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The *casus irreducibilis* in Cardano's *Ars Magna* and *De Regula Aliza*

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Abstract In Cardano's classification in the *Ars Magna* (1545, 1570), the cubic equations were arranged in thirteen families. This paper examines the well-known solution methods for the families $x^3 + a_1x = a_0$ and $x^3 = a_1x + a_0$ and then considers thoroughly the systematic interconnections between these two families and the remaining ones and provides a diagram to visualize the results clearly. In the analysis of these solution methods, we pay particular attention to the appearance of the square roots of negative numbers even when all the solutions are real—the so-called *casus irreducibilis*. The structure that comes to light enables us to fully appreciate the impact that the difficulty entailed by the *casus irreducibilis* had on Cardano's construction in the *Ars Magna*. Cardano tried to patch matters first in the *Ars Magna* itself and then in the *De Regula Aliza* (1570). We sketch the former briefly and analyze the latter in detail because Cardano considered it the ultimate solution. In particular, we examine one widespread technique that is based on what I have called splittings.

1 Introduction

At the middle of the sixteenth century in Europe, equations were dealt with in the context of algebra treatises. During the early Renaissance, the Arabic algebraic tradition had already reached Europe through the mercantile paths, and its knowledge of equa-

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tions had been assimilated. For many centuries, scholars could solve any quadratic equation provided that it has real solutions. On the other hand, solving cubic equations in general by radicals still constituted an obstacle. Here and there, even before the 16th century, some cubic equations with numerical coefficients had been solved, but particular methods (that is, methods that depend on the coefficients and cannot be generalized to any cubic equation) were employed. Some approximation methods had also been developed to deal with cubic equations.

The topic of cubic equations was successfully approached for the first time in the context set by the *abaco* tradition in what is today northern Italy. There a certain number of scholars, Scipione del Ferro, Tartaglia, Girolamo Cardano, and Ludovico Ferrari among them, worked on this subject. As it is well known, the paternity of the solution methods for cubic equations was highly disputed. Allegedly, Scipione del Ferro discovered some of the solution methods in Bologna before his death in 1526, but he kept the lid on his results until the very end of his life. A few years later, Tartaglia, during a mathematical public duel with del Ferro's pupil, came to some comparable results, and the news reached Cardano. In 1539, he managed to make Tartaglia reveal his methods to him, and six years later, he published an improved and augmented version of these. This gave rise to the well-known dispute between Tartaglia, who felt robbed, and Cardano's pupil Ferrari, who had by now also come to the solution of quartic equations.

In the following, we focus on Cardano's algebraic treatises. We deal in particular with the solution methods for cubic equations that are delivered in the middle of his masterpiece, the *Ars Magna*, and with the (expected) patch to these methods that should have been contained in the *De Regula Aliza*.

2 A precision concerning Cardano's cubic equations

In order to best grasp Cardano's treatment of cubic equations, we need to start from the very beginning. In this section, I would like to take stock of equations in his algebraic works,[1] and in particular of the way in which cubic equations are expressed. This will let us see that Cardano did not have a terminology available for speaking of cubic equation in the abstract—that is, when the coefficients are completely unspecified—and so, he needed to take into account many different families of cubic equations and classify them according to some criteria.

[1] Namely, they are the *Practica Arithmeticae* (1539), the *Ars Magna Arithmeticae* (1663), the *Ars Magna* (1545, 1570), and the *De Regula Aliza* (1570).
Concerning in particular the *Ars Magna Arithmeticae*, we remark that the only edition that we have available is the one posthumously printed in Cardano's *Opera Omnia*. For a long time, it has been considered as a minor work and in any case as a late work (see Bortolotti 1926; Loria 1950, p. 298). Cardano himself did not mention this treatise in his autobiographies and never sent it to be printed. Moreover, the (supposedly chronological) position of the published version in the framework of the 1663 *Opera Omnia* contributed to the misunderstanding because the *Ars Magna Arithmeticae* was wrongly placed after the *Ars Magna*. Thanks to some recent reappraisals of Cardano's mathematical writings (see Tamborini 2003, pp. 177–179; Gavagna 2012), it is nowadays commonly agreed that the *Ars Magna Arithmeticae* was conceived before—or at least at the same time as—the *Ars Magna*.

In agreement with Renaissance habits, Cardano wrote equations in words, employing the common Latin terminology. The coefficient of the term of degree zero—as we call it nowadays[2]—is called a 'number (*numerus*)', the unknown a 'thing (*res*)' or 'position (*positio*)', the unknown to the second power a 'square (*quadratus*)' or '*census*', and the unknown to the third power a 'cube (*cubus*)'.[3] We can consider that for Cardano, an equation is written in the standard form when it is monic. Indeed, in the great majority of the cases, he directly took into account monic equations and, when the equation is not, the first step is to reduce it to a monic one. When the coefficients of the unknown or of its powers are different from 1 and Cardano did not want to specify their numerical value, he employed plural forms, like 'some things (*res*)' or 'some squares (*quadrati*)'. We remark that even though Cardano had available this work around for speaking about families of equations where the numerical value of the coefficients is not pointed out, he did not have any particular preference to employ it rather than to specify the coefficients. Indeed, as it was common at that time, one could do mathematics in a general way, but yet not in the abstract.

Abbreviations, like '*p:*' or '*p.*' for '*plus*', are frequently used, but they are not always stable (for instance, 'cube' is sometimes abbreviated '*cub.*' or '*cu.*' and sometimes written in full words). The role of these shorthand expressions, far from being an established formalism, is to shorten the text, in the same manner as the accents in Renaissance Latin did.

Cardano was of course interested in finding the real solutions of the equations that he considered. He usually preferred the positive ones [called 'true (*vera*)'], but it sporadically happens that also the negative ones [called 'fictitious (*ficta*)'] were mentioned.[4]

At this point, one is quite naturally led to wonder what, for Cardano, are the quantities that I have called 'coefficients' above. The status of these quantities depends to a great extent on what Cardano considered to be a *numerus*. In comparison to our sets of

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[2] I have deliberately chosen to expound Cardano's mathematics using some terminology and symbols that he did not employ. Of course, this betrays to a certain extent the text and can lead to over-interpreting it. However, I believe that the advantages for the reader in terms of grasping the contents precisely are worth the risk. For the same reason and for the sake of brevity, I use certain notations, especially the discriminant Δ_3, that refer to Cardano's equations. In this case, the discriminant is not to be understood in the modern sense, that is, as a symmetric function of the roots of the corresponding polynomial. Rather it is a shorthand for the number that is under the square root in Cardano's formulae (even though this number has the same value up to a constant as the symmetric function evaluated at an appropriate point).

[3] Here and there in Cardano's mathematical treatises, we also find equations of higher degrees. The prime degrees of the unknown greater than three that Cardano mentioned are the 'first related (*relatus primus*)' for degree five and the 'second related (*relatus secundus*)' for degree seven. The non-prime degrees are obtained by multiplication of prime factors, such as for instance the 'square of a square (*quadrati quadratus*)' for degree four, the 'cube of a square (*quadrati cubus*)' for degree six, the 'square of a square of a square (*quadrati quadrati quadratus*)' for degree eight, the 'cube of a cube (*cubi cubus*)' for degree nine.

[4] Interestingly, in *Ars Magna*, Chapter XXXVII Cardano gave a justification of the fact that one is also entitled to take into account negative solutions. Indeed, a negative solution of a certain family of equations is admissible as long as it corresponds to a positive solution of another family of equations where some of the coefficients are opposite in sign. For instance, this happens for $x^2 = a_1x + a_0$ and $y^2 + a_1y = a_0$, see Cardano (1545, Chapter XXXVII, rule I, p. 65v) but also for $x^3 + a_0 = a_1x$ and $y^3 = a_1y + a_0$, and for $x^3 + a_2x^2 = a_0$ and $y^3 + a_0 = a_2y^2$ (see Cardano 1545, Chapter I, Paragraphs 5–6 and 8, pp. 4r–4v and 5r–5v).

numbers, he had available only an incomplete system of numbers that was not closed under every algebraic operation. More precisely, he took the freedom to consider that when he performed some algebraic operations on rational numbers, other numbers that he knew how to deal with were produced. For instance, the operation of taking the square root applied to 2 gives $\sqrt{2}$, and Cardano knew how to further perform algebraic operations on this number. This remark opens a broad-ranging issue: how can we characterize Cardano's numbers? And, moreover, was Cardano's conception of numbers common at that time?

These fundamental questions are far from being naive, and trying for an answer is not the aim of this paper. Therefore, rather than attempting for an interpretation, I provide only a descriptive account. In Cardano's algebraic treatises, these coefficients are always nonzero, positive numbers. Wherever we would write a negative coefficient, we find that in Cardano's equations, it has been moved on the other side of the equality.[5] In the great majority of the examples, the coefficients are rational (even natural) numbers, but in a small handful of cases, irrational numbers also appear (see, for instance, Footnote 19).

Concerning cubic equations in particular, Cardano classified them according to the following criteria. He took into account only those that cannot be solved in elementary ways, that is, equations that cannot be solved by simply taking the third root and that do not come down in degree by means of a substitution. Therefore, when it comes to solving cubic equations, the families $x^3 = a_0$, $x^3 = a_1x$, $x^3 = a_2x^2$, $x^3 = a_2x^2 + a_1x$, $x^3 + a_2x^2 = a_1x$, and $x^3 + a_1x = a_2x^2$ are not considered. Moreover, since Cardano considered equations with positive (real) solutions, the families of equations $x^3 + a_1x + a_0 = 0$, $x^3 + a_2x + a_0 = 0$, and $x^3 + a_2x^2 + a_1x + a_0 = 0$ do not appear in the classification either. In the end, Cardano had thirteen families of cubic equations divided into:

- depressed equations lacking a second degree term:

$$\begin{aligned} x^3 + a_1x &= a_0 \quad \text{or} \quad \textit{cubus et res aequales numero},\\ x^3 &= a_1x + a_0 \quad \text{or} \quad \textit{cubus aequalis rebus et numero},\\ x^3 + a_0 &= a_1x \quad \text{or} \quad \textit{cubus et numerus aequales rebus}; \end{aligned}$$

- equations lacking a first degree term:

$$\begin{aligned} x^3 &= a_2x^2 + a_0 \quad \text{or} \quad \textit{cubus aequalis quadratis et numero},\\ x^3 + a_2x^2 &= a_0 \quad \text{or} \quad \textit{cubus et quadrati aequales numero},\\ x^3 + a_0 &= a_2x^2 \quad \text{or} \quad \textit{cubus et numerus aequales quadratis}; \end{aligned}$$

[5] Sometimes, Cardano wrote an equation such as $x^3 = a_2x^2 + a_0$ in the form $x^2(x - a_2) = a_0$. Despite the fact that a minus sign appears, we observe that this does not mean that Cardano allowed negative numbers in an equation since $a_0 > 0$ implies that $x - a_2 > 0$ (but it would have been so if he had written $x^3 - a_2x^2 = a_0$, which he never did).

– complete equations:

$$
\begin{array}{lll}
x^3 + a_2x^2 + a_1x = a_0 & \text{or} & \textit{cubus, quadrati et res aequales numero,} \\
x^3 + a_1x = a_2x^2 + a_0 & \text{or} & \textit{cubus et res aequales quadratis et numero,} \\
x^3 + a_2x^2 = a_1x + a_0 & \text{or} & \textit{cubus et quadrati aequales rebus et numero,} \\
x^3 = a_2x^2 + a_1x + a_0 & \text{or} & \textit{cubus aequalis quadratis, rebus et numero,} \\
x^3 + a_0 = a_2x^2 + a_1x & \text{or} & \textit{cubus et numerus aequalis quadratis et rebus,} \\
x^3 + a_1x + a_0 = a_2x^2 & \text{or} & \textit{cubus, res et numerus aequales quadratis,} \\
x^3 + a_2x^2 + a_0 = a_1x & \text{or} & \textit{cubus, quadrati et numerus aequales rebus.}
\end{array}
$$

This provides an effective way of arranging cubic equations in families so that it is easier to speak about them. In the main, Cardano's classification conveys a combinatory oriented, linguistic concern, rather than being governed by purely mathematical requirements.[6]

3 Cardano solves cubic equations in the *Ars magna*

The so-called *Ars Magna* appeared in 1545,[7] where the most advanced results of that time on equations were released. The importance of the *Ars Magna* can only be understood by contrasting it with what came before: for the first time, all the scattered procedures, techniques, and particular cases that had been discovered were replaced by a highly systematic treatment of cubic equations, which also included proofs. This does not mean, however, that the *Ars Magna* itself is a systematic treatise: even though we can identify in the middle of the book (Chapters XI–XXIII) the consistent block that delivers the famous solution methods for cubic equations, what comes before and after is still a miscellany of rules. As such, the tricky contents of the *Ars Magna* also moved away from the usual considerations of the algebra treatises of that time.

Every chapter in the middle block has the same structure. For each one of the thirteen families of cubic equations, Cardano first provided a proof (*demonstratio*) of its solution method,[8] from which he subsequently deduced a rule (*regula*), that

[6] We remark that a classification in the same vein can already be found, for instance, in al-Khayyām's algebra.

