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Source: *Archive for History of Exact Sciences*, Vol. 64, No. 5 (September 2010), pp. 525-559

Published by: Springer

Stable URL: <https://www.jstor.org/stable/41342430>

Accessed: 19-05-2020 12:36 UTC

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From classical to Voigt's molecular models in elasticity

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Received: 16 June 2009 / Published online: 23 June 2010
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Abstract In the first decades of the nineteenth century the French mechanicians—Cauchy and Poisson amongst them—developed a theory of linear elasticity according to which matter is composed of material points. They believed that these points interact by means of opposite central forces, whose magnitude depends on the length of the segment joining the particles. This theory suggested that homogeneous isotropic materials were characterized by a unique elastic constant. Later experiments, however, showed that two elastic constants were necessary. These results undermined the corpuscular model of matter as well as the interpretation of elasticity in terms of central intermolecular actions. The continuous theory of Green, based on the postulate that a potential function exists, gained fresh consensus in light of these experiments. These opposite views continued throughout the nineteenth century until Woldemar Voigt proposed a molecular model confirmed by experiments. This article presents the theories of each of these scientists and describes the contrasting views of nineteenth-century mechanicians.

1 Introduction

The modern theory of linear elasticity dates back to the work of the French mechanicians of the early nineteenth century. Most of their studies tended to use a molecular

Communicated by Umberto Bottazzini.

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model of matter.¹ In his studies of 1821 on the propagation of light in solids, Fresnel² was one of the first to introduce body-points, generally called molecules, to characterize linear elasticity.

In the same year Navier presented a memoir to the Academy of Sciences in Paris (Navier 1827)³ where he obtained the differential equations of the motion of a linear isotropic elastic body in terms of a unique material constant. Navier modelled an elastic solid body as an assemblage of material molecules that are very close to each other. These body-points interact by means of central forces depending on the distance between pairs of molecules, even if the force law is actually unknown. The magnitude of these forces is perceivable for close molecules, but quickly vanishes as the distance increases. Such an approach originated in Laplace's and Boscovich's studies on the composition of matter.

In contrast with the molecular model presented by Fresnel and Navier, in a short note Cauchy (1823) presented a brief outline of the results that he had obtained by adopting a continuous model of matter. He clearly stated that an internal force, called tension or pressure, could be defined as being similar to the one exerted by a fluid on the surface of a solid body. This was further expounded in a series of later publications, where he formulated the famous theorems on stress (Cauchy 1827a) and obtained results on both the strain measures of a continuous medium and the differential balance equations between inner and outer forces, regardless of the body's mechanical properties (Cauchy 1827b,c).

In a subsequent article, Cauchy wrote down the differential equations of motion for an isotropic linear elastic body in terms of a single material constant (Cauchy 1828a, p. 204). In the same article, however, Cauchy obtained other equations of motion for isotropic bodies in terms of two elastic constants (Cauchy 1828a, pp. 215–218) and remarked that for a certain fixed ratio of the two constants he was able to obtain the same equations as Navier and Poisson (Cauchy 1828a, p. 221).

The continuous model of matter adopted by Cauchy in these articles appears to be an exception: by the end of the 1820s the molecular model assumed a leading role in the studies of elasticity and remained a reference tool for some time to come, especially in France. On October 1st, 1827, Cauchy and Poisson presented similar memoirs to the Academy of Sciences in Paris, in which Navier's molecular approach was re-considered.⁴ In 1828 and 1829 Poisson presented two more extensive memoirs to the Academy, published between 1829 and 1831, which were to become milestones in the theory of linear elasticity.

¹ For the origins and the development of linear elasticity, see, amongst others (Todhunter and Pearson 1886; Love 1907; Marcolongo 1907; Whittaker 1910; Dugas 1950; Timoshenko 1953; Truesdell and Toupin 1960; Fichera 1979; Belhoste 1991; Benvenuto 1991; Dahan Dalmedico 1992; Capecchi et al. 2006).

² See A. Fresnel, *Oeuvres complètes*, Paris, Imprimerie Impériale, 1866–1870; tome 2, pp. 261–368.

³ An abstract with the same title appeared in the *Bulletin de la Société philomathique*, 1822, pp. 177–181.

⁴ Cauchy's memoir was published later in his *Exercices*. We refer to it as Cauchy (1828b); see Navier (1864, clv). As for Poisson's memoir, apart from two excerpts published in the *Annales de chimie et de physique*, 1827, and in the *Bulletin des sciences (Férussac)*, t. IX, 1828, it was published in 1829. We refer to it as Poisson (1828); see Navier (1864, clix).

Poisson established the very foundations of the molecular theory of elasticity: in his article of 1828 he rejected Lagrange's *Mécanique Analytique* by claiming that any theory of mechanics should be based on molecular actions, they alone being responsible for equilibrium and the transmission of given external forces. He introduced the concept of a molecular sphere of action and obtained the equations of motion for isotropic bodies in terms of a single elastic constant, remarking that his results coincided with Navier's.

In two articles of 1828 Cauchy (1828b,c) reflected Poisson's view of matter to some extent. Whilst in his previous studies he had clearly conceived of a body as continuous, here he proved that the same results could be obtained by supposing the matter to be an assemblage of material points subjected to central forces.⁵ The main difference between Poisson's and Cauchy's molecular models is given by their different views of molecules: corpuscles for Poisson, points without extension for Cauchy.

The constitutive relations obtained by both Cauchy's and Poisson's molecular theories were characterized by 15 constants in the case of general linear elasticity, and by only one in the case of isotropic bodies. This confirmed the early results obtained by Navier.⁶ The first experimental results on the constitutive properties of elastic materials were not particularly precise, and did not question the validity of the results based on the molecular model of matter. As measurements became more precise, however, it was apparent that experience contradicted the predictions of the theory. The doubt also arose that, at least for isotropic bodies, linear elasticity should not be characterized by two material constants.

In Green (1839) a completely different approach and a continuous model of matter are adopted. Since Green thought that the actual law of molecular interactions is unknown, he postulated that elastic forces are described by a potential, the existence of which was proved later on in Thomson (1855) on thermodynamic grounds. Green obtained two elastic constants, as observed in experiments. In spite of this, there was a certain reluctance, especially in France, to adopt Green's model, as it abandoned the interpretation *per causas* of elastic phenomena. Indeed, not even Lamé, Clapeyron and Saint-Venant were able to indicate where the traditional molecular theory of elasticity, so elegant and precise, could have gone wrong.

The contrast amongst the molecular and continuous theorists of elasticity continued until the mid-nineteenth century, and studies on crystals shed no light on the problem.⁷ The studies of Bravais on crystallography (Bravais 1866)⁸ marked a turning point. According to Bravais, molecules are not simply body-points but polyhedra at the vertices of which are the atoms of the material under consideration. The polyhedra-

⁵ In a subsequent article Cauchy (1829) considered a continuous distribution of matter and obtained the equations of motion for isotropic linear elastic bodies in terms of two elastic constants.

⁶ We point out that Cauchy was somewhat ambiguous and on some occasions introduced two constants for linear elastic isotropic bodies, claiming that the equations so obtained were "more general". This quite weak position was applied to a very particular situation, hence it is beyond the scope of this article.

⁷ This subject has not been dealt with in depth in the historical literature and, to the authors' knowledge, it is only mentioned in passing in the standard studies on the history of elasticity such as Todhunter and Pearson (1886), Timoshenko (1953) and Benvenuto (1991).

⁸ These studies were presented at the Academy of Sciences in Paris in 1849 but published posthumously in 1866.

molecules are arranged in regular lattices which repeat themselves in space according to precise symmetry groups, which were studied and classified by Bravais. Such an approach used a discrete, ‘refined’ description of matter, yet was completely different from the body-point conception of Cauchy and Poisson. It constituted the starting point for Woldemar Voigt, whose studies were able to reconcile the molecular and the continuous approaches to the linear theory of elasticity.

In this article we briefly present the molecular theory of elasticity as founded by Navier, Cauchy and Poisson. We point out its advantages and drawbacks, and we show how the latter gave rise to a debate amongst mechanicians that lasted for decades. This introductory part deals with well-known aspects of the theory of elasticity from an unusual point of view. Attention is focussed more on the drawbacks of the molecular model of linear elasticity than on other aspects. Our aim is to show how Voigt’s approach, molecular but not rejecting ‘abstract’ energetic techniques, can reconcile the attempt at justifying phenomena *per causas* and the wish for strong continua theories. This technique is still up-to-date, and it is useful for studying mechanics of ‘complex’ bodies.

2 The molecular model and its shortcomings

2.1 The molecular model of Navier, Cauchy and Poisson

Although Navier, Cauchy and Poisson used different analytical techniques to obtain their results, their models of matter were similar and clearly molecular. Matter was thought of as composed of body-points that are very close to each other, and the standard notions of central attractive–repulsive forces depending on the length could be applied.

We consider a solid body as an assemblage of material molecules placed at a very small distance [from each other]. These molecules exert two opposite actions on each other, that is, a proper attractive force, and a repulsive force due to the principle of heat. [...] we assume here as a principle that the [...] forces π' , developed by the change of shape of the body between any two material molecules M, M' , and which alone must balance the forces applied to this body, are respectively proportional to the amount of variation of the distance MM' of the two molecules due to the change of shape (supposed very small). The force π' is an attraction if the distance M' has increased; it is a repulsion if that distance has decreased. On the other hand, we consider the discussed molecular actions as existing only between very close molecules, and having values which are very quickly decreasing, according to an unknown law, for molecules which are more and more distant.⁹ (Navier 1827, pp. 375–377)

⁹ On regarde un corps solide élastique comme un assemblage de molécules matérielles placées à des distances extrêmement petites. Ces molécules exercent les unes sur les autres deux actions opposées, savoir, une force propre d’attraction, et une force de répulsion due au principe de la chaleur. [...] nous prenons ici pour principe que ces [...] forces π' , développées par le changement de figure du corps entre deux molécules matérielles quelconques M, M' , et qui doivent seules faire équilibre aux forces appliquées à ces corps, sont respectivement proportionnelles à la quantité dont le changement de figure (supposé très-petit) a fait

Navier's idea originated in Laplace's concept of matter. To describe natural phenomena Laplace proposed a model of particles of very small dimensions that exchanged forces,¹⁰ thus elaborating on the ideas expressed by Newton in his *Opticks*. The description of the model of matter introduced by Navier was almost exactly reproduced by Poisson, who sought to explain elasticity by means of the laws of molecular interaction. Poisson emphasized the existence of a characteristic length for the intermolecular action, which he called the radius of the molecular activity (Poisson 1828, pp. 368–369). In 1828 Cauchy, too, proved his results using Navier's and Poisson's view of matter as a bunch of dimensionless molecules (Cauchy 1828b, p. 227).

The molecular models of Navier, Cauchy and Poisson share some common features:

- a. the molecules are body-points subjected to opposite central forces;
- b. the total effect of the interacting forces is obtained by the vector sum of all forces;
- c. the molecular interaction is a smooth function of the distance between the molecules and may be linearized;
- d. there is a sphere of molecular activity, the radius of which is much larger than the intermolecular distances; within this sphere the molecular interaction is sensible but quickly decreasing, and totally disappears outside;
- e. the displacements of body-points are described by a smooth field, the domain of which is the continuum in which the system of molecules is embedded;
- f. the system of body-points can be assumed homogeneous and in particular isotropic.

Once the expression of intermolecular forces is known, a suitable definition of stress provides its constitutive relations in terms of the strain and of the displacement components. By applying linearization procedures Cauchy and Poisson obtained constitutive relations of stress components in terms of strain components which generally depended on 36 material constants. They showed that there are 21 symmetry relations (later on named after Cauchy and Poisson) amongst the constants. Thus, the number of independent elastic constants was 15 and, in the case of an isotropic body, reduced to only one.

