Arch. Hist. Exact Sci. (2011) 65:567–587
DOI 10.1007/s00407-011-0085-2

Proposition 10, Book 2, in the *Principia*, revisited

Michael Nauenberg

Received: 1 April 2011 / Published online: 8 September 2011

Abstract In Proposition 10, Book 2 of the *Principia*, Newton applied his geometrical calculus and power series expansion to calculate motion in a resistive medium under the action of gravity. In the first edition of the *Principia*, however, he made an error in his treatment which lead to a faulty solution that was noticed by Johann Bernoulli and communicated to him while the second edition was already at the printer. This episode has been discussed in the past, and the source of Newton's initial error, which Bernoulli was unable to find, has been clarified by Lagrange and is reviewed here. But there are also problems in Newton's corrected version in the second edition of the *Principia* that have been ignored in the past, which are discussed in detail here.

1 Introduction

In his guide to Newton's *Principia*, I. B. Cohen writes that "anyone studying the history of the *Principia* will find Book 2, Proposition 10 to be of special interest. The problem is to find the density of the medium that makes a body move in any given curve under the supposition that gravity is uniform and of constant direction, and that the resistance of the medium varies jointly as the density of that medium and the square of the velocity... The proposition is notable, among other things, for a display of Newtonian fluxions (or "moments")" (Cohen 1999, 167). In an unpublished preface to the *Principia*, written after the publication of the second edition, and during his priority dispute with Leibniz on the development of the calculus (Hall 1980),

Communicated by: Niccolò Guicciardini.

M. Nauenberg (✉)
Physics Department, University of California, Santa Cruz, CA 95064, USA
e-mail: michael@physics.ucsc.edu

Newton called attention to this proposition and related ones,[1] to give explicit examples where he had applied his method of analysis.[2] It turns out, however, that his approach was solely based on his geometrical approach to the calculus and on his power series expansion, without any reference to fluxions.[3] Moreover, in the first edition of the *Principia*, he made a mistake in this proposition, which came back to haunt him during this dispute. On September 1712, Newton received Nikolaus Bernoulli, who came with some upsetting news from his uncle Johann Bernoulli. The elder Bernoulli had found a "serious error" in the first example that Newton gave for Proposition 10, Book 2 in the case that the motion occurred along a circular path. For this case, Newton had found that the component of gravity along the motion exactly balanced the resistance, which implied that the velocity is a constant. But this result is inconsistent with his calculation that the velocity decreases as the square root of the vertical distance along the circular path. Both Bernoullis obtained the correct solution to this problem by applying the differential calculus of Leibniz, and they concluded that Newton's failure demonstrated that he had not mastered the application of higher-order differentials in his fluxional calculus. They were unable, however, to find the source of Newton's error which later was pointed out by Lagrange (1797) who showed that Newton had treated incorrectly the third-order differentials that appeared in his approach.

After frantically working on this problem, Newton was able to find the correct solution by modifying the diagram that he had used in his original approach. Although the second edition of the *Principia* was already in the printer's hand, he managed to insert the revised solution, but he failed to acknowledge Bernoulli's contribution which offended the latter. Later on, this episode played a significant role in the priority dispute that emerged between Newton and Leibniz on the discovery of the calculus, and Johann Bernoulli argued, anonymously, that the failure of Proposition 10, Book 2, demonstrated that at the time Newton wrote the first edition of the *Principia* he had not yet mastered the differential calculus. In a letter printed in the *Acta Eruditorum* for July 1716, Bernoulli wrote

> It is firmly established that Newton at the time he wrote his *Principia Philos. Mathematica*, still had not understood the method of differentiating differentials (Bernoulli 1716)

It will be shown that there is some validity in Bernoulli's statement, and that even in the revised version of Proposition 10, Book 2, Newton continued to make errors in the

[1] For example, in Propositions 15–17, Book 2, Newton considered the effect of resistance on the motion of a body under the action of inverse square force (Brackenridge and Nauenberg 2002).

[2] For a detailed discussion of Newton's mathematical methods see (Guicciardini 2009).

[3] In a letter to Keill, "who was puzzled to know what Bernoulli was getting at" in his contention that Newton had not mastered the differential calculus, Newton wrote

> It appears thereby that I did not understand y^e 2nd fluxions when I wrote that Scholium [i.e. to Proposition 10] because (as he thinks) I take the second terms of the series of the first fluxions, the third terms for the second fluxions & so on. But he [Bernoulli] is mightily mistaken when he thinks that I here make use of the method of fluxions. Tis only a branch of y^e method of converging series that I there make uses of (Hall 1958, 296).

application of third-order differentials, but fortuitously these errors did not prevent him from obtaining the correct solution to this proposition.

In a section of his textbook *Théorie des Fonctions Analytiques*, Lagrange (1797) presented an analysis of Proposition 10, Book 2, applying Leibniz's differential calculus to describe the source of Newton's error. He concluded that

> We believe that it was not useless to show how the method of series is applied, and that it would be granted to us to shed light at the same time on a point of analysis on which the greatest geometers made mistakes, and which may be of interest to the history of the birth of the new calculus.[4]

Newton's problems with Proposition 10, Book 2, illustrate some of the limitations of Newton's geometrical-fluxional method, and some of the advantages of the algorithmic approach of Leibniz.

In Proposition 6, Book 1, Newton gave a mathematical description of the motion of a body under the influence of a central force. For a small but finite interval of time t, he postulated that the motion can be composed of an inertial or constant velocity contribution along the tangent of the orbit proportional to t and an accelerated motion along the instantaneous direction of the force proportional to t^2. It is important to recognize, however, that in a series expansion in powers of t, Newton's composition is valid only up to quadratic terms in t. In Proposition 10, Book 2, Newton considered an additional force due to the resistance of the medium acting along the tangent of the orbit. In this case, the motion on a line element along tangent is not inertial leading to the complications in his treatment of this problem discussed here.

In Sect. 2, Proposition 10, Book 2, is presented in the form given in the first edition of the *Principia*, and the source of Newton's error is described. In Sect. 3, the revised version of this proposition in the second edition of the *Principia* is discussed following closely Newton's verbal description of this proposition, but applying his fluxional calculus in a novel form that avoids his inconsistent treatment of cubic terms in his power expansions. In "Appendix I", the relation of Proposition 6, Book 1, to Proposition 10, Book 2, is presented, and following Lagrange's work (Lagrange 1797), in "Appendix II" Newton's missing term in cubic order in powers of the time interval t is derived.

