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Source: *Archive for History of Exact Sciences*, Vol. 68, No. 1 (January 2014), pp. 35-65
Published by: Springer
Stable URL: https://www.jstor.org/stable/24569612

Arch. Hist. Exact Sci. (2014) 68:35–65
DOI 10.1007/s00407-013-0123-3

# Two problems in Aristarchus's treatise *on the sizes and distances of the sun and moon*

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Received: 15 April 2013 / Published online: 14 July 2013


**Abstract** The book of Aristarchus of Samos, *On the distances and sizes of the sun and moon*, is one of the few pre-Ptolemaic astronomical works that have come down to us in complete or nearly complete form. The simplicity and cleverness of the basic ideas behind the calculations are often obscured in the reading of the treatise by the complexity of the calculations and reasoning. Part of the complexity could be explained by the lack of trigonometry and part by the fact that Aristarchus appears unwilling to make some simplifications that could be simply taken for granted. But an important part of the complexities is due to some unnecessary inconsistencies, as recently discovered by Berggren and Sidoli (Arch Hist Exact Sci 61:213–254, 2007). In the first part of this paper, I will try to show that some of these inconsistencies are just apparent. But the complexity of the calculations and reasoning is not the only reason that could disturb a reader of the treatise. The great inaccuracy—even for the measurement methods and instruments available at those times—of one of the three input values of the treatise is really astonishing. In the sixth and last hypothesis, Aristarchus states that the moon's apparent size is equal to 2°, while the correct value is one-fourth of that. Some attempts have been made in order to explain such a big value, but all of them have problems. In the second part of this paper, I will propose a new speculative but plausible explanation of the origin of this value.

## 1 Introduction

The book of Aristarchus of Samos, *On the distances and sizes of the sun and moon*, is one of the few pre-Ptolemaic astronomical works that have come down to us in

Communicated by: Len Berggren.

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complete or nearly complete form. Undoubtedly, part of this fortune is due to the fact that it was included in the *Little Astronomy*, a list of works that had usually been studied before the *Almagest* (see Heath 1913, 317–321).

As it is clear from the title of the book, the purpose of the treatise is to obtain values for the sizes and distances of the sun and moon. The work consists of 18 propositions and could be divided into two parts. In the first one, Aristarchus compares the distances and sizes of the sun and moon (propositions 1–10). In the second one, he compares the sizes of both sun and moon with the earth (propositions 11–18).[1]

The ideas behind the calculations of these two parts are very simple and powerful. For calculating the proportion between the distances of the sun and moon, Aristarchus invites us to imagine the sun, moon, and earth at the exact moment of a quadrature forming a right triangle with its right angle at the moon. Calculating the elongation of the moon at the very same moment, we will have two angles of the triangle. Aristarchus asserted in hypothesis 4 that the elongation is "less than a quadrant by one-thirtieth of a quadrant" (Heath 1913, 353), i.e., 87°. Therefore, we could calculate the ratio of its sides, i.e., the proportion between the distances.

For calculating the proportion of the sizes of the moon, sun and earth, Aristarchus applied, again, a very simple and powerful idea: it is possible to know the size of an object by analyzing the size of its shadow and the distances between the illuminated object, the light source and the screen in which the shadow is projected. But we know that in a lunar eclipse, the earth's shadow produced by the sun's light is projected on the moon's surface, so it is possible to calculate the sizes.

The simplicity and cleverness of the basic ideas are often obscured in the reading of the treatise by the complexity of the calculations and reasoning. Part of the complexity could be explained, on one hand, by the lack of trigonometry. What can be solved in just one operation applying trigonometric functions, requires, in Aristarchus's case complex reasoning that often implies the analysis of new triangles, sides and angles. On the other hand, part of the complexity could also be explained by the fact that Aristarchus appears unwilling to make some simplifications that could be simply taken for granted.

But an important part of the complexities is due to some unnecessary inconsistencies in this treatise, as recently discovered by Berggren and Sidoli (2007). In the first part of this paper, we will try to show that some of these inconsistencies are just apparent.

But the complexity of the calculations and reasoning is not the only reason that could disturb a reader of the treatise. The great inaccuracy—even for the measurement methods and instruments available at those times—of one of the three input values of the treatise is really astonishing. As we said before, in the first part of the work, Aristarchus used 87° as the value of the moon elongation at a quadrature. This value is not too accurate, but the difficulties in measuring it—particularly in determining the exact moment of the quadrature—could excuse Aristarchus. On the second part of the work, he uses two other values. On hypothesis 5, he asserts that the earth shadow at the distance of the moon is equal to two moons. This value is accurate enough. But, in

[1] See also Heath (1913) for the only complete English translation, he also offers a long introduction at pp. 299–350. See also Neugebauer (1975, 634–643). But the most updated and serious study is Berggren and Sidoli (2007). For more references, see the note 1 of this work.

the sixth and last hypothesis, Aristarchus states that the moon's apparent size is equal to 2°, while the correct value is one-fourth of that. Some attempts have been made in order to explain such a big value, but all of them have problems. In the second part of this paper, we will propose a new speculative but plausible explanation of the origin of this value.

## 2 First part: the problem with proposition 13

### 2.1 The problem

Hypothesis 5 of Aristarchus's treatise states "that the breadth of the (earth's) shadow is (that) of two moons (Τὸ τῆς σκιᾶς πλάτος σεληνῶν εἶναι δύο) (Heath 1913, 352–353). For Ptolemy, this value is a bit bigger: 2;36° (Toomer 1998, 254), and he interpreted this by asserting that line EF which shows the size of the shadow when crossing the moon's center (M) is 2;36 times line GH, which represents the moon's diameter (Fig. 1).

But Aristarchus has a very peculiar way of interpreting this statement that implies, according to a recent study of Berggren and Sidoli (2007) and Sidoli (2007), some inconsistencies.

As we said before, in order to calculate the ratios between the sizes of both sun and moon with the size of the earth, Aristarchus had to know the ratio of the sizes of sun and moon with respect to the earth's shadow. The aim of proposition 13 is to obtain these proportions. To accomplish this, Aristarchus invites us to represent the sun, earth and moon in a lunar eclipse, "when the eclipse first becomes total through the moon having fallen wholly within the earth's shadow" (τελείας οὔσης τῆς ἐκλείψεως καὶ πρώτως ὅλης ἐμπεπτωκυίας εἰς τὸ τῆς γῆς σκίασμα (Heath 1913, 392).

In Fig. 2, we omitted the sun in order to zoom in to the relevant zone, but it would be located at the left of the figure, with its center on line ΒΛ. Β is the earth's center, and Γ is the moon's center. Lines ΚΝ and ΗΞ show the limits of the earth's shadow cone. ΛΝ is the *dividing line*. Arc ΝΞ, centered at Β, is the arc in which the endpoints of the dividing line—i.e., points Λ and Ν—move as the eclipse progresses; and line ΝΞ is the chord of the arc that we will call the *endpoint arc*. Line ΒΝ joins the center of the earth with Ν, the endpoint of the dividing line that is also in the shadow limits line ΚΝ. Finally, ΛΟ represents the moon's diameter, passing through its center, Γ.

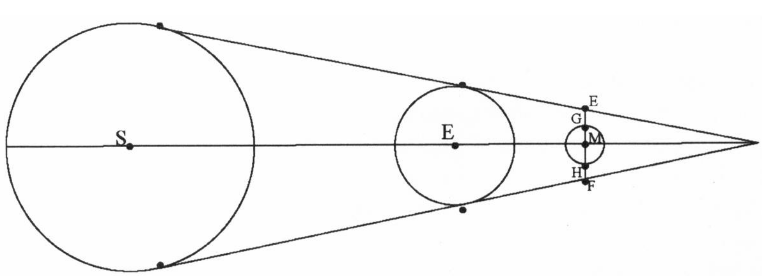



**Fig. 1** Sun, earth and moon at a lunar eclipse: the size of the earth's shadow is equal to two moons

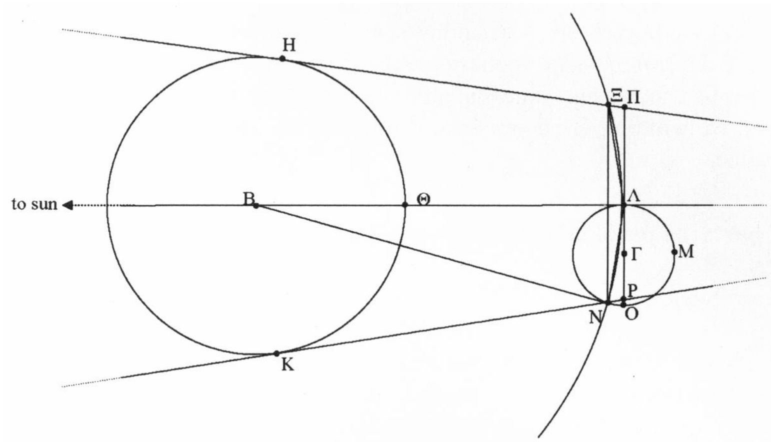



**Fig. 2** The endpoint arc (ΞΛN) carries the dividing line (ΛN) around B, the center of the earth

Let us add to Aristarchus's figure two points, Π and P. These points are the intersection of line ΓΛ with lines HΞ and KN, respectively. Therefore, line ΠP shows the diameter of the earth's shadow at the distance of the moon's center (Γ). So, according to Ptolemy, line ΠP is 2;36 times bigger than the moon, i.e., than line ΛO. But Aristarchus does not assert that line ΠP is twice as large as the moon, nor does he claim that chord NΞ is twice as large as the moon. For Aristarchus, to say that the breadth of the shadow is that of two moons means that two and only two dividing lines fit in the endpoint arc, which is limited by the shadow cone. This is, undoubtedly, a very odd way of interpreting the hypothesis. But, before analyzing the possible reasons for this particular interpretation, let us enumerate some inconsistencies involved in the construction of the figure that Sidoli (2007, 235–236) has recently brought to light.