2.2 Shortcomings of the molecular model

The conclusions of Navier, Cauchy and Poisson were widely accepted, especially in France, given the undoubted simplicity of the underlying theory and because of their appeal from a physical point of view. On the other hand, they were gradually undermined by experimental results,¹¹ which showed, especially with to the refinement of the experimental techniques, that two elastic constants were needed to characterize

Footnote 9 continued

varier la distance MM' des deux molécules. La force π' est une attraction si la distance MM' a augmenté; elle est une répulsion si cette distance a diminué. Nous regardons d'ailleurs les actions moléculaires dont il s'agit comme ne subsistant qu'entre des molécules très-voisines, et comme ayant des valeurs qui décroissent très-rapidement, suivant une loi inconnue, pour des molécules de plus en plus éloignées l'une de l'autre.

¹⁰ See for instance his *Exposition du système du monde*, in *Œuvres complètes*, vol. 6, pp. 349–350.

¹¹ For instance, those widely illustrated in Wertheim (1844).

linear elastic isotropic bodies. The more reliable were the experimental results,¹² the more the theoretical predictions by Cauchy and Poisson went wrong, and no one could explain why.

A first step to match the 'classical' molecular model with the experimental results was to get rid of a number of hypotheses on which the theory was based. Poisson was amongst the first timidly to suggest a variation. In his memoir presented at the Academy of Sciences in 1829 (Poisson 1831), Poisson advanced the hypothesis that the molecules are not body-points and that solids are crystallized, with a regular disposition of the molecules. The notion of central forces was therefore somewhat undermined.

One supposes that in bodies of this nature the molecules are regularly distributed, and that they attract or repel each other differently on their different sides. For this reason, in calculating the action exerted by a part of the body on another one it is not permitted anymore to consider the mutual action between two molecules as a simple function of the distance separating them [...]. If one deals with a homogeneous body which is in its natural state, where it is not subjected to any external force, one may consider it as an assemblage of molecules of the same kind and *shape*, of which the homologous sections are parallel to each other [...].¹³ (Poisson 1831, p. 69)

According to Poisson, the 21 relations reducing the number of independent constants to 15 no longer hold in linear elastic crystallized bodies (Poisson 1831, p. 85). On the other hand, in non-crystallized bodies, or in bodies with weak or irregular crystallization, even if molecules are not body-points, the mechanical behaviour continues as if the exchanged forces were central, due to a series of compensatory causes (Poisson 1831, pp. 7–8).

Even Cauchy expressed doubts about the validity of the 'classical' molecular model in a series of memoirs presented in 1839¹⁴ and in a review¹⁵ of experimental memoirs by Wertheim on elastic constants (March 3rd 1851). There he stated that in crystallized bodies the molecules should not be considered as body-points but instead as very small particles composed of atoms. Since there is a regular arrangement of molecules in crystals, the elastic moduli should not be considered as constants but rather as periodic functions of the spatial coordinates. To obtain a constitutive relation with constant coefficients, Cauchy found it necessary to enlarge the

¹² For instance, those presented in Kohlrausch and Loomis (1870).

¹³ On suppose que dans les corps de cette nature, les molécules sont régulièrement distribuées, et quelles s'attirent ou se repoussent inégalement par leurs différens cotés. Par cette raison il n'est plus permis, en calculant l'action exercée par une partie du corps sur une autre, de regarder l'action mutuelle de deux molécules comme une simple fonction de la distance qui les sépare [...]. S'il s'agit d'un corps homogène qui soit dans son état naturel, où il n'est soumis à aucune force étrangère, on pourra le considérer comme un assemblage de molécules de même nature et de même *forme*, dont les sections homologues seront parallèles entre elles [...].

¹⁴ *Œuvres* ser. 2, t. XI, pp. 11–27, 51–74, 134–172.

¹⁵ *Comptes Rendus de l'Académie des sciences*, vol. 32 (1851), pp. 326–330.

number of elastic moduli, eventually obtaining two elastic constants for isotropic bodies.¹⁶

In his studies on elasticity¹⁷ Lamé advanced a series of doubts on the subject. For instance, in Lamé (1859) most of the 20th lesson is devoted to his doubts regarding what a molecule actually is, the exact postulate concerning mutual interactions, the acceptable form of the law of the intermolecular action, and the right direction of the molecular force. According to Lamé, the hypothesis of homogeneous matter, whilst permitting symmetry considerations, is not admissible:

Such is the method followed by Navier and other geometers to obtain the general equations of elasticity in solid media. But this method clearly supposes the continuity of matter, an inadmissible hypothesis. Poisson believed he had eliminated this difficulty, [...] but [...] in reality only replaced the sign \int by the sign Σ [...]. The method which we have followed [...] finds its origins in the studies by Cauchy¹⁸, [and] seems to us to be free of any objection.¹⁹ (Lamé 1866, p. 38)

Though the results of the molecular theory of elasticity were clearly queried by the followers of the French school of mechanics, the validity of the molecular approach was not. Barré de Saint-Venant was amongst the main supporters of the molecular theory and his ideas on the subject, apart from his memoirs, can be found in the vast amount of notes, comments and appendices attached to Navier (1864) and Clebsch (1883). Saint-Venant was perfectly aware that experimental results contradicted the conclusion of the 'classical' molecular models attributing one elastic constant only to isotropic bodies. However, since he found no evident fault in the traditional molecular theory of elasticity, he concluded that there are no isotropic bodies in nature:

Yet experiences [...] and the simple consideration of the way which cooling and solidification take place in bodies, prove that isotropy is quite rare [...]. So, instead of using, in place of the equations [...] with one coefficient only, the formulæ [...] with two coefficients [...], which hold, like these others, only for perfectly isotropic bodies, it will be convenient to use as many times as possible the formulæ [...] relative to the more general case of different elasticity in two or three directions.²⁰ (Navier 1864, p. 583)

¹⁶ A detailed and commented reconstruction of Cauchy's reasoning may be found in Saint-Venant's Appendix V of Navier (1864, pp. 691–706).

¹⁷ We have considered Lamé (1859, 1866), the first edition of which appeared in 1852.

¹⁸ Lamé clearly refers to the studies in which Cauchy did not follow a molecular approach, but rather a 'continuous' one, such as, Cauchy (1828a, 1829).

¹⁹ Telle est la méthode suivie par Navier et autres géomètres, pour obtenir les équations générales de l'élasticité dans les milieux solides. Mais cette méthode suppose évidemment la continuité de la matière, hypothèse inadmissible. Poisson croit lever cette difficulté, [...] mais [...] il ne fait, en réalité, que substituer le signe Σ au signe \int [...]. La méthode que nous avons suivie [...] dont on trouve l'origine dans les travaux de Cauchy, nous paraît à l'abri de toute objection [...].

²⁰ Mais les expériences [...] et la simple considération de la manière dont s'opèrent le refroidissement et la solidification des corps, prouvent que l'isotropie est fort rare [...]. Aussi, plutôt que de prendre, au lieu des formules [...] à un seul coefficient [...], les formules [...] à deux coefficients [...], qui ne sont composées comme celles-ci que pour des corps parfaitement isotropes, il conviendra de se servir le plus qu'on pourra des formules [...] relatives au cas plus général d'une élasticité inégale dans deux ou trois sens.

In the over 200 pages of notes and appendices attached to Navier (1864), Saint-Venant presented experimental results and theoretical explanations of the paradox, showing an amazing knowledge of the literature (amongst others he quoted Savart, Wertheim, Hodgkinson, Regnault, Oersted, Green, Clebsch, Kirchhoff, Rankine, and William Thomson).

However, there was no agreement amongst the contemporaries of the great French mechanician, and the question still remained: as it was apparent that two elastic constants were needed, where could the failure of such a seemingly perfect theory as that by Navier, Cauchy and Poisson reside?

Discussion was further enriched by the contributions of Green and Bravais. These generated completely different schools of elasticity in England and Germany, not to mention different ways of studying crystallography.

3 Green's energetic model and Bravais's crystals

In his search for a model for the luminiferous aether, in 1839 George Green published an article (Green 1839) which was completely opposed to the principles of the French school of mechanics. Obviously aware of the criticism directed at the traditional molecular model, Green abandoned the primitive role of force in favour of the role of virtual work, and later of energy. He considered a continuous mathematical model of matter, and admitted that inner forces can be characterized only by the fact that their virtual work is an exact differential:

M. Cauchy seems to have been the first who saw fully the utility of applying to the Theory of Light those formulæ which represent the motions of a system of molecules acting on each other by mutually attractive and repulsive forces; [...however,] we are so perfectly ignorant of the mode of action of the elements of the luminiferous aether on each other, that it would seem a safer method to take some general physical principle as the basis of our reasoning, rather than assume certain modes of action, which, after all, may be widely different from the mechanism employed by nature; more especially if this principle include in itself, as a particular case, those before used by M. Cauchy and others, and also lead to a much more simple process of calculation. The principle selected as the basis of the reasoning contained in the following article is this: *In whatever way the elements of any material system may act upon each other, if all the internal forces exerted be multiplied by the elements of their respective directions, the total sum for any assigned portion of the mass will always be the exact differential of some function.* (Green 1839, pp. 245–246)²¹

Green's approach was juxtaposed to Poisson's desire to found "Physical mechanics", based only on molecular forces as the ultimate descriptors of the inner exchanges of action. In other words, Green completely abandoned the attempt at finding an explanation *per causas* of elasticity. We point out, however, that, whilst he assumed

²¹ The use of italics was chosen by the authors to highlight the relevant passage.

a continuous mathematical model of matter, Green did not query the existence of molecules:

Let us conceive a mass composed of an immense number of molecules acting on each other by any kind of molecular forces, but which are sensible only at insensible distances [...]. (Green 1839, p. 248)

What distinguished Green's treatment from that of the French school of mechanics was the fundamental equation he assumed to describe any motion:

[...] we get, by combining D'Alembert's principle with that of virtual velocities,

$$\Sigma Dm \left\{ \frac{d^2u}{dt^2} \delta u + \frac{d^2v}{dt^2} \delta v + \frac{d^2w}{dt^2} \delta w \right\} = \Sigma Dv \delta \phi \dots\dots (1)$$

Dm and Dv being exceedingly small corresponding elements of the mass and volume of the medium, but which nevertheless contain a very great number of molecules, and $\delta\phi$ the exact differential of some function and entirely due to the internal actions of the particles of the medium on each other. Indeed, if $\delta\phi$ were not an exact differential, a perpetual motion would be possible [...]. (Green 1839, p. 248)

Green's empirical assumption of the existence of the function ϕ , which could be a weak point, was later given a rational background and proved by William Thomson (Lord Kelvin) on thermodynamic bases (Thomson 1855).²² Green observed that the function ϕ must maintain its value when any arbitrary small volume element remains undeformed during the process, hence for small deformations ϕ must depend on the first-order quantities expressing strain. Green wrote the infinitesimal strain measures, which he denoted $s_1, s_2, s_3, \alpha, \beta, \gamma$, in the form we know them nowadays.²³ Since Green assumed small kinematical increments, it was reasonable to consider a series expansion of ϕ ,

But $s_1, s_2, s_3, \alpha, \beta, \gamma$ being very small quantities of the first order, we may expand ϕ in a very convergent series of the form

$$\phi = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \&c. :$$

$\phi_0, \phi_1, \phi_2, \&c.$ being homogeneous functions of the six quantities $\alpha, \beta, \gamma, s_1, s_2, s_3$ of the degrees 0, 1, 2, &c. each of which is very great compared with the next following one. (Green 1839, p. 249)

Since in his Eq. 1 the first-order variation of ϕ appears, it is obvious that the constant ϕ_0 is immaterial. Green remarked (Green 1839, pp. 249–250) that the first-order variation of ϕ_1 must also vanish, since it represents the first-order variation of the

²² Lord Kelvin called the function ϕ "potential" and proved its existence in isothermal, non-dissipative processes.

