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# Embedding semigroups in groups: not as simple as it might seem

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**Abstract** We consider the investigation of the embedding of semigroups in groups, a problem which spans the early-twentieth-century development of abstract algebra. Although this is a simple problem to state, it has proved rather harder to solve, and its apparent simplicity caused some of its would-be solvers to go awry. We begin with the analogous problem for rings, as dealt with by Ernst Steinitz, B. L. van der Waerden and Øystein Ore. After disposing of A. K. Sushkevich's erroneous contribution in this area, we present A. I. Maltsev's example of a cancellative semigroup which may not be embedded in a group, which showed for the first time that such an embedding is not possible in general. We then look at the various conditions that were derived for such an embedding to take place: the sufficient conditions of Paul Dubreil and others, and the necessary and sufficient conditions obtained by A. I. Maltsev, Vlastimil Pták and Joachim Lambek. We conclude with some comments on the place of this problem within the theory of semigroups, and also within abstract algebra more generally.

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## Contents



## 1 Introduction

In a previous paper (Hollings 2009a), I considered the early development of the (algebraic) theory of semigroups—that is, the abstract theory of sets with a single associative binary operation. In that paper, I identified what I deemed to be the first 'independent' semigroup-theoretic result, published by the American mathematician A. H. Clifford in 1941: a theorem with no direct analogue in either group or ring theory. As noted in Hollings (2009a, Sect. 2), the latter two theories had a heavy influence on the emerging study of semigroups during the 1930s: semigroups were viewed either as groups for which certain axioms had been negated, or as rings in which an entire operation (namely, addition) had been dropped. In the present paper, I turn my attention to one particular semigroup-theoretic problem which emerged in the 1930s by direct analogy with a problem from ring theory, but which was touched upon only briefly in Hollings (2009a, Appendix B): the question of when it is possible to embed a semigroup in a group. Over the course of this paper, a particular result connected with this problem will emerge as an alternative contender for 'first independent semigroup theorem'. Moreover, we will also see that efforts to solve this problem spanned the final, twentieth-century, phase of the development of abstract algebra.

The problem of embedding a semigroup in a group is one of the first topics to have been considered in the early days of semigroup theory. It concerns the derivation of conditions under which an isomorphic copy of a given semigroup may be located inside some group. It is thus, in some sense, a 'converse' to the study of semigroups via their subgroups (as carried out, for example, by the Russian mathematician A. K. Sushkevich[1] in the 1920s and 1930s—see Hollings 2009b or Hollings 2009a, Sect. 5). A typical approach to the problem has been to take a semigroup and attempt to augment it by the introduction of an identity, as well as an inverse for each element, and then to extend the original binary operation to these new elements in such a way that it remains associative. As we will see, a set of necessary and sufficient conditions for

[1] А. К. Сушкевич. Besides Russian and Ukrainian, Sushkevich also published papers in English, German and French, in which he used a German transliteration of his name: 'Suschkewitsch'. Certain of his relevant publications may therefore be found in the bibliography under this version of his name.

such an embedding to take place was obtained as early as 1939, and yet the problem has seen a great deal of study ever since—perhaps because the solution of 1939 is not a particularly *practical* one. This seemingly innocuous problem is by no means as simple as it might appear: not every semigroup can be embedded in a group, and, indeed, not even every *cancellative* semigroup[2] can be embedded in a group, as we might naively suspect, given that any subsemigroup of a group is necessarily cancellative.

The inspiration for the study of the embedding problem for semigroups came chiefly from the analogous problem of embedding a ring in a field, or, more generally, in a skew field (also known as a 'division ring' or a 'non-commutative field': a system with two binary operations satisfying all the axioms of a field, except perhaps for commutativity of multiplication). Some of the authors who considered these problems, however, realised that since much of the associated theory was phrased entirely in terms of multiplication, it might therefore be appropriate to seek a solution in the same terms. The operation of addition was thus dropped, and the problem of embedding semigroups in groups (or in other semigroups) emerged alongside that of embedding rings in (skew) fields (or in other rings). Although a great range of semigroup embedding problems has been considered over the decades, the focus here will be upon embeddings into *groups*: these have seen the most study and are, I believe, the most historically significant, having been the first such embeddings to have been studied in the burgeoning semigroup theory of the 1930s.

The ring embeddings that eventually gave impetus to their semigroup counterparts first arose in the early years of the twentieth-century as part of the development of abstract algebra. The notion of a field had already appeared in the nineteenth century in the work of Leopold Kronecker and Richard Dedekind (Katz 2009, Sect. 21.5). That of a ring followed in a 1914 paper by Abraham (Adolf) Fraenkel (see Corry 2000a, b, and also Hollings 2014a), with the subsequent theory of rings seeing much development in the 1920s at the hands of Emmy Noether and her school. Before this, however, comprehensive theories had emerged for the older notions of integral domains and fields, by analogy with the properties of the integers and the rational numbers. One of the major pieces of work in this direction was Ernst Steinitz' 1910 paper 'Algebraische Theorie der Körper' (Steinitz 1910).

In his paper (on which, see Corry 1996, 2nd ed., Sect. 4.2), Steinitz set about the construction of a comprehensive abstract theory of fields. In the early sections, for example, he derived a range of basic properties for both fields and integral domains. Amongst these is a result that is of great significance for our present purposes: the so-called 'Quotientenbildung', namely, the construction of the field of fractions of an abstract integral domain, by analogy with the construction of the rationals from the integers. We will discuss Steinitz' method in Sect. 2.

I have referred above to embedding 'problems'. However, Steinitz did not phrase his treatment of the 'Quotientenbildung' in such terms, simply because, for him, there was no problem: any integral domain admits such a construction. The explicit formulation of 'embedding problems' in the sense in which we are interested here appears to have

[2] A semigroup $S$ in which either $ac = bc$ or $ca = cb$ implies that $a = b$, for any $a, b, c \in S$. This does not, of course, guarantee that $c$ has an inverse element in the traditional group-theoretic sense. If $S$ has a zero element, then we must make the additional requirement that $c$ be nonzero.

been due to B. L. van der Waerden in his seminal text *Moderne Algebra* of 1930. We know that van der Waerden read Steinitz' paper at Noether's behest (see Waerden 1975, p. 33): Steinitz's 'Quotientenbildung' appears in van der Waerden's book as the proof of the theorem that every integral domain may be embedded in a field (presented here as Theorem 1 in Sect. 2). Indeed, van der Waerden went further by posing the additional question in the non-commutative case: can every non-commutative ring without zero divisors be embedded in a skew field? Several of the mathematicians who are mentioned in the present article were inspired by this question, which we will refer to as *van der Waerden's problem*.

An early contribution to the solution of van der Waerden's problem came from Øystein Ore in a paper of 1931 (on which, see Corry 1996, 2nd ed., p. 264). Ore was interested in the development of a non-commutative theory of determinants and the application of such a theory to the solution of systems of linear equations with coefficients from a skew field. In the course of his paper, Ore extended the 'Quotientenbildung' to the non-commutative case, thereby embedding a non-commutative ring without zero divisors in a skew field of fractions (presented here as Theorem 2 in Sect. 2). However, Ore was unable to carry out his construction in general; he was forced to assume a further condition which had emerged from his study of determinants: that every pair of elements of the ring in question must have a common right multiple (for all elements $a, b$ in the ring, we can find elements $m, n$ such that $am = bn$). The 'common right multiples' condition is therefore a sufficient condition for the embedding of a non-commutative ring without zero divisors in a skew field.

It is difficult to know quite when mathematicians first began to consider the question of *semigroup* embeddings. As we will see in Sect. 2, the adaptation of the methods of both Steinitz and Ore to the purely multiplicative case is in fact quite straightforward. Indeed, the extension of these constructions was evidently so clear that it was some time before explicit proofs (or even statements of the relevant semigroup embedding results) appeared in print. Nevertheless, they quickly passed into 'mathematical folklore' and, as we will see, were often cited by subsequent authors as though they were obvious and widely known facts.

The first explicit semigroup-theoretic treatment of the embedding problem was that of the Russian-born, but Ukraine-based, mathematician A. K. Sushkevich; he appears to have been influenced by van der Waerden, but not by Ore. In a paper of 1935, Sushkevich claimed to have proved that any cancellative semigroup may be embedded in a group. As noted above, however, this is not the case: there is a flaw in Sushkevich's argument. He appears to have fallen into the trap of assuming that the embedding problem is straightforward.

The first published indication that Sushkevich had given an erroneous result came from another Russian mathematician, A. I. Maltsev,[3] whose name is now inextricably linked with the embedding problem. In a paper of 1937, Maltsev presented a counterexample to show that not every cancellative semigroup may be embedded in a group.

[3] А. И. Мальцев. I choose to omit the silent Cyrillic letter ь from my transliteration, although its presence is sometimes indicated (after transliteration into Latin letters) by '. Note that Maltsev's name has also been transliterated as 'Malcev'. Indeed, the 1937 paper that will be of interest to us in Sect. 3.2 was published under this name, and so it appears in this form in the bibliography.

In fact, Maltsev's semigroup counterexample is merely a means to an end: in the final section of his paper, he used his semigroup to construct a particular non-commutative ring without zero divisors, which may not be embedded in a skew field. Maltsev therefore gave a negative solution to van der Waerden's problem. Nevertheless, he continued to study the embedding problem in the semigroup context and, in a celebrated paper of 1939 (together with a sequel of 1940), gave necessary and sufficient conditions for a cancellative semigroup to be embedded in a group. These conditions consist of a countably infinite set of implications.

In spite of Maltsev's apparently comprehensive solution to the semigroup embedding problem, it did not go away—perhaps because a countably infinite set of conditions is not so easy to use. Thus, throughout the 1940s, a number of further *sufficient* conditions were obtained by various authors, many of them inspired by Ore's work. There also appeared further sets of necessary and sufficient conditions. V. Pták gave a necessary and sufficient condition in terms of normal subgroups of the given semigroup, thereby giving a rather more manageable criterion for embedding than that of Maltsev. We mention also the 'polyhedral conditions' of J. Lambek: these are much like Maltsev's conditions, but, as the name suggests, they are a little easier to handle because they have a geometrical interpretation; they are certainly much easier to present.

The problem of embedding semigroups in groups continues to be studied from various different viewpoints. However, I do not propose to try to cover all approaches here. I will cover simply those mentioned above; for a range of other embedding problems, I refer the reader to the various available survey articles, including, but not limited to Bokut (1987), Shutov (1966) and Thibault (1953a, b). One point that I will not address here is the question of 'universality' of embeddings: speaking informally, the question of whether a given embedding is the most 'economical' possible. For some comments on this issue, see Clifford and Preston (1967, Sect. 12.2). 'Universality' is an important property in making embeddings useful, but, apart from a brief observation by Maltsev (1940, Sect. 4), it was not a significant concern of the early authors who attacked semigroup embedding problems—in the interests of saving space, we will therefore make little mention of such properties.

A knowledge of the basics of both group theory and ring theory will be assumed of the reader, but no semigroup theory will be supposed—surprisingly little introductory theory is required for an understanding of the mathematics to be discussed here, which perhaps explains why this is a subject which was studied in the early days of semigroup theory. Any relevant notions will be introduced as we go along.

This article consists of the following sections. In Sect. 2, I begin by giving details of Steinitz' construction of the field of fractions of a given integral domain, and of Ore's corresponding method in the non-commutative case. In Sect. 3, I pick apart Sushkevich's erroneous contributions in this area and give Maltsev's counterexample of a cancellative semigroup which may not be embedded in a group. Section 4 deals with a number of sufficient conditions that came after Ore's 'common right multiples' condition. In particular, I consider the work of the French mathematicians Paul Dubreil and Raouf Doss, as well as that of Thoralf Skolem. In the following three Sects. 5, 6 and 7, I discuss the necessary and sufficient conditions of Maltsev, Pták and Lambek, respectively. In Sect. 8, I compare the conditions of Maltsev and Lambek, and detail work that was carried out to identify the conditions they had in common. At the end of the paper (Sect. 9), I make some concluding remarks on the nature of the embedding problem for semigroups and its place within abstract algebra.

We note here the curious fact that the problem of embedding a semigroup in a group seems to have been one on which many young mathematicians 'cut their teeth'; most of the major contributions to this area were made by people who were just beginning their careers and who, in many cases, never returned to it: Maltsev (Sects. 3, 5) did his embedding work early in his career, returning to it only briefly some years later (in Maltsev 1953); Doss (Sect. 4) and Pták (Sect. 6) also considered embedding problems at the beginning of their careers, before abandoning this for other areas—Fourier analysis for the former and functional analysis for the latter; Lambek and Bush (Sect. 7) both wrote PhD theses on embedding questions and, save for publishing parts of their theses as papers, did no further work in this area; Jackson (also Sect. 7) wrote a master's thesis on embeddings, which contained new results, and yet, despite pursuing an academic career, he never returned to them or made any effort to publish them. The embedding theorems are certainly easy to state, but their proofs are far from elementary, so it is difficult to see why these results should be particularly suitable for early-career mathematicians. On the other hand, perhaps there is in fact no real pattern here: Steinitz, Ore (Sect. 2), Sushkevich (Sect. 3), Dubreil and Skolem (Sect. 4) were all quite well established at the time of carrying out their embedding work.

A number of brief mathematical comments are in order before we proceed. First of all, it is worth noting that the object into which a *non-commutative* ring or semigroup is embedded must also be non-commutative, for, if it were not, this would contradict the non-commutativity of the original ring or semigroup. In the ring context, we observe also that, since a skew field has no zero divisors, the lack of zero divisors is a necessary condition for this embedding to take place. Equivalently, it is necessary that the multiplication of the original ring be cancellative.[4] Indeed, cancellation is a necessary condition in both the commutative and non-commutative cases. The equivalence of cancellation and 'lack of zero divisors' no longer holds in the semigroup case, since a non-cancellative semigroup may lack zero divisors simply by not having a zero element. Obviously, if a semigroup with zero is cancellative, then it does not have any zero divisors. Thus, adapting the above comments on rings, cancellation is clearly a necessary condition for the embedding of a semigroup in a group.

I conclude this introduction with a note on terminology: rather than 'embed'/'embedding'/'embeddability', some authors (for example, both Maltsev and Lambek) have used the terms 'immerse'/'immersion'/'immersibility'. However, I have largely adopted the former terminology here, in the interests of a uniform presentation. The only places where I have not are in translations of the titles of certain papers for which the term 'immersibility' etc. seemed more appropriate.

[4] For a ring has no zero divisors if and only if its multiplication is cancellative (in the sense of note 2). Suppose first of all that a ring $R$ has no zero divisors, and that $ac = bc$, for $a, b, c \in R$, with $c \neq 0$. Then $ac - bc = (a - b)c = 0$, from which it follows that $a - b = 0$, whence $a = b$. The other part of the cancellation law is similar. Conversely, suppose that $R$ is cancellative. For $a, b, c \in R$, with $c \neq 0$, we take the equality $ac = bc$ (from which it follows that $a = b$) and set $b = 0$. □

## 2 Embeddings for rings

### 2.1 The commutative case: Steinitz' Theorem

As noted in the introduction, an embedding problem for rings was given its first explicit formulation by van der Waerden in his *Moderne Algebra* (van der Waerden 1930, Chap. III, Sect. 12) in connection with the notion of the field of fractions of a ring. Van der Waerden began his treatment of this by making the simple observation that if a given commutative ring $\mathfrak{R}$ is already contained within a field $\Omega$, then we may, in a very natural manner, construct *fractions* (or *quotients*) in $\Omega$ of elements of $\mathfrak{R}$:

$$\frac{a}{b} = ab^{-1} = b^{-1}a \quad (b \neq 0),$$

where $b^{-1}$ is the inverse of $b$ in $\Omega$. It is a straightforward observation that equivalence of such fractions is governed by the condition

$$\frac{a}{b} = \frac{c}{d} \Longleftrightarrow ad = bc, \tag{1}$$

and that they may be added and multiplied according to the rules

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}. \tag{2}$$

It may be shown that the collection of all fractions forms a field under the operations in (2): the *field of fractions* (or *field of quotients*) of the original ring.[5]

Having thus dealt with commutative rings that are already located inside a field, van der Waerden next turned his attention to a more general question:

> Which commutative rings possess a field of quotients? Or, what amounts to the same thing, which, in general, can be embedded in a field?[6]

As we have noted, a simple necessary condition for such an embedding to take place is that the ring in question contain no zero divisors. Building upon this observation, van der Waerden (1930, p. 47) presented the following theorem, which we will refer to as *Steinitz' Theorem*, since it first appeared, in essence, in Sect. 3 of Steinitz' paper of 1910:

**Theorem 1** (Steinitz' Theorem) *Every integral domain can be embedded in a field.*

The proof that van der Waerden gave of this theorem was simply Steinitz' 'Quotientenbildung'. Indeed, this is nothing other than the still-familiar construction of the field of fractions of a given integral domain $\mathfrak{R}$: we take the set of all ordered pairs

[5] For a general discussion of fields of fractions, see Cohn (1995, Chap. 1).

[6] 'Welche kommutativen Ringe besitzen einen Quotientenkörper? Oder, was auf dasselbe hinauskommt, welche lassen sich überhaupt in einen Körper einbetten?' (van der Waerden 1930, p. 47).

