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The dimensions of the magnetic pole: a controversy at the heart of early dimensional analysis

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Abstract The rise of dimensional analysis in the latter part of the nineteenth century occurred largely in the context of electromagnetism. It soon appeared that the subject, albeit seemingly straightforward, was in fact wrought with difficulties. These revealed deep conceptual issues regarding the character of physical quantities. Usually, whether or not these problems actually constituted inconsistencies was itself a matter of debate. In one instance, however, regarding the electrostatic dimensions of the magnetic pole, all protagonists agreed that the matter required attention. A controversy ensued in 1882. Its resolution partly relied on the realization that it arose from differences between the scientific cultures prevalent on the Continent versus in Great Britain. These cultural differences concerned the possible relevance of the medium in which interactions involving magnetic poles take place, as well as the understanding of permeability in Ampère's model of magnetism. The controversy around the electrostatic dimensions of the magnetic pole entailed crucial issues that were soon to play just as central a part in a wider debate about the dimensions of electromagnetic quantities in different systems of units. Why the latter topic was never raised during the 1882 controversy provides insights into the early understanding of dimensional analysis.

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1 Introduction

1.1 What is dimensional analysis

Dimensional analysis is a method used in physics and engineering to investigate the relations between physical quantities. It first ascribes to the latter so-called dimensions. These can then be compared, notably to ensure that physical equations obey the expected rule of "dimensional homogeneity." The key concept, i.e., "dimensions," is related to the concept of units, and indeed dimensional analysis in its modern sense arose out of a concern for conversions between different systems of units.

Physics requires that units be defined for all physical quantities. However, all these units are not chosen independently from one another—hence the phrase "*system* of units." Indeed, some quantities are chosen whose units are deemed "fundamental" or "base units." All other physical quantities then see their units defined with reference to these fundamental ones, and for this reason, these are referred to as "derived units."

For instance in the SI system, one of the base units is that of length: the meter.[1] The unit of area is then defined with reference to it as the square meter.

Base quantities and their units are defined arbitrarily, primarily for reasons of convenience and accuracy. In our example, the meter is the distance that light travels in vacuum in a time interval of 1/299,792,458 s.[2]

Derived units on the other hand are defined from base units by using physical laws that relate the corresponding physical quantities, or to be more precise the values that measure these physical quantities. To actually fix the definition of a derived unit can be more than a matter of formalism: because concerns for precision involve experimental considerations, it may require to have an actual setup in mind in which the physical law can be embodied.[3] However, the relationship between a derived unit and the base units can be found simply by formal inspection of the chosen physical law. In our example, the area of a surface is given by squaring the length of the side of a square surface of the same size, i.e., area = length^2, and the same goes for the units of these quantities: square meter = (meter)^2.

This is where dimensional analysis comes in. It assigns symbols to the fundamental units: notably, L represents the base unit of length, T that of time, and M that of mass. In practice, one would not write "square meter = (meter)^2," but instead, using the

[1] SI stands for "Système International d'Unités." The others are the units of: time (the second), mass (the kilogram), temperature (the kelvin), electric current (the ampere), light intensity (the candela) and amount of substance (the mole).

[2] As if to highlight the arbitrary character of its definition, the meter went through several: originally, in 1793, it was defined as one ten-millionth of the distance from the Earth's equator to its North Pole; then in 1889 it became the length of a prototype bar, and in 1960 the 11th "Conférence Générale des Poids et Mesures" (CGPM) redefined this as 1,650,763.73 times the wavelength of the emission line of krypton-86. The current definition has been in use since 1983, since it was decided at the 17th CGPM.

[3] For example, Wilhelm Weber defined the unit of *e.m.f.* as follows: "As an absolute unit of measure of electromotive force, may be understood that electromotive force which the unit of measure of the earth's magnetism exerts upon a closed conductor, if the latter is so turned that the area of its projection on a plane normal to the direction of the earth's magnetism increases or decreases during the unit of time by the unit of surface" (Weber and Wilhelm Eduard 1861, p. 227).

language of dimensional analysis: [area] $= L^2$, where the square brackets stand for "dimensions of." To take a somewhat less trivial example, the dimensions of force $[F]$ would be derived from Newton's second law as follows. Newton's second law states:

$$F = m \times \frac{d^2x}{dt^2}$$

where F stands for the force on the object, m its mass, x its position and t for time—all understood as the physical quantities themselves.[4] The quantities on the right-hand side—which I conveniently chose to be base ones—have dimensions: $[m] = M$, $[x] = L$, $[t] = T$.

The dimensions of a derivative are the same as that of a normal ratio,[5]; therefore, the dimensions of force are $[F] = MLT^{-2}$.

Once the dimensions of all known quantities have been established in this way, dimensional analysis can be used as a check on derivations. Indeed all the terms that appear in an equation are required to be "(dimensionally) homogeneous," i.e., to have the same dimensions as one another.[6] When a result is found by inspection to violate this rule, this signals that a mistake must have been made in the course of the derivation. This is systematically used as a first check on results, and it seems fair to say that it constitutes the most widespread use of dimensional analysis in physics today.[7]

In anticipation of the issues that shall be raised below, it is important to point out how the relation between dimensions and systems of units is understood nowadays. Some common systems of units differ by their base units, yet share the same base quantities. For instance, the CGS system differs from the metric system in so far that it uses centimeters instead of meters and grams instead of kilograms as the base units of length and mass, respectively, but both systems use L, T and M as base quantities. The same is true for the systems of units between which one would typically have to convert in everyday life due to their being used to various degrees in different countries: SI (meter, km, kg...), imperial units and US customary units (foot, mile, pound...). However, different systems of units can also have different base quantities. For instance, particle physics uses the so-called natural units, which use three base units (unit of length, the speed of light c and Planck's constant h), three fundamental definitions that enable us to relate these to units of length, time and mass,[8] and only one dimension, L, because both c and h are considered dimensionless.[9]

[4] That is, a numerical value and the unit, such as "4 kg".

[5] That is, taking a single derivative with respect to a quantity yields the same dimensions as dividing by it, and taking the double derivative, the same dimensions as dividing by its square.

[6] Note that this is trivially the case in the examples given above: there are only two terms, one on each side of the equal sign, and both have the same dimensions.

[7] It is by no means the only one however. Notably, in quantum field theory dimensional analysis is used to determine what terms can rightfully appear in a theory. See footnote 8 below for more details.

[8] These relations are:
- $L = ct$, where L is the distance traveled by a photon, t its time of travel and c the speed of light in vacuum.
- $E = hf$, where E is the energy of a photon of frequency f, and h is Planck's constant.
- $E = mc^2$, where E is the energy of a particle of mass m (and c still the speed of light).

[9] Both are also given the numerical value of 1. Since $L = ct$, $[L] = L$ implies $[t] = L$. Since $E = hf$

As we shall see below, at the time of the controversy which is the topic of the present paper, two systems of units were in use in the field of electromagnetism. These had the same base quantities (L, M and T) and base units. The difference between them lay in how they obtained the units of derived quantities: some of the physical relations used for this purpose were not the same in the two systems, and the dimensions obtained for the derived quantities were not either.

1.2 The emergence of dimensional analysis

Dimensional analysis in its modern sense can be traced back to Jean Baptiste Joseph Fourier's *Théorie analytique de la chaleur* of 1822. Concern over homogeneity, in the sense of the conditions required for a relationship between physical quantities to be meaningful, had existed since Antiquity. For instance, quantities could only be added that were homogeneous, by which was meant, etymologically enough, "of the same nature," and Euclid stated something similar for ratios. These quantities were usually thought as geometrical—lines or surfaces (Euclid, book 5, lemma 3; see Mueller 1970, pp. 1–6). The introduction of derived quantities by Isaac Newton in the seventeenth century eventually led to a numerical conception of equations in the eighteenth century (see Macagno 1971, p. 392). With all terms and quantities now being pure numbers, the issue of homogeneity seemed a moot point.[10] In the early nineteenth century, however, Fourier recast the issue into a different problematic. If terms represented numbers, then what remained of an equation when those numbers were altered by a change of units? Under what conditions would an equation remain true independently of the system of units considered? Fourier defined these conditions by means of the relationship between the conversion factors of the derived quantities and those of a set of basic quantities. Terms were homogeneous whose units had the same relationship to the base units (Fourier 1822, pp. 154–158).

Fourier's work did not have an immediate impact, and in the following decades before dimensional analysis aroused interest, Wilhelm Eduard Weber introduced three systems of units in the context of electromagnetic theory. A same physical quantity usually had different *dimensions* in these different systems of *units*.

As well as Carl Friedrich Gauss, Weber wished to define new, stable units for electric and magnetic quantities, and developed the notion of absolute measurement.[11] The

Footnote 9 continued
and f is as always given by inverse of a time, this in turn implies $[\mathrm{E}] = \mathrm{L}^{-1}$. $E = mc^2$ then implies $[\mathrm{M}] = [\mathrm{E}] = \mathrm{L}^{-1}$. Because particle physics deals more with energies than lengths, it has become customary to use M rather than L as the base quantity: $[L] = [t] = \mathrm{M}^{-1}$.
An important application of this is the following. The action, which is the time integral of an energy, is dimensionless: $\mathrm{M}^{-1}\mathrm{M} = \mathrm{M}^0$. The Lagrangian density $\mathcal{L}$ is related to the action S by: $S = \int d^4x\mathcal{L}$, where the integral is over spacetime coordinates. Since both time and space have dimension M^{-1}, each term in $\mathcal{L}$ must have dimension M^4. Taking a scalar theory as an example, that is, a theory where the field, ϕ, is a scalar quantity, one of these terms is $\frac{1}{2}(\partial_t\varphi)^2$, which implies that $[\varphi] = \mathrm{M}$. Knowing this puts constraints on what form the other terms in $\mathcal{L}$ can take, i.e., what functions of the field they can be, since they all are required to have dimension M^4.

[10] See notably Roche (1998, pp. 88–190) and Charbonneau (1996, pp. 15–38).

[11] See Roche (1998, pp. 197–202). The period was characterized by a preoccupation for standards: the metric system endured after the Napoleonic wars and motivated several German states to organize reforms

work of Gauss and Weber was based on the mechanist credo that the units of all physical quantities, notably electromagnetic ones, could be expressed in terms of the units of mechanical quantities. They regarded a few mechanical units as fundamental, i.e., length, time and mass, and from these they derived "absolute units" for electric and magnetic quantities.[12] More specifically, Gauss was interested in defining an absolute unit for magnetic field intensity, and Weber for electrical resistance (Gauss 1833; Weber 1851). This led Weber to define three different "systems of absolute units": the "electrostatic, "electromagnetic" and "electrodynamic" systems. The first two especially gained wide acceptance. They were notably described by James Clerk Maxwell in his famous *Treatise on Electricity and Magnetism* (Maxwell 1873, Vol. 2, chapter 10).