[7] The full title is *Hieronymi Cardani, praestantissimi mathematici, philosophi, ac medici, artis magnae sive de regulis algebraicis, liber unus, qui et* [sic] *totius operis de arithmetica, quod opus perfectum inscripsit, est in ordine decimus*, see Cardano (1545). This first edition was published in Nuremberg in 1545 by Iohannes Petreium. The second edition was published in Basel in 1570 by Oficina Henricpetrina and was a joint volume together with the *De Proportionibus* and the *De Regula Aliza*. There is, moreover, a third posthumous edition in the *Opera Omnia*. The second edition contains some additions, while the third edition is virtually the same as the second one. We also have the available manuscript Plimpton 510/1700: s.a., which is entitled *L'algebra*, at the Columbia University Library in New York.
According to the title, the *Ars Magna* should have filled up the tenth volume of a certain *Opus Arithmeticae Perfectum*, which was never realized. This should have been an encyclopedic work on mathematics composed of fourteen books and was probably conceived between the 1530s and the 1560s. For an extensive explanation of this reference, see Gavagna (2010, p. 65) and Cardano (2004, pp. 64–66).

[8] Whether, and if so to what extent, these proofs contain geometrical arguments is up to discussion.

is to say, a step-by-step description of the operations that one has to perform on the coefficients of the equation under consideration in order to get a solution. This is a numerical value that substituted in the equation returns a true equality. In three cases (namely, in Chapters XI, XII, and XIV), the rule is in fact a formula that allows one to write a solution by combining the coefficients of the equations directly through the usual algebraic operations. In the other cases, the rules involve some substitutions for transforming the equation at issue into another one, the solution method of which has already been explained. After that, we sometimes find an additional and/or alternative proof and always some numerical examples.

3.1 The two pristine solution methods and the *casus irreducibilis*

In Chapters, XI and XII Cardano dealt, respectively, with the pristine family of equations $x^3 + a_1x = a_0$ and $x^3 = a_1x + a_0$. Indeed, as we will see in Sect. 4, the solution methods for all the families of cubic equations can be reduced to one of these two. Many accounts have been given in the secondary literature,[9] and for our purpose, it is enough to sketch the overall structure of the arguments. In these chapters, Cardano argued in a completely analogous, though independent way. First, as he always did, he translated the algebraic setting into a geometrical environment, where I use 'geometrical' merely to mean that Cardano spoke about points, segments, surfaces, and solids.[10] The tricky parts of the proofs in these chapters follow from the key idea, which is to write, respectively, $x = y - z$ and $x = y + z$ and to take, respectively, the systems

$$\begin{cases} a_0 = y^3 - z^3 \\ \frac{a_1}{3} = yz \end{cases}$$

and

$$\begin{cases} a_0 = y^3 + z^3 \\ \frac{a_1}{3} = yz, \end{cases}$$

where the coefficients of the two equations are associated to some functions of y and z. Then, Cardano used the development of the cube of a binomial and the properties of the operations on the measures of segments to show that such xs are solutions of the equations. Indeed, speaking about geometrical objects, he performed the actions that

[9] These solution methods are indeed the most taken into account in the secondary literature: see, among others, Cantor (1892), Bortolotti (1926), Boyer (1991), Bashmakova and Smirnova (2000) and Maracchia (2005). For a detailed account of these two solution methods as well as all the others, see also Confalonieri (2013, pp. 103–151).

[10] In other words, I employ the term 'geometrical' only on a superficial linguistic level, and I do not associate to it a precise interpretation of what geometry was for Cardano. Note in particular that the fact that Cardano spoke about points and other such objects does not necessarily mean that he also employed the positional properties of these objects. Concerning the general relation between geometry and positional arguments, see Panza (2007).

are equivalent to replacing such assumed values of x, a_1, and a_0 in the equations. These proofs, like all the ones in the middle block of the *Ars Magna*, give no hint of how Cardano got the key idea—in these two cases, to use the substitutions $x = y \pm z$ and the above systems.[11] We, moreover, remark that once one has made the substitutions and has the systems, one also has at hand the rules, which—as has been said—are descriptions of the cubic formulae in algorithmic terms. Indeed, the formulae are obtained by solving the systems depending on y^3, z^3 (which is possible since solving these systems is equivalent to solve two quadratic equations), taking the (real) cubic roots of their solutions, and exploiting their internal symmetries to reduce the number of solutions.

There is nevertheless a very remarkable difference between the solution methods for the families of equations $x^3 + a_1x = a_0$ and $x^3 = a_1x + a_0$: in the first case, the cubic formula never contains square roots of negative numbers, whereas this can happen in the second case. To verify this, it is enough to have a look at the two formulae

$$x = \sqrt[3]{\sqrt{\left(\frac{a_0}{2}\right)^2 + \left(\frac{a_1}{3}\right)^3} + \frac{a_0}{2}} + \sqrt[3]{\sqrt{\left(\frac{a_0}{2}\right)^2 + \left(\frac{a_1}{3}\right)^3} - \frac{a_0}{2}}$$

[11] Historically speaking, it seems very likely that the idea came straight from Tartaglia's poem (up to a change of variables). Indeed, its first two verses describe the substitutions and the systems:

When the cube with the things next to it
    is made equal to some other discrete number
    find two others the difference of which [is] this.
Hereafter, you will consider this customarily
    that their product always will be exactly equal
    to the third of the cube of the things.
Its general remainder then
    of their cube sides appropriately subtracted
    will be the value of your principal unknown.

In the second of these acts
    when the cube remains alone,
    you will observe these other arrangements,
you will immediately make of the number two such parts
    that the one times the other will produce straightforward
    the third of the cube of the things
of which then by common precept
    you will take the cube sides joined together
    and this sum will be your concept.

(*Quando che'l cubo con le cose appresso\se agguaglia à qualche numero discreto\trovan dui altri differenti in esso.\Da poi terrai questo per consueto\che 'l lor produtto sempre sia eguale\al terzo cubo delle cose neto.\El residuo poi suo generale\delli lor lati cubi ben sottratti\varra la tua cosa principale.\In el secondo de codesti atti\quandi che 'l cubo restasse lui solo\tu osservarai quest'altri contratti,\del numero farai due tal part'àl volo\che l'una in l'altra si produca schietto\el terzo cubo delle cose in stolo\delle qual poi, per comun precetto\torrai li lati cubi insieme giontì\e tal somma sara il tuo concetto*), see Tartaglia (1959, Quesito XXXIII, p. 119). Nevertheless, this only displaces the difficulty to finding out how Tartaglia (or whoever else) got the key idea.

and

$$x = \sqrt[3]{\frac{a_0}{2} + \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}} + \sqrt[3]{\frac{a_0}{2} - \sqrt{\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_1}{3}\right)^3}},$$

respectively, for $x^3 + a_1 x = a_0$ and $x^3 = a_1 x + a_0$ in Chapters XI and XII. Nowadays we know, as an elementary consequence of Galois theory, that when a cubic equation has three real distinct solutions, imaginary numbers necessarily appear in the cubic formula. This means that—roughly speaking—in order to arrive at a simple solution, one cannot avoid passing through a more complicated formula. Down the centuries, this has been called the *casus irreducibilis*.[12] Since Cardano did not allow square roots of negative numbers,[13] he had to count them out by imposing a condition on the coefficients of $x^3 = a_1 x + a_0$. In 1545 *Ars Magna*, the condition is stated as follows:

> when the cube of the third part of the number of the things exceeds the square of the half of the number of the equality, which happens whenever the number of the equality is less than $\frac{3}{4}$ of this cube or when a number greater than the number of the equality is produced from $\frac{2}{3}$ the number of the things times the root of $\frac{1}{2}$ the same number, then this can be solved by the aliza question, which is discussed in the book on the geometrical questions. But if you wish to avoid such a difficulty, you may be satisfied for the most part by Chapter XXV of this (*[a]t ut cubis tertiae partis numeris rerum, excedat quadratum dimidij numeri, aequationis, quod accedit quandocunque numerus aequationis est minor $\frac{3}{4}$ cubi illius, vel ubi ex $\frac{2}{3}$ numerum rerum, producitur in* R $\frac{1}{2}$ *eiusdem numeri maior numerus numero aequationis, tunc hoc dissolvitur per quaestionem alizam, de*

12 We do not know exactly when this term came into use. The first occurrence of which I am aware is in Lagny (1697). I thank David Rabouin for this reference.

13 This also happened in the case of quadratic equations. Considering $x^2 + a_0 = a_1 x$ in Chapter V, Cardano asked for a condition on the coefficients (namely, $\left(\frac{a_1}{2}\right)^2 - a_0 > 0$) so that negative numbers never appear under square roots. More precisely, the condition is "[i]f that subtraction of the number from the square of the half of the number of the things cannot be done, the question itself is false and what has been proposed cannot be ("*[q]uod si detractio ipsa numeri a quadrato dimidii numeri rerum fieri nequit, quæstio ipsa est falsa, necesse potest quod proponitu*)", see Cardano (1545, Chapter V, p. 11v).
Nevertheless, there is an isolated passage in the *Ars Magna* where Cardano tried to justify the square roots of negative numbers. This is in Chapter XXXVII, while discussing "the rules for assuming a false (*regula falsum ponendi*)" and in particular in the proof of Rule II (see Cardano 1545, p. 66r). This is about solving a certain system equivalent to a quadratic equation with two complex, nonreal solutions. Cardano used the quadratic formula to find their (correct) value (namely, "*5 p: R m: 5* and *5 m: R m: 5*", that is, $5 \pm \sqrt{-15}$), and then provided a proof, which is in short a geometrical interpretation of the quadratic formula. There, he had a try at linking the expression $\sqrt{-15}$ to a geometrical object, namely to the difference between a surface and a segment (where the surface is called AD and measures 25 and the segment AB that measures 10 is taken four times: "*imaginaberis R m: 15, id est differentiae AD et quadrupli AB*"). A few lines later, Cardano commented that this is "sophisticated" since one cannot carry it out with the same operations with a negative quantity ("*quae vere est sophistica, quoniam per eam, non ut in puro m: nec in aliis, operationes exercere licet*"). In my opinion, this amounts to one of the earliest investigations of the square roots of negative numbers. At the same time, it also shows how far Cardano was from being acquainted with these kinds of numbers. It is in fact significant that this chapter comes long after the ones that deliver the solution methods for quadratic and cubic equations and that it was a marginal comment.

> *qua in libro de quaestionibus geometricis dictum est, sed si libet tantam effugere difficultatem, plerumque capitulum 25m huius tibi satisfaciet*),[14]

while in 1570, the last part of the quotation was modified into "then consult the book of aliza here appended (*tunc consules librum Alizae hic adjectum*)", see Cardano (1570a, Chapter XII, p. 62).

3.2 The other solution methods

In Chapter XIII, Cardano dealt with the family of equations $x^3+a_0 = a_1x$. Its solution method, as it is expounded in the proof, consists in transforming the equation at issue into $y^3 = a_1y + a_0$ through one of the substitutions[15]

$$x = \frac{y}{2} \pm \sqrt{a_1 - 3\left(\frac{y}{2}\right)^2}.$$

As before, no hint is given about how the substitutions were discovered, and in this case, Tartaglia's poem only tells about a certain connection with $y^3 = a_1y + a_0$, without going any further. Cardano provided also an additional proof to show how, given a (real) solution of the equation at issue, one can obtain a second solution (when a real one exists).

All the remaining solution methods consist in giving a suitable substitution for transforming the equation at issue either into one of the three discussed above or into an equation that Cardano knew how to solve without using a cubic formula (namely, into $x^3 = a_0$ or $x^3 = a_1x$). In Chapters XIV–XVI, Cardano dealt with the cubic equations lacking in the first degree term and in Chapters XVII–XXIII with the complete ones.

Let us consider an example. Chapter XIV is about the family of equations $x^3 = a_2x^2 + a_0$. This chapter is quite short, and in this sense, it is illustrative of Cardano's ways of proceeding: it starts with a brief remark, followed by a proof (which fills up most of the chapter), then the rule derived from it, and then one example. The structure of this chapter is simple because the family of equations $x^3 = a_2x^2 + a_0$, as was already the case with $x^3+a_1x = a_0$, never falls into the *casus irreducibilis*, so that Cardano neither needed to add a condition nor to provide many examples. The opening remark points out that the equation at issue can be changed into[16] $y^3 = a_1y + a_0$. The

[14] See Cardano (1545, Chapter XII, p. 31v). The condition means that Cardano took into account only the case where

$$\left(\frac{a_0}{2}\right)^2 > \left(\frac{a_1}{3}\right)^3,$$

which is equivalent (up to a constant) to ask for $\Delta_3 > 0$, see Footnote 2.

[15] From the modern viewpoint, these are not real changes of variable since the maps $v \mapsto v^2$ and $v \mapsto \sqrt{v}$ are not bijections. This is sometimes the case in Cardano's substitutions. Anyway, if we consider that Cardano favored positive numbers, his substitutions are invertible. Indeed, in the examples that he provided, he used the substitutions in both directions, see for instance Footnote 19.

