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The language of the "Givens": its forms and its use as a deductive tool in Greek mathematics

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Abstract The aim of this article is to present and discuss the language of the «givens», a typical stylistic resource of Greek mathematics and one of the major features of the proof format of analysis and synthesis. I shall analyze its expressive function and its peculiarities, as well as its general role as a deductive tool, explaining at the same time its particular applications in subgenres of a geometrical proposition like the *locus* theorems and the so-called «porisms». The main interpretative theses of this study are the following: the language of the «givens» (1) is the standard idiom in which "existence and uniqueness" of a mathematical object was proved, (2) was conceived as an unified framework reducing to a strictly deductive format disparate argumentative steps such as deductions, constructions, and calculations.

1 Introduction

The aim of this article is to present and discuss the language of the «givens», a typical stylistic resource of Greek mathematics and one of the major features of the proof format of analysis and synthesis. I shall analyze its expressive function and its peculiarities, as well as its general role as a deductive tool, explaining at the same time its particular applications in subgenres of a geometrical proposition like the *locus* theorems and the so-called «porisms». This article is organized as follows. Section 2 presents the basic features of the language of the «givens», most notably the "species" of the predicate «given». I shall set the language in the context of the several kinds of propositions where it is applied and of the archetypal definitions (*Data* def. 1–4) were it was first introduced. Section 3 comments on these definitions and provides

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a preliminary logical setup capturing an essential aspect of the idiom: the species of «givens» are the predicative counterparts of some fundamental relations: identity, equality, congruence, similarity, and parallelism. Section 4 deals with the question of the ontological status of a «given» object, namely, whether it is a particular or a generic mathematical entity. In particular, the phenomenon of the use of the article in identifying given objects is discussed on the basis of a large sample of textual data. Section 5 presents some alternative idioms to the one of the «givens»; among other things, it highlights the peculiar demonstrative format displayed by the propositions pertaining to the research field of "gnomonics" (the theory of sundials). Section 6 clarifies the role played, in a proof by analysis and synthesis, by what I shall call a «chain of givens». In Sect. 7 it is explained how they were used to validate calculations. Section 8 expounds the deductive function of these chains; it contains also a detailed account of their application in contexts other than the "canonical" analytical proofs, namely, Hero's *Metrica*, Ptolemy's *Almagest*, Diophantus' *De polygonis numeris*, and the explication of an unexpected link between a Heronian evolution of the analytical method and the Stoic analysis of syllogisms. This section presents also the main interpretative theses of this study: the language of the «givens» (1) is the standard idiom in which "existence and uniqueness" of a mathematical object was proved, (2) was conceived as an unified framework reducing to a strictly deductive format disparate argumentative steps such as deductions, constructions, and calculations.

2 The species of the «givens» and their use in different kinds of propositions

The basic features of the language of the «givens» are best understood if we read the enunciations of some typical kinds of propositions where it is used. No interpretative problems are raised by the enunciations of problems, where the predicate «given» qualifies the geometrical objects starting from which the required construction must be performed, as in *El.* I.23 (*EE* I, 32.12–14):

> πρὸς τῇ <u>δοθείσῃ</u> εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῇ <u>δοθείσῃ</u> γωνίᾳ εὐθυγράμμῳ ἴσην γωνίαν εὐθύγραμμον συστήσασθαι.
> On a <u>given</u> straight line and at a point on it to construct a rectilineal angle equal to a <u>given</u> rectilineal angle.

A little effort is required to follow the enunciation of a *locus* theorem, for instance one out of the three by Charmandrus added as preliminary to Apollonius' *Plane loci*. It is the transposition of the definition of a circle as a geometrical *locus*; a point is said «to touch» a line when it is contained in it (*Coll.* VII.24):

> ἐὰν εὐθείας <u>τῷ μεγέθει δεδομένης</u> τὸ ἓν πέρας ᾖ <u>δεδομένον</u>, τὸ ἕτερον ἅψεται <u>θέσει δεδομένης</u> περιφερείας κοίλης.
> If one end of a straight line <u>given in magnitude</u> be <u>given</u>, the other will touch a concave arc <u>given in position</u>.

Some difficulties arise with the enunciation of a theorem in Euclid's *Data* (prop. 40), where it is apparently required to prove that a triangle is «given in form» (*EOO* VI, 70.2–3):

> ἐὰν τριγώνου ἑκάστη τῶν γωνιῶν δεδομένη ᾖ τῷ μεγέθει, δέδοται τὸ τρίγωνον τῷ εἴδει.
> If each of the angles of a triangle be given in magnitude, the triangle is given in form.

A similar request is formulated in theorems having a more transparent geometrical content, such as Pappus, *Coll.* IV.11:

> τετράπλευρον τὸ ΑΒΓΔ ὀρθὴν ἔχον τὴν ὑπὸ ΑΒΓ γωνίαν καὶ δοθεῖσαν ἑκάστην τῶν ΑΒ ΒΓ ΓΔ ΔΑ εὐθειῶν. δεῖξαι δοθεῖσαν τὴν ἐπιζευγνύουσαν τὰ Δ Β σημεῖα.
> <Let it be> a quadrilateral ΑΒΓΔ having angle ΑΒΓ right, and <having> each of the straight lines ΑΒ ΒΓ ΓΔ ΔΑ given: to show that the <straight line> joining points Δ Β is given.

A problem entirely formulated in the idiom of the «givens» is Pappus, *Coll.* VII.294; it introduces the technical clause παρὰ θέσει, where the predicate «given» is understood:

> θέσει δεδομένων τῶν ΑΒ ΑΓ, ἀγαγεῖν παρὰθέσει τὴν ΔΕ καὶ ποιεῖν δοθεῖσαν τὴν ΔΕ.
> ΑΒ ΓΔ being <given> in position, to draw ΔΕ parallel to <a line> in position and to make ΔΕ given.

Serious difficulties may arise with an enunciation from Euclid's *Porisms*, in the strongly elliptical form transmitted by Pappus, *Coll.* VII.18:

> ἐὰν ἀπὸ δύο δεδομένων σημείων πρὸς θέσει δεδομένην εὐθεῖαι κλασθῶσιν, ἀποτέμνῃ δὲ μία ἀπὸ θέσει δεδομένης εὐθείας πρὸς τῷ ἐπ' αὐτῆς δεδομένῳ σημείῳ, ἀποτεμεῖ καὶ ἡ ἑτέρα ἀπὸ ἑτέρας λόγον ἔχουσαν δοθέντα.
> If straight lines from two given points be inflected on a line given in position, and one cuts off <a segment> from a straight line given in position up to a given point on it, the other too will cut off from another <line given in position a segment> having a given ratio <to the first>.

These enunciations bear out two facts.

(1) The predicate «given» has two functions: an object can be given (*a*) because it is assigned by hypothesis or (*b*) because can be obtained from the assigned objects by means of geometrical constructions or of theorems. In the latter case the object is *proven* given. It is not always straightforward to distinguish between the two functions, especially in the case of porisms (see Sect. 6 *infra*).

(2) Several species of «being given» are specified, namely, «in magnitude», «in position», or «in form», depending on the geometrical object to which the predicate is applied and on the point of view from which the former is considered. This point is clarified if we read the archetypal definitions *Data* 1–4, to which I add def. 9 for the sake of comparison (*EOO* VI, 2.4–10 and 4.3–5—notice that

in all definitions the dative of respect has the article; this often disappears in the applications, as we shall repeatedly see):

> δεδομένα τῷ μεγέθει λέγεται χωρία τε καὶ γραμμαὶ καὶ γωνίαι, οἷς δυνάμεθα ἴσα πορίσασθαι.
> λόγος δεδόσθαι λέγεται, ᾧ δυνάμεθα τὸν αὐτὸν πορίσασθαι.
> εὐθύγραμμα σχήματα τῷ εἴδει δεδόσθαι λέγεται, ὧν αἵ τε γωνίαι δεδομέναι εἰσὶ κατὰ μίαν καὶ οἱ λόγοι τῶν πλευρῶν πρὸς ἀλλήλας δεδομένοι.
> τῇ θέσει δεδόσθαι λέγονται σημεῖά τε καὶ γραμμαὶ καὶ γωνίαι, ἃ τὸν αὐτὸν ἀεὶ τόπον ἐπέχει.
> μέγεθος μεγέθους δοθέντι μεῖζόν ἐστιν, ὅταν, ἀφαιρεθέντος τοῦ δοθέντος, τὸ λοιπὸν τῷ αὐτῷ ἴσον ᾖ.
> Given in magnitude are said figures, lines, and angles for which we can provide equals.
> A ratio is said to be given for which we can provide the same.
> Rectilineal figures are said to be given in form for which are given both the angles one by one and the ratios of the sides to one another.
> To be given in position are said points, lines, and angles which always hold the same place.
> A magnitude is greater than a magnitude by a given when, if the given one is subtracted, the remainder be equal to the same.

In the same way, a circle is given in magnitude only when its radius is given in magnitude; it is given in position and magnitude when its center is given in position and its radius in magnitude (*Data* def. 5–6); similar definitions apply to segments of circles, which can be given in magnitude (def. 7) or in position and magnitude (def. 8). A triangle can be given independently in position, magnitude, or form.

3 The "givens" as predicative counterparts of relations

The definitions allow us to highlight the following, fundamental points:

(1) What is at issue in the definitions is introducing a predicate: the verbal form λέγεται «is said» is in fact typical of the definitions of predicates. In the case when terms or relations are defined, «to be» (cf. def. 9 just read) or «to call» are used; the two verbs are equivalent, as is borne out by a comparison of *El.* VII.def.17–8. The only exceptions to this practice can be found in *El.* V.def.11 and X.def.1. Furthermore, the verb πορίσασθαι «provide» alludes to constructive issues (notice the middle form), whereas the modal connotation δυνάμεθα «we can» underlines the existential import and to the role of the mathematician. Finally, the definitions must be referred to all forms of the verb «to give», even if in defs. 1–4 only nominal forms of the perfect tense appear (this tense is here to be taken in its aspectual value of «accomplished present»). In the course of the treatise, however, one finds quite frequently the aorist participle or finite verbal forms.

(2) The language of the givens is the formal "predicative" *pendant* of some relations, which are of crucial importance both in geometry and in number theory and whose transformations make up the deductive fabric of any proof. The relations at issue are equality (to which «given in magnitude» corresponds), identity («given» for ratios), similitude («given in form»), congruence, intended as coincidence by superposition («given in position»). The mechanism of formation of such predicates, which is explicit in the definitions, is the following. Each of them is obtained from the corresponding relation by "saturating" one of its entries: in symbols $g(^*) \equiv \exists a{:}R(a,^*)$, where the existential connotation is, as we have seen, expressly formulated in the definition. For instance, to predicate «given in magnitude» of an angle A amounts to claiming «there is an angle that has been made equal to A», where the verb «to make» generically refers to the same galaxy of unspecified operations alluded to by the verb «to provide». A similar mechanism of partial saturation is at work in the notions of «exprimability» (relation of commensurability saturated by a reference straight line, said «exprimable»; cf. *El.* X.def.1 and 3) and of «right angle». The latter is a predicate, but this ultimate subspecies of angle is defined as the one such that its adjacent is equal to it (*El.* I.def.10), and the definition itself gives the mode of «providing» the two (right) angles involved. As a consequence, to assert that a right angle is given is a paradigmatic application of *Data* def. 1: for this reason, and not because it has a well-defined "value", a right angle is always given.

(3) Ratios deserve a separate definition because they are «given» *tout court* and, most importantly, because two ratios are said to be «the same», not «equal»: the predicate «given» applied to them comes from the saturation of the relation of identity, not of equality. It is a mistake to say, as Proclus and Marinus do (*iE*, 205.13–206.11, and *EOO* VI, 256.12), that two magnitudes are «given in ratio».

(4) The definition of «given in form» has its model in *El.* VI.def.1: «Similar rectilineal figures are such as have both their angles severally equal and the sides about the equal angles in proportion». This fact is *prima facie* surprising, since the redactor might well have included a direct reference to similitude, in line with what we read in def. 1: *«Given in form are said figures for which we can provide similars». The attested definition, which obviously results from combining *El.* VI.def.1 and def. 1*, was probably suggested by concerns of deductive economy, since it does nothing but reducing the notion of «given in form» to the predicates «given in magnitude» and «given» for ratios. However, in the applications similitude can directly feature in the inferences, as we shall see in the next section.