2 Proposition 10, Book 2, in the first edition of the *Principia*

This section describes Newton's discussion of motion in a resisting medium under the action of a constant gravitational force given in Proposition 10, Book 2 of the first edition of the *Principia*, when the geometrical curve for the trajectory is given, and it explains why this description is flawed along the lines first discussed by Lagrange (1797).

[4] Nous avons cru qu'il n'ètait pás inutile de faire voir comment la méthode des séries pouvait y conduire, et qu'on nous saurait gré d'éclaircir, en même temps, un point d' analyse sul lequel les plus grands géomètres' s'étoient trompés, etqui peut intérreser l'histoire de la naissance des nouveaux calculs (Lagrange 1797, 251).

Referring to the curve ACK and tangent line hCH at C, Fig. 1, Newton started:

> let CH and Ch be equal rectilinear lengths which bodies moving away from place C would describe in these times [equal time intervals t] without the action of the medium or gravity (Cohen 1999, 655).

Without the action of the medium or gravity, a body moves with constant velocity, and therefore during an interval of time t

$$CH = Ch = vt, \tag{1}$$

where v is the velocity along the tangent line at C. Newton continued:

> Through the resistance of the medium it comes about that the body as it moves forward describes instead of length CH, only length CF, and through the force of gravity the body is transferred from F to G, and thus line element HF and line element FG are generated simultaneously, the first by the force of resistance and the second by the force of gravity (Cohen 1999, 656).

Let g be the acceleration of gravity, acting along the vertical direction, and r de-acceleration due to the resistance of the medium acting along the tangent of the motion. Then, Newton asserted that

$$HF = CH - CF = (1/2)rt^2, \tag{2}$$
$$FG = IG - IF = (1/2)gt^2, \tag{3}$$

and concluded

> And hence the resistance comes to be as HF directly and FG inversely, or as HF/FG.

Indeed, according to the Eqs. 2 and 3

$$HF/FG = r/g \tag{4}$$

Newton's decomposition of motion into a tangential and vertical component can be verified from the solution of the differential equations of motion in Cartesian coordinates x, y to *second order* in the time interval t. We have

$$\ddot{x} = -r\dot{x}/v \tag{5}$$
$$\ddot{y} = -r\dot{y}/v - g \tag{6}$$

where $v = \sqrt{\dot{x}^2 + \dot{y}^2}$ is the velocity, r is the resistance,[5] and g is the gravitational constant . Taking the origin at O, the Taylor series expansion of the solution of

[5] In the second edition of the *Principia* Newton assumed that r is proportional to v^2, but this condition is not relevant to his derivation.

Eqs. 5 and 6 to second order in t is

$$x = OB + \left(vt - (1/2)rt^2\right)\cos(\alpha), \tag{7}$$

and

$$y = BC - \left(vt - (1/2)rt^2\right)\sin(\alpha) - (1/2)gt^2 \tag{8}$$

where v is the velocity at C, and $\tan(\alpha) = -\dot{y}/\dot{x}$. The angle α gives the inclination of the tangent line at C relative to the horizontal, i.e. $\tan(\alpha) = IF/CI$. These two relations demonstrate the validity, to second order in the time interval t, of Newton's decomposition of the motion along the arc CG, namely, a component along the tangential line

$$CF = vt - (1/2)rt^2, \tag{9}$$

where $CH = vt$, and another vertical component

$$FG = (1/2)gt^2, \tag{10}$$

These two relations, however, cannot be applied to determine the ratio r/g from Newton's relation, Eq. 4, because although the line element FG is determined by the geometry of the curve AGK, this is not the case for the line element HF. Therefore, Newton proceeded to obtain a second expression for r/g. But it will be shown that the resulting expression is not valid because it depended on quantities that are of cubic order in t that could not be calculated from Newton's geometrical construction.

For this purpose, Newton considered the *time-reversed* motion in which case the medium *accelerates* the body along the tangent. Taking a segment Ck of the line element Cf to be equal to CF,

$$Ck = vt' + (1/2)rt'^2 = vt - (1/2)rt^2 \tag{11}$$

where t' is the time interval to traverse the line element Ck in the absence of gravity. Hence, to *second order* in powers of t.

$$t' - t = -(r/v)t^2, \tag{12}$$

and the vertical line element kl is then

$$kl = (1/2)gt'^2. \tag{13}$$

It follows that

$$FG - kl \approx g(t - t')t = (gr/v)t^3. \tag{14}$$

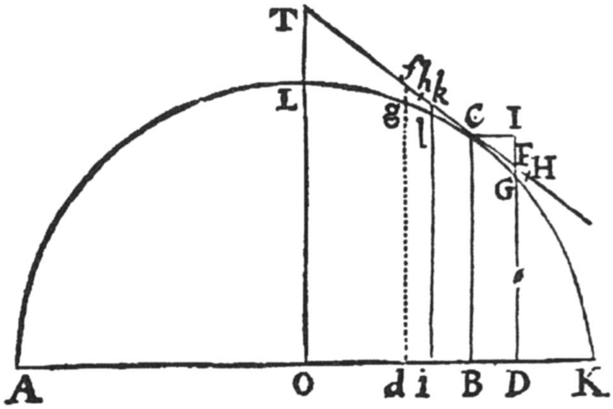

Fig. 1 Newton's diagram for Prop. 10, Book 2 in the first edition of the *Principia*

It is clear, however, that this relation cannot be valid, because Newton had determined the line elements FG, Eq. 10, and kl, Eq. 13, only up to second order in t, while the difference between these line elements is found to be *third order* in t. Unaware of this difficulty, Newton applied this relation to obtain an alternate expression that depends on the resistance r, and in Corollary 2 to Proposition 10, he obtained result

$$r/g = (FG - kl)CF/4FG^2 \tag{15}$$

which follows from Eq. 14 by substituting for v, the approximation $v \approx CF/t$, Eq. 9, and for t, $t = \sqrt{2FG/g}$, Eq. 10. In this form, Newton's relation for r depends on line elements that can be determined purely from the geometry of curve.