The defining difference between these two systems was that they were founded on different force laws—or equivalently different field laws. The electrostatic system of units was based on Coulomb's law for the force between electrostatic charges:

$$f = \frac{e_1 e_2}{r^2} \tag{1}$$

where e stands for charge, f for the force and r for the distance between the two charges. In terms of dimensions it implies:

$$[f] = \frac{[e]^2}{\mathrm{L}^2}, \tag{2}$$

and with $[f] = \mathrm{LT^{-2}M}$ this yields:

$$[e_\mathrm{s}] = \mathrm{L^{3/2}T^{-1}M^{1/2}}, \tag{3}$$

where the subscript "s" refers to dimensions in the electrostatic system.[13]

As will be discussed in greater detail in Sect. 2.1 below, Maxwell actually used the field rather than forces. His defining equation for the electrostatic system was:

$$\mathcal{F} = \frac{e}{r^2}, \tag{4}$$

where $\mathcal{F}$ was the electric field (Maxwell 1873, § 625). This would seem to first require finding the dimensions for the field in terms of L, M, T. Instead he substituted for $[\mathcal{F}]$

Footnote 11 continued
of weights and measures. To this end, they enlisted the help of scientists, notably Gauss himself who in 1836 was put in charge of such a program in Hanover. Defining new standards of measurement called for precision instruments, and also the respect of well-defined procedures, both when duplicating material standards and when actually making measurements. See notably Olesko (1995) and Olesko (1996).

[12] Gauss: original Carl Friedrich Gauss, Goettingue, 1833.
See Weber (1851).

[13] See notably Everett (1879), chapters X and XI. An analogous description in terms of units rather than dimensions ran as follows: "In the *electrostatic* system the most important unit, which serves as the basis of all the others, is the *unit of electricity*. This is determined by the following definition: *The unit of electricity is that amount of electricity which exerts the unit of force upon an equal amount of electricity at the unit of distance*" (Clausius 1882, p. 383).

and $[e]$ relations in terms of T, L, M, e or T, L M, m, which held independently of the system concerned—notably in so far that their derivation did not require appealing to Coulomb's force laws.[14]

In contrast the electromagnetic system was defined on the basis of Coulomb's law between magnetic poles:

$$f = \frac{m_1 m_2}{r^2}, \tag{5}$$

where m stands for the strength of the pole, and again f for the force and r for the distance between the two poles. This relation was then in common use, on the same footing as its electrostatic counterpart. The concept of pole used at the time corresponded to a pole of an idealized magnet, i.e., uniformly magnetized throughout its length, and in the limit that this magnet becomes infinitely long and thin (so its ends become point-like).[15] Equation 5 leads to the dimensional relation:

$$[f] = \frac{[m]^2}{\mathrm{L}^2}, \tag{6}$$

and with $[f] = \mathrm{LT^{-2}M}$ this results in:

$$[m_{\mathrm{m}}] = \mathrm{L^{3/2}T^{-1}M^{1/2}}, \tag{7}$$

where the subscript "m" stands for the electromagnetic system. Comparison with Eq. 3 shows that Coulomb's law for magnetism yields the same dimensions for m in the electromagnetic system as its electrostatic counterpart does for e in the electrostatic one—as could indeed be expected from the structural similarity between the two laws.[16]

Again, instead of the force law Maxwell actually used:

$$\mathcal{H} = \frac{\mathrm{m}}{\mathrm{L}^2}, \tag{8}$$

as the basis of the electromagnetic system, where H was the magnetic field (Maxwell 1873, § 625; see Sect. 2.1 below).[17]

[14] He naturally attributed to r^2 the dimensions L^2.

[15] Maxwell (1873, Vol. 2, chapter 1, § 373). Magnetism was attributed to microscopic electric currents. In 1882 (the time when the controversy of interest took place) Lodge believed "universally accepted" that "the properties of magnetic substances of all kinds [were] explained by molecular electric currents, and no magnets or magnetic substances other than those consisting of current-conveying molecules exist." (Lodge 1882, p. 363)

[16] Again, this can be expressed in terms of units in the following way:
"*The unit of magnetism is that amount of magnetism which exerts upon an equal amount of magnetism at the unit of distance the unit of force.*" (Clausius 1882, p. 384)
Clausius actually favored referring to this system as the "electrodynamic system," because he agreed with Ampère that magnetism arose from microscopic electric *currents*, hence was electrodynamic in nature, and consequently uses the subscript "d" instead of "m"; however, he made clear that it was standardly referred to as the electromagnetic system. (Clausius 1882, p. 383)

[17] What we would now call the auxiliary magnetic field.
As for the electrodynamic system, it was based on Ampere's law between current elements (Weber 1851).

Until 1883, no one appears to have expressed dissatisfaction with the fact that a quantity generally had different *dimensions* in different systems of *units*. However, a given physical quantity was expected to have a unique set of dimensions in any given system. In 1882, this requirement gave rise to a controversy regarding the dimensions of a magnetic pole. The debate took place mostly in Great Britain, in the *Philosophical Magazine*, sparked by a German publication by Rudolf Clausius in March 1882 soon translated in the British magazine (Clausius 1882).

2 The electrostatic dimensions of the magnetic pole according to Maxwell and Clausius

2.1 The dimensions of electromagnetic quantities in Maxwell's treatise on electricity and magnetism

In order to appreciate the issues at stake, it is necessary to remember what the British viewpoint was at the time, as heralded by James Clerk Maxwell. And indeed it can fairly be said that Maxwell's *Treatise on Electricity and Magnetism* (Maxwell 1873) had been responsible for sparking interest in dimensional analysis both in and beyond Great Britain: the *Treatise* had been very influential, and Maxwell had devoted an entire chapter of it to the topic (Maxwell 1873, Vol. 2, chapter 10). The importance he afforded dimensional analysis can be partly attributed to his general interest in the classification of physico-mathematical quantities (Darrigol 2000; Harman 1887), but it was also related, once again, to the prevalent concern for measurement standards. In fact, ten years earlier Maxwell had co-authored a contribution on the topic to the *Committee on electrical standards* that had been appointed by the *British Association for the Advancement of Science* with the view of establishing a standard of electric resistance.[18] The wish to improve electromagnetic measurement standards was in turn strongly driven by the needs of cable telegraphy.[19]

In his treatise, Maxwell focused on relations independent of any system of units. He gave fifteen dimensional relations deemed fundamental, either products of quantities that had the dimensions of energy (Eq. 9 below), energy density (Eq. 10),[20] or ratios that led to especially simple dimensions in terms of the base units of time and length (T, L, L^2, Eqs.11–13).[21]

[18] British Association for the Advancement of Science (1863).

[19] See notably, Hunt (1994, 1996, 1997, 2003, 2005) and Schaffer (1997) as well as Atten (1992) for the case of France.

[20] Energy and work have the same dimensions, as required by the work–energy theorem. Since work is given by force × displacement, it has dimensions: $[\text{work}] = [\text{force}] \times L = LT^{-2}M \times L = L^2T^{-2}M$. Energy density then has dimensions: $[\text{energy density}] = [\text{energy}] \div [\text{volume}] = L^2T^{-2}M \div L^3 = L^{-1}T^{-2}M$.

[21] Following the standard usage I shall use italics for quantities (for instance "*e*" for quantity of electricity), square brackets to denote the dimensions of a derived quantity (such as [*e*]) and no brackets, nor italics for the fundamental quantities in terms of which the dimensions are expressed (for instance "L" for length). (Maxwell 1873, Vol. 2, chapter 10, § 621–622.)

$$[eE] = [m\Omega] = [pC] = \frac{\mathrm{L^2M}}{\mathrm{T^2}} \tag{9}$$

$$[\mathcal{D}\mathcal{F}] = [\mathcal{B}\mathcal{H}] = [\mathcal{C}\mathcal{U}] = \frac{\mathrm{M}}{\mathrm{LT^2}} \tag{10}$$

$$\left[\frac{e}{C}\right] = \left[\frac{p}{E}\right] = \left[\frac{\mathcal{U}}{\mathcal{F}}\right] = \mathrm{T} \tag{11}$$

$$\left[\frac{E}{\mathcal{F}}\right] = \left[\frac{\Omega}{\mathcal{H}}\right] = \left[\frac{p}{\mathcal{U}}\right] = \mathrm{L} \tag{12}$$

$$\left[\frac{e}{\mathcal{D}}\right] = \left[\frac{C}{\mathcal{C}}\right] = \left[\frac{m}{\mathcal{B}}\right] = \mathrm{L}^2 \tag{13}$$

with e the quantity of electricity; E the line integral of electromotive intensity; m the quantity of magnetism; Ω the magnetic potential; p the electrokinetic momentum of a circuit; C the electric current; $\mathcal{D}$ the electric displacement; $\mathcal{F}$ the electromotive intensity; $\mathcal{B}$ the magnetic induction; $\mathcal{H}$ the auxiliary magnetic field; $\mathcal{C}$ the strength of the current at a point; $\mathcal{U}$ the vector potential.

Maxwell did not specify how he reached Eqs. 9 and 10. He appears to have simply deemed physically obvious that the products involved had dimensions of energy and energy density, respectively. He justified Eq. 11 by the fact that it involved the ratios of the time integral of quantities to the quantities themselves. For instance, the quantity of electricity e is the time integral of the electric current C. Dimensionally, integrating over a quantity entails multiplying by its dimensions, so:

$$\left[\frac{e}{C}\right] = \left[\frac{\int C\mathrm{d}t}{C}\right] = \mathrm{T}. \tag{14}$$

Similarly Eqs. 12 and 13 were ratios of, respectively, line and surface integrals to the quantities themselves, thereby yielding dimensions of L and L^2.

Maxwell then used these fifteen relations in order to obtain the dimensions of the major electric and magnetic quantities. There was a problem however: although these relations only involved twelve quantities, they were not all independent, so that in fact another one would have been required in order to uniquely determine the dimensions of all twelve quantities in terms of L, M and T alone.

To deal with this difficulty, Maxwell temporarily added a fourth base unit—or rather, a choice of two. He gave a list of dimensions in terms of L, M, T and the unit of static charge e, and another list in terms of L, M, T and the unit of magnetic pole m. Both these sets of dimensions were valid irrespective of the system of units considered—electrostatic or electromagnetic system notably.

Of greatest import in Maxwell's derivation were $[e]$, $[m]$ and the electric field $[\mathcal{F}]$ for which he found, in terms of e and m, respectively:

$$[m] = \frac{\mathrm{L^2M}}{\mathrm{eT}} = \mathrm{m} \tag{15}$$

$$[e] = \mathrm{e} = \frac{\mathrm{L^2M}}{\mathrm{mT}} \tag{16}$$

$$[\mathcal{F}] = \frac{\mathrm{LM}}{\mathrm{eT^2}} = \frac{\mathrm{m}}{\mathrm{LT}} \tag{17}$$

Importantly, Maxwell did not explain how he obtained these dimensions from the fifteen dimensional relationships previously established, and this is the step for which disagreements were going to occur regarding the interpretation of his work.

