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How do the earliest known mathematical writings highlight the state's management of grains in early imperial China?

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Abstract The earliest extant mathematical books from China contain a lot of problems and data about grains. They also betray a close relationship with imperial bureaucracy in this respect. Indeed, these texts quote administrative regulations about grains. For instance, the *Book on mathematical procedures* 筭數書, found in a tomb sealed *ca.* 186 BCE, has a section in common with the "regulations on granaries" from the *Qin statutes in eighteen domains*, known thanks to slips excavated at Shuihudi. Mathematical writings also deal with official vessels used to measure grains. They cast light on statements from, and practices evidenced by, official histories and administrative documents. This article addresses the following issues.

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Which information about the concrete management of grains can we derive from mathematical writings in relation to administrative documents? Which data can we find in these writings about continuities and changes in the management of grains in the time span between the Qin and Han dynasties? In particular, how can we account for the fact that in a later mathematical text, namely, *The Nine Chapters*, probably completed in the first century CE, there was a change in the form in which the data about grain equivalences were given, by comparison with the "regulations on granaries"? Finally, what do our conclusions imply with respect to the nature of the earliest extant mathematical writings. In this article, we gather the various types of statement that mathematical writings contain about grains and offer several elements of interpretation for the "regulations on granaries" and the related text in *The Nine Chapters*. From this perspective, we offer several hypotheses about the management of grain in the Qin and Han dynasties.

A Pascal Crozet

Grains were an essential product for the state economy in early China.[1] The croplands owned by the state were allotted to people who had to pay taxes in return mainly in grain. Salaries were generally paid to officials in grain. Soldiers and corvée laborers also regularly received food rations in grain.[2] These few pieces of information suffice to highlight the importance, for the state, of managing grains. They also explain why it is essential for historians of early China to understand the ways in which the state organized the management of grains. Historians who have discussed this issue in early China have mainly relied on documents produced by the administration. The main aim of this article is to show how the earliest extant mathematical documents from early China provide unique evidence that complements the administrative documents and thereby allows us to understand features of the management of grains that would otherwise remain unattainable. The features we focus on are the measuring units with which amounts of grain were measured at different moments of the management process. Clarification in this respect allows us to interpret administrative statutes dealing with grain. Further, on this basis, we suggest hypotheses on the actual processes of managing grains, from the payment of taxes up to the payment of salaries or rations. Throughout, the relationship between the mathematical writings discussed and the actual activities of the administration in charge of grains will appear to be quite close. In conclusion, we return to this issue

[1] When we speak of early China in this article, this refers mainly to the period between the fourth century B.C.E. and the third century C.E. During this time period, the state of Qin (one of the "Warring States") defeated the other major states and unified the Chinese empire, establishing in 221 B.C.E. the so-called Qin dynasty. This dynasty was short-lived—it lasted between 221 B.C.E. and 206 B.C.E.—and was quickly overturned, giving way to the Han dynasty (206 B.C.E.–220 C.E.). The regent Wang Mang 王莽 (45 B.C.E.–23 C.E.) attempted to overthrow the Han dynasty and establish a new dynastic rule, the Xin 新 dynasty. The latter was short-lived (9–23 C.E.), and Han rule was reestablished in 25 C.E. This event divides the Han dynasty into two periods: the "Western Han dynasty" (206 B.C.E.–9 C.E.) and the Eastern Han dynasty (25–220 C.E.).

[2] See, for instance, (Loewe 1961/1962, 1967).

and suggest how the study of the management of grains can shed some light on the nature of these mathematical writings.

The mathematical documents in question contain, as we see below, data about the management of grains. Whenever they overlap with administrative sources, they appear to accord with them. Mathematical documents also include mathematical procedures as well as problems related to grains, which will prove important for our argument. In the first section of this article, we present these documents as well as the related evidence on which we are relying, and we offer a preliminary analysis of the information on grain that they provide. Interestingly, some of these mathematical writings discuss questions related to official measuring vessels.[3] We shall see the intimate connection between these vessels and the measurement of grain. Most importantly, one of these books, the *Book of Mathematical Procedures* (*Suanshu shu* 筭數書), found in a tomb sealed in around 186 B.C.E.,[4] actually has a paragraph virtually identical to an administrative statute from the Qin statutes and edicts, and more specifically from the "Statutes on granaries 倉律 *cang lü*."[5] This latter fact is a key element in our argument. Interestingly, this fact reflects a wider phenomenon, i.e., the close relationship between mathematical writings and certain administrative documents, namely statutes, which are essential for us here. Aside from grain, mathematical writings mirror questions raised, and use data provided by, texts of statutes.[6] Moreover, there is a crucial material connection between these two types of sources: in the contexts in which manuscripts dealing with mathematics have been discovered, texts of administrative statutes were also discovered.[7]

Our argument develops along the following three main lines. The second section of this article focuses on a table dealing with grains which is contained in *The Nine Chapters on Mathematical Procedures* (hereafter, *The Nine Chapters*), a book probably compiled in the first century C.E., a fact to which we shall return.[8] We will show that this table is closely connected to the statute whose text occurs in the *Book of Mathematical Procedures*. Further, the context of *The Nine Chapters*, that is, the problems and procedures contained in the book, allows us to establish that this table relates to a phenomenon that affects the largest measuring unit for capacity used in the book, namely the *hu* 斛. In fact, in *The Nine Chapters*, this measuring unit has two distinct meanings. In some contexts, it refers, as usual, to a unit of capacity.

[3] (Bai Shangshu 白尚恕 1982, Guo Shuchun 郭書春 1988, Volkov 1985, 1995).

[4] Let us recall the fact that the mathematical manuscripts that were excavated, including those discussed in this article, were all written on bamboo slips. Originally, the slips were interlinked by vegetable fiber cords to make a book. However, the cords decomposed during the two thousand years they remained buried underground. A key operation that archeologists face in order to publish critical editions of these documents is to attempt to recover the order in which the slips were originally arranged.

[5] Qin statutes, slips 41–43, see (Peng Hao 彭浩 2001, 80), note 1.

[6] Peng Hao (2001), *passim*, made this point clear in relation to the first manuscript discovered, see below. Zou Dahai 鄒大海 (2005), translated into English as Zou (2007), explores the relationship between the Qin edicts and statutes found at Shuihudi and mathematical activities in ancient China.

[7] When we present the various manuscripts below, we shall provide reference to publications that provide the basis for making this point.

[8] For a critical edition and a translation into French of *The Nine Chapters*, as well as the commentaries handed down with the book through the written tradition, (see Chemla and Guo Shuchun 郭書春 2004).

However, in other contexts, it represents a unit of value, in a sense we discuss below. This unit of value corresponded to different amounts of capacity, depending on the grain measured. In this latter use, *The Nine Chapters* provides evidence showing that the various *hu* were embodied in vessels, precisely the vessels that mathematical documents discuss.[9]

On this basis, in the third section, we shall return to the statute whose text occurs in the *Book of Mathematical Procedures* and establish, by relying on the problems and procedures contained in the book as well as evidence found in other mathematical documents, that the same phenomenon affected the largest measuring unit for capacity used before it was replaced by the *hu*, namely the *dan* 石, whose name occurs in this statute. In other words, the *dan* is used in the text of this statute both as a unit of value and a unit of capacity. This fact has remained unnoticed so far, but it has far-reaching consequences for the interpretation of administrative documents. This would suffice to show the relevance of mathematical documents for general history. In fact, things are even more complex in the very same text of the statute, as the term *dan* is also used to designate the highest-order measuring unit for weight. Illuminating this fact draws our attention to the part played by one of the various states in which grain is mentioned in these texts, namely *mi* "husked grain," or *li mi* 糲米, "coarsely husked grain."

The fourth part returns to the table contained in *The Nine Chapters* to show that in fact "*li mi* 糲米 coarsely husked grain" plays a key part in the arrangement of the grains displayed in the table. This provides us with an important insight to show that the same holds true for the statute contained in the "Statutes on granaries," whose structure and meaning we can then fully explain. As a result, we can see that despite important differences in the type of numerical values used, these two documents, namely the table inserted in *The Nine Chapters* and the text common to the "Statutes on granaries" and the *Book of Mathematical Procedures*, actually have the same meaning. In addition, these results allow us to draw some conclusions about the origin of the fact that the same name, 石 *dan*, could refer to the largest measuring unit for both weight and capacity.

In conclusion, with respect to the management of grains, we suggest that contrary to what was previously believed, it was not the state of unhusked grain but that of *mi*, which we suggest should be translated as "standard husked grain," that played the most fundamental role. We can also rely on evidence provided by mathematical writings to offer further hypotheses on the actual use of various types of measuring units in the management of grains. Lastly, as regards the mathematical manuscripts excavated and books handed down, the discussion allows us to express some views on their status as well as on their connection to administrative activities.

1 Sources and issues about grains

Let us first describe the evidence used in this article and the information it provides on measuring units, on the one hand, and the management of grains on the other. To

9 On these points, see K. Chemla, introduction to chapter 2, in Chemla and Guo Shuchun (2004, 201–205).

begin with, we shall concentrate on *The Nine Chapters*, the earliest extant book devoted to mathematics that was handed down through the written tradition—as opposed to manuscripts that have been excavated, and which we shall consider below.

1.1 *Dealing with grains according to* The Nine Chapters

As an important element of background for this article, we shall say a few words about the units of measurement used in *The Nine Chapters*. In fact, these are precisely the units described in the *History of the [Western] Han [Dynasty]* (*Hanshu* 漢書, abbreviated *History of the Han*), an official history written mainly by Ban Gu 班固 (32–92) (Hulsewé 1993). The system of measuring units presented in the chapter of this official history entitled "Monograph on Pitch Pipes and the Calendar (*Lü li zhi* 律曆志)" should denote a system enacted by state institutions (Ban Gu etal. 1962, 955–972). However, note that, as far as we know, the system of measuring units described in the *History of the Han* was not one that was used by Western Han institutions. When exactly this system began to be used is not completely clear. We return to this question below. However, it is certain that when the reformer Wang Mang 王莽 (45 B.C.E.—23 C.E.) established the Xin 新 dynasty (9–23), this system was enacted.

The units of length as described in the *History of the Han* form the following system, the highest-order unit of which (the *yin*) is not mentioned in *The Nine Chapters*:

(1) "The standards for measuring length are the *fen*, the *cun*, the *chi*, the *zhang*, the *yin*.
(…) Ten *fen* make a *cun*, ten *cun* make a *chi*, ten *chi* make a *zhang*, ten *zhang* make a *yin* and the five standards for measuring length are understood。
度者, 分丶寸丶尺丶丈丶引也 (…) 十分為寸, 十寸為尺, 十尺為丈, 十丈為引, 而五度審矣。".[10]

The successive units have a decimal relationship, as is the case for the system of measuring units for capacity, which the "Monograph on Pitch Pipes and the Calendar" of the *History of the Han* describes as follows:

(2) "The standards for measuring capacities are the *yue*, the *ge*, the *sheng*, the *dou*, the *hu*. (…) Coupling *yue*'s makes a *ge*, ten *ge* make a *sheng*, ten *sheng* make a *dou*, ten *dou* make a *hu* and the five standards for measuring capacities are excellent.
量者, 龠丶合丶升丶斗丶斛也 (…) 合龠為合, 十合為升, 十升為斗, 十斗為斛, 而五量嘉矣。" (Ban Gu etal. 1962, 967).

[10] (Ban Gu etal. 1962, 966–967) deals with measuring units of length; the whole system of measuring units is presented in chapter 21a. In the present article, we shall number all the passages quoted to enable us to refer easily to documents under discussion.

The units of capacity used in *The Nine Chapters* are taken from this list. They include the *hu*, the *dou*, and the *sheng*, whose relationships can be represented as follows:

$$1\,hu = 10\,dou = 100\,sheng (1\,sheng \text{ was roughly speaking } 0.2 \text{ liter})$$

In contrast to these two systems, the system for weight that is described is not decimal, as is clear from the following quotation of the same monograph:

(3) "The standards for weighing are the *zhu*, the *liang*, the *jin*, the *jun*, the *dan*. (…) Twenty-four *zhu* make a *liang*, sixteen *liang* make a *jin*, thirty *jin* make a *jun*, four *jun* make a *dan*. 權者, 銖、兩、斤、鈞、石也 (…) 二十四銖為兩, 十六兩為斤, 三十斤為鈞, 四鈞為石。" (Ban Gu etal. 1962, 969).

$$24\,zhu = 1\,liang,\ 16\,liang = 1\,jin,\ 30\,jin = 1\,jun,\ 4\,jun = 1\,dan.$$

As will become clear below, the systems of measuring units presented so far are valid not only for *The Nine Chapters* but also for all the documents mentioned here, including the earliest ones dating from the third century B.C.E. This fact, which points to the stability of official systems of measuring units over quite long a time period, holds true aside from one exception that will prove essential for us, namely, the highest-order unit of capacity, to which we return.

As a book handed down through the written tradition, *The Nine Chapters* was the object of commentaries, some of which were selected to be handed down with the main text. These commentaries include the one completed by Liu Hui 劉徽 in 263. They also include the subcommentary written by a collective of scholars under Li Chunfeng's supervision and presented to the throne in 656. Although the label is inaccurate, we refer to this subcommentary below as "Li Chunfeng's subcommentary."[11] In fact, at the beginning of the Tang dynasty (618–907 C.E.), Li Chunfeng also composed the chapters entitled "Monograph on Pitch Pipes and the Calendar" for both the *History of the Jin [Dynasty]* and the *History of the Sui [Dynasty]* (hereafter *History of the Sui*). In these monographs, Li Chunfeng dealt in minute detail with questions related to grain, making use of quotes from both *The Nine Chapters* and Liu Hui's commentary. The facts disclose a connection between the production of documents related to administrative activities and mathematical books, the same author contributing to the composition of both types of texts and quoting the latter sources in his writings on the former range of activities. Moreover, it is certainly revealing that this connection occurs precisely on the topic of the management of grains. Again, we shall return to this question below.

Let us now turn to the main document about grains in *The Nine Chapters*, which will be a fundamental piece of evidence in our article. It consists of the following table, placed at the beginning of chapter 2, which is entitled "UNHUSKED AND HUSKED GRAINS 粟米".

[11] In what follows, when we quote *The Nine Chapters*, we use small capitals to distinguish the book from the commentaries, which we render in lower case letters.

