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Integral equations between theory and practice: the cases of Italy and France to 1920

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Abstract In 1899, Ivar Fredholm discovered how to treat an integral equation using conceptual methods from linear algebra and use these ideas to solve certain classes of boundary value problems. He formulated a theory allowing him both to unify large classes of problems and to attack several problems fruitfully. The historical literature on the theory of integral equations has concentrated largely on the unification that was afforded by Hilbert and his school, but has not throughly investigated the roots of the subject in the older theory of partial differential equations, as developed for instance by Fredholm himself but also by Volterra and Levi-Civita. By concentrating on work issuing from this older tradition, in particular on French and Italian work, the paper shows how the new theory of integral equations was enthusiastically received, especially for its fruitful applications to areas of mathematical physics such as hydrodynamics, elasticity, and heat theory.

> La théorie des équations intégrales, née d'hier, est d'ores et déjà classique. Elle a fait son entrée dans plusieurs de nos enseignements. Nul doute que—peut-être à

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la faveur de nouveaux perfectionnements—elle ne s'impose bientôt à la pratique courante de calcul. C'est une fortune rare parmi les doctrines mathématiques, si souvent destinées à rester des objets de musée.
J. Hadamard, Préface to (Heywood and Fréchet 1912, v).

1 Introduction and background

1.1 Integral equations, functional analysis, and boundary value problems

For the mathematics student of today, the 1900 result of Ivar Fredholm will typically appear in a fairly advanced introduction to analysis, in a chapter on compact operators, for example as a corollary to Atkinson's theorem.[1] Fredholm's more limited result, in modern language, states roughly that the spectrum of a compact operator T on a Hilbert space consists of $\{0\}$ and the eigenvalues for T, and is a countable subset of the complex plane with 0 as the only possible accumulation point (Pedersen 1988, 111). This tidy description obscures the pivotal position of this theorem in early twentieth-century analysis. The theorem was conceived as a contribution to the theory of functional equations involving integrals, though the author, Ivar Fredholm, saw at once its usefulness in demonstrating the existence of solutions to certain boundary value problems involving partial differential equations. In this paper, we explore the context in which the theorem was developed, and discuss its reception.

This approach contrasts with the context in which this work has often been seen by mathematicians and historians until recently. Already in the *Enzyklopädie der mathematischen Wissenschaften II, 3*, for example, in the 1927 article on integral equations by Hellinger and Toeplitz (1927), all of this work was placed in the light of subsequent developments in abstract functional analysis. The various traditions in the treatment of concrete differential equations have thus been somewhat submerged by the story of functional analysis. Hellinger and Toeplitz were in fact quite historically ambitious, pushing the antecedents of the operator approach back beyond Fredholm to Poisson, Fourier, and Daniel Bernoulli. The survey due to Hadamard at the Bologna congress of 1928 (Hadamard 1928) appears to be the only older mathematical survey in which some trace of the roots of the subject in solving boundary value problems may be found, not surprising given the role he played himself.[2]

The solution of such boundary value problems, and the question of the existence of solutions, occupied a large number of mathematicians throughout Europe at the time, and the techniques provided by Fredholm's result garnered an enthusiastic audience. Indeed, the "integral equation method" for the study of differential equations became a standard feature of the mathematical landscape by around 1915, with the appearance of several textbooks and expository accounts of the theory as well as its inclusion in lecture courses on analysis. Hilbert's response to the theorem was deeper and had

[1] Atkinson's theorem (due to F. V. Atkinson in 1950) states that an operator T in the set of bounded operators on a Hilbert space H is a Fredholm operator (that is, the kernels of T and its adjoint are finite dimensional and its range is closed) if and only if it is invertible modulo compact perturbation, i.e., for some bounded operator S and compact operators C_1 and C_2, $TS = I + C1$ and $ST = I + C2$.

[2] Mathematical treatments of the subject of integral equations nonetheless often retain contact with the roots of the subject in partial differential equations. Here, we mention in particular (Tricomi 1957).

broader consequences. By 1904, Hilbert had already grasped analogies between the study of certain cases of Fredholm's result and the theory of quadratic forms. Together with a number of students, notably Erhard Schmidt, Hilbert formulated the basic ideas of what are now know as Hilbert spaces and linear operators on them with decisive effect for the future of mathematics (See for example Hellinger, 1935). The impact of Hilbert's work took several decades to be fully felt. While other threads fed into Hilbert's understanding of this area, the insight afforded by the Fredholm's result provides the reader of today with one of the clearest points of entry into the study of the origins of functional analysis.

There are a number of historical treatments which investigate this and related issues in some detail, including Dieudonné (1981) and Siegmund-Schultze (1982). In particular, Siegmund-Schultze (2003) draws attention to what he terms a split into three directions in the beginnings of functional analysis: abstract operator theory, the abstract theory of spaces, and the theory of concrete functionals. Here, we focus instead on the reception, largely Franco-Italian, that remains closer to roots of the problems in mathematical physics, adopting more modern viewpoints somewhat tentatively.

Fredholm's work had been preceded by related investigations due to Vito Volterra, who appears to have devised essentially the same method by 1895, and already much earlier noted that methods for "inverting" integral equations—following an idea of Abel—would be extremely useful in the solution of physical problems. Volterra corresponded with Tullio Levi-Civita about this and was somewhat disappointed by the fact that he had not published the method before Fredholm's work appeared.

In this paper, we concentrate on the response to Fredholm's work within the community of researchers on differential equations in France and Italy, where the results were understood above all as a fruitful and powerful method for proving the existence of solutions to boundary value problems. In contrast to the work of Hilbert and his students, this work appears conservative and lacks the abstract, generalizing stamp which was to become a hallmark of twentieth-century mathematics and that is specifically associated with the German context. An exception to this conservative bent is provided by the Italian mathematician Giuseppe Lauricella, who skillfully adopted the new methods coming from some of the strongest members of Hilbert's school: Erhard Schmidt, Friedrich Riesz, and Hermann Weyl. The reception of Fredholm's work nonetheless marks an important moment in the development of research in analysis in these two countries, as we shall discuss below. Furthermore, the case provides insight into the ways in which innovative work is received in different national and institutional contexts.

This work is part of the background leading to the development of functional analysis as a free-standing entity and clearly identified research specialty. However, it should not be imagined that all these researchers were consciously involved in the construction of such a research specialty, nor were they intentionally carrying out specific elements of what were to be its later research programs. It is of course true that ultimately one can see specific results from this period as special cases of functional analytic results. But, there was nothing called functional analysis at this point of time (all our discussions are limited to the period before 1915).

In what follows, we begin with an account of some background developments in the theory of differential equations. We then proceed to a discussion of Fredholm's result and various aspects of its reception. The work of Hilbert and his school will be treated in the background, insofar as it provided points of reflection for researchers in France and Italy. Many of the papers in this general area were very definitely seen in an applied context by most readers, as is shown for example by their treatment by the *Jahrbuch über die Fortschritte der Mathematik*, where they were routinely indexed with work in partial differential equations, elasticity theory, or potential theory as well as under functional equations.

Italian researchers in this field were particularly numerous, and their contributions, like those of many of their conationals in other fields, are less well known today than those of their German or French contemporaries. The emergence of the Italian research school on boundary value problems may be traced to the mid-nineteenth century, with Enrico Betti and Eugenio Beltrami acting as mentors to a large number of future researchers in areas such as elasticity theory, hydrodynamics, and the theory and applications of partial differential equations more generally. By the late nineteenth-century, Levi-Civita and Volterra had emerged in a leadership role, and most of the Italian mathematicians we discuss had specific links to one or both of these men, as we shall see. Even a certain amount of the French work was directly connected with developments in Italy; here, we note the strong Italian connections of Picard and Hadamard, and Volterra's close involvement with these and other French colleagues.[3]

One of the significant contributions of the Italian group was a work related to the vibration of elastic plates done by Tommaso Boggio, in the context of the 1906–1907 competition for the Prix Vaillant of the Paris Académie des Sciences. This work remained unpublished, and we provide here in an "Appendix" some of the more significant portions of the memoir that are related directly to our story.

1.2 Background: the Dirichlet problem for the Laplace equation

Integral equations are, in a sense, nearly as old as integrals. However, for our purposes, they are a nineteenth-century development. Early work by Abel and Liouville has been described well in (Lützen 1990). Some immediate background activity related to our discussion was due to Carl Neumann and Poincaré in one direction, to Picard in another, and to Volterra.

In order to introduce the different approaches to the theory of integral equations, we go back to the Dirichlet problem, a central problem in nineteenth-century mathematical analysis. The Dirichlet problem can be reduced to a special class of integral equations and, under appropriate assumptions, solved by using the methods of Fredholm's theory.

Peter Gustav Lejeune Dirichlet stated the following problem in his lectures on potential theory held in 1856–1857, to find a harmonic function in a closed region with values given continuously on the boundary of the region. A function f is harmonic in a domain R if f satisfies the Laplace equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

3 See Mazliak and Tazzioli (2009).

at the interior points of R. This question, which later came to be called a Dirichlet problem, plays an important role in potential theory, and hence in many questions involving gravitational or electromagnetic forces.[4] Several authors, including Karl Weierstrass, Leopold Kronecker, Carl Neumann, Felice Casorati, Heinrich Weber, and Hermann Amandus Schwarz, sought to solve the problem in a general and rigorous way. In 1870, Schwarz developed his own method—called the alternating method—of solving the Dirichlet problem for special boundaries and for restricted conditions on the given boundary functions.[5] Also in the 1870s, Carl Neumann devised a method for solving the Dirichlet problem for the Laplace equation which was to be of considerable importance for many writers, notably Poincaré and Fredholm. Neumann devised his so-called method of the arithmetic mean to permit series computation of a solution to the Dirichlet problem. The rather complicated method he employed involves constructing a function at each interior point of a bounded region defined as a kind of "mean effect" of the boundary values at the point in question. An iterative procedure is devised to replace the initial boundary value by the mean and then computing the new "mean" once again.[6]

1.3 Picard and Poincaré

Of more immediate relevance to Fredholm's work is work of Picard and Poincaré in the 1890s.

Emile Picard became involved in research in differential equations early in his career, largely through his contact with his mentor Charles Hermite. Following in Hermite's footsteps, and influenced by contemporary French work by writers such as Tannery, Flocquet, and Poincaré, Picard specialized at first in questions concerning what can be said about the nature of the solutions of a differential equation based on formal characteristics of the equation (such as the periodicity of the coefficients). By the 1880s, this had evolved into a general interest in existence issues, and around 1887 or 1888, he focussed his attention on existence results of Schwarz and Carl Neumann from the 1870s. The work of Schwarz and Neumann was intended to provide solutions to the Dirichlet problem, partly to replace Riemann's use of the Dirichlet principle. Both were iterative series methods, and it may be that this inspired Picard to devise his method of successive approximations, a constructive existence method for proving existence of solutions, widely applicable to both ordinary and partial differential equations.

As a trivial example, consider:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g\,(x(t), x(0))$$

[4] Dirichlet's problem and its role in nineteenth-century mathematics figure frequently in the literature. See Bottazzini (1986), Monna (1975), Tazzioli (2001).

[5] For more details see Tazzioli (1994).

[6] More details about Neumann's method are in (Dieudonné 1981, 39–46).

Let $x_{n+1}(t) = \int_0^t g(\tau, x_n(\tau))\mathrm{d}\tau$. If the sequence $x_n(t), n = 1, 2, \ldots$ converges, it converges to a local solution, which under certain conditions may be extended to a global solution. In this case, the series that results is easily identifiable.

Picard established conditions for local convergence in the case of the 2-dimensional Laplace equation and used Schwarz's procedure to assemble local solutions, proving global existence. In so doing, he shows his mastery of the Schwarz–Weierstrass language for analysis. These results were the most powerful existence methods available at the time and rapidly became part of the standard repertoire. As Lützen has pointed out (Lützen 1990), the basic method was already known to Liouville, but Picard's discovery appears thoroughly independent.

Simultaneously with Picard's efforts, Henri Poincaré made fundamental contributions to the field of partial differential equations which were of immediate and long-term consequence.[7] His interest in the field dated already to his doctoral thesis, on the functions defined by partial differential equations, a difficult work which was received without much understanding. In two astonishingly rich papers of 1890 and 1894, he created a variety of tools and approaches which have had tremendous influence (Poincaré 1890, 1894).

In 1890, he gave the first complete proof of the existence and uniqueness of solutions to the Laplace equation with continuous boundary conditions, for a large class of three-dimensional regions. Where Neumann had defined a sequence of functions satisfying the Laplace equation, converging to one with the correct boundary condition, Poincaré instead employed a sequence of functions which are not harmonic, but have the right boundary values, and devised a method to make the sequence converge to a harmonic function via the method of "balayage" (sweeping out). He first showed that if such a Dirichlet problem for the Laplace equation can be solved when the values on the boundary are given by a polynomial in three variables, it can be solved when they are given by any continuous function. To solve the problem when the boundary value is a polynomial p, Poincaré defined a countable covering of the interior of the region by spheres $S_1, S_2, \ldots$, and used the known solution for the Dirichlet problem on a sphere to replace p by the harmonic function f given by this solution. A new function f_1 is now defined, equal to f inside the first sphere, and equal to p elsewhere. Proceeding to the second sphere, we likewise "sweep" it by solving the Dirichlet problem, likewise getting a function f_2 which satisfies the Laplace equation inside S_2 and is equal to p elsewhere. We now need to go back to the first sphere, so we continue this process in the order $S_1, S_2, S_1, S_2, S_3, \ldots$, passing through each sphere infinitely often while retaining the boundary values for the region. Poincaré was able to show that this process leads to a function with the correct boundary values which is harmonic in the entire interior of the sphere.

In the same paper of 1890, Poincaré began to look at eigenvalues. H. A. Schwarz and Picard had found the first and second eigenvalues of the Laplace operator for Dirichlet boundary conditions in 1885 and 1893, respectively (Schwarz 1885; Picard 1893). Poincaré, in 1894, found the infinite sequence of eigenvalues and their corresponding

[7] For studies of Poincaré's work on partial differential equations, see for example Mawhin (2006/2010) and Gray (2013).

eigenfunctions, to be identified later in terms of singular values as we discuss below, probably soon after reading of Schwarz's work in Picard's paper. This is the beginning of spectral theory, a fundamental tool of functional analysis in the twentieth century. A fine account of this work is to be found in (Dieudonné 1981).

We mention two other innovations due to Poincaré. One of these is the so-called continuity method. In 1898, Poincaré had used Picard's successive approximation method to obtain a solution for the equation $\Delta u = e^u$. Here, he had the idea of approaching the solution of other nonlinear equations by starting with a problem with known solution and then continuing the existence result along a parameter to a more complicated equation. This method was to some degree foreshadowed in the 1890 paper, as was a second important tool, that of the a priori estimate of which good use was soon to be made by Sergei Bernstein in his work on the Hilbert problems 19 and 20 (Bernstein 1904).

Poincaré also built on Carl Neumann's work in a way which is part of the immediate background to Fredholm, Hilbert, and Picard, in particular with two papers (Poincaré 1894, 1897). In these papers, the notions of eigenvalues and eigenfunctions for a particular problem are introduced and termed "valeurs fondamentales" and "fonctions fondamentales." Poincaré obtains improved hypotheses over Neumann, for example getting rid of convexity. The beginning of the paper is scrupulously rigorous, but changes gears in the middle, where he states "Jusqu'ici j'ai cherché à être parfaitement rigoureuse," and then goes on to discuss in non-rigorous terms the fact that one can construct an infinite series of real eigenvalues (parameter values for which there is a solution to a DE expressed as an integral using Neumann's method). The use of a parameter we will see below in Fredholm's work. Poincaré notes the possibility of eigenfunction expansions, but cannot prove it, remarking: "une fois que l'on connaîtrait les fonctions fondamentales, il serait aisé de résoudre le problème de Dirichlet". The entire approach is not infrequently referred to as the "Poincaré–Neumann method," as we will see below.

1.4 Volterra

If Picard and Poincaré were working on elements of the theory of boundary value problems that were to be important for the application of integral equations, Vito Volterra explicitly studies integral equations and was already using them in some contexts for exactly such research. Though Fredholm was the first to publish a theoretical justification of a method for solving integral equations, Volterra had already treated such questions in print some years previously. Early in his career, in an 1884 paper on electrostatics, he had investigated the problem of "inverting" an integral of the form

$$\phi(x) = \int_0^a f(\alpha)F(\alpha, x)\mathrm{d}\alpha$$

to solve for f when ϕ and F are known, and F depends also on α (Volterra 1884). Over 10 years later, Volterra's colleague Tullio Levi-Civita posed the question more generally:

> In some research in pure analysis and in a great many problems of physics and mechanics it is useful to invert some definite integral. We can even affirm that there is no branch of physics in which difficulties of this kind are not encountered.[8] (Levi-Civita 1895–96, 159)

The interest of Levi-Civita spurred Volterra to return to this area. In a series of papers written in 1896, he investigates the more general problem and presented a number of results of the following general kind (Volterra 1896a,b,c). If we have the functional equation

$$f(y) - f(\alpha) = \int_{\alpha}^{y} \phi(x) H(x, y) \mathrm{d}x$$

then under certain hypotheses on the functions and the region then exactly one continuous function exists ϕ such that the given equation is satisfied. This is given by the formula

$$\phi(y) = \frac{f'(y)}{H(y, y)} - \frac{1}{H(y, y)} \int_{\alpha}^{y} f'(x) \sum_{i=0}^{\infty} S_i(x, y) \mathrm{d}x$$

where

$$S_i(x, y) = \int_{y}^{x} S_0(\xi, y) S_{i-1}(x, \xi) \mathrm{d}\xi$$

$$S_0(x, y) = \frac{\partial H / \partial y(x, y)}{H(x, x)}.$$

Volterra did not indicate how he had obtained this expression, but merely proved that it satisfied the original equation. He extended these results to more general cases, including a case involving multiple integrals.

Volterra was to claim later, very credibly, that he was in possession of the method of Fredholm and that this was more or less exactly what he had used to get the solution. In fact, Volterra (1895) had used a determinantal method on a finite system of differential equations, so clearly the method could have been in his mind. Furthermore as Tricomi remarks:

> Volterra, instead of deducing his results by the same methods he used for their discovery (which were identical to those employed later so successfully by Fredholm), simply published a verification of his solution. This was told to me by Volterra himself when, in 1923–1924, I lectured for the first time on integral equations at the University of Rome (Tricomi 1957).

[8] In alcune ricerche di analisi pura e in moltissimi problemi de fisica e di meccanica fa d'uopo invertire qualche integrale definito. Si può anzi affermare che non v'è ramo della della fisica, in cui non si incontrino difficoltà di questa natura.

Volterra was to work further on related matters before Fredholm's publication. This was connected with later work by his students Lauricella and Almansi, a point we return to later. While many of his students were to work in this area in the immediately ensuing period, Volterra himself returned to it only later.

