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Felice Casorati's work on finite differences and its influence on Salvatore Pincherle

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Abstract This paper, which is mainly based on unpublished material, focuses on the scientific influence that Felice Casorati exerted on Salvatore Pincherle. This influence can be traced, in particular, in Casorati's work on the finite-difference calculus as conceived and published between 1879 and 1880 when Pincherle was living in Pavia. Casorati's work has an interesting *back story* related to his entry to the 1880 *Grand Prix* of the French *Académie des Sciences* that helps us in understanding Casorati's personality. Moreover, the correspondence that Casorati exchanged with other mathematicians on his work reveals that some of the results contained in Casorati (Annali di Matematica pura ed applicata 10(S. II):10–45, 1880b) had been obtained—though in a narrower context—in an early paper by Christoffel. Finally, the letters between Casorati and Pincherle contain a short unpublished note by Pincherle on a paper by Jules Tannery (Ann Sci. l'École Norm Super 4(S. II):113–182, 1875). This note offers the first evidence of the influence of Casorati (Annali di Matematica pura ed applicata 10(S. II):10–45, 1880b) on Pincherle's work on the finite-difference calculus.

1 Introduction

Felice Casorati (1835–1890) and Salvatore Pincherle (1853–1936) are representatives of two successive generations of Italian mathematicians: Casorati, jointly with Francesco Brioschi, Enrico Betti, Eugenio Beltrami, and Ulisse Dini, played an important rôle in the renaissance of Italian mathematics that intertwined with the political

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events leading to unification of Italy. An important step in this process was Italy's entry into the main stream of European mathematical research as cultivated in Paris, Göttingen, and Berlin. In this sense, the journey through Europe that Casorati, jointly with Betti and Brioschi, performed in 1858 cannot be underestimated, since first-hand information on the main trends in mathematics were gained and then diffused in Italy (Volterra 1902). Casorati's relevant efforts culminated with the publication of the volume *Teorica delle funzioni di variabili complesse* (Casorati 1868) where he expounded the main results in complex analysis, with particular attention to Riemann's approach (see Bottazzini 1994, Chaps. IV and VI).

Pincherle, jointly with Luigi Bianchi, Ernesto Cesàro, Giuseppe Peano, Vito Volterra, and Tullio Levi-Civita, belonged to the subsequent generation of mathematicians who contributed to the strengthening of Italy's position within the international scientific community. His main contributions were in the field of functional (linear) calculus where he worked for more than thirty years along a route distinct from the one that Vito Volterra followed in the same period.

Pincherle graduated from Pisa as a student of the *Scuola Normale Superiore* and spent his academic life entirely in Bologna. His first scientific interest was in the theory of capillarity which was the subject of his thesis (*Tesi di Laurea*) published in the *Nuovo Cimento* (Pincherle 1874) with a short addition (Pincherle 1875). In 1875, when he moved to Pavia to teach at the *Liceo Ginnasio* "Ugo Foscolo", he was recommended to Felice Casorati by Ulisse Dini:[1]

Dear Casorati

November 5th, 75

Busy as I am, I cannot write at length to you. Just a couple of words to introduce you D^{r}. Salvatore Pincherle, one of the best students of both our University and our Scuola, at present Professor at the Liceo in Pavia. You will find in him a studious young man, willing and very ingenious. Please, help him and advise him in his studies.

Yours sincerely

U. Dini

1

Caro Casorati

5. 9bre 75

Occupato come sono, non posso in questo momento scriverti a lungo. Ti scrivo soltanto due parole per presentarti il D^{r}. Salvatore Pincherle, uno dei migliori allievi della nostra Università e della nostra Scuola, attualmente Professore al Liceo di Pavia. Troverai in Lui un giovane studioso e volenteroso di fare e con molto ingegno. Ajutalo e consiglialo nei suoi studi.

Credimi

Tuo aff. amico,
U. Dini

This short letter opened the way to Casorati's and Pincherle's personal relationship that lasted for 15 years, up to the untimely death of Casorati in 1890. Their correspondence amounts to some 80 letters that have been kept in Casorati's archive.[2] These letters provide some further information about Casorati's influence on Pincherle. At the very beginning of his stay in Pavia, Pincherle had devoted his interests to the theory of minimal surfaces that formed the subject of three notes (Pincherle 1876a, b, c) published in 1876, but the interaction with Casorati turned his attention to differential equations. Shortly thereafter Pincherle obtained a fellowship to spend a year in Berlin, where he met Weierstrass, whose methods were to exert a decisive influence on his scientific activity (Bottazzini 1994, Chap. VI). Direct contacts with Casorati were resumed in 1878 when Pincherle came back to Pavia from Berlin. At the beginning of 1879, Pincherle applied for the habilitation (*libera docenza*) in higher calculus at the University of Pavia. A local committee examined his application and, the judgement being positive, on May 9 the Faculty approved his habilitation, which was ratified by the Ministry of Education at the end of 1879.[3] Consequently, in the fall semester 1879–1880, Pincherle gave a course in the higher calculus (*Analisi superiore*) that Beltrami and Casorati among others attended,[4] **where he offered an exposition of Weierstrass's views on the theory of analytic functions. Pincherle's lectures and, more importantly, his long account of them published in Battaglini's *Giornale di Matematiche* (Pincherle 1880) contributed significantly to the spread of Weierstrass's ideas in Italy.[5] In the period 1878–1880 Pincherle also attended Casorati's lectures on higher calculus, as he reminded in the following letter addressed to Casorati in 1888:

Bologna, li 25/3/88

Dear Professor

During the researches on functional operations represented by definite integrals and on their inversion, on which I have been working for a while, it occurred to me to study linear differential equations, whose solution is a problem that arises as a particular case in the inversion of definite integrals; on this occasion, I rejoiced at having attended the lectures you gave on this subject, sparing to your audience the reading, and the not-too-easy digestion, *of the Memoirs by Fuchs and Thomé* (...).[6]

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[2] Hereafter denoted as his *Nachlass*, after (Neuenschwander 1978), it is preserved by Casorati's descendants in Pavia. It has been made available to me by Prof. Alberto Gabba, to whom I address my warmest thanks.

[3] The original documents concerning Pincherle's habilitation are kept in the Historical Archive of the University of Pavia.

[4] See Bortolotti (1937).

[5] See (Bottazzini, 1994, Chap. VI).

[6] Chiarissimo Sig. Professore

Nel corso delle ricerche che ho intraprese già da parecchio tempo sulle operazioni funzionali rappresentate da integrali definiti e sulla loro inversione, mi è capitato di dover studiare le equazioni differenziali lineari, la cui risoluzione è un problema compreso come caso particolare nell'inversione degli integrali definiti; ed in questa occasione ho avuto più di una volta da rallegrarmi di avere assistito alle lezioni che Ella ha fatto in proposito, risparmiando ai Suoi uditori la lettura e la *digestione* non troppo agevole delle Memorie di Fuchs e Thomé.

Actually, since the mid-1870s, Casorati had turned his attention to Fuchs's theory of linear differential equations in the complex plane that was consonant with his early concern with complex analysis,[7] and in the academic years 1878–1879 and 1879–1880 Casorati based his course in higher calculus on this topic, too. As a comment to the opening lecture, held on 16 November 1879, Casorati wrote in his log-book:

I declare that I will carry out the theory of linear differential equations according to Fuchs, etc. as the first topic. They [the students] will find the original memoirs in Borchardt's Journal.[8] (Casorati 1879).

Casorati's influence on Pincherle had three aspects. Firstly, there was what could be called a *political* influence: Casorati knew personally most of the leading mathematicians in Europe and, together with Betti, introduced Pincherle to Kronecker and Weierstrass when he got a fellowship to spend the academic year 1877–1878 in Berlin. Moreover, Casorati was a member of both the Committee that declared Pincherle eligible for the chair of Infinitesimal Analysis in 1880 at Bologna *ex aequo* with Cesare Arzelà, who was preferred to Pincherle on that occasion, and the Committee that promoted Pincherle to a full professorship in 1888. In addition, Casorati supported Pincherle's election as a corresponding member (*Socio Corrispondente*) of the Accademia dei Lincei in 1887. Secondly, there was an *editorial* influence. From 1868, Casorati was an effective member of the Royal *Istituto Lombardo di Lettere, Scienze ed Arti* based in Milan, and as such he presented some early papers by Pincherle to this local Academy for publication. More importantly, Casorati was a member of the editorial board of the *Annali di Matematica pura ed applicata* that had Francesco Brioschi as the editor in chief. In this capacity, Casorati presented Pincherle's first important memoir (Pincherle 1884a) on series expansions for publication in the *Annali*. Thirdly, there was a *scientific* influence that stemmed from Casorati's 1880 paper on the interpretation of finite difference (Casorati 1880b), to which I will devote more attention in the present paper.

An early acknowledgement of the close connection between Casorati's and Pincherle's researches on that topic was given in the foreword of (Guldberg and Wallenberg 1911) where the authors stated (p. vi).

In the meanwhile, another area of our discipline underwent a powerful development, precisely the formal side of the theory of linear difference equations, due in particular to the analogy with algebraic equations, and to the extension of the analogies with linear differential equations. Here, after Casorati, Pincherle with his students Bortolotti[9] *and Amaldi—above all—has built the foundations and, in particular, in*

[7] For a historical account of the rôle of complex function theory on the research in differential equation, we refer the reader to Chap. 7 of Bottazzini and Gray (2013).

[8] Dichiaro che svolgerò per primo argomento la teoria delle equazioni differenziali lineari secondo Fuchs, ecc. Troveranno le memorie originali nel Giornale di Borchardt.

[9] Ettore Bortolotti, better known as a historian of mathematics, in his early days devoted several papers to the formal theory of finite differences, (see for instance Bortolotti 1895).

Chap. 10 of his book on distributive operations,[10] *he has built the frame, so to say, for the edification of such a theory.*[11]

In his [1880b], Casorati had proved the necessary and sufficient condition for n functions of a variable that undergoes discrete increments to be linearly independent by introducing a suitably defined determinant, now known as the Casorati determinant, or the Casoratian. However, the aspect that Casorati emphasized there was the wide spectrum of applications of the classical finite-difference calculus by showing that, after suitable interpretations, classical results could be translated into complex function theory: in particular, Casorati noticed that the local behaviour of the solution of a linear differential equation around a singular point (Fuchs 1866) could be easily obtained by adapting finite-difference calculus appropriately.

Casorati attached great importance to the results contained in his paper. Also urged by Brioschi, with great expectation, he decided to apply for the 1880 Grand Prix de Mathématiques announced by the French Academy of Sciences, and devoted to the theory of differential equations. However, his paper found in Halphen's memoir (Halphen 1884) an invincible obstacle to winning the prize, and Casorati had to content himself with a special mention shared with the young Henri Poincaré. Casorati's (1880b) paper also lies at the heart of a harsh argument that Casorati had with Ludwig Stickelberger, who was working somehow on the same topics. The paper (Stickelberger 1881), published a few months after (Casorati 1880b), appeared to Casorati as an attempt to question, if not to deny, his priority in the applications of finite differences to complex analysis. Their controversy reached its *climax* with the publication of a short letter in the *Annali* (Casorati 1880c) where Casorati staunchly defended his primacy in bringing new life to finite-difference calculus.

In Casorati's *Nachlass*, there is a folder where Casorati kept most of his correspondence with several mathematicians concerning both (Casorati 1880b) and the Grand Prix. The content of those letters provides us with a vivid account of his personality and informs us of the unpredictable events that contributed to hamper his work. Moreover, copies of letters that Stickelberger and Casorati sent to Brioschi after the publication of (Casorati 1880c) also contain some valuable information, not only because they offer an idea of Stickelberger's point of view, but also because there Stickelberger pointed out that Casorati's criterion of linear dependence among functions of a discrete variable had already been established by Christoffel (1858).

In Italy, the main results of Casorati (1880b) were made known to a broader audience through a detailed account that appeared in the *Giornale di Matematiche* by Paolo Cazzaniga, a relative of Casorati's, who presented them as a chapter of the symbolic calculus (Cazzaniga 1882). As we shall see, Casorati had denoted the operation mapping $f(x)$ into $f(x+1)$ by θ, and he had emphasized the importance of conceiving

[10] We will discuss below in detail the content of Chap. X of the monograph (Pincherle and Amaldi 1901).

[11] Inzwischen erfuhr aber ein anderes Gebiet unserer Disziplin eine mächtige Förderung, nämlich die *formale* Seite der Theorie der linearen Differenzengleichungen, insbesondere soweit sie sich auf die Analogien mit den algebraischen Gleichungen und auf die entsprechenden Analogien mit den linearen Differentialgleichungen erstreckt. Hier hat nach *Casorati* hauptsächlich *Pincherle* mit seinen Schülern *Bortolotti* und *Amaldi* der Grundstein gelegt und besonders im 10. Kapitel seiens Buches über die distributiven Operationen gewissermaßen das Gerüst für den Aufbau einer solcher Theorie errichtet.

it as an operator mapping *functions* into *functions*. This is the aspect that most influenced Pincherle's subsequent treatment of the θ operator. He first applied Casorati's operator to linear differential equations in a short note written in 1885. Even though this note was meant to appear in the *Rendiconti* of the Istituto Lombardo, it was left unpublished. There, resorting to Casorati's (1880b) approach, Pincherle re-obtained a theorem proved in Tannery (1875) by which Tannery had given a characterization of the integrals of a linear differential equation with uniform coefficients in the complex plane: a theorem that is the "pleasant converse to Fuchs's main theorem" (Gray 1986, p. 90). Pincherle explained his result to Casorati who, though appreciating the simplicity of the latter's result, suggested him to add some more applications. Actually, he asked Pincherle to correct a point in Tannery's paper but, as Pincherle could not find any relevant mistake at the point indicated by Casorati, he did not insist in the publication, all the more so because he was diverted from this issue by the publication of a paper (Mellin 1886) where Mellin had bridged a differential and a difference equation by employing a transformation that was a particular case of a more general class just obtained by Pincherle: this urged the latter to write a note (Pincherle 1886a) to secure his priority on this topic.

A few years later, in the academic year 1893–1894, Pincherle went back to Casorati's θ operator that in his hands became the prototype of linear operators acting upon sets of analytic functions, i.e., the functions that Pincherle considered in his theory of linear operators where the symbolic calculus rose to the rank of a rigorous mathematical theory and ceased to be only a useful but dubious compact notation.

The present paper is organized as follows. In Sect. 2, we describe Casorati scientific activity in differential equations in the period 1875–1877 when he first interacted with Pincherle and then, in Sect. 3, we illustrate the content of Casorati (1880b). Section 4 is entirely devoted to the history of Casorati's application to the Grand Prix des Mathématiques by examining the letters he exchanged with both Italian and foreign mathematicians (the original letters are transcribed in "Appendix 1"). Intertwined with this topic, there is the argument with Stickelberger that will be described mainly in its lesser-known aspects through the letters that both he and Casorati addressed to Brioschi after the publication of (Casorati 1880c): some excerpts of these letters have been also transcribed in "Appendix 1". Then, in Sect. 5, Pincherle will be back on stage as we examine the letters he exchanged with Casorati concerning his note devoted to Tannery's theorem: in particular, Pincherle's unpublished note will be translated into English, while the original, relevant letters have been transcribed in "Appendix 2". Finally, in Sect. 6 we will examine the impact that (Casorati 1880b) had on Pincherle's research in the last decade of the nineteenth century up to the publication of his treatise (Pincherle and Amaldi 1901). In a conclusive section, we will resume the outcomes of the analysis contained in this paper and point to directions for future work.

2 Casorati and Pincherle interactions on differential equations

During the period that Pincherle spent in Pavia, which spans from 1875 to 1880 with the exclusion of his Berlin year, Casorati's main scientific activity was in the field of ordinary differential equations. In fact, Casorati had resumed scientific activity in analysis in 1874 after a long break. As he wrote in a letter to Gaston Darboux on 3 August 1879.

...For a lot of different reasons, I seldom read [the Darboux Bulletin] in the years 1872, 73, 74. In those years, as in the previous years 70, 71, I lived as a stranger to mathematics.[12]

In the period there alluded to, Casorati was intensely involved in teaching both in Pavia and in Milan, at Brioschi's *Regio Istituto Tecnico Superiore*—today the Milan Polytechnic—but in his personal life, Casorati also suffered several grievous events that impaired him to the point of hampering his research. He lost his beloved father—the renowned physician Francesco Casorati—in 1859; then, his mother, and his life-mate. The following passage from a 1880 letter to Joseph Bertrand (February 6th) clarifies Casorati's feelings:

After that a cruel illness mercilessly led my father to death—he, who was the sun of my life, the man to whose goodness I will never be able to think without being completely affected—my heart and my health were no longer as before; that inexplicable grief and that cruel reality depressed my forces, annihilated the poetry of life, in short, they destroyed my person both physically and morally. Moreover, I had my mother suffering from cancer for six consecutive years, when our family was reduced to the two of us and, finally, she died; it will not be hard to believe for you that, for several years, I remained as suspended between life and death. Finally, I improved; but my soul has remained, and will always remain, ready to be frightened, to get disheartened, to overstate any misfortune, to suffer enormously.[13]

Looking at Casorati's scientific production on differential equations, we can single out two main periods, each of them with its own *leitmotif.* In the first period up to 1878, Casorati's main concern was the research on *singular* solutions of differential equations, a topic on which mathematicians such as Cayley and Darboux also devoted their attention. Since 1878, Casorati was mostly influenced by Fuchs's theory of linear differential equations in the complex plane. Since Casorati's activity in the former period had little impact on Pincherle, I will limit myself to a few remarks.

Following the definition given by Boole in his treatise (Boole 1859), a textbook on which Casorati often based part of his lectures on infinitesimal calculus,

a singular solution of a differential equation is a relation between x and y, which satisfies the differential equation "by means of the value it gives to the differential coefficients" $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and *c., but is not included in the complete primitive* (Boole 1859, p. 139).

12 ..., per un cumulo di cause diverse, avvenne che negli anni 1872, 1873, 1874 non lo osservassi quasi mai. In quegli anni, come nei precedenti 70 e 71, ho vissuto da straniero alla matematica.

13 Dopoché un morbo crudele trasse spietatamente alla tomba mio padre, il sole della mia vita, l'uomo alla cui bontà non potrò mai pensare senza commuovermi tutto, l'animo mio e la mia salute non furono più quelli di prima; quell'indicibile dolore, quella feroce realtà abbatterono le mie forze, annientarono la poesia della vita, demolirono insomma tutta la mia persona fisica e morale. Avuta, per giunta, mia madre ammalata di cancro per sei anni consecutivi, mentre la famiglia riducevasi a noi due soli, ed essa infine soccombeva; non stenterete a credere come io dovessi per lunga serie di anni rimanerne come tra questo e l'altro mondo. Finalmente migliorai; ma l'anima mia è rimasta e rimarrà sempre pronta a sgomentarsi, ad avvilirsi, ad esagerarsi ogni contrarietà, a soffrire senza misura.

As an example, Boole considered the equation (Boole 1859, p. 150)

$$y^2 - 2xy\frac{\mathrm{d}y}{\mathrm{d}x} + (1+x^2)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 \tag{2.1}$$

whose general solution (or complete primitive) is given by

$$y = cx + \sqrt{1-c^2}, \tag{2.2}$$

where c is an arbitrary constant. However, it is also solved by

$$y = \sqrt{x^2+1} \tag{2.3}$$

which is a singular solution of (2.1) since there is no value of c for which (2.2) yields (2.3).

Casorati focused his attention on algebraic differential equations whose coefficients are polynomials in both the dependent and the independent variables:

A first-order differential equation between two variables u and v is called algebraic-differential *when it yields a constraint among* u, v, $\mathrm{d}u$ *and* $\mathrm{d}v$ *that can be expressed by a* finite *number of* algebraic *operations.*[14] (Casorati 1875 ≡ (Casorati 1952, p. 10). If $\varphi_k(u, v)$ are $m+1$ polynomials, then it is always possible to reduce such an equation to the form

$$\sum_{k=0}^{m} \binom{m}{k} \varphi_{m-k}(u, v)(\mathrm{d}u)^k(\mathrm{d}v)^{m-k} = 0$$

where $\binom{m}{k}$ are binomial coefficients and m determines the *degree* of the equation.