(4) "粟米之法

粟率五十	糲米三十
粺米二十七	糳米二十四
御米二十一	小䵂十三半
大䵂五十四	糲飯七十五
粺飯五十四	糳飯四十八
御飯四十二	菽、荅、麻、麥各四十五
稻六十	豉六十三
飧九十	熟菽一百三半
糵一百七十五	

NORMS/DIVISORS FOR UNHUSKED AND HUSKED GRAINS

THE *LÜ* FOR UNHUSKED GRAIN (MILLET) IS FIFTY,
FOR COARSELY HUSKED GRAIN THIRTY,
FOR FAIRLY HUSKED GRAIN (MILLET) TWENTY-SEVEN,
FOR FINELY HUSKED GRAIN (MILLET) TWENTY-FOUR,
FOR SUPREMELY HUSKED GRAIN (MILLET) TWENTY-ONE,
FOR SMALL GRITS OF BARLEY THIRTEEN AND A HALF,
FOR LARGE GRITS OF BARLEY FIFTY-FOUR,[12]
FOR COARSELY HUSKED GRAIN COOKED SEVENTY-FIVE,
FOR FAIRLY HUSKED GRAIN (MILLET) COOKED FIFTY-FOUR,
FOR FINELY HUSKED GRAIN (MILLET) COOKED FORTY-EIGHT,
FOR SUPREMELY HUSKED GRAIN (MILLET) COOKED FORTY-TWO,
FOR SOY, MUNG, AND HEMP BEANS, BARLEY, RESPECTIVELY, FORTY-FIVE
FOR PADDY SIXTY,
FOR FERMENTED SOY BEANS (*chi*3) SIXTY-THREE,
FOR COOKED RICE IN WATER (*sun*1) NINETY,
FOR COOKED SOY BEANS ONE HUNDRED AND THREE AND A HALF
FOR SPROUTS OF BARLEY OR BEANS[13] ONE HUNDRED AND SEVENTY-FIVE" (Chemla and Guo Shuchun 2004, 222–223).

Let us observe some of the main features of this table. The basic grains that explicitly appear in it are paddy rice (unhusked rice), and soy, mung, and hemp beans, as well as barley. However, the beginning of the table refers to a state of grain, "unhusked grain," rather than to a specific grain. Given that millet was the

[12] This identification was provided by Wang Guowei 王國維, "*Shi er* 釋二" in (Luo Zhenyu 羅振玉 and Wang Guowei 1993, 164). Wang Guowei offered this interpretation on the basis of occurrences in bamboo slips from the Han dynasty (Juyan). Yang Lien-sheng 楊聯陞 (1950), quoted from (Yang Lien-sheng 2006, 8), confirmed this interpretation and established its relationship to this passage of *The Nine Chapters*.

[13] The *Grand Dictionary of Chinese characters* 漢語大字典, p. 3163 (Sichuan cishu chubanshe and Hubei cishu chubanshe, 1984–1990) gives 糵 *nie* as equivalent to 糱 *nie*, relying on the *Shuowen jiezi* 說文解字 dictionary and its 1815 commentary by Duan Yucai 段玉裁. This suggests the interpretation that the character refers to sprouts of either barley or beans. The value with which the state of the grain is associated in the table seems to support this interpretation, as we can state on the basis of the meaning of the table expounded below.

most common crop in Northern China at the time,[14] it is most probable that the first term of the table, "unhusked grain," referred to millet (that is, to unhusked millet).[15] This is what we suggest in the translation. This feature, namely that the most generic terms are used in relation to millet, might somehow reflect the fact that the table is a document used by people from the north, for whom millet was the most common grain.[16] Another explanation for this fact is that millet was a basis for the definition of official standards and thus perceived as the most fundamental of all crops. This status of millet is reflected by the use of this grain to define the basic measuring units in the "Monograph on pitch pipes and the calendar" of the *History of the Han*.[17] We prefer the second explanation, for reasons to be made clear below. In any event, millet is clearly the grain that appears most often in *The Nine Chapters*. And it is always designated by generic terms. These facts might, among other things, be an indication of the higher status of a book like *The Nine Chapters* in comparison to the manuscripts that have been discovered recently.

In addition to the basic grains listed above, the table mentions a huge variety of states of these grains: unhusked or husked, in different degrees of fineness, cooked, fermented, and in the form of grits or sprouts. Moreover, the table associates each type of grain or each state of grain to an integer, and no unit of measurement is attached to its expression. Two exceptions must be mentioned, since for two numerical values the fraction "one half" appears. This can be accounted for easily if we note that the table apparently aims to offer numerical values stating equivalences between all forms of grain by means of integers that are all relatively prime to each other. The twice-repeated use of "one half" thus enables the retention of smaller integers for all the other numerical values.

The first question to be asked is: What was the meaning and function of this table? We just stated that it provides equivalences between forms of grain. However, we must clarify the meaning of these equivalences and explain how the numerical values in the table were understood and used. Some authors have suggested that the table expressed rates of exchange between grains on markets or in general barter.[18] Is this the case? We shall see that answering the questions just raised provides an essential key to understand the management of grains in ancient China.

In *The Nine Chapters*, the table opens chapter 2, which is titled "Unhusked and husked grains." Immediately following the table, a general and abstract formulation

[14] For a study of the evolution of crops in northern China, see (Li Qiufang 李秋芳 2012). Although millet was the most common plant in the north, note that both millet and paddy could be found in the north from a very early period onwards.

[15] In contexts in which we believe that *su* also designates the unhusked state of millet in particular, we shall translate as follows "unhusked grain (millet)." Whether *li mi*, *bai mi* and *zuo mi* were states of grain specifically attached to millet or could be attached to other types of grain is a topic awaiting further research. We shall thus translate them as states of grain, as is done in the table, adding "(millet)" in brackets. Zou Dahai (2005, 538–540) discusses the various terms used to designate these states of grain in different documents and attempts to account for why *The Nine Chapters* differs from several other documents in this respect.

[16] Peng Hao (2012, 196) obtained a similar conclusion by studying the occurrence of the same term in various contexts in the mathematical manuscripts. We return to this question below.

[17] Ban Gu etal. (1962, 966–970). Vogel (1994) discusses this feature.

[18] See, for instance, (Shen Kangshen etal. 1999, 141–142), in particular note 1.

of the rule of three, an operation called "suppose" in *The Nine Chapters*, is provided, outside the context of any problem. It reads as follows:

(5) "今有 (... commentary ...)
術曰:以所有數乘所求率爲實。以所有率爲法。(... commentary ...)實如法而一。"

"SUPPOSE. (... commentary ...)
PROCEDURE: ONE MULTIPLIES, BY THE QUANTITY OF WHAT ONE HAS, THE *LÜ* OF WHAT ONE SEEKS, WHICH MAKES THE DIVIDEND; ONE TAKES THE *LÜ* OF WHAT ONE HAS AS DIVISOR. (... commentary ...). DIVIDING THE DIVIDEND BY THE DIVISOR MAKES THE RESULT" (Chemla and Guo Shuchun 2004, 222–225).

A few words about the meaning of the procedure will prove useful below. The procedure refers to the three numerical values entering a rule of three using a special terminology that distinguishes between the data in two ways. On the one hand, it refers to the data related to the thing that the rule of three has to transform using the expression "what one has," in contrast to the other data, referred to as "what one seeks." On the other hand, the terminology opposes quantities to *lü*. Quantities, that is, the "quantity of what one has" and the quantity of what one seeks, are actual amounts, in contrast to *lü*, which are numerical values defined only relative to each other. Thus, the two numerical values used to express the equivalence between "what one has" and "what one seeks" in a problem solved by a rule of three are referred to as *lü* ("*lü* of what one has," "*lü* of what one seeks"). In relation to their designation by the term *lü*, the values can be transformed in correlation with each other, both, for instance, being transformed through adequate operations into mutually prime integers, before performing the operation.

In *The Nine Chapters,* this abstract rule of three is followed by thirty-two problems devoted to grains, all similar to the first one, which we translate as follows:

(6) (Problem 1 from chapter 2)

"今有粟一斗, 欲爲糲米。問得幾何。
荅曰:爲糲米六升。
術曰:以粟求糲米, 三之, 五而一。

SUPPOSE THAT, HAVING ONE *DOU* OF UNHUSKED GRAIN (MILLET), ONE WANTS TO MAKE COARSELY HUSKED GRAIN (MILLET). ONE ASKS HOW MUCH IT YIELDS.
ANSWER: IT MAKES SIX *SHENG* OF COARSELY HUSKED GRAIN (MILLET).
PROCEDURE : IF, HAVING UNHUSKED GRAIN (MILLET), ONE SEEKS COARSELY HUSKED GRAIN (MILLET), ONE THREE-FOLDS THIS, AND ONE DIVIDES BY FIVE" (Chemla and Guo Shuchun 2004, 224–225).

The seventh-century commentator Li Chunfeng elucidates the relationship between the table, the abstract procedure for the operation "suppose," and the specific procedure for solving this problem. First, in this problem, the unhusked grain is what one has, whereas the coarsely husked grain is what one seeks. The data for unhusked and coarsely husked millet are taken from the table: They are, respectively, 50 and 30. These are, Li Chunfeng explains, the *lü* of the unhusked grain of millet and the *lü* of the husked grain. As *lü*, they can then be transformed in correlation with each other, here into 5 and 3. This is how, here and below, Li Chunfeng accounts for the specific procedures following the first thirty-two problems of the chapter as well as for the numerical values they use, in relation to the table.

With *The Nine Chapters*, we have examined the earliest known mathematical document handed down through the written tradition, which we agree to date from the first century CE, a point to which we return below. However, in recent decades, archeological excavations in China, combined, unfortunately, with the antiquities market, have provided us with an entirely new kind of textual evidence, since they have unearthed documents that had remained untouched for more than two thousand years. These documents give grains pride of place, and their treatment of grains corresponds closely with what we find in *The Nine Chapters*. Let us now examine evidence they contain in this respect. They will give us subsequent opportunities to complement the information contained in *The Nine Chapters* about the management of grains.

1.2 *Dealing with grains according to the* Book of Mathematical Procedures

The first mathematical book excavated was the *Book of Mathematical Procedures*, unearthed in 1984 from tomb 247 at Zhangjiashan (Hubei Province). The other documents buried in the same tomb indicate that the book could not have been written later than around 186 B.C.E. (Peng Hao 2001). It thus belongs at the latest to the early period of the Western Han dynasty. In the same tomb as this mathematical writing, archeologists found a set of statutes of the early Western Han dynasty, entitled *Statutes and edicts of year* 2 (*of Empress Lü*) (*Ernian lü ling* 二年律令), and other books which all seem to indicate that the owner of the tomb could have been an official.[19]

The *Book of Mathematical Procedures* has measuring units all identical to those used in *The Nine Chapters*, except for one key difference, mentioned above. Instead of using the *hu* 斛 as the largest measuring unit for capacity, the *Book of Mathematical Procedures* uses the *dan* 石, that is, a measuring unit whose name is the same as the largest measuring unit for weight. This is an essential point to which we return.

We mentioned above another important feature of the book, which will play a central role in our article: The *Book of Mathematical Procedures* contains a passage concerning grains in common with another document unearthed in 1975 at Shuihudi (Hubei province) and containing administrative laws and statutes from the Qin dynasty.[20] This latter document is of the same type as the one found in the same

[19] On these statutes, see the introduction of (Zhangjiashan er si qi hao Han mu zhujian zhengli xiaozu 張家山二四七號漢墓竹簡整理小組 Group of editors of the bamboo strips from the Han tomb 247 at Zhangjiashan 2001, 1), a book containing an annotated critical edition of the text (pp. 131–210).

[20] As mentioned above, the parallel was first noticed in (Peng Hao 2001). See the edition of *Edicts and statutes of the Qin dynasty*, especially the slips 41–43, in (Shuihudi Qin mu zhujian zhengli xiaozu 睡虎地秦墓竹簡整理小組 Group of editors of the bamboo strips from Qin tombs at Shuihudi 1990, 29–30). Hulsewé (1985) contains a complete English translation of these statutes. (Ikeda Yûichi 池田雄一 2008, 147–195) (chapter "Kohoku Unbō Suikochi shinbo kanken 湖北雲夢睡虎地秦墓管見) (Personal ideas about the Qin tombs at Shuihudi (Yunmeng, Hubei)") suggested that *Edicts and statutes of the Qin dynasty* deals with regulations related to officials' management of several types of activities as well as to duties they had to carry out. Ma Biao 馬彪 (2013, 88–102) (chapter 6 "Unbô Soôjô no nijyû no seikaku 雲夢楚王城の二重の性格 (The double nature of the Chu royal city at Yunmeng") further puts forward

tomb as the *Book of Mathematical Procedures*. However, *Statutes and edicts of year 2 (of Empress Lü)*, and the Qin statutes from Shuihudi, cover completely different aspects of regulations.

Let us quote this common passage in the version found in the *Book of Mathematical Procedures*. Note that this version bears the mark that its text was checked by a certain "Wang," who put his signature at the bottom of the first of the three slips on which the section is written. In fact, the text of the statute as found in the mathematical text seems to be much closer to the original statute than the version known through the copy of the Qin edicts and statutes found in tomb 11 at Shuihudi. The publication of the *Book of Mathematical Procedures* has helped towards finding a new approach to the issue of the edition of the statute. The correctness of the mathematical book in this respect, in contrast to the legal document, is a striking phenomenon that still requires further discussion. In quoting and translating the text now, we shall point out the nature of measuring units as they are commonly understood—a point to which we return later:

(7) "程禾　程曰:禾黍一石為粟十六斗泰(大)半斗，舂之為糲=米=一=石=，(糲米一石)為糳=米=九[=][21]斗=，(糳米[九]斗)為毀(毇)米八斗。　王/88/

程曰：稻禾一石為粟廿斗，舂之為米十斗，為毀（毇）粲米六斗泰(大)半斗。麥十斗䵂 (*zhi*) 三斗。/89/

程曰:麥、菽、荅、麻十五斗一石，稟毀(毇)、糳[22]者，以十斗為一石。/90/

Footnote 20 continued

the idea that the *Edicts and statutes of the Qin dynasty* are almost all about statutes and regulations for local administrative offices dispatched by the court.