Furthermore, despite what Maxwell seemed to imply, the dimensions in terms of L, T, M, e and L, T, M, m of the various electromagnetic quantities that he provided cannot all be derived from the fifteen dimensional relations that he gives in the text. Some can be and even with far less than his fifteen relations. Notably, the first equality in Eq. 17 can be easily obtained from $[eE] = \frac{\mathrm{L^2M}}{\mathrm{T^2}}$ and $\left[\frac{E}{\mathcal{F}}\right] = \mathrm{L}$ (see Eqs. 9 and 12, respectively), or alternatively from $[\mathcal{DF}] = \frac{\mathrm{M}}{\mathrm{LT^2}}$ and $\left[\frac{e}{\mathcal{D}}\right] = \mathrm{L^2}$ (see Eqs. 10 and 13). However, in many cases, an additional relation is required. Crucially, this is true even for $[m] = \frac{\mathrm{L^2M}}{\mathrm{eT}}$ (Eq. 15) and $[e] = \frac{\mathrm{L^2M}}{\mathrm{mT}}$ (Eq. 16). Either one can been obtained directly from the other, simply by solving for either e or m, respectively. However, neither can be derived using only the fifteen relations given by Maxwell.[22]

For this reason, Clausius later argued that Maxwell used $[p] = [m]$. This is conceivable, for it does lead to all the dimensions Maxwell gave, and he indeed wrote it explicitly when listing them. However, the way he did so rather suggests that he had derived the dimensions of p and m independently and happened to find them the same.[23] In contrast, Maxwell did provide an additional relation in the very section where he listed the main fifteen ones, although he only did so in a footnote and offered no clue regarding its origin: $\left[\frac{\mathcal{U}}{\mathcal{B}}\right] = \mathrm{L}$.[24] In combination with the others, this relation (as does $[p] = [m]$)[25] indeed yields all the dimensions that Maxwell gave, including $[m] = \frac{\mathrm{L^2M}}{\mathrm{eT}}$ and $[e] = \frac{\mathrm{L^2M}}{\mathrm{mT}}$. Using the relation $\left[\frac{\mathcal{U}}{\mathcal{B}}\right] = \mathrm{L}$ instead of $[p] = [m]$

22 Neither can be the second equality in Eq. 17, $[\mathcal{F}] = \frac{\mathrm{m}}{\mathrm{LT}}$. See footnote 26 below.

23 Indeed Maxwell wrote:
(3) and (5) $[p] = [m] = \left[\frac{\mathrm{L^2M}}{\mathrm{eT}}\right] = [m]$,
where the "(3)" and "(5)" refer to the numbering of m and p, respectively, when he listed earlier the electromagnetic quantities themselves. Furthermore, he never listed separately (i.e., on different lines) quantities that have the same dimensions. The situation occurs for only m and p, and C and Ω, and in both cases he presented the results the same way, i.e., again with:
(4) and (6) $[C] = [O] = \left[\frac{\mathrm{e}}{\mathrm{T}}\right] = \left[\frac{\mathrm{L^2M}}{\mathrm{mT^2}}\right]$.
If $[C] = [O]$ were to be also interpreted as a relation additional to the fifteen he gave explicitly, we would now have an abundance of riches, for either $[p] = [m]$ or $[C] = [O]$ is necessary to find all the dimensions, but not both (they do yield the same results).

24 Neither Maxwell nor Clausius comments on why $[p]$ should be equal to $[m]$, either.

25 And as does $[C] = [\Omega]$.

however, even the simplest ways to get them require many more equations[26]:

$$\left[\frac{m}{\mathcal{B}}\right] = \mathrm{L}^2, \quad [C\mathcal{U}] = \frac{\mathrm{M}}{\mathrm{LT}^2}, \quad \left[\frac{C}{\mathcal{C}}\right] = \mathrm{L}^2, \quad \left[\frac{e}{\mathcal{C}}\right] = \mathrm{T} \text{ with } \left[\frac{\mathcal{U}}{\mathcal{B}}\right] = \mathrm{L} \tag{18}$$

In contrast, using $[p] = [m]$ they can be obtained from:[27]

$$[eE] = \frac{\mathrm{L}^2\mathrm{M}}{\mathrm{T}^2}, \left[\frac{p}{E}\right] = \mathrm{T} \quad \text{with } [p] = [m]. \tag{19}$$

This probably explains why Clausius favored $[p] = [m]$ as the relation most likely used by Maxwell.

After listing the dimensions in terms of L, T, M, e and L, T, M, m of all the quantities of interest, Maxwell then derived their dimensions in the electrostatic and in the electromagnetic systems of units (hence in terms of L, M and T only) using *either* of these two lists. He obtained the electrostatic system of units by substituting in either list the electrostatic dimensions of the electric charge e and of the magnetic pole m—and similarly used their electromagnetic dimensions to get the electromagnetic system. The electrostatic dimensions of the magnetic pole thereby affected the electrostatic dimensions of all magnetic quantities, as the latter involved m.

In turn, Maxwell defined the electrostatic dimensions of e and m by comparing the dimensions of the electric field obtained above to those obtained from Coulomb's law between electrostatic charges:[28]

$$\mathcal{F} = \frac{\text{electrostatic charge}}{(\text{distance from the charge})^2} \Rightarrow [\mathcal{F}] = \frac{\mathrm{e}}{\mathrm{L}^2}. \tag{20}$$

Having obtained above $[\mathcal{F}] = \frac{\mathrm{m}}{\mathrm{LT}}$ and $[e] = \frac{\mathrm{L}^2\mathrm{M}}{\mathrm{mT}}$,[29] this yielded for the electrostatic dimensions of the magnetic pole: $[m] = \mathrm{L}^{1/2}\mathrm{M}^{1/2}$.[30]

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[26] Again there are many other possible combinations, including another four that involve the same number of equations, with $\left[\frac{\mathcal{U}}{\mathcal{B}}\right] = \mathrm{L}$. The reason why using $\left[\frac{\mathcal{U}}{\mathcal{B}}\right] = \mathrm{L}$ as opposed to $[p] = [m]$ demands the use of more equations for the derivation of the relation between e and m may be related to the fact that three of Maxwell's fifteen dimensional relations involve p, whereas only two contain $\mathcal{U}$ (both m and $\mathcal{B}$ appear twice).

The second equality in Eq. 17, $[\mathcal{F}] = \frac{\mathrm{m}}{\mathrm{LT}}$, can be derived equally simply with $\left[\frac{\mathcal{U}}{\mathcal{B}}\right] = \mathrm{L}$ as with $[p] = [m]$:

$\left[\frac{E}{\mathcal{F}}\right] = \mathrm{L}\left[\frac{p}{E}\right] = \mathrm{T}$, with $[p] = [m]$,

or: $\left[\frac{m}{\mathcal{B}}\right] = \mathrm{L}^2, \left[\frac{\mathcal{U}}{\mathcal{F}}\right] = \mathrm{T}$, with $\left[\frac{\mathcal{U}}{\mathcal{B}}\right] = \mathrm{L}$.

[27] There are many other possibilities, most more complicated. To give but one by way of example:

$[\mathcal{D}\mathcal{F}] = \frac{\mathrm{M}}{\mathrm{LT}^2}, \quad \left[\frac{p}{E}\right] = \mathrm{T}, \left[\frac{\mathrm{E}}{\mathcal{F}}\right] = \mathrm{L}, \left[\frac{e}{\mathcal{D}}\right] = \mathrm{L}^2$ with $[p] = [m]$.

[28] Analogously, the electromagnetic system was based on Coulomb's law for magnetism: $\mathcal{H} = \frac{m}{L^2}$, where $\mathcal{H}$ was the "magnetic force" on a magnetic pole of unit magnitude.

[29] See respectively Eqs. 17 and 16.

[30] $[\mathcal{F}] = \frac{\mathrm{e}}{\mathrm{L}^2} = \frac{\mathrm{L}^2\mathrm{M}}{\mathrm{mT}} \times \frac{1}{\mathrm{L}^2} = \frac{\mathrm{M}}{\mathrm{mT}}$ and $[\mathcal{F}] = \frac{\mathrm{m}}{\mathrm{LT}}$ imply $\frac{\mathrm{M}}{\mathrm{mT}} = \frac{\mathrm{m}}{\mathrm{LT}}$, so that $[\mathrm{m}] = \mathrm{L}^{1/2}\mathrm{M}^{1/2}$.

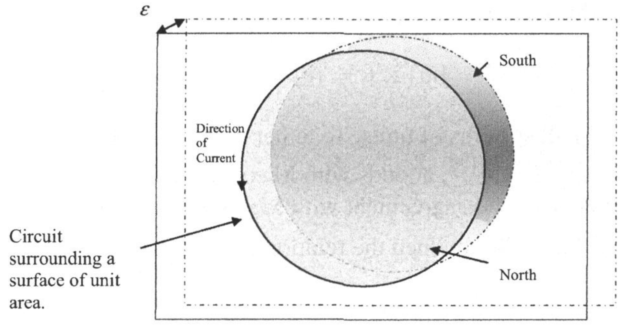

Fig. 1 Equivalence of a current loop and a magnet

2.2 The electrostatic dimensions of the magnetic pole according to Clausius

Clausius too wished to find the dimensions of all major electric and magnetic quantities in both the electrostatic and electromagnetic systems of units. However, his derivation differed from Maxwell's and was somewhat more straightforward.

Clausius first determined the dimensions of the electric charge in the electrostatic system and of the magnetic pole in the electromagnetic system, using the relevant Coulomb's law in each case, and attributing to force its usual mechanical dimensions, $\mathrm{ML\,T^{-2}}$. As discussed above, this led to the dimensions $[e_\mathrm{s}] = \mathrm{L^{3/2}T^{-1}M^{1/2}}$ and $[m_\mathrm{m}] = \mathrm{L^{3/2}T^{-1}M^{1/2}}$, wherein the subscripts "s" and "m" refer to the electrostatic and electromagnetic systems, respectively—results that agreed with Maxwell's (Clausius 1882, pp. 382–383).

In order to find the dimensions of these quantities in the other system, i.e., $[e_\mathrm{m}]$ and $[m_\mathrm{s}]$, Clausius reasoned that all that was needed was a relationship between e and m that held true in both systems of units. He appealed to Ampère's hypothesis that an electric current loop is equivalent to a magnet—equivalence that Maxwell endorsed, but which he had not used in his dimensional derivations. Clausius posited a plane circuit surrounding a surface of unit area, and a second surface of the same shape, parallel to the circuit, separated by an infinitesimal distance e. The two surfaces were deemed to contain the same amount of magnetism, north on the first and south on the second, thereby forming the magnet thought equivalent to the current loop (Clausius 1882, pp. 382–383) (Fig. 1).

For the magnetic sheets to produce forces of the same intensity as the current, the physical quantities involved had to satisfy the following relation:[31]

magnitude of the magnetism on one sheet × distance between the two sheets
= magnitude of the current × surface area enclosed by the current loop.

Footnote 30 continued
Similarly, having obtained above in Eq. 17 $[\mathcal{F}] = \frac{\mathrm{LM}}{\mathrm{eT^2}}$, and using $\mathcal{F} = \frac{\mathrm{e}}{\mathrm{L^2}}$ yielded for the electrostatic dimensions of the electric charge: $[\mathrm{e}] = \mathrm{L^{3/2}M^{1/2}T^{-1}}$.

[31] Nowadays this would be expressed as the requirement that the magnet and the circuit must have the same magnetic moments.

Dimensionally, this meant:

$$[m] \times \mathrm{L} = [e]\mathrm{T}^{-1} \times \mathrm{L}^2, \tag{21}$$

or $\frac{[m]}{[e]} = \mathrm{LT}^{-1}$, in all systems of units. Together with $[e_s] = \mathrm{L}^{3/2}\mathrm{T}^{-1}\mathrm{M}^{1/2}$ this result implied $[m_s] = \mathrm{L}^{5/2}\mathrm{T}^{-2}\mathrm{M}^{1/2}$, at odds with Maxwell's result $[m_s] = \mathrm{L}^{1/2}\mathrm{M}^{1/2}$.