(5) The only relations one finds in the definitions of the *Data* are: «to be greater/less by a given» (def. 9–10), and «to be greater/less by a given to ... than in ratio» (def. 11–12). The latter can be written in symbols in the following way: A is by a given magnitude C greater than in ratio to B when $(A - C){:}B$ is a given ratio. All of these are two-place relations, and the initial segment of the *Data* (up to prop. 21) proposes a systematic treatment of the "group properties" of these relations with respect to some basic operations. Here is a concise list. Stability of the predicate «given» under composition or subtraction of given magnitudes (3–4). Stability of the relation «to have a given ratio» under *separando* (5) and *componendo* (6). Transitivity of the relation «to have a given ratio» (8); its

extension to equal multiplicities of magnitudes in a given ratio (9). If two magnitudes are added to the same, and the sums are given, either the magnitudes are equal or they differ by a given magnitude (12). Prop. 10–11 and 13–21: detailed analysis of the relation «magnitude by a given greater than in ratio to a magnitude» (=**1**). Its stability under *componendo* and *separando* (10–11); its connections with the relation «to have a given ratio» (=**2**): transitivity of their combination (13). If given magnitudes are added (14) or subtracted (15) to magnitudes in a given ratio, the resulting magnitudes are either in relation **1** or in relation **2**; if, on the other hand, from one of them a magnitude is subtracted and to the other a different magnitude is added, then only relation **1** may hold (16). If two magnitudes are in relation **1** with respect to the same magnitude, they are to each other either in relation **1** or in relation **2** (17). If one magnitude is in relation **1** with respect to two magnitudes, the latter are to each other either in relation **1** or in relation **2** (18). Transitivity of relation **1** (19). If magnitudes in a given ratio be added (20) or subtracted (21) from two given magnitudes, the remainders are to each other either in relation **1** or in relation **2**.

It appears, thus, that picking out **1** as a significant mathematical relation is linked with the requirement of completing the "group properties" of the pivotal relation **2** under the two basic manipulations of *componendo* and *separando*.

(6) The above abstract network can be set up only for magnitudes, to which issues of constructibility do not pertain. In order to treat geometrical objects one must introduce the concept of position. However, the correspondence relation/predicate set by def. 4 cannot be placed on the same logical level as the others. As we shall see, the move amounts to an attempt, necessarily unsuccessful even if it can be read as an adaptation of the relation of identity to a geometrical context, at "formalizing" the constructions as logical objects. The difficulties resulting from this move immediately show up in the *Data*: they include the embarrassing presence of the adverb ἀεί «always» in def. 4, the fact that, in the applications, the redundant negation of the predicate «given in position» is introduced by means of the verb μεταπίπτειν «to change position» (prop. 25–30, 38, and 42), the inconsistency of propositions such as 25–30, that simply amount to a rewriting of def. 4.

(7) Of particular interest are defs. 13–15, that an anonymous scholiast ascribes to Apollonius (*EOO* VI, 264.2–3). They read (*ibid.*, 4.15–20):

> κατηγμένη ἐστὶν ἡ ἀπὸ δεδομένου σημείου ἐπὶ θέσει εὐθεῖαν ἀγομένη εὐθεῖα ἐν δεδομένῃ γωνίᾳ.
> ἀνηγμένη ἐστὶν ἡ ἀπὸ δεδομένου σημείου πρὸς θέσει εὐθείᾳ ἀγομένη εὐθεῖα ἐν δεδομένῃ γωνίᾳ.
> παρὰ θέσει ἐστὶν ἡ διὰ δεδομένου σημείου θέσει εὐθείᾳ παράλληλος ἀγομένη.
> A straight line drawn in a given angle from a given point to a straight line in position is dropped.
> A straight line drawn in a given angle from a given point on a straight line in position is raised.
> A line drawn through a given point parallel to a straight line in position is parallel to a line in position.

The fact that the defined terms are not used in the *Data* (but for the verb κατάγειν «to drop» see *Data* 37 and 67 *aliter*—admittedly very dubious occurrences) does not entail that they are surely spurious, even if one should add that defs. 13 and 14 do not define (species of) «givens». A strong case of inauthenticity can be set up, on internal grounds, for def. 15: the need for such a definition can be felt only as a consequence of a misunderstanding of a peculiar expression occurring, for instance, in *Data* 28 and 35–8. Yet, in the light of the considerations developed in the preceding points it is easy to see the rationale behind the introduction of the notion of παρὰ θέσει: to offer a predicate identifying a class of parallel straight lines, insofar as they have the same "direction", as a counterpart to the relation of «parallelism». Of course, one must emend the received definition 15: the qualification «through a given point» must be eliminated as spurious, and probably arising from a contamination with the statements in *Data* 28 and 35–8 themselves. Otherwise, the definition identifies a single straight line, which is given in position (this is exactly what is established in *Data* 28). To show that a certain straight line is παρὰ θέσει features among the «things sought» in the last proposition of book III of the Euclidean *Porisms* (*Coll.* VII.20). The expression is also attested in Apollonius, *Con.* II. 46 and 49 (*bis*), where it is always question of a straight line drawn παρὰ θέσει *and* through a given point, and at several places in Pappus, *Coll.* IV.51–2 (these are the transcriptions of two attempts at defining the quadratrix as the projection of a surface locus), VII.26 (one of the enunciations of the second book of Apollonius' *Plane loci*), 294, 312 (the latter is a lemma to Euclid's *Loci on a surface*). We see thus that the idiom παρὰ θέσει is an ancient one; the redactor of def. 15 probably regarded it as necessary to introduce this notion among the definitions in order to canonize a widespread linguistic practice.

4 Is a «given» object particular or generic? The use of the article in identifying given objects

One might well wonder whether a «given» object is (and was considered) particular or generic. In order to assess this problem of ontological commitment, and in the absence of any explicit hint on the side of the Greek mathematicians, one must turn to linguistic features. In our instance, everything hinges upon the presence or absence of an article in front of the predicate «given».

Manifold examples of this phenomenon come from the enunciations of problems: the object that must be constructed has no article, but the «givens» always have one, both in the enunciation and in the setting out, as we read in *El.* I.1 (*EE* I, 7.2–6—I use the definite article in the translation too):

> ἐπὶ <u>τῆς</u> δοθείσης εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συστήσασθαι.
> ἔστω <u>ἡ</u> δοθεῖσα εὐθεῖα πεπερασμένη ἡ AB.
> δεῖ δὴ ἐπὶ τῆς AB εὐθείας τρίγωνον ἰσόπλευρον συστήσασθαι.
>
> On <u>the</u> given finite straight line to construct an equilateral triangle.
> Let it be <u>the</u> given finite straight line AB.
> Thus it is required to construct an equilateral triangle on the straight line AB.

One might argue that we are here facing a defect of generality, since the given straight line is after all a generic one (and for this reasons I used the indefinite article in the translations offered in the preceding sections). Why not give an indefinite linguistic expression to the designation of a generic object? The problem can be given two complementary solutions: one of them is purely grammatical (Federspiel 1995), the other is semantical and contextual.

The grammatical reason is that, when a substantive is qualified by a participle in attributive position further determined by a complement in prepositional form, the participle must be preceded by the article, as in ἡ εἰς τὰς παραλλήλους εὐθείας εὐθεῖα ἐμπίπτουσα «a straight line falling on parallel straight lines» (enunciation of *El.* I.29, *EE* I, 41.6). The predicate δοθέν, however, is never preceded by such a prepositional complement. Therefore, as the construct without the article is in principle possible, the solution adopted as canonical is the consequence of a stylistic choice, consistently pursued in the whole ancient *corpus*.

The second answer is complementary to the first. The propositions of the *Elements* where an object is qualified as «given» are invariably problems of construction, and what is given is the object starting from which the construction must be performed. The main verb of the enunciation of a problem is in the infinitive, with a jussive connotation. The context is a dialectical one, where a questioner proposes a geometrical object to be acted upon and a respondent must perform the construction. Therefore, the object given in a problem is «present» to the active mathematician in a stronger sense than an object assigned in a theorem. To better assess this point, recall that πρόβλημα «problem» is a deverbal from προβάλλειν and has therefore the proper meaning of «what is proposed», or, better still, of «what is put forward»; see *in primis* Aristotele, *Top.* A 4. An "etymological" usage of this verb in a mathematical context can be found in the prefatory letter to Archimedes *Con. sph.*, *AOO* I, 246.6, 248.11, 250.24, 254.21, the last three occurrences in the beautiful terminological short circuit προβάλλεται/προεβάλλετο τάδε θεωρῆσαι «it is/was proposed to investigate this». In a mathematical text all of this is a fiction, but the linguistic canon still bears traces of it.

It is a surprising fact prima facie that the article is always absent in the *Data*, as we see with prop. 78 (*EOO* VI, 150.2–8):

> ἐὰν δοθὲν εἶδος πρός τι ὀρθογώνιον λόγον ἔχῃ δεδομένον, καὶ μία πλευρὰ πρὸς μίαν πλευρὰν λόγον ἔχῃ δοθέντα, δέδοται τὸ ὀρθογώνιον τῷ εἴδει.
> δοθὲν γὰρ εἶδος τὸ ΑΖΒ πρός τι ὀρθογώνιον τὸ ΓΔ λόγον ἐχέτω δεδομένον, καὶ ἔστω λόγος τῆς ΖΒ πρὸς τὴν ΕΔ δοθείς. λέγω, ὅτι δέδοται τὸ ΓΔ τῷ εἴδει.
> If a given form have a given ratio to some rectangle, and one side have a given ratio to one side, the rectangle is given in form.
> In fact, let a given form AZB have a given ratio to some rectangle ΓΔ, and let the ratio of ZB to EΔ be given: I say that ΓΔ is given in form.

What has been said above makes it clear why the article is absent: this is a theorem and not a problem—no proposition of the *Data* can be a problem—and the status of the geometrical data is altogether different. What is at issue is transferring the predicate «given» from some objects to others; the fact that «given» qualifies geometric entities is incidental: the *Data* deal with mathematical objects *qua* given, not with mathematical objects an incidental trait of which is to be given, as is the case with the «givens» of a problem. What happens in the *Elements* is exactly the opposite.

An analysis of the occurrences of the predicate δοθέν allows us to better assess the issue. Let us start with the differentiated use of the participles δοθέν/δεδομένον. The former is a passive aorist, the second a middle perfect with passive value; the former expresses the punctual aspect of an action and is more suited to feature in suppositions (namely, the specific parts of a proposition whose main verb is at the imperative: setting out, construction); the latter, which in the whole Euclidean *corpus* can be found only in the *Data*, has an aspectual value of «accomplished present», without any temporal connotation. This aspectual trait is prevalent over the resultative one; otherwise one should expect that the perfect be the preferred form in the inferences, both in the intermediate steps and in the conclusion, or whenever reference is made to a magnitude that was stated or proven given—yet the distinction has no value in the *Data*, where the two forms are used indifferently. We find only the aorist in the *Elements*, where δοθέν «given» is predicated exclusively within suppositions. As for the two occurrences of a present participle in the corollary to *El.* IV.5 and in the determination of VI.28 (*EE* I, 159.15, and II, 90.6), the first is surely spurious, the second probably is, and at any rate should be corrected to a perfect participle. In the *Data* we find both aorist and perfect participle, in a ratio 1,128/509; this value is somewhat an exception, but it is easily explained by the different character of the propositions of the *Data*, and the present participle is absent. In Apollonius' *Conica* the ratio aorist participle/perfect participle is 180/11, with two occurrences of present participle, one within a partial setting out in II.49 and one in the determination of II.53 (*AGE* I, 276.23 and 310.22). The only other occurrence in a determination (*Con.* II.50) is a perfect participle. In the Archimedean *corpus* the ratio is 119/15 (of the latter, 14 items are found in *Spir.*, the remaining one being in *Fluit.*), with two occurrences of present participle, one in the setting out of *Sph. cyl.* II.1 and one in the determination of II.7 (*AOO* I, 258.13 and 208.13). Even if the sample is very reduced, it can be that the present participle was felt suited for the determinations. If one is willing to propose emendations, then, one should correct the perfect participle in *Con.* II.50 (the opposite proposal was made in Federspiel 2000, pp. 374–376). In Hero's *Metrica* one finds 339/0, with two occurrences of a present participle, one in the setting out of III.13 and one in the determination closing II.15 (*HOO* III, 164.15 and 132.11).

As for the presence or absence of the article, I have checked, in Apollonius and in Archimedes, the occurrences of δοθέν analogous to those in the *Elements*, namely, such as to mark assigned magnitudes at their first occurrence in an enunciation or in a setting out (of course, one must exclude the items in absolute genitives, where the article in front of the participle is necessarily absent).

