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Root extraction by Al-Kashi and Stevin

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Abstract In this paper, we study the extraction of roots as presented by Al-Kashi in his 1427 book "Key to Arithmetic" and Stevin in his 1585 book "Arithmetic". In analyzing their methods, we note that Stevin's technique contains some flaws that we amend to present a coherent algorithm. We then show that the underlying algorithm for the methods of both Al-Kashi and Stevin is the same.

1 Introduction

In 1427, Jamshid Al-Kashi completed a monumental book in Arabic on arithmetic called *Miftāh al-hisāb*, "Key to Arithmetic" that encompasses arithmetic, algebra, and measurement. In this book, among other things, Al-Kashi provides a method of extracting nth roots and apply it to numbers in the trillions. Given a positive integer N, his algorithm produces an approximation $R = r\frac{u}{v}$ to $\sqrt[n]{N}$ such that $R^n \leq N$. Moreover, when N is a perfect nth power, it produces the exact root.

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In 1585, Simon Stevin published an influential book in French called *L'arithmétique*, "Arithmetic". Actually, it is two books put together, with definitions in the first book and operations that include rules of ordinary arithmetic, radicals, polynomials, and equations in the second book. In the second part of this book, Stevin dedicated a section to extracting roots. He applied the method he presented to the extraction of roots of integers and fractions to any degree of accuracy.

The problem of the extraction of roots has been extensively studied, and many methods have been developed throughout the centuries (Dakhel 1960; Rashed 1978, 1994). Here, we look at the two previously mentioned examples emanating from two mathematicians who had great influence in their respective regions at different periods of time. In exploring Stevin's method, we found that it presents some flaws mainly due to oversimplifying some optimizations. We provide amendments that make Stevin's method viable. Then, we show that although the presentations of the methods as given by Al-Kashi and Stevin look different, the underlying algorithm is in fact the same. We note that, to the best of our knowledge, there is no proof of direct link between Al-Kashi and Stevin. Following Berggren (1986), we will provide a justification for the extraction of roots.

2 Brief history

Jamshid Al-Kashi (1380–1429) was an Islamic scientist who was born in Kashan (modern day Iran) around 1380 and passed away in Samarkand (modern day Uzbekistan) on June 22, 1429. His accomplishments, mostly in mathematics and astronomy, have been well documented and dated. His best work was achieved under the patronage of Ulugh Beg who was a great scientist in his own right and the ruler of Samarkand at the time. As Samarkand grew into a world-class scientific center, Al-Kashi was considered as one of the leading scientists in the city. His work spanned many areas of mathematics and astronomy, dealing with arithmetic, algebra, trigonometric functions, astronomical tables, and numerical approximations. In terms of findings and exposition, he was well ahead of his time. For example, Al-Kashi used a fine pedagogy (Taani 2014), explicitly and thoroughly explained decimal fractions in his work more than a hundred and fifty years before Stevin (Luckey 1951), and obtained an approximation of π correct to 16 decimal places (Hogendijk 2009) that was unsurpassed for nearly 200 years. Without doubt, Al-Kashi's most remarkable book is *Miftāh al-hisāb*, "Key to Arithmetic", which he finished on March 2, 1427. It is a part of this book that we are concerned with in this paper.

Simon Stevin (1548–1620) was a Flemish scientist born in Bruges in present-day Belgium around 1548 and passed away in The Hague on February 1620. His work covered many areas such as architecture, geography, mathematics, music, and navigation. Many of his scientific accomplishments came out while he was working for the Prince Maurice, Count of Nassau, first as an advisor and later at different other positions as a public officer. Stevin was a renowned scientist who contributed to the dissemination and acceptance of many mathematical concepts in Europe. For example, it is through the English translation of his book "La Theinde", that Thomas Jefferson proposed a decimal currency in the USA. Stevin's introduction and popularization of many mathematical notations that are still in use today, such as the square root symbol, is another legacy of his. Just like Al-Kashi, Stevin usually provides the reader with many examples and methods for solving a given problem. This is a striking similarity with Al-Kashi that we are bound to observe in his book *L'arithmétique*, "Arithmetic" which he published in 1585. It is a part of this book that we are dealing with in this paper.

3 Root extraction in Al-Kashi's work

In *Miftah*, Al-Kashi starts the process of extracting roots by defining roots. He introduces and explains the idea of cycles that is central in his method of extracting roots. Then, he describes the square root algorithm and its layout in general for an arbitrary number. The algorithm is executed using a table, and his description is purely in prose. He illustrates the algorithm with a few examples, but he does not provide a formal justification as to why it works. Dakhel (1960) and Berggren (1986) give a theoretical basis and justification for the algorithm. Al-Kashi then goes on to describing a general algorithm to approximate a root of any degree. The general algorithm is a natural generalization of the square root algorithm. Hence, we start by describing the square root algorithm in some detail and then, we provide the general case of extraction of roots.

3.1 The square root algorithm of Al-Kashi

In chapter five of Al-Kashi (1977), Al-Kashi describes his square root algorithm in general but entirely in words. Here, we closely follow his description although in the language and notation of modern mathematics. To provide a clear and concise account without loss of generality, we illustrate the algorithm using an integer with a given number of digits. The general case of an integer with any number of digits is then obvious, and we undertake that task in the description of the extraction of nth roots that follows.

Given an integer N, its digits are divided into groups of two that Al-Kashi calls "cycles" starting from the units digit. Let us assume, for the sake of illustration, that N is a 6-digit integer, say $N = abcdef$ (the argument applies to integers of any number of digits). So, for a 6-digit number, the cycles are ab, cd, and ef. (If N was a 5-digit integer, the last cycle would have contained one digit). Among their other uses in the algorithm, the cycles determine the number of digits in the integer part of the square root. So, for example, since in our case, the number N has 3 cycles; the square root will be formed by 3 digits, that is, a number in the hundreds. After creating the cycles, Al-Kashi looks for $\sqrt{N}$ one digit at a time starting form the left. First, he looks for the integer part of the root of the left most cycle. So, Al-Kashi will be looking for the square root of ab first. He considers the greatest integer A whose square is less than or equal to the number formed by the cycle; $A^2 \leq ab$. A will be the left most digit of the integer part of $\sqrt{N}$. He then computes the difference $P_0 = ab - A^2$ and forms a new number P_0cd, where P_0cd is the concatenation of P_0 and cd. Observe that, since ab is a number with two digits, P_0 is a digit and the number P_0cd is in the hundreds.