2 Fredholm

This brings us to Fredholm.[9]

Fredholm's letter of Aug. 8, 1899 to Mittag–Leffler announced a method for solving integral equations, seen as functional equations (Fredholm 1899, 1900). Fredholm gave an immediate application to the proof of existence theorems for the solution of boundary value problems. This was originally published 1900 in Swedish, but communicated to Poincaré[10] already in Dec. 1899. Fredholm had also lectured on it at the Paris congress of 1900, though only the title is noted in the proceedings. A French summary was published 1902 in *Comptes Rendus*,[11] with the full version appearing in 1903 in *Acta Mathematica*, in a volume in honor of Abel (Fredholm 1903).

Fredholm noted that Neumann's "double-layer" method had shown how the solution to the Dirichlet problem for the Laplace equation in two or three dimensions could be expressed as an integral, which Neumann could then find using series. Poincaré had extended Neumann's method (1894, 1896), improving the hypotheses.

Fredholm in turn considered the functional equation

$$\phi(x) + \lambda \int_0^1 f(x, y)\phi(y)\mathrm{d}y = \psi(x)$$

where we are solving for ϕ. The resemblance to the problems considered by Neumann and Poincaré is obvious—the λ is Poincaré's parameter.

Fredholm noted:

> Most problems of mathematical physics which lead to linear differential equations are translated into functional equations [of this form, possibly with more variables].

Fredholm's basic insight consisted of the following. In the equation

$$\phi(x) + \lambda \int_0^1 f(x, y)\phi(y)\mathrm{d}y = \psi(x)$$

we may consider the analogy with a system of linear equations with ϕ as the variable. One can then get a kind of analogy with Cramer's rule, with "determinants" and "minors" expressed as series expansions in the parameter λ, the coefficients involving

9 For a biographical sketch of Fredholm, see (Zeilon 1930)

10 See Nabonnand (1999).

11 For more details on this result see Tricomi (1957).

functional determinants of the kernel $f(x, y)$. The expression is easily seen to formally satisfy the integral equation; convergence of the series is guaranteed by an 1893 result of Hadamard (1893) (rediscovered independently by Fredholm). The singularities of the determinantal expression are what we would now term the eigenvalues of the corresponding boundary value problem. In this regard, we note that while Fredholm understood many properties which we now interpret in a linear algebraic way, this viewpoint is not central to his mode of addressing the subject, nor is it used systematically.

In Fredholm (1902), began by noting that, in the two-variable case, one can write many problems of mathematical physics in the form

$$\phi(x) + \int_0^1 f(x, y)\phi(y)\mathrm{d}y = \psi(x), \quad 0 \leq x \leq 1 \tag{2.1}$$

and that the left side may be denoted $A_f\phi(x)$ for brevity. The idea of an integral as a transformation was not new, but we note its explicit use here. He then further explicitly remarked that the Eq. (2.1) is "a limiting case of the theory of linear equations," in which we have "all the results of the theory of determinants." (Fredholm 1902, 219). Both the analogy with linear equations and the concept of determinant employed in the infinitary setting are left implicit by Fredholm.

Fredholm then considered (without stating it clearly) a partition of the unit square given by

$$0 < x_1 < x_2 < \ldots < x_n \leq 1, \quad 0 < y_1 < y_2 < \ldots < y_n \leq 1$$

and denoted the determinant of the n^2 quantities $f(x_i, y_k)$ by

$$f\begin{pmatrix} x_1\ x_2 \ldots x_n \\ y_1\ y_2 \ldots y_n \end{pmatrix}.$$

He defined the expression he will use as the determinant for the integral equation as

$$D_f = \sum_{n=1}^{\infty} \frac{1}{n!} \int \int \ldots \int f\begin{pmatrix} x_1\ x_2 \ldots x_n \\ y_1\ y_2 \ldots y_n \end{pmatrix} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n,$$

which is defined to be equal to 1 for $n = 0$. He then further defined kth order minors of this expression,

$$D_f\begin{pmatrix} \xi_1\ \xi_2 \ldots \xi_n \\ \eta_1\ \eta_2 \ldots \eta_n \end{pmatrix}$$
$$= \sum_{n=1}^{\infty} \frac{1}{k!} \int \int \ldots \int f\begin{pmatrix} \xi_1\ \xi_2 \ldots \xi_n,\ x_1 \ldots x_n \\ \eta_1\ \eta_2 \ldots \underset{n}{\eta},\ y_1 \ldots y_n \end{pmatrix} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n.$$

All integrals in these expressions are taken from 0 to 1.

If we can accept for a moment that these expressions actually correspond to the determinant and minors of a linear system and that the series converge, then the next statement by Fredholm is unsurprising. Calling on Cramer's rule, if $D_f \neq 0$, we get as unique solution

$$\phi(x) = A_g \psi(x)$$

where the kernel g is given by

$$g(x, y) = -\frac{D_f \begin{pmatrix} x \\ y \end{pmatrix}}{D_f}.$$

What Fredholm and many of his readers found interesting here was the possibility of taking a given boundary value problem for a partial differential equation and translating into an integral equation that could then be solved by this method. The method employed for doing this depends on the particular problem, and a detailed explanation would take us into a lengthy technical excursus. For examples, one can look at (Heywood and Fréchet 1912, 25). Given the high degree of interest, both theoretical and applied, in the boundary value problems of mathematical physics, such a promising approach turned out to ensure a wide audience.

3 Hilbert, Picard, and the Fredholm theory to 1906

In fact, the response to Fredholm's result was electric. Most writers seized on it as a method for solving classes of boundary value problems that had formerly been impossible, as we discuss in detail below. Hilbert's response went further. First off the mark in grasping the value of Fredholm's theory, he embarked on a deep investigation of integral equation methods, lecturing on them already in 1901–1902.[12] Picard, on the other hand, saw relations to his own earlier work.

3.1 Hilbert and integral equations

In 1898–1899, after an extended period of research on the theory of numbers, Hilbert became interested in existence theory for PDE's. He announced that he had "saved" the Dirichlet principle for Laplace equation 1899, though this was not published until 1904 (Hilbert 1900). Here, Hilbert found hypotheses he thought sufficient to guarantee the existence of a solution to the variational problem associated with the Dirichlet problem for the Laplace equation, the solution having been assumed by Riemann (for example, in connection with his proof of the Riemann mapping theorem). As (Brezis and Browder 1998) have pointed out, the methods used by Hilbert were complex and hard to follow, and received considerable clarification by other authors. In his 1900 Paris lecture, Hilbert's problems 19 and 20 bear on related issues.

[12] For details see Siegmund-Schultze (1982).

Many of Hilbert's 50 Ph.D. students between 1898 and 1910 worked in the general area of partial differential equations and integral equations: Hedrick, Noble, Lutkemeyer, and others worked on classical aspects of the theory. Theses on integral equations began to appear in 1902, with the thesis of O. D. Kellogg, Mason (1903), Schmidt (1905), W. D. A. Westfall (1905), A. Myller (1906), W. Lebedeff (1906), C. Haseman (1907), W. D. Cairns (1907), R. König (1907), E. Hellinger (1907), Weyl (1909), A. Haar (1910), W. A. Hurwitz (1910), H. Bolza (1913), and K. Schellenberg (1915). The work of Schmidt and Weyl in particular took a turn that gave prominence to a general viewpoint of the kind Leo Corry has referred to as "structural" in nature. Using the idea of what we would now term an inner product of two functions in a function space, defined as an integral, Schmidt and others were able to provide a geometry for function spaces, leading ultimately to the structure now referred to as a Hilbert space. However, many of these writers worked on applied problems of one sort or another.[13]

Between 1902 and 1904, Hilbert realized that for integral equations of Fredholm type (Hilbert's type II, in the classification still used, with real symmetric kernel), there would be a theory analogous to that of orthogonal transformations of quadratic forms. He saw this as a unified viewpoint for "Schwingungslehre," as he termed linear problems in analysis, namely boundary value problems, and the associated study of eigenvalues and eigenfunction expansions for the corresponding operators. He rapidly moved from the determinantal viewpoint to a "function space" viewpoint, which is noticeable also in the work of his students, beginning with Schmidt. We also observe the usual Hilbertian emphasis on conventionalism in constructing a theory.

Hilbert had looked at the question of the existence of "normal functions" (that is, eigenfunctions associated with particular equations) in the theory of differential equations, and the development of arbitrary functions in terms of them, via the introduction of Green's functions. In his 1904 *Nachricht* (Hilbert 1904), stimulated by Fredholm's work, had reduced this (if reduced is the appropriate term) to the question of finding eigenfunctions corresponding to a symmetric kernel and found the laws governing the development of arbitrary functions with respect to them.

A very rapid development in Hilbert's circle between 1904 and 1910 led to an increasingly general viewpoint, which was to mature fully in 1920s. While we will discuss further some aspects of this research direction, our main aims here are to present it in contrast to the more classical turn and to discuss how this German work informed the Italian and French contexts, to which we now return.

3.2 Picard's work on integral equations to 1906

As early as 1902, Picard saw Fredholm's *Comptes rendus* article, and also the PhD thesis of Max Mason (or at least, the brief account of it in the *Jahrbuch*), which he described as follows (Picard 1904a, 687):

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[13] See (Hilbert Ges. Abh.), III, 432.

> The study of periodic solutions of linear ordinary differential equations has been the object of much research for a long time, and just recently, in an interesting thesis, M. Mason has arrived at important results by taking as a point of departure the remarkable method of M. Freedholm [sic].[14]

Mason had applied Hilbert's idea of using Fredholm's results to get existence of solutions for a large class of problems, e.g., the equation of vibrating plates, and the isoperimetric problem, by combining the Dirichlet principle idea with Fredholm's approach. In Picard (1904a), restated results that he had already obtained together with certain extensions of them that the Fredholm method is well-adapted to solve. He did not, however, provide the solution using those methods. Picard also was soon to situate the results of Fredholm in the context of the work of Volterra in 1895–1896; and of the thesis work of his own student Le Roux (1895, 1903). In his first papers on the subject, Picard (1904a) and (1904b), he saw this subject above all as a place where his own method of successive approximation could be put to good use, since once the problem is reformulated as an integral equation the method of successive approximations can be applied to the integral version of the problem. This, indeed, had been the contribution of Le Roux in his study of the functional equation

$$\phi(y) = \int_0^y f(x)P(x, y)\mathrm{d}x.$$

Grasping the importance of the results, Picard already devoted "quelques leçons"[15] to this equation in 1903–1904. Picard employed a parameter λ in the method of solution; this harks back to Neumann and was likewise used by Volterra in 1896, as well as in Fredholm's 1903 paper (for the first time by Fredholm). Picard's reference to Abel strongly suggests that he had read Fredholm's 1903 article by this time. For all of them, the aim of the parameter is to obtain a power series expansion, by "une manière d'opérer par approximations successives" according to Picard (1904b, 314). Struck by the power of Fredholm's method, Picard lectured on "Fredholm's equation"—significantly, no longer on results of Le Roux—by 1905–1906.[16]

In Picard (1906b), he gave a fuller account via the *Rendiconti* of Palermo, for some time a favored vehicle (like *Acta Mathematica*) for reaching a broad international audience with a longer paper. The paper begins with an exposition of the basic results of Fredholm and then takes on the task of formulating various previously studied problems as Fredholm equations. The first group consists of single- and double-layer potential problems, following Fredholm. Double-layer potentials arise from attempting to solve a Dirichlet problem for the Laplace equation on the interior of closed surfaces. (By a double-layer potential is meant a dipole layer on a surface that is asso-

[14] L'étude de solutions périodiques des équations différentielles linéaires ordinaires a fait depuis longtemps l'objet de bien des recherches, et tout récemment, dans une thèse intéressant ...M. Mason est arrivé à d'importants résultats, en prenant comme point de départ la remarquable méthode de M. Freedholm [*sic*] pour la résolution de certaines équations fonctionelles.

[15] (Picard 1904b, 313).

[16] See (Picard 1906b, 323).

ciated with a potential function, for example by thinking of the dipole layer as the density of a magnetic fluid on the surface and the associated potential as the potential of the corresponding "Newtonian" magnetic force.) Poincaré (1897) had posed the related problem of finding a single layer potential on a closed surface satisfying a certain equation. In particular, if such a potential is defined as

$$V = \int\int \frac{\rho}{r} \mathrm{d}\sigma,$$

where ρ is the density, r the distance between the surface element $\mathrm{d}\sigma$ and the point at which the potential is being evaluated, then Picard denotes by

$$\frac{\mathrm{d}V}{\mathrm{d}n} \text{ and } \frac{\mathrm{d}V'}{\mathrm{d}n}$$

the interior and exterior normal derivatives, respectively. Picard states the identities

$$\frac{1}{2}\left[\frac{\mathrm{d}V'}{\mathrm{d}n} - \frac{\mathrm{d}V}{\mathrm{d}n}\right] = 2\pi\rho$$
$$\frac{1}{2}\left[\frac{\mathrm{d}V'}{\mathrm{d}n} + \frac{\mathrm{d}V}{\mathrm{d}n}\right] = \int\int r\frac{\cos\psi}{r^2}\mathrm{d}\sigma$$

Here, ψ is the angle between the normal and r. With this notation, the equation Poincaré had sought to solve was

$$\frac{\mathrm{d}V'}{\mathrm{d}n} + \mu\frac{\mathrm{d}V}{\mathrm{d}n} = 0,$$

which was transformed by Picard to the Fredholm equation

$$\rho + \frac{1+\mu}{1-\mu}\int\int \rho\frac{\cos\psi}{2\pi r^2}\mathrm{d}\sigma = 0.$$

After translating several such classical problems, Picard illustrated the real power of the method by using it to consider the generalized Dirichlet problem for linear elliptic equations of form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = f,$$

where a, b, and c are functions of x and y. This has been reformulated, Picard noted, by "Hilbert and his students" (Picard 1906b, 333) though Picard has devised a more direct method for doing so. Assuming the existence of a solution u, Picard used standard Green's function methods and integration by parts to show that u must satisfy

$$u(x,y)+\frac{1}{2\pi}\int\int\left[\frac{\partial(aG)}{\partial\xi}+\frac{\partial(bG)}{\partial\eta}-cG\right]u)\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta$$
$$=-\frac{1}{2\pi}\int\int f(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta.$$

Here, $G(\xi,\eta;x,y)$ is the Green's function evaluated at the point (x,y). This has the Fredholm form, and after arguing that it does indeed provide a solution Picard takes $a=b=0$ to investigate the question of a vibrating membrane. This is a convincing test case for the theory, since it had a long pedigree. Schwarz had already established the existence of the first singular value, and Poincaré in 1894 had shown that the integral has an infinite set of singular values that are simple poles, as we remarked earlier (Siegmund-Schultze 1982). Picard, using the Fredholm results, obtained this in a few lines.

4 Italian work and the Prix Vaillant

The fact that Picard's work appeared in the *Rendiconti del Circolo Matematico di Palermo* guaranteed it a wide audience, particularly in Italy. In Italy, elasticity theory had a long and venerable tradition, and many researchers of all levels could read this work with understanding and interest. The result was a rather large response to Picard's paper, both direct and indirect, and the paper and its offshoots were to stimulate central theoretical developments, notably in the hands of Lauricella and Volterra.

4.1 The Italian tradition in elasticity theory

The Italian tradition in elasticity theory began with Enrico Betti. Betti and his somewhat junior colleague Eugenio Beltrami (1835–1900) were the founders of an Italian school of elasticity which was highly active for many years.[17] Many researchers were involved, of whom the most successful was certainly Volterra, who began his research career in this field. This loosely knit group became dominant in the area during the late nineteenth and early twentieth centuries. In their biography of Hadamard, V. Mazya and T. Shaposhnikova write (Maz'ya and Shaposhnikova 1998, 347):

> In the beginning of the 1880s there was also the brilliant work of Poincaré, but toward the end of the nineteenth century the French school was no longer dominant. The Italian school in the theory of elasticity, founded by Betti and Beltrami, and represented by Lauricella, Levi-Civita, Volterra, and others, had surpassed it.

A letter from Lauricella to Volterra,[18] dated August 18, 1897, indicates a certain self-awareness on the apart of this Italian group and also indicates efforts to make their work better known:

[17] Here, the term school is used informally, but in fact, there are direct educational and mentoring links among many of the members, as well as significant overlaps in method.

[18] This letter is contained in Volterra's Archive, Archivio dell'Accademia dei Lincei, Roma.

Very illustrious Professor,
I received your very nice letter, followed by your postcard today. I shall send to Prof. Ivar Fredholm of Stockholm the bibliography of Italian works on elasticity, as you wrote me.[19]

In fact, Betti's main contribution to the theory of partial differential equations was developed in the context of the mathematical theory of elasticity. This concerns a particular method for integrating the equations of elastic equilibrium, based on what we now term Betti's reciprocity theorem (Betti 1872–73).[20]

This theorem states that two given systems of forces—volume forces (X, Y, Z), (X', Y', Z') and pressures (L, M, N), (L', M', N')—acting on an elastic and homogeneous body R with boundary σ and density ρ produce two systems of displacements (u, v, w), (u', v', w'), which are not independent of one another, but are linked by the following formulae:

$$\int_{\sigma} (L', M', N') \cdot (u, v, w) \mathrm{d}\sigma + \rho \int_{R} (X', Y', Z') \cdot (u, v, w) \mathrm{d}v$$
$$= \int_{\sigma} (L, M, N) \cdot (u', v', w') \mathrm{d}\sigma + \rho \int_{R} (X, Y, Z) \cdot (u', v', w') \mathrm{d}v.$$

From this reciprocity theorem, Betti derived functions, similar to Green's functions, which allowed him to describe fundamental properties of elastic bodies. In particular, he was able to express the dilatation of an elastic and isotropic body by means of the displacements of the given forces.[21]

This was a surprising and fruitful result. Betti's student Orazio Tedone, later wrote (Tedone 1907, 43):

> Betti's really admirable Memoir on the equations of elasticity cast a new and unexpected light on them, and was the precursor, particularly in Italy, to a blossoming of work that few other memoirs can vaunt themselves for having produced. His reciprocity theorem was like a revelation. By using very simple methods it at once yielded results and allowed us to penetrate the analytical properties of the equations at issue.[22]

[19] Chiar.mo Sig. Professore, Ho ricevuto la di Lei cortesissima lettera, seguita oggi da una sua cartolina. Invierò al Prof. Ivar Fredholm a Stockholm la bibliografia delle pubblicazioni italiane sulla elasticità, per come Ella mi scrive.

[20] See the introduction to (Capecchi et al. 2006).

[21] Green developed a direct procedure, now called the Green's function method, for solving the Dirichlet problem. This method requires finding a particular function, Green's function. For details see Tazzioli (2001).

[22] La Memoria, veramente mirabile, del Betti sulle equazioni della elasticità gettò su queste un fascio di luce nuova, inattesa, e preparò, specialmente in Italia, una fioritura di lavori quale poche altre memorie possono vantarsi di aver prodotto. Il suo teorema di reciprocità dovette sembrare una rivelazione. Con mezzi semplicissimi dava già una folla di risultati e permetteva di penetrare addentro nelle proprietà analitiche delle equazioni di cui si tratta.