This is not a large sample. As both authors solve a number of problems by analysis and synthesis, most of the occurrences of participial forms recorded above are within chains of givens, where the participle is invariably in a predicative position and hence without the article. In Apollonius the relevant occurrences are in the problems *Con.* I.52–60, II.4, II.44–7, 49–51, 53 (I exclude the four occurrences in I.8 and the two in II.14, in expressions like «every given straight line», where the participle has the article). If we exclude also II.4, which is surely an Eutocean interpolation, the other problems present such linguistic peculiarities as to suggest that they are earlier redactions integrated by Apollonius in his treatise (Federspiel 2008). Within this small *corpus* one finds a usage at variance with the one in the *Elements* only in the enunciations of I.52 and 56 (*AGE* I, 158.21, 174.28): the expressions are ἐν δοθείσῃ γωνίᾳ and ἐν γωνίᾳ δοθείσῃ «in a given angle», to be compared with the analogous ἐν τῇ δοθείσῃ γωνίᾳ in *El.* I.42, 44–5. Other occurrences without the article in the same propositions are either further qualified by a determinative of indefiniteness τινος or by a dative of respect θέσει «in position», or have the perfect participle δεδομένον (*ibid.*, 158.24, 25, 176.11). Therefore, the sample is not comparable to the one of the *Elements*, even if the last occurrence corresponds to the one at 174.28 (the latter in the enunciation, the former in the setting out). However, the two eccentric designations in *Con.* I.52 and 56 occur within identical clauses having a strong formulaic connotation, involve the substantive γωνία, that is notoriously recalcitrant at receiving the article (Federspiel 1995, pp. 290–293), and, most importantly, are at variance with the formulation of three identical clauses contained in propositions I.54, 59–60, where one finds the article (*AGE* I, 166.19, setting out, 184.28–186.1, enunciation, 188.24, instantiated citation of I.59). Therefore, in I.52 and 56 one is led to suspect scribal slips responsible for the missing article. As for Archimedes, the situation is even worse than with Apollonius: only one occurrence without the article, in the prefatory epistle to *Con. sph.* (*AOO* I, 258.13).

An analysis of the sample of occurrences of δεδομένον leads to the same conclusions. Only two occurrences in Apollonius have the article (I.55 and the second one in II.50), but the absence in the others is forced either by the fact that the participle is in predicative position (II.47), or by their being inserted in citations of enunciations of the *Data* (II.46, 49, first and third one in 50, within citations of *Data* 28, 28, 29, and 90, respectively), or by the fact that they are the subject of clauses in ἐκκείσθω, whose character is necessarily indefinite (II.51 *bis*, in the former strengthened by a τις), or, finally, by its being qualified by the dative of respect θέσει «in position» (I.52), that imposes an indefinite form. The only residual occurrence without the article is in the setting out of I.56, dictated by the parallel formulation of the enunciation. In Archimedes the only occurrence without the article is the one at the beginning of the setting out of *Spir.* 8.

I conclude that the variant with δοθέν without an article in presenting the givens of a problem, although permitted by Greek language, was not regarded as admissible. Granted, it is possible that the presence of the article is merely a stylistic trait, but it remains that this choice provides the objects given in a problem with a particularizing connotation that cannot be found elsewhere in the *corpus*.

5 Alternative idioms. The testimony of Marinus of Neapolis

The late fifth-century Neoplatonic commentator Marinus of Neapolis, a pupil of Proclus, wrote a short introduction to Euclid's *Data* in the form of *Prolegomena* (edited in *EOO* VI, 234–256). Marinus discusses in detail three notions to which, on the basis of definitions by synonymy, earlier authors tried to reduce the predicate «given», namely, τεταγμένον «ordered» (ascribed to Apollonius, in his *General treatise* and in the *Neuseis*), γνώριμον «known» (ascribed to the renowned gnomonist Diodorus), ῥητόν «exprimable» (to Ptolemy). Marinus added to them the notion of πόριμον «provided», whereas he introduced and quickly dismissed as simplistic and arising from a usage limited to the *Elements* the following definition: given is «what is established in a supposition from the proposer of a problem».

Marinus first sets out definitions of the alternative notions and of their opposites, clarified by mathematical examples. A discussion follows of the mutual relationships among the extensions of the four notions. The six possible pairings are systematically reviewed. The result is that the extensions of the four notions can be put in a decreasing order in this way: «known» > «provided» > «exprimable», while «ordered» can only be said to have a non-empty intersection with the others (the discussion in Michaux 1947, pp. 27–33, about an alleged logical mistake committed by Marinus, is wrong: the text is unambiguous and does not admit Michaux' interpretation). By using coextensivity with καταληπτόν «cognitively accessible» as a criterion, Marinus claims that the best definition where only one predicate is involved is the one employing as a synonym πόριμον «provided». His choice is *prima facie* surprising, since this is without doubt the notion that is less easily grasped, but it is enough to recall that *Data* def. 1–2 contain in the *definiens* the verb πορίζεσθαι «to provide» to realize that Marinus had no real alternatives.

He then passes on to the definitions containing a conjunction of two of the basic notions. The tradition seems to exclude «exprimable», and Marinus discusses the residual three pairings, adding also the combination «ordered and exprimable». By using the criterion that in a good definition the *definiens* is coextensive with the *definiendum* or obtainable from it by conversion, he finally establishes that the best definition of this kind is «known as well as provided» (which is of course redundant on the very grounds of his own analysis of the extensions of these notions).

Only a few resonances with Marinus' analysis can be found, preserved in authors preceding him. Proclus' discussion of the notion of «given» is nothing but a list of four subspecies (in position, in ratio, in magnitude, in form), followed by uninteresting examples (*iE*, 205.13–206.11, repeated at 277.5–15). Marinus does not seem to draw anything from here, unless it be the erroneous idea that something can be given in ratio. In other contexts, we find conjunctions of the predicates introduced by Marinus in Proclus, *iE* 48.12–13: «what is exprimable and knowable [γνωστόν] in [geometry] can be determined by arithmetic ratios» and 204.9–10, where what is sought in a geometric proposition is further qualified as «what one has to know [γνῶναι] or provide [πορίσασθαι]».

In his description of the analytical *corpus*, Pappus introduces a link between ποριστόν and δοθέν; because of the presence of the predicate «possible» and of the double modal connotation (ποριστόν instead of πόριμον and «possible») this is a variant of the one-predicate definition adopted by Marinus (*Coll.* VII.2):

> In the case of the problematic kind, we assume what is proposed as something we know, then, proceeding through its consequences, as if true, to something established, if what is established <u>is possible and can be provided</u> [δυνατὸν καὶ

> πoριστóν]—what the mathematicians call "given"—what is proposed will also be possible [...].

Pappus' formulation suggests that he is reporting a well-known definition. Two characterizations in the *Prolegomena* can be likened to Pappus' (Marinus ends his tract by referring to «Pappus' notes to the book [*scil.* the *Data*]»: *EOO* VI, 256.25). In the first of them, Marinus claims that the term «provided» «was also used by Euclid when he described all species of what is given». Since Pappus and Marinus employ different terms (that is, «provided» and «<what> can be provided»), and Marinus makes every effort to differentiate them, it is unlikely that they refer to the same description. It is more likely that what Pappus said corresponds to a second definition reported by Marinus and ascribed to unknown authorities: «it is given what we can provide [πoρίσασθαι δυνάμεθα] thanks to what we have set out in the suppositions and in the principles». In the latter case, then, an Euclidean authorship must be excluded, and in fact, the simplest interpretation is that the *first* characterization by Marinus actually refers to defs. 1–2 of the *Data*, where the verb πoρίζεσθαι «to provide», further qualified by a modal operator, features in the *definiens*. As a consequence, neither Marinus' second characterization nor Pappus' could be taken as a rewriting of the first two definitions of the *Data* in the form of an abstract definition of «given» (which, we must recall, does not appear in the *Data*): they would thus constitute the only trace of such a direct definition.

I shall now discuss in detail the textual evidence related to the use of the four notions introduced by Marinus as synonyms of «given». A difficulty lies in the fact that three of the terms he introduces may or may not be read as carrying a modal shade: for instance, πόριμoν can mean either «provided» or «<what> can be provided». I have chosen the first translation since Marinus himself makes a difference between πόριμoν and πoριστόν, the latter carrying only the modal meaning. I have accordingly translated γνώριμoν as «known» and not as «knowable». As for ῥητόν «exprimable», I have decided not to depart from a long-standing tradition of translation of this term, typical of the Euclidean theory of the irrationals.

(**1**) τεταγμένoν «ordered»
This is the term offering the richest historical record.

(*i*) The verb ἐπιτάσσειν is not frequently found in the geometrical domain: it appears in the canonical clause γεγoνὸς ἂν εἴη τὸ ἐπιταχθέν «what was assigned would have come to be», closing a branch of a division of a problem into cases and whose first occurrence in the *Elements* is in II.14 (the others are in IV.1, VI.28, VII.31 (*bis*), 32, XI.11). Only one occurrence in a similar clause is found in Archimedes, *Sph. cyl.* I.5; we find, on the other hand, a few references to the proposed problem as an ἐπίταγμα «assignment» in *Sph. cyl.* I.2 (*bis*), 3, 4, *Con. sph.* preface. This substantive designates in Pappus the several «assignments» of a complex problem; it features mainly in book VII of the *Collectio* (34 occurrences). Both this verb and the basic form τάσσειν are overwhelmingly abundant in Diophantus' *Arithmetica*, always in the sense of «setting», «giving» a number, without any shade of ordering; the expression is sometimes resumed in the form «the given number». An analogous meaning can be recorded in *El.* VI.9 (*bis*), where one has to subtract from a given straight line an «assigned» part of it, and in VIII.2 (*ter*), where it is required to «find minimal numbers in continuous proportion, as many as one assigns»; see also Pappus, *Coll.* III.27 (*ter*), 66 (*bis*), 67, 72 (*bis*), 74, IV.74 (an expression analogous to the one in *El.* VIII.2, and the same holds for the one in III.67), VIII.25, 26. In Pappus it is frequently a ratio that is «assigned». Forms of τάσσειν can be found in two alternative proofs of *El.* X (n. 19 and 22). In Apollonius' *Conica*, τάσσειν is employed in the first person to «assign» two straight lines in position (in II.44), a perpendicular (46) or a point as given (47): *AGE* I, 264.26–7, 266.23–4, 270.1.

(*ii*) Let us be given two sequences of three terms: A, B, Γ and Δ, E, Z, and let the following proportions hold: A:B::E:Z and B:Γ::Δ:E. In this case (*El.* V.def.18) the second proportion is said to be τεταραγμένη «perturbed» (cf. *El.* V.def.18, 21 (*ter*), 23 (*quater*) VII.8 (*bis*)). One might expect that in the complementary state of affairs, namely, the one described by the *ex equali* theorems V.20 and 22 and in which A:B::Δ:E and B:Γ::E:Z hold, the proportion be said «ordered», and in fact this qualification is found in a definition alternative to V.def.17 preserved by some manuscripts (*EE* II, 3.17 *in app.*). That this was the original usage is attested by the occurrences in Archimedes, where the «perturbed disposition» is met in expressions like ἀνομοίως τεταγμένων τῶν λόγων «once the ratios are ordered in dissimilar ways» and the terms in non-perturbed dispositions of sequences with indefinite multiplicities are said ὁμοίως τεταγμένα «ordered in similar ways» (*AOO* I, 374.18, 414.2, II, 194.22, 196.15, 198.5; I, p. 260.28 and II, p. 434.6, respectively). On the other hand, the clauses by which the «perturbed» dispositions are expressly qualified as such in the Archimedean texts look very much like later glosses inserted in the text exactly with the aim to make the link with the Euclidean jargon explicit (*AOO* I, 194.10, II, 194.22, 196.15, 198.5). In Archimedes, however, τεταγμένον was very likely not a technical term, insofar as its meaning is near to the current one. Once «perturbed», whose meaning is non-transparent and hence strictly technical, came to be introduced, τεταγμένον as a designation of the non-perturbed disposition fell out of usage, the latter being simply identified by the absence of further qualifications. The presence of the expression in Eutocius' commentary to Archimedes' *Aequil.* (*AOO* III, 300.18 and 308.7) can easily be explained by adherence to the style of the main text.

(*iii*) A passage in Proclus (*iE*, 219.18–221.6) clarifies the notion of «ordered» as applied to problems, and shows at the same time that the term does not relate to issues of ordering only, as in the preceding example, but also of uniqueness as in Marinus' exposition. Commenting on *El.* I.1, Proclus expounded some constructions of an isosceles and a scalene triangle. The interest of such constructions, he points out, lies in the fact that they exemplify a general classification of problems, that he ascribes to Amphinomous, a contemporary of Speusippus and Menaechmus: according to the number of the essentially different constructions solving it, a problem can be ordered, intermediate, or unordered.

(*iv*) Apollonius wrote a whole treatise on ἄτακτα «unordered» irrationals. The meaning to be given to the expression, very much as in the preceding point and as Proclus again suggests, is «not univocally determined by the construction». It is not clear whether «ordered» as applied to the irrationals studied in the *Elements* is an original denomination; maybe it was introduced by Apollonius.

