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The mathematics in the structures of Stonehenge

Albert Kainzinger

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Abstract The development of ancient civilizations and their achievements in sciences such as mathematics and astronomy are well researched for script-using civilizations. On the basis of oral tradition and mnemonic artifacts illiterate ancient civilizations were able to attain an adequate level of knowledge. The Neolithic and Bronze Age earthworks and circles are such mnemonic artifacts. Explanatory models are given for the shape of the stone formations and the ditch of Stonehenge reflecting the circular and specific non-circular shapes of these structures. The basic mathematical concepts are Pythagorean triangles, thus adopting the construction procedures of the Neolithic circular ditches of Central Europe in the fifth Millennium BC and later earthworks and stone circles in Britain and Brittany. This knowledge was extended with new elliptical concepts. Approximations for the values of π and the square root of 2 are encoded in the henge. All constructions were performed using a standardized "Babylonian" metrology that shows a remarkable consistency and comprehensible development over some 14 centuries.

1 Introduction

In ancient script-using civilizations, the achievements in mathematics and astronomy have been investigated to an adequate degree (e.g., Neugebauer 1969, 1975; van der Waerden 1954, 1973, 1983). In illiterate ancient European civilizations, there is

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some evidence of basic astronomical knowledge due to prehistoric monuments and earthworks (Thom and Thom 1980), but there are almost no secured findings about the level of mathematical knowledge. Because there are many identical solutions for mathematical problems in the ancient civilizations of Mesopotamia, Egypt, India, China and Greece, van der Waerden assumed a common origin for the basic mathematical concepts, which he placed in Neolithic Europe (van der Waerden 1983). Two examples should highlight the basis of this supposition: (a) in all the above-mentioned civilizations, the Pythagorean Theorem was used, and a common set of Pythagorean Triangles was known and/or calculated in the same way; (b) in India and in Greece, comparable ritual geometric constructions were performed (generally for the shapes of altars). This assumption was supported by earlier articles of Seidenberg (Seidenberg 1961, 1978, 1981). A figure summarizing these findings of van der Waerden is included as Electronic Supplementary Material (ESM Fig. 1). In illiterate civilizations, knowledge is generally transferred by oral tradition to subsequent generations. Mathews proposed an ancient core of a Neolithic oral tradition of mathematics (Mathews 1985). In addition, the knowledge can be documented by the depiction of illustrating pictures or symbols on mnemonic objects (e.g., Vansina 1985; Haarmann 1991) or by the construction of appropriate devices and monuments. The majority of the Neolithic and Bronze Age earthworks and stone/wood rings are object of this mnemonic procedure. The encoding of the mathematical concepts in these monuments was performed by the appropriate construction of ditches and rings which are composed by circles and circular arcs. With their centers and endpoints, these elements determine the geometrical models. This procedure is distinct from the documentation of geometric problem solutions nowadays. With the present method, linear geometric models are depicted directly by their line segments, while in the ancient monuments, the linear geometric models are represented exclusively by the respective vertices.

The principal objective of the investigation of ancient earthwork and stone/wood rings now is to determine the original construction concept. So much effort in the construction of these outstanding monuments could not have been expended without a comprehensible and consistent plan. A determination procedure comprising six steps was established to work out highly probable concepts of the ancient plans (see Sect. 8, Fig. 13). Two steps have to be explained at this point to familiarize the reader with the often-used terms "backward construction" and "forward construction." A backward construction is the deduction of a potential construction concept from the available data of the plan of the ancient monument's present remains. There might be several attempts and different results in this step for the search of the original concept. The six-step determination procedure aims to reduce these options to preferably one plausible plan. The forward construction is the exact construction based on the selected mathematical concept and the established metrology.

This article is concerned exclusively with the construction concepts of the Stonehenge structures. Some efforts have been there to uncover the principles of the Stonehenge constructions, e.g., the early attempt by the use of computers in the early 1960s (Hawkins 1965). However, this statistical approach and other methods produced no satisfactory results. A new, sound approach was published by Johnson

(2008), although the author cannot agree with the proposed mathematical explanation models (Johnson 2008).

2 Z and Y holes

We begin with the last phases of Stonehenge. Although the construction of the Z and Y holes document in some way a decline of the mathematical knowledge of the Stonehenge society, the construction concepts of these ring-like structures reflect the beginning in an especially descriptive manner. The Y and Z holes obviously do not form an exact circle. We encounter this characteristic in nearly all early rondels (circular ditches) and wood and stone "circles" in Central Europe and Britain. In the case of the exact circles in Stonehenge (Aubrey holes, Sarsen circle), we will show a clear reasoning for these constructions. The basis of the construction of these ringlike structures is always a mathematical concept. In the majority the construction starts with a Pythagorean triangle. The knowledge of Pythagorean triangles traces back to the fifth millennium BC (rondels/"Kreisgräben" in Central Europe).

2.1 Z holes

The construction of the Z holes is an excellent example of this procedure. To exemplify the a.m. determination procedure, we will depict the results of the backward construction for the Z holes (Fig. 1a) and also the forward construction as well (Fig. 2). (For the remaining structures, a combination of the sampling points and the resulting forward construction is displayed, with the exception of the Bluestone horseshoe.) At first, an isosceles triangle ($Z_{31}Z_{32}Z_{33}$) composed of two mirrored Pythagorean triangles of the shape (3,4,5) is constructed by sharing the leg "4." Then, another isosceles triangle, ($Z_{31}Z_{33}Z_{34}$), also composed of two mirrored Pythagorean triangles of the shape (3,4,5) is attached, but in this case sharing the leg "3." The outline of the overall geometry is another single Pythagorean triangle ($Z_{32}Z_{33}Z_{34}$) of the shape (3,4,5) (Fig. 1c). From a mathematical point of view, the basis for a Pythagorean triangle is a Pythagorean triple of three integers a, b, and c fulfilling the Pythagorean theorem $a^2+b^2=c^2$. If these integers have no common factor, then they are called "primitive Pythagorean triples."[1] In the depiction of the construction concepts by Pythagorean triangles, the integers i of the corresponding primitive Pythagorean triples are denoted by squared brackets: $[i]$. The position of the vertices Z31–Z34 (which are the centers of the respective circular arcs) are calculated by a nonlinear regression analysis (see Sect. 8); the sampling points for the regression analysis are depicted in Fig. 1a by a small cross symbol. The construction was performed on the basis of a consistent metrology (see also Sect. 7). The concrete length of the legs of the respective Pythagorean triangles are 6, 8, and 10 cubits; the resulting overall shape is a Pythagorean triangle with legs of 12, 16, and 20 cubits. The position of vertex Z31 is near the generic center of the Stonehenge structures.

[1] For example, (6,8,10) is a Pythagorean triple; (3,4,5) is the corresponding primitive Pythagorean triple.

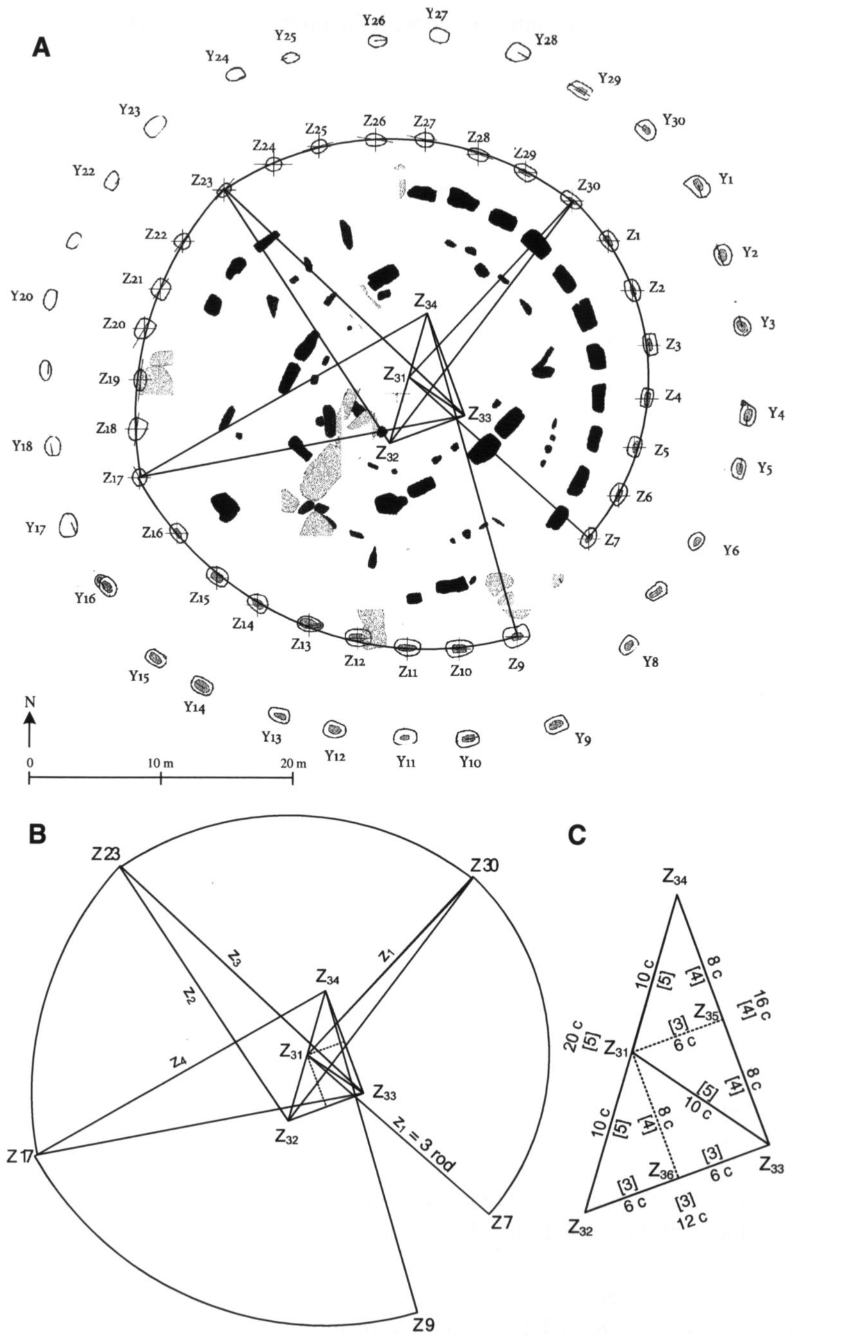


Fig. 1 Z holes. **a** The points Z_{31}–Z_{34} are the center points of the respective circular arcs provided by circular regression analysis. **b** The arbitrary radius z_1 is 36 cubits long which equals 6 reeds or 3 rods. **c** The triangles $Z_{31}Z_{32}Z_{33}$ and $Z_{31}Z_{33}Z_{34}$ are isosceles triangles composed of two mirrored Pythagorean triangles of the shape (3,4,5). The basic Pythagorean triangles of the shape (3,4,5) have concrete legs of 6, 8, and 10 cubits