1. The first inconsistency is related to the *orientation* of the dividing line. Because lines ΛB and NB are tangents to the moon's circle, the dividing line ΛN is perpendicular to the line joining the earth's center and the moon's center. However, the dividing line should be always perpendicular to the line joining the moon's center and the sun's—not the earth's—center. It is perpendicular also to the line to the earth's center when the three bodies are aligned, i.e., when the moon is in the middle of an eclipse, not in its beginning. Thus, *while the dividing line should be perpendicular to the line to the sun's center, actually it is perpendicular to the line to the earth's center.*
2. The second inconsistency is related to the *position* of the dividing line. Aristarchus proved in proposition 2 that, because the moon is smaller than the sun, the portion illuminated in the moon will be greater than half its surface. This implies, obviously, that the dividing line should always be farther away from the sun than the center of the moon, i.e., that the center of the moon should be between the sun and the dividing line. But, in the figure, Aristarchus located the dividing line between the sun and the center of the moon, i.e., he located the dividing line at the wrong place. *Thus, while the dividing line should be farther away to the sun than the moon's center, actually it is closer.*

3. Aristarchus asserts that the endpoints of the dividing line are carried by a circle centered in the earth. But, actually, while it is true that the center of the dividing line is carried by a circle centered at the earth, this is not true for the endpoints because the dividing line is always perpendicular to the line to the sun and not to the line to the earth. This can only be true if the dividing line is always facing the earth and not the sun. *Thus, while the endpoints of the dividing line could not be carried by a single circle centered on the earth unless the dividing line is always facing the earth, they are actually carried by such a circle.*
4. Fourth, Aristarchus requires at the same time two things that are incompatible: on the one hand, he asks "that the eclipse first becomes total through the moon having fallen wholly within the earth's shadow", but, on the other hand, he also asks that the lower endpoint of the dividing line, point B, be at the shadow limit line KN. Nevertheless, as is clear looking at the figure, it is impossible to have at the same time point N at line KN and the moon totally eclipsed: a small portion of the moon necessarily will protrude. *Thus, Aristarchus asks that the moon be totally and not totally eclipsed at the same time.*
5. Finally, a detail could be added, which even if it could not be considered an inconsistency, is clearly at least an oddity in Aristarchus's style. In proposition 4, Aristarchus had shown that even in the worst case, i.e., when the difference between the moon's diameter and the dividing line reaches its maximum, the difference is still imperceptible to an observer on the earth. This is very helpful in the analysis of the first part, for Aristarchus can suppose that the dividing line is passing through the center of the moon at quadrature, making it possible to build the right-angled triangle. In the second part of the treatise, Aristarchus will need the proportion between the dividing line and the moon diameter. However, this time, he doesn't treat them as equal. Rather, in proposition 12, he will calculate the proportion, again, in the worst scenario, when sun, moon and earth are aligned in that order. He obtains that the proportion between the dividing line and the lunar diameter is greater than 89/90 but less than one. Because it has been measured in the worst scenario, the limits are still true in any other possible situation.

In the figure of proposition 13, in which Aristarchus will use the limits obtained in proposition 12, the moon is close to the best scenario, i.e., in which the difference between the dividing line and the moon diameter reaches its minimum, because, now, the moon is at the other side of the earth, i.e., close to its maximum distance to the sun.

Therefore, the dividing line used in proposition 13 is greater than the dividing line of proposition 12, but, of course, it is still inside the limits obtained in this proposition. So, the use of the limits obtained in proposition 12 in the calculations of proposition 13 is totally reasonable. But what constitutes an oddity is that, every time that a ratio originally calculated for some value is applied by Aristarchus to a lower or greater value that nevertheless is inside the same limits, no matter how small the difference between them is, he makes the move explicitly using the formulas "πολλῷ ἐλάσσων" (*a fortiori* smaller) or "πολλῷ μείζων" (*a fortiori* greater), letting us know that even if the value is not exactly the same for which the proportion has been calculated, the

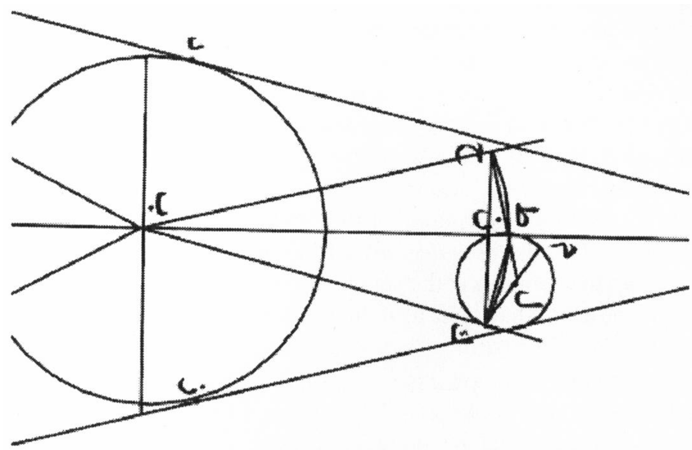


**Fig. 3** Detail of the diagram for Proposition 13 in the Kraus Ms of Thābit's revision (from Sidoli 2007, 540)

proportion is still valid. In this case, however, Aristarchus doesn't use this expression, inviting the reader to think that the dividing line has the same size in both propositions.

The first four problems lead Sidoli (2007, 534) to assert that "none of these assumptions are strictly true, and it is not actually possible to draw a mathematically coherent diagram of the configuration the proposition demands". Sidoli doesn't propose any explanation for these inconsistencies, but shows in a very clever way, that, analyzing the different configurations that the figure of proposition 13 took on in the various medieval manuscripts and modern editions, it is possible to realize that many of medieval and modern scholars were aware of these inconsistencies and tried to solve them. As one example, in the Kraus Ms of Thābit's revision, it was decided to introduce the entire moon within the shadow, but then the line BN does not touch the limit of the shadow cone. See Fig. 3.

The fourth problem is by far the most notorious and, probably, the most disturbing, because it can be easily detected as an inconsistency between the text and the figure. This inconsistency not only can be found in the clever analysis of the figures in manuscripts and earlier editions that Berggren and Sidoli propose, but it is also present in one scholion. In a thirteenth-century manuscript, there can be read close to the figure: "The figure is falsified, because the whole circle ΛMN should be inside the cone" ("Ἔσφαλται ἡ καταγραφή· ὅλος γὰρ ὁ ΛΜΝ κύκλος ἐντος τοῦ κώνου ὀφείλει εἶναι) (Noack 1992, 368) (Fig. 4).

The first thing to be said in defense of Aristarchus is that these inconsistencies do not alter the results significantly. In fact, in Proposition 4, Aristarchus has proved that the difference between the diameter of the moon and the dividing line is perceptually indistinguishable, so to locate it at one side or at the other of the moon would not make any difference, because it would always remain imperceptibly indistinguishable from the diameter. For the same reason, to be perpendicular to the line to the center of the sun or to the line to the center of the earth, when they are almost aligned, does not imply any significant difference in values. However, the inconsistencies have to

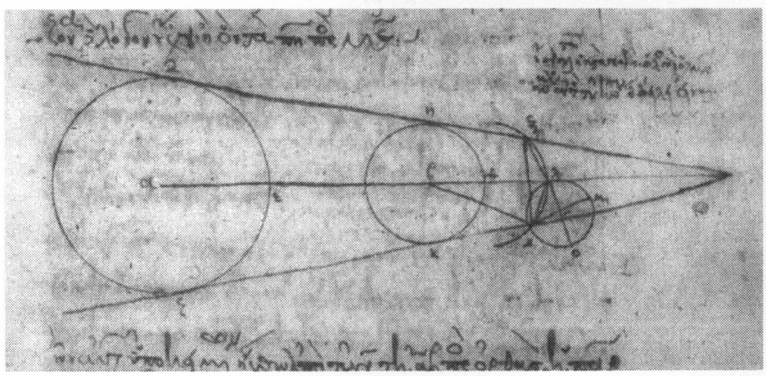


**Fig. 4** Diagram for Proposition 13 at Ms. Vat.Gr. 202. The scholion is at the upper right corner

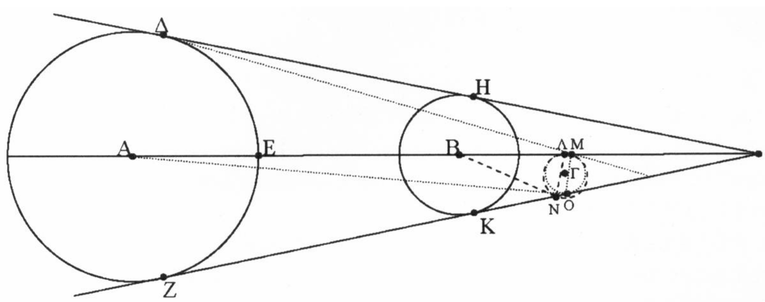



**Fig. 5** With *dashed lines*, the *dividing line* ΛN according to the figure of proposition 13. In this case, *line* ΛN is facing the center of the earth; with *dotted lines*, the *dividing line* MO as it should be if it were facing the center of the sun

have some reason. Why would Aristarchus incorporate them, if there were no strong reason?

Undoubtedly, the most natural reason would be that incorporating these inconsistencies would significantly simplify the calculations, as has happened when, in proposition 4, he asserted that the difference between the dividing line and the diameter of the moon is imperceptible. But even if this was the reason, an explicit enunciation would still have been legitimately expected from Aristarchus of the simplifications that he was making. After all, to justify just one simplification in the first part of the treatise, i.e., the line that divides the dark and bright portions of the moon is not exactly the same as but sensibly equal to the moon diameter, he spends three propositions. But let us see if, indeed, the calculations are simpler with these assumptions.

To do this, let us introduce the situation without any simplification. In Figs. 5 and 6, we have superimposed to the actual figure of Proposition 13—in which the dividing line ΛN (with dashed lines) is facing the center of the earth—the *correct* situation—in which the dividing line MO (with dotted lines) is facing the center of the sun. First, take a look to the dashed triangle (BΛN). Since the angle at B is 2° (the apparent size of the moon) and the triangle is isosceles, the other two angles each measure 89°. Therefore, the dividing line ΛN is tilted with respect to the perpendicular line

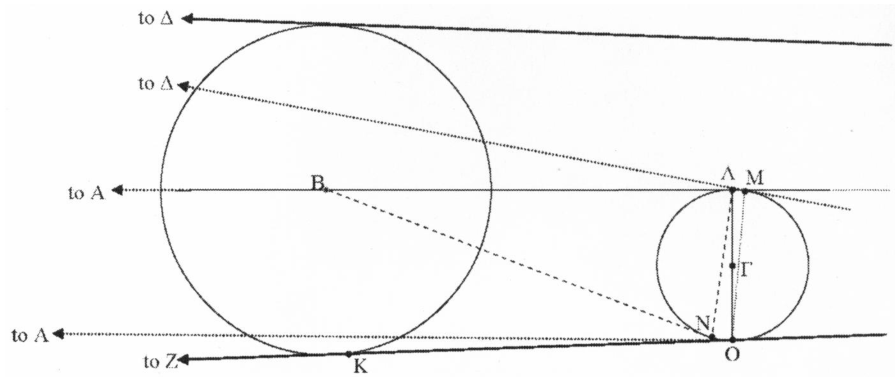



**Fig. 6** A detail of Fig. 5. In this figure, I decided to locate the entire moon in the *shadow* cone, just as in Thābit's interpretation, in order to avoid *drawing* two different moons, one for each *triangle*, but this does not affect the analysis

ΛO by just 1°, i.e., angle NΛO is one degree. Of course, Line ΛO is the diameter of the moon passing through the moon center, and so, it is the line used by Ptolemy for comparing the moon and shadow sizes. This is the difference between Aristarchus's and Ptolemy's approach. So this degree is the reason that justifies all the efforts made in propositions 11, 12, 13 and 14.