[21] Peng Hao (2001, 80), note 3, comments that the repetition mark was omitted. Evidence for this is provided by the parallel text in the Qin statute. Unless otherwise stated, here we follow Peng Hao's edition of the text.

[22] Peng Hao (2001, 81), note 7, explains that he reads here on the slip another character (糳), which was erroneously copied instead of *zuo*, which we use here. On p. 5, however, he seems to suggest that the former is borrowed to write down the latter. Cullen (2004, 130) takes up the former character, without mentioning a difference. Chôka zan kankan Sansûsho kenkyûkai 張家山漢簡『算數書』研究会編 The research group on the Han bamboo strips from Zhangjiashan *Book of Mathematical Procedures* (2006, 52) suggests reading the character on the slip as 糳. Dauben (2008, 137) quotes the same text for this section of the slip. We also follow the reading of the Japanese edition. The latter three publications provide translations. Note that we do not punctuate in the same way as previous authors, even though the Japanese translation, the Chinese one, and the two English translations follow this punctuation. All these authors discuss the meaning of the grains and compare these slips, on the one hand, with the text of the Qin regulations and, on the other hand, with *The Nine Chapters* (Chôka zan kankan Sansûsho kenkyûkai 2006, 52–56, Cullen 2004, 66–68, Dauben 2008, 137–139). In particular, they quote the corresponding text of the Qin regulation for the last sentence here. It reads "稟毇粺者，以十斗為石 *bing hui bai zhe, yi shi dou wei dan*." In this parallel text, the character of the *Book of Mathematical Procedures* under discussion is, in fact, replaced by *bai* 粺. Peng Hao (2001, 80–81), note 3, observes that, except for this set of slips quoted here, *zuo* never recurs in the *Book of Mathematical Procedures*. Elsewhere the state of millet corresponding to the same degree of fineness is always referred to as *bai*, as in the version of the sentence under discussion found in the Qin regulation.

Regulation for cereal plants. **The regulation** says: one ***dan*** **(unit of weight)**[23] of broomcorn millet (that is, the cereal just harvested) makes sixteen *dou* two-thirds of a *dou* **(unit of capacity)** of unhusked grain (millet). Husking this makes one ***dan*** **(unit of capacity)** of coarsely husked grain (millet) (*li mi*); one ***dan*** **(unit of capacity)** of coarsely husked grain (millet) makes nine *dou* of finely husked grain (millet) (*zuo mi*); nine *dou* of finely husked grain (millet) makes eight *dou* of highly finely husked grain (millet) (*hui mi*).[24] (text checked by Wang).

The regulation says: One **dan (unit of weight)** of paddy stalks (that is, cereal just harvested) makes twenty *dou* **(unit of capacity)** of unhusked grain (rice).[25] Husking this makes ten *dou* **(unit of capacity)** of husked grain (rice), makes six *dou* two-thirds of a *dou* of highly refined rice (*hui can mi,* compare Hulsewé 1985, 42).[26] Ten *dou* of barley is three *dou* of grits.

The regulation says: Fifteen *dou* **(unit of capacity)** of barley, or soy, mung, and hemp beans is one *dan* **(unit of ???)**. If one grants highly refined or finely husked grain, one takes ten *dou* as one *dan* (???)."[27]

[23] Peng Hao (2001, 80), note 2, rightly suggests that this measuring unit *dan*, as well as the corresponding one below, designate a measuring unit for weight.

[24] We see here that the relationship between three different degrees of fineness in the husking, namely, *li mi*, *zuo mi*, *hui mi*, is defined by the ratios between, respectively, 10, 9, and 8 *dou*. The table included in *The Nine Chapters* contains numerical values with the same ratios for, respectively, *li mi*, *bai mi*, and *zuo mi* (namely, 30, 27, 24). It thus seems that the definition of *zuo mi* changed between the time of the Qin statute and the moment when *The Nine Chapters* was composed (Peng Hao 2001, 80–81, note 3, and Zou Dahai 2005). However, it is worth noticing that these occurrences of the expression *zuo mi* are the only occurrences in all the manuscripts. By contrast, in all statements that assert a ratio of 10 to 9 between two degrees of husking in the mathematical manuscripts, from the oldest one known, these states of grain are referred to as, respectively, *li mi* and *bai mi*, that is, by means of the same expressions as those found in *The Nine Chapters*. See for instance, the *Book of Mathematical Procedures*, slip 98 (Peng Hao 2001, 84), or the Qin manuscript discussed below, *Mathematics,* (Zhu Hanmin 朱漢民 and Chen Songchang 陳松長 zhubian 主編 (gen. ed.) 2011), slip 0822 (editors' number 94), in which one reads: "one *sheng* of *li* (*mi*) makes nine tenths *sheng* of *bai* (*mi*) 糲(糲)一升為粺十分升九 " (see also Xiao Can 肖燦 2011, 56). These facts indicate that further research is required on the history of the use of the term *zuo mi* and the state of grain it designated at different time periods.

[25] This "unhusked grain *su*" must be "rice." We encounter evidence below that the expressions "unhusked rice 稻粟" and "husked rice 稻米" were used.

[26] Peng Hao (2012, 195) shows that the text as preserved in the *Book of Mathematical Procedures* is correct in this respect. He points out further that in contrast to the system of states of grain deriving from millet, the system of states of grain deriving from paddy had only two degrees: one called *mi* (*Mathematics,* slip 0756, calls this state 稻米 *dao mi* "husked rice") and a second one called *hui can mi* or simply *hui mi.*

[27] Peng Hao (2001, 80–81) provides an annotated edition of the text quoted above, and as mentioned, in his note 1, he notes the parallel between the Qin statute contained in slips 41–43 in question and the *Book of Mathematical Procedures*. Peng Hao (2012, 194–196) summarizes different scholars' contributions to the establishment of the text of this Qin statute and shows how one can rely on the *Book of Mathematical Procedures* to restore the text as transmitted in the extant copy of Qin statutes. See (Hulsewé 1985, 42) for a translation into English of the Qin statute parallel to the paragraph of the *Book of Mathematical Procedures* under discussion. Regarding the final sentence of the text, (Peng Hao 2012, 199), in conformity with his punctuation of the text, suggests following the editors of the Qin regulations in reading *huizuo* as a single word, designating states of millet with finer degrees of husking. He interprets this sentence in the context of Qin regulations, as referring to special ways of measuring rations of grains for officials in mission, in relation to their degrees in the hierarchy. Chôka zan kankan Sansûsho kenkyûkai (2006, 52), Cullen (2004, 67–68), and Dauben (2008, 139) interpret the text as referring to two types of

Like the table quoted above, this passage deals with equivalences between different kinds of grain. The parallel between the two excavated documents, an administrative one and a mathematical one, indicates that this passage of the *Book of Mathematical Procedures* was related to the context of granaries. The way in which the scribe who wrote the mathematical book perceived the status of the text he or she was quoting is made explicit, since the clauses of the passage are all introduced by the same expression: "The regulation says 程曰." The parallel also suggests that these documents must have been linked in practice. It is striking that the two types of documentation present the most intimate connection in relation to the topic of determining equivalent amounts of different grains. Also note that both unearthed writings were found in tombs of men who were apparently local officials. This suggests that these documents were written and used in relation to the state bureaucracy.

In the last paragraph of the passage quoted, we indicated by the insertion of repeated question marks problems raised by the usual interpretation of measuring units in this text. Indeed, in both of its last occurrences, *dan* does not seem capable of designating a unit of measurement of capacity as is usually understood.

If that were the case, to begin with, how could one make sense of a statement asserting that 15 *dou* makes a *dan,* when according to the official system, 10 *dou* should make a *dan*? To underscore the first point, we have to be more precise about the system of measuring units used during the Qin dynasty. In the same set of statutes as the one containing the passage from the "Statutes on granaries" translated above, we find another section making clear the legal system of measuring units that officials had to use. Inserted in the set of statutes entitled "*Xiao lü* Statutes on checking," this section describes the measuring units by listing the fines to be taken from officials using inaccurate measuring standards. The fines are determined in relation to the degree of inaccuracy of the standard used. In this context, the largest vessel used to measure capacity—and at the same time the largest measuring unit for capacity—is referred to as *tong* 桶.[28] Moreover, according to this section of the Qin statutes, *dan* was only a measuring unit for weight and not one for capacity. Did we make a mistake when we asserted that the term *dan* also referred to the largest unit in the capacity system?

In fact, this is not the case, because *at the same time*, and for reasons that need to be explained, actual statutes, included in the *same* set of slips and mentioning administrative practice, *also* use *dan* as the highest-order measuring unit for capacity. This is what can be deduced from the following text of a statute, which

Footnote 27 continued
grain. Cullen admits that the text is difficult to understand, whereas Dauben suggests that the two types of grain could refer more specifically to millet. Peng Hao (2012, 195–196) establishes that the finer state of rice can be designated as *hui*, that is, with the same term as the finer state for millet (see previous footnote). The two terms *zuo* and *hui* could thus either designate all states of rice and millet finer than the coarsely husked state, or alternatively, in each of the series deriving one from rice and the other from millet, respectively, the second state of grain after the state of "coarsely husked grain." For the interpretation of *dan*, see below.

28 *Xiao lü*, slips 3–7, in (Hulsewé 1985, 93–94, Shuihudi Qin mu zhujian zhengli xiaozu 1990, 69–70). Qiu Guangming 丘光明 etal. (2001, 215–217) also describe a vessel named *tong*, whose inscription records that one *tong* has a capacity of 10 *dou*.

prescribes monthly rations to be distributed to various types of workers: "小妾、舂作者，月禾一石二斗半斗; Small bond-women and grain-pounders who are working receive a millet grain ration of one *dan* two and a half *dou* per month." (Shuihudi Qin mu zhujian zhengli xiaozu (1990, 32) and Hulsewé (1985, 31) whose English translation is slightly revised). In this clause, the expression of the quantity combines the *dan* with the *dou*, which is undoubtedly a measuring unit for capacity. The same type of phenomena occurs in the *Book of Mathematical Procedures,* for instance in slip 96, and in this context, there is no doubt that the measuring unit *dan* makes 10 *dou*. This establishes beyond doubt why there is a problem with the statement, "fifteen *dou* … is a *dan*," read on the last slip of the passage translated above, if we interpret *dan* as a measuring unit for capacity.

The last sentence of the same text raises similar difficulties. If it deals with *dan* as a measuring unit for capacity and it is the case that one *dan* is ten *dou*—a fact used in the previous clauses of the same text—what is the significance of stating this common fact again? Clearly, there are phenomena here related to the measuring unit *dan* that we do not understand. One of the aims of this article is to clarify these questions. To the questions on *dan* listed above, we can add many others. Why, although *dan* seems to have been only a measuring unit for weight, do we also find it in administrative practice as a measuring unit for capacity? What is the origin of this phenomenon? And why was *dan* later replaced, precisely in its role as a measuring unit for capacity and only there, by 斛 *hu*? When did that change occur? Resolving this question is essential for dating *The Nine Chapters* since this book makes use only of *hu*, whereas all the manuscripts excavated never use *hu* and mainly use *dan* as the highest-order measuring unit for capacity.[29]

If we now compare the text inserted in both the *Book of Mathematical Procedures* and the Qin statutes (Text 7) with the table from *The Nine Chapters* quoted above (Text 4), we observe some interesting similarities and differences. To begin with, if we focus on the types of grain mentioned in both documents, the cereals occurring in the two texts are the same: millet, rice, and soy, mung, and hemp beans, as well as barley. However, the table in *The Nine Chapters* contains a much greater variety of states of grain, when compared to the other document. There is one exception to be mentioned: Rice seems to be considered in a greater number of states in the *Book of Mathematical Procedures* than in *The Nine Chapters*. We return to this point below. The fact is that in contrast to the table in *The Nine Chapters,* in Text 7 millet is mentioned explicitly. It is also mentioned before paddy. In other words, the main crops from the north and the south both appear, but the millet, placed in the first position, is granted a higher status. This fact evokes the fundamental character millet has in *The Nine Chapters*. The *Book of Mathematical Procedures* contains procedures and problems dealing with millet as well as rice, even though millet occurs more frequently. Moreover, often—but not always—generic terms are used to refer to millet in its different states. This fact recalls the practice in *The Nine Chapters.* To conclude on this point, it appears that millet plays a more fundamental

[29] This question was raised by K. Chemla in the introduction to chapter 6 (Chemla and Guo Shuchun 2004, 477–478).

part in *The Nine Chapters*, which, as we suggested above, might indicate a comparatively more official status for the book.

The statute quoted in the *Book of Mathematical Procedures* and the table in *The Nine Chapters* are also quite different with respect to the nature of numerical values they use to state equivalences. In contrast to *The Nine Chapters,* which only uses integers (with the exception of the fraction one half) without measuring units, the text in the *Book of Mathematical Procedures* makes use of quantities expressed by means of integers and fractions both associated with various types of measuring units. So far, in this text, we have identified measuring units for capacity and for weight. We shall soon establish that the situation is even more complex.

Lastly, the structure of the two texts is apparently different. The table in *The Nine Chapters* seems to be putting all types of grain and all kinds of states on the same level. By contrast, in the text from the *Book of Mathematical Procedures*, one has four enumerations of quantities of grain, each enumeration stating equivalent quantities between different states of the same grain.

However, despite all these differences, most of the relationships stated in the *Book of Mathematical Procedures* are quantitatively the same as those stated in the table of *The Nine Chapters*. So, it seems that the same numerical relationships were expressed by means of two completely different types of texts. What accounts for this difference between textual forms? This is a question we shall address in what follows.

In addition to the text just analyzed, the *Book of Mathematical Procedures* contains several mathematical procedures related to grains or presented in relation to grains. Let us mention a few examples, which will also illustrate the close relationship between these procedures and those presented in *The Nine Chapters* with respect to grains. A first pair of slips illustrates a rule enabling the transformation of unhusked grain into husked grain. The section recorded on these slips has a title indicating the task to be carried out. It is followed by a rule that is formulated outside the context of any problem and enables one to execute the task. This rule is exactly the procedure used in *The Nine Chapters* after problem 2.1 quoted above. The use of the same terms (*su, mi*) and the same values (5, 3) indicates that the states of grain meant are both associated to millet, a point that we indicate in the translation. Lastly, the rule is followed by a punctuation sign and then a problem, itself followed by another related procedure.[30] The beginning of the section reads as follows:

(8) "粟求米 粟求米, 因而三之, 五而成一。•今有粟一升七分[31]三, 當為米幾何L?曰:為米七分升六。術曰:.../113/"

"Having unhusked grain (millet), one seeks husked grain (millet). If, having unhusked grain (millet), one looks for husked grain (millet), one three-folds it (i.e., the quantity of unhusked grain) by the *yin* multiplication and five becomes 1 (i.e., one divides by 5). • Suppose there is one *sheng* and

[30] The structure of the text is interesting, but this is the topic of another publication.