Dissatisfied with the procedures he found in the relevant literature, Casorati proposed a new theory for dealing with singular solutions of algebraic differential equations of first order and second degree (Casorati 1876). Specifically, the equations considered by Casorati are

$$\alpha(u, v)(\mathrm{d}u)^2 + 2\beta(u, v)\mathrm{d}u\mathrm{d}v + \gamma(u, v)\mathrm{d}v^2 = 0 \tag{2.4}$$

where the polynomials α, β and γ have no common factors. Casorati made the hypothesis that (2.4) has an algebraic complete primitive.

For an equation of the first order and the first degree, that is,

$$A(x, y)\mathrm{d}x + B(x, y)\mathrm{d}y = 0 \tag{2.5}$$

[14] Una equazione differenziale di primo ordine tra due variabili u e v va detta *algebrico-differenziale* quando consista in un legame tra u, v, $\mathrm{d}u$ e $\mathrm{d}v$ esprimibile mediante un numero *finito* di operazioni *algebriche*.

Jakob Rosanes had worked out a condition under which this is the case, in Rosanes (1871). It is precisely to equations like (2.5) that Pincherle turned his attention before his leaving for Berlin. He explained his reflections on algebraic differential equations to Casorati in a meeting (*Conferenza*) on 5 February 1877, a few weeks before presenting the note (Pincherle 1877) at the Istituto Lombardo. Like Rosanes, Pincherle emphasized the geometric meaning of (2.5) that can be generated by taking a one-parameter family of algebraic curves

$$G(x, y) + \lambda H(x, y) = 0 \tag{2.6}$$

where G and H are irreducible polynomials of degree n, and λ is a parameter. By using the differential of this equation to get rid of λ, an equation of the class (2.5) is obtained. Pincherle resorted to the geometric meaning of (2.5) to obtain *analytically* a result geometrically established by Cremona 15 years before:

In a family of curves of order m, there are $3(m-1)^2$ *double points* (Cremona 1862, p. 67).

In the second part of Pincherle (1877), Pincherle followed the opposite direction: under the hypothesis that (2.6) is an integral of (2.5), he aimed at finding its explicit expression in terms of A and B. The results obtained in this part of the paper are subject to restrictions—as Pincherle himself stated in his 1887 curriculum vitae submitted for the election at the *Lincei*. Actually, these restrictions were pointed out to Pincherle by Casorati on September 1878, that is, almost eighteen months after the presentation of the note. These limitations concern some terms that were omitted by Pincherle, but taken into account by Casorati in his research on this subject. All this might be surprising, as Casorati had himself read the note to the members of the *Istituto Lombardo* but, presumably, on that occasion, he did not take too much care over the details of Pincherle's note since he relied on the meeting he had had with the latter a few weeks before. On the other hand, in 1878, when Casorati was working on equations like (2.5) (Casorati 1878b), he read Pincherle's paper more accurately, and pointed him out that the generality of his conclusions had to be considerably reduced.[15]

3 Casorati's paper on finite differences

In the history of mathematics, Casorati is best remembered for the Casorati–Weierstrass–(Sokhotskii) theorem he published in (Casorati 1868). However, his name has also been linked to a special determinant—the Casoratian or the Casorati determinant—that plays a rôle in linear difference equations analogue to the one played by the Wronskian in the theory of linear differential equations. This determinant was introduced in Casorati (1880b), the paper by Casorati that mostly influenced Pincherle. This long memoir had a history that I review in the next section, after having examined its content.

[15] Casorati met Rosanes in Parpan (Switzerland) on 27 August 1879, and on that occasion, they discussed of research on the equation $A\mathrm{d}x + B\mathrm{d}y = 0$.

Casorati applied the formalism of finite differences to study the behaviour of a function y of a complex variable x in the neighbourhood of a point x_1. By taking a point x in a circular annulus surrounding x_1 within which y is free of singularities, and by performing a simple closed contour starting at x, Casorati set

$$\Delta y(x) := y(x + e^{2\pi i}) - y(x)$$

and he noted that

$$t(x) := \frac{\log(x - x_1)}{2\pi i}$$

is such that $\Delta t = 1$. Moreover single-valued (monodromic) functions are such that $\Delta y = 0$ and so they play the same rôle as that played in standard calculus of differences by periodic functions f whose period is equal to the constant increment $h = 1$ of the independent variable. Casorati introduced an operator—the θ operator—defined by

$$\theta y(x) := \Delta y(x) + y(x) = (\Delta + 1)y(x) \tag{3.1}$$

which is clearly the counterpart of the operator mapping $f(t)$ into $f(t+1)$. In this new interpretation θy yields the value of y after that the closed contour has been completed once. Casorati remarked:

It is well known that, in the Calculus of differences, together with the differences Δy, $\Delta^2 y$ of a function, it is also important to consider the values that it attains at the values $t + 1, t + 2, \ldots$ of the variable. These values are mostly denoted as follows:

$$y_{t+1}, \quad y_{t+2}, \ldots$$

But there is a great advantage in denoting these values with another notation, such as the following

$$\theta y, \quad \theta^2 y$$

that makes it possible to consider θ as a symbol of an operation, that is, of that operation which yields y_{t+1} when it is performed on y_t[16] (Casorati 1880b, p. 13; Casorati 1951, p. 320).

[16] Si sa che nel *Calcolo delle differenze* importa di considerare insieme con le differenze Δy, $\Delta^2 y$ di una funzione, anche separatamente i valori che essa prende corrispondentemente ai valori $t + 1, t + 2, \ldots$ della variabile. Questi valori vengono per lo più significati come segue:

$$y_{t+1}, \quad y_{t+2}, \ldots$$

Ma vi ha grande vantaggio a significare questi valori con altra notazione, come la seguente

$$\theta y, \quad \theta^2 y$$

la quale si presta a lasciar riguardare θ come un simbolo di operazione, cioè di quell'operazione che eseguita su y_t dà per risultato y_{t+1}.

Such an *operator* interpretation is in the spirit of Pincherle's subsequent views as clearly stated, for instance, in his major memoir on Laplace transform (Pincherle 1887) where he considered the function

$$f(x) := \int e^{xy}\varphi(y)\mathrm{d}y,$$

the integration being performed along a line in the complex plane. Pincherle observed that $f(x)$ is an analytic function if $\varphi(y)$ is analytic as well, and hence,
in this paper, we will consider this transformation under a new point of view: that is, as a functional operation (...) defined by some characteristic properties[17] (Pincherle 1887, p. 125).
The most remarkable property of the operator θ is that it is *eminently distributive* (*eminentemente distributivo*), (Casorati 1880b, p. 13; Casorati 1951, p. 321) since it is not only linear but it also obeys

$$\theta(u(x)v(x)) = \theta(u(x))\theta(v(x)) \quad \theta\left(\frac{u(x)}{v(x)}\right) = \frac{\theta u(x)}{\theta v(x)} \tag{3.2}$$

and

$$\theta F(u(x), v(x), w(x), \ldots) = F(\theta u(x), \theta v(x), \theta w(x), \ldots), \tag{3.3}$$

for any operator F that yields a unique result when applied on functions $u(x)$, $v(x)$, $w(x), \ldots$. The main result contained in Casorati (1880b) is the following theorem, proved in Chap. 2:

Theorem *Among the functions $y_1(x), y_2(x), \ldots, y_n(x)$ of a complex variable x there exists or not a linear, homogeneous relation with coefficients that are single-valued (monotropi) in an annulus centered at $x = x_1$ according to whether the determinant*

$$\Theta = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ \theta y_1 & \theta y_2 & \ldots & \theta y_n \\ \ldots & \ldots & \ldots & \ldots \\ \theta^{n-1} y_1 & \theta^{n-1} y_2 & \ldots & \theta^{n-1} y_n \end{vmatrix}$$

vanishes or not (Casorati 1880b, p. 19; Casorati 1951, p. 329).

The direct theorem is easy to prove since, assuming that

$$\sum_{i=1}^{n} \varphi_i y_i = 0$$

[17] In questo lavoro si vuole considerare questa trasformazione sotto un punto di vista nuovo: si vuole cioè riguardarla come una operazione funzionale (...) definita da alcune sue proprietà caratteristiche.

and observing that $\theta\varphi_i = 0$ as these functions are single-valued, then $\theta(\varphi_i y_i) = \theta\varphi_i\theta y_i = 0$. Thus, also

$$\sum_{i=1}^{n} \varphi_i \theta^k y_i = 0 \quad \forall k = 0, \dots, n-1$$

and so the determinant Θ vanishes. Casorati proved the converse by induction on the number of functions y_i. When $n = 2$, from

$$\det\begin{pmatrix} y_1 & y_2 \\ \theta y_1 & \theta y_2 \end{pmatrix} = 0$$

it follows that

$$\frac{\theta y_1}{\theta y_2} = \frac{y_1}{y_2}$$

and since $\theta\left(\frac{y_1}{y_2}\right) = \frac{\theta y_1}{\theta y_2}$, he concluded that the ratio $\frac{y_1}{y_2}$ goes back to its original value, when its argument has performed a complete tour. Hence it is a single-valued function, say $-\frac{\varphi_2}{\varphi_1}$, and the relation

$$\varphi_1 y_1 + \varphi_2 y_2 = 0$$

holds. Induction from $n-1$ to n is then performed by resorting to a transformation of determinants given in Hermite (1849). We note that, as a result of the new interpretation of finite-difference formalism, the linear, homogeneous relation between the functions $y_i(x)$ has *variable* coefficients, contrary to its counterpart based upon the Wronskian.

As a first application in Chap. 3 Casorati employed his formalism to study the structure of the roots of the algebraic equation

$$z^n + \psi_1 z^{n-1} + \cdots + \psi_{n-1} z + \psi_n = 0,$$

where ψ_i are single-valued functions of a complex variable x. In doing this, Casorati simplified considerably the analysis contained in Puiseux's long memoir (Puiseux 1850). In Chap. 4 Casorati applied the theorem on the Θ determinant—that is the Casoratian—to the linear differential equation with single-valued coefficients $p_i(x)$ in the complex plane

$$D^m y + p_1 D^{m-1} y + \cdots + p_{m-1} D y + p_m y = 0, \tag{3.4}$$

where $D := \frac{\mathrm{d}}{\mathrm{d}x}$. Given a point x_1, Casorati associated to the Eq. (3.4) a homogeneous difference equation that is linear and of the mth order as well but with *constant* coefficients, whose general solution he had recalled in Chap. 6. Actually, Casorati observed that, when y denotes the general integral of (3.4), then (3.4) itself can be viewed as a linear, homogeneous relation with single-valued coefficients p_i among the functions $y, Dy, \ldots, D^m y$, and so the corresponding Θ determinant vanishes, that is

$$H = \begin{vmatrix} y & Dy & \ldots & D^m y \\ \theta y & \theta Dy & \ldots & \theta D^m y \\ \ldots & \ldots & \ldots & \ldots \\ \theta^m y & \theta^m Dy & \ldots & \theta^m D^m y \end{vmatrix} = 0.$$

However, since θ commutes with D,

$$H = \begin{vmatrix} y & Dy & \ldots & D^m y \\ \theta y & D\theta y & \ldots & D^m \theta y \\ \ldots & \ldots & \ldots & \ldots \\ \theta^m y & D\theta^m y & \ldots & D^m \theta^m y \end{vmatrix} = 0$$

follows, that can be viewed as the Wronskian determinant of the functions $y, \theta y, \ldots, \theta^m y$. Hence, it follows that these functions obey a linear difference equation with constant coefficients

$$A_0\theta^m y + A_1\theta^{m-1} y + \cdots + A_{m-1}\theta y + A_m y = 0 \tag{3.5}$$

that can be used in turn to gain information on the behaviour of the function y around the point $x = x_1$. When $x = x_1$ is a singular point of (3.4), Casorati applied the correspondence between (3.4) and (3.5) to revisit §3 of Fuchs (1866), and to obtain quickly the local form of the canonical fundamental system of solutions of (3.4) with a cursory mention to the fact that also the structure of the integrals forming Hamburger's subgroups (Hamburger 1873) can be similarly obtained, an issue that will be the core of the subsequent paper (Casorati 1881a).

The operator θ admits other interpretations. For instance, by setting

$$\theta f(x) = f(x + \omega)$$

it is possible to view periodic functions with period ω as the fixed points of θ. By introducing another operator θ' such that

$$\theta' f(x) = f(x + \omega')$$

it is possible to study systems of difference equations

$$\begin{cases} A_0\theta^m y(x) + A_1\theta^{m-1} y(x) + \cdots + A_{m-1}\theta y(x) + A_m y(x) = 0 \\ B_0\theta'^n y(x) + B_1\theta'^{n-1} y(x) + \cdots + B_{n-1}\theta' y(x) + B_n y(x) = 0 \end{cases} \tag{3.6}$$

where the mn coefficients A_i, B_j are constants. As a special case, Eqs. (3.6) embody those arising in a problem just tackled by Picard (1879) where the latter generalized the concept, due to Hermite, of doubly periodic functions of the second kind with periods $2K$ and $2\iota K'$, by looking for uniform functions $f(x)$ such that

$$f(x+4K)=Af(x)+Bf(x+2K) \quad \text{and} \quad f(x+4\iota K')=A'f(x)+B'f(x+2\iota K').$$

The same path followed to associate (3.5)–(3.4) also works for a linear differential equation with periodic or doubly periodic coefficients, and Casorati devoted Chap. 7 of Casorati (1880b) to investigate the structure of the general solution of these equations that will been studied intensively also by Floquet (1883, 1884).

As a final interpretation of the θ-operator, in Chap. 8 of his paper, Casorati considered functions in p independent variables that are multiply periodic, so that

$$f(x_1, x_2, \ldots, x_p) = f(x_1+\omega_1, x_2+\omega_2, \ldots, x_p+\omega_p)$$

holds for a set of periods $\omega_1, \ldots, \omega_p$ and now θ is interpreted as the operator

$$\theta: \quad f \mapsto \theta f(x_1, x_2, \ldots, x_p) = f(x_1+\omega_1, x_2+\omega_2, \ldots, x_p+\omega_p).$$

4 Chronicle of a missed prize

From the introduction of Casorati (1880b) we realize the importance that Casorati attached to his paper:

Researches that employ complex variability rely—at least in part—in an essential way upon relations that stem from considering the way in which functions of a complex variable behave when the variable turns around a particular value. This remark prompted me to investigate and determine once and for all, in advance and independently of any particular study or aim, the properties and the general formulae that it is important, and possible, to obtain to the benefit of further, particular researches; in such a way to form a theory in itself from which all those who begin investigations on functions with complex variables from now on can avail themselves as a common instrument[18] (Casorati 1880b, p. 10 $\equiv$ Casorati 1951, p. 317).

At the beginning of November 1879, Casorati, influenced by Brioschi, decided to apply for the *Grand Prix des Sciences Mathématiques* that the French Academy of Sciences proposed for the year 1880, whose theme perfectly matched the topic of his researches:

[18] Le ricerche che si fanno colla variabilità complessa si basano, almeno in parte, essenzialmente sulle relazioni che scaturiscono dal considerare i modi di comportarsi delle funzioni di una variabile complessa al girare di questa intorno a suoi valori particolari. Questa osservazione mi fece nascere il pensiero di investigare e stabilire, una volta per sempre in anticipazione ed indipendentemente da ogni studio o scopo particolare, le proprietà e le formole generali che da siffatte relazioni fondamentali importa ed è possibile di ricavare a beneficio delle singole ricerche ulteriori; così da costituire una teorica a sè, della quale, come di strumento comune, possano valersi tutti coloro che d'ora innanzi vorranno intraprendere studî sulle funzioni di variabili complesse.

Improve in some important point the theory of linear differential equations in only one variable.[19]

Casorati kept all the correspondence concerning (Casorati 1880b) and the prize, and in this section, we outline the content of part of these letters whose original texts are displayed in "Appendix 1".[20] We should note that at the beginning of the correspondence Casorati had not published yet either (Casorati 1880b) or anything else on the interpretation of finite differences. We also recall that the letters sent by Casorati are known through his minutes, whereas those addressed to him are the originals. They cover more than one year: the first letter is dated 27 December 1879 and the last letter was written on 17 April 1881. Letters 1–5 contain the correspondence between Casorati and Joseph Bertrand, the *Sécretaire perpetuel* of the French Academy. Casorati asked for general information on the competition and, curiously enough, at the beginning of the correspondence, he pretended that the applicant was a fellow-countryman of his, probably to secure anonymity: it was only after the long-waited reply from Bertrand that Casorati disclosed the identity of the applicant. From these letters, we learn that the topic for the prize was chosen by Bertrand who wished to reward Edmond Nicolas Laguerre for the results on linear differential equations that had impressed him so much. As an aside, we note that, apparently, Laguerre did not apply for the prize. Casorati was not willing to wait the end of the competition to publish his results but, from Bertrand replies, we also realize that published papers could be eligible for the prize, provided that no member of the judging Committee raised formal objections.

A second group of letters (Letters 6–9) are concerned with the turbulent editorial process related to the publication of Casorati (1880b), whose content had been presented to the *Istituto Lombardo* in February 1880. In fact, a strike of the typographers was unduly delaying the publication of his notes: besides (Casorati 1880b), Casorati illustrated applications both to Fuchs's theory and to Floquet's theory of differential equations with periodic coefficients in Casorati (1880a). He then wrote to his friend Luigi Cremona, asking him to present a revised version of the manuscript to the *Annali* at the Accademia dei Lincei in Rome. Casorati not only called attention to applications to differential equations, but also to algebraic equations that might seem more interesting to Cremona who, in turn, presented the manuscript on March 7th[21]. The long strike in Milan went on, and the deadline of June 1 for applications to the *Grand Prix* was approaching: Casorati withdrew the first version of Casorati (1880b) from the *Annali* and worked on the manuscript that was finally sent to the Académie on May 22nd. As to the epigraph, he chose a passage from Blaise Pascal's *Pensées*: *Nous sommes si malheureux, que nous ne pouvons prendre plaisir à une chose qu'à condition de nous fâcher si elle réussit mal* (Pascal 1835, Chap. IV, Art. V. p. 153).

This epigraph was inserted in the *billet cacheté* together with an explanatory note where Casorati summarized the history of the manuscript that could not be sent in a printed form. We remark that the judging committee for the prize was formed by

[19] *Perfectionner en quelque point important la théorie des équations différentielles linéaires à une seule variable indépendante.*

[20] We did not transcribe Fuchs's letter to Casorati, dated March 30th, 1881 on the Stickelberger controversy, as it has been already published in Neuenschwander (1978).

[21] *Memorie della Reale Accademia dei Lincei*, **5**, S.III, 195–208, (1880).

Bertrand, together with Ossian Bonnet, Charles Hermite, Victor Puiseux, and Jean-Claude Bouquet, as communicated in the *Comptes Rendus* of April 12th, 1880.

Casorati applied to another competition, the second edition of the *Premio Bressa* proposed by the Royal Academy of Sciences in Turin (Letter 10). With respect to the *Grand Prix*, the Bressa prize had a broader spectrum, as it aimed at rewarding the best work written by an Italian scholar in the period 1877–1880 on a subject ranging from pure and applied mathematics to pathology and geology. To have a chance, it was then important that the mathematicians in the Turin Academy could appreciate the importance of Casorati's work. For this reason, since the principal analyst in Turin at that time was Angelo Genocchi, Casorati wrote him a letter (June 21st, 1880, not published here) asking for an appointment, since he was going to visit Turin with Beltrami. In this occasion, he would have explained the importance he ascribed to his findings. The meeting with Genocchi took place at the end of June and, according to Casorati's short account of it to Brioschi, Genocchi was favourably impressed. Thus Casorati asked Brioschi and Betti, both members of the Turin Academy, to support his paper: in spite of these efforts, however, the 2nd Bressa prize was awarded to the natural scientist Luigi Maria d'Albertis.

At the beginning of August, with his manuscript finally published in the *Annali*, Casorati sent offprints of it to many mathematicians, both in Italy–for instance to Francesco Siacci—and abroad, including Fuchs, Klein, Lindemann, Mittag-Leffler, Dillner, Kronecker, Picard, and Weierstrass. In all his accompanying letters, he stressed the importance of the theorem on the Θ determinant. As an example, letters 11 and 12 of "Appendix 1" are those that Casorati sent to, and received by, Émile Picard.