Clausius explained his disagreement with Maxwell's results in the following way. He believed that Maxwell obtained the relation $\mathrm{e} = \frac{\mathrm{L^2M}}{\mathrm{mT}}$ (Eq. 16 above) from:

$$[pC] = \frac{\mathrm{L^2M}}{\mathrm{T^2}} \quad \text{and} \quad \left[\frac{e}{C}\right] = \mathrm{T} \tag{22}$$

and by then identifying $[p]$ and $[m]$.[32] Substituting m for p, and multiplying both equations one gets $[em] = \mathrm{ML^2T^{-1}}$ instead of $\frac{[m]}{[e]} = \mathrm{LT}^{-1}$. Clausius judged this reasoning incorrect because, he argued, it implies taking into account the force between a magnetic pole and a current. This force is *electrodynamic* in nature, so it cannot rightfully be used to derive *electrostatic* dimensions, Clausius concluded.

3 The controversy: strategies and motivations

The controversy begun in May 1882 with an article published in the *Philosophical Magazine* by Joseph David Everett, then professor of natural philosophy at Queen's College, Belfast (Everett 1882a, pp. 376–377).[33] Everett briefly summarized Clausius's reasoning, stating that it was based on the following equation, true in all systems of units:

$$\text{Pole} \times \text{Length} = \text{Current} \times (\text{Length})^2\,, \tag{23}$$

hence

$$\text{Pole} = \text{Current} \times \text{Length}, \tag{24}$$

where the quantities were meant to represent their dimensions. He then noted that the electrostatic dimensions of current were $\mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-2}$, obviously from Coulomb's electrostatic law, and noted that these two relations implied the electrostatic dimensions of the magnetic pole to be $\mathrm{M}^{1/2}\mathrm{L}^{5/2}\mathrm{T}^{-2}$.[34]

32 As was discussed above in Sect. 2.1 (Maxwell 1873, Vol. 2, chapter 10, § 623).

33 Joseph David Everett (1831–1904) had been secretary of the Committee for the selection and nomenclature of dynamical and electrical units, which had been appointed by the *British Association* (British Association for the Advancement of Science 1875), and had published two treatises entirely devoted to units and dimensions (Everett 1875, 1879). He was professor of natural philosophy at Queen's College, Belfast for most of his career, from 1867 until 1897.

34 From Coulomb's law between electrostatic charges, $f = \frac{e_1 e_2}{r^2}$, we have in terms of dimensions:
$[e^2] = [f][r^2] = \mathrm{M\,L\,T^{-2}} \times \mathrm{L}^2 = \mathrm{M\,L^3T^{-2}}$,
i.e.$[e] = \mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-1}$.
Current has dimensions of charge divided by time, therefore in the electrostatic system of units: [current] $= \mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-2}$.
Using the relation Pole = Current × Length then implies $[pole] = \mathrm{M}^{1/2}\mathrm{L}^{5/2}\mathrm{T}^{-2}$.

Everett deemed Clausius's analysis "unimpeachable" and went on to give another argument in its favor. The electromagnetic dimensions of a pole (using the relation for the force between two magnetic poles) are $M^{1/2}L^{3/2}T^{-1}$ and those of a current $M^{1/2}L^{1/2}T^{-1}$.[35] This again yields [Pole] = [Current] × [Length]. Finding the same relation in the electromagnetic system as the one reached by Clausius in the electrostatic system seemed to Everett an argument in Clausius's favor—which may seem odd considering that this result is the direct consequence of the equivalence between a circuit and a magnet, and therefore independent from the system considered (as Everett himself states). More importantly perhaps, Everett declared being unsure how Maxwell actually derived the result in his treatise and invited his readers to clarify this point.

Everett's request did not go unheard. The following month, the reactions of three scientists were published in the *Philosophical Magazine*—as well as a translation of Clausius's paper: Joseph John Thomson and the Irish theorist Joseph Larmor each provided an article (Thomson 1882a; Larmor 1882), and in addition George Francis FitzGerald (then Professor of natural philosophy at Trinity College in Dublin) contacted Everett, who discussed Fitzgerald's contribution in his own paper (Everett 1882b).

In response to Everett's plea, Larmor and Fitzgerald provided derivations of Maxwell's result—according to which the electrostatic dimensions of the magnetic pole were $M^{1/2}L^{1/2}$. After discussing their suggestions, Everett compared them to Clausius's work, with a view to resolving the contradiction.

In his article, Larmor had argued in favor of using the dimensional relation for the force between a magnet and a current (Larmor 1882, p. 430):[36]

$$\text{current} \times \mathrm{L} \times \text{pole} \times \mathrm{L}^{-2} = \text{force}, \tag{25}$$

where "current," "pole" and "force" in fact stood for the dimensions of these physical quantities. This was the dimensional version of the following relation:

$$\text{force} = \frac{\text{length of wire} \times \text{current} \times \text{pole}}{(\text{distance between the wire and the pole})^2}. \tag{26}$$

[35] These were the electromagnetic dimensions of current given by Maxwell (Maxwell 1873, Vol. 2, chapter 10, § 626). In order to obtain them he doubtless divided by T the electromagnetic dimensions for the electrostatic charge, $[e_{\mathrm{m}}] = M^{1/2}L^{1/2}$. In turn, the latter were derived by taking the dimensional form of the expression for the magnetic form of Coulomb's law, i.e., $\mathcal{H} = \frac{m}{L^2}$, and substituting in it the dimensional expressions for $[m]$ and $[\mathcal{H}]$ in terms of e valid in all systems of units:
$[m] = \frac{L^2M}{eT}$(Eq. 15 above); $[\mathcal{H}] = \frac{e}{LT}$.
How Maxwell obtained the latter from his fifteen dimensional relations (Eqs. 9–13) is again unclear.
$[\mathcal{H}] = \left[\frac{m}{L^2}\right]$ then led to $\frac{e}{LT} = \frac{L^2M}{eT} \times \frac{1}{L^2}$, and solving this for e indeed results in $[e_{\mathrm{m}}] = M^{1/2}L^{1/2}$.

[36] This force had been discovered by Oersted and studied by Ampère.

Solving for the pole, and using as the electrostatic dimensions of current the unanimously accepted $M^{1/2}L^{3/2}T^{-2}$,[37] Larmor obtained Maxwell's result:

$$[\text{pole}] = \frac{MLT^{-2} \times L^2}{L \times M^{1/2}L^{3/2}T^{-2}} = M^{1/2}\frac{L^3}{L^{5/2}} = M^{1/2}L^{1/2}. \tag{27}$$

When discussing Larmor's work, Everett gave the more explicit relation but for one important difference: for reasons that will soon become clear, he introduced a constant k_1 (Everett 1882b, pp. 431–432):

$$\text{force} = k_1 \frac{\text{length of wire} \times \text{current} \times \text{pole}}{(\text{distance between the wire and the pole})^2} \tag{28}$$

Clausius's criticism that Maxwell used an electrodynamic relation to build the electrostatic system is clearly relevant here. Everett's reaction to this is interesting: this may be so, he argued, but so did all other definitions of the unit of magnetic pole in the electrostatic system, including the one used by Clausius (Everett 1882b, p. 432). This is only partly true: although Clausius did take a current into account in his demonstration, the resulting equation embodied the *equivalence* between a circuit and a magnet; Larmor, on the other hand, expressed the *force* between them.

The second derivation discussed by Everett, which had been sent to him by Fitzgerald, another Irish follower of Maxwell's, also yielded $M^{1/2}L^{1/2}$ for the electrostatic dimensions of the magnetic pole (Everett 1882b, p. 432). However, it was based on yet a different relation: the "magneto-electric law" according to which the *e.m.f.* due to the motion of a conductor in a magnetic field is proportional to the length of the conductor, its speed and the field's intensity:

$$\text{e.m.f.} \propto L \times v \times B. \tag{29}$$

If one then appealed to Coulomb's magnetic force law,[38] this yielded:

$$[\text{e.m.f.}] = L \times LT^{-1} \times [\text{pole}] \times L^{-2}, \tag{30}$$

(or, according to Everett, $[\text{e.m.f.}] = [k_2] \times L \times LT^{-1} \times [\text{pole}] \times L^{-2}$, with $k_2 = 1$), hence:

$$[\text{pole}] = \frac{[\text{e.m.f.}]}{L \times LT^{-1} \times L^{-2}} = \frac{[\text{e.m.f.}]}{T^{-1}} = [\text{e.m.f.}] \times T \tag{31}$$

The e.m.f. being the line integral of the electric field, its dimensions in the electrostatic system are $M^{1/2}L^{1/2}T^{-1}$,[39] hence, again, Maxwell's result:

$$[\text{pole}] = M^{1/2}L^{1/2}T^{-1} \times T = M^{1/2}L^{1/2} \tag{32}$$

[37] From Coulomb's electrostatic law the dimensions of charge in the electrostatic system of units are $[e] = L^{3/2}M^{1/2}T^{-1}$ (see footnote 30 above), and those of current are simply these divided by T.

[38] More precisely, the relation that this force law implies for the magnetic intensity, i.e., $[B] = [\text{pole}] \times L^{-2}$.

[39] Recall from Eq. 17 that Maxwell had somehow obtained for the electric field $\mathcal{F}$ the dimensional

Together with Clausius's work, these reactions inspired Everett the following comment:

> the definitions of the unit quantity of electricity, the unit current, the unit electromotive force (or difference of potential), and the unit resistance (all of which may be called purely electrostatic definitions), are not in question, but are accepted by all parties. The divergence begins when we attempt to express magnetic quantities in an electrostatic system; and different results may be obtained according to the particular relation between magnetism and electricity which we select as the guiding principle in our definitions. In a strict sense there is no such thing as an electrostatic unit of any magnetic quantity; since magnetism and its relations to electricity lie outside the domain of electrostatics (Everett 1882b, p. 431).

Everett compared the derivations offered by Larmor and Fitzgerald to Clausius's work, reframing the latter in a way that allowed him to draw his own conclusions (Everett 1882b, pp. 432–434). As noted above, in anticipation of his own arguments, Everett had introduced multiplicative constants in the equations used by Larmor and FitzGerald. He then took the same step with the relation used by Clausius that expressed the equivalence between a current loop and a magnet and added to it a constant k_3:

$$\text{Pole} \times \text{Length} = k_3 \times \text{Current} \times \text{Area} \tag{33}$$

Everett then attributed the disagreements between on the one hand Larmor and Fitzgerald (hence Maxwell), and on the other Clausius, to an incompatibility between their implicit choices for the dimensions of these constants:

> [Clausius's result] disagrees with that of the two preceding investigations; and hence the two equivalent assumptions k_1= 1, k_2= 1 are inconsistent with the assumption k_3= 1 (Everett 1882b, p. 433).