[31] Peng Hao (2001, 89), note 1, thinks that after the character "part *fen* 分," used to express the fraction, the measuring unit of capacity 升 *sheng*, which occurs after the integral part of the result, was omitted.

> three-sevenths of (a *sheng*) (3/7 *sheng*, literally, three (parts) of (a *sheng*) divided into seven parts) of unhusked grain (millet), how much husked grain (millet) must this make? ∟ It is said: this makes six sevenths of a *sheng* of husked grain (millet). Procedure: ..."[32]

A few remarks on this text will highlight the relationships between the various documents mentioned. First, despite the fact that the statute stating relationships between grains included in the *Book of Mathematical Procedures* (7) has numerical values of a nature different from those in table (4) of *The Nine Chapters*, integers without measuring units and mutually prime still occur in the *Book of Mathematical Procedures*. Secondly, both the states of grain mentioned, the equivalences between them and the numerical values used in the rules of three are exactly the same in all our documents.

Other slips in the *Book of Mathematical Procedures* offer sets of rules of three to transform a state or a type of grain into other states or other types of grain. Again, they correspond perfectly with both the procedures, the numerical values, and the types of grain mentioned in *The Nine Chapters*. Some of these procedures, however, present a slight deviation with respect to the terms encountered in *The Nine Chapters*. Let us quote the following example:

(9) "粟為米 麻、麥、菽、荅三而當米二∟, 九而當粟十∟。粟五為米三∟, 米十為粺九∟, 為毀(毇)八∟。麥三而當稻粟四, 禾粟 楊/109/
五為稻粟六。"/110/

> "If unhusked grain (millet) makes husked grain (millet). Three of hemp beans, barley, soy, or mung beans are equivalent to two of husked grain (millet). ∟ Nine are equivalent to ten of unhusked grain (millet). ∟ Five of unhusked grain (millet) make three of husked grain (millet). ∟ Ten of husked grain (millet) make nine offinely husked grain (millet). ∟ make eight of highly finely husked grain (millet). ∟ Three of barley are equivalent to four of **unhusked rice**, of **unhusked millet** (*hesu*) (text checked by) Yang/109/five make six of unhusked rice./110/"[33]

As above, the section has a title indicating the type of rules it contains. In contrast to *The Nine Chapters*, it provides sequences of equivalences between grains outside the framework of problems, and their clauses are separated by punctuation marks. As a result, the presentation is more compact than in *The Nine Chapters*. However, no connection is established with concepts like that of *lü*, contrary to what we find in *The Nine Chapters*. The numerical values allowing the reader to transform grains into each other are strictly the same as in *The Nine Chapters*. They all correspond to the statements of the Qin statute quoted above. In these slips of the *Book of Mathematical Procedures*, however, the numerical values are now expressed in the form of integers without measuring units. They were simplified in the same way as in the corresponding procedures that followed problems in *The Nine Chapters*. One feature, however, is specific to this text of the *Book of Mathematical Procedures* and does not occur in *The Nine Chapters*—the mention of specific types of unhusked

[32] See the whole text in Peng Hao (2001, 89), slips 113–114.

[33] Peng Hao (2001, 88–89), slips 109–110. We use the same convention as above for the translation of names of grains.

grain, which we emphasized with bold characters: unhusked rice and unhusked millet. In fact, the way of computing equivalences between these two types of unhusked grain is the topic of the last clause. This information will prove valuable below.

As above, this sketchy presentation does not exhaust the wealth of information contained about the management of grains in the *Book of Mathematical Procedures*. At an appropriate future moment, we shall quote other pieces of evidence and add further remarks. Let us now turn to a second manuscript whose text became available only recently, namely *Mathematics*.

1.3 *Dealing with grains according to* Mathematics

The mathematical book entitled *Shu* 數 (*Mathematics*) was dated by its editors from the Qin dynasty (221 B.C.E.—206 B.C.E.).[34] This book was bought in December 2007 on the Hong Kong antiquities market, being the product of illegal excavations. We believe it is authentic.[35] It is now kept in the Yuelu Academy 嶽麓書院 (Hunan University) along with other documents that were bought in the same lot (more than 1,300 bamboo slips altogether) and are believed to come from the same site (Chen Songchang 陳松長 2009, 75). In August 2008, a collector from Hong Kong bought a second lot (more than 30 slips) and donated it to the Yuelu Academy 嶽麓書院 (Hunan University). On the basis of the shape of the slips, the writing, and the content, the two lots are believed to have come from the same tomb (Chen Songchang 陳松長 2009, 75).[36] The set of slips among which *Mathematics* was discovered include other texts of the same kind as those excavated from the tomb in which the *Book of Mathematical Procedures* was found. In particular, a set of statutes belongs to the purchased lots and is in the process of being published.[37] These facts again indicate the probable relation between the owner of the slips and the bureaucracy. They also confirm the link between the use of mathematical

[34] On these two pieces of information, see (Chen Songchang 2009, 75, 85), respectively. The title is on the verso of slip 0956 (editors' number 1 verso), and its photograph is reproduced in (Zhu Hanmin and Chen Songchang (gen. ed.) 2011, 3).

[35] One key reason for this is that it clearly uses the character 券 *quan* in a way quite similar to the use of *quan* by another mathematical manuscript, and only this one. However, the slips of this other manuscript showing that use of *quan* were only published in 2008. See (Chemla and MA 2011).

[36] Several papers devoted to different aspects of the book *Mathematics* appeared in 2009: (Xiao Can and Zhu Hanmin 2009a, b, Zhu Hanmin and Xiao Can 2009). In 2011, the contents of the book were made public through (Xiao Can 2011) and the critical edition of *Mathematics*, with reproductions and annotations, published in (Zhu Hanmin and Chen Songchang (gen. ed.) 2011). Xiao Can and Zhu Hanmin (2009b) focus in particular on the units of measurement for quantities of grain in Qin China. Since the topic is close to that of our article, let us sketch their relevant results. In their terms, they mainly discuss the conversion, for grains, between measuring units of volumes and weights. Their conclusion is that *Mathematics* records many coefficients expressing the ratios between volume and weight for grains. Moreover, they suggest that in the third century B.C.E., before the unification of the Chinese empire, people usually measured grains by means of measuring units of volume and then used these coefficients to compute the related weight. We show below why we have come to different conclusions.

[37] See (Chen Songchang 2009, 86–87). According to the description given there, the statutes seem to be comparable to those found in Shuihudi and mentioned above.

writings and the practice of regulation in ancient China. This will prove to be quite important in relation to grains.

Let us now turn more specifically to the evidence provided by *Mathematics* on the management of grains. As we already emphasized, grains are granted pride of place in the book. In this respect, the statements and procedures included in *Mathematics* have many similarities with both the *Book of Mathematical Procedures* and *The Nine Chapters*. In fact, the topic of grain is one key common topic to all these early mathematical books. Moreover, despite differences, the common characteristics in this respect between the three books reveals a close relationship between them, representing one of the major links between them. Let us examine some similarities and differences in greater detail.

First, the systems of measuring units used in *Mathematics* are the same as those used in the *Book of Mathematical Procedures* and *The Nine Chapters*. However, the same exception as above can be noted. Like the *Book of Mathematical Procedures*, and thus unlike *The Nine Chapters*, *Mathematics* uses *dan* not only as the largest measuring unit for weight, but also the largest measuring unit for capacity.[38]

The extant set of slips gives access to what is probably only part of the contents of the book entitled *Mathematics* and contains no paragraph comparable to the table from *The Nine Chapters* or the statute on grains quoted in the *Book of Mathematical Procedures*. However, the grain types mentioned in *Mathematics* are exactly the same as those in both the *Book of Mathematical Procedures* and *The Nine Chapters*. The list of states of the various grains occurring in *Mathematics* is closer to that found in the *Book of Mathematical Procedures* than it is to *The Nine Chapters*. Furthermore, the standards of equivalences between grains and states of grain to which *Mathematics* bears witness presents no contradiction with what is stated in either the *Book of Mathematical Procedures* or *The Nine Chapters*. These facts, with others presented below, show that with respect to grains, *Mathematics* reflects a system of management of grains that corresponds closely to the systems reflected by the other mathematical writings.

Similarities can also be noted in the types of procedures included in *Mathematics*. For instance, slips 0756 and 0776 record rules of three quite comparable to those found in the *Book of Mathematical Procedures* or in *The Nine Chapters* (texts 5, 8, 9 quoted above).[39] Let us quote some of these rules to allow the reader to see these similarities and also to be able to refer

(10) "以稻米求毀(毇)粲米, 三母倍𧷢（實）(...)./excavation number 0756; editors' numbering 87/[40]

[38] See, for example, *Mathematics*, slip 0939 (editors' number 11) (Zhu Hanmin and Chen Songchang (gen. ed.) 2011, 4). This slip also confirms that in this system one *dan* is equal to ten *dou*.

[39] This article does not provide the scope to analyze extensively the various formulations of the rules of three contained in these books (several different formulations in the same book, as well as similarities and differences across books), their relationships with each other, and what these facts reveal in terms of conceptual history. Such a systematic study is interesting and important. It will be the topic of another publication.

[40] Zhu Hanmin and Chen Songchang (gen. ed.) (2011, 13).

If, having **husked rice**, one seeks **highly refined rice**, one three-folds the denominator and doubles the dividend."

The statement of the rule, outside the context of any problem, reminds us of the quotations from the *Book of Mathematical Procedures* (8). The terminology and the type of numerical values used to carry out the transformation are similar to both *The Nine Chapters* (6) and the *Book of Mathematical Procedures* (8). However, the terms found here to refer to types of grain are not found in either the *Book of Mathematical Procedures* or *The Nine Chapters*. More precisely, the latter statement holds true, if we exclude Text 7, which the *Book of Mathematical Procedures* shares with the Qin statutes. But, importantly enough, these terms occur in Text 7. In addition, despite differences between the formulations of numerical values, the equivalence that the Qin statute provides between these two states of rice is quantitatively the same as that employed in the rule of three contained in *Mathematics*. This is an extremely important piece of evidence since it suggests a relationship between Qin statutes and *Mathematics*.

Another rule of three provided by *Mathematics* formulates an equivalence that can also be found in the *Book of Mathematical Procedures*:

(11) "(…) 以稻粟求叔(菽)、荅、麥，三之，四成一。(…)/excavation number 0776; editors' numbering 101/

If, having **unhusked rice**, one seeks **soy or mung beans, or barley**, one three-folds this and makes four become one (i.e., one divides by four)." Zhu Hanmin and Chen Songchang (gen. ed.) (2011, 15)

Text 9 from the *Book of Mathematical Procedures*, slip 109, states the same equivalence between unhusked rice and barley. If we add to this the information stated in Qin statute (Text 7), namely that the terms soy or mung beans or barley occur in equivalences in exactly the same way, we see that the statements in both books refer to the same transformation. At first sight, the Qin statute or *The Nine Chapters* do not seem to provide this information. However, our conclusions allow us to show that in fact both texts give it. This point will become clear below. In conclusion, these statements of *Mathematics* correspond perfectly with the evidence of the other documents.

In addition to procedures, *Mathematics* also contains tables of clauses stating other types of equivalences related to grains. These are of the utmost importance for the argument made in this article. Let us first look at a table which assembles statements of a kind that are only documented in *Mathematics*. There is no doubt that for a given amount of a single type of grain and a single state of grain, each of these statements provides equivalences between its weight and the corresponding capacity. To illustrate such clauses, let us mention some that will prove useful below:

(12) "黍粟廿三斗六升重一石。(…)。 䊪(糲) 米廿斗重一石。
麥廿一斗二升重一石。/excavation number 0780; editors' numbering 103/
(…) 稻粟廿七斗六升重一石。(…)/excavation number 0981; editors' numbering 104/

Twenty-three *dou* six *sheng* of unhusked millet (*shu su*)[41] has a weight of one *dan.* (…)

Twenty *dou* of coarsely husked grain (millet) has a weight of one *dan.*

Twenty-one *dou* two *sheng* of barley has a weight of one *dan.*/excavation number 0780; editors' numbering 103/

(…) Twenty-seven *dou* six *sheng* of unhusked rice has a weight of one *dan.* (…)/excavation number 0981; editors' numbering 104/" (Zhu Hanmin and Chen Songchang (gen. ed.) 2011, 15)

We can see that for each type of grain, these statements invariably give the measurement in capacity units corresponding to the same unit of weight. These statements reveal a very important fact. Various types of measuring units were used to measure a given amount of grain. Accordingly, specific types of statements were shaped, and listed here in tables, to deal with conversions between different systems of measuring units. Moreover, the type of conversion depended on the grain used.[42] Note that no such clause occurs either in the *Book of Mathematical Procedures* or in *The Nine Chapters*. In our conclusions, we try to interpret this fact.

In *Mathematics,* another table systematically gathers a second type of statement quite similar to the latter in shape, but different in terms of the relationships stated. At first sight, one may be tempted to believe that, taking each clause for one type of grain and a given state of that grain, they provide the volume of grain corresponding to the largest unit of capacity. Let us read one of the clauses:

(13) "麥二尺四寸一石。/excavation number 0760; editors' numbering 107/
(A volume of) two *chi* four *cun* of barley is ***one dan***" (Zhu Hanmin and Chen Songchang (gen. ed.) 2011, 16).

Before discussing these clauses, let us mention the fact that in addition to the table, such clauses also appear in problems. Since these problems provide important information that will be useful below, let us translate one of them:

(14) "倉廣五丈, 袤七丈, 童高二丈, 今粟在中, 盈與童平, 粟一石居二尺七寸, 問倉積尺及容粟各幾/excavation number 0801; editors' numbering 177/
可(何)?曰:積尺七萬尺, 容粟二萬五千九百廿五石廿七分石廿五。述(術)曰:廣袤相乘, 有(又)以高乘之, 即尺。以二尺/excavation number 0784; editors' numbering 178/

[41] The expression by means of which one refers to unhusked millet here is different from that used in the *Book of Mathematical Procedures* (Text 9). Note that the manuscript writes down these tables by means of registers, which we indicate with spaces. In the translation, each clause is translated in a separate paragraph.