Now, Al-Kashi looks for the greatest integer B such that $(2A \cdot 10 + B) \cdot B \leq P_0cd$, where $2A$ is the double of A. That number B is the second digit of $\sqrt{N}$. To look for the third digit, Al-Kashi considers the difference $P_1 = P_0cd - (2A \cdot 10 + B) \cdot B$ and forms the number P_1ef, where $2A$ is the double of A and P_1ef is the concatenation of P_1 and ef. At this point, he proceeds as he did previously to find A and B. He looks for the largest integer C such that $(2AB \cdot 10 + C) \cdot C \leq P_1ef$, where AB is the concatenation of A and B, and $2AB$ is the double of AB. This digit C is the third digit (from the left) of the integer part of $\sqrt{N}$. At last, he computes the difference $P_2 = P_1ef - (2AB \cdot 10 + C) \cdot C$. If $P_2 = 0$, then the process is finished and $ABC = \sqrt{N}$. Otherwise, Al-Kashi takes P_2 as the numerator of the fractional part of the square root and $(2AB \cdot 10 + C) + C + 1 = 2ABC + 1$ as the denominator. Thus, $\frac{P_2}{2ABC+1}$ will be the fractional part of the square root. In other words, $ABC\frac{P_2}{2ABC+1}$ is an approximation to $\sqrt{N}$.

We remark that Al-Kashi made great use of approximate square roots in his remarkable approximation of π (Hogendijk 2009).

3.2 Example

Here, we provide an example of extraction of a square root, taken from *Miftah* (Al-Kashi 1969, 1977), and show how to execute the algorithm described above using a table as presented by Al-Kashi. The example is important as it provides the practical framework [it is also explained in Berggren (1986)]. We use the same notation as in the general description for further clarity. We would like to extract the square root of the number 331781. Al-Kashi divides this number into cycles of two digits $ab = 33$, $cd = 17$ and $ef = 81$ using a table as follows.

33	17	81

First, find the greatest integer A whose square is less than or equal to the number formed by the cycle 33. $A^2 \leq 33$; $A = 5$ and place it on the top and bottom of the table to get the next table.

5		
33	17	81
8		
5		

Now, we compute $P_0 = 33 - 5^2$; $P_0 = 8$ and put it underneath the 33. We consider the number 817. We also double 5; $2 \cdot 5 = 10$ and write it above the bottom 5 shifted one place to the right

5		
33	**17**	**81**
8		
1	0	
5		

Next, look for the greatest number B such that $(2 \cdot 5 \cdot 10 + B) \cdot B \leq 817$. By inspection, we find that $B = 7$ which we write on the top and the bottom of the second column.

5	7	
33	**17**	**81**
8		
1	07	
5		

We compute the difference $P_1 = 817 - (2 \cdot 5 \cdot 10 + 7) \cdot 7$, so, $P_1 = 68$. We put it in the second column below 17 and form the number 6881. We also double the last digit of 107 which becomes 114 and write it above the 107 shifted one place to the right.

5	7	
33	**17**	**81**
8	68	
	11	4
1	07	
5		

Now, we look for the greatest number C such that $(114 \cdot 10 + C) \cdot C \leq 6881$. We find, $C = 6$ which write on the top and the bottom of the third column.

5	7	6
33	**17**	**81**
8	68	
	11	46
1	07	
5		

Next, $P_2 = 6881 - (2 \cdot 57 \cdot 10 + 6) \cdot 6$ That is, $P_2 = 5$. Put it in the third column below 81. The integer part is found, and $P_2 = 5 \neq 0$ is the numerator of the fractional part of the root. Doubling the last digit of 1146, it becomes 1152. The denominator is $1152 + 1 = 1153$.

So, we found $\sqrt{331781} \approx 576\frac{5}{1153}$.

3.3 Extraction of higher order roots in Al-Kashi's work

Although the computations become more tedious, the same algorithm can be naturally generalized to obtain approximate roots of higher orders. Al-Kashi gives a general description of his algorithm for extraction of roots of any order, again purely in prose. He illustrates his method (Al-Kashi 1969, 1977) by finding the fifth root of the number $N = 44240899506197$. We will consider this example after we describe Al-Kashi's general method using the language and notation of modern mathematics. We keep the notations of the previous section.

Let $N = a_1 a_2 \ldots a_m$, $m \geq 2$ be an m-digit integer. To extract its nth root, Al-Kashi divides the digits of N into cycles of length n starting from the right. He looks at the left most cycle of N, let us say $a_1 a_2 \ldots a_k$, where $k \leq n$ and finds the integer part of its nth root. That is the largest integer A whose nth power is less than or equal to the number formed by the cycle; $A^n \leq a_1 a_2 \ldots a_k$. A will be the left most digit of the integer part of $\sqrt[n]{N}$. To look for the next digit of the root, Al-Kashi computes the difference $P_0 = a_1 a_2 \ldots a_k - A^n$ and forms a new number $P_0 a_{k+1} \ldots a_{k+n}$, where $a_{k+1} \ldots a_{k+n}$ is the second (from the left) cycle of N, and $P_0 a_{k+1} \ldots a_{k+n}$ is the concatenation of P_0 and $a_{k+1} \ldots a_{k+n}$. He then considers the quantity

$$f_1(B) = \left(\binom{n}{1} 10^{n-1} A^{n-1} B^0 + \cdots + \binom{n}{n-1} 10^1 A^1 B^{n-2} + \binom{n}{n} 10^0 A^0 B^{n-1}\right) B.$$