Now, take a look at the dotted triangle (AMO), which corresponds to the non-simplified proposal. We will calculate the equal angles at M and O. Suppose that, accepting proposition 4, the base of both triangles is the same, i.e., MO is equal to ΛN. We know that the equal sides of the dotted triangle are approximately 20 times larger than the equal sides of the dashed triangle BΛN, since the sun is approximately 19 times farther than the moon, but in this case, the moon is on the other side of the earth, and so we have to add one earth–moon distance. So, line AM is 20 times line BM. In proposition 11, Aristarchus calculated the ratio between the diameter of the moon and the earth–moon distance, asserting that it is greater than 1/30 but less than 2/45. Averaging, we could say that the moon is 26 moon diameters from the earth. If, again, assuming proposition 4, we accept that the dividing lines are equal to the moon diameter, we can say that side BΛ is 26 times ΛN, the base of the triangle. But the equal sides of the dotted triangle are 20 times greater, and so they will be 520 times greater than their base, i.e., AM is 520 times MO. With these side proportions, the equal angles of the dotted triangle measure 89.94°. Therefore, the difference with the perpendicular line ΛO is only a little bit more than 3 min, i.e., angle ΛOM measures just $3'$.

Now, If Aristarchus wants to make some approximation to simplify the calculations, it would certainly be more justified to treat MO as identical to the lunar diameter perpendicular to the axis of the cone, i.e., line ΛO and not treat MO as equal to ΛN. First, because MO is much closer to ΛO than to ΛN: 3 min versus around 1°. But, second and more important, the first equality would surely greatly simplify the calculations, for the new configuration would virtually be the same as Ptolemy's. In this case, Aristarchus could have avoided propositions 11–14, dispensing with around a third of the treatise. Of course, in this case—and also if he had decided to work directly with MO without any simplification—there would not be any portion of the moon protruding, because point O is in the limiting shadow line. In contrast, there is no simplification at all in equating MO to ΛN, and, as we have shown, many problems.

Thus, there seems to be no reason to justify either the inconsistencies or the curious interpretation given to hypothesis 5. Aristarchus, without any apparent reason, says that the hypothesis according to which the earth's shadow is equal to two moons should be interpreted as saying that two and only two misplaced, misdirected and perhaps wrongly sized dividing lines inserted in a badly built arc fit with the cone of earth's shadow when the moon is wholly immersed in the cone, even if the moon actually could not be wholly immersed. Definitely, proposition 13 tarnishes the genius of Aristarchus.

## 2.2 Possible solution

However, we think there is a way to justify the curious interpretation of hypothesis 5 that also removes all the five inconsistencies that we have enumerated.

Let us take a look again at proposition 4, in which Aristarchus wants to show that the difference between the dividing line and the lunar diameter is imperceptible to the human eye. The only value used in this calculation is the apparent size of the moon that, according to hypothesis 6, is equal to 2°. See Fig. 7. For Aristarchus, this means that angle ΔAΓ and not angle ZAE measures 2°. In turn, ΔΓ, the base of ΔAΓ triangle, is the dividing line. However, the light of the moon depends on the position of the sun and the proportion between the moon's and sun's sizes, but the eye's position doesn't play any role. On the other hand, the apparent size of the moon depends on the observer's position, not the sun's position. What is the link between the apparent size of the moon and the dividing line that allows us to assert that the base of the triangle having the apparent size of the moon as the unequal angle is the dividing line? The link is very clear: the same cone comprehending both the sun and the moon has its vertex at our eye. This means that the dividing line is not only dividing the dark and the bright portions in the moon, but also the visible and the invisible portions of the moon from the earth.

But of course, these two divisions do not always coincide on the same line. So, from now on, we will stop using the ambiguous name *dividing line* for the line that divides the dark and bright portions of the moon and we will call it the *light line*. On the other hand, the line that divides the visible and the invisible portions of the

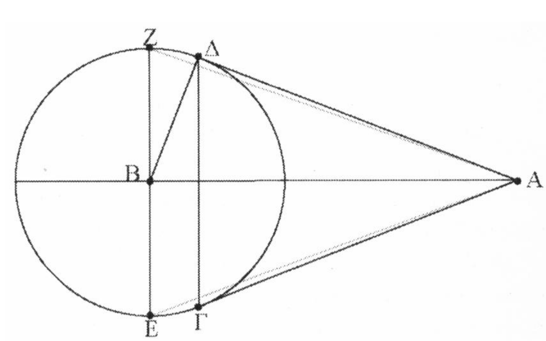



**Fig. 7** The *dividing line* when the vertex of the cone comprehending the sun and moon is at our eye

moon from the earth will be called the *visibility line*. Now, the *light line* depends only on the relative position of the moon and the sun, and the earth's position plays no role at all in it. Its orientation, therefore, is always perpendicular to the line joining the centers of the sun and the moon. Besides, because the sun–moon distance varies, the size of the light line varies reaching its maximum difference with a lunar diameter—and so, its minimum size—when sun, moon and earth are aligned, in that order. On the other hand, the *visibility line* depends only on the relative position of the moon and earth, being always perpendicular to the line connecting the center of the moon to the center of the earth. Besides, because the distance between them does not change (because the moon rotates in a circular orbit around the earth), the visibility line always has the same size. The size of the visibility line is the same as that of the light line when the light line reaches its maximum difference with a lunar diameter, that is, in the configuration of a total eclipse of the sun, described in proposition 12. In this configuration, the orientation of both dividing lines coincides. But, for example, in a quadrature, their sizes are different and their orientations perpendicular between each other.

Cleomedes distinguished two circles in *The Heavens* for explaining moon phases, and he asserted that both circles are smaller than a great circle even if, of course, the difference is not perceivable for us. He talks about circles and not lines, because he is thinking in three-dimensions. Our dividing lines are the two-dimensional expression of these circles, because the circles are always perpendicular to the plane of the figures. He also asserts that they change their position (generating the phases). The only detail that Cleomedes seems to have overlooked is that the light line changes its size, for he asserts that both have the same size.

> Two circles are conceived of in the moon: A, the one by which its dark part is separated from its illuminated part (εἷς μέν, ᾧ διακρίνεται τὸ σκιερὸν αὐτῆς ἀπὸ τοῦ πεφωτισμένου); B, the one by which the part visible to us is separated from the part that is in-visible (ἕτερος δέ, ᾧ χωρίζεται τὸ ὁρώμενον ὑφ′ ἡμῶν αὐτῆς [ἀπὸ] τοῦ μὴ ὁρωμένου). Each of these circles is smaller than C, the circle that can divide the moon into two equal parts, that is, its great circle. Because the sun is larger than the moon, it illuminates more than half of it, and thus A (the circle that separates the dark from the illuminated part) is smaller than C (the great circle of the moon). B (the circle in our line of sight) is, by the same token, necessarily smaller than C (its great circle), since we see less than half of the moon. (...) So since B divides the moon not into equal, but into unequal, parts, it too is smaller than C, the great circle. Both A and B, however, appear as great circles relative to our perception, and while they always have the same size, they still do not maintain the same fixed position, but cause numerous interchanges and configurations relative to one another...(Cleomedes II.5., Bowen and Todd 2004, 147)

Now, once we have conceptually distinguished between both dividing lines, if we turn back to the figure of proposition 13 (see again Fig. 2), we will see that line ΛN, which is called by Aristarchus the line which divides the dark and the bright portions of the moon is actually not the light line, but the visibility line. Indeed, the line connects the two endpoints of the lines that starting at the center of the earth are tangent to the

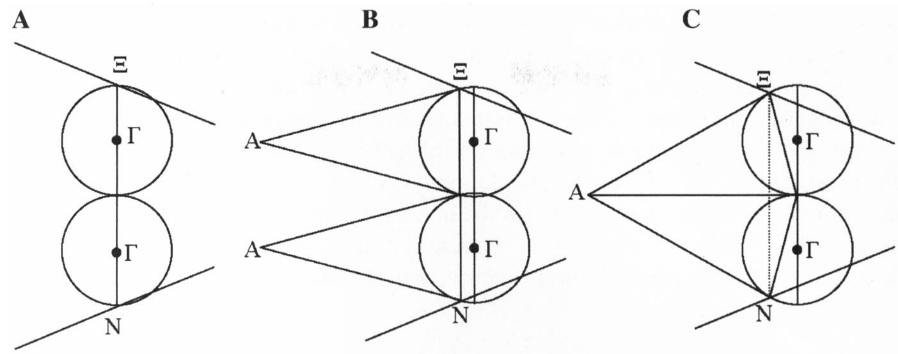



**Fig. 8** Three different ways of interpreting that the earth shadow visible in a lunar eclipse is equal to two moons. In **A**, the line that measures the size of the *shadow* crosses through the center of the moon, in **B** coincides with two visibility *lines* parallel to each other; in **C**, the two visibility *lines* are facing the same center, **A**

moon circle. So, while in Fig. 6, line MO is the light line, line ΛN is the visibility line. It is true that if line ΛN is interpreted as the light line, it is misplaced, misdirected and perhaps with the wrong size. But if we consider that line ΛN is the visibility line, then, it is well located, well orientated and has the appropriate size. Indeed, the visibility line is perpendicular to the line joining the earth's center with the moon's center, because the visibility line always faces the earth (and so, inconsistency 1 disappears); it is also between the center of the moon and the earth, because we see less than a half moon (and so inconsistency 2 disappears); therefore, the circle carrying the endpoints of the visibility line is a circle centered at the earth (and so, inconsistency 3 disappears); and, finally, the visibility line has the same size that the light line has in the configuration of proposition 12, having therefore, the correct size (and so, inconsistency 5 disappears).