Referring to his diagram, Fig. 1, Newton expressed these line elements by a series expansion in powers of the space interval $o = BD = Bi$, up to cubic powers in o. Setting $P = BC$, the ordinate DG is

$$DG = P - Qo - Ro^2 - So^3, \tag{16}$$

where Q, R and S are coefficients with values at the abscissa OD that are determined[6] by the *given* curve $ALGK$, Fig. 1. Since

$$DF = P - Qo, \tag{17}$$

and

$$FG = DF - DG = Ro^2 + SO^3. \tag{18}$$

[6] In Cartesian coordinate $x = OD$, $y = DG$, and according to the Taylor series expansion, in Leibniz's notation $P = -\mathrm{d}y/\mathrm{d}x$, $R = -(1/2!)\mathrm{d}^2y/\mathrm{d}x^2$, and $S = -(1/3!)\mathrm{d}^3y/\mathrm{d}x^3$. At least 24 years earlier, however, in Corollary 3 of the revised *De Quadratura* (1691), Newton already had obtained these relations for the coefficient of a power series expansion, expressed in his dot notation for higher-order fluxions, e.g $\dot{y} = \mathrm{d}y/\mathrm{d}x$, $\ddot{y} = \mathrm{d}^2y/\mathrm{d}x^2$, etc. (Whiteside 1976). I thank N. Guicciardini for this reference.

Likewise, the ordinate il is

$$il = P + Qo - Ro^2 + So^3, \tag{19}$$

and

$$ik = P + Qo. \tag{20}$$

Hence

$$kl = ik - il = Ro^2 - So^3, \tag{21}$$

and

$$FG - kl = 2So^3. \tag{22}$$

Finally, substituting in Newton's relation, Eq. 15, $CF = o\sqrt{1 + Q^2}$ and $FG^2 = R^2o^4$, one obtains

$$r/g = (S/2R^2)\sqrt{1 + Q^2}. \tag{23}$$

For the first application of this relation, Newton consider the case that the given curve ACK, Fig. 1, is a semicircle of radius n. Introducing the symbol a for the abscissa and e for the ordinate of this curve, he obtained $\sqrt{1 + Q^2} = n/e$, $R = n^2/2e^3$, and $S = an^2/2e^3$, which

$$r/g = a/n. \tag{24}$$

Since the tangential component of gravity is ga/n, Newton's result implied that the total tangential acceleration $r - ga/n$ vanishes. Hence, the velocity must be a *constant*, but he found, on the other hand, that the velocity varied with the square root of the ordinate of the semicircle:

> departing from C along the straight line CF, could subsequently move in a parabola having diameter CB and latus rectum $(1 + Q^2)/R \ldots$ (Cohen 1999, 660)

For parabolic motion, the velocity v is given by $v^2 = ga/2$, where a is the latus rectum, and therefore $v = \sqrt{(g/2R)(1 + Q^2)}$ which implies in this case that

$$v = \sqrt{ge}. \tag{25}$$

This relation for the velocity turns out to be correct, but it contradicts Newton's previous result, Eq. 24, that the acceleration due to gravity along a circular path is exactly compensated by the de-acceleration due to the resistance of the medium. Evidently this contradiction escaped Newton's attention, but it was noticed by Johann Bernoulli when he read Proposition 10, Book 2.

In Theorem VI of his paper in the *Mémoires* of 1711 and the *Acta Eruditorum* of 1713, Johann Bernoulli wrote, Hall (1958)

> In order that a body moving in a resisting medium may describe a circle LCK, on the supposition hat a uniform gravity tends directly towards the horizontal, I say that the resistance in each point C will be to the gravity as $3OG$ to $2OK$

Here OK is the radius n of the circle, and OG is the abscissa a in Newton's notation. Hence Bernoulli's correct result[7] for the ratio of resistance to gravity, $r/g = (3/2)OG/OK$, differs from Newton's, Eq. 24, by a factor 3/2.

Newton' analysis is flawed because the *difference* in the Galilean time intervals t and t' to fall under the action of gravity along the line elements FG and kl, respectively, depends on the third power of t, but he could calculated these line elements only up to second power of t. Applying the differential equations of motion, Eq. 5 and 6 to obtain the power series expansion of x, y to cubic order in t, first carried out by Lagrange (1797), yields (see "Appendix II")

$$FG = (1/2)gt^2 - (1/6)(rg/v)t^3. \tag{26}$$

Correspondingly

$$kl = (1/2)gt'^2 + (1/6)(rg/v)t^3, \tag{27}$$

which together with Eq. 12, now gives the correct value

$$FG - kl = (2/3)(gr/v)t^3. \tag{28}$$

valid to third order in t. Hence, instead of Newton's relation for r/g, Eq. 15, we have

$$r/g = (3/2)(FG - kl)CF/4FG^2 \tag{29}$$

and substituting $FG - kl = 2So^3$, Eq. 22, $CF = o\sqrt{1+Q^2}$ and $FG^2 = R^2o^4$ yields the correct value[8]

$$r/g = (3S/4R^2)\sqrt{1+Q^2}. \tag{30}$$

[7] In a scholium following his theorem, Bernoulli criticized Newton's result as follows:

> Lest any one who is unable to examine these matters more deeply should wonder whether perhaps we were mistaken in confuting what has been disclosed with so much labour by this most acute man, I will demonstrate here that this Newtonian ratio leads to a contradiction. For if the resistance be to the gravity as OG to OK as Newton has it, then since the gravity itself is to the tangential force as OC or OK to OG, equally the resistance would be to the tangential or motive force as OG to OG, therefore the resistance will be equal to the motive force, and the velocity at any point C uniform, whereas we previously showed it to be $\sqrt{CG}$, and consequently non-uniform, as Newton himself agrees (Hall 1958).

[8] Johann Bernoulli suggested that Newton obtained the wrong result for the ratio r/g, 23, because he had erroneously assumed that $R = -d^2y/dx^2$, and $S = -d^3y/dx^3$ (Whiteside 1981). But Bernoulli's suggestion is incorrect, and it is a coincidence that replacing in Eq. 23, $2R$ for R, and $6S$ for S, yields the missing factor 3/2.