Al-Kashi computes $f_1(B)$ in several steps and by the help of a table [see Berggren (1986) for more details] and looks for the largest integer B such that $f(B) \leq P_0 a_{k+1} \ldots a_{k+n}$. That number B is the second digit of (integer part of) $\sqrt[n]{N}$. To look for the third digit, Al-Kashi considers the difference $P_1 = P_0 a_{k+1} \ldots a_{k+n} - f_1(B)$ and forms the number $P_1 a_{k+n+1} \ldots a_{k+2n}$, where $a_{k+n+1} \ldots a_{k+2n}$ is the third (from the left) cycle of N, and $P_1 a_{k+n+1} \dots a_{k+2n}$ is the concatenation of P_1 and $a_{k+n+1} \dots a_{k+2n}$. Now, he considers the quantity

$$f_2(C) = \left(\binom{n}{1} 10^{n-1}(AB)^{n-1}C^0 + \cdots + \binom{n}{n-1} 10^1 (AB)^1 C^{n-2} + \binom{n}{n} 10^0 (AB)^0 C^{n-1}\right) C$$

and looks for the largest number C such that $f_2(C) \leq P_1 a_{k+n+1} \dots a_{k+2n}$. This digit C is the third digit (from the left) of the integer part of $\sqrt[n]{N}$. He continues in this way until he gets all the digits of $\sqrt[n]{N}$. The last difference is $P_{\lceil \frac{m}{n} \rceil}$, where $\lceil \frac{m}{n} \rceil$ is the smallest integer not less than $\frac{m}{n}$. If $P_{\lceil \frac{m}{n} \rceil} = 0$, then the process is finished, and $\sqrt[n]{N}$ is an integer $r = ABC\dots$ Otherwise, $\sqrt[n]{N} \approx r\frac{u}{v}$ is an underestimation of the root, where $r = ABC\dots$ is the largest integer such that $r^n \leq N$, $u = P_{\lceil \frac{m}{n} \rceil} = N - r^n$ and $v = (r+1)^n - r^n$.

According to Rashed (1994), Al-Kashi's algorithm for extracting nth roots is a special case of what is known as the Ruffini–Horner method, which Dakhel states is a misleading name because "the method was already used by Islamic and Chinese mathematicians long before either Ruffini or Horner lived" (Dakhel 1960). Rashed also states that Al-Kashi's algorithm comes from Al-Karaji school, after the tenth-century mathematician Abu Bakr ibn Muhammad Ibn Al-Husayn Al-Karaji (953–1029) who is well known for his work in algebra (Rashed 1978, 1994).

3.3.1 Example

Al-Kashi illustrates his method on the example of finding the fifth root of $N = 44240899506197$. From the description of the general algorithm, we know apriori that his approximation will produce $R = r\frac{u}{v}$, where r is the largest integer such that $r^5 \leq N$, $u = N - r^5$ and $v = (r+1)^5 - r^5$. Therefore, the outcome on this particular example will be $536\frac{21}{414237740281}$. The digits of r are again obtained one at a time using the cycles. In the case of the 5th root, the cycles have 5 digits (in the general case of nth root, cycles have n digits). The cycles count the numbers of $10^0, 10^5, 10^{10}, \dots$ in N (each being a perfect 5th power), and the number of cycles determines the number of digits in r. In this example, $r = ABC$, each digit corresponding to one of the cycles 4424, 08995, and 06197. The first step in the algorithm is to find the largest integer (digit) A such that $A^5 \leq 4424$. By inspection, $A = 5$. In finding other digits, Al-Kashi makes use of the identity $(x+y)^5 - y^5 = (((((\binom{5}{5})y + \binom{5}{4}x)y + \binom{5}{3}x^2)y + \binom{5}{2}x^3)y + \binom{5}{1}x^4)y)y$. He arranges his work in a more elaborate table to facilitate the computation of the quantity on the right side of the last equation. Even though computational details are more involved, the main idea behind the procedure remains the same. More details can be found in Berggren (1986).

4 Root extraction in Stevin's work

In *L'arithmétique*, Stevin dedicates 16 examples of a variety of extractions of roots. He starts by laying out the grounds for a general scheme in problem XVIII: "Given a simple geometric number, find its required root." To solve this problem, Stevin produces Pascal-like triangles:

				2				
			3	•	3			
		4	•	•	•	4		
	5	•	•	•	•	•	5	
6	•	•	•	•	•	•	•	6

				20				
			3	•	30			
		4	•	6	•	40		
	5	•	10	•	10	•	50	
6	•	15	•	20	•	15	•	60

Then, he uses them in the extraction of square roots, cube roots, fourth roots, and fifth roots. Just as in Al-Kashi's examples, there is no proof or justification of why the algorithm works. There are, however, checks that the roots found are the desired ones. Moreover, the general algorithm here is a natural generalization of the square root algorithm, just like Al-Kashi's. We start by describing the square root algorithm. It is worth mentioning that all the examples that Stevin treats are numerical with no variables.

4.1 The square root algorithm of Stevin

In our investigation of the algorithm, we analyze Stevin's first examples as described in his book (Stevin 1625) and describe it in modern mathematics. By following Stevin's method word for word, we found a flaw in his method and noticed that it does not work in general. For example, Stevin extracts the square root of the number 186624 and finds the correct root which is 432, but if we apply the same algorithm to the number 331781 from Al-Kashi's work (given above) then Stevin's method does not work. We illustrate this with the two numbers just mentioned and pinpoint where Stevin's method fails. We will then show how to remedy the situation.

4.1.1 Stevin's example of square root extraction

This is the authors' translation from the French of Stevin's first example in pages 102–103 of his book (Stevin 1625). Stevin's second example is similar to the first one and runs into the same problems.

Beginning of translation.

First example of extraction of square root of an integer with its commensurable root.

Explanation of the given. Given a square number 186624. *Explanation of the required.* One must find its square root. *Construction.* We draw under the given number two lines and between them, we put a dot under the first digit to the right, namely under the 4.