In order to obtain consistent results, one had to take either $[k_3] = \mathrm{L}^{-2}\mathrm{T}^2$, in which case the dimensions of the pole were $\mathrm{M}^{1/2}\mathrm{L}^{1/2}$, or $[k_1] = [k_2] = \mathrm{L}^{-2}\mathrm{T}^2$, and they then became $\mathrm{M}^{1/2}\mathrm{L}^{5/2}\mathrm{T}^{-2}$.[40] k_3 in the former case, and k_1 and k_2 in the latter were attributed the dimensions of $\frac{1}{v^2}$, where v is the ratio of the unit of electric charge in the electromagnetic system of units to its unit in the electrostatic system and has

Footnote 39 continued
relation $[\mathcal{F}] = \frac{\mathrm{LM}}{\mathrm{eT}^2}$, valid in all systems of units. In the electrostatic system charge has dimensions $[\mathrm{e}] = \mathrm{L}^{3/2}\mathrm{M}^{1/2}\mathrm{T}^{-1}$ (see footnote 30 notably), so in that system the electric field has dimensions:
$[\mathcal{F}] = \frac{1}{\mathrm{L}^{3/2}\mathrm{M}^{1/2}\mathrm{T}^{-1}} \times \frac{\mathrm{LM}}{\mathrm{T}^2} = \mathrm{L}^{-1/2}\mathrm{M}^{1/2}\mathrm{T}^{-1}$,
and therefore the electrostatic dimensions of its line integral are $[\text{e.m.f.}] = \mathrm{L}^{-1/2}\mathrm{M}^{1/2}\mathrm{T}^{-1} \times \mathrm{L} = \mathrm{L}^{1/2}\mathrm{M}^{1/2}\mathrm{T}^{-1}$.

[40] $[\text{Pole}] \times [\text{Length}] = [k_3] \times [\text{Current}] \times [\text{Area}]$ yields in the electrostatic system of units, with $[\text{Current}] = \mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-2}$ (see notably footnote 37):
$[\text{Pole}] \times \mathrm{L} = [k_3] \times \mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-2} \times \mathrm{L}^2 \Rightarrow [\text{Pole}] = [k_3] \times \mathrm{M}^{1/2}\mathrm{L}^{5/2}\mathrm{T}^{-2}$,
so with $[k_3] = \mathrm{L}^{-2}\mathrm{T}^2$: $[\text{Pole}] = \mathrm{L}^{-2}\mathrm{T}^2 \times \mathrm{M}^{1/2}\mathrm{L}^{5/2}\mathrm{T}^{-2} = \mathrm{M}^{1/2}\mathrm{L}^{1/2}$,
but when $[k_1] = [k_2] = \mathrm{L}^{-2}\mathrm{T}^2$, i.e. $[k_3] = 1$: $[\text{Pole}] = \mathrm{M}^{1/2}\mathrm{L}^{5/2}\mathrm{T}^{-2}$.

Table 1 Dimensions of the force laws constants

Constants	Electrostatic system		Electromagnetic system
	Maxwell	Clausius	
k_1 (force between a magnet and a current)	Dimensionless	$L^{-2}T^2$	Dimensionless
k_2 (force on a moving conductor in a magnetic field)	Dimensionless	$L^{-2}T^2$	Dimensionless
k_3 (equivalence between a current loop and a magnet)	$L^{-2}T^2$	Dimensionless	Dimensionless
k_4 (force between two parallel currents)	$L^{-2}T^2$	$L^{-2}T^2$	Dimensionless
k_5 (force between two magnetic poles)	$L^{-2}T^2$	$L^{-2}T^2$	Dimensionless
k_6 (force between two electric charges)	Dimensionless	Dimensionless	$L^{-2}T^2$

dimensions of speed LT^{-1}. Everett favored the former, but only for the practical reason that it was more convenient in calculations.

Everett did not stop there however: he also discussed the electromagnetic system of units, more convenient for electromagnetic calculations. These considerations led him to introduce three equations with three other constants (Everett 1882b, pp. 433–434):

- for the force between two parallel currents in wires of length l:

$$F = k_4 \times (\text{current } 1) \times (\text{current } 2) \times 2l/\text{distance}, \tag{34}$$

- for the force between two magnetic poles:

$$F = k_5 \times (\text{pole } 1) \times (\text{pole } 2)/(\text{distance between the poles})^2, \tag{35}$$

- for the force between two electric charges:

$$F = k_6 \times (\text{charge } 1) \times (\text{charge } 2)/(\text{distance between the charges})^2. \tag{36}$$

The dimensions of these constants in each system of unit can be summarized as in Table 1).

Recall that Fourier had introduced dimensional considerations to express the invariance of physical equations. Without actually finding himself at odds with Fourier, Everett, by contrast, insisted on the conventional character of dimensional equations: although invariant under change of units within a given system, they could differ from one system to the next—where "system" here refers to a specific choice of fundamental units and the set of relations used in order to obtain the dimensions of derived quantities from these fundamental units (for instance, the electrostatic system), and not simply in Fourier's sense of a given set of units for the basic quantities (like L, M, T in the present context).

> A comparison of the foregoing six equations shows how cautious we ought to be in asserting that a particular dimensional relation "must hold in every system of units." The laws of nature which connect dissimilar quantities are laws of *proportion*, and it is only by convention that they can be stated as laws of *equality*.[41]

In other words, physical laws involved proportionality constants which could have dimensions. This issue was to become the focus of much attention, beyond the present controversy.[42]

Everett was not alone in inserting potentially dimensioned constants in fundamental equations to resolve the question. In the same issue of the *Philosophical magazine*, Joseph John Thomson responded to Everett's first description of Clausius's work by appealing to similar considerations (Thomson 1882a).[43] Thomson's main point was that in the equation:

$$\text{Pole} \times \text{Length} = \text{Current} \times \; (\text{Length})^2, \tag{37}$$

$$\text{i.e. magnetic moment } = \text{current} \times \text{area}, \tag{38}$$

Clausius should have taken into account the magnetic permeability μ of the material (Thomson 1882a, p. 427):

$$\text{magnetic moment } = \mu \times \text{current} \times \text{area}. \tag{39}$$

Recall that in this relation, the left-hand side refers to a magnetic pole and the right-hand side to a circuit deemed equivalent, in the sense that it produces the same magnetic force as the pole (when acting on another pole of the same strength). Thomson argued that the magnetic force due to a current does not depend on the surrounding medium, whereas the force due to a pole is inversely proportional to its magnetic permeability of the medium (Thomson 1882a, pp. 428–429).[44] From this he deduced the relation he advocated, Eq. 37.[45]

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[41] Everett (1882b, p. 434). Italics in the original.

[42] See concluding remarks.

[43] This article was published in the same issue as Everett's second paper, i.e., Everett (1882b).

[44] More specifically, Thomson discussed the magnetic force between a current and a pole, and the force between two poles. In order to conclude from these considerations their implications regarding Clausius's relation, what is at stake is the effect of, respectively, the circuit and of one of the two poles; one could describe the other pole involved in these two forces as a "test pole," and what Thomson calls "forces" actually corresponds to our field H—and indeed, the H generated by a current-carrying wire is independent of the permeability μ.

[45] Indeed, although Thomson did not make this explicit, his results implied:
$F_{\text{current loop}} \; \alpha$ current $\times$ area $\quad F_{\text{pole}} \; \alpha \; \frac{1}{\mu} \times$ magnetic moment
so $F_{\text{current loop}} = F_{\text{pole}}$ requires:
current $\times$ area $\; \alpha \; \frac{1}{\mu} \times$ magnetic moment,
which motivated Thomson's modification of the relation magnetic moment = current $\times$ area used by Clausius.

Thomson further derived that μ had dimensions $L^{-2}T^{2}$ (Thomson 1882a, p. 428), so that the revised relation he advocated implied:[46]

$$[\mathrm{pole_s}] = \left[\frac{\mu \times \text{current} \times \text{area}}{\text{length}}\right] = \frac{L^{-2}T^{2} \times M^{1/2}L^{3/2}T^{-2}xL^{2}}{L} = M^{1/2}L^{1/2}, \tag{40}$$

in agreement with Maxwell's result.[47] Thomson therefore recovered what Everett deemed necessary to get Maxwell's result with Clausius's approach, namely a constant factor of dimensions $\left[\frac{1}{v^2}\right] = L^{-2}T^{2}$ (Thomson 1882a, p. 429).

A few months later, Thomson published a second paper in which he further developed his argument (Thomson 1882b). He noted that Clausius's assumption implied that

[46] Thomson's demonstration of this point was somewhat circular, in so far that the way he obtained the electrostatic dimensions of the permeability μ required using $M^{1/2}L^{1/2}$ for the electrostatic dimensions of the magnetic pole (Thomson 1882a, p. 428). Indeed, μ is given by the ratio of the "magnetic induction" (i.e., our modern B) and the "magnetic force" (our field H, which he represented by F); now Thomson took the "magnetic induction" to be given by m/r^2, which clearly involves the pole m:

$[\mu] = \frac{[B]}{[H]} = \frac{[m]}{[H]}L^{-2}$

Thomson had already derived the electrostatic dimensions of the magnetic pole to be $M^{1/2}L^{1/2}$, without referring to the relation of interest (i.e., magnetic moment = $\mu \times$ current$\times$ area): from the expression for the force between a current and a pole, he deduced that the product current $\times$ pole must have dimensions of energy. Indeed, he took for what we would now call H:

$H = 2\frac{\text{current}}{\text{distance from wire}}$,

so that the work required to move a magnetic pole along a circle around the wire must be:

work $= H \times$ pole $\times$ distance moved $= 2\frac{\text{current}}{\text{distance from wire}} \times$ pole $\times 2\pi \times$ distance from wire.

Hence the dimensions of the product current $\times$ pole must be those of work, i.e., energy: $ML^{2}T^{-2}$. Since the electrostatic dimensions of current (from the relation between current and charge, and Coulomb's force law between electrostatic charges) were $M^{1/2}L^{3/2}T^{-2}$, it follows that the pole had electrostatic dimensions:

$[m] = \frac{[\text{energy}]}{[\text{current}]} = \frac{ML^{2}T^{-2}}{M^{1/2}L^{3/2}T^{-2}} = M^{1/2}L^{1/2}.$

This indeed implied that the dimensions of μ were $L^{-2}T^{-2}$ as noted in the main text:

$[\mu] = \frac{[m]}{[H]}L^{-2} = \frac{[m]}{\left[\frac{\text{current}}{\text{distance from wire}}\right]}L^{-2} = \frac{M^{1/2}L^{1/2}}{\left[\frac{M^{1/2}L^{3/2}T^{-2}}{L}\right]}L^{-2} = L^{-2}T^{-2}$

It is worth noting, however, that the dimensions of μ can also be obtained in a way that does not rely upon the dimensions of m: in the electrostatic system the dimensions of the force between two charges are $\left[\frac{qq'}{d^2}\right]$, and the dimensions of the force between two current-carrying wires are $[\mu i i']$, hence $\left[\mu\frac{qq'}{tt'}\right]$. Therefore, requiring that these forces have the same dimensions implies $\left[\mu\frac{qq'}{tt'}\right] = \left[\frac{qq'}{d^2}\right]$, and $[\mu] = L^{-2}T^{2}$.