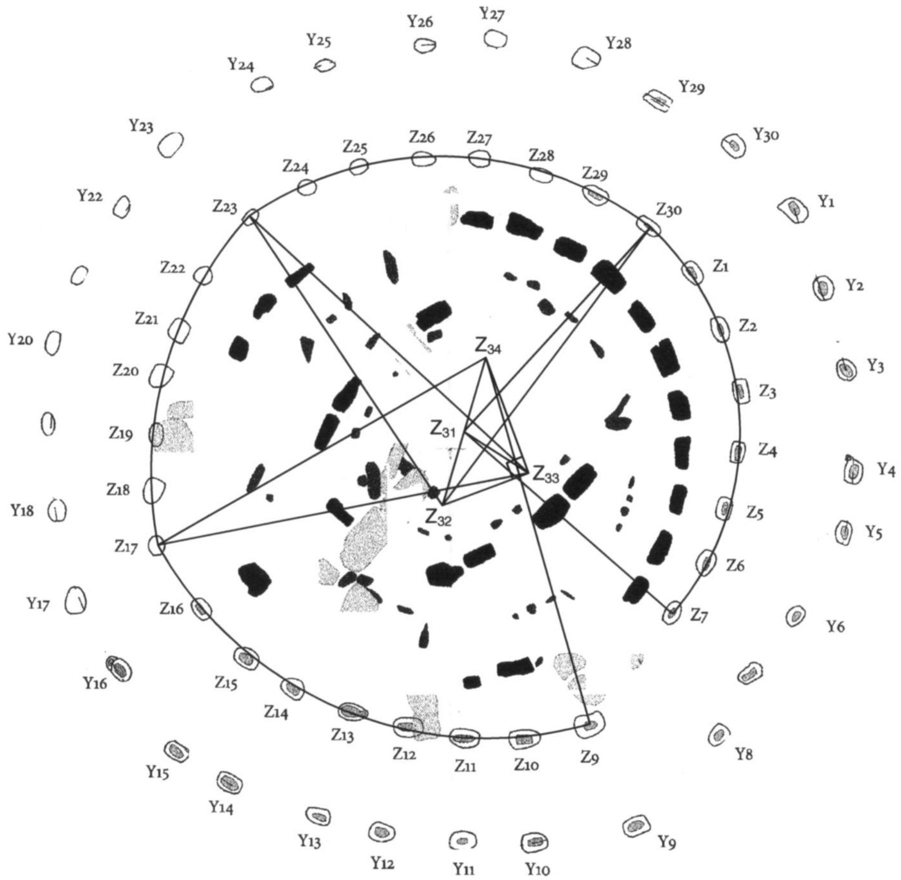


Fig. 2 Forward construction of the Z holes. The isosceles triangles $Z_{31}Z_{32}Z_{33}$ and $Z_{31}Z_{33}Z_{34}$ and the circular arc with center Z_{31} are computed by a mathematical exact construction. The resulting curve for the Z holes is totally smooth at the stones Z17, Z23, and Z30 compared to the one achieved by the backward construction

The vertices Z_{31}–Z_{34} serve as centers of circular arcs on which the stones of the Z holes were set (Fig. 1b). The four circular arcs are composed in a reasonably smoothed curve starting with hole/stone Z7 and ending with hole/stone Z9 anticlockwise. The length of the radius of the first circular arc with center Z_{31} was arbitrary and was set 36 cubits which equals six reeds or three rods. The lengths of the radii of the remaining circular arcs result from the smoothing procedure; the ring so defined was, therefore, not smoothly closed between the holes Z7 and Z9. In the backward construction (Fig. 1b), we still see minor gaps at the connection points Z30, Z23, and Z17. In the exact forward construction, the position of the starting vertex Z_{31} was set very slightly lower. This construction results in a totally smooth curve (in terms of mathematical continuity) at all connection points as is shown in Fig. 2. The lengths of the circular arcs could be determined by either the angle of the arc or the heading of the two legs. This procedure for defining the construction of the Z holes provides a blue print for the construction of all the ring-like structures of Stonehenge.

2.2 Y holes

As we will show later, the construction concept of the Z holes is somewhat simpler than that of the preceding phases 3ii and 3iv of Stonehenge. The Y holes show a further decline of the construction complexity. The base of the construction is only one Pythagorean triangle ($Y_{31}Y_{32}Y_{33}$) of the type (5,12,13); this is the simplest construction concept of all the ring-like structures of Stonehenge (Fig. 3b). The concrete lengths of the legs are 10, 24, and 26 cubits, thus starting the construction with a Pythagorean triangle of the shape (10,24,26). Once again the vertices of the triangle serve as the centers of the three circular arcs on which the stones were set. Vertex Y_{31} again is very near to the general center of Stonehenge.

The length of the radius y_1 of the dominant circular arc with center Y_{31} seems to be set arbitrarily to 50 cubits which equals five poles. The leg $Y_{31}Y_{32}$ of the Pythagorean triangle and the radius $Y_{32}Y3$ of the circular arc with center Y_{32} form a straight line; as the leg $Y_{31}Y_{32}$ is 10 cubits, the length of this latter radius is exactly 40 cubits which equals four poles. The length of the radius with center Y_{33} results from the smoothing procedure at hole Y15. In the backward construction (Fig. 3a), these two arcs meet each other at hole Y15 with a small gap, while the arc with center Y_{32} misses the main arc in the segment between the holes Y2 and Y3. In general, the holes/stones of the Y ring were not set with the accuracy of the Z ring. The shapes of three holes are outside the exact circular arc (Y20, Y25, Y29), and others do not hit the arc centrically. For the position of the hole/stone Y8, we have no mathematical concept. It could be an arbitrary determination to fill the gap between the holes Y7 and Y9. This additionally reflects the decline in the geometrical methods and accuracy for ring constructions in this last phase of the Stonehenge constructions.

For the forward construction, the mean of the measures of the radii y_1 and y_2 were used; the radius y_1 provides the cubit measure with the least tolerance. In this forward construction, the position of vertex Y_{33} by an exact construction of the Pythagorean triangle (10,24,26) becomes a bit higher compared with the backward construction. This forward construction now results in a smooth curve at hole Y15 and hole Y3 as well (Fig. 3c).

2.3 Deduced measures

‘ The above mentioned cubit measures are ideal values according to the selected mathematical concept—in these cases, the underlying Pythagorean triangles. The actual length values extracted from the plan are summarized in Table 1A. For the Z holes, the resulting five cubit measures vary between 49.9003 and 51.4553 cm and have a mean of 50.7228 cm. The mean cubit measures of the Y holes are 52.3558 cm (backward construction) and 53.7403 cm (forward construction).

The arbitrary radius z_1 of the circular arc with center Z_{31} with the supposed length of 36 cubits (= 6 reeds = 3 rods) results in a cubit measure of 50.4836 cm which is very close to the above established mean value for the Z holes. The radius y_1 of the circular arc with center Y_{31} seems to be set arbitrarily to 50 cubits (= 10 poles) and results in a cubit measure of 53.9227 cm which is again very close to the above established mean value for the Y holes (forward construction).

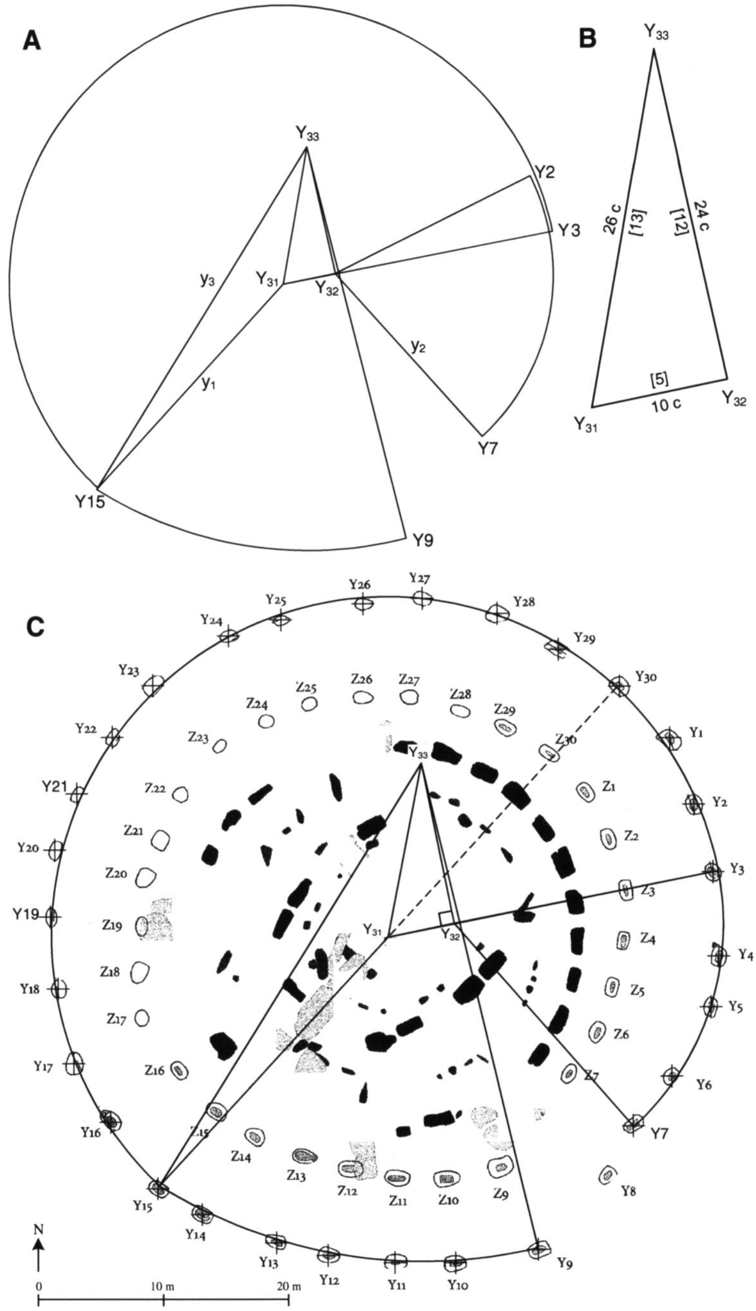


Fig. 3 Y holes. **a** The points Y_{31}–Y_{33} are the center points of the respective circular arcs provided by circular regression analysis (backward construction). The radius y_1 is 50 cubits long which equals five poles. **b** The triangle $Y_{31}Y_{32}Y_{33}$ is a Pythagorean triangle of the shape (5,12,13) and has concrete legs of 10, 24, and 26 cubits. **c** Forward construction. The Pythagorean triangle $Y_{31}Y_{32}Y_{33}$ and the circular arc with center Y_{31} are computed by a mathematical exact construction. The resulting curve for the Z holes is totally smooth at holes Y3 and Y15 compared to the one achieved by the backward construction

Table 1 Measures of the Y and Z holes, the ditch, and the Aubrey holes. "length figures" are the measures of the respective segments in the computer program. "length henge" results from the scale in the figure. "result. cubit measure" is calculated as "length henge" divided by "no. cubits"

Segment	Length figure (mm)	Length henge (m)	No. cubits	Result. cubit measure (cm)
A: Y holes and Z holes measures				
Measure	58.4667	20.00000		
Z holes				
Z_{31}–Z_{32}	15.0421	5.14553	10	51.4553
Z_{31}–Z_{33}	14.8960	5.09555	10	50.9555
Z_{32}–Z_{33}	17.6184	6.02682	12	50.2235
Z_{31}–Z_{34}	14.5884	4.99033	10	49.9033
Z_{33}–Z_{34}	23.8901	8.17221	16	51.0763
			Mean:	50.7228
z_1	53.129	18.17411	36[a]	50.4836
z_2	66.471	22.73807	45	50.5290
z_3	72.916	24.94275	50	49.8855
z_4	73.462	25.12952	50	50.2590
Y holes				
Y_{31}–Y_{32}	15.2718	5.22410	10	52.2410
Y_{32}–Y_{33}	36.9047	12.62418	24	52.6007
Y_{31}–Y_{33}	39.6949	13.57864	26	52.2255
			Mean:	52.3558
y_1	78.8170	26.96133	50[b]	53.9227
y_2	62.9270	21.42313	40[c]	53.5578
y_3	114.7840	39.26474	75	52.3530
Y_{31}–Y_{32}^{d}	15.7101	5.37403	10	53.7403
B: ditch and Aubrey Holes measures				
Measure	30.9000	20.00000		
Ditch				
D_1D_2	8.0347	5.20045	10	52.0045
D_1D_3	7.6574	4.95625	10	49.5625
D_2D_3	12.3383	7.98595	16	49.9122
			Mean:	50,4931
S_1	90.528	58.59417	120[e]	48.8285
S_2	79.408	51.39676	100	51.3968
S_3	83.412	53.98835	110	49.0803
S_4	83.624	54.12557	110	49.2051
Aubrey Holes				
r_A	68.9710	44.64142	90[f]	49.6016

Table 1 Continued

Segment	Length figure (mm)	Length henge (m)	No. cubits	Result. cubit measure (cm)
C: Station stones, Heel stone measures				
91-93	139.5024	90.29282	182	49.6114
92-94	139.0235	89.98285	182	49.4411
91-92	53.8535	34.85663	70	49.7952
94-93	52.2805	33.83851	70	48.3407
91-94	129.2255	83.64110	168	49.7864
92-93	128.2930	83.03754	168	49.4271
91-96	119.3471	77.24731	159	48.5832
94-96	120.3418	77.89113	159	48.9881
			Mean:	49.2467

[a] 36 cubits = 6 reeds = 3 rods
[b] 50 cubits = 5 poles
[c] 40 cubits = 4 poles
[d] forward construction
[e] 120 cubits = 20 reeds = 10 rods = 2 ropes
[f] 90 cubits = 9 poles = 1 1/2 ropes

3 Ditch and Aubrey Holes

The ditch and the Aubrey Holes belong to Stonehenge phase 1 (ca. 2950 BC).