Similarly, a dot under the 6, leaving a digit in two. Then, a dot under 8, similarly leaving a digit in two. Similarly, we put more dots if there are more digits. These three dots indicate the three positions of the three digits which will come out of the required root, so that the disposition is as follows. Then, it is necessary to take the square root in integer part of 18, as close as possible but lower which is 4. We put it on the first dot under the 8.

$$\begin{array}{ccc}
18 & 66 & 24 \\
\hline
\bullet & \bullet & \bullet \\
\hline
\end{array}$$

Then, we subtract the square of 4 which is 16 from 18. It remains 2 which we put above the 8 and cross 16 and 18. Their disposition is then as follows.

$$\begin{array}{ccc}
2 & & \\
\cancel{18} & 66 & 24 \\
\hline
4 & \bullet & \bullet \\
\hline
\cancel{16} & & \\
\end{array}$$

But to find the second digit of the root, one must put aside 20 (which is the number serving in general in the extraction of any square root, as we said above in the note of this problem). In front of this same 20, one must put what comes as a root, namely 4. We multiply the 20 by 4 to get 80: Similarly, we need to divide the 266 and find as a quotient 3, which is put on the second dot between the lines and also joining the 20 which are put aside: Also, we put under the 3 its square 9, and their disposition is as follows.

$$\begin{array}{ccc}
4. & 20. & 3. \\
 & & 9. \\
\end{array}
\qquad
\begin{array}{ccc}
2 & & \\
\cancel{18} & 66 & 24 \\
\hline
4 & 3 & \bullet \\
\hline
\cancel{16} & & \\
\end{array}$$

Then, it is necessary to multiply the 20 by the 4 which is 80, and this same number by 3 to get 240 which we put by the 3 and add 9 to get 249. We put it also below the 266 and subtract them. The remainder is 17, and their disposition will be then as follows.

$$\begin{array}{cccc}
4. & 20. & 3. & 240. \\
 & & 9. & 9. \\
\hline
 & & & 249. \\
\end{array}
\qquad
\begin{array}{ccc}
\cancel{2} & 17 & \\
\cancel{18} & \cancel{66} & 24 \\
\hline
4 & 3 & \bullet \\
\hline
\cancel{16} & \cancel{49} & \\
\cancel{2} & & \\
\end{array}$$

But to find the third digit of the root, one must proceed in the same way as we did in finding the second digit. We put then, as above, aside 20 (namely the 20 which generally helps in any extraction of square root) and in front of this same 20, we put what comes out as a root, namely 43. By multiplying the 20 by 43, we get 860 by which is divided 1734 [probably a typo, should be 1724] to find 2 as quotient which we put on the third dot between the lines and also joining the 20 which are put aside. Also, we put below the said 2 its square 4 and the disposition is such.

$$\begin{array}{rrrr} 4. & 20. & 3. & 240. \\ & & 9. & 9. \\ \hline & & & 249. \\ 43. & 20. & 2. & \\ & & 4. & \end{array} \qquad \begin{array}{ccc} \cancel{2} & 17 & \\ \cancel{18} & \cancel{66} & 24 \\ \hline 4 & 3 & 2 \\ \hline \cancel{16} & \cancel{49} & \\ \cancel{2} & & \end{array}$$

Then, we multiply the 20 by 43 which make 860 and this same number by 2 to get 1720 which are put next to the 2 and by adding to it again 4, the sum becomes 1724. We put it also under the 1724 of the extraction and subtract them to find zero as a reminder. Then, the achieved disposition is such that.

$$\begin{array}{rrrr} 4. & 20. & 3. & 240. \\ & & 9. & 9. \\ \hline & & & 249. \\ 43. & 20. & 2. & 1720. \\ & & 4. & 4. \\ \hline & & & 1724. \end{array} \qquad \begin{array}{ccc} \cancel{2} & \cancel{17} & \\ \cancel{18} & \cancel{66} & 24 \\ \hline 4 & 3 & 2 \\ \hline \cancel{16} & \cancel{49} & \cancel{24} \\ \cancel{2} & \cancel{17} & \end{array}$$

I say that 432 is the required root. *Proof.* By multiplying 432 by itself, we get the product 186624 which is equal to the given square number. So, 432 is the true root which is required to prove.

End of translation.

4.1.2 Counterexample to Stevin's square root extraction

The above example of Stevin's provides the correct square root of the number 186624, $\sqrt{186624} = 432$. However, it has a flaw that one can detect with the following counterexample. Let us follow Stevin's method as described above and try to find the square root of the number 331781 from Al-Kashi. We found using Al-Kashi's method that $\sqrt{331781} \approx 576\frac{5}{1153}$.

To extract the square root of the number 331781 using Stevin's method, divide it into cycles of two digits, by spacing them and drawing horizontal lines with dots between them at the beginning of each cycle.

33	17	81
•	•	•

Let us find the greatest integer A whose square is less than or equal to the number formed by the cycle 33. $A^2 \leq 33$; $A = 5$ and place it on the first dot under the 3. We compute $33 - 5^2 = 8$. Cross 25 and 33 to get the next table.

8		
$\cancel{33}$	17	81
5	•	•
$\cancel{25}$		

To find the second digit of the square root, we put aside 20 and in front of it 5. We multiply $20 \cdot 5 = 100$. We consider the number 817 and divide it by 100. Its integer part is 8 which we put as the second digit of the root on the second dot.

5.	20.	8.	8		
		64.	$\cancel{33}$	17	81
			5	8	•
			$\cancel{25}$		

We can readily see that the process is giving us the wrong square root of 331781 as the second digit of the square root is found to be 8 instead of the correct digit 7. Moreover, if we continue the process, then we would have to subtract $20 \cdot 5 \cdot 8 + 64 = 864$ from 817 which is not possible for such an algorithm (Stevin never gets negative numbers in the execution of the algorithm). The problem is mainly due to the fact that in looking for the second digit of the square root, Stevin uses a simple division operation instead of the more complex constraint equation (inequality) used in Al-Kashi's method. This appears to be the case at all stages where satisfying such a constraint is required. Actually, right after that glitch in the process, Stevin goes on to subtracting those same quantities that should have been optimized in the previous step. So, in fact, Stevin is using formulas and quantities in his process of subtraction that should have been used earlier. For example, in this counterexample, one can look for the second digit of the root as the number (digit) that satisfies the constraint: the greatest number B such that $20 \cdot 5 \cdot B + B^2 \leq 817$. This yields $B = 7$. In considering the division of 817 by 20, Stevin misses out the B^2 part. That did not matter in Stevin's example as $B^2 = 9$ is small compared to Al-Kashi's example where $B^2 = 49$. Moreover, in the next step of the process of finding the square root, had Stevin found the correct B, he would have considered the subtraction $817 - (20 \cdot 5 \cdot B + B^2)$ which would have been correct. In conclusion, the only problem that we found in Stevin's algorithm is his failing to use the optimization at all the stages where it is required. This becomes more tedious in the extraction of higher roots.