Besides, if we consider that Aristarchus had in mind the visibility line, the oddity of his way in interpreting hypothesis 5 turns into a subtlety worthy of a mathematical genius. As noted above, the simplest interpretation of hypothesis 5 would say that the diameter of the shadow is equal to two lunar diameters, as Ptolemy did. This is represented in Fig. 8a.

But this is not Aristarchus's choice. Aristarchus could have reasoned in this way: since we do not see half the moon, but only the portion covered by the visibility line (which is smaller than the moon's diameter) when we see that the earth's shadow is equal to two moons, what we see is that the earth's shadow is equal to two visibility lines, as shown in Fig. 8b, where A is the center of the earth.

However, Fig. 8b is not entirely correct, because the center of the earth (point A) is duplicated. If we join both points A, the figure turns into Fig. 8c, which is the figure that Aristarchus finally adopted.

Thus, when Aristarchus asserted in hypothesis 5 that the breadth of the shadow is equal to two moons, he is always referring to angles measuring apparent sizes: indeed, it is true in his figure that the apparent size of the shadow seen from the center of the earth is equal to the apparent size of two moons, also seen from the center of the earth.

Finally, this interpretation is also supported by the way it naturally dissolves the most disturbing of the inconsistencies: i.e., the fact that Aristarchus's configuration assumes that the moon is totally immersed in the earth shadow, and at the same time, the same configuration implies the opposite, as the figure of proposition 13 shows. The solution is simple. We must ask ourselves which is the first moment of a total eclipse: is it when the entire moon is immersed in the earth shadow or when the entire visible portion of the moon is immersed in the earth shadow? Obviously, the second option is the correct one: we will see the entire moon eclipsed when the entire moon that we see is immersed in the shadow. This is the first instant of the total eclipse. Therefore, the portion of the moon that, in the figures, protrudes from the shadow cone does not affect the totality of the eclipse because, actually, that portion of the moon is not visible from the earth.

## 2.3 Textual evidence

It still remains unexplained why Aristarchus did not use two different expressions for distinguishing the two dividing lines. But maybe he did. The expression "the circle dividing the dark and the bright portions in the moon" (or similar expressions) is mentioned in the formulation of hypothesis 3 (once), in proposition 3 (four times), in proposition 4 (seven times), in proposition 12 (twice), in proposition 13 (seven times) and in proposition 14 (three times). According to our proposal, until proposition 12, when Aristarchus talks about the circle corresponding to the dividing line, he is thinking of the circle corresponding to the light line; in proposition 12, he would be demonstrating that the visibility line is equal to the light line when the light line is the smallest possible, and in propositions 13 and 14, every time Aristarchus mentions the dividing circle he is thinking of the circle corresponding to the visibility line. Every time Aristarchus mentions the circle in propositions 3 and 4, he talks about the circle itself, i.e., he uses the expression: *the circle in the moon which divides the dark and the bright portions* (ʻΟ διορίζων κύκλος ἐν τῇ σελήνῃ τό τε σκιερὸν καὶ τὸ λαμπρὸν) or a very similar expression, while, every time he mentions the circle from proposition 12 onwards, he talks about the diameter of that circle, but never about the circle itself. He uses expressions like: *the diameter of the circle which divides in the moon the dark and the bright portions* (ἡ διάμετρός τοῦ κύκλου τοῦ διορίζοντος ἐν τῇ σελήνῃ τὸ σκιερὸν καὶ τὸ λαμπρόν). He never says, for example that the circle drawn around these diameters is the circle that divides the dark and the bright portions in the moon. That is to say, he affirms that the diameter is the diameter of the circle that divides, but he does not say that this is the circle that divides. Because the two dividing lines are equal (in the condition established in proposition 12), it is not wrong to say that this line is (equal to) the diameter of the circle that divides, even if the circle is not the light circle, but the visibility circle.

There might be some textual evidence that Aristarchus had this distinction in mind. Against this idea, however, one could argue that the reason for talking about the circle first and about the diameter later is that, while the subject of propositions 3 and 4 is the circle itself, the subject of propositions 13 and 14 is the diameter, because Aristarchus wants to establish some numerical relationships between the sizes of the circles.

Nevertheless, there could be some other textual indication that Aristarchus was aware of the distinction between the dividing lines. The word ἡ διάμετρος is used

107 times in Aristarchus's treatise. It is mainly used, of course, to refer to the diameters of the moon, sun and earth. If we leave aside the ten instances in which it refers to the diameter of the circle that divides the dark from the illuminated part in the moon in propositions 13 and 14, we have 97 instances. In all of them, except in three instances, the word διάμετρος is accompanied by the definite article. The three cases in which the word is alone appear in the first paragraph of propositions 16, 17 and 18, where Aristarchus is identifying the lines of the figures with the diameters of the moon, sun and earth in order to cube them and obtain the proportion between the sizes of the bodies. In all three cases, the figures are only lines, with no circles around them. It seems clear that Aristarchus is not using the definite article because he does not mean that these lines are actually the diameter of the circles, which is not necessary. He simply means that they have the same magnitude as the diameters and, therefore, that they represent any possible diameter of the bodies, as opposed to a particular diameter in a particular diagram. Thus, for example, at the beginning of proposition 16, Aristarchus says that line A is (equal to) one diameter of the sun and not that line A is *the* diameter of the sun. The three times that διάμετρος is used in proposition 14, it is not accompanied by the definite article. In two of these cases, Aristarchus says that the line MO is (equal to) one diameter of the circle that divides. In the third one, he says that both lines MO and ΞΛ are (equal to) one diameter of the dividing circle. Had Aristarchus meant that MO is the diameter of the dividing circle, he would have introduced the definitive article, because line MO would have represented a particular diameter in a particular diagram. Therefore, its absence could be considered as textual evidence that Aristarchus had the distinction between the two dividing lines in mind. Unfortunately, the case is not so clear in proposition 13 in which the word is not accompanied by the definitive article only once. There is, however, at least one case in which it does not need the article, but it is there.

Suppose, then, that we cannot affirm that there is any strong textual evidence for the conceptual distinction between the two lines. Does this mean that the conceptual distinction does not exist? certainly not. There is at least one case in which Aristarchus uses exactly the same expression referring to two different concepts. In the treatise, Aristarchus talks about the moon, sun and earth as spheres, and so he talks about the *sphere of the moon*, *the sphere of the sun* and *the sphere of the earth*. But he also talks about the orbit of the moon and the sun, which for Aristarchus is also a sphere. Sometimes he uses the more accurate expression: "the sphere on which the center of the [sun or moon] moves", but in the case of the moon, he often calls it "the sphere of the moon". This is the case, among others, of the formulation of hypothesis 2, in which Aristarchus says that "the earth is in the relation of a point and center to the *sphere of the moon*". The expression is so confusing and the meaning so clear by context that heath decided to translate "the sphere in which the moon moves". We have, then, the same expression: *the sphere of the moon*, referring to two clearly different objects: the moon as a sphere and the orbit of the moon.

In the case of the two meanings of *sphere*, the context is so clear, and the two refer to things which are so different, that the confusion is almost impossible. Nevertheless, we cannot say the same about the two dividing lines. How can a reader distinguish between them if Aristarchus uses the same expression? We must remember, as Reviel Netz has shown, that diagrams play a crucial role in Greek mathematics, in particular

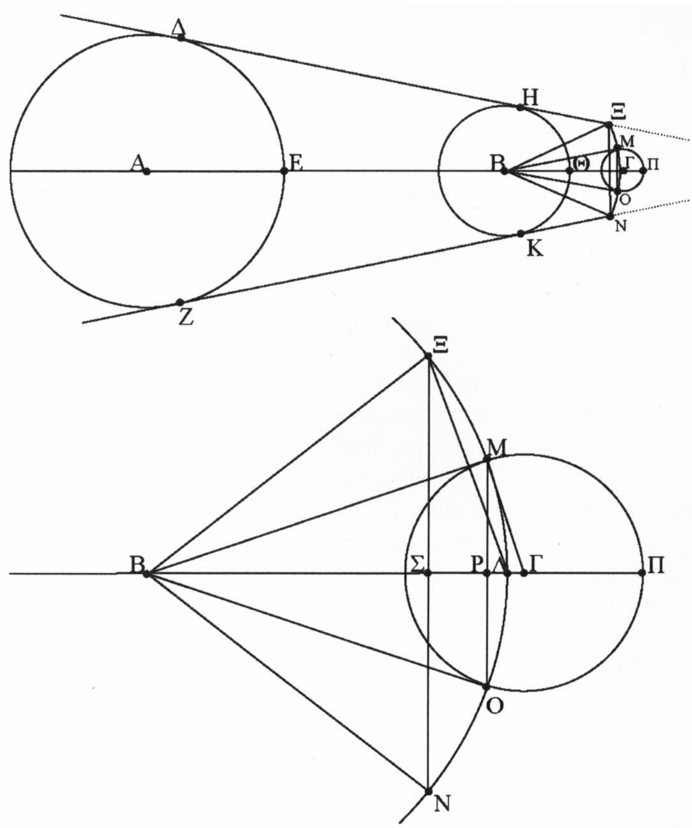



**Fig. 9** Figure of proposition 14. **A** is the center of the sun; **B**, the *center* of the earth and Γ is the *center* of the moon. The moon is in the *middle* of a lunar eclipse. The *lower* figure is a detail of the *upper* one

for fixing the reference of expressions that the text leaves unspecified. Netz talks specifically of letters, but we see no reason not to extend this idea to some kind of expressions which are so repeated in the same way, that they constitute formulae. As Netz pointed out, "very often—most often—letters are not completely specified. So, how do we know what they stand for? Very simple: we see this in the diagram" (Netz 2004, 23). In the diagram of proposition 13, it is evident that the dividing line could not be the light line but the visibility line, for it is located between the center of the moon and the sun, what is impossible for the light line, given that the sun is bigger than the moon, as Aristarchus explicitly asserts in proposition 2. This is even clearer in the figure of proposition 14 (see Fig. 9).