$(a, b)$ $(b \neq 0)$ of elements of $\mathfrak{R}$ and define an (equivalence) relation $\sim$ on such pairs by the rule that

$$(a, b) \sim (c, d) \Longleftrightarrow ad = bc \tag{3}$$

(cf. (1)). Each pair $(a, b)$ is next replaced by the formal symbol $\frac{a}{b}$. It is then necessary to show that the collection of all such $\frac{a}{b}$ forms a field under the operations given in (2). The original integral domain $\mathfrak{R}$ is finally seen to be included in this field, via the mapping $a \mapsto \frac{ab}{b}$, for some fixed $b \in \mathfrak{R}\backslash\{0\}$. We note that commutativity has an important role to play in this construction. For comparison with what appears below, I include, for example, the proof that relation (3) is transitive:

*Proof* Suppose that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $ad = bc$ and $cf = de$. Taking the first equality and multiplying on the right by $f$, we see that $adf = bcf = bde$, whence $afd = bed$, by commutativity. It follows by cancellation that $af = be$, in which case, $(a, b) \sim (e, f)$. □

Van der Waerden did not confine his attention merely to embeddings for integral domains. He observed, for example, that the above construction is easily modified for commutative rings *with* zero divisors: the denominator of a formal fraction is required to be a non-divisor of zero (van der Waerden 1930, p. 49). He turned his attention also to the non-commutative case, although in this instance he had no solution to offer:

> The possibility of embedding non-commutative rings without zero divisors into full fields is an unsolved problem, except in very special cases.[7]

The problem of whether we can embed a non-commutative ring without zero divisors in a (skew) field is what we have termed *van der Waerden's problem.* We will see Maltsev's negative solution of this problem in Sect. 5. Subsequent versions of van der Waerden's book acknowledge Maltsev's work in a footnote.

## 2.2 The non-commutative case: Ore's Theorem

We turn now to an approach to the embedding problem for *non-commutative* rings—that of Øystein Ore. As we will see (particularly in Sect. 4), Ore's ring-theoretic investigations exerted a strong influence on subsequent work on the embedding problem for semigroups.

Ore's is a familiar name within twentieth-century mathematics.[8] This Norwegian-born mathematician spent most of his life in the USA at Yale University. He appears to have been greatly influenced by Noether and her school, both in his PhD thesis (supervised by Thoralf Skolem, who will appear briefly in Sect. 4) and in his subsequent work, having spent time in Göttingen in the mid-1920s. Indeed, S. C. Coutinho (2004, p. 257) says of Ore's work that it displays

> ... very clearly the influence of the abstract algebra movement that was taking shape under the leadership of Emmy Noether and Emil Artin.

[7] 'Die Möglichkeit der Einbettung nichtkommutativer Ringe ohne Nullteiler in einen sie umfassenden Körper bildet ein ungelöstes Problem, außer in ganz speziellen Fällen' (van der Waerden 1930, p. 49).

[8] For biographies of Ore, see Anon (1970) and Aubert (1970).

This influence is certainly in evidence in the 1931 paper, 'Linear equations in non-commutative fields', that is of interest to us here. A very good account of this paper may also be found in Coutinho (2004, Sect. 2.1).

In the paper in question, Ore considered the generalisation of the theory of determinants to skew fields, with the goal of solving systems of linear equations with coefficients from these. This was a problem that had seen much study since the mid-1920s, but Ore was critical of some of the earlier efforts of other mathematicians. For example, the notion of determinant adopted by both Heyting (1927) and Richardson (1926, 1928) was not defined for all values of the coefficients of the corresponding system of linear equation—to Ore's mind, this limited the usefulness of such determinants (Ore 1931, p. 463). He was also critical of a lack of symmetry in Richardson's definition. The goal of Ore's 1931 paper was to devise a satisfactory definition of determinant in the non-commutative case. It was not his purpose to treat any embedding problem, but one such problem emerged along the way. For the time being, we note that Ore was certainly aware of van der Waerden's problem:

> In the commutative case all domains of integrity (rings without divisors of zero) have a uniquely defined quotient-field, which is the least field containing the ring. For the non-commutative case v. d. Waerden ... has recently indicated this problem as unsolved (Ore 1931, p. 464).

The basic system with which Ore began was simply a non-commutative ring, for which he offered 'algebra' as an alternative term. However, it quickly emerged that this notion was inadequate for his purposes, since it did not allow for all the operations needed when solving systems of linear equations:

> We shall in the following consider systems of linear equations with coefficients which are elements of such a ring. In order to perform an elimination to obtain a solution of a linear system, it seems necessary that the coefficients should satisfy the axioms mentioned ... The main operation for the usual elimination is however to multiply one equation by a factor and another equation by another factor to make the coefficients of one of the unknowns equal in the two equations (Ore 1931, p. 465).

Ore therefore augmented his non-commutative ring with the following further condition (Ore 1931, p. 465):[9]

$M_V$. *Existence of common multiplum.* When $a, b \neq 0$ are two arbitrary elements of a ring $S$, then it is always possible to determine two other elements $m, n \neq 0$ such that $an = bm$.

It is easy to see that this new condition makes possible the operation indicated above. Notice that $M_V$ holds immediately in the commutative case, since $ab = ba$, for any elements $a, b$. To any (non-commutative) ring which satisfies $M_V$ and which has no zero divisors, Ore gave the name *regular ring*;[10] he did not assume the existence of a

[9] Ore labelled his condition in this manner because it was the fifth multiplicative condition in his list.

[10] Not to be confused with the notion of a (*von Neumann*) *regular ring* (see, for example, Goodearl 1979). What Ore called a regular ring (with identity) is now termed a *right Ore domain* (Coutinho 2004, p. 258).

multiplicative identity. Adopting the term 'non-commutative field' for what we have called a skew field, Ore proved the following result, which we will refer to as *Ore's Theorem* (Ore 1931, Theorem 1):

**Theorem 2** (Ore's Theorem) *Any regular ring can be considered as a subring (more exactly: is isomorphic to a subring) of a non-commutative field.*

Thus, any non-commutative ring without zero divisors, and in which any two elements have a common right multiple, can be embedded in a skew field. Ore proved this theorem via a reasonably straightforward generalisation of Steinitz' 'Quotientenbildung'. In Ore's case, however, the formal fractions form not a field, but a skew field. Complications not present in the commutative case emerge in the course of the proof, but condition $M_V$ serves to resolve these.

As already noted, Ore began by forming fractions $\frac{a}{b}$ for $b \neq 0$. The first point that it was necessary for him to address was that of *equality* of fractions, that is, the derivation of a non-commutative version of condition (1). Let $\frac{a}{b}$ and $\frac{a_1}{b_1}$ be two fractions, with $b, b_1 \neq 0$. By $M_V$, we can find $\beta, \beta_1 \neq 0$ such that

$$b\beta_1 = b_1\beta. \tag{4}$$

We then say that

$$\frac{a}{b} = \frac{a_1}{b_1} \Longleftrightarrow a\beta_1 = a_1\beta. \tag{5}$$

The elements $\beta_1, \beta$ will not in general be the unique values satisfying (4), but it is an easy exercise to show that (5) does not depend on the choice of these elements (see Ore 1931, p. 466). The proof that the notion of equality defined in (5) is transitive needs special care, but is again an easy exercise (see Ore 1931, p. 467). I include it here for comparison with the corresponding proof in the commutative case (see Sect. 2.1):

*Proof* Suppose that $\frac{a}{b} = \frac{a_1}{b_1}$ and $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, where $b, b_1, b_2 \neq 0$. Then there exist $\beta, \beta_1, \beta_1', \beta_2 \neq 0$ such that $b\beta_1 = b_1\beta$, $a\beta_1 = a_1\beta$, $b_1\beta_2 = b_2\beta_1'$ and $a_1\beta_2 = a_2\beta_1'$. By $M_V$, there exist $r, s \neq 0$ such that $\beta r = \beta_2 s$. It then follows easily from the listed equations that $b(\beta_1 r) = b_2(\beta_1' s)$ and $a(\beta_1 r) = b_2(\beta_1' s)$; hence, $\frac{a}{b} = \frac{a_2}{b_2}$. □

In the non-commutative case, addition of fractions is defined by

$$\frac{a}{b} + \frac{a_1}{b_1} = \frac{a\beta_1 + a_1\beta}{b\beta_1} = \frac{a\beta_1 + a_1\beta}{b_1\beta},$$

where $\beta, \beta_1$ are as in (4), whilst multiplication is given by

$$\frac{a}{b} \cdot \frac{a_1}{b_1} = \frac{a\alpha_1}{b_1\beta}, \tag{6}$$

where $b\alpha_1 = a_1\beta$. Ore showed that these operations do not depend on the choices of $\beta, \beta_1, \alpha_1$, that they are well-defined, and that they satisfy the other properties required for a skew field. The original ring may be embedded in the skew field of fractions via the mapping $a \mapsto \frac{ac}{c}$, for any fixed $c \neq 0$. Condition $M_V$ is thus shown to be sufficient for the embedding of a non-commutative ring in a skew field.

As observed above, the main purpose of Ore's work was to find a suitable definition of determinant in the non-commutative case. In particular, he aimed to find those rings in which it is possible to define such a determinant, and in which we may therefore solve systems of linear equations by elimination. Regular rings are precisely the rings that Ore sought. In the introduction to the paper, he commented:

> ... I discuss the properties of rings in which the elimination can be performed; these rings must satisfy a certain axiom $M_V$ and this is, as I show, equivalent to the fact, that the ring can be completed to a non-commutative field ('Quotientenkörper') by the introduction of formal quotients of elements in the ring (Ore 1931, p. 464).

Theorem 2, together with some associated results that I do not reproduce here, provides a proof of this assertion. Further, we can see that condition $M_V$ is required for Ore's notion of determinant if we consider the following pair of linear equations:

$$x_1a_{11} + x_2a_{12} = b_1, \qquad x_1a_{21} + x_2a_{22} = b_2, \tag{7}$$

where the $a_{ij}$ and $b_k$ belong to some regular ring $S$. By $M_V$, we can find $A_{12}, A_{22} \in S$ such that $a_{12}A_{22} = a_{22}A_{12}$. We then multiply the first equation by $A_{22}$, and the second by $A_{12}$ (both on the right), before subtracting one from the other, to obtain:

$$x_1(a_{11}A_{22} - a_{21}A_{12}) = b_1A_{22} - b_2A_{12},$$

which Ore wrote as

$$x_1 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{Vmatrix}.$$

The quantities between the vertical lines are termed *right-hand determinants of second order*; the determinant on the left-hand side of the equality will be denoted, following Ore, by $\Delta_{12}^{(12)}$. I omit all further details of Ore's work, except to note the following pleasing result (Ore 1931, p. 472):

**Theorem 3** *The system of Eq.* (7) *has a unique solution if and only if the determinant* $\Delta_{12}^{(12)}$ *does not vanish.*

Thus, Ore's notion of determinant appears to be the 'correct' one, since it behaves, at least to the above extent, as a determinant should. In the final section of the paper, Ore extended these ideas to determinants of $n$th order.

### 2.3 Extension to semigroups

We have seen that an integral domain, and also a certain type of non-commutative ring without zero divisors, may easily be embedded in a (skew) field simply by the construction of the corresponding (skew) field of fractions. Moreover, the methods employed by Steinitz and Ore were almost entirely multiplicative in nature, and may therefore be adapted immediately to the semigroup case. In this way, we obtain two theorems concerning the embedding of a commutative cancellative semigroup, and a particular non-commutative cancellative semigroup, in a group, namely, its group of fractions: the multiplicative group of the (skew) field of fractions obtained via the construction of Steinitz (or Ore). As noted in the introduction, however, this adaptation to the semigroup case seems to have been regarded as so obvious that it was a long time before any mathematician took the trouble to write down the theorems rigorously. The earliest explicit statement and proof of these results came in the work of Dubreil, which we will see in Sect. 4. For future reference, however, we record here the semigroup versions of both Steinitz' and Ore's Theorems:

**Theorem 4** (Semigroup version of Steinitz' Theorem) *Every commutative cancellative semigroup may be embedded in a group, namely, its group of fractions.*[11]

**Theorem 5** (Semigroup version of Ore's Theorem) *Any cancellative semigroup in which every pair of elements has a common right multiple may be embedded in a group, namely, its group of fractions.*

## 3 Early results for semigroups

### 3.1 Sushkevich goes awry

As observed in the introduction, the first author to give an explicit treatment of the embedding problem for semigroups was the Russian mathematician A. K. Sushkevich. This was in a 1935 paper in Ukrainian, entitled 'On the extension of a semigroup to a whole group' ('Про поширення півгрупи до цілої групи'). Unlike some of the authors we will meet in later sections, Sushkevich gave no explicit indication of what led him to consider the embedding problem, although the final part of his paper does mention the embedding of integral domains in fields, and contains a passing reference to van der Waerden's *Moderne Algebra*. The goal of the above-mentioned paper was the construction of a group 'around' any given (cancellative) semigroup, thereby 'proving' that any such semigroup may be embedded in a group.

The main technique employed in Sushkevich's 1935 embeddings paper had its origins in some work of the previous year. In an earlier (German) paper, 'Über Semigruppen' (Suschkewitsch 1934), Sushkevich had given a decomposition of an arbitrary semigroup as the union of two disjoint parts. To modern eyes, this decomposition is somewhat simple-minded, for it is merely the separation of a semigroup into its group of units and the two-sided ideal formed of all non-units.[12] Sushkevich termed the former the *group part* (*Gruppenteil*, or *групова частина* in the Ukrainian of the 1935 paper) of the semigroup, and the latter the *principal part* (*Hauptteil*; *головна частина*). Using this terminology, he had been able to make some elementary observations, such as the fact that the group part of a semigroup is nonempty if and only if

[11] This may in fact be stated as a slightly stronger result: a commutative semigroup can be embedded in a group if and only if it is cancellative (see Clifford and Preston 1961, Sect. 1.10).

[12] A two-sided ideal in a semigroup is defined in the same way as for a ring, but without any mention of addition; see Hollings (2009a, p. 502).

the semigroup has an identity element. It should be noted that, in both the 1934 and 1935 papers, all of Sushkevich's semigroups were cancellative: his decision to adopt the German term 'Semigruppe' for his objects of interest in the 1934 paper, which he had not previously employed in his earlier studies of not-necessarily-cancellative semigroups, may very well have been influenced by L. E. Dickson's use of 'semigroup' to denote what we would now term a cancellative semigroup (see Hollings 2009a, Sect. 3).[13] Sushkevich assumed implicitly that none of his (cancellative) semigroups were groups. Since any finite cancellative semigroup is necessarily a group,[14] all of Sushkevich's 'Semigruppen' were thus assumed to be infinite. In particular, the principal part of a given cancellative semigroup, as a cancellative semigroup itself, was necessarily infinite; there was no such restriction on the group part.

In the paper of 1935, Sushkevich began with a cancellative semigroup $\mathfrak{S}$; the group part of $\mathfrak{S}$ was denoted by $\mathfrak{G}$, and its principal part by $\mathfrak{H}$. Sushkevich's goal was the construction of a group into which $\mathfrak{S}$ may be embedded. To this end, he considered separately the case where $\mathfrak{S}$ has an identity element, and that where it does not. I give here a sketch of his method in each instance:

(1) $\mathfrak{S}$ *has no identity element.* By the above comments, $\mathfrak{G} = \emptyset$, and so $\mathfrak{S} = \mathfrak{H}$. For each $X \in \mathfrak{H}$, introduce a new element $\overline{X}$, and denote the collection of all such by $\overline{\mathfrak{H}}$. Define a multiplication in $\overline{\mathfrak{H}}$ by setting $\overline{Q}\,\overline{P} = \overline{R}$ in $\overline{\mathfrak{H}}$ whenever $PQ = R$ in $\mathfrak{H}$. It is clear that $\overline{\mathfrak{H}}$ forms a cancellative semigroup, anti-isomorphic to $\mathfrak{H}$.

Next, introduce another new element $E$ and define it to be a two-sided identity for all elements of both $\mathfrak{H}$ and $\overline{\mathfrak{H}}$. Further, for any $X \in \mathfrak{H}$ and the corresponding $\overline{X} \in \overline{\mathfrak{H}}$, set

$$X\overline{X} = \overline{X}X = E, \tag{8}$$

thereby making each $\overline{X}$ a formal inverse for the corresponding $X$.

Finally, construct all alternating products of elements from $\mathfrak{H}$ with elements from $\overline{\mathfrak{H}}$:

$$P\overline{Q},\ \overline{P}Q,\ P\overline{Q}R,\ \overline{P}Q\overline{R},\ P\overline{Q}R\overline{S},\ \overline{P}Q\overline{R}S,\ \ldots \tag{9}$$

Products of the forms in (9) may be multiplied together to obtain other such alternating products simply by applying the rules for multiplication in $\mathfrak{H}$ and $\overline{\mathfrak{H}}$, with further simplification afforded by the application of (8). Two alternating products

[13] Sushkevich is one of only two authors I have found who used the term 'Semigruppe' (at least in German), the modern German term for a semigroup being 'Halbgruppe'. The other author was Fritz Klein-Barmen (1943). Moreover, in the Ukrainian of his 1935 embeddings paper, Sushkevich used the term 'півгрупа', in contrast to the modern Ukrainian 'напівгрупа' ('пів-' and 'напів-' both being Ukrainian prefixes denoting 'half-' or 'semi-'). We will see similarly unusual terminology in the work of Vlastimil Pták in Sect. 6 (see, in particular, note 44).