Let us now apply the above proposed remedy to Stevin's algorithm and describe it in modern mathematics. Since most of the algorithm works well, we will clearly specify our amendment below. Then, we will briefly take on Al-Kashi's example to show how this amended algorithm works.

4.1.3 The square root algorithm of Stevin, amended

We adopt analogous notations to Al-Kashi's extraction of roots method described above.

Given a number N, Stevin divides the digits constituting N into pairs starting from the right. If the number of digits forming N is odd, then the last cycle in the left will have only one digit. This subdivision of the digits corresponds to the cycles in Al-Kashi's work and provides the number of digits in the square root. Stevin looks for (integer part of) $\sqrt{N}$ one digit at a time starting form the left. Each cycle will provide a digit of the root. Without loss of generality, and following our investigation of Al-Kashi's square root extraction above, we let $N = abcdef$. A generalization of the process to an integer with any number of digits follows naturally. To find the first digit of the square root, Stevin considers the greatest integer A with $A^2 \leq ab$. A will be the left digit of the integer part of $\sqrt{N}$. He then computes the difference $\text{٩}_0 = ab - A^2$ and considers the number $20A$.

(He then, looks for the integer B of the division of $\text{٩}_0 cd$ by $20A$, where $\text{٩}_0 cd$ is the concatenation of ٩_0 and cd.)

Amendment: He then, looks for the greatest integer B such that $20AB + B^2 \leq \text{٩}_0 cd$, where AB is the concatenation of A and B, $20AB$ is 20 times the number AB and $\text{٩}_0 cd$ is the concatenation of ٩_0 and cd.

That number B is the second digit of $\sqrt{N}$. Now, Stevin considers the difference $\text{٩}_1 = \text{٩}_0 cd - \left(20AB + B^2\right)$ and forms the number $\text{٩}_1 ef$, where $\text{٩}_1 ef$ is the concatenation of ٩_1 and ef.

(Then, as he did previously, he proceeds to find the number C as the integer part of the division of $\text{٩}_1 ef$ by $20AB$, where $\text{٩}_1 ef$ is the concatenation of ٩_1 and ef.)

Amendment: Then, as he did previously, he proceeds to find the greatest integer C with the property $20ABC + C^2 \leq \text{٩}_1 ef$, where ABC is the concatenation of A, B and C and $\text{٩}_1 ef$ is the concatenation of ٩_1 and ef.

This number C is the third digit (from the left) of the integer part of $\sqrt{N}$. Finally, he computes the difference $\text{٩}_2 = \text{٩}_1 ef - \left(20ABC + C^2\right)$. If $\text{٩}_2 = 0$, then the process is finished and $ABC = \sqrt{N}$. Otherwise, $\sqrt{N} \approx ABC\frac{\text{٩}_2}{2ABC+1}$ is an approximation of the square root.

4.1.4 Example

Here, we illustrate the amended Stevin's method using a specific example. To draw a parallel with Al-Kashi's method, we use the same number as in Al-Kashi's, namely 331781. Moreover, this number will help us tackle Stevin's method of finding the fractional part of the square root. In his book, Stevin uses 186624 which is a perfect square number of the same order. Thus, in one setting, we are showing two examples from Stevin's book, extraction of the integer part of the square root and finding its fractional part. The example below is important as it presents the practical framework.

To extract the square root of the number 331781, Stevin divides it into cycles of two digits, $ab = 33$, $cd = 17$ and $ef = 81$ by spacing them and drawing horizontal lines with dots between them at the beginning of each cycle.

$$\begin{array}{ccc} 33 & 17 & 81 \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$$

Let us find the greatest integer A whose square is less than or equal to the number formed by the cycle 33. $A^2 \leq 33$; $A = 5$ and place it on the first dot under the 3. Compute $\P_0 = 33 - 5^2 = 8$. Cross 25 and 33 to get the next table.

$$\begin{array}{ccc} 8 & & \\ \cancel{33} & 17 & 81 \\ \hline 5 & \bullet & \bullet \\ \hline \cancel{25} & & \end{array}$$

To find the second digit of the root, we put aside 20 and in front of it 5. We multiply $20 \cdot 5 = 100$ and consider the number 817 to look for the greatest integer B such that $20 \cdot 5 \cdot B + B^2 \leq 817$. $B = 7$, which is put on the second dot between the lines and also joining the 20. We also put under the 7, its square 49 and their disposition is as follows.

$$\begin{array}{ccc} 5. & 20. & 7. \\ & & 49. \end{array} \qquad \begin{array}{ccc} 8 & & \\ \cancel{33} & 17 & 81 \\ \hline 5 & 7 & \bullet \\ \hline \cancel{25} & & \end{array}$$

Then, compute $\P_1$. Put $20 \cdot 5 \cdot 7 = 700$ next to 7. Then, we add $700 + 49 = 749$ which we put below the 817 also. We get $\P_1 = 817 - 749 = 68$.

$$\begin{array}{llll} 5. & 20. & 7. & 700. \\ & & 49. & 49. \\ \hline & & & 749. \end{array} \qquad \begin{array}{ccc} \cancel{8} & 68 & \\ \cancel{33} & \cancel{17} & 81 \\ \hline 5 & 7 & \bullet \\ \hline \cancel{25} & \cancel{49} & \\ \cancel{7} & & \end{array}$$

Now, we look for the third digit. We put aside 20 and in front of it, we put 57. We multiply $20 \cdot 57 = 114$ and consider the number 6881 to look for the greatest integer C, with $20 \cdot 57 \cdot C + C^2 \leq 6881$. $C = 6$, which is put on the second dot between the lines and also joining the 20. We also put under the 6, its square 36 and their disposition is as follows.

$$\begin{array}{llll} 5. & 20. & 7. & 700. \\ & & 49. & 49. \\ \hline & & & 749. \\ 57. & 20. & 6. & \\ & & 36. & \end{array} \qquad \begin{array}{ccc} \cancel{8} & 68 & \\ \cancel{33} & \cancel{17} & 81 \\ \hline 5 & 7 & 6 \\ \hline \cancel{25} & \cancel{49} & \\ \cancel{7} & & \end{array}$$

Then, we compute $\P_2$. We start by computing $20 \cdot 57 \cdot 6 = 6840$ and put it next to the 6. Then, we add $6840 + 36 = 6876$ which we put below the 6881 also.