## 2.4 Other problems

Of course, this proposal does not solve all the problems present in the treatise. For example, the problematic implications of the second hypothesis still remain unexplained. Aristarchus asserts in this hypothesis that "the earth is in the relation of a point and center to the sphere in which the moon moves" (Heath 1913, 353).

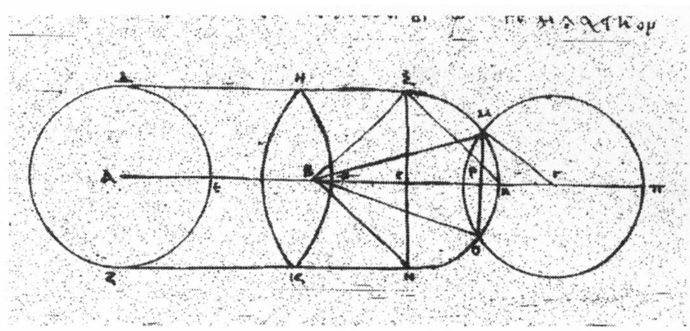


**Fig. 10** Figure of proposition 14 in Mss.Vat.Gr.204

This implies, however, as Berggren and Sidoli (2007, 216–127) have emphasized, that the fact that the earth is taken as having no radius, "denies the possibility of relating the lunar distance to the radius of the earth, and establishing terrestrial distances of the luminaries". In addition, "it implies that no extended terrestrial shadow will fall on the orbit of the moon", contradicting hypothesis 5 and all the propositions based on it.

This contradiction is not solved by assuming the two dividing lines. And we do not pretend to solve it. It is worth mentioning, however, that the illustrator's awareness of this contradiction could explain some odd features of the figure of proposition 14. In this proposition, Aristarchus invites us to draw the same figure of proposition 13 but, now, he asks that the moon be placed so "that its center is on the axis of the cone comprehending both the sun and the earth", i.e., in this case, we are in the middle and not at the beginning of the eclipse. So, the figure would be like Fig. 9.

The figures in the manuscripts are shown in Fig. 10, reproduced from the oldest surviving manuscript.

The first thing to note is that the three bodies have the same size, making a cylinder rather than a cone. It is an oddity which could probably be explained by the necessity of having a big moon in order to make the lines clear. There is, however, another oddity, even more difficult to explain: the earth shape (centered at B) is not a circle, but some kind of lens-shape object. In figures of other texts, even in the same manuscript, this lens shape is used for drawing ellipses or circles in perspective, as in Eutocius's commentary on Apollonius's *Conics* 1, where the drawings represent the various kinds of conic section (see Fig. 11).

Therefore, the earth seems to be represented with an ellipse or a circle in perspective, as is shown in Fig. 12.

This could be an ingenious way to try to solve the contradiction of supposing the earth to be a point that nevertheless casts some shadow. In effect, in this case, the earth would be considered a great circle, perpendicular to the plane of the cone's axis. In proposition 14, Aristarchus have to assume that the earth has the relation of a point and center to the moon's orbit, because the observer must be located at the center of the moon's orbit that coincides with the center of the earth and not with its surface. But, again, assuming that, there would be no shadow. The situation could be partially solved if the sphere of the earth is replaced by a great circle perpendicular to the axis of the cone because there would be no problem with having the eye at

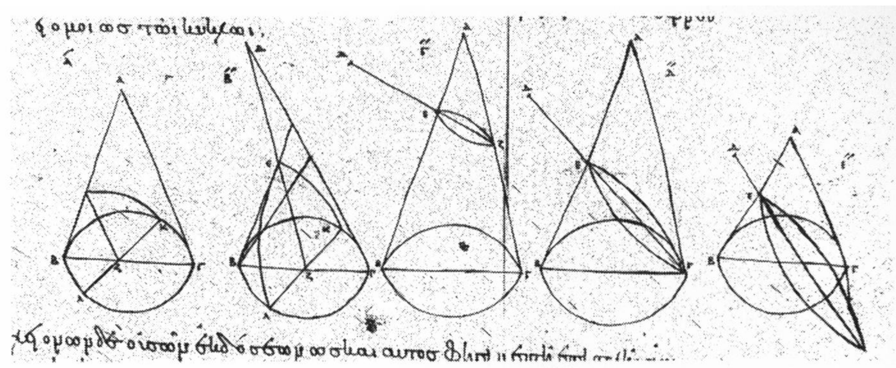


**Fig. 11** Figure of Eutocious's commentary on Apollonius Conics I in Mss.Vat.Gr.204

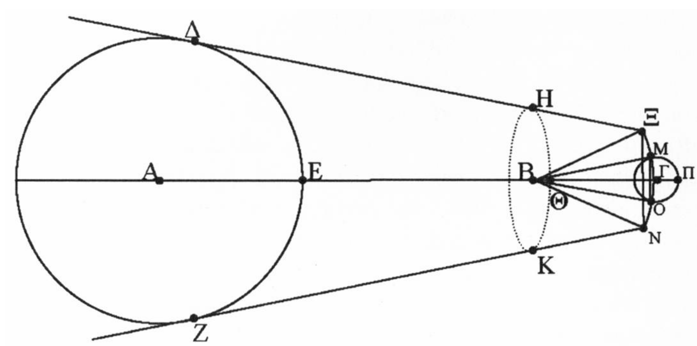



**Fig. 12** A possible representation of Fig. 14 of the treatise, in which the earth is represented just with a *circle* in perspective

the center of the circle and the circle casting a shadow. It is hard to know whether this detail could be traced back up to the original figure drawn by Aristarchus. Nevertheless, what can be asserted with certainty is that there are no traces of this detail in the text because Aristarchus is asking us to draw the same figure as in proposition 13 in which the earth is a sphere. Maybe, it was just a way the illustrator found to direct the attention of the reader to the problem. But the same situation is represented in the figure of proposition 13. Nevertheless, in this figure, the earth is a circle, not an ellipse. Therefore, why would the illustrator modify the figure of proposition 14 and not the figure of proposition 13? Maybe, a reason for this could be that in proposition 14 (but not in 13), hypothesis 2 is explicitly mentioned: *because the point B is the center of the earth, and the earth has the relation of a point and center to the sphere in which the moon moves* (διὰ τὸ τὸ Β σημεῖον κέντρον εἶναι τῆς γῆς, καὶ <τὴν γῆν> σημείου καὶ κέντρου λόγον ἔχειν πρὸς τὴν τῆς σελήνης σφαῖραν) (Heath 1913, 403).

There are still many unsolved questions in the treatise, but the distinction between the visibility and the light line at least solves the five inconsistencies that we have previously enumerated.

## 3 Second part: a possible explanation for the 2° of the moon's apparent size in Aristarchus

### 3.1 Introduction

As we said before, one of the most intriguing mysteries in Aristarchus's treatise is the value of the moon's apparent diameter. Aristarchus asserts in hypothesis 6 that the moon subtends one-fifteenth part of a zodiac sign, i.e., its apparent diameter is 2°. The real value—which could be obtained by a very simple procedure—is around four times smaller. So, there seems to be no reason for so big a number. Moreover, in *Sand-Reckoner*, Archimedes says that Aristarchus discovered that the sun's apparent radius (which is equal to that of the moon) is 1/720 of the zodiac, i.e., 0.5° (Heath 1897, 223).

Attempts have been made to justify this big value. Manitius (1909, 292) suggested that it could be a transcription error, and that should be read it as 1/50 (πεντηκοστόν μέρος) and not 1/15 (πεντεκαιδέκατον μέρος), so being around 0.6°. But all the calculations of the treatise suppose the 2° value. Previously, Tannery (1883, 241) proposed that Aristarchus deliberately chose a false value in order to show that "his treatise was mainly intended to give a specimen of calculations which require to be made on the basis of more exact experimental observations, and to show at the same time that, for the solution of the problem, one of the data could be chosen almost arbitrarily" (Tannery 1883, 241, translation taken from Heath 1913, 311–312).

Actually, it is true that the modification of the moon's apparent size is almost irrelevant in the main results of the treatise.[2] It is only involved in three propositions. In proposition four, it is used to show that, even if the light line does not exactly cross the center of the moon, the difference is imperceptible. As we already saw, Aristarchus needs to treat these two lines as equivalent in order to guarantee the geometrical construction of the quadrature that is the basis of the distance proportion calculation. Then, it is used in proposition 15 for calculating the upper limit of the proportion between the sun's and earth's diameters. And consequently, it is also used in proposition 17, for calculating the lower limit of the proportion between the earth's and moon's diameters.[3] In both cases, nevertheless, its role is almost negligible. It is so irrelevant that Neugebauer (1975, 643), who usually is moderate in his judgments, says that "it is pure mathematical pedantry". Actually, if the lunar diameter was absolutely 0, the upper limit of the proportion between the earth's and sun's diameters would change from 7.16 to 7, and the lower limit of the proportion between the earth's and moon's diameters would go from 2.51 to 2.57. If, at the other extreme, the ridiculous value of 15° was adopted for the moon's apparent size, the values would be 8.51 and 2.11, respectively. This clearly shows that Tannery is right when he says that "one of

[2] Wall (1975, 208–209) mistakenly says that "the entire treatise is an attempt to fix the sizes and distances of the sun and moon within limits which are then completely falsified by changing this value".

[3] In both cases, it is also used for calculating the upper in the first case and the lower in the second case of the proportion between the volumes, for he simply *cubes* the diameter proportions in order to obtain the volume proportions.

the data could be chosen almost arbitrarily". Maybe, as Heath (1913, 311) suggested, Tannery's explanation is "perhaps too ingenious". Why would Aristarchus stop at 2° if he wants to introduce a clearly exaggerated value? Actually, 2° is not too exaggerated if we remember that, for example, Macrobius value was around 1.66°.

James Evans (1998, 72) has proposed a very interesting and simple way to explain the origin of the other inaccurate value of Aristarchus's treatise: the 87° for the moon's elongation at a quadrature. Evans shows that the 87° could be understood as the result of assuming the largest imperceptible inequality between two parts of the lunar month: one that starts with the conjunction and ends with quadrature (or vice versa), and one that starts with quadrature and ends with the full moon (or vice versa). If the angle were smaller than 87°, we would necessarily perceive that these two parts of the month have a different length. But we do not perceive this difference, so this is the limit.

## 3.2 A possible solution

The hypothesis that we propose is in the line of Evans's proposal. It asserts that the moon's apparent size used by Aristarchus is the greatest possible value that still allows one to assert what proposition 4 asserts, i.e., the difference between the light line and the moon's diameter is imperceptible to the human eye.