Newton, however, could not have obtained this result, which differs from his result by an extra factor $3/2$, because it required the expansion of line elements to third order in the time interval t. But he could not have calculated the coefficient of this term from his geometrical—fluxional approach, which was confined to the second order in t.

After obtaining his first relation for r/g, Eq. 4, Newton warned the reader that

> This is so in the case of nascent line elements. For in the case of line elements of finite magnitude these ratios are not accurate (Cohen 1999, 656).

But then he proceeded to ignore his own warning, and derived a second relation for r/g, Eq. 23, that is incorrect and leads to a result for r/g that misses a factor $3/2$.

3 Proposition 10, Book 2, in the second and third edition of the *Principia*

After receiving Niklaus Bernoulli warning from his uncle Johann Bernoulli that his result in Proposition 10 in Book 2 for motion in a medium under the action of gravity led to an inconsistent result for circular motion, Newton approached the problem in a different manner which he managed to insert, in the last moment, in the second edition of the *Principia*. He abandoned his previous approach based on time-reversed motion, and instead, he now determined the required differential changes in line elements and arcs at two adjacent points on a given curve describing the motion of a body moving only forward in time. He described his new diagram, Fig. 2, as follows:

> Let PQ be the plane of the horizon perpendicular to the plane of the figure; $PFHQ$ a curved line meeting this plane in points P and G; G, H, I, and k four places of the body as it goes in the curve from F to Q; and GB, HC, ID and KE four parallel ordinates dropped from these points to the horizon and standing upon the horizontal line PQ at points B, C, D, and E; and let BC, CD and DE be distances between the ordinates equal to one another. From the points G and H draw the straight lines GL and HN touching the curve at G and H and

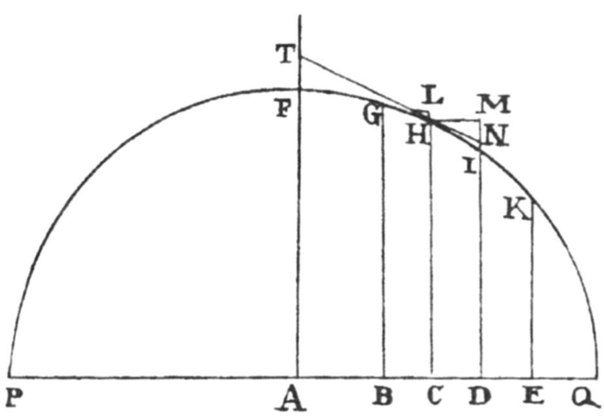

Fig. 2 Newton's diagram for Prop. 10, Book 2 in the second and third editions of the *Principia*

meeting at L and N the ordinates CH and DI produced upwards and complete the parallelogram $HCDM$ (Cohen 1999, 655).

Newton proceeded again to describe analytically the arcs and the line elements associated with the curve in his new diagram by expanding locally the ordinates, separated equally by a small distance o, in a power series up to *cubic order* in o. But he continued to assert, as before, that the different time intervals associated with these space intervals are proportional to the square root of the deviations from tangential motion due to gravity, although this proportionality is correct only to *first order* in o.

Then the times $[T, t]$ in which the body describes arcs GH and HI will be as the square roots of the distances LH and NI which the body would describe in those times by falling from the tangents (Cohen 1999, 656).

The "distances" LH and NI replaced the analogous line elements FG and kl in his previous diagram, Fig. 1, but again Newton set these line elements proportional to the square of the time intervals t and T, although his expansion of these line elements in powers of the space interval o required also cubic terms. Moreover, Newton also assumed that these distances were proportional to the corresponding sagittas associated with the description of this curve, that he evaluated to cubic order in o. But this equivalence[9] is valid only up to second order in o.

Therefore, the question arises why, in spite of requiring the expansion of line elements to cubic powers in o, Newton nevertheless succeed in obtained the correct expression for the resistance of the medium. It will be shown that the answer to this question is that Newton's geometrical descriptions can be implemented by restricting the calculation of line elements to power series expansions up to *second order* in o, and that Newton's application of cubic terms in o, erroneously linked in this order to his time intervals t and T, fortuitously, did not affect his calculations.

Referring to his diagram, Fig. 2, Newton explained that

the velocities will be directly as GH and HI and inversely as the times. Represent the times by T and t, and the velocities by GH/T and HI/t, and the decrement of the velocity occurring in time t will be represented by $GH/T - HI/t$ (Cohen 1999, 656).

The ratios GH/T and HI/t, however, are the *mean* velocities over the arcs GH and HI, while for his calculation, Newton required only the *instantaneous velocities* at G and H. These velocities are given by the ratios GL/T and HN/t, respectively. Expanding the ordinate DI in a power series to cubic order in o

$$DI = P - Qo - Ro^2 - So^3, \tag{31}$$

[9] For motion under the action of a central force, Newton described in Proposition 1, Book 1 of the *Principia* the deviation from inertial motion by the sagitta of small arcs of a curve that can be associated with this proposition, while in Proposition 6, Book 1, the corresponding deviation was described by a small line element from the tangent to the orbit. But these two deviations are equal, only to *second order* in a series expansion in powers of the differential time interval associated with these two deviations.

where $P = CH$, and the coefficients Q, R and S define the nature of the curve at H,

$$HN = o\sqrt{1 + Q^2}, \tag{32}$$

and

$$NI = Ro^2 + So^3. \tag{33}$$

Up to first order in o, we have

$$t = \sqrt{(2/g)NI} = o\sqrt{(2R/g)}, \tag{34}$$

where g is the constant of gravity, and the velocity at H is[10]

$$v = HN/t = \sqrt{(g/2R)(1 + Q^2)}. \tag{35}$$

Newton obtained this result which he phrased as follows:

> And the velocity is that with which a body going forth from any place H along the tangent HN can then move in a vacuum in a parabola having a diameter[11] HC and a latus rectum HN^2/NI or $(1 + Q^2/R)$ (*Cohen* 1999, 658).
>
> Likewise, the velocity at the adjacent point G is

$$v' = GL/T = \sqrt{(g/2R')(1 + Q'^2)} \tag{36}$$

where, to first order in o, the coefficients Q' and R' are

$$Q' = Q - 2Ro, \tag{37}$$
$$R' = R - 3So, \tag{38}$$

Newton wrote:

> and the decrement of the velocity occurring in time t will be represented by $GH/T - HI/t$ (Cohen 1999, 657).