[47] Thomson also obtained $M^{1/2}L^{1/2}$ for the electrostatic dimensions of the magnetic pole without referring to the relation of interest (i.e., magnetic moment = $\mu \times$ current $\times$ area): from the expression for the force between a current and a pole, he deduced that the product current $\times$ pole must have dimensions of energy. Indeed, he had found for what we would now call the magnetic field H (which he called magnetic force F):

$H = 2\frac{\text{current}}{\text{distance from wire}}$,

so that the work required to move a magnetic pole a full circle around the wire must be:

work $= H \times$ pole $\times$ distance moved $= 2\frac{\text{current}}{\text{distance from wire}} \times$ pole $\times 2\pi \times$ distance from wire.

Hence the dimensions of the product current $\times$ pole must be those of work, i.e., energy: $ML^{2}T^{-2}$. Since the electrostatic dimensions of current (from the relation between current and charge, and Coulomb's force law between electrostatic charges) are $M^{1/2}L^{3/2}T^{-2}$, it follows that the pole has electrostatic dimensions:

$[m] = \frac{[\text{energy}]}{[\text{current}]} = \frac{ML^{2}T^{-2}}{M^{1/2}L^{3/2}T^{-2}} = M^{1/2}L^{1/2}.$

the dimensions of the force between poles were different from those of a mechanical force: indeed, inserting Clausius's electrostatic dimensions for the pole in Coulomb's law yields:

$$[F]=\left[\frac{m_S m'_S}{r^2}\right]=\frac{\left(\mathrm{M}^{1/2}\mathrm{L}^{5/2}\mathrm{T}^{-2}\right)^2}{\mathrm{L}^2}=\frac{\mathrm{ML}^5\mathrm{T}^{-4}}{\mathrm{L}^2}=\mathrm{ML}^3\mathrm{T}^{-4}, \tag{41}$$

which can only be reconciled with the accepted dimensions of mechanical force by inserting a properly dimensioned factor. Thomson commented that Maxwell's approach avoided such *ad hoc* adjustments, since it warranted homogeneity through the constants that express the influence of the medium. Indeed, using $\frac{ee'}{\kappa r^2}$ for the electrostatic force between two charges, where κ is the electric permittivity of the medium, and $\frac{mm'}{\mu r^2}$ for the force between two poles, where μ is its permeability, resulted in those forces having the usual mechanical dimensions, as shown in Table 2.[48]

Despite the formal similarity of their analysis, Thomson did not share Everett's neutrality toward Clausius's work. In his view, the Maxwellian necessity of the medium decided in favor of Maxwell's choice (Thomson 1882b, p. 429).

The strategies and motivations of the other protagonists were quite different from Everett's and Thomson's. Larmor recovered Maxwell's result without introducing any additional factor in the equations (Larmor 1882, pp. 429–430). His strategy was dictated by the notion that some relationships are in some sense more fundamental and more natural than others. In his view, the relation used by Clausius,

$$\text{Magnetic moment} = \text{current} \times \text{area}, \tag{42}$$

was inferior in these respects to the expression for the force between a current and a magnet—and to Coulomb's force law between magnetic poles (Larmor 1882, p. 429). Firstly, Larmor argued, one always made use of these expressions, if only tacitly. Secondly, Clausius's relation relied more on an analogy than on an actual phenomenon:

> The somewhat recondite fact that the magnetic action of a small circuit carrying a current may be represented as due to two *fictitious* magnetic poles, does not seem to possess any claims to supplant the natural statement of the only fundamental relation which makes *natural* poles play a part in electric theory at all. That relation is, of course, itself in its very nature electromagnetic.
>
> It might be said that the system objected to built up the theory from electric foundations, inasmuch as its fundamental magnet is a small electric current. But, in reply, it is the existence of actual magnets which introduces the idea of pole at all, other than as in Ampère's purely mathematical directrix of electrodynamic action (Larmor 1882, p. 430).

and again:

[48] "These factors introduce themselves naturally through symbols representing some physical property of the body or medium." (Thomson 1882b, p. 226)

Table 2 Dimensions of the electric and magnetic forces

System	e	κ	$\frac{ee'}{\kappa r^2}$	m	μ[a]	$\frac{mm'}{\mu r^2}$
Electrostatic	$M^{1/2}L^{3/2}T^{-1}$	No dimensions	$\frac{\left(M^{1/2}L^{3/2}T^{-1}\right)^2}{L^2} = MLT^{-2}$	$M^{1/2}L^{1/2}$	$L^{-2}T^2$	$\frac{\left(M^{1/2}L^{1/2}\right)^2}{L^{-2}T^2L^2} = MLT^{-2}$
Electromagnetic	$M^{1/2}L^{1/2}$	$L^{-2}T^2$	$\frac{\left(M^{1/2}L^{1/2}\right)^2}{L^{-2}T^2L^2} = MLT^{-2}$	$M^{1/2}L^{3/2}T^{-1}$	No dimensions	$\frac{\left(M^{1/2}L^{3/2}T^{-1}\right)^2}{L^2} = MLT^{-2}$

[a] The dimensions of μ in the electrostatic system are the inverse of the dimensions of the factor obtained previously because μ is in the denominator

> The only simple fact of nature which connects pole with current is Oersted's phenomenon (as developed by Ampère) of the existence of a mutual force between them [...]. Hence by this relation must we deduce the dimensions of pole already found (Larmor 1882, p. 430).

Thus Larmor loathed Clausius's choice precisely for the reason that made it legitimate in Clausius's eyes: it expressed an *equivalence* between two distinct situations rather than an *influence* between two entities.[49]

Larmor's concerns were largely shared by Charles Kasson Wead, physics Professor at the University of Michigan. Wead proposed yet a different derivation of Maxwell's results, which he attributed to Hermann Herwig (Wead 1882, pp. 530–533).[50] This demonstration was based on the expression for the torque on a magnet placed in an external field, as being proportional to the intensity I of the magnetic field and the magnetic moment μ_M (Wead simply represented the latter by μ but it is best to avoid his notation since it was used above for permeability):[51,52]

$$\text{torque} = I \times \mu_M \tag{43}$$

$$[\mu_{\mathrm{Ms}}] = \frac{[\text{torque}]}{[I_{\mathrm{s}}]} = \frac{\mathrm{ML^2T^{-2}}}{\mathrm{M^{1/2}L^{1/2}T^{-2}}} = \mathrm{M^{1/2}L^{3/2}}, \tag{44}$$

and since $[\mu_M] = [m][L]$, one gets (Wead 1882):

$$[m_{\mathrm{s}}] = \frac{[\mu_{\mathrm{Ms}}]}{\mathrm{L}} = \mathrm{M^{1/2}L^{3/2}} \times \frac{1}{\mathrm{L}} = \mathrm{M^{1/2}L^{1/2}}. \tag{45}$$

Like Larmor, Wead favored his demonstration over Clausius's because of the status he conferred to the analogy between current and magnet:

> [According to Clausius mL] must represent a magnet; consequently [the current] $\times\, \mathrm{L}^2$ is put equal to a magnet. But the passage of the current ordinarily produces effects, such as the movement of a galvanometer-needle, which we explain more naturally by saying that the circular current produces a magnetic field at its center, than by saying that the current is, or makes, or even is equivalent to, a magnet. Herwig's derivation, therefore, in which a magnet placed in a field experiences a couple, conforms to the ordinary way of thinking better than the way of Clausius, and is the way used in the derivation of the electromagnetic system... It may be noted that, in all discussions except that of Clausius, the magnet pole P is introduced into the field. Clausius produces a field by the pole (Wead 1882, p. 531).

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[49] It is also interesting to note that Larmor not only viewed some relations as more fundamental than others, but he also deemed the electrostatic system of units, and electrostatic phenomena at large, to be more fundamental than their electromagnetic counterparts.

[50] Herwig was then Professor of physics at Darmstadt.

[51] The electrostatic dimensions Wead attributed to I agreed with those for the field H in Maxwell's treatise (Maxwell 1873, chapter 10).

[52] The dimensions of torque are given by: [torque] = [force] × [lever arm] = $\mathrm{MLT^{-2}} \times \mathrm{L} = \mathrm{ML^2T^{-2}}$.

Wead deemed that "to write " $C \times L^2 = P \times L$ in any consistent system" is simply begging the whole question." (Wead 1882, p. 532)

In addition to preferring a relation expressing a force law to one based on an analogy, Wead also granted special status to two equations concerning magnetism: those relating the moment of a magnet suspended in the Earth's magnetic field and the horizontal component of the latter field.[53] He noted that these equations led to the following dimensional relations:

$$[\mu_M B] = \frac{\mathrm{ML}^2}{\mathrm{T}^2} \quad \left[\frac{\mu_M}{B}\right] = \frac{\mathrm{L}^5}{\mathrm{L}^2} = \mathrm{L}^3 \tag{46}$$

So that μ_M had dimensions $\mathrm{M}^{1/2}\mathrm{L}^{5/2}\mathrm{T}^{-1}$, and B had $\mathrm{M}^{1/2}\mathrm{L}^{-1/2}\mathrm{T}^{-1}$, in all systems of units. Yet Clausius's equations implied that the first was not true, and according to Maxwell the second one was not. What motivated Wead to favor the two equations on which he based this argument were experimental considerations: "no physical formulae are better established" he wrote (Wead 1882, p. 532).

Wead even presented a third line of reasoning, based on the order in which relations were used and the dimensions of quantities were found in the different systems of units. In the electromagnetic system the dimensions of the pole m were found first, then from them those of the magnetic moment μ_M, then those of the magnetic field B (using the torque), those of the current C, and finally those of the electric charge:

$$\frac{[m][m]}{\mathrm{L}^2} = [F] \quad [m]\mathrm{L} = [\mu_M]; \quad [B][\mu_M] = [\text{torque}]; \quad \frac{[C]\mathrm{L}}{\mathrm{L}^2} = [B]; [e] = [C]\mathrm{T} \tag{47}$$

In the electrostatic system as advocated by Maxwell, the dimensions of the charge e were determined first, then those of the current C, then those of the magnetic field B, those of the magnetic moment μ through the torque and finally those of the pole:

$$\frac{[e][e]}{\mathrm{L}^2} = [F], \frac{[e]}{\mathrm{T}} = [C], \frac{[C]\mathrm{L}}{\mathrm{L}^2} = [B], [B]\,[\mu_M] = [\text{torque}], [\mu_M] = [m]\mathrm{L}. \tag{48}$$

The two approaches differed by the physical quantity they took as primary, m or e: once the dimensions of either m or e had been obtained, those of the other quantities were established in a different order, but using the same equations.