So far, we followed only the correspondence concerning the *Grand Prix*, but at this point, the events concerning the prize intertwined with a polemics involving Casorati and Ludwig Stickelberger. By the end of 1880, Casorati received a memoir—an *Akademische Antrittsschrift* (Stickelberger 1881)—that accompanied Stickelberger's entry as a professor at the University of Freiburg. There the author employed finite differences to study the behaviour of the solutions of a linear differential equation in the neighborhood of its singular points, thus obtaining results in the same vein as those contained in Chap. 4 of Casorati (1880b). Casorati was really hurt by a sentence placed in the Introduction to the *Antrittschrift*, where (Stickelberger, 1881, p. 4) had referred to Casorati's procedure as *inadvisable* (*nicht rathsam*). Moreover, since Stickelberger referred to papers by Riemann (1857) and Hamburger (1873) where the use of the logarithm of the independent variable had been made, Casorati felt that Stickelberger was attempting to diminish the importance of his own findings. Hence, at the end of 1880, he sent a letter to Brioschi to be published in the *Annali* (Casorati 1880c). This letter is firm in making it clear that Casorati's memoir had a broader aim than Stickelberger's and that the applications to differential equations were just an example to show, in a significant topic, the versatility of his ideas.

Admittedly, Casorati reply was excessive and, in my opinion, the reasons for his attitude could have been the following: (a) he was concerned with the priority of the theorem, now known as the Casorati–Weierstrass theorem, which he had proved in §88 of Casorati (1868), and had then been published by Weierstrass in (Weierstrass 1876) without any reference to Casorati's work (Neuenschwander 1978) [it should be noted, however, that there is some evidence that Weierstrass had known the theorem since

1863 (see Neuenschwander 1978; Bottazzini and Gray 2013, p. 436)]; (b) the great expectations he had about the impact of his work that were now brought into question by Stickelberger's remark; (c) the familiar concerns (see Letter 6 of "Appendix 1") he had to face when working on the final version of Casorati (1880b) and that interfered strongly with his scientific activity. The short letter (Casorati 1880c) was probably also written in a hurry, and Casorati did not properly separate relevant scientific comments from personal feelings. A few months later, on 6 February 1881, Casorati wrote a long letter to Ossian Bonnet (Letter 13) where he separated personal animosity from scientific judgements. There Casorati was appreciative of some aspects of Stickelberger's paper so that his judgement of (Stickelberger 1881) contained in the central part of the letter to Bonnet resembles a standard referee's report. First, he praises Stickelberger's use of the results obtained by Cauchy (1827), a paper devoted to symbolic calculus. There, using the notation

$$\Delta y = \Delta_x y := f(x + \Delta x) - f(x)$$

for the finite difference of a function $y = f(x)$, and Df for $\frac{\mathrm{d}f}{\mathrm{d}x}$, Cauchy had also proved (Cauchy 1827, pp. 228–230) that, if $F(x)$ is a polynomial with *constant* coefficients having only *simple* roots $r_1, r_2, \ldots, r_n$, then the general solution of the differential [difference] equation

$$F(D)y(x) = f(x) \quad [F(\Delta)y(x) = f(x)], \tag{4.1}$$

could be rewritten as

$$y(x) = \sum_{k=1}^{n} \frac{1}{F'(r_k)} \frac{f(x)}{D - r_k} \tag{4.2}$$

where $F'(r_k)$ is the shorthand for $\frac{\mathrm{d}F}{\mathrm{d}r}\big|_{r_k}$ and $\frac{f(x)}{D-r_k}$ denotes symbolically the solution of the *first-order* equation $(D - r_k)y(x) = f(x)$, for which the representation

$$y(x) = \mathrm{e}^{r_k x} \int \mathrm{e}^{-r_k t} f(t)\mathrm{d}t$$

exists. On the contrary, if $r_1 = r_2 = \cdots = r_m = \varrho$, the first m terms in the expression (4.2) had to be replaced by the *residue* of the function

$$\frac{1}{(r - z)F(z)}$$

at $z = \varrho$, so that the contribution of these terms to the solution of (4.1) was

$$\sum_{k=1}^{m} R_{m-k} \frac{f(x)}{(D - \varrho)^k}$$

where

$$R_{m-k} = \frac{1}{(m-k)!} \frac{\partial^{m-k}}{\partial \varepsilon^{m-k}} \left(\frac{\varepsilon^m}{f(\varrho + \varepsilon)} \right)_{\varepsilon=0}$$

and, again, $\frac{f(x)}{(D-\varrho)^k}$ represented symbolically the solution of $(D-\varrho)^k y(x) = f(x)$, given by

$$\mathrm{e}^{r_k x} \int \cdots \int \mathrm{e}^{-r_k t} f(t) \mathrm{d}t^k.$$

Actually Cauchy's version of the theorem was more general, but this application is enough for our purposes, as it is this classification of the solution according to the presence or not of coincident roots in the characteristic equation $F(x) = 0$ that bridges Cauchy's and Fuchs's theories. However, in all the equations considered by Cauchy the coefficients of $F(x)$ are constant, and so the worlds of differential and difference equations remained distinct, though parallel, in his work. As noted before, it is the change of variable $u := \frac{\log x}{2\pi \iota}$ that makes it possible to export methods of finite-difference calculus to linear differential equations with variable coefficients.

The second remark made by Casorati on (Stickelberger 1881) concerned the resort to Weierstrass's theory of elementary divisors (*Elementartheiler*) to classify solutions of (3.4). Casorati did not think that Stickelberger's approach is optimal, and this view was the starting point of the notes he alluded to in this letter to Bonnet, and that also appeared in the *Comptes Rendus* (Casorati 1881a).

Casorati's disappointment with Stickelberger's behaviour reappears strongly in a letter, dated 22 February 1881, he intended to write to Frobenius, with whom Stickelberger had scientific connections, but that was never sent. This sudden change of mind might have been caused by the final decision of the Committee charged to award the Grand Prix, communicated on February 14th: Georges Halphen was the recipient of the prize, and two special mentions—*mentions très honorables*—were granted to two other memoirs: to Poincaré's essay on Fuchsian functions that remained unpublished until 1923 (Poincaré 1923) (see Chap. 3 of (Gray 2013), for a detailed account of Poincaré's achievements)— and to Casorati's who, unlike Poincaré, decided to remain anonymous (see vol. **92** of the *Comptes Rendus*, pp. 551–554 for the reports on the three best memoirs among the six manuscripts sent to the Academy). The outcome of the competition did not satisfy Casorati who feared that his prestige as a professor could suffer a serious blow (Letter 15), despite of the appreciation that his work received from Bertrand (Letter 14) and Darboux. In particular, the attitude of Casorati, who was prone to be bewildered in front of misfortune, appears in his letter to Bertrand. We note that, on writing the report on behalf of the Committee, Hermite meant to point out a preference for Poincaré's memoir over Casorati's since the order of the memoires by Casorati (N. 3) and Poincaré (N. 5) was shifted from the natural one: Casorati asked Bertrand (on February 26th) to correct it *unless* it meant a difference in quality between the two memoirs.

From the historical point of view, Letters 16 and 17 are the more significant. Stickelberger replied to Casorati (1880c) by writing to Brioschi twice to have his rebuttal published in the *Annali*: Letter 16 is the second one written by Stickelberger. Since these letters provoked another bitter reply from Casorati, Brioschi finally decided not to proceed with publishing Stickelberger's defence. Stickelberger stated he received the paper (Casorati 1880b) from Ferdinand von Lindemann in August 1880, when he had already obtained his main results but, more importantly, he quoted a 1858 paper by Christoffel (1858) where Casorati's condition of linear independence—with *constant* coefficients—among functions depending on a variable undergoing discrete increments had already been obtained. Casorati admitted to Brioschi that he was unaware of Christoffel's result and promised to quote it as soon as possible—a promise he did not fulfill as he ceased to work on this topic.

Christoffel's paper—which also had a continuation first published in his Collected papers (Christoffel 1910)—is divided in two parts: for our purposes, the discussion in Sects. 1–2 of Christoffel (1858) is the most important. There he considered $n+1$ functions $\{f(m), f_1(m), \dots, f_n(m)\}$ with $f_i : \mathbb{Z} \mapsto \mathbb{R}$ and, supposing that they are linear dependent for the values of m ranging from m_0 to m_0+n+p, with $p \geq 0$, he distinguished three cases according to whether, for these values of m,

(a) only a relation $Af(m) + A_1 f_1(m) + \cdots + A_n f_n(m) = 0$ holds;

(b) at least two relations $Af(m) + A_1 f_1(m) + \cdots + A_n f_n(m) = 0$, $Bf(m) + B_1 f_1(m) + \cdots + B_n f_n(m) = 0$ hold for the *same* values of m, and so the *rank* of the matrix

$$\boldsymbol{A} := \begin{pmatrix} f(m) & f_1(m) & \cdots & f_n(m) \\ f(m+1) & f_1(m+1) & \cdots & f_n(m+1) \\ \cdots & \cdots & \cdots & \cdots \\ f(m+n) & f_1(m+n) & \cdots & f_n(m+n) \end{pmatrix}$$

is strictly lower than in case (a);

(c) it is possible to single out $q+r$ intervals, each consisting of at least n elements, such that in q of them, case (a) holds, whereas in remaining r intervals case (b) holds. (Casorati 1880b) is consistent with case (a) of Christoffel's treatment. Here Christoffel wrote down the linear system

$$\begin{cases} Af(m) + A_1 f_1(m) + \cdots + A_n f_n(m) = 0 \\ Af(m+1) + A_1 f_1(m+1) + \cdots + A_n f_n(m+1) = 0 \\ \cdots \\ Af(m+n) + A_1 f_1(m+n) + \cdots + A_n f_n(m+n) = 0 : \end{cases}$$

denoting the determinant of the matrix $\boldsymbol{A}$ by $\Delta(m)$, and the minor obtained by removing the μth column and the last line of $\boldsymbol{A}$ by $\Delta_\mu(m)$, he obtained

$$A_\mu = \omega \Delta_\mu(m)$$

where the factor ω is undetermined, while $\Delta(m) = 0$ guarantees existence of the solution in the unknowns A_μ. To prove the converse statement (Christoffel 1858, §2) supposed that $\Delta(m)$ vanishes at $m = m_0, m_0+1, \dots, m_0+p$, but that it does not vanish at $m_0 - 1$ and at $m_0 + p + 1$: then, the functions $\{f, f_1, \dots, f_n\}$ are linearly dependent at $m = m_0, m_0 + 1, \dots, m_0 + p + n$. Christoffel set

$$\varphi(m) := Af(m) + A_1 f_1(m) + \cdots + A_n f_n(m)$$

and selected the $n + 1$ coefficients A_i such that the n equations

$$\varphi(m') = \varphi(m' + 1) = \cdots = \varphi(m' + n - 1) = 0$$

hold, for a yet unspecified value m' of m. By adding to these equations either

$$Af(m' - 1) + A_1 f_1(m' - 1) + \cdots + A_n f_n(m' - 1) = \varphi(m' - 1)$$

or

$$Af(m' + n) + A_1 f_1(m' + n) + \cdots + A_n f_n(m' + n) = \varphi(m' + n),$$

then

$$A_\mu \Delta(m' - 1) = (-1)^n \varphi(m' - 1)\Delta_\mu(m'), \quad \text{or} \quad A_\mu \Delta(m') = \varphi(m' + n)\Delta_\mu(m') \quad (4.3)$$

follows, respectively. Hence, when m' is set equal to one of the values $m_0, m_0 + 1, \dots, m_0 + p$, the left-hand side of $(4.3)_2$ vanishes by hypothesis and so

$$\Delta_\mu(m')\varphi(m' + n) = 0:$$

if at least one of the minors $\Delta_\mu(m')$ does not vanish, then $\varphi(m' + n) = 0$, and linear dependence of $\{f, f_1, \dots, f_n\}$ is guaranteed. As a consequence of his analysis, Christoffel obtained that

The $n + 1$ assigned functions $\{f(m), f_1(m), \dots, f_n(m)\}$ are linearly independent for any value of m if and only if their determinant $\Delta(m)$ is different from zero for all values of m[22] (Christoffel 1858, p. 288).

Christoffel's paper is not mentioned even in treatises on finite differences that have rather extensive bibliographies like (Nörlund 1924; Guldberg and Wallenberg 1911), and (Pascal 1897a). It was quoted in passing by Boole (Boole 1880, p. 232) as a *remarkable paper* though for a different reason, and it did not escape the encyclopedic history of determinants by Muir who actually reviewed both Christoffel's (Muir 1911, pp. 225–228) and Casorati's papers (Muir 1923, pp. 242–243), though he did not feel the need to note any interrelation between them, apart from inserting both of them in chapters on Wronskians. We note in passing that Ernesto Pascal made a short reference to Christoffel's paper in another book on determinants, for a theorem on Wronskians (Pascal 1897b). Looking at the outline (Butzer 1981) of the main achievements by Christoffel in mathematics, (Christoffel 1858) does not occupy the first rank, and the

[22] Damit gegebene $n + 1$ Functionen $\{f(m), f_1(m), \dots, f_n(m)\}$ für alle Werthe von m linearunabhängig sind, ist erforderlich und hinreichend, dass ihre Determinante $\Delta(m)$ für alle Werthe von m von Null verschieden ist.

episode discussed in this section certifies once more a certain difficulty in appreciating the content of Christoffel's paper by the scientific community.

We can conclude that the novelty of Casorati's approach is the *interpretation* he gave to the formalism of finite differences. Curiously enough, his theorem on the Θ determinant has been later applied as a tool in the *standard* theory of finite-difference calculus where, however, Casorati had been partially anticipated by Christoffel.

5 An unpublished note by Pincherle

In 1875, Jules Tannery wrote a long memoir (Tannery 1875) in which he aimed at clearly explaining the main principles upon which the theory of linear differential equations rests. In doing this, he had as models the relevant papers by Thomé and, more importantly, by Fuchs. The memoir was actually Tannery's thesis, which was discussed at the Faculty of Sciences in Paris on 28 November 1874. It is divided into five parts: the first and the second part illustrate general basic properties of the solutions of linear differential equations while the third part, on which we will focus attention, is concerned with the behaviour of a fundamental set of solutions in the neighbourhood of a singular point; finally, the fourth part is concerned with a special class of differential equations, namely those that only admit integrals with poles of finite order at the singular points of the equations and at infinity, and the closing fifth section is devoted to applications. Casorati, who was given a copy of the thesis by Tannery himself, expounded parts of Tannery (1875) in his 1878–79 course devoted to differential equations (Casorati 1878a) and, as we shall see later on, he made some critical remarks on (Tannery 1875) that played a rôle in the history of Pincherle's unpublished note. Tannery's paper was noted also by Goursat in Goursat (1883) where he generalized the representation of the hypergeometric series as definite integrals depending on parameters. Actually, both Goursat and Pincherle were interested in the same theorem (Tannery 1875, pp. 130–132):

Let $y_1, y_2, \ldots, y_m$ be m functions of x that are continuous everywhere, except at some singular points isolated from each other, and that are uniform within the portions of the plane (or of the sphere) with a simple boundary that do not contain singular points: if, whenever the variable makes a complete tour around a singular point, the new values

$$[y_1]', [y_2]', \ldots, [y_n]'$$

of these functions are related to the previous ones by linear equations with constant coefficients, (...), these functions are the integrals of a linear differential equation with uniform coefficients.[23]

[23] Soient $y_1, y_2, \ldots, y_m$ m fonctions de x continues, sauf pour des points singuliers isolés les uns des autres, uniformes dans les portions de plan (ou de la sphère) à contour simple qui ne contiennent pas de points singuliers: si, lorsque la variable fait le tour d'un point singulier, les nouvelles valeurs

$$[y_1]', [y_2]', \ldots, [y_n]'$$

Since the converse of this theorem is true (Tannery 1875, pp. 129–130) the property analysed by Tannery *characterizes the integrals of a linear, homogeneous differential equation, with uniform coefficients* (Goursat 1883, p. 21). With this remarks, we can now follow the letters between Casorati and Pincherle on this topic in the period 1885–1886: the original texts are available in "Appendix 2".

On 21 July 1885, Pincherle communicated to Casorati (see Letter 1) that he could prove Tannery's theorem quickly by resorting to the formalism of the θ operator proposed in (Casorati 1880b). We note that Casorati had lectured on some results then published in Casorati (1880b) during the course of Higher Calculus in the academic year 1879–1880, when Pincherle was in Pavia. Pincherle had only an indirect knowledge of Tannery (1875) via Goursat's memoir (Goursat 1883) that he was studying during his researches *on integrals*.[24] Actually, Pincherle's work on the θ operator is an *impromptu* since at that time he was more concerned with the representation of analytic functions by definite integrals that form the content of Pincherle (1885b, 1886b) (Letters 3 and 5). In any case, Casorati (Letter 2) suggested that Pincherle read (Tannery 1875) because he might find new applications therein for the θ operator. While in Letter 2, Casorati seemed to consider the presence of these applications as not so stringent a requirement if he were to present Pincherle's note for publication, he then changed his mind (Letter 4) and his doubts cooled Pincherle who, however, was still convinced that Casorati's method provided the right way to obtain an easier proof of the theorem (Letter 5). At this point Casorati felt that Pincherle had taken his doubts too seriously, and encouraged him in writing the note (Letter 6). As a result, Pincherle asked Casorati for the volume of the *Annales* containing (Tannery 1875) that was unavailable in Bologna, and that might give him new ideas for a possible expansion of his note (Letter 7).

Letter 8, dated 9 January 1886, is the most important letter of this correspondence as it contains the note that, as we shall see, will remain unpublished. Pincherle proved Tannery's theorem under slightly weaker hypotheses, since he assumed that the coefficients could also be multi-valued functions, though of a specific type: uniform functions of an analytic point.[25] Here is the translation of Pincherle's note attached to that letter.

Analysis: On a theorem of Mr. Tannery

Footnote 23 continued
de ces fonctions sont liées aux prèmieres par des équations linéaires à coefficients constants (...), ces fonctions sont les intègrales d'une équation différentielle linéaire à coefficients uniformes.

[24] Pincherle started reading Goursat's memoir on June 18th, 1885 as noted in Vol. V of his *Ricerche e Saggi*, an impressive collection of notebooks on which Pincherle annotated his reflections on the scientific literature, and wrote down drafts of his own papers and letters. The notebooks are kept in the library of the Mathematical Department of the University of Bologna.

[25] These functions are uniform on their Riemann surface which, in turn, consists of a finite number of sheets. Here Pincherle follows Paul Appell who introduced this concept in his Appell (1882a, b). Giulio Vivanti, who also studied with Pincherle, devoted (Vivanti 1887) to study their properties. In this paper, Vivanti also gave this informal definition (p. 54): *Uniform functions of an analytic point are uniform functions in two variables that are related by an algebraic equation.*

This short note aims to show, by a new application, the advantages in the study of analytic functions, and of their behaviour in the neighborhood of given points, of the method envisaged by prof. Casorati and exposed in his Memoir: "Il calcolo delle differenze finite interpretato[26] etc, Annali di Matematica, S. II, T. X." Here I apply this method to prove a theorem stated by Mr. Tannery in his Memoir "Propriétés des intégrales des équations différentielles linéaires, Annales de l'Ec. Normale, S. II, T. IV, p. 130;" this theorem is remarkable both in its own, and for an important application given by Mr. Goursat.[27]

The theorem is as follows, though in a modified form: "Let $E_1, E_2, \dots, E_n$ be n analytic functions that are regular in the neighborhood of any point of a simply connected set T, apart from a finite number of points $a_1, a_2, \dots, a_p$, and among which there is no linear relation with constant coefficients; if, when the variable moves around any of the points $a_1, a_2, \dots, a_p$ without leaving the set T, the new values of the functions are related to the former ones by linear relations with constant coefficients, then these functions are the solutions of a linear differential equation with coefficients that are single-valued in the set T."