(*v*) In the theory of conic sections the adverb τεταγμένως «ordinatewise» is introduced to designate, within a formulaic expression that is quite complex and often formulated in severely abridged forms, what is still called the «ordinates» of a conic. Let us read the definition placed at the very beginning of Apollonius' *Conica*, that as a matter of fact refers to any line (*AGE* I, 6.23–29):

> πάσης καμπύλης γραμμῆς, ἥτις ἐστὶν ἐν ἑνὶ ἐπιπέδῳ, διάμετρον μὲν καλῶ εὐθεῖαν, ἥτις ἠγμένη ἀπὸ τῆς καμπύλης γραμμῆς πάσας τὰς ἀγομένας ἐν τῇ γραμμῇ εὐθείας εὐθείᾳ τινὶ παραλλήλους δίχα διαιρεῖ, κορυφὴν δὲ τῆς γραμμῆς τὸ πέρας τῆς εὐθείας τὸ πρὸς τῇ γραμμῇ, τεταγμένως δὲ ἐπὶ τὴν διάμετρον κατῆχθαι ἑκάστην τῶν παραλλήλων.
> Of any curved line which is in one plane, I call that straight line the diameter which, drawn from the curved line, bisects all straight lines drawn in this line parallel to some straight line, and I call the extreme of that straight line on the line the vertex of the line, and I say that each of the parallels is drawn ordinatewise to the diameter.

The underlined expression assumes in the course of the *Conica* a variety of verbal and nominal forms, such as ἡ τεταγμένως καταγμένη «the <straight line> drawn down ordinatewise». Notice that it is false that a line having the property of the diameter exists for every line, and Apollonius must show that every conic section has at least one diameter (*Con.* I.7), and actually infinitely many (I.46 and II.44), whereas the ordinates relative to a diameter are parallel to the tangent passing through the vertex parallel to that diameter (I.17 and 32). Even if it is usually held that the origin of the expression, in fact quite obscure (it is not clear whether the adverb τεταγμένως was introduced in the theory of conic sections by Apollonius or before him and whether the three Archimedean occurrences at *AOO* II, 206.10, 436.2, 456.8, are spurious or not), is linked with the ordering of the «ordinates», the focal meaning is in fact «univocally determined» (insofar as the direction of all ordinates is referred to one and the same diameter). In order to formulate this, admittedly quite complex, state of affairs a new term was needed, since the Euclidean species of «given» were not adequate: a possibility was to resort to a participial form of διδόναι qualified by a dative of respect such as τάξει «in order», but Apollonius or others choose the adverb τεταγμένως, probably because, if the other choice had been made, the formulation of the nominal designations of the ordinates would have contained a participle qualifying another participle, a construction that is not idiomatic in Greek.

(*vi*) Other two testimonies before Apollonius confirm that the focal meaning of τεταγμένος in the mathematical domain was «univocally determined». The first one is contained in the peripatetic tract *De lineis insecabilibus*, 968b18:

> ἀλλὰ μὴν εἰ μετρηθήσεται μέτρῳ τινὶ τεταγμένῃ καὶ ὡρισμένῃ γραμμή
> But, in truth, if a determined and well-defined line will be measured by some measure

The second one is in Archimedes, who repeatedly employed, at the beginning of *Spir.*, the expression τὸν ταχθέντα λόγον ἔχειν «to have the assigned ratio» (*AOO* II, 4.14.17.26, 20.20, 22.19, 24.16, 26.5). The first three occurrences, placed in the prefatory letter, repeat the enunciations of prop. 3–4, 7 of *Sph. cyl.* II, changing δοθέντα (or τὸν αὐτὸν τῷ δοθέντι) in ταχθέντα. The other four occurrences can be found in the enunciations of prop. 6–9, within 4 identical clauses; the same ratio, in immediately subsequent clauses, is qualified as δοθείς.

(2) γνώριμον «known»
In the Euclidean *corpus*, the term is attested only in the *Optica* redaction **A**, in the peculiar problems 18–21. Marinus mentions Diodorus, the supporter of the synonymy of this predicate with «given», and asserts that he exemplified his position by references to rays and angles, objects that are typical of the optical domain. But there is more to the issue. This Diodorus must be identified with the renowned gnomonist, the author of an *Analemma* on which Pappus wrote a commentary (*Coll.* IV.40). An *Analemma* is a treatise describing a method to determine univocally the angular coordinates of a celestial object, for instance the sun, once the latitude of the observer and the hour are known. This was effected by a series of plane constructions performed on three planes defined with respect to three independent directions; the planes were then rotated in such a way that all the arcs on the celestial sphere get projected, for a given configuration of hour and latitude, on one and the same plane. On this plane, trigonometrical methods were then used to calculate the arcs (cf. Neugebauer 1975, pp. 848–856). The methods finds its origin in gnomonics, the theory of sundials.

Of ancient *Analemmas*, we can read only Ptolemy's, which was transmitted in a palimpsest. An application of the method can be found in Hero, *Dioptra* 35, to determine the distance between two cities along a great circle, once the parameters of a lunar eclipse observed in both places are known (add to these documents a short description in Vitruvius, *Arch.* IX.7). Ptolemy' *Analemma* contains (*POO* II, 203–210) some propositions in analytical format: these consist of chains of givens, the omitted syntheses being the determination of the numerical values of the sought-for magnitudes; the proof scheme is the same as the one we shall see at work in the metrological tradition (Sect. 7). Furthermore, an analemma is a method of projection: it deals primarily with rays and angles. Therefore, I submit that Diodorus might have proposed in his *Analemma* a definition of «given» as «what is known [...], even if it were not exprimable» (that is, not assignable in terms of numerical ratios); maybe he also used γνώριμον in place of δεδομένον within analyses analogous to those proposed by Ptolemy. The latter consistently employed δεδομένον in his treatise (the presence of «notum» in Ptolemy' *Planispherium*, *POO* II, 227–59, is not significant since the text has been transmitted only as a Latin translation from Arabic), but it is not said that Ptolemy should have adhered to the linguistic idiosyncracies of his predecessors. On the contrary, a corroboration of my hypothesis about the Diodorean use on «known» in place of «given» comes from *Opt.* 18 **A**:

> τὸ δοθὲν ὕψος γνῶναι, πηλίκον ἐστίν, ἡλίου φαίνοντος.
> ἔστω τὸ δοθὲν ὕψος τὸ ΑΒ, καὶ δέον αὐτὸ γνῶναι, πηλίκον ἐστίν. ἔστω μὲν ὄμμα τὸ Δ, ἡλίου δὲ ἀκτὶς ἡ ΓΑ συμβάλλουσα τῷ πέρατι τοῦ ΑΒ μεγέθους καὶ διήχθω μέχρι τοῦ Δ ὄμματος. ἔστω δὲ σκιὰ ἡ ΔΒ τοῦ ΑΒ, καὶ κείσθω ἕτερόν τι μέγεθος τὸ ΕΖ συμβάλλον τῇ ἀκτῖνι μὴ πάντως καταυγαζόμενον ὑπ' αὐτῆς κατὰ τὸ Ζ πέρας.

> ἥρμοσται οὖν εἰς τὸ ΑΒΔ τρίγωνον ἕτερόν τι τρίγωνον τὸ ΕΖΔ. ἔστιν ἄρα, ὡς ἡ ΔΕ πρὸς τὴν ΖΕ, οὕτως ἡ ΔΒ πρὸς τὴν ΒΑ. ἀλλ' ὁ τῆς ΔΕ πρὸς τὴν ΕΖ λόγος ἐστὶ γνώριμος καὶ ὁ τῆς ΔΒ ἄρα πρὸς τὴν ΒΑ λόγος ἐστὶ γνώριμος. γνώριμον δὲ τὸ ΔΒ. γνώριμον ἄρα καὶ τὸ ΑΒ.
> To know the value of a given height, if the sun appears.
> Let it be a given height AB, and let it be required to know its value. Let it be an eye Δ, a sun ray ΓA falling together with the extreme of magnitude AB, and let <the ray> be drawn across as far as the eye Δ. Let ΔB be a shadow of AB, and let some other magnitude EZ be placed falling together with the ray, not totally enlightened by it, at extreme Z.
> Therefore, on triangle ABΔ some other triangle EZΔ is fitted. Therefore, it is as ΔE to ZE, so ΔB to BA. But the ratio of ΔE to EZ is known: Therefore the ratio of ΔB to BA is also known. And ΔB is known. Therefore AB is also known.

It is clear that this proposition can be considered a very elementary theorem in gnomonics. Its seemingly bizarre format amounts in fact to an analysis, whose aim is validating a procedure producing the πηλικότης «value» of the shadow (cf. Sect. 7). This proof is thus a relic attesting for an approach typical of the marginal research domain of gnomonics.

The last relevant occurrence of γνώριμον as a synonym of «given» is in Archimedes, *Aequil.* II.1–8, whose aim is to determine, by means of the method of exhaustion, the center of gravity of a parabola. Let us read the passage, at the beginning of *Aequil.* II.2, where the class of approximating figures is defined (*AOO* II, 168.2–9):

> εἴ κα εἰς τμᾶμα περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογωνίου κώνου τομᾶς τρίγωνον ἐγγραφῇ τὰν αὐτὰν βάσιν ἔχον τῷ τμάματι καὶ ὕψος ἴσον, καὶ πάλιν εἰς τὰ καταλειπόμενα τμάματα τρίγωνα ἐγγραφέωντι τὰς αὐτὰς βάσιας ἔχοντα τοῖς τμαμάτεσσιν καὶ ὕψος ἴσον, καὶ ἀεὶ εἰς τὰ καταλειπόμενα τμάματα τρίγωνα ἐγγραφέωντι τὸν αὐτὸν τρόπον, τὸ γενόμενον σχῆμα ἐν τῷ τμάματι γνωρίμως ἐγγράφεσθαι λεγέσθω.
> If in a segment contained by a straight line and a section of a right-angled cone be inscribed a triangle having the same base as the segment and equal height, and again, in the segments that are left out, triangles be inscribed having the same bases as the segments and equal height, and in succession triangles be inscribed in the same way in the segments that are left out, let the resulting figure be said to be inscribed in the segment in a known way.

Using only one term, the adverb γνωρίμως, Archimedes defines the generic object of a class of figures, that can be obtained recursively from the first one: a triangle that, of course, is itself inscribed γνωρίμως. The choice of the word is in line with what we have seen so far, since the procedure determines univocally, for a given segment, each figure that is inscribed γνωρίμως. The fact that the number of vertices is arbitrary is crucial, since exactly this is required by the method of exhaustion: insofar as the figure inscribed γνωρίμως is arbitrary within a well-defined class, it is univocally determined.

(**3**) πόριμον «provided»
The term does not appear in the Euclidean *corpus*, but it is apparently connected with the meaning of «porism» alluded to in the title of the lost Euclidean treatise of the *Porisms*. It seems to refer to the notion of existence, in a wider sense that does not require the construction of the «provided» object: in the *Porisms*, in fact, it is shown that some element of a geometric configuration is given, but its construction is not performed (see Sect. 6 below). In Apollonius, *Con.* I.49, 50 (*bis*), 51 (*bis*) (*AGE* I, 146.12, 150.2 and 6, 154.23 and 26), the «provided» object is a straight line determined as a fourth proportional: it is provided already in the enunciation but it is neither constructed nor features among the givens of the problems. Of some interest is also the use made by Hero, in *Dioptra* 13–14 and 25–30, of forms of the verb πορίζειν in the sense of «to calculate», «to univocally determine» (about 20 occurrences at *HOO* III, 230–6 and 268–80). They are often supplemented with a modal connotation making it certain the synonymy with «given», via the intermediation of the *definientes* of *Data* def. 1–2.

(**4**) ῥητόν «exprimable»
This is a technical term of the theory of irrationals, and it is ubiquitous in book X of the *Elements*. The line assigned as a reference is said to be the «exprimable », as are said also all straight lines commensurable with it, either in length or in power only. Also a domain commensurable with the square on the reference line is said to be exprimable. Once the exprimable is fixed, the complementary concept is ἄλογος «irrational». Marinus' allusion to «to find [. . .] three exprimable straight lines commensurable in power only» sets the intended meaning in line with the one in the *Elements*, whereas his reference to Ptolemy seems to entail a conception of exprimability of a metrological kind, taking into account only lines commensurable in length as exprimable. This is confirmed by a subsequent discussion, where Marinus claims that «exprimable by itself is what we know by means of some number with respect to the measure of position, a palm, as it were, or a finger» and that the ratio of the diameter of a square to its side is not exprimable» (in the Euclidean theory it is so).

The same conception is behind the definitions of «exprimable» found in the pseudo-Heronian *Definitiones* 136.34–5 and in a number of scholia to *El.* X (for instance n. X.94 and 141, *EE* V,2, 127.13–25 and 145.4–8). In *Definitiones* 136.35 one finds also the adjective γνώριμος, whose connection with calculatory procedures we have just seen. Also in this characterization «known» works as a name of genre, very much like in the definition of «given» preferred by Marinus (*HOO* IV, 140.7–17; the opening definition is identical with the one in scholium n. X.9, *EE* V,2, 93.17–18):

> They define a exprimable <line> also thus: exprimable is the one which is known [γνωρίμη] by numbers. This is not a definition of exprimable, but an incidental character of it. For when, for the sake of the argument, exprimables be fixed among those <resulting> from the one-foot exprimable, we know of each of them how many palms or fingers it is; for this reason, from incidental characters we say that an exprimable is known by numbers. «Exprimable» differs from «given» insofar as the exprimable is altogether given, whereas a given is not

> necessarily exprimable. The exprimable is known in value and quantity, the given in value and magnitude only: and in fact, there are given irrational <lines>.

