999, p. 40). This applies to all the rectangles inside the four squares in Fig. 1. Nevertheless, the equality of complements might be of special relevance in the case of the last two figures. The two pellets on one of the rectangles of Fig. 1c or the corresponding double stripes on Fig. 1d may be interpreted as symbols of doubleness in the sense of equality. Their appearance is quite appropriate on a geometrical complement which always has its equal counterpart on the other side of the diagonal. There are several specimens of this type of coins and they seem to appear in this way especially when the design clearly does not comply with the '$3a^2$'-constraint as discussed above.[6] This suggests to ask for the intended message of such symbols. Maybe the message is just a stress on equality: the beholder of the coin should appreciate that the coin design expresses an equality of 'value'. The obvious geometrical equality is here that the two complements have equal areas. But coins are meant for commerce and monetary exchange. Maybe there is here also an allusion to monetary equivalence—a point which we will pursue further below in Sect. 5.2.

It should be noted that there are several coins which, in addition to respecting more or less accurately the '$3a^2$' constraint, incorporate in their design some letters, in particular the letters 'A' or 'I'as shown schematically in Fig. 1a, b. [7] This is remarkable because as the designs are drawn, these letters stand in that part which has a unit value.

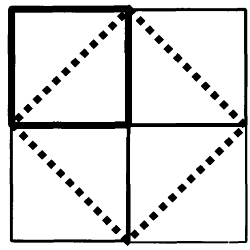
A unit value for the area of the small square follows implicitly from Aboav's interpretation where that square has the value of $1 \times a^2$ and the large inner square has the area value of $2 \times a^2$ since it contains two triangles, each being as large as the small square. According to Heath (1921, pp. 30–31), the Greek notation for unity is either their letter *Alpha* or, in the older 'Herodianic' system, the sign '|' which resembles the letter 'I'. One can therefore suppose that these letters were put there with the intended implication of signalling a unit value in some sense, and in a geometrical context the

6 For an example on the internet see e.g. http://www.s110120695.websitehome.co.uk/PHP/SNG_PHP/04_03_Reply.php?Series=SNGuk&AccessionNo=0300_2003, and some others on this site, URL visited on 09 April 2012.

7 In addition to the references written beneath the figures see also http://numismatics.org/collection/1944.100.36407 for an approximately fitting example containing the letter 'A' (visited on 9 April 2012, American Numismatic Society site). The 'Descriptive Information' on that web-page wrongly lists the letters as being 'A, I, Pi'. But the last letter on the coin clearly is Γ (*Gamma*), the three letters abbreviating the island's name Aegina in Greek spelling as 'A IG'-(INA) with 'A' being in the unit square, according to our interpretation.



(a) The Aegina coins approach



(b) Plato's *Menon* approach

Fig. 2 Doubling the square

intended sense could be that this square should be taken as unit of reference. That fits Aboav's interpretation. It fits also our following interpretation which treats the small square as being of unit value. There are many cases, however, when the coin design is left to speak for itself without any letters or symbols (Aboav 2008, p. 604, Fig. 1b; Artmann 1990, p. 47, Fig. 18). [8] But it is open to much speculation what exactly that design was meant to say, as we will see below.

3 Constructing a square under the '$3a^2$' constraint

Under Aboav's perspective, the main feature of the 'type A' squares on the Aeginetan coins is that they obey the '$3a^2$' constraint mentioned in Sect. 1. This implies that the larger square must have exactly the double area of the smaller one. A geometrical construction of this particular case of doubling the unit square is given below by Fig. 2a.

This construction proceeds as follows:[9] Draw a circle with arbitrary radius r. Quarter the circle with two perpendiculars crossing at the midpoint. Insert a square into one quarter of the circle so that the square's diagonal is given by the radius r, one corner of the square being at the midpoint of the circle and the opposite corner being at its

[8] For a specimen of an Aeginetan coin with perpendicular transversals without any other symbols see also http://www-img.fitzmuseum.cam.ac.uk/img/cm/cm6/CM.LS.685-R(2).jpg , URL last visited on 13 April 2012.

[9] For an alternative construction see Aboav (2008, pp. 695–696) which he chooses 'for its simplicity' (ibid.). It has, however, the complication that it starts from defining the sides of the outer square as being of unit length (his line AB with sections AF and FB). With this choice of the unit length, he ends with showing that the length of the side of the small upper square is $1/\sqrt{2}$ in relation to the rest of the outer square (his AF/FB). He then proceeds (606) to show that 'AP/PC $= 1/\sqrt{2}$' holds as well, this time referring to the diagonal of his unit square. But its diagonal necessarily is larger than its side. It is far simpler to define the sides of the small square (his line AF) as being of unit value and to let all the other measures develop from there. This is what happens in the construction of our Fig. 2a.

circumference. Assign the unit value to the area of the square just drawn.[10] Since the areal value of a square with unit value is generated by multiplying two of its sides so that $1 \times 1 = 1^2$, it is clear that by defining the area of the square just constructed as unity, therefore the sides of this square also have the unit value. The diagonal of the unit square has the value $r = \sqrt{2}$ (Pythagoras' theorem).

Draw now a second square opposite to the first one. Let two of its sides be given by perpendicular radii of the circle just drawn. We know now that their respective value is $r = \sqrt{2}$. Since the new square is constructed with two radii as its sides, one of its corners must also be the midpoint of the circle. Its diagonally opposite corner lies outside the circle. This is so because since the sides of the second square are $r = \sqrt{2}$ each, then, by Pythagoras' theorem, the square of the diagonal must be $(\sqrt{2})^2 + (\sqrt{2})^2 = 4$. The diagonal then has the value 2 and this is larger than the radius.

The area of the second square is $\sqrt{2} \times \sqrt{2} = r^2 = 2$. Thus, its area is double that of the first square which had the value of unity. Draw the diagonal from the midpoint of the circle so that the second square is halved into two isosceles right triangles. Assign the symbol a to the sides of the first square. The first square has the areal value $a^2 = 1$. The second square has the double areal value of the first, hence $2a^2 = 2$. It therefore is clear that each triangular half of the second square has the value of $a^2 = 1$.

Extend now the sides of the two squares just drawn so that a square circumference is formed for the two squares. The resulting outer square is of the desired 'type A', the small square and the two triangles all having the same area. The '$3a^2$' constraint is fulfilled. Under the perspective of the outer square, the perpendicular lines dividing the square are now two transversals which cut the square eccentrically. The larger inside square simply doubles the area of the other, namely of the unit square which stands in the left upper part of the construction.

There are two non-contradictory ways of looking at the total construction and its constituent parts. Apart from the two ab-type rectangles mentioned in the Sect. 1, it has (*i*) three areas equal to $a^2 = 1$; or (*ii*) two squares, one of the area $a^2 = 1$, the other of the area $2a^2$. Whereas aspect (*i*) is rarely considered as an interesting topic in the history of mathematics there is much literature about (*ii*), the mathematics of doubling the square. As we mentioned in the Sect. 1, the first 'extended piece of evidence about Greek mathematics' (Fowler), namely Socrates' lesson in Plato's dialogue *Menon*, is about this very topic, the doubling of a square. It is true that the construction suggested by Socrates is different from ours. Figure 2b shows that the main difference between the present approach and the one which was discussed by Socrates in *Menon* is that he suggested to build the double square directly on the diagonal of the initial square as shown by the square with the broken lines of Fig. 2b where the initial square is the one with the somewhat thicker drawn-out lines in the north-west corner of Fig. 2b.[11] In our alternative construction as given by Fig. 2a, two sides of the doubled square are drawn as radii of a circle and hence the square resulting from these sides has more of a life of its own and it appears in the context of a circular construction. But a doubling

[10] The diagonal just described for the small square is not drawn in Fig. 2a in order to avoid the impression that the unit square is bisected in the course of this construction.