3.1 Ditch

The ditch was excavated by Hawley in the early 1920s in the eastern and southern parts. As far as the original course and shape of the ditch is concerned, the excavation results have to be questioned. The excavated cross sections and the irregularities of both the width and the depth along the ditch ring are not in line with earlier and coeval rondels and ditches. In general, the bottom of the ditch should define a comprehensible line or small band that reflects the original construction concept. The eastern ditch terminal at the entrance causeway is a good example for this critical analysis. Johnson assumed that "it is more than likely therefore that the workmen had mistakenly cut a little beyond the actual end of the ditch, slicing through the postholes on the edge of the causeway" (Johnson 2008, p. 103). It seems that the workmen had overdone their task in the rest of the ditch also. This open question can be resolved by an adequate investigation and cautious excavation of the remaining part of the ditch including non-destructive methods. For the generation of a potential construction concept, we have to cope with an unsettled database as far as the excavated ditch is concerned. The situation for the remaining non-excavated part of the ditch is at the present time somewhat better but not ideal; here, we can assume the construction curve in the middle of the still evident depression of the ditch. However, this can be an indication only due to the unknown refilling conditions. Despite this soft database, the backward construction provides a comprehensible mathematical construction concept (Fig. 4).

For each of the two sections of the non-excavated part of the ditch, eight reading points were chosen. The reading points of the two sections of the excavated part are sampled on the basis of the sampling theorem at equidistant spaces of 5° as shown in Fig. 4c. This procedure provides 17 and 15 sampling points, respectively. The construction starts with the isosceles triangle $D_1D_2D_3$ composed of two mirrored Pythagorean triangles of the shape (3,4,5) in this case sharing the leg "3" (Fig. 4b). This geometric figure (including the one sharing the leg "4") can be assessed as one with the highest appreciation and is therefore quite probable for the starting phase of the Stonehenge structures. These shapes of isosceles triangles were also the basis of many explanation models for the construction of stone rings in Britain and Brittany by Thom and Thom (1980). The construction of center D_4 is not apparently due to the soft database for the respective circular arcs; this center could be identical either to the point D_1 or the center A_1 of the Aubrey Holes (see below).

The lengths of the legs of the triangle $D_1D_2D_3$ are 6, 8, and 10 cubits. The lengths of the radius s_1 are the most reliable data for the deduction of a cubit measure. This length can be assumed 120 cubits which equals 20 reeds or 10 rods and results in a cubit measure of 48.8285 cm that fits to the overall measuring scheme (see Sect. 7).

However,owing to the a.m. database situation, the determination procedure according Fig. 13 has to be assessed as non-successful. As both the construction concept and the deduced measuring unit fit in the overall explanation model of the Stonehenge structures, the ditch results are worthwhile to be documented.

3.2 Aubrey Holes

The 56 Aubrey Holes define an exact circle (Fig. 4a, c). The circle was calculated by circular regression analysis involving all the 56 holes. The center of this circle A_1 is inside the triangle $D_1D_2D_3$ (Fig. 4b). The construction of the 56 holes/stones is a good example for the encoding of mathematical concepts in earthworks by illiterate societies; thereby an approximation of the circle constant π is represented in the henge:

$$\begin{aligned}
&\text{radius of the Aubrey holes circle } r_\mathrm{A} = 90 \text{ cubits} = 9 \text{ poles (Table 1B)}\\
&\quad\rightarrow \text{diameter } d_\mathrm{A} = 180 \text{ cubits} = 18 \text{ poles;}\\
&\text{circumference of the Aubrey holes circle } c_\mathrm{A} = 56 \text{ poles;}\\
&\text{circle constant } \pi := \frac{c_\mathrm{A}}{d_\mathrm{A}} = \frac{56 \text{ poles}}{18 \text{ poles}} = \frac{28}{9} = 3.1111\ldots.
\end{aligned}$$

The mathematical concept is based on a regular polygon of 56 sides inscribed in a circle. The respective circular area A_C could be established by the method described by Johannes KEPLER (Wußing 2008, p. 439). This method also starts with an inscribed polygon composed of isosceles triangles providing the formula

$$A_\mathrm{C} = \text{circumference} \times \text{radius}/2$$

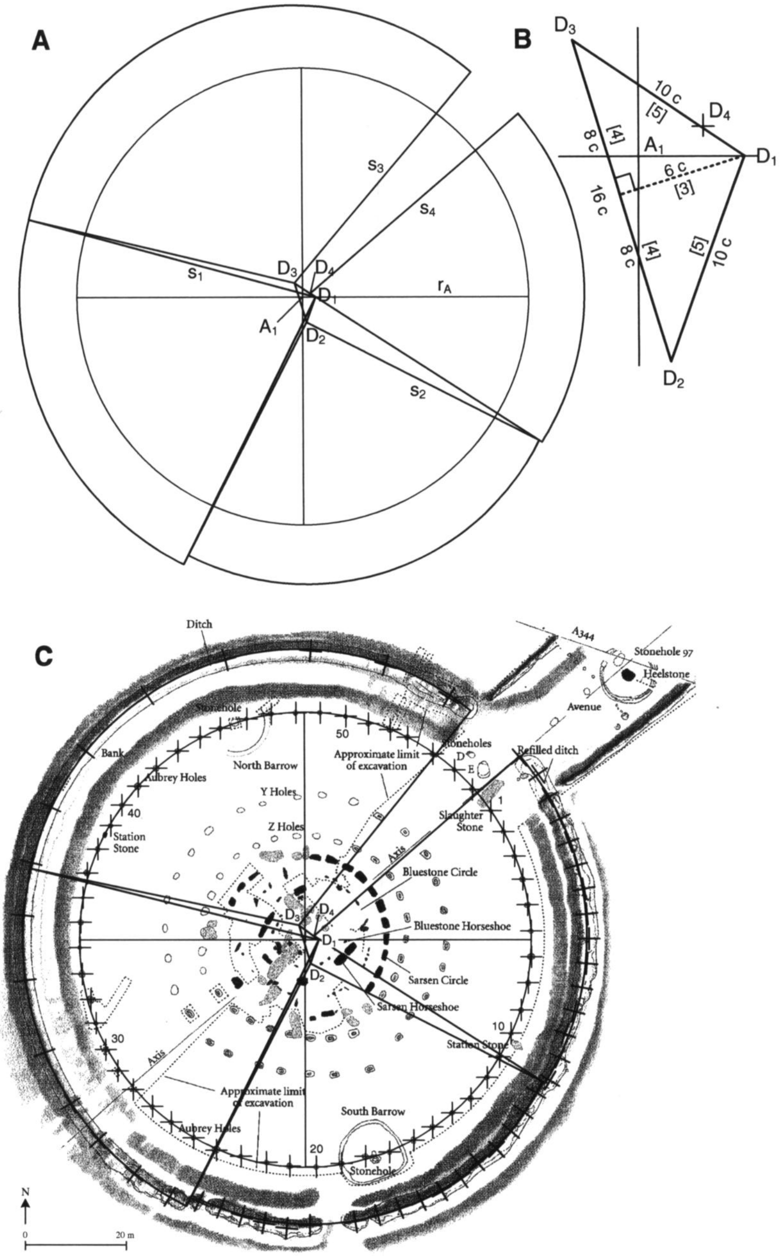


Fig. 4 Ditch and Aubrey Holes. **a** The points D_1–D_4 are the center points of the respective circular arcs of the ditch provided by circular regression analysis. The center of the Aubrey Holes A_1 was calculated by circular regression analysis also. The radius s_1 is 120 cubits long which equals 20 reeds, 10 rods, or 2 ropes. The radius of the Aubrey holes circle r_A equals 90 cubits or nine poles. **b** The triangle $D_1D_2D_3$ is an isosceles triangle composed of two mirrored Pythagorean triangles of the shape (3,4,5). The basic Pythagorean triangles (3,4,5) have concrete legs of 6, 8, and 10 cubits.**c** Forward construction of the Ditch and the Aubrey Holes. The isosceles triangle $D_1D_2D_3$ and the circular arc with center D_1 are computed by a mathematical exact construction

This formula is equivalent to $A_C = r^2\pi$ but does not need π explicitly. We get $A_C = 9^2 \times 28/9 = 252$ pole^2 $= 36 \times 7$ pole^2. We have no knowledge of the algebraic achievements of the Stonehenge people. An approximation of $\sqrt{7}$ could, however, be established by the formula $\sqrt{a^2 \pm b} \approx a \pm b/2a$, which was used in Babylonian, Chinese, and Indian mathematics. "It seems that the [a.m.] approximations ... were already known in pre-Babylonian mathematics" (van der Waerden 1983, p. 47); q.v. (Folkerts 2006, p. II 29–33). For $\sqrt{7}$, we get $\sqrt{7} = \sqrt{3^2 - 2} \approx 2\,2/3$ and thereby $\sqrt{252} \approx 6 \times 2\,2/3 = 16$. Therefore, we have a solution for the problem "squaring the circle" in whole numbers: the area of a circle with the diameter 18 equals the area of a square with the sides 16. This proportion 18/16, alternatively, 9/8 is also applied in problem 50 of the Rhind papyrus (ca. 1650 BC, copied from an earlier papyrus of ca. 1860–1814 BC) for the tasks "squaring the circle" and "circling the square" (van der Waerden 1983, p. 170).

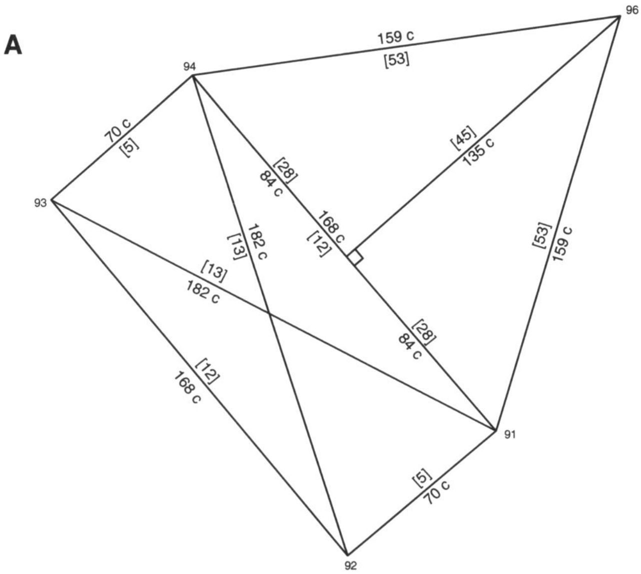
The deduced value 28/9 is the most accurate approximation for π at the beginning of the third millennium BC known so far. The establishment of this figure is an admirable mathematical achievement for this time horizon. "The Babylonians ... always used $\pi = 3$. This is also the value given by Vitruvius; and is found again in the Chinese literature." (van der Waerden 1954, p. 32). We will meet the approximation $\pi = 3$ in the structures of Stonehenge also (see Sect. 5.1.).

3.3 Deduced measures

The mean cubit measure for the ditch is 50.4931 cm (Table 1B). The arbitrary radius s_1 of the circular arc with center D_1 with the supposed length of 120 cubits or 2 ropes results in a cubit measure of 48.8285 cm. The radius r_A of the Aubrey holes was set arbitrarily to 90 cubits or nine poles and results in a cubit measure of 49.6016 cm. Of all the measures deduced from the structures of Stonehenge, this measure is the one with the lowest range of error or the highest confidence as we have a complete circle with 56 sampling points all lying on an exact circle.