A scholium to the *Data* establishes a relation between the extensions of the notions of «given» and «exprimable» in the same way as Marinus and *Def.* 136.35 do, but ascribes the idea to Pappus (*EOO* VI, 262.1–7). In Pappus' commentary to *El.* X nothing about this issue can be found.

Marinus' description presents a variety of idioms replacing the one of the «givens», and this seems to reflect to some extent an uneasiness with the several meanings of «given» offered by the mathematical practice. When we pass from lexical issues to the actual use of the «givens», the situation gets even more complicated, as we shall see in the remaining sections.

6 Proving by the «givens». Analysis and synthesis

Until now, we have discussed the definitions of the several species of «givens». But how is a proof framed that employs the language of the givens? The purest form of application of the idiom as a deductive tool is found, most naturally, in the *Data*: let us read only in translation the proof of prop. 40 (*EOO* VI, 70.4–23), corresponding to the enunciation transcribed in the Sect. 2; references to propositions or to definitions of the *Data* themselves, or of the *Elements*, are added in brackets.

> For, let each of the angles of triangle ΑΒΓ be given in magnitude: I say that triangle ΑΒΓ is given in form.
>
> For, let a straight line ΔΕ be set out given in position and magnitude, and on ΔΕ at the points on it ΔΕ let a rectilineal angle ΕΔΖ be constructed equal to angle ΓΒΑ (*El. I.23*), and ΔΕΖ equal to ΑΓΒ (*El. I.23*): therefore ΒΑΓ as a remainder is equal to ΔΖΕ as a remainder (*El. I.32*).
>
> And each of the <angles> at Α Β Γ is given; therefore each of those at Δ Ε Ζ is also given (*def. 1*).
>
> Since then, on a straight line given in position ΔΕ and at a given point on it Δ a straight line ΔΖ is drawn making a given angle, the one at Δ, therefore ΔΖ is in position (29). Exactly for the same reason also ΕΖ is in position (29): therefore point Ζ is given (25). And each of ΔΕ is also given (27): therefore each of ΔΖ ΔΕ ΕΖ is given in position and magnitude (26): therefore triangle ΔΖΕ is given in form (39). And it is similar to triangle ΑΒΓ (*El. VI.4*): therefore triangle ΑΒΓ is also given in form (*def. 1 and 3, El. VI.def.1*).

Two remarks impose themselves.

(*a*) The main steps of the proof are the following: a chance straight line is «set out» and two angles are constructed on it that are equal to those of the triangle that is given in magnitude. This amounts to providing an "alias" of the triangle; it is then shown that some of the elements of the "alias" are given: its angles first, then two sides in position, and finally the intersection of these sides and the remaining two extremes of them. Since their extremes are given, the sides themselves are also given in magnitude. As a consequence, the form of the "alias" triangle is given, and therefore, by similitude and def. 3, also the form of the original triangle given in magnitude is given (cf. remark 4 of Sect. 3). The proof may seem uselessly complicated and roundabout, but its form is in fact the only possible one once the fundamental definitions *Data* 1–3 are formulated in the way they are (props. 25, 27, 29 applied in the text are, as we have seen, nothing but a rewriting of def. 4). The creation of the "alias" figure is the analytic counterpart of the "actualization" of the assigned magnitudes by means of the production of duplicates one finds in synthetic proofs such as the one of *El.* I.22: the three straight lines with which to construct a triangle are first set out as segments αἱ Α Β Γ, then "reproduced in duplicate" as the sides of the triangle.

In *El.* I.22 as well as in *Data* 40, in order to construct the "alias", one must set out an object that is fixed by stipulation and is free of constraints induced by the assigned objects (in *El.* I.22 a base straight line from which to "cut off" two of the three sides of the triangle, in *Data* 40 the straight line ἡ ΔΕ): the verb ἐκκεῖσθαι is exactly what is needed to fill this function. In the analyses, the same function is assumed by the initializing verbal form γεγονέτω (cf. remarks 1 and 4 *infra*).

(*b*) The inferences proving that an object is given are expressed in the language of the givens. It is not permitted to qualify an object as «given» simply starting from assigned entities, performing constructions on them or setting up demonstrative steps in a synthetic language, and finally attaching the label «given» to the object so obtained. As a consequence, it is of primary importance to identify and collect a series of deductive rules operating on «givens» and producing «givens» as output: the *Data* do exactly that, while leaving undecided what the term «given» must be taken to mean. This is the reason why such a meaning is never defined: what is really useful is to set out, in the form of definitions, zero-level rules of inference allowing to transfer the predicate «given» from an object to another by equality or identity. This is the primary function of *Data* def. 1–2; they are so cleverly framed that the first propositions of the *Data* do not have the axiomatic character that, for instance, *El.* I.4 has. Propositions with this drawback can be found only from prop. 25, where «given in position» is first introduced.

The universe of the *Data* is self-referential; the ones of the «porisms» and of the *locus* theorems have a clear geometrical rooting. The similarities between the enunciations of such kinds of propositions, both requiring to show that some object or some configuration is given, point to deep interrelationships, that are worth explaining in detail.

Euclid's *Porisms* is lost and we can get an idea of its contents only from the information provided by Pappus: a broad outline of the treatise (*Coll.* VII.13–20) and 38 lemmas that in the *Porisms* were apparently taken for granted (VII.193–232). A «porism» is a proposition in which it is required to show that some elements of an assigned geometrical configuration are given. Such "elements" can be ratios, domains contained by straight lines, lines insofar as geometric *loci*. The configuration is assigned in the broad sense that some of its elements are given, whereas others are subjected to constraints expressed in terms of the given elements. In short, a porism is a geometrical configuration with degrees of freedom, of which one must identify some invariants. As in a theorem of the *Data*, a proposition of the *Porisms* shows that some object is given. Therefore, the propositions of the *Data* are a subclass of those of the *Porisms* (namely, the ones in which all constraints are rigid), and we must expect that the structure of a proof of a porism and of a proposition of the *Data* be similar.

The first who understood this fact among Western scholars was Newton (Whiteside 1967–81 VII, p. 262), who correctly identified the *Porisms* as the key to understand ancient analysis. In the *Data* as well as in the *Porisms*, ambiguities might arise from the fact that an object can be «given» either because it is assigned or because it must be proven given. For instance, let us take again the enunciation of the porism read in Sect. 2:

> If straight lines from two given points be inflected on a line given in position, and one cuts off <a segment> from a straight line given in position up to a given point on it, the other too will cut off from another <line given in position a segment> having a given ratio <to the first>.

It is not clear whether the second segment, which is part of a straight line and has an extreme in a point that must undisputably be proven to be given, has to have with the first segment an assigned ratio, or it is enough to show that, once the straight line and the point are proven given, the ratio is uniquely determined for all segments. Clearly, the former interpretation, in which more restrictive conditions are imposed, is more general than the latter, and in fact implies it. Simson first proposed (1723) a reconstruction of the porism reported by Pappus in line with the second interpretation, but then (1776, a work published *post mortem*) he realized that the first one could work as well; the straight lines corresponding to different ratios are all parallel.

In a *locus* theorem the assigned objects are *all* given. In a porism some objects are allowed to undergo changes, within a class identified by well-defined constraints. In the case of our porism, two straight lines may move, even if they must intersect on a line given in position and each of them must pivot around an assigned point. Therefore, the idea that «a porism is what is lacking of a supposition in order to be a local theorem» (this characterization is reported with disapproval by Pappus, *Coll.* VII.14, because it was formulated «on the basis of an incidental property») very likely refers to the "incomplete" assignments relating to some of the objects present in the configuration of a porism: they are only constrained, not given. We must conclude that also the *locus* theorems are a subspecies of porisms, as Pappus already recognized: «the form of this class of porisms is the loci, and these abound in the analytical *corpus*» (*ibid.*).

The story does not end here. The language of the givens is applied also in propositions or contexts that do not require to show that some object or configuration is given. This works on two levels. The first can be found, in its most celebrated form, in the proof format of analysis and synthesis. The second is linked with the first but has a wider structural import: the idiom provides an unified framework able to formalize deductions operating on both relations and constructions. We shall discuss the first function presently; the following sections will deal specifically with the second.

Let us read, in translation only, a problem solved by analysis and synthesis: Apollonius, *Con.* II.50 (*AGE* I, 286.26–288.23). The only notion of the theory of conic sections needed to follow this proof is the so-called "property of the subtangent" to a parabola, that can be formulated in this way. From a point C of a parabola of vertex *A* drop the perpendicular to the axis and draw the tangent to the parabola; both of them fall on the axis of the parabola, say in points *B* and *D*, respectively. The segment *BD* of the axis between the two points of intersection is called subtangent. The "property of the subtangent" asserts that *BD* is bisected by the vertex *A* of the parabola (*Con.* I.35): *DA* = *AB*. The system of bold, italics and underlining with which I have presented the text gives prominence to the correspondences between steps of the analysis and of the synthesis.

> To draw a <straight line> tangent to a given section of a cone, which will make on the axis, on the same side as the section, an angle equal to a given acute angle.
>
> Let it be first as a section of a cone a parabola, whose axis is AB: then it is required to draw a <straight line> tangent to the section, which will make on the axis AB, on the same side as the section, an angle equal to a given acute angle.
>
> Let it have come to be, and let it be ΓΔ: therefore, angle BΔΓ is given. **Let a <straight line> BΓ be drawn perpendicular**: then the angle at B is also given. Therefore the ratio of ΔB to BΓ is given. *And the ratio of BΔ to BA is given:* therefore the ratio of AB to BΓ is also given. And the angle at B is given: therefore BAΓ is also given. And it is on a <straight line> BA <given> in position and at a given point A: therefore ΓA is in position. And also the section is in position: therefore Γ is given. And ΓΔ is tangent: therefore ΓΔ is in position.
>
> Then the problem will be synthesized thus. Let it be first as a section of a cone a parabola, whose axis is AB and the given acute angle EZH, and let a point E be taken on EZ, **and let a <straight line> EH be drawn perpendicular**, *and let ZH be bisected by Θ*, and let ΘE be joined, and let equal to angle HΘE be constructed an <angle> BAΓ, **and let a <straight line> BΓ be drawn perpendicular**, *and let equal to BA be placed a <straight line> AΔ*, and let ΓΔ be joined. Therefore ΓΔ is tangent to the section.
>
> I say now that ΓΔB is equal to EZH.
>
> For, since it is as ZH to HΘ, so ΔB to BA, and it is also as ΘH to HE, so AB to BΓ, therefore, ex aequali, as ZH to HE, so ΔB to BΓ. And the angles at H, B are right: therefore angle Z is equal to angle Δ.

The last statement of the "chain of givens" with which the analysis ends claims that the predicate «given in position» applies to the same object that at the very beginning was supposed (imperative γεγονέτω) to feature in the final configuration of the problem; we might say that this object is "potentially constructible", and to have established its "existence and uniqueness". However, the analysis has a disturbing feature: it opens and closes by asserting two different properties of the same object. As a consequence, the synthesis cannot simply consist in an inversion of the analysis. Furthermore, the connection between a chain of givens and a sequence of constructions solving the problem is a priori ineffable. What is certain is that the *Data* does not tell us how to do it.