This "new and unexpected light" attracted other researchers interested in the theory of elasticity, not all of whom were under Betti's direct influence in Pisa. One of these was Valentino Cerruti who graduated from the University of Turin in 1873 and soon moved to the University of Rome, where Cremona and Beltrami had been appointed professors a short time before. In 1877, he became professor of mechanics in Rome, succeeding Beltrami—who moved to the University of Pavia—and from then on he did research on the theory of elastic bodies. Cerruti was typical of Italian researchers who worked on the theory of elasticity during the decade 1880–1890, in that he based his work on Betti's results.[23]

Carlo Somigliana also made important contributions to elasticity theory in the Betti tradition. Somigliana studied at the University of Pavia, where Beltrami was professor. He then moved to the Scuola Normale of Pisa, where he graduated in 1881, having had Betti and Dini as teachers. His first papers concern the theory of elasticity and followed Betti's approach. In Somigliana (1890), he deduced the results now known as Somigliana's formulae, using Green's function methods. These formulae express the displacements of an elastic body by means of the forces and the displacements acting on the surface of the body. In elasticity theory, Somigliana's formulae play essentially the same role as Green's formula plays in potential theory.

Somigliana's results were considerably extended by Giuseppe Lauricella, who studied at the Scuola Normale in Pisa and subsequently became full professor of analysis at the University of Catania, working not only in pure analysis but in mathematical physics. In his dissertation, supervised by Volterra, Lauricella used a version of Neumann's method of the arithmetic mean in order to prove the existence of solutions for the equations of equilibrium of an elastic body (Lauricella 1895). Later Lauricella extended this result, valid for special values of the elasticity constants only, to the general case (Lauricella 1905), displaying a deep knowledge of contemporary papers on the Dirichlet problem published by Liapunov (1898), Poincaré (1894, 1897), Stekloff (1900), Korn (1899) and Zaremba (1902). We return to aspects of this in what follows.

Italian mathematicians working in elasticity thus became adept at tailoring the Green's function method and related results to their advantage, particularly in the context of investigating elastic equilibrium. They developed similar procedures in order to deal with and solve another problem of elasticity theory, the vibrating membrane problem, leading to a generalization of Dirichlet's problem—the Dirichlet problem for biharmonic differential equations.

4.2 Biharmonic differential equations and the theory of elasticity

In order to introduce the Dirichlet problem for biharmonic functions, let us begin with the following definition:

[23] Cerruti deduced a new method for integrating the equations of elastic equilibrium, which improved Betti's procedure. It is now described as the Betti–Cerruti method. Cerruti developed his new integration method (Cerruti 1882) by using Betti's reciprocity theorem and showed the efficacy of the Betti–Cerruti method by integrating the elastic equations of a homogeneous solid body bounded by a half-plane. Later on, this problem came to be called the Boussinesq-Cerruti problem.

A function $U = U(x_1, x_2, \ldots, x_m)$ of class $C^2(D)$, where D is a regular region, is polyharmonic or, more precisely, n-harmonic if

$$\Delta_{2n} U = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_m^2} \right)^n U = 0.$$

If $n = 1$ then U is a harmonic function in the usual sense; if $n = 2$, then U is called a biharmonic function.

In 1869, Emile Mathieu had already noticed the importance of biharmonic functions in the theory of elasticity (Mathieu 1869, 378–379):

> In fact when a solid homogeneous elastically isotropic body is subjected to pressures on its surface that maintain it in equilibrium, the projections of the displacement of an arbitrary point of the interior of the body on the coordinate axes satisfy the equation $\Delta\Delta U = 0$; and the same is true of the components of elastic force that that act on plane elements parallel to the coordinate planes.[24]

He had also formulated the Dirichlet problem for biharmonic functions (Mathieu 1869, 396):

> There is further a function, and only one, that satisfies the equation $\Delta\Delta U = 0$ in the interior of the surface σ, varying there continuously together with its derivatives up to order three, of which the value is given at the surface as is that of $\frac{\partial U}{\partial \nu}$.[25]

A generalization of the usual Green's formula to the case of biharmonic functions can in fact be deduced easily; for a biharmonic function U in a region D in 3-space, with boundary σ and external normal ν, we have

$$U(P) = \frac{1}{8\pi} \int\limits_{\sigma} \left(U \frac{\partial \Delta_2 (r - G_2)}{\partial \nu} - \Delta_2 (r - G_2) \frac{U}{\nu} \right) d\sigma.$$

Here, the function G_2 (called the Green's function of the second kind) is sought, subject to the condition that it is biharmonic in D and satisfies the boundary conditions $G_2 = r$ and $\frac{\partial G_2}{\partial \nu} = \frac{\partial r}{\partial \nu}$. This formula gives us the function U at each point P of the region D, provided that the values of U and of its partial derivative $\frac{\partial U}{\partial \nu}$ are given on the boundary σ. The resemblance to the Dirichlet problem for the Laplace equation is evident.

[24] En effet, quand un corps solide, homogène, et dont l'élasticité est la même dans tous les sens, est soumis à sa surface à des pressions qui le maintiennent en équilibre, les projections du déplacement d'un point quelconque de l'intérieur du corps sur les axes des coordonnées satisfont à l'équation $\Delta\Delta U = 0$; et il en est encore de même des composantes des forces élastiques qui s'exercent sur des éléments plans parallèles aux plans de coordonnées.

[25] Il existe encore une fonction, et une seule, qui satisfait à l'équation $\Delta\Delta U = 0$ dans l'intérieur de la surface σ, qui y varie d'une manière continue avec ses dérivées des trois premiers ordres, et dont la valeur est donnée à la surface ainsi que celle de $\frac{\partial U}{\partial \nu}$.

The Dirichlet problem for n-harmonic functions (with $n > 2$) may be stated in a similar way.[26] We notice that, similarly to what happens for the usual Dirichlet problem, Green's functions of the second kind (and, generally, of the nth kind) are very difficult to find. Mathematicians succeeded for special shapes of the region D only—when D is a half-plane, a circle, a sphere, a cube, or a parallelepiped. The case of the Dirichlet problem for polyharmonic functions when D is an n-sphere received particular attention from Italian mathematicians in the years 1900–1906, among them Tommaso Boggio, Roberto Marcolongo, Luciano Orlando and Pierina Quintili.[27]

In the classical theory of thin plates, a biharmonic equation describes the deflection of the middle part of the surface of an elastic isotropic flat plate, and the boundary conditions usually impose that the tangent plane at each point of the boundary is still while the plate moves.[28] The theory of biharmonic equations is also connected to research in hydrodynamics concerning the very slow motion of a body in a viscous fluid, as V.V. Meleshko explains in his paper (Meleshko 2003, 36):

> Two-dimensional creeping flow of a viscous incompressible fluid can also be described in terms of the biharmonic problem. If the motion is assumed to be so slow that the inertial terms involving the squares of the velocities may be omitted compared with the viscous terms, the stream function ψ satisfies the $2D$ biharmonic equation $\Delta_2\psi = 0$. This type of flow is also called the low-Reynolds-number flow (Happel and Brenner 1965) or slow viscous flow (Langlois 1964). It is also named the Stokes flow after the famous Stokes' memoir (Stokes 1850) devoted to estimate of the frictional damping of the motion of a spherical pendulum blob due to air resistance.

This problem for the sphere was studied by several writers, for example by Boggio (see below).[29]

[26] Details about the generalized Dirichlet problem for n-harmonic functions can be found for example in Marcolongo's lectures on the theory of elasticity, Marcolongo (1903).

[27] Tommaso Boggio was full professor of rational mechanics at the University of Turin. His works concern potential theory, harmonic and biharmonic functions, integral equations, the theory of elasticity, and hydrodynamics. Roberto Marcolongo was full professor of rational mechanics at the Universities of Messina and Naples. He wrote important treatises on many subjects of mathematical physics and also researched on history of mathematics. Luciano Orlando died in the First World War, having served as an artillery officer. His main works concern the theory of elasticity and the theory of integral equations, where he recognized the importance of the so-called Goursat kernels. Pierina Quintili was a student of Volterra and graduated in mathematics from the University of Rome in 1906. Afterward, she taught at secondary schools in Rome. See Nastasi and Taazzioli (2004).

[28] On the history of biharmonic equations see the long paper (Meleshko 2003).

[29] The connections between hydrodynamics and integral equations are studied in detail in (Nastasi and Tazzioli 2006). In particular, we refer to some interesting exchanges of letters between Boggio and Levi-Civita, and Henri Villat and Levi-Civita on questions concerning hydrodynamics and their reformulations in terms of Fredholm's integral equations. See especially the letter by Villat to Levi-Civita on June 24, 1911 published in (Nastasi and Tazzioli 2006, pp. 103–104).

4.3 A turning point: 1906

In the two decades from 1895 to 1915, Italian mathematicians studied the theory of harmonic and polyharmonic functions deeply. Until 1906, they applied a variety of methods to the study of the (generalized) Dirichlet problem—among them the method of Green's functions, Picard's approximating method, and two other methods (due to Almansi and Levi-Civita respectively). In 1906, Picard's paper in the *Rendiconti del Circolo Matematico di Palermo* markedly changed the direction of research (Picard 1906a). After that, Italian mathematicians regularly employed the theory of integral equations in approaching the generalized Dirichlet problem. The competition for the *Prix Vaillant*, announced in 1906, makes this shift particularly evident.

Emilio Almansi was one of Volterra's students at the University of Turin, where he became Volterra's assistant. In 1903, he moved to the University of Pavia and in 1922 to the University of Rome. In his dissertation, which treats the deformation of an elastic sphere, he applied the methods of Betti, Cerruti, and Somigliana, by then well known. Starting from these studies, he then investigated biharmonic and then polyharmonic equations. In a letter to Volterra on November 7, 1895, he describes his idea of finding the solution of a polyharmonic equation by splitting it into a certain number of harmonic equations, and then by calculating successive integrations of harmonic equations. He developed this idea in (Almansi 1899) where he found a *formal solution*, though an *explicit solution* was exhibited for special cases only. In 1896, he had already solved the Dirichlet problem for biharmonic functions in some special regions—the circle, sphere, and half-plane (Almansi 1896). Applying these results, Almansi was able to calculate the displacement of each point of an elastic sphere when displacements and deformations are given on the boundary (Almansi 1897). Volterra noticed these results: In a letter on February 1, 1904, he asked Almansi to send him a copy of the dissertation (published in Almansi 1897), since he had used it in his lectures on mathematical physics.

Almansi's method for biharmonic equations was also applied by Boggio in order to deduce the deformations of an elastic, isotropic, and plane lamina vibrating under the action of some forces on the boundary, if the displacements on the boundary are given (Boggio 1900a). He solved the problem, which was mathematically described by a biharmonic equation, in the case where the membrane could be conformally mapped to a circle by means of polynomials alone. In Boggio (1900d), he extended his own results to any plane simply connected lamina using a method developed by Levi-Civita (see below). In the same year, he published another paper, Boggio (1901b), on the equilibrium of elastic plates. There he used the classic method of successive approximations due to Picard. Here, he considered the generalized Green's function of the second kind and conjectured that this function had always to be a positive function. This assumption was called by Hadamard "Boggio's conjecture."[30]

Volterra had an important role in the development of the theory of integral equations in Italy, even if we only think of his mentoring activity: Almansi, Lauricella, Orlando, and Quintili were his students, and did research in directions inspired by him. In

[30] This conjecture is false, as Englij and Peetre proved in 1995 [for details see (Maz'ya and Shaposhnikova 1998, 438–439)].

the years 1900–1906, Volterra had started thinking about possible applications of mathematics to biology and economics (Volterra 1901). In this period, Volterra also published some papers on mathematical physics and gave his celebrated lectures at the University of Stockholm in 1906 (Volterra 1906). Volterra and several of his students were especially interested in the theory of elasticity, and more generally in applying Fredholm's theory. Volterra's main works on these subjects, where he applied the theory of integral equations to biology and economic models, only appeared after the First World War. We return to this and related issues in Sect. 5.2 below.

Levi-Civita had a similar role; he was interested in integral equations quite early, published some influential papers, and had an extensive correspondence on the subject with Boggio, Lauricella, Orlando, Guido Fubini and Eugenio Elia Levi.[31] Even before Fredholm's work, Levi-Civita had tried to integrate a biharmonic equation in a simply connected plane region without using the (generalized) Green's function (Levi-Civita 1898a,b). He broke the problem into two steps:

(1) To find the conformal representation of the given region over a circle.
(2) To solve a system Ω of an infinite number of differential equations in infinitely many variables.

From a theoretical point of view, this solved the problem, but not in a practical sense. To find the solutions of the doubly infinite system Ω poses great difficulties and Levi-Civita was aware of that. He wrote to Volterra on April 13, 1898:

> I show that formally the boundary value problem for the equation $\Delta_2\Delta_2 = 0$ can be reduced under certain boundaries to the solution of infinitely many algebraic linear equations with infinitely many unknowns. The theory of infinite determinants does not shed any light on the system in question; but we can show with complete rigour that there is a rather broad class of boundaries for which the equation can be solved by a method of successive approximations. In such a case the function u that we seek may be expressed in a quite transparent form by means of definite integrals, determined by the given system, in which the unknowns appear linearly. I am writing a note and in due time I will allow myself to subject it to your judgement and ask if you will recommend it to the Accademia delle Scienze.[32]

[31] Guido Fubini graduated from the University of Pisa in 1900 and became professor at the University of Catania in 1901. He moved to the University of Genoa in 1906, to the Polytechnic of Turin in 1908, and during the Fascism, he escaped to USA. where he taught at Princeton and at the New York University. He gave important contributions to analysis and mathematical physics. In addition, he founded a new research field, the projective differential geometry. Eugenio Elia Levi graduated from the University of Pisa in 1904 and in 1909 was appointed professor of analysis at the University of Genoa. He died in the First World War in the defeat of Caporetto. He published important papers on the theory of partial differential equations and on the calculus of variations.

[32] Io mostro che formalmente il "Randwerthaufgabe" per l'equazione $\Delta_2\Delta_2 = 0$ si può ridurre, per qualunque contorno, alla risoluzione di un sistema di infinite equazioni algebriche lineari con infinite incognite. La teoria dei determinanti infiniti non getta alcun lume sul sistema in questione; si può per altro far vedere con tutto rigore che vi è una categoria assai ampia di contorni, per cui le equazioni si lasciano risolvere con un metodo di approssimazioni successive. In tal caso la funzione cercata u si esprime sotto forma assai perspicua, per mezzo di integrali definiti, in cui entrano linearmente le incognite, determinate dal sistema anzidetto. Sto redigendo una nota e mi permetterò a suo tempo di sottoporla al Suo giudizio, sollecitando per essa il Suo patrocinio presso la Accademia delle Scienze. (See Levi-Civita 1898a).

Volterra's positive answer on April 19, 1898 allowed Levi-Civita to publish his paper immediately:

> What you wrote me about $\Delta_2\Delta_2 = 0$ interested me greatly. On the same subject, treated by Lauricella, there is a note of Dr. Almansi in the Accademia of Torino (Almansi 1896). Presently a memoir of Dr. Almansi is going to be published in the Annali di matematica on $\Delta_2\Delta_2 \ldots \Delta_2 = 0$ (Almansi 1899), but from what I remember of this it is not in the direction of your research.
> With regard to the question of the solution of infinitely many first-degree equations in infinitely many unknowns, I have already for some time been in possession of a method of solution that I have never published, and which may be valid in cases where the procedure of infinite determinants is not valid. I allow myself to communicate to you what I remember of this, not having the details here, in case that may turn out to be useful.[33]

Volterra used a method of approximation which was, essentially, the same as the one employed by Levi-Civita in his paper (Levi-Civita 1898a), as Levi-Civita himself pointed out.

As mentioned earlier, Lauricella had been interested in elasticity theory since the beginning of his scientific career. In his papers on elastic plates (Volterra 1896a; Lauricella 1896b, 1898a), Lauricella introduced an auxiliary function for a biharmonic equation analogous to the Green's function and solved the generalized Dirichlet problem for special regions (a circle or a ring). His solution for a circle was equivalent to Almansi's formula, as Volterra pointed out in his letter to Levi-Civita. In some subsequent papers (Lauricella 1901a,b, 1903, 1904), Lauricella derived the same formulae obtained by Almansi (1897) for the deformations of an elastic and isotropic sphere when strains or displacements on the surface are given. The methods were different—while Almansi used his own method for integrating biharmonic functions, Lauricella developed other methods (generally involving Green's function).[34]

Fubini, who was a colleague of Lauricella at the University of Catania, extended Levi-Civita's method to regions of any dimensions by using a procedure which did not involve conformal representation. He wrote to Levi-Civita (16 February 1905):

> Esteemed Professor

[33] Quanto Ella mi ha scritto per rapporto al $\Delta_2\Delta_2 = 0$ mi ha molto interessato.
Sul soggetto stesso trattato da Lauricella vi è una nota del D.r Almansi nell'Accademia di Torino, pure inserita negli Atti di Torino. [See Almansi 1896] Ora il D.r Almansi ha in corso di stampa una Memoria negli Annali di matematica sul $\Delta_2\Delta_2 \ldots \Delta_2 = 0$ [see Almansi 1899], ma ritengo da quanto ricordo, che egli non sia nel suo indirizzo di ricerche.
Riguardo alla questione della soluzione di un numero infinito di equazioni di primo grado con infinite incognite sono in possesso già da vario tempo di un metodo di soluzione, che non ho mai pubblicato, e che può valere in casi nei quali il procedimento dei determinanti infiniti può non essere valido. Mi permetto di comunicarle, ciò che mi ricordo di esso, non avendo qui gli appunti, pel caso che potesse riuscirle utile.

[34] Somigliana also had devised a new procedure in (Somigliana 1894). There he considered the equations of elastic equilibrium and introduced three new functions which satisfied a biharmonic equation and were the components of an elastic deformation. The expression of these functions was found using Green's method, and this allowed to find the displacement of the body. Somigliana explicitly solved the original problem if the region was a sphere or a half-plane.

> Prof Lauricella recently solved the problem of $\Delta_2\Delta_2$ for plane regions. I was therefore led to think about polyharmonic functions and related problems that I had abandoned for some time. I re-read your memoir on biharmonic functions that was published in Torino in 1898 (Levi-Civita 1898b); and here are some simple observations, that I permit myself to communicate to you, and which besides work for the problem of Δ_{2n}. For brevity I will speak only about Δ_4 and I'll refer to the notation you used.[35]

His method was based on Picard's method of successive approximations; Fubini also quoted the "recent methods by Hilbert" in the theory of integral equations. Fubini published these results in (Fubini 1905) as Levi-Civita had suggested.[36]

In Boggio (1900d), Boggio also used Levi-Civita's method for improving a result already obtained in (Boggio 1900a) concerning the deformations of elastic plates. Boggio used several different procedures in order to solve a biharmonic equation and the corresponding Dirichlet problem, including Almansi's method, Green's method, Levi-Civita's method.