In fact, let us consider the function E

$$E = c_1 E_1 + c_2 E_2 + \cdots + c_n E_n, \tag{1}$$

where the coefficients c are arbitrary constants; if the variable varies following a contour enclosing a_1, and if we denote by θ the Casorati operator, it follows that

$$\theta E = c_1 \theta E_1 + c_2 \theta E_2 + \cdots + c_n \theta E_n$$

and, by hypothesis,

$$\theta E = k_{1,1} E_1 + k_{1,2} E_2 + \cdots + k_{1,n} E_n; \tag{2}$$

similarly

$$\begin{cases} \theta^2 E = k_{2,1} E_1 + k_{2,2} E_2 + \cdots + k_{2,n} E_n, \\ \dots\dots\dots \\ \theta^n E = k_{n,1} E_1 + k_{n,2} E_2 + \cdots + k_{n,n} E_n; \end{cases} \tag{3}$$

if we eliminate $1, E_1, E_2, \dots, E_n$ in Eqs. (1), (2), and (3), an equation

$$K_n \theta^n E + K_{n-1} \theta^{n-1} E + \cdots + K_0 E = 0$$

is obtained; this equation, however (Casorati, loc. cit. §10), shows that E obeys a linear differential equation

$$\varphi_0 E^{(n)} + \varphi_1 E^{(n-1)} + \cdots + \varphi_n E = 0,$$

[26] *See a detailed account of this Memoir in "Bulletin de Darboux, S. II, T. VI, 1882"*

[27] In the Memoir "Sur une classe des fonctions representés per des intégrales définies. Acta Mathematica, T.II".

with coefficients that are single-valued in a neighborhood of a_1 and having E as its general integral. From the same proof, it also follows that $\varphi_0, \varphi_1, \dots, \varphi_n$ are single-valued in a neighborhood of $a_2, a_3, \dots, a_p$, whence the theorem follows.

If the set T covers all the sphere once, the functions $\varphi_0, \varphi_1, \dots, \varphi_n$ are uniform. The case in which the set T is a Riemann surface, reduced to be simply connected by means of appropriate cuts, is not ruled out; in this case, the functions $\varphi_0, \varphi_1, \dots, \varphi_n$ would be uniform functions of an analytic point.

S. Pincherle

The important application by Goursat alluded to here is Theorem I of Goursat (1883). There Goursat considered functions $f(x, u)$, where both x and u are complex variables such that, for any fixed value of x, $f(x, u)$ admits only a finite number $m = n + p$ of singular values $v_1, \dots, v_m$, some of which $(a_1, \dots, a_n)$ are independent of x, whereas the remaining ones $(u_1, \dots, u_p)$ do depend on x. For instance, the function

$$f(x, u) := (u - a_1)^{b_1 - 1}(u - a_2)^{b_2 - 1} \cdots (u - a_n)^{b_n - 1}(u - x)^{\lambda - 1} \tag{5.1}$$

has $v_1 = a_1, \dots, v_n = a_n, v_{n+1} = u_1 = x$ as its critical values. Goursat considered the functions

$$(v_i v_h)(x) := \int_{v_i}^{v_h} f(x, u)\mathrm{d}u$$

where the integration is performed along the straight line joining v_i to v_h. To define these functions, one has first to make sure that none of the singular points v_i fall on the path of integration : if, in (5.1), b_i and b_h have a positive real part, in defining $(a_i a_h) = \int_{a_i}^{a_h} f(x, u)\mathrm{d}u$ we need to suppose that none of the remaining points a_j lie on the segment joining a_i with a_h, and the x-plane must have cuts (*coupures*, after Hermite 1881) along the segment (a_i, a_k), to ensure that the singularity u_1 does not enter the domain of integration. However, this is not enough to guarantee that (a_i, a_h) is a uniform (single-valued) function of x since, when x makes a complete turn along a close curve surrounding the segment (a_i, a_h), then the integral $(a_i a_h)$ turns into $(a_i, a_h)\mathrm{e}^{\pm 2\pi\iota\lambda}$ that, in general, does not coincide with $(a_i a_h)$: new *coupures* must be introduced to guarantee that $(a_i a_h)$ is uniform. Goursat applied Tannery's theorem to prove that *all* the integrals $(v_i v_h)(x)$ that can be formed from a function $f(x, u)$ obey one and the same linear differential equation with uniform coefficients of order $m - 1$.

Pincherle's proof simplified Tannery's proof, and Casorati himself had considered the possibility of applying the results of Casorati (1880b) to simplify Tannery's analysis, though at another point. In fact, in a footnote on p. 336 of Casorati (1951), the editors of Casorati's collected papers—either Luigi Berzolari or Silvio Cinquini—report on an unpublished manuscript where Casorati commented upon the advantages of (3.5) to reobtain both the main results of Fuchs (1866) as well as Tannery's results on the local behaviour of the general solution of (3.4) as exposed on pp. 139–140 of Tannery (1875). However, as we mentioned in Sect. 4, after the outcome of the Grand Prix, and the stress induced by Stickleberger's controversy, Casorati avoided any publication somehow related to Casorati (1880b).

With Pincherle's note at his disposal, Casorati changed his mind again. On the envelope containing Letter 8, he wrote:

On going to Modena, I have to take Tannery [memoir] with me, to tell Pincherle that it would be a good thing that the note to be published should contain the correction of pp. 132–133 of Tannery's [memoir]. Without this or without something more, I would not publish the Note.[28]

Casorati and Pincherle met shortly in Modena, not far from Bologna where Pincherle taught, at the beginning of 1886, and it is clear from Letter 9 that Pincherle followed Casorati's advice, though he was unable to find errors on the pages indicated by Casorati, apart from a trivial misprint. Casorati remarked (Letter 10) that he had in mind something more serious than a typo. To understand his doubts, we have to summarize the content of (Tannery 1875, pp. 132–133), where Tannery aimed at obtaining the linear differential equation satisfied by the roots of an algebraic equation $f(x, y) = 0$, f being a polynomial of the mth degree in y, with the coefficient of y^m a constant, for simplicity. The m roots of this equation are functions of x that, when x moves around a singular point, are simply shuffled so that by Tannery's theorem they have to satisfy a linear differential equation of order m at most, the order being strictly less than m when the roots of $f(x, y) = 0$ satisfy a linear relation with constant coefficients. To find this differential equation Tannery eliminated y between the equations

$$f(x, y) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

thus obtaining

$$\varphi(x) = f(x, y)A(x, y) + \frac{\partial f}{\partial y}B(x, y) = 0,$$

for suitable polynomials A and B of degree $m-1$ and $m-2$ in y, respectively. By taking y as a function of x, the implicit function theorem yields

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{B\frac{\partial f}{\partial x}}{B\frac{\partial f}{\partial y}} = -\frac{B\frac{\partial f}{\partial x}}{Af + B\frac{\partial f}{\partial y}} = -\frac{B\frac{\partial f}{\partial x}}{\varphi(x)} = 0$$

where $f(x, y) = 0$ has been used to write $B\frac{\partial f}{\partial y} = Af + B\frac{\partial f}{\partial y}$ and to arrange $B\frac{\partial f}{\partial x}$ so that it contains y up to the power $m-1$. Iteration of the procedure makes it possible to conclude that

$$\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = \frac{P_k(x, y)}{[\varphi(x)]^k} \quad \forall k = 1, \ldots, m \tag{5.2}$$

[28] Andando a Modena, portar meco il Tannery, per dire al Pincherle che starebbe bene far sì che la Nota a pubblicarsi contenesse la correzione delle pp. 132–133 del Tannery. Senza di ciò o di qualche altra cosa, io non pubblicherei la Nota.

where all the polynomials $P_k(x, y)$ contain y up to the power $m - 1$. From equation (5.2) Tannery concluded that *clearly* (*visiblement*) the following equation is satisfied by $y(x)$:

$$\frac{\mathrm{d}^m y}{\mathrm{d}x^m} + \sum_{k=1}^{m} \frac{Q_k(x)}{[\varphi(x)]^k} \frac{\mathrm{d}^{m-k} y}{\mathrm{d}x^{m-k}} = 0, \tag{5.3}$$

where Q_k are polynomials in x only, since all the powers of y from y^2 to y^{m-1} can be eliminated. Casorati criticized this argument by producing a counterexample, in the case $m = 2$, by setting

$$f(x, y) = ay^2 + by + c$$

where a, b, and c are polynomials in x. Hence, by standard manipulations he wrote

$$\varphi(x) = ac - b^2 = (ay^2 + 2by + c) \times a + (2ay + 2b)\left(-\frac{a}{2}y - \frac{b}{2}\right)$$

whence, following Tannery's recipe he obtained

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\alpha y + \beta}{a\varphi(x)}, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\gamma y + \delta}{a^2 \varphi^2}$$

where $\alpha, \beta, \gamma, \delta$ are polynomial functions of a, b, c, and of their first and second derivatives with respect to x. If the terms not containing y are eliminated, the equation

$$\beta \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{\delta}{a\varphi} \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\alpha\delta - \beta\gamma}{a^2\varphi^2} y = 0$$

arises where, contrary to Tannery's claim, the coefficient β fails to be a constant.

Actually, Casorati had had the same doubt a few years before in 1879, when he studied Tannery's memoir (p. 1286 of his unpublished mathematical diary). We note that Casorati's example concerns the case $m = 2$ for which there are no powers of y with exponent larger than 1 in (5.2). Essentially, to keep the order of the equation equal to 2 he eliminated from the expressions of $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$ the terms independent of y. In Pincherle's *Memorie e Saggi* for the year 1891, Pincherle went back to Tannery's method to obtain a differential equation satisfied by an algebraic function. He repeated all the passages up to (5.2) and then he remarked that:

It is possible to eliminate $y^2, y^3, \ldots, y^{m-1}$ among the m equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{P_1}{\varphi(x)}, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{P_2(x, y)}{[\varphi(x)]^2}, \ldots, \frac{\mathrm{d}^{m-1} y}{\mathrm{d}x^{m-1}} = \frac{P_{m-1}(x, y)}{[\varphi(x)]^{m-1}},$$

whence an equation like

$$\varphi(x)^{m-1} y^{(m-1)} + a_1 \varphi(x)^{m-2} y^{(m-2)} + \cdots + a_m y = R.$$

is obtained. To eliminate R one has to further differentiate this equation.[29]
The outcome, however, is similar to Casorati's remark because, after differentiations to get rid of the polynomial $R(x)$ one is led to an equation with a constant coefficient in front of the highest derivative, as in Tannery (1875), but now the structure of the denominators in the remaining terms differs from that suggested by Tannery. More importantly, the order of the differential equation is larger than m.

In Letter 9 Pincherle attempted to prove a generalization of Tannery's result: any algebraic function satisfies a linear differential equation but, again, his results eluded the question asked by Casorati that concerned the *structure* of the coefficients.

In my opinion, the example proposed by Casorati played an important rôle in Pincherle's decision not to publish his note: in fact, he could not find the error in Tannery's paper, and looking for other interesting applications of Casorati's method would have probably taken him too much time. Moreover, Pincherle was soon urged to write another short note (Pincherle 1886a), and he again resorted to Casorati (Letter 11) to have it published, if possible. In this note, Pincherle introduced the function

$$f(x) := \int_{(\varrho)} \varphi(y) y^{x-1} \mathrm{d}y \tag{5.4}$$

where $\varphi(y)$ is an analytic function without singularities along the curve ϱ of the complex plane. By performing an integration by parts, and selecting ϱ so that the boundary terms disappear, he obtained

$$-(x-1) f(x-1) = \int_{(\varrho)} \varphi'(y) y^{x-1} \mathrm{d}y$$

and, by iterating the procedure, he arrived at

$$\int_{(\varrho)} \varphi^{(r)}(y) y^{\lambda} y^{x-1} \mathrm{d}y = (-1)^r (x+\lambda-1)(x+\lambda-2)\cdots(x+\lambda-r) f(x+\lambda-r).$$

When (5.4) is applied to the linear differential equation

$$\sum_{r=0}^{n} \sum_{\lambda=0}^{m} a_{\lambda,r} y^{\lambda} \varphi^{(r)}(y) = 0,$$

[29] *Fra le m equazioni*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{P_1}{\varphi(x)}, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{P_2(x,y)}{[\varphi(x)]^2}, \ldots, \frac{\mathrm{d}^{m-1} y}{\mathrm{d}x^{m-1}} = \frac{P_{m-1}(x,y)}{[\varphi(x)]^{m-1}}$$

si può eliminare $y^2, y^3, \ldots, y^{m-1}$ *e viene un'equaz.[ione] della forma*

$$\varphi(x)^{m-1} y^{(m-1)} + a_1 \varphi(x)^{m-2} y^{(m-2)} + \cdots + a_m y = R.$$

Per eliminare R bisogna derivare ancora.

then it is transformed into the linear difference equation

$$\sum_{r=0}^{n}\sum_{\lambda=0}^{m}(-1)^r a_{\lambda,r}(x+\lambda-1)(x+\lambda-2)\cdots(x+\lambda-r)f(x+\lambda-r)\theta^{\lambda-r}f(x)=0$$

that Pincherle wrote down by resorting to Casorati's operator θ. Pincherle noted that, if the boundary terms do not drop out in (5.4), then the latter transformation embodies that adopted by Mellin (1886, pp. 79–80), in which he obtained a correspondence between fundamental sets of solutions of a linear homogeneous differential equation and of a functional equation. Actually, the publication of Mellin (1886) prompted Pincherle to have (Pincherle 1886a) published in order to secure his priority (Letter 12).

For a while, the interest in Casorati (1880b) remained quiescent, but in the next section, we will see how Pincherle expounded and then extended Casorati's approach to finite differences in his papers written after Casorati's death.

6 The influence of (Casorati 1880b) on Pincherle's work

Pincherle's interest in finite-difference equations and, more generally, in functional equations remained alive during all his long career. In a conference held in 1926, and published in Pincherle (1926), considering the lack of attention to the calculus of finite differences in textbooks of mathematical analysis, he wrote:

The development of differential and integral calculus, with its numberless applications; the supremacy of the concept of limit and the criticism that, raised by the need to make more precise this concept, led to an unavoidable revision of the foundations of calculus; the interest originated by the seminal results of the theory of functions of one real variable on one hand, and of analytic functions, on the other hand, and the appealing questions posed by the theory of aggregates [set theory]: all this left little room to a branch of science in which the concept of limit is almost foreign at a first sight, and also those branches of analysis in which that concept plays a minor rôle, I mean higher algebra, Galois's theory, the study of finite groups, did not seem to offer any relation with the calculus of finite differences.

However, a more careful examination not only suggests that this branch of science is still worthy of attention, but [it suggests] also the expectation that, thanks to the new paths recently opened, it can tend to a more brilliant future[30] (Pincherle 1926, pp. 233–234).

[30] Lo sviluppo del calcolo differenziale ed integrale, colle infinite sue applicazioni; il signoreggiare del concetto di limite e le critiche, che, nate dalla necessità di precisare questo concetto, conducevano alla necessaria revisione dei fondamenti del Calcolo; l'interesse destato dai fecondi risultati della teoria delle funzioni di variabile reale d'una parte, delle analitiche dall'altra, e dalle attraenti questioni della teoria degli aggregati: tutto ciò lasciava poco spazio ad un capitolo della scienza in cui il concetto di limite appariva a prima vista pressochè estraneo, mentre anche quei rami dell'analisi in cui quel concetto ha minore parte, intendo dire l'algebra superiore, la teoria di Galois, gli studi sui gruppi finiti, non parevano offrire, col calcolo delle differenze finite, alcun addentellato. Però, un esame più attento fa nascere non solo il pensiero che questo ramo della scienza sia ancora degno di interesse, ma la fiducia che, in grazia delle nuove vie che di recente gli vennero aperte, possa giustamente aspirare ad un più brillante avvenire.

In my opinion, Casorati's views on the calculus of finite differences belong to these *new paths* that were given to this discipline. In this section, I will consider only those of Pincherle's results that can be related to Casorati (1880b): essentially, they were published in the last decade of the nineteenth century and they form the backbone of Chap. X of the monograph on functional calculus written by Pincherle and Ugo Amaldi in 1901. It should be clear that Pincherle also contributed to other areas of finite differences. Notably, the close relation between difference and differential equations that had been revealed new aspects after the publication of Poincaré (1885) soon captured Pincherle's attention, leading him to the concept of a *distinguished integral* of a difference equation (Pincherle 1892): for the relations between Poincaré and Pincherle I refer the reader to Chap. VI of Bottazzini (1994) and to Dugac (1989), where a couple of letters between the two mathematicians have been published. Moreover, it is from finite-difference equations that Pincherle was led to generalize the algorithm of continued fractions (see, for instance, Pincherle 1890), a topic on which he had a correspondence with Charles Hermite.

Turning back to the θ operator, we start from the Academic year 1893–1894, when Pincherle gave a course on hypergeometric functions, a long account of which was then printed in Pincherle (1894). There, for pedagogical reasons, Pincherle inserted several sections devoted to difference and differential equations where he gave a detailed exposition of Casorati (1880b), reproducing part of the results and qualifying Casorati's method as *extremely ingenious* (genialissimo) because

[it] offers an unrivalled simplicity and an interest that is not only scientific but also pedagogical[31] (Pincherle 1894, pp. 210–211 ≡ Pincherle 1954, p. 275).

In Chap. 2 of Pincherle (1894), devoted to linear difference equations, Pincherle emphasized the importance of the theorem on the Θ determinant, of which he gave a proof that coincides, apart from a slight change of notation, with the original one proposed by Casorati. Then, in Chap. IV (§27) Pincherle reviewed the content of Chaper 2 of Casorati (1880b) where applications to Fuchs's theory of differential equations had been illustrated. It is interesting to remark that whenever Pincherle considered the distinctions of solutions in subgroups, he never mentioned (Casorati 1881a), neither in Pincherle (1894) nor in Pincherle and Amaldi (1901) whose Chap. XII is devoted to this subject.

If in Pincherle (1894), there are no original results on the θ operator, in 1895 Pincherle published a series of papers (Pincherle 1895a, b, c) where the *operatorial* nature of θ plays a major rôle. In these papers, Pincherle laid the basis of his functional calculus, based upon the properties of linear operators (*operazioni distributive*) acting on a set of *analytic* functions, or to a properly delimited subset.

In Pincherle (1895a) Pincherle, soon after the definition of θ, quoted Casorati to emphasize the large spectrum of distributivity endowed by this operator, recalling Eqs. (3.2) and (3.3). Then, in §2, he started to study the properties of the set of operators

$$F := a_0(x) + a_1(x)\theta + a_2(x)\theta^2 + \cdots + a_m(x)\theta^m, \tag{6.1}$$

[31] Offre una semplicità senza pari ed interesse scientifico non meno che didattico.

where $\{a_k(x)\}$ are given functions. Pincherle called operators like (6.1) linear difference forms (*forme lineari alle differenze*) of the mth order. They map a function $f(x)$ into

$$f \mapsto F(f) := a_0(x)f(x) + a_1(x)f(x+1) + a_2(x)f(x+2) + \cdots + a_m(x)f(x+m).$$

Up to §20, Pincherle (1895a) is devoted to forms of finite order and there Pincherle did not insist on the nature of the functions $a_i(x)$: $a_0(x)$ is only supposed not to vanish at the points $x+k$, so that after a rescaling, it can be set equal to unity. On the contrary, in the second part of Pincherle (1895a), devoted to series in θ, he sketched the elements of the systematic presentation of the theory contained in Pincherle and Amaldi (1901) where much attention is devoted to characterize a suitable general functional space $\mathcal{D}$ where θ is defined. Pincherle considered a region $\boldsymbol{a}_0$ of the complex plane lying in the strip $\Re(x) \in [0, 1]$ and then he defined the region

$$\boldsymbol{a} := \bigcup_{k=-\infty}^{\infty} \boldsymbol{a}_k$$

where $\boldsymbol{a}_k$ is obtained by translating $\boldsymbol{a}_0$ of k units along the real axis. In any $\boldsymbol{a}_k$ Pincherle considered a meromorphic function $\alpha_k(x)$ and defined another meromorphic function $\alpha(x)$ in $\boldsymbol{a}$ such that

$$\alpha(x) = \alpha_k(x) \quad \text{if } x \in \boldsymbol{a}_k :$$

the set $\mathcal{D}$ of all meromorphic functions defined in this way is, by construction, an invariant set for the operator θ and Pincherle considered forms (6.1) with coefficients belonging to this set. [In particular, if $\alpha_k(x)$ is independent of k, we have linear difference forms with constant (periodic) coefficients].

Pincherle was careful about the algebraic properties of the sets of operators he considered, not only those in the class (6.1) but, in general, throughout his functional calculus. As an early example of this attitude, we can quote the papers (Pincherle 1884b, 1885a) that were markedly influenced by the works of Kronecker (1882) and Dedekind and Weber (1882). To give a flavour of his attitude in the present context, we now look at: (a) the extension of the *Ruffini rule* to form the quotient and the remainder obtained when a form F is divided by the first-order form $E_a := \theta - a(x)$; (b) a criterion to decide when a form B divides another form A. We follow the presentation as given in Pincherle and Amaldi (1901), which is more complete than the original treatment in Pincherle (1895a).