4 Station stones and Heelstone

The Station stones and/or holes display no circular or ring-like structure (Fig. 5a). The Station stones 91-94 determine a rectangle that is composed of two Pythagorean triangles (91-92-93 and 91-93-94 or vice versa 91-92-94 and 92-93-94); these triangles have the primitive shape (5,12,13) and are combined with the leg "13." The concrete length of the legs are 70, 168, and 182 cubits thus having a Pythagorean triangles of the real shape (70,168,182) and a multiplication factor of 14 compared to the primitive triangle. Another isosceles triangle composed of two mirrored Pythagorean triangles of the primitive shape (28,45,53) is added at the leg 91-94. This isosceles triangle has therefore the shape (56,53,53); the Heelstone 96 was placed at the top of this triangle. The concrete lengths of the legs of the Pythagorean triangles (28,45,53) are 84, 135, and 159 cubits; thus, the base leg of the isosceles triangle 91-94-96 is 168 cubits. This geometry represents the algebraic concept of the least common multiple (LCM) as 168 is the LCM of the integers 12 and 56.



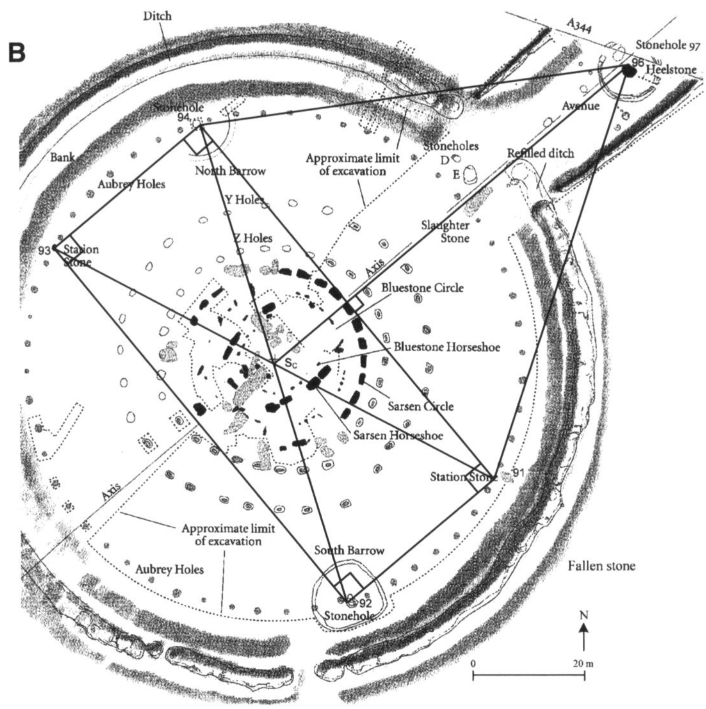


Fig. 5 Construction of the Station stones and the Heelstone. **a** The Station stones and the Heelstone form a geometric figure assembled by a rectangle and an isosceles triangle. Both the rectangle and the isosceles triangle are composed of Pythagorean triangles of the shape (5,12,13) and (28,45,53), respectively. The lengths of concrete legs are 70, 168 and 182 cubits and 84, 135, and 159 cubits. The leg 91-92-93-94 is 168 cubits which is the LCM of the integers 12 and $56 = 2 \times 28$. **c** Forward construction for the Station stones and the Heelstone. The rectangle 91-94 and the isosceles triangle 91-94-96 are computed by a mathematical exact construction

Presently, the Heelstone 96 is leaning considerably toward SSE so we can assume an ancient construction point at the northwestern rim of the Heelstone's present image. The exact forward construction was rotated adequately to cope with this fact (Fig. 5b). The bearing from the center of the rectangle to the Heelstone S_C-96 has an angle to East of 39.65°. The deduced cubit measure is 49.2467 cm (Table 1C).

5 Sarsen stones

The mathematics involved in the construction of the Sarsen stones shows a great leap forward compared with the preceding phases of Stonehenge. In this stage (ca. 2550 BC) as well as in the later phase of the Bluestones, elliptical concepts were adopted.

5.1 Sarsen circle

The Sarsen circle is a completely exact circle. For the circular regression analysis, all the 17 still standing stones were incorporated. The reading points were taken at the innermost edge of the stones. This approach provided a best fit of the regression circle with the exception of stones 10 and 11 which are not as exactly on the circle line as do all others (Fig. 6c). The radius of this circle with center S_1 is 30 cubits which equals five reeds. This supports the assumption that the inner faces of the stones were the basis for the construction concept as well as the fact that the inner faces of the uprights were made considerably smoother than the outer faces.

By the construction of the 30 Sarsen circle uprights, we have another approximation of the circle constant π encoded in the Stonehenge structures. We have identified two options of constructions: (a) if we apply the concept of the Aubrey holes circle, then the circumference of the Sarsen circle was taken as 30 reeds:

$$\begin{aligned}
&\text{radius of the Sarsen circle } r_S = 30 \text{ cubits} = 5 \text{ reeds (Table 2A)}\\
&\quad \rightarrow \text{diameter } d_S = 60 \text{ cubits} = 10 \text{ reeds} = 1 \text{ rope};\\
&\text{circumference of the Sarsen circle } c_S = 30 \text{ reeds} = 3 \text{ ropes};\\
&\text{circle constant is } \pi := \frac{c_S}{d_S} = \frac{30 \text{ reeds}}{10 \text{ reeds}} = \frac{3 \text{ ropes}}{1 \text{ rope}} = 3.
\end{aligned}$$

The mathematical concept bases again on a regular polygon inscribed in a circle, in this case with 30 sides, resulting in an integer value for π. For the area of the circle, we get $A_C = 5^2 \times 3 = 75$ reed^2. Applying the a.m. approximation for the calculation of a square root, we get $\sqrt{3} \approx 1\,3/4$ and for $\sqrt{75} = \sqrt{5^2 \times 3} \approx 5 \times 1\,3/4 = 8\,3/4$. The solution for the problem "squaring the circle" in this case is: the area of a circle with the diameter 10 equals the area of a square with the side 8 3/4. This Sarsen circle approximation for π is worse than that established by the Aubrey holes circle some 400 years ago, but in accordance with the practiced standards in Babylonia. Either the Aubrey holes' value of 28/9 was not communicated by oral tradition and it had fallen into oblivion, or it was a conscious decision to operate from now on with this simple integer value. The latter assumption could be motivated by having a more

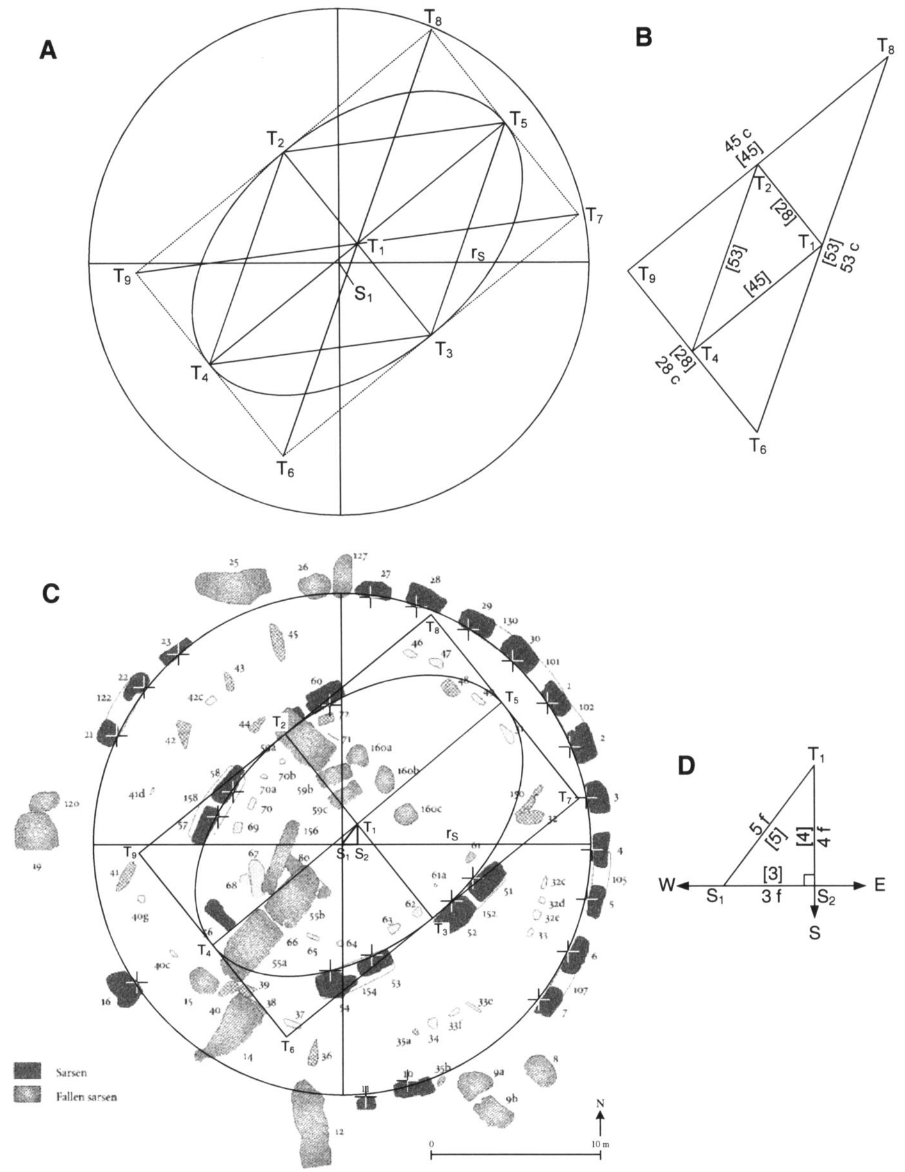


Fig. 6 Sarsen circle and horseshoe. **a** The ellipse defined by the center T_1, the major and minor axes and the tilt angle (points T_2 through T_5) are calculated by elliptical regression analysis. The center of the Sarsen circle S_1 was calculated by circular regression analysis. The ellipse defines a rectangle and a regular lozenge. **b** The lozenge is composed of four Pythagorean triangles of the shape (28,45,53); the rectangle $T_6T_7T_8T_9$ is composed of two Pythagorean triangles ($T_6T_8T_9$, and $T_6T_7T_8$ or $T_6T_7T_9$ and $T_7T_8T_9$, respectively) of the shape (28,45,53); each of these Pythagorean triangles has concrete legs of 28, 45, and 53 cubits. **c** Forward construction of the Sarsen circle and horseshoe. The ellipse is computed by a mathematical exact construction. **d** The centers S_1 and T_1 of the Sarsen circle and horseshoe aligned with the point of the compass define a Pythagorean triangle $S_1S_2T_1$ of the shape (3,4,5); this Pythagorean triangle has concrete legs of 3, 4, and 5 feet

practical figure for circle calculations or because this change was related to ritual traditions; Seidenberg et al. showed that in Greece and India "geometrical constructions were regarded important for ritual purposes" (Seidenberg 1961, 1978, 1981; van der Waerden 1983). (b) In the above mentioned concept, the angle of a singular isosceles triangle with base 1 reed and legs 5 reeds has a central angle of 11.478° which is significantly smaller than the theoretical value of $360°/30 = 12°$; relating to the full circle, we get a difference by the 30 triangles of 15.65° which could doubtless be recognized by the builders of the Sarsen circle. A construction which fits the 30-side polygon better into the full circle could be based on an isosceles triangle with base 1 reed + 2 palm = 38 palms:

$$\begin{aligned}
&\text{radius of the Sarsen circle } r_S = 30 \text{ cubits} = 5 \text{ reeds (Table 2A)}\\
&\qquad \rightarrow \text{diameter } d_S = 60 \text{ cubits} = 360 \text{ palms}\\
&\text{circumference of the Sarsen circle } c_S = 30 \times 38 \text{ palms} = 1140 \text{ palms;}\\
&\text{circle constant is } \pi := \frac{c_S}{d_S} = \frac{1140 \text{ palms}}{360 \text{ palms}} = \frac{57}{18} = 3.1666\ldots.
\end{aligned}$$

This value approximates π better than the value 28/9 = 56/18 established by the Aubrey holes, and can be seen as a direct improvement of the latter value by raising the numerator from 56 to 57. For the area of the circle, we get $A_C = 5^2 \times 57/18 = 475/6 = 9^2 - 11/6 \approx 9^2 - 2 \text{ reed}^2$. Applying the a.m. approximation for the calculation of a square root, we get $\sqrt{9^2 - 2} \approx 9 - 2/18 = 8\,8/9$. The solution for the problem "squaring the circle" in this case is: the area of a circle with the diameter 10 equals the area of a square with the side 8 8/9.