Like most of his Italian colleagues, Boggio changed his approach to the study of Dirichlet's problem after the publication of Picard's memoir (Picard 1906b). This memoir made most Italian mathematicians aware of very recent work on integral equations published by Fredholm, Hilbert, and E. Schmidt from 1905 onward, in which the Dirichlet problem was tackled using the theory of integral equations. In this regard, Boggio wrote to Levi-Civita (November 24, 1906):

> Reading the clear and interesting memoir of Picard: Sur quelques applications de l'équation fonctionelle, published in the last issue of the Rendiconti de Palermo (Picard 1906b) motivated me to study in a bit of depth the theory of integral equations. It was then because of this that I studied the Memoir of Fredholm in the Acta Math. (Fredholm 1903), and that of Hilbert in the Nachrichten of Göttingen (Hilbert 1904).
> I translate by "perno" [pivot] the meaning of Hilbert's "Kern" [kernel]. It seems to me that the idea of a pivot corresponds closely enough to the idea that Hilbert wanted to express by kernel.[37]

At around the same time, Orlando referred directly to the "important memoirs by D. Hilbert" in his paper devoted to the solution of the biharmonic function in a

35 Chiar.mo Sig. Professore,
Il Prof. Lauricella ha risoluto in questi giorni il problema del $\Delta_2\Delta_2$ per aree piane. Così io mi sono rimesso a pensare alle funzioni poliarmoniche e ai problemi relativi, che da tanto tempo avevo abbandonato. Rilessi la Sua Memoria, relativa alle funzioni biarmoniche, pubblicata a Torino nel 1898 (Levi-Civita 1898b): ed ecco qui alcune facili osservazioni, che mi permetto comunicarle, e che del resto valgono anche per il problema del Δ_{2n}. Per brevità parlerò solo del Δ_4 e mi riferirò alle notazioni da Lei usate.

36 See the letter by Fubini to Levi-Civita dated 23 February 1905.

37 La lettura della chiara ed interessante memoria di Picard: Sur quelques applications de l'équation fonctionnelle, comparsa nell'ultimo fascicolo dei Rendiconti di Palermo mi ha invogliato a studiare un po' a fondo la teoria delle equazioni integrali; ed ho appunto perciò studiato la Memoria di Fredholm negli Acta Math. e quelle di Hilbert delle Nachrichten di Gottinga.
[...] Traduco con *perno* il senso di *Kern* di Hilbert; mi pare che l'idea di perno si avvicini abbastanza bene all'idea che Hilbert ha voluto rappresentare con Kern.

rectangular parallelepiped (Orlando 1906); after that, Orlando continued by applying Fredholm's method to the solution of problems in elasticity theory. In the following year, Marcolongo pointed out the importance of Fredholm's theory in potential theory and analysis, and above all in the theory of elasticity (see Marcolongo 1907).

Besides Boggio, Orlando, and Fubini, many other mathematicians, including E.E. Levi and Marcolongo, wrote to Levi-Civita and asked him for suggestions about the new theory of integral equations and its applications to mathematical physics. Levi-Civita and Volterra were the key figures at that time in Italy and both had a fundamental role in the spreading of Fredholm's theory through Italian mathematicians.

4.4 The *Prix Vaillant*

A full discussion of the French and Italian reception of Fredholm's theory would be incomplete without an account of work on the vibration of elastic plates. This problem was posed by the Académie des Sciences as the *Prix Vaillant* problem in 1906, and solved by Hadamard, Lauricella, Boggio, and Korn who shared the prize in 1907.[38] The problem stated by the Paris Academy was the following:

> To improve in an important detail a problem of analysis of the equilibrium of clamped elastic plates, namely the problem of integrating the equation
>
> $$\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 y^2} + \frac{\partial^4 U}{\partial y^4} = f(x, y)$$
>
> subject to the condition that the function u and its normal derivative on the boundary are null. Examine more particularly the case of a rectangular contour.[39]

As Boggio himself explained in a letter to Volterra (April 17, 1907), he used Fredholm's method in his memoir and was able to solve the original problem in a satisfactory and simple way:

> Last December I sent a memoir to the Paris Academy in the competition for the Prix Vaillant (L. 4000). The theme was the integration of the double Laplace equation in a plane region. I solved, more generally, for any plane area, the problem of deformation of an elastic membrane, for given displacements or tensions applied on the boundary. I obtained the solution using the solution of a Fredholm integral. I solved as well, very simply, the problem of equilibrium of an elastic solid, for given surface displacements, reducing it to the solution of a rather simple Fredholm equation. And I also solved using a Fredholm equation,

[38] Some of the details are discussed in Maz'ya and Shaposhnikova (1998) and Meleshko (2003). Arthur Korn (1870–1945) obtained a Ph.D. in Leipzig in 1890 and then went to Paris, where he attended the courses of Picard and Poincaré. From 1895 to 1908, he taught theoretical physics at the University of Munich. In 1903, he discovered an early forerunner of the fax machine.

[39] Perfectionner en un point important le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées; c'est-à-dire le problème de l'intégration de l'équation ... avec les conditions que la fonction u et sa dérivée suivant la normale au contour de la plaque soient nulles. Examiner plus spécialement le cas d'un contour rectangulaire. See *Comptes Rendus*, vol. 145, 1907, p. 983.

> the boundary value problem for equations of order $2m$, and for an arbitrary plane area, while in my Palermo memoir (Boggio 1906), where I treated the question with the method of successive approximations of Picard, I needed the condition that the area be sufficiently small.
> As you see, I have solved the problem posed by the Academy, and even more general question, and as a result, I hope part of the prize will be awarded to me.
> I submitted, a few days ago, the program of the *corso libero* for the next year here at the university, where I will treat Fredholm's equation and its applications to mathematical physics.[40]

Thus, already in 1907, Boggio was using Fredholm's theory and realized its importance for mathematical physics. Poincaré wrote the report on Boggio's memoir; Poincaré noted that: (1) Boggio's work was not completely original, since he often used results already known and obtained by others (Mathieu, Almansi, Picard); (2) he applied different methods (the method of Green's functions, Picard's method of successive approximations, and Fredholm's theory). The report suggested him to publish the memoir in *Acta Mathematica*, as Boggio wrote to Volterra (December 15, 1907):

> I am very happy that Mittag–Leffler will print my work. It is very long, consisting of over 200 pages, in parts of which results are reproduced that I already published earlier, with simplifications and additions, and in other parts new results are described.
> Mittag–Leffler can get the manuscript from the Paris academy, because I don't have time to recopy my Memoir. I decided to compete only in the last 4 months of last year, and working rapidly I managed to complete it by the 27 December. I took it to Milan myself to be sure that it would arrive at the secretariat of the Academy by Dec. 31, as the announcement of the contest prescribed.
> You will already have seen, if you read the Dec. 2 issue of the *Comptes Rendus*, that I had 1000 lire, Lauricella and Korn 2000 lire each, and Hadamard 3000 lire.[41]

[40] Nel Dicembre scorso ho mandato all'Accademia di Parigi una Memoria per il concorso al premio Vaillant (L. 4000). Il tema era l'integrazione della doppia equazione di Laplace in un'area piana. Io ho risolto, più in generale, per qualunque area piana, il problema della deformazione di una membrana elastica, per dati spostamenti o tensioni applicate sul contorno; la soluzione la ottenni mediante la risoluzione di un'equazione integrale di Fredholm. Ho pure risolto, molto semplicemente, il problema dell'equilibrio di un solido elastico, per dati spostamenti in superficie, riducendolo alla risoluzione di un'eq. di Fredholm assai semplice. Ho pure risolto, mediante un'equazione di Fredholm, il problema dei valori al contorno per le equazioni d'ordine 2m, e per un'area piana qualsiasi, mentre invece nella mia Memoria di Palermo, ove trattavo la questione col metodo delle approssimazioni successive di Picard, dovevo porre la condizione che l'area fosse sufficientemente piccola. [On the subject see (Nastasi and Taazzioli 2004)].
Come vede, io ho risolto il problema posto dall'Accad., anzi ho risolto questioni ancor più generali, perciò spero che una parte di premio mi sarà accordata.
Ho consegnato, giorni sono, qui all'Università il programma del corso libero per l'anno prossimo, ove tratterò delle equazioni di Fredholm e delle loro applicazioni alla Fisica Matematica.

[41] Sono molto contento che il Mittag–Leffler stampi il mio lavoro; esso è molto lungo, perché consta di oltre 200 pagine, in parte delle quali sono riprodotti, spesso con semplificazioni ed aggiunte, risultati che avevo già pubblicato precedentemente, e in parte sono esposti risultati nuovi.
Il Mittag–Leffler potrà avere il manoscritto dall'Accademia di Parigi, perché io non ebbi tempo di ricopiare la mia Memoria. Io mi decisi a concorrere solo negli ultimi 4 mesi dell'anno scorso, e lavorando alacremente

Boggio went on to complain, since his work—in his view—was no worse than Lauricella's.

> However it seems that Lauricella gave only one solution (without making the extensions to the equation ...), based on integral equations. It is deduced from the problem of elastic plates for two dimensions. ... However I already gave this in my first Turin note on the equilibrium of elastic membranes. Besides, it appears that the formulae obtained by Lauricella for the case of plates are analogous to those obtained by Fredholm himself in 1905, for the case of 3-dimensional bodies, in one of his notes in the Arkiv für Mathematik etc. in Stockholm (Fredholm 1905). Poincaré says that I should have treated the integration of the equation for the rectangle in a direct way. This would certainly have been easy for me to do, because it was enough to make developments analogous to those adopted by Mathieu to solve a problem for the rectangular prism. However, I thought to be more conscientious by showing that, as a consequence of a general theorem on plates that I gave in my note (Boggio 1904), the integration of the equation for the rectangle, so dear to the Academy, could be reduced to the case of the prism, solved by Mathieu some 30 years ago. I, however, took the solution of Mathieu simplifying it here and there. But Lauricella, more adroit than me, integrated it directly for the rectangle, imitating the solution of Mathieu.[42]

However, Boggio's work was not published in *Acta Mathematica*, for reasons unknown. He may simply have assumed that the recommendation he received would be enough to guarantee publication. Clearly, the length of the paper would have been an obstacle. He did publish a number of papers on this matter in other journals; see for example (Boggio 1907a,b). Indeed, Boggio's memoir was never published in that form. It remains in the Archives of the Académie des Sciences in Paris. The "Appendix" below contains a portion of the paper.

In his winning memoir, Lauricella provided a different approach to the solution of the biharmonic problem. His work was published in *Acta Mathematica* in 1909

Footnote 41 continued
riuscii a terminare la memoria il 27 Dicembre; la portai io stesso sino a Milano, per essere sicuro che arrivasse alla Segreteria dell'Accademia entro il 31 X.bre, come era prescritto dal bando di concorso. Ella avrà già visto, leggendo il n^o del 2 Dicembre dei *Comptes Rendus*, che io ebbi 1000 lire, Lauricella e Korn 2000 lire ciascuno, ed Hadamard 3000 lire.

[42] Invece pare che il Lauricella abbia dato una sola soluzione (senza fare estensioni all'equazione ...), fondata sulle equazioni integrali; essa è dedotta dalla soluzione del problema dell'equilibrio elastico per 2 dimensioni ... Questo, per altro, l'avevo pure detto già anch'io nella mia prima nota di Torino sull'equilibrio delle membrane elastiche (a. 1900) (Boggio 1900b). Inoltre pare che le formule ottenute dal Lauricella per il caso delle piastre, siano analoghe a quelle date già nel 1905 da Fredholm stesso, per il caso di corpi a 3 dimensioni, in una sua nota degli "Archiv für Mathematik ecc." di Stoccolma (Fredholm 1905). Il Poincaré dice che io dovevo trattare l'integrazione della equazione per il rettangolo in modo diretto; questo certo mi era assai facile di fare, perché bastava fare sviluppi analoghi a quelli adoperati dal Mathieu, per risolvere un problema sul prisma rettangolare. Io invece ho creduto di essere più coscienzioso facendo vedere che applicando un teorema generale sulle piastre, da me dato nella Nota "Sulla deformazione delle piastre cilindriche ecc." (Lincei, 1904; Boggio 1904) l'integrazione della equazione per il rettangolo, che tanto stava a cuore all'Accademia, si poteva ricondurre al problema anzidetto sul prisma, risolto da Mathieu un 30 anni fa. Io però ho ripreso la soluzione di Mathieu, semplificandola qua e là. Invece il Lauricella, più accorto di me, ha integrato direttamente la pel rettangolo, imitando la soluzione del Mathieu.

(Lauricella 1909a). He introduced two new functions u and v, where $u = \partial U/\partial x,\ v = \partial U/\partial y$ and an auxiliary function $\Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \Delta U$.

Lauricella considered the equations

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x},\ \Delta\Theta = 0$$

inside the region, with special boundary conditions. After extensive transformations, Lauricella reduced this problem to a system of two integral equations in two unknowns. He proved that the solution is unique for the case of a finite region with smooth contour.[43] In particular, Lauricella was able to give an interesting physical interpretation to his procedure by using the theory of the double sheet (u and v are known as the internal and the external potential of the double sheet).[44]

In contrast to Boggio's laments, Lauricella expressed himself as being very happy to have won (a part of) the Prize Vaillant, as he wrote to Volterra (November 24, 1907):

> I am, of course, extremely happy with the prize I obtained, and I feel at the moment a great need to express to you all my gratitude for the warm interest that you have always shown toward me.
> I will prepare a resumé of my results for the Lincei, and I will keep the plan to publish the entire memoir.
> Please accept, dear Professor, the most distinguished expression of your disciple's obligation.
> G. Lauricella[45]

As for the other winner, A. Korn, his account was later published in *Annales de l'Ecole Normale Supérieure* (Korn 1907). Korn did not use Fredholm's methods; in addition, the work appears not to have been influential in the history of biharmonic equations.

On the memoirs of the Italian mathematicians Picard remarked to Volterra in a letter on June 5, 1907:

> The competition for the Prix Vaillant on the equation $\Delta\Delta u = f(x, y)$ of elasticity theory has brought a very large number of memoirs to our Academy, among which several Italian works are to be found. All of these memoirs appear interesting to me, and it will be a lot of work to rank them.[46]

[43] See (Meleshko 2003, p. 46).

[44] In 1940, Sherman established a similar integral equation for a complex function (usually known as the Lauricella-Sherman equation). See (Meleshko 2003, 46) and (Sherman, 1940).

[45] Naturalmente sono contentissimo del premio ottenuto, ed in questo momento sento maggiormente il bisogno di esprimerLe tutta la mia gratitudine per l'affettuoso interessamento che mi ha sempre addimostrato. Preparerò un sunto dei miei risultati per i Lincei, e mi riservo di pubblicare per disteso la mia Memorie. Gradisca, Chiar. Sig. Professore, i più distinti ossequi dal Suo obblig.mo discepolo.

[46] Le concours du prix Vaillant relatif à l'équation $\Delta\Delta u = f(x, y)$ de la théorie de l'élasticité a amené à notre Académie un très grand nombre de mémoires, parmi lesquels se trouvent plusieurs travaux italiens. Tous ces mémoires paraissent intéressants, et ce sera un grand labeur que de faire un classement.

In the end, Hadamard's memoir (Hadamard 1908, 1909, 1910) placed first. His main goal was to study the Green's function of the second kind connected to the (biharmonic) Dirichlet problem in question. He also studied the variational properties of the Green's function and introduced the Hadamard–Boggio conjecture which we mentioned above. Though this was an important memoir, it did not deal with the theory of integral equations.

4.5 Applications of Fredholm's theory to analysis 1907–1910

From 1907 onward, mathematicians often applied Fredholm's theory in order to solve problems of mathematical physics, for the most part concerning elasticity, heat conduction, and hydrodynamics. In this context, Picard wrote to Volterra on December 29, 1909, complaining to some extent about the influx of newcomers:

> I read your articles on integro-differential equations in the Lincei with great interest; it's a collection of ideas that interests me very much. I have thought for a long time that it is this kind of equation, more general than differential equations, that will come up in mathematical physics more and more. If you would like to give me an overview of your research for the Annales of the École Normale, I would be very happy for it. This literature on integral equations is becoming something frightening; unfortunately not everything is equally good, and many beginners apply themselves to it in order to follow the fashion.[47]

This fashion was international in scope, as we discuss later, but many Italian mathematicians followed the new trend and published papers on this subject. We have already mentioned work by Almansi, Lauricella, and Boggio on the generalized Dirichlet problem, and by Orlando and Marcolongo. They approached the problems initially by using different methods; but after 1906, all of them employed Fredholm's theory of integral equations. In particular, Marcolongo wrote that the "new and very important" Fredholm theory had "very remarkable applications to potential theory and analysis" as well as to the theory of elasticity (Marcolongo 1907, 742).

We notice that the new Fredholm theory was successfully applied to pure mathematics too. In Levi (1907) recognized that the results obtained by Fredholm and Hilbert could be applied to the theory of ordinary and partial differential equations. It was not by chance that Levi-Civita asked E.E. Levi to resolve a "difficulty"—to reduce a special complex function to an integral equation. This "difficulty" was soon solved by Levi who wrote a long and technical letter to Levi-Civita (May 5, 1908), which was then published in a note in the *Nachrichten* of the Academy of Sciences of Göttingen, the same journal where Hilbert had published his notes on integral equations (Levi 1908). This paper was one of the most significant works Levi was to publish (see for example Dieudonné 1981) and is remarkable, along with

47 Je lis avec grand intérêt dans les Lincei vos articles sur les équations intégro-différentielles; c'est un ordre d'idées qui m'intéresse beaucoup. Il y a longtemps que je pense que c'est ce type plus générale que les équations différentielles, qui se présentera de plus en plus en physique mathématique. Si vous vouliez me donner pour les Annales de l'École Normale un aperçu de vos recherches, j'en serais très heureux. Cette littérature des équations intégrales devient quelque chose d'effrayant; tout n'y est pas également bon malheureusement, et beaucoup de débutants se mettent là dessus pour suivre la mode.

the work of Lauricella, for its success in understanding the approach of the Hilbert school.

4.5.1 The Poincaré lemma

Another interesting application of Fredholm's theory to a problem of pure mathematics concerns the so-called Poincaré lemma, which Lauricella extended to non-convex domains. In the third paragraph of his 1894 memoir (Poincaré 1894, 70ff.), Poincaré argued that If V is a function of x, y, z in a region D such that $\int V d\tau = 0$, if $A = \int V^2 d\tau$, $B = \int \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] d\tau$ then $\frac{B}{A} > \frac{16}{9l^2}$ if D is convex and l is its maximum distance between two points of D.

The result is linked both to the *balayage* method and to the problem of eigenfunction expansions, and in fact, it was a central tool for Poincaré in proving the existence of such expansions. In fact, Poincaré returned to the result repeatedly, having already discussed it in (Poincaré 1890). One key feature of this work is that Poincaré himself believed that he had established the lemma for any convex region. However, he made the tacit assumption that a region could be appropriately decomposed into convex regions, and in fact, this was false without further restriction. The lemma is used again in (Poincaré 1897).[48]

Lauricella engaged himself in finding the proof of Poincaré's lemma for any region, since it was connected with other research in mathematical physics that interested him. He wrote to Volterra (April 28, 1901):

> The work on the propagation of heat might appear inopportune since the work on the same problem by Stekloff (1900); but Stekloff used Poincaré's lemma (Poincaré 1894) maintaining it was proved for non-convex surfaces. While the proof of Poincaré requires essentially that the surfaces be convex; I have not either succeeded in extending it to non-convex surfaces.[49]

And in another letter (April 7, 1904):

> The study of the conditions that the density of a double layer has to satisfy for the existence of its normal derivatives is important because of the fact that the Poincaré–Neumann method for solving the Dirichlet problem requires that the given function of the points of the surface be such that the corresponding double layer have a finite normal derivative.
> With regard to the method of Poincaré–Neumann it will be known to you that Zaremba in a Note (Zaremba 1901) has succeeded in overcoming, so to speak, the last of the difficulties that this method presented. Zaremba's result is based on the known lemma of Poincaré, proved only for convex regions. In his memoir

[48] Note that this is not related to the "Poincaré Lemma" about closed and exact differential forms.