We get $\P_2 = 6881 - 6876 = 5$. Then, the achieved disposition is such that.

$$\begin{array}{llll} 5. & 20. & 7. & 700. \\ & & 49. & 49. \\ \hline & & & 749. \\ 57. & 20. & 6. & 6840. \\ & & 36. & 36. \\ \hline & & & 6876. \end{array} \qquad \begin{array}{ccc} \cancel{8} & \cancel{68} & \\ \cancel{33} & \cancel{17} & 81 \\ \hline 5 & 7 & 6 \\ \hline \cancel{25} & \cancel{49} & \cancel{76} \\ \cancel{7} & \cancel{68} & \end{array}$$

Finally, we found that the integer part of the square root is 576 and since the last difference $\P_2 = 5 \neq 0$, the square root has a fractional part with a numerator of 5 and a denominator of $2 \cdot 576 + 1$. Thus, $\sqrt{331781} \approx 576\frac{5}{1153}$.

4.2 Extraction of higher order roots in Stevin's work

In extracting cube roots, fourth roots, and fifth roots, Stevin uses the same method as in square root extraction. In the previous section, we detected some problems in his algorithm which we amended to provide a correct one. That same exact problem of using a simple division to find the digits of the root instead of the far more complex optimization process persists in all of his extractions of roots at all stages where such a process is called. Since all of the other stages of Stevin's algorithm are correct, we propose amendments to provide a correct root extraction algorithm. We follow the square root algorithm description and use analogous notations. Moreover, we keep our account of the process close to the process of Al-Kashi to better compare the two algorithms. We then provide a brief example showing how the algorithm effectively works.

4.2.1 Extraction of higher order roots in Stevin's work, amended

Consider an m-digit integer $N = a_1a_2\dots a_m$, $m \geq 2$. The aim is to extract its nth root. To that end, Stevin divides the digits of N into cycles of length n starting from the right. Assuming that $a_1a_2\dots a_k$, $k \leq n$ is the left most cycle of N, Stevin finds the greatest integer A such that $A^n \leq a_1a_2\dots a_k$. This integer A is the left most digit of the integer part of $\sqrt[n]{N}$. He then computes the difference $\P_0 = a_1a_2\dots a_k - A^n$ and considers the number $10^{n-1}nA^{n-1}$.

(He then, looks for the integer B of the division of $\P_0a_{k+1}\dots a_{k+n}$ by $10^{n-1}nA^{n-1}$, where $a_{k+1}\dots a_{k+n}$ is the second (from the left) cycle of N, and $\P_0a_{k+1}\dots a_{k+n}$ is the concatenation of $\P_0$ and $a_{k+1}\dots a_{k+n}$.)

Amendment: He then, looks for the greatest integer B such that $g_1(B) \leq \P_0a_{k+1}\dots a_{k+n}$, where $a_{k+1}\dots a_{k+n}$ is the second (from the left) cycle of N, $\P_0a_{k+1}\dots a_{k+n}$ is the concatenation of $\P_0$ and $a_{k+1}\dots a_{k+n}$, and

$$g_1(B)=\binom{n}{1}10^{n-1}A^{n-1}B^1+\dots+\binom{n}{n-1}10^1A^1B^{n-1}+\binom{n}{n}10^0A^0B^n.$$

That number B is the second digit of $\sqrt[n]{N}$. Now, Stevin computes the difference $\P_1 = \P_0a_{k+1}\dots a_{k+n} - g_1(B)$, where $a_{k+1}\dots a_{k+n}$ is the second (from the left) cycle of N, and $\P_0a_{k+1}\dots a_{k+n}$ is the concatenation of $\P_0$ and $a_{k+1}\dots a_{k+n}$. He then forms the number $\P_1a_{k+n+1}\dots a_{k+2n}$, where $a_{k+n+1}\dots a_{k+2n}$ is the third (from the left) cycle of N, and $\P_1a_{k+n+1}\dots a_{k+2n}$ is the concatenation of $\P_1$ and $a_{k+n+1}\dots a_{k+2n}$.

(Then, as he did previously, he proceeds to find the number C as the integer part of the division of $\P_1a_{k+n+1}\dots a_{k+2n}$ by $10^{n-1}n(AB)^{n-1}$.)

Amendment: Then, as he did previously, he proceeds to find the greatest integer C with the property $g_2(C) \leq \P_1a_{k+n+1}\dots a_{k+2n}$, where

$$g_2(C) = \binom{n}{1}10^{n-1}(AB)^{n-1}C^1+\dots+\binom{n}{n-1}10^1(AB)^1C^{n-1}$$
$$+\binom{n}{n}10^0(AB)^0C^n.$$

This integer C is the third digit (from the left) of the integer part of $\sqrt[n]{N}$. His next step is to compute the difference $\P_2 = \P_1a_{k+n+1}\dots a_{k+2n} - g_2(C)$ and he proceeds as before to find the next digit constituting the integer part of $\sqrt[n]{N}$. One has to do so with the appropriate amendments in mind. He continues in this way until he gets all the digits of the integer part of $\sqrt[n]{N}$. The last difference is $¶_{\lceil \frac{m}{n} \rceil}$, where $\lceil \frac{m}{n} \rceil$ is the smallest integer not less than $\frac{m}{n}$. If $¶_{\lceil \frac{m}{n} \rceil} = 0$, then the process stops and $\sqrt[n]{N}$ is an integer $r = ABC \cdots$ Otherwise, $\sqrt[n]{N} \approx r\frac{u}{v}$ is an underestimation of the root, where $r = ABC\ldots$ is the largest integer such that $r^n \leq N$, $u = ¶_{\lceil \frac{m}{n} \rceil} = N - r^n$ and $v = (r+1)^n - r^n$.