Data def.1 is crucial in making the analytical engine start to work: the angle ἡ ὑπὸ BΔΓ is given exactly because, in the final configuration supposed as realized, it is equal to another angle (a given one, but this fact is totally irrelevant), not because the final configuration is «given», which is exactly what the analysis must *prove*. It is crucial, then, that pairs of "identical" objects are displayed during the proof: this is the reason way the enunciation requires that the tangent «make on the axis an angle *equal to* a given acute angle» and not «make on the axis a given acute angle». The subsequent inferences, constituting the chain of givens, will be transformed and rearranged to make up the synthesis. The most notable features are the following:

(1) The verbal form that initializes the analysis, γεγονέτω «let it have come to be», is an invariant stylistic trait with a strong mathematical meaning. It is a perfect tense with state value that *does not* assume that the problem is solved, but only that the final configuration has come to be realized. The problem is in fact solved by performing the construction, not by "displaying" the final configuration. (It is important to stress that to assume the realization of the final configuration does not characterize the proofs by analysis and synthesis, as is commonly held: no *locus* theorem could begin in this way.)
(2) Some of the constructions of the synthesis correspond in the analysis to steps that are included in *deductive* chains formulated in the language of the givens. The direction of the deductive progression and the ordering of the sequence of constructions is the same. (Broken underlining and sentences in italics.)
(3) The truly deductive steps of the synthesis can be found also in the analysis, in (approximately) inverted order and again expressed in the idiom of the givens. (Unbroken underlining.)
(4) The fictitious object featuring in *Data* def. 1–4 in order to "saturate" a relation is actualized in the synthesis as a «given» of the problem or as an immediate evolution of it (for instance, triangle τὸ ΕΖΘΗ). In its turn, the latter will be duplicated in the proof as triangle τὸ ΓΔΑΒ, in order to «provide» the sought-for object. In a similar way, logical or geometrical objects featuring only once in the analysis can be found in duplicate in the synthesis. (In bold the construction of the perpendicular, ἡ ΒΓ or ἡ ΕΗ/ΒΓ, in italics the doubling of the property of the subtangent, transformed in two constructions.)
(5) Furthermore, what in the analysis is qualified as «given», features in the synthesis within equalities or identities of ratios: the analysis displays predicates, the synthesis relations. This fact shows that it is meaningless to claim that some chain of inferences in the analysis is the inverse of some one in the synthesis. At best, an isomorphism can be established. Yet, the correspondence remains misleading, as propositions in the *Data* and in the *Elements* cannot be related exactly, and, what is worse, those in the *Data* are usually proven by means of the corresponding ones in the *Elements*. For instance, there is an obvious symmetry between the steps applying *Data* 40 in the analysis and *El.* VI.4 in the synthesis or *Data* 41 in the analysis and *El.* VI.6 in the synthesis, but (*i*) *Data* 40 is proven by means of *El.* VI.4 and *Data* 41 by means of *El.* VI.6; (*ii*) in order to show that the relevant ratios and angles are given one must, in the analysis, apply def. 3 to disentangle them from the notion of a figure «given in form», whereas the synthesis may refer directly to VI.4 and VI.6 (where the notion of similar triangles does not appear). It follows that a parallel can be set at best between VI.6 and VI.4. Yet they are not each inverse of the other, but only partial converses. There is more: the inference *ex aequali* licensed by *El.* V.22 corresponds to what? To

Data 8, which is proved, with an explicit reference to *ex aequali*, by means of V.22 itself. This very inference, and not a sort of inverse of it, is used in *Con.* II.50, only formulated in language of the givens. In short, the deduction in *Con.* II.50 is a logical labyrinth: it arranges the "isomorphic" propositions of the *Data* and of the *Elements* in the same order, thanks to a clever permutation of the angles from which analysis and synthesis start and to the fact that VI.4 and VI.6 are partial converses. To appreciate that a true inversion of deductive order is at work one has to check the position of the «ratio of ΔB to BΓ» within the two chains.

(6) The final part of the analysis, not underlined in the text, gives rise in the synthesis to a sequence of constructions: these are the "geometric" passages of the chain of givens. For instance, *Data* 29 corresponds to *El.* I.23 and both are applied to the same angle, but, on the other hand, what corresponds to *Data* 25 remains unstated (it is for instance a consequence of *Con.* I.17). The steps related to the property of the subtangent, even if they are contained in the underlined portion, become two constructions: a bisection (within triangle τὸ ZHE) and a doubling (within triangle τὸ ΔBΓ) of a straight line; the converse of the property is then applied in the form of *Con.* I.33. The only auxiliary construction (dropping perpendicular ἡ BΓ) is also provided in double in the synthesis, once for each triangle. If it is not immediate to set an isomorphism between the chain of the givens and the synthetic proof, it is simply meaningless to claim that a sequence of constructions has the same or the opposite direction as a chain of givens, or simply that the former corresponds to the latter. Two constructions cannot be each a consequence of the other; they can at best come one after the other, the former giving rise to a geometric configuration suited to perform the second. In the chains of givens, on the contrary, what is at stake is always a deductive sequence. The only residual correspondence is that, with respect to the actual development of the proof, the chain of givens and the construction have the same direction: see for instance the ordering of the sequence "straight line ἡ ΓA (that is, angle ἡ ὑπὸ BAΓ) → straight line ἡ ΓΔ" both at the end of the analysis and in the construction.

All of this confirms the remarkable versatility of the predicate «given», which can be applied to the determination of any mathematical object, either by constructive or deductive means, for instance through chains of equalities. I shall further substantiate this view in the following sections.

7 Calculations validated by "chains of givens"

The chains of givens have a prominent algorithmic connotation: this feature can be naturally linked with the function of the analysis as a demonstrative format able to settle problems of existence and uniqueness (see next section). A whole tradition of Greek mathematicians (Hero, Ptolemy, Diophantus) read the analysis in an algorithmic perspective: in these authors, «given» has the meaning of «univocally determined» and hence «calculable» from the numerical assignments of a problem. Accordingly, a chain of givens came to be straightforwardly viewed as validating either (*a*) a "formula" devised to express some magnitude, whose value will be calculated in the synthesis by a procedure conforming to the formula itself—take for instance *Metrica* I.14 to be read below, of which the enunciation in Pappus, *Coll.* IV.11, read in Sect. 2 is a variation in pure geometrical style—or (*b*) a procedure extracting one of the magnitudes involved in a formula (*Alm.* I.10). Let us see the issue in more details.

The aim of Hero' *Metrica* is to set demonstrative grounds to the procedures of calculation of areas (book I) and volumes (II) of some basic geometrical figures and to problems of division of plane domains (III): he offers rigorous geometrical proofs validating algorithms. Paradigmatic calculations are then performed, by using ad hoc numerical values. The geometrical proofs are sometimes in synthetic form, but they are most often in analytical form. That the sought-for geometrical magnitude can be determined is proven in strictly geometrical terms by means of an analysis, the magnitude itself being the final product of a chain of givens. In the synthesis the calculation of the same magnitude is performed, either as a description of the procedure supported by numerical examples, or on the grounds of a paradigmatic "raw" calculation. Both of them are exactly parallel to the chain of givens.

Validating a calculation by a chain of givens requires very few tools, which often help with extracting a magnitude from a "formula". These are some definitions and theorems of the *Data*, which correspond to operations on the numerical values assigned to the given magnitudes, as we see in the following table. In the right column, the signs a and b denote magnitudes having an explicitly expressed numerical value; the arrow separates the input from the output of the operation.

Proposition of the *data*	Operation
def. 1, 2. Given in magnitude are said figures, lines, and angles for which we can provide equals. A ratio is said to be given for which we can provide the same.	$a, a = b \rightarrow b$
1. The ratio of given magnitudes to one another is given.	$a, b \rightarrow a{:}b$
2. If a given magnitude have a given ratio to some other magnitude, the other is also given in magnitude.	$a, a{:}b \rightarrow b$
3. If any number of given magnitudes be added together, the magnitude composed of them will also be given.	$a, b \rightarrow a + b$
4. If a given magnitude be subtracted from a given magnitude, the remainder will be given.	$a, b \rightarrow a - b$
5. If a magnitude have a given ratio to some part of itself, it will also have a given ratio to the remainder.	$a{:}b \rightarrow a{:}(a - b)$
6. If two magnitudes having a given ratio to one another be added together, the whole will also have a given ratio to each of them.	$a{:}b \rightarrow (a + b){:}a, (a + b){:}b$
52. If on a straight line given in magnitude a form given in form be described, the <described> form is given in magnitude.	$a \rightarrow q(a)$
55. If a figure be given in form and in magnitude, its sides will be given in magnitude.	$q(a) \rightarrow a$
57. If a given <domain> be applied to a given <straight line> in a given angle, the width of the application is also given.	$ab, a \rightarrow b$
85. If two straight lines contain a given area in a given angle, and their sum be given, each of them will be given.	$ab, a + b \rightarrow a, b$
5 + 8. <Magnitudes> which have a given ratio to the same, will also have a given ratio to one another.	$a{:}b \rightarrow (a - b){:}b$

Notice that the *separando* manipulation of ratios requires combining two theorems of the *Data*, 5 and 8. Other theorems validate geometrical constructions having an immediate operative import in the metrological domain, most notably in the divisions of surfaces of *Metrica* III.

Proposition of the *data*	Construction
26. If the extremities of a straight line be given in position, the line is given in position and in magnitude.	Join two points by a segment of a given length
27. If the one extremity of a straight line given in position and in magnitude is given, the other will also be given.	Cut off a segment of a given length

In the *Metrica* these are taken as operations and not as constructions: this is confirmed by the fact that they are formulated with the aorist imperative: see for instance τοσούτων ἀπόλαβε τὴν ΑΕ καὶ ἐπίζευξον τὴν ΔΕ at *Metrica* III.2 or, in the same proposition, the clause in suppositive mode with an aorist subjunctive ὥστε ἐὰν ἀπολάβωμεν τὴν ΑΔ μονάδων ια δʹ καὶ παράλληλον ἀγάγωμεν τὴν ΔΕ, ἔσται τὸ προκείμενον (*HOO* III, 144.29–30 and 144.12–14, respectively). That the first operation is sometimes formulated with a perfect imperative (ἀπειλήφθω, for instance in III.5–6) is probably due to the stylistic inertia of the model of geometrical constructions.

Notice also that the calculation of the area of triangle of given sides $a, b, c \rightarrow tr(a, b, c)$, whose synthetic validation we read in *Metrica* I.8, has the combination of *Data* 39 and 52 as an immediate analytic counterpart: if the three sides a, b, c of a triangle are given in magnitude, the triangle is also given in magnitude.

The crucial point of Hero's approach is that the calculation, very much as a geometrical synthesis, is considered accessible only after the analysis has shown that the sought-for object is univocally determined by the assignments of the proposition: in metrical contexts, «given» is synonymous with «given in magnitude» and has the numerical determination of that «magnitude» as a "synthetic" counterpart. This guarantees that one can proceed to actually «providing» it, by means of either a calculation or a construction. This is the reason why the analytical format is identical in a metrical and in a geometrical proposition. Let us read in parallel the validating chain of givens and the calculation-synthesis of *Metrica* I.14 (*HOO* III, 36.23–40.10); one has to determine the area of a quadrilateral τὸ ΑΒΓΔ having the angle at τὸ Γ right, no pairs of side parallel, and given sides ἡ ΑΒ of 13 units, ἡ ΒΓ of 10, ἡ ΓΔ of 20, ἡ ΔΑ of 17. The procedure splits the quadrilateral in two triangles and adds their areas. The calculation of the area of the right-angled triangle τὸ ΒΓΔ is immediate; as for triangle τὸ ΑΒΔ, one has to determine first the height ἡ ΑΕ by means of *El.* II.13.

To show that its area is given.	And as a consequence of the analysis it will be synthesized thus.
Since both of ΒΓ ΓΔ are given and the angle at Γ is right, therefore triangle ΒΓΔ is given,	The 10 by the 20: it results 200; and half of these: it results 100.

and again, the <square> on BΔ will be given	And again, the 10 by themselves: it results 100. And the 20 by themselves: it results 400. Add: it results 500.
– for it is of 500 units –;	
but also the one on AB is given:	And the 13 by themselves: itresults 169;
thus, the ones on AB BΔ are given;	these with the 500: it results 669;
and they are greater than the one on AΔ: therefore angle ABΔ is acute: therefore the <squares> on AB BΔ are greater than the one on AΔ by twice the <rectangle contained> by ΔB BE:	
therefore twice the one by Δ B BE is given:	subtract the 17 by themselves: 380 as a remainder;
so that once the one by ΔB BE is also given;	half of these: it results 190;
and it is a side of the <square> on BΔ by the one on BE:	
therefore the one on BΔ by the one on BE is also given;	these by themselves: it results 36,100;
and the one on BΔ is given: therefore the one on BE is also given;	these applied to the 500: it results 72 1/5;
but also the one on EA	subtract these from 169: it results 96 1/2 1/5 1/10 as a remainder;
by the one on BΔ;	these by the 500: it results 48,400;
and its side is the <rectangle contained> by BΔ AE:	
therefore the one by BΔ AE is also given;	A side of these: it results 220;
and it is the double of triangle ABΔ:	
therefore triangle ABΔ is also given;	half of these: it results 110. That much will be the area of ABΔ.
but also BΓΔ was:	But, also, the <area> of B ΓΔ is of 100 units:
so that the quadrilateral ABΓΔ will be given as a whole.	therefore, the area of quadrilateral AB ΓΔ will be of 210.

An analogous approach can be found in *Liber de canonio* 4 (Clagett and Moody 1960, pp. 72–74).

We find the same format in a purely geometrical problem: in *Metrica* III.4 Hero deals with a problem of division of a figure. This is solved by a canonical analysis, after which we might expect to find a geometrical synthesis, but Hero directly calculates the positions, with respect to the vertices of the figure, of the points at which the sides must be cut in order to have the required division performed. The analysis shows that such points are univocally determined by the assignments of the problem, and the calculation of their position can be effected without further ado.