[11] For a detailed discussion of this figure and Socrates' associated argumentation see e.g. Szabó (1978, p. 93). See also the following footnote concerning the way the initial square was divided in *Meno*.

of the unit square takes place in both cases [12] and hence many comments which we find concerning the mathematics of the *Menon* case will be of interest in the present case as well—a point to which we will have occasion to return to a few more times in the following text.

We mentioned in the Sect. 1 already that the construction of 'type A' squares was only an initial step for Aboav (2008) whence he proceeded to the construction of 'type B' squares. For the details of his construction, we refer the reader to Aboav's article. A verbal characterization is that, instead of the two perpendicular transversals as just drawn, the 'type B' design has two skewed transversals. Like the perpendicular ones they intersect eccentrically but in such a way that, together with the 45° line in the right corner which we encountered as a diagonal in the 'type A' Fig. 2a already, they divide the area of the square into five equal parts according to Aboav. The construction of 'type B' squares appears as being a rather eccentric task, also in a figurative sense. Aboav's reason for undertaking this task is the existence of an earlier Aeginetan coin design which was characteristic for the entire fifth century until Aegina lost its independence through Athens and the population was temporarily expelled from the island in 431 BC until, after Athens' defeat in the Peloponnesian War, they could return in 404 BC and resume the minting of coins, henceforth with the 'type A'-based design as just discussed in connection with Fig. 2a.

4 Aeginetan coin design: a case of regression in mathematical sophistication?

Aboav (2008, p. 606) explained the disappearance of 'type B'-based coin design in the fourth century BC with the Aeginetans' temporary expulsion from their island and that they 'on their return have forgotten how it (the 'type B' square, GMA) had been drawn'. Thus, we seem to have here a striking manifestation of a regression of mathematical sophistication in a whole community. But we regard this hypothesis with scepticism.

Aboav's explanation is not very plausible, even in the framework of his own assumption, namely that the Aeginetans' coin design with the skewed transversals of 'type B' was introduced in commemoration of Pythagoras and shortly after the presumed date of his death around 490 BC in Metapontum. If it was indeed Pythagorean, then the mathematical knowledge behind the coin design was an imported one since it was in the Greek colonies of southern Italy that Pythagoras' doctrine flourished at that time and Aegina was in the Aegean Sea, a neighbour of Athens. If the once imported knowledge had been lost for all relevant members of the Aeginetan community by the time of their return after expulsion, then that knowledge could have been revived by new importation if that was what was wanted. After all, the old design was still around

[12] Szabó (1978, p. 342, Fig. 17) points out that the basic square in *Meno* is *not* of unit length, however, Socrates assigns *two* feet to the sides of the initial square. His square is thus divisible in two ways: along the diagonal, and along the middle of the sides, giving two parts in the former case and four parts in the latter. Socrates thereby anticipates Euclid's proposition *Elements* I.34 about the diameter bisecting the area of a rectangle (Artmann 1999, 37, p. 37) and prop. II.4 about the quartering of a square (Artmann 1999,
Footnote 12 continued
p. 62)—the same proposition which Artmann sees behind the perpendicular transversals of Aeginetan coin design.

to be seen on the old coins. It is more likely that the Aeginetans changed the design of their coins not out of ignorance but on purpose. But what could that purpose have been?

It is unknown why some ancient Greek communities chose geometrical design for their coins in the first place and also why there was some persistent change of the basic geometrical design in the coinage of Aegina. Thus, any consideration of these issues must be based on speculation. But as already mentioned in Sect. 1, there is some fairly contemporary ancient literature about the mathematical issues which were discussed at that time and we will see in the following that those changes in design can be fruitfully related to this literature.

There is a remarkable trait of constancy in the Aeginetan coin design in spite of much change. It is the eccentric intersection of the transversals, no matter whether they are skewed, as in the fifth century BC ('type B' pattern) or whether they are perpendicular as they are in the fourth century BC ('type A' pattern). It is even more remarkable that if the partial diagonal of the respective squares were completed, then the ratio of its upper—normally not drawn—section to the drawn-out lower section is $\sqrt{2}$:2 or, what amounts to the same ratio, it is 1:$\sqrt{2}$ as Aboav (2008, p. 605) expressly points out, using the latter version.

The surviving Aeginetan coins make it abundantly clear that an eccentric intersection of transversals was indeed intended over many decades of the fifth and fourth centuries BC. The proportions of the area segments realized on the coins were not always exactly the same as the ones stated and constructed by Aboav. There is nevertheless a remarkable constancy in spite of great variety of details. This suggests that there must have been some very strong unifying conception which could command adherence in very varied circumstance of the community. Behind such constancy over centuries there must have been more than the mere question how to divide a square plot (or cake) into five pieces under the condition that the pieces have one meeting point, but not in the middle of the square. According to Aboav's interpretation, this is the unifying question behind the Aeginetan coin design of the fifth and fourth centuries BC, namely the division of areas by means of geometrical constructions. But it is most unlikely that this view does justice to the driving force behind the Aeginetans' tenacity to their coin design.

A simple and maybe indeed the primordial way of constructing the ratio $\sqrt{2}$:2 might stem from the attempt to order square root expressions. This is shown in Fig. 3a in the context of a rudimentary application of Pythagoras' theorem. The figure depicts the $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$ in a coherent scheme.[13] But $\sqrt{1} = 1$ and $\sqrt{4} = 2$ are, of course, trivial cases. This leaves the $\sqrt{2}$ and $\sqrt{3}$ as non-trivial cases and these were indeed of utmost interest in ancient Greek mathematical thought as we know from, among other places, Plato's dialogue *Timaeus*, written probably around 360 BC. These particular roots are implied by the discussion of squares and equilateral triangles which the philosopher Timaeus invokes in the course of this dialogue in a search for the geometrical figures representing the cosmological elements earth, water, air and fire.

[13] An alternative system of ordering lines expressing these square roots would be the famous 'Theodorus spiral', named after a mathematician in Plato's dialogue *Theaetetus*. It has a series of right triangles with one leg always of unit length, and arranged in spiral form. But as Knorr (1975, p. 69) observed: 'no Greek mathematical work, for instance, contains the recursive construction of square roots of the spiral-type'.

In this context, the well-known philosopher of science Popper (1950, p. 528) once observed

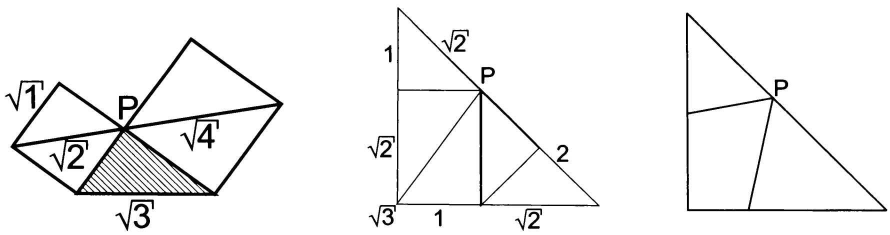
> [Plato] would not have introduced the two irrationalities $\sqrt{2}$ and $\sqrt{3}$ (which he explicitly mentions in 54b) *had he not been anxious to introduce precisely these irrationalities as irreducible elements into his world.* [Popper's emphasis and his round brackets][14]

These irrational numbers appear also in Fig. 3a where we have a (shaded) rectangular triangle with unit length of the short leg and the long leg having length $\sqrt{2}$. It then follows from Pythagoras' theorem that the hypothenuse is $\sqrt{3}$ long as marked in Fig. 3a. The corresponding squares over the legs have diagonals with the values $\sqrt{2}$ and $\sqrt{4} = 2$ as shown. The line connecting these diagonals goes through a point 'P' which divides the total length of the line into a segment marked $\sqrt{2}$ and a segment marked $\sqrt{4} = 2$—that is: in the proportions just mentioned as underlying the diagonal in the 'type A' as well as in the 'type B' squares. Hence, it underlies centuries of Aeginetan coin design—in principle. There are coins on which this principle is not well respected but that does not invalidate the desire to detect the reason for the noted principle.