[49] Il lavoro sulla propagazione del calore potrebbe parere inopportuno dopo il lavoro sullo stesso argomento dello Stekloff (Annali di Tolosa) (Stekloff 1900); però lo Stekloff fa uso del lemma del Poincaré (Mem. cit. del Circ. Mat. p. III) (Poincaré 1894) ritenendolo dimostrato per le superficie non convesse; mentre che la dimostrazione del Poincaré richiede essenzialmente che la superficie sia convessa, né a me è riuscito estenderlo alle superficie non convesse.

in the Circolo Matematico (p. 97) (Poincaré 1894) and the one in the American Journal (Poincaré 1890), Poincaré, it is true, says that non-convex regions can be decomposed into convex regions by a finite number of cuts. But this observation does not hold for Zaremba's case, because a non-convex surface that admits a tangent plane at each of its points can't be decomposed into a finite number of convex regions. Whence it seems to me that one ought to extend the lemma of Poincaré to non-convex regions. I thought about this, but did not succeed; and I would have wished to take the opportunity of this note that I am sending to you, to say that the method of Zaremba contains this lacuna, but I fear that the means are not the most opportune.[50]

Lauricella produced a satisfactory solution to this problem in 1908 (Lauricella 1908c) by using Fredholm's method. After quoting Picard and Hilbert, he wrote in the Introduction of his paper (Lauricella 1908c, 114):

> The extension of Poincaré's results to non-convex regions has been the object of various important works, among which those of Stekloff (1900, 1902) and Zaremba (1905) stand in the first rank. But the methods developed for such aims all depend on Poincaré's lemma. This lemma has been extended, it is true, to non-convex regions (Korn 1902; Levi 1906) but with some restrictions on the surfaces bounding the regions, and not with that generality which, as we shall see, the application of Fredholm's theory provides.[51]

[50] Lo studio delle condizioni, a cui deve soddisfare la densità di un doppio strato per l'esistenza delle sue derivate normali, è importante anche per il fatto che il metodo Neumann-Poincaré, per risolvere il problema di Dirichlet, richiede che la funzione data dei punti della superfici sia tale che il corrispondente doppio strato abbia la derivata normale finita. [...] A proposito del metodo Poincaré–Neumann Le sarà noto che lo Zaremba in una Nota del Bulletin Internatonal de l'Académie des Sc. de Cracovie, N 3 del 1901 (Zaremba 1901) è riuscito a superare, diciamo così, l'ultima delle difficoltà che il metodo stesso presentava. Il risultato dello Zaremba si fonda sul noto lemma di Poincaré, dimostrato solo per i campi convessi. Il Poincaré, è vero, nella sua Mem. del Circolo Matematico (p. 97) (Poincaré 1894) e in quella dell'American Journal (Poincaré 1890), relativamente ai campi non convessi dice che essi, con un numero finito di tagli, si possono decomporre in campi convessi; però questa osservazione non vale per il caso dello Zaremba; perché una superficie non convessa, che ammette un piano tangente in ogni suo punto, non si può decomporre in uno numero finito di campi convessi. Quindi a me pare che si debba estendere il lemma del Poincaré ai campi non convessi. Io ci ho pensato, ma non ci sono riuscito; ed avrei volto prendere occasione da questa Nota che Le invio, per dire che il metodo dello Zaremba presenta questa lacuna; ma temo che il mezzo non sia il più opportuno.

[51] L'estensione dei risultati del Poincaré ai campi non convessi è stata oggetto di vari importanti lavori, tra i quali stanno in prima linea quelli dello Stekloff (*Sur les problèmes fondamentaux de la physique mathématique* (Annales de l'Ecole Normale supérieure, 1902); *Théorie générale des fonctions fondamentales* (Annales de la Faculté des Sc. de Toulouse, 2 s., VI). N.d.A.) e dello Zaremba (*Solution générale du problème de Fourier* (Bulletin international de l'Acad. des sc. de Cracovie ; Février 1905). N.d.A.); però i metodi a tal uopo escogitati fin qui, oltre di essere poco semplici, dipendono sempre dal lemma di Poincaré. Questo lemma è stato esteso, è vero, ai campi non convessi (A. Korn, *Abhandlungen zur Potentialtheorie*, 4 (Berlin, 1902). E.E. Levi, *Su un lemma del Poincaré* (Rendiconti della R. Acc. dei Lincei, vol. XV). N.d.A.); ma con alcune restrizioni sulla natura della superficie limitante il campo, e non con quella generalità che, come si vedrà, apporta l'applicazione della teoria di Fredholm.

4.5.2 The Picard–Lauricella criterion

Another fundamental mathematical result obtained by Lauricella (and Picard) is the so-called Picard–Lauricella criterion. This is a case where, independently, the two mathematicians sought a necessary (and ideally sufficient) condition for solvability of certain kinds of equations. In order to achieve this, it was necessary for both to grasp the essential features of the recent work of the Hilbert School, most notably that of Erhard Schmidt and Friedrich Riesz.

This was achieved by Picard by 1909. While Hilbert had restricted his investigation to the case of continuous kernels, Schmidt generalized this to square-integrable kernels in his 1905 thesis (Schmidt 1905). Schmidt likewise gave a geometric interpretation to Hilbert's results in his thesis, as Hilbert had not. Functions with singularities could thus serve as kernels, and in consequence of this, Schmidt presented a spectral theory for non-symmetric operators. It was this theory that was used by Picard, as (Groetsch 2003) points out.[52] In so doing, Picard clarified what is now seen as the essential difference between integral equations of first and second type and gave an explicit criterion for the possibility of solving equations of type one. Picard developed this theory in 1909 and presented it in his lectures of that year as well as giving a short account in the *Comptes rendus* of June 14, 1909 (Picard 1909a). The same result was found independently by Lauricella in the same year (Lauricella 1909b). No priority dispute resulted, and the issue of independent discovery seems clear, the more likely since the question is obvious and the result not difficult at the level of generality employed by Picard and Lauricella.

In this research, both Picard and Lauricella demonstrated their mastery of essential results due to Schmidt and F. Riesz (Riesz 1907). Schmidt's result, from his dissertation of July 1905, had been published in the *Mathematische Annalen* in 1907. Schmidt's result states the following. Suppose we are given two "conjugate" integral equations

$$\phi(x) = \lambda \int_a^b K(x, y)\psi(y)\mathrm{d}y \tag{4.1}$$

$$\psi(x) = \lambda \int_a^b K(y, x)\phi(y)\mathrm{d}y \tag{4.2}$$

and suppose that, in Hilbert's terminology, $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of the problem, that is the values of λ that correspond to non-trivial solutions ϕ_i and ψ_i. Then, the ϕs form an orthonormal system, as we would now say (Picard says orthogonal and normal, following Schmidt). Picard combined this with a version of Riesz's result, the theorem we now know as the Riesz–Fischer theorem, likewise from 1907. In Picard's statement, this says the following: a kernel function $f(x)$ can be found such that a given sequence a_n can correspond to the "Fourier" coefficients with respect to a given

52 Interestingly, Groetsch does not mention Lauricella's independent discovery of the same criterion.

orthonormal collection of functions if and only if the sequence is square-summable, that is, $\sum a_n^2$ is convergent.

The system $(\phi_n, \psi_n, \lambda_n)$ is what would now be called a *singular system* of the kernel K. To understand what Picard and Lauricella had, it is useful to state a modern version of the theorem, as it is given in the classic work (Smithies 1965), with some changes in notation.

Theorem 4.1 *Let $(\phi_n, \psi_n, \lambda_n)$ be a singular system of the $\mathcal{L}^2$ kernel $K(s,t)$ and let $y(s)$ be a given $\mathcal{L}^2$ function. Then the equation*

$$y(s) \approx \int K(s,t)x(t)\mathrm{d}t$$

has an $\mathcal{L}^2$ solution $x(t)$ if and only if (a)

$$\sum_{n=1}^{\infty} \lambda_n^2 |(y, \phi_n)|^2 < +\infty,$$

and (b) $(y, \phi) = 0$ for every $\mathcal{L}^2$ function ϕ such that $K^*\phi \approx 0$.

The $\approx$ here indicates equality except on a set of measure 0, and the (,) refers to the usual operation of projection onto the basis functions.[53]

How does Picard treat this in his original version? This appears in brief in the *Comptes rendus* of June 14, 1909, and more elaborately in a fuller version in the Palermo *Rendiconti*, submitted in July 1909 and appearing in the following year.

Picard assumes in addition that the functions ϕ_i form what he, again following Schmidt, terms a closed sequence (*fermé*), which means that there is no function $h(x)$ other than the zero function such that $\int_a^b h(x)\phi_n(x)\mathrm{d}x = 0$. Nowadays, we would note that this means the sequence is complete as a basis for the entire function space, that is, the null space of the operator adjoint to the one defined by the integral equation is trivial. Under these circumstances, Picard was able to show that a necessary and sufficient condition that an integral equation of type 1 be solved is the criterion that

$$\sum \lambda_n^2 a_n^2$$

is convergent, where a_n are the components of the kernel with respect to the given basis. These are referred to as Fourier coefficients by Picard, following Riesz. The

[53] See (Tricomi 1957) on this result, see in particular (Tricomi 1957, 88–90) on Riesz–Fischer 1907 (see Riesz 1907; Fischer 1907) and (Tricomi 1957, 143–150) on this criterion in the symmetric and non-symmetric cases. Tricomi casts all this in terms of convergence in the mean. In that case, the so-called Weyl lemma, the analog of the Cauchy convergence criterion for convergence in the mean, is required. The Weyl lemma is from a 1909 paper by Weyl (1909). Tricomi (1957, 88) observes that the Riesz–Fischer result is "one of the first brilliant successes of the concept of the Lebesgue integral." This is because the function f corresponding to a sequence a_i of Fourier coefficients with respect to the system ϕ_i with $\sum a_i^2 < \infty$ need not be Riemann integrable. Weyl was aware of this early on, as we see in his thesis p. 16 footnote (Weyl 1908). Lebesgue's *Leçons* appeared in 1904 (Lebesgue 1904).

proof is straightforward if one uses the ideas of Schmidt and Riesz (and Hilbert), and thus, this demonstrates nicely that by 1909 Picard had indeed grasped these ideas fully and was able to put them to good use.[54]

Lauricella, on the other hand, was able to take this further than Picard. In a series of papers between 1906 and 1912, he published a collection of important results which show his capacities and his good grasp of contemporary literature. In 1908, he already indicates that he knows the Hilbert and Schmidt results (Lauricella 1908a). Among his own findings in this series of papers the criterion of Picard likewise appears, generalized in (Lauricella 1909b) to kernels that are more general than the closed kernels required by Picard. In particular, they are explicitly required to be $\mathcal{L}^2$ and integrable except on a set of measure zero. In this hypothesis, using (Schmidt 1907), he showed that the pairs of conjugate equations correspond to pairs of eigenfunction solutions and eigenvalues, so the criterion was shown by him to be valid in something very close to the form we stated.[55] Lauricella thus already understood a good part of the Schmidt–Riesz work as well as being familiar with the notion of almost everywhere convergence. His paper was published only a few months after the note of Picard in the *Comptes Rendus*. In (Lauricella 1909b), he uses the term *funzione caratteristica* for "Kern," changing to "nucleo" in 1911 (Lauricella 1911). In this 1911 paper, Lauricella studied conditions for the existence of a solution of an equation of the first kind, employing the results of Weyl that went beyond the earlier work of Schmidt and Riesz to deal with non-symmetric kernels. From this, Lauricella was able to find necessary and sufficient conditions for the existence of an $\mathcal{L}^2$, and even $\mathcal{L}^1$ solution in a closed and bounded region under certain conditions. The full version of this now-classic result, obtained in 1912 (Lauricella 1912) and going far beyond any work in France of the time, is discussed in detail in (Tricomi 1957).

While it would be possible to continue with a story of ongoing theoretical developments and applications of these techniques, the change in the mathematical world brought about by the onset of the first world war marks a place where instead it is reasonable to pause and look at the broader reception of this work. For not only was it well-received by leading researchers; it was seen as a set of tools that should be in the hands of every analyst.

5 Diffusion and reception of the theory to 1915

The rapid development of the theory of integral equations, and its immediate important application to the solution of boundary value problems, led to many presentations in lectures in Germany, France, Italy, USA, and elsewhere. Between 1909 and 1915 Maxime Bôcher, Adolph Kneser, Trajan Lalesco, the duo of H. B. Heywood and M. Fréchet, Édouard Goursat, Vito Volterra and Robert Adhémar all gave textbook accounts, most of them based on lectures. (Plancherel 1912) gives an expository account for a broad mathematical audience. Some of these treatments were incorpo-

[54] Picard certainly steers clear of issues involving sets of measure 0 and so on, however, by his hypotheses. Thus, the theorem is a bit simpler than in more recent statements such as that of Smithies, p. 164.

[55] This result had already be published in (Lauricella 1908b).

rated in longer works on analysis (for example, Goursat's, which nonetheless runs to 220 pages (Goursat 1907, 1915)), while others were free-standing. Furthermore, the general interest of the subject made it a topic for not only short lecture courses, but also for international meetings. One of these is the Rome congress, only the second following the announcement by Fredholm in Paris in 1900.

5.1 Integral equations at the International Congress in Rome 1908

With excitement about the potential of integral equations still growing, it is not surprising that we find them mentioned repeatedly both in the plenary lectures and in the scientific contributions at this meeting. Poincaré, Picard, and A.R. Forsyth all draw attention to these developments, in rather different ways.

Poincaré had recently turned his attention to Fredholm's work and its outgrowths (Poincaré 1908a,b) Poincaré both describes the method of Fredholm and places it in the context of his conventionalist ideas about the use of mathematics to describe the natural world:

> Our knowledge of partial differential equations has recently made a considerable step forward as a result of the discoveries of M. Fredholm ... one sees that [his discoveries] consisted in modeling this difficult theory on another, much simpler, that of determinants and systems of first-degree equations....
> One can take a partial differential equation, representing so to speak an infinity of equations, and use it to determine an unknown function, representing a continuous infinity of unknowns. One thus has other infinite determinants, which are to ordinary determinants as integrals are to finite sums. This is what Fredholm did.[56] (Poincaré 1908c, 178).

Picard, on the other hand, saw the potential of extending the integral equation method to permit modeling of phenomena far beyond those accessible via the usual partial differential equations and cites biology and the study of "hereditary phenomena" (those that do not depend on immediate initial conditions in a simple way) as possible realms of application:

> We know with what success a particular type of functional equations has been studied in recent times ... a memorable example, amid many others, of the ease that a more general viewpoint can bring to the solution of a particular problem. For certain boundary conditions there is even great interest in substituting functional equations for differential equations... a systematic study of more and more complicated functional equations should therefore demand the effort of

[56] Notre connaissance des équations aux dérivées partielles a fait récemment un progrès considérable par suite des découvertes de M. FREDHOLM ...on voit que [les découvertes de Fredholm] ont consisté à modeler cette théorie difficile [des équations diff. partielles] sur une autre beaucoup plus simple, celle des déterminants et des systèmes d'équations du 1er degré. ... On peut prendre une équation aux dérivées partielles, représentant pour ainsi dire une infinité déquations, et s'en servir pour déterminer une fonction inconnue, représentant une infinité continue d'inconnues. On a alors d'autres déterminants infinis, qui sont aux déterminants ordinaires ce que les intégrales sont aux sommes finies. C'est là ce qu'a fait Fredholm...

researchers... we can dream of more complicated functional equations that will bring in the role of heredity...[57] (Picard 1908, 194).

Here, heredity refers not to the biological phenomenon; the term was used by Volterra, Picard, and others to describe physical processes in which the time evolution of the phenomenon depends not only on "initial" conditions at a single point in time but (typically) on other states prior to that; we return to this shortly.

While Picard's predictions about the realms of application were not to be fully realized, his observation about the ways in which integral equations would extend the domain of analysis ring true:

> A domain more vast than that of differential equations, containing them as a special case, opens before us. We will not take a path there by chance, but guided by mechanics and physics in the choice of forms to treat.[58] (Picard 1908, 194)

A. R. Forsyth, writing on the theory of formal integration of partial differential equations in the same *Atti*, specifically mentions the importance of these recent developments, which he expressly omits. However, his perception of the need to acknowledge these developments speaks to their importance:

> The choice [of subjects for the article]... involves a study of processes which belong to the earlier developments of the theory, and leads me, for the present purpose, to omit references to many of the quite modern contributions. It is from no lack of appreciation of the work of Mr. Dini, Mr. Hilbert, Mr. Picard, Mr. Poincaré, and Mr. Volterra ...that my choice has been made. (Forsyth 1908, 87).

In the detailed communications, we also find many papers treating integral equations both in themselves and for the purposes of application to the solution of boundary value problems. Again, sometimes these remarks are simply passing mentions indicating the perceived importance of Fredholm's methods. This is the case, for example, with the paper of Lauricella, who limits himself for simplicity to a case where Fredholm's method is not needed, while noting its power in more general cases (Lauricella 1908d, 33). Boggio makes reference to results obtained by this method, in the theory of elasticity, citing particularly the work of Lauricella and Picard (Lauricella 1907; Picard 1906b). He treated these papers as well known, as they certainly were in his immediate circles. More theoretical work on integral equations was presented by d'Adhémar and Orlando, d'Adhémar speaking about equations of Volterra type, and Orlando proposing

[57] On sait avec quel succès un type particulier d'équations fonctionelles a été étudié dans ces derniers temps ... mémorable exemple, après tant d'autres, de la facilité que peut apporté à la solution d'un problème particulier un point de vue plus générale. Pour certaines conditions á limites, il y a même grand intéret à substituer des équations fonctionelles à des équations differentielles ... une étude systématique d'équations fonctionelles de plus en plus compliquées devra donc dans un prochain avenir solliciter l'effort des chercheurs... nous pouvons rêver déquations fonctionelles plus compliquées ...qui apporteront la part de l'hérédité...

[58] Un domaine plus vaste que celui des équations différentielles, et les comprenant comme cas particuliers, s'ouvre devant nous. Nous n'y marcherons pas au hasard, guidés dans le choix des formes à traiter par la Mécanique et la Physique.

a method for a different class of equations, based on Fredholm's ideas (d'Adhémar 1908; Orlando 1908).

5.2 Volterra's Rome lectures

We do not propose to give an account of all the textbooks and lectures that address the subject of integral equations in the prewar period. However, the lectures of Volterra were given in various versions in Stockholm, Paris, and Italy, and we have already remarked on Volterra's extensive influence in the Italian arena, as well as on his close association with Picard. Hence, we will discuss the version of this subject that is recorded in these lectures.