[16] Note that despite the fact that the equation in x has always $\Delta_3 > 0$, this is not a priori the case for the transformed equation in y. However, if we go and check the discriminant of the transformed

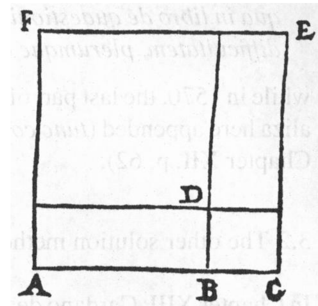

Fig. 1 Diagram from the proof in *Ars Magna*, Chapter XII (also referred to in Chapter XIV). Kind courtesy of Biblioteca Virtual del Patrimonio Bibliográfico and Universidad de Sevilla

proof is as follows. Cardano transformed the algebraic setting of the problem (that is, to solve the abstract family of equations $x^3 = a_2x^2 + a_0$) into a specified geometrical environment (see Fig. 1), and considered the equation

$$AC^3 = 6AC^2 + 100.$$

He implicitly assumed that $BC = \frac{6}{3} = 2$ and wanted to show that the above equation can be changed into

$$AB^3 = 12AB + 116$$

through the substitution $AC = AB + BC$. By the square of a binomial, Cardano knew that $AC^2 = AB^2 + 2AB\,BC + BC^2$, and therefore, $AC^2 = AB^2 + 4AB + 4$, which in turn implies that

$$AC^3 = 6(AB^2 + 4AB + 4) + 100.$$

On the other hand, by the cube of a binomial, Cardano got

$$AC^3 = (AB + BC)^3 = AB^3 + 3AB^2\,BC + 3AB\,BC^2 + BC^3.$$

Finally, by comparison and developing the calculations, he obtained the desired equation

$$AB^3 = 12AB + 116.$$

At this point, since Cardano knew how to solve the transformed equation in AB by Chapter XII, the relation $AC = AB + BC$ yields the solution of the equation in AC.

footnote16 continued
equation, we get

$$\Delta_3 = \frac{a_0^2}{4} + \frac{1}{27}a_0a_2^3,$$

which is always positive since we took a_0, a_2 positive.

We remark that if we take $AC = x$ and $AB = y$, Cardano is in truth performing the substitution $x = y + \frac{a_2}{3}$ in a geometrical fashion. The proof consists essentially in showing that by replacing the assumed value of x in the equation $x^3 = a_2x^2 + a_0$, one gets the equation $y^3 = a_1y + a_0$. Afterwards, Cardano summed up the proof in the following rule:

> Add the cube of the third part of the number of the squares to the half of the number of the equality and multiply the whole that is made from there to the square. Take away the cube of the square of the third part of the number of the squares, add or decrease the root of the remainder to or from the aggregate that you had multiplied by itself. You will have the *binomium* and the *apotome*. Join their cubic roots and add to them the third part of the number of the squares, and the whole that is composed is the value of the thing (*[a]dde cubum tertiae partis numeri quadratorum dimidio numeri aequationis, et totum quod inde fit in se ducito a quadrato. Abice cubum quadrati tertiae partis numeri quadratorum, residui radicem adde et minuo dimidio aggregati quod in se duxeras, habebis binomium et apotomen, cuius R cubicam iunge et eis adde tertiam partem numeri quadratorum, et totum quod conflatur est rei aestimatio*),[17]

which is the formula

$$x = \sqrt[3]{\left(\frac{a_2}{3}\right)^3 + \frac{a_0}{2} + \sqrt{\left(\left(\frac{a_2}{3}\right)^3 + \frac{a_0}{2}\right)^2 - \left(\frac{a_2}{3}\right)^6}}$$
$$+ \sqrt[3]{\left(\frac{a_2}{3}\right)^3 + \frac{a_0}{2} - \sqrt{\left(\left(\frac{a_2}{3}\right)^3 + \frac{a_0}{2}\right)^2 - \left(\frac{a_2}{3}\right)^6}} + \frac{a_2}{3}.$$

Indeed, if we consider the equation $x^3 = a_2x^2 + a_0$ and apply to it the substitution $x = y + \frac{a_2}{3}$, we get the transformed equation $y^3 = \frac{a_2^2}{3}y + (\frac{2}{27}a_2^3 + a_0)$. Then, using the cubic formula in Chapter XII, we immediately have the above formula. We remark that, since a_0 and a_2 are always positive, the number under the square root is always positive.

It can happen that after having performed the substitution in the equation at issue, the sign of the coefficient(s) of the transformed equation is not a priori determined. More precisely, in Chapters XV, XIX, XXI, and XXII, the sign of a_0 is not determined, and therefore, Cardano needed to discuss some cases. In Chapters XVII and XVIII, the signs of neither a_0 nor a_1 are determined, so Cardano also needed to discuss some subcases. In Chapter XV, we exceptionally find two substitutions for $x^2 + a_2x^2 = a_0$: the first one (which is $x = y - \frac{a_2}{3}$) leads to several cases, while the second one (which is $x = y^2 - a_2$) leads to a unique case. In Chapter XVI, as for Chapter XIII, an additional proof for obtaining a second solution of the considered equations is provided (when a real solution exists), given

[17] See [Cardano (1545, Chapter XIV, p. 33v). The second '*dimidio*' in "*residui radicem adde et minuo dimidio aggregati quod in se duxeras*" is a typo (emended in none of the editions).

Table 1 Substitutions in the solution methods for cubic equations in *Ars Magna*, Chapters XI–XXIII

Chapter	Equation	Substitution
XI	$x^3 + a_1x = a_0$	
XII	$x^3 = a_1x + a_0$	
XIII	$x^3 + a_0 = a_1x$	$x = \frac{y}{2} \pm \sqrt{a_1 - 3\left(\frac{y}{2}\right)^2}$
XIV	$x^3 = a_2x^2 + a_0$	$x = y + \frac{a_2}{3}$
XV	$x^3 + a_2x^2 = a_0$	$x = y - \frac{a_2}{3}$ and $x = y^2 + a_2$
XVI	$x^3 + a_0 = a_2x^2$	$x = \frac{\left(\sqrt[3]{a_0}\right)^2}{y}$
XVII	$x^3 + a_2x^2 + a_1x = a_0$	$x = y - \frac{a_2}{3}$
XVIII	$x^3 + a_1x = a_2x^2 + a_0$	$x = y + \frac{a_2}{3}$
XIX	$x^3 + a_2x^2 = a_1x + a_0$	$x = y - \frac{a_2}{3}$
XX	$x^3 = a_2x^2 + a_1x + a_0$	$x = y + \frac{a_2}{3}$
XXI	$x^3 + a_0 = a_2x^2 + a_1x$	$x = y + \frac{a_2}{3}$
XXII	$x^3 + a_1x + a_0 = a_2x^2$	$x = y + \frac{a_2}{3}$
XXIII	$x^3 + a_2x^2 + a_0 = a_1x$	$x = y - \frac{a_2}{3}$

a first one. In Chapter XVIII, Cardano quoted a second rule by Ferrari with its proof. We finally notice that as for the families of equations $x^3 + a_1x = a_0$ and $x^3 = a_2x^2 + a_0$, two cases in Chapters XVII and XVIII never fall into the *casus irreducibilis*.

Substitutions were, therefore, a fundamental tool for dealing with cubic equations for Cardano. In Table 1, the substitutions in the solution methods in *Ars Magna*, Chapters XI–XXIII are summarized. We note that the two most elaborate ones are $x = y \pm \frac{a_2}{3}$. At that time, a few scholars seemed to be aware of these substitutions that (always) make the term of second degree vanish.[18] In general, Cardano employed substitutions widely to deal with equations—this is one of the most frequently used tools in his algebraic treatises. In particular, in the *Ars Magna*, we find two other remarkable substitutions (at least, as regards our aim in this paper, as we will see in the next section): they are in Chapter VII, which is entirely devoted to substitutions and allow us to pass from $x^3 + a_1x = a_0$ to $y^3 = a_2'y^2 + a_0'$ and *vice versa*.[19]

[18] See Franci (1985).

[19] See Cardano (1545, Chapter VII, Paragraphs 3–4 and 8–9, pp. 18r–18v and 19v–20r). More precisely, the substitutions are $x = \frac{\sqrt[3]{a_0}}{y}$ and $x = y^2 + a_0$. As said, Cardano's substitutions can be considered invertible as long as he only considers the positive roots while taking the $2n$-th roots. We remark that the substitution $x = y^2 + a_0$ is also in Chapter XV, but is applied to a different family of equations. Concerning the first substitution, Cardano provided two examples: $x^3 + 18x = 8$, which is transformed into $x^3 = 36x^2 + 8$ (but Cardano wrongly wrote $x^3 = 9x^2 + 8$), and $x^3 = 6x^2 + 16$, which is transformed into $x^3 = \sqrt[3]{3456}x + 16$.

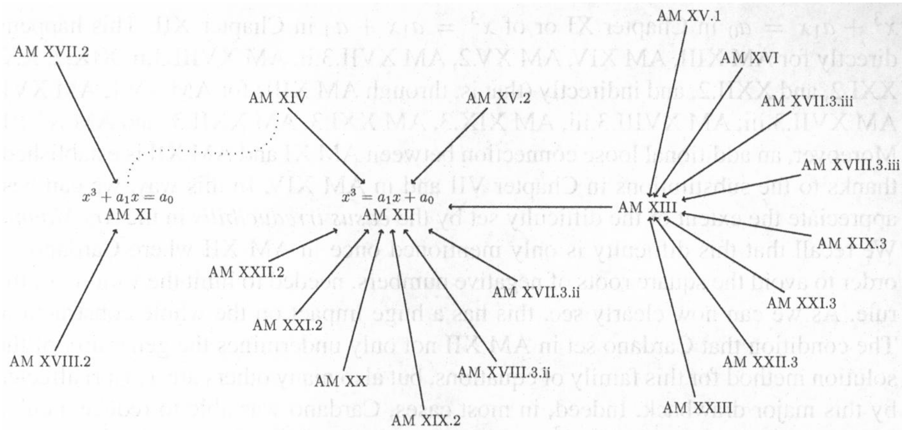

Fig. 2 Interconnections between the solution methods in *Ars Magna*, Chapters XI–XXIII

4 Interconnections of the solution methods and consequences for the *casus irreducibilis*

We are now ready to take the stock of the situation. Considering the thirteen solution methods for cubic equations that Cardano presented in the *Ars Magna*, we notice that they are all interconnected, as clearly turns out from the diagram in Fig. 2. There I write 'AM ⟨*chapter*⟩' to refer to the solution method for the family of equations considered in *Ars Magna*, Chapter ⟨*chapter*⟩. When there are also some cases and subcases, I write, respectively, 'AM ⟨*chapter*⟩.⟨*case*⟩' and 'AM ⟨*chapter*⟩.⟨*case*⟩.⟨*subcase*⟩'. These are numbered according to their order of appearance in Cardano's text. Note that I left out the cases and subcases where no cubic formula is necessary. More precisely, these are AM XVII.1 and AM XVIII.1, in which the families of equations being considered are reduced to $x^3 = a_0$, and AM XV.3, AM XVII.3.i, AM XVIII.3.i, AM XIX.1, AM XXI.1, AM XXII.1, in which the families of equations being considered are reduced to $x^3 = a_1x$. I have not represented the second substitution in Chapter XV (that is, $x = y^2 - a_2$ for $x^3 + a_2x^2 = a_0$) either, which—as noted above—leads to a unique case linked to AM XII.

In the diagram, I use two kinds of arrows. The solid ones mean that the connection between the two solution methods in the chapters named at its extremities is directly expounded in one of the solution methods themselves (more precisely, in the later chapter). The dotted arrow means that the connection is described elsewhere (more precisely, in Chapter VII). As we have seen, the connections are performed by means of some substitutions. Though potentially, all of Cardano's substitutions can be used in both directions,[20] I put the head of the arrow in the same direction as Cardano effectively used the substitution in the *Ars Magna*.

Let us take a closer look at the diagram. We observe that ultimately, all the solution methods are brought back to the solution methods either of the family of equations

[20] See Footnote 15.

$x^3 + a_1x = a_0$ in Chapter XI or of $x^3 = a_1x + a_0$ in Chapter XII. This happens directly for AM XIII, AM XIV, AM XV.2, AM XVII.3.ii, AM XVIII.3.ii, XIX.2, XX, XXI.2, and XXII.2, and indirectly (that is, through AM XIII) for AM XV.1, AM XVI, AM XVII.3.iii, AM XVIII.3.iii, AM XIX.3, AM XXI.3, AM XXII.3, and AM XXIII. Moreover, an additional loose connection between AM XI and AM XII is established, thanks to the substitutions in Chapter VII and in AM XIV. In this way, we can best appreciate the extent of the difficulty set by the *casus irreducibilis* in the *Ars Magna*. We recall that this difficulty is only mentioned once in AM XII where Cardano, in order to avoid the square roots of negative numbers, needed to limit the validity of the rule. As we can now clearly see, this has a huge impact on the whole construction. The condition that Cardano set in AM XII not only undermines the generality of the solution method for this family of equations, but also many others are a priori affected by this major drawback. Indeed, in most cases, Cardano was able to reduce a cubic equation to one of the family $x^3 = a_1x + a_0$, but then it can happen that he was not able to solve this last equation.