In Fig. 3b, we extend the scope of the preceding figure by doubling the shaded triangle of Fig. 3a to form now an (unshaded) rectangle in Fig. 3b. Then combine one side of this rectangle with one side of the adjacent square with the unit side length and a second side of the rectangle with the side of the other adjacent square so that the lines thus combined together with the line through point P in Fig. 3a become the circumference of a new rectangular and isosceles triangle as shown. Now there is a horizontal line through P and a parallel line at the bottom of Fig. 3b forming two parallels. The hypothenuse and the vertical leg may then be treated as two transversals for which the intercept theorem applies. In this case, we have then a geometrical statement that the ratio of $\sqrt{2}$:2 along the hypothenuse is the same as the ratio 1:$\sqrt{2}$ along the corresponding leg, or transversal—a claim which we just stated verbally on the previous page and which can now be read off the two transversals with the corresponding markings.

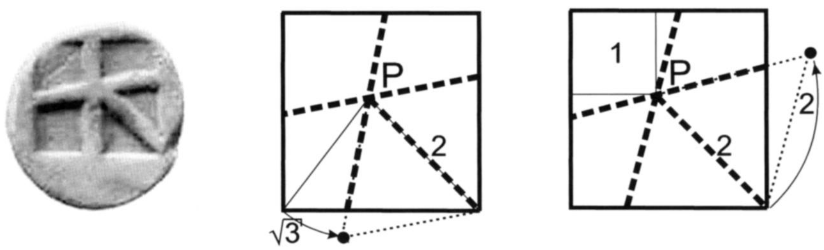
'Type B' squares as proposed by Aboav are symmetrical across the diagonal which contains the 45° line (see also Fig. 4b, c below). If we halve such a square along the axis of symmetry, we get Fig. 3c with a point P in its proper place. What is remarkable in this alternative rendering of the crucial point 'P' is that all the markings have disappeared which cluttered the previous lines and line segments in Fig. 3a, b. The reason for their disappearance is that the whole cut of lines and segments has changed so that we just have no basis for definite markings. Of course, if we have a point 'P' which marks the segments of the axis of symmetry in the stated proportions, then we know that the section from P upwards must have the value $\sqrt{2}$. But this line segment is just a part of the axis of symmetry—it will not appear as being visible in the completed

[14] Plato uses in '54b' the Greek term '*dynamis*' which does *not* mean 'square root', in spite of Popper's just quoted claim concerning square roots. Taylor (1928. p. 372) comments '54b' by observing, contrary to Popper, that 'Timaeus ...speaks only of the "squares," never of their "square roots"'. But in geometrical substance one can agree with Popper in the construction of the associated quadrangles and rectangles.



(a) Primordial square roots **(b)** Incommensurability and disorder **(c)** Disorder concealed

Fig. 3 A primordial Pythagorean setup and problems of commensurability



(a) Head (1911) Pl.XXIV, nr. 3 **(b)** Constructing a quinquepartite square **(c)** The "*Timaeus* variant"

Fig. 4 'Type B' design revisited

square. Thus, the only visible line segment inside the square which would have inherited a marking from the previous constructions would be that segment of the diagonal which was marked '2'. In the context of Fig. 3c, it is out of the question to propose the application of the intercept theorem as we did in the preceding paragraph because now we have no sections for the required parallels. This might be intentional because in a sense the formulation of such proportions relating integers and square roots is downright preposterous, because the latter are 'irrationals'. Their very name sets them apart from ratios.

We might have herein an explanation for the appeal of Fig. 3c or of a square containing this figure for the early mathematicians designing the Aeginetan coins with skewed transversals: the altered cut of lines as illustrated by Fig. 3c abolishes the troublesome segments with square root expressions. With this remark, we are now back at the topic of 'incommensurability' which we briefly mentioned above in connection with Plato's dialogue *Meno*. The term refers to the fact that often the square roots of a number cannot be depicted as an integer or as a ratio—the square root of two being a special focus of attention since this corresponds to the length of the diagonal in a unit square. In fact, the $\sqrt{2}$ is the unit of measurement of a diagonal of a square with any side length as mankind knew from 1600 BC on. This is known from Old Babylonian cuneiform tablets with the corresponding geometrical constructions and arithmetical calculations (Neugebauer 1945, p. 42).[15] So, when in Fig. 3 we combined a line marked '2' with a line marked '$\sqrt{2}$', we were combining magnitudes which are incomparable from an arithmetical perspective.

The Old Babylonians demonstrated around 1600 BC that there was a fundamental arithmetic difference between integer magnitudes on the one hand and $\sqrt{2}$-magnitudes on the other hand and they made the corresponding calculations with remarkable precision. According to many authors, the Pythagoreans knew about the possibility of incommensurability between integers and square roots by maybe 450 BC.[16] It is not known whether this knowledge reached the ancient Greeks directly from the Babylonians or through other channels. As Høyrup (1990, p. 210, n. 11) observed: 'Next to nothing is known about the transmission of Babylonian mathematics after the end of the Old Babylonian period (ca. 1600 BC), but that transmission took place is sure'. The Aeginetans as dominant international traders in archaic Greece certainly were predestined to come into contact with spatially distant knowledge. This is not just a speculation but the overseas communication of the Aeginetan traders was an outspoken contemporary theme when, around 485 BC and in praise of the athlete Pytheas, the poet 'Pindar introduces the idea of publicizing the fame of his client by having his song sail on all the merchant ships from Aegina' (Figueira 1981, p. 323). It should be remembered that it is known that Aeginetan coins were interred in Persia, Persepolis (around 510 BC, Kraay 1976, p. 43), and, around 500–475 BC, in four places in Egypt (Kroll and Waggoner 1984, p. 337, Table II). It is most unlikely that in their international long distance trade the Aeginetans never exchanged ideas and never were interested in new methods of calculation—especially since they are known to have been highly experienced tradesmen. As such they must have been accustomed to extended arithmetical calculations—and they had a long tradition of geometrical designs on their coins.

These considerations are important in order to free the mathematics of Aeginetan coin design from too narrow a connection with the Pythagorean activities.[17] Both might have had a common root which had its beginnings in Old Babylon and the offsprings of which reached the Pythagoreans and the Aeginetans at about the same time. In short: it seems likely that by 510 BC the Aeginetans knew some geometrical method of doubling the square and some intricacies of incommensurability. These are speculations, but let us remember that the speculations have an incontrovertible material basis: the eccentric meeting point of all the transversals on Aeginetan coins after about 500 BC which, when interpreted mathematically, gives the diagonal proportions of type $\sqrt{2}$:2. It must be granted that between mathematical design and practical execution of coinage there are many steps where much can go wrong, thus blurring the basic idea of the designer's master-mind. But if the basic idea is accepted that

[15] On the internet, the site http://www.math.ubc.ca/~cass/euclid/ybc/ybc.html might be inspected for an instructive presentation of the astounding tablet YBC 7289 with a *Menon*-type square (visited on 20 April 2012). For a detailed discussion of its context see Kaplan and Kaplan (2010, p. 29).

[16] See Knorr (1975, p. 55): 'a date of ca. 450 BC for that discovery [of incommensurability, GMA]...accords with the view of most others, for instance, Heller, von Fritz and van der Waerden'.