But, how could Aristarchus know what portion of the moon is just beyond human perception? Aristarchus surely was aware that only in the instant of the new moon is the illuminated part of the moon totally opposite to the earth, but, after a negligibly brief time, a small part of the illuminated portion would be facing the earth. Nevertheless, we do not see the illuminated part of the moon just after (or before) the opposition. Actually, we do not see the moon during at least two consecutive nights, though of course there must be an illuminated part facing the earth on at least one of those nights and probably both. This means that the human eye cannot perceive any angle smaller than the angle of the illuminated part of the moon during around 24 h after and before new moon.

There is some textual evidence supporting the idea that Aristarchus wrote about this issue. In a difficult passage in a commentary written in the second century AD on book 20 of Homer's Odyssey, the anonymous author apparently quotes Aristarchus and says: "Aristarchus of Samos makes clear when he writes... Heraclitus (said that the moon) does not appear during (a period of) 3 consecutive days... [but] sometimes the moon changes sides in fewer days and sometimes in more" (Bowen and Goldstein 1994, 693). This text is important for two reasons: first of all shows that Aristarchus explicitly analyzed the duration of the moon's occultation and, second, Aristarchus says that the occultation could take fewer or more than three days. In our case, of course, it is the lower limit that is relevant, so, we know from this passage that for Aristarchus, the lower limit is smaller than three days. Of course, the text doesn't exclude one day, but it is plausible to suppose that Aristarchus is thinking of two and not just one day of occultation.

Of course, it could be objected that the reason for the invisibility of the moon is not just the size of the angle of the illuminated part facing the earth, but also—or even mainly—the proximity of the sun in apparent position: the moon would be so close to the sun after the conjunction that it will be lost in the glare of the sun.[4] We are not sure that Aristarchus would take this objection too seriously, for as the style of the entire treatise shows, he is dealing with geometrical problems, not with real observations. But, even if he did take this objection into account, we think that he would also have noted that Venus is visible at an elongation smaller than 12°. Actually, according to Ptolemy, the first visibility of Venus as an evening star is at 5 2/3° (Toomer 1998, 638), around half the elongation that the moon has to reach on a night after conjunction when it starts to be visible. So, the reason for the invisibility of the moon could not be the glare of the sun, but just the size of the illuminated part facing the earth.

Therefore, if Aristarchus wants to guarantee that an angle in the moon is imperceptible, he has to show that this angle is equal to or smaller than the angle that reaches the illuminated part of the moon when it is located one day apart from the sun. But, this angle depends not just on the elongation of the moon, but also on its apparent size. Therefore, the problem for Aristarchus is to find a value for the apparent size so that the angle of the difference between the light line and the lunar diameter that is parallel to it ($\beta$ from now on) is equal to or smaller than the angle of the illuminated part of the moon seen from the earth one day apart from the conjunction with the sun ($\alpha$ from now on).

Let us first calculate this value using modern trigonometry. Refer to Fig. 13. S is the center of the sun; M, the center of the moon; and O, the center of the earth; line IJ is the light line while line AB is the visibility line. Therefore, arc AJ is the illuminated portion of the moon visible from the earth. Let us call $k$ the fraction of the apparent lunar diameter that is visible from the earth, i.e., arc AJ/arc AB. According to Meeus (1991, 315–316), the illuminated fraction $k$ of the disk of the moon depends on the angle (SMO), centered at the moon, between the earth and the sun, according to this formula:

$$k = \frac{1 + \cos(\mathrm{SMO})}{2} \tag{1}$$

And the angle SMO depends on the geocentric elongation of moon ($\eta$), the earth–sun distance (SO) and the earth–moon distance (MO), according to the next formula:

$$\tan \mathrm{SMO} = \frac{\mathrm{SO} \sin \eta}{\mathrm{SO} - \mathrm{MO} \cos \eta} \tag{2}$$

So, as we already said, $k$ will give us the fraction of the apparent lunar diameter that is visible from the earth. Taking 19 as the proportion between the earth–sun and earth–moon distances and keeping constant, the elongation of the moon at 12.2°, $k$ will be 0.0125, i.e., the illuminated portion of the moon would be around 1.25 % of the total disk. If, for example, the moon's apparent diameter is 2°, then k would be around one and a half minutes. So $\alpha$, the angle of the apparent size of the illuminated part of the moon, is the angle of the apparent size of $k$, as seen from the earth. Then, of course, keeping $k$ constant, $\alpha$ depends only on the apparent size of the moon.

[4] I owe this objection to Dennis Duke.

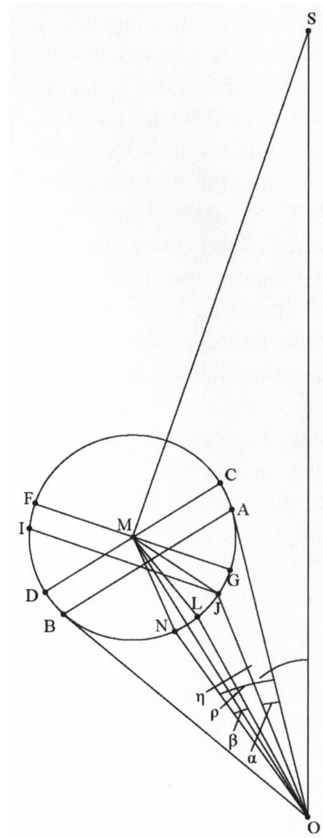



**Fig. 13** The moon (M) is far enough from the sun (S) to be visible from earth (O)

Now, we have to calculate $\beta$, i.e., the angle as seen from the earth of the difference between the light line and the lunar diameter parallel to it.

Refer again to Fig. 13. Aristarchus proved that angle CMA is equal to $\rho$, the apparent radius of the moon. And, by construction, CMA is equal to LMN. So, we also know that LMN $= \rho$. Therefore:

$$\tan\beta = \frac{\tan\rho}{\frac{1}{\sin\rho} - 1} \tag{3}$$

So, $\beta$ depends only on the apparent size of the moon. In the following graph, therefore, we plot both $\alpha$ and $\beta$ in relation to $\rho$. The graph shows that when $\rho$ is smaller than $1.4^\circ, \alpha > \beta$, but around $\rho = 1.4^\circ, \beta > \alpha$. This means that, only when the moon radius is smaller than $1.4^\circ$, angle $\beta$ will be unperceivable for the human eye (Fig. 14).

Of course, Aristarchus couldn't perform the same kind of calculation, for it involves trigonometry, which was not available to him. Aristarchus would instead calculate an upper limit for the moon's apparent radius that still guarantees that $\beta < \alpha$. So, he presumably found a value for $\rho$ smaller than $1.4^\circ$.

Actually, taking, when possible, the results obtained in the treatise and imitating his style when not, we can calculate the apparent radius of the moon that fulfills

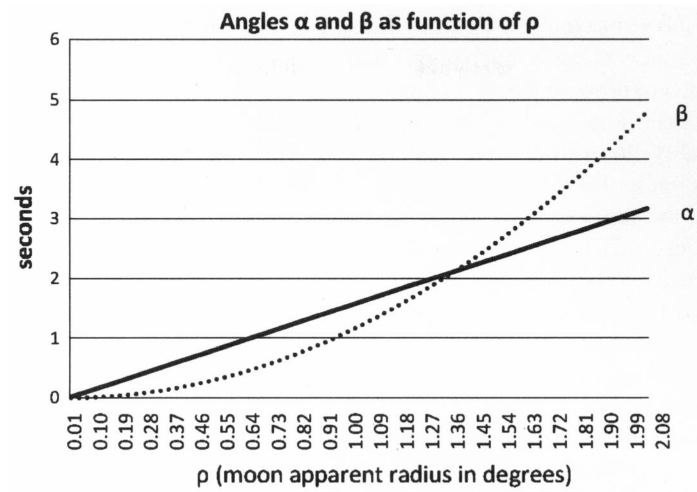



**Fig. 14** Angles $\alpha$ and $\beta$ as function of the moon's apparent radius

this requirement, imitating as closely as possible Aristarchus's way of proceeding. The value calculated in this way should be smaller than 1.09° (if the elongation of the moon for one day is 12.20°) or 1.018° (if the elongation is 12.19°). We offer the calculation in the appendix. So, it is totally reasonable that Aristarchus took 1° as the radius.

Consequently, according to this proposal, Aristarchus took the greatest value for the moon's apparent size that still guarantees that the difference between the dividing line and the diameter of the moon is imperceptible and so allows him the geometrical construction for the distance proportion calculations. If this is the case, it doesn't seem too odd that Aristarchus, probably later, discovered that the value is 0.5°, and so much smaller than the value used in the treatise, for here he is using the upper limit. (In "Appendix" we develop the calculation in detail).

## 3.3 The role of the hypotheses

Aristarchus's treatise constitutes the earliest surviving example of a work of empirical sciences applying mathematical method. This is the first example of a work in which the human mind used numerical data as input for obtaining, through the clever and rigorous application of geometry, accurate results related to facts as far from our daily experience as the solar and lunar distances or sizes. Therefore, it is very important to try to reveal which criteria and methods were used for selecting the input values.

If my proposal is correct—and Evans's too—the selection of at least two of the three input values was not based purely on crude or naïve observations, but neither was it absolutely arbitrary. The selection of these particular values was the result of the complex interaction of some qualitative observations and theoretical considerations.

The value of hypothesis 4 plays a central role in the determination of the ratio between the sun's and moon's distances from the earth. Presumably, Aristarchus wanted to show that the sun is much farther away than usually asserted and, consequently, to show that it is much bigger than it appears. In this way, Aristarchus could have intended to propose a new argument against of Epicurus's claims that the sun is "no bigger than it appears". Or perhaps, Aristarchus wanted to show that the moon is considerably closer to the earth than the sun. After all, Eudoxus model gives us no reason to suppose that there is any significant difference between the distances of the nearest and the farthest of the heavenly bodies from the earth, because the models assume no eccentricity. Remember that Archimedes asserts in *The Sand-Reckoner* that the distance to the sun was generally considered to be the radius of the Cosmos, which makes no sense unless the planets and stars are negligibly further from us than the sun. Before Aristarchus's argument, one might have said that the distance to the moon is effectively the radius of the Cosmos.[5] In any case, the bigger the elongation, the bigger the proportion between the distances. So, it would have been reasonable for Aristarchus to choose the smallest possible elongation in order to make his case, even in the worst situation. If the elongation is too small, however, the time from first quarter to third quarter would be perceivably different than the time from third quarter to first quarter. Given that we do not perceive any difference, which a difference of one day would be perceivable, and that with values smaller than 87°, the difference would be bigger than one day, the value must be equal to or bigger than 87°. Aristarchus was working with the smallest value. Thus, his value would be 87°. There is a theoretical consideration in the selection of the value (Aristarchus wants to show that the sun is farther away) that determines the direction of the limit of the value (it should be the smallest possible), and a very crude observation (that we do not perceive a difference between the time inter-quadratures) that constitutes the limit of this direction.