²³ This is the reason why, for instance, Truesdell in (Truesdell 1991) gave the name "Green—Saint-Venant" to the expression of the infinitesimal strain components.

virtual work in a neighbourhood of the equilibrium configuration, assumed to be the natural one. Thus, the first meaningful term of the function ϕ is ϕ_2 ; moreover, one may ignore higher-order terms, and

If now we can obtain the value of ϕ_2 , we shall only have to apply the general methods given in the *Mécanique Analytique*. But ϕ_2 being a homogeneous function of six quantities of the second degree, will in its most general form contain 21 arbitrary coefficients. The proper value to be assigned to each depend on the internal constitution of the medium. If, however, the medium be a non-crystallized one²⁴, the form of ϕ_2 will remain the same, whatever be the directions of the coordinate axes in space. Applying this last consideration, we shall find that the most general form of ϕ_2 for non-crystallized bodies contains only two arbitrary coefficients. (Green 1839, p. 250)

Green then expressed ϕ_2 in terms of the displacement components, and by means of standard calculus of variations, obtained the balance and boundary equations in terms of two elastic constants (Green 1839, pp. 255–256). As already remarked, these equations hold for the luminiferous aether in Green's intentions, but fit any other linear elastic isotropic body perfectly.

Green's approach was widely followed, especially in England,²⁵ and Germany²⁶ but at times it was also strongly opposed. We have already seen to what extent it was rejected by Saint-Venant, and even in England an outstanding scientist like Stokes searched for a refinement of the molecular model instead of adopting Green's energetic approach (Stokes 1845).

In 1866 a posthumous book by Auguste Bravais appeared (Bravais 1866), containing memoirs that he had read at the Academy of Sciences in Paris. These studies mainly dealt with crystallography, and with rigorous organization of crystals in groups of symmetry. They contained assumptions which were to be essential for the work of the mechanicians who wished to overcome the *empasse* related to the 'true' number of elastic constants. That Bravais' ideas shed a new light on the molecular theory of the composition of matter was quite clear to the editors of the posthumous volume, who pointed out in the Introduction that according to Bravais:

[...] the crystals are assemblages of identical molecules with the same orientation, which, reduced by thought to a unique point, i.e., their centre of gravity, are disposed in rectilinear and parallel ranges, in which the distance of two points is uniform.²⁷ (Bravais 1866, VII)

²⁴ It is apparent, from further examination of the article, that by the term "non-crystallized" Green intended "isotropic"; indeed, on p. 253 he affirmed that "[...] non-crystallized bodies, [are those] in which it is perfectly indifferent in what directions the rectangular axes are placed".

²⁵ For instance, one may consider the monograph Thomson and Tait (1867).

²⁶ For instance, one may consider Clebsch (1862), Kirchhoff (1876).

²⁷ [...] les cristaux sont des assemblages de molécules identiques entre elles et semblablement orientées, qui, réduites par la pensée à un point unique, leur centre de gravité, sont disposées en rangées rectilignes et parallèles, dans chacune desquelles la distance de deux points est constante.

In the language of a contemporary mechanician the introduction of an orientation in the internal structure of matter endowed the ultimate components of matter with a local structure, that of a rigid body:

[...] no longer considering the molecules as points but as small [rigid] bodies.²⁸ (Bravais 1866, VIII)

The words used by Bravais in his memoirs clearly indicate what his thinking was:

It is admitted today by all physicists that massive bodies are aggregations of molecules of the same kind, kept at a distance by attractive or repulsive forces the resultant of which is nil, for each of these molecules, since these bodies have achieved their state of internal equilibrium. By the term «molecules of the same kind» we subsume not only that the chemical composition shall be the same, but also that the geometrical disposition of the constituting atoms is the same around the centre of mass of each molecule. [...] In non-crystallized solid bodies, the relative disposition of the centres of the molecules does not seem to follow any fixed rule, and corresponding lines of molecular polyhedra are directed towards any direction in space with no preference. If we isolate a cylinder of such a substance to determine its characteristic constants, [...] one finds constant results, independent of the direction that the axis of the cylinder, thus separated, occupies in the interior of the mass. On the contrary, in crystallized bodies the results obtained vary in general with the direction of the axis; a kind of molecular *polarity* appears, and one is also led to admit that molecular polyhedra are similarly rotated, their corresponding lines being parallel.

The molecules of a crystal are not only similar and similarly oriented; they are, moreover, arranged in rectilinear rows, and separated from each other, in any row, by equal intervals.²⁹ (Bravais 1866, pp. 101–102)

Bravais advanced experimental evidence of his hypothesis based on cleavage specimens. He did not question the existence of ideal isotropic non-crystallized media (those modelled by Navier, Cauchy and Poisson), made up of undifferentiated spheres. He wished to stress that crystallized media are in principle composed of different mole-

²⁸ [...] cessant de regarder les molécules comme des points et le considérant comme des petits corps.

²⁹ Il est admis aujourd'hui, par tous les physiciens, que les corps pondérables sont des agrégations de molécules de même espèce, tenues à distance par des forces attractives ou répulsives dont la résultante est nulle, pour chacune de ces molécules, dès que ces corps sont parvenus à leur état d'équilibre interne. Par le terme de «molécules de même espèce» on sous-entend non-seulement que la composition chimique doit être identique, mais encore que la disposition géométrique des atomes constitutants est la même autour du centre de gravité de chaque molécule. [...] Dans les corps solides non cristallisés, l'arrangement relatif des centres des molécules ne paraît soumis à aucun règle fixe, et les lignes homologues des polyèdres moléculaires sont indistinctement dirigées vers toutes les régions de l'espace. Si l'on isole un cylindre d'une tel substance pour déterminer ses constantes spécifiques, [...] on trouve des résultats constants et indépendants de la direction qu'occupait dans l'intérieur de la masse général l'axe du cylindre ainsi séparé. Dans les corps cristallisés, au contraire, le résultats obtenus varient en général suivant la direction de cet axe; une sorte de *polarité* moléculaire se manifeste, et l'on est ainsi conduit à admettre que les polyèdres moléculaires sont semblablement tournés, leurs lignes homologues étant parallèles. Non-seulement les molécules d'un crystal sont semblablee et semblablement orientées; elles sont, de plus, disposées en files rectilignes, et séparées l'une de l'autre, sur chaque file, par des intervalles égaux.

cules with an orientation. Indeed, after having examined the geometrical symmetries of the relative disposition of the centres of gravity of molecules, Bravais went on to examine the molecular structure, that is, the geometry of the elements constituting molecules around their centres of gravity. For this purpose, he observed that:

The exterior signs that let us single out the molecular structure, if it is compound, or at least its shape, if it is simple, cannot be but the result of attractive and repulsive forces exerted by the molecule along different directions. If these forces spherically radiated from the centre, if they were the same at a given distance in all directions, if, in a word, all were similar around the centre, the molecule would act as if it were spherical, or, still better, as a simple point, a pole of forces; its intimate structure would not reveal any sensible phenomenon, and could be deduced only by theoretical considerations, for instance, the atomistic laws of chemistry.

However, this is not the case; the forces exerted by the molecules are not the same in all directions; the reactions that they are subjected to do not always pass through their centre of gravity and the effects of their inequality, when studied properly, allow us to determine their internal organisation.³⁰ (Bravais 1866, p. 194)

In the crystallization process, when a fluid slowly becomes solid, the molecules must necessarily follow the crystal symmetry group: hence a moment must be present and one shall suppose a different geometrical structure for the molecules of crystallized bodies:

The molecules of crystallized bodies will henceforth be polyhedra, the vertices of which, distributed at will around the centre of gravity, will be the centres, or poles, of the forces exerted by the molecule.³¹ (Bravais 1866, p. 196)

This view of matter would in time lead to Voigt's molecular model.

4 Voigt's molecular model

Woldemar Voigt (1850–1919) attended the university of Königsberg, where, under Franz Neumann's direction, he studied mineralogy and crystallography. In the early

³⁰ Les signes extérieurs capables de trahir au dehors la structure de la molécule, si elle est composée, ou du moins sa forme, si elle est simple, ne peuvent être que le résultat de forces attractives ou répulsives exercées par la molécule suivant différentes directions. Si ces forces allaient en s'irradiant sphériquement à partir du centre, si elles étaient les mêmes pour une distance donnée suivant toutes les directions, si en un mot tout était semblable autour de se centre, la molécule agirait comme étant sphérique, ou, mieux encore, comme un simple point, pôle de forces; sa structure propre ne se traduirait par aucun phénomène sensible, et ne pourrait se déduire que de considérations théoriques, par exemple de lois atomistiques de la chimie.

Mais les choses ne se passent point ainsi; les forces émanées de la molécule ne sont pas les mêmes dans tous les sens; les réactions qui la sollicitent ne passent pas toujours par son centre de gravité et les effets de ses inégalité permettent, lorsque'on les étudie convenablement, de pénétrer dans son organisation intime.

³¹ Les molécules des corps cristallisés seront donc pour nous dorénavant des polyèdres dont les sommets, distribués d'une manière quelconque autour du centre de gravité, seront les centres, ou pôles, des forces émanées de la molécule.

1880s Voigt published a series of fundamental contributions on crystallography and theory of elasticity, reconciling the results of the corpuscular and the continuous models of matter. The importance of his results was immediately acknowledged by his contemporaries.³² In 1902 Marcolongo provided a brief but clear account of Voigt's ideas and procedures:

[...] By supposing that the body is formed by an aggregate of corpuscles (hence that the matter constituting the body is discrete), and by supposing that each corpuscle is subjected by the others to actions reducible to a force and a couple, infinitely decreasing with distance, Voigt (1887) found the general equations of elasticity to be the same as those obtained by the theory of potential, without necessarily verifying the relations of Cauchy and Poisson.³³ (Marcolongo 1902, p. 97)

The article to which Marcolongo refers (Voigt 1887) appeared in the annals of the Society of Sciences of Göttingen, where Voigt presented his ideas systematically for the first time.

These ideas were re-considered and better expounded in a memoir (Voigt 1900) presented at the International Congress of Physics in Paris in 1900.³⁴ After the introduction, Voigt began to tackle the molecular theory:

[...] elasticity was first investigated by means of a particular conception of the mechanism of [natural] phenomena which we will call *the molecular hypothesis*. According to this hypothesis, the body is supposed to be built by distinct elementary masses, the molecules, in the wide sense of the word, being kept in their positions or displaced by mutual actions having sensible magnitude only at insensible distances. In particular, for a homogeneous undeformed crystal, its molecules are identical, oriented in the same way and regularly distributed so that each molecule is surrounded by the others in the same way. The distances at which these mutual actions are exerted are considered much larger compared to the distance of near molecules, and deformations occur so slowly in space that, even in the deformed body, the disposition of the molecules can be considered regular, in the sense specified above, within the range of the sphere of molecular activity.³⁵ (Voigt 1900, pp. 287–288)

³² See for instance Poincaré (1892), Marcolongo (1902) and Hellinger (1914).

³³ [...] il Voigt (1887) supponendo il corpo come formato da un aggregato di corpuscoli (e quindi discontinua la materia costituente il corpo): supponendo che ogni corpuscolo risenta dagli altri delle azioni riducibili ad una forza ed una coppia, decrescenti indefinitamente col crescere della distanza; ha ritrovate le equazioni generali della elasticità sotto la stessa forma ottenuta dalla teoria del potenziale, senza che siano necessariamente verificate le relazioni di Cauchy-Poisson.

³⁴ In this memoir Voigt claimed to have been the first to introduce in a previous study (W. Voigt, *Die fundamentalen physikalischen Eigenschaften der Krystalle*, Leipzig, Von Veit, 1898) the terms *tensor* and *triple tensor* to mean algebraic quantities undergoing certain laws of variation under changes of coordinates. These have a strong correspondence with the quantities nowadays called 'tensors'.