After defining the non commutative product of a form A of order m time a form B of order n as the form AB mapping $f(x) \mapsto A[B(f(x))]$, Pincherle aimed at extending the classical Euclidean algorithm to linear difference forms. By assuming $n \leq m$ he looked for a third form Γ, of order $m-n$, such that

$$A - \Gamma B$$

is of order $n-1$, at most. By setting

$$A := \sum_{h=0}^{m} \alpha_h(x)\theta^h, \quad B := \sum_{k=0}^{n} \beta_k(x)\theta^k, \quad \text{and} \quad \Gamma := \sum_{i=0}^{m-n} \gamma_i(x)\theta^i,$$

the requirement on Γ yields the following linear system for the $m-n+1$ coefficients $\gamma_i(x)$:

$$\begin{cases} \alpha_m(x) - \gamma_{m-n}(x)\beta_n(x+m-n) = 0 \\ \alpha_{m-1}(x) - \gamma_{m-n}(x)\beta_{n-1}(x+m-n) - \gamma_{m-n-1}(x)\beta_n(x+m-n-1) = 0 \\ \cdots \\ \alpha_{m-r}(x) - \sum_{j=0}^{r} \gamma_{m-n-j}(x)\beta_{n-r+j}(x+m-n-j) = 0 \\ \cdots \end{cases} \tag{6.2}$$

that can be solved provided that

$$\beta_n(x)\beta_n(x+1)\cdots\beta_n(x+m-n-1)\beta_n(x+m-n) \neq 0, \tag{6.3}$$

a condition automatically satisfied in Pincherle (1895a) where $\beta_n(x) \equiv 1$. When (6.3) is satisfied, the form Γ is uniquely determined, and it can be called the *quotient* of the forms A and B. When the remainder vanishes one has $A = \Gamma B$ so that B *divides* A. When $B = \theta - \gamma$ the conditions (6.2) become

$$\begin{cases} \beta_{m-1}(x) = \alpha_m(x) \\ \beta_{m-2}(x) = \alpha_{m-1}(x) + \alpha_m(x)\gamma(x+m-1) \\ \beta_{m-3}(x) = \alpha_{m-2}(x) + \alpha_{m-1}(x)\gamma(x+m-2) + \alpha_m(x)\gamma(x+m-1)\gamma(x+m-2) \\ \cdots \\ \beta_0(x) = \alpha_1(x) + \alpha_1(x)\gamma(x+1) + \cdots + \alpha_m(x)\gamma(x+1)\cdots\gamma(x+m-1), \end{cases} \tag{6.4}$$

that is the analogue of the classical Ruffini's rule to obtain the coefficients of the quotient between a polynomial and binomial $x-a$.

In terms of higher algebra, Pincherle defined irreducibility of a linear difference form A with coefficients in a set Ω that is a field $\mathcal{D}$ say, (*campo di razionalità*, i.e the Italian term for *Rationalitätsbereich*): A is irreducible in Ω if there is no linear difference form with coefficients in Ω that divides A (Pincherle 1895a, §6, Pincherle and Amaldi 1901, §295). In §296 of Pincherle and Amaldi (1901), the analogue of a lemma employed repeatedly by Galois is stated:

Let Φ be a form of the order n, irreducible in a certain field, and let Φ_1 another form of the order $m \geq n$ that shares a root φ with Φ, and whose coefficients belong to the same field. Then, Φ_1 is necessarily divided by Φ.[32]

[32] Sia una forma Φ, d'ordine n, irriducibile in un determinato campo di razionalità, e sia Φ_1 una forma di ordine $m \geq n$, i cui coefficienti appartengono a quel campo, e che abbia una radice φ comune con Φ. La Φ_1 è allora necessariamente divisibile per Φ.

We recall that in Pincherle's language a root of a linear difference form (6.1) is an element $\omega(x)$ of $\mathcal{D}$ such that $F(\omega(x)) = 0$. Finally, in §297 of Pincherle and Amaldi (1901) the dependence of irreducibility of a form on its domain is mentioned, together with the fact that irreducible forms can be made reducible by adjoining suitable functions to the field where the coefficients lie.

We now turn to the second part of Pincherle (1895a) where Pincherle introduced *series* in θ. There he clearly stated a difference between the traditional way of conceiving symbolic calculus and his own. As for series like

$$\sum_{n=0}^{\infty} \alpha_n(x)\theta^n \tag{6.5}$$

he remarked (Pincherle 1895a, pp. 107–108) that

These series (...) have been conceived more as a concise style of symbolic representation rather than a means apt to yield the element of a genuine theory: here, we aim at showing how, by placing suitable constraints on the functions that are acted upon by these series, they can be taken as the basis of a truly rigorous calculus.[33]

As a consequence of this programme, Pincherle explored sufficient conditions for the expressions (6.5) to make sense. He considered functions $\alpha_n(x)$ that belong to $\mathcal{D}$, and defined the *functional field of convergence* (*campo funzionale di convergenza*) of (6.5) referred to the point $x_0 \in \boldsymbol{a}$ as the subset of $\mathcal{D}$ such that (6.5) converges absolutely in a neighbourhood of x_0 proving that any series (6.5) admits a non-empty field of convergence. As an example of an elementary condition guaranteeing the convergence of (6.5) Pincherle (1895a, p. 109) (see also §311 of Pincherle and Amaldi 1901, p. 250) stated that:

Let $\{\lambda_n(x)\}$ be a sequence of functions that are finite in the set $\boldsymbol{a}_0$. If the series

$$\sum_{n=0}^{\infty} \alpha_n(x)\lambda_n(x)$$

is absolutely convergent, and if the element $\varphi(x)$ is such that, $\forall x \in \boldsymbol{a}_0$,

$$\varphi(x+n) \leq \gamma(x)|\lambda_n(x)|$$

for $n \geq n_0$, where $\gamma(x)$ is both positive and finite in a neighbourhood of a point $x = x_0 \in \boldsymbol{a}_0$, then $\varphi(x)$ belongs to the functional field of convergence of (6.5), *relative to the point x_0.*

Let us now turn attention to the concept of *functional derivative A'* of a linear operator A defined by Pincherle (1895b) as

$$A'(\varphi) := A(x\varphi) - xA(\varphi), \tag{6.6}$$

[33] Tali serie (...) sono state considerate più come un modo conciso di rappresentazione simbolica che come atte fornire l'elemento di una vera teoria: noi qui ci proponiamo di mostrare come, assoggettando a convenienti limitazioni le funzioni su cui si opera, queste serie siano suscettibili di essere prese a fondamento di un calcolo perfettamente rigoroso.

provided that $\varphi(x)$ and $x\varphi(x)$ belong to the domain of A. The Casorati operator θ coincides with its derivative or, equivalently, it satisfies

$$\theta' = \theta$$

and so it plays a distinguished rôle, analogue to the exponential function in classical calculus. Actually, in Pincherle (1895b), Pincherle showed that an operator A satisfying $A' = A$ has the form $A = M\theta$, where M is the multiplication by a suitable function. In fact, by setting $\varepsilon(x) = A(1)$ and by using (6.6), the requirement $A' = A$ implies that

$$A(x) = A(x \cdot 1) = (x+1)\varepsilon(x), \ldots, A(x^n) = (x+1)^n \varepsilon(x)$$

so that, *if* $\varphi(x)$ belongs to the set $\mathcal{S}^1$ of functions whose series expansions centered at the origin have radius of convergence larger than 1, then

$$A(\varphi(x)) = \varepsilon(x)\varphi(x+1) = \varepsilon(x)\theta.$$

As we have already seen, Casorati emphasized that the operator θ is distributive also with respect to other operations besides sum. Pincherle deepened this property in (Pincherle 1895b, c) where he introduced the operation of ***substitution*** (sostituzione), denoted with S or S_a, that amounts at replacing $\varphi(x)$ with $\varphi(a(x))$, where $a(x)$ is an assigned function. By definition, the operation S_a satisfies

$$S(\varphi\psi) = S(\varphi)S(\psi) \quad \text{and} \quad S\left(\frac{\varphi}{\psi}\right) = \frac{S(\varphi)}{S(\psi)}$$

like the operator θ that clearly corresponds to the particular choice $a(x) = x + 1$ (see also §122 of Pincherle and Amaldi 1901). In fact, it satisfies the first order *symbolic* differential equation

$$S_a' = (a(x) - x)S_a \tag{6.7}$$

which, as we have just seen for the operator θ, is a characteristic property of substitutions since any solution of (6.7) is of the form MS, where S is a substitution, and M is the multiplication by a certain function. Proceeding in parallel with Fuchs's theory of linear differential equations, Pincherle considered nth order symbolic differential equations

$$\lambda_0(x)A^{(n)} + \lambda_1(x)A^{(n-1)} + \cdots + \lambda_n(x)A = 0 \tag{6.8}$$

and he proved that n operators $A_1, \ldots, A_n$ solving (6.8) are linearly dependent if and only if the determinant

$$\Delta := \begin{vmatrix} A_1 & A_2 & \ldots & A_n \\ A_1' & A_2' & \ldots & A_n' \\ \ldots & \ldots & \ldots & \ldots \\ A_1^{(n-1)} & A_2^{(n-1)} & \ldots & A_n^{(n-1)} \end{vmatrix} \tag{6.9}$$

is equal to zero (see also Chap. VI of Grévy 1894 for an analogue statement). Then he proved that it is possible to build a fundamental system of solutions of (6.8) starting from a particular set of substitution operators, defined as S_{a_n}, where the n functions $a_1(x), \ldots, a_n(x)$ are the roots of the characteristic equation

$$f(z) := \lambda_0(x)(z-x)^n + \lambda_1(x)(z-x)^{n-1} + \cdots + \lambda_{n-1}(x)(z-x) + \lambda_n(x) = 0.$$

Chapter XIV of Pincherle and Amaldi (1901) is devoted to a detailed study of the expressions

$$\sum_{k=0}^{m} \alpha_k(x) S_\mu^k$$

called linear substitution forms (*forme lineari alle sostituzioni*). There, a parallel of the theory to the one already illustrated for linear difference forms is drawn by referring to results by Grévy (1894).

Given that, besides the Wronskian and the Casoratian determinants, there is also (6.9), the question arises *of looking for larger classes of operations for which a theorem, analogous to that of the Wronskian, holds*[34] (Pincherle and Amaldi 1901, p. 430).

Chapter XV of Pincherle and Amaldi (1901) is devoted to answering this question by looking at the common structure of the "Wronskian-like" theorems: to find out a necessary and sufficient condition for n regular enough functions $\varphi_1, \ldots, \varphi_n$ to satisfy a relation

$$c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n = 0$$

where not all the coefficients c_i vanish. The coefficients c_i are *constant* with respect to a suitable *linear* operator A: if $A = D = \frac{\mathrm{d}}{\mathrm{d}x}$ then c_i are really constants; if $A = \theta$ then c_i are periodic functions with period equal to 1; if $A = S_\mu$, then $c_i(x) = c_i(\mu(x))$. Pincherle and Amaldi note that the crucial steps to obtain a Wronskian theorem are: (a) to show that

$$A(c_i) = 0$$

[34] Ricercare se esistano classi più ampie di operazioni, per le quali sussista un teorema analogo a quello del Wronskiano.

and (*b*) to show that a *multiplication theorem* holds, that is, a particular relation among $A(\varphi\psi)$, $A(\varphi)$ and $A(\psi)$ whose meaning, however, slightly varies on passing from D to θ and to S_μ:

$$D(\varphi\psi) = D(\varphi)\psi + \varphi D(\psi) \quad \theta(\varphi\psi) = \theta(\varphi)\theta(\psi) \quad S_\mu(\varphi\psi) = S_\mu(\varphi)S_\mu(\psi).$$

As a consequence, Pincherle and Amaldi restricted their attention to single-valued linear operators such that

$$\begin{aligned} A(\varphi\psi) &= \alpha_{00}\varphi\psi + \alpha_{01}\varphi A(\psi) + \alpha_{10}\psi A(\varphi) + \alpha_{11}A(\varphi)A(\psi) \\ &\quad + \beta_1\varphi A(\varphi) + \beta_2\psi A(\psi), \end{aligned} \tag{6.10}$$

where α_{ij} and β_i are functions that are not completely independent. In fact, by imposing that $A(\varphi\psi) = A(\psi\varphi)$ and then testing (6.10) on $\psi = 1$ they got the simpler expression

$$A(\varphi\psi) = \xi(\alpha\xi - 1)\varphi\psi + (1 - \alpha\xi)[\varphi A(\psi) + \psi A(\varphi)] + \alpha A(\varphi)A(\psi) \tag{6.11}$$

where $\alpha := \alpha_{11}$ and $\xi := A(1)$ are independent from one another. If $\alpha = 0$, the most general linear operator satisfying (6.11) is

$$A = (\xi_1 - \xi x)D + M_\xi,$$

where $\xi_1 = A(x)$ and M_ξ represents pointwise multiplication by ξ. On the other hand, if $\alpha \neq 0$ such a rôle is played by

$$A = \frac{1}{\alpha}S_\mu + M_{\xi - \frac{1}{\alpha}}:$$

hence, linear differential forms and substitution forms exhaust the class of linear operators satisfying (6.11). For this class of operations, an analogue of the Θ-determinant holds (see Pincherle and Amaldi 1901, §455). With Pincherle's work the results of Casorati (1880b) found their natural place within functional calculus.

7 Conclusions

The main obituaries of Pincherle, namely Amaldi (1937) and Tonelli (1937) refer to the influence exerted by Casorati on some aspects of Pincherle's research. Resorting to unpublished documents kept in the Casorati *Nachlass*, we have examined some aspects of their relationship. In particular, we have found an unpublished note by Pincherle centered on a paper by Tannery (1875) where Casorati's θ operator is used in the new interpretation envisaged by Casorati himself to export finite-difference methods in the realm of complex analysis. Casorati's paper (1880b) had the most influence on Pincherle, as we showed in Sect. 6 through an overview of Pincherle's work on this topic, culminating in the publication of his monograph (Pincherle and Amaldi 1901). Moreover, other documents of the *Nachlass* made it possible to investigate the complex history behind Casorati's participation in the 1880 *Grand Prix de Sciences Mathématiques* and to add some detail on the controversy he had with Stickelberger. In particular, we have seen that Stickelberger claimed that the theorem on the now-called Casorati's determinant had already been proved by Christoffel more than 20 years before the publication of Casorati (1880b), though in the context of classical finite-difference calculus. We plan to deepen other aspects of Pincherle's early career in future publications.

Acknowledgments As already remarked in the text, most of the original documents are preserved in the Casorati *Nachlass* at Pavia. It is a pleasure to thank once more Prof. Alberto Gabba for his kindness in disclosing the *Nachlass* to me: without him, this paper would not have been written. I would like to thank also prof. Umberto Bottazzini, prof. Salvatore Cohen, and prof. Otto Liess for useful information on Pincherle's unpublished *Ricerche e Saggi* preserved in the historical section of the library of the Department of Mathematics in Bologna; dr. Anna Biavati and dr. Claudio Cappelletti who helped me during my visit at this library. I also acknowledge the kind assistance of dr. Alessandra Baretta and dr. Maria Piera Milani during my visits at the historical archive of the University of Pavia. Finally, I express my gratitude to prof. Mario Ferrari for his constant encouragement.

Appendix 1: Original letters concerning the 1880 Grand Prix des Mathématiques

Letter 1: Casorati to Bertrand

Pavia, 27 dicembre 1879

Caro ed illustre Amico

Un mio compaesano, discontinuo ma passionato cultore dei nostri studî, è riuscito a scoperte analitiche di grande momento, segnatamente efficaci nelle moderne ricerche basate sulla variabilità complessa. (...) Egli ha già potuto impossessarsi dei lavori [che sono] stati fatti in Germania, e vi ha subito introdotto belle ed importanti semplificazioni, così da avere la certezza di figurare con onore tra concorrenti anche fortunatissimi. Ma poiché i suoi concepimenti hanno un'importanza generale, e ripeto grande, e non subordinata al tema su citato, così egli, giustamente nell'interesse dei suoi studî e nel suo, non vorrebbe tenerli segreti sino alla fine del 1880. Però d'altra parte, egli vorrebbe comportarsi in modo da conseguire la grande soddisfazione di un premio dell'Accademia di Parigi, la quale vedrà con piacere uscire in luce le nuove idee per mezzo delle sue pubblicazioni.[35] Interessatissimo pel mio compaesano, pensai di ricorrere in segreta confidenza a voi per consiglio; fidente nell'interessamento che voi avete sempre altamente mostrato per i nostri studî, e nella benevolenza che avete sempre accordato a chi vi scrive. Io dunque chiedo non al Segretario Perpetuo, ma

[35] Casorati added the following comment, that was not included in the letter: *Qualunque sia per essere l'esito del concorso, deploro di aver scritto questi pensieri. Essi erano naturali quando un fatale inganno mi fece credere commutativi due simboli d'operazione e mi condusse quindi a conseguenze d'incredibile importanza. Ma non dovevo esternarli subito, come feci, ad uno straniero; dovevo lasciar passare almeno qualche giorno, secondo le buone regole, che non si violano impunemente. Che cosa potrebbe aver detto Bertrand, se avesse letta bene la mia lettera? E perché, dopo riconosciuto il mio traviamento, non gli scrissi per avvertirvelo? Ciò che rimaneva di accertato nelle mie ricerche era pur sempre meritevole, a mio giudizio, di molta considerazione.*

all'amico benevolo, come dovrebbe il mio compaesano comportarsi? Se stampasse anonime una parte delle cose sue, potrebbero essere ancora valutate in un manoscritto presentato al Concorso? [...]

Il vostro devotissimo
F. Casorati
della Università di Pavia

Letter 2: Casorati to Bertrand

Pavia, 28 genn. 80

Ch. mo Prof. (Bertrand)

(...) Questo silenzio, ch'io credo non meritato, ha già naturalmente prodotto il suo triste effetto, di scoraggiare chi forse meritava incoraggiamento, di dissuaderlo dal continuare in quelle applicazioni che aveva cominciato felicemente, con quell'ardore che guida sempre a qualche risultato non volgare.

Ho sempre creduto che, come la grande maggioranza delle persone colte guarda a Parigi con ammirazione ed affetto, come alla gran madre d'ogni progresso dell'umanità, così voi Parigini, e voi particolarmente membri dell'Istituto, dobbiate guardare con benevolo affetto tutti gli operai del progresso, più o meno distinti che sieno, giovani o no, purché onesti, purché sinceramente intenti al meglio dell'umanità.

Ma non aggiungerò altro, perché, rattristato come sono, vi riuscirei troppo molesto.

Obbl.$^{\text{mo}}$ vostro
F. Casorati
dell'Università di Pavia

Letter 3: Bertrand to Casorati

Paris, 31 janvier 1880

Très cher monsieur

(...) Je ne puis malhereusement engagé à l'avance ni l'académie, ni la Commission, qui n'est pas même nommée et qui sera souveraine. Elle fera tout, j'en ai la convinction, pour courronner une mémoire importante et les questions du forme, celles même du publications antérieure, n'ont pas l'habitude du prévaloir sur le mérite du fond. Je peut vous citer l'example du Kummer qui a obtenu le prix proponi pour le théorème de Fermat, *sans avoir concouru*, son mémoir publié en langue allemande depuis plusieurs annés, et traduit en Francais depuis plus d'un an, a été jugé digne du prix et l'a obtenu au grand étonnement du l'auteur qui n'y songeait pas. Je conseillerai donc à votre ami d'imprimer en langue italienne les principaux résultats et d'envoyer un mémoire détaillé qui, j'en ai la convinction, sera lu et jugé avec le même empressement que si tout etait inédit. Le secret qui doit être gardé sur le nom de l'auteur, sera, il est vrai, violé: un juge formaliste pourrait y voir une difficulté et je n'ai pas le droit *d'affirmer* que cela n'arrivera pas, mais je ne pense pas, d'après mon habitudes et les ideés que je connais à mon Confrères, que cela soit à craindre.

Je ne veux pas vous cacher que le sujet a été proposé pour moi et que j'avais pensé en le rédigeant à la possibilité de récompenser les travaux de Mr. Laguerre sur les équations differentiélles lineaires et particulièrement sur les invariants. Mr. Laguerre lui même ignore d'ailleurs cette arrière pensée dont je n'ai pas fait part à mes Confrères, mais si lui vient l'idée de concourir, il sera, vous le voyer, dans le même cas que votre ami. (...)