[14] For, suppose that $S$ is a finite cancellative semigroup. It follows immediately from cancellation that $sS = S = Ss$, for any $s \in S$. From this, we conclude that, for a fixed $s$, there exists $e \in S$ such that $se = s$. But $e$ is in fact a right identity for any element of $S$, since any $t \in S$ may be written in the form $t = us$. It follows further that $e^2 = e$. Consider the product $es$, for any $s \in S$. We may write $s$ as $eu$, for some $u \in S$. Then $es = e^2u = eu = s$. Thus, $e$ is a two-sided identity for $S$. As to inverses, it follows from $sS = S$ that there exists $s' \in S$ such that $ss' = e$. Furthermore, $ss's = es = s = se$, and so it follows from cancellation that $s's = e$. □

are deemed equivalent if one may be obtained from the other by the application of the rules for multiplication in $\mathfrak{H}$ or $\overline{\mathfrak{H}}$, or through the insertion or deletion of factors of the form $X\overline{X}$ or $\overline{X}X$; otherwise, two such products are distinct.

Sushkevich denoted by $\mathfrak{H}_1$ the system formed from the union $\mathfrak{H} \cup \overline{\mathfrak{H}} \cup \{E\}$, together with all products from (9). He argued briefly that $\mathfrak{H}_1$ is in fact a group. The semigroup $\mathfrak{H}$ is thus, apparently, embedded in the group $\mathfrak{H}_1$.

(2) $\mathfrak{S}$ *has an identity element.* Starting from the semigroup $\mathfrak{S} = \mathfrak{G} \cup \mathfrak{H}$, apply the construction from case (1) to the principal part $\mathfrak{H}$, thereby constructing the 'group' $\mathfrak{H}_1$. Identify the identity $E$ of $\mathfrak{H}_1$ with that of $\mathfrak{G}$ and, with this done, form the union of $\mathfrak{H}_1$ and $\mathfrak{G}$. The only products that are not already defined within this union are those of elements from $\overline{\mathfrak{H}}$ with elements from $\mathfrak{G}$. Let $A \in \mathfrak{G}$ and $P, Q, R \in \mathfrak{H}$; the inverse of $A$ in $\mathfrak{G}$ is denoted by $A^{-1}$. Products of elements from $\overline{\mathfrak{H}}$ with elements from $\mathfrak{G}$ are then defined by the following rules:

$$A\overline{P} = \overline{Q}, \text{ if } PA^{-1} = Q; \tag{10}$$
$$\overline{P}A = \overline{R}, \text{ if } A^{-1}P = R. \tag{11}$$

In fact, these rules have an immediate and somewhat surprising consequence: if the first equality in (10) is multiplied on the right by $P$, and the first equality in (11) on the left by $P$, then:

$$A = \overline{Q}P = P\overline{R}. \tag{12}$$

The products $\overline{Q}P$ and $P\overline{R}$ are clearly of the forms given in (9), and so they both belong to $\mathfrak{H}_1$. Moreover, given any $A \in \mathfrak{G}$ and any $P \in \mathfrak{H}$, it is always possible to find unique elements $Q, R \in \mathfrak{H}$ which satisfy the second equalities in (10) and (11). This means that *any* $A \in \mathfrak{G}$ can be written in the form (12). It follows that $\mathfrak{G}$ is contained in $\mathfrak{H}_1$. At this point, Sushkevich had thus, apparently, embedded an arbitrary cancellative semigroup in a group—indeed, the same group as in case (1).

We know from the comments in the introduction (and will see in more detail in the next subsection) that Sushkevich must have gone wrong somewhere—indeed, the error lies in his definition of $\mathfrak{H}_1$, as we will see below. Following his presentation of the above methods, Sushkevich went on to 'prove' this embedding in a slightly different manner. However, since this second method also employs the above construction for $\mathfrak{H}_1$, we need not go into it here. The paper concludes with what would have been a nice application of Sushkevich's results, had the preceding parts of the paper been error-free. Specifically, Sushkevich offered up an extension of his methods for semigroups as a new proof Steinitz' Theorem, distinct from the 'Quotientenbildung'. In connection with the latter, Sushkevich cited van der Waerden's *Moderne Algebra* (as well as its Russian translation).[15]

Regarding the question of what went wrong with Sushkevich's proof, we need look no further than A. G. Kurosh's review (Zbl 0013.05502) of the paper for *Zentralblatt für*

[15] Not long after this paper was published, Sushkevich presented the 'Quotientenbildung' in both his short Ukrainian textbook *Elements of new algebra* (*Елементи нової алгебри*) (Sushkevich 1937b, Sect. 14) and in the 3rd Russian edition of his *Foundations of higher algebra* (*Основы высшей алгебры*) (Sushkevich 1937a, Sect. 236). In neither case, however, did he mention the semigroup case.

*Mathematik und ihre Grenzgebiete*. Kurosh did not identity the error precisely, but he did note that Sushkevich had failed to prove adequately that the multiplication in $\mathfrak{H}_1$ is both well-defined and associative, something that Kurosh considered to be 'certainly not trivial'.[16] Indeed, this is where I believe the error to be: despite Sushkevich's readiness to believe that it was so, it is not at all clear why the multiplication of the products (9) should be well-defined.

Sushkevich's 'On the extension of a semigroup to a whole group' is not a very widely cited paper. This is perhaps because, as we will see in the next subsection, the error was picked up very quickly. One of the very few references that I have found to it is in Chap. IV of Sushkevich's own 1937 monograph *Theory of generalised groups* (*Теория обобщенных групп*), in which he set out in a coherent form the majority of his semigroup-theoretic investigations. The relevant passages of the monograph are, however, somewhat muddled, and appear to contain a (needless to say, unsuccessful) attempt to patch up the proof of the embedding theorem. At the same time, we find an acknowledgment of the work of Maltsev that demonstrated the error in Sushkevich's methods:

> The example presented by Maltsev refutes my previous assertion that *any* [cancellative] semigroup can be extended to a group.[17]

We turn our attention to Maltsev's 1937 counterexample in the following subsection.

### 3.2 Maltsev's counterexample

As I will discuss briefly in Sect. 5, it appears to have been A. N. Kolmogorov who first drew Maltsev's attention to van der Waerden's problem (see Aleksandrov et al. 1968, English translation, p. 157). The (negative) solution of the problem was the real goal of Maltsev's 1937 paper: the construction of the corresponding semigroup counterexample was for him simply a means to an end.

At the time of submitting his counterexample to *Mathematische Annalen* (in his paper 'On the immersion of an algebraic ring into a field': Malcev 1937), Maltsev was teaching mathematics at a high school near Moscow. He had graduated with a first degree from Moscow State University in 1931 but retained a Moscow University affiliation in connection with the research work that he was engaged in by the mid-1930s (see Sect. 5). Indeed, S. I. Nikolskii (1972, English translation, p. 179) later recalled that around this period, Maltsev 'haunted the mathematics library' of Moscow State University. We might therefore suppose a broad familiarity on Maltsev's part with the mathematical literature of the time. I believe, however, that there is some small doubt as to whether he had seen Sushkevich's 1935 embeddings paper—I suggest that he knew it only through the *Zentralblatt* review that we saw in the preceding subsection. I make this assertion because of the style in which Maltsev subsequently cited Sushkevich in his own paper: rather than using Ukrainian (surely an easy option

[16] 'gewiß nicht trivial'.

[17] 'Пример, приводимый Мальцевым, опровергает мое прежнее утверждение о том, что в с я к у ю полугруппу можно дополнить до группы' (Sushkevich 1937c, p. 91, footnote 1).

for a native Russian speaker?), Maltsev listed Sushkevich's paper under the German version of its title ('Über die Erweiterung der Semigruppe bis zur ganzen Gruppe'), which had appeared in the German summary at the end of the paper, and which was also used in the *Zentralblatt* review. Moreover, Maltsev used the German version of Sushkevich's name ('Suschkewitsch'), as employed in the German summary, and by *Zentralblatt*. Maltsev also gave the language of Sushkevich's paper as being Russian—*Zentralblatt* makes the same mistake. I contend that Maltsev would not have made this error himself, had he seen the paper. Of course, the mistakes may not have been Maltsev's—they could have been the work of an over-zealous editor who, with the best of intentions, sought to make Maltsev's references consistent with *Zentralblatt*.

Whether or not Maltsev had seen Sushkevich's paper, he made no detailed comments on its content, nor did he need to: it was enough for him to state Sushkevich's erroneous result, and then present his own counterexample. As already commented, Maltsev's 1937 paper was published in *Mathematische Annalen*, although it did not appear in German, but, rather, in English. It begins, like Sushkevich's paper, by defining a 'semigroup' to correspond to the modern notion of a *cancellative* semigroup. As a reference for his notion of 'semigroup', Maltsev cited O. Yu. Schmidt's monograph *Abstract theory of groups* (*Абстрактная теория групп*) of 1916; an examination of the latter text reveals that, like Sushkevich later on, Schmidt derived his sense of the word 'semigroup' from Dickson (Schmidt 1916, p. 55). Thus, although Schmidt employed the modern Russian term for a general semigroup ('полугруппа'), he used it to mean a *cancellative* semigroup.

Further down the first page of his paper, Maltsev made the following observations, which include his only direct reference to Sushkevich (with the full citation in a footnote, which I do not reproduce here):

> It can be easily proved that every commutative semigroup can be „immersed" (eingebettet) into a group... However, the analogous question concerning non-commutative semigroups, as far as I know, remained unsolved.
>
> Prof. A. Suschkewitsch has published a proof... that every semigroup can be immersed into a group. However, we shall construct ...a semigroup which can not be immersed into a group; thus, Professor Suschkewitsch's result fails to be true (Malcev 1937, p. 686).

The first sentence in this quotation is of course a statement of the semigroup version of Steinitz' Theorem (our Theorem 4). Maltsev attached a footnote to the end of this sentence, explaining that the result may be proved in much the same way as for integral domains: via the construction of the group of fractions. We thus have, seemingly for the first time, an explicit statement of Theorem 4, if not an explicit proof. In connection with the original 'Quotientenbildung', Maltsev cited van der Waerden's *Moderne Algebra*. It is therefore not unreasonable to suggest that the inspiration for his 1937 work came from a reading of van der Waerden. The parenthetic German term in the above quotation (as well as in one below) is also quite suggestive.

In spite of Maltsev's initial discussion of semigroups (which he probably included because it became important later on, and was unlikely to have been entirely familiar to his readers), we can see from his title that the ring case was his main object of interest. After stating the embedding problem for cancellative semigroups, he commented:

> An analogous problem exists for rings, viz. can every ring without divisors of zero (Nullteilern) be immersed in a field... ? (Malcev 1937, p. 686)

This is of course van der Waerden's problem, Maltsev's main concern: he constructed a counterexample to demonstrate that the problem has a negative solution. The corresponding counterexample for semigroups appeared as part of Maltsev's ring counterexample: he first constructed a cancellative semigroup $\mathfrak{H}$ that cannot be embedded in a group, and then constructed a ring without zero divisors around this. Since the multiplicative semigroup of this ring contains $\mathfrak{H}$, it cannot be embedded in a skew field. Thus, in Maltsev's words, 'a problem mentioned by van der Waerden finds its solution' (Malcev 1937, p. 687).

The key to Maltsev's counterexample(s) is his 'condition Z', which he introduced in the first section of the paper:

*Condition* Z. For a cancellative semigroup $\mathfrak{H}$, if $A, B, C, D, X, Y, U, V \in \mathfrak{H}$ and

$$AX = BY, \qquad CX = DY, \qquad AU = BV, \tag{13}$$

then

$$CU = DV. \tag{14}$$

Maltsev proved immediately that condition Z is necessary for the embedding of a cancellative semigroup in a group (Malcev 1937, p. 687):

*Proof* Suppose that $\mathfrak{H}$ may be embedded in a group. We may therefore speak of inverses of elements of $\mathfrak{H}$ (within the group). This, in turn, means that we may rearrange the expressions in (13), to obtain

$$B^{-1}A = YX^{-1}, \qquad D^{-1}C = YX^{-1}, \qquad B^{-1}A = VU^{-1},$$

which combine to give $D^{-1}C = VU^{-1}$. Further rearrangement yields (14). □

In general, however, condition Z is not sufficient (see, for example, the later counterexample of Holvoet 1959). Nevertheless, with the necessity of condition Z established, Maltsev was able to outline his procedure for the construction of the required counterexample:

> Hence [it] follows that if a [cancellative] semigroup $\mathfrak{H}$ does not satisfy the condition Z then this semigroup can not be immersed into a group. In the next [section] we shall construct a [cancellative] semigroup not satisfying the condition Z (Malcev 1937, p. 687).

Note that Maltsev's counterexample also demonstrates that whilst cancellation may be necessary for embedding, it is not, in general, sufficient.

Since it is reasonably straightforward, I outline here Maltsev's recipe for constructing his counterexample. First of all, we take all possible strings of the eight letters $a, b, c, d, x, y, u, v$ and identify the following 'corresponding pairs' of letters:

$$ax \longleftrightarrow by, \qquad cx \longleftrightarrow dy, \qquad au \longleftrightarrow bv.$$

We may then use these relations to transform one string into another, by replacing any of the above pairs of letters with their corresponding pair; Maltsev referred to this procedure as an 'elementary transformation'. Two strings are deemed 'equivalent' (denoted $\alpha \sim \beta$, for strings $\alpha, \beta$) if we may get from one to the other via a finite sequence of elementary transformations; it is clear that this notion of 'equivalence' is indeed an equivalence relation. Moreover, it is a *congruence*: if, for strings $\alpha, \beta, \gamma, \delta$, we have $\alpha \sim \beta$ and $\gamma \sim \delta$, then $\alpha\gamma = \beta\delta$, where the binary operation on strings is concatenation. The congruence property then makes it possible for us to write down a well-defined multiplication of equivalence classes:[18] $(\alpha)(\beta) = (\alpha\beta)$, where $(\alpha)$ denotes the equivalence class of a string $\alpha$. With respect to this operation, the collection $\mathfrak{H}$ of all equivalence classes thus forms a semigroup, indeed, a cancellative semigroup. Focusing, in the first instance, on equivalence classes of singleton strings, Maltsev observed that condition Z does not hold in $\mathfrak{H}$: $(a)(x) = (b)(y)$, $(c)(x) = (d)(y)$ and $(a)(u) = (b)(v)$, but $(c)(u) \neq (d)(v)$. It follows that $\mathfrak{H}$ may not be embedded in a group. The counterexample in the ring case then consists of all linear forms $\sum_i k_i X_i$, where $X_i \in \mathfrak{H}$ and $k_i \in \mathbb{Q}$, and only a finite number of the $k_i$ are non-zero. Maltsev thus constructed a non-commutative ring without zero divisors which is not embeddable in a skew field, and whose multiplicative semigroup is not embeddable in a group. This later gave rise to a more subtle problem: are there rings which do not embed in skew fields but whose multiplicative semigroups embed in groups? On this problem, see Hollings (2014b, Sect. 5.5).

Maltsev's paper of 1937 merely marked the beginning of his comprehensive work in this area. Indeed, it appears that even as early as 12th April 1936 (the date at the end of the paper), Maltsev had already obtained the deeper results that we will consider in Sect. 5, for we find the following in his introduction:

> We have also found the necessary and sufficient conditions for the possibility of immersion of a semigroup into a group. However these are too complicated to be included in this paper (Malcev 1937, p. 687).

Although Maltsev's necessary and sufficient conditions are, in essence, of much the same form as condition Z, the theory required to set them up is indeed rather complicated, as we will see.

### 3.3 Sushkevich revisited

We conclude this section by returning briefly to Sushkevich. Following his material on embeddings in *Theory of generalised groups*, Sushkevich does not appear to have attempted to make any further contributions in this area. Quite how Sushkevich reacted to Maltsev's disproof of his 'theorem' is something we will never know, although it is interesting to note that the two of them almost certainly met on at least one occasion, in

[18] Indeed, this is what is going on in the background of the 'Quotientenbildung'. For more on this idea, see Howie (1995, Sect. 1.5) or Hollings (2007).

1939. On 13th–17th November of that year, an All-Union Conference on Algebra was held in Moscow (see Anon 1940). During this conference, both Sushkevich and Maltsev delivered lectures in the afternoon session of 16th November, one after the other. Sushkevich began the session with a talk entitled, somewhat vaguely, 'On one type of generalised group' ('Об одном типе обобщенных групп'). Maltsev followed with 'On extensions of associative systems' ('О расширениях ассоциативных систем'), in which he reported, amongst other things, his necessary and sufficient conditions for the embeddability of a semigroup in a group. It would therefore be rather surprising if Sushkevich had not been exposed to Maltsev's wider work here, even if he had not encountered it earlier.

As a final comment on this matter, we note that there is in fact a small piece of (admittedly, circumstantial) evidence to suggest that, in later years, Sushkevich tried to disown his 1935 paper: it is the only significant omission from an otherwise quite comprehensive publications list, in Sushkevich's own hand, that once comprised part of his Kharkov State University personnel file, and which is now held by the Ukrainian State Archives (Kharkiv Region).[19]

## 4 Further sufficient conditions

We have already seen two conditions for the embeddability of a semigroup in a group: Ore's sufficient condition $M_V$ and Maltsev's necessary condition Z. Before considering Maltsev's later work in this area, we temporarily abandon our hitherto chronological account, and pause to consider some further sufficient conditions which were obtained by various authors in the 1940s, the inspiration for several of which came from Ore's original sufficient condition $M_V$.