However, there are few exceptions on the left side of the diagram, namely AM XI, AM XIV, AM XVII.2, and AM XVIII.2. We have already seen that the families of equations dealt with in AM XI and AM XIV can never fall into the *casus irreducibilis*. As a consequence, this also holds for the two cases in AM XVII and AM XVIII that refer to the first of these. We further note the outstanding situation of AM XIV. Even if Cardano also had available two substitutions in *Ars Magna*, Chapter VII that enable the connection with AM XI, its solution method is reduced to the one in AM XII, which is sometimes problematic.[21] Unfortunately, I cannot explain why Cardano favored the first link over the second, especially considering that he used one of the substitutions in Chapter VII also in AM XV.

5 Cardano's first attempt to put a patch in *Ars Magna*, Chapter XXV

It seems to be very likely that Cardano understood that it was possible systematically to make vanish the square roots of negative numbers that appear while solving the family of equations $x^3 = a_1x + a_0$ in *Ars Magna*, Chapter XII. Possibly, he was aware of some cases in which one or more natural solutions of a cubic equation were known independently from the solution method in the *Ars Magna*, but in applying this solution method, square roots of negative numbers were returned. If we trust Tartaglia's account,[22] Cardano asked him (unsuccessfully) for a general rule for taking the cubic root in cases such as $\sqrt[3]{\sqrt{108} + 10} = \sqrt{3} + 1$ and $\sqrt[3]{\sqrt{108} - 10} = \sqrt{3} - 1$. We remark that those cubic roots are exactly the ones that the solution method in *Ars Magna*, Chapter XI returns for the equation $x^3 + 6x = 20$ (which, by the way, is mentioned in Chapter XI). It is, therefore, reasonable that Cardano might also have thought that a similar calculation holds when, for instance, -108 instead of 108 is under the square root.

21 Nevertheless, see Footnote 16.

22 See Tartaglia (1959, Libro IX, Quesito XL, pp. 125–126).

Speculations aside, Cardano had a few attempts at actively finding a patch for the difficulty entailed by the *casus irreducibilis*. As we have seen,[23] Cardano explicitly acknowledged it in the 1545 *Ars Magna* saying that, when an equation of the family $x^3 = a_1x + a_0$ falls into the *casus irreducibilis*, one should better refer to the "aliza question (*quaestio aliza*)" or to *Ars Magna*, Chapter XXV. Afterwards, in 1570 and 1663 editions, only the *De Regula Aliza*, which appeared for the first time in 1570 as a published book together with the *Ars Magna*, is mentioned.

Concerning the reference to the *Ars Magna*, in Chapter XXV, Cardano set out 18 propositions dealing with the families of incomplete cubic equations that can fall into the *casus irreducibilis* (this is not explicitly stated, but almost all the numerical examples provided are in this case). In the main, he provided some ways to rewrite the equation at issue in the form of a system using two or more auxiliary quantities, in the hope that it will then be easier to find a solution. The first eight propositions deal with the family of equations $x^3 = a_1x + a_0$. For instance,[24] in the first proposition Cardano stated that, if

$$\begin{cases} a_1 = f + g \\ a_0 = f\sqrt{g}, \end{cases}$$

[23] See above Sect. 3.1.

[24] For a detailed account, see Confalonieri (2013, pp. 84–99). There I guess one possible origin of all these propositions. Let us limit to the first two propositions concerning the family of equations $x^3 = a_1x + a_0$ since the line of reasoning for the other cases is analogous. If Cardano was aware of Vieta's formulae

$$\begin{cases} 0 = -x_1 - x_2 - x_3 \\ -a_1 = x_1x_2 + x_2x_3 + x_1x_3 \\ -a_0 = -x_1x_2x_3, \end{cases}$$

he could have rewritten them as

$$\begin{cases} a_1 = F + G^2 \\ -a_0 = FG \end{cases}$$

with $G = x_1$ and $F = x_1x_2 + x_2^2$. Solving the above system is equivalent to solve

$$\begin{cases} F = a_1 - G^2 \\ G^3 - a_1G - a_0 = 0, \end{cases}$$

which in turn is equivalent to solve the equation $x^3 = a_1x + a_0$. The solutions are $x_1 = G$, together with the solutions of $x_2^2 + Gx_2 - F = 0$. In the end, we get

$$x_1 = G, \quad x_2 = -\frac{G}{2} + \sqrt{\frac{G^2}{4} + F}, \quad \text{and} \quad x_3 = -\frac{G}{2} - \sqrt{\frac{G^2}{4} + F}.$$

We remark that if we take $F = f$ and $G = -\sqrt{g}$ (of course, from a modern viewpoint, these are not invertible substitutions, see Footnote 15), we obtain the system and the solution in the first proposition (where only x_2 is considered since the other two solutions are always negative), and if we take $F = -f$ and $G = g$, we obtain the system and the solution in the second proposition (where only x_1, which is always positive, is considered).

then a solution is $x = \sqrt{f + \frac{g}{4}} + \frac{\sqrt{g}}{2}$. In the second proposition he stated that if

$$\begin{cases} g = \sqrt{a_1 + f} \\ a_0 = fg, \end{cases}$$

then a solution is $x = \sqrt{a_1 + f}$. An earlier version of some of these propositions can be also found in *Ars Magna Arithmeticae*, Chapters XXXI, XXXIII, XXXV, and XXXVII.[25]

Even if in some cases, this strategy eventually enables one to avoid the use of the cubic formulae, for instance for some well-chosen coefficients, this does not settle the difficulty at all. Indeed, being systematically able to solve these systems is as difficult as solving the cubic equations at issue, so that in the end, this cannot help in finding a theoretical solution to the difficulty entailed by the *casus irreducibilis*. However, even if the above-mentioned propositions are no step toward finding a cubic formula that works every time, they make explicit Cardano's willing to bypass the *casus irreducibilis*, at least by trying to put on a first temporary patch. Indeed, these systems can help the reader's intuition, in that they possibly provide a way of writing of the equation that is friendly enough to make it more easy to guess a solution. Of course, what 'enough' is still depends on the numerical coefficients of the equation, which is the reason why this is not a systematic solution.

6 Getting acquainted with the *De Regula Aliza*

According to what Cardano stated in the *Ars Magna*, a more effective patch to the *casus irreducibilis* should have been contained in the *De Regula Aliza*.[26] This is a quite unfamiliar treatise the contents of which deserve to be better known.[27]

25 See Confalonieri (2013, pp. 215–220).

26 We have two editions of the *De Regula Aliza* available. The first one, printed in 1570 in Basel, is entitled *Hieronymi Cardani mediolanensis, civisque bononiensis, medici ac mathematici praeclarissimi, de aliza regula, libellus, hoc est operi perfecti sui sive algebraicae logisticae, numeros recondita numerandi subtilitate, secundum geometricas quantitate inquirendis, necessaria coronis, nunc demum in lucem editæ*. As said, it is coupled with the *De Proportionibus* and the second edition of the *Ars Magna*. The second edition of the *Aliza*, the title of which is simply *De regula aliza libellus*, is in the fourth volume of the 1663 edition of Cardano's *Opera Omnia*, edited by Charles Spon almost a century after Cardano's death. A few miscalculations and typos are emended, and the punctuation is here and there improved, but as a whole, this contains no major changes compared to the first edition. As far as I know [see the website www.cardano.unimi.it (last checked February 14, 2015] associated to the project of the edition of Cardano's works supported by Consiglio Nazionale delle Ricerche in Italy and by the university of Milan), no manuscript of the *Aliza* is available.

27 Over the centuries, the *Aliza* had a handful of readers. Among the generations of scholars contemporary to Cardano, Commandino (1572, Book X, Proposition 34, Theorem III, p. 149), Stevin (1585, Book II, p. 309), and Thomas Harriot [see Schemmel and Stedall (last checked February 14, 2015], Add. Ms. 6783, folio 121) studied few of its pages. It is also possible (though unlikely) that Rafael Bombelli got in touch with this treatise. During the nineteenth century and until the beginning of the twentieth century, Hutton (1812, pp. 219–224), Cantor (1892, pp. 532–537), and Loria (1950, pp. 298–299) reported on the *Aliza* from an historical viewpoint. More recently, Tanner (1980), Maracchia (2005, pp. 227, 331–335), and Stedall (2011, p. 10) discussed (parts of) this book.

The reader who gets acquainted with this book of Cardano's is initially disappointed: This is such a cryptic work, starting with the title. Indeed, the term '*aliza*' is neither a common Latin word at that time nor was it one of the brand new mathematical terms that were word-by-word translations from the vernacular. Moreover, this term does not seem to belong to Cardano's own mathematical terminology. He does not even care to explain this rare word: In the whole collection of Cardano's mathematical writings, there are only three other occurrences,[28] in addition to the title page of the *Aliza*, and they all refer to the title of the book itself. A few scholars have tried for an interpretation,[29] but none of them provided a precise etymology. Very recently, Paolo D'Alessandro has confirmed Cossali's hypothesis.[30] Likely, the term '*aliza*', or '*aluza*', is a misspelling based on the Byzantine pronunciation 'alidʒja' of the Greek word 'ἀλυθεῖα', composed by the negative particle 'α' and by the feminine singular passive aorist participle of the verb 'λύω', which means 'to unbind', 'to unfasten', 'to loosen', 'to dissolve', 'to break up', 'to undo', 'to solve'. Thus, '*aliza*' means 'non-solvable'. We remark that this etymology soundly agrees with Cardano's words in *Ars Magna*, Chapter XII (see above Sect. 3.1).

Footnote27 continued

As far as I know, the one scholar who studied the *Aliza* in detail is the Italian mathematician and historian of mathematics Pietro Cossali (Verona 1748, Padua 1815). He was priest in Milan, professor at the university of Padua, member of the Società Italiana delle Scienze, and pensionnaire of the Reale Istituto Italiano di Scienze, Lettere e Arti. Mostly the *Aliza*, as well as the *Ars Magna Arithmeticae*, were handled on the same footing as the *Ars Magna* in the second volume of his history of algebra (see Cossali 1799a) and in the *Storia del Caso Irriducibile* (see Cossali 1966, which is the commented transcription by Romano Gatto of the unpublished manuscript). Nevertheless, it must be said that for the most of the time, Cossali's accuracy as historian is not up to modern standards: He was doing mathematics from a historical starting point rather than history of mathematics. Indeed, Cossali was a mathematician, and this could partially explain his attitude.

In 1781, the Academy of Padua announced a competition to prove either that the cubic formula could be freed from imaginary numbers or the contrary, but in the end, the prize was never assigned. However, this renewed mathematicians' interest in the *casus irreducibilis*. Cossali was absorbed by this topic [his very first work in 1799 is devoted to it, and in 1813, he came back to the same issue, see Cossali (1799b) and Cossali (1813)]. He also wanted to take part in the 1781 competition, but he could not finish his contribution on time; he subsequently published it in 1782 (see Cossali 1782).

28 In chronological order, we have already seen the first reference in *Ars Magna*, Chapter XII in Sect. 3.1. Unfortunately, the other occurrences step outside the context of cubic equations. The second reference is in the 1554 *De subtilitate* (and following editions), where Cardano dealt with the "reflexive ratio (*proportio reflexa*)", see Cardano (1554, Book XVI, p. 428). The last reference is in the *Sermo de plus et minus*, which contains a specific mention of the *Aliza*, Chapter XXII, see Cardano (1663a, p. 435). Thereafter the term is mentioned nowhere else.

29 In 1799 Pietro Cossali hinted that the term '*aliza*' means 'unsolvable': "De Regula Aliza, *cioè* De regula Irresolubili", see Cossali (1799a, volume II, p. 441) or Cossali (1966, Chapter I, Paragraph 2, p. 26). In 1892 Moritz Cantor related that Armin Wittstein suggests that this term comes from a wrong transcription of the Arabic word '*a'izzâ*' and means 'difficult to do', 'laborious', 'arduous': "*Titel* De regula Aliza, *der durch unrichtige Transkription aus dem arabischen Worte* a'izzâ *(schwierig anzustellen, mühselig, beschwerlich) entstanden sein kann [Diese Vermutung rührt von H. Armin Wittstein her.], und alsdann Regel der schwierigen Fälle bedeuten würde*", see Cantor (1892, p. 532). Finally in 1929, Gino Loria advanced as a common opinion that the term comes from a certain Arabic word that means 'difficult': "*il titolo, sinora inesplicato* De Regula Aliza *(secondo alcuni* aliza *deriverebbe da una parola araba significante difficile)*", see Loria (1950, p. 298).

30 By personal communication.

The *Aliza* is composed of 60 highly sketchy chapters. In the 1570 edition, it fills up 111 *folia*. The chapters vary considerably in length, from half a *folio* up to six or seven *folia*. They all have their own heading and sometimes display an inner organization. At least one diagram is present in each chapter and sometimes also a table where the calculation at issue is displayed. In the margins and in the text, we find a good number of references, pointing in most cases at the *Aliza* itself or at the two other treatises, the *Ars Magna* and the *De Proportionibus*, that appeared in the same joined edition. There are also many references to Euclid's *Elements*,[31] one reference to the *Conics* by Apollonius, and one reference to the *On the sphere and cylinder* by Archimedes. Even though Cardano claims that none of his works were rewritten less than three times,[32] the *Aliza* seems not to have enjoyed this benefit. Probably, Cardano barely proofread the text or proofread it very quickly. However, common the typos and mistakes in calculations were at the time; the book is particularly full of them. In addition to this, we must recall that Cardano did not write a simple Latin. Even more noteworthy, it is quite hard to detect an inner structure in the treatise—it would be better to say that there is no global structure at all. A topic is usually scattered throughout the book, with repetitions and logical gaps, so that in the end, the reader feels highly disoriented. Under these conditions, the *Aliza* is more pragmatically to be regarded not as a unitary treatise but as a miscellany of mathematical writings, notes, remarks, and observations. Its *raison d'être*, as the author himself said in the *Ars Magna*, is to overcome the difficulty entailed by the *casus irreducibilis*. This is fundamental in order to understand how Cardano may have considered the *Aliza*, that is, a treatise to amend the *Ars Magna*.