J. Bertrand

Letter 4: Casorati to Bertrand

Pavia, 6 febbraio 1880

Carissimo Signore (Bertrand)

(...) Ora vi faccio una confessione. Il mio compaesano è quegli stesso che vi scrive. Ma venendo ormai all'argomento, vi dirò, che, ritornato con lena agli studî dopo un lungo soggiorno sulle Alpi, fui condotto ad interpretare il calcolo delle differenze finite, diretto ed inverso, in modo da farlo diventare un potente ausiliario delle moderne ricerche basate sulla variabilità complessa. Naturalmente, a questa interpretazione aggiunsi alcuni teoremi affatto nuovi. Allorché poi l'utilità di queste mie cose mi parve largamente dimostrata dall'applicazione allo studio, ora tanto in voga, delle equazioni differenziali lineari, pensai che avrei potuto concorrere al premio proposto dalla vostra Accademia su quest'argomento, e mi risolvetti a scrivervi la mia prima lettera.

Però voi avete interpretato questa lettera in un senso che io era lontanissimo dal darle. Dite "Je ne puis malhereusement engagé à l'avance ni l'Académie, ni la Commission, qui n'est pas même nommée et qui sera souveraine." Io lo credo bene! E non avrei mai imaginato [sic!] di pretendere cose ingiuste. Ma tutto ciò sia come non detto. L'essenziale ora si è che la vostra lettera mi ha rialzato e soddisfatto pienamente.

Accettando il vostro consiglio, farò stampare subito negli *Annali di Matematica* di Milano una parte della interpretazione suddetta con l'applicazione per ora alle proprietà fondamentali degli integrali delle equazioni differenziali lineari a coeff.[icienti] monodromi. E non manderò alcun manoscritto; parendomi non necessario giacché mi basta di sapere dalla vostra lettera, che la Comm.[issione] potrebbe anche prendere in considerazione lavori pubblicati per la stampa. Io non avrei nessun gusto di fare concorrenza al sig. Laguerre. Sono abbastanza pago di poter credere che ove il sig. Laguerre non si risolvesse a concorrere, potrebbe essere presa in considerazione anche la Memoria che io adesso pubblicherò, e quelle altre sulle equazioni lineari ancora e su altre ricerche con variabilità complessa che avessi agio di pure redigere in netto e stampare.

Perdonatemi tutto questo disturbo, pur troppo grande per voi certamente accerchiato da mille cose, ed aggradendo l'attestazione della profonda mia stima e del mio affetto, vogliate conservarmi la vostra preziosa benevolenza.

Dev.^{mo} vostro
F. C.

Letter 5: Bertrand to Casorati

Paris, 15 février 1880

Cher monsieur,

J'ai recu votre lettre avec grand plaisir très heureux de voir qu'il ne reste rien du malentendue dont j'ai été, bien sans inténtions, la cause.

Je ne puis, comme je vous j'ai dit, prendre aucun engagement au nom d'une Commission qui n'existe pas encore et vous l'avez parfaitement compris mais je crois de vous avoir laissé prendre pour une habitude de l'académie ce qui a été fait une fois or deux seulement dans des circumstances exceptionelles. Les commissions *peuvent* décerner le prix à des mémoirs imprimé et non envoyé au concours mais elles ne l'ont fait que très rarement, en l'absence de plis régulierement envoyés au concours et digne de disputer le prix et lorsqu'aucune membre de la Commission n'envoquait la lettre des programmes pour s'y opposer.

Si donc vous avez obtenu ....[36] des résultats importants et si votre désir bien naturél est de les faire courronner par l'académie des Sciences de Paris, il est prudent de les lui envoyer et s'ils ont été imprimés par entier, vous pouver citer le pubblications, mais en mettant sous les yeux des commissaires un[37]... complet de votre travail. En procedent autrement, vous ne rendrez pas le succès impossible parce qu'il y a des examples qui sont tout semblables, mais vous en diminuerez singulierement les chances. [...]

J. Bertrand

Letter 6: Casorati to Cremona

Pavia, 25 febb. 1880

Caro Cremona,

Sono scampato anch'io da una grave disgrazia. La mia Eugenia veniva assalita da pleurite con minaccia di tifo, così da metterne per qualche giorno in pericolo la vita. Fortunatamente il tifo non si sviluppò e la pleurite va attenuandosi. Benché sventata, questa minaccia ha tanto prostrato le mie forze che non sono peranco capace di applicarmi per un'ora allo studio. L'essersi ammalata in Milano, e il dovervisi naturalmente fermare ancora molto tempo fa sì che non posso più vedermi nella solitaria casa di Pavia.

Al principio di questo mese avevo fatto comunicar una mia lettera all'Ist.[ituto] Lombardo alla quale tenevo assai. Ma nella seduta ultima mi mancava ogni lena per spiegare come avrei desiderato ai colleghi mat.[ematici] le cose mie. Ora poi lo sciopero degli operai tipografi impedisce la stampa dei Rendiconti. Perciò avrei piacere di fare ai Lincei la Comunicazione suddetta, e per mezzo tuo, che sai sostenere così bene i tuoi clienti.

La comunicazione ha per titolo: "Il calcolo delle differenze finite interpretato ed accresciuto di nuovi teoremi a sussidio principalmente delle odierne ricerche basate sulla variabilità complessa".

[36] It was hard to decode Bertrand's manuscript here. It seems to me: *comme j'en suis content*

[37] Same remark as before. It looks like: *essai*.

In virtù della medesima moltissime ricerche che costarono gran fatica a matematici valenti diventano ovvie traduzioni di quanto già da tempo è stato fatto nel calcolo delle differenze finite. In questa prima Comunicazione dimostro la importanza della mia interpretazione facendone applicazione a due argomenti: allo studio delle equazioni algebriche a coefficienti monodromi; ed a quello ora di moda delle eq.[uazioni] diff.[erenziali] lineari.

Potrai vedere nella 1ª applicazione come riducasi a poche linee una metà delle celebri e lunghe "Recherches sur les fonctions alg.[ébriques] del sig. Puiseux" e nella 2ª come scendano immediatamente dalle note formole d'integrazione delle equazioni alle differenze finite lineari e coi coeff.[icienti] costanti le proprietà e le espressioni degli integrali delle equaz.[ioni] differenziali lineari con coefficienti monodromi che costarono tanta fatica al sig. Fuchs (1ª sua Memoria) e diedero da fare ai signori Thomé Frobenius Hamburger, Jurgens.

Quel teorema sul determ.[inante] Θ parvemi veramente bello e di grande uso. Utile pure il determ.[inante] H.

Ero molto soddisfatto di questa interpretazione che sembrami di poter qualificare come una scoperta analitica di grande fecondità, e m'avviavo a farne molte altre applicazioni, quando la malattia dell'Eugenia venne a spaventarmi ed istupidirmi.

Se non ti dispiace di fare per me questa comunicazione nella prossima seduta Lincea ti manderò il manoscritto, che in parte feci già leggere a Beltrami.

Non t'avrei dato questo disturbo se il ministero t'avesse trascinato già sino d'ora, come si cominciava a dire, nel suo vortice.

Mille affettuosi e rispettosi saluti alla sig. Elisa ed alle tue care figlie, ed a te una cordiale stretta di mano.

Tuo F. C.

Letter 7: Casorati to Bertrand

Pavia, 3 aprile 80

Car.^{mo} ed. illustr. Signore

(......) Mi sarei deciso di ritirare il manoscritto destinato agli Annali, ampliarlo e riscriverlo meglio che potrò, e spedirlo al Segretario della vostra Accad.[emia] entro il maggio, conformemente alle riflessioni della vostra ultima lettera. Ma devo pregarvi di una risposta alla seguente domanda: Posso mandare un manoscritto in lingua italiana, o devo tradurlo in francese? Se aveste la bontà di darmi questa risposta, ve ne sarò sempre più obbligato, e spero che sarà l'ultimo disturbo che vi avrò arrecato.

Però non posso chiudere la lettera senza pregarvi vivamente, per quando riceverete il mio articolo stampato dai Lincei, di volergli dare un'occhiata. Voi troverete tutto facilissimo, e mi è caro sperare che il teorema pel determin.[ante] Θ, e le applicazioni che ho indicate alla fine, e la evidente molteplicità delle altre applicazioni possibili, non abbiano a parervi indegne della vostra attenzione.

Vostro devotissimo
F. C.
Univ.[ersità] di Pavia

Letter 8: Bertrand to Casorati

Paris, 7 avril 1880
Cher monsieur,

Il n'y a pas de temps perdu, aucun mémoire n'a été envoyé jousqu'ici et la Commission qui doit les juger n'est pas encore nommée.

Un manuscrit en langue italienne sara certainement reçu et la lecture, j'en suis certains, sera facile à tous les commissaires. Je pense cependant que si cela ne vous est trop pènible, il vaut mieux écrire en Francais; vous serait jugé plus facilement et sans porter atteinte ou secret qui, éxigé pour les noms des concurrents, ne l'est pas pour leur nationalité, mais reste cependent préferible. [...]

Votre amis très dévoué,
J. Bertrand

Letter 9: Casorati in the *billet cacheté*

ce 22 mai 1880

Félix Casorati
Professeur à l'Université de Pavie-Italie

Le manuscrit concernant les interprétations du *Calcul aux différences* dont j'ai fait mention dans la préface a été comuniqué à l'*Institut Lombard de sciences et lettres* dans sa séance du 19 février, et devait être publié tout-de-suite dans les *Annali di Matematica (Milano)*. Mais une longue trève des ouvrièrs interrompit tout travail dans l'imprimerie des Annali. Alors j'ai comuniqué une copie de ce manuscrit à l'*Académie R. des Lincei* dans sa séance du 7 mars. Mais l'excès des travaux académiques à imprimer et une grave malheur domestique, qui détournait mon esprit des études, retardèrent aussi l'impression de cette comunication. Les deux manuscrits, maintenant quelque peu amplifiés, vont enfin paraître publiés dans peu de jours. Mais leur lecture serait tout-à-fait inutile pour qui connaît le manuscrit présenté avec ce billet au concours.

Letter 10: Casorati to Brioschi

Madesimo presso Pianazzo
sullo Spluga, 31 luglio 80

Carissimo sig. Direttore (Brioschi)

Già le dissi in Milano di aver fatto qualche pensiero sul premio Bressa da conferirsi stavolta all'Italiano che nel quadriennio 1877–1880 a giudizio dell'Accad.[emia] di Torino avrà fatto il miglior lavoro "sulle scienze fisiche e sperimentali, storia naturale, matematiche pure ed appl.[icate], chimica, fisiologia e patologia, non escluse la geologia, la storia, la geografia e la statistica".

In verità, per la moltitudine delle materie ammesse a concorso, è assai poco probabile che il premio venga dato ad un matematico. Però, dappoiché ciò è possibile, e dappoiché a me sembra che la Memoria, che ora le dirigo (Il calcolo delle diff.[erenze]

ecc.), sia la più importante tra le apparse nel quadriennio, in primo luogo per la estesa applicabilità delle idee che vi sono esposte, in secondo luogo per la importanza dei teoremi (specialmente quello del Θ o Δ) e delle applicazioni che già in questo primo saggio potei abbastanza minutamente indicare, penso che sarebbe quasi colpevole negligenza in me che ho famiglia, il nulla fare per mettermi in vista all'Accademia. Sentito il parere di Beltrami, ne parlai con Genocchi, il quale aggradì moltissimo l'esposizione che gli feci di una parte della Memoria, allora non per anco uscita dalla Tipografia. Ma a portare vie più l'attenzione dell'Accademia sulla medesima, servirebbe egregiamente la proposta di Lei, quale socio dell'Accad.[emia], come pure quella di Betti. E pertanto, lo scopo della presente lettera è pregarla a voler dare un'occhiata alla Memoria, ed a voler poi, nel caso che ne ricevesse impressione abbastanza favorevole, scrivere due semplici righe all'Accademia, come già altri fecero pel 1° premio (...)

Ed ora devo pregarla di scusarmi, se, per questa eccezionale circostanza dovetti lodare un mio lavoro. L'assicuro che questa necessità mi spiace assai, e che non penso di prendere da qui le mosse per seguire le consuetudini dei francesi. (...)

il suo aff.$^{\text{mo}}$ F. Casorati

Letter 11: Casorati to Picard

Pavie, ce 11 décembre 1880

Monsieur (E. Picard)

Vos travaux, et tout ce que m'a dit des vous M. Mittag-Leffler, en me visitant il y a quelque mois, et la nouvelle de vôtre prochain marriage avec une fille du grand Géomètre[38], qui est une des plus pures et plus hautes gloires de la France, tout cela m'ispire une vive estime et sympathie et, par conséquent, le désir d'établir avec vous quelques rapports personnels. C'est pour cela que je me suis décidé à vous envoyer tous les exemplaires qu'il m'est possible, des mes travaux de mathém.[atique] pure, (deux paquets) avec cette lettre, en vous priant d'avoir la bonté de les agréer et de vous souvenir possiblement de moi, lorsque vous dispenserez les exemplaires de vos propres travaux.

Je vous enverrai dans une autre occasion quelques autre Mémoires, dont une partie est déjà imprimée, contenants bon nombre d'applications du *Calcul des différences finies*, lesquelles vous interesseront, j'èspere, beaucoup. Dans pluiseurs d'entre elles joue un rôle très important et très-élégant le detérminant

$$\begin{vmatrix} f_1 & f_2 & \dots & f_n \\ \Delta f_1 & \Delta f_2 & \dots & \Delta f_n \\ \dots & \dots & \dots & \dots \\ \Delta^{n-1} f_1 & \Delta^{n-1} f_2 & \dots & \Delta^{n-1} f_n \end{vmatrix}$$

38 Picard was going to marry Charles Hermite's daughter.

formé avec les différences successives de pluiseurs fonctions $f_1, \ldots, f_n$, d'une même variable t, le quel j'écris aussi comme il suit

$$\begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ \theta f_1 & \theta f_2 & \ldots & \theta f_n \\ \ldots & \ldots & \ldots & \ldots \\ \theta^{n-1} f_1 & \theta^{n-1} f_2 & \ldots & \theta^{n-1} f_n \end{vmatrix}$$

en écrivant $\theta f, \theta^2 f, \ldots$ au lieu de $f(t+1), f(t+2), \ldots$. De telle manière, on peut regarder θ comme un symbole d'opération, ce qui est, comme vous verrez, très utile.

En envisageant la variabilité de t de plusieurs point des vue différents et en relations avec d'autres variables qui jouent elles-mêmes le rôle de variables indépendantes, on obtient plusieurs interprétations différentes du symbole θ, et autant de manières corréspondantes de traduire toutes les formules ou propositions, qui ont été acquiséz jusqu'ici dans l'analyse directe et inverse des différences finies, dans des résultats concernants d'autres branches de l'analyse mathématique. Le Calcul des différences finies si negligé dans ces dernièrs temps acquiert par là une importance très-grande et inattendue même pour l'analyse infinitésimale.

Mais je ne veux pas être trop indiscret, en vous entretenant plus longtemps sur mes travaux. Veuillez donc agrée, je vous en prie de nouveau, les séntiments d'éstime et de sympathie qui m'ont fait écrire cette lettre, et soyez si bon de me pas m'oublier tout-à-fait.

Votre serviteur
F. Casorati
Prof. à l'Université de Pavie
Italie

Je vous serai très-obligé si vous voudrez m'assurer par quelques mots d'avoir reçu les paquets.

Letter 12: Picard to Casorati

Toulouse, ce 17 décembre 1880.

Monsieur

J'ai été très flatté de votre aimable lettre et je serai très heureux d'établir avec vous de relations amicales. Je vous remercie vivement des exemplaires de vos travaux que vous m'avez envoyés; la plupart d'entre eux m'étaient déja connus, et quant à votre traité sur les fonctions d'une variable complexe c'est un livre excellent et que j'ai pu apprécier d'autant mieux que les sujets qui y sont traités me sont extrêmements familiers. Tout ce que vous me dites des différences finies et notamment du déterminant des Δ, m'intéresse extrêmement, et j'attends avec impatience les résultats complets de vos études sur ces importantes questions.

Je vous enverrai très prochainement les mémoires ou notes qu'ai publiés et dont j'ai encore entre le mains de tirages et vous pouvez être assuré que dans l'avenir je me ferai un devoir et un plaisir de vous réserver toujours un exemplaire de ce que je pourrai faire. Ce sont surtout les questions de la théorie générale des fonctions de variables complexes qui m'ont beaucoup occupé depuis quelque temps. Les beaux résultats de M. Weierstrass ont été le point de départ des mes propres récherches et vous trouverez dans mon mémoire sur les fonctions entières un théorème qui, si je ne me trompe, a peut être quelque importance. Je viens de corriger en ce moment les épreuves d'un mémoire qui va prochainement paraître dans le Journal de Crelle,[39] et qui est relatif aux équations linéaires à coefficients doublement périodiques; j'espère qu'il vous intéressera. Ce sont les recherches de M. Hermite sur l'équation de Lamé qui m'ont jeté sur la voie des résultats auxquels je suis parvenu.

En ce moment, monsieur, mon mariage est la grand action qui me préocupe; j'ai du revenir à Toulouse pour faire quelques cours à la Faculté et j'attends avec impatience mon départ pour Paris, qui aura lieu dans trois jours.

Soyez persuadé que je recevrai toujours vos lettres avec joie et je serai heureux de toutes les communications mathématiques que vous voudrez bien me faire. Permettez de vous assurer de sentiments d'estime et de sympathie que je conserve pour vous.

Emil Picard
professeur à la Faculté des Sciences
de Toulouse.

Letter 13: Casorati to Bonnet

Pavie, ce 6 février 1881

Cher Monsieur et ami (Ossian Bonnet)

Avec ces lignes je vous envoye la traduction française d'une lettre que j'ai fait insérer dans les Annali de M. Brioschi, à cause d'une Note publiée récemment par M. Stickelberger;[40] et je profite de l'amitié dont vous m'avez toujours honoré pour vous prier de vouloir l'observer, et de compatir mon irritation, en égard à la peine que j'ai dû essayer en voyant malveillance et, peut-être, mauvoise foi de la part d'un collègue.

Mon maître bien-aimé M. Brioschi me conseillez d'adresser à M. Hermite une Note, pour être insérée aux Comptes Rendus, dans le but de faire toujours mieux constater ma priorité dans l'application du Calcul des différences à la théorie des équations différentielles linéaires. Cependant dans cette Note je n'ai rien dit qui pût avoir l'aspect de personnalité, en me bornant à rappeller mon Memoire, et du rest completant la détermination des sous-groupes d'intégrales faite par M. Hamburger; ce qui peut intéresser indépendamment de toute question personnelle.

Mais dans une lettre confidentielle à un ami, quelque manifestation de mécontentement personnel pourra être pardonneé.

L'année dernière je m'étais occupé avec ardeur de la theorie des équations différentielles linéaires, mais une maladie très-grave de ma fille ainée, étant survenue [à] troubler ma tête et deranger mes plans, je n'ai pu ne m'a permis de rediger mes recherches qu'en partie d'une manière convenable pour être imprimées. Comme si cela n'etait pas assez pour me fâcher, M. Stickelberger vient maintenent essayer d'approprier à soi

[39] It is the paper (Picard 1881).

[40] This French version of Casorati (1880c) is (Casorati 1881b).

même et à Riemann et a M. Hamburger une partie des idées qu'il m'a été cependant possible de publier dans cette année malheureuse. Mais sa tentative ne peut qu'échouer. Je n'ai pas la pretension que personne ne s'occupe de l'application du Calcul des différences à la dite théorie ou à d'autres branches de l'analyse de la variabilité continue. Au contraire je dois voir de bon gré paraître des travaux fondé sur les idées que j'ai conçues et chercher à expliquer. Mais je ne puis pas être content que l'on cherche à m'ôter ce qui m'appartient.

En redigeant sa Note avec calme, sans l'idée injuste de la faire passer pour contemporaine de mon Mémoire, il aurait pu la livrer au public sous une forme beaucoup meilleure et il n'aurait pas certainement suscité de récriminations de ma parte. La méthode de Cauchy à la résolution du système d'equations aux différences, qui est l'object de la première partie du §1 est bien faite; et la combination des formules de résolution ainsi obtenues avec la formule de Weierstrass, effectué dans la première partie du §3, vient à-propos; quoique ce tour ne soit necessaire, comm'il semble croire, pour arriver à la forme la plus simple de la distinction des integrales en sous-groupes. Mais, dans l'empressement de faire paraître sa Note, il ne s'est pas aperçu que le seconde partie de ces deux paragraphes et les paragraphes restants (hormis le dernier qui se rapport à un autre point) sont trop remplis de répétitions et des calculs superflus. Le §2 particulierment, tout plein de calculs assez ennuyeuse, pouvait se remplacer par quelques observations brèves, claires et sans calculs.