### 4.1 Right regularity: Paul Dubreil

The first author whom we will consider here is the French mathematician Paul Dubreil.[20] The study of semigroups in France owes it initial impetus to Dubreil's work (see Hollings 2014b, Chap. 7); it is one of his early semigroup-theoretic papers that concerns us here. In this paper, Dubreil considered not merely the *possibility* of embedding a semigroup in a group, but also the *manner* in which the embedding may be realised. A. H. Clifford and G. B. Preston, in their 1961 textbook *The algebraic theory of semigroups*, commented:

> Although [the semigroup version of] Ore's Theorem is phrased as a sufficient condition for embeddability in a group, it was noted by Dubreil ... that [Ore's condition] is nonetheless a necessary and sufficient condition for the embedding to be of [a] simple type (Clifford and Preston 1961, p. 36).

[19] Ф.Р-2782, оп. 20, спр. 572, арк. 10–12.

[20] For biographies of Dubreil, see Lesieur (1994) and Lallement (1995), and also the autobiographical material contained in Dubreil (1981, 1983).

Dubreil's study of embeddings began with a very short communication, 'Sur les problèmes d'immersion et la théorie des modules', in *Comptes rendus hebdomadaires des séances de l'Académie des Sciences de Paris*. Early in the paper, Dubreil acknowledged Maltsev's use of the necessary condition Z, before commenting in passing about his necessary and sufficient conditions:

> Subsequently, this author has given necessary and sufficient conditions for there to exist a group containing a given [cancellative] semigroup.[21]

We can in fact say with some certainty that Dubreil's knowledge of Maltsev's work came from van der Waerden and H. Richter, for Dubreil made the following acknowledgment:

> This memoir [that is, Maltsev's paper of 1939] and its translation were kindly communicated to me by Messrs. B. L. van der Waerden and H. Richter, to whom I express my sincere thanks.[22]

The translation mentioned here was presumably an ad hoc, privately circulated one, since it pre-dated the systematic translation of Soviet mathematical work by many years.

Maltsev's necessary and sufficient conditions, however, were not Dubreil's immediate concern:

> But another result of intermediate generality, and of particular interest for its ease of handling and possibilities for application, was given in 1931 by O. Ore ...[23]

Dubreil next quoted Ore's Theorem (our Theorem 2), though he replaced Ore's 'regular' by the new term *right regular* (*régulier à droite*); *left regularity* may be defined similarly, in terms of common *left* multiples. Amongst Dubreil's observations concerning this condition is the fact that Maltsev's condition Z follows from right (or left) regularity.

The major semigroup-theoretic result of Dubreil's paper, which he stated without proof, follows, in part, from Theorem 2 (Dubreil 1943, p. 626):

**Theorem 6** *For a cancellative semigroup $S$ to be contained in a group $G$, and for every element $\xi$ of $G$ to admit at least one representation in the form $\xi = a \cdot b^{-1}$, for $a, b \in S$, it is necessary and sufficient that $S$ be right regular.*

Note that since the semigroup in this theorem is non-commutative in general, an element of the form $a \cdot b^{-1}$ (a *left fraction*/*fraction à gauche*—so called for seemingly

[21] 'Ultérieurement, cet auteur a donné des conditions nécessaire et suffisantes pour qu'il existe un groupe contenant un semi-group donné' (Dubreil 1943, p. 626). Note that in his use of the term 'semi-groupe' to mean a cancellative semigroup, Dubreil was almost certainly following de Séguier (1904), who had originally coined the term for this notion (see Hollings 2009a, Sect. 3); Dubreil termed a general semigroup a 'demi-groupe'.

[22] 'Ce Mémoire et sa traduction m'ont été aimablement communiqués par MM. B. L. van der Waerden et H. Richter, auxquels j'exprime mes sincères remerciements' (Dubreil 1943, p. 626, footnote 2).

[23] 'Mais un autre résultat, d'un degré de généralité intermédiaire, et particulièrement intéressant par sa maniabilité et ses possibilités d'application, a été donné dès 1931 par O. Ore ...' (Dubreil 1943, p. 626).

arbitrary reasons) is an entity distinct from an element of the form $a^{-1} \cdot b$ (a *right fraction/fraction à droite*).

With regard to the above quotation from Clifford and Preston, Theorem 6 gives the remarked-upon 'simple type' of an embedding: an embedding into the group of left fractions. Thus, although right regularity is only sufficient for embedding in general, it is both necessary *and* sufficient for embedding specifically into a group of left fractions.

Dubreil's paper concludes with the statement of four theorems on right regular rings, with a view, like in Ore's paper, towards the study of systems of linear equations over a right regular ring; this goes some way towards explaining the second part of Dubreil's title: 'the theory of modules'.

Such material on embeddings also appears in Dubreil's 1946 textbook *Algèbre* (Chap. V, Part A, Sect. 1). The discussion here also takes in right regularity and condition Z, but the presentation is a little different, if only cosmetically. Perhaps with the intention of mirroring Maltsev, Dubreil (1946, p. 138) introduced the following condition:

*Condition* $\mathcal{C}$. Given a group $G$ with subsemigroup $S$, every element $\xi \in G$ may be written as a left fraction of elements of $S$: $\xi = ab^{-1}$, for $a, b \in S$.

Theorem 6 may thus be rephrased more succinctly in terms of condition $\mathcal{C}$.

Later in his section on semigroup embeddings, Dubreil considered formal right fractions of semigroup elements in a slightly different setting, showing that they exhibit the structure, not of a group, but of a *Brandt groupoid*—see Coutinho (2004, p. 272). We note also that Dubreil's *Algèbre* contains what appears to be the first explicit proof of Theorem 5.

## 4.2 Left quasi-regularity: Raouf Doss

The next sufficient condition to be considered here is another that was derived from Ore's original, and was due to Raouf Doss. Doss is one of our examples of a mathematician who studied the embedding problem early in his career, and then never returned to it again (the majority of his subsequent mathematical work was in Fourier analysis). Moreover, he gave no indication of what inspired him to carry out this work, or what led him to the condition contained therein.

Doss' contribution to this subject may be found in a paper entitled 'Sur l'immersion d'un semi-groupe dans un groupe', published in 1948.[24] Like other authors in this area, Doss began with a brief summary of the previous embedding conditions obtained by other authors, including Ore's sufficient condition, and Maltsev's necessary and sufficient conditions. He then gave a new condition, based upon that of Ore: a semigroup $S$ is called *left quasi-regular* (*quasi-régulier à gauche*) if, whenever two elements $a, b \in S$ have a common left multiple, $ra = sb$, for some $r, s \in S$, it is possible to find $r', s' \in S$ such that $r'a = s'b$ and at least one of $r', s'$ has a common left multiple

[24] A detailed account of this paper, as well as a general discussion of the embedding problem, may be found in a series of articles by Faisant (1971a, b, c, 1972).

with every other element of $S$. The main, indeed only, result of Doss' paper is the following (Doss 1948, p. 139):

**Theorem 7** *Every cancellative left quasi-regular semigroup can be embedded in a group.*

Since left quasi-regularity follows from left regularity, Theorem 6 may be obtained as a corollary to Theorem 7. Doss' proof of the above theorem involves the verification of a system of Maltsev conditions (see Sect. 5), and so it was necessary for him to give a brief account of Maltsev's criteria for embeddability. To the best of my knowledge, besides Maltsev's own, this is the earliest detailed account of Maltsev's conditions, and it was praised by A. H. Clifford in *Mathematical Reviews* (MR0029384): 'The author's clear statement of these conditions is very welcome'.

As already noted, Doss made no comment on his motivation for this study. His condition is certainly a generalisation of Ore's, so perhaps Doss set out to find a more general sufficient condition than that employed by Ore and Dubreil. He certainly had new tools at his disposal for such an endeavour: his appears to be the first paper to have made extensive use of Maltsev's conditions.

### 4.3 Common multiples revisited: Thoralf Skolem

I conclude this section with the contributions of Thoralf Skolem to the embedding problem. However, no new conditions will be given here: Skolem rediscovered Ore's condition apparently independently.

Skolem's first statement of an embedding theorem for semigroups appears at the beginning of his paper 'Some remarks on semi-groups' (Skolem 1951). Unfortunately, it is a 'theorem' which, as we well know by now, is not true: Skolem asserted that every cancellative semigroup can be embedded in a group. He attempted no detailed proof of this result, merely stating: 'I will only give a hint of the proof of the ... assertion' (Skolem 1951, p. 43). His 'hint', sadly, is rather familiar to us from Sect. 3: he suggested the adjunction of formal inverses to the given semigroup, and arrived at alternating products of the form:

$$a^{-1}bc^{-1}\cdots, ab^{-1}c\cdots.$$

Thus, we see that Skolem fell into the same trap as Sushkevich: for Skolem, just as for Sushkevich, there is no guarantee that the multiplication of such alternating products is well-defined.

By the following year, however, Skolem appears to have spotted his mistake; he gave the following result at the beginning of his paper 'A theorem on some semi-groups' (Skolem 1952, p. 72):

**Theorem 8** *Let $H$ be a semigroup with left and right cancellation, and suppose that every pair of elements has a common left (or right) multiple. Then $H$ can be embedded in a group.*

This is of course a result which we know to be true, thanks to our study of the work of both Ore and Dubreil. Perhaps not one to make the same mistake twice, Skolem provided a full proof of this theorem. He proved it, as we might expect, via the construction of a group of fractions. Indeed, Skolem's proof is much the same as Ore's. Any hint of motivation is, unfortunately, lacking. Skolem may not have had access to a particularly wide range of foreign sources,[25] but it is hard to believe that he did not know about Ore's Theorem—not only because this seems to have been a well-known result, and is perhaps the only common thread running through much of the work considered so far, but also because Skolem had been Ore's supervisor, and it would have been perfectly natural for him to have kept track of his former student's work. I suggest therefore that, although he made no reference to Ore (indeed, Skolem does not appear to have been one for extensive bibliographies), Skolem's proof of Theorem 8 was an attempt to fill what he perceived as a gap in the literature: a semigroup version of Ore's Theorem. Given his apparent isolation, it is not unreasonable to suppose that, apart perhaps from Ore's, Skolem had not seen any of the work we have studied so far. That he had not seen Maltsev's is strongly suggested by a comment made after his proof of Theorem 8, in reference to his earlier error:

> In a previous paper... I set forth this theorem without the restriction that two elements always possess a common left (or right) multiple. I doubt, however, whether the theorem is generally true without the said restriction (Skolem 1952, p. 77).

A further remark tells us that Skolem's work was indeed carried out in ignorance of Dubreil's:

> After this note was printed I got acquainted with the fact that the above theorem has been proved before. Indeed a treatment of this matter can be found in P. Dubreil Algèbre... It is a pity that I did not know this earlier, because I would then instead of this note have written something about the more difficult case when common left or right multiples do not always exist (Skolem 1952, p. 77).

Skolem does not appear to have returned to this subject ever again (at least not in print)—perhaps he subsequently 'got acquainted' with Maltsev's work as well?

## 5 Maltsev's embeddability conditions

Anatolii Ivanovich Maltsev (1909–1967) was born near Moscow, and studied mathematics at Moscow State University in the early 1930s.[26] Upon graduation, he took a teaching post at Ivanovo State Pedagogical Institute (now Ivanovo State University), at which institution he remained until 1960, when he moved to the newly founded

[25] His papers were published mostly in Norwegian journals, and have thus not been widely available internationally (see Fenstad 1996); we may speculate that the opposite was also true and that Skolem did not have wide-ranging access to foreign sources.

[26] A non-exhaustive list of biographical articles on Maltsev is: Aleksandrov et al. (1968), Anon (1989), Dimitrić (1992), Gainov et al. (1989), Khalezov (1984), Kurosh (1959), Malcev (2010) and Nikolskii (1972). Maltsev also features in Sinai (2003, pp. 559–560).

Novosibirsk State University. Ivanovo's proximity to Moscow enabled Maltsev to travel regularly into the city to discuss his burgeoning research work, principally on mathematical logic, with A. N. Kolmogorov, who was eventually able to arrange a Moscow research studentship for Maltsev. Biographies of Maltsev stress the influence that Kolmogorov had upon him. In particular, Aleksandrov et al. (1968, English translation, p. 157) note that Maltsev's 1937 paper, which we discussed in Sect. 3, was 'the solution of a problem raised by Kolmogorov'.

The necessary and sufficient conditions (for the embedding of a cancellative semigroup in a group) which Maltsev merely hinted at in his 1937 paper appeared instead in a Russian paper of 1939, entitled 'On the immersion of associative systems in groups' ('О включении ассоциативных систем в группы').[27] As the title suggests, semigroups were now studied for their own sake, rather than as a means to an end, as had been the case in the earlier paper.

An interesting feature of Maltsev's 1939 paper is his choice of terminology. Rather than following Sushkevich (and Schmidt), as he might have done, by taking the Russian equivalent of the term 'semigroup' (= 'полугруппа'), he instead adopted the name that appears in the title of his paper: 'associative system' ('ассоциативная система'). Some years later, this term would be adopted, for a short period, by other Russian authors for the modern general notion of a semigroup (see Hollings 2014b, Chap. 9), but for Maltsev it denoted a *cancellative* semigroup. He justified his interest in such objects by pointing to their appearance as the purely multiplicative parts of certain rings, and also made the following remarks concerning groups:

> Some problems of the theory of groups are connected with the properties of associative systems. However, for a solution of these problems, a more thorough study is needed of the conditions under which a given associative system may be considered as part of some group.[28]

However, Maltsev gave no indication of the 'problems of the theory of groups' that he had in mind.

With the context of earlier work in this area established, I will now give a brief sketch of the necessary and sufficient conditions set forth by Maltsev for the embeddability of a cancellative semigroup in a group. However, since Maltsev's presentation is somewhat brief in places, I do not propose to follow his original exposition slavishly. Instead, I will follow that of a Canadian mathematician, George C. Bush, which appeared in a PhD thesis of 1961 and then in a paper of 1963. Bush's presentation is particularly clear,

[27] Although, as noted in the introduction, I use the terms 'embed/embedding/embeddability' elsewhere, I nevertheless choose to translate 'включение' as 'immersion' here, rather than 'inclusion' or 'insertion', in order to mirror Maltsev's own English terminology in Malcev (1937). Another often-used Russian word for 'embedding'/'imbedding' is 'вмещение' ('containment'), but this word only seems to have come into use in this context with later papers (see Schein 1961, for example). Other terms are 'погружение' ('immersion'), as used in Lyapin (1960), and 'вложение' ('enclosure'), as used in the Russian translation of Clifford and Preston (1961).

[28] 'Некотоые проблемы теории групп также связаны со свойствами ассоциативных систем. Однако, для решение этих проблем необходимо более тщательное изучение условий, при которых данная ассоциативная система может быть рассматриваема как часть некоторой группы' (Maltsev 1939, p. 331).

and has the virtue of using much of Maltsev's original notation and terminology.[29] Aside from his amplification of certain parts of Maltsev's proofs, Bush also made some original contributions in this area, which we will come to in due course (Sect. 8).

Let $\mathfrak{A}$ be a cancellative semigroup with identity element $e$. For each $x \in \mathfrak{A}$, we introduce new elements $x^+$ and $x^-$, not in $\mathfrak{A}$. In Maltsev's original paper, $x^-$ was an *ideal element of the first kind* (*идеальный элемент первого рода*), whilst $x^+$ was an *ideal element of the second kind* (*идеальный элемент второго рода*); to Bush, $x^+$ was the *formal right element* associated with $x$, whilst $x^-$ was the *formal left element*.

Let $S$ denote the collection of all finite strings made up of elements from $\mathfrak{A}$ and of ideal elements of both kinds. We permit certain *elementary transformations* (*элементарные преобразования*) on strings in $S$, for any $x \in \mathfrak{A}$:

($\alpha$) we may insert either $x^-x$ or $xx^+$ between any two adjacent entries in a string;
($\beta$) we may delete $x^-x$ and $xx^+$ from a string;
($\gamma$) for $a, b \in \mathfrak{A}$ with $ab = c$, if $a$ and $b$ appear as adjacent entries in a string, then we may replace them by $c$;
($\delta$) if $a \in \mathfrak{A}$, and $a = bc$, for some $b, c \in \mathfrak{A}$, then we may replace any appearance of $a$ in a string by $bc$.

As in Sect. 3.2, two strings are deemed *equivalent* if we can get from one to the other via a finite sequence of elementary transformations; Maltsev termed such a sequence of transformations a *chain* (*цепочка*). Once again, this 'equivalence' is in fact a congruence, and so we may multiply equivalence classes in a sensible way. It may then be shown that the collection $\mathfrak{G}$ of all equivalence classes forms a group with respect to this multiplication. This notion of equivalence may also be applied to elements of $\mathfrak{A}$: these are simply the strings in $S$ that contain no ideal elements of either kind. Moreover, by ($\gamma$) above, any such string may be reduced to a singleton string. In this connection, Maltsev (1939, p. 332) noted the following lemma for future use:

**Lemma 1** *The cancellative semigroup $\mathfrak{A}$ may be embedded in a group if and only if distinct elements of $\mathfrak{A}$ are not equivalent.*

Bush (1963a, Lemma 1) later turned this result around and stated it as: $\mathfrak{A}$ may be embedded in a group if and only if equivalent elements of $\mathfrak{A}$ are also equal. We can see that if equivalent elements are indeed equal, then $\mathfrak{G}$ contains a copy of $\mathfrak{A}$.

Maltsev recognised, however, that, when considering chains of elementary transformations, we may in fact restrict our attention to those of a particular type, which he termed a *normal chain of transformations* (*нормальная цепочка преобразований*): a chain of elementary transformations, during whose application no transformations occur to the left of any ideal element of the first kind, or to the right of any ideal element of the second kind. Introducing a new notion of equivalence (one for chains, rather than strings), Maltsev considered two chains to be *equivalent* if they have the same initial and final strings. Combining this with the notion of a normal chain, we have the following:

[29] More specifically, the notation used in Bush (1961) is more or less identical to that of Maltsev; the notation used in Bush (1963a) is somewhat simpler. Other accounts of Maltsev's conditions may be found in Clifford and Preston (1967, Sect. 12.6) and Cohn (1965, Sect. VII.3).