To settle for one of the cases (a) or (b), we need more secured data about exact and regular stone or wood circles in coeval earthworks.

5.2 Sarsen horseshoe

The seven Trilithons 51–54, 57, 58, and 60 form an ellipse. Five points in a plane always form one and only one conic section—in this convex situation exactly one ellipse. Hence, the accuracy of the seven reading points on the ellipse is remarkable (Fig. 6c). Once again the Trilithons were smoother at the inner faces, and therefore the reading points were also taken at the innermost edge of the stones. The fallen and broken Trilithon 59 was not incorporated in the computing as there are not sufficient data on the original position. However, the remaining eventually displaced base matches with the computed ellipse. The resulting major and minor axes of the ellipse generate the rhombus $T_2T_4T_3T_5$ and the rectangle $T_6T_7T_8T_9$ (Fig. 6a). Both geometric objects are composed of Pythagorean triangles of the shape (28,45,53). In the case of the triangle $T_6T_8T_9$ (and the congruent triangles $T_6T_7T_8$, $T_6T_7T_9$ and $T_7T_8T_9$ alike), this fact is very impressive: these triangles have the concrete legs of 28, 45, and 53 cubits (Fig. 6b). The results of the elliptical regression analysis including the standard deviation values are provided as ESM.

The Trilithon 56 was not obviously constructed at a position on the elliptical curve. This upright was straightened in 1901, and its partner stone 55 has fallen and broken.

For a later comment, we want to state here that the depiction of Trilithon 56 in Fig. 8a shows the alignment of the smoother outer face not perpendicular to the major axis of the ellipse but rotated some 10° anticlockwise. If we assume that stone 55 has fallen in a right angle in relation to its original alignment, then this statement holds for stone 55 also. There are still reasonable and debatable arguments about the original position of both stones (Johnson 2008, pp. 136–140). Therefore, the determination procedure according to Fig. 13 breaks down after step 1.

The exact mathematical forward construction of the ellipse matches with that of the plan with a high degree of precision (Fig. 6c). Surprisingly, the centers of the Sarsen circle and the Sarsen ellipse are not identical. However, there is a comprehensible mathematical concept to explain this difference (Fig. 6d). The a.m. centers and the virtual point S_2 configure the Pythagorean triangle $S_1 S_2 T_1$ of the shape (3,4,5). The lengths of the legs are 3, 4, and 5 feet (Table 2A). The deduced foot measure is 29.8918 cm. The ancient foot measures vary considerably, even on the basis of a stable cubit measure depending on different subdivisions of the cubit and the palm (Rottländer 1979).[2] The deduced foot measure is therefore absolutely within the deduced overall measuring scheme. The displacement of the center T_1 of the ellipse might be because the Sarsen horseshoe was thereby placed better in the middle of the Sarsen circle.

5.3 Deduced measures

Once again, the above mentioned cubit measures are ideal values according to the selected mathematical concept—in these cases, the underlying circle and the Pythagorean triangles. The actual length values extracted from the plan are summarized in Table 2A. The cubit measure for the Sarsen circle is 48.8733 cm and that for the Sarsen horseshoe, 49.3194 cm; the mean of all Sarsen stones' cubit measures is 49.2079 cm.

6 Bluestones

The Bluestone circle and horseshoe were constructed some 450 years after the setting of the Sarsen stones.

6.1 Bluestone circle

The present record of the remains of the Bluestone circle comprises 27 stones of which 12 have fallen. Hence, we can involve 15 stones in the determination procedure for the original construction plan. The original amount of stones generating the Bluestone circle is estimated between 44 and 62 uprights (Johnson 2008, p. 158); thus, we can imagine a very dense ringlike structure. Despite there being only 15 secured original positions, the nonlinear regression analysis provides a clear picture (Fig. 7c). We obtain four circular arcs with centers B_{21}–B_{24} with 5, 6, 3, and 3 reading points,

[2] Typical variations: foot on the statue of Gudea = 26.45 cm; 18 digit foot from the Nippur cubit = 33.199 cm.

Table 2 Measures of the Sarsen stones and the Bluestones

Segment	Length figure (mm)	Length henge (m)	No. cubits	Result. cubit measure (cm)
A: Sarsen stones measures				
Measure	52.0250	10.00000		
Sarsen circle				
Radius	76.279	14.66199	30[a]	48.8733
Sarsen horseshoe/Trilithons				
$T_6 T_9$	71.222	13.68996	28	48.8927
$T_8 T_9$	116.204	22.33618	45	49.6360
$T_6 T_8$	136.2936	26.19771	53	49.4296
			Mean:	49.3194
Mean of all Sarsen stones:				49.2079
Circle–horseshoe relation				
			No. feet	Foot meas.
$S_1 S_2$	4.7230	0.90783	3	30.2611
$S_2 T_1$	6.1575	1.18357	4	29.5891
$S_1 T_1$	7.7583	1.49126	8	29.8253
			Mean:	29.8918
B: Bluestones measures				
Bluestone circle				
Measure	52.0250	10.00000		
$B_{21} B_{22}$	5.2062	1.00071	12	50.0356
$B_{21} B_{23}$	14.7238	2.83014	35	48.5167
$B_{22} B_{23}$	15.9908	3.07368	37	49.8434
$B_{21} B_{24}$	37.3092	7.17140	84	51.2243
$B_{23} B_{24}$	39.8863	7.66676	91	50.5500
			Mean:	50.0340
v_1	66.2110	12.72677	25	50.9071
v_2	54.1300	10.40461	20	52.0231
v_3	62.6860	12.04921	24[b]	50.2050
v_4	98.1440	18.86478	36	52.4022
Bluestones horseshoe				
Measure	65.3750	10.00000		
$B_1 B_6$	16.1440	2.46945	5	49.3889
$B_1 B_{11}$	16.3149	2.49559	5	49.9117
$B_1 B_2$	39.1920	5.99495	12	49.9579
$B_2 B_6$	42.0928	6.43867	13	49.5282
$B_2 B_{11}$	43.0451	6.58434	13	50.6487
$B_3 B_6$	42.6787	6.52829	13	50.2176
$B_3 B_{11}$	41.8508	6.40165	13	49.2435
B_1-79	39.4354	6.03218	12	50.2682

Table 2 Continued

Segment	Length figure (mm)	Length henge (m)	No. cubits	Result. cubit measure (cm)
			Mean:	49.9506
Circle and horeseshoe mean:				49.9923
B_2-79	55.4680	8.48459	Forward construction	
B_6-79	55.5642	8.49930		

[a] 30 cubits = 5 reeds → diameter = 10 reeds = 1 rope
[b] 24 cubits = 4 reeds = 2 rods

correspondingly; the stones 40g and 47 are connecting points of the adjacent arcs. The circular arcs with centers B_{23} and B_{24} are calculated on the basis of three stone positions. As three points in a plane define one and only one circle, we get an exact solution for the respective centers. However, this accuracy is deceptive because the regression analysis is calculated without redundancy. In this case, a defective position of one reading point (stone) results directly in a defective position of the center, while the regression calculation of a circle on the basis of more than three points compensates much better for slight position errors. Therefore, the circular arc with center B_{24} has the lowest probability in our assessment procedure as this arc has additionally a very small central angle from stone 47 to stone 31. Nevertheless, the results in combination with the deduced measure units pass our determination procedure according to Fig. 13—with reservations as far as the center B_{24} is concerned.

The construction starts with the Pythagorean triangle $B_{21}B_{22}B_{23}$ of the shape (12,35,37) and the size 12, 35, and 37 palms (Fig. 7b). Again, this primitive form of the Pythagorean triangle is a strong support in favor of this explanation model. Another Pythagorean triangle of the shape (5,12,13) is added to the first at the leg "35." As 35 is a multiple of 5, we get this triangle in terms of palms in the concrete shape of (35,84,91).

The calculated radius of the circle with center B_{23} is 24 cubits which equals 4 reeds or 2 rods (see ESM Table 2). The construction of the ring on which the stones of the Bluestone circle were set started probably with this circular arc. The additional arcs were attached in a way to get a smooth curve; the ring does not close smoothly between the stones 31 and 32c (Fig. 7a). The exact forward construction (Fig. 7c) matches with the plan perfectly: the connection of the adjacent arcs at stone 40g is now totally smooth compared with the backward construction where the calculation provided a minor gap (Fig. 7a); the gap between stone 31 and 32c remains.

6.2 Bluestone horseshoe

As far as mathematics is concerned, the complexity of Stonehenge's construction concepts reaches its point of culmination with the Bluestone horseshoe. In combination with the holes 73–78 the stones/holes of the Bluestone horseshoe create a ring-like structure. The nonlinear regression analysis provides three ellipses (Fig. 8a): a large ellipse with center B_1, a smaller ellipse in SW with center B_6 and another smaller

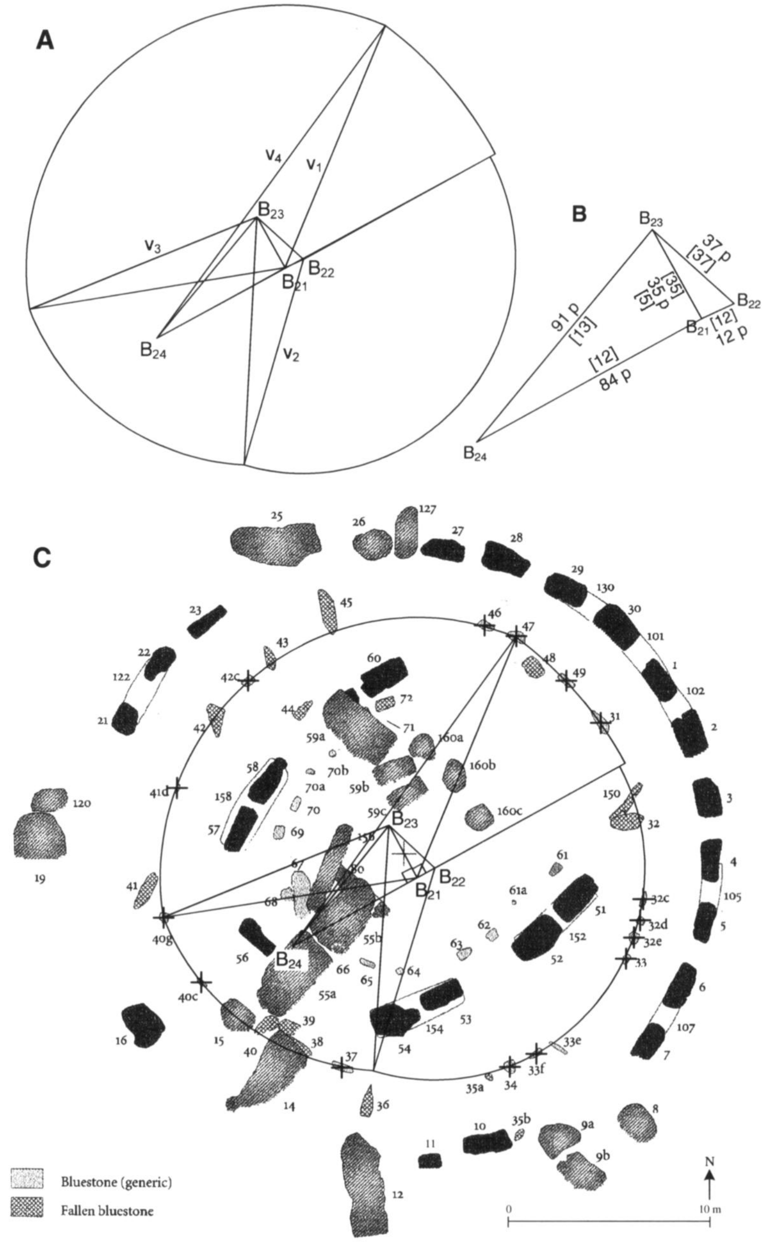


Fig. 7 Bluestone circle. **a** The points B_{21}–B_{24} are the center points of the respective circular arcs provided by circular regression analysis. The arbitrary radius v_3 is 24 cubits long which equals 4 reeds or 2 rods. **b** The triangle $B_{21}B_{22}B_{23}$ is a Pythagorean triangle of the shape (12,35,37), and the triangle $B_{21}B_{23}B_{24}$ is a Pythagorean triangles of the shape (5,12,13). The basic Pythagorean triangle $B_{21}B_{22}B_{23}$ of the shape (12,35,37) has concrete legs of 12, 35, and 37 palms (p). **c** Forward construction of the Bluestone circle. The right-angled triangles $B_{21}B_{22}B_{23}$ and $B_{21}B_{23}B_{24}$ and the circular arc with center B_{23} are computed by a mathematical exact construction. The resulting curve of the Bluestone circle is totally smooth at the stones 40g and 47 compared with the one achieved by the backward construction. The gap between the stones 31 and 32c remains