³⁵ [...] l'élasticité a été explorée d'abord à l'aide d'une conception particulière du mécanisme des phénomènes que nous appellerons l'*hypothèse moléculaire*. Dans cette hypothèse, l'édifice du corps est supposé construit de masses élémentaires distinctes, les molécules, au sens large du mot, maintenues dans leurs positions, ou déplacées par des actions mutuelles qui n'ont de grandeur sensible qu'à des distances

Voigt gave credit to the French mechanicians for founding the molecular theory of elasticity, but at the same time he emphasized that their theory, even if correct from a deductive point of view, was contradicted by numerous and sufficiently accurate experimental results:

The molecular theory, or theory of actions at a distance, proposed by Navier, Cauchy and Poisson [...] made the elastic properties of isotropic bodies, depend indeed on a single parameter, while numerous observations did not seem to agree with this result.³⁶ (Voigt 1900, p. 288)

On the other hand, Voigt remarked that a continuous theory of matter yielded results which contradicted those of Navier, Cauchy and Poisson, but which were confirmed by experimental evidence:

A new theory has for some time been generally adopted [...] that supposes that matter is continuous and that the mutual actions between near portions of matter are localized at their separation surface [...]. This theory, which we will call the *theory of immediate actions*, provides, contrary to the former, two characteristic constants for isotropic media, and all these results agree with observation.³⁷ (Voigt 1900, p. 288)

Voigt, however, was not satisfied simply with the fact that a theory matched the experimental results. He insisted on finding the causes that underpinned the fallacies of the molecular theory:

If the only aim is to obtain the elementary laws of elasticity in a form responding to reality, one can be satisfied with the theory of immediate actions, without asking *why* the molecular hypothesis leads to incorrect results. But, if one seeks a general and coherent theory of the subject [...] the search for the underlying reasons for this failure will be instructive [...].³⁸ (Voigt 1900, p. 289)

Footnote 35 continued

insensibles. Plus particulièrement, pour un cristal homogène, non déformé, on admet que ses molécules sont identiques, orientées de la même manière et distribuées régulièrement de façon que chaque molécule soit entourée par les autres de la même manière. Les distances auxquelles s'exercent ces actions mutuelles sont considérées comme très grandes par rapport à les distances des molécules voisines, et l'on convient que les déformations varient avec une lenteur telle dans l'espace que, même dans le corps déformé, la répartition des molécules puisse être considérée comme régulière, au sens indiqué plus haut, dans l'étendue de la sphère d'activité moléculaire.

³⁶ La théorie moléculaire ou des actions à distance, fondée par Navier, Cauchy et Poisson [...] fait dépendre, en effet, les propriétés élastiques des corps isotropes d'un seul paramètre, alors que de nombreuses observations semblant être en désaccord avec ce résultat.

³⁷ C'est alors que fut généralement adoptée pendant quelque temps une nouvelle théorie [...] en supposant la matière continue et les actions mutuelles entre les portions de matière voisines localisées dans leur surface de séparation [...]. Cette théorie, que nous appellerons *théorie des actions immédiates*, donne, contrairement à la précédente, deux constantes caractéristiques des milieux isotropes, et tous ses résultats se sont trouvés d'accord avec l'observation.

³⁸ Si l'on ne se propose d'autre but que d'obtenir les lois élémentaires de l'élasticité sous une forme répondant à la réalité, on se déclarera satisfait par les résultats de la théorie des actions immédiates, sans demander *pourquoi* l'hypothèse moléculaire conduit à des résultats inexacts. Mais si l'on tend vers une

According to Voigt, this failure could be traced to the fact that the traditional molecular theory was “unnecessarily specialized”:

[...] the previous molecular theory of elasticity starts from an unnecessarily specialized fundamental conception, that is, the hypothesis of central molecular actions depending only on distance [...].³⁹ (Voigt 1900, p. 289)

Experimental observations on crystals, suggested Voigt, provided a hint as to which corrections needed to be made:

[...] the regular formation of a crystal [...] cannot be understood unless an *orienting moment* acts on the particle which unites with the crystal and gives it [the particle] an orientation parallel to those already part of the building. But, according to the principle of conservation of energy, mutual actions directed along the line of the centres and depending only on the distance are incompatible with such moments. Indeed, the existence of moments requires a potential of the mutual action depending on the orientation, and this leads to forces varying with the orientation of the molecules and, in general, not coinciding with the line of centres.⁴⁰ (Voigt 1900, p. 289)

Voigt's starting point is clear: if material corpuscles have extension and orientation, it is useless to limit oneself by reducing mutual actions to forces only, and, more importantly from the constitutive point of view, the mutual actions shall in principle depend on orientation, too. This is a strong assumption, clearly departing from all the previous attempts to improve the ancient molecular theory and reconcile its results with experiments:

Poisson has already built [...] a molecular theory that tries to take these circumstances into account; but he still made some restrictive hypotheses and his results are, as a consequence, more particular than those of the theory of immediate actions and in part incompatible with experience.⁴¹ (Voigt 1900, pp. 289–290)

It is interesting to remark that Voigt pointed out that the same results of his refined molecular theory could also be obtained by means of the techniques of the continuous model of matter, that is, by the introduction of a potential of inner actions:

Footnote 38 continued

théorie générale et conséquente de la matière [...] on tiendra pour instructive la recherche des raisons qui les firent échouer [...].

³⁹ [...] la théorie moléculaire ancienne de l'élasticité part d'une conception fondamentale inutilement spécialisée, à savoir l'hypothèse d'actions moléculaires centrales et ne dépendant que de la distance [...].

⁴⁰ [...] la formation régulière d'un cristal [...] n'est compréhensible que si un *moment directeur* agit sur la particule qui va s'associer au cristal en lui donnant une orientation parallèle à celles qui font déjà partie de l'édifice. Mais, d'après le principe de la conservation de l'énergie, des actions mutuelles dirigées suivant la ligne des centres et ne dépendant que de la distance sont incompatibles avec de semblables moments. Car l'existence des moments exige un potentiel de l'action mutuelle dépendant de l'orientation, et celui-ci conduit à des forces qui varient avec l'orientation de la molécule et, en général, ne coïncident pas avec la ligne des centres.

⁴¹ Poisson a déjà édifié [...] une théorie moléculaire qui cherche à tenir compte de ces circonstances; mais il fait encore quelques hypothèses restrictives et ses résultats sont, en conséquence, plus particuliers que ceux de la théorie des actions immédiates et en partie incompatibles avec l'expérience.

[...] the method based on the theory of immediate actions leads to analogous results. [...] The work [...] is the (negative) differential of [...] the *general elastic potential* [...].⁴² (Voigt 1900, pp. 298–299)

The same assumptions were later considered again and more widely expounded by Voigt in a comprehensive monograph on crystal physics (Voigt 1910), in which he gave credit to Bravais and put forth the key points at the basis of his theory, that is, kinematics and the existence of a potential:

A molecular theory of the elastic phenomena that offers the perspective to explain all observations shall have such a general basis as *Bravais'* structural theory [...] provides. According to this, the crystal shall be thought of as composed by identical and parallel oriented bricks or elementary masses, arranged so that each of these is surrounded by the others in the same way inside the sphere of [molecular] action. There is no need to make special assumptions on the nature of these bricks; these could in some way be identified with the chemical molecules of the substance, with the only limitation being that the construction shall show the characteristic symmetry of the crystal. As a matter of fact, an assumption is only necessary regarding the forces that the elementary masses exert on each other, and about the displacements that each of these masses is subjected to.

Considering the former [the forces], we will not restrict generality by introducing any other limitations except that the mutual actions shall have potential; considering the latter [the displacements], in order to report the elastic phenomena we will allow to consider the elementary masses as rigid bodies. By this we do not say that these maintain their rigidity under all circumstances; we will simply consider their changes of configuration as not particularly noticeable in the phenomena of interest to us.

The global effect of a deformation on the system of masses of the crystal therefore consists in an alteration of the length and angle of the spatial grid of the elementary masses and in a change of the orientation of the elementary masses compared to the same [spatial grid]. We may assume that both changes have sensibly uniform values inside the range of molecular action, and also that the modified system is again sensibly homogeneous in the [above] quoted range.⁴³ (Voigt 1910, pp. 596–597)

⁴² [...] la méthode fondée sur la théorie des actions immédiates conduit à des résultats analogues. [...] Le travail [...] soit la différentielle (négative) [...] du *potentiel général élastique* [...].

⁴³ Eine molekulare Theorie der elastischen Vorgänge, die Aussicht bietet, alle Beobachtungen zu erklären, wird eine so allgemeine Grundlage verlangen, wie sie etwa der *Bravais'sche* Strukturtheorie [...] an die Hand gibt. Der Kristall ist nach ihr aus unter einander identischen und parallel orientierten Bausteinen oder Elementarmassen aufgeführt zu denken, die so angeordnet sind, daß jeder von diesen innerhalb der Wirkungssphäre in gleicher Weise von anderen umgeben ist. Über die Natur dieser Bausteine braucht man spezielle Annahmen nicht zu machen; dieselben können irgendwie aus den chemischen Molekeln der Substanz zusammengesetzt sein, mit der einzigen Beschränkung, daß das Gebilde die charakteristische Symmetrie des Kristalles aufweisen muß. Wesentlich ist aber eine Annahme über die Kräfte, welche die Elementarmassen aufeinander ausüben, und über die Bewegungen, die eine jede dieser Massen auszuführen vermag.

Voigt did not abandon a discrete description of matter (Navier, Cauchy and Poisson) but chose to adopt the powerful tools provided by the equation of virtual work (Lagrange) and by the potential theory (Green). The hypothesis that an elastic potential exists has a constitutive background and implicitly leads to symmetries in the constitutive relations.

Voigt highlighted that a molecular description shall not consider a single elementary mass but rather what in contemporary language is called a representative volume element:

Following the fundamental studies of Cauchy and Poisson, it is common to base the molecular theory of elasticity not on the consideration of a single molecule, but on those [molecules] of a volume element, which is large compared to the sphere of molecular action, and containing, as a consequence, an extremely large amount of molecules. The molecules surrounding the volume element therefore act only on the inner molecules immediately next to the surface [of the volume element] and combine with each other to give the components of pressure.⁴⁴ (Voigt 1900, p. 290)

4.1 The pressure of forces

It is remarkable how Voigt, following the tradition dating back to the French mechanicians of the first decades of 1800, adopted the term “pressure” (*pression* in French, *Druck* in German) to denote the representative of the inner actions, while at that time the scientific community had begun to use the more neutral term “stress” (*tension* in French, *Spannung* in German). It is also remarkable how Voigt, following again the school of Navier, Cauchy and Poisson, made a clear distinction between the idea of the actions exchanged amongst corpuscles (a well-known and established concept of the mechanics of body-points and rigid bodies) and that of stress, a surface density of the total force exchanged amongst the molecules of the volume element and the surrounding ones compared to the area of the surface of separation. In the memoir

Footnote 43 continued

In bezug auf ersteres wollen wir der Allgemeinheit halber keine andere Beschränkung einführen, als daß die Wechselwirkungen ein Potential haben sollen; in bezug auf letzteres wird es zur Wiedergabe der elastischen Erscheinungen erlaubt sein, die Elementarmassen wie starre Körperchen zu behandeln. Damit ist nicht ausgesagt, daß dieselben sich unter allen Umständen starr verhalten; wir wollen nur ihre Konfigurationsänderungen als bei den uns interessierenden Vorgängen nicht merklich wirksam betrachten.

Die Gesamtwirkung einer Deformation auf das Massensystem des Kristalles besteht hiernach in einer Veränderung der Längen und der Winkel des Raumgitters der Elementarmassen und in einer Änderung der Orientierung der Elementarmassen gegen dasselbe. Wir dürfen annehmen, daß beide Änderung innerhalb des sehr kleinen Bereiches molekularer Wirkung merklich konstante Größen besitzen, daß also das veränderte System in dem genannten Bereich wieder merklich homogen ist.