[42] It is this development that (Xiao Can and Zhu Hanmin 2009b) interpreted as the introduction in third century B.C.E. China of the concept of specific gravity.

A granary has a width of five *zhang*, a length of seven *zhang*, and a height of the heap[43] of two *zhang*. Suppose there is unhusked grain (millet) in it, in such a way that it fills the granary up to the highest level. **One *dan* of unhusked grain (millet) occupying two *chi* seven *cun***, one asks the amount, respectively, of the *chi* of the volume of the granary and the unhusked grain (millet) it contains.[44]/ excavation number 0801; editors' numbering 177/ It is said: the *chi* of the volume are seventy thousand *chi*, and it contains twenty-five thousand nine hundred and twenty-five *dan* twenty-five twenty-sevenths *dan*. Procedure: width and length being multiplied by each other, one further multiplies this (result) by the height, hence the *chi* (of the volume). One takes two *chi* ..."[45]

We highlighted the clause in which we are interested in bold type. Before discussing the main problematic feature of these clauses, namely the interpretation of *dan*, we shall first explain the system used to express volumes.[46] The measuring units *chi* and *cun* are basically designed to express lengths (see Text 1). In fact, they are also used to express volumes, as is the case here, by means of the following principle. A basic unit is chosen, in this case the *chi*. The length "two *chi* seven *cun*" designates the height of a parallelepiped whose square base has a side of length one *chi*. The numerical value "two *chi* seven *cun*" also expresses a volume equal to that of this parallelepiped. In this system, whether *cun* represents a unit of length or a unit of volume, its relation to the unit *chi* is the same: 1 *chi* equals 10 *cun*. One could alternatively take the *cun* as the basic unit; the height expressed would then be that of a parallelepiped with a square basis of 1 *cun* side, and the relation between the *chi* and the *cun* would remain invariable. In the latter case, the same volume as above would be expressed by means of the quantity 2,700 *cun*, which occurs in our documents, as shown below.

Let us now turn to the main problem, namely, the interpretation of *dan* in these statements. If we compare the two clauses quoted above (texts 13 and 14), we see that we have here two different volumes associated with a single measuring unit *dan*, depending on whether we are dealing with barley or unhusked grain, which we interpret here as millet for reasons to be explained. So far, we only have two possibilities.

43 The term *tong* 童 that we translate as "heap" and interpret as "granary" occurs in *The Nine Chapters* as well as in the *Book of Mathematical Procedures*. Xiao Can (2011, 85), note 1, and Zhu Hanmin and Chen Songchang (gen. ed.) (2011, 127), note 1, suggest relying on the third century commentator Liu Hui's gloss on this character in *The Nine Chapters* to interpret it. Liu Hui writes: "凡積芻有上下廣曰童 Every time when, piling up forage grass, in the (shape constructed) there is a higher width and a lower width, one calls (the shape) *tong*." We suggest that the character *tong* could be a character borrowed to write *chong* 重 "heap." The meaning could thus have been extended to designate the "height of the granary" itself.

44 For the moment, we translate *rong* loosely. We shall return to this concept below, as its interpretation demands revision. Note that we also translate "*jichi* the *chi* of the volume" loosely. The precise interpretation is the topic of another publication.

45 Zhu Hanmin and Chen Songchang (gen. ed.) (2011, 25). The procedure solving the problem is not complete, since the editors could not identify the following slip among the extant slips. A similar mathematical problem can be found on slips 175 and 176 (editors' numbers), Zhu Hanmin and Chen Songchang (gen. ed.) (2011, 24, 126).

46 The explanation of the system was first published in (Li Jimin 李繼閔 1998, 768–778). We are grateful to Zhu Yiwen for having called our attention to the first publication explaining a feature now universally accepted.

Could *dan* be the measuring unit for weight? It is clear, from the problem contained in *Mathematics* quoted above (Text 14), that this is not possible. The term used in the answer to the problem to refer to the value obtained with respect to the unit *dan* is 容 *rong*, which we have so far translated loosely as "contain." Whatever the interpretation of this term may be, it is not the term used in *Mathematics* to refer to weight: Text 12 clearly uses the term *zhong* 重 for this purpose.[47]

Could *dan* then be the measuring unit for capacity? If, in each of these contexts, *dan* was the unit of capacity of the same name, how would it be possible that the volume corresponding to it changes in relation to the grain? To formulate the puzzle in contemporary terms for the sake of clarity, this would involve a vessel containing 1 liter or 2 liters, depending on the liquid poured into it. This is clearly a phenomenon that awaits explanation, and this is the key question that the present article aims at solving. This question is all more intriguing in that we encounter exactly the same phenomenon, relating to the same type of clauses, in *The Nine Chapters*.

In chapter 5, titled "Discussing works 商功 *shang gong*," we find the statement of three clauses of the same type as the ones examined: They relate a volume and what appears to be the largest measuring unit of capacity, namely the *hu*. Note that in this other context, the phenomenon affects the measuring unit for capacity unambiguously, since, as we explained above, the name of the highest-order measuring unit for capacity in *The Nine Chapters* is no longer the same as that for the weight system. That is, it is no longer *dan*, but *hu* 斛. Also note that in *The Nine Chapters,* these clauses are referred to as 程 *cheng*, which we have translated so far as "regulation." In other words, the term used to designate their status is the same as the one used by the scribe who copied the text of the "Regulation for cereal plants" (Text 7) in the *Book of Mathematical Procedures.* Let us represent the three clauses inserted in *The Nine Chapters* in the form of the following table.

	Problem 5.23	Problem 5.24	Problem 5.25
Grain	Unhusked grain (millet) 粟	Soy beans 菽	Husked grain (millet) 米
The volume of 1 斛 *hu* is	2 *chi* 7 *cun*	2 *chi* 4 *cun* 3/10 *cun*	1 *chi* 6 *cun* 1/5 *cun*

[47] Peng Hao (2012, 199–201) endorses this interpretation, and this is a key point where our explanations of the measuring units used for grains diverge. In fact, the weights he believes these statements refer to are not coherent. On p. 200, he suggests at the same time that the expression in Text 14 refers to the quantity of unhusked millet corresponding to a weight of the plant with stalks and leaves (he establishes that *heshu* 禾黍 refers to this on p. 195) and also that the weight is that of the unhusked grain (1 *dan* would be the weight of 2 *chi* 7 *cun* or 16 *dou* 2/3 *dou* of unhusked millet). However, on p. 196, he showed these weights were different (a weight of 1 *dan* of harvested plant yields 16 *dou* 2/3 *dou* of unhusked millet, after deletion of stalks and leaves). As a result, Peng is compelled to conclude that the similar clauses found in *The Nine Chapters* either derive from compilers' mistakes or refer to artificial problems. Further, he is compelled to think that the commentaries on *The Nine Chapters* also made the same mistake. In the interpretation we suggest, all these sources are perfectly clear and coherent.

Not only is the shape of the statement quite similar, but also the numerical values are basically the same. If we observe the volume associated with one *hu* of unhusked grain (which we again interpret as millet), we find it is exactly the same as that quoted in the problem of *Mathematics* (Text 14). In fact, the same relationship between this amount of volume and 1 *dan* is also quoted in a problem in the *Book of Mathematical Procedures* (Peng Hao 2001, slips 146–147, 105–106). We thus have quite a strong congruence between all mathematical documents with respect to another feature related to grains: Not only do we have the same type of clauses in the three books, linking volumes and the largest measuring unit for capacity, but also the same actual numerical value. Note, however, that the value provided for soybeans in *The Nine Chapters*, which, for reasons explained above, should be the same as that for barley, presents a slight difference when compared to the clause contained in *Mathematics* (Text 13). The fraction does not appear in *Mathematics*, a fact that awaits further research.

However, the most striking element of correspondence lies elsewhere. Again, for the name of the highest-order measuring unit of capacity, we have different volumes associated with it, depending on the grain dealt with. This is precisely the phenomenon underlined above on the basis of the evidence provided by *Mathematics*. Despite the fact that the name of the measuring unit for capacity was changed between the third century B.C.E. and the moment when *The Nine Chapters* was compiled, we encounter the same striking use of this name. Supposedly, according to the context, the volume of a measuring unit for capacity differed depending on the grain measured.

Note that we are thus facing a phenomenon relating to the highest-order measuring unit of capacity and the measurement of grain that persisted for more than four centuries. Moreover, the recurrence of the phenomenon in an excavated document as well as a source handed down by the written tradition helps us to dismiss the easy interpretation that this is due to mistakes made by scribes who did not understand what they were writing.

The present article offers a solution to this puzzle by establishing a third meaning for the term *dan*—and accordingly, a second meaning for the term *hu*.[48]

[48] Zou Dahai (2009, 513–515) is the publication that went farthest in the direction explored in this article. As we suggest here, Zou Dahai understood that in addition to being a measuring unit for weight and capacity, *dan* had a third meaning in relation to the measurement of cereals and on the basis of capacity. However, he did not address the question of the *meaning* of this measuring unit. Nor did he establish the link with the similar phenomenon related to *hu*. Further, Zou Dahai rightly perceived that a state of husked millet and husked rice allowed this third meaning of *dan* to correspond with the capacity amount of 1 *dan*, without, however, bringing to light the specific part played by "standard husked grain" in structuring the system of grains (see below). Zou seems also not to understand the use of this measuring unit as we do for states of grain finer than the "standard husked grain." Finally, because Zou treats volume and capacity together, he did not address the question of the part played by volume, as opposed to capacity, in the statement of norms relating to grains. Various clues indicate that the management of grain in early imperial China made use of distinct types of measuring units in various instances. Zou Dahai seems to think that capacity measurement played a central role, a point we question below.

1.4 The problem with *dan* (or *hu*) and its resonance with the enigma of "small *dan*"/"large *dan*"

The phenomena described revealed a complex use of measuring units called *dan*, which is reflected in mathematical sources. Such a conclusion recalls a problem related to documents of the practice that has been discussed in many publications since the 1950s and is still not satisfactorily solved. Indeed, administrative accounting books were found in the northwestern part of China, at 居延 Juyan and 敦煌 Dunhuang. Most of them date from the Western Han dynasty, although some probably date from Wang Mang's time or the Eastern Han dynasty. That is to say, these documents were produced during a time period that partially overlapped with the period during which the mathematical documents discussed above were produced. Finally, these documents were found in a desert area, at that time at the margins of the Chinese empire.

The key point for us is the *nature* of the unsolved problem these documents raised, since it relates precisely to measuring units called *dan*. In these documents, one regularly finds not only *dan*, in relation to measures of capacity, but also "小石 *xiao dan* small *dan*" and "大石 *da dan* large *dan*," that is, different types of *dan*. However, no agreement has been reached on the meaning and nature of these measuring units, or regarding how they were used. The clarification on *dan* that this article offers on the basis of the earliest extant mathematical documents in China also helps us to elucidate this other puzzle.

Before we present our solution to the problems outlined above, let us recapitulate the data and information collected about the measurement of grain in the various documents we have mentioned, mainly mathematical writings. We have emphasized several times that on the topic of grain, the three mathematical works referred to displayed strong correspondences. In our summary, we shall emphasize similarities and differences.

We have listed several types of data, and several kinds of statements related to grains.

First, we have seen that except for the case of *Mathematics*, mathematical books like the *Book of Mathematical Procedures* and *The Nine Chapters* contain texts stating numerical values expressing equivalences between different types and states of grain. The two books deal with the same list of grains, although they do not mention the same variety of states for these grains.[49] Moreover, as a whole, the numerical values state the same relationship between the types and states of grain that are common to both books. However, the nature of the quantities used to express these equivalences, as well as the structure of the texts in which they occur, differ depending on the book they appear in.

Second, we have discussed the fact that our three mathematical writings contain sets of rules of three to compute equivalent amounts of different types and states of grain, according to the standards formulated in the texts just mentioned. Despite differences between the three books concerning the way the rules of three were

[49] Many states of the grain that appear in the table contained in *The Nine Chapters* (Text 4) are not mentioned in the *Book of Mathematical Procedures*. On the other hand, the state of "highly refined rice *hui can mi*" seems not to occur in *The Nine Chapters*.

formulated, a point we leave for another article, the procedures, and the numerical values they use are basically the same. Moreover, the values used in the rules of three are of the same type.

Third, we have seen that *Mathematics* was the sole book to offer clauses of the type "Twenty-seven *dou* six *sheng* of unhusked rice has a weight of one *dan*," (Text 12) which allow conversion of a measurement of grain using measuring units of capacity into measuring units of weight. However, we should add that in slips 48 and 49 of the *Book of Mathematical Procedures* (Peng Hao 2001, 61), apparently a mixture of measuring units of weight and measuring units of capacity is used to express quantities of unhusked grain. By contrast, *The Nine Chapters* do not seem to refer to the use of weight measuring units to assess quantities of grain.

Fourth, it appears that all three books provide clauses of the same type, and generally with identical numerical values, for transforming an amount of grain expressed by means of measuring units of volumes into what appeared to be measuring units for capacity. The key example for this is the statement that we find in *Mathematics*: "one *dan* of unhusked grain (millet) occupies two *chi* seven *cun*" (Text 14). Note that the numerous rules of three contained in our three mathematical books are all devoted to the computation of equivalences between grains, but that none relates to the other types of conversion (between weight and capacity, or between volume and what seems to be capacity) for which the mathematical books provide standards.

Altogether, the evidence gathered shows that different types of measuring units (weight, volume, capacity...) were used to assess amounts of grain. Why did actors need to use such a variety of measuring units? In which phase of the management of grains was each of these measuring units used? Do the various mathematical books echo the same practice of grain management or not? Which features of the management of grains are the same and which could be different? Given that we do not have the answer to these questions yet, we must be careful in using the evidence the books provide. In relation to the use of weight measuring units, does the difference between *Mathematics* and the *Book of Mathematical Procedures,* on the one hand, and *The Nine Chapters* on the other, reflect another way of handling grain or the fact that the books are of different types? How do the various books relate to the evidence provided in documents of practice? Which light do they shed on the management of grains and the mathematical knowledge required to carry out the tasks it involved? Finally, in addition to the complexity of the measuring units used, we have revealed another type of complexity, which relates to the measuring units called *dan* and later *hu*. According to our sources, these units are affected by very strange phenomena that need to be explained. This is the task to which we now turn.