In the period 1909–1911, Volterra published a series of papers in *Rendiconti della Reale Accademia dei Lincei* concerning applications of integral equations—which Volterra named integro-differential equations (*equazioni integro-differenziali*)—to theory of elasticity, electrodynamics and, especially, to what he termed hereditary phenomena (*fenomeni ereditari*).

This refers to a distinction made by Picard, but dating back to Boltzmann or even further, between phenomena that depend only on the present (or infinitesimally near) state of a system (non-hereditary) and those in which some "memory" exists that may surface to cause changes that are not easily modeled using differential equations and boundary conditions (Boltzmann 1874, 1876). Picard's discussion of this matter occurs at the end of a paper titled *La mécanique classique et ses approximations successives*. The venue of this paper is interesting: It appears in the first issue of the first volume of a journal called *Rivista di Scienza*, published in Bologna with an editorial board including F. Enriques. The journal has a philosophical spin, with its subtitle indicating it is a *Revue internationale de synthèse scientifique*. (The articles are in French, Italian, English, and German.) As Picard noted,

> The laws expressing our ideas on motion find themselves condensed into differential equations, that is, relations between variables and their derivatives. We must not forget that we have definitively formulated a principle of *non-heredity*, in supposing that the future of a system at a given moment depends only on its present state, or more generally (if we consider forces as being able also to depend on velocities) that this future depends on the preceding infinitely near state. This is a restrictive hypothesis, and one that, in appearance at least, many facts contradict. There are numerous examples in which the future of a system seems to depend on anterior states. In such complex cases one tells oneself that perhaps we must abandon *differential* equations and think of *functional* equations, in which integrals figure that are taken from a very distant time up to the present, integrals that will be the legacy of this heredity.[59] (Picard 1907, 13–14).

[59] ...les lois exprimant nos idées sur le mouvement se sont trouvées condensées dans des équations differentielles, c'est à dire des relations entres les variables et leurs dérivées. Il ne faut pas oublier que nous avons en définitive formulé un principe de *non-hérédité*, en supposant que l'avenir d'un système ne dépend à un moment donné que de son état actuel, ou d'une manière plus générale (si on regarde les forces comme pouvant aussi dépendre des vitesses) que cet avenir depend de l'état infiniment voisin qui précède. C'est une hypothèse restrictive et que, en apparence au moins, bien de faits contredisent. Les exemples sont nombreux, où l'avenir d'un système semble dépendre des états antérieurs. Dans des cas aussi complexes, on se

In the introduction to the Rome lectures, Volterra went to some lengths also to discuss the history of his work from his own point of view, perhaps hoping to establish his priority. Noting that his work on functions of lines dates to 1883, he stated specifically that he had begun to work on the subject systematically from 1887. Further, he had linked problems in the calculus of variations to integral equations with symmetric kernel already in 1884. The idea of inverting definite integrals by using infinite determinants he claims to have mentioned in private conversations in Zürich in 1897, and in a paper on standing waves in lakes (*seiches*) in 1898 in *Nuovo Cimento* (Volterra 1898) (see Volterra 1913a, v–vi).

In the next few years, Volterra likewise extended the connection between integral equations, his complementary work on functions of lines, and various applications. Some of this work consisted in working out technical details, but he also made an effort to explain to a broader public what he saw as the future of his program.To this end his final lecture in the series, he gave at the Sorbonne was published in Émile and Marguerite Borel's *Revue du Mois* (Volterra 1912). In particular, the obvious issue of whether such "hereditary" effects could always be reduced to deterministic models was debated. Citing an example given by Painlevé (Painlevé 1909), Volterra noted that we may have two macroscopically identical nails, one of which has many hammer blows applied to it, with results for the microstructure. In such a case, the imperfect knowledge of the initial data is merely replaced or compensated for by using a hereditary framework.

> I think it reasonable to admit hereditary procedures as the only ones possible, at least in the present state of science, to include hereditary processes like those we have mentioned. There may be a difference of principle, but both those who regard the previous postulate as a truth beyond all discussion as well as those who doubt it, thinking that to neglect heredity is just a simple approximation, must agree on the practical terrain and the methods that must be followed[60] (Volterra 1912, 564–565)

and again

> I believe that I have demonstrated to you that hereditary theories ... may be studied by the methods of integro-differential equations... it is useful to leave the constants undetermined ... it is because of this that the importance of the

Footntoe 59 continued
dit qu'il faudra peut-être abandonner les équations *différentielles* et envisager des équations *fonctionelles*, où figureront des intégrales prises depuis un temps trés lointain jusq'au temps actuel, intégrales qui seront le part de cette hérédité.

60 Au contraire, je pense qu'il est raisonnable d'admettre les procédés d'hérédité comme les seuls possibles, au moins dans le moment actuel de la science, pour embrasser les [hénomènes dont nous avons parlé. Il peut y avoir une différence de principe, mais, autant ceux qui admettent comme une vérité hors de toute discussion le postulat précédent, et ceux qui en doutent, en considérant que négliger l'hérédité n'est qu'une simple approximation, doivent être d'accord sur le terrain pratique et sur les méthodes qu'il faut suivre.

application of algebra to questions of nature has increased always.[61] (Volterra 1912, 573)

Volterra's Rome lectures provide an account of the Fredholm's theory and its applications, originally given in Rome in 1909–1910, and then published in French in 1913 (Volterra 1913a). M. Tomassetti and F.S. Zarlatti, who edited Volterra's lectures, added some results by Picard and Lalesco to Volterra's original lectures. Volterra's integral equations are defined, with Fredholm's equations, as two chapters of the same theory—the treatise is divided into four chapters. Some applications to the Dirichlet problem, liquid oscillations, vibrations, stationary waves, and hereditary electrodynamics and elasticity are described in detail.

In these lectures, there is no systematic *use* of Fredholm's theory—in fact, Fredholm's works are not quoted at all—in the treatment of such problems; instead Volterra often applied Green's function methods for solving PDEs. In particular, in the 10th and 11th lectures, devoted to the equation of heat propagation, he based his results on the work of several of his students—such as Boggio, E.E. Levi and G. Picciati.[62]

This publication was intended to accompany a second volume, on functions of lines, which contained the lectures he had given at the Sorbonne from January to March of 1912. This second book was edited by Joseph Pérès, a student of Volterra's at the University of Rome (Volterra 1913b). Both appeared in Borel's book series on theory of functions. These lectures on integral equations concentrate much on the theory of what are here labeled the Volterra and Fredholm , with applications playing a smaller role in comparison to the lectures held at the University of Rome. The last chapter (the 14th) is completely devoted to "hereditary" phenomena, as he describes hysteresis.

5.3 Some other textbook treatments and reception elsewhere

As already mentioned, a considerable number of expository versions of the theory of integral equations were written very early. Here, we restrict ourself to a quick description of three more: one by Maxime Bôcher, of interest in part because of its priority and because of the location of publication in the USA. The others, published in France at the same time, give competing versions associated with Picard and Hadamard.

5.3.1 Bôcher and the USA

Maxime Bôcher studied at Harvard, then went to Göttingen where he completed a PhD in 1891. He arrived in Göttingen when Klein was interested in potential theory, and his dissertation concerns series expansions of potential functions (i.e., functions

[61] ... je crois vous avoir démontré que les théories héréditaires ... peuvent être étudiées par les méthodes des équations. intégro-différentielles ... il est utile de laisser... indéterminé les constantes ... c'est à cause de cela que l'importance de l'application de l'algèbre aux question naturelles a toujours grandi.

[62] Picciati graduated from Pisa and became a secondary school teacher in Padua in 1896. He taught at the University of Padua occasionally but never had a professorial position. His research interests in mathematical physics were broad, though he is best known for a 1907 paper on the application of integral equations to the descent of a sphere in viscous fluid. Boggio supplied a simplification of the proof of the main result soon after (Picciati 1907; Boggio 1907c)

satisfying Laplace's equation), employing the *cyclides* of Darboux as a device to unify various kinds of special functions which are employed depending on the geometry of the problem. He was unable to solve the problem of convergence of these expansions, which remained open for many years, and his interest in integral equations may have arisen partly from the fact the Hilbert's methods offered the possibility of progress in that direction.[63]

Bôcher completed his *Introduction to the Study of Integral Equations* in November of 1908. It appeared in the series *Cambridge Tracts in Mathematics and Mathematical Physics* edited by G. H. Hardy and E. Cunningham and had a second edition in 1914. This seventy-page account pushes the subject back to Fourier, following Hermann Weyl in considering Fourier an unconscious user of integral equations. Bôcher looks at the historical roots of integral functional equations in the work of Abel and Dirichlet, and his account devotes quite a bit of attention to historical developments. Bôcher argues that the prospective importance of the theory was already noted by Du Bois-Reymond (1888) or even earlier by Rouché (1860). As for what the importance consists of, Bôcher is clear:

> ...like so many other branches of analysis the theory was called into being by specific problems in mechanics and mathematical physics. This was true not merely in the early days of Abel and Liouville, but also more recently in the cases of Volterra and Fredholm. Such applications of the theory, together with its relations to other branches of analysis, are what give the subject its great importance (Bôcher 1909, 2).

He is not specific about the issue of "relations to other branches of analysis," stating in a footnote: "cf., for instance, much of Hilbert's work.".

The book is a very thorough elementary presentation, starting with work by Abel and Liouville, continuing with Volterra and Fredholm, and the work of Hilbert and Schmidt on equations with symmetric kernel. Particular attention is paid to giving a rigorous account of Fredholm's work. There is nothing original as far as the results are concerned, and most of the description is based on the work of others, notably Fredholm, Hilbert, and Volterra. No account is given at all of the application to boundary value problems, though there is some mention of the relationship to series expansions, notably the 1907 work of Kneser (1907).

Despite the importance of applications, he expressly excludes them from his account, which means that this book would seem to have had at most an indirect effect on research in partial differential equations. Though it is useful as an introduction to the subject in English, it appeared a bit prematurely given the rapid development of the theory.

Bôcher was not the only writer in the American world to take an interest in the field. Wally Hurwitz had done a thesis in a related area with Hilbert at the University of Göttingen and then took a position at the University of Cornell in 1910. It was through the intermediary of Wally Hurwitz that Hu Mingfu, a Chinese student, had come to the U. S. sponsored by the Boxer reparation monies. Hu studied at Cornell

63 At any rate, Osgood, in an obituary of Bôcher, mentions the fact that Hilbert's work is relevant here (Osgood 1919, 238).

University from 1910 to 1914 and then went to Harvard, where he completed a PhD in 1917 under Bôcher. Hu was thus the first Chinese student to obtain a western PhD. The thesis, *Linear Integro-Differential Equations with a Boundary Condition* appeared in the *Transactions of American Mathematical Society*, v. 24.

5.3.2 Lalesco and Heywood & Fréchet

Probably more mainstream was the various textbook treatments in France. Trajan Lalesco was a student of Picard, completing a doctoral thesis in Paris on integral equations of the first kind in 1908.[64] The aim of Lalesco's book (Lalesco 1912) is to give an account of the theory (and not the applications) to that date. Accordingly, he treats the Volterra and Fredholm equations, discusses the work of Hilbert on symmetric kernels and Schmidt's extension of that theory. The corresponding eigenfunction expansions are treated in some detail. The concluding portion of the book gives a preliminary discussion of singular integral equations, following the work of Hermann Weyl; however, it is not a complete account of that theory. Lalesco likewise treats recent work of Volterra concerning adjoint pairs of integral equations (*fonctions permutables* in Volterra's terminology).

In contrast, the book by Heywood and Fréchet restricts itself to the Fredholm equation and specifically aims at giving an account of applications to mathematical physics. This book comes from the orbit of Hadamard and Picard. Fréchet was very much a protégé of Hadamard, Maz'ya and Shaposhnikova (1998). Horace Bryon Heywood had completed a doctoral thesis on the Fredholm equation and its applications in 1908 (Heywood 1908). The theory presented in this book is fully attached to the applications. In fact, the first chapter sets up a considerable number of boundary value problems as integral equations, beginning with an introduction to potential theory and the idea of a Green's function. Once the appropriate methods are developed, the final chapter of the book concerns the solution of these problems. The procedure is done in a very step-by-step fashion and indicates the author's faith in the notion that the Fredholm approach is one which will be generally used by anyone interested in the solution of such equations, notably physicists.

6 Concluding remarks

The distinct approaches belonging to different national "schools" were becoming merged in the textbook tradition by 1914, and as we observed French and Italian writers were beginning to be more cognizant of Hilbert's methods and approaches by that time. The story was then complicated by the advent of World War I.

The picture we are left with is one in which the abstract and theoretical approach led by the Hilbert school had a limited reception outside Germany in the context of work on integral equations. While Lauricella, and to some degree E. E. Levi, found it possible

[64] His name is more properly taken as Traian Lalescu, as it is written in Rumanian. The French spelling was used in most of the early literature, however.

to assimilate this point of view, they seem to have been substantially alone (outside the Hilbert sphere) in their success at grasping the potential of the new framework. The strong Italian tradition of applied work, particularly in elasticity but also in fluid dynamics, was able to make good use of these tools and appears to have continued to build on this tradition after the war, Tricomi being a notable (if late) example. Further research in this direction could be interesting in illuminating this national context.

The French case is to some degree intermediate between the two positions, in part perhaps because the two leading researchers, Picard and Hadamard, took rather different approaches to the interest of integral equations. While the subject was seen as important, there is no "majority" view, a situation indicated by the varying textbook treatments of Lalesco, Heywood & Fréchet, Goursat, D'Adhémar, and so on.

Volterra himself was later to claim that he had had a very general view of integral equations from early on, and the coincidence in time of his lectures on functions of lines and those on integral equations is only one indicator of their association in his mind. Writing in 1936, on the occasion of a new edition of his 1909 lectures, Volterra noted (Volterra and Përès 1936, vii):

> the concept of treating integral and integro-differential equations as chapters in the theory of functionals was already taken up in my *Leçons* of 1913. The development of these theories has revealed ever-tighter links between the different branches ...[65]

However, the differing views of the disciplinary status of this field has not been our main subject. Rather it is the explosive reception of a relatively simple insight in a variety of milieux that drew our attention to this work, and that, as Hadamard noted in the passage we cited at the beginning, is such a rare fate for a mathematical discovery.

Appendix: excerpt from Boggio's submission to the competition for the Prix Vaillant

Boggio's memoir has never been published, not in *Acta Mathematica*, to which it was submitted, or anywhere else. However, a copy of Boggio's work is contained at the Archives of the Academy of Sciences in Paris (Dossier "Prix Vaillant 1907"). In the same Dossier, all the memoirs of the candidates to the Prize can be found. Apart from Boggio they are (the names in brackets refer to the "rapporteurs" assigned): P. Debye of Monaco (Humbert), Boris Coialowitsch of St. Petersburg (Jordan), Henri Willotte from Quimper (Maurice Levy), John Dougall from Scotland (Darboux), Frederich Hollister Sofford of the USA (Darboux), Walter Ritz (Appell), Jow Simic (Darboux and Appell), Giuseppe Lauricella (Picard, Poincaré), Arthur Korn (Appell), Hadamard (Painlevé), Stanislaw Zaremba (Poincaré, Maurice Levy).

[65] le concept de traiter les équations intégrales et intégro-différentielles comme des chapitres de la Théorie des fonctionelles était déjà amorcé dans mes *Leçons* de 1913. Le développement de ces théories a révélé des rapports toujours plus étroits entre leurs diverses branches ...

Boggio's memoir is entitled: *Mémoire sur l'intégration de l'équation* $\Delta_{2m} = 0$ and we present in what follows the introduction to the paper.

Introduction

Le problème posé par l'Académie, consistant dans la détermination du déplacement transversal des points d'une plaque élastique plane, encastré au bord, soumise à des forces quelconques, est un des problèmes les plus intéressants de la théorie mathématique de l'Elasticité. Analytiquement il revient à déterminer la fonction w, régulière dans une aire donnée, qui satisfait aux points de cette aire, à l'équation indéfinie:

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + 2\frac{\mathrm{d}^4 w}{\mathrm{d}x^2 \mathrm{d}y^2} + \frac{\mathrm{d}^4 w}{\mathrm{d}y^4} = f(x, y) \tag{1}$$

$[f(x, y)$ étant une fonction donnée$]$ et qui s'annule sur le bord, ainsi que sa dérivée relative à la normale intérieure, c'est-à-dire:

$$u = 0, \frac{\mathrm{d}u}{\mathrm{d}n} = 0 \tag{2}$$

Un cas particulier de ce problème est celui où la plaque, supposée horizontale, est tirée par un seul poids, suspendu à un point quelconque de la plaque. Ce problème est très intéressant aussi pour les applications; cette importance a été reconnue déjà depuis longtemps : par ex. dans l'ouvrage classique *Théorie de l'elasticité des corps solides*[66] (traduct. française) Clebsch, après avoir résolu le problème pour une plaque circulaire, écrit (p. 777): *c'est un problème sur l'importance duquel, au point de vue des applications, il convient d'appeler l'attention des géomètres-physiciens, ainsi que sur la méthode au moyen de laquelle on réussira peut-être à trouver la solution pour d'autres formes que la circulaire.*

Donc non seulement est importante la résolution du problème, mais aussi la méthode employée pour le résoudre.

Pour traiter ce problème, pour le cas d'une aire circulaire, Clebsch emploie des développements en série, et ses calculs résultent bien compliqués ; il écrie en effet (p. 777): *c'est à des formules d'une telle complication que conduit un problème dont l'analogue, pour les tiges minces, passe à juste titre pour un des plus élémentaires. Et même on n'est arrivé à ces formules qu'en supposant la plaque circulaire.*

Au point de vue analytique ce problème, comme je le démontre au n. 5, se réduit à la détermination d'une particulière fonction, que l'on appelle *deuxième fonction de Green*, et qui pour une aire circulaire se détermine immédiatement sans recourir aux séries. Le problème général de l'intégration des équations (1), (2) peut toujours être nommé à la recherche de la deuxième fonction de Green, par conséquent lorsqu'on connaît cette fonction particulière, le problème en question peut être regardé comme résolu.

[66] Dans cet ouvrage on trouvera d'intéressants renseignéments historiques, de M. de Saint-Venant, sur l'équation (1), qui a été rencontrée, le premier, par Lagrange.

Comme on reconnaît immédiatement, l'intégration des équations (1), (2) peut aussi être ramenée à l'intégration des équations:

$$\frac{d^4u}{dx^4}+2\frac{d^4u}{dx^2dy^2}+\frac{d^4u}{dy^4}=0 \tag{3}$$

(dans l'aire donnée)

$$u=\varphi,\ \frac{du}{dn}=\psi \tag{4}$$

(sur le bord)

où φ, ψ désignent des fonctions données sur le bord de l'aire considérée.

Si l'on pose, pour abréger:

$$\Delta_2=\frac{d^2}{dx^2}+\frac{d^2}{dy^2}$$

on peut écrire l'équation (3) aussi ainsi $\Delta_2(\Delta_2 u)=0$, ou bien:

$$(\Delta_4 u)=0 \tag{3$'$}$$

L'intégration des équations (3′), (4) est un cas particulier de l'intégration des équations:

$$\begin{aligned}&\Delta_{2m}=0 && (5)\\ &\frac{d^iu}{dx^i}=\varphi \quad \text{(sur le bord)}\ (i=0,1,\ldots)\end{aligned}$$

où φ_i est une fonction donnée sur le bord.