⁴⁴ Depuis les travaux fondamentaux de Cauchy et de Poisson, il est d'un usage général de fonder la théorie moléculaire de l'élasticité, non sur la considération d'une molécule unique, mais sur celle d'un élément de volume, grand par rapport à la sphère d'action moléculaire et contenant, par conséquent, un nombre extrêmement grand de molécules. Les molécules environnant l'élément de volume n'agissent alors que sur les molécules intérieures immédiatement voisines de la surface de séparation et se combinent pour donner les composantes de la pression.

of 1900 Voigt introduced the components of stress directly, referring to Poisson⁴⁵ to derive the stress components starting from intermolecular forces. In the more extended monograph of 1910, the primitive quantities were the kinematics of the elementary masses and the potential. The latter was a linear functional of the first-order increments of the possible displacements of the elementary masses. Intermolecular forces were then obtained by Voigt as first-order derivatives of the potential with respect to the kinematical quantities (Voigt 1910, pp. 597–598). Irrespective of these differences, stress was in both cases defined by Voigt following Poisson:

[The components of pressure] have been defined by Cauchy and Poisson as the sums of the parallel components [of molecular forces] that all the molecules (i) inside a right cylinder elevated on a surface element q are exerted on by all the molecules (e) located at the other side of q , divided by q , that is, with respect to the unit of area.⁴⁶ (Voigt 1900, p. 290)

According to Voigt's definition, the component of stress in direction S on the surface element q having normal n is provided by Eq. 27 in Voigt (1900, p. 291):

$$q S_n = \sum_i \sum_e S_{ie} \quad (1)$$

where S_{ie} is the component in direction S of the force f_{ie} . Following a notation originally introduced by Kirchhoff, Voigt adopted the symbols $X_x, X_y, X_z, Y_x, Y_y, Y_z, Z_x, Z_y, Z_z$ to indicate the nine components of stress with respect to a Cartesian set of coordinates.

Since Voigt considered elementary masses as rigid particles, he could not reduce their interaction, in general, to mutual opposite forces, but rather to a force and a couple:

The interactions occurring between two rigid bodies (h) and (k) give origin to both global [force] components X_{hk}, X_{kh}, \dots and also rotation moments

⁴⁵ In particular to Poisson (1828) and the article "Mémoire sur l'équilibre et le mouvement des corps cristallisés" (1839), *Mémoires de l'Académie des Sciences de l'Institut de France*, vol. 18, pp. 3–152, 1842.

⁴⁶ [Les composantes de la pression] ont été définies par Cauchy et Poisson comme les sommes de composantes parallèles que toutes les molécules (i) à l'intérieur d'un cylindre droit élevé sur un élément de surface q éprouvent de la part de toutes les molécules (e) situées de l'autre côté de q , divisées par q , c'est-à-dire rapportées à l'unité de surface.

This is the definition of "pression" according to Poisson:

[...] soit M un point situé dans l'intérieur du corps, à une distance sensible de la surface. Par ce point menons un plan qui partage le corps en deux parties, et que nous supposons horizontal [...]. Appelons A la partie supérieure et A' la partie inférieure, dans laquelle nous comprendrons les points matériels appartenant au plan même. Du point M comme centre, décrivons une sphère qui comprenne un très-grand nombre de molécules, mais dont le rayon soit ce-pendant insensible par rapport au rayon d'activité des forces moléculaires. Soit ω l'aire de sa section horizontale; sur cette section élevons dans A un cylindre vertical, dont la hauteur soit au moins égale au rayon d'activité des molécules; appelons B ce cylindre: l'action des molécules de A' sur celles de B , divisée par ω , sera la *pression* exercée par A' sur A , rapportée à l'unité de surface et relative au point M (Poisson 1831, p. 29).

L_{hk}, L_{kh}, \dots , which we will calculate around the centre of gravity of the considered body. We assume analogous actions occurring between two elementary masses of our crystal model.⁴⁷ (Voigt 1910, pp. 597–598)

Since in principle there is no reason to distinguish between interactions represented by a force and others represented by a couple, Voigt remarked that all considerations on the interactions of the kind ‘force’ must hold also for the interaction of the kind ‘couple’. Hence, intermolecular torques depend on the distance between the centres of gravity of elementary masses, there is a sphere of molecular action for the couple, and so on:

[...] we have considered only the pressure of forces acting between molecules, but the moments, or *couples*, acting between them may be considered in the same way.⁴⁸ (Voigt 1900, p. 293)

Voigt suggested that, in analogy with the stress induced by forces, he would define the stress induced by moments (“pressions de moments”) according to the relation expressed by his Eq. 32 in Voigt (1900, p. 293)

$$q D_n = \sum_i \sum_e D_{ie} \quad (2)$$

where D_{ie} is the component in direction D of the moment that the molecule i inside the cylinder is exerted upon by the molecule e outside the cylinder. Voigt commented that

[...] one has the nine special components of the stress-moments

$$L_x, L_y, L_z, \quad M_x, M_y, M_z, \quad N_x, N_y, N_z,$$

which exactly correspond to the X_x, \dots, Z_z .⁴⁹ (Voigt 1900, p. 293)

4.2 The vanishing of the pressure of moments

In order to provide an expression for the stress-moments, Voigt needed to calculate the moments exchanged between elementary masses and to establish the basic laws for these mechanical quantities. Hence, Voigt build a potential of molecular interactions

⁴⁷ Die zwischen zwei starren Körper (h) und (k) stattfindenden Wechselwirkungen geben Veranlassung sowohl zu Gesamtkomponenten X_{hk}, X_{kh}, \dots als auch zu Drehungsmomenten L_{hk}, L_{kh}, \dots , die wir je um den Schwerpunkt des betreffenden Körpers rechnen wollen. Analoge Wirkungen nehmen wir auch zwischen zwei Elementarmassen unseres Kristallmodelles als stattfindend an.

⁴⁸ [...] nous avons considéré exclusivement les pressions des forces agissant entre les molécules, mais il est que les moments, ou les *couples*, qui agissent entre elles peuvent être traités de la même manière.

⁴⁹ [...] on a les neuf pressions de moments particulières

$$L_x, L_y, L_z, \quad M_x, M_y, M_z, \quad N_x, N_y, N_z,$$

qui correspondent exactement aux X_x, \dots, Z_z .

made up of sums of products of forces and first-order displacements as well as of moments and first-order rotations:

If we denote by Φ_{hk} the potential of mutual actions, [...] for any possible [virtual] displacement of both elementary masses it must be equal to (Voigt 1910, 598)⁵⁰

$$-\delta\Phi_{hk} = X_{hk}\delta x_h + \dots + L_{hk}\delta l'_h + \dots + X_{kh}\delta x_k + \dots + L_{kh}\delta l'_k + \dots \quad (3)$$

where δx_h , δx_k and the like for y , z are the variations of the Cartesian coordinates of the centres of gravity of the elementary masses (h), (k), respectively; $\delta l'_h$, $\delta l'_k$ and the like for m , n are variations of the rotations, with respect to a fixed system of Cartesian axes, of the elementary masses (h), (k), respectively. By means of the equation of virtual displacements, Voigt obtained the balance equations, expressing the 'law of action and reaction' for forces and moments between elementary masses:

Relations exist among the [force] components and the moments, which derive from the fact that the potential must be a function of the relative position of the interacting masses, and so must remain unaltered if one dislocates both elementary masses in a rigid link, i.e., [if one introduces] for the variations

$$\delta x_h, \dots \delta x_k, \dots \delta l'_h, \dots \delta l'_k, \dots$$

the values that express such a dislocation.

Such values are easy to build [...] and lead to the following six general relations⁵¹ (Voigt 1910, p. 598):

$$\begin{aligned} X_{hk} + X_{kh} &= 0, \dots \\ L_{hk} + L_{kh} + Z_{hk}(y_h - y_k) - Y_{hk}(z_h - z_k) &= 0, \dots \end{aligned} \quad (4)$$

⁵⁰ Wird das Potential der Wechselwirkungen mit Φ_{hk} bezeichnet, so muß [...] bei jeder zulässigen Dislokation beider Elementarmassen (h) und (k) gelten

$$-\delta\Phi_{hk} = X_{hk}\delta x_h + \dots + L_{hk}\delta l'_h + \dots + X_{kh}\delta x_k + \dots + L_{kh}\delta l'_k + \dots \quad (71)$$

⁵¹ Zwischen den Komponenten und den Momenten bestehen Relationen, die daraus fließen, daß das Potential eine Funktion nur der relativen Lage der aufeinander wirkenden Massen sein kann, also ungeändert bleiben muß, wenn man beide Elementarmassen in starrer Verbindung miteinander disloziert, d. h., für die Variationen,

$$\delta x_h, \dots \delta x_k, \dots \delta l'_h, \dots \delta l'_k, \dots$$

Werte einführt, die solche Dislokation ausdrücken.

Derartige Werte sind [...] leicht zu bilden und ergeben folgende sechs allgemeine Relationen:

$$\begin{aligned} X_{hk} + X_{kh} &= 0, \dots \\ L_{hk} + L_{kh} + Z_{hk}(y_h - y_k) - Y_{hk}(z_h - z_k) &= 0, \dots \end{aligned} \quad (72)$$

The reduced expression for the potential of molecular interactions thus is expressed by Eq. 74 in Voigt (1910, p. 598):

$$\begin{aligned}
 -\delta\Phi_{hk} = & X_{hk} \left(\delta x_{hk} + y_{hk} \delta \frac{n'_h + n'_k}{2} - z_{hk} \delta \frac{m'_h + m'_k}{2} \right) \\
 & + \dots + (L_{hk} - L_{kh}) \delta \frac{l'_h - l'_k}{2} \dots
 \end{aligned} \quad (5)$$

where $x_{hk} = x_h - x_k$ and the like (Voigt did not actually write down the passages which lead from his Eqs. 71 and 72—ours (3) and (4)—to his Eq. 74—ours (5)).

At this point Voigt made a basic assumption on the kinematics of his model:

If we now introduce the above quoted assumption, that when dealing with real displacements inside the range of molecular action the elementary masses rotate by sensibly equal angles, because of $l'_h - l'_k = \dots = 0$ we obtain for the same [expression of the potential]

$$-d\Phi_{hk} = X_{hk} (dx_{hk} + y_{hk} dn' - z_{hk} dm') + \dots \quad (6)$$

where the now common components of the rotation dl' , dm' , dn' are introduced without subscript.⁵² (Voigt 1910, p. 599)

That is, Voigt considered sensibly uniform values of the rotation of the molecules inside the molecular sphere of action; therefore, the potential of molecular interactions (5) was further reduced to the expression (6), depending only on intermolecular forces. This is a strong deduction: in modern language, we may say that an inner constraint is imposed and as a consequence this implies that constitutive prescriptions shall be produced for intermolecular forces only.

Voigt added some considerations with the intention to explain the structure of the L_{hk} , ... better and further prove that the only meaningful stresses are those of force. Indeed, both in the report of 1900 and in the monograph of 1910, Voigt claimed that symmetry considerations yield the equivalence of the moment exerted by the elementary mass (h) on the elementary mass (k) and vice-versa:

[...] because of the parallel orientation of (h) and (k), from symmetry [it follows that] $L_{hk} = L_{kh}, \dots$ ⁵³ (Voigt 1910, p. 599)

⁵² Indem wir nun die oben signalisierte Annahme einführen, daß innerhalb der Weite molekularer Wirkung die Elementarmassen sich bei den wirklichen Veränderungen um merklich gleiche Winkel drehen, gelangen wir für dergleichen wegen $l'_h - l'_k = \dots = 0$ zu

$$-d\Phi_{hk} = X_{hk} (dx_{hk} + y_{hk} dn' - z_{hk} dm') + \dots \quad (75)$$

Dabei sind die jetzt gemeinsamen Drehungskomponenten dl' , dm' , dn' ohne Indizes geführt.

⁵³ [...] da bei parallelen Orientierungen von (h) und (k) nach Symmetrie $L_{hk} = L_{kh}, \dots$

If one considers that $L_{hk} = L_{kh}$ and the like, the balance equations for the moments, Eq. 3-2, let Voigt obtain his Eq. 76 in Voigt (1910, p. 599)

$$2L_{hk} = Y_{hk}z_{hk} - Z_{hk}y_{hk} \dots \quad (7)$$

showing that mutual couples are actually moments of intermolecular forces, and the constitutive characterization must be provided only for the latter, whilst the former are obtained by balance equations as is standard today when one introduces an inner constraint.