In contrast, as discussed above in Clausius's version of the electrostatic system, the dimensions of the magnetic moment, and hence of the pole, were obtained from a

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53 i.e., $\mu_M H = \frac{\pi K}{t^2(1+\theta)} \quad \frac{\mu_M}{H} = \frac{1}{2}\frac{\gamma^5 \tan\varphi - \gamma'^5 \tan\varphi'}{\gamma^2 - \gamma'^2}$
where μ_M: magnetic moment of the magnet H: horizontal component of the Earth's magnetic field
K: moment of inertia t: time γ and γ': lengths
Moment of inertia has dimensions M L^2. Indeed from the angular version of Newton's second law, torque = moment of inertia × angular acceleration:
[moment of inertia] = [torque] × [angular acceleration]$^{-1}$ = $\mathrm{ML}^2\mathrm{T}^{-2} \times (\mathrm{T}^{-2})^{-1} = \mathrm{ML}^2$
(see previous footnote for the dimensions of torque).
Therefore the two equations yield the dimensional relations:

$$[\mu_M H] = \left[\frac{\pi K}{t^2(1+\theta)}\right] = \frac{[K]}{[t^2]} = \frac{\mathrm{ML}^2}{\mathrm{T}^2} \quad \frac{\mu_M}{H} = \frac{1}{2}\frac{\gamma^5 \tan\varphi - \gamma'^5 \tan\varphi'}{\gamma^2 - \gamma'^2} = \frac{[\gamma^5]}{[\gamma^2]} = \frac{\mathrm{L}^5}{\mathrm{L}^2} = \mathrm{L}^3$$

different equation, $CL^2 = \mu_M$—the equivalence of a magnet and a current loop:

$$\frac{[e][e]}{L^2} = [F], \frac{[e]}{T} = [C], [C]L^2 = [\mu_M] = [m]L. \tag{49}$$

Thus Wead can be said to have favored a solution over the other on the basis of symmetry between the electrostatic and electromagnetic systems. As he noted, this symmetry was reflected in the relations between the dimensions of the primary quantities (Wead 1882, p. 532):[54]

$$[e_m] = \frac{ML^2T^{-1}}{[m_m]} \tag{50}$$

$$[m_s] = \frac{ML^2T^{-1}}{[e_s]}, \tag{51}$$

but this was not the case with Clausius's result,[55]

$$[m_{s\ \text{Clausius}}] = \frac{LT^{-1}}{[e_{s\ \text{Clausius}}]}. \tag{52}$$

Yet for Wead, regarding some equations as more "natural" or "fundamental" than others was largely subjective, so that Clausius's reasoning was not actually incorrect. Wead was mostly concerned with practicality and the fact that the electromagnetic system is widely used. He argued:

> If P be changed, three other quantities of the twelve that Maxwell discusses must have their dimensions changed, and confusion would be introduced into his system, that is based on fifteen equations, in each of which the second member is some simple mechanical quantity, as work, time, etc. Until it has been clearly shown how this system will be affected by the proposed change, and why the new expression is to be preferred to the older one, that has been "unimpeached" for some twenty years, is it not clearly better to write $P_e = M^{1/2}L^{1/2}$ It is not a question merely of correctness, but of consistency, simplicity, and usefulness; and on all these grounds Maxwell's expression seems to the writer to deserve the preference (Wead 1882, p. 533).

He even went as far as stating:

> In deducing the dimensions of physical quantities, there is much that is as arbitrary as the order in which several numbers shall be multiplied together (Wead 1882, p. 533).

For Wead, in the end the choice was purely one of convenience.

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54 These two equations can be obtained directly from Maxwell's system-independent relations, which take $[m_m] = $ m and $[e_s] = $ e: Eq. 50 is essentially Eq. 16 above, and Eq. 50 corresponds to Eq. 15.

55 Since $[m_{s\ \text{Clausius}}] = L^{5/2}T^{-2}M^{1/2}$ and $[e_{s\ \text{Clausius}}] = L^{3/2}T^{-1}M^{1/2}$ as usual.

4 The resolution of the controversy

The controversy ended in November with the intervention of the Maxwellian experimentalist Oliver Joseph Lodge (Lodge 1882). It was probably largely inspired by Thomson's September paper, in which the latter insisted that homogeneity was satisfied through the constants already present in Maxwell's work (Thomson 1882b).[56]

Lodge's approach is most interesting, both because it offers a synthetic view of the debate and because Lodge attributed the disagreements to the scientific culture of the protagonists, largely related to their nationality:

> The discussion which has been carried on in your pages respecting the dimensions of a magnetic pole serves to illustrate the divergency of thought between those in this country who have been brought up, electrically, under Faraday and Maxwell, and the continental philosophers so eminently represented by Prof. Clausius. From one point of view the discussion may be said to have been roused by a simple mare's nest constructed by dropping a factor out of one side of an equation...; but from another point of view it is the natural and inevitable consequence of the different aspects from which these matters can be regarded: the English standpoint, in which the medium is recognized as the active agent, and is continually present both in the mind and in the formulæ; and the continental standpoint, from which the medium is perceived as so much empty space, and is taken account as such in the formulæ (Lodge 1882, p. 357).

As Thomson had, Lodge argued for the need to take the electric inductive capacity K into account in the equation for the force between two charges and explained that Coulomb's electrostatic law – developed on the continent—was first meant as a law of proportionality. Faraday later established that it held only if the charges remained in the same medium and made Coulomb's law an equality by introducing the constant K to characterize this medium:

$$F = \frac{ee'}{Kr^2}. \tag{53}$$

Lodge then offered a dimensional demonstration reminiscent of Thomson's, but instead of assuming the dimensions of the various quantities to be known, he derived them from those of e and K. He noted from Eq. 53 that $\frac{e^2}{K}$ has the same dimensions as Fr^2 in all systems of units so:

$$[e] = \mathrm{L}[KF]^{1/2}. \tag{54}$$

The electrostatic system resulted when K was arbitrarily taken to be dimensionless (Lodge 1882, p. 358).

56 Lodge's paper was followed by an article by E.B. Sargant, but the latter mostly expressed his agreement with Lodge (Sargant 1882, pp. 395–396). Sargant is probably Edmund Beale Sargant (1855–1938), but this is not certain; the latter was educated in Cambridge and later distinguished himself through his contributions to education, notably in South Africa.

Turning his attention to the controversial case, Lodge added the quantity μ to express the influence of the medium on the force between two magnetic poles (Lodge 1882, p. 358):

$$F = \frac{mm'}{\mu r^2}, \tag{55}$$

so that

$$[m] = \mathrm{L}\,[\mu F]^{1/2}, \tag{56}$$

and

$$\frac{[e]}{[\sqrt{K}]} = \frac{[m]}{[\sqrt{\mu}]} = \mathrm{L}\ [F]^{1/2} = \mathrm{L}\ \left[\mathrm{MLT}^{-2}\right]^{1/2} = \mathrm{L}^{3/2}\mathrm{T}^{-1}\mathrm{M}^{1/2}. \tag{57}$$

Determining dimensions for the pole then required to arbitrarily set those of μ. This choice, however, was a priori independent from that which fixes the dimensions of e. Lodge argued that whatever relationship may exist between m and e should become the focus of experimental research, notably by considering the equivalence between a magnet and a current loop:

$$\text{Magnetic moment} \propto \text{current} \times \text{area enclosed by the current loop.} \tag{58}$$

The question was then whether this relationship was an equality or whether the medium had to be accounted for.

By rephrasing the debate in this way, Lodge not only clarified it but also accounted for it by a difference in scientific culture. However, Lodge went beyond this in two ways: first, he offered an explanation as to why this specific cultural difference caused confusion (Lodge 1882, pp. 363–365), and second he argued that the issue could be resolved, through experiments.

Indeed at the end of his paper Lodge gave himself the task to understand why Clausius had so easily accepted a result that differed from the standard one. Lodge noted:

> according to Weber's extension of Ampère's theory (an extension I [Lodge] suppose universally accepted), the properties of magnetic substances of all kinds are explained by molecular electric currents, and no magnets or magnetic substances other than those consisting of current-conveying molecules exist. [...]
> The coefficient μ is thus foreign to Ampère's theory applied universally; and this is how it has happened that Prof. Clausius has failed to recognize its existence and has been led into error." (Lodge 1882, p. 363)

Lodge therefore thought that the salient difference between the British and continental scientific cultures was related to different understandings of Ampère's theory. He defended the British standpoint, but in fact contrary to what the above quotation might suggest he did not reckon the concept of permeability to be incompatible with an Ampèrian model of magnetism. What Lodge deemed nonsensical was a model where microscopic currents would be bathed in an external medium itself characterized by a permeability μ. Instead, he thought Ampère's model fit to provide a microscopic account for permeability, understood as a macroscopic property:

> The effect of the medium is a physical fact; and no theory can presume really to dispense with the constant μ. All that the Ampèrian theory does is to give a physical interpretation to it, and to render one independent of it so soon as one takes account of every current-conveying circuit, whether molecular or other, existing in the field, and does not arbitrarily elect to deal only with those gross solenoids which we can excite and immediately control by batteries. (Lodge 1882, p. 365)

Rephrased in modern terms, one could say that Lodge thought permeability to be an emergent property of Ampèrian currents. In this respect he found himself in agreement with Maxwell:

> According to Ampère's hypothesis... the properties of what we call magnetized matter are due to electric currents, so that it is only when we regard the substance in large masses that our theory of magnetization is applicable. (quoted by Lodge 1882, p. 365, Maxwell 1873, Chapter 9, Art 615)

So Lodge pointed out that the disagreements could be explained by a difference in scientific culture, but having said this he then consciously endeavored to deepen our understanding of this difference, and then used the resulting insights to in turn help clarify the original issue.

Ultimately though, Lodge stressed that experiments were needed to settle whether the equivalence relationship at stake (Eq. 56) should involve permeability: the magnetic effects of a current and a magnet in media of different permeabilities had to be compared experimentally, because the results were still open to debate.[57] What was at stake was to *confirm* (or invalidate) the predictions of Maxwell's theory regarding the comparative behavior of poles and macroscopic currents in media of different permeabilities—not to explore issues not already covered in Maxwell's work.

J.J. Thomson had already made a similar suggestion (Thomson 1882a, p. 429, Lodge 1882, p. 359) but unlike Thomson, Lodge drew the distinction between medium in the region surrounded by the current and medium in the region surrounding it (consequently he wished to use solenoids). The scenarios he proposed to investigate and the results he predicted on the basis of Maxwell's work are summarized in Table 3 (Lodge 1882, pp. 361–362):

Lodge wished to know whether the fields due to a solenoid and a magnet remained the same irrespective of the permeability of the medium. If not, the equivalence relation between magnet and current (Eq. 58) should be modified accordingly to include μ. Like Thomson, he expected it should be. However, Thomson's treatment implied that

[57] Indeed after referring to Clausius's choice as an "error," Lodge stated in a footnote:
"In using the term "error" here, I would be understood to mean rather "divergency from opinions commonly held in this country" than absolute incorrectness as to matter of fact. For it would not be becoming to apply the latter term to views held by Prof. Clausius when the experimental foundation of opposing views is confessedly incomplete. The views held by Prof. Clausius are no doubt perfectly consistent and would probably be in accord with fact if only the medium produced no effect such as it is here [i.e., in Great Britain] commonly supposed to produce; and whether the medium does or does not produce such an effect appears to some extent at present a subject of legitimate debate and a matter for experimental investigation." (Lodge 1882, p. 364)