Dating the *Aliza* miscellany, even if approximately, is a thorny task. We cannot trust the (supposedly) chronological order in which the treatises are displayed in the fourth volume of the 1663 *Opera Omnia*. It *Aliza* comes immediately after the *Ars Magna Arithmeticae*, which in turn follows the *Ars Magna*, but these last two treatises are listed in the wrong order (see Gavagna 2012).[33] In Cardano's mathematical treatises, the *Aliza* is first mentioned—as we have seen—in the 1545 *Ars Magna* (see above Sect. 3.1). This means that Cardano had in mind a certain "*aliza* problem" around 1545—which makes sense since he already had to came across the *casus irreducibilis* developing the cubic formulae in the *Ars Magna Arithmeticae* and in the *Ars Magna*. Some other hints can be found in Cardano's nonmathematical treatises and more precisely in the 1544, 1550, 1557, 1562, and 1576 versions of his autobiography. There Cardano recalled his personal life, but also his career and his writings. He set down in particular the titles, sizes, *incipit*, contents, and the main topics of all of his treatises. The *Aliza* as a published book is, of course, mentioned only once, in 1576.[34]

31 It is very likely that the edition that Cardano had at hand was the one by Jacques Lefèvre d'Étaples printed in Paris in 1516. Indeed, it was explicitly mentioned by Cardano in the short preface to the manuscript *Commentaria in Euclidis Elementa* (see Gavagna 2003, p. 134). An accurate study to determine whether Cardano followed Campano or rather Zamberti's interpretation is still lacking.

32 "*[N]ullus liber minus quam ter scriptum est*", see Cardano (1557, p. 78).

33 See also Footnote 1.

34 See Cardano (1663b, p. 40).

For our purposes, it is however interesting to consider, among the listed writings, the *Opus Arithmeticae Perfectum*. This should have been an encyclopedic work on mathematics, composed of fourteen books, and was probably conceived between the 1530s and the 1560s, but it was never realized. Today, all we have is Cardano's description in his autobiography. In 1544, he wrote that "the tenth [book of the *Opus Arithmeticae Perfectum*] has written Ars magna on it [and] contains 67 chapters (*decimus inscribitur Ars magna, continet sexaginta septem capitula*)",[35] whereas Books XIII and XIV "are dedicated to the arithmetical and geometrical questions (*quaestionibus arithmeticis et geometricis destinantur*)".[36] The *Aliza* is not explicitly mentioned, but according to the quotation from the 1545 *Ars Magna* where the "*aliza* question" is coupled with a certain "book on geometrical questions", it should be associated to a certain geometrical problem in the last book of the *Opus Arithmeticae Perfectum*. In 1550, Book X came to be devoted to "all the chapters on the quadratic [equations] with the *aliza* rule (*omnia capitula supra quadratum cum regula aliza*)",[37] Book XIV still concerned geometry, but was restricted "to what pertain the measure of figures that are called geometrical (*ad mensuram figurarum pertinentia, quae geometrica vocantur*)",[38] and any reference to the *Ars Magna* disappeared. From 1557 on, the *Aliza* is no longer mentioned, so the 1544 state of affairs is restored.[39] Finally, in 1576, Cardano said that in 1568, he had joined the *Aliza* and the *De Proportionibus* to the *Ars Magna* and sent them to the printers.[40] Therefore, a certain "*aliza*" was first mentioned in 1545 (but we can reasonably stretch this time limit up to 1544) and sank into oblivion after 1557.

We can conjecture that the miscellany of writings that compose the *Aliza* (or their original core) was first conceived at worst between 1544 and 1545 around the *casus irreducibilis* and a certain "geometrical question". From that moment on, the topic at issue grew more and more in importance and attained its culmination around 1550, when the *Aliza* replaced the *Ars Magna* in the plan of the *Opus Arithmeticae Perfectum*. Twenty years later, the *Aliza* surfaced again and was published. It is still an open question why Cardano decided to publish it. The changes in the editorial project of the *Opus Arithmeticae Perfectum* could, therefore, attest to Cardano's hope of finding

[35] See Cardano (1544, p. 426). This quotation raises the incidental question of how to know what that "Ars magna" was of which Cardano is speaking because all the editions that we have with that title have only forty chapters. Tamborini (2003, pp. 178–179), Ian Maclean in Cardano (2004, p. 65), and Gavagna (2012) discuss this incongruity, and I refer to them for an accurate discussion.

[36] See Cardano (1544, p. 426).

[37] See Cardano (1998, p. 9v).

[38] See Cardano (1998, p. 9v).

[39] Indeed, "*[d]ecimus de regulis magnis, atque ideo ars magna vocatur: atque hic solus ex omnibus editus est*" and "*[t]ertiusdecimus quaestiones arithmeticas, ut quartusdecimus geometricas*", see Cardano (1557, pp. 37–38) and Cardano (1562, p. 16).

[40] "*De proportionibus, et aliza regula addidi anno MDLXVIII ad librum artis magnae et edidi*", see Cardano (1663b, p. 41).
This agrees with what Ercole Bottrigari (1531–1612) wrote in his *La mascara overo della fabbrica de teatri e dello apparato delle scene tragisatirocomiche*: "two or three years" before 1570, Bottrigari questioned Cardano about the *Aliza* and Cardano, avoiding answering, dropped that the *Aliza* was going to be printed in Germany. The reference can be found in Betti (2009, p. 163), and I thank Veronica Gavagna for it.

some new results relating to the *casus irreducibilis*. The lapse of time during which only the *Aliza* appeared as a part of the *Opus Arithmeticae Perfectum* could in particular correspond to the lapse of time during which Cardano believed that he would have eventually manage to avoid the *casus irreducibilis*, namely to the few years from 1550 to (at the latest) 1557. However, when the *Aliza* was printed in 1570, its inner title page recalled its origins as a question since it contains the term '*libellus*', which means 'pamphlet', 'booklet', 'small book': not substantial enough at that time to replace the *Ars Magna*. Moreover, in the light of the quotation in Sect. 3.1, we remark that the *Aliza* likely arose around a certain delimited problem and then grew bigger and bigger until it became a book in 1570. This could (at least partially) explain the lack of order in its structure. It would be extremely useful to be able to provide a relative chronology of the pieces that compose the miscellany. Unluckily, this can only be achieved locally here and there.[41]

7 The method of splittings in *Aliza*, Chapter I

Having set the context for this unfamiliar treatise, we are now ready to deal with the one widespread thread in it. The *Aliza* being a miscellany, it is not surprising that the techniques that it contains are not systematically organized. We rather see how Cardano groped his way forward and find some attempts in different directions, such as, for instance, a proof in a geometrical style of the existence of a solution for the family of equations $x^3 = a_1x + a_0$ or an alternative sign rule for multiplication by means of which Cardano had hoped (unsuccessfully) eventually to avoid imaginary numbers in the formulae. The importance of the *Aliza* also lies in the fact that it shows how Cardano was accustomed to deal with equations, which does not appear in his other algebraic treatises.

The pattern to which Cardano accorded the most credit which is widespread in the *Aliza*[42] is what I call splittings. This term is a word-by-word translation from the Italian '*spezzamenti*' that Cossali devised (see Cossali 1799a; Cossali 1966). As we have seen in Sect. 4, all the solution methods for cubic equations in the *Ars Magna* are reduced to the solution methods either for the family of equations $x^3 + a_1x = a_0$ or for $x^3 = a_1x + a_0$. This correspond to Cardano's description in the first lines of the *De Regula Aliza*, where he argued as follows.

> Since I have already demonstrated in the *Ars Magna* that all the chapters are transformed, provided that the two principal [chapters] had been discovered [to be] general and not by transformation, it is plain that having discovered another general chapter in addition to the chapter of the cube and some things equal to a number and [to the chapter] of the cube equal to some squares and a number, which is deduced by transformation from the preceding [one], even if [the other general chapter] was general; all the chapters either of three or of four terms would not only be known in general, but also demonstrated, provided that

41 For more details, see Confalonieri (2013, p. 235).

42 This pattern is dealt with directly in Chapters I, II, VII, X, XVIII, XLV, XLVI, XLVIII, LIII, LX, and indirectly in many others.

> this very same [chapter] is discovered by a demonstration. (*Cum iam in arte magna demonstraverimus omnia capitula converti, modo duo principalia nec iam ex conversione inventa generalia fuerint, manifestum est, invento alio capitulo generali, praeter capitula cubi et rerum æqualium numero et cubi aequalis quadratis et numero, quod ex priore per conversionem deducitur, et si generale sit, omnia capita seu ex tribus seu ex quatuor nominibus generaliter non solum cognita esse, sed et demonstrata, modo hoc ipsum demonstratione inventum sit*),

see Cardano (1570b, Chapter I, p. 1). This is the sole declaration of intent that can be found in the entire *Aliza*, and significantly, Cardano put it in the very first paragraph of the book. It is an issue, though, to understand its meaning clearly. I interpret the term 'chapter' as generously as possible: Everything that concerns the solution of a certain family of equations, even in a loose way, so that the statement of the equation itself, the numerical examples, the corollaries, and *a fortiori* the solution method are addressed by this term. In the quotation, an overall recurrent strategy to deal with the *casus irreducibilis* is explained. Since all the solution methods are linked, one should look for a family of equations the solution method of which is generally proved, that is, it works for every numerical equation that belongs to that family. According to our analysis of the *Ars Magna*, the "two principal chapters" mentioned in the quotation are $x^3 + a_1x = a_0$ and $x^3 = a_1x + a_0$. Cardano specified that the family of equation sought for should not be $x^3+a_1x = a_0$ or $x^3 = a_2x^2+a_0$ (this last family being linked to the first one by substitution, see the above Fig. 2). Indeed, we have seen that, even if these two families of equations can never fall into the *casus irreducibilis*, still the difficulty entailed by it can be displayed in other families, primarily in $x^3 = a_1x+a_0$. This is why Cardano devoted the rest of Chapter I (and many other passages) to it.

Let us consider the family of equations $x^3 = a_1x + a_0$ and substitute $x = y + z$. Developing the calculations, we come to the equality

$$y^3 + 3y^2z + 3yz^2 + z^3 = a_1(y + z) + a_0.$$

This is what Cardano implicitly did in the paragraphs of *Aliza*, Chapter I that follow the quotation. Afterwards, his strategy is to arrange the left hand side $y^3+3y^2z+3yz^2+z^3$ of the equality as the sum of two parts both of which are going to be matched either to $a_1(y + z)$ or to a_0 on the right hand side. We remark that choosing one of the two assignments; for instance, the one for a_0 completely determines the choice for the other. Whenever possible (that is, when the polynomial assigned to $a_1(y + z)$ can be divided by $y + z$ without remainder), Cardano also matched this term directly to the given coefficient a_1. The name 'splitting' is appropriate since the left side of the equality is split in two parts. At first, Cardano considered extensively the splittings in the following seven propositions.

A I 1 "[T]he number is assigned to the cubes [...], and the things themselves are equal to the six parallelepipeds that remain (*numerus tribuatur cubis [...] et res ipsæ æquantur parallelipedis sex*)", that is,

$$\begin{cases} a_0 = y^3 + z^3 \\ a_1x = 3y^2z + 3yz^2. \end{cases}$$

A I 2 "[A]ll the parallelepipeds are given to the number and the cubes to the things (*parallelipeda omnia dentur numero et cubi rebus*)", that is,

$$\begin{cases} a_0 = 3y^2z + 3yz^2 \\ a_1x = y^3 + z^3. \end{cases}$$

A I 3 "[F]our parallelepipeds are given to the number, the remaining two with the cubes to the things. But in these bodies, this is particular that the ratio of those bodies is as [the ratio] of the squares of the parts simultaneously joined to twice the product (*quatuor parallelipeda numero dantur, reliqua duo cum cubis rebus. Est autem hoc inter corpora illa præcipuum, quod proportio ipsorum corporum est ut quadratorum partium simul iunctorum ad ambo producta*)", that is,[43]

$$\begin{cases} a_0 = 2y^2z + 2yz^2 \\ a_1x = y^3 + y^2z + yz^2 + z^3. \end{cases}$$

A I 4 "[T]he number is given only to the two mutual parallelepipeds (*numerus datur duobus tantum parallelipedis mutuis*)" that is[44]

$$\begin{cases} a_0 = y^2z + yz^2 \\ a_1x = y^3 + 2y^2z + 2yz^2 + z^3. \end{cases}$$

A I 5 "[W]e give the number to one cube, the remaining seven bodies to the things (*damus numerum uni cubo, reliqua septem corpora rebus*)", that is,[45]

$$\begin{cases} a_0 = y^3 \\ a_1x = 3y^2z + 3yz^2 + z^3 \end{cases} \quad \text{or} \quad \begin{cases} a_0 = z^3 \\ a_1x = y^3 + 3y^2z + 3yz^2. \end{cases}$$

[43] The first sentence in the description identifies three possible splittings, namely

$$\begin{cases} a_0 = 2y^2z + 2yz^2 \\ a_1x = y^3 + y^2z + yz^2 + z^3 \end{cases} \quad \text{or} \quad \begin{cases} a_0 = y^2z + 3yz^2 \\ a_1x = y^3 + 2y^2z + z^3 \end{cases} \quad \text{or} \quad \begin{cases} a_0 = 3y^2z + yz^2 \\ a_1x = y^3 + 2yz^2 + z^3 \end{cases}$$

The second sentence definitely points at the first splitting since only the following proportion holds

$$(y^3 + y^2z + yz^2 + z^3) : (2y^2z + 2yz^2) = y^2 + z^2 : 2yz.$$

[44] The numerical example that follows this description enables to clarify what a "connected" parallelepiped (to a cube) is. More generally (using the same formalism as above), in the *Aliza*, a parallelepiped y^2z is "connected (*coherens, proximum*)" to the cube y^3, a parallelepiped yz^2 is "opposite (*adversum, alternum, altrinsecum, remotum*)" to the cube y^3, and the two parallelepipeds y^2z and yz^2 are "mutual (*mutua*)".