**Theorem 9** *Every chain of transformations whose initial and final strings are elements of $\mathfrak{A}$ (that is, strings which contain no ideal elements of either kind) is equivalent to some normal chain* (Maltsev 1939, p. 332).

It is clear that if a chain of elementary transformations is applied to an element of $\mathfrak{A}$, with the result also being an element of $\mathfrak{A}$, then any ideal elements that are inserted during this process must at some later point be deleted. A chain of elementary transformations therefore determines a sequence of insertions and deletions of ideal elements, a sequence which, as we will see below, assumes a particularly simple form in the case of a *normal* chain.

At this point in his paper, Maltsev modified his notation. He introduced the symbols $A_i, a_i, B_i, b_i, R_i, r_i, L_i, l_i$ ($i = 1, 2, \ldots$), with the assumption that these may be used to represent all elements of the given cancellative semigroup $\mathfrak{A}$. Notice that, although Maltsev made no explicit statement to this effect, his new notation implies that $\mathfrak{A}$ is countable. In addition to the above symbols, Maltsev introduced the symbols $l_i^*$, $L_i^*$ ($i = 1, 2, \ldots$) to stand for ideal elements of the first and second kinds, respectively. Maltsev's intention was to use this new notation to represent any sequence of insertions or deletions of ideal elements; this sequence would then, in turn, define a (normal) chain of elementary transformations. With the restriction to *normal* chains, it is clear that any such sequence of insertions and deletions must contain an even number elements, since there must be the same number of insertions of ideal elements as deletions. A sequence of $n$ insertions and $n$ deletions in a normal chain may be represented by a $2n$-tuple via the following method:[30]

- an insertion of $L_i^* L_i$ is denoted by $L_i$;
- an insertion of $l_i l_i^*$ is denoted by $l_i$;
- a deletion of $L_i^* L_i$ is denoted by $L_i^*$;
- a deletion of $l_i l_i^*$ is denoted by $l_i^*$.

Thus, to reproduce an example given by Bush (1961, p. 22) the sequence

$$\text{insert } L_i^* L_i, \quad \text{insert } l_i l_i^*, \quad \text{delete } L_i^* L_i, \quad \text{delete } l_i l_i^*,$$

is represented by the quadruple $(L_i, l_i, L_i^*, l_i^*)$. To any $2n$-tuple constructed in this way, Maltsev gave the name *l-chain* (*l-цепочка*). For later use, we observe that $l$-chains are subject to the following restrictions:

(1) no $L_i$ or $l_i$ can appear more than once;
(2) $L_i^*$ appears if and only if $L_i$ appears; $l_i^*$ appears if and only if $l_i$ appears;
(3) $L_i^*$ cannot appear before $L_i$; $l_i^*$ cannot appear before $l_i$;
(4) if $L_j^*$ appears, and $L_k$ is the nearest $L_i$ to the left of $L_j^*$ for which $L_i^*$ has not yet appeared, then $j = k$; similarly for $l_j^*$.

The above comments outline a procedure for the construction of $l$-chains from normal chains of elementary transformations. The reverse construction is also possible. Recall that $S$ denotes the collection of all strings which feature elements of the

[30] This part of the discussion is drawn from Bush (1961). Bush termed these $2n$-tuples *l-sequences*; Clifford and Preston (1967, Sect. 12.6) referred to them as *Malcev sequences*.

cancellative semigroup $\mathfrak{A}$ together with ideal elements of both kinds. The *$\mathfrak{A}$-operable portion* of a string in $S$ is that segment of the string which lies between (but does not include) the rightmost ideal element of the first kind and the leftmost ideal element of the second kind.[31] If the string in question contains no ideal elements of the first kind, then the $\mathfrak{A}$-operable portion extends to the end of the string; if it contains no ideal elements of the second kind, then the $\mathfrak{A}$-operable portion extends to the beginning of the string.

Suppose now that we are given an $l$-chain of insertions and deletions of ideal elements; this is of course subject to restrictions (1)–(4) above. The procedure for constructing a chain of elementary transformations is as follows, where we consider each of the entries in the $l$-chain in turn, from left to right:

- whenever $L_i$ appears in the $l$-chain, we replace the $\mathfrak{A}$-operable portion of the current string by $R_iA_i$ and insert $L_i^*L_i$ to obtain $R_iL_i^*L_iA_i$;
- whenever $l_i$ appears in the $l$-chain, we replace the $\mathfrak{A}$-operable portion of the current string by $a_ir_i$ and insert $l_il_i^*$ to obtain $a_il_il_i^*r_i$;
- whenever $L_i^*$ appears in the $l$-chain, we replace the $\mathfrak{A}$-operable portion of the current string by $L_iB_i$ to get $R_iL_i^*L_iB_i$, and then delete $L_i^*L_i$ to obtain $R_iB_i$;
- whenever $l_i^*$ appears in the $l$-chain, we replace the $\mathfrak{A}$-operable portion of the current string by $b_il_i$ to get $b_il_il_i^*r_i$, and then delete $l_il_i^*$ to obtain $b_ir_i$.

This mutual constructibility of $l$-chains and chains of elementary transformations is in fact one of the aspects of Maltsev's work that Bush felt the need to amplify: these ideas are implicit in Maltsev's paper, but were only made explicit by Bush. Nevertheless, Maltsev gave an example of the above procedure by applying it to the $l$-chain $(l_1, L_1, L_2, l_1^*, L_2^*, L_1^*)$; he took the empty string as the starting point for the corresponding (normal) chain of elementary transformations (Maltsev 1939, p. 335; see also Bush 1961, p. 24):

$$
\begin{array}{ccccc}
a_1r_1 & \to & a_1l_1l_1^*r_1 & \to & R_1A_1l_1^*r_1 \\
& \to & R_1L_1^*L_1A_1l_1^*r_1 & \to & R_1L_1^*R_2A_2l_1^*r_1 \\
& \to & R_1L_1^*R_2L_2^*L_2A_2l_1^*r_1 & \to & R_1L_1^*R_2L_2^*b_1l_1l_1^*r_1 \\
& \to & R_1L_1^*R_2L_2^*b_1r_1 & \to & R_1L_1^*R_2L_2^*L_2B_2 \\
& \to & R_1L_1^*R_2B_2 & \to & R_1L_1^*L_1B_1 \\
& \to & R_1B_1 & &
\end{array}
$$

Although we have yet to take any account of their nature as elements of $\mathfrak{A}$, we nevertheless recall at this point that this is precisely what the symbols $A_i, a_i, B_i, b_i, R_i, r_i, L_i, l_i$ represent. We recall also that any chain of elementary transformations, normal or otherwise, may contain applications of the rule for multiplication for $\mathfrak{A}$, or, to put this a different way, instances of transformations of the types $(\gamma)$ and $(\delta)$ in our earlier list. In fact, if we look at the above example, we see changes taking place that may be interpreted as just such transformations. For example, in moving from the second string $(a_1l_1l_1^*r_1)$ to the third $(R_1A_1l_1^*r_1)$, $a_1l_1$ has been transformed into $R_1A_1$. Similar transformations occur in the transition from the fourth string to the fifth, the

31 It should be noted that this terminology was introduced by Bush, and was not used by Maltsev.

sixth to the seventh, the eighth to the ninth, and the tenth to the eleventh. Thus, if the symbols $A_i, a_i, B_i, b_i, R_i, r_i, L_i, l_i$ are indeed to represent elements of $\mathfrak{A}$, then we need to assume the following relationships amongst them:

$$a_1 l_1 = R_1 A_1, \ L_1 A_1 = R_2 A_2, \ L_2 A_2 = b_1 l_1, \ b_1 r_1 = L_2 B_2, \ R_2 B_2 = L_1 B_1. \quad (15)$$

Another simple consequence of the above example is the fact that $a_1 r_1$ and $R_1 B_1$ are in fact *equivalent*, in the sense that they are connected by a chain of elementary transformations. It therefore follows from Lemma 1 that if $\mathfrak{A}$ can be embedded in a group, then

$$R_1 B_1 = a_1 r_1. \quad (16)$$

For Maltsev, any collection of equations of the form (15) (that is, a system of equations corresponding to a normal chain of elementary transformations) was a *normal system of equations* (*нормальная система равенств*); the related equation (16) was the *completing equation* (*замыканое равенство*).[32] If $\mathfrak{A}$ can be embedded in a group, then the completing equation follows from the normal system of equations. Later writers have often adopted the term *Maltsev* (or, indeed, *Malcev*) *condition* for the implication of a completing equation by a normal system of equations.

Maltsev's 1939 paper contained no general proof of the methods outlined above; as already indicated, it was necessary for Bush later to fill in many of the details. Another process that received only an outline in Maltsev's work was a shorter procedure for constructing a normal system of equations from a given $l$-chain, which avoided the need to write out chains of elementary transformations explicitly. However, I do not reproduce this method here—I instead refer the interested reader to Bush (1961, pp. 26–27).

To summarise those of Maltsev's ideas that I have outlined here: if a cancellative semigroup $\mathfrak{A}$ can be embedded in a group, and if some collection of elements of $\mathfrak{A}$ satisfies a normal system of equations, then those same elements must also satisfy the corresponding completing equation. Conversely, if some elements of $\mathfrak{A}$ satisfy a normal system of equations, then they also satisfy the completing equation. But then for any normal chain of transformations, the initial and final strings are equal. In other words, equivalent elements are equal. By Lemma 1 and Theorem 9, this is a sufficient condition for $\mathfrak{A}$ to be embedded in a group. We therefore have the following:[33]

**Theorem 10** (Maltsev's Embeddability Theorem) *For a cancellative semigroup $\mathfrak{A}$ to be embedded in a group, it is both necessary and sufficient that, whenever certain elements of $\mathfrak{A}$ satisfy some normal system of equations, the corresponding completing equation is also satisfied.*

Since there are infinitely many possible $l$-chains, each giving rise to a Maltsev condition, it follows that a cancellative semigroup must satisfy an infinite set of conditions

[32] Clifford and Preston (1967, Sect. 12.6) subsequently referred to this as the *locked equation*; to them, a normal system of equations was a *Malcev system*.

[33] Maltsev (1939, p. 335); see also Bush (1963a, Theorem 2) and Clifford and Preston (1967, Theorem 12.17).

in order to be embeddable in a group. This is indeed a far cry from the simplicity of Sushkevich's supposed solution to the problem.

Maltsev did little but sketch the above ideas in his 1939 paper. However, in his concluding paragraphs, he briefly suggested an interesting way in which Maltsev conditions may be used as a means of classifying semigroups. With reference to the infinite set of conditions needed to ensure embeddability, he made the following remark:

> If it is demanded that only part of these conditions are fulfilled, we obtain an associative system, more or less approximating a group.[34]

By way of illustration, he considered the conditions arising from the simplest possible $l$-chains. That derived from $(l, l^*)$, for example, is simply the cancellation law. Indeed, this is the simplest possible Maltsev condition. Moreover, the Maltsev condition that arises from the $l$-chain $(l, L, l^*, L^*)$ is none other than Maltsev's 1937 condition Z (see Sect. 3.2). The cosmetic similarity that the reader may have noticed between condition Z and the general form of a Maltsev condition is therefore no coincidence.

Maltsev touched only briefly upon this possible classification of semigroups via Maltsev conditions, for he recognised that an important component of any such theory was missing:

> For a rigorous execution of the classification outlined here, it is necessary to investigate the independence of the given conditions. Such independence is easily studied for the simplest chains, for example, those containing only one ideal element of the 1st kind. However, in its general form, the question remains open.[35]

Indeed, the question of the independence of Maltsev conditions was addressed by Maltsev in a paper published the following year: 'On the immersion of associative systems in groups II'. Also considered in this subsequent work was the problem of finding a *minimal extension* (*минимальное расширение*) of a given cancellative semigroup: if a cancellative semigroup $\mathfrak{A}$ can be embedded in a group $\mathfrak{G}$, then the embedding is said to be *minimal* if $\mathfrak{A}$ is not contained in any proper subgroup of $\mathfrak{G}$. However, I will not go into this material here, but will focus instead on the question of independence.

Let $\mathfrak{A}$ be a cancellative semigroup that is defined by a presentation[36] $\langle R | S \rangle$. For Maltsev, a *simple consequence* (*простое следствие*) of $S$ was any relationship between products of positive powers of generators from $R$ that holds in $\mathfrak{A}$. Thus, for a semigroup that can be embedded in a group, the completing equation is a simple consequence of the corresponding normal system of equations.

[34] 'Если потребовать, чтобы выполнялась только часть этих условий, то получится ассоциативная система, более или менее приближающаяся к группе' (Maltsev 1939, p. 336).

[35] 'Для строгого проведения намеченной здесь классификации необходимо исследовать независимость указанных условий. Такая независимость легко изучается для простейших цепочек, например, содержащих только один идеальный элемент 1-го рода. Однако, в общем виде вопрос остается открытым' (Maltsev 1939, p. 336).

[36] Note that the '$\langle \cdot | \cdot \rangle$' notation was not used by Maltsev.

As in the 1939 paper, $l$-chains, now renamed *schemes* (*схемы*), were Maltsev's main tools for approaching Maltsev conditions. We shall say that a scheme is *fulfilled* (*выполненный*) if the corresponding Maltsev condition holds. Moreover, the *normal semigroup* (*нормальная полугруппа*) of a scheme is the semigroup that is generated by the symbols appearing in the normal system of equations corresponding to that scheme, and which has those equations as its defining relations. A scheme is *irreducible* (*неприводимый*) if none of its proper segments is a scheme. Finally, an *irreducible normal system* (*неприводимая нормальная система*) is the normal system of equations associated with an irreducible scheme. Using these various notions, Maltsev proved the following (Maltsev 1940, Theorem 3(1); see also Bush 1961, Lemma 6.6):

**Lemma 2** *The completing equation of an irreducible normal system is not a simple consequence of the system: it does not hold in the corresponding normal semigroup.*

But if the normal semigroup in question is to be embedded in a group, then the completing equation needs to be a simple consequence of the corresponding normal system. Hence, (see Bush 1961, Lemma 6.7):

**Lemma 3** *The normal semigroup of an irreducible scheme cannot be embedded in a group.*

Recall that schemes ($l$-chains) were introduced earlier as $2n$-tuples, since they contain an equal number of insertions and deletions of ideal elements. Maltsev defined the *length* (*длина*) of such a scheme to be $n$, and employed this notion in the following result, whose proof calls upon Lemma 2 (Maltsev 1940, Lemma 4; see also Bush 1961, Theorem 6.1):

**Theorem 11** *No irreducible scheme of length $n$ can be a consequence of schemes whose lengths are less than $n/2$.*

The essential purpose towards which Maltsev was building with these various ideas was the consideration of *finite* sets of Maltsev conditions. Suppose that we have such a set of conditions. Each condition may be represented by the fulfilment of some scheme. We suppose that the longest scheme amongst these has length $n$. Let $\mathfrak{A}$ be the normal semigroup of some scheme of length $2n + 2$. By Lemma 3, $\mathfrak{A}$ may not be embedded in a group. However, all schemes of lengths less than or equal to $n$ must be fulfilled in $\mathfrak{A}$ (this may be proved in a similar manner to Theorem 11). We see then that although $\mathfrak{A}$ satisfies a finite set of Maltsev conditions, it cannot be embedded in a group. In this way, Maltsev arrived at the following:[37]

**Theorem 12** *No finite set of Maltsev conditions is sufficient for the embeddability of a cancellative semigroup in a group.*

We have seen then, in what is perhaps the most exhausting section of this paper, that Maltsev's conditions for embeddability are far from simple: not because the individual Maltsev conditions are complicated (they are certainly not), but because there are

[37] Maltsev (1940, Theorem 4); see also Bush (1961, Theorem 6.2) and Clifford and Preston (1967, Sect. 12.8).

infinitely many of them. Moreover, Theorem 12 tells us that this is a feature of Maltsev's embedding results that cannot be bypassed. Indeed, this has had an effect on the 'usability' of Maltsev's work; it led, for example, to the search for more manageable (sufficient) conditions, such as those that we saw in Sect. 4. This is a point to which I will return in Sect. 9. However, it is possible to find necessary and sufficient conditions in a more 'closed' form, as we will see in the next section.

## 6 Pták's embeddability condition

In this section, we will consider a slightly lesser-known contribution to the problem of embedding semigroups in groups: that of Vlastimil Pták. This work is not widely cited by other authors—it was published in Czechoslovakia and may therefore not have been easily available to other researchers internationally (despite Pták's obvious efforts to spread awareness of his work—see below). The work of Lambek, for example, which we will see in the next section, appeared after Pták's but contains no reference to it. However, Clifford and Preston were well aware of Pták's work and included a summary of it in their chapter on embeddings (Clifford and Preston 1967, Sect. 12.3). Pták's work is a little easier to grasp than that of Maltsev, and relies largely upon familiar group-theoretic notions, particularly normal subgroups. At least superficially, Pták's solution is 'algebraic', whereas Maltsev's is 'logical'.