The value in hypothesis 6 plays a central role on the determination of the distance between the center of the moon and the line subtending the arc of the earth shadow. As we already have shown, this distance, in turn, affects very little the upper limit of the proportion between the sun's and earth's diameters and the lower limit of the proportion between the earth's and moon's diameters. Presumably, Aristarchus wanted to show that the influence of this value is absolutely irrelevant for the final results. The bigger the value, the bigger its influence. Thus, it would be reasonable to choose the biggest possible apparent lunar diameter in order to make his case even in the worst situation. If the apparent diameter of the moon is too big, then the difference between the light line and the diameter of the moon would not be imperceptible. We do not perceive any difference between the light line and the diameter of the moon. If we are right, with values bigger than 2° for the moon's apparent diameter, the difference would start to be perceptible. Therefore, the value must be smaller than 2°. But Aristarchus was looking at the biggest one, so it would be exactly 2°. There is a theoretical consideration in the selection of the value (Aristarchus wants to show that the distance between the center of the moon and the line that subtends the earth shadow arc is irrelevant) that determines the direction of the limit of the value (it should be, therefore, the biggest possible) and a very crude observation (that we do not perceive

[5] I owe this proposal to Alexander Jones.

that the illuminated part of the moon at a quadrature is smaller than half moon) that constitutes the limit of this direction. In this case, however, there is also a theoretical limit. If the two lines could not be considered to be at the same plane, Aristarchus could not build the diagram needed for proposition 7. Therefore, the limit could arise from a crude observation, theoretical necessity, or both.

The third empirical datum used in the hypotheses seems to be the exception. In fact, the value adopted by Aristarchus for the earth's shadow size seem not need any explanation, since it is a value close enough to the right one. One could guess that he obtained this value through some more or less simple observations. Of course, this could be the case. Nevertheless, if two of the three empirical values of the treatise have not been obtained only through observation, but also due to theoretical considerations, we can suspect that the third value was as well obtained through a similar process. The argument goes like this. The earth's shadow is observed in lunar eclipses. If the proportion between the earth's shadow and moon radius was 1, the earth's shadow and the moon would have had the same apparent size, just like the sun and the moon. This would imply that the moment of a total lunar eclipse would only be an instant, as happens with solar eclipses. However, unlike the case of solar eclipses, it is evident that the moon remains eclipsed for more than an instant. Thus, this could be considered a lower limit. The value could not be equal to or smaller than 1. At the other extreme, if the proportion was 3, the totality of the eclipse would last one entire night (12 h), since the earth's shadow would subtend 6° and it takes the moon 2 h to move 1° relative to the sun. This would mean that the moon could sometimes not be visible for an entire night at full moon. But we do not observe this. Thus, this could be an upper limit: the value could not be equal to or bigger than 3. Then, the value must be bigger than 1 and smaller than 3. Two seems to be a good candidate.[6] We can see the same style of reasoning in proposition 8, but now applied to solar eclipses. In this proposition, Aristarchus shows that the moon's and sun's apparent sizes are equal, discarding either possible inequality on the basis of observation. Since, he asserts, the sun is sometimes totally eclipsed (which would not happen if the apparent size of the sun were bigger than that of the moon), but the sun never remains entirely eclipsed

[6] To have a proportion of exactly 2 is also very convenient for the geometrical construction. In proposition 14, Aristarchus calculates the distance between the "straight line subtending the portion intercepted with the earth shadow of the arc of the circle in which the extremities of the diameter of the circle dividing the dark and the bright portions in the moon move" and the moon's center. Refer to Fig. 9. A fundamental step in the calculation is to establish the equivalence between $\Sigma\Xi/\mathrm{M}\mathrm{P}$ and $\Sigma\Lambda/\mathrm{P}\Gamma$, and for doing so, Aristarchus must assert that the triangles $\Lambda\Xi\Sigma$ and $\mathrm{MP}\Gamma$ are similar. In turn, for proving the triangles to be similar, $\Xi\Lambda$ must be perpendicular to BM. But this is true only in two cases: if $\Xi\Lambda$ is equal to MO or if $\Xi\Lambda$ is equal to $\mathrm{M}\Gamma$. The first case would only take place if the earth's shadow is exactly two moons. The second, if the earth shadow is exactly one moon, and therefore, the two triangles $\Lambda\Xi\Sigma$ and $\mathrm{MP}\Gamma$ would not be simply similar but identical. If the proportion value were not 1 or 2, however, the distance between these two points could be calculated by another method, totally within Aristarchus's reach. Look at the triangle $\mathrm{B}\Xi\Sigma$. One has a right angle at $\Sigma$ and one knows the angle at B, since it is the apparent radius of the earth's shadow. One can then calculate $\mathrm{B}\Sigma$ as a function of the hypotenuse, $\mathrm{B}\Xi$, using the cosine. We know that Aristarchus could perform geometrical constructions for arriving at a cosine (for this is what he does in proposition 7 when he calculates the cosine of 87°). But he wants the proportion between $\mathrm{B}\Sigma$ and $\mathrm{B}\Gamma$, not between $\mathrm{B}\Sigma$ and $\mathrm{B}\Xi$. The proportion between $\mathrm{B}\Gamma$ and $\mathrm{B}\Xi$, however, is easy to obtain. Aristarchus can calculate the relation between $\mathrm{B}\Gamma$ and $\Gamma\mathrm{P}$ using the similarity between triangles $\mathrm{BM}\Gamma$ and $\mathrm{PM}\Gamma$. $\Lambda\Gamma$ is half of $\Gamma\mathrm{P}$, so he knows $\Lambda\Gamma$ in function of $\mathrm{B}\Gamma$. $\mathrm{B}\Xi$ is equal to $\Lambda\Gamma-\mathrm{B}\Gamma$, so he can go from $\mathrm{B}\Xi$ to $\mathrm{B}\Gamma$.

for more than an instant (which would happen if the apparent size of the sun were smaller than that of the moon). "It [the sun] is in fact totally eclipsed and does not remain eclipsed: for this is manifest from observation". Therefore, both apparent sizes are equal.

Coming back to the analysis of the size of the earth's shadow at the distance of the moon, the size of the shadow reflects the size of the earth relative to those of the moon and sun: if the shadow is smaller, the earth's size would be closer to the moon's size; if bigger, it would be closer to the sun's size. We can guess that Aristarchus wanted to show that the sun is much bigger than the earth, in which case, again, he would have preferred the smallest value for making his case even in the worst situation. In this case, the limit would be the observation that the moon is totally eclipsed for a while and so the value should be the smallest possible value larger than one. If, on the contrary, Aristarchus wanted to show that the moon is not too small with respect to the earth, he would have preferred the biggest value, and in this case, the limit would be the observation that the moon is not occulted an entire night during an eclipse. Or maybe he was indifferent to any of these two possibilities, and he simply took the mean value between the upper and lower limits.

**Acknowledgments** I am very grateful to Dennis Duke, Alexander Jones, Nathan Sidoli, Lennart Berggren for their comments on previous versions of this paper and to Rodolfo Buzón because these ideas came out in the context of our joint translation of Aristarchus's treatise. I would also like to thank Ignacio Silva who helped me to improve the English of some paragraphs.

## Appendix: The calculation

In what follows, we reproduce the way Aristarchus could have calculated the value.

In the Fig. 15, S is the center of the sun; O, the center of the earth; and M, the center of the moon. So, angle SOM is the elongation of the moon ($\eta$). Line CD is a diameter of the moon perpendicular to line MO, which is the line that joins the center of the earth with the center of the moon. BA is parallel to but smaller than CD and represents the line that divides the visible from the invisible portion of the moon, seen from the earth. Therefore, angle AOB is the moon's apparent diameter and AOM the moon's apparent radius ($\rho$).

Line FG is a diameter of the moon perpendicular to the line MS, which is the line that joins the center of the sun with the center of the moon. IJ is parallel to but smaller than FG and represents the line that divides the illuminated from the obscure portion of the moon. Therefore, arc IFGJ is the illuminated portion of the moon, a portion a bit bigger than half the moon, facing the sun. And arc AJ is the illuminated portion of the moon that is also visible from the earth. Consequently, angle AOJ represents the apparent size of the illuminated part of the moon visible from the earth ($\alpha$). If this angle $\alpha$ is perceptible for the human eye, then the moon would be visible, if not, not.

Arc CA in the lunar sphere is the difference between the moon's visibility line (AB) and the moon's diameter (CD). In proposition 4, Aristarchus asks us to move this arc in order for it to be in front of the earth, i.e., to arc LN. So, by construction, arc CA is equal to arc LN. Aristarchus also shows that angle CMA is equal to angle MOA, i.e., to $\rho$, the moon's apparent radius. So, angle LMN also would be equal to moon's

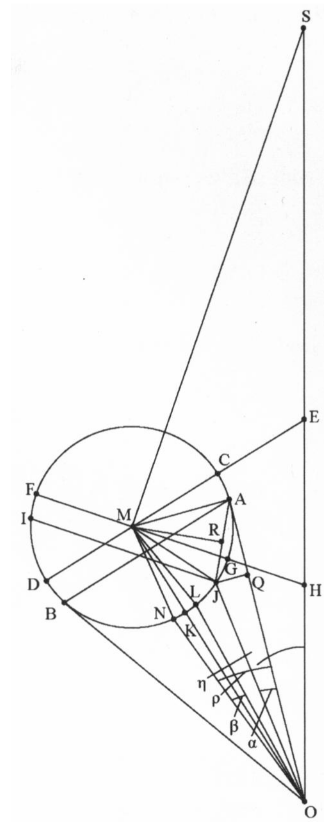



**Fig. 15** The moon (M) is far enough from the sun (S) to be visible from earth (O)

apparent radius. Consequently, angle LON is the angle that represents the apparent size of the difference between the light line and the apparent diameter. Aristarchus shows that when the moon's diameter is 2°, this angle is smaller than 1/44 of a degree (or, as he says, 1/3,960 of a right angle). Let us call angle LON $\beta$.