4.2.2 Example

In this example, we briefly discuss Stevin's method applied to the extraction of the fifth root of the number 44240899506197. He starts by making two columns of the 9 digits and their fifth powers starting with 1 and ending with 9. On the other hand, he divides the number into cycles of 5 digits starting from the right. So, he gets the cycles 4424, 08995, and 06197. The first digit of fifth root of the number is the greatest integer A such that $A^5 \leq 4424$. By looking at the table of fifth powers, he finds $A = 5$; $5^5 = 3125$. He computes the difference $¶_0 = 4424 - 3125 = 1299$ and considers the number 129908995. Also, he considers $A^4 = 625$ which he did compute to find the table of fifth powers, and then multiply $50000 \cdot 625 = 31250000$. This is just $10^4 \cdot 5 \cdot A^4$. Stevin would then divide 129908995/31250000 and take the integer part as the second digit of the fifth root. In this case, we will find 4 which is not the correct digit of the fifth root as seen from Al-Kashi's example above. So, we must work with the amended version and compute the whole quantity

$$\begin{aligned} g_1(B) = {} & 10^4 \cdot 5 \cdot 5^4 \cdot B + 10^3 \cdot 10 \cdot 5^3 \cdot B^2 + 10^2 \cdot 10 \cdot 5^2 \cdot B^3 + 10^1 \cdot 5 \cdot 5^1 \cdot B^4 \\ & + 10^0 \cdot 1 \cdot 5^0 \cdot B^5. \end{aligned}$$

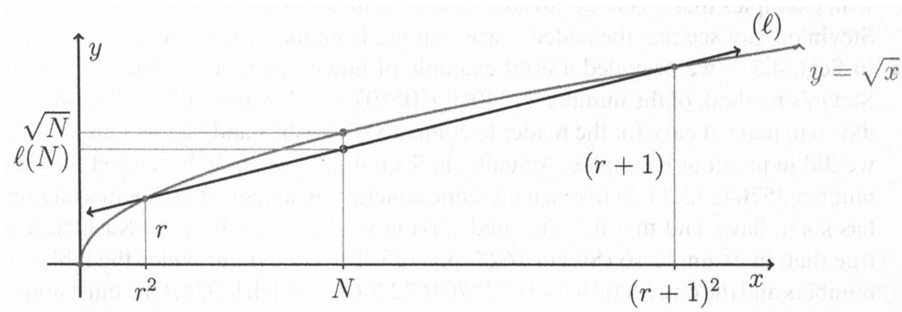
So, $g_1(B) = 31250000B + 1250000B^2 + 25000B^3 + 250B^4 + B^5$. Now, we look for the greatest integer B such that $g_1(B) \leq 129908995$ which is found to be 3. Stevin's process can be continued in this way to find the third digit 6. The computation of the fractional part is similar to that of Al-Kashi and it is $\frac{21}{4142377740281}$. Thus, $\sqrt[5]{44240899506197} \approx 536\frac{21}{4142377740281}$.

5 Justification of the root extraction algorithms

Our justification of Al-Kashi's square roots algorithm closely follows Berggren (1986). The same justification applies to Stevin's algorithm as well.

Given an integer N, its digits are divided into groups of two, called "cycles" starting from the units digit. Let us assume, for the sake of illustration, that N is a 6-digit integer, say $N = abcdef$ (the argument applies to integers of any number of digits). So, for a 6-digit number, the cycles are ab, cd, and ef. (If N was a 5-digit integer, the last cycle would have contained one digit). The cycles are useful because the first cycle counts the number of units in N, the second cycle counts the number of 100's in N, etc., and the numbers 1, 100, 10000 are perfect squares. This means we can regard N as $N = ab \cdot 10^4 + cd \cdot 10^2 + ef \cdot 10^0$. In the case, N is a perfect square, $\sqrt{N}$ is another integer, otherwise it is an irrational number. Al-Kashi's algorithm finds the exact square root in the former case (say $r = \sqrt{N}$) and produces a rational number $R = r\frac{u}{v}$, where r, u, v are positive integers, as an approximation to $\sqrt{N}$ in the latter case which is much more common. His approximation is always an underestimation, that is $R^2 < N$, when N is not a perfect square. He knows that r will contain 3 digits, say $r = ABC$. He determines the digits one at a time. The digit A is chosen as the largest digit such that $A^2 \leq ab$. From this, it is not difficult to show that $((A + 1)100)^2 > N$, which means $A + 1$ cannot be the hundreds digit of r. Then, he computes the difference $D_1 := N - (A \cdot 100)^2 = (ab - A^2) \cdot 100^2 + cd \cdot 10^2 + ef$. To determine the next digit B, Al-Kashi notes that $(100 \cdot A + 10 \cdot B)^2$ must be as close to N as possible without exceeding it, i.e., the difference $N - (100 \cdot A + 10 \cdot B)^2$ must be positive and as small as possible. This difference can be written as $D_2 := D_1 - (2A \cdot 10 + B)B \cdot 100$. In the execution of the algorithm with the help of a table, Al-Kashi makes use of this expression. Finally, he determines the units digit of r from the condition that $N - (100 \cdot A + 10 \cdot B + C)^2$ must be positive and as small as possible. This quantity can be expressed as $D_3 := D_2 - (2 \cdot 100 \cdot A + 2 \cdot 10 \cdot B + C) \cdot C$. Al-Kashi takes advantage of the fact that previous differences are already computed in the table. He also makes use of place values to simplify certain computations.