A fundamental consequence of this is that the pressure of moments, or stress-moment, vanishes. Indeed, let us introduce the expressions (7) into the definition (2) and let $D_{ie} = L_{ie}, M_{ie}, N_{ie}$. Voigt observed that in (2)–(7) only the molecules inside the sphere of molecular action should be taken into account. Thus, since this sphere is infinitesimal and the lever arms x_{ie}, y_{ie}, z_{ie} are smaller than the radius of molecular activity, D_n , and hence L_n, M_n, N_n are infinitesimals of higher order compared to S_n and are hence negligible. More precisely,

On the one hand, one obtains a comparison between the relative orders of magnitude of the X_n, \dots and of the L_n, \dots . Because of (76) [our equation (7)] one can indeed write

$$L_n = \frac{1}{2f} \mathbf{S}_i \mathbf{S}_a (z_{ia}Y_{ia} - y_{ia}Z_{ia}), \dots \quad (8)$$

Now, since the relative coordinates x_{ia}, y_{ia}, z_{ia} must be smaller than the range of action of the molecular forces, the sums expressing the L_n, \dots are in general vanishingly small in comparison with those [sums] which express the X_n, \dots .⁵⁴ (Voigt 1910, p. 602)

In the quotation above, the symbol f (*Flächenelement* in German) has the same role as q in our Eqs. 1 and 2; the subscript a (*außer* in German) has the same role as e in the same equations; the symbol \mathbf{S} stands for a sum.

The use of two different symmetry considerations (the first for the rotations of the elementary masses inside the molecular sphere of action, the second for the intermolecular couples) does not seem quite clear. From the contemporary point of view one could instead probably make the following constitutive reasoning: since Voigt had assumed the rotation to be uniform inside the molecular sphere of action, the relative rotation between two molecules vanishes. The constitutive function for the

⁵⁴ Einerseits gewinnt man daraus eine Beurteilung der relativen Größenordnungen der X_n, \dots und der L_n, \dots . Nach (76) kann man nämlich schreiben

$$L_n = \frac{1}{2f} \mathbf{S}_i \mathbf{S}_a (z_{ia}Y_{ia} - y_{ia}Z_{ia}), \dots \quad (86)$$

Da nun die relativen Koordinaten x_{ia}, y_{ia}, z_{ia} kleiner als die Wirkungsweite der Molekularkräfte sein müssen, so sind die Summen, welche die L_n, \dots ausdrücken, im allgemeinen verschwindend klein gegen diejenigen, welche die X_n, \dots bestimmen.

intermolecular couples depends in general on the relative rotation between molecules. Since the latter vanishes, the intermolecular couples cannot depend on it and cannot be different.

After these premises, Voigt obtained the expressions for pressure in terms of sums of forces.⁵⁵ Owing to the supposed homogeneity of the distribution of the molecules, the number of molecules interacting in pairs from both sides of the elementary surface f is $\zeta f n_{ia}$, provided ζ is the number of molecules in the volume unit and n_{ia} is the projection along n of the distance between the centres of gravity of two corresponding molecules on opposite sides of f (which cannot be greater than the radius of molecular action) (Voigt 1910, p. 603).⁵⁶

Moreover, again because of homogeneity, Voigt stated that it was immaterial to exchange the role of the molecules inside the cylinder with those outside. Hence, one could in fact consider only one half of the total number of pairs of interacting molecules, and the subscripts denoting inner and outer molecules could be omitted. By considering n parallel to the coordinate axes, it assumes the values x , y , z of the relative coordinates of the centres of gravity of the corresponding molecules with respect to the assumed coordinate frame. Eventually, the components of stress are provided by Eq. 89 in Voigt (1910, p. 604)

$$X_x = \frac{1}{2} \zeta \mathbf{S}_x X, \quad Y_x = \frac{1}{2} \zeta \mathbf{S}_x Y, \quad Z_x = \frac{1}{2} \zeta \mathbf{S}_x Z, \quad (9)$$

and the like for the other components.

Owing to these fundamental results, Voigt had to provide constitutive relations only for ordinary stresses, thus recovering one of the basic assumptions of the French mechanicians of the first decades of 1800. Voigt derived the constitutive relations by means of standard techniques.

4.3 Constitutive laws—first approach

In the memoir of 1900, Voigt adopted what a reader of our times would interpret as a procedure of linearization of the equations of interest (that is, Voigt considered only infinitesimal displacements). He did not actually show the passages of his procedure, but referred to his previous article (Voigt 1887). Roughly speaking, he assumed the expressions (9) as given, then he expressed x , y , z as well as ζ in terms of the first-order variations of the displacement components starting from a natural state. Voigt then performed the symbolic sums and products indicated in (9) and retained only linear terms. Thus, each component of stress turned out to depend, in principle, on the first-order increments of displacement and rotation:

[...] admitting that the body is deformed, starting from a state in which all pressures are zero, by continuous and infinitely small displacements with com-

⁵⁵ In Voigt (1900) he limited himself to quote the same result as obtained by Poisson (1831).

⁵⁶ The surface element was called q in Voigt (1900), f in Voigt (1910).

ponents u , v , w , and by continuous and infinitely small molecular rotations with components l , m , n , one may develop the sums appearing in those formulæ.

The result of the calculation is a system of nine equations of the kind

$$\begin{aligned} -X_x = & C_{11} \frac{\partial u}{\partial x} + C_{12} \left(\frac{\partial u}{\partial y} + n \right) + C_{13} \left(\frac{\partial u}{\partial z} - m \right) \\ & + C_{14} \left(\frac{\partial v}{\partial x} - n \right) + C_{15} \frac{\partial v}{\partial y} + C_{16} \left(\frac{\partial v}{\partial z} + l \right) \\ & + C_{17} \left(\frac{\partial w}{\partial x} + m \right) + C_{18} \left(\frac{\partial w}{\partial y} - l \right) + C_{19} \frac{\partial w}{\partial z}, \end{aligned} \quad (10)$$

in which the C_{hk} are characteristic parameters of the body (in the undeformed state) and of the orientation of the coordinate axes, which are determined, on one hand, by the law of elementary actions and, on the other hand, by the distribution of the molecules in the space; the relations $C_{hk} = C_{kh}$ take place among them. The number of independent constants C_{hk} is, as a consequence, 45.⁵⁷ (Voigt 1900, p. 296)

Voigt did not clarify the reasons why the symmetry relation $C_{hk} = C_{kh}$ hold; one might infer from the context and from a comparison with (Voigt 1887) that this derives from the existence of an elastic potential.

Voigt claimed that the local balance equations for force and torque he had written in Voigt (1900, p. 295) were well known. At this point, he advanced the hypothesis that no couple densities were distributed inside the body. Therefore, the standard symmetry relations for six of the components of the stress held, and these represented a set of linear relations which permitted the elimination of the components of the rotation l , m , n from the constitutive relations (10). Thus, only six independent components of stress remained, depending on six arguments which consist only in the partial derivatives of the displacement components:

⁵⁷ [...] en admettent que le corps est déformé, à partir d'un état dans lequel toutes les pressions sont nulles par des déplacements infiniment petits et continus dont les composantes sont u , v , w , et des rotations moléculaires infiniment petites et continues dont les composantes sont l , m , n , on peut développer les sommes qui figurent dans ces formules.

Le résultat du calcul est un système de neuf équations de la forme

$$\begin{aligned} -X_x = & C_{11} \frac{\partial u}{\partial x} + C_{12} \left(\frac{\partial u}{\partial y} + n \right) + C_{13} \left(\frac{\partial u}{\partial z} - m \right) \\ & + C_{14} \left(\frac{\partial v}{\partial x} - n \right) + C_{15} \frac{\partial v}{\partial y} + C_{16} \left(\frac{\partial v}{\partial z} + l \right) \\ & + C_{17} \left(\frac{\partial w}{\partial x} + m \right) + C_{18} \left(\frac{\partial w}{\partial y} - l \right) + C_{19} \frac{\partial w}{\partial z}, \end{aligned} \quad (39)$$

dans lesquelles les C_{hk} sont des paramètres caractéristiques du corps (à l'état non déformé) et de l'orientation des axes de coordonnées, qui sont déterminés, d'une part, par la loi des actions élémentaires et, de l'autre, par la distribution des molécules dans l'espace et entre lesquels les relations $C_{hk} = C_{kh}$ ont lieu. Le nombre des constantes C_{hk} indépendantes est, par suite, 45.

If there are no moments in the volume [...] one has [...]

$$Y_z = Z_y, \quad Z_x = Y_z, \quad X_y = Y_x,$$

one can [...] eliminate the components of the rotation l , m , n in the formulæ for the components of pressure.

The result of this operation consists in six expressions for the components of pressure

$$X_x, Y_y, Z_z, \quad Y_z = Z_y, \quad Z_x = Y_z, \quad X_y = Y_x$$

which are linear and homogeneous with respect to the six arguments.⁵⁸ (Voigt 1900, p. 297)

$$\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial y}, \quad \frac{\partial w}{\partial z}, \quad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (11)$$

Voigt arranged both the six components of stress and those of strain so as to form a list:

We will pose, following Kirchhoff,

$$\begin{aligned} \frac{\partial u}{\partial x} &= x_x, \quad \frac{\partial v}{\partial y} = y_y, \quad \frac{\partial w}{\partial z} = z_z, \quad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = y_z = z_y, \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} &= z_x = x_z, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = x_y = y_x; \end{aligned} \quad (12)$$

[...] These six arguments, x_x , ..., x_y , will always be arranged in the same order in the following, corresponding to that of the components of pressure.

The result of the above indicated elimination [that of the components of the rotation from the variables on which the components of stress depend] are thus six equations of the kind

⁵⁸ S'il n'y a pas de moments dans le volume [...] on a [...]

$$Y_z = Z_y, \quad Z_x = Y_z, \quad X_y = Y_x,$$

on peut [...] éliminer les composantes de la rotation l , m , n dans les formules des composantes de la pression.

Le résultat de cette opération consiste en six expressions pour les composantes de la pression

$$X_x, Y_y, Z_z, \quad Y_z = Z_y, \quad Z_x = Y_z, \quad X_y = Y_x$$

qui sont linéaires et homogènes par rapport aux six arguments

$$\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial y}, \quad \frac{\partial w}{\partial z}, \quad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

$$-X_x = c_{11}x_x + c_{12}y_y + c_{13}z_z + c_{14}y_z + c_{15}z_x + c_{16}x_y, \quad (13)$$

where c_{hk} are parameters characteristic of the substance of the body and of the orientation of the coordinate system. Among them one has the relations

$$c_{hk} = c_{kh}; \quad (14)$$

the number of independent parameters c_{hk} is thus 21.⁵⁹ (Voigt 1900, pp. 297–298)

Voigt obtained the linear elastic constitutive equations in matrix form, since both stress and strain were written as vector-like lists of six independent components.⁶⁰

Hence, Voigt proved that the constitutive moduli were in general 36 in number, and because of the symmetry of the tangential components of stress this number was reduced to 21.⁶¹ In the memoir of 1900, Voigt proved this result also by means of thermodynamics. Like Green, he showed that the elastic potential, being a homogeneous second-order polynomial of the six strain independent components, is completely represented by means of 21 coefficients (Voigt 1900, pp. 298–300).

Voigt remarked how his refined molecular approach, built improving the model by Navier, Cauchy and Poisson, provided results in complete agreement with those provided by assuming a continuous model and the existence of an elastic potential, as Green did:

But the theory of actions at a distance lets us bypass the results of the theory of immediate actions from an important point of view. Since, according to this [the theory of actions at a distance], the parameters C_{hk} , and as a consequence c_{hk} ,

⁵⁹ Nous poserons, avec Kirchhoff,

$$\begin{aligned} \frac{\partial u}{\partial x} = x_x, \quad \frac{\partial v}{\partial y} = y_y, \quad \frac{\partial w}{\partial z} = z_z, \quad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = y_z = z_y, \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = z_x = x_z, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = x_y = y_x; \end{aligned} \quad (40)$$

[...] Ces six arguments, x_x, \dots, x_y , seront dans la suite constamment rangés dans cet ordre, correspondant à celui des composantes des pressions.

Le résultat de l'élimination susindiquée est alors six équations de la forme

$$-X_x = c_{11}x_x + c_{12}y_y + c_{13}z_z + c_{14}y_z + c_{15}z_x + c_{16}x_y, \quad (41)$$

où c_{hk} sont des paramètres caractéristiques de la substance du corps et de l'orientation du système de coordonnées. On a entre eux les relations

$$c_{hk} = c_{kh}; \quad (42)$$

le nombre des paramètres c_{hk} indépendants est donc 21.