L'intégration des équations (3′), (4) peut aussi être regardée comme un cas particulier de l'intégration des équations:

$$\Delta_2 U+k\frac{d}{dx}\left(\frac{dU}{dx}+\frac{dU}{dy}\right)=0 \tag{6}$$

(dans l'aire donnée)

$$\Delta_2 V+k\frac{d}{dx}\left(\frac{dV}{dx}+\frac{dV}{dy}\right)=0$$

$$U=\varphi,\ V=\psi \quad \text{(sur le bord),}$$

où k est une constante, et φ, ψ sont des fonctions données sur le bord.

Dans mon Mémoire j'étudie précisément l'intégration des systèmes (5) et (6). Voici rapidement le contenu des différents Chapitres.

Chap. I J'établie que la fonction u, qui vérifie les équations (5), est unique, et ensuite je démontre deux formules, qui sont pour l'opération Δ_{2m}, les analogues de celles de Green pour l'opération Δ_2. Je définie en outre les fonctions G_m de Green de l'ordre m, en montrant aussi que l'intégrale des équations (5) est comme si l'on connaît la fonction G_m. Des formules établies j'en fais une intéressante application en démontrant le théorème de réciprocité pour les fonctions G_m, théorème analogue à celui que Riemann a démontré pour l'ordinaire fonction de Green. Je termine en montrant l'application de la deuxième fonction de Green à la résolution du problème sur les plaques, représenté par l'intégration des équations (1), (2), et au cas particulier de la plaquer tirée par un seul poids ; de la formule que je trouve pour ce cas (et qui constitue la véritable interprétation physique de la deuxième fonction de Green) j'en fais plusieurs applications: entre autres j'en déduis une interprétation physique de mon théorème de réciprocité.

Chap. II Je démontre plusieurs théorème qui donnent la représentation d'une fonction polyharmonique (c'est-à-dire qui vérifie l'équation $\Delta_{2m} = 0$) au moyen de fonctions harmoniques. Ces théorèmes sont déjà connus, mais je les démontre ici avec des hypothèses moins restrictives par rapport à l'aire à laquelle ils se rapportent. Je démontre ensuite des théorèmes sur l'intégrale générale d'une équation linéaire, à coefficients constants, dont le premier nombre est le produit de plusieurs expressions différentielles ; les théorèmes descendent facilement d'un autre théorème qui donne la condition nécessaire et suffisante parce que deux équations différentielles linéaires, à coefficients constants, dont une seule est homogène, aient des intégrales communes.

Chap. III Je cherche l'intégrale de l'équation $\Delta_{2m}U = F(x, y)$, $[F(x, y)$ étant une fonction donnée, que l'on suppose polyharmonique], qui est régulière dans ne aire circulaire, et qui s'annule sur le bord ainsi que ses dérivés normales successives des $m - 1$ premiers ordres; la fonction cherchée est exprimée au moyen d'intégrales simples (quadratures).

Je détermine ensuite l'intégrale des équations (1), (2), pour une aire elliptique, et sous l'hypothèse que la fonction donnée f soit un polynôme du degré n; je démontre alors que la fonction cherchée u sera aussi un polynôme du degré n, dont les coefficients se calculent facilement. Ce résultat, que j'étends aussi au cas de l'équation $\Delta_{2m}u = f$, je l'établis à l'aide d'un lemme général qui peut appliqué à plusieurs questions de Physique-mathématique. Je cherche enfin (toujours pour une ellipse) les intégrales des équations (6) qui sur le bord coïncident avec deux polynômes donnés : ces intégrales sont aussi des polynômes.

Chap. IV La recherche de la fonction de Green G_m de l'ordre m, pour une aire circulaire est l'objet de ce chapitre. Je détermine d'abord la fonction préliminaire de Green de l'ordre m par un procédé très simple, au moyen de calculs algébriques et de dérivations ; et ensuite il en résulte l'expression de la fonction cherché G_m. Cette expression peut s'écrire sous la forme d'une intégrale définie, très simple, de laquelle on déduit que la fonction G_m a un signe constant dans le cercle; la formule établie s'étend immédiatement au cas d'un nombre quelconque de variables, et montre que, dans ce cas aussi, la fonction G_m a un signe constant; en outre elle permet de démontrer pour la fonction G_m un théorème analogue à un théorème donné par M. Poincaré pour l'ordinaire fonction de Green.

Je termine en donnant des limites supérieures pour les dérivées successives de la fonction G_m; ces inégalités seront très utiles pour l'application de la méthode des approximations successives de M. Picard à l'équation $\Delta_{2m}u = f(u)$.

Chap. V Il est facile de trouver, pour une aire circulaire, l'intégrale des équations (5); il suffit en effet de considérer la fonction G_m déterminée au Chap. IV et d'appliquer une formule établie au Chap. I. On peut aussi déterminer l'intégrale des équations (5) par une méthode directe et très simple, indiquée par MM. Almansi et Volterra.

Ensuite je cherche l'intégrale des équations (3), (4) pour une couronne circulaire, en appliquant la méthode de Venske, complétée par M. Almansi; mais tandis que M. Almansi doit faire des calcules pénibles pour prouver la compatibilité des équations que l'on trouve entre les coefficients à déterminer, j'applique un lemme établi au Chap. IV duquel il résulte immédiatement, sans aucun calcul, la compatibilité de ces équations.

Chap. VI Je démontre un théorème intéressant établi par MM. Levi-Civita et Volterra, et ensuite un théorème que l'on peut regarder comme le réciproque de celui de M. Volterra, et une généralisation d'un théorème bien connu, de M. Painlevé, sur les fonctions harmoniques.

Chap. VII L'intégration des équations (3), (4) pour une aire elliptique a été abordée par M. Mathieu, mais ses calculs sont très compliqués et ne sont pas complets. Je donne, au contraire, une méthode bien simple par résoudre ce problème, et je démontre deux théorèmes analogues à des théorèmes bien connus sur les séries de puissances, qui permettent d'établir la convergence des séries employées.

Je donne ensuite une méthode très simple pour intégrer les équations (3), (4), ou bien (5), dans l'aire extérieure à une ellipse donnée ; la solution que je trouve pour ce cas est composée seulement par des intégrales définies.

Chap. VIII J'intègre les équations (6) pour les aires dont on peut faire la représentation conforme sur un cercle par des polynômes, ou plus en général par des fonctions rationnelles. Au premier groupe d'aires appartiennent les épicycloïdes qui ont des points d'inflexion comme points singuliers. La solution dans les deux cas est composée seulement par des intégrales définies.

Comme application je traite le cas du limaçon de Pascal qui ne passe pas par son pole.

Chap. IX Par un simple procédé d'intégration je réduis le problème de l'intégration des équation (6), pour une aire quelconque, représentable sur un cercle d'une manière conforme, à la résolution d'un système d'infinies équations linéaires avec infinies inconnues. Je résoudre ce système par la méthode des approximations successives, qui est plus simple que celle employée, pour un système analogue, par M. Levi-Civita.

Chap. X J'applique la méthode des approximations successives de M. Picard, pour résoudre le problème généralisé de Dirichlet pour de certains types d'équations aux dérivées partielles de l'ordre $2m$. Je considère d'abord le cas d'une aire circulaire et ensuite le cas d'une aire quelconque simplement connexe. Si l'aire donnée est suffisamment petite la méthode des approximations successives donne toujours la solution du problème donné, aussi dans le cas des équations linéaires que dans le cas des équations non linéaires. J'étends en outre ces résultats au cas d'un systèmes de deux équations linéaires de l'ordre $2m$.

Chap. XI Je démontre un théorème qui ramène l'intégration des équations de la déformation des plaques élastiques cylindriques à l'intégration des équations (3), (4). Des formules que j'établis pour ce cas on déduit d'intéressants propriétés : entre autres celle que les intégrales des équations de la déformation des plaques élastiques cylindriques sont des fonctions régulières de deux certains paramètres. M. Lauricella a ensuite remarqué qu'il subsiste le théorème réciproque de mon théorème.

Chap. XII Je fais une remarquable application des théorèmes établis au Chapitre précédent, à l'intégration des équations (3), (4) pour une plaque rectangulaire ; ces théorèmes en effet permettent de ramener l'intégration de ces équation à un problème qui a été résolu par Mathieu. J'expose la solution de Mathieu en la simplifiant un peu, notamment pour ce qui est relatif à la résolution d'un système d'infinies équations linéaires avec infinies inconnues, que je traite par la méthode des approximations successives.

Chap. XIII J'expose dans ce dernier Chapitre quelques applications des équations intégrales à la Physique mathématique, notamment aux questions qui ont des relations avec le problème posé par l'Académie.

Après avoir simplifié la démonstration de M. Hilbert, je réduis le problème général de la détermination de la déformation d'un corps élastique isotrope, quand on connaît la déformation de sa surface, à la résolution d'une équation intégrale du type de Fredholm. Je démontre que cette équation a une solution et une seule pourvu qu'un certain paramètre soit $\neq \geq -1$.

Dans le cas de deux variables, cette méthode donne l'intégrale des équations (6) et par conséquent, comme cas particulier, la résolution du problème posé par l'Académie.

J'étudie ensuite la détermination de la déformation d'un corps élastique, isotrope, quand on connaît les tensions appliquées à sa surface; dans le cas de deux variables cette méthode donne une nouvelle solution du problème posé par l'Académie. De ce problème j'en donne aussi d'autres solutions, valides pour un nombre quelconque de variables.

Enfin j'étudie, par les équations intégrales, l'intégration de certaines équations, aux dérivées partielles, de l'ordre $2m$, et j'en trouve la solution pour une aire quelconque, simplement connexe. En terminant je traite l'existence des solutions singulières pour de certaines équations de l'ordre $2m$, qui sont une généralisation des équations des membranes et plaques vibrantes.

References

Almansi, Emilio. 1896. Sulla integrazione dell'equazione differenziale $\Delta^2\Delta^2 = 0$. *Atti R. Acc. Scienze di Torino* 31: 881–888.

Almansi, Emilio. 1897. Sulla deformazione della sfera elastica. *Memorie R. Acc. Scienze di Torino s. II* 47: 103–125.

Almansi, Emilio. 1899a. Sull'integrazione dell'equazione differenziale $\Delta^{2n} = 0$. *Ann. matem. pura ed appl. s. III* 2: 1–51.

Atti del IV Congresso Internazionale dei Matematici. Roma, Accademia dei Lincei. 3 v. 1909.

Bernstein, Sergej. 1904. Sur la nature analytique des solutions des équations aux dérivées partielles du second ordre. *Math. Annalen* 59: 20–76.

Betti, Enrico. 1872–73. *Teoria della elasticità*. Pisa: Tip. Pieraccini, 1874; *Nuovo Cimento s. II*, vol. 7–10. Reprinted in (Capecchi, Ruta and Tazzioli 2006).

Bôcher, Maxim. 1909. *An introduction to the theory of integral equations*. Cambridge: Cambridge University Press.

Boggio, Tommaso. 1900a. Sull'equilibrio delle membrane elastiche piane. *Nota I. Atti Acc. Scienze di Torino* 35: 219–239; Nota II, Ibid., 239 (1901a); see also *Nuovo Cimento* s. IV, vol. 12, 1900, 170–190; s. V, vol. 1, 1901, 161–178.

Boggio, Tommaso. 1900b. Sull'equilibrio delle membrane elastiche piane. *Atti Acc. Scienze Torino* 35: 219–239.

Boggio, Tommaso. 1900c. Un teorema di reciprocità sulle funzioni di Green d'ordine qualunque. *Atti Acc. Scienze di Torino* 35: 498–509.

Boggio, Tommaso. 1900d. Integrazione dell'equazione in una corona circolare e in uno strato sferico. *Atti Ist. Veneto di Scienze, Lettere ed Arti* 59(II): 497–508.

Boggio, Tommaso. 1901a. Integrazione dell'equazione in un'area ellittica. *Atti Ist. Veneto di Scienze, Lettere ed Arti* 60(II): 591–609.

Boggio, Tommaso. 1901b. Sull'equilibrio delle piastre elastiche incastrate. *Rend. R. Acc. dei Lincei s. V* 10(I): 197–205.

Boggio, Tommaso. 1904. Sulla deformazione delle piastre elastiche cilindriche di grossezza qualunque. *Rend. R. Acc. dei Lincei s. V* 13(II): 419–427.

Boggio, Tommaso. 1906. Risoluzione del problema dei valori al contorno per alcuni casi di equazioni alle derivate parziale. *Rend. Circolo Mat. Palermo* 21: 283–306.

Boggio, Tommaso. 1907a. Nuova risoluzione di un problema fondamentale della teoria dell'elasticità. *Rend. R. Acc. dei Lincei* 16(II): 248–255.

Boggio, Tommaso. 1907b. Sull'equazione del moto vibratorio delle membrane elastiche. *Rend. R. Acc. dei Lincei (5)* 16(II): 386–393.

Boggio, Tommaso. 1907c. Integrazione dell'equazione funzionali che reggie la caduta di una sfera in un liquido viscoso. *Rend. R. Acc. dei Lincei (5)* 16(II): 613–620, 730–737.

Boltzmann, Ludwig. 1874. Theorie der elastischen Nachwirkung. *Ber. K. K. Akad. Wien* 70: 275–306.

Boltzmann, Ludwig. 1876. Zur Theorie der elastischen Nachwirkung. *Pogg Ann. Erg*. 7: 624.

Bottazzini, Umberto. 1986. *The higher calculus. A history of real and complex analysis from Euler to Weierstrass*. New York: Springer.

Brezis, Chaim and F. Browder. 1998. Partial differential equations in the 20th century. *Advances in Mathematics* 135: 76–144.

Capecchi, Danilo, Ruta Giuseppe, and Rossana Tazzioli. 2006. *Enrico Betti:Teoria della Elasticità*. Benevento: Hevelius.

Charpentier', E., É. Ghys, and A. Lesne. 2010. *L'heritage scientifique de Poincaré*. Paris: Belin . English translation *The Scientific Legacy of Poincaré*. Providence: AMS.

Cerruti, Valentino. 1882. Ricerche intorno all'equilibrio dei corpi elastici isotropi. *Memorie R. Acc. dei Lincei s. III* 13: 81–123.

d'Adhémar, Robert. 1908. Sur les équations intégrales de M. Volterra. In *(Atti Roma 1909), II*, 115–121.

Dieudonné, Jean. 1981. *History of functional analysis*. Amsterdam: North Holland.

Du Bois-Reymond, Paul. 1888. Bemerkungen über $\Delta z = 0$. *Jour. f ür Math*. 103: 204–229.

Fischer, Ernst. 1907. Sur la convergence en moyenne. *Comptes rendus de l'Académie des Sciences* 144: 1022–1024.

Forsyth, A.R. 1908. On the present condition of partial differential equations of the second order as regards formal integration. In *(Atti Roma 1908)* I, 87–103.

Fredholm, Ivar. 1899. Sur une classe d'équations aux dérivées partielles. *Comptes rendus de l'Académie des Sciences* 129: 32–34.

Fredholm, Ivar. 1900. Sur une nouvelle méthode pour la résolution du problème de Dirichlet. *Stockholm Öfver* 57: 39–46.

Fredholm, Ivar. 1902. Sur une classe de transformations rationnelles. *Comptes rendus de l'Académie des Sciences* 134: 219–222.

Fredholm, Ivar. 1903. Sur une classe d'équations fonctionelles. *Acta Math.* 27: 365–390.

Fredholm, Ivar. 1905. Solution d'un prolblème fondamentale de la théorie de l'élasticité. *Arkiv för Mat. fys, och ast.* 2(28): 8

Fubini, Guido. 1905. Un'osservazione sulla teoria delle funzioni poliarmoniche. *Rendiconti Istit. Lombardo* 38: 449–453.

Goursat, Édouard. 1907. Sur un cas élémentaire de l'équation de Fredholm. *Bulletin de la Soc. math. de France* 35: 163–173.

Goursat, Édouard. 1915. *Traité d'analyse*. 2nd edn., Paris. On Integral equations see v. 3, chaps 30–33, 324–544.

Gray, Jeremy. 2013. *Henri Poincaré: A scientific biography*. Princeton: Princeton University Press

Groetsch, Charles W. 2003. The delayed emergence of regularization theory. *Bollettino Storiadelle Scienze Matemetiche* 23: 105–120.

Hadamard, Jacques. 1893. Sur le module maximum qui puisse atteindre un déterminant. *Comptes rendus de l'Académie des Sciences* 116: 1500–1501; in Oeuvres (vol. 4, Paris, CNRS, 1968) vol. 1, 237–238.

Hadamard, Jacques. 1908. Sur certains cas intéressants du problème biharmonique. *Atti del IV Congresso dei Matematici* 2: 61–63. Roma: Accademia dei Lincei; in Oeuvres vol. 3, 643–645.

Hadamard, Jacques. 1909. Sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées. *Mémoires savants étrang. s. II* 33(4); in Oeuvres vol. 2, 515–629. Prix Vaillant.

Hadamard, Jacques. 1910. Sur les ondes liquides Ière et IIéme notes. *Comptes rendus de l'Académie des Sciences* 150: 609–611, 772–774; in Oeuvres vol. 3, 1301–1303, 1317–1320.

Hadamard, Jacques. 1928. Le développement et le rôle scientifique du calcul fonctionnel. *Atti del Congresso Internazionale dei Matematici*, Bologna, Zanichelli, Bologna 1929, vol. 1, 143–161; In *Oeuvres* vol. 1, 435–453.

Happel, J., and H. Brenner. 1965. *Low Reynolds number hydrodynamics*. Englewood Cliffs NY: Prentica Hall.

Hellinger, Ernst. 1935. Hilberts Arbeiten über Integralgleichungen und unendliche Gleichungssysteme. In (*Hilbert Gesammelte Abhandlungen*) vol. III, 95–145.

Hellinger, Ernst and Otto Toeplitz. 1927. Integralgleichungen und Gleichungen mit unendlichviele unbekannten. *Enzyklopädie der mathematischen Wissenschaften II.3.2, section IIC.* 13: 1335–1597.

Heywood, Horace B. 1908. *L'équation de Fredholm et quelques-unes de ses applications*. Thèse, Université de Paris.

Heywood, Horace B., and Maurice Fréchet. 1912. *L'équation de Fredholm et ses application á la physique mathématique*. Préface de J. Hadamard. Paris, Hermann

Hilbert, David. 1900. Sur le principe de Dirichlet. Trans. M. Laugel. *Nouvelles Annales de Mathématiques s. III* 19: 337–344.

Hilbert, David. 1904. Grundzüge einer allgemeinen Theorie der linearen Integralgkleichung. *Gött. Nachr.* 49–91: 213–259.

Hilbert, David. 1935. *Gesammelte Abhandlungen*. Berlin: Springer. Cited here from the Second edition, 1970.

Korn, Arthur. 1899. *Lehrbuch der Potentialtheorie, allgemeine Theorie des Potentials und der Potentialfunktionen im Raume*. Berlin: F. Dümmler.