[17] See Kaplan and Kaplan (2010, p. 48) who write about 'the startling paradox delivered by modern scholarship: Pythagoras had nothing to do with the theorem that bears his name!'.

the persistent geometrical eccentricity of the Aeginetan coin design has a systematic background, then human curiosity ought to be permitted to ask for an explanation.

We have mentioned three different contexts so far from where the proportion emerged which is characteristic for point 'P': (*i*) Aboav's idea of an anticipation of Euclid's *On Divisions [of figures]*. (*ii*) This section's constructions of triangles producing geometrical representations of square roots. (*iii*) Doubling the square according to the previous section's Fig. 2a. Since all three constructions produce this proportion, it might well be that they all were relevant. But we consider the variant as represented by Fig. 2a to be the most elegant and the one which is most directly related to the general framework of Aeginetan coin design, but there are still other possibilities of construction as we will see presently.

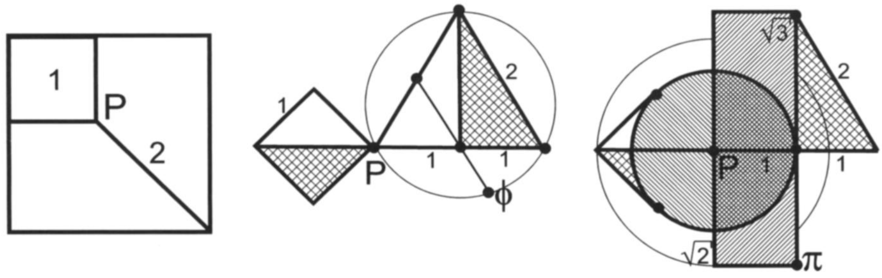
What is common to all constructions leading to the generation of this important point 'P' is that they all deal with lines standing for integer numbers—either 1 or 2—and then they combine them with 'incommensurables', lines representing $\sqrt{2}$ or $\sqrt{3}$. There seems to be largely agreement that in the Greek history of mathematics there were two important epochs concerning these 'incommensurables': in a first epoch, they were recognized as incompatible with integers or rational numbers and they were treated as 'taboo', the Greeks having other terms for this eeriness, of course. Referring to 'the *Greater Hippias* 303b/c' and to '*Republic* , 546c' Popper (1950, p. 526) writes that 'Plato still calls the irrational at first "*arrhētos*," the secret, the unmentionable mystery'. In a similar vein, Szabó (1978, p. 93) stresses that in Plato's *Menon* 83e, when Socrates gets the uneducated slave boy to perform the doubling of the square, he tells the boy to 'just show' the relevant diagonal, and Szabó comments: 'These words are a quiet hint to the informed reader that the length ...cannot be given a numerical value, because it is an *arrheton*' (Greek letters are transliterated, GMA), thus alluding to the same word as Popper did as characterization for the ancient Greek 'incommensurables'.

According to some ancient Greek historical anecdotes, there was indeed a veil of secrecy surrounding the incommensurability of square roots. But that relates probably to an era when Pythagoras still lived, maybe around 510 BC. An entirely different attitude existed as far as Plato himself is concerned. Popper (1950, p. 527) observes: 'Plato's great interest in the problem of irrationality is shown especially in...*Laws* 819d–822d, where Plato declares that he is ashamed of the Greeks for not being alive to the great problem of incommensurable magnitudes'. Plato's complaint is not that the public knows too much about incommensurability but that it knows too little. Again and again Plato deals openly with incommensurability and likewise did his erstwhile disciple Aristotle. Fowler (1999, p. 290) gave a detailed list of more than 30 occurrences of the concept of 'incommensurability' in Aristotle's writings. Artmann (1999, p. 229) notes that 'Aristotle ...quotes incommensurability as the prototype of a scientific discovery'.

This shift in ancient Greek attitude towards incommensurability probably occurred around 450 BC because that is the time when the first mathematical proofs of its existence were invented. Since the first Aeginetan coins with skewed transversals depicting or suggesting 'type B' design appeared well before that date, it seems plausible to assume that this design had traits of its era-specific secretive attitude to incommensurability. But we do know from the above that the basic coin design is such that in its constructed form it does imply the appearance of lines representing incommensurable magnitudes, therefore a connection with incommensurability is clearly present. We can presume that at the end of the sixth century when we had the first appearances of this coin design there was indeed already awareness of incommensurable magnitudes among Greek mathematicians but that this awareness consisted in a practical knowledge about possibly endless calculations when irrational numbers appeared. The difference to the subsequent period of dealing with incommensurability in the way which is documented in the writings of Plato and Aristotle of the fourth century BC probably manifested itself in a lack of knowledge of methods of mathematically *proving* incommensurability and in a wariness of using proportions involving incommensurable magnitudes since proportions were initially thought to involve 'numbers' as integers and not irrational 'unmentionables'. An era-specific explanation of Aeginetan 'type B' coin design then might have two aspects: On the one hand, its inventors may be assumed to have known about incommensurability, as witnessed by the proportions associated with a well-constructed point 'P' of this design. On the other hand, these inventors seem to have consciously avoided drawing lines which could be seen as standing for incommensurable magnitudes, since these were 'unmentionables'. Reasons for such avoidance could have been respect for the aforementioned taboo associated with such magnitudes or an uneasiness to deal with them for practical reasons of tedious and endless approximations, or still other reasons beyond our present imagination.

A drawing which concealed lines standing for 'unmentionable' incommensurable magnitudes was shown above in the context of the discussion of Fig. 3 in the course of which we had the juxtaposition of Fig. 3b which was full with markings for lines or line segments containing square roots on the one hand with Fig. 3c on the other hand where these markings disappeared because the line segments were cut differently. But that still leaves open the mathematical method of this different cut of lines, or rather: its geometrical method of construction. As already mentioned, Aboav (2008) believed that the relevant method was an anticipation of geometrical constructions which later found their expression in the last proposition of Euclid's book *On Divisions*. But if we inspect the surviving coins of that time, we see that many of them do not comply to Aboav's assumption that the five sections of the coins' square are of equal area. Some even have a mixture of a perpendicular and a skewed transversal as reproduced below in Fig. 4a. But many coins do show at least a symmetry across the diagonal. Such cases may be constructed as shown in Fig. 4b.

Figure 4b presents an alternative to the constructions of Aboav's Figs. 4 and 5. It is based on the $\sqrt{3}$ line of triangle 3b. Focus there the line of length $\sqrt{3}$ running from point P. Use the length of this line in order to construct an isosceles triangle over the diagonal section marked '2' as base. The resulting triangle is shown in the lower part of Fig. 4b by the two legs in thin broken lines running from the basis, the line segment '2'. In Fig. 4b, the total triangle of Fig. 3b has been doubled to form a square as shown. Of the isosceles triangle just constructed on the basis '2' take the leg which is mostly inside the square, extend its line through P, and mark its course from one side of the square to the other side. Focus the other leg of the triangle just constructed. Its thin broken line lies outside the square at the bottom of the figure. Draw a parallel with a thick broken line running through P and across the square.



(**a**) Reconstructing " P " (**b**) *Timaeus*' geometric 'elements' (**c**) Approximating π

Fig. 5 The Timaeus approach

There are now five thick broken lines running from P to the sides of the square. They divide the area of the square into five parts. Since the same construction could be done from the opposite corner of the square, the subdivision is symmetric. The diagonal line of length $\sqrt{2}$ halving the upper quintile has disappeared. It is now covered by the area of that quintile and thus it is out of sight now. The lines representing irrational numbers are, of course, still underlying this construction but they are not *obviously* there. They cannot be simply pointed at as in Fig. 3b or in a square which is based on doubling that triangle. So, if it was indeed the aim of the early Greek mathematicians to conceal the phenomenon of incommensurability, then a 'type B' figure like Fig. 4b might have had some appeal. But under the perspective of incommensurability it is a device to conceal and not to reveal.