⁶⁰ Such a representation is still attributed to Voigt in many textbooks on strength of materials.

⁶¹ We point out that there are *two* different symmetry reasonings: the first is due to the fact that an elastic potential exists, the second holds only when there are no distributed couples and hence the stress is symmetric. In modern language, the first assumption implies the so-called main symmetry of the elasticity tensor $E_{ijhk} = E_{hki j}$, whereas the second assumption holds then the so-called secondary symmetry exists, $E_{ijhk} = E_{jihk} = E_{ijkh}$.

may be calculated based on the law of molecular actions and the arrangement of molecules in space, one can, by means of particular hypotheses on such a law and such an arrangement, establish particular relations among the parameters of elasticity and compare the results with experience. *Observations may also be useful for examining certain particular hypotheses and, as a consequence, for developing our ideas on molecular actions.*

The most important specialization of the general hypotheses on molecular actions has already been presented; it is the hypothesis of *forces coinciding with the straight line joining the centres and function of the distances only*. It leads, as Poisson and Cauchy showed [...], without making any particular assumption on the arrangement of the molecules, to the following six relations among the parameters c_{hk} [...] which we will call [...] of *Poisson and Cauchy*.

[...] the observations are, in general, in contradiction with the relations of *Poisson and Cauchy* [...] and, as a consequence, with the concepts at their basis; one cannot therefore refuse to make molecular actions depend on the relative orientation of the molecules. Yet, it seems possible that specializations of other kinds lead to relations which are confirmed by experience, at least in certain cases.⁶² (Voigt 1900, pp. 300–302)

Voigt remarked that if he assumed intermolecular forces to be central he was also able to recover the conditions of Cauchy and Poisson, reducing the number of independent elastic coefficients from 21 to 15. By introducing the further specializations of orthotropy and complete isotropy, he was finally able to prove that the number of independent elastic coefficients is 2 for the isotropic case (1 assuming that forces are central).

Voigt replaced the traditional definition of isotropy (that is, equal arrangement of molecules in all directions of space) with that of quasi-isotropy; this overcame the paradoxes inside the old definition of isotropy provided by Cauchy and Poisson:

All metals, almost all compact rocks seem at first to be bunches of crystals of varying extension according to circumstances, juxtaposed in all possible orientations. [...] One may then admit that a structure of this kind, called *quasi-isotropic* by the author, is the rule in nature. [...] The elastic proprieties of isotropic bodies

⁶² Mais la théorie des actions à distance permet de dépasser à un point de vue important ces résultats de la théorie des actions immédiates. Puisque, dans celle-là, les paramètres C_{hk} , et par suite aussi c_{hk} , peuvent être calculés en partant de la loi des actions moléculaires et de la répartition des molécules dans l'espace, on peut, par des hypothèses particulières sur cette loi et cette répartition, établir des relations particulières entre les paramètres élastiques et comparer les résultats avec l'expérience. *Les observations peuvent ainsi servir à l'examen de certaines hypothèses particulières et, par suite, au développement de nos idées sur les actions moléculaires.*

La plus importante des spécialisations des hypothèses générales sur les actions moléculaires a déjà été indiquée; c'est l'hypothèse des *forces coïncidant avec la droite joignant les molécules et fonctions des distances seulement*. Elle conduit, comme Poisson et Cauchy l'ont montré [...], sans faire aucune hypothèse particulière sur la disposition des molécules, aux six relations suivantes entre les paramètres c_{hk} [...] que nous appellerons [...] de *Poisson et Cauchy*.

[...] les observations sont, en général, en contradiction avec les relations de *Poisson et Cauchy* [...] et, par suite, avec les conceptions qui sont à leur base; on ne peut donc renoncer à faire dépendre les actions mutuelles de l'orientation relative des molécules. Mais il semble possible que des spécialisations d'autre espèce conduisent à des relations qui soient confirmées par l'expérience, au moins dans certains cas.

cannot be explained by means of the molecular hypothesis if one does not admit that bodies are composed of fragments of crystals.⁶³ (Voigt 1900, pp. 339, 343)

4.4 Constitutive laws—second approach

Analogous results were revealed by Voigt in a more extended form in the monograph of 1910, where he adopted a different approach and obtained a more precise identification of the number of independent elastic coefficients.

Voigt remarked that if the coordinate axes solidal with the two molecules (h) and (k), denoted A, B, C , rotate by a very small angle of components l', m', n' with respect to the fixed coordinate system of axes, denoted X, Y, Z ,⁶⁴ his relations (77) in Voigt (1910, p. 599), characteristic of a rigid infinitesimal motion, hold

$$\begin{aligned} a &= x + yn' - zm', \\ b &= y + zl' - xn', \\ c &= z + xm' - yl', \end{aligned} \quad (15)$$

Therefore, in the reduced expression (6) for the potential of inner actions the terms multiplying the X_{hk} , ... coincide with the first-order increment of the relative quantities a, b, c in Eqs. 15 as related to the molecules (h) and (k). Hence, the reduced expression for the potential can assume the simple form expressed by his Eqs. 82, 83 in Voigt (1910, p. 600)

$$-d\Phi_{hk} = A_{hk} da_{hk} + B_{hk} db_{hk} + C_{hk} dc_{hk}, \quad (16)$$

where A_{hk} and the like denote the intermolecular actions as referred to the coordinate axes A, B, C .⁶⁵ Voigt then operated a linearization procedure:

The components [of force] A, B, C are [...] functions of the relative coordinates of the elementary masses on which the mutual actions act; because of the deformation their variations therefore consist only in the modification of these quantities. Let us denote a, b, c the coordinates of a mass (a) relative to a mass (i) with respect to the axes A, B, C , thus their variations are denoted $\Delta a, \Delta b, \Delta c$. Due to the smallness of these variations one may pose

$$A = A^0 + \left(\frac{\partial A}{\partial a}\right)^0 \Delta a + \left(\frac{\partial A}{\partial b}\right)^0 \Delta b + \left(\frac{\partial A}{\partial c}\right)^0 \Delta c, \dots, \quad (17)$$

⁶³ Tous les métaux, presque toutes les roches compactes se présentent immédiatement comme des accumulations de cristaux de grandeur variant avec les circonstances, qui sont juxtaposés dans toute les orientations possibles [...]. On peut donc admettre qu'une structure de cette espèce, appelée *quasi-isotrope* par le rapporteur, est la règle dans la nature. [...] Les propriétés élastiques des corps isotropes ne peuvent être expliquées par l'hypothèse moléculaire que si l'on admet que ces corps sont composés de fragments de cristaux.

⁶⁴ It is indeed remarkable that Voigt used a notation which induces confusion in the reader, since X is both a coordinate axis and a component of intermolecular force.

⁶⁵ We point out, as already noted above, the possible confusion induced by Voigt, who denoted both coordinate axes and force components by the same symbol.

where the superscripts characterize the function in the original, undeformed state.⁶⁶ (Voigt 1910, p. 605)

Since the axes A, B, C were supposed as slightly shifted and rotated with respect to the fixed axes X, Y, Z , Voigt easily found the laws relating the $\Delta a, \dots$ to the $\Delta x, \dots$ and to the components of the small rotation l', m', n' , since they are those of a rigid motion. He thus obtained from his Eq. 77 (our Eq. 15) the expressions provided by the unnumbered equation in Voigt (1910, p. 605)

$$\Delta a = \Delta x + n'y^0 - m'z^0 = \Delta x + n'b^0 - m'c^0, \dots \quad (18)$$

but, since he supposed the displacements to be small, he also obtained his Eq. 94 in Voigt (1910, p. 605)

$$\Delta x = \frac{\partial u}{\partial x}x^0 + \frac{\partial u}{\partial y}y^0 + \frac{\partial u}{\partial z}z^0 = \frac{\partial u}{\partial a}a^0 + \frac{\partial u}{\partial b}b^0 + \frac{\partial u}{\partial c}c^0, \dots \quad (19)$$

and, by combining Eqs. 18 and 19,⁶⁷

$$\Delta a = \frac{\partial u}{\partial x}a^0 + \left(\frac{\partial u}{\partial y} + n'\right)b^0 + \left(\frac{\partial u}{\partial z} - m'\right)c^0, \dots \quad (20)$$

$$x = \left(1 + \frac{\partial u}{\partial x}\right)a^0 + \frac{\partial u}{\partial y}b^0 + \frac{\partial u}{\partial z}c^0, \dots \quad (21)$$

and expressed in his Eq. 97 in Voigt (1910, p. 605) the number of molecules in the unit volume in terms of the deformation:

$$\zeta = \zeta^0 \left(1 - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right). \quad (22)$$

By inserting Eqs. 20–22 into Eq. 16, performing all the lengthy operations and retaining only linear terms, Voigt obtained the desired constitutive relations, assuming that the reference configuration is a natural, stress-free state:

By means of these [relations, i.e. our equations (20)–(22)] all the quantities [...providing] X_x, Y_x, Z_x are expressed through the partial derviatives

⁶⁶ Die Komponenten A, B, C sind [...] Funktionen der relativen Koordinaten der Elementarmassen, auf deren Wechselwirkungen sie sich beziehen; ihre Änderungen infolge der Deformation beruht also ausschließlich auf der Veränderung dieser Größen. Die relativen Koordinaten der Massen (a) gegen eine (i) nach den Achsen A, B, C sind jetzt kurz a, b, c zu nennen, ihre Änderungen mögen demgemäß mit $\Delta a, \Delta b, \Delta c$ bezeichnet werden. Wegen der Kleinheit dieser Änderungen kann man setzen

$$A = A^0 + \left(\frac{\partial A}{\partial a}\right)^0 \Delta a + \left(\frac{\partial A}{\partial b}\right)^0 \Delta b + \left(\frac{\partial A}{\partial c}\right)^0 \Delta c, \dots, \quad (93)$$

wobei die oberen Indizes die auf der ursprünglichen, undeformierten Zustand bezogenen Funktionen charakterisieren.

⁶⁷ These correspond to Eqs. 95, 96 in Voigt (1910, p. 605).

of u, v, w with respect to the coordinates, through l', m', n' and through parameters that relate to the original state. The obtained values are then inserted and all multiplications are performed [retaining] only the members until the first order with respect to the variations [induced] by the deformation.

The result of this operation is very complicated. It becomes significantly simpler if one remembers that under our assumption the deformation takes place starting from the natural, stress-free state, so that for vanishing u, v, w, l', m', n' the components of pressure must also vanish. [...]

The thus obtained result, which one may complete through the use of the expressions for $X_y, Y_y, Z_y, X_z, Y_z, Z_z$, yields all the components of pressure which are linear in the nine arguments

$$\begin{aligned} \frac{\partial u}{\partial x} = x_x, \quad \frac{\partial u}{\partial y} + n' = x'_y, \quad \frac{\partial u}{\partial z} - m' = x'_z, \\ \frac{\partial v}{\partial x} - n' = y'_z, \quad \frac{\partial v}{\partial y} = y_y, \quad \frac{\partial v}{\partial z} + l' = z'_z, \\ \frac{\partial w}{\partial x} + m' = z'_x, \quad \frac{\partial w}{\partial y} - l' = z'_y, \quad \frac{\partial w}{\partial z} = z_z. \end{aligned} \tag{23}$$

Here the x'_y, \dots are new quantities, of which y'_z is not equal to z'_y , in the same way as Y_z is not equal to Z_y .

The coefficients of these expressions are sums of the kind

$$\frac{1}{2} \zeta^0 \mathbf{S} a^0 c^0 \left(\frac{\partial^2 \Phi}{\partial b \partial c} \right)^0 = C_{23}^{13}, \tag{24}$$

which will be shortened in the suggested [following] way; the superscripts of the C stand for the coordinates present in the numerator, the subscripts for those in the denominator.