Vlastimil Pták was born in Prague in 1925.[38] He studied mathematics and physics at Charles University from 1945 to 1948, at the same time working as an assistant at the Czech Technical University. Upon completion of his undergraduate studies, Pták became a research student of the topologist Miroslav Katětov. In 1952, he joined the Mathematical Institute of the Czechoslovak Academy of Sciences (or the Central Mathematical Institute, as it was then called); he obtained his candidate degree[39] in 1955 (which appeared in print as Pták 1953b, c), followed by his doctoral degree in 1963. Pták rose through the academic ranks and became a full professor at Charles University in 1965. From 1960 until his retirement, he was head of the Department of Functional Analysis at the Mathematical Institute. Pták died in 1999.

The vast majority of Pták's work was in functional analysis; in 1966 he was awarded the Czechoslovak Federal Prize for his work in this area. Indeed, this is the work upon which his various biographers focus—his study of embeddings is not mentioned at all in any of the biographies I have seen. As with many of the other mathematicians mentioned in this article, Pták only considered the problem of embedding semigroups in groups at the very beginning of his career, and then only briefly. Unlike some of the others, however, Pták's work in this direction does not appear to have originated in a dissertation. In his writing, he gave no indication of what brought him to this area, although he began his account of the problem by mentioning van der Waerden's

[38] There is a particularly large number of biographical articles on Pták, including: Anon (2000), Fiedler (2000), Fiedler and Müller (2000a, b), Vavřín (1995, 1996a, b) and Vrbová (1985).

[39] The candidate of sciences degree is an Eastern European qualification, equivalent to a Western PhD; it is often followed, as in Pták's case, by a higher research degree, the doctor of sciences, which is sometimes compared with a DSc, or a German Habilitation.

*Moderne Algebra*, so this may have been his starting point, as it seems to have been for Maltsev. In a 1953 summary of Pták's work on embeddings, we find:

> This problem is very important in algebra simply because any ring without divisors of zero, after removing the element 0, becomes a semigroup with respect to multiplication.[40]

In the same paragraph, Pták hinted at possible applications in analysis: he mentioned, somewhat vaguely, and without specific reference, an embedding theorem due to Wintner concerning a semigroup connected with linear differential equations with analytic coefficients. He noted that:

> The importance of similar results for analysis is obvious.[41]

However, I have been unable to identify the relevant work of Wintner.

Pták's contributions to the problem of embedding semigroups in groups are contained in three papers (Pták 1949, 1952, 1953a). However, the content of the three is more or less the same. The original material appeared in English in the first of the three (1949), whilst the second and third papers are Russian and Czech versions, respectively, of the first.[42] Pták evidently wanted his work to reach a wide audience. The noteworthy thing about Pták's work is that, although it was published in 1949, a full decade after Maltsev's embeddability conditions, Pták was still not aware of the full extent of Maltsev's work—he seemingly knew only about the counterexample of 1937.

Pták began his 1949 paper, 'Immersibility of semigroups', with what would appear to be a summary of his knowledge of the subject, going back to the problem's origins with van der Waerden:

> In elementary textbooks on algebra a well-known theorem is proved, stating that every commutative ring without divisors of zero may be imbedded in a field. In the first edition of his Moderne Algebra VAN DER WAERDEN mentioned the following problem: Is the same possible even for non-commutative rings without divisors of zero?...
>
> This problem is evidently closely connected with the question concerning the existence of a group containing a given non-commutative semigroup. This second problem was solved by M. MALCEV [*sic*]... (Pták 1949, p. 3).

[40] 'Tento problém je dåuležitý v algebře samé již proto, že každý okruh bez dělitelåu nuly se stane po odstranění prvku 0 semigrupou vzhledem k násobení' (Pták 1953a, p. 259). In fact, the removal of 0 is not necessary.

[41] 'Význam podobných výsledkåu pro analysu je nasnadě' (Pták 1953a, p. 260).

[42] All three papers carry (essentially) the same title, but in different languages: 'Immersibility of semigroups' in English, 'О включении семигрупп' in Russian, and 'Vnořitelnost semigrup' in Czech (for some comments on the terminology used in these titles, see note 44 below). Excluding blank pages, the English version is 14 pages long. The Russian version is 22 pages in length—the extra pages are accounted for by the fact that this version was expanded very slightly (in particular, some comments were added which draw connections with Maltsev's work), and the fact that the typeface is slightly larger with respect to the page than in the English version. The Russian version also features a further 3 pages of English summary. The Czech version is a slightly modified Czech translation of the English summary from the Russian version. We note also the existence of a 1 page French summary of Pták's work, based upon the Russian version: Thibault (1953b, p. 10).

The work by Maltsev that is cited here is of course that of his 1937 paper, which we saw in Sect. 3.2. The incorrect initial given to Maltsev in the above quotation may be an indication that Pták was merely *aware* of Maltsev's work, but had not actually *seen* it.[43] However, there is an indication that something very specific in Maltsev's work provided inspiration for Pták's. He noted that

> MALCEV found a necessary condition that a semigroup be immersible and constructed a semigroup not satisfying that condition (Pták 1949, p. 3).

Pták was of course referring to condition Z here. He did not quote this condition in his original work of 1949, but it does appear in the 1952 Russian version, labelled as 'condition (M)'. This condition was apparently a source of inspiration for Pták; in the English summary appended to the 1952 paper, he commented:

> The author's condition is a generalization of the condition (M). By generalizing the condition of Malcev it is easy to see, that the following condition is necessary for a semigroup to be immersible.
>
> (M′) If [a cancellative semigroup] $S$ contains several elements $a_1, \ldots, a_n$ which fulfill some relations, then any relation between the elements $a_1, \ldots, a_n$ which may be deduced from the given relations *in a group*, must hold in $S$, too.
>
> If this condition is violated, the semigroup $S$ evidently cannot be imbedded in a group. The purpose of the [1949] paper... was to show that this is the only way how that can happen, that is to show that the condition mentioned is sufficient as well (Pták 1952, p. 269).

This rather vague statement of intent was made precise in due course.

Returning our attention to the 1949 paper, we see that Pták went on:

> On the other hand, there is a sufficient condition found by O. Ore ..., which, as we shall see, is not necessary. As far as we know, sufficient and necessary conditions have not been published (Pták 1949, p. 3).

The second sentence here is clear evidence of the fact that Pták had not seen Maltsev's further work. This is something which Pták addressed directly in the English summary at the end of the 1952 Russian version of his work:

> [My 1949 paper was] written in June 1947, that is at a time when our knowledge of Russian literature published during the war was still a very incomplete one.... the author did not know of the existence of the papers... of Malcev, which contain a solution of the problem... Nevertheless both the method used and the result obtained by Malcev are different from those of the author, so that—after some hesitation—the author decided to publish [the current] extract from the paper... containing his own solution (Pták 1952, p. 268).

[43] My first thought upon seeing the 'M.' here was that Pták might have meant 'Monsieur Maltsev'. However, leaving aside the question of why Pták would have addressed Maltsev in this manner, a glance through the paper suggests that this is not in fact the case: none of the other authors is referred to as 'Monsieur'—Ore appears as 'O. Ore' and Dubreil as 'P. Dubreil'; van der Waerden is not afforded any initials at all.

Thus, it seems that by 1952, Pták had finally come across Maltsev's results and had decided to try to bring his own to a wider Russian-reading audience.

The introductory passage of the 1949 paper concludes with the following:

> In the present paper the necessary and sufficient conditions that a semigroup be immersible are given; for that purpose a method is developed which can also be applied in other investigations (Pták 1949, p. 3).

Pták began, as we have seen other authors do, by taking the word 'semigroup' to refer to a cancellative semigroup.[44] He assumed further that his semigroups always contained an identity because he considered there to be 'no essential difference' (Pták 1949, p. 5) between the case where the semigroup has an identity and the case where it does not, since if it has none, we may simply adjoin one (in the manner described by Howie 1995, p. 2).

The notion of the free group played an important role in Pták's considerations, and so he spent some time developing this and other ideas; these are presumably what 'can also be applied in other investigations'. Pták denoted by $\mathfrak{G}$ the free group on a set of generators $\Gamma$. The free semigroup on $\Gamma$ is denoted by $\mathfrak{S}$. Bearing in mind the comments above on Pták's terminology, $\mathfrak{S}$ is in fact, in modern terminology, the free cancellative monoid on $\Gamma$. Pták observed that if $S$ is a semigroup with system of generators $\Gamma$, and $\mathfrak{S}$ is the free semigroup on $\Gamma$, then there is a canonical homomorphism $H : \mathfrak{S} \to S$, namely, the mapping which takes any string in $\mathfrak{S}$ to the corresponding product in $S$.

We note that Pták made use of a universal algebraic notion which is very familiar in modern semigroup theory: the congruence which is termed the *kernel* of a given homomorphism (see Howie 1995, p. 20). Given any homomorphism $\theta : S \to T$ between semigroups $S$ and $T$, the kernel, $\rho$, is the congruence on $S$ defined by the rule that

$$s \,\rho\, t \Longleftrightarrow \theta(s) = \theta(t).$$

Upon factoring the semigroup $S$ by the congruence $\rho$ (see Sect. 3.2), we have that $S/\rho \cong \operatorname{im}\theta$. This is semigroup theory's version of the First Isomorphism Theorem, and it was noted by Pták, although not under this (or any other) name. Pták referred to the kernel of a homomorphism $H$ as the *induced equivalence* and typically denoted it by $E$. Perhaps following French authors, such as Dubreil (1946, Chap. IV), Pták referred to congruences as *regular equivalences* (in French: *équivalences régulières*).

After setting up the basic notions just described, Pták began to consider conditions for embeddability. In particular, he arrived at the following (Pták 1949, Theorems 3.2 and 3.3):

**Theorem 13** *Let $S$, $\mathfrak{S}$, $\mathfrak{G}$ and $E$ be as above. Then $S$ may be embedded in a group if and only if $E$ is induced on $\mathfrak{S}$ by a regular equivalence on $\mathfrak{G}$.*

[44] Curiously, the Russian version of Pták's work employs the term 'семигруппа' ('semigruppa') for semigroup, rather than the usual Russian term 'полугруппа' ('polugruppa'). Perhaps he adopted this term to mirror Dickson's notion of 'semigroup', that is, to refer specifically to a *cancellative* object—much as Sushkevich appears to have done with his German term 'Semigruppe' (see note 13). I have found no other author who uses the term 'семигруппа'. Noteworthy terminology also appears in the Czech version of Pták's work, where he used the term 'semigrupa'; the modern Czech word for 'semigroup' is 'pologrupa'.

In other words, a necessary and sufficient condition for the embeddability of $S$ in a group is that the $E$-class of each $\alpha \in \mathfrak{S}$ be the intersection of $\mathfrak{S}$ with the $E'$-class of $\alpha$ in $\mathfrak{G}$, where $E'$ is some regular equivalence on $\mathfrak{G}$. We note that any regular equivalence (congruence) on a group has a special form, in terms of which is it usually more convenient to work: the congruence class of the identity is a normal subgroup, and all other congruences classes are cosets of that subgroup. Thus, if we factor out a group by a congruence, we obtain the factor group in the usual sense, that is, the group factored by the corresponding normal subgroup.

Pták's proof of Theorem 13 is fairly straightforward. For the necessity of the above condition, he took $\mathfrak{N}$ to be the normal subgroup of $\mathfrak{G}$ which corresponds to the canonical homomorphism $H : \mathfrak{G} \to G$ (that is, its 'kernel' in the usual group-theoretic sense), and showed that the $E$-class of $\alpha \in \mathfrak{S}$ is equal to $\alpha\mathfrak{N} \cap \mathfrak{S}$. For sufficiency, he took $\mathfrak{N}$ to be the normal subgroup of $\mathfrak{G}$ which corresponds to the regular equivalence on $\mathfrak{G}$, that is, the equivalence class of the identity. It was then a simple matter to show that $\mathfrak{G}/\mathfrak{N}$ contains a subsemigroup which is isomorphic to $S$. Pták gave the following as an immediate consequence of Theorem 13, essentially a restatement of the comments just made (Pták 1949, Theorem 3.4):

**Corollary 1** *Let $S$, $\Gamma$, $\mathfrak{G}$ and $E$ be as above. Then a necessary and sufficient condition for $S$ to be embeddable in a group is that $\mathfrak{G}$ contain a normal subgroup $\mathfrak{N}$ with the following property: for any two finite sequences $(a_1, \dots, a_r)$ and $(b_1, \dots, b_s)$ of elements of $\Gamma$, we have for the corresponding products:*

$$a_1 \cdots a_r \; E \; b_1 \cdots b_s \Longleftrightarrow a_1 \cdots a_r \equiv b_1 \cdots b_s \pmod{\mathfrak{N}}. \tag{17}$$

Suppose now that $a_1 \cdots a_r = \alpha$ and $b_1 \cdots b_s = \beta$ in $\mathfrak{S}$. Then, using basic facts about cosets, (17) may be rewritten in the form:

$$\alpha \; E \; \beta \Longleftrightarrow \alpha\beta^{-1} \in \mathfrak{N}.$$

This observation appears to have inspired Pták to define the following subsets of $\mathfrak{G}$:

$$\mathfrak{M}_r = \{\alpha\beta^{-1} \in \mathfrak{G} : \alpha, \beta \in \mathfrak{S} \text{ and } \alpha \; E \; \beta\};$$
$$\mathfrak{M}_l = \{\alpha^{-1}\beta \in \mathfrak{G} : \alpha, \beta \in \mathfrak{S} \text{ and } \alpha \; E \; \beta\}.$$

We put $\mathfrak{M}_0 = \mathfrak{M}_r \cup \mathfrak{M}_l$. Let $\{\mathfrak{M}_x\}$ denote the intersection of all subgroups of $\mathfrak{G}$ which contain $\mathfrak{M}_x$, and let $[\mathfrak{M}_x]$ denote the intersection of all *normal* subgroups of $\mathfrak{G}$ which contain $\mathfrak{M}_x$, where $x = 0, r, l$. By manipulating the $\mathfrak{M}_x$ in certain ways, Pták arrived ultimately at an embeddability condition in terms of these:[45]

**Theorem 14** *Let $S$, $\mathfrak{S}$, $\mathfrak{G}$ and $E$ be as above. Then $S$ may be embedded in a group if and only if, for any $\alpha, \beta \in \mathfrak{S}$,*

$$\alpha\beta^{-1} \in [\mathfrak{M}_r] \Longrightarrow \alpha \; E \; \beta. \tag{18}$$

[45] Pták (1949, Theorem 3.6); see also Clifford and Preston (1967, Corollary 12.8). An equivalent version of this result, stated in terms of congruences rather than normal subgroups, may be found in Clifford and Preston (1967, Theorem 12.7).

In 1952, Pták described this theorem as being 'an exact formulation of the (rather vaguely formulated) condition (M′)' (Pták 1952, p. 270). We note that, as with Sushkevich's work in Sect. 3, Pták's work contains the implicit assumption that all semigroups under consideration are infinite: all of Pták's semigroups were cancellative, and so the infinite assumption ensures that, in general, they are not already groups. Thus, as in Maltsev's case, we once again have an infinite number of implications as our embeddability conditions, this time of the form (18).

A simple consequence of Theorem 14 provided Pták with a sufficient condition for embeddability (Pták 1949, Theorem 4,3; see also Clifford and Preston 1967, Theorem 12.10):

**Corollary 2** *Let $S$, $\mathfrak{S}$, $\mathfrak{G}$ and $E$ be as above. If $\{\mathfrak{M}_r\}$ is a normal subgroup of $\mathfrak{G}$, then $S$ is embeddable.*

Pták went on to use Corollary 2 to obtain a new proof of Ore's Theorem for semigroups. Given a semigroup $S$, he wrote down the following condition: for every ordered pair $(a, b)$ of elements of $S$, there exist elements $a_l, b_l \in S$ such that $a_l a = b_l b$. This is of course (the left-hand version of) Ore's condition $M_{\text{V}}$ from Sect. 2.2. To any (cancellative) semigroup satisfying this condition, Pták gave the name *Ore semigroup*. He then used Corollary 2 to prove that any Ore semigroup may be embedded in a group, thereby giving a new proof of Theorem 5. We note that although Pták included Dubreil's *Algèbre* (1946) in his bibliography, he cited it only as a reference in connection with the properties of congruences on semigroups, and not in relation to embedding problems. He attributed the proof of the semigroup version of Ore's Theorem to Ore himself.

## 7 Lambek's polyhedral condition

In this section, I describe the last major approach to the embedding problem to be considered here: that of Joachim Lambek. The conditions laid down by Lambek (in a 1950 Ph.D. thesis *The immersibility of a semigroup into a group*, as well as a 1951 paper of the same name[46]) are wonderfully simple to present. Lambek's solution is as comprehensive as Maltsev's, and yet it is much easier to grasp, thanks to Lambek's use of the pictorial aid of *polyhedra*.

Joachim Lambek was born in Leipzig in 1922, but escaped Germany for Britain in the late 1930s.[47] During the Second World War, however, he was transported to Canada as an enemy alien, and it was here that he ultimately settled. Lambek entered McGill University in Montreal; he obtained a first degree in 1945 and a master's a year later. Staying at McGill, Lambek embarked upon mathematical research, under the supervision of H. J. Zassenhaus. He completed his PhD in 1950, producing two dissertations: *Biquaternion vectorfields over Minkowski space* and *The immersibility of a semigroup into a group*.[48] It is of course the second dissertation which will

[46] We note also the existence of what is effectively a French translation of this paper: Thibault (1953b).

[47] These biographical details on Lambek are drawn from Barr (1999) and Barr et al. (2000).