La personne à laquelle je fais allusion, au commencement de ma lettre imprimée, est M. Frobenius, qui a travaillé et publié plusieurs articles en communion avec M. Stickelberger. C'est pour cela que j'ai terminé cette lettre en rappellant mon chapître sur le determinant

$$\Theta = \begin{vmatrix} y_1 & \dots & y_n \\ \dots & \dots & \dots \\ \theta^{n-1}y_1 & \dots & \theta^{n-1}y_n \end{vmatrix} \quad \text{ou son égal} \quad \Delta = \begin{vmatrix} y_1 & \dots & y_n \\ \dots & \dots & \dots \\ \Delta^{n-1}y_1 & \dots & \Delta^{n-1}y_n \end{vmatrix}.$$

M. Frobenius tient fort beaucoup à ce qu'on le regarde comme l'auteur qui a le plus soigné le determinant

$$D = \begin{vmatrix} y_1 & \dots & y_n \\ \dots & \dots & \dots \\ D^{n-1}y_1 & \dots & D^{n-1}y_n \end{vmatrix}$$

et M. Stickelberger déclare pompeusement dans son dernier § qu'il désignera, d'apres M. Frobenius, par $D(y_1, y_2, \dots, y_n)$. N'est ce donc pas étonnant qu'il puisse oublier mon chapître susdit, qui comprend, comme cas particulier, tout ce que l'on peut dire sur le determinant D? Mais le temps fera justice, car les applications nouvelles, qu'il m'arrive de faire chaque fois que je reviens à ce determinant Θ, ne me laissent plus douter qu'il soit pour prender une place distinguée parmi les instruments les plus utiles et les plus élégants de l'analyse. Permettez-moi d'ajouter, qui je partage la répugnance de la plus part des géomètre contre l'utilisation de notations ou dénominations nouvelles. Mais l'introduction d'un symbole (tel que θ), pour désigner par θf ce que une fonction f devient par effet de la variation d'une nature prefixée, de certaines éléments dont elle dépend, m'a paru trop utile pour y renoncer. Parmi les advantages ce n'est pas petit celui-ci, de pouvoir comprendre sous un même enoncé des propositions qui se rapportent à des sphères des recherches très différentes. Dans les unes, la variation, supposée par θ pourra consister dans le tour qu'une variable ou plusieurs variables imaginaires, dont nos fonctions dépendront, doivent faire autour d'un ou plusieurs systèmes des points singuliers; dans d'autres, la variation pourra consister dans un système d'accroissement que les éléments variables doivent prendre dans d'autres, enfin, la variation pourra consister dans tout autre chose. Et il va sans dire qu'il est très utile, de pouvoir aussi employer plusieurs symboles $\theta, \theta', \ldots$ simultanément.

Je voudrais bien vous entretenir un peu plus sur cet argument, mais je crois d'avoir absorbé déja trop de temps à vos occupations. Et la note de M. Stickelberger aurez vous eu le temps et l'envie de la lire? Dans le cas négatif je serais encore plus coupable d'indiscretion! Mais votre bonté a été toujours très grande; j'èspere donc, même dans ce cas, votre pardon. Et plus ancore, je compte aussi sur votre bienveillance pour l'avenir, de sorte que pourrai toujours me souségner

votre ami dévoué
F. Casorati

Soyez aussi indulgent pour les
fautes que j'aurais commisés
en ecrivant en francais

Letter 14: Bertrand to Casorati

Paris 15 février 1881

Je vous remercie bien cordialement, très cher Monsieur, de la nouvelle, pour moi si flattante, que vous voulez bien m'annoncer. Je ne suis pas digne de succeder à mon illustre ami M$^{r.}$ Chalses et n'en dois que plus de recoinnaissance à ceux qui ont bien voulu songer à moi.[41] Vous voudrez bien leur transmettre l'expression de mes sentiments les plus sincèrement et entièrement devoués.

L'Académie a décerné hier le Prix de Mathématiques pour la question des équations linéaires. Un très beau et ingenieux mémoire dans laquel l'étude des point critiques est éclairci et simplifiée par l'emploi des équations aux différences, avait attiré l'attention de la Commission qui n'a pu cependant lui accorder qu'une mention très honorable. Que le savant auteur veuille bien attendre pour condamner ses juges, la lecture du mémoire très considérable qui a été couronné et dont l'auteur est M^{r} Halphen; notre compatriote s'y montre beaucoup plus grand géomètre encore que les excellents debuts ne l'avaient promis. Il étudie les changements d'une équation linéaire quand on remplace la variable independante par une autre ou quand on multiplie l'inconnue par une fonction choisie de manière à simplifier l'équation qui ne cesse pas d'être lineaire. Il me semble qu'il équisse le sujet en tirant un très grand partie des invariants introduits par M. Laguerre qui je crois n'a pas concours.

Veuillez recevoir, cher Monsieur Casorati, l'assurance de mes sentiments les plus affectueuse.

J. Bertrand

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41 Bertrand refers to his election to the *Istituto Lombardo* on Febraury 10th, 1881.

Letter 15: Casorati to Bertrand

Pavie, ce 21 février 1881

Cher Monsieur

je vous remercie de votre comunication, adressée à Milan m'est parvenue en ritard, du 15 février, et particulierement des expressions affectueuses dont vous confortez mon esprit dans une circostance [sic!] où il en a vraiment tout le besoin. Car, je l'avait dit avec Pascal dans ma devise, nous ne pouvons prendre plaisir à une chose qu'à condition de nous fâcher si elle ne réussit pas. Et je ne puis à présent m'empêcher d'être fâché douloureusement d'autant plus que j'ignore les terms précis du rapport. Vos Comptes Rendus ne sont pas seulement la pubblication la plus observée par les professeurs de mon Université, mais ils sont lus aussi par les étudiants qui sont inscrits dans l'école normale annexée maintenent à cette Facultés des Sciences math.[ématiques] etc.

Un professeur de cette même Université, feu M. Codazzi, qui avait concouru au prix relatif à la théorie des surfaces applicables l'une sur l'autre, a pu rester satisfait de la simple mention qu'on lui avait accordée en Mars 1861. Mais alors le rapporteur, que vous connaissez très-bien, avait écrit: "Les trois autres Mémoires, inscrits sous les nos. 1, 2, et 5, *remplissent complètement le programme tracé par l'Académie. Si l'un quelconque des trois avait été présenté seul à notre examen, nous lui aurions sans hésiter accordé le prix.*"[42]

Maintenent, qui sera le rapporteur?

Et connaîtra-t-il assez une position scolastique pour prendre bon soin de la ménager par quelques expression de la valeur de celles que je vient de citer? Ce serait pour lui d'autant plus facile que mon travail et celui de M. Halphen n'admettent pas, il me semble, une comparaison rigoureuse; leur points de départ et leur directions étant differents; et mon travail contienant aussi incontestablement des perfectionnements importants. Je ne doute pas de la haute valeur du Mémoire du jeune géomètre que vous avez couronné; mais son mérite n'exclut pas que mes applications du Calcul aux differences soient tout-à-fait nouvelles, et que leur portée ne soit née à l'étude des intégrales autour des points critiques, faite par M. Fuchs, comme on peut perçevoir par mon manuscrit et comme on verra toujours mieux dans la suite.

A la vérité l'expression *très honorable* contenue dans votre lettre devrait me rassurer; mais je vous prie de compatir ces craintes, dont je ne sais pas me delivrer sur le moment. Du reste, je vous prie aussi de croire que je ne voudrai pas mal à mes juges, ni à mon maître bien aimé, M. Brioschi, qui m'a inspiré l'idée du concours. Je ne ferai que répéter ce que j'ai pensé dans d'autres moments de ma vie: d'être né sous un étoile assez malheureuse.

Pardonnéz moi, cher Monsieur Bertrand, cette lettre trop longue et trop triste et veuillez me conserver votre bienveuillance toujours précieuse.

F. Casorati

[42] In fact, Bertand had written this report.

Letter 16: Stickelberger to Brioschi.[43]

Schaffhause [sic!], 9 avril 1881

Monsieur le directeur

Puisque vous avez accordé quelque pages de vos annales à la lettre de M. Casorati, ecrite à l'occasion de mon mémoire "Zur Theorie der linearen Differentialgleichungen", j'éspère que vous voudrez bien inséré dans le même recueil une response de ma part, qui, du reste, ne sera pas longue.

M. Casorati part de la supposition que j'aie commencé mon travail ayant conaissance de son mémoire; il me reproche d'avoir employé sa méthode sans en indiquer l'origine. Voici ce que je pourrais lui répondre. C'est seulement après avoir trouvé les principaux résultats des paragraphes 1 à 4 de mon écrit, que j'en parlai avec M. Lindemann en faisant mention spéciale du calcul aux differences finies, et c'etait précisement cette mention qui l'engagea à me faire parvenir le mémoire de M. Casorati que celui-ci venait de lui envoyer. Mais je crois ne pas avoir besoin de repondre au [sic!] long à une imputation qui ne trouvera aucun credit auprès de tout ceux qui me connaissent.

Au reste il est evident que M. Casorati n'a pas pris la peine de lire mon mémoire en entier avant d'écrire sa lettre. S'il avait fait, il aurait remarqué que ce qui, pour lui, est une idée cardinale, n'est pour moi qu'un moyen pour arriver à un but determiné; il ne m'aurait pas demandé, pourquoi j'ai passé sous silence les recherches contenues dans son chapître deuxième et, en particulier, le théorème relatif à la condition pour l'existence d'une équation linéaire à coefficients périodiques entre plusieurs fonctions d'une seule variable indépendante. D'ailleurs, s'il avait en lieu d'employer ce théorème, je ne l'aurais pas emprunté à M. Casorati, mais à un Mémoire de M. Christoffel "Über die lineare Abhängigkeit von Functionen einer einzigen Verändlichen" publié en 1858 dans le Tome 55^{me} du journal de Borchardt, où ce suject est traité d'une manière aussi elegante que complète. J'ai coutume de citer, dans toutes les questions importants, les auteurs qui, à ma connaissance, ont proposé les prèmiers les théorèmes dont je fais usage.

M. Casorati m'accuse d'avoir manqué de politesse vis-à-vis avec lui. Je laisse aux lecteurs des Annali le soin de juger, s'il est plus courtois d'employer, dans une critique purement objective, les mots "nicht rathsam", ou bien de se servir, dans une résponse toute personnelle, d'éxpressions telles que "disgusto, malvolere, cattivi pensieri".

Agréz, Monsieur, l'expression de mon plus profond respect

Votre dévué
L. Stickelberger

Letter 17: Casorati to Brioschi

Pavia, 17 aprile 1881

Carissimo sig. Direttore (Brioschi)

[43] In the *Nachlass* there is a copy written by Casorati's daughter, Eugenia.

Ho letto la Memoria di Christoffel del 1858 e vi trovai infatti il teorema che la eguaglianza

$$\sum \pm f(m) f_1(m+1) \cdots f_n(m+n) = 0$$

è condizione necessaria e sufficiente affinché le funzioni

$$f(m),\ f_1(m), \ldots, f_n(m)$$

della variabile discreta m abbiano tra loro una relazione lineare a coefficienti costanti. Se avessi conosciuto questa Memoria, non avrei mancato di citare il Christoffel nell'occasione del mio determinante Θ come non mancherò di ricordarlo in una ventura occasione. Ma il Christoffel non allude mai ai *coefficienti periodici* e molto meno a coefficienti che sieno *funzioni di una variabile complessa, monodrome intorno ad un punto*. Non può essere che il *malvolere* dello Stickelberger capace di insinuare che queste idee fossero implicite in quella Memoria. (...)

Appendix 2: Original letters concerning Pincherle's unpublished note on Tannery's theorem

Letter 1: Pincherle to Casorati.

Bologna, li 21/7/85.

Chiarissimo Sig. Professore

Nella Memoria del Sig. Goursat (Sur une classe de fonctions representeés par des intégrales définies—Acta Math., 1883) ho trovato l'enunciato di una proposizione data dal Sig. Tannery, che è la seguente:

Se n funzioni sono tali che facendo qualunque giro colla variabile intorno a p punti $a_1, a_2, \ldots, a_p$, *si ritrovano sempre combinazioni lineari a coefficienti costanti delle n funzioni, esse funzioni soddisfano ad un'equazione differenziale lineare omogenea a coefficienti monotropi rispetto ai punti* $a_1, a_2, \ldots, a_p$.

Non ho potuto vedere quale sia la dimostrazione che il Sig. Tannery dà di questo teorema, essendo esso pubblicato nelle [sic!] *Annales de l'Ecole [Sic!] Normale,* che non si trovano nella Biblioteca di Bologna.

Però questa dimostrazione risulta come conseguenza immediata dal di Lei metodo per lo studio dei valori di una funzione nell'intorno di un punto singolare, contenuto nella Sua memoria *Il calcolo delle differenze finite ecc.* (Annali di Matematica, t. X); e spero che non le sarà discaro di leggere questa dimostrazione. Eccola in due parole:

Formiamo colle n funzioni date, e con n costanti arbitrarie, la funzione

$$E = c_1 E_1 + c_2 E_2 + \cdots + c_n E_n;$$

eseguendo un giro intorno ad a_1, ed essendo θ l'operazione da Lei introdotta nella citata Memoria, sarà

$$\theta E = \sum_{i=1}^{n} c_i \theta E_i$$

e per l'ipotesi fatta:

$$\theta E = \sum_{i=1}^{n} k_{1i} E_i$$

e così

$$\theta^r E = \sum_{i=1}^{n} k_{ri} E_i, \quad (r = 2, 3, \dots, n):$$

da queste eliminando le $E_1, E_2, \dots, E_n$, viene:

$$K_n \theta^n E + K_{n-1} \theta^{n-1} E + \cdots + K_1 \theta E + K_0 E = 0,$$

dove le K_i sono costanti. Ma questa equazione (Casorati, mem.[oria] citata, §10) indica che la E soddisfa ad un'equaz.[ione] differenziale lineare a coefficienti monotropi in a_1, sia

$$\varphi_0 E^{(n)} + \varphi_1 E^{(n-1)} + \cdots + \varphi_{n-1} E' + \varphi_n E = 0;$$

e di questa sarà E l'integrale generale. Ma lo stesso ragionamento dimostrerebbe che la E è l'integrale generale di un'equaz.[ione] a coefficienti monotropi in a_2, di un'equaz.[ione] a coeff.[icienti] monotropi in a_3, ecc., il che non può essere che se

$$\varphi_0, \ \varphi_1, \dots, \varphi_n$$

sono monotropi in $a_1, a_2, \dots, a_p$; c.d.d.

Io credo che questa dimostrazione così semplice di una proposizione così importante non sarà senza interesse. In ogni modo La lascio giudice di decidere se sia il caso di farne una comunicazione a qualche Accademia; a me basta che risulti una volta di più l'importanza di quella Sua bella Memoria.

Mi conservi la Sua preziosa benevolenza, e mi creda, ch$^{\underline{\text{mo}}}$ Sig. Professore

di Lei dev$^{\underline{\text{mo}}}$
S. Pincherle

Letter 2: Casorati to Pincherle.

Hôtel Piora sopra Airolo (Svizzera)
30 luglio 1885

Caro Pincherle

Ricevo adesso da Pavia la sua lettera del 21 e rispondo come posso.

Sono assai distratto dagli studî e lontano dai libri, e non ebbi occasione di vedere la Mem.[oria] di Tannery contenente la proposizione enunciata nella Mem.[oria] del Goursat. Nondimeno convengo con Lei che tale proposizione non si possa dimostrare in maniera più semplice dell'indicata nella sua lettera. Naturalmente, a me fa piacere di veder presente in Lei la mia Mem.[oria] "Il calc.[olo] delle differ.[enze] ecc." e farebbe pur piacere di vederla per di Lei mezzo ricordata agli studiosi che credo potrebbero farne facilmente molte applicazioni. Ma sembrerebbemi conveniente ch'Ella desse un'occhiata a quella Mem.[oria] Se fossi a Pavia, gliela manderei; ma così bisognerebbe ch'Ella dimandasse il vol.[ume] degli Annali de l'Ecole [sic!] normale a qualche collega di Milano, che potrebbe ritirarlo da quel collegio degli Ingegneri. Tale occhiata potrebbe suggerirle anche altre applicazioni della stessa mia Mem.[oria] da presentarsi insieme con la già scrittami ai Lincei o ad altra Accad.[emia]

Quando avrò il piacere di vederLa, Le dirò dei non pochi progetti che feci anch'io di correggere, semplificare, ricordare cose nostre [ai] matem.[atici] francesi in questi ultimi anni.

Ma Ella, ch'è giovane, fa bene a non contentarsi del semplice *fare progetti*.

Se non potrà avere il Tannery da Milano, me lo scriva, che glielo manderò poi io da Pavia.

Continui ad amar

l'aff.[ezionatissimo] suo F.C.

Letter 3: Pincherle to Casorati.

Bologna, li 3 Ottobre 1885

Ch$^{\underline{\text{mo}}}$ Sig. Professore

Secondo il suo Consiglio, ho aspettato a redigere la Nota sulle equazioni differenziali, per cercare prima la Memoria del Tannery: ma non mi è stato possibile avere quel volume degli *Annales de l'Ec.*, e non ho voluto disturbarla per questo finché Ella si trovava presumibilmente in campagna.

Ora ella sarà forse tornato a Pavia: in questo caso Le sarei gratissimo se mi potesse far avere quel volume (Serie 2, Tomo 4) degli Annali; e dopo letta la memoria del Tannery, redigerò (se sarà ancora del caso) e Le spedirò la noticina, rimettendo a Lei di decidere se convenga o no presentarla a qualche Accademia.

In queste vacanze ho lavorato ad una Memoria sugl'integrali definiti atti a rappresentare funzioni analitiche, nella quale spero di dare alcuni risultati nuovi: ma ci vorrà forse del tempo perché sia in ordine per la stampa, e l'argomento mi si fa sempre più vasto. Mi consiglia Ella a sospendere per qualche tempo ogni pubblicazione, lavorando intanto per il Premio dei Lincei dell'89, o è troppa presunzione la mia?[44]

Sarei ben lieto che si presentasse quell'occasione, che Ella accenna nella sua lettera, di poterla vedere qui; è un desiderio che nutro da molti anni, perché da molti anni non

[44] Pincherle shared this prize with Luigi Bianchi.

ho potuto discorrere dei miei studî con persona competente e perché voglio ringraziarla a voce della benevolenza di cui Ella mi è stato prodigo in questo periodo di tempo.

E con questa speranza, La prego di credere all'affetto ed alla devozione del suo

Salvatore Pincherle

Letter 4: Casorati to Pincherle.

Pavia, 17 ott. 1885

Caro Pincherle

Circa il premio linceo io non trovo niente affatto presuntuoso il progetto suo di concorrervi. Di più non vorrei dire; se non altro, perché impedito dalla lontananza di seco conversare distesamente.

Rileggendo la Sua lettera del 21 luglio, vedo che la proposizione ivi considerata appartiene all'antica (1874) [sic!] Memoria del Tannery sulle equaz.[ioni] diff.[erenziali] lineari; Memoria che a suo tempo lessi con attenzione. E però, pur seguitando a credere che la di Lei dimostrazione sia della massima semplicità non oserei consigliarle il sacrificio del tempo occorrente alla redazione di una Nota per la stampa, se nella Nota non dovesse entrare che la pura dimostrazione suddetta. Faccia dunque Ella ciò che le pare conveniente; è certo ch'io ne resterò in ogni modo soddisfatto. Quanto al volume contenente la Mem.[oria] del Tannery, glielo spedirò appena sarà arrivata una delle tre persone che tengono le chiavi della libreria della Scuola Normale di questa Università.

Le vacanze hanno ristorato assai le mie forze fisiche; ma sono tuttora sbalordito dall'inaspettata perdita di mio fratello[45]. Era il solo superstite della mia famiglia paterna, aveva per me indicibile affetto ed io per lui tenerezza e venerazione ad un tempo.

Mi ricordi ai colleghi e mi voglia sempre bene.

Suo F. Casorati

Letter 5: Pincherle to Casorati.