We know that the moon is not visible for at least two nights, i.e., at an elongation of at least 12.2° from conjunction. So, if the figure represents an elongation of 12.2° degrees ($\eta = 12.2°$), then angle $\beta$ must be equal or smaller than angle $\alpha$, in order to be sure that it is imperceptible to the human eye. Therefore, the limit of perceptibility for $\beta$ is when $\alpha = \beta$. But Aristarchus works with inequalities, so we have to find a $\rho$ for which *the greatest possible b is still smaller than the smallest possible a.*

We are looking for the greatest possible $\rho$, for which the difference between the light line and the diameter is still imperceptible, so we have to find the upper limit of $\rho$, i.e., *a value smaller than the* $\rho$ *for which the greatest possible* $\beta$ *is still smaller than the smallest possible* $\alpha$.

In the figure, line JQ is perpendicular to AO. So, we know (see Berggren and Sidoli 2007, 224) that:

$$\frac{\mathrm{OQ}}{\mathrm{AQ}} > \frac{\mathrm{JAO}}{\alpha} \tag{4}$$

We also know from Aristarchus (proposition 4), leaving $\rho$ variable that:

$$\beta < \frac{\rho}{(45 - \rho)} \tag{5}$$

From (4), we know that:

$$\alpha > \frac{\mathrm{JAO} \cdot \mathrm{AQ}}{\mathrm{JQ}} \tag{6}$$

Now, we want that:

$$\beta \leq \alpha \tag{7}$$

But, because $\frac{\rho}{(45-\rho)}$ is bigger than $\beta$ and $\frac{\mathrm{JAO} \cdot \mathrm{AQ}}{\mathrm{JQ}}$ is smaller than $\alpha$, we may be sure that $\beta \leqq \alpha$ if:

$$\frac{\rho}{(45 - \rho)} \leq \frac{\mathrm{JAO} \cdot \mathrm{AQ}}{\mathrm{OQ}} \tag{8}$$

In what follows, we will first calculate angle JAO, then lines AQ and OQ.

Angle JAO

We know that angle MAO is right, for AO is tangent to the moon's circle. So:

$$\mathrm{JAO} = 90 - \mathrm{MAJ} \tag{9}$$

Now, let us look at triangle MAJ. Sides MA and MJ are equal and equal to the moon's radius ($r$, from now on). So, triangle MAJ is isosceles and angles MAJ and MJA are equal. Therefore:

$$\mathrm{MAJ} = 90 - \frac{\mathrm{AMJ}}{2} \tag{10}$$

Now, from (9) and (10), we have:

$$\mathrm{JAO} = \frac{\mathrm{AMJ}}{2} \tag{11}$$

So, we have to calculate angle AMJ. But we also know that,

$$\mathrm{AMJ} = \mathrm{CMG} \tag{12}$$

because CD and BA are parallel, as well as FG and IJ and the distance between the parallel lines is equal.[7] So, let us produce line MC until line SO, and call E the point of intersection. Also, let us produce line MG again to meet line SO and call H the point of intersection. Now, we look at triangle EMH. The angle at M is angle CMG, which is, of course:

$$\mathrm{EMH} = 180 - \mathrm{MEH} - \mathrm{EHM} \tag{13}$$

Now, see triangle EMO. Angle MEO is MEH. But the angle at M is right (because ME and MO are perpendicular) and angle at O is the lunar elongation ($\eta$), so:

$$\mathrm{MEH} = \mathrm{MEO} = 90 - \eta \tag{14}$$

And something similar happens with angle SHM, regarding triangle OMH. Angle at M is right (because MH is perpendicular to MS) and angle at S (MSO) is the elongation of the moon from the earth seen from the sun. So:

$$\mathrm{MHS} = \mathrm{EHM} = 90 - \mathrm{MSH} \tag{15}$$

But, because SM represents the sun–moon distance and MO the earth–moon distance, we can establish a relationship between SM and MO. We know that if the moon were at line OS (at conjunction), then the proportion would be smaller than 19 but bigger than 17 (Because SM = SO – OM). Now, the closer the sun, the bigger the angle AMJ and, therefore, the bigger the angle $\alpha$. But, we are looking for the smallest possible $\alpha$, so, we should choose the furthest distance from the sun, i.e., the value of 19. Now, at the position illustrated in the figure, OM is greater than OM at conjunction, but the difference is so small (being the angle between them a bit bigger than half a degree) that they could be considered equal. So:

$$\frac{\mathrm{SM}}{\mathrm{MO}} > \frac{\eta}{\mathrm{MSO}} \tag{16}$$

Therefore:

$$\mathrm{MSO} < \frac{\mathrm{MO}}{\mathrm{SM}} \cdot \eta \tag{17}$$

And, because MO/SM is 1/19:

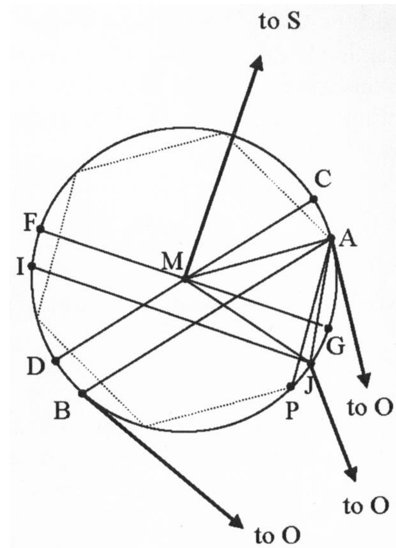
$$\mathrm{MSO} < \frac{\eta}{19} \tag{18}$$

Finally, from (13), (14), (15) and (18), we have that:

$$\mathrm{AMJ} = \eta + \frac{\eta}{19} = \frac{20}{19}\eta \tag{19}$$

[7] Actually, the distance between AB and CD should be a bit bigger than the distance between FG and IJ, because they are exactly the same only at conjunction. But the difference is absolutely negligible.

**Fig. 16** A hexagon inscribed in the moon sphere




Now, from (11) and (19), we know that:

$$\mathrm{JAO} = \frac{10}{19}\eta \tag{20}$$

Line AQ

In order to obtain line AQ, we first have to find LA.

Let us draw a hexagon inside the moon's circle. Refer to Fig. 16. Line AP is a side of the hexagon and so, it is equal to the radius of the circle, $r$. So, we know that:

$$\frac{\widehat{\mathrm{AP}}}{\widehat{\mathrm{AJ}}} > \frac{r}{\mathrm{AJ}} \tag{21}$$

From here, we can infer that:

$$\mathrm{AJ} > \frac{\widehat{\mathrm{JA}} \cdot r}{\widehat{\mathrm{AP}}} \tag{22}$$

If JA is smaller, then, AQ would also be smaller, and a value smaller than AQ will still keep the inequality of Eq. 6. for the final value will be even smaller than $\alpha$.

But arc AP is 60° and arc AJ is angle AMJ, and so, $\frac{20}{19}\eta$ (Eq. 19), therefore:

$$\mathrm{JA} > \frac{20}{60 \cdot 19}\eta \tag{23}$$

Refer again to Fig. 13. Now, draw a line from M to the middle of AJ, bisecting triangle MAJ. Angle AMR would be half of AMJ, but we know that also angle JAO is half of AMJ (Eq. 11). So AMR = JAQ. In triangle JAQ, we know that angle AQJ is right, as is also angle MRA in triangle MRA, so triangle MAR is similar to triangle JAQ. Therefore, we can establish the following proportion between their sides:

$$\frac{\mathrm{MA}}{\mathrm{MR}} = \frac{\mathrm{AJ}}{\mathrm{AQ}} \tag{24}$$

Now, we know that RA is half of JA, so:

$$\mathrm{MA}^2 = \mathrm{MR}^2 + \left(\frac{\mathrm{JA}}{2}\right)^2 \tag{25}$$

But, because MA is $r$:

$$\mathrm{MR} = \sqrt{r^2 - \left(\frac{\mathrm{JA}}{2}\right)^2} \tag{26}$$

It follows, from Eqs. (24) and (26), that:

$$\mathrm{AQ} = \frac{\sqrt{r^2 - \left(\frac{\mathrm{JA}}{2}\right)^2} \cdot \mathrm{JA}}{r} \tag{27}$$

or:

$$\mathrm{AQ} = \sqrt{1 - \left(\frac{\mathrm{JA}}{2}\right)^2} \cdot \mathrm{JA} \tag{28}$$

Line OQ

If we want to keep the inequality of Eq. (6), we need a value equal or bigger than OQ.

We know that OQ is a bit smaller than OJ, since OJ is the hypotenuse in triangle JQO. And we know that OJ is between OM and OM-$r$. We need, again, the upper limit, for it is obvious from the figure that OQ is bigger than OM-$r$ = OK.

In proposition 11, Aristarchus calculated that the relationship between OM and r is smaller than 60. Now, if AO is smaller than AB, would also be smaller than 60. Actually, in this calculation, $\rho$ has been taken as 1°; if we keep $\rho$ as a variable, we have

$$\mathrm{AB} < \frac{60}{\rho} \tag{29}$$

So, we have also:

$$\mathrm{OQ} < \frac{60}{\rho} \tag{30}$$

Finally, from (8) and (29) we have:

$$\frac{\rho}{(45-\rho)} \leq \frac{\mathrm{JAO} \cdot \mathrm{AQ}}{\frac{60}{\rho}} \tag{31}$$

Consequently:

$$\rho \leq \frac{45 \cdot \mathrm{JAO} \cdot \mathrm{AQ} - 60}{\mathrm{JAO} \cdot \mathrm{AQ}} \tag{32}$$

And, from (20), (23) and (28):

$$\mathrm{JAO} \cdot \mathrm{AQ} = \frac{10}{19}\eta \cdot \sqrt{1 - \left(\frac{20}{60 \cdot 19}\eta\right)^2} \cdot \frac{20}{60 \cdot 19}\eta = 1.3664\ldots \tag{33}$$

Being $\eta = 12.2$ (and $r = 1$), $\mathrm{JAO} \cdot \mathrm{JA} = 1.3664\ldots$, so:

$$\rho \leq \frac{45 \cdot 1.3664 - 60}{1.3664} = \frac{1.4897}{1.3664} = 1.09 \tag{34}$$

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