In obtaining the fractional part of the approximate square root, Al-Kashi uses a linear approximation. In the case when N is not a perfect square, it is clear that $r^2 < N < (r + 1)^2$, equivalently, $r < \sqrt{N} < r + 1$. Consider the secant line ℓ to the curve $y = \sqrt{x}$ that goes through the points (r^2, r) and $((r + 1)^2, r + 1)$. The idea of the approximation is to use the value on this line whose x-coordinate is N (whereas the exact value lies on the curve $y = \sqrt{x}$ with the same abscissa, see figure below.) The equation of this line is $\ell(x) = r + \frac{x-r^2}{2r+1}$, hence $\ell(N) = r + \frac{N-r^2}{2r+1}$. This is exactly what is obtained from the algorithm where r is the integer part. By doubling each new digit obtained, Al-Kashi keeps $2r$ in the bottom of the table. At the very end, he adds 1 to that quantity because the $2r + 1$ is required for this linear approximation. Note that since the function $y = \sqrt{x}$ is concave down, this secant line is below the curve, hence the approximation is an underestimate. Also, note that in the case when N is a perfect square, the fractional part will be 0, hence the exact value is obtained.



It is actually possible to increase the accuracy of this approximation by using the identity $\sqrt[n]{N} = \frac{\sqrt[n]{N \cdot 10^{nk}}}{10^k}$ and the fact that the difference between the curve $y = \sqrt{x}$ and the line $y = \ell(x)$ on the interval $[r^2, (r+1)^2]$ gets smaller as r (hence x) increases. In fact, it is easy to show that the maximum of the difference $D(x) = \sqrt{x} - \ell(x)$ on the interval $[r^2, (r+1)^2]$ is $\frac{1}{8r+4}$, which goes to 0 as $r \to \infty$. Hence, the error in the approximation can be made as small as desired by taking k large enough. According to Berggren (1986), the identity $\sqrt[n]{N} = \frac{\sqrt[n]{N \cdot 10^{nk}}}{10^k}$ was known as early as al-Khwarizmi's time (d. 850 CE).

The justification of the algorithm for higher order roots is very similar.

6 Concluding remarks

- In studying the methods of Al-Kashi and Stevin for extracting roots, we found that both mathematicians use a fine pedagogy with many examples and variations of the methods whenever possible. It is, however, often the case that what they call another method is actually just a variation in executing some of the steps of the same algorithm instead of it being a substantially new method or algorithm. Another noticeable similarity is the intentions of Al-Kashi and Stevin to write their arithmetic books for beginners and for real-life practitioners of other disciplines.
- Concerning the root extraction per se, Al-Kashi and Stevin methods use the binomial expansion implicitly with Stevin making Pascal's triangle more apparent. We also found that neither of them explain or justify why their methods work. They only check that the roots found are the required roots.
- Another similarity is the handling of the fractional part of the root. Both Al-Kashi and Stevin find the same fractional part with a small difference in the approach. It is, however, worth pointing out that the authors felt that Stevin did not fully grasp the addition of 1 to find the denominator because in his book Stevin (1625) on pages 117–118, he writes a note[1] regarding this addition. He noticed that Nicolas Tartaglia does not add 1 while he does, just like Juan De Moya. He further argues with examples that it is better to add 1 rather than not to add 1, which suggests that Stevin did not see that the added 1 was coming from the expression $(r+1)^n - r^n$.
- In Sect. 4.2.2, we provided a brief example of how to extract the fifth root, using Stevin's method, of the number 44240899506197 which was used by Al-Kashi. We did so to make it easy for the reader to compare Al-Kashi's and Stevin's methods as we did in previous examples. Actually, in Sect. 4.2.2, we could have used Stevin's number 3570467226624 to reach the same conclusion, namely that Stevin's method has some flaws and that the amended version works similarly to Al-Kashi's. It is true that, in example 16 (Stevin 1625, pp. 125–126), Stevin provides the tables of numbers and the correct fifth root of 3570467226624, which is 324. One must notice

[1] Stevin (1625 p. 118) " ... by wanting to give a general rule (method), there is even more reason to add it than to leave it. It is true that there is no example which we can approximate it better by leaving it, but there also others which are to the contrary can be approached better by adding it."

though that he does not provide his usual steps to find the root.[2] Actually, if we follow his method as it was presented in the extraction of square roots, cube roots, and fourth roots, then we encounter the same problem in finding the second digit as discussed in Sect. 4 above. Given the correct setup and correct root in Stevin's work, it is the authors' opinion that Stevin may have not actually done all the computations and may have just used the fact that $324^5 = 3570467226624$. This is confirmed by the fact that Stevin computed, in his examples 8 and 9, $\sqrt[3]{34012224} = 324$ and in his example 15, $\sqrt[4]{11019960576} = 324$ where his method, although flawed, gives the correct root. The fact that his tables in his example 16 are correct is understandable as Stevin's method has no problems in the setup and its relevant computations, but rather in the use of a simple division instead of a far more complex optimization to find the digits of the root after finding the first one.

- Although decimal fractions are not our main focus in this work, it is interesting to note the following fact that adds another dimension to possible connections between Al-Kashi and Stevin. Reportedly, many historians held the view that Stevin was the first to introduce decimal fractions (O'Conner and Robertson 1999; Rashed 1994), which was proven false when Luckey (1951) showed that in *Miftah* Al-Kashi gives as clear a description of decimal fractions. However, Al-Kashi was not actually the original inventor of decimal fractions because according to best available records Al-Uqlidisi (around 952 CE) used them some five centuries before Al-Kashi (Berggren 1986; Saidan 1966, 1978). According to Berggren, "Stevin's awkward notation was nowhere near so good as Al-Uqlidisi's" (Berggren 1986). From what we have seen in the original works of Al-Kashi and Stevin and related facts about the origin of the root extraction reported by Rashed (1994), something similar to the development of decimal fractions may have taken place in the case of root extraction as well.
- At last, but not the least, it is of particular interest to explore possible links between Al-Kashi and Stevin in light of the existence of many similarities.

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[2] Stevin (1625 p. 125) "We will not give in this construction any verbal explanation but only the disposition of digits of the achieved Operation ...".

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