[48] These two are bound together into a single volume in the McGill University Library, with the dissertations labelled as parts A and B, respectively.

concern us here. Lambek's mathematical work went on to encompass a wide range of areas, most notably (like Maltsev) mathematical logic and category theory; much of his work has concerned mathematical linguistics (see, for example, Lambek 2006). In algebra, Lambek has worked on the theories of rings and modules, but the material to be considered below represents his only significant foray into semigroup theory.

Lambek recalls his introduction to the embedding problem, and the fact that he did not yet know of Maltsev's 1939/1940 work:

> I became interested in the [problem of embedding semigroups in groups] as a graduate student, when my supervisor Hans Zassenhaus mentioned it in the first lecture of a graduate course and I realized that here was a problem I could work on without any further prerequisites. Having solved it, I was disappointed to learn that Maltsev had done so ..., but consoled by the fact that the two solutions were different.[49]

We note that in the acknowledgments at the beginning of his thesis on semigroup embeddings, Lambek thanked not only Zassenhaus 'for his encouragement and criticism', but also H. S. M. Coxeter, for 'for his kind and valuable suggestions' (Lambek 1950, p. 0).

Lambek's embedding results were written up into a thesis at a time when he was still ignorant of Maltsev's 1939/1940 work. In his introduction, he noted Maltsev's comment on having found necessary and sufficient conditions but their being too complicated for publication in the 1937 paper. Nevertheless, Lambek seems to have had some idea of what form the conditions should take, perhaps based upon Maltsev's condition Z:

> It is in fact not difficult to find such conditions. They state that certain systems of equations are not independent. This means that if all but one of the equations are given, the remaining equation can be deduced. However, as soon as we wish to give verbal utterance to these conditions, it becomes desirable to label the equations and the variables contained in them. This is where things begin to get involved (Lambek 1950, p. 1).

Lambek noted that his method involving polyhedra was a way around this complication.

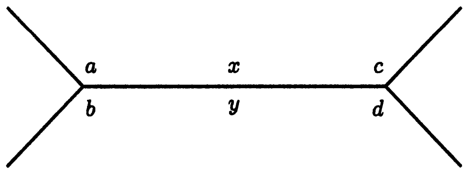
By the time Lambek came to write up his results into a paper, he had learnt of Maltsev's work and therefore presented his own as being an easier alternative to Maltsev's; in reference to Maltsev's normal systems of equations, he noted:

> Malcev offers a rather complicated construction for obtaining such chains of equations. In the present paper I have tried to give a simpler construction, by the device of using parts of polyhedra, rather than natural numbers, for labelling the equations and variables contained in these conditions (Lambek 1951, p. 34).

Indeed, Lambek's work provided an entirely fresh approach to the subject, independent of Maltsev's:

[49] Joachim Lambek, private communication, 14th June 2011.

**Fig. 1** An edge in a Lambek polyhedron




> Acquaintance with Malcev's work will not be expected from the reader (Lambek 1951, p. 34).

I base the following account of Lambek's polyhedral conditions on that of Clifford and Preston (1967, Sect. 12.5), which roughly follows Lambek's original. However, I do not present Lambek's methods with full rigour, as the following comment from Clifford and Preston (1967, p. 303) also applies in our case:

> It will suffice for our purposes to treat polyhedra informally and to illustrate the argument by drawing pictures.

George C. Bush (whose account of Maltsev's methods we followed in Sect. 5) presented an alternative version of Lambek's method, in purely algebraic terms.[50] As already observed, Lambek presented his results in both a thesis and a paper. Since the content of these is virtually identical, I will refer to the more easily accessed paper.

Lambek's conditions for the embeddability[51] of a semigroup in a group are stated in terms of Eulerian polyhedra, that is, polyhedra for which $V + F = E + 2$ (where $V$ is the number of vertices, $F$ is the number of faces, and $E$ is the number of edges) and which are homeomorphic to the 2-sphere in Euclidean 2-space.

Each edge in a polyhedron is an edge to two faces, and we say that each edge has two *sides*: one on each face. As we might expect, each edge has two vertices. At the point where an edge enters a vertex, the edge is considered to have two *angles*: one on each side of the edge. For example, Fig. 1 shows an edge with two sides $x$ and $y$; $a$ and $b$ are the angles at the left-hand vertex of the edge, whilst $c$ and $d$ are the angles at the right-hand vertex.

Each edge is assumed further to consist of two *half-edges*: one for each vertex. It is clear that each half-edge has two sides and two angles, which it inherits from the edge of which it is a part. The left-hand half-edge of the edge illustrated in Fig. 1 clearly has sides $x$ and $y$, and angles $a$ and $b$; the *half-edge equation* of this half-edge is $xa = yb$.

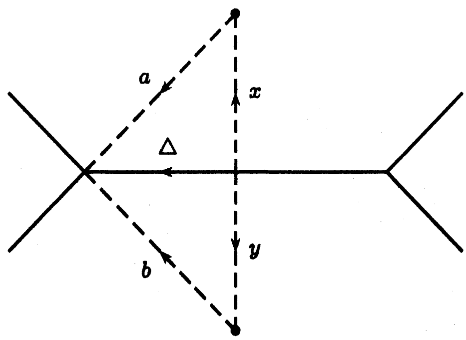
[50] Bush (1961, 1963a). Indeed, Bush's approach differed considerably in presentation from Lambek's. As Bush noted:

> We use polyhedra as a convenient model and mnemonic device, but not as part of the proof (Bush 1961, p. 30).

Thus, in contrast to the situation with Maltsev's conditions, it is inappropriate for us to follow Bush's exposition if we are to explore Lambek's original conditions.

[51] Unlike Maltsev, who referred to the 'immersibility' of a semigroup, and spoke of 'immersing' it in a group, Lambek used a mixture of terminology: for him, a semigroup that was 'immersible' could be 'embedded' in a group. I have standardised the terminology as 'embed'/'embedding'/'embeddability' throughout this section.

**Fig. 2** Triangulation of an edge in a Lambek polyhedron




Let $S$ be a semigroup. We say that $S$ satisfies the *polyhedral condition* if, for any polyhedron whose sides and angles are labelled by elements of $S$, any half-edge equation is a consequence of all the other half-edge equations. We note that there are $2E$ half-edge equations in total. Modifying slightly the terminology adopted by Bush, the implication of one half-edge equation by the $2E - 1$ others will be referred to as a *Lambek condition*;[52] notice that a Lambek condition is of a very similar form to a Maltsev condition.

Let $P$ be any polyhedron whose sides and angles are labelled by elements of $S$. We *triangulate P* by taking a point (the *centre*) in the interior of each face and drawing a line from this point to each vertex of that face; we also draw a line from an interior point (the *mid-point*) of each edge of that face to its centre. The triangulation is oriented by directing the lines from centre to vertex, mid-point to centre, and mid-point to vertex.

The labelling of the polyhedron determines a labelling of its triangulation. Consider the edge in Fig. 1—in particular, the left-hand half-edge (with angles $a$ and $b$). The labels of this half-edge are transferred to the newly drawn lines in the triangulation of the half-edge, as illustrated in Fig. 2. The half-edge itself (labelled $\Delta$ in Fig. 2) may be labelled in two different ways: one for each triangle adjoining it. Using the upper triangle, the label would be $xa$; for the lower, it would be $yb$. Labelling in this way, it is easy to see that the half-edge has a unique label if and only if the corresponding half-edge equation holds.

Returning to the general situation of the polyhedron $P$, we suppose that all half-edge equations, except perhaps one, hold in the semigroup $S$. Then, in the triangulation of $P$, all half-edges except one can be labelled uniquely. Suppose that this one half-edge is that shown in Fig. 2, and that $yb = p$. We attach the label $p$, arbitrarily, to the half-edge $\Delta$.

We assume now that the semigroup $S$ may be embedded (by inclusion) in some group $G$; let 1 denote the identity in $G$. Each triangle in the triangulation of $P$ corresponds to an equation of the form $yb = p$. Since $S$ is assumed to be contained in a group, this equation may also be written in any of six different forms for which the

[52] Bush (1963a, p. 52). Bush called this a *Lambek polyhedral condition* but we drop the 'polyhedral' not only to avoid confusion with *the* polyhedral condition, but also for reasons of symmetry with the 'Maltsev conditions' of Sect. 5, with which these Lambek conditions will shortly be connected. Lambek himself gave no special name to these individual conditions.

right-hand side is 1, for example, $ybp^{-1} = 1$, $bp^{-1}y = 1$, etc. Each of these six forms represents a different manner of traversing the edges of the corresponding triangle.

More generally, we consider any connected path $C$ consisting of edges $E_1, E_2, \ldots, E_n$ (in that order) in the triangulation. Suppose that the edge $E_i$ is labelled by $x_i \in S$. Then the path $C$ determines a product $f(C) = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}$, where $\varepsilon_i = +1$ if the edge $E_i$ is traversed in the given direction, and $\varepsilon_i = -1$ otherwise.[53] We saw above that a simple closed path bounding a single triangle determines an equation whose right-hand side is 1. In fact, although we omit the details, it follows by induction on the number of triangles bounded that $f(C) = 1$ for any simple closed path $C$ which bounds any simply connected set of triangles.

Now suppose that $C$ is the (simple, closed) path which bounds all of the triangles in the triangulation of $P$, with the exception of that triangle for which the corresponding half-edge equation is not assumed to hold. But $C$ is also a path along the edges of the excluded triangle. Therefore, $f(C) = 1$ implies that $xa = p = yb$. We have shown that if $S$ can be embedded in a group, then an arbitrary half-edge equation follows from all the rest, that is, the polyhedral condition holds. Indeed, we have effectively demonstrated the necessity part of the following theorem:[54]

**Theorem 15** (Lambek's embeddability theorem) *A semigroup may be embedded in a group if and only if the cancellation law and the polyhedral condition are both satisfied.*

The sufficiency proof is considerably more complicated so I do not present it here.[55] We see, however, that even though Lambek's Embeddability Theorem is easier to state than Maltsev's, the proof is far from simple. Lambek concluded his paper by testing the polyhedral condition: he took a commutative, cancellative semigroup (which we already know may be embedded in a group) and, using methods similar to those employed in the necessity proof above, demonstrated that the polyhedral condition does indeed hold in such a semigroup.

We see then that Lambek's Embeddability Theorem, like Maltsev's, involves an infinite number of conditions. At the end of his thesis on embeddings, before he had become aware of Maltsev's conditions, Lambek made the comment:

> … it remains an open question whether a finite number of conditions may not do. I conjecture that this question is to be answered in the negative (Lambek 1950, p. 18).

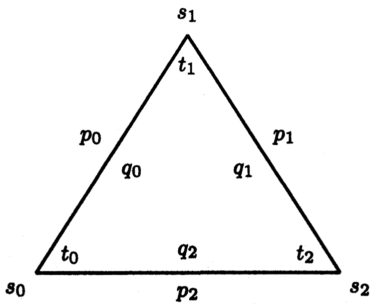
Indeed, Lambek is correct: as for Maltsev conditions, no finite set of Lambek conditions will suffice.

53 The '$f(C)$' used here is Lambek's notation; Clifford and Preston used '$C\alpha$'.

54 Lambek (1951, p. 35); see also Clifford and Preston (1967, Theorem 12.16) and Bush (1963a, Theorem 3).

55 For the full details of necessity, and for the proof of sufficiency, see Clifford and Preston (1967, Sect. 12.5).

**Fig. 3** A polyhedron which yields a Lambek condition which is not a Maltsev condition; vertices are labelled $s_i$, angles $t_i$, and sides $p_i$ and $q_i$, where $i = 0, 1, 2$




## 8 Maltsev's and Lambek's conditions compared

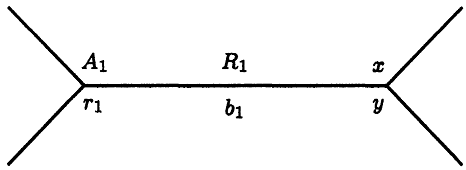
Having now seen both Maltsev's and Lambek's conditions for the embeddability of a semigroup in a group, both of which comprise infinite sets of conditions, it is natural to wonder whether there is any connection between the two. Throughout Sect. 5 and, to a lesser extent, Sect. 7, I cited the work of George C. Bush on the embedding of semigroups in groups. So far, apart from commenting that Bush derived an algebraic version of the polyhedral condition, I have only mentioned Bush's surveys of the existing work of Maltsev and Lambek. However, it is when we turn to the comparison of the conditions of Maltsev and Lambek that we see some original work due to Bush.

Bush's work in this direction originated in his PhD thesis, *On embedding a semigroup in a group*, which he completed at Queen's University, Ontario, in 1961, under the supervision of I. Halperin. Results from this thesis appeared in two papers: one in 1963, entitled 'The embedding theorems of Malcev and Lambek', and one rather later, in 1971, entitled 'The embeddability of a semigroup—conditions common to Mal'cev and Lambek'. Bush does not appear to have gone on to pursue a career in research mathematics, and the papers cited here represent almost the full extent of his mathematical work.[56]

Bush began his comparison of the conditions of Maltsev and Lambek by first showing that the two sets of conditions are formally distinct, that is, that there exist Lambek conditions which are not Maltsev conditions, and vice versa. He achieved this by presenting two examples. The first is based upon the polyhedron in Fig. 3. This polyhedron gives rise to the Lambek condition:

[56] *Mathematical Reviews* contains reviews of five papers by Bush: the two already cited, a further paper concerning an embedding theorem given by Adyan (Bush 1963b), an article containing proposals for the standardisation of certain terminology relating to functions and binary relations (Bush 1969), and a much later paper on a topic related to computer science, published in a Turkish journal (Bush 1989). A casual online search reveals also a precursor to the cited 1969 article (Bush 1967), an elementary mathematics textbook (Bush and Obreanu 1965), and a problem contributed to *The American Mathematical Monthly* (Bush 1962; Heuer 1963). Furthermore, the search also yields the information that Bush was elected to ordinary membership of the American Mathematical Society in April 1965 (Green and Pitcher 1965, p. 590).

**Fig. 4** An edge of a polyhedron arising from the Maltsev condition (20)




$$\{p_0s_1 = q_0t_1,\ p_0s_0 = q_0t_0,\ p_1s_1 = q_1t_1,\ p_1s_2 = q_1t_2,\ p_2s_0 = q_2t_0\} \\ \Longrightarrow p_2s_2 = q_2t_2. \tag{19}$$

By considering, case by case, the possible forms that the corresponding scheme and completing equation would have to take, and obtaining a contradiction in each instance, Bush showed that no Maltsev condition can have the form (19). Therefore, (Bush 1961, Lemma 5.1):

**Lemma 4** *There exist Lambek conditions which are not Maltsev conditions.*

For the second of his two examples, Bush considered the scheme

$$(L_1, l_1, l_2, L_1^*, l_2^*, l_1^*),$$

with associated Maltsev condition:

$$\{L_1A_1 = a_1r_1,\ a_1L_1 = a_2r_2,\ a_2l_2 = L_1B_1,\ R_1B_1 = b_2l_2,\ b_2r_2 = b_1l_1\} \\ \Longrightarrow b_1r_1 = R_1A_1. \tag{20}$$

He observed that if this were identical to a Lambek condition, then the completing equation would force us to have, for some edge, the situation illustrated in Fig. 4, for some values $x, y$. But then we would have to have an equation of the form $R_1x = b_1y$ in the normal system of (20), which is not the case. Hence, (Bush 1961, Lemma 5.2):

**Lemma 5** *There exist Maltsev conditions which are not Lambek conditions.*

Bush thus showed in particular that neither of the sets of conditions of Maltsev or Lambek is a subset of the other. The problem which he turned to next was the question of whether the two sets have any conditions in common. He gave, for example, a complete characterisation (in terms of their schemes) of those Maltsev conditions which are also Lambek conditions (Bush 1961, Theorem 5.1). He also proved the following general theorem (Bush 1961, Theorem 5.3; see also Bush 1963a, Theorem 6):

**Theorem 16** *Let $K$ be a system of equations and $M$ be a single equation. If the condition $K \Rightarrow M$ is satisfied in every semigroup that is embeddable in a group, then $K \Rightarrow M$ is a consequence of some Maltsev condition.*

Since every Lambek condition is of the form $K \Rightarrow M$, we have (Bush 1961, Corollary 5.1; see also Bush 1963a, Corollary 1):

**Corollary 3** *Each Lambek condition is a consequence of a Maltsev condition.*

In fact, there is nothing special about the appearance of the Maltsev condition in Theorem 16, nor about the relative positions of the Maltsev and Lambek conditions in Corollary 3, as Bush went on to show (Bush 1961, Theorem 5.4 and Corollary 5.2; see also Bush 1963a, Theorem 7 and Corollary 2):[57]

**Theorem 17** *Let $K$ be a system of equations and $M$ be a single equation. If the condition $K \Rightarrow M$ is satisfied in every semigroup that is embeddable in a group, then $K \Rightarrow M$ is a consequence of some Lambek condition.*

**Corollary 4** *Each Maltsev condition is a consequence of a Lambek condition.*

We see then that Bush managed to establish some nice interrelations between the sets of Maltsev and Lambek conditions. However, we have not yet made any mention of the conditions *common* to Maltsev and Lambek. In order to describe these, we need another type of condition that we have not seen so far: the so-called *lunar conditions*.