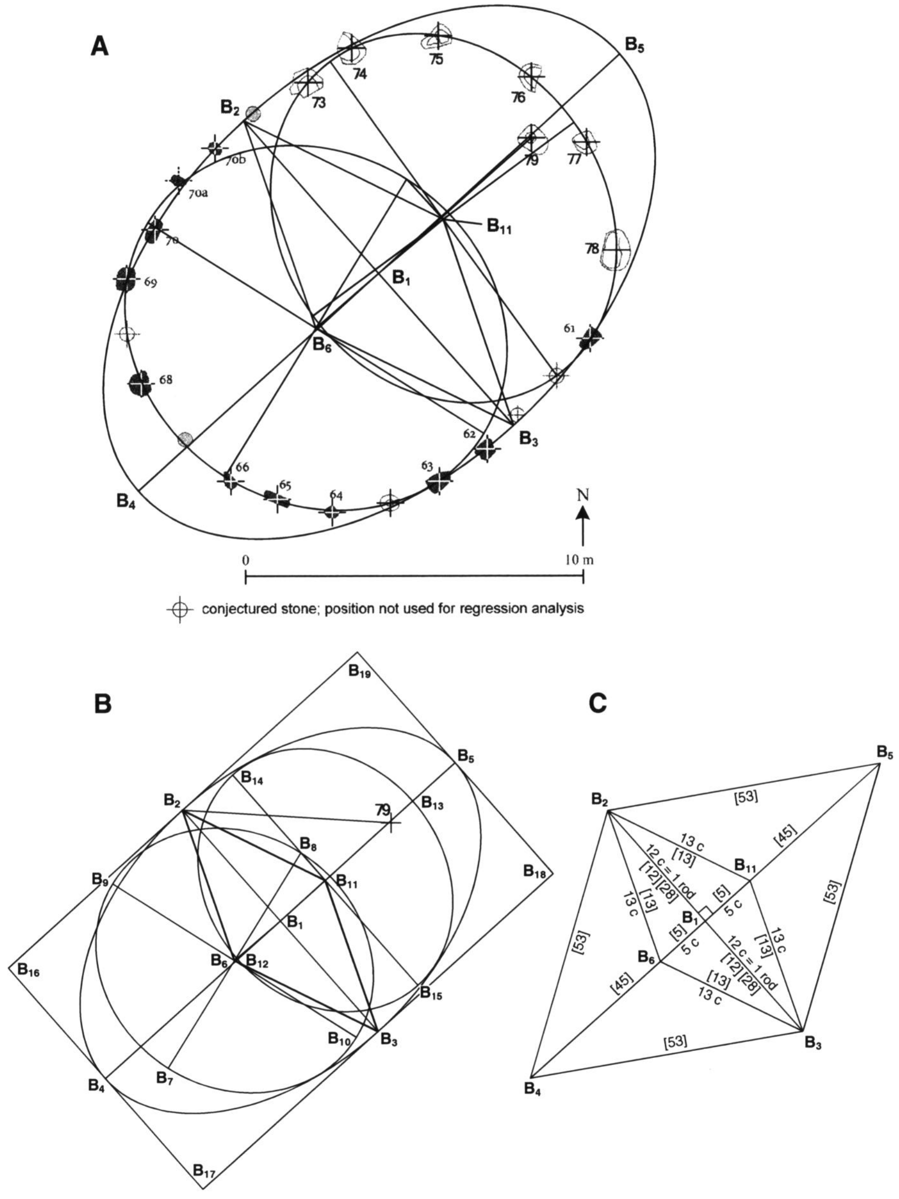


Fig. 8 Bluestone horseshoe. **a** The ellipses with center B_1 (large ellipse), center B_6 (small ellipse SW), and center B_{11} (small ellipse NE) are calculated by elliptical regression analysis. **b** In the depiction of the construction concept, the small ellipse NE was shifted perpendicular to the major axis of the large ellipse so that the center B_{11} rests on this major axis; additionally, this small ellipse was rotated anticlockwise so that the axes coincide. **c** The basis of the construction concept of the Bluestone horseshoe is the lozenge $B_2B_6B_3B_{11}$ composed of four Pythagorean triangles of the shape (5,12,13); each of these Pythagorean triangles has concrete legs of 5, 12, and 13 cubits. The large ellipse with center B_1 defines the rectangle $B_{16}B_{17}B_{18}B_{19}$ (see Fig. 13b) and a regular lozenge $B_2B_4B_3B_5$

ellipse in NE with center B_{11}. In the depiction of the construction concept in Fig. 8b, the center B_{11} of the small ellipse NE was shifted perpendicular to the major axis of the large ellipse so that this center rests on this major axis (length of shift 9.2 cm); additionally this small ellipse was rotated anticlockwise so that the axes coincide (rotation 5.67°). The slight shift and rotation are within the respective confidence intervals of the regression analysis due to the input data situation: the original position of the now disappeared stones 73–78 cannot be established with the accuracy of the other elements of the Bluestone horseshoe.

The basis of the construction concept of the Bluestone horseshoe is the lozenge $B_2B_6B_3B_{11}$ composed of four Pythagorean triangles of the shape (5,12,13); each of these Pythagorean triangles has concrete legs of 5, 12, and 13 cubits. The large ellipse with center B_1 defines the rectangle $B_{16}B_{17}B_{18}B_{19}$ and a regular lozenge $B_2B_4B_3B_5$ composed of four triangles similar to Pythagorean triangles of the shape (28,45,53). The proportion of the legs "45" and "28" is $45/28 = 1.607\ldots$, and therefore also a good approximation for the Golden Ratio of $1.618\ldots$ (Fig. 8b, c). The small ellipse SW with center B_6 defines the regular lozenge $B_7B_{10}B_8B_9$ composed of four right-angled triangles being similar to a Pythagorean triangle of the shape (48,55,73) (Fig. 9a). The ellipse $B_7B_{10}B_8B_9$ is significantly rotated anticlockwise by 16.362° in relation to the major axis of the large ellipse. We had stated above that the alignment of the great Trilithons of the Sarsen horseshoe might have been rotated the same way. This rotation of in the inner segment of both horseshoes may have a reason for astronomical or ritual purposes. The small ellipse NE with center B_{11} defines the regular lozenge $B_{12}B_{15}B_{13}B_{14}$ composed of four right-angled triangles also being similar to a Pythagorean triangle of the shape (48,55,73) (Fig. 9b). Both small ellipses are tangential to the larger ellipse—the small ellipse SW with one tangent point and the small ellipse NE with two tangent points; hence, we can assume that the intention of the Bluestone horseshoe designers was to demonstrate this geometrical achievement.

The positioning of hole/stone 79 gives another insight to the mathematics applied by the builders of Stonehenge in phase 3iv. (a) The line segment B_1-79 has the same length as the semi-minor axis B_1B_2 (12 cubits = 1 rod) thus giving with segment B_2-79, the construction of the diagonal of a unit square or a geometrical representation of $\sqrt{2}$ (Figs. 8b, 9c). On the other hand, this diagonal equals the line segment B_6-79 with a difference of 1.471 cm as established in the forward construction (Table 2B). B_1B_6 equals five cubits, and so we get an approximation for the square root of 2 by $17/12 = 1.41666\ldots$ (exact value $1.41421\ldots$). "This approximation (1;25 in the Babylonian sexagesimal notation) frequently occurs in Babylonian texts" (van der Waerden 1983, p. 47). (b) We have shown above that the proportion of the major and minor semi-axes of the large Bluestone ellipse is an approximation for the Golden Ratio; the point 79 therefore divides the major semi-axis B_1B_5 in the Golden Section (Fig. 8b). We had assumed that the large ellipse was constructed on the basis of a regular lozenge based upon Pythagorean triangles of the shape (28,45,53). The construction of a Golden Section geometry as a first step cannot be excluded totally either. A construction method for the Golden Section as also described in the *Elements* by EUCLID is depicted in Fig. 9d. This open question could be resolved if the remains of the respective construction pegs would ever be found in the soil of Stonehenge. (c) For the special case of an ellipse with the axes in the proportion of the Golden Ratio,

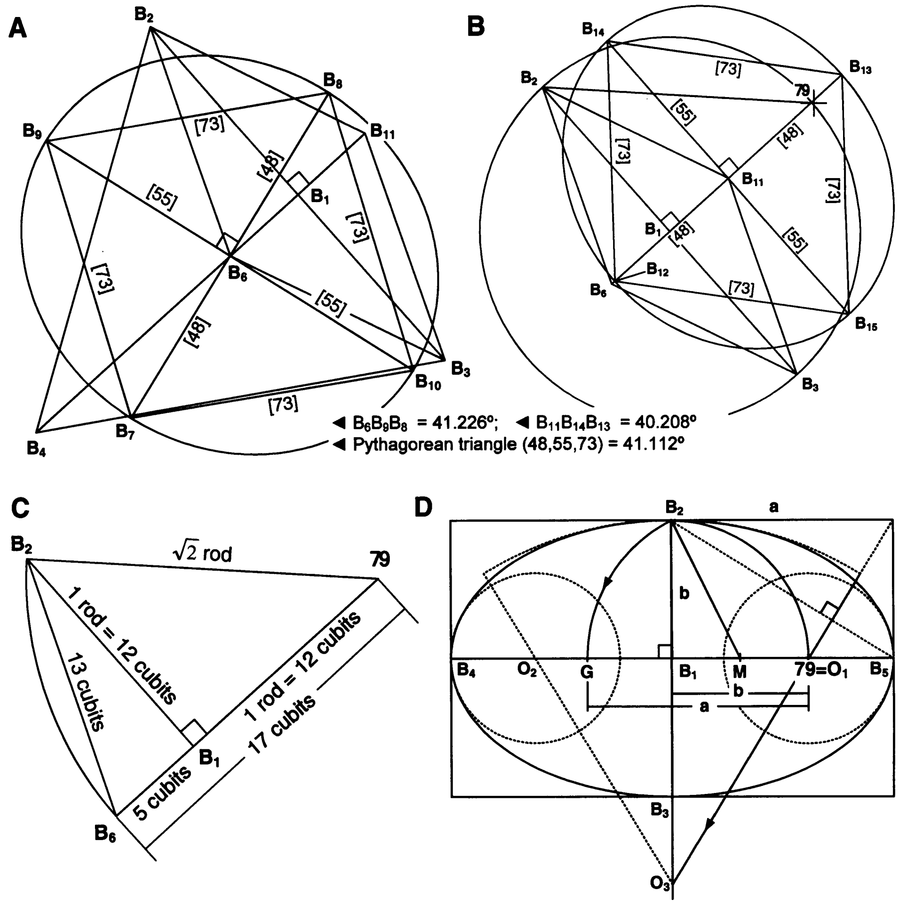


Fig. 9 Bluestone horseshoe (II). **a** The small ellipse SW with center B_6 defines the regular lozenge $B_7B_{10}B_8B_9$ composed of four right-angled triangles being similar to a Pythagorean triangle of the shape (48,55,73). **b** The small ellipse NE with center B_{11} defines the regular lozenge $B_{12}B_{15}B_{13}B_{14}$ composed of four right-angled triangles being similar to a Pythagorean triangle of the shape (48,55,73). The semi-major axis B_1B_2 (or B_1B_3) of the lozenge $B_2B_6B_3B_{11}$ has equal length as the line segment B_1-79. **c** The line segments B_6-79 and B_2-79 have equal length thus having an approximation for the square root of 2 by 17/12. **d** Construction of a rectangle having the legs in the ratio of the Golden Section and construction of the osculating circles with centers O_1, O_2 and O_3

the point 79 is also the center of the minor osculating circle at the vertex (Fig. 9d). The construction of the four osculating circles at the vertices is a good first step for an ellipse approximation. The knowledge and use of this concept is absolutely within the scope of a society constructing ellipses. Once again, traces of ancient construction pegs of the remaining three centers would resolve this option.