Figure 4c shows, however, that there are other, seemingly similar, ways of creating such skewed transversals. The alternative offered there is based on an isosceles triangle with sides of length two based on the 45° line which is always characteristic of Aeginetan coin design. Thus, we have here an equilateral triangle with sides of length two whence we can proceed in a way analogous to the one just discussed in connection with Fig. 4b in order to arrive at the required symmetrical structure with skewed transversals. But this construction is maybe only seemingly similar because it might result from quite a different basic geometrical conception. Such an alternative is shown below in Fig. 5a. The pivotal point 'P' of the coin design appears there simply as that point which we get from a unit square when we add a line of length two at one of its corners in the way shown. The only unit of measurement in this context is '1' and an addition of two 'ones' to give the cipher 'two' and a line representing that cipher.

We have here two very basic—one might say 'archaic'—geometrical concepts, namely 'area' and 'line' and the very basic ciphers 'one' and 'two'. Yet, in Plato's *Timaeus*, this setup is the basis for an entire cosmology. The beginnings of that argument can be traced if we now regard Fig. 5b which reproduces parts of Fig. 4c in that the unit square is now seen in connection with an equilateral triangle on the basis of the line '2', only that the construction of Fig. 4c is now tilted 45° to the left. This construction may be broken down into right triangles as shown, where the two right triangles generated from halving the square have a hypothenuse of length $\sqrt{2}$ and the two triangles constituting the equilateral have both a leg of length $\sqrt{3}$ so that, again following Pythagoras' theorem, the square of this length and the square of the other leg, namely $3 + 1 = 4$ give the square of the equilateral's sides which have the length '2'. Thus, we arrive at the two square roots $\sqrt{2}$ and $\sqrt{3}$ which Popper (1950, p. 528) considered to be the 'irreducible elements' of Plato's 'world' as was quoted above. But in fact it is the two right-angled triangles just drawn which Timaeus proclaims to be 'the two irreducible "elements"', namely for producing the four primary bodies of Timaeus' cosmos (Cornford 1935, p. 211). It would lead too far, however, if we followed Plato's cosmological arguments further. Nevertheless, the geometrical similarity is interesting which exists between Plato's 'elementary' construction of Fig. 5b on the one side and the potential geometrical basis for the skewed Aeginetan coin design of Fig. 4c on the other side. There is the possibility that this similarity had not just a geometrical aspect but also a cosmological one.

A further aspect of Timaeus' elementary construction is its association with geometrical 'beauty'. Timaeus (54 a) gives as reason for his choice of triangles that they give the most 'beautiful' appearance since they are 'constituent of the equilateral triangle' (Gregory and Waterfield 2008, p. 47). Further criteria are not given but there is an aspect of Fig. 5b which we can appreciate only after the 1980s. It is only then that it was proven that a line drawn through two midpoints of the legs of an isosceles triangle to the circumference of a circle inscribing that triangle represents the 'golden section' (Walser 2001, p. 127, Fig. 128b).[18] In Fig. 5b, the ratio of the total line marked 'Φ' to that section of the line which lies inside the triangle expresses the 'golden ratio' $\Phi \approx 1.61803$. Whether this was known at Plato's time and in which way it was known is totally unclear, but the 'golden section', or the 'mean and extreme ratio' as it was called at that time, is often regarded as aesthetically appealing. Hence, this aspect might have been implied in some way when Timaeus invoked beautiful appearance as justification for his choice.

There is a still further speculative aspect associated with Timaeus' elementary geometry. Its existence is stressed by Popper (1950, p. 530) and it is represented by Fig. 5c. The figure features two circles. The inner circle has a radius of unity, cutting the unit square at two corners and covering the rest of that square. The outer circle has a radius equal to the diagonal of the unit square and thus it represents $\sqrt{2}$ as marked. The line halving the equilateral triangle has length $\sqrt{3}$ as also marked. A rectangle with unit width and extending over these two sections has an area of $1 \times (\sqrt{2}+\sqrt{3}) = 3.146\ldots$. The circle with the unit radius has the area $1 \times \pi = 3.141\ldots$. Thus, the circle and the rectangle in Fig. 5c have approximately the same area of about 3.14 units. It is possible that a keen observer could have known this long before Plato's time. In any case, it is Popper's contention that Plato was aware of this approximation of π.[19] Maybe this was a further reason why Plato let Timaeus propose his combination of the unit square and the '2'-based equilateral.

[18] Concerning its definition see Walser (2001, p. 2) 'We say that a line segment is divided in the *ratio of the Golden Section*, or in the *Golden Ratio*, if the larger subsegment is related to the smaller exactly as the whole segment is related to the larger segment.'

[19] Popper (1950, p. 529): 'Plato may have found out the approximate equality of $\sqrt{2} + \sqrt{3} \approx \pi$ ...Plato knew of this equation, but was unable to prove whether or not it was a strict equality or only an approximation'.

One aspect of the 'beauty' of Timaeus' 'elementary' conception can be seen in that it offers an integrated framework for representing several irrational magnitudes: $\sqrt{2}$, $\sqrt{3}$, the 'golden' number Φ, and the circle number π in an approximation, while involving only the measurement of the unit for the unit square and the doubling of this unit for the basis of an associated equilateral triangle. But it still is true that the structure of Fig. 5b can be simultaneously alluded to and concealed, namely by spreading over it a square with skew transversals in the manner of Fig. 4c. The skewed transversals of Aeginetan coin design can be seen as implying the knowledge of incommensurable magnitudes but this will reveal itself only to the initiated beholders of the coin design.

If there was indeed a latent desire to conceal the existence of incommensurability from outsiders in the discussion of these constructions, then this suggests that chronologically they belong not to Plato's fourth century BC but rather to the time of the early Pythagoreans. Indeed, it seems that Plato's real Timaeus has to be placed into the preceding, the fifth century BC if he really was Socrates' mature conversation partner as it is claimed in the dialogue. Maybe even more relevant in this context is the ancient slander that Plato plagiarized the dialogue *Timaeus* from earlier work by Philolaus the Pythagorean (ca. 470 to ca. 385 BC). Even if that charge by Diogenes Laertius is dismissed, 'it is entirely reasonable to suppose that a work of Philolaus' acted as a source book for Plato's *Timaeus*' Olsen (1985, p. 76). If we assume that Philolaus himself might have had some earlier sources for his constructions, then this might well move the origin of these constructions to the time around 500 BC when the Aeginetan skewed coin design was first used (Kraay 1976, p. 44).

At the end of the discussion of the Aeginetan skewed coin design, we see no reason to modify our prior suggestion that its appeal for the inventors of this design might well have been that it can simultaneously reveal knowledge about incommensurability while at the same time it conceals this phenomenon from the uninitiated viewers. But there are many patterns possible which the more or less skewed transversals can form and it will always be open to debate whether cases not fitting a particular geometrical interpretation were expression of a different geometrical conception, whether they were the fault of incompetent craftsmen making the real coin, or whether there was absence of any formal intention. We can just marvel at the strange structure of these coins' design but we can also marvel at the richness of its possibly intentional implications.

Concealment of incommensurability became obsolete once this phenomenon lost its mental horror. Plato's dialogue *Meno* gives literary proof that by his time it had become an intellectual enjoyment to show how to deal with incommensurability, and the doubling of the unit square played an important role in this context. As Artmann (1990, p. 44) observed, some coins of the island of Melos gave *physical* proof that this development occurred long before Plato's dialogue:

> This component of elementary instruction in geometry seems to have been so popular [by 420 BC, GMA] that it found its way into the decoration of coins.

We propose to see the change of Aeginetan coin design after 404 BC in this context. Then the 'type A' design appears not as intellectual regression but as a sign of progress in mathematical thinking.