2 First decoding of the table "Norms/divisors for unhusked and husked grains"

The key piece of evidence for clarifying the question of *hu* comes from *The Nine Chapters*. It is associated with a set of problems (problems 23–25 in chapter 5) which play an important role in our argument. Let us thus first quote them:

(15) "今有委粟平地，下周一十二丈，高二丈，問積及爲粟幾何。
苔曰:
積八千尺。(...commentary...)
爲粟二千九百六十二斛二十七分斛之二十六。(...commentary...)
今有委菽依垣，下周三丈，高七尺，問積及爲菽各幾何。
苔曰:
積三百五十尺。(...commentary...)
爲菽一百四十四斛二百四十三分斛之八。(...commentary...)
今有委米依垣内角，下周八尺，高五尺　問積及爲米各幾何。
苔曰:
積三十五尺九分尺之五。(...commentary...)
爲米二十一斛七百二十九分斛之六百九十一。(...commentary...)

SUPPOSE THERE IS UNHUSKED GRAIN (MILLET) PILED UP ON A FLAT GROUND (SO TO MAKE A SOLID WHOSE) LOWER CIRCUMFERENCE IS TWELVE *ZHANG* AND WHOSE HEIGHT IS TWO *ZHANG*. ONE ASKS HOW MUCH THE VOLUME IS AND HOW MUCH UNHUSKED GRAIN (MILLET) IT MAKES.

ANSWER:
THE VOLUME IS EIGHT THOUSAND *CHI*.
IT MAKES, IN UNHUSKED GRAIN (MILLET), TWO THOUSAND NINE HUNDRED SIXTY-TWO *HU* TWENTY-SIX TWENTY-SEVENTHS OF A *HU*.

SUPPOSE THERE IS A HEAP OF SOY BEANS PILED UP AGAINST A WALL (SO AS TO MAKE A SOLID WHOSE) LOWER CIRCUMFERENCE[50] IS THREE *ZHANG* AND WHOSE HEIGHT IS SEVEN *CHI*. ONE ASKS, RESPECTIVELY, HOW MUCH THE VOLUME IS AND HOW MUCH SOY BEANS IT MAKES.

ANSWER:
THE VOLUME IS THREE HUNDRED FIFTY *CHI*.
IT MAKES, IN SOY BEANS, ONE HUNDRED FORTY-FOUR *HU* EIGHT TWO HUNDRED FORTY-THIRDS OF A *HU*.

SUPPOSE THERE IS A HEAP OF HUSKED GRAIN (MILLET) PILED UP AGAINST THE CORNER OF A WALL (SO AS TO MAKE A SOLID WHOSE) LOWER CIRCUMFERENCE[51] IS EIGHT *CHI* AND WHOSE HEIGHT IS FIVE *CHI*. ONE ASKS, RESPECTIVELY, HOW MUCH THE VOLUME IS AND HOW MUCH HUSKED GRAIN (MILLET) IT MAKES.

ANSWER:
THE VOLUME IS THIRTY-FIVE *CHI* FIVE NINTHS OF A *CHI*.
IT MAKES, IN HUSKED GRAIN (MILLET), TWENTY-ONE *HU* SIX HUNDRED NINETY-ONE SEVEN HUNDRED TWENTY-NINTHS OF A *HU*."[52]

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[50] As the commentator explains later on, the circumference here designates the half-circumference, which is the lower dimension of the heap, which has the shape of a half-cone.

[51] As the commentator explains later on, the circumference here designates a quarter of the circumference of a cone. It is the lower dimension of the heap, which now has the shape of a quarter of a cone.

[52] A critical edition with annotations can be found in (Chemla and Guo Shuchun 2004, 446–453). Here, we simply summarize the main point of the argument, since it provides a foundation for the following part of the article. See the discussion in relation to the problem of *hu* in K. Chemla, introduction to chapter 2, in (Chemla and Guo Shuchun 2004, 201–205).

These three problems each deal with grain piled up in such a way that the heap constitutes, respectively, a cone, half a cone, or a quarter of a cone. The grain used for each problem differs, which is the important point here. Note first that the lengths of two dimensions of the solid created are given, namely the lower circumference and the height. They are naturally expressed using measuring units for length. This illustrates how measuring units for length can be used to assess an amount of grain. The procedure for solving the first part of the problems, namely that yielding the volume of the heap, enters these two data into the algorithm, given earlier in *The Nine Chapters*, that computes the volume of a cone. We do not need to look at that part of the procedure except to emphasize the fact that in the end, the amount of grain is measured using measuring units of volume, as explained above. Let us summarize the data and first result of each problem in the following table.

Problem	Problem 5.23	Problem 5.24	Problem 5.25
Grain	Unhusked grain (millet) 粟	Soy beans 菽	husked grain (millet) 米
Shape of the Pile	Cone	A half-cone	A quarter cone
Circumference and height allow determining the volume in *chi*			
Amount of Volume	8,000 *chi*	350 *chi*	35 *chi* 5/9 *chi*
Capacity in *hu*	2,962 *hu* 26/27 *hu*	144 *hu* 8/243 *hu*	21 *hu* 691/729 *hu*

It is in this context, and at this point, that the text of *The Nine Chapters* mentions the clauses explaining how to transform measuring units of volume into *hu*, which is precisely the task to be carried out in the second part of each of the solutions to the respective problems. Let us quote these clauses, which we already mentioned above, adding Liu Hui's commentary on them. They read as follows:

(16) "程粟一斛積二尺七寸；二尺七寸者，謂方一尺，深二尺七寸，凡積二千七百寸。其米一斛積一尺六寸五分寸之一；謂積一千六百二十寸。其菽、荅、麻、麥一斛皆二尺四寸十分寸之三。謂積二千四百三十寸。

ONE *HU* OF UNHUSKED GRAIN (MILLET), BY REGULATION, HAS A VOLUME OF TWO *CHI* SEVEN *CUN*. Two *chi* seven *cun* means a parallelepiped with a square base and a side of one *chi* and a depth of two *chi* seven *cun*, making the total volume two thousand seven hundred *cun*. ONE *HU* OF THE HUSKED GRAIN (MILLET), BY REGULATION, HAS A VOLUME OF ONE *CHI* SIX *CUN* ONE-FIFTH OF A *CUN*, which means a volume of one thousand six hundred twenty *cun*. ONE *HU* OF SOY, MUNG, HEMP BEANS AND BARLEY, BY REGULATION, ARE ALL TWO *CHI* FOUR *CUN* THREE TENTHS OF A *CUN*, which means a volume of two thousand four hundred thirty *cun*." (Chemla and Guo Shuchun 2004, 450–453)

Our discussion of such clauses, above, referred to the problem with understanding *hu* as a measuring unit for capacity. If *hu* were the largest measuring unit for capacity here, how could we account for the fact that the associated volume varies in relation to the kind of grain measured, as was made clear above, using the following table?

	Problem 5.23	Problem 5.24	Problem 5.25
Grain	Unhusked grain (millet) 粟	Soy beans 菽	Husked grain (millet) 米
The volume of1 斛 *hu* is	2 *chi* 7 *cun*	2 *chi* 4 *cun* 3/10 *cun*	1 *chi* 6 *cun* 1/5 *cun*

In any event, the second part of the solution to each of the problems 5.23 to 5.25 requires that the volume of the heap be divided by the volume associated with the *hu* linked to the grain dealt with, in order to yield the number of *hu* stated in the answer. The way the results are given thus implies that the different volumes associated with the *hu* in relation to the different grains are put into play.

The issue becomes even more complex when we notice that in other contexts, the measuring unit *hu* is used in a completely different way, even though exactly the same types of grain are involved. To make this point, it suffices to mention the following problem, in which the answers and the procedure make clear that the same measuring unit for capacity *hu*, that is, a *hu* with the same volume throughout, is used to express measurements for each of these grains. Problem 6 of chapter 6 reads as follows:

(17) "今有人當稟粟二斛 倉無粟，欲與米一、菽二，以當所稟粟 問各幾何。

苔曰:

米五斗一升七分升之三,

菽一斛二升七分升之六。

SUPPOSE A PERSON MUST GRANT TWO *HU* OF UNHUSKED GRAIN (MILLET). SINCE IN THE STATE GRANARY, THERE IS NO UNHUSKED GRAIN (MILLET), HE WANTS TO GRANT HUSKED GRAIN (MILLET) AND SOY BEANS IN A PROPORTION OF ONE TO TWO IN SUCH A WAY THAT THIS AMOUNTS TO THE UNHUSKED GRAIN (MILLET) THAT SHOULD BE GRANTED. ONE ASKS HOW MUCH FROM EACH (THE PERSON GRANTS).

ANSWER:

FIVE *DOU* ONE *SHENG* THREE-SEVENTHS OF A *SHENG* OF HUSKED GRAIN (MILLET);

ONE *HU* TWO *SHENG* SIX SEVENTHS *SHENG* OF SOY BEANS." (Chemla and Guo Shuchun 2004, 506–507)

After the procedure presented in *The Nine Chapters,* the commentary written under Li Chunfeng's supervision explains the reasons why that procedure is correct. In particular, it shows that the numerical values associated with unhusked grain (millet), husked grain (millet), and soy beans in "Norms/divisors for unhusked and husked grains 粟米之法" (Text 4) were used to compute equivalences between the various grains and complete the solution of the problem.

What makes the difference between these two uses of *hu*? Why is it that in some contexts, the volume associated with the *hu* varies depending on the grain and, in others, it does not? To get insight into the solution to these puzzles, we must rely on two sets of documents. One of these is Liu Hui's commentary, completed in 263, in particular the commentary on Text 16. The other set consists of the chapters "Monograph on Pitch Pipes and the Calendar" that the second commentator, Li

Chunfeng, composed for insertion in the *History of the Jin* and the *History of the Sui*, respectively. In these monographs, Li Chunfeng quotes Liu Hui's commentary precisely on topics related to grains and provides us with important pieces of evidence. On this basis, we can establish that the *hu* mentioned in problems 5.23 to 5.25 are each associated with a standard vessel (Liu Hui uses the term 器 *qi*) also called *hu*. In other words, each type of grain or each state of a grain has a vessel associated with it whose capacity is expressed as "one *hu*," and whose volume is given by the clauses quoted in Text 16 from *The Nine Chapters*.

The key to this use of *hu* is given by Li Chunfeng, when in the context of the "Monographs on Pitch Pipes and Calendar," that is, in the context of official historical writing on systems of measuring units and their history, he comments on these different types of *hu* mentioned in *The Nine Chapters*. Let us note in passing that this fact alone shows that Li Chunfeng believes the evidence provided by *The Nine Chapters* on these *hu*, and Liu Hui's interpretation of them actually constitutes reliable historical information. This highlights that Li Chunfeng regards these two earlier mathematical writings as closely related to official institutions.

In the context of the "Monographs on Pitch Pipes and the Calendar," Li Chunfeng states the following about this use of *hu*: "(These *hu*) cause the **values** to be at the same level/be equal, but do not equalize the amount of *cun* of the volumes of these vessels. (...) 使價齊而不等其器之積寸也。"[53] In other words, according to Li Chunfeng's explanation, when using *hu* in this way, one *hu* of a given grain measured with the vessel associated with this grain has the *same value*, whichever grain is considered. This is what lies behind the variability of the unit *hu* 斛: In these contexts, and in fact in these contexts only in the cited passages in *The Nine Chapters*, *hu* is a *unit of value*, and the different volumes of grain corresponding to a *hu*, depending on the grain considered, correspond to the same value. Note the correlation between value and volumes in Li Chunfeng's statement, which precisely echoes the relation between value and volumes stated by the clauses inserted in Text 16. We have thus established a second meaning of *hu* as a measuring unit: As *The Nine Chapters* evidences, the term *hu* is not only used as a measuring unit for capacity, but also refers to a measuring unit for value.

Our first conclusion is thus that the clauses quoted above, in Text 16, express the volume of a given type of grain or a given state of grain that is needed to make a unit of value *hu*. Now, we have seen that such clauses also occur in *Mathematics* (see texts 13 and 14). For instance, *Mathematics* has the following statement, quoted above: "One *dan* of unhusked grain (millet) occupying two *chi* seven *cun*." Moreover, we mentioned that the *Book of Mathematical Procedures* also uses a clause of this type, with exactly the same value (slip 146).

In these other cases, the clauses link what seems to be the largest measure of capacity, namely, in these other contexts, *dan*, and a volume. Further, different clauses, for different types or states of grain, provide different volumes for the *dan*. We can thus assume that the continuity in the type of statement as well as the

[53] Our emphasis. In fact, we could interpret the term 價 as "price" as well as "value." However, we believe that the interpretation as "value" is here more accurate (Fang Xuanling et al. 房玄齡等 1973, 492, Wei Zheng et al. 魏徵等 1973, 409).

continuity in the amount across all mathematical sources is most probably correlated with continuity in meaning and in practice. If true, this would indicate that the same phenomenon affected *dan* from the Qin dynasty onward. In other words, *dan* was not only the name of a measuring unit for weight and capacity, but also for value. We provide evidence below to support this assumption. We shall see how such an interpretation of *dan* reveals the strong coherence of the system it enables us to perceive. Note, for the moment, that if these clauses were defining units of value, this would easily explain the stability in the quantities that occur in these statements.

To conclude on this use of the measurement unit *hu* 斛 (and its precursor 石 *dan*), we have seen that in such contexts as those of problems 5.23 to 5.25 (respectively, the related problems from the *Book of Mathematical Procedures* and from *Mathematics*), actors wanted 1 *hu* 斛 (resp., 1 *dan* 石) of a given grain to correspond to the same value whatever the grain under consideration. This is what the unit *hu* measures in this context, and it corresponds to a system of vessels—at least, this is what *The Nine Chapters* seems to indicate and how its commentators interpreted its text.