In the exact forward construction in Fig. 10, the size of the two small ellipses based on Pythagorean triangles of the shape (48,55,73) was performed in such a way that these ellipses have the indicated tangent points with the large ellipse. This method provides the best matching with the Bluestone horseshoe plan. The geometrical

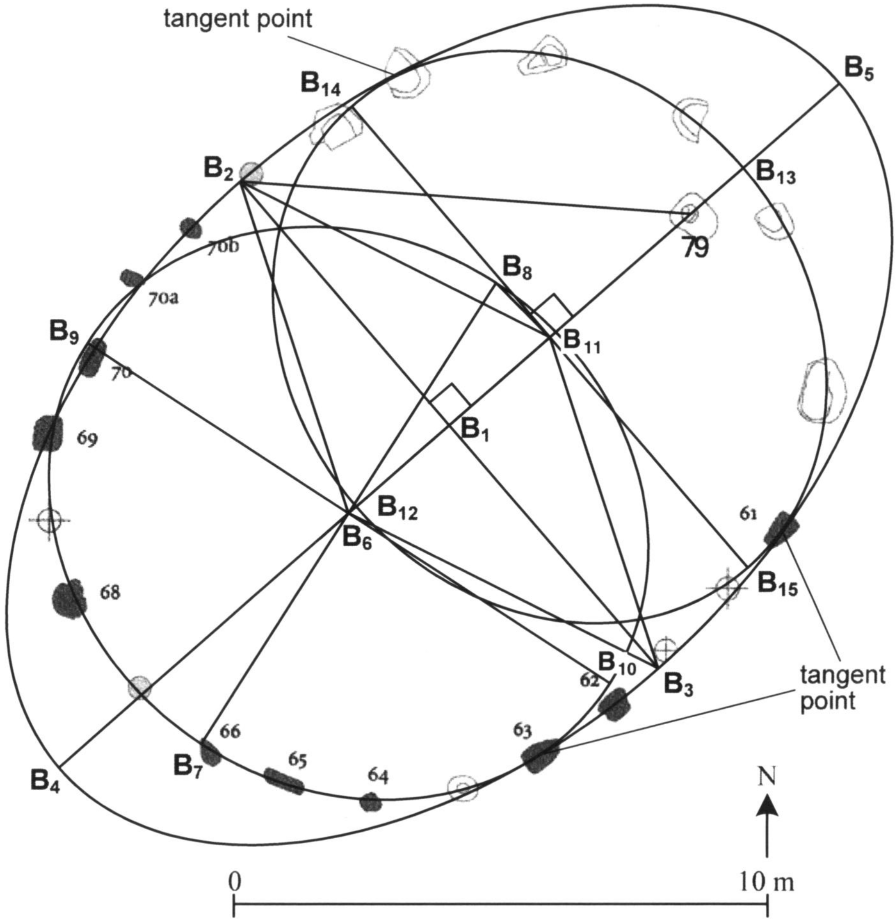


Fig. 10 Forward construction for the Bluestone horseshoe. The small ellipses with corresponding centers B_6 and B_{11} are constructed similar to a Pythagorean triangle of the shape (48,55,73) and in the way that the indicated tangent points are achieved

construction of an ellipse being tangent to another given ellipse can be achieved by an approximation procedure.

We can interpret the structures of ancient rings and earthworks as large mnemonic artifacts. Fortunately, there are also handy ancient artifacts discovered in the proximity of Stonehenge which underscore the layout concept of the Bluestone horseshoe: the Bush Barrow gold lozenges. The large lozenge displays the underlying lozenge of the two small Bluestone horseshoe ellipses based upon a Pythagorean triangle of the shape (48,55,73). For the investigation, we use the scaled graphic rendering in Fig. 11a. The angles 41.121° in the determining triangle $L_1 L_3 L_2$ matches the respective angles 41.112° in the Pythagorean triangle (48,55,73) nearly exactly (Fig. 11b). However, there is another way to confirm this explanatory model: when applying the determination procedure according to Fig. 13 to the gold lozenge also, we have to establish a consistent metrology for the manufacture of the lozenge. The lozenge is now 18.55 cm

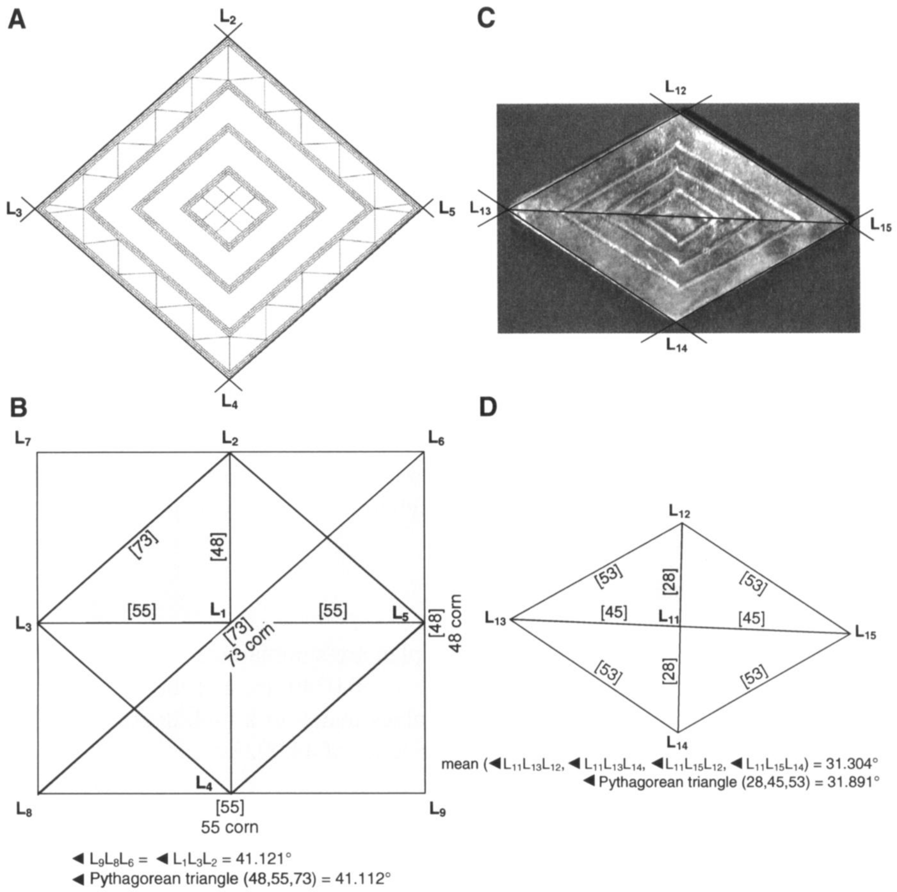


Fig. 11 Bush Barrow gold lozenges. **a** Bush Barrow large lozenge. The lozenge defines the regular rhombus $L_2L_3L_4L_5$. **b** The rhombus is composed of four Pythagorean triangles of the shape (48,55,73). Hence, the triangle $L_8L_9L_6$ is a Pythagorean triangle of the shape (48,55,73) also, and this triangle has concrete legs of 48, 55, and 73 corns as well as the axes of the rhombus $L_2L_3L_4L_5$ having the lengths of 48 and 55 corns. **c** Bush Barrow small lozenge. The lozenge defines the regular rhombus $L_{12}L_{13}L_{14}L_{15}$. **d** The rhombus is composed of four Pythagorean triangles of the shape (28,45,53)

long; it is considerably compressed at the corners, and the original size should be slightly larger due to the crumbling of the plate. Let us, therefore, assume as working hypotheses that the original distance between the vertices L_3 and L_5 was 19 cm. 19 cm divided by 55 provides 0.34545 cm; taking this measure as a unit for the "corn" measure, we have to multiply by 144 and get a cubit measure of 49.7455 cm. This cubit is equivalent to the established cubit for the Bluestone horseshoe of 49.9506 cm (2B). Hence, the triangles $L_8L_9L_6$ (and $L_6L_7L_8$ alike) are Pythagorean triangles with the concrete legs of 48, 55 and 73 corns and the axes of the lozenge have a length of 48 and 55 corns, respectively.

The small Bush Barrow gold lozenge was not manufactured with the precision and artistry of the large one. The production of this tiny plate may date back to the early phases of the manufacture of gold plates. This plate could have been passed over generations to the man to whom it was attached in his tomb. The marvelous large lozenge seems to reflect the culmination of that kind of plate. Nevertheless, the small gold lozenge could have played a comparable role. For the backward construction, we use a photo as no adequate scaled rendering is available (Fig. 11c); therefore, we have to take into account potential slight photographic distortions. The backward construction is very close to a regular rhombus composed of Pythagorean triangles of the shape (28,45,53) (Fig. 11d). This Pythagorean triangle is the basis of the construction of the large ellipse of the Sarsen horseshoe, and the large ellipse of the Bluestone horseshoe as well. The length of the major axis of the small lozenge is 3.1 cm; the division by 45 yields a measure unit of 0.06888 cm—far below the unit of a corn. We have no secured sources for a subdivision of the unit corn in antiquity. Following the old English unit "(shoe) iron" of 0.0529167 cm (1/48 inch), let us denote this potential unit an "iron". A subdivision of a corn by six irons yields a cubit measure of 59.5199 cm and a subdivision by five irons of 49.5999 cm. The latter measure matches with the established cubit measure scheme for Stonehenge, although this can only be an indication.

6.3 Deduced measures

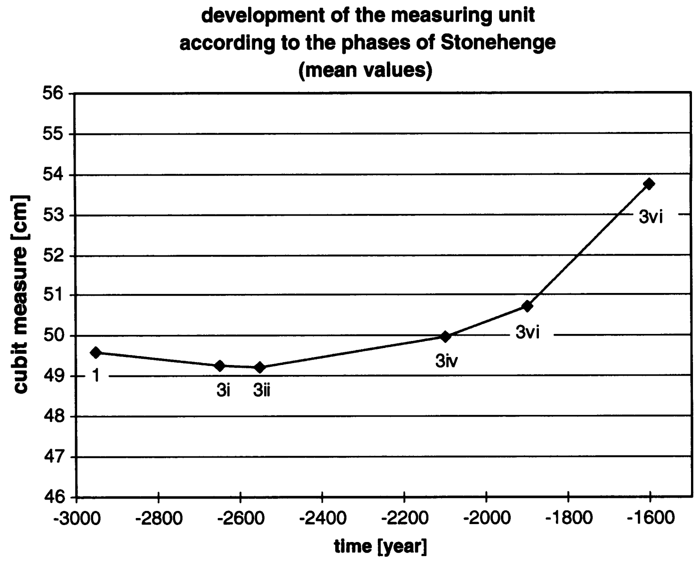
The actual length values extracted from the plan are summarized in Table 2B. The mean cubit measure for the Bluestone circle is 50.0340 cm, and the mean for the Bluestone horseshoe 49.9506 cm. As both values match to a high degree, we have established a mean cubit measure for all Bluestones of 49.9923 cm.