[10] Newton had shown that the square of the velocity satisfies the relation $v^2/\rho = g_n$, where ρ is the radius of curvature at H, and g_n is the component of gravity normal to the tangent (Proposition 6, Corollary 3). Moreover, he had obtained the general mathematical expression for ρ in Cartesian coordinates , $\rho = (1 + Q^2)^{3/2}/2R$, and $g_n = g\cos(\alpha)$, where α is the angle of the tangent line HN with the abscissa, i.e. $\cos(\alpha) = 1/\sqrt{1 + Q^2}$. These relations lead to an alternative derivation of Eq. 35.

[11] The latus rectum for parabolic motion in a vacuum under the action of gravity is $2v^2/g$, and setting its value equal to $(1 + Q^2)/R$ leads to Eq. 35. But the diameter of this parabola is not HC, but is directed, instead, along the line from H perpendicular to the tangent line HN.

Actually, this decrement of the velocity can be obtained more readily by replacing the arcs GH and HI by the tangent lines GL and HN, respectively.[12] Hence, applying Eqs. 35, 36, 37 and 38, one obtains.[13]

$$v' - v = o\sqrt{2gR}\left[-Q/\sqrt{(1+Q^2)} + (3S/4R^2)\sqrt{(1+Q^2)}\right]. \tag{39}$$

where $v' - v = GL/T - HN/t$

This is an example of the "the method of differentiating differentials", which Johann Bernoulli later asserted, that Newton had not understood. Instead, Newton continued as follows:

> This decrement arises from the resistance retarding the body and from the gravity accelerating the body ... but in a body describing arc HI, gravity increases the arc by only the length $HI - HN$ or $MI \times NI/HI$, and thus generates only the velocity $2MI \times NI/t \times HI$, (*Cohen* 1999, 657).

Substituting

$$MI/HI \approx MN/HN = Q/\sqrt{1+Q^2}, \tag{40}$$

and

$$NI/t = o\sqrt{gR/2} \tag{41}$$

one obtains, to first order in o,

$$2MI \times NI/t \times HI = o\sqrt{2gR}\left(Q/\sqrt{(1+Q^2)}\right), \tag{42}$$

which corresponds to the first term on the right hand side of Eq. 39.

> Add this velocity to the above decrement and the result is the decrement of the velocity arising from the resistance alone namely $GH/T - HI/t + 2MI \times NI/(t \times HI)$, (Cohen 1999, 657).

[12] To obtain his results, Newton had to determine the arcs HI and GH to second order in o: $HI = o\sqrt{1+Q^2} - QRo^2/\sqrt{1+Q^2}$, and $GH = o\sqrt{1+Q^2} + QRo^2/\sqrt{1+Q^2}$. But the difference between these arc lengths $GH - HI = 2QRo^2/\sqrt{1+Q^2}$ is equal to the difference between the tangent lines $GL - HN$, where $HN = o\sqrt{1+Q^2}$, Eq. 32, and $GL = o\sqrt{1+Q'^2}$, with $Q' = Q - 2Ro$, Eq. 37.

[13] This geometrical relation can be verified by applying the equations of motion in differential form, Eqs. 5 and 6. We have $\dot{x}\ddot{x} + \dot{y}\ddot{y} = v\dot{v} = -rv - g\dot{y}/v$, and substituting $\dot{y} = v\sin(\alpha)$, one obtains $\dot{v} = -r + g\sin(\alpha)$, the total acceleration along the tangent. In Eq. 39, $\dot{v} = (v - v')/t$, $t = o\sqrt{2R/g}$, Eq. 34, and $\sin(\alpha) = Q/(1+Q^2)$. Hence $r = g(3S/4R^2)\sqrt{1+Q^2}$, which is Newton's result for the resistance of the medium in terms of the coefficients Q, R and S that determine the local properties of the given curve.

Hence, this decrement of the velocity "arising from the resistance alone" is the second term on the right hand side of Eq. 39, and

$$GL/T - HN/t + 2MN \times NI/(t \times HN) = o\sqrt{g/2R}\left[(3S/2R)\sqrt{(1+Q^2)}\right]. \tag{43}$$

> And accordingly since gravity generates the velocity $2NI/t$ in the same time in a fallen body, the resistance will be to the gravity as $GH \times t/T - HI + 2MI \times NI/HI$ to $2NI$ (Cohen 1999, 657).

The velocity generated by gravity in a fallen body is

$$gt = 2NI/t = o\sqrt{2gR}, \tag{44}$$

and dividing Eq. 43 by this velocity yields Newton's relation for the ratio r/g of the resistance r of the medium to the acceleration g of gravity

$$r/g = \left(3S/4R^2\right)\sqrt{1+Q^2} \tag{45}$$

where $r/g = (GL \times t/T - HN)/2NI + MN/HN$.

Newton, however, proceeded in a different manner to evaluate the ratio r/g by expressing analytically his geometrical line elements in a power series expansion up to *cubic* powers in o. But his relations for the time intervals T and t in term of the line elements LH and NI, e.g. Eq. 34, are only valid to *first* order in o. Moreover, Newton calculated these time intervals from the corresponding *sagittas* of the arcs GI and HK which in third order in o *differ* from the expansion of the line elements LH and NI. He stated,

> Furthermore, if from ordinate CH half the sum of ordinates BG and DI are subtracted and from ordinate DI half the sum of ordinates CH and EK are subtracted, the remainder will be sagittas Ro^2 and $Ro^2 + 3So^3$ of arcs GI and HK. And these are proportional to the line elements LH and NI, and thus as the square of the infinitesimal small time T and t (Cohen 1999, 658).