Howard L. Jackson was another student, this time a master's student, of I. Halperin at Queen's University. He completed his MA in 1956, with a dissertation entitled simply *The embedding of a semigroup in a group*. We have already seen that Bush did not begin his studies at Queen's University until 1958, so it is difficult to say whether he and Jackson ever crossed paths.[58] Nevertheless, even if Jackson and Bush never met, they were certainly connected by their supervisor, and Bush's work was very much a continuation of that of Jackson. However, whereas Bush's work was quite wide-ranging, taking in the embedding conditions of Maltsev and Lambek (and, indeed, Jackson, as we will see shortly), the focus of Jackson's work was much narrower; he concentrated on a certain special type of Lambek condition: the lunar conditions, mentioned above. At the beginning of the thesis, Jackson credited Halperin with the discovery of these conditions:

> Prof. Halperin gave me these conditions verbally. As yet he has published no paper on them (Jackson 1956, p. 3, footnote 1).

Indeed, Halperin does not appear to have published anything relating to groups or semigroups. As we will appreciate once we have seen their definition, to have delivered the lunar conditions verbally, at least in the form presented by Jackson, was quite a feat. Bush (1961, p. 55) indicated further that the *name* 'lunar conditions' is also due to Halperin.

Jackson's dissertation begins with the very careful definition of both semigroups and groups, together with the proof that any finite cancellative semigroup is in fact

[57] In fact, Bush proved our Theorem 17 and Corollary 4 in a slightly more specific instance, with 'Lambek condition' replaced by 'Lambek associative condition', where a 'Lambek associative condition' is a special type of Lambek condition, identified by Bush, which gives a pictorial representation of the associative law. This arose from Bush's observation that, whilst Lambek had proved the necessity of *all* Lambek conditions for the embeddability of a semigroup in a group, his proof of sufficiency used only those Lambek conditions of a certain form, namely, the Lambek associative conditions. As a result, Bush worked almost exclusively in terms of Lambek associative conditions; in particular, he proved the necessity part of Lambek's Embeddability Theorem for Lambek associative conditions only. Since Bush's definition of a Lambek associative condition is rather involved, I have avoided it here. The interested reader is referred to Bush's original exposition: see Bush (1961, Sect. 3.3) or Bush (1963a, Sect. 2.2).

[58] Unlike Bush, Jackson pursued mathematical research beyond his MA dissertation: *Mathematical Reviews* lists 16 publications, ranging from 1965 to 1985, most of them on potential theory.

a group.[59] Jackson's first chapter also contains the counterexample of Maltsev that we saw in Sect. 3.2. Then, without any further preamble, Jackson launched into a discussion of the lunar conditions in his second chapter.

For any $4n+4$ symbols $x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_n, a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n$, let $S$ denote the set containing the following equations:

$$\begin{gathered} x_0a_0 = y_0a_1,\ x_0b_0 = y_0b_1,\ x_1a_1 = y_1a_2,\ x_1b_1 = y_1b_2,\ \ldots, \\ x_{n-1}a_{n-1} = y_{n-1}a_n,\ x_{n-1}b_{n-1} = y_{n-1}b_n,\ x_na_0 = y_na_n. \end{gathered} \tag{21}$$

The *nth lunar condition* is the following: whenever the equations in $S$ hold, the equation $x_nb_0 = y_nb_n$ must also hold. The $n$th lunar condition is denoted by $C_n$. Jackson also gave an alternative definition of $C_n$ in terms of the notion of an *equaliser*: if $xa = yb$, then the pair $(x, y)$ is called an *equaliser* of the pair $(a, b)$. The $n$th lunar condition may now be restated thus: if $(x_i, y_i)$ is a common equaliser of $(a_i, a_{i+1})$ and $(b_i, b_{i+1})$, for all $i \in \{0, 1, 2, \ldots, n-1\}$, and $(x_n, y_n)$ is an equaliser of $(a_0, a_n)$, then $(x_n, y_n)$ is also an equaliser of $(b_0, b_n)$.

We observe that Jackson's $C_1$ is in fact Maltsev's condition Z, and so we know that $C_1$ is a necessary condition for the embeddability of a semigroup in a group. In fact, more generally, $C_n$ is a necessary condition, as Jackson showed in very short order (Jackson 1956, Chap. II, Sect. 2):

*Proof* Suppose that a semigroup $S$ may be embedded in a group $G$. Then every element of $S$ has an inverse in $G$. From the equations (21), we have:

$$\begin{gathered} y_0^{-1}x_0 = a_1a_0^{-1} = b_1b_0^{-1},\ y_1^{-1}x_1 = a_2a_1^{-1} = b_2b_1^{-1},\ \ldots, \\ y_{n-1}^{-1}x_{n-1} = a_na_{n-1}^{-1} = b_nb_{n-1}^{-1},\ y_n^{-1}x_n = a_na_0^{-1}. \end{gathered}$$

In particular, we take the equality $a_na_{n-1}^{-1} = b_nb_{n-1}^{-1}$ and relabel and rearrange to obtain:

$$a_{n-1} = b_{n-1}b_{n-2}^{-1}a_{n-2}. \tag{22}$$

In order to derive the final equation, we show that $y_n^{-1}x_n = b_nb_0^{-1}$. We have

$$\begin{aligned} a_na_0^{-1} &= b_nb_{n-1}^{-1}a_{n-1}a_1^{-1}b_1b_0^{-1} = b_nb_{n-1}^{-1}\left(b_{n-1}b_{n-2}^{-1}a_{n-2}\right)a_1^{-1}b_1b_0^{-1} \qquad \text{(by (22))} \\ &= b_nb_{n-2}^{-1}a_{n-2}a_1^{-1}b_1b_0^{-1} = \cdots = b_nb_{n-i}^{-1}a_{n-i}a_1^{-1}b_1b_0^{-1}, \end{aligned}$$

for $1 \le i \le n-1$. Now, for $i = n-1$:

$$a_na_0^{-1} = b_nb_1^{-1}\left(a_1a_1^{-1}\right)b_1b_0^{-1} = b_n\left(b_1^{-1}b_1\right)b_0^{-1} = b_nb_0^{-1}.$$

Hence, $y_n^{-1}x_n = a_na_0^{-1} = b_nb_0^{-1}$, or $x_nb_0 = y_nb_n$. □

[59] See note 14.

What Jackson did not comment upon, however, is the fact that since the lunar conditions are special cases of Lambek conditions, their necessity follows automatically from Lambek's results. Nevertheless, Jackson's proof is a little easier than Lambek's.

Jackson followed his proof with an example, quite similar to Maltsev's, of a semigroup which satisfies $C_1$ but not $C_2$, thereby showing that $C_2$ is effectively stronger than $C_1$. In fact, this example represented Jackson limbering up to tackle the more general problem of whether $C_{n+1}$ is effectively stronger than $C_n$: he went on to attempt to prove that this is indeed the case by presenting a suitable example; however, Jackson's proof is incomplete (see below). Nevertheless, he concluded that if a semigroup satisfies $C_n$, then it necessarily satisfies $C_1, C_2, \ldots, C_{n-1}$. The third and final chapter of Jackson's dissertation is taken up with a straightforward presentation of Lambek's polyhedral condition and a proof (essentially a reproduction of Lambek's own) of the corresponding embedding theorem.

These discussions do little to bring out the reasons for the name 'lunar conditions', nor has much been said to justify the assertion that these are a special type of Lambek condition. Jackson did attempt to explain the name by noting in his introduction that

> ... [t]hese are called *lunar* conditions because they result when a sphere is divided into simple lunes ... (Jackson 1956, p. 3)

where a *lune* (at least in the two-dimensional pictures used here) is a region bounded by two arcs. Later on, after having defined Lambek's polyhedra, Jackson commented further:

> ... if the polyhedron chosen consists of exactly $n + 1$ lunes then the polyhedral condition in this case is the $n$th lunar condition (Jackson 1956, p. 21).

What is missing here is a diagram to *illustrate* the lunar conditions and their interpretation as Lambek conditions. After all, the ability to visualise the conditions is the major selling-point of Lambek's work. Fortunately, in his thesis, Bush noted that

> ... [t]he condition $C_n$ is the Lambek polyhedral condition ... for a polyhedron with 2 vertices and $n + 1$ edges (Bush 1961, p. 56)

and provided the picture given in Fig. 5 to illustrate the condition $C_2$.

In his chapter on the lunar conditions, Bush pointed out the error (noted above) in Jackson's proof that $C_{n+1}$ is essentially stronger than $C_n$: 'the set of cases he considers is not exhaustive' (Bush 1961, p. 58). However, the result is true nevertheless, and Bush patched up Jackson's proof.[60]

Notice that we have had a great deal to say about the necessity of the lunar conditions for the embeddability of a semigroup in a group, but we have so far made no comment on their sufficiency. This is because they are insufficient, as Bush showed (Bush 1961,

[60] Bush (1961, Sect. 4.3). It is also worth noting that, in spite of our earlier observation that their necessity follows from that of the Lambek conditions, Bush presented Jackson's proof of the necessity of the lunar conditions (in his Sect. 4.2). This is because of the comment made in note 57 above, concerning the 'Lambek associative conditions': Bush did not prove the necessity of arbitrary Lambek conditions, only of the so-called Lambek associative conditions. The lunar conditions are Lambek conditions, but they are not Lambek *associative* conditions, so their necessity does not follow from the earlier necessity proof in Bush's presentation. He therefore needed to prove this separately.

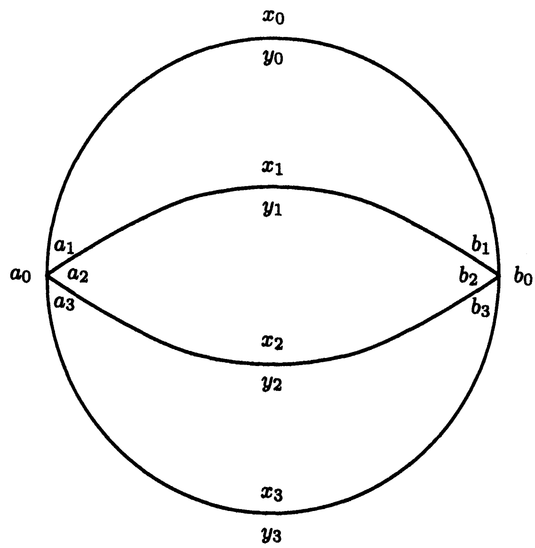



**Fig. 5** The polyhedron corresponding to the lunar condition $C_2$

Sect. 4.5). He even tried to rectify this by introducing the *dual* lunar conditions (Bush 1961, Sect. 4.4), that is, the conditions given by reversing all the products in (21), as well as in the final equation $x_n b_0 = y_n b_n$. But even taken together, the lunar conditions and dual lunar conditions are still not sufficient.[61]

The lunar conditions also appear later in Bush's thesis, in connection with his comparison of Maltsev and Lambek conditions. Thus, we come finally to the promised characterisation of the conditions common to Maltsev and Lambek. As noted earlier, Bush obtained a complete characterisation, in terms of their schemes, of those Maltsev conditions which are also Lambek conditions. He went on to prove that the Maltsev conditions associated with these particular schemes are in fact lunar conditions (Bush 1961, Lemma 5.7), that is, those Maltsev conditions which are also Lambek conditions are necessarily lunar conditions. Indeed, the result is even stronger (Bush 1961, Theorem 5.2):

**Theorem 18** *The only conditions which are both Maltsev conditions and Lambek conditions are the lunar conditions.*

In accordance with our comments above on the insufficiency of the lunar conditions, we have the following theorem noted by Bush (1961, Theorem 5.3):

**Theorem 19** *The set of conditions common to Maltsev and Lambek is insufficient for the embeddability of a semigroup in a group.*

We have recorded here several results from Jackson's master's thesis and Bush's Ph.D. thesis. However, as is often the case with theses, this material did not reach a very wide audience. Jackson does not appear to have published any parts of his thesis, whilst Bush published only two short papers, 8 years apart, only one of which would have been available to Clifford and Preston when they were preparing the second volume of their textbook *The algebraic theory of semigroups*, which was published in

[61] The *necessity* of the dual lunar conditions may be shown in much the same way as for the lunar conditions.

1967. Clifford and Preston did not list Bush's thesis in their bibliography and therefore probably had not seen it. The result of this was the unwitting duplication by Clifford and Preston of some of Bush's work—in particular, Theorem 18 above.

In their chapter on the embedding of semigroups in groups, Clifford and Preston, like Bush, included a section (Sect. 12.7) that is devoted to the comparison of Maltsev's and Lambek's conditions. As we have already noted, Clifford and Preston would (probably) only have had access to those elements of Bush's work which had been published in his 1963 paper; in connection with the comparison of conditions, this would have amounted to the results that we have recorded here as Theorems 16 and 17, and Corollaries 3 and 4. Thus, Clifford and Preston did not know the full extent of Bush's work when they proved the following (Clifford and Preston 1967, Theorem 12.21):[62]

**Theorem 20** *The Maltsev conditions which are also Lambek conditions are precisely those Lambek conditions which arise from polyhedra with two vertices.*

In essence, this is Theorem 18: Lambek conditions arising from two-vertex polyhedra are of course lunar conditions; diagrams very much like our Fig. 5 appeared in Clifford and Preston's proof of the above theorem.

Despite his having hitherto made no apparent effort to publish further material from his thesis, the appearance of Theorem 20 in the second volume of *The algebraic theory of semigroups* seems to have spurred Bush into action. The result was the appearance in June 1971 of the second paper mentioned earlier: 'The embeddability of a semigroup—conditions common to Mal'cev and Lambek'.

The purpose of this paper was, essentially, to present Theorem 19 to the world, which, Bush hoped, would

> ... take on new interest in the light of Clifford and Preston's work (Bush 1971, p. 437).

With admirable modesty, Bush made no attempt to claim priority for Theorem 20, but simply presented a slightly modified version of his earlier discussion of the lunar conditions; the paper concludes with Theorem 19.

We see then that there is indeed some overlap between the work of Maltsev and that of Lambek on the embedding of semigroups in groups. However, as Bush showed, their approaches are essentially distinct. The geometrical approach of Lambek is considerably easier to present that the equational approach of Maltsev, but it is not simpler in any real sense: the complications which are so patent in Maltsev's approach are merely hidden behind the diagrams in Lambek's. Nevertheless, Lambek's methods are intuitively rather pleasing, and so they mark a good place to end our main discussion of the embedding problem.[63]

62 Clifford and Preston stated this theorem in rather more complicated (and precise) terms, but this simplified statement will suffice for our purposes, not least because their notation for Maltsev conditions differs from that used here.

63 We note that, besides the work on lunar conditions, there have been other attempts to unify the approaches of Maltsev and Lambek. For example, the work of Krstić (1985), which gave a new geometrical interpretation of Maltsev's conditions, and used this to draw connections with Lambek's. Johnstone (2008) united the approaches of Maltsev and Lambek in a single categorical method, and then extended this to the embedding of categories in groupoids.

## 9 Concluding remarks

Like so many problems in mathematics, that of seeking to embed a semigroup in a group has a simple statement, but proved rather harder to solve. Indeed, as we have seen, the apparent simplicity of the problem led some of its attackers astray. The comprehensive solution, when it did come, turned out to be not entirely usable: Maltsev's necessary and sufficient conditions may be easy to state, but they are a little more complicated to wield, and so other authors continued to seek simpler conditions that were merely sufficient (Sect. 4). Indeed, the semigroup theorist Peter Trotter has said of Maltsev's conditions that

> ... they are difficult to apply ... It is therefore of interest to find simple conditions satisfied by large classes of semigroups that are sufficient to ensure embeddability in groups (Trotter 1972, p. 1).

As part of this 'practical' approach, Trotter studied a more simply defined class of semigroups, which he labelled '$L$-semigroups'. These $L$-semigroup can be embedded in groups and are defined via a generalisation of conditions earlier studied by Doss (Sect. 4.2).

I want now to make some comments on the 'independence' of Maltsev's work. Recall from the introduction that I have argued elsewhere (specifically, in Hollings 2009a) that the first truly 'independent' theorem on semigroups was one proved by Clifford in a paper of 1941, concerning certain semigroups that arise as disjoint unions of groups. By 'independent', I meant that this theorem has no direct analogue in, nor was driven by considerations from, either group or ring theory. In fact, Maltsev's Embeddability Theorem might displace Clifford's work from this exalted position, depending upon one's point of view. As we have seen, Maltsev's study of embeddings for semigroups grew out of that of embeddings for rings, but his derivation of his necessary and sufficient conditions did not parallel anything within the ring theory of the time; necessary and sufficient conditions for the embeddability of a ring in a field were not derived until about 30 years later (see Cohn 1971, Sect. 7.6, Corollary 1). Indeed, in this instance, it is the results on semigroups that drove the search for corresponding results on rings.

One final observation that might be made about the embedding problem is that it spans the final critical decades in the formation of what we now term 'abstract algebra'. We have traced its earliest origins to Steinitz' abstraction of the construction of the rationals from the integers (his 'Quotientenbildung') and seen it appear in van der Waerden's highly influential *Moderne Algebra*. The realisation that we might throw away the additive part of the rings under consideration and thus treat an embedding problem purely for semigroups is almost the quintessence of abstract algebra—the adaptation of the problem was motivated not by any 'concrete' factors, but merely by abstract concerns.[64] In the various solutions to the problem, we have seen traditional group-theoretic tools brought to bear (such as the normal subgroups employed by Pták), and also notions that now, at their most general, come under the banner of

[64] Indeed, the same might also be said of Ore's adaptation of the 'Quotientenbildung' to the non-commutative case.

universal algebra (such as congruences). Finally, in Maltsev's comprehensive solution of the embedding problem, we see him on the verge of one of the most abstract (and, indeed, logical) of branches of modern mathematics: model theory, a field in which Maltsev made important contributions (see Dimitrić 1992). Thus, this seemingly simple problem may even have had some small influence on the twentieth-century drive towards abstract algebra.

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