[45] We remark that as long as one do not put any further condition on y, z, they are a priori interchangeable in Cardano's phrasing. For example, the expression 'one cube' addresses both y^3 and z^3, or 'one cube and one opposite parallelepiped' addresses both $y^3 + yz^2$ and $z^3 + y^2z$. A few splittings, namely the ones where the assignation for a_0 (and for a_1x) are symmetrical in y, z, are defined in a univocal way. This is the case for instance in A I 1—A I 4. For the sake of clarity, I choose in the other cases to explicitly write all the combinations of y, z.

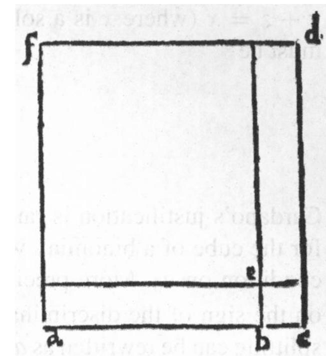

Fig. 3 Diagram from *De Regula Aliza*, Chapter I. Kind courtesy of Biblioteca Virtual del Patrimonio Bibliográfico and Universidad de Sevilla

A I 6 "[W]e give the number to one cube and three parallelepipeds that are made by the side of that cube times the square of the side of the other cube and the remaining four bodies, that is, the cube with three parallelepipeds, to the things (*demus numerum uni cubo et tribus parallelipedis quæ fiunt ex latere cubi illius in quadrata lateris alterius cubi et reliqua quatuor corpora, scilicet cubum cum tribus parallelipedis, rebus*)", that is,

$$\begin{cases} a_0 = y^3 + 3yz^2 \\ a_1x = 3y^2z + z^3 \end{cases} \quad \text{or} \quad \begin{cases} a_0 = 3y^2z + z^3 \\ a_1x = y^3 + 3yz^2. \end{cases}$$

A I 7 "[W]e give the number to the aggregate from the cube and two connected parallelepipeds with the opposite [one] and the things to the remaining four bodies (*numerum damus aggregato ex cubo et duobus cohærentibus parallelipedis cum uno adverso et res reliquis quatuor corporibus*)", that is,

$$\begin{cases} a_0 = y^3 + 2y^2z + yz^2 \\ a_1x = y^2z + 2yz^2 + z^3 \end{cases} \quad \text{or} \quad \begin{cases} a_0 = y^2z + 2yz^2 + z^3 \\ a_1x = y^3 + 2y^2z + yz^2. \end{cases}$$

7.1 The splitting in A I 1

Let us consider more in details the splittings in the proposition A I 1. Referring to Fig. 3, Cardano stated the following condition:

> since the cubes of AB and BC can never be smaller than one-fourth of the whole cube AD, and indeed, this does not happen except when AC will be divided in equal [parts] by B; therefore, being the number smaller than one-fourth of the whole cube AD, it will not be able to be equal to the cubes of AB, BC (*quoniam cubi ab et bc nunque possunt esse minores quarta parte totius cubi ad, et hoc etiam non contingit nisi cum fuerit ac divisa per æqualia in b. Cum igitur numerus fuerit minor quadrante cubi totius ad, non poterit æquari cubis ab, bc*)

and added that "the chapter is not general in this part (*capitulum hac in parte non fuit generale*)". Let us call $AB = y$ and $BC = z$. The condition implies that, if

$y + z = x$ (where x is a solution of $x^3 = a_1x + a_0$), since $y^3 + z^3 \geq \frac{1}{4}x^3$, then it must be

$$a_0 \geq \frac{1}{4}x^3.$$

Cardano's justification is rather vague.[46] First of all, we remark that by the formula for the cube of a binomial, we can immediately derive a condition on a_1x, given the condition on a_0. More precisely, the inequality $y^3 + z^3 \geq \frac{1}{4}x^3$ implies a condition on the sign of the discriminant.[47] We, moreover, remark that the second line of the splitting can be rewritten as $a_1 = 3yz$ and therefore, that the splitting itself is basically the system from *Ars Magna*, Chapter XII (see above). Not surprisingly, it turns out that if one knows the splitting in A I 1, it should also be able to derive the cubic formula for $x^3 = a_1x + a_0$ since there are no major conceptual gaps to be filled: A calculation that involves a quadratic equation is enough. Given the resemblances with the *Ars Magna*, it seems extremely unlikely that Cardano was not aware of the link with the already expounded solution method for the family of equations under consideration. This splitting can be seen in truth as a reformulation of Cardano's solution method for $x^3 = a_1x + a_0$ in the *Ars Magna*. From this point of view, the condition $a_0 \geq \frac{1}{4}x^3$ is nothing other than a translation of the condition (on the discriminant) that Cardano added there in Chapter XII.

[46] In the margin, Cardano made reference to "the 9th [Proposition] of the second [Book] of Euclid" and to "some dialectic rules (*[p]er 9 secundi Elementorum et regula Dialec.*)". Note that both 1570 and 1663 editions have '*regula*'. Since the preposition '*per*' takes the accusative case, we would have expected to find either '*regulam*' or '*regulas*', which is not the case. Therefore, I have chosen to translate this term in the plural to leave the interpretation as loose as possible. The mention of the 'dialectic rules' seems to be a blurry reference to some classical logical rules. Cardano also wrote a *Dialectica*, printed in 1566 in Basel [in the second volume of Cardano (1566)], which is a logic treatise where the rules are sometimes expounded through mathematical examples. Since *Elements* II.9 (both in Campano's and Zamberti's versions) implies that $y^2 + z^2 \geq \frac{1}{2}x^2$, we are searching for one or more rules that enable to pass from the inequality with the squares to the inequality with the cubes, but I could find none in the *Dialectica*. It is nevertheless not hard to derive it by hand.
Note, moreover, that in order to verify the inequality $a_0 \geq \frac{1}{4}x^3$, we must assume both y and z are positive. Indeed, let us suppose for a contradiction that $y \geq 0$ and $z \leq 0$, for example, $y = 1$ and $z = -2$. Then, $-7 < -\frac{1}{4}$, which contradicts $y^3 + z^3 \geq \frac{1}{4}x^3$.

[47] The condition yields to the following system

$$\begin{cases} a_0 \geq \frac{1}{4}x^3 \\ a_1 \leq \frac{3}{4}x^2, \end{cases}$$

which in turn entails that

$$\begin{cases} \frac{1}{4}a_0^2 \geq \frac{1}{4}\left(\frac{1}{4}x^3\right)^2 \\ a_1^3 \leq \left(\frac{3}{4}x^2\right)^3 \end{cases},$$

that is $\frac{1}{27}a_1^3 \leq \frac{1}{4^3}x^6 \leq \frac{1}{4}a_0^2$. Then, $\frac{1}{4}a_0^2 - \frac{1}{27}a_1^3 \geq 0$, or $\Delta_3 \geq 0$.

Cossali maintained—very reasonably—that this splitting came directly from Tartaglia's poem (see above Footnote 11) and that, once discovered it, Cardano looked for other similar splittings.[48]

7.2 The splittings in A I 2–A I 7

We now consider the remaining splittings. Cardano also stated analogous conditions for A I 2–A I 4. Namely, with regard to a_0, the conditions, respectively, are

$$a_0 \leq \frac{3}{4}x^3, \quad a_0 \leq \frac{1}{2}x^3, \quad a_0 \leq \frac{1}{4}x^3.$$

These conditions delimit, for each splitting, a range of validity complementary to the one for A I 1. Concerning in particular A I 2, Cardano himself remarked that "since neither the first [A I 1] nor the second [A I 2] of these chapters can be general by itself, nevertheless, both of them joined constitute a general chapter (*[c]um ergo neutrum istorum capitulorum possit esse generale per se, ambo tamen iuncta constituit*[49] *capitulum generale*)". In my interpretation, especially considering the link between A I 1 and the solution method in *Ars Magna*, Chapter XII, Cardano was eventually looking for a complementary solution method, that is, for a solution method, for the same equation that holds when one cannot apply the one in *Ars Magna*, Chapter XII. The privileged way to discover this new solution method is passing by another splitting. Hopefully, this could have also led to a complementary formula.

Let us now consider A I 5, A I 6, and A I 7. Cardano gave no condition[50] on the splittings here, but he called the one in A I 5 "the worst of all (*deterius omnibus*)" and the ones in A I 6 and A I 7 "misshaped (*difformia*)". These expressions are justified as follows. With regard to A I 5, Cardano observed that "it falls back on a chapter of four terms, thence from that to the principal [*redit ad capitulum quatuor nominum, inde ex eo ad primum*]". Indeed, if we try to solve the system in y, z, we bump into a complete equation that falls into the *casus irreducibilis*. Deleting the second degree term thanks to one of the substitutions that Cardano knew, we recover again an equation of the family $x^3 = a_1x + a_0$. With regard to A I 6 (and this also holds for A I 7), Cardano remarked that "similar [*similia*]" aggregates, as the cube and three parallelepipeds like in $y^3 + 3yz^2$ and $3y^2z + z^3$, are made equal to "dissimilar [*dissimilia*]" ones, as the number a_0 and the things a_1x. According to Cardano's reasoning, since none of the splittings in A I 5–A I 7 is subject to a condition, if one could conceive a solution method starting from one of them, this would be general and could replace the one in A I 1.

[48] "*Giusta il metodo di Tartaglia questa equazione* ($y^3 + 3y^2z + 3yz^2 + z^3 - p(y+z) - q = 0$) *si spezza nelle due* $y^3 + z^3 = q$, $3y^2z + 3yz^2 = p(y+z)$. *È egli di natura sua generale questo spezzamento? È egli l'unico? E qui stendendo Cadano lo sguardo su le varie combinazioni de' termini, moltissimi gli si presentarono alla mente i supposti possibili a farsi*", see Cossali (1966, Chapter I, Paragraph 3, pp. 27–28).

[49] The 1663 edition has "*constituunt*".

[50] Obviously it is not true that—strictly speaking—no condition on a_0 is entailed. Indeed, there is at least the trivial condition $a_0 \leq x^3$ given by the equation $x^3 = a_1x + a_0$ itself. Therefore, we better account for Cardano's "no condition" as "no non-trivial condition".

Unfortunately, the difficulty with the splittings in A I 2–A I 7 (and actually with any other splitting) is exactly that one cannot conceive a solution method for the considered equation starting from them. Indeed, they all lead to equations (namely, to the equations in one unknown associated to the systems in the splittings) that are never of a lower degree than three.[51] Therefore, they cannot be used to derive a solution for the family of equations $x^3 = a_1x + a_0$, as was instead the case for A I 1.

7.3 The other splittings

In the first seven splittings, which Cardano addressed as the "easiest (*faciliora*)", he was possibly trying to derive, albeit unsuccessfully, either a solution method complementary to A I 1 or an alternative one. It is natural then to wonder on the basis of what criteria did Cardano choose to describe in details the splittings A I 2–A I 7.

According to Cardano's requirements (that is, always considering positive coefficients and writing no equality with zero), there are 62 possible splittings. To count them all, we recall that splitting $y^3 + 3y^2z + 3yz^3 + z^3$ in two parts and assigning one to a_0, completely determines the choice for the part assigned to a_1x. We consider then all the possible combinations of the eight terms in $y^3 + 3y^2z + 3yz^3 + z^3$. We have two choices for each term y^3, z^3 (since we can have one or none of them) and four choices for each term y^2z, yz^2 (since we can have three, two, one, or none of them and we cannot distinguish them three by three). Moreover, we have to exclude the two limit cases, where all the terms or no term are assigned to a_0. The most effective way to classify the combinations is on the basis of how many terms are associated to a_0 as in Table 2. Note that the dotted line points to the y, z-symmetry axis. For the sake of brevity, in Table 2 I omit the terms assigned to a_1x. We remark anyway that one can also directly read them in the table. For instance, when one term of $y^3 + 3y^2z + 3yz^3 + z^3$ is associated to a_0, seven terms remain for a_1x, and they are exactly the ones listed in the seven-term column for a_0. In the same way, the two-term column is associated to the six-term column, the three-term column is associated to the five-term column, and the four-term column is associated with itself.