In this case, the determination of amounts of grain having the same value could be carried out by mere manipulation of the vessels. This raises two sets of questions. First, were there similar vessels at the time this passage of *The Nine Chapters* was written? Were there similar systems of vessels at the time the *Book of Mathematical Procedures* and *Mathematics* were composed? We know that vessels that could fulfill these criteria were found. However, we have not yet been able to do research on them to establish this point. Secondly, which grain in *The Nine Chapters* corresponded to the *hu* as the regular measure of capacity? The first set of questions is still open. In what follows, we shall suggest an answer to the second.

We have reached the essential point. In fact, in order to determine amounts of different types of grain or different states of these grains having the *same value*, one could choose between three distinct practices for which *The Nine Chapters* provides pieces of evidence. First, for each type or each state of the grains involved, one could use the vessel specifically associated with it. Taking the same number of times the amount of grain corresponding to the capacity of the vessel associated with a given grain, and repeating the operation for the different grains each time with the appropriate vessel, yields amounts of grain having the *same value*. In this context, the equivalent amounts of grain yielded for these different grains have *different volumes*.

Alternatively, for each of the grain types, one could convert the given value, expressed using *hu* as a measuring unit of value, into the volume corresponding to it. To do this, one could simply use clauses of the type evidenced in Text 16. One thus determines the volume corresponding to a given value for each grain and measures these amounts for each of these grains using volume measurement. Again, in this case, the volumes differ, but the value is the same.

Finally, and this is what is shown in problem 6.6 (Text 17), one could use *hu* as a measuring unit for capacity and thus only use a single vessel for all grains. In this context, the same amount of *hu* refers to the *same volume* of the various grains dealt with. However, since these same volumes do *not* have the *same value*, to determine amounts of grain with the same value one picks out the numbers related to the grains dealt with from the table "Norms/divisors for unhusked and husked grains" (Text 4). Using rules of three of the type evidenced in our three mathematical books (see texts 6, 8, 10, 11), one determines the amount of any type or state of grain equivalent to a given amount of any other type or state of grain, all expressed with capacity measuring units. The key reason why this alternative method works in the same way as the two others mentioned above is that the numbers inserted in the table "Norms/divisors for unhusked and husked grains" have the same ratio as the ratio between the volumes provided by the clauses. To illustrate this claim, let us give the example of unhusked and coarsely husked millet. In Text 16 from *The Nine Chapters,* and in Liu Hui's commentary, the volumes attached to these two states of grain are, respectively, "2,700 *cun*," and "1,620 *cun,*" and their ratio is the same as 50 and 30, as shown in the table. This last remark highlights the essential connection between clauses of the type listed in Text 16 and the table "Norms/divisors for unhusked and husked grains," both found in *The Nine Chapters*. Before we conclude what this reveals with respect to the table, let us refer to a piece of evidence showing that the connection just established was meaningful for actors in early China.

In fact, the scholar Zheng Xuan 鄭玄 (127–200) associates the title of the table, "Norms/divisors for unhusked and husked grains 粟米之法," with a relationship of the type listed in Text 16. This fact appears clearly in Zheng Xuan's commentary on the vase *fu* 鬴 mentioned in the canonical work called the *Rites of the Zhou*. In this book, the *fu* is said to have a capacity of 6 *dou* and 4 *sheng*, whereas in the *Commentary by Master Zuo,* it is said to have a volume of 1,000 *cun.*[54] It is by relying on what he calls "Norms/divisors for unhusked and husked grains" that Zheng Xuan states the amount expressed in capacity by which these two numerical values fail to correspond to each other. When the subcommentator Jia Gongyan 賈公彥 accounts for the result stated by Zheng Xuan, he explicitly states that the correspondence thereby indicated by Zheng Xuan relates one *dan* to 1,620 *cun.* We have seen earlier that this was precisely the clause corresponding to coarsely husked grain (millet) in *The Nine Chapters*. What is important for us is that Zheng Xuan's use of the expression "Norms/divisors for unhusked and husked grains 粟米之法" to refer to such relations confirms the reading we propose for the table.[55]

To conclude on *The Nine Chapters,* problems 5.23 to 5.25 provide evidence of a system of vessels, each measuring a unit of value *hu* for a specific grain. As a whole, this system embodies the fact that *hu* is a measuring unit of value. On the other hand, *The Nine Chapters* testifies to a system of rules of three (texts 5, 6, 17), which reflects the numbers attached to each type of grain in the table "Norms/divisors for unhusked and husked grains." These two facts indicate the meaning of the table: It assembles numbers that can be used to express any ratio between the volumes of the various measuring units for value called *hu* associated with the different grains. The table is thus intimately linked to the concept of value introduced by Li Chunfeng in the "Monograph on pitch-pipes and the calendar." Note that these ratios between

[54] On the history of these two canonical books, compare (Boltz 1993, Cheng 程艾藍 1993).

[55] On the analysis of Jia Gongyan's computation in the context of the subcommentary on Zheng Xuan's commentary on the *Rites of Zhou,* see Zhu Yiwen, forthcoming. We are grateful to Zhu Yiwen for drawing our attention to this piece of evidence. We shall see below that in the case of "coarsely husked millet," the unit of capacity *hu* coincides with the unit of value *hu.*

volumes of the same value generally remain stable between the time when *Mathematics* was composed and the time when *The Nine Chapters* was compiled.[56] Further, note that in the table "Norms/divisors for unhusked and husked grains," these relationships are expressed by means of abstract "integers."

We have thus proved, on the basis of the evidence provided by *The Nine Chapters* and its commentators, that the measuring unit *hu*, the use of which these writings testify to, was both a measuring unit for capacity *and* a measuring unit for value. How did actors differentiate between the two uses when they read the character *hu* in a document? And how did they decide which *hu* should be used in each different context? These are some of the questions that await further research. We shall now proceed to establish that the same conclusion holds true for *dan* in the earlier time period.

3 First decoding of the "Regulation for cereal plants" and the issue of "small *dan*"/"large *dan*"

We shall now analyze in greater detail the section entitled "Regulation for cereal plants." The *Book of Mathematical Procedures* shares this text with the "Granary statutes" from Qin statutes and edicts (Text 7). Our attention will be mainly focused on the meaning of the measuring units that appear in the text.

Slips 88 and 89 from the *Book of Mathematical Procedures* begin as follows: "The regulation says: one *dan* of broomcorn millet (that is, the cereal just harvested) makes ..." and "The regulation says: One *dan* of paddy stalks (that is, cereal just harvested) makes..." Given the state of the plants designated by the terms, there is no doubt that the *dan* occurring in these statements refers to a measuring unit of weight. This point was emphasized by Peng Hao. All present-day authors agree on it. Moreover, as all historians noted, the relevant entries of the dictionary *Shuowen jiezi*, completed around 100 by Xu Shen 許慎, confirm this interpretation. It is worth noting that the parallel part played by weight measuring units in the two series deriving from millet and rice is made clear in the structure of the text of the statute as quoted in the *Book of Mathematical Procedures*. However, the damaged version kept in the copy of the Qin statutes buried in tomb 11 at Shuihudi concealed this parallel.

After these initial statements, measuring units for capacity, mainly the *dou*, clearly occur. Naturally, the *dan* also occurs. However, on the basis of the explanations above, we have reason to be careful about the interpretation of this character. Note, in particular, that every time the measuring unit *dou* occurs in Text 7, only the *dou* is used, even for quantities higher than 10 *dou*, i.e., for quantities for which the *dan* could be used. For example, the quantity "sixteen *dou* two-thirds of a *dou*" is not expressed as "one *dan* six *dou* two-thirds of a *dou*." However, we quoted earlier another passage from a Qin statute in which both *dan* and *dou* are used to

[56] More precisely, the values used in the rules of three have exactly the same ratio in all the books. However, as we stressed above, the volume that *Mathematics* associates with one *dan* of barley slightly differs from the one *The Nine Chapters* mentions.

express an amount of capacity: "Small bond-women and grain-pounders who are working receive a ration of millet grain of one *dan* two and a half *dou* per month." Moreover, we encounter a similar phenomenon in the *Book of Mathematical Procedures* (slip 76, Peng Hao 2001, 74.)

As we emphasized above, Text 7 begins with two sequences of processing of grain starting, respectively, with "millet 禾黍" and "paddy 稻禾." For each of these kinds of grain, in the related slip, there is a state of the grain called "unhusked grain *su* 粟," which refers to the grain obtained through threshing. In Text 9 from the *Book of Mathematical Procedures*, we saw that this state of "unhusked grain *su* 粟" for the two types of grain (millet and paddy) occurs in the rules of three. However, we also emphasized that when the character *su* occurs alone, it generally refers to this state of the grain for millet.

After "unhusked grain," several other states are mentioned in slips 88 and 89. The question we want to address now is: What is the meaning of the equivalences presented in Text 7?

In slips 88 and 89, the same phenomenon occurs: the state of the grain that is listed following "unhusked grain," namely, "(coarsely) husked grain," is constantly associated with 1 *dan* (once formulated as 10 *dou*). Note that this state is followed by other states of further refined husking. This means there is one degree of husking for which the related measure is precisely 1 *dan*, and this holds true in both slips. For both slips, whether we deal with millet or paddy, there is a state of the grain whose associated measure is 1 *dan*, in relation to the fact that the harvested amount of plants from which it derives weighed 1 *dan*—the latter *dan* being a weight measuring unit.[57] We shall return to this remark. For the moment, let us concentrate on the overall meaning of the table.

For millet, the ratio between 1 *dan* of this state, namely (coarsely) husked grain, and the 16 *dou* 2/3 *dou* of unhusked grain is exactly the same as the ratio recorded for these two states of the grain in the table in *The Nine Chapters* (Text 4). In all these cases, it is 3 to 5. The same ratio is also found in all the rules of three related to these two states of millet. We mentioned earlier that both *Mathematics* and the *Book of Mathematical Procedures* give the following clause to state *dan* in relation to measuring units for volume for unhusked millet: "One *dan* of unhusked grain (millet) occupying two *chi* seven *cun*" (Text 14). We have seen that the ratio between this volume and that associated with husked grain was also 5 to 3 in *The Nine Chapters*. This was a key element for establishing that *hu* was a measuring unit of *value*.

An analysis of Text 7 and other documents with the help of measuring units of volume now proves essential. Let us thus first consider the evidence we have about how the official measuring units for capacity were related to amounts of volume. In the fourth century before the Common Era, the minister of the State of Qin, Shang Yang, carried out key reforms in relation to war and agriculture.[58] One of the important measures attached to the latter domain, and more particularly to the tax system and the management of grains, was the reform of measuring units. Part of the

[57] Peng Hao (2012, 196) noticed the relationship between these two data. He emphasized the simplicity in the management of grains that such a relationship entailed.

[58] On Shang Yang, see, for instance, (Vandermeersch 1965), in particular chapter 1.

reform consisted in instituting a bronze vessel as standard for one of the central measuring units for capacity: the *sheng* (one tenth of the *dou* and one hundredth of the *dan*). An example of the standard, called "Shang Yang's parallelepipedic *sheng* 商鞅方升,"[59] is now kept in Shanghai museum 上海博物館. It bears an inscription from Shang Yang's time which indicates the date 344 B.C.E., which is crucial for our purpose. The inscription contains the following statement about the measurement of the vessel: "(...) 鞅爰積十六尊 (寸) 五分尊 (寸) 壹為升 [Shang] Yang took the volume of sixteen *cun* (one) fifth of a *cun* as a *sheng*." The measurement of the volume is expressed using the same principle as that explained above, namely, it utilizes measuring units of length. Moreover, it uses fractions. Using a decimal representation of the amount of *cun* yields a volume of 16.2 *cun* as a *sheng*. The corresponding unit of capacity *dan* is accordingly related to the volume 1,620 *cun*. Note that this numerical value for the volume corresponds to that associated with coarsely husked grain in *The Nine Chapters*, in which it is expressed as "one *chi* six *cun* one-fifth of a *cun*" (Text 16), i.e., in a way quite close to the content of the inscription quoted above. We shall return to this remark below. More than a century later, during the Qin dynasty, another inscription was added to the base of Shang Yang's parallelepiped *sheng*. It refers to the standardization of measuring units for weights and measures carried out by the first emperor, Qin Shihuang, after the unification of the Chinese empire. This addition of an official inscription indicates that the Qin rule continued using the same measuring standard for the *sheng*. More than two centuries later, Wang Mang, the emperor who attempted to overthrow the Han dynasty and enacted a new system of measuring units, had a bronze measuring vessel made to embody the system of measurement units for capacity. This vessel, called by Liu Hui in 263 and by later scholars "Wang Mang's bronze *hu*," was thus named after the largest measuring unit for capacity, at that time *hu*. It also bore a complex inscription, discussed through the following centuries by scholars versed in mathematics.[60] Only one point of this inscription is of interest to us here, namely, the statement of the relationship between the *hu* and measuring units for volumes. The corresponding statement reads: "律嘉量斛方尺而圜其外, 庣旁九釐五毫, 冥(冪)百六十二寸, 深尺, 積千六百二十寸, 容十斗。The most common standard capacity vessel *hu*, enacted by edict, has a (fictitious) square of side one *chi* (inside) and a circle outside of it. The distance between the interior and the exterior[61] is 9 *li* 5 *hao*. The area is 162 *cun*, the depth 1 *chi*, the **volume 1,620 *cun***, the capacity (*rong*) 10 *dou*."[62]

[59] For a discussion of Shang Yang's vessel with a photograph of the vessel and its inscriptions, as well as a transcription of the inscribed text, see (Qiu Guangming etal. 2001, 165–166).

[60] See the translation and annotation of passages in (Chemla and Guo Shuchun 2004, 182–185, 452–453, 456–457).

[61] This translates the term *tiaopang*, which commentators interpreted as the distance between the inner circumference of the vessel and the vertices of the fictitious square inscribed in its center ("what exceeds/goes beyond on the side"). Note that the *hao* is about 0.02 mm, which implies that the order of magnitude of accuracy probably has no relation to the actual design of the vessel.

[62] For a photograph of the vessel and a discussion of its inscription, see (Qiu Guangming etal. 2001, 221–222). We return below to the term *rong* used here to designate the capacity.