5 Aegina's coin design as hallmark of Greek scientific genius

5.1 The mastering of incommensurability

A central aspect of the Aeginetans' newer coin design was the explicit demonstration of the doubling of the square as we argued above in connection with Fig. 2a. Doubling the square has occupied many historians of mathematics, not least because with Plato's *Menon* 82b–85b we have an excellent account of every single step in such a procedure. The geometrical counterpart of these passages is not identical with the Aeginetan coin design but, as already noted, it is in principle that of Fig. 2b. Nevertheless, the argumentation is in substance the same and we are dependent on Plato's text and its interpretation since there is no other relevant literary evidence from that time.

Maybe the strongest emphasis on the significance of this type of geometrical exercise comes from Szabó (1978, p. 93) who, in view of these passages in *Meno*, proclaimed:

> Without a doubt the problem of doubling the square...led to the problem of the diagonal of the square and thence to the problem of linear incommensurability.

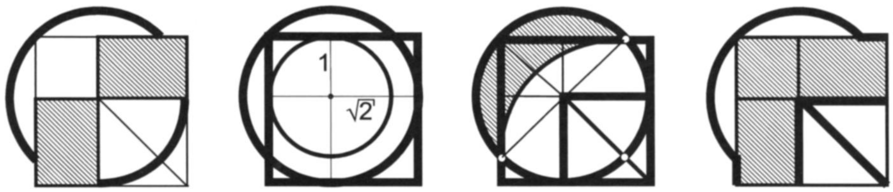
Szabó goes on to claim (ibid.) that in this context we may also see the first Pythagorean proof for the existence of incommensurability and Knorr (1975, p. 26) gives the corresponding details in a Fig. 2b-type scheme. However, there are many ways to prove incommensurability (Artmann 1999, pp. 229–253, Chap. 24) and there is controversy whether the most relevant figure for the ancient proof really was the square and its diagonal or whether it was not rather the pentagon (von Fritz 1945).

In any case, the ancient Greeks' 'genius for and their tenacity in the pursuit of scientific theory' became apparent 'amazingly soon after the stunning discovery of incommensurability' as Fritz (1945, p. 261) wrote in greatest praise. The conscious treatment of incommensurability was so important in this regard because in its wake there appeared a new conception of proportionality, namely one which included incommensurable magnitudes as well as integers. This change enabled Hippocrates of Chios to extend conceptions of proportionality to segments of circles and to entire circles as Fritz (1945, p. 261, n.84) briefly points out.

A reference to mathematical work on circles might appear to be a bit far-fetched in connection with the *Menon*-type discussion of the doubling of the square since this deals only with interlacing squares. But we may briefly return to Fig. 2 and, regarding them as a group, we may remind ourselves that virtually the same doubling of the square occurs in both of them. The advantage of the Aeginetan approach to the self-same topic of doubling the square is now that the problem is solved with explicit reference to a circle. As we will see below, this construction will offer a direct venue to the just mentioned remarkable work of Hippocrates of Chios.

5.2 Complements, equivalence and circular flow

Although the explicit doubling of the square is a central aspect of the perpendicular Aeginetan coin design, it is not the only one. In our typology of such coins in Sect. 2



(a) Circular flow (b) Double doubling (c) Square and circle (d) Stress on *gnomon*

Fig. 6 Aspects of Aeginetan coin design

above, we juxtaposed the 'doubling' patterns of Fig. 1a, b to those of Fig. 1c, d where it is plain that such doubling is not depicted. Instead, particular stress seems to be put on equality, namely equality of the respective ab-rectangles of Artmann's parlance which was mentioned in the Sect. 1. Such equality of the two rectangles is a general feature of the diagonal-based rectangular division of any square, even if it does incorporate the doubling of the square as the types (Fig. 1a, b) did. It seems that with the remarkable markers of doubleness in types (Fig. 1c, d)—either in form of two pebbles or in form of two lines—the designers of such coins wanted to stress an aspect which they felt to be particularly important.

If we see this construction in the light of Euclid's *Elements*, then it comes to mind that such an equality of rectangles is expressly formulated as

> Prop. I.43
> In any rectangle the complements of the rectangle about the diagonal are equal to one another. (Artmann 1999, p. 40)

As briefly mentioned in the discussion of Fig. 1, it is these 'complements' which we encounter in the Aeginetan coin design as upper and lower rectangle, their equality being assured by mathematical proof if we consider this construction as an anticipation of Euclid.

Proposition I.43 'has many useful consequences' as Artmann (1999, p. 40) comments, in particular the consequence of preparing for the next two propositions which, in Artmann's further comments, amount to the algebraic problem of first defining a value $A = ab$ where a and b are sides of an inscribed rectangle inside the outer rectangle or, in the present case an outer square as illustrated by Fig. 6a and, say, the lower shaded rectangle, its area standing for such a value A. Given such a multiplicative value A which is made up of two components we can then formulate the task to find a multiplicand x so that for a value R we get $R = ax$. Thus, if $R = A$ and a is as before, then the value x is the same as b before. In this special case, it is not only the value of the multiplication which is the same but also the two multiplicands and the algebraic formulation corresponds to the geometrical equality of the two shaded rectangles in Fig. 6a.

The geometrical Fig. 6a is a special case which follows from the fact that we put the application of Euclid's propositions I.43 to I.45 into the special figure of an encompassing square and not into that of a rectangle. It is of the more general case of complements consisting of two rectangles with unequal sides but identical values that we must think of when invoking propositions I.43–45 in the present context.[20] We are discussing here the design of coins. Its symbolism is likely to have been meant to relate to the context of coins and this is economic exchange, of course. Now, in economic context, a 'value' like the A or the R in Artmann's comment is a money value and money value as a result of a multiplication is the result of the two multiplicands: 'price' and 'quantity'. They correspond to the a and b just used. Thus, the geometrical equality of area-values as shown by the complements in Fig. 6a, if interpreted in a context of economic exchange, means: '$\text{price}_1 \times \text{quantity}_1 = M$' and '$M = \text{price}_2 \times \text{quantity}_2$' where the indices signify good '1' and good '2' and where M is a specific money value, this value being represented by the shaded area of a rectangle and this rectangle having a complement of exactly the same area value—and monetary value—in a similar way as shown in Fig. 6a.

It might seem that the 'economistic' interpretation just suggested is anachronistic. After all, at the time of the coin design under discussion, the Aeginetans had coined money for maybe <150 years and such equations of money values might have never occurred to them. But such an objection appears to be unfounded if we think of the famous epigram D90 of Heraclitus of Ephesus (ca. 600 BC):

> There is turnover[21] of fire against everything and of everything against fire, like gold against goods and goods against gold.

There is much debate among commentators about the appropriate interpretation but it is mostly accepted that monetary turnover is taken here as a metaphor for the cosmos. The 'monetary form of exchange', according to Shell (1993, p. 60) 'informs many fragments of Heraclitus'. It is in this sense that Seaford (2004, p. 94) explains: 'gold here as a universal means of exchange is a model for the cosmic role of fire, which is for Heraclitus constitutive of the cosmos', elaborating then: 'Just as the circulation of goods (money–goods–money, etc.) is driven by…money, so cosmic circulation (fire–things–fire, etc.)' (Seaford 2004, p. 232).[22]

It seems quite clear that from Heraclitus' time on there existed the perspective that monetary exchange can be seen as an exchange of equivalents and that there is a circular flow of currency. In this connection, Shell (1993, p. 60) quotes also Heraclitus' fragment D60, reading 'The way up and the way down are one and the same' and he comments: 'The way up and the way down refer to sale and purchase'. The only figure in the text (apart from the later plates) is when Shell (1993, p. 61) draws the circular flow of goods and gold between Ephesus and Sardis, the capital of Lydia from where we have indeed the oldest gold coins.