Kneser, Arthur. 1907. Die Theorie der Integralgleichungen und die Darstellung willkürlicher Funktionen in der mathematischen Physik. *Math. Annalen* 63: 477–524.

Korn, Arthur. 1902. *Fünf Abhandlungen zur Potentialtheorie*. Berlin: F. Dümmler.

Korn, Arthur. 1907. Sur les équations de l'élasticité. *Ann. Sci. Ecole Normale Sup. s. III* 24: 9–75.

Lalesco, Trajan. 1912. *Introduction à la théorie des équations intégrales*. Préface de E. Picard. Paris, Hermann.

Langlois, W.E. 1964. *Slow viscous flow*. New York: Macmillan.

Lauricella, Giuseppe. 1895. Equilibrio dei corpi elastici isotropi. *Annali R. Scuola Norm. Sup. di Pisa* 7: 1–119.

Lauricella, Giuseppe. 1895–96. Integrazione dell'equazione in un campo di forma circolare. *Atti Acc. Scienze Torino* 31: 1010–1018.

Lauricella, Giuseppe. 1896. Sull'equazione delle vibrazioni delle placche elastiche incastrate. *Memorie Acc. Scienze Torino s. II* 46: 65–92.

Lauricella, Giuseppe. 1898. Integrazione della doppia equazione di Laplace in un campo a forma di corona circolare. *Atti Ist. Veneto di Scienze, Lettere ed Arti* 57: 236–250.

Lauricella, Giuseppe. 1901a. Sulle funzioni biarmoniche. *Rend. R. Acc. dei Lincei s. V* 10: 147–150.

Lauricella, Giuseppe. 1901b. Sulla deformazione di una sfera elastica isotropa per dati spostamenti in superficie. *Ann. matem. pura ed appl. s. III* 6: 289–299.

Lauricella, Giuseppe. 1903. Sulla deformazione di una sfera elastica isotropa per date tensioni in superficie. *Nuovo Cimento s. V* 5: 5–26.

Lauricella, Giuseppe. 1904. Sulle formole che danno la deformazione di una sfera elastica isotropo. *Rend. R. Acc. dei Lincei s. V* 13(II): 583–590.

Lauricella, Giuseppe. 1905. Sull'integrazione delle equazioni dell'equilibrio dei corpi elastici isotropi. *Ann. matem. pura ed appl. s. III* 11: 269–283.

Lauricella, Giuseppe. 1907. Alcune applicazioni della teoria delle equazioni funzionali alla fisica matematica. *Nuovo Cimento s. V* 13: 104–118, 155–174, 237–262, 501–518.

Lauricella, Giuseppe. 1908a. Sulla vibrazione delle placche elastiche incastrate. *Rend. R. Acc. dei Lincei s. V* 17(II): 193–204.

Lauricella, Giuseppe. 1908b. Sopra alcune equazioni integrali. *Rend. R. Acc. dei Lincei s. V* 17(I): 775–786.
Lauricella, Giuseppe. 1908c. Applicazione della teoria di Fredholm al problema del raffreddamento dei corpi. *Ann. matem. pura ed appl. s. III* 14: 143–169.
Lauricella, Giuseppe. 1908d. Sull'equazione $\Delta^{2i}V = 0$ e su alcune estensioni delle equazioni dell'equilibrio dei corpi elastici isotropi. In (*Atti Roma 1908d*) vol. III, 33–59.
Lauricella, Giuseppe. 1909a. Sur l'intégration de l'équation relative à l'équilibre des plaques élastiques encastrées. *Acta Mathematica* 32: 201–256.
Lauricella, Giuseppe. 1909b. Sull'equazione integrale di prima specie. *Rend. R. Acc. dei Lincei s. V* 18(II): 71–75.
Lauricella, Giuseppe. 1911. Sulla risoluzione dell'equazione integrale di prima specie. *Rend. R. Acc. dei Lincei s. V* 20: 528–536.
Lauricella, Giuseppe. 1912. Sulla chiusura di sistemi di funzioni ortogonali e dei nuclei delle funzioni integrali. *Rend. R. Acc. dei Lincei s. V* 21(I): 675–685.
Lebesgue, Henri. 1904. *Leçons sur l'intégration et la recherche de fonctions primitives*. Paris: Gauthier-Villars.
Le Roux, Jean Marie. 1895. Sur les intégrales des équations linéaires. *Ann. Sci. ENS s. III* 12: 227–316.
Le Roux, Jean Marie. 1903. Recherches sur les équations aux dérivées partielles. *Journal de mathématiques pures et appliquées S. V.* 9: 403–455.
Levi, Eugenio Elia. 1906. Su un lemma del Poincaré. *Rend. R. Acc. dei Lincei s. V* 14: 83–89, 353–358.
Levi, Eugenio Elia. 1907. Sulle equazioni integrali. *Rend. R. Acc. dei Lincei s. V* 16(II): 604–6012.
Levi, Eugenio Elia. 1908. Sur l'application des équations intégrales au problème de Riemann. *Gött. Nachr.*, 249–252.
Levi-Civita, Tullio. 1954–1973. *Opere Matematiche* vol. 6. Zanichelli: Bologna.
Levi-Civita, Tullio. 1895–96. Sull'inversione degli integrali definiti nel campo reale. *Atti Acc. Scienze di Torino* 31: 25–51; In (*Levi-Civita 1954–1973*) vol. 1, 159–184.
Levi-Civita, Tullio. 1897–98. Sopra una trasformazione in se stessa dell'equazione $\Delta_2\Delta_2 = 0$. *Atti Ist. Veneto di Scienze, Lettere ed Arti* 56: 1399–1410; In *(Levi-Civita 1954–1973)* vol. 1, 357–367.
Levi-Civita, Tullio. 1898. Sulla integrazione dell'equazione $\Delta_2\Delta_2 = 0$. *Atti Acc. Scienze di Torino* 33: 932–956; In *(Levi-Civita 1954–1973)* vol. 1, 331–355.
Liapunov, Alexander Mikhaylovich. 1898. Sur certaines questions qui se rattachent au problème de Dirichlet. *Journ. de math. pures et appl. s. V* 4: 241–311.
Lützen, Jesper. 1990. *Joseph Liouville 1809–1882: Master of pure and applied mathematics*. New York: Springer.
Marcolongo, Roberto. 1903. *Teoria matematica della elasticità: lezioni dettate nella R. Università di Messina, anno scolastico 1902–1903*. Messina.
Marcolongo, Roberto. 1907. La teoria delle equazioni integrali e le sue applicazioni alla Fisica matematica. *Rend. R. Acc. dei Lincei s. V* 6(I): 742–749.
Mason, Charles Max. 1903. *Randwertaufgaben bei gewöhnlichen Differentialgleichungen*. Ph. D. thesis, Göttingen.
Mathieu, Emile. 1869. Mémoire sur les équations aux différences partielles du quatrième ordre $\Delta\Delta u = 0$ et sur l'équilibre d'élasticité d'un corps solide. *Journ. Math. pures et appliquées, s. II* 14: 378–421.
Mawhin, Jean. 2006/2010. Henri Poincaré and the partial differential equations of mathematical physics. In *Charpentier, Ghys, and Lesne*, 257–278.
Mazliak, Laurent, and Rossana Tazzioli. 2009. *Mathematicians at war: Volterra and his French colleagues in World War I*. New York: Springer.
Maz'ya, Vladimir, and Tatyana Shaposhnikova. 1998. *Jacques Hadamard, a universal mathematician*. Providence: American Mathematical Society.
Meleshko, V.V. 2003. Selected Topics in the history of the two-dimensional biharmonic problem. *Applied Mechanics Reviews* 56: 33–85.
Monna, A.F. 1975. *Dirichlet's principle: A mathematical comedy of errors*. Oosthoek, Scheltena & Hoeltema.
Nabonnand, Philippe (ed.). 1999. *La correspondance entre Henri Poincaré et Gösta Mittag-Leffler*. Basel: Birkhäuser.
Nastasi, Pietro, and Rossana Tazzioli. 2004. Sulla determinazione della m.ma funzione di Green e questioni connesse. *Rend. Circ. Matem. Palermo Suppl. s. II* 74: 71–101.
Nastasi, Pietro, and Rossana Tazzioli. 2006. Problems of method in Levi-Civita's contributions to hydrodynamics. *Revue d'histoire des mathématiques* 12: 81–118.

Orlando, Luciano. 1906. Sull'integrazione della Δ_4 in un parallelepipedo rettangolo. *Rend. Circolo Matem. Palermo* 21: 316–344.

Orlando, Luciano. 1908. Sulla risoluzione delle equazioni integrali. In *(Atti Roma 1908)* II, 1908, 122–128.

Osgood, William Fogg. 1919. The life and services of Maxime Bôcher. *Bulletin of the American Mathematical Society* 25: 337–350.

Painlevé, Paul. 1909. Mécanique: Le genèse de la mécanique et ses influences sur les autres sciences. *De la méthode dans les sciences, par MM. les professeurs Bouasse [et al]*. Paris, F. Alcan, 363–409.

Pedersen, Gert K. 1988. *Analysis Now*. New York: Springer.

Picard, Émile. 1893. *Traité d'Analyse*, vol. 3. Paris: Gauthier-Villars.

Picard, Émile. 1904a. Sur certaines solutions doublement périodiques de quelques équations aux dérivées partielles. *Comptes rendus de l'Académie des Sciences* vol. 138, 181–183; in (Picard 1981) vol. 2, 687–689.

Picard, Émile. 1904b. Sur une équation fonctionelle. *Comptes rendus de l'Académie des Sciences* 139: 245–248; in *(Picard 1981)* vol. 4, 313–316.

Picard, Émile. 1906a. Sur quelques problèmes de Physique mathématique se rattachant à l'équation de M. Fredholm. *Comptes rendus de l'Académie des Sciences* 142: 861–865; in *(Picard 1981)* vol. 4, 317–321.

Picard, Émile. 1906b. Sur quelques applications de l'équation fonctionelle de M. Fredholm. *Rendiconti Circolo matem. Palermo* 22: 1906, 241–259; in *(Picard 1981)*, vol. 4, 323–341.

Picard, Émile. 1907. La mécanique classique et ses approximations successives. *Rivista di Scienza* 1(1): 14.

Picard, Émile. 1908. La mathématique dans ses rapports avec la physique, in *(Atti Roma 1908)*, I, 183–195.

Picard, Émile. 1909a. Quelques remarques sur les équations intégrales de première espèce et sur certains problèmes de Physique mathématique. *Comptes rendus de l'Académie des Sciences* 148: 1909, 1563–1568; in *(Picard 1981)* vol. 4, 379–384.

Picard, Émile. 1909b. Sur les équations intégrales de première espèce. *Comptes rendus de l'Académie des Sciences* 148: 1707–1708; in *(Picard 1981)* vol. 4, 385–386.

Picard, Émile. 1910. Sur un théoréme général relatif aux équations intégrales de première espèce et sur quelques problémes de physique mathématique. *Rendiconti Circolo Matem. Palermo* 29: 79–97.

Picard, Émile. 1981. *Œuvres de Charles-Émile Picard*, vol. 4. Paris: CNRS.

Picciati, Giuseppe. 1907. Integrazione dell'equazione funzionali che reggie la caduta di una sfera in un liquido viscoso. *Rend. R. Acc. dei Lincei* (5) 16(II): 45–50.

Plancherel, M. 1912. La théorie des équations intégrales. *Enseignement mathématique* 14: 89–107.

Poincaré, Henri. 1913–1965. *Œuvres*, vol. 11. Paris: Gauthier-Villars.

Poincaré, Henri. 1890. Sur les équations aux dérivées partielles de la physique mathématique. *American Journal of Mathematics* 1: 211–294; in *(Poincaré 1913–1965)* vol. 9, 28–113.

Poincaré, Henri. 1894. Sur les équations de la physique mathématique. *Rendiconti Circ. Matem. Palermo* 8: 57–156; in *(Poincaré 1913–1965)* vol. 9, 123–196.

Poincaré, Henri. 1897. La méthode de Neumann et le problème de Dirichlet. *Acta Math* 20: 59–142; in *(Poincaré 1913–1965)* vol. 9, 202–272.

Poincaré, Henri. 1908a. Remarques sur l'équation de Fredholm. *Comptes rendus de l'Académie des Sciences* 147: 1367–1371; In *(Poincaré 1913–1965)* vol. 3, 540–544.

Poincaré, Henri. 1908b. Sur quelques applications de la méthode de Fredholm. *Comptes rendus de l'Académie des Sciences* 148: 125–126; In (*Poincaré 1913–1965*) vol. 3, 545–546.

Poincaré, Henri. 1908c. L'avenir des mathématiques, in *(Atti Roma 1908)*, vol. I, 167–182.

Riesz, Friedrich. 1907. Ueber orthogonale Funktionsysteme, *Gött. Nachr.*, 116–122.

Rouché, Eugène. 1860. Sur le calcul inverse des intégrales définis. *Comptes rendus de l'Académie des Sciences* 51: 126–128.

Schmidt, Erhard. 1905. *Entwicklung willkürlicher Funktionen nach Systemen vorgeschriebener Funktionen*. Göttingen.

Schmidt, Erhard. 1907. Zur Theorie der linearen und nichtlinearen Integralgleichungen. *Math. Annalen* 63: 433–476.

Schwarz, Hermann Amandus. 1885. Ueber ein die Flächen kleinsten Flächeninhalts betreffendes Problem der Variationsrechnung. *Acta Soc. Sci. Finn.* 15: 315–362. In *Festschrift zum siebzigsten Geburtstage der Herrn Karl Weierstrass*. Helsingfors, Druck der Finnischen Literatur-Gesellschaft

Sherman, D.I. 1940. On the solution of a plane static problem of the theory of elasticity under prescribed forces. *Dokl. Akad. Nauk SSSR* 28: 25–28.

Siegmund-Schultze, Reinhard. 1982. Die Anfänge der Funktionalanalysis und ihr Platz im Umwälzungsprozeß der Mathematik um 1900. *Arch. Hist. Exact Sciences* 26: 13–71.

Siegmund-Schultze, Reinhard. 2003. The origins of functional analysis. In *A history of analysis*, ed. H.N. Jahnke, 385–407. Providence: American Mathematical Society.

Smithies, Frank. 1965. *Integral equations*. Cambridge: Cambridge University Press.

Somigliana, Carlo. 1890. Sulle equazioni della elasticità. *Ann. matem. pura ed appl. s. II* 17: 37–64.

Somigliana, Carlo. 1894. Sopra gli integrali delle equazioni della isotropia elastica. *Nuovo Cimento s. III* 36(29–39): 113–126.

Stekloff, Vladimir. 1900. Mémoire sur les fonctions harmoniques de M. Poincaré. *Toulouse Ann. s. II* 2: 273–303.

Stekloff, Vladimir. 1902. Sur les problèmes fondamentaux de la physique mathématique. *Annales de l'Ec. Norm. Sup. s. III* 19: 191–259, 455–490.

Stokes, George Gabriel. 1850. On the effect of the internal friction of fluids on the motion of pendulums. *Transactions Cambridge Philosophical Society* 9: 8–106.

Tazzioli, Rossana. 1994. Il teorema di rappresentazione di Riemann: Critica e Interpretazione di Schwarz. *Rend. Circolo Matem. Palermo Suppl. s. II* 34: 95–132.

Tazzioli, Rossana. 2001. Green's Function in some contributions of 19th century mathematicians. *Historia Mathematica* 28: 232–252.

Tedone, Orazio. 1907. Sui metodi della fisica matematica. *Atti del Primo Congresso della S.I.P.S.*, 33–47. Parma.

Tricomi, Francesco. 1957. *Integral equations*. New York: Interscience. Page references here from the 1985 reprint by Dover, New York.

Volterra, Vito. 1954–1960. *Opere matematiche*, vol. 4. Roma: Bardi.

Volterra, Vito. 1884. Sopra un problema di elettrostatica. *Nuovo Cimento s. III* 16: 49–57; in *(Volterra 1954–1960)* vol. 1, 188–195.

Volterra, Vito. 1895. Sopra un sistema di equazioni differenziali. *Atti R. Acc. Scienze di Torino* 30: 445–454; in *(Volterra 1954–1960)* vol. 2, 122–128.

Volterra, Vito. 1896a. Sulla inversione degli integrali definiti. *Atti R. Acc. Scienze di Torino* 31: 311–323, 400–408, 557–567, 693–708; in *(Volterra 1954–1960)* vol. 2, 216–254.

Volterra, Vito. 1896b. Sulla inversione degli integrali definiti. *Rend. R. Acc. dei Lincei s. V* 5: 177–185; in *(Volterra 1954–1960)* vol. 2, 255–262.

Volterra, Vito. 1896c. Sulla inversione degli integrali multipli. *Rend. R. Acc. dei Lincei s. V* 6: 289–300; in *(Volterra 1954–1960)* vol. 2, 263–275.

Volterra, Vito. 1898. Sul fenomeno delle seiches. Conf. tenuta al Congr. della Soc. italiana di Fisica in Torino il 23 settembre 1892. *Nuovo Cimento s. IV* 8: 270–272; in *(Volterra 1954–1960)* vol. 2, 370–372.

Volterra, Vito. 1901–02. Sui tentativi di applicazione delle matematiche alle scienze biologiche e sociali. *Ann. R. Univ. di Roma*, 3–28; in *(Volterra 1954–1960)* vol. 3, 14–29.

Volterra, Vito. 1906. Leçons sur l'intégration des équations différentielles aux dérivées partielles; in *(Volterra 1954–1960)* vol. 3, 63–141; publ. also Paris, Hermann, 1912.

Volterra, Vito. 1912. L'applicazione del calcolo ai fenomeni di eredità, French version *Revue du Mois*, 556–575 and Italian version in Saggi Scientifici, 1920, Bologna, Zanichelli, 189–218; in *(Volterra 1954–1960)* vol. 3, 554–568.

Volterra, Vito. 1913a. *Leçons sur les équations intégrales et les équations intégro-différentielles*. Paris: Gauthier-Villars.

Volterra, Vito. 1913b. *Leçons sur les fonctions de lignes*. Paris: Gauthier-Villars.

Volterra, Vito, and Joseph Përès. 1936. *Théorie générale des fonctionelles*. Paris: Gauthier-Villart.

Weyl, Hermann. 1908. *Singuläre Integralgleichungen mit besonderer Berücksichtigung des Fourierschen Integraltheorems*. Göttingen.

Weyl, Hermann. 1909. Ueber die Konvergenz von Reihen, die nach Orthogonalfunktionen fortschreiten. *Math. Annalen* 67: 225–245.

Zaremba, Stanislaw. 1901. Ueber die Laplacesche Gleichung und die Methoden von Neumann und Robin. *Krak. Anz*, 171–189.

Zaremba, Stanislaw. 1902. Sur l'intégration de l'équation $\Delta u + u = 0$. *Journ. de Math. pures et appl. s. V* 8: 59–117.

Zaremba, Stanislaw. 1905. Solution générale du problème de Fourier. *Krak. Anz*, 69–168.

Zeilon, Nils. 1930. Biographie de Ivar Fredholm. *Acta Mathematica* 54: I–XVI.