The coefficient system in the nine components of pressures assumes the following global expression:

	x_x	x'_y	x'_z	y'_x	y_y	y'_z	z'_x	z'_y	z_z
$-X_x$	C_{11}^{11}	C_{11}^{12}	C_{11}^{13}	C_{12}^{11}	C_{12}^{12}	C_{12}^{13}	C_{13}^{11}	C_{13}^{12}	C_{13}^{13}
$-X_y$	C_{11}^{21}	C_{11}^{22}	C_{11}^{23}	C_{12}^{21}	C_{12}^{22}	C_{12}^{23}	C_{13}^{21}	C_{13}^{22}	C_{13}^{23}
$-X_z$	C_{11}^{31}	C_{11}^{32}	C_{11}^{33}	C_{12}^{31}	C_{12}^{32}	C_{12}^{33}	C_{13}^{31}	C_{13}^{32}	C_{13}^{33}
$-Y_x$	C_{21}^{11}	C_{21}^{12}	C_{21}^{13}	C_{22}^{11}	C_{22}^{12}	C_{22}^{13}	C_{23}^{11}	C_{23}^{12}	C_{23}^{13}
$-Y_y$	C_{21}^{21}	C_{21}^{22}	C_{21}^{23}	C_{22}^{21}	C_{22}^{22}	C_{22}^{23}	C_{23}^{21}	C_{23}^{22}	C_{23}^{23}
$-Y_z$	C_{21}^{31}	C_{21}^{32}	C_{21}^{33}	C_{22}^{31}	C_{22}^{32}	C_{22}^{33}	C_{23}^{31}	C_{23}^{32}	C_{23}^{33}
$-Z_x$	C_{31}^{11}	C_{31}^{12}	C_{31}^{13}	C_{32}^{11}	C_{32}^{12}	C_{32}^{13}	C_{33}^{11}	C_{33}^{12}	C_{33}^{13}
$-Z_y$	C_{31}^{21}	C_{31}^{22}	C_{31}^{23}	C_{32}^{21}	C_{32}^{22}	C_{32}^{23}	C_{33}^{21}	C_{33}^{22}	C_{33}^{23}
$-Z_z$	C_{31}^{31}	C_{31}^{32}	C_{31}^{33}	C_{32}^{31}	C_{32}^{32}	C_{32}^{33}	C_{33}^{31}	C_{33}^{32}	C_{33}^{33}

(25)

The total amount of coefficients in this scheme is not 81, as it would seem at first sight, but 36, because according to (100) [our (24)] the relationships

$$C_{hk}^{mn} = C_{hk}^{nm} = C_{kh}^{mn} = C_{kh}^{nm} \quad (26)$$

exist, and only six combinations of the subscripts or of the superscripts occur respectively.⁶⁸ (Voigt 1910, pp. 606–607)

⁶⁸ Hiermit sind alle [...] für X_x, Y_x, Z_x auftretenden Größen durch die Differentialquotienten von u, v, w nach den Koordinaten, durch l', m', n' und durch Parameter, die sich auf den ursprünglichen Zustand beziehen, ausgedrückt. Die gefundenen Werte sind nunmehr einzusetzen und alle Multiplikationen nur bis auf die Glieder erster Ordnung bezüglich der Veränderungen durch die Deformation auszuführen. Das Resultat dieser Operation ist sehr kompliziert. Es vereinfacht sich erheblich, wenn man einmal berücksichtigt, daß nach unserer Annahme die Deformation von dem natürlichen, spannungsfreie Zustand aus stattfinden soll, daß also bei verschwindenden u, v, w, l', m', n' auch die Druckkomponenten verschwinden müssen. [...]

Das so gewonnene Resultat, das man durch Heranziehung der Ausdrücke für $X_y, Y_y, Z_y, X_z, Y_z, Z_z$ vervollständigen kann, ergibt die sämtlichen Druckkomponenten linear in den neun Argumenten

$$\begin{aligned} \frac{\partial u}{\partial x} = x_x, \quad \frac{\partial u}{\partial y} + n' = x'_y, \quad \frac{\partial u}{\partial z} - m' = x'_z, \\ \frac{\partial v}{\partial x} - n' = y'_x, \quad \frac{\partial v}{\partial y} = y_y, \quad \frac{\partial v}{\partial z} + l' = z'_z, \\ \frac{\partial w}{\partial x} + m' = z'_x, \quad \frac{\partial w}{\partial y} - l' = z'_y, \quad \frac{\partial w}{\partial z} = z_z. \end{aligned} \quad (99)$$

Hierin sind die x'_y, \dots neue Bezeichnungen, bei denen y'_z ebenso wenig gleich z'_y ist, wie jetzt Y_z gleich Z_y .

Die Parameter dieser Ausdrücke sind Summen von dem Typ

$$\frac{1}{2} \zeta^0 \mathbf{S}_a^0 c^0 \left(\frac{\partial^2 \Phi}{\partial b \partial c} \right)^0 = C_{23}^{13}, \quad (100)$$

die in der angedeuteten Weise abgekürzt werden sollen; die oberen Indizes der C weisen dabei auf die Koordinaten hin, die im Faktor, die unteren auf diejenigen, die im Nenner auftreten.

Das Parametersystem in den neun Druckkomponenten besitzt folgende Gesetzmäßigkeit:

	x_x	x'_y	x'_z	y'_x	y_y	y'_z	z'_x	z'_y	z_z
$-X_x$	C_{11}^{11}	C_{11}^{12}	C_{11}^{13}	C_{12}^{11}	C_{12}^{12}	C_{12}^{13}	C_{13}^{11}	C_{13}^{12}	C_{13}^{13}
$-X_y$	C_{11}^{21}	C_{11}^{22}	C_{11}^{23}	C_{12}^{21}	C_{12}^{22}	C_{12}^{23}	C_{13}^{21}	C_{13}^{22}	C_{13}^{23}
$-X_z$	C_{11}^{31}	C_{11}^{32}	C_{11}^{33}	C_{12}^{31}	C_{12}^{32}	C_{12}^{33}	C_{13}^{31}	C_{13}^{32}	C_{13}^{33}
$-Y_x$	C_{21}^{11}	C_{21}^{12}	C_{21}^{13}	C_{22}^{11}	C_{22}^{12}	C_{22}^{13}	C_{23}^{11}	C_{23}^{12}	C_{23}^{13}
$-Y_y$	C_{21}^{21}	C_{21}^{22}	C_{21}^{23}	C_{22}^{21}	C_{22}^{22}	C_{22}^{23}	C_{23}^{21}	C_{23}^{22}	C_{23}^{23}
$-Y_z$	C_{21}^{31}	C_{21}^{32}	C_{21}^{33}	C_{22}^{31}	C_{22}^{32}	C_{22}^{33}	C_{23}^{31}	C_{23}^{32}	C_{23}^{33}
$-Z_x$	C_{31}^{11}	C_{31}^{12}	C_{31}^{13}	C_{32}^{11}	C_{32}^{12}	C_{32}^{13}	C_{33}^{11}	C_{33}^{12}	C_{33}^{13}
$-Z_y$	C_{31}^{21}	C_{31}^{22}	C_{31}^{23}	C_{32}^{21}	C_{32}^{22}	C_{32}^{23}	C_{33}^{21}	C_{33}^{22}	C_{33}^{23}
$-Z_z$	C_{31}^{31}	C_{31}^{32}	C_{31}^{33}	C_{32}^{31}	C_{32}^{32}	C_{32}^{33}	C_{33}^{31}	C_{33}^{32}	C_{33}^{33}

Die Gesamtzahl der in diesem Schema auftretenden Parameter ist nicht 81, wie auf den ersten Blick scheinen möchte, sondern 36, da nach (100) die Relationen bestehen

We point out that the symmetry of the subscripts in Eq. 26 derives from the hypothesis that a regular enough elastic potential exists, hence its mixed second-order derivatives coincide. Voigt did not say so explicitly; yet, it is apparent that the symmetry of the superscripts is of constitutive origin, since it states that it is immaterial to change the role of the relative positions of the centres of gravity of the elementary masses.

Voigt's matrix representation (25) is easy to grasp and still up-to-date; in principle the elasticities are 36 in number, hence Voigt's results are more general than those presented by the French mechanicians at the beginning of the 1800s. Voigt then proved that for central mutual actions he obtained the relations of Poisson and Cauchy (Voigt 1910, pp. 607–608).

Voigt completed the study of linear elasticity in his monograph of 1910, passing from the elementary masses inside the molecular sphere of action to a volume element, small enough to apply the techniques of differential calculus but also large enough to encompass a vast amount of elementary masses (a representative volume element, RVE). Voigt obtained the local balance equations by means of the so-called 'principle of solidification':

One easily obtains the balance equations for one part of a deformable body through the hypothesis that equilibrium cannot be altered if one lets the considered part become rigid.⁶⁹ (Voigt 1910, p. 609)

It is remarkable that in principle he obtained equations both for force and torque.⁷⁰

In the end, Voigt presented the particular hypothesis that no external volume couples are present: in this case, the local balance equations for the moments reduce to the symmetry of the components of the stress.⁷¹ This reflected on constitutive relations, since some coefficients shall equal others because of the coincidence of some stress components, and Voigt concluded that

[...] the result of this operation are expressions for the six components of pressure [...] with 21 constants, as was supposed by earlier hypotheses.⁷² (Voigt 1910, pp. 613–614)

Thus, Voigt was once again able to reconcile the seemingly irreducible discrete, force-based, and continuous, energy-based, theories of mechanics, which had caused such great debate amongst the mechanicians of the 1800s.

Footnote 68 continued

$$C_{hk}^{mn} = C_{hk}^{nm} = C_{kh}^{mn} = C_{kh}^{nm}, \quad (101)$$

und nur sechs Kombinationen oberer resp. unterer Indizes vorkommen.

⁶⁹ Die Gleichgewichtsbedingungen für einen Teil eines deformierbaren Körpers erhält man sehr einfach durch die Überlegung, daß das Gleichgewicht nicht gestört werden kann, wenn man den betreffenden Teil starr werden läßt.

⁷⁰ These are provided by Eq. 103 in Voigt (1910, p. 609).

⁷¹ Expressed by Eq. 117 in Voigt (1910, p. 613).

⁷² [...] das Resultat dieser Operation sind Ausdrücke für die sechs Druckkomponenten [...] mit 21 Konstanten, wie sie den früheren Überlegungen zugrunde gelegt war.

5 Final remarks

The French mechanicians of the first decades of the 1800s, amongst them Cauchy and Poisson, developed a theory of linear elastic phenomena based on a model of matter made of point-particles interacting by mutual central forces depending only on the distance between the body-points. Such a model brought results which did not agree with experience, since the model predicted one elastic modulus for isotropic materials, whilst experience suggested two. To justify such a result, the theory proposed by Green, based on a continuous model of matter and the existence of an elastic potential, gained credit.

On the basis of the crystallographic studies carried out by Bravais, Voigt proposed a mechanical theory for linear elasticity which is more general than the “unnecessarily specialized” one by Cauchy and Poisson. The ultimate components of matter are rigid corpuscles oriented in space, exchanging systems of forces reducible to a resultant through the centre of gravity of the elementary mass and a couple. Indeed, the formation of crystalline lattices requires an orienting moment to allow the particle to respect the orientation of the lattice. Voigt postulated that moments as well as forces depend on the distance between the centres of gravity of the particles and on the orientation of the particles. By admitting that inside the sphere of molecular action the particles rotate according to the same angle, he found that the exchanged moments are actually moments of forces, and the only constitutive relations needed are those for the intermolecular force. This result can be traced to Cauchy and Poisson's, if one assumed the central force hypothesis, and the more general constitutive relations for linear elasticity found by Voigt agreed with experimental results. Voigt also proposed a more general definition of isotropy, which completed the agreement between theory and experiments.

The study of Voigt, which sometimes seems to have been forgotten in the textbooks on strength of materials and continuum mechanics, still appears to be of great interest: it shows how a discrete approach, if suitably refined, is able to find a justification *per causas* of the phenomena of interest. This may still be of great importance nowadays when studying ‘complex’ or ‘new’ materials, for which reasonable constitutive relations must be investigated starting from a refined, ‘microscopic’ approach.

Acknowledgements The authors gratefully acknowledge the support of the grant “Progetti di ricerca d'Ateneo” of “Sapienza” University of Rome for the year 2007.

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