Bologna, li 19/10/85

Chiarissimo Sig. Professore

Con vivo rincrescimento rilevo dalla Sua lettera la notizia della perdita del di Lei fratello: io non ho avuto l'onore di conoscerlo, ma ne ho spesso sentito parlare con deferenza e stima. Le auguro che il tempo e l'affetto dei suoi possano lenire il Suo dolore, cui mi associo.

In questa circostanza non voglio più tediarLa con cose mie, che non possono avere per Lei che un meschino interesse, e ringraziandoLa dell'offerta di spedirmi il volume degli *Annales*, credo che Ella possa risparmiarsi questo disturbo se non vede l'opportunità di pubblicare quella dimostrazione. Certamente quella Memoria, essendo

[45] Luigi Casorati was a jurist. He was born in Pavia on January 26th, 1834 and died in Rome on August 4th, 1885.

del 1874 (cosa che non sapevo) non può contenere risultati che non siano noti dopo i più recenti lavori sulle equazioni differenziali.

Come già Le dissi, ho trovato quel Teorema citato dal Goursat, in una Memoria che studiavo per i miei lavori sugl'integrali; e mi venne subito alla mente l'idea che la dimostrazione di quella proposizione si dovesse ottenere nel modo più semplice applicando il Suo metodo (l'operazione che Ella chiama θ.) D'altronde questa osservazione non ha attinenza diretta coll'argomento che ora sto studiando, e di cui Ella vedrà fra poco un primo saggio in una "Note sur une intégrale définie" che uscirà in uno dei prossimi fascicoli degli *Acta Mathematica.*

Mi auguro che questo lavoro sia per incontrare la Sua approvazione, alla quale tengo sopra tutto; e pregandola di riverire per me i Sig. Professori Beltrami, Bertini e Maggi[46], La prego, ch$^{\underline{\text{mo}}}$ signor Professore, di tenermi sempre per

Suo dev$^{\underline{\text{mo}}}$ e aff$^{\underline{\text{mo}}}$
S. Pincherle

Letter 6: Casorati to Pincherle.

24 dic. 85

Mio caro Pincherle

Voglia aggrad.[ire] coi miei aug.[uri] l'artic.[olo] che le invio.

Avrei voluto scriverle assai prima per dirle che ha dato repentinamente troppo peso alla mia opin.[ione] circa quella tale dimostr.[azione] Del resto, io non potevo vederne che con piacere la pubbl.[icazione], che Ella ad ogni modo avrebbe potuto fare dicendo che proponevasi ricordare agli studiosi una Memoria forse meritevole di qualche attenzione (e non di essere negletta, come si fa dai Francesi, non ostante la lunga recensione fattane da non so chi nel Bulletin di Darboux[47]).

Ami sempre il suo aff. F.C.

Letter 7: Pincherle to Casorati.

Bologna, li 29 X$^{\text{bre}}$ 85

Chiarissimo Sig. Professore.

Prima di tutto la Prego di accogliere i miei più sinceri auguri di felicità per il prossimo anno: ed i miei ringraziamenti per la Memoria che Ella mi ha favorito.[48]

Questa Memoria è destinata certamente ad avere una grande importanza, e tocca uno degli argomenti puì vitali della Analisi. Anche a me—per quanto poco possa

46 Eugenio Bertini was a distinguished algebraic geometer who taught in Pavia in the period 1880-1892. Bertini theorems are still of use in current research in algebraic geometry. Gian-Antonio Maggi was Casorati's son in law and taught Rational mechanics and Mathematical physics in Modena, Messina, and Milan. A set of equations to study non-holonomic systems is named after him.

47 The review appeared in 1882 (Darboux et al. 1882).

48 Pincherle presumably refers here to the preliminary version of Casorati (1886a, b) that was printed in Milan in 1885.

valere la mia opinione,—era sempre sembrato naturale che si dovesse poter trovare proprietà della $\int_0^z \frac{dz}{\sqrt{R(z)}}$ o della sua inversa anche se $R(z)$ è di grado superiore al $4^{\underline{\text{to}}}$; e credo che se ciò non è stato fatto si deve attribuire in parte alla grande fecondità delle idee di Abel e Jacobi, di considerare le inverse come funzioni di più variabili, che non ha fatto cercare una estensione della Teoria delle f.[unzioni] Ellittiche nel campo di una variabile sola. Sono persuaso che in questo ordine d'idee Ella giungerà a dei risultati di somma importanza. E così, la Teoria delle f.[unzioni] Ellittiche verrà anch'essa ad acquistare maggiore importanza, trovandosi per così dire all'*intersezione* delle trascendenti Abeliane da una parte, delle Sue nuove trascendenti dall'altra.

Io avevo rinunziato all'idea di pubblicare quella Nota sulla Mem.[oria] di Tannery, più di tutto per non darLe il disturbo di spedirmi il volume degli Annales de l'Ec.[ole] Normale. Se Ella però ha la compiacenza di farmelo avere, vedrò di stendere quella Nota, cercando anche d'applicare il Suo metodo ad altri Teoremi se è possibile, e quando la Nota sia redatta, gliela spedirò ed Ella giudicherà se sia o no da pubblicare.

Desidererei pure dalla Sua Cortesia un'altro [sic!] favore, che spero Ella non mi vorrà negare; e sarebbe (con tutto suo comodo) l'indicazione del tomo del American Journal nel quale trovansi le memorie d'un certo Daniell sulla Teoria delle f.[unzioni] Ellittiche secondo il Weierstrass.[49] Se potessi avere quella indicazione, farei venire quel volume per la via di questa Biblioteca, dalla Biblioteca Universitaria di Pavia, dove si trova a quanto credo, l'American Journal.

Ho avuto il piacere di vedere poco fa il prof. Maggi, che mi ha dato le di Lei notizie e mi ha fatto sperare una Sua gita a Modena e a Bologna. Sarei lietissimo di questa circostanza, che mi auguro già da 5 anni.

Pregandola d'accogliere di nuovo i miei auguri ed i miei ringraziamenti, e di riverire per me il Prof. Beltrami colla Sua Signora, ed i di Lei colleghi di costì

mi creda, Ch<u>mo</u> Signor
Professore, di Lei dev<u>mo</u>
S. Pincherle

Letter 8: Pincherle to Casorati.

Bologna, li 9 Gennajo 1886

Ch<u>mo</u> Signor Professore

Le accludo la nota relativa al teorema di Tannery, lasciandola giudice dell'opportunità di presentarla all'Istituto Lombardo. L'enunciato che ho dato è un po' più generale di quello di Tannery, ed i coefficienti dell'equaz.[ione] differenziale possono anche essere f.[unzioni] polidrome di specie determinata (f.[unzione] uniforme d'un punto analitico) riguardando il campo T come una superficie di Riemann ridotta, con tagli; semplicemente connessa.

Le spedisco pure, raccomandato, il volume degli *Annali*: la Memoria di Tannery, che ho letto in questa occasione, non mi è sembrata nulla di straordinario, una compilazione

[49] The author's name is misprinted: the notes on Weierstrass theory of elliptic functions were written by Daniels (1883a,b, 1884).

dei lavori di Fuchs fatta discretamente e nulla più. Non vi è di notevole che il Teorema che mi ha ispirato questa Nota.

La ringrazio nuovamente per il disturbo che Ella s'è preso di spedirmi il libro, e La prego di credermi sempre

il Suo dev.$^{\underline{\text{mo}}}$ ed obblig$^{\underline{\text{mo}}}$
S. Pincherle

Analisi: Sopra un teorema del Sig. Tannery

Scopo di questa breve Nota è di mostrare con una nuova applicazione, i vantaggi che può arrecare nello studio delle funzioni analitiche e del loro modo di comportarsi nell'intorno di punti determinati, il metodo ideato dal prof. Casorati ed esposto nella Sua Memoria: "Il calcolo delle differenze finite interpretato[50] ecc., Annali di Matematica, S.II, T. X." Ne faccio qui l'applicazione alla dimostrazione di un teorema enunciato dal Sig. Tannery nella Memoria "Propriétés des intégrales des équations différentielles linéaires, Annales ce l'Ec. Normale, S. II, T. IV, p. 130;" teorema notevole per sé, e per una applicazione importante che ne ha fatto il Sig. Goursat.[51]

Il teorema è il seguente, sotto un enunciato un po' modificato: "Siano n funzioni analitiche $E_1, E_2, \ldots, E_n$ a carattere regolare nell'intorno di ogni punto di un campo T semplicemente connesso, eccettuati i punti $a_1, a_2, \ldots, a_p$ in numero finito e fra le quali non passa alcuna relazione lineare a coefficienti costanti; se quando la variabile gira senza uscire dal campo T intorno ad uno qualunque dei punti $a_1, a_2,..a_p$, i nuovi valori delle funzioni sono legati ai primitivi da relazioni lineari a coefficienti costanti, queste funzioni sono gl'integrali di un'equazione differenziale lineare a coefficienti monodromi in T."

Infatti, formiamo la funzione E i cui coefficienti c siano costanti arbitrarie

$$E = c_1E_1 + c_2E_2 + \cdots + c_nE_n; \tag{1}$$

eseguendo colla variabile un giro intorno ad a_1, ed essendo θ l'operazione di Casorati, viene

$$\theta E = c_1\theta E_1 + c_2\theta E_2 + \cdots + c_n\theta E_n$$

e per l'ipotesi fatta

$$\theta E = k_{1,1}E_1 + k_{1,2}E_2 + \cdots + k_{1,n}E_n; \tag{2}$$

[50] V. un estratto particolareggiato di questa Memoria nel "Bulletin de Darboux, S. II, T. VI, 1882".

[51] Nella Memoria "Sur une classe des fonctions representés per des intégrales définies. Acta Mathematica, T.II".

analogamente

$$\begin{cases} \theta^2 E = k_{2,1}E_1 + k_{2,2}E_2 + \cdots + k_{2,n}E_n, \\ \dots \\ \theta^n E = k_{n,1}E_1 + k_{n,2}E_2 + \cdots + k_{n,n}E_n; \end{cases} \tag{3}$$

eliminando $1, E_1, E_2, \dots, E_n$ fra le (1), (2) e (3), viene una equazione

$$K_n\theta^n E + K_{n-1}\theta^{n-1}E + \cdots K_0 E = 0;$$

ma questa (Casorati, loc. cit. §10) indica appunto che la E soddisfa ad un'equazione differenziale lineare a coefficienti monodromi nell'intorno di a_1

$$\varphi_0 E^{(n)} + \varphi_1 E^{(n-1)} + \cdots + \varphi_n E = 0,$$

di cui E è l'integrale generale. La stessa dimostrazione farà conoscere che $\varphi_0, \varphi_1, \dots, \varphi_n$ sono monodromi nell'intorno di $a_2, a_3, \dots, a_p$; onde il teorema è dimostrato.

Se il campo T ricopre tutta la sfera una sol [sic!] volta, le funzioni $\varphi_0, \varphi_1, \dots, \varphi_n$ sono uniformi. Non è escluso che il campo T possa essere una superficie di Riemann ridotta semplicemente connessa coi tagli opportuni; nel qual caso le $\varphi_0, \varphi_1, \dots, \varphi_n$ sarebbero funzioni uniforme [sic!] d'un punto analitico.

S. Pincherle

Letter 9: Pincherle to Casorati.

Bologna, li 27/2/86

Chiarissimo Signor Professore

Mi permetto di manifestarle di nuovo il mio compiacimento per aver potuto passare qualche ora nella Sua compagnia: e sono ben grato al prof. Maggi di avermi procurato col suo gentile invito l'occasione di quella visita che ho forse prolungata oltre i limiti della discretezza, ma che a me è sembrata troppo breve. Spero che nell'entrante Marzo Ella farà, come ci ha fatto sperare, la sua gita a Roma passando per Bologna e mi procurerà così il piacere di rivederLa.

Ho letto attentamente il paragrafo della Memoria di Tannery che Ella mi ha indicato, ma non ho potuto trovarvi una inesattezza. La dimostrazione è però redatta poco bene, e vi è un errore (di stampa probabilmente) alla linea 10 (p. 133) dove va letto m invece di $m - 1$.

Le rimando il volume degli *Annales*; ho copiato il paragrafo in discorso per cui, anche senza tenere più a lungo il volume presso di me, potrò ripensare meglio a quella dimostrazione: la quale si può forse redigere come ho tentato nel poscritto seguente.

I professori Boschi[52], Villari[53] ed Arzelà mi hanno incaricato di contraccambiare i Suoi saluti; insieme a questi, accolga ch$^{\underline{\text{mo}}}$ Professore, i saluti ed i sensi di affetto

del Suo dev$^{\underline{\text{mo}}}$
S. Pincherle

La prego di riverire per me il prof. Beltrami

Tentativo di dimostrazione del teorema che ogni funzione algebrica soddisfa ad un [sic!] equaz.[ione] differenziale lineare.

Sia l'equaz[ione].

$$f(x, y) = 0 \tag{1}$$

e sia ξ una funzione razionale di x ed y, la quale come è noto, si può scrivere:

$$\xi = a_{0,0} + a_{0,1}y + a_{0,2}y^2 + \cdots + a_{0,n-1}y^{n-1} \tag{2}$$

dove le $a_{h,k}$ sono funzioni razionali di x. Si ha dalla (1)

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' = 0, \tag{3}$$

onde y' è una funzione della forma (2). Ora derivando la (2), viene ponendo per y il suo valore dalla (3):

$$\begin{cases} \frac{d\xi}{dx} = a_{1,0} + a_{1,1}y + \cdots + a_{1,n-1}y^{n-1} \\ \frac{d^2\xi}{dx^2} = a_{2,0} + a_{2,1}y + \cdots + a_{2,n-1}y^{n-1} \\ \cdots \\ \frac{d^n\xi}{dx^n} = a_{n,0} + a_{n,1}y + \cdots + a_{n,n-1}y^{n-1} : \end{cases} \tag{4}$$

e moltiplicando le (2) e (4) per i reciproci della prima colonna del determinante

$$\begin{vmatrix} 1 & a_{0,0} & a_{0,1} & \dots & a_{0,n-1} \\ 1 & a_{1,0} & a_{0,1} & \dots & a_{1,n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_{n,0} & a_{n,1} & \dots & a_{n,n-1} \end{vmatrix}$$

[52] Pietro Boschi (Rome, 7/6/1833–Bologna, 4/11/1887) taught projective and descriptive geometry in Bologna.

[53] Emilio Villari (Naples, 25/9/1836–20/8/1904) was a chemist and a physicist who taught experimental physics in Bologna from 1871 to 1889, when he moved to Naples. He discovered the variation in magnetic susceptibility of a sample subject to mechanic tension (the Villari effect).

supposto diverso da zero, viene un'eq.[uazione] della forma

$$A_0\xi + A_1\frac{\mathrm{d}\xi}{\mathrm{d}x} \dots A_n\frac{\mathrm{d}^n\xi}{\mathrm{d}x^n} = 0.$$

Nel caso speciale che la ξ sia la stessa y, viene l'equaz.[ione] di Tannery.[54]

Letter 10: Casorati to Pincherle.

Pavia, 10 marzo 1886

Caro Pincherle

....

Di matematica non dovrei scrivere nulla, perché non ho la testa a posto. Nondimeno osserverò, rispetto alla dimostrazione del Tannery che l'errore di stampa ($m-1$ invece di m) della linea 10 di p. 133 (dove parmi eziandio doversi leggere y^0 insieme con $y^2, y^3, \dots, y^{m-1}$) non costituisca il difetto a cui io alludevo. Questo difetto si riferisce, se la memoria non m'inganna, alle linee 12 e 13.

Prendiamo il caso di $m = 2$.

$$f(x, y) = ay^2 + by + c,$$
$$\varphi(x) = ac - b^2 = (ay^2 + 2by + c) \times a + (2ay + 2b)\left(-\frac{a}{2}y - \frac{b}{2}\right)$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\alpha y + \beta}{a\varphi(x)}, \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\gamma y + \delta}{a^2\varphi^2}$$

dove $\alpha, \beta, \gamma, \delta$ sono funzioni intere di a, b, c e loro derivate 1^e e 2^e. L'eliminazione di y^0 fra queste espressioni dà

$$\beta\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\delta}{a\varphi}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\alpha\delta - \beta\gamma}{a^2\varphi^2}y = 0;$$

[54] If $A_{i,j}$ is the $n \times n$ minor associated with the (ij) element of the matrix (determinante) A written above, Pincherle actually considers

$$A_{1,1}\xi + A_{2,1}\frac{\mathrm{d}\xi}{\mathrm{d}x} + \dots + A_{n+1,1}\frac{\mathrm{d}^n\xi}{\mathrm{d}x^n}$$

and, due to the identity (see, e.g. Pincherle 1909, p. 61)

$$\sum_{j=1}^{n+1} a_{jr}A_{j,s} = 0$$

that holds whenever $r \neq s$ if a_{jr} is an element of A, he concludes that

$$A_0\xi + A_1\frac{\mathrm{d}\xi}{\mathrm{d}x} \cdots A_n\frac{\mathrm{d}^n\xi}{\mathrm{d}x^n} = 0:$$

by setting $\xi = y$, Pincherle thinks that Tannery's equation (5.3) is recovered.

ma non è così *visibile*, come dice Tannery, che pure supponendo a costante si possa ritenere costante il coeff[iciente]. di $\frac{d^2y}{dx^2}$, che Tannery riduce ad 1.[55]

Sono il suo aff.$\underline{^{mo}}$ F.C.

Letter 11: Pincherle to Casorati.

Bologna, 15/5/86

Chiarissimo Sig. Professore

Eccomi ancora una volta a ricorrere alla di Lei gentilezza. Nel continuare le mie ricerche sulle *operazioni funzionali rappresentabili da integrali definiti*, ricerche di cui la prima parte è in corso di stampa, mi si è presentata una famiglia di trasformazioni, che credo nuova e che permette di passare da qualunque equazione lineare differenziale a coefficienti razionali, in una equazione di forma simile alle differenze finite. Prendo la libertà di unirle una breve nota[56] in proposito e Le sarei gratissimo se Ella volesse gettarvi un'occhiata. Nel caso poi che questo risultato Le sembrasse nuovo e degno d'interesse, ed ove ciò non dovesse recarle disturbo, sarei ben lieto se Ella si compiacesse di presentarla sia all'Istituto Lombardo, sia ai Lincei.

[...]

Nella speranza che ella vorrà perdonare il nuovo disturbo che le reco, e pregandola di salutare per me i Sig. professori Beltrami e Bertini, La prego, ch$\underline{^{mo}}$ sig. professore, di credere ai sensi di profondo rispetto e di vivo affetto del Suo dev$\underline{^{mo}}$

S. Pincherle

Possiamo sperare di vedere fra poco un ampliamento ed un seguito delle Sue ricerche sulle funzioni a più di due periodi?

Letter 12: Pincherle to Casorati.

Bologna, 14/6/86.

Chiarissimo Signor Professore.

La ringrazio vivamente per la premura che Ella si è data circa alla mia nota sulle equazioni differenziali, e sono lieto che Le sia sembrata interessante. Ciò che mi ha

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55 On the back of this minute, Casorati wrote: "Noto qui, per future notizie che potrò dare a Pincherle, che, sulla formaz.[ione] delle equaz.[ioni] diff.[erenziali] soddisfatte dalle radici di equaz.[ioni] algebriche, havvi da leggere in:

Besso *Sull'eq.[uazione] del* 5° *grado* V.[ol] XIX dele Mem.[orie] lincee (Anno 1883–1884), e i successivi lavori da lui presentati con questo al premio minist.[eriale] scaduto il 30 aprile 1885.

Heymann *Ueber die Integration der Diff[erential]gl.[eichungen]* $\frac{d^r y}{dx^r} + A_m \frac{d^m y}{d(\ell x)^m} + A_{m-1} \frac{d^{m-1} y}{d(\ell x)^{m-1}} + \cdots + \frac{dy}{d(\ell x)} + A_0 y = 0$. Math. Annalen XXVI B[and]. 4. H[eft]. Anno 1886."

56 Casorati inserted in a footnote the title of the note, by writing: *dal titolo "Sopra una trasformazione delle eq.[uazioni] diff.[erenziali] lin.[eari] in equaz.[ioni] lin.[eari] alle differenze, e viceversa" F.C.*

spinto a redigerla è stato il vedere che il Sig. Mellin, nell'ultimo fascicolo degli Acta pubblicò un teorema che è un caso molto, ma molto particolare della trasformazione che ho indicata.

....

Suo S. Pincherle

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