Table 3 Experiments proposed by lodge and their expected results

	Field produced by the magnet	Field produced by the solenoid	Comparison of the fields produced by the magnet and solenoid
Case 1: reference case	$\mathcal{H}_{air}$	$\mathcal{H}_{\text{air}}$	Equal to one another
Magnet: air			
Solenoid:			
Outer region: air			
Inner region: air			
Case 2	$\frac{1}{\mu_{\text{outer}}} \times \mathcal{H}_{\text{air}}$	$\frac{1}{\mu_{\text{outer}}} \times \mathcal{H}_{\text{air}}$	Equal to one another
Magnet: medium[a]			
Solenoid:			
Outer region: medium			
Inner region: air			
Case 3: considered by Thomson[b]	$\frac{1}{\mu_{\text{outer}}} \times \mathcal{H}_{air}$	$\frac{\mu_{\text{inner}}}{\mu_{\text{outer}}} \times \mathcal{H}_{\text{air}}$ with $\mu_{\text{inner}} = \mu_{\text{outer}}$: $\mathcal{H}_{\text{air}}$	Field due to solenoid $= \mu_{\text{inner}} \times$ Field due to magnet
Magnet:medium			
Solenoid:			
Outer region: medium			
Inner region:medium			
Case 4	$\mathcal{H}_{\text{air}}$	$\mu_{\text{inner}} \times \mathcal{H}_{\text{air}}$	Field due to solenoid $= \mu_{\text{inner}} \times$ field due to magnet
Magnet: air			
Solenoid:			
Outer region: air			
Inner region: medium			

[a] Here the term "medium" refers to one of permeability μ different from 1

[b] Lodge stated regarding the third scenario: "This latter seems to be the kind of experiment which Mr. J.J. Thomson suggests in his June letter (p. 429), and which, he says, Mr. Sargant then intended (and I hope still intends) to carry out" (Lodge 1882, p. 362). And indeed since Thomson did not distinguish between regions outside versus inside the current, he would not have taken the extra precautions necessary to prevent the interior of the solenoid to fill with the medium surrounding it, which corresponds to case 3

the relevant permeability was that of the outer region.[58] Lodge insisted that the latter had no effect (as case 2 was meant to show) and that it was the inner one that mattered.[59]

[58] The situation discussed by Thomson was equivalent to setting $\mu_{\text{inner}} = \mu_{\text{outer}}$ in case 3, so Thomson stated that the magnetic force due to a circuit was the same as in air, hence independent of the medium. He argued that the magnetic force due to a magnet goes as the inverse of the permeability of the medium, but having no way of distinguishing:
Field due to current = $\mu_{\text{inner}} \times$ Field due to magnet
from:
Field due to current = $\mu_{\text{outer}} \times$ Field due to magnet,
he argued for the latter.

[59] Lodge quoted Maxwell to justify his predictions:

Lodge certainly seems to have convinced his British colleagues, for the controversy ended with an article by E.B. Sargant in the same issue of the Philosophical magazine, in which the author concurred:[60]

> His [Lodge's] suggestion seems to me to reconcile the views of Prof. Clausius and Mr. J.J. Thomson on this subject. A model of the magnetic system must be made in a substance of the same magnetic permeability as the medium that is to surround the current-system, and must be substituted in the place of the magnets, before any comparison can be effected. The two systems, current- and magnetic, will then be always equivalent if once equivalent. (Sargant 1882, p. 395)

Nevertheless Lodge's suggestions calls for a number of remarks. For the purpose of figuring out the electrostatic dimensions of m, the macroscopic experiments proposed do seem to fulfill their duty. In so far that a magnetic pole and a current giving rise to the same magnetic force justifies viewing them as equivalent, the experiments do tell us about what form this equivalence should take.

However, it is doubtful that Lodge intended his macroscopic solenoids to serve as *models* for the behavior of Ampèrian currents.[61] As Lodge himself insisted, since the latter are responsible for the permittivity effect of the medium (i.e., essentially constitute the latter), a model where the microscopic currents themselves would be bathed in such a medium is nonsensical. The solenoids discussed were merely meant to establish the equivalence relationship at the *macroscopic* level.

In fact, Lodge did not expect the equivalence relation[62] to involve permeability at the fundamental, microscopic level of description, precisely because the latter involved only currents and no medium:

> We may accept [...] without hesitation Clausius's presentation of Maxwell's views, viz. both that a small magnet *is* an electric current, and that *magnetic current* ALWAYS *equals simply integral current* × *area*—remembering, however, that there exist currents in molecules besides the gross and artificial currents in our copper wires, that these are directed by our artificial currents and add their

Footnote 59 continuned

> I venture with diffidence to think that Maxwell would have drawn a distinction between the medium inside the region of the solenoid corresponding to the substance of the magnetic shell, and that outside. He over and over again lays stress upon the fact that artificial solenoids can only be compared with magnetic shells for the space outside the shells... (Lodge 1882, pp. 360–361)

The purpose of the proposed experiments was to verify this.

[60] Sargant also stated that an experiment that he had performed at the Cavendish Laboratory in June was in agreement with Lodge's view (Sargant 1882), but unfortunately he did not give any details. In the same issue of November Lodge had also reported that Sargant had expressed the intent to carry out what I called case 3 above, thereby implying that he believed Sargant had not yet done so.

[61] Whether Sargant actually had this in mind when he spoke of "model of the magnetic system" is unclear. In any case, the fact that Lodge's experiments involved two types of solenoids ("open" ones whose internal medium changed with the external one and "closed" solenoids with fixed internal medium) made it tempting to assume these could be meant as alternative models for the behavior of Ampèrian currents.

[62] At least, when understood as a relationship between physical quantities, as opposed to dimensions.

effects, and that in all cases they are most distinctly to be taken into account.[63] (Lodge 1882, p. 364)

Again, this was in agreement with Maxwell's views:

> ... if our mathematical methods are supposed capable of taking account of what goes on within the individual molecules they will discover nothing but electric circuits, and we shall find the magnetic force and the magnetic induction everywhere identical. (quoted by Lodge 1882, p. 365, Maxwell 1873, Chapter 9, Art 615)

That is to say, at the fundamental level $\mathcal{B} = \mu \times H$ becomes $\mathcal{B} = \mathcal{H}$.

It is only at the macroscopic level that Lodge held the equivalence relation took the form:

$$\text{Magnetic moment} = \mu \times \text{current} \times \text{area enclosed by the current loop,} \tag{59}$$

At that level, microscopic currents conspired to form the macroscopic μ according to:[64]

$$\mathcal{B} = \mathcal{B}_{\substack{\text{macro.}\\\text{currents}}} + \mathcal{B}_{\text{medium}} = \mathcal{H} + 4\pi\mathcal{I} = \mathcal{H} + 4\pi\,(\kappa\mathcal{H}) = (1 + 4\pi\kappa) \times \mathcal{H} = \mu \times \mathcal{H}, \tag{60}$$

where $\mathcal{I}$ is the magnetization and the "coefficient of induced magnetization" (Maxwell 1973, Vol.2, § 426). In contrast, at the fundamental, microscopic level:

$$\mathcal{B} = \mathcal{B}_{\text{currents}} = \mathcal{H}. \tag{61}$$

However, it is not the fundamental level that Lodge deemed relevant for the electrostatic dimensions of the magnetic pole, but the macroscopic one. Why is unclear, but one can readily find two possible (and mutually compatible) explanations.

Perhaps Lodge held that μ could not be neglected for dimensional purposes even at the microscopic level. Indeed eliminating μ and from the dimensional counterparts of Eqs. (59) and (60) implies that the dimensions of several physical quantities differ in the microscopic and the macroscopic realms.[65] Perhaps this would have seemed perfectly acceptable to Lodge (after all, dimensions differed in different systems of units), but then again perhaps not.

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[63] The earlier quotation "All that the Ampèrian theory does is to give a physical interpretation to it, and to render one independent of it so soon as one takes account of every current-conveying circuit, whether molecular or other, existing in the field, and does not arbitrarily elect to deal only with those gross solenoids which we can excite and immediately control by batteries" implies something similar.

[64] Lodge stated: "... in media consisting of Ampèrian molecules there is an extra magnetic induction [in addition to that due to macroscopic currents], due to the pointing of these along the lines of force, which is 4π times the magnetization, and which has to be added to the other, thus making the total magnetic induction at any point μ times the magnetic force (i.e., $\mathcal{H}$) there." (Lodge 1882, pp. 364–365)

[65] That is the dimensions of either $\mathcal{B}$ or $\mathcal{H}$ [see Eq. (60)] and those of the magnetic moment and, of course, m [Eq. (59)].

In any event, the systems of units had been developed on the basis of macroscopic phenomena (notably since the latter were involved in absolute measurements). Hence for the issue at hand, the relevant description had to be the macroscopic one.

5 Concluding remarks

The controversy regarding the dimensions of the magnetic pole in the electrostatic system therefore gave rise to different strategies, which often corresponded to different criteria for selecting the defining relation of a quantity. Some argued that a relation should be used because it expressed an analogy between two situations rather than a force between entities (Clausius). Others used the opposite criterion (Larmor, Wead). Some took certain equations, or even phenomena, to be more fundamental than others (Larmor, Wead). Some also presented symmetry arguments (Wead). Still others suggested the introduction of dimensioned constants in the equations (Everett, Thomson, Lodge). The latter strategy is the one that ultimately resolved the controversy, when Lodge argued that the disagreement was related to differences in scientific cultures, insofar as the said constants expressed the influence of the medium as imagined by British natural philosophers, whereas the medium was traditionally ignored on the Continent.

In contrast to this diversity in their approach, most of the participants certainly seem to have expected that there was indeed a fact of the matter regarding the electrostatic dimensions of the magnetic pole. Only Wead can be said to have displayed more pragmatic motives.

It is noteworthy that throughout the controversy, none of the protagonists expressed dissatisfaction with the fact that electromagnetic quantities had different dimensions in the different systems of units. Concerns over the matter only begun to be raised the following year, i.e., in 1883, in France. Although heavily influenced by Maxwell in other respects, two telegraph engineers, Ernest Mercadier (1839–1906) and Aimé Vaschy (1857–1899), strongly condemned this *status quo* in a series of papers (1883a, 1883b, 1883c, 1883d). To what extent their protest had been inspired, or merely influenced, by the 1882 controversy surrounding the electrostatic dimensions of the magnetic pole discussed here is unclear. Mercadier and Vaschy's timing is of course suggestive. Furthermore, they defended the view that the resolution of the ambiguity laid in determining the dimensions of the constants in the force laws and stressed that the issue could only be ultimately resolved by experimental investigations. However, they did not refer to the British debate, and even more significantly, they took for granted the relation expressing the equivalence between magnet and current, without referring to the issue of the permeability of the medium.

That the multiplicity of dimensions in the *different* systems of units was not brought up during the British controversy may seem surprising, notably because the two issues had much in common. As briefly mentioned above, they notably raised the question of what dimensions should be attributed to the constants in the force laws. Most of the British controversy focused on μ, i.e., k_3 in J.J. Thomson's classification, and the latter also called attention on the importance of k_6 and k_1; Mercadier and Vaschy focused on k_6 and k_4. In and of itself, this difference between the two debates certainly does not seem very profound.

In contrast, Mercadier and Vaschy were motivated by very different views. They held a conception of homogeneity faithful to the etymological meaning of the term: they argued that a given physical quantity should have a unique legitimate set of dimensions because the latter were determined by the very nature of this physical quantity. This conception is to be contrasted with that championed by Fourier and Maxwell whereby dimensions were purely defined in terms of units—and tension between the two views underlay much of the debates regarding dimensional analysis for the rest of the century. That the British protagonists of the 1882 controversy did not question the legitimacy of having different dimensions in different systems of units for the same physical quantity certainly goes to show how natural such plurality seemed to them after dimensions had been defined in terms of conversion factors. The pluralism of units led people to expect an analogous one for dimensions.

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