7 Metrology

All measurements of the Stonehenge structures are based on a consistent metrology. This metrology is in line with practiced standards in Babylonia concerning both the principal system of measures and the concrete measures.

Verifications of mathematical constructions without written sources get an additional affirmation if they are based on a consistent and coherent metrology and correspond with coeval verified systems of measures. In the construction of the Stonehenge structures, length measures were used as we encounter them in Babylonia. A preferred unit to compare ancient measures of length is the cubit. In the common Babylonian system of measures, a cubit was subdivided into six palms, a palm into four (st. five) fingers, and a finger into six corns; six cubits equal one reed and 12 cubits equal one rod; 10 cubits equal one pole, and 10 reeds equal one rope (Trapp and Wallerus 2006).[3] The unit 10 palms was also a used measure; this unit is commonly denoted as Megalithic yard (my) (see also ESM Fig. 2). The system of measures used in the geometrical construction of Stonehenge conforms to this Babylonian system. We have identified respective cubit measures for the phases of Stonehenge under investiga-

[3] The Babylonian corn measure differs from the old English barleycorn measure. In the latter case, the basis for the measure is the length of a barleycorn while a Babylonian corn (*še*) is about the third of this measure and seems to represent the width of a corn.



measure units			
structure	phase	time	cubit measure
Aubrey Holes	1	-2950	49.6016
Station stones, Heel stone	3i	-2650[1)]	49.2467
Sarsen circle and horseshoe	3ii	-2550	49.2079
Bluestone circle and horseshoe	3iv	-2100	49.9923
Z holes	3vi	-1900	50.7228
Y holes	3vi	-1600	53.7403
[1)] currently undated, 'before the central stones' (Johnson 2008, p. 136)			

Fig. 12 System of measures. The system of measures corresponds with practiced standards in Babylonia

tion. The length of the cubit has a comprehensible development scheme from phase 1 (ca. 2950 BC) to the end of phase 3vi (ca. 1600 BC) (Fig. 12). A considerable increase in length measures in times of decline is an often-noticed fact. The cubit measure 49.99 cm established for Stonehenge phase 3iv (ca. 2100 BC) corresponds with the coeval Babylonian cubit of Lagash of 49.61 cm.[4] This Lagash value is identical to the cubit measure of the Aubrey Holes of 49.60 cm, which we had identified as the cubit measure with the lowest range of error. Hence, the consistency of the Stonehenge measuring system is a strong support favoring the established mathematical concepts encoded in the structures of Stonehenge.

[4] The cubit of Lagash was documented on a statue of Gudea of Lagash. The current status of research does not provide exact data for the ruling time of Gudea. New approaches vary from 2140 to 2060 BC.

8 Methods

The construction of exact circles in ancient earthworks and stone/wood rings can be verified easily. The deduction of construction concepts of noncircular, ringlike structures without written sources is a complex task with a considerable number of pitfalls. To avoid misinterpretations to a great extent and to work out a highly probable solution a six-step determination procedure was established (Fig. 13). In terms of science theory, this procedure should be taken as final conclusions as "scientific proofs" for concepts of illiterate societies are hardly to be established. An explanatory model for a potential construction concept can be accepted only if all steps of the procedure are passed successfully. The procedure can break down at every step—except step 4.

Step 1 If the plan of the present situation of the monument does not contain sufficient data, then the procedure should be canceled at this point. Secured correction can be made, e.g., if the original position of a leaning stone is obvious.
Step 2 To establish circular or elliptical arcs in the backward calculations in a reproducible manner, nonlinear regression analysis methods are inevitable.
Step 3 If the selected construction concept does not match with a consistent system of measurements, the procedure should be canceled at this point (exception: similar transformations).
Step 4 The forward construction yields an exact mathematical construction model.
Step 5 Owing to both the decay of ditches/stones/holes and the potential imprecise construction of individual elements of the original monument, the concept of step 2 is no exact model in general. Therefore, slight modifications of the measure, position

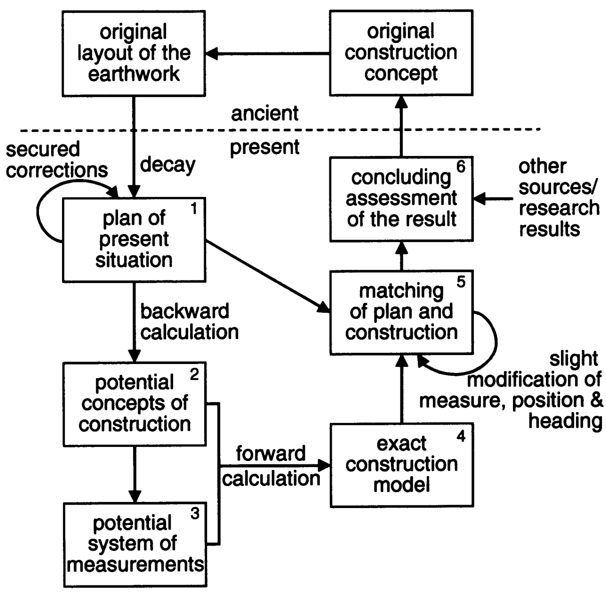


Fig. 13 Determination procedure. The determination of the original construction concept and potential earth work layout is performed on the basis of a six-step procedure. The procedure can break down at all steps except step 4

and alignment of the step 4 model can be made in step 5. If this forward construction model results in a worse matching compared to the backward construction, then the procedure should be canceled at this point in general.
Step 6 The potential solution of step 5 should pass an overall assessment taking into account also superordinate arguments.

The circles and circular/elliptical arcs in the backward calculation were computed by a nonlinear regression analysis. For the analysis of line elements (ditch, bank), preferable eight or more points of the respective section of the curve should be identified. In case of particular irregularities of curved lines, the reading points should be taken on the basis of the sampling theorem, as performed in the line determination of the excavated part of the ditch. In case of stones and holes, the number of reading points is predefined. Centers of stones and holes were established by a visual approximation. The center of gravity of shapes should be taken if the shape has an irregular noncircular image and no additional information about the original center is available. The results of the regression analysis including the standard deviations are provided as Supplementary Material (ESM Tables 1–4).

9 Summary

All the stone structures of the Stonehenge enclosure and probably also the ditch were designed, constructed, and set on the basis of mathematical concepts. We could identify only one exception: in times of an obvious decline of the mathematical traditions,

Table 3 List of the first 12 primitive Pythagorean triangles. "rope length" denotes the sum of the lengths of the three legs

First 12 primitive Pythagorean triangles (ordered by rope length)			
Primitive Pythagorean triangle	Rope length	Used in constructions	Element
(3,4,5)	12	II (I)	Sarsen circle/horseshoe, Z holes, (ditch)
(5,12,13)	30	IIII	Station stones, Bluestone circle, Bluestone horseshoe, Y holes
(8,15,17)	40		
(7,24,25)	56		
(20,21,29)	70		
(12,35,37)	84	I	Bluestone circle
(9,40,41)	90		
(28,45,53)	126	III	Heelstone, Sarsen horseshoe, Bluestone horseshoe
(11,60,61)	132		
(16,63,65)	144		
(33,56,65)	154		
(48,55,73)	176	II	Bluestone horseshoe (II)

the stone indicated by hole Y8 of the Y holes was set presumably arbitrarily. The major mathematical concept in the design of the Stonehenge structures is the application of Pythagorean triangles. This is in line with earlier and coeval earthworks and stone rings. Among the outstanding Pythagorean triangle (3,4,5), specific Pythagorean triangles were repeatedly applied in the construction concepts of Stonehenge: (5,12,13), (28,45,53), and (48,55,73)—the last two with considerable large rope lengths (see Table 3). The Pythagorean triangle (28,45,53) is applied for the design of both the ellipse of the Sarsen horseshoe and the large ellipse of the Bluestone horseshoe. The Pythagorean triangle (48,55,73) seems to be discovered in the forefront of the construction of the Bluestone horseshoe, and the design of the two smaller ellipses was dedicated to this discovery, while the shape of the large ellipse was designed in the tradition of the Sarsen horseshoe ellipse, indicating a practice bequeathed over centuries. The dedication of the Pythagorean triangle (48,55,73) to Stonehenge is underscored by the identical design of the Bush Borrow large lozenge. In general, Pythagorean triangles were composed to form rectangles, isosceles triangles, rhombi, and general polygons.

The standard concept for the design of ringlike structures was the composition of circular arcs or—in the case of the horseshoes—elliptical arcs to predominantly smooth lines; we could notice the total closing of the line (mathematical continuity) in the Bluestone horseshoe only. In this phase, the application of mathematics with the involvement of elliptical concepts was at its peak. As far as geometry is concerned the construction of ellipses being tangent to a given ellipse is the highlight of the Stonehenge mathematical designs. The concrete construction method of the ellipses could not be disclosed. The encoding of approximations for the circle constant π (28/9, 3 and 57/18 respectively) for the task "squaring the circle" and of the approximation 17/12 for the square root of 2 in the construction of Stonehenge is another important achievement as is the application of the law of similitude. The a.m. concepts are encoded in the structures of Stonehenge. It can be assumed that the general level of mathematical achievements in the Stonehenge society was remarkably of a high standard—with a distinct culmination in phase 3iv and an obvious time of decline in phase 3vi (Y holes).

The used length metrology corresponds with common Babylonian systems of measurement based on the unit of a cubit, broken down till the measure of a corn and summed up to the measure units reed, pole, and rope. Systems of measurement change rarely; we can, therefore, assume a common origin of these metrologies noticeable before the time under consideration. In addition, this consistent metrology establishes high confidence in the described mathematical concepts.

References

Folkerts, Menso. 2006. *The development of mathematics in Medieval Europe*. Aldershot: Ashgate.
Haarmann, Harald. 1991. *Universalgeschichte der Schrift*, 38–50. Frankfurt-New York: Campus.
Hawkins, Gerald S. 1965. *Stonehenge decoded.* Garden City: Doubleday.
Johnson, Anthony. 2008. *Solving Stonehenge*. London: Thames & Hudson.
Mathews, Jerold. 1985. A neolithic oral tradition for the van der Waerden/Seidenberg origin of mathematics. *Archive for History of Exact Sciences* 34: 193–220.
Neugebauer, Otto. 1969. *The exact sciences in antiquity*, 2nd ed. Mineola: Dover Publication.
Neugebauer, Otto. 1975. *A history of ancient mathematical astronomy part one-three*. New York: Springer.

Rottländer, Rolf C.A. 1979. *Antike Längenmaße*, 40–43. Braunschweig/Wiesbaden: Vieweg.
Seidenberg, Abraham. 1961. The ritual origin of geometry. *Archive for History of Exact Sciences* 1: 488–527.
Seidenberg, Abraham. 1978. The origin of mathematics. *Archive for History of Exact Sciences* 18: 301–342.
Seidenberg, Abraham. 1981. The ritual origin of the circle and square. *Archive for History of Exact Sciences* 25: 270–327.
Thom, Alexander, and A.S. Thom. 1980. Megalithic rings: plans and data for 229 Monuments in Britain. British Archaeological Reports, Br. Series 81.
Trapp, Wolfgang, and Heinz Wallerus. 2006. *Handbuch der Maße, Zahlen, Gewichte und der Zeitrechnung*, 193–194. Ditzingen: Reclam.
van der Waerden, Bartel L. 1954. *Science awakening*. Groningen: Nordhoff.
van der Waerden, Bartel L. 1973. Science awakening II. In *Birth of Astronomy*. New York: Springer.
van der Waerden, Bartel L. 1983. *Geometry and algebra in ancient civilizations*. New York: Springer.
Vansina, Jan. 1985. *Oral tradition as history*, 44–45. Wisconsin: University of Wisconsin Press.
Wußing, Hans. 2008. *6000 Jahre Mathematik*. Berlin: Springer.