But to third order in o

$$LH = Ro^2 - 2So^3 \tag{46}$$

and

$$NI = Ro^2 + So^3 \tag{47}$$

which, contrary to Newton's assertion, are not proportional to the corresponding sagittas when the expansion is carried out to the cubic order in o. This discrepancy raises the concern that, like in his earlier version of Proposition 10, Book 2, Newton's would have obtained incorrectly cubic terms in the expansion of T^2 and t^2 in power of o. But it turns out that the quantity required for Newton's new calculation for r/g, Eq. 45, requires only the ratio T/t up to first order in o which is proportional to the square root of ratio between the line elements NI/LH. And to first order in o, this ratio is also equal to the ratio between the corresponding sagittas, namely[14]

$$T/t = \sqrt{NI/LH} = 1 - (3/2)So. \tag{48}$$

4 Concluding remarks

In a private memorandum, written on 7 June 1713, Newton accounted for the mistake in the first version of Proposition 10, Book 2 as follows:

> Mr. Newton corrected the error himself, shewed him [Nickolas Bernoulli] the correction & told him that the Proposition should be reprinted in the new Edition which was then coming abroad. The Tangents [GL and HN] of the Arcs GH and HI are first moments of the arcs FG & GH [that] should have been drawn the same way with the motion describing those arcs, whereas through *inadvertency* [my italics] one of them had been drawn the contrary way, & this occasioned the error in the conclusion (Hall 1958)

Actually, it has been shown here that in this memorandum, Newton gave a misleading account of the origin of his error in the first version of this proposition. There was nothing "inadvertent" in his drawing a tangent line, labelled Cf in the first edition of the *Principia*, Fig. 1, in the "contrary way" to the tangent line CF, because his drawing, introduced specifically to solve his proposition, described the time-reversed motion at C. Evidently, at the time Newton failed to realize that a solution by his geometrical calculus approach required that the power series expansion associated with the trajectory curve had to be taken to cubic order in the time interval. But the fundamental principles of orbital motion under the action of external forces that he had enunciated in Propositions 1 and 6, Book 1, were based on such expansions only up to second order in differential time and spatial intervals. After obtaining his first relation for the ratio of resistance to gravity, r/g, Eq. 4, Newton warned the reader that

> This is so in the case of nascent line elements. For in the case of line elements of finite magnitude these ratios are not accurate (Cohen 1999, 656).

But then he proceeded to ignore his own warning to derive a relation for r/g, Eq. 23, that is incorrect by a factor $3/2$. In the revised version of Proposition 10, Book 2, he also applied power series expansions to cubic order in the space interval o, but as has been shown here, in this case only expansions to second order in o are required, which explains why, somewhat fortuitously, he obtained the missing factor $3/2$.

[14] A simple way to see how this relation saved Newton from repeating his previous error is to suppose that $t^2 = Ro^2 + Co^3$ where C is an undetermined coefficient. Then the expansion of T^2 must be calculated with the corresponding value $R' = R - 3So$, while the change in C gives a fourth order change in T^2 that can be neglected. Hence $T/t = 1 - (3S/2R)o$, Eq. 48, independent of the value of C that Newton evaluated incorrectly.

During the bitter priority dispute with Leibniz on the invention of the calculus, Newton called attention to Proposition 10, Book 2, to exhibit an example of an early application of his fluxion version of the calculus to solve problems solution *Principia*. But this proposition was a poor choice, because he had made a "serious error" in it, and after he corrected his mistake, he failed to acknowledge Johann Bernoulli's contribution, who was bitter and became, albeit anonymously, his most effective critic.

Appendix I: Relation of Proposition 10, Book 2, to Proposition 6, Book 1

In Proposition 6, Book 1 in the *Principia* Newton gave a geometrical expression for the acceleration of a body moving under the action of a central force in terms of a line element, QR, directed towards the centre of force, that describes the deviation from inertial motion along a line element PR, tangent at P, Fig. 3. The magnitude of PR is proportional to the elapse time interval t, and the deviation QR is proportional t^2, in accordance with Lemma 10, Book 1. Moreover, in Proposition 1, Book 1, Newton had shown that for central forces t is proportional to the area element $SP \times QT$ swept by the radial line SP, leading to the geometrical expression for the force or acceleration, $QT/(SP \times QT)^2$. Such a description, however, not valid in the presence of non-central Newton was able to generalize his description for non-central forces, but in the presence of a tangential force however, such a description is incomplete, because the resulting acceleration is directed along the instantaneous direction of motion. In the original version of the Proposition 10, Book 2, the line elements FG and CF, Fig. 1, are the analogs of QR and PR, respectively, while in the revised version in the second edition of the *Principia*, the corresponding line elements are NI and HN. Newton continued to assert that the line elements FG and NI are proportional to t^2, and to take into account the resistance of the medium, Newton had to assume that the motion along the tangent was the sum of an inertial component proportion to t and a component proportional to t^2. But the resulting geometrical relations were incomplete and Newton searched for additional conditions that would enable him to obtain a geometrical relation for the motion. In the first edition of the *Principia*, he obtained an additional constraint by considering the time-reversed motion starting from the same initial point. In this case, the medium accelerates the motion along the tangent, but unfortunately, his application of Lemma 10 for this problem was insufficient, because as Lagrange, later demonstrated, a contribution to the lime element of cubic powers of t is required for a correct solution of this problem. Newton's neglect of this contribution led him to an incorrect solution which was noticed by Johann Bernoulli and pointed out by his nephew, Nikolaus Bernoulli to Newton. Newton's evaluation for the instantaneous velocity at H, $v = \sqrt{(g/2R)(1+Q^2)}$, Eq. 62, corresponds to the relation $g_n = v^2/\rho$, where $g_n = g\cos(\alpha)$ is the component of gravity normal to the tangent, and $\rho = (1+Q^2)^{3/2}/R$ is the curvature at H. In Corollary 3 of Proposition 6, Book 1, this relation was given in a related form $g = l^2/SY^2 \times PV$ for a central force g, where $PV = 2\rho\cos(\alpha)$ and $l = SPv\cos(\alpha)$ are the angular momentum about the centre of force.