Variations on this theme can be found in *Alm.* I.10, where Ptolemy calculates the values of a table of chords with a step of 1/2 degree. In order to do that, he needs procedures for calculating the chords of complementary arcs, and the analogs of our bisection, addition, and subtraction formulas. His proof of how to calculate the chord of the arc difference of the arcs relative to two given chords starts from the assigned chords αἱ AB, AΓ and from the diameter ἡ AΔ; the quadrilateral τὸ ABΓΔ, which is inscribed in a semicircle, is then completed. Here is the enunciation (*POO* I.1, 37.19–38.3):

> This being set out as a preliminary, let it be a semicircle ABΓΔ on a diameter AΔ, and from A let two lines AB AΓ be drawn across, and let each of them be

> given in magnitude, such that the diameter is given as 120, and let ΒΓ be joined. I say that this too is given.

Now, "Ptolemy's theorem", proven in another form in *Data* 93, tells us that for quadrilaterals inscribed in a semicircle $r(\mathrm{AB},\Gamma\Delta) + r(\mathrm{A}\Delta,\mathrm{B}\Gamma) = r(\mathrm{A}\Gamma,\mathrm{B}\Delta)$ holds. But all magnitudes in the expression, the straight line ἡ ΒΓ excepted, are assigned by the problem, ἡ ΓΔ and ἡ ΒΔ as chords complementary to assigned chords. The chain of givens set forth by Ptolemy simply explains how to extract ἡ ΒΓ from the expression: what is at issue is to explicitate a variable contained in a closed formula, and this is what the enunciation claims. The corresponding calculation is not performed but it is clear that the goal is to validate a procedure aiming at providing a numerical value. The same approach of setting out only the general framework of the chain of givens, the calculation being performed when specific numerical data are at hand, can be found again in Ptolemy, *Analemma* 5 (*POO* II, 206.17–209.3), and Hero, *Dioptra* 14 (*HOO* III, 234.19–238.2).

Some ancient commentators on Ptolemy's *Almagest* were apparently uncomfortable with calculations performed without any explanations of why exactly the adopted procedure was followed. For instance, at *Alm.* V.5 Ptolemy calculates the direction of the mean apogee of the moon. He only partly describes the related diagram, proceeding directly to perform the calculation with the actual numerical data. Pappus writes in his commentary: «we shall analyze the 5° theorem of the *Composition* in this way» (*iA*, 35.21–2). The analysis Pappus provides is a chain of givens exactly parallel to Ptolemy's calculations, does not include a clarification of the related diagram, and lasts 36 lines in Rome's edition. Such an alleged "analysis" is nothing but a reformulation of Ptolemy's procedure without references to numerical values. When Pappus comes to show that intermediate quantities with some interest are given, he stops for a while and comments: «if the numbers are inserted one proves...» and calculates the numerical value of the quantity arrived at.

We find a format analogous to Ptolemy's in Diophantus'*De polygonis numeris*: the only difference is that the explicit calculation is replaced by the description of a procedure. Diophantus wants to determine the polygonal number with an assigned multiplicity of angles once the side is known. The basis of Diophantus' procedure is the closed expression linking a polygonal number to its side and to the multiplicity of its angles (*DOO* I, 472.16–9):

> Every polygonal multiplied by eight times the <number> less by a dyad than the multiplicity of angles, and taking in addition the <square> on the <number> less by a tetrad than the multiplicity of angles, makes a square.

We would write in the following way, with P a polygonal number, s its side, v the multiplicity of its angles (the latter an assigned parameter): $8P(v-2) + (v-4)^2 = (2 + (v-2)(2s-1))^2$ or, as Diophantus says, $8P(v-2) + (v-4)^2 = \textit{square}$.

Let us read how Diophantus formulates the validating chain of givens and the procedure determining a polygonal number once its side is known (*DOO* I, 472.20–474.20); this amounts to extracting P from the above formula (*legenda*: $\mathrm{H}\Theta = s$ and therefore $\Theta\mathrm{M} = s-1$, $\mathrm{KB} = v-2$, $\mathrm{NK} = 2$, and the rest follows):

The Hypsiclean definition and the present one of polygonal <numbers> being thus simultaneously proved, it remains to show how, a side being given, the assigned polygonal is found.

In fact, having given a side of some polygonal, HΘ, and having also the multiplicity of its angles, we also have KB among the givens, so that we shall also have given the <rectangle contained> by HΘ ΘM, the one and the other together, and KB, which is equal to Nξ: so that we shall also have given Kξ—since NK is a dyad —: so that we shall also have given the <square> on Kξ, and subtracting from this the square on NB, which is given, we shall also have given the remainder, which is a multiple of the sought for polygonal by eight times KB: so that also the sought for polygonal can be found. [...]

We shall describe it in a more pedagogical way, also for those who aim at quickly understanding, by means of procedures, what is sought for.

In fact, taking the side of the polygonal, always doubling it we shall subtract a unit, and multiplying the remainder by the <number> less by a dyad than the multiplicity of the angles we shall always add a dyad to the result, and taking the square on the result we shall subtract from it the <square> on the <number> less by a tetrad than the multiplicity of the angles, and dividing the remainder by eight times the <number> less by a dyad than the multiplicity of the angles we shall find the sought for polygonal.

8 The deductive function of the "chains of givens"

I shall sum up in this section the results of the previous discussion, combining them in a unitary interpretative framework. I shall argue that the chains of givens have two main functions:

(1) To formulate issues of existence and uniqueness (for the latter see also Taisbak 2003, p. 95).
(2) To formulate as a deduction disparate demonstrative steps such as deductions, constructions, calculations, chains of preconditions (namely, the inferences of the synthesis viewed "backward").

(1) *Existence and uniqueness*
Let us see in detail the first function on the example of the *locus* theorems. Their enunciation, in a conditional form that attests for their being theorems and not problems, employs the language of the givens to describe the constraints and to identify the line solving the locus. The proof is framed as an analysis and synthesis. The analysis, again expressed in the idiom of the givens, consists in identifying as given the line that a point subjected to the assigned constraints comes to ἅψεσθαι «touch»; the nature of the solution, it is to be stressed, is already made explicit in the enunciation. In other terms, if a point satisfies the constraints, then it belongs to a well-determined curve—this corresponds to establishing both the existence and the uniqueness of the *locus*, even if, in the enunciation of a *locus* theorem, what is sought is formulated as a predication: «a point A touches such-and-such line». Let us read the *locus* in Pappus,

Coll. IV.78 (the point is on a hyperbola because of a characteristic property that we read in *Con.* I.21):

> A straight line AB in position, and from a given point Γ let some <straight line> fall on ΓΔ, and let ΔE be at right <angles> with AB, and the ratio of ΓΔ to ΔE be <given>: that E is on a hyperbola.
> Let a <straight line> ΓZ be drawn through Γ parallel to the one at right <angles>: therefore Z is given. And parallel to AB a <straight line> EH, and let the ratio of ΓZ to any of ZΘ ZK be the same as the one of ΓΔ to ΔE: therefore any of Θ K is given. Since then it is as the <square> on ΓΔ <to the one on ΔE, so> the one on ΓZ to the one on ZΘ, therefore the ratio of the one on ZΔ as a remainder, that is the one on EH, to the one on KHΘ as a remainder is also given. And K Θ are given: therefore E is on a hyperbola through Θ E.

The scanty and late evidence on the issue of existence and uniqueness in the analytical domain puts emphasis on existence only, probably because uniqueness was taken for granted (for instance, a characteristic property univocally determines a conic section with assigned parameters). This view is corroborated by a passing remark by Pappus about some recent authors who, by applying Euclid's *Porisms*, neglected problems of constructibility, «proving only that what is sought exists without providing it [*verb* πορίσασθαι]» (*Coll.* VII.14). Since the *Porisms* obviously contained only analyses, one must conclude that the authors mentioned by Pappus had it clear that this kind of propositions could be interpreted as existence proofs.

To do the synthesis of a *locus* means to construct the sought-for line on the basis of the data of the theorem and to show that it satisfies the assigned constraints. Since the solution of a *locus* has to be a previously known line (straight line, circumference, conic section, ...), many *locus* theorems are nothing but the inverse of known characterizations of such lines, namely, the theorems in which the constraint is proved to be a property of them: Proclus calls «locus theorems» exactly the cluster *El.* I.35–8 (*iE,* 394.11–396.9, and passing mentions at 405.4–6, 412.5–7 and 431.23).

Locus theorems such as we find in Pappus usually have a synthesis. However, the format of these propositions underwent an evolution. In a first phase, the *loci* were only analyzed. This view is corroborated by the lemmas Pappus proved for the Euclidean *Loci on a surface*, that actually are the syntheses of some *loci*, and by some problems and *loci* that he very likely extracted from Aristaeus' *Solid loci*, where the syntheses are either lacking or are later appendages (Jones 1986, pp. 582–584, 591–595): this suggests that these works contained only analyses. Furthermore, the enunciation itself of a *locus* theorem requires one to show that a point touches a line «given in position», and the attested analyses end exactly with a proof of this fact: the synthesis performs a construction that is simply not required by the enunciation. In a second phase, the analysis was supplemented by a synthesis where the line solving the *locus* is constructed starting from the initial data. Apollonius is representative of this phase: Arabic sources confirm that he was used to synthetize the *loci* (Hogendijk 1986, pp. 206–218). A third phase is attested for only in late texts, such as for instance the Eutocean "transcription" of the Apollonian solution of a problem already mentioned in Aristotle's *Meteorologica* (*AGE* II, 180.11–184.20; cf. Vitrac 2002). This texts adheres to a strictly synthetic format: therefore, it must provide a synthetic proof of the uniqueness of the solution (that normally pertains to the analysis), showing that no point that does not belong to it satisfies the constraints. The synthetic mode also makes it necessary to change the manner of identification of the solution in the enunciation: one must prove that «it is possible to trace» the line.

It is not surprising that the demonstrative format underwent an evolution, while the formulation of the enunciation was kept fixed: the latter is what makes a *locus* theorem immediately recognizable as such. One might wonder why the Greek geometers became dissatisfied with the analysis of the *loci*. Several phenomena may have contributed to the perception that it was necessary to add a synthesis to a *locus* theorem, and maybe Apollonius himself is responsible for this reform. Requiring the construction of the solution, that can be performed only in a synthesis, is among the "foundational" concerns that are typical of him (Acerbi 2010). On the other hand, one must be able to complete such syntheses. They normally apply crucial results of the theory of conic sections; a suitable supply of such tools was at the mathematicians' disposal only with Apollonius, as he himself pointed out in the prefatory letters to *Con.* I and IV (*AGE* I, 4.5–17, and II, 4.5–7, 4.16–17). The generality of his approach, in particular the "discovery" of the opposite sections, made the number of tractable (solid) *loci* considerably larger than in earlier approaches; his complete treatment of the intersections between conics eased the analyses of the diorisms.

(2) *Transforming non-inferential propositional chains into deductive sequences*

Let us come to the second function. As I have pointed out in Sect. 6, the properly deductive steps of a synthesis can be found in the analysis, in an approximately reversed order and expressed in the language of the givens. In this way, a sequence of preconditions is formulated as a "forward" deduction. This is a nontrivial solution to a nontrivial problem. To see this, it is enough to compare it with the highly contrived solution to the same problem adopted in the so-called theorematic analyses, namely, those preliminary to the synthetic solution of a *theorem*, not to the constructive solution of a problem. In a theorematic analysis the chain of givens is absent: since it is not required to construct an object, there is no place to set up the chain corresponding to the construction. This fact probably suggested also that the chain of givens corresponding to the deductive passages could be replaced by the structure we are going to read on the example of the theorematic analysis in *Coll.* IV.17:

> Let it be a semicircle ABΓ, and let ABΔ be inflected, and let AB be equal to BΔ, and let a <straight line> ΔE be drawn at right <angles> with BΔ, and let BE be joined, and let a <straight line> EZ be drawn at right <angles> with it, and the center be H, and let it be as AH to HΔ, so ΔΘ to ΘZ, and let ΘE be joined: that angle BEΔ is equal to angle ΔEΘ.
>
> Form H let a <straight line> HK be drawn at right <angles> with BE: therefore BK is equal to KE. And BΔE is right: therefore the three BK, KΔ, KE are equal to each other. And HK is parallel to EZ, and since angle KEΔ equal to angle ΔEΘ was sought for, and ΔK is equal to KE, therefore that (ὅτι ἄρα) angle KEΔ is equal to KΔE, therefore that KΔE is equal to ΔEΘ, therefore that ΔK is parallel to EΘ.
>
> Let also a <straight line> KΛ be drawn parallel to ΔE, and let ΓΔ be produced as far as Λ, and let BΛ be joined. Thus, since KΛ is parallel to ΔE, and KH to

EZ, and KΔ parallel to EΘ was sought for, therefore that—insofar as triangle KΛH is equiangular to triangle EΔZ, and ΔKH to EΘZ—it is as ΛH to HK, ΔZ to ZE, and as KH to HΔ, EZ to ZΘ: therefore that also as ΛH to HΔ, so ΔZ to ZΘ (for *ex aequali*): therefore that also as ΛΔ to ΔH, so ΔΘ to ΘZ (for *separando*). And it was also supposed as ΔΘ to ΘZ, so AH to HΔ: therefore that as ΛΔ to ΔH, so ΔΘ to ΘZ, that is AH to HΔ: therefore that ΛΔ is equal to AH: therefore that also ΛA is equal to ΔH. But also AB is equal to BΔ: therefore that also ΛB is equal to BH. But BH is equal to both of ΛΔ AH: therefore that also BΛ is equal to ΛΔ. And it is: for since KΛ is parallel to ΔE, and ΔK is equal to KE, also angle BKΛ is equal to ΛKΔ. Thus, since BK is equal to KΔ and angle BKΛ is equal to angle ΔKΛ, therefore also BΛ is equal to ΛΔ.