At the very end of *Aliza*, Chapter I Cardano briefly listed (some of) the remaining splittings.

> The remaining compositions are anomalous, as if we had given the number to one parallelepiped, to three [parallelepipeds], to five [parallelepipeds], to two [parallelepipeds] non-mutual, to four [parallelepipeds] two of which are non-mutual, or again to one cube and one parallelepiped, [to one cube and] to two [parallelepipeds] or [to one cube and] to three [parallelepipeds] not of the same kind. The others are useless, as if we had given the number to the aggregate from both cubes and two or four parallelepipeds in whatever way ("*[r]eliquæ autem compositiones aut*[52] *sunt anomale, velut si daremus numerum uni parallelipedo,*

[51] See Confalonieri (2013, pp. 254–255).

[52] Its correlative '*aut*' has been omitted and was—in my understanding—originally referred to "*inutiles*" in the next sentence. Moreover, it was common at that time to mix up '*vel*' and '*aut*' especially in listing, which should have happened in the two lists referred to the anomalous and useless splittings.

Table 2 The sixty-two possible splittings in *Aliza*, Chapter I classified according to the number of terms associated to a_0

Number of terms associated to a_0 in the splittings			
One-term	Two-term	Three-term	Four-term
	$y^3 + z^3$ (A I 1)		$y^3 + y^2z + yz^2 + z^3$
	$y^2z + yz^2$ (A I 4)		$2y^2 + 2yz^2$ (A I 3)
		$y^3 + y^2z + yz^2$	$y^3 + 2y^2z + yz^2$ (A I 7)
		$y^3 + y^2z + z^3$	$y^3 + 2y^2z + z^3$
		$y^3 + 2y^2z$	$y^3 + 3y^2z$
	$y^3 + yz^2$	$y^3 + 2yz^2$	$y^3 + y^2z + 2yz^2$
y^3 (A I 5)	$y^3 + y^2z$	$y^2z + 2yz^2$	$y^3 + 3yz^2$ (A I 6)
y^2z	$2y^2z$	$3y^2z$	$y^2z + 3yz^2$
yz^2	$2yz^2$	$3yz^2$	$3y^2z + yz^2$
z^3 (A I 5)	$y^2z + z^3$	$2y^2z + yz^2$	$3y^2z + z^3$ (A I 6)
	$yz^2 + z^3$	$2y^2z + z^3$	$2y^2z + yz^2 + z^3$
		$2yz^2 + z^3$	$3yz^2 + z^3$
		$y^3 + yz^2 + z^3$	$y^3 + 2yz^2 + z^3$
		$y^2z + yz^2 + z^3$	$y^2z + 2yz^2 + z^3$ (A I 7)

Number of terms associated to a_0 in the splittings		
Five-term	Six-term	Seven-term
	$y^3 + 2y^2z + 2yz^2 + z^3$	
	$3y^2z + 3yz^2$ (A I 2)	
$y^2 + 3y^2z + z^3$		
$y^3 + y^2z + 2yz^2 + z^3$		
$y^3 + y^2z + 3yz^2$		
$y^3 + 3y^2z + yz^2$	$y^3 + 3y^2z + yz^2 + z^3$	
$3y^2z + 2yz^2$	$y^3 + 2y^2z + 3yz^2$	$y^3 + 3y^2z + 2yz^2 + z^3$
$y^3 + 2y^2z + 2yz^2$	$y^3 + 3y^2z + 2yz^2$	$y^3 + 3y^2z + 3yz^2$
$2y^2z + 2yz^2 + z^3$	$2y^2z + 3yz^2 + z^3$	$3y^2z + 3yz^2 + z^3$
$2y^2z + 3yz^2$	$3y^2z + 2yz^2 + z^3$	$y^3 + 2y^2z + 2yz^2 + z^3$
$y^2z + 3yz^2 + z^3$	$y^3 + y^2z + 3yz^2 + z^3$	
$3y^2z + yz^2 + z^3$		
$y^3 + 2y^2z + yz^2 + z^3$		
$y^3 + 3yz^2 + z^3$		

vel tribus vel quinque vel duobus non mutuis, aut quatuor ex quibus duo mutua non essent, aut uni cubo et uni parallelipedo, vel duobus vel tribus non eiusdem generis. Aliæ sunt inutiles, velut si daremus numerum aggregato ex ambobus cubis et duobus parallelipedis aut quatuor quomodocumque), see Cardano (1570b, Chapter I, p. 4). Among the "anomalous" splittings Cardano mentioned, in this order, the splittings determined by the following assignations:

$$
\begin{aligned}
&a_0 = y^2z, \quad a_0 = yz^2,\\
&a_0 = 3y^2z, \quad a_0 = 3yz^2,\\
&a_0 = y^2z + 2yz^2, \quad a_0 = 2y^2z + yz^2,\\
&a_0 = 3y^2z + 2yz^2, \quad a_0 = 2y^2z + 3yz^2,\\
&a_0 = 2y^2z, \quad a_0 = 2yz^2\\
&a_0 = y^2z + 3yz^2, \quad a_0 = 3y^2z + yz^2,\\
&a_0 = y^3 + y^2z, \quad a_0 = yz^2 + z^3,\\
&a_0 = y^3 + yz^2, \quad a_0 = y^2z + z^3\\
&a_0 = y^3 + y^2z + yz^2, \quad a_0 = y^2z + yz^2 + z^3,\\
&a_0 = y^3 + 2y^2z, \quad a_0 = 2yz^2 + z^3,\\
&a_0 = y^3 + 2yz^2, \quad a_0 = 2y^2z + z^3\\
&a_0 = y^3 + 2y^2z + yz^2\,(\text{A I 7}), \quad a_0 = y^2z + 2yz^2 + z^3\,(\text{A I 7}),\\
&a_0 = y^3 + y^2z + 2yz^2, \quad a_0 = 2y^2z + yz^2 + z^3.
\end{aligned}
$$

Note that I interpret "four [parallelepipeds] two of which are non-mutual" in the previous quotation as pointing only to the splittings determined by $a_0 = y^2z + 3yz^2$ and $a_0 = 3y^2z + yz^2$, but not to the one in A I 3 determined by $a_0 = 2y^2z + 2yz^2$. Indeed, taking into account four parallelepipeds, the expression 'two of which are non-mutual' is superfluous since Cardano had only six parallelepipeds available that are three by three of the same kind. In my interpretation, Cardano added this expression to distinguish these splittings from the one in A I 3. Moreover, in "[as if we had given the number to one cube and] to two [parallelepipeds] or [to one cube and] to three [parallelepipeds] not of the same kind" I loosely interpret "not of the same kind" to refer only to "three [parallelepipeds]" but not to "two [parallelepipeds]". We finally remark that, since the splittings in A I 7 also appear above, the labels 'misshaped' and 'anomalous' are not exclusive. According to Cardano's description, the splittings determined by the assignations $a_0 = y^3 + y^2z + 2yz^2$ and $a_0 = 2y^2z + yz^2 + z^3$ are also misshaped. We have no hint as to why Cardano called these splittings 'anomalous'. I would tend toward a literal interpretation of '*anomalus*' as 'without name', meaning that it was not worthwhile giving a name to those splittings. Of course, this interpretation leaves open the question of why Cardano decided not to give a name to these splittings.

Among the "useless" splittings Cardano mentioned, in this order, the splittings determined by the following assignations:

$$
\begin{aligned}
&a_0 = y^3 + y^2z + yz^2 + z^3,\\
&a_0 = y^3 + 2y^2z + z^3, \quad a_0 = y^3 + 2yz^2 + z^3,
\end{aligned}
$$

$$a_0 = y^3 + 3y^2z + yz^2 + z^3, \quad a_0 = y^3 + y^2z + 3yz^2 + z^3,$$
$$a_0 = y^3 + 2y^2z + 2yz^2 + z^3.$$

In the last sentence of *Aliza*, Chapter I, Cardano added a crucial observation. With regard to the useless splittings, he argued that

> if it [the number a_0] is not enough for the aggregate of the cubes since the number is small, in what way is it enough for the same if the parallelepipeds are added? (*nam si numerus cum parvus sit, non sufficit aggregato cuborum, quomodo sufficit eidem si addantur parallelipeda*),

see Cardano (1570b, Chapter I, p. 4). This is a key sentence for making sense of the splittings. In the *Ars Magna* (and in A I 1), Cardano assumed that $a_0 = y^3 + z^3$, and as a consequence, he needed to put the condition on $y^3 + z^3$–or, more precisely, on a_0—in which he asked that this number be large. In the *Aliza*, Cardano was trying to fix this difficulty. As we have argued, it is very reasonable that he was looking for some splittings where the condition was not satisfied, that is, where the number a_0 is small, hoping that they could lead to another solution method (either to complement or to replace the one in A I 1). This is the reason why Cardano called the above four splittings useless: there, the number a_0 is greater than in A I 1 since it is equal to the two cubes y^3 and z^3 plus some other parallelepipeds (and we recall that $a_0 = y^3 + z^3$ was already too large). In the list of the useless splittings, Cardano did not include the ones determined by the following assignments:

$$a_0 = y^3 + y^2z + z^3, \quad a_0 = y^3 + yz^2 + z^3,$$
$$a_0 = y^3 + y^2z + 2yz^2 + z^3, \quad a_0 = y^3 + 2y^2z + yz^2 + z^3,$$
$$a_0 = y^3 + 3y^2z + z^3, \quad a_0 = y^3 + 3yz^2 + z^3,$$
$$a_0 = y^3 + 3y^2z + 2yz^2 + z^3, \quad a_0 = y^3 + 2y^2z + 3yz^2 + z^3.$$

Actually there is no doubt that they should have been included because there also the number is made equal to the two cubes y^3 and z^3 plus some other parallelepipeds, that is, positive quantities.

According to the above interpretation, in *Aliza*, Chapter I forty-eight splittings are mentioned in all. In almost all the remaining ones, a_0 is matched with five, six, or seven terms among which there is always one (and only one) cube. The splitting determined by the assignation $a_0 = y^3 + 3y^2z$ and its symmetrical $a_0 = 3yz^2 + z^3$ are the exception since a_0 is matched with relatively few terms. We remark that indeed there is a striking correspondence between them and the splittings in A I 6: the first ones involve one cube and its "connected" parallelepipeds while the ones in A I 6 involve one cube and its "opposite" parallelepipeds (see Footnote 44). They seem, therefore, to be similar in a way to the splittings in[53] A I 6. In the end, twelve splittings remained unmentioned:

[53] By the way, according to Cardano's description, these are all misshaped splittings.

$$\begin{aligned}
a_0 &= y^3 + y^2z + 3yz^2, \quad a_0 = 3y^2z + yz^2 + z^3,\\
a_0 &= y^3 + 3y^2z + yz^2, \quad a_0 = y^2z + 3yz^2 + z^3,\\
a_0 &= y^3 + 2y^2z + 2yz^2, \quad a_0 = 2y^2z + 2yz^2 + z^3,\\
a_0 &= y^3 + 2y^2z + 3yz^2, \quad a_0 = 3y^2z + 2yz^2 + z^3,\\
a_0 &= y^3 + 3y^2z + 2yz^2, \quad a_0 = 2y^2z + 3yz^2 + z^3\\
a_0 &= y^3 + 3y^2z + 3yz^2, \quad a_0 = 3y^2z + 3yz^2 + z^3.
\end{aligned}$$

My hypothesis is that these are not mentioned because Cardano believed that they were in an obvious way uninteresting. 'Obvious' could mean that, as in the case of the useless splittings, a_0 is matched with a quantity that is too large compared to $y^3 + z^3$. Indeed, assuming $y \ll z$, we can immediately discard half of the above splittings, namely the ones with z^3, as the quantity assigned to a_0 is too large. But the six splittings where y^3 appears are also too large since we are interested in a (very) small a_0 and we choose z arbitrarily large (for instance such that $3y^2z + 3yz^2 > z^3$).

If this interpretation holds, it also sheds a posteriori a new light on the splittings in A I 2–A I 7. Indeed, in the whole *Aliza*, Chapter I, Cardano could have been adopting the criterion of "a small a_0". This is explicit in the case of the useless splittings. Moreover, we have already argued that, in A I 2–A I 4, Cardano was looking for a solution method the condition of validity of which is complementary to the ones in A I 1. There, he found that the quantities matched with a_0 were relatively small (this is what is stated in the conditions). It seems, therefore, very likely that Cardano continued listing splittings in which a_0 complies with the above criterion, and A I 5–A I 7 fits it. At best, Cardano could hope to find, thanks to a fortunate splitting, a solution method to replace the one in A I 1, and this would have been the case if this hypothetical splitting had no conditions; at worst, he could hope for a complementary solution method.