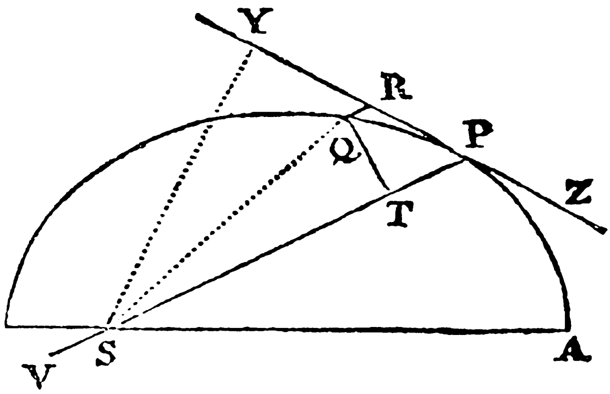

Fig. 3 Newton's diagram associated with Prop. 6, Book 1 in the *Principia*

Appendix II: Lagrange's analysis of Proposition 10, Book 2

Lagrange was the first mathematician to obtain the correct explanation for Newton's failure to obtain the correct relation for motion in a resistive medium (Lagrange 1797). Lagrange' s analysis did not deal directly with Newton's geometrical approach, but a similar analysis is presented here evaluating Newton's line element NI in Proposition 10, Book 2, to cubic order in the time interval t.

The equations of motion for this problem in differential form are

$$\ddot{x} = -r\cos(\alpha) \tag{49}$$

$$\ddot{y} = r\sin(\alpha) - g, \tag{50}$$

where $\ddot{x} = \mathrm{d}^2x/\mathrm{d}t^2$, $\ddot{y} - \mathrm{d}^2y/\mathrm{d}t^2$, and $\tan(\alpha) = -\mathrm{d}y/\mathrm{d}x$. For the power series expansion, we require also the third-order derivative

$$\mathrm{d}^3x/\mathrm{d}t^3 = r\sin(\alpha)\dot{\alpha} - \cos(\alpha)\dot{r}, \tag{51}$$

and

$$\mathrm{d}^3y/\mathrm{d}t^3 = r\cos(\alpha)\dot{\alpha} + \sin(\alpha)\dot{r}, \tag{52}$$

where $\dot{\alpha} = \mathrm{d}\alpha/\mathrm{d}t$, and $\dot{r} = \mathrm{d}r/\mathrm{d}t$.

Expanding Newton's line elements in powers of a small interval t up to cubic order, we have

$$HM = x = v\cos(\alpha)t - (1/2)r\cos(\alpha)t^2 + (1/6)(r\sin(\alpha)\dot{\alpha} - \cos(\alpha)\dot{r})t^3, \tag{53}$$

and

$$MI{=}HC - y{=}v\sin(\alpha)t - (1/2)(r\sin(\alpha) - g)t^2 - (1/6)(r\cos(\alpha)\dot{\alpha}{+}\sin(\alpha)\dot{r})t^3. \tag{54}$$

Since $MN = HM\tan(\alpha)$,

$$MN = v\sin(\alpha)t - (1/2)r\sin(\alpha)t^2 + (1/6)(r\sin(\alpha)\tan(\alpha)\dot{\alpha} - \sin(\alpha)\dot{r})t^3, \quad (55)$$

and $NI = MI - MN$

$$NI = (1/2)gt^2 - (1/6)(r/\cos(\alpha))\dot{\alpha})t^3, \quad (56)$$

which is independent of $\dot{r}$. We have

$$\mathrm{d}\tan(\alpha)\mathrm{d}t = (1/\cos^2(\alpha))\dot{\alpha} = (\dot{y}\ddot{x} - \dot{x}\ddot{y})/\dot{x}^2 = g/\dot{x} \quad (57)$$

which yields

$$\dot{\alpha} = (g/v)\cos(\alpha) \quad (58)$$

Hence, to cubic order in t[15]

$$NI = (1/2)gt^2 - (1/6)(rg/v)t^3, \quad (59)$$

Appendix III: Proposition 10, Book 2, expressed in terms of differential equations, and Johann Bernoulli's solution

The problem of motion in a resisting medium under the action of a constant gravitational force g is treated by expressing the equations of motion in differential form. In Cartesian coordinates x, y,

$$\ddot{x} = -r\dot{x}/v, \quad (60)$$

and

$$\ddot{y} = -r\dot{y}/v - g, \quad (61)$$

where r is the resistance, and v is the velocity

$$v = \sqrt{\dot{x}^2 + \dot{y}^2}. \quad (62)$$

Hence

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} = v\dot{v} = -rv - g\dot{y}, \quad (63)$$

[15] Following Lagrange, this result was also obtained by Whiteside; see footnote (6) in (Whiteside 1981, 375).

and

$$r/g = -(\dot{v}/g + \dot{y}/v) \tag{64}$$

To obtain an expression for this ratio in terms of the geometrical curve for the trajectory, the time must be replaced by x as an independent variable. Since

$$\dot{y} = y'\dot{x} \tag{65}$$

and

$$\ddot{y} = y'\ddot{x} + y''\dot{x}^2 \tag{66}$$

then, according to Eqs. 60 and 61, we have

$$\ddot{y} - y'\ddot{x} = -g = y''\dot{x}^2 \tag{67}$$

or

$$\dot{x} = \sqrt{-g/y''}. \tag{68}$$

Hence, the velocity $v = \dot{x}\sqrt{1 + y'^2}$ can be expressed in terms of the first- and second-order spatial derivatives y' and y'',

$$v = \sqrt{-(g/y'')(1 + y'^2)}. \tag{69}$$

Hence

$$\dot{v} = g\left(-\frac{y'}{\sqrt{1 + y'^2}} + \frac{1}{2}\frac{\sqrt{1 + y'^2}}{y''^2}y'\right), \tag{70}$$

and

$$\frac{\dot{y}}{v} = \frac{y'}{\sqrt{1 + y'^2}}. \tag{71}$$

Finally, substituting Eqs. 70 and 71 in Eq. 64,

$$\frac{r}{g} = -\frac{\sqrt{1 + y'^2}}{2y''^2}y''', \tag{72}$$

In Newton's notation $y' = -Q$, $y'' = -2R$, $y''' = -6S$, and

$$v = \sqrt{(g/2R)(1 + Q^2)}, \tag{73}$$

$$\frac{r}{g} = \frac{3S}{4R^2}\sqrt{1+Q^2}, \tag{74}$$

For a solution to be possible, R and S must be positive numbers.