The Aeginetan coin design accords well with such a perspective: the two rectangles are complements in the sense of Euclid and as such they represent mathematically

[20] See Artmann (1999, pp. 40–41, Figs. 4.18, 4.19).

[21] In this translation of Heraclitus' *antamoibē* I follow Diels' 1901 German translation which uses the term *Umsatz* = 'turnover'.

[22] Similarly Shell (1993, p. 56): 'Heraclitus …splits the barter transaction into two opposite operations, namely, sale and purchase. This split presupposes an intermediating third term, money…In a money economy, one thing is not exchanged directly for another, but is first exchanged for money which seems to represent or be all things'.

proven equality. As rectangles they represent price×quantity values. Embedded in a square they can associate a 'square deal' in accordance with the alleged Pythagorean conception of justice as a square.[23] The fact that the equal (money) values of the rectangles are connected by segments of one circle (see Fig. 6a) suggests an invocation of a circular flow of money values in the tradition of Heraclitus' famous metaphor which likens the economic circular flow to that of the circular flow of fire in the all-involving cosmos.

5.3 Doubling the circle

It was mentioned above that one of the important spin-offs of doubling the square was the reformulation of conceptions of proportionality so that they could cover cases of incommensurability and nonlinearity, in particular cases involving circles. We may now carry on in this vein by hypothesizing that in such an extended view of proportionality there must be a narrow correspondence between the doubling of the square and the doubling of the circle. Thus, if we have two square areas $A_{\Box 1}$ and $A_{\Box 2}$ with known proportionality to each other then there must be a corresponding proportionality between two circular areas $A_{\bigcirc 1}$ and $A_{\bigcirc 2}$ so that we have

$$A_{\Box 1} : A_{\Box 2} = (1)^2 : (\sqrt{2})^2 = A_{\bigcirc 1} : A_{\bigcirc 2}$$

The corresponding circles are drawn in Fig. 6b where the area of the outer circle is double that of the inner circle in the same way as the square with side $\sqrt{2}$ is double the square with side 1. This result could be derived directly from Euclid's *Elements* where we have

Prop. XII.2
Circles are to one another as the squares on the diameters (Artmann 1999, p. 272)

which means that for two circular areas we have the general case of $A_{\bigcirc 1}:A_{\bigcirc 2} = r_1^2:r_2^2$ where r stands for the radius (half the diameter) of a circle.

We have here again the problem of chronology, namely that the putative Aeginetan construction antedates Euclid by about 100 years. But there are ancient authors' reports that Hippocrates' of Chios work (around 430 BC) indicates familiarity with 'a form of the theorem that circles are as the squares on their diameters (cf. XII, 2)' as Knorr (1975, p. 40) relates, his reference being to the just quoted proposition. It is thus not implausible that around 400 BC there was awareness among some of the makers and the users of the newer Aeginetan coins that their design implied not only the doubling of a square but also the doubling of a circle as shown in Fig. 6b.

[23] Cf. Rucker (1982, p. 54) who lists the Pythagorean identifications of numbers with concepts and registers: '4 was justice (a "square" deal)'.

5.4 Squaring the *lunule*

The basic outline of the newer Aeginetan coin design is that of a circle and a square interwoven with each other in a way which could easily lead to the question whether we have here not also a contribution to that popular ancient problem of 'squaring the circle'.[24] Good eyesight would immediately give a negative answer, confirmed by computation which shows that $A_{\square} \approx 5.83 < A_{\bigcirc} \approx 6.28$. But there is also a positive answer in that the Aeginetan design incorporates one of the most astonishing ancient contributions to the debate of this problem, namely Hippocrates' of Chios squaring of the *lunule*. This is shown by Fig. 6c where the area of the moon sickle-shaped shaded area on the upper left is the same as that of the double unit square on the lower right.

The construction of Fig. 6c requires that a circle is drawn with radius 2, the double of the side of the unit square. Its midpoint is the intersection of the circumference of the initial circle with the diagonal of the doubled square as shown. The resulting intersections of the new circumference with the outer square mark a segment of that circle which gives the lower border of the shaded moon sickle. One of Hippocrates' theorems now states that a right isosceles triangle with a hypothenuse given by the line connecting the two corners of the moon sickle has the same area as the moon.[25] But such a triangle consists in this case of the unit square enlarged at its right side by one of its halves and likewise enlarged at its lower side. The total area is then the double of the unit square, hence it is the same as that of the lower square.

Hippocrates of Chios, the putative inventor of this type of construction, taught in Athens around 430 BC. He is reported to have written the first *Elements* of mathematics, but they have not survived. It is plausible that by ca. 404 BC, the earliest possible date for the new Aeginetan coins, knowledge of this type of construction had reached the mathematically interested Aeginetans. Those among them who conceived the new coin design might well have been aware of an implication of this design which is represented by our Fig. 6c.

5.5 The *gnomon* as instrument and as metaphor

In presenting the geometry of the newer Aeginetan coins as precursor of Euclid's proposition II.4, Artmann (1990, p. 48) remarked that there is a slight difference, commenting this difference as follows:

> Similarly subdivided rectangles and parallelograms abound in the *Elements*; the so-called 'Gnomon' is a familiar tool in geometrical (and arithmetical) proofs. The missing diagonal in the small square could be a means to emphasize the gnomon

Artmann (1999, p. 61) then reproduced Euclid's definition (originally emphasized):

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[24] Artmann (1999, p. 73): 'It must have been popular in the intellectual circles in Athens about 440–400 B.C.E. Evidence for this is a line from the comedy *The Birds* written by Aristophanes'. The comedy is dated 414 BC and features a grotesque city planner who proposes the squaring of the circle.

[25] For a detailed construction along these lines consult O'Connor and Robertson (1999).

II. Def. 2. (For rectangles) Let any one of the rectangles around the diameter of any rectangular area (together) with the two complements be called a gnomon.

whence it emerges that the 'tilted L'-shaped structure surrounding the doubled square of the Aeginetan coins is such a *gnomon*. The fact that this *gnomon* itself is embraced by a semicircle in the present construction as illustrated by Fig. 6d above may be interpreted as a reinforcement of the emphasis on the *gnomon* which Artmann has emphasized already. But it is rather unclear why the designers of the Aeginetan coins put so much emphasis on such a geometrical shape. Artmann (1999, p. 62) relates that the *gnomon* 'was originally a sort of primitive sundial', in literal translation 'an instrument for knowledge', as we may add, but both pieces of information contribute little to a better understanding of the significance of such a design on Aeginetan coins.

There are some passages in Heath (1921, p. 78) which enlighten us further about this term and its ancient epistemological significance (emphasis as in the original):

> The term is used in a fragment of Philolaus [the Pythagorean, ca. 470 – ca. 385 BC, GMA] where he says that 'number makes all things knowable and mutually agreeing in the way characteristic of the *gnomon*'. Presumably, as Boeckh (1819, p. 144) says, the connection between the *gnomon* and the square to which it is added was regarded as symbolical of union and agreement, and Philolaus used the idea to explain the knowledge of things, making the *knowing* embrace the *known* as the *gnomon* does the square.

In short, the geometrical figure of a *gnomon* is not only mathematically very fruitful as we may learn from Artmann (1999, *passim*), but it may also be seen as a sign of friendly embrace which produces self-similar 'offspring' from the embraced square. It is in this latter sense that Aristotle, in his *Categories* (ca. 14 ; 15a29–33) describes the *gnomon* as follows:

> But there are some things which are increased without being altered; for example, a square is increased (i.e. enlarged) when a gnomon is placed round it, but it is not any the more "altered" (i.e. changed in shape) thereby, and so in other similar cases. (Heath 1949, p. 20)

Philolaus' and Aristotle's allusion to a clasping or embracing visual appearance of the *gnomon* gives this noun an ambiguity of meaning which one might liken to the one associated with the English verb 'to grasp something'. It can mean 'to understand' as well as 'to make something one's own'. The latter association is, of course, quite appropriate in connection with coins since they are meant to be used for acquisitive commercial purposes. It is quite congenial for a commercial context to have a means of payment with a symbol which can be understood as meaning 'knowledge' as well as 'seizure', or 'augmentation' as well as 'friendly embrace'. It seems that all of these meanings could have been implied when the coin design was interpreted as emphasizing a *gnomon*. An accentuating version of the coin design as represented by the above Fig. 6d therefore could indeed have been very much to the liking of its initial inventors.