8 Conclusions

In the *Ars Magna*, Cardano provided a highly systematic procedure for solving cubic equations, but he did not succeed in strongly grounding the whole construction. The overview that we have given of the connections between all the solution methods lets us take the stock of the situation. First of all, we notice that the substitutions are a privileged method for Cardano to deal with equations. Unfortunately, they do not have a heuristic value, so that it is extremely difficult to say which kind of reasoning they come from. The overview that we have given makes clear how the difficulty entailed by the *casus irreducibilis*, which in Cardano's text seems to concern only one family of equations, is in truth inherited by most of the families. As such, this becomes a fundamental issue and must be overcome, if one wants to have general solution methods.

Cardano planned to deal with this difficulty in two ways. On the one hand, in the *Ars Magna* itself, he gave a handful of propositions for putting a temporary patch in some particular cases. This is, however, no step toward finding a general solution theoretically.

On the other hand, the *De Regula Aliza* is referred to as the treatise that should have permanently corrected the problem. Due to its many inconsistencies, this is a quite unfamiliar treatise. It comes from a project that gradually grew bigger and bigger and likely attained its summit around 1550. Nevertheless the treatise was not printed until 1570, by which time probably Cardano no longer hoped to overcome the difficulty entailed by the *casus irreducibilis*. A peculiar topic in the *Aliza* is the so-called splittings for the family of equations $x^3 = a_1x + a_0$. These seem to be the way that Cardano trusted most, and they are present all through the treatise. The idea is to use the substitution $x = y+z$ in the family of equations at issue and to split the equality thus obtained by different assignments of its terms to the given coefficients. It is likely that Cardano was exploring all the possible assignments to a_0. He first considered the splitting that allows the cubic formula to be altered since it bears a condition on a_0, namely that a_0 must be large. Afterwards, Cardano displayed some splittings (with or without conditions) where the number a_0 is small. Eventually he was hoping to derive from one of these splittings another solution method for $x^3 = a_1x + a_0$. If the splitting that possibly lead to the solution method had a condition, Cardano would have obtained a technique that complemented the cubic formula since it would have been valid where the cubic formula fails. If the splitting that possibly lead to the solution method had no condition, Cardano would have obtained an alternative technique, and if possible an alternative cubic formula. In both cases, the difficulty entailed by the *casus irreducibilis* would have been solved.

Unluckily for him, this was impossible. Nowadays we know that when a cubic equation has three real different solutions imaginary numbers cannot be avoided in the formula. This also means that, in certain cases, Cardano could not have found a way to express the solutions without square roots of negative numbers. Still, the *Aliza* deserves to be better known inasmuch as it shows in what ways Cardano was used to deal with equations. This is especially true when it was a matter of overcoming the difficulty presented by the solution methods for cubic equations that he had found.

References

Bashmakova, I., and G. Smirnova. 2000. *The beginnings and evolution of algebra*. Washington, DC: The Mathematical Association of America.

Betti, G.L. 2009. Cardano a Bologna e la sua polemica con il Tartaglia nel ricordo di un contemporaneo. *Bruniana e campanelliana* 15: 159–169.

Bortolotti, E. 1926. I contributi del Tartaglia, del Cardano, del Ferrari e della scuola matematica bolognese alla teoria algebrica delle equazioni cubiche. *Studi e memorie per la storia dell'Università di Bologna* 10: 55–108.

Boyer, C.B. 1991. *A history of mathematics*, 2nd ed. New York: Wiley.

Cantor, M. 1892. *Vorlesungen ber die Geschichte der Mathematik*, vol. 2. Leipzig: Teubner.

Cardano, G. 1544. De sapientia libri quinque. Eiusdem De consolatione libri tres, alias æditi, sed nunc ab eodem authore recogniti. Eiusdem De libris propriis, liber unus. Omnia locupleti indice decorata, Johannes Petreius, Nuremberg, chap Libellus de libris propriis, cuius titulus est ephemerus, 419–431.

Cardano, G. 1545. Hieronymi Cardani, præstantissimi mathematici, philosophi, ac medici, Artis magnae sive de regulis algebraicis, lib. unus. Qui et totius operis de arithmetica, quod Opus perfectum inscripsit, est in ordine decimus. Iohannes Petrius, Nuremberg.

Cardano, G. 1554. *Hieronymi Cardani mediolanensis medici De subtilitate libri XXI nunc demum recogniti atque perfecti*. Basel: Ludovicum Licium.

Cardano, G. 1557. *Liber De libris propriis, eorumque ordine et usu, ac de mirabilibus operibus in arte medica per ipsum factis*. Lyon: Guillaume Rouill.

Cardano, G. 1562. Somniorum synestesiorum, omnis generis insomnia explicantes libri IIII. Quibus accedunt eiusdem hæc etiam: De libris propriis. De curationibus et prædictionibus admirandis. Neronis encomium. Geometriæ encomium. De uno. Actio in Thessalicum medicum. De secretis. De gemmis et coloribus. Dialogus de morte. [Dialogus] de humanis consiliis, tetim inscriptus. Item ad somniorum libros pertinentia: De minimis et propinquis. De summo bono, Heinrich Petri, Basel, chap De libris propriis, eorumque usu, liber recognitus, 1–116.

Cardano, G. 1566. *Heorinymi Cardani mediolanensis civisque bononiensis, Ars curandi parva, quae est absolutiss. medend. methodus, et alia, nunc primum aedita, opera, in duos tomos diuisa, quae versa pagina indicabit; omnia autem qualia sint autoris epistola vere praedicat*. Basel: Officina Henricpetrina.

Cardano, G. 1570a. Hieronymi Cardani mediolanensis, civisque bononiensis, philosophi, medici et mathematici clarissimi, Opus novum de proportionibus numerorum, motuum, ponderum, sonorum, aliarumque rerum mensurandarum, non solum geometrico more stabilitum, sed etiam varijs experimentis et observationibus rerum in natura, solerti demonstratione illustratum, ad multiplices usus accommodatum, et in V libros digestum. Prætera Artis magnæ, sive de regulis algebraicis, liber unus, abstrusissimus et inexhaustus plane totius arithmeticæ thesaurus, ab authore recens multis in locis recognitus et auctus. Item De aliza regula liber, hoc est, algebraicæ logisticæ suæ, numeros recondita numerandi subtilitate, secundum geometricas quantitates inquirentis, necessaria coronis, nunc demum in lucem edita, Oficina Henricpetrina, Basel, chap Artis magnae sive de regulis algebraicis, lib. unus. Qui et totius operis de arithmetica, quod Opus perfectum inscripsit, est in ordine decimus. Ars magna, quam volgo cossam vocant, sive regulas algebraicas, per D. Hieronymum Cardanum in quadraginta capitula redacta, et est liber decimus suæ Arithmeticæ.

Cardano, G. 1570b. Hieronymi Cardani mediolanensis, civisque bononiensis, philosophi, medici et mathematici clarissimi, Opus novum de proportionibus numerorum, motuum, ponderum, sonorum, aliarumque rerum mensurandarum, non solum geometrico more stabilitum, sed etiam varijs experimentis et observationibus rerum in natura, solerti demonstratione illustratum, ad multiplices usus accommodatum, et in V libros digestum. Prætera Artis magnæ, sive de regulis algebraicis, liber unus, abstrusissimus et inexhaustus plane totius arithmeticæ thesaurus, ab authore recens multis in locis recognitus et auctus. Item De aliza regula liber, hoc est, algebraicæ logisticæ suæ, numeros recondita numerandi subtilitate, secundum geometricas quantitates inquirentis, necessaria coronis, nunc demum in lucem edita, Oficina Henricpetrina, Basel, chap De aliza regula libellus, hoc est Operis perfecti sui sive algebraicæ Logisticæ, numeros recondita numerandi subtilitate, secundum geometricas quantitates inquirenti, necessaria coronis, nunc demum in lucem editæ.

Cardano, G. 1663a. Hieronymi Cardani mediolanensis Opera omnia in decem tomos digesta, vol 4, Ioannis Antonii Huguetan and Marci Antonii Ravaud, Lyon, chap Sermo de plus et minus. Edited by Charles Spon.

Cardano, G. 1663b. Hieronymi Cardani mediolanensis Opera omnia in decem tomos digesta, vol 1, Ioannis Antonii Huguetan and Marci Antonii Ravaud, Lyon, chap "*Liber a me conscripti, quo tempore, cur, quid acciderit*", Hieronymi Cardani De propria vita, liber (1576), pp 1–54. Edited by Charles Spon.

Cardano, G. 1998. De libris propriis [MS 1550], *edited by Baldi, Marialuisa and Canziani, Guido*. Rivista di storia della filosofia 4:767–98, with the foliation (1–12r) of the MS F II.38 Nr 1, Öffentliche Bibliothek der Universität Basel.

Cardano, G. 2004. *De libris propriis. The editions of 1544, 1550, 1557, 1562, with supplementary material*. Milan: Franco Angeli. Edited by Ian Maclean.

Commandino, F. 1572. *Euclidis Elementorum libri XV*. Pisa: Jacobus Chriegber.

Confalonieri, S. 2013. The telling of the unattainable attempt to avoid the *casus irreducibilis* for cubic equations: Cardano's *De Regula Aliza*. With a compared transcription of 1570 and 1663 editions and a partial English translation. PhD thesis, Universit Diderot Paris 7, online at http://tel.archives-ouvertes.fr/tel-00875863 and http://www2.math.uni-wuppertal.de/confalon/. To be published with Springer in 2015.

Cossali, P. 1782. Sul quesito analitico proposto all'Accademia di Padova per il premio dell'anno 1781 di una assoluta dimostrazione della irriducibilit del binomio cubico. per gli eredi di Marco Moroni.

Cossali, P. 1799a. Origine, trasporto in Italia, primi progressi in essa dell'Algebra. Reale Tipografia Parmense, two volumes.

Cossali, P. 1799b. Particularis methodi de cubicarum æquationum solutione a Cardano luci traditæ. Generalis posteriorum analystarum usus ex cap. I De Regula Aliza ipsius Cardani vitio luculentissime evictus. Atque mysterium casus irreducibilis post duo sœcula prorsus retecta causa sublatum specimen analyticum primum. Caroli Palesii.

Cossali, P. 1813. Disquisizione sui varj metodi di eliminazione con il componimento di uno nuovo. Memorie di matematica e di fisica della Società italiana delle scienze XVI, first part:272–330.

Cossali, P. 1966. Storia del caso irriducibile. Istituto veneto di scienze, lettere ed arti, edited by Romano Gatto.

Franci, R. 1985. Contributi alla risoluzione delle equazioni di terzo grado. In *Mathemata: Festschrift fr Helmuth Gericke*, ed. M. Folkerts, and U. Lindgren, 221–228. Stuttgart: Steiner.

Gavagna, V. 2003. Cardano legge Euclide. I Commentaria in Euclidis Elementa. In *Cardano e la tradizione dei saperi. Atti del Convegno internazionale di studi*, ed. M. Baldi, and G. Canziani, 125–144. Milano: Franco Angeli.

Gavagna, V. 2010. Medieval heritage and new perspectives in Cardano's *Practica arithmeticæ. Bollettino di storia delle scienze matematiche* 1: 61–80.

Gavagna, V. 2012. Dalla Practica arithmeticæ all'Ars magna. Lo sviluppo dell'algebra nel pensiero di Cardano. In: *Pluralité de l'algèbre à la Renaissance, Champion*, ed. by Maria-Rosa Massa-Esteve, Sabine Rommevaux, and Maryvonne Spiesser, 237–268.

Hutton, C. 1812. Tracts on mathematical and philosophical subjects, F. C. and J. Rivington, chap Tract 33. History of algebra, 219–224.

Lagny, T.F.d. 1697. *Nouveaux lemens d'arithmtique et d'algbre ou introduction aux mathmatiques*. Jean Jombert.

Loria, G. 1950. *Storia delle matematiche. Dall'alba della civiltà al secolo XIX*, 2nd ed. Milan: Hoepli.

Maracchia, S. 2005. *Storia dell'algebra*. Liguori: Liguori.

Panza, M. 2007. What is new and what is old in Viète's *analysis restituta* and *algebra nova*, and where do they come from? Some reflections on the relations between algebra and analysis before Viète. *Revue d'histoire des mathématiques* 13: 85–153.

Schemmel, M., and J. Stedall. 2015. Topics in the manuscripts of Thomas Harriot (1560–1621). Online at http://echo.mpiwg-berlin.mpg.de/content/scientific_revolution/harriot/harriot-bl/maps/2.4.14_cardano.pt

Stedall, J. 2011. From Cardano's great art to Lagrange's reflections. Filling a gap in the history of algebra. EMS.

Stevin, S. 1585. *L'arithmtique*. Leiden: Christophle Plantin.

Tamborini, M. 2003. Per una storia dell'Opus Arithmeticæ Perfectum. In *Cardano e la tradizione dei saperi. Atti del Convegno internazionale di studi*, ed. Marialuisa Baldi, and Guido Canziani, 157–190. Milano: Franco Angeli.

Tanner, R.C.H. 1980. The alien realm of the minus. Deviatory mathematics in Cardano's writings. *Annals of Science* 37: 159–178.

Tartaglia, N. 1959. Quesiti et inventioni diverse. Ateneo di Brescia, facsimile reproduction of the 1554 second edition by Arnaldo Masotti.

www.cardano.unimi.it (last checked February 14, 2015) Sito Girolamo Cardano. Strumenti per storia del Rinascimento nell'Italia settentrionale. Online at http://www.cardano.unimi.it/