Example 1 When the curve is a semicircle of radius a

$$Q = \frac{x}{y}, \quad R = 2\frac{a^2}{y^3}, \quad S = 18\frac{a^2 x}{y^5} \tag{75}$$

Hence

$$\frac{r}{g} = \frac{3x}{2a}, \tag{76}$$

and

$$v = \sqrt{gy} \tag{77}$$

In particular, $ra/gx = 3/2$ is the constant ratio of frictional force to the tangential gravitational force, which accounts for the decrease in velocity as y decreases.

In the first edition of the *Principia*, the factor $3/2$ was absent, and Johann Bernoulli observed that in this case, the resistance force was equal to the gravitational acceleration along the curve, indicating that the velocity should be a constant—"la vitesse de ce mobile seroit ici toujours la mesme & uniforme"—along the curve, leading to an inconsistency in Newton's result—"Ce que est la contradiction que j'avois à demontrer"[16]

Johann Bernoulli treated the motion in a resisting medium somewhat differently, by separating the differential equations of motion into its tangential and normal components. He considered only the special case of circular motion, but I will treat also the motion along a general curve.

Let

$$\mathrm{d}v = (F_T - r)\mathrm{d}t \tag{78}$$

and

$$\frac{v^2}{\rho} = F_N \tag{79}$$

where $F_T = -g\dot{y}/v = gy'/\sqrt{1+y'^2}$, and $F_N = g\dot{x}/v = g/\sqrt{1+y'^2}$ are the tangential and normal components of force, respectively, and r is the resistance force.

[16] Extrait d' une Lettre de M. Bernoulli, écrite de Basle le 10. Janvier 1711, touchant la maniere de trouver les force centrales me milieux resistans en raisons composée de leurs densités & des puissances quelconques de vitesses du mobile. in Memoires de L' Academie Royale des Sciences.

Bernoulli writes r in the form $r = \gamma v^n$, but his treatment does not depend on this particular form for r. Setting

$$\mathrm{d}t = \frac{\mathrm{d}s}{v} \tag{80}$$

where ds is a differential arc length, Bernoulli obtains and equation between spatial differential

$$r\mathrm{d}s = -v\mathrm{d}v - g\mathrm{d}y \tag{81}$$

Assuming that the motion is along semicircle of radius a, $\rho = a$ and Eq. 79 takes the form

$$v^2 = gy. \tag{82}$$

Hence

$$v\mathrm{d}v = \frac{1}{2}g\mathrm{d}y, \tag{83}$$

which substituted in Eq. 81 leads to

$$r\mathrm{d}s = -\frac{3}{2}g\mathrm{d}y, \tag{84}$$

and since $\mathrm{d}s/\mathrm{d}y = -x/a$,

$$\frac{r}{g} = \frac{3x}{2a} \tag{85}$$

—"ce qu'il faloit encore a demontrer"—in accordance with Newton's result in Example 1 in the second edition of the *Principia*.

For a general curve, substituting for ρ in Eq. 79, $\rho = -(1 + y^2)^{3/2}/y''$, I obtain

$$v^2 = -g\frac{(1 + y'^2)}{y''} \tag{86}$$

which is equal to the expression for v, Eq. 69, obtained previously from Newton's formulation of this problem.

Hence

$$v\mathrm{d}v = -g\left(\mathrm{d}y - \frac{1}{2}\frac{(1 + y'^2)y'''}{y''^2}\mathrm{d}x\right) \tag{87}$$

and substituting this expression in Eq. 81 with $\mathrm{d}s/\mathrm{d}x = \sqrt{1+y'^2}$ yields

$$\frac{r}{g} = \frac{\sqrt{1+y'^2}}{2y''^2} y''' \tag{88}$$

which corresponds to Eq. 72.

Acknowledgments I would like to dedicate this paper to the memory of Derek (Tom) Whiteside, whom I had the privilege to know and benefit from his numerous criticisms of my earlier work. Tom devoted over hundred pages of his last volume of *Newton's Mathematical Papers* to describe the manuscripts in which Newton made frantic efforts to correct the initial errors he had made in Proposition 10, Book 2. Recently, he warned against creating "monsters" when framing Newton's ideas in vector form, "which felled even the mighty Lagrange in his attempt wholly to understand it" (Whiteside 2002), but such monsters are still being created up to the present time. I also would like to thank Niccolo Guicciardini for bringing to my attention the controversies associated with Newton's formulation of Proposition 10, Book 2, and for an uncountable number of exchanges that helped to clarify my ideas on this subject, and to Gary Weisel for calling my attention to Whiteside's above apropos comment.

References

Bernoulli, J. 1716. In *The correspondence of Isaac Newton*, vol. 6, ed. A. Rupert Hall and Laura Tilling, p. 303. Cambridge: Cambridge University Press.

Brackenridge, J.B., and M. Nauenberg 2002. Curvature in Newton's dynamics. In *The Cambridge Companion to Newton*, ed. I.B. Cohen and G.E. Smith, pp. 85–137. Cambridge: Cambridge University Press.

Cohen, I.B. 1999. *Isaac Newton the Prncipia.* A new translation by Cohen, I.B., and Anne Whittman, Preceeded by a guide to Newton's *Principia*, p. 167. Berkeley: University of California Press.

Guicciardini, N. 2009. *Isaac Newton on mathematical certainty and method.* Cambridge: MIT Press.

Hall, A.R. 1958. Correcting the Principia. *Osiris* 13: 291–326.

Hall, A.R. 1980. *Philosophers at war, the quarrel between Newton and Leibniz.* Cambridge: Cambridge University Press.

Lagrange, J.L. 1797. *Théorie des Fonctions Analytiques, contennant Les Principes du Calcul Différentiel*, pp. 244–251. Paris: De l'Imprimerie de la République.

Whiteside, D.T. 1976. *The Mathematical Papers of Isaac Newton* vol. 7, ed. T. Whiteside, pp. 96–97. Cambridge: Cambridge University Press.

Whiteside, D.T. 1970. The mathematical principles underlying Newton's Principia mathematica. *Journal for the History of Astronomy* 1: 116–138.

Whiteside, D.T. 1981. *The mathematical papers of Isaac Newton*, vol. 8, ed. Derek T. Whiteside, pp. 312–419. Cambridge: Cambridge University Press

Whiteside, D.T. 2002. The math forum @ drexel. Posted July 5 http://mathforum.org.