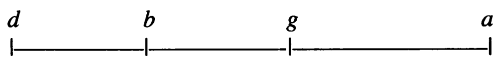
In an analysis of this kind two registers overlap: (1) the "forward" inference, directly operating on the object language and whose progression is marked by the connector ἄρα «therefore» and by transformations such as the passage in *separando*; (2) the logical arrow implicit in the ὅτι «that» (understood «it is to be shown» or the like), acting as a second-order language and providing a direction to the sequence of "backward" inferences, that is, to the chain of preconditions. This overlap of stylistical levels is the projection of a fundamental ambiguity: a theorematic analysis, because of the absence of the chain of givens, quite frequently has a strong algorithmic connotation—series of equivalences, manipulations of proportions—and its steps can normally be inverted: therefore, it is a question of personal taste whether to formulate it as a sequence of consequents or of antecedents. The ancient mathematicians (theorematic analyses can already be found in Apollonius, *Con.* III.24–26, propositions that are surely pre-Eutocean) left the ambiguity unresolved, resorting to the bizarre format of stating conclusions in a declarative form (ὅτι ἄρα «therefore that»): the stylistic inertia of the synthetic (and hence "forward") format is unable to conceal the "mathematical" perception that one is proceeding backward with respect to the "natural" logical ordering.

There is more to the issue. The solution of Greek mathematicians to the problem of giving the cogency of some inferences an adequate linguistic representation was dictated by the perception that lexical and stylistic conventions, when rigidly pursued, have an actual mathematical import. The rigidity of formulation in a natural language replaces the formalism. This tendency can be viewed as a struggle toward minimizing the "intuitive" component at work in a mathematical proof. As a matter of fact, "intuitions" are required:

(*a*) At every conclusion, be it the conclusion of an inference or the conclusion we draw from the whole argument of the proof of a proposition (that is, its enunciation). The answer to the second problem, which is the one of mathematical generality, was constituted by the indefinite structure I have studied elsewhere (Acerbi 2011a). The first problem is partly answered by the rigidity of the formulaic system. Some particular deductions, most notably the ones involving relations, can be said to conclude in virtue of their form: in a sense, the conclusion shows itself in a suitable notation (Acerbi 2011b, Sect. 1.5.1).

(*b*) In the auxiliary constructions. How is one to understand which auxiliary constructions must be performed? There is no answer, of course. However, one may introduce tools that are able to minimize the intuitive component. First of all, the constructions are formulated in a rigid matricial structure, that does not interact with the deductions but that in some sense reproduces their formulation. In a strong linguistic sense, then, *El.* I.post.1 has as a correlate only one operation of joining a straight line; it is a two-entry relation of sorts, the free variables being simply the denotative letters of the extremes. Second, and most importantly, the chains of givens transform the constructive sequences into deductive sequences. To repeat, in an analysis constructions and deductions are placed at the same level.

(*c*) At every deductively self-contained step. Who gives us a rule to understand which passage is to follow a given one? No one, of course, but one can do it in such a way as to reduce the number of possible choices. Paradigmatic examples are the analyses elaborated by Hero as alternative proofs of *El.* II.2–10 (Tummers 1994, pp. 73.25–86.5). In the first place, they purposely, and admittedly, lack the auxiliary constructions: as a consequence, their analytic part bears no signs of the language of the givens. Second, they are framed as pure reductions that operate on two levels. At a first level one takes two expressions constructed starting from the objects featuring in a geometric configuration; one must show that the expressions are equal. The mathematical "facts" one deals with are some configurations of geometrical objects, not the relations between them. In the Heronian procedure, one of the configurations is assumed as the starting point, the other as the end point; the former is reduced to the latter by means of theorems in the same sequence II.2–10 (II.1 features as a "principle"). At a second level, however, the declared goal of the whole analysis is exactly to set up a complete list of the theorems in the sequence II.2–10 to which the one at issue is reduced: this is again an analysis. The proof is conceived as a process of explication of the passages that are implicit in it, both as a deployment of the deductive structure in its complete form and as a decomposition of the geometric configuration in its ultimate components. To clarify the issue, it is worth reading in the first analysis of Hero, *El.* II.8 *aliter* (Tummers 1994, pp. 81.20–82.10)—the proof is really made up of two analyses, even if the second one is called *compositio*:



Ponam lineam ***ab*** quam super punctum ***g*** dividam qualitercumque contingat divisio, et adiungam ei lineam ***bd*** equalem linee ***bg***. Cum ergo resolverimus quadratum linee ***ad*** resolvetur in probationem figure quarte huius partis. Quod ideo erit quoniam quadratum factum ex linea ***ad*** est equale duplo superficiei quam continent due linee ***ab***, ***bd***, cum duobus quadratis factis ex duabus lineis ***ab***, ***bd***. Et quia ***bd*** posita est equalis sectioni ***bg***, ergo duplum superficiei que continetur a duabus lineis ***ab***, ***bg***, cum duobus quadratis factis ex duabus lineis ***ab***, ***bg*** est equale quadrato facto ex linea ***ad***. Secundum probationem vero figure septime huius partis erunt duo quadrata facta ex duabus lineis ***ab***, ***bg*** equalia duplo superficiei que continetur a duabus lineis ***ab***, ***bg***, cum quadrato ***ag***. Cum ergo illud coniungetur, erit quadruplum superficiei que continetur a duabus lineis *ab*, *bg*, cum quadrato *ag* equale duplo superficiei que continetur a duabus lineis *ab*, *bg*, cum duobus quadratis factis ex lineis *ab*, *bd*. Sed iam ostendimus quod ista sunt equalia quadrato facto ex linea *ad*: ergo quadruplum superficiei que continetur a duabus lineis *ab*, *bg*, cum quadrato *ag* est equale quadrato *ad*. Ergo iam resolutum est hoc in figuram quartam prius, post in figuram septimam. Et illud est, quod demonstrare voluimus.

I shall place a line *ab* that I shall divide at point *g* with a chance division, and I shall add to it line *bd* equal to line *bg*. Therefore, once we shall analyze the square of line *ad*, it shall be analyzed in the proof of the fourth figure of this part. This is because the square made from line *ad* is equal to the double of the superficies that contain the two lines *ab*, *bd*, together with the two squares made from the two lines *ab*, *bd*. And since *bd* is placed equal to segment *bg*, therefore the double of the superficies that is contained by the two lines *ab*, *bg*, together with the two squares made from the two lines *ab*, *bg* is equal to the square made from line *ad*. And by the proof of the seventh figure of this part, the two squares made from the two lines *ab*, *bg* will be equal to the double of the superficies that is contained by the two lines *ab*, *bg*, together with square *ag*. Therefore, when this is conjoined, the quadruple of the superficies that is contained by the two lines *ab*, *bg*, together with square *ag*, is equal to the double of the superficies that is contained by the two lines *ab*, *bg*, together with the two squares made from the lines *ab*, *bd*. But we have proved that these are equal to the square made from line *ad*: therefore, the quadruple of the superficies that is contained by the two lines *ab*, *bg*, together with square *ag*, is equal to square *ad*. Therefore, this is analyzed in the fourth figure first, then in the seventh figure. And this is what we wanted to prove.

Hero's achievement finds an ideal domain of application in the linear lemmas of *El.* II, but an interesting parallel can also be made with a particular logical doctrine. This is the analysis of those formally valid arguments that in the Stoic tradition got the name of «non-simple undemonstrated <syllogisms>» (Sextus Empiricus, *M*. VIII.228). Simple undemonstrated are such inferences as it is «immediately clear that they conclude validly, namely, that for them the conclusion is validly deduced from the premises» (*ibid.*). Chrysippus established five basic kinds, among which forms of *modus ponens*, *modus tollens*, etc. can be recognized. The arguments made up of chains of simple undemonstrated were said non-simple undemonstrated. The latter too are valid arguments, but this can be appreciated only after they are reduced to their simple components. The Stoics called «analysis» this procedure, which was performed by means of rules to manipulate deductions, called *themata*. «Theorems» were also elaborated, summing up several *themata* in a single prescription; the Stoics called their version «dialectical theorem»: «when we have the premisses which imply some conclusion, we have also the conclusion which lies potentially in them, even if this is not expressly stated» (*M*. VIII.229). A fundamental feature of the Stoic analysis of a syllogism is that the reduction ends exactly when the simple undemonstrated compounding the syllogism are made explicit. It is not a proof but only a reduction method; it works by inserting intermediate elementary inferences and by deploying the complete formulation of an argument; once this is done, the validity of the argument is immediately grasped, insofar it is a well-formed sequence of valid arguments. The "result" of the reduction, namely, the non-simple undemonstrated to be reduced, is taken for granted from the very outset: the method is analytic. The deductive progression is "backward", but it does not look for preconditions: the operation is metalogical, and does not have a formal structure. Well, both in the Heronian rewriting of *El.* II.2–10 and in Stoic analysis, the irreducible inferential structures are known and finite in number (just a few, in fact): they are the preceding theorems in the sequence *El.* II.2–10 and the five simple undemonstrated, respectively. This makes the choice of the "subsequent passage" easy.

(d) In order to "see" the proof. The analysis came to be considered as a heuristic tool. This is only a myth, but the illusion was allegedly supported by the idea that there exists only a finite supply of deductive structures. The issue is discussed in a companion paper, and see also Acerbi, submitted.

Summing up, the language of the givens is applied to the determination of any mathematical object, either by constructive or by abstract approaches (for instance by means of chains of equalities). The same language is an attempt at unifying, under the logical form of a predicate whose rules of inference are established in the *Data*, the following argumentative patterns:

(i) *Deductions*, *in primis* manipulations of relations reduced by saturation to subspecies of the predicate «given».
(ii) *Constructions*, for which a deductive ordering replaces a non-deductive sequence of prescriptions. A chain of givens to which a construction corresponds without any accompanying (synthetic) proof can be found in Hero, *De speculis* 16 (*HOO* II, 352.3–356.10 = [22] Jones 2001, pp. 163–164).
(iii) *Calculations*, as we have seen in the previous section.

Add to this that the same language formulates (=formalizes) several kinds of enunciations, contributing in a decisive way to creating stable domains of research: this is a metalinguistic function that induces a partition of the mathematical propositions that is transverse to the canonical one between theorems and problems. Such concerns are strictly related to ontological issues, in this case the demarcation between existence and constructibility. One must also recognize that the unifying character of this language is also its main weakness, insofar as it conflates under the same lexical range a galaxy of demonstrative and enunciative practices that the ancient tradition tried to disentangle, as Marinus attests.

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Sources and their sigla

The abbreviations of the titles of the writings of mathematical authors are self-explanatory, as for instance *El.* = *Elements*, *Con.* = *Conica*, *Coll.* = *Collectio*; in most cases they coincide with the abbreviations found in the the Liddell-Scott-Jones lexicon. Propositions are referred to by book and number, as for instance «*El.* III.15». Pappus' *Collectio* is cited by book and chapter. Other sources are cited according to the following sigla:

AGE: *Apollonii Pergaei quae graece exstant, cum commentariis antiquis*. 2 vol. Leipzig, B.G. Teubner 1891–1893.
AOO: *Archimedis opera omnia, cum commentariis Eutocii*. 3 vol. Leipzig, B.G. Teubner 1910–1915.
DOO: *Diophanti Alexandrini opera omnia cum graeciis commentariis*. 2 vol. Leipzig, B.G. Teubner 1893–1895.
EE: *Euclidis Elementa*. 5 vol. Leipzig, B.G. Teubner 1969–1977.
EOO: *Euclidis opera omnia*. 10 vol. Leipzig, B.G. Teubner 1883–1916.
HOO: *Heronis Alexandrini opera quae supersunt omnia*. 5 vol. Leipzig, B.G. Teubner 1899–1914.
iA: *Commentaires de Pappus et de Théon d'Alexandrie sur l'Almageste*. 3 vol. Roma, Biblioteca Apostolica Vaticana 1931–1943.
iE: *Procli Diadochi in primum Euclidis Elementorum librum commentarii*. Leipzig, B.G. Teubner 1873.
POO: *Claudii Ptolemaei opera quae exstant omnia*. 3 vol. in five parts. Stuttgart und Leipzig, B.G. Teubner 1898–1998.

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