6 Summary and conclusion

There are some ancient Greek coins from the island state of Aigina which depict peculiar geometric designs. They come in basically two interesting versions, one from the beginning of the fifth century BC, the other from its end. From the viewpoint of mathematics, this century was a 'golden age' as van der Waerden (1963) once remarked in his treatise on *Science Awakening*. We do not know why and by whom the coin designs were conceived as we know little about Aeginetan society and policy in general (Figueira 1981). Even the geometric constructions which generated these designs are in dispute (Aboav 2008).

We do know, however, that the usage and the knowledge of the Aeginetan coins was wide-spread, that there was 'Aiginetan currency domination' (von Reden 2010, p. 72). This started to change from about 480 BC on—the time when Athens started to assert its hegemony and vied for monetary dominance herself. But in spite of the Athenian hegemony during much of that time, the fifth century ended with the monetary standard of Aegina being used in a large part of the ancient Greek world (Figueira 1981, p. 82). About 50 years later Aristotle treated Aegina as proverbial for something 'necessary' in monetary matters (*Metaphysics* 1015a25), but it is not quite clear what that necessity was.

As today there are people who are curious about the significance of the design on Aegina's coins, such curiosity must have been far greater among the ancient users of the coins. It should be remembered that for many centuries coins were the only imprinted mass medium (Shell 1993, p. 64). Their message certainly was noted and discussed intensively by their contemporary users. Hence, it is quite likely that we would have an improved understanding of the ancient Greek mind-set if we knew more about the mental background of the Aeginetan coins. But since there is no documentary evidence about them, we must rely on speculation and on considerations of plausibility in our quest for a better understanding.

Differing considerably from previous interpretations (Artmann 1990, 1999; Aboav 2008), we proposed to see the Aeginetan coin design under the perspective of concealing *versus* revealing 'incommensurability'. Their older versions with skewed transversals fall under the topic of concealing incommensurability. It does show knowledge of incommensurability, namely through the choice of an eccentric meeting point of the construction's transversals which divides the diagonal into two sections with the proportion $\sqrt{2}$:2. Yet that design conceals incommensurability by hiding any line or section of a line which could be easily identified as representing an irrational number, including that part of the diagonal which represents the $\sqrt{2}$.

The present interpretation of the construction of the older Aeginetan coin design with skewed transversals is in agreement with some ancient reports about the early Pythagoreans according to which 'traitors' who revealed the 'secret' of incommensurability were expelled from the brotherhood and met an untimely death (for a 'mental movie' on this theme see Rucker 1982, p. 57). It is impossible to tell whether the Aeginetans chose this design out of reverence (Aboav 2008, p. 611) or out of deference against the teachings of Pythagoras who may have died shortly before the appearance of this design. A publication via coin design of the Pythagorean secret of incommensurability, even in a veiled form, was hardly in the interest of a secretive Pythagorean

Fig. 7 Artmann's coin sketch overlaid

Brotherhood. But it is possible that the Aeginetans got knowledge of incommensurability and 'eccentric' divisions of squares through their far-reaching overseas trading contacts for which they were well known. The seemingly old 'Pythagorean' knowledge that the diagonal of squares had to be measured in terms of the 'irrational' $\sqrt{2}$ was definitely Old Babylonian wisdom of about 1600 BC as is well documented through clay tablets of that time.

The Pythagorean secret of incommensurability had been lifted by the time when this phenomenon became the *explicit* topic of coin design. There are two cases which one can claim as being in point: one case is given by some coins from the island of Melos (ca. 420 BC), the other case is given by a number of coins from Aegina (from 404 BC on). Important literary evidence for assessing these two cases comes from a time which is much later than their physical manifestation in minted form. Most important is a famous passage in Plato's dialogue *Meno* (written probably 385 BC) in which Socrates gives a geometry lesson on 'doubling the square'. This doubling occurs by taking the diagonal of the original square as the side of the square with double the area of the first. But Pythagoras' Theorem about right triangles shows that when the square's sides are given by integers, then such a diagonal must be measured in units of the $\sqrt{2}$, i.e. in irrational numbers, the diagonal thus being incommensurable with a side of the square which is measured in integers. Thus, incommensurability is revealed by doubling the square (Szabó 1978, p. 93); but at the same time it is also healed: in a square with a side of numerical length the incommensurable diagonal line, when squared, gives an area measurable in integers.

It is clearly stated in modern literature that the Melian coin design of ca. 420 BC is the visual forerunner of the corresponding literary description in the *Meno* dialogue of 385 BC (Artmann 1990, p.44). But it is less clear that the typical Aeginetan coin design of 404 BC onwards deals with the self-same problem as the Melian one, albeit on the basis of a very different construction. Its main feature is a circle in which the radius gives the diagonal of the initial square and the side of the double square. Figure 7 shows that such a construction does cover the historical coins quite well, but there is the problem that the circle of the construction is not congruent with the circle of the coin when it is supposed to include the whole of the double square. The solution which we see here is that the coin is struck not as a circle but as an oval (Artmann 1999, p. 63, Fig. 7.3). This shape is a 'compromise' between the two relevant circles. At the left upper edge, its rounding follows the circle of the construction. At the right lower edge, its rounding catches the extruding part of the doubled square. The shape confirms the plausibility of the geometrical interpretation given in our Fig. 2a in the above.

Establishing connections between the Aeginetan coin designs and Plato's writings opens for us a voluminous modern secondary literature on ancient Greek mathematics. We established some links between Popper's (1950) geometrical interpretations of Plato's *Timaeus* and the older Aeginetan coin design with skewed transversals. We just mentioned that the newer Aeginetan design with perpendicular transversals has clear connections with Plato's *Meno*, the mathematics of which having been much discussed in several treatises on the history of mathematics (Knorr 1975; Szabó 1978; Fowler 1999). There is a difference to Plato, however, in so far as the Aeginetan approach to doubling the square is more fertile than the one which is described in *Meno*. The Aeginetan construction opens the perspective on a number of geometrical figures which are unrelated to or not so clear in the *Meno* approach, namely (*i*) the circle which is totally absent there, (*ii*) the *gnomon* which is implicitly there but more openly in the Aeginetan approach, and (*iii*) the complements, for which the same comment applies.

Apart from their obvious significance for the developments in the 'golden age' of ancient Greek mathematics to which these concepts and constructions belong, they are also metaphors which are likely to be appreciated in the commercial context of coins: the complements symbolize equivalence of monetary values; the *gnomon* alludes to penetrating knowledge and to friendly embrace and fertile partnership; the circle recalls Heraclitus' famous reference to the circular flow of money as a metaphor for the role of fire in the ever-revolving cosmos.

One can surely debate at length whether all the implications which we got out of the Aeginetan coin designs were put consciously into them by their ancient inventors. But let us see these designs not only as documents of the intellect which went into them but also as having been meant to tease the intellect of the beholder. Seen in this light, the coins' significance lies not just in the thoughts which went into their design but also in the ones which they provoked. Whatever the original intentions behind them, these coins may appear as snapshots of Greek mathematical ingenuity as well as lasting vestiges of their way of thinking beyond the narrower problems of geometry.

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