We see that again the volume attached to the largest measuring unit of capacity is still 1,620 *cun*, despite the fact that its name was changed to *hu*. Finally, this is also, as we have seen, the numerical value that Zheng Xuan uses in his commentary on the *Rites of Zhou* to connect the measuring unit of capacity and those of volumes. On this basis, we can assume that the relationship has remained unchanged over the whole period we consider. In particular, it held true at the time when the statute whose interpretation we are discussing was used in the Qin statutes or in the *Book of Mathematical Procedures*. Let us now use this numerical value to analyze the volumes corresponding to some of the capacities mentioned in Text 7. We show the results of the analysis by replacing some of the capacity measurements in the translation by the corresponding volumes:

(7′) The regulation says: one *dan* (unit of weight) of broomcorn millet (that is, the cereal just harvested) makes **2,700 *cun*** of unhusked grain (millet). Husking this makes one ***dan*** **(1,620 *cun*)** of coarsely husked grain (millet) (*li mi*) (…) The regulation says: **2,430 *cun*** of barley, or soy, mung, and hemp beans is one *dan* (…)[63]

This conversion brings to light several important features of the text.

First, we see that the measures of capacity attached to each of the grains we kept from the text are related to amounts of volumes we know quite well and have already encountered above, some in the *Book of Mathematical Procedures,* some in *Mathematics*, and all in *The Nine Chapters*. These amounts of volumes are precisely those with which units of value for the *same* grains are defined in *The Nine Chapters*.

Second, whether we look at the ratios between the volumes, or between the related capacities, for the grains underlined, these ratios are the same as those involved in the rules of three inserted in all the three books that enable the determination of equivalent amounts of grain. In fact, if we now rely on the text as a whole and focus on the ratios between the capacity measures it mentions for most of the types of grain and most of the states of grain included, these ratios do indeed reproduce the ratios used in the related rules of three.[64] It is thus probable that in the same way as "Norms/Divisors for unhusked and husked grains" (Text 4) provided numbers whose ratios reproduced the ratios between the volumes of the *hu* as

[63] Peng Hao (2012, 199–200, 201) also uses the tool provided by volumes to analyze the text. In Peng Hao (2013, 30-01-2013, at 10:42), he comes back to this point. For this, in both publications, he relies on Shang Yang's parallelepipedic *sheng*. However, our conclusions are different. With respect to our conclusions, see the explanations above and below. Peng Hao, for his part, uses the relationship between the first measuring unit for weight *dan* and the corresponding volume of unhusked grain to propose a hypothesis: In his view, unhusked grain was the foundation from which the whole system of grain was derived on the basis of determining conversions between unhusked grain and all the other grains and states of grain. One key difference, as will emerge below, is that he does not establish any connection between Shang Yang's parallelepipedic *sheng* and coarsely husked grain.

[64] This statement is generally true, with the exceptions mentioned earlier. First, the state the text calls *zuo mi* (finely husked grain (millet)) is called *bai mi* in the rules of three inserted in all three texts. However, in *The Nine Chapters,* the term *zuo mi* is used to refer to what is in this text called *hui mi*. Secondly, in fact the volume associated with barley, or soy, mung, and hemp beans in *Mathematics* is 2,400 *cun* and not 2,430 *cun*, as it is the case in Text 7 and Text 16 from *The Nine Chapters*. Finally, for the moment, the final statement of Text 7 cannot be explained in these terms.

measuring unit of value, Text 7, namely the text shared by the *Book of Mathematical Procedures* and the Qin statutes, expresses, by means of amounts of capacity, the ratios between the different volumes associated with each type or each state of grain and having the same value.

Lastly, and most importantly, if, on the basis of the previous discussion, we return to a sentence that appeared difficult to understand in Text 7, namely "Fifteen *dou* (unit of capacity) of barley, or soy, mung, and hemp beans is one *dan*," the analysis developed allows us to interpret it quite naturally. It further brings to light an essential fact. We mentioned above the problem of interpreting this statement, by raising the following question: If *dan* and *dou* are both measuring units of capacity, the *dan* being 10 *dou*, how can one explain that in this sentence, 15 *dou* is associated with 1 *dan*? Seen from the viewpoint of the volume associated with the 15 *dou*, the statement becomes "2,430 *cun* of barley, or soy, mung, and hemp beans, is one *dan*." This statement is clearly identical, up to some details, to the statement with which *The Nine Chapters* defines the unit of value *hu* for the same grains. From this perspective, we can now suggest two interpretations. The first is that in this statement of the Qin statute, *dan* is a measuring unit of value, exactly as we have proved that *hu* is. This interpretation makes the statement crystal clear. Moreover, this conclusion indicates that some of the *dan* that occur in this text must be interpreted as units of value. The alternative interpretation is that the statement provides the relationship between unhusked barley, or soy, mung, and hemp beans, on the one hand, and the related husked grains, which would have one *dan* of capacity and thus a volume of 1,620 *cun*. In this case, the statement would establish an equivalence between two states of the grain. It may be the case that some occurrences of barley in Juyan slips refer to an opposition between unhusked and husked barley. However, we need more research on this topic. If this is confirmed, we would then adopt the latter branch of the alternative, which is the least favorable for the argument we want to make that here *dan* means a unit of value. Let us stress that in the latter case, in Text 7, for the husked grain here and in the parallel statements, the measure in capacity units would constantly be the same as the measure in value units. We shall return to this point later. If we opt for the former branch of the alternative, in one occurrence, the *dan* associated with the husked grains in all statements has the meaning of unit of value only, and not the double meaning, and could also assume this meaning elsewhere in the text.

Given the inconclusive nature of the argument at this point, let us prove by other means that in the earliest manuscripts, *dan* not only has the meaning of measuring unit for weight or capacity, but also for value. To establish this point, we shall first return to Text 14, which deals with a question of a granary in the context of *Mathematics*. The outline of the problem provides the dimensions of a parallelepipedic granary. In addition, it makes clear that "one *dan* of unhusked grain (millet) occupies two *chi* seven *cun*." On this basis, the problem asks for the volume of the granary and the number of *dan* it contains. The results given make clear that the volume is computed by the usual means and expressed in the measuring unit for volume *chi*. The fact that the volume is divided by 2 *chi* 7 *cun* to yield the number of *dan* contained excludes the possibility that the *dan* computed refer to a capacity. We have seen that if such were the case, the volume should have to be divided by 1 *chi* 6 *cun* 1/5 *cun*. We have already shown (see note 47) that here *dan* could not be a measuring unit for weight. The amount of *dan* given as a result can thus only be assessing the number of measuring units of value contained in the granary. Note that this amount is given in *dan* and fractions of *dan*, and thus without using *dou* or *sheng*. Also, note that the amount is designated by the term *rong* 容. This term is usually interpreted as referring to the capacity. However, in this case, it must refer to the amount in terms of value.[65] We shall return to this problem later.

The *Book of Mathematical Procedures* offers a problem that is similar to this one and unambiguously requires that we also interpret the *dan* occurring there as referring to a unit of value. The problem reads as follows:

(18) "旋粟　旋粟高五尺, 下周三丈, 積百廿(二十)五尺。•二尺七寸而一石, 為粟卌(四十)六石廿(二十)七分石之八。其述(術)曰:下周自乘, 以高,/146/乘之, 卅六成一。•大積四千五百尺。/147/[66]

Making a cone of unhusked grain (millet). Making a cone of unhusked grain (millet) with a height of five *chi* and a lower circumference of three *zhang*, the volume is one hundred and twenty-five *chi*. • two *chi* seven *cun* making one *dan*, this makes forty-six *dan* eight twenty-sevenths of a *dan*. The corresponding procedure: the lower circumference being multiplied by itself, with the height/ 146/, one multiplies this, thirty-six generates one (i.e., one divides by thirty-six). • The volume of the large (solid)[67] is four thousand five hundred *chi*./147/"

Again, the transformation derives from dividing the volume of the cone by 2 *chi* 7 *cun*. As above, the result expressed in *dan* assesses the number of units of value in the heap of unhusked grain (millet) by counting one such unit for each 2 *chi* 7 *cun*.[68] Note that all these cases are strictly identical to problems 5.23 to 5.25 from *The Nine Chapters*. They all require determining an amount of value on the basis of, and in relation to, a volume. They all make use of the statements linking volume and value that occur in all the books. Finally, they are clearly all used in the context of the management of granaries. In all these cases, whether we are dealing with *dan* or *hu*, the meaning of the measuring unit is clearly neither weight nor capacity, but can only be value. Note that in all these cases, the use of *dan* as a measuring unit of value is intimately connected to its use as a measuring unit for capacity. This fact brings us back to Text 7, which occurs precisely, we recall, among the Qin statutes in the "Statutes on granaries."

[65] The same conclusion holds true in *The Nine Chapters*. If we observe the problem similar to this one, which is included in Chapter 5 (problem 5.27), the problem deals with a granary and relates volume and measuring units of value. It is again the term *rong* that designates the amount in measuring units of value, this time designated as *hu*.

[66] Peng Hao (2001, 105–106). We opt for the interpretation of *xuan* suggested by Peng Hao in first page note.

[67] This expression refers to the volume of the parallelepiped, which is to be divided by 36 to yield the volume of the cone sought.

[68] This is one of the main points where our interpretation differs from that presented in Peng Hao (2012, 199–200). In that passage, Peng Hao explains why he believes that these *dan* must be interpreted as measuring units for weight. As we explained above (see note 47), we think this interpretation is not feasible. We do not understand how this interpretation fits with Peng Hao's interpretation of *dan* as referring to "capacity" in assessing millet (p. 197).

Now that we have established that *dan* was also used as a measuring unit for value at the time our manuscripts were composed, one can state the following: Whichever branch of the alternative for the sentence discussed above from Text 7 we adopt, we are forced to the same conclusion about the nature of the text. In Text 7, all the grains opening the given lists, and all the states of grain deriving from them, are associated with the amount of capacity that expresses a unit value of 1 *dan* for the corresponding grain. There is therefore no doubt about the reference of Text 7. From this, we must, however, exclude two final statements of the text. First, the final equivalence in slip 89 states: "Ten *dou* of barley is three *dou* of grits." This statement, as we confirm below, associates equivalent amounts of barley and grits of barley, the value of which is not equal to one *dan*. Second, the last statement of Text 7 seems to state something about the measuring unit *dan* that we cannot yet interpret. However, if we exclude these two statements, the reference of Text 7 is clear.

We believe one can go a step further and state that this is what Text 7 explicitly states, by making use of *dan* as a measuring unit of value as well. The reason why we believe *dan* can be interpreted as a measuring unit of value when it can be also interpreted as a measuring unit of capacity in Text 7 is that, clearly, when this was not the case, the authors of the statute preferred to use 10 *dou* instead of 1 *dan* (remember the quantities we emphasized above: 16 *dou* 2/3 *dou*, or 20 *dou*). The opposition between cases in which they used 10 *dou* and cases in which they used 1 *dan* is, we suggest, meaningful. It can be interpreted as an emphasis on the double meaning of *dan,* when it occurs. In conclusion, we can thus reproduce Text 7 with a new analysis of the measuring units appearing in it as follows:

(7″) "Regulation for cereal plants. **The regulation** says: one ***dan*** **(unit of weight)** of broomcorn millet (that is, the cereal just harvested) makes sixteen *dou* two-thirds of a *dou* **(unit of capacity)** of unhusked grain (millet). Husking this makes one ***dan*** **(unit of capacity and unit of value)** of coarsely husked grain (millet) (*li mi*); one ***dan*** **(unit of capacity and unit of value)** of coarsely husked grain (millet) makes nine *dou* of finely husked grain (millet) (*zuo mi*); nine *dou* of finely husked grain (millet) makes eight *dou* of highly finely husked grain (millet) (*hui mi*). (Text checked by) Wang.

The regulation says: One **dan (unit of weight)** of paddy stalks (that is, cereal just harvested) makes twenty *dou* **(unit of capacity)** of unhusked grain (rice). Husking this makes ten *dou* **(unit of capacity, equivalent to unit of value)** of husked grain (rice), makes six *dou* two-thirds of a *dou* of highly refined rice. Ten *dou* **(unit of capacity, not equivalent to unit of value)** of barley is three *dou* of grits.

The regulation says: Fifteen *dou* **(unit of capacity)** of barley, or soy, mung, and hemp beans is one *dan* **(unit of value, and, in the context of the second interpretation, also capacity)**. If one grants highly refined or finely husked grain, one takes ten *dou* as one *dan* (unit of value)."

In other words, the meaning of this statute is quite similar to the meaning of the table "Norms/Divisors for unhusked and husked grains" (Text 4) in *The Nine Chapters*. It is the key to the definition of equivalences between values for the different grains. Moreover, we have seen how this text was connected to such statements as "one *dan* of unhusked grain (millet) occupies two *chi* seven *cun*," namely, statements closely associated with the formulation of value from the time when *Mathematics* was composed until the time of *The Nine Chapters*.

Let us recapitulate what we have discovered about the use of the term *dan* during the periods when *Mathematics* and the *Book of Mathematical Procedures* were composed. *Dan* could be the name of a measuring unit for weight, for capacity, or for value, the latter two being related to each other. In relation to the assessment of value, one needed different amounts of capacity, possibly expressed in terms of *dan*, to obtain the same value, depending on the grains associated with them. With respect to capacity and value, the use of *dan* is quite similar to that of *hu*, as reflected in *The Nine Chapters*. Further, only in the case of husked grain was there an identity between the *dan* measuring unit of capacity and the *dan* measuring unit of value.

As we said earlier, the complexity of the use of the measuring unit *dan* just revealed echoes problems related to the interpretation of *dan* in documents of practice from Juyan and Dunhuang. Now that on the basis of mathematical documents, we have a better grasp of the interpretation of *dan*, how can we offer an interpretation of the phenomena related to *dan* in these other documents? The first point worth noting is that the phenomena referred to in relation to *dan* in the documents of practice also seem to indicate that when used in contexts unconnected with weight, *dan* sometimes relates to different volumes or, in other words, different capacities. This is precisely the situation the terms *xiao dan* "small *dan*" and *da dan* "large *dan*" explicitly refer to. In other words, the phenomena whose interpretation raise problems in these documents of practice are of the same kind as those dealt with above. Moreover, the ratio between *da dan* "large *dan*" and *xiao dan* "small *dan*" that the documents manifest is that of, respectively, 5 to 3.[69] In the context of the previous discussion, it is indeed quite natural

69 See some samples of evidence in slips 13 or 16, in document W2 from Wayen-torei (Loewe 1967, 334–335) (volume 1, plate 41, and see pp. 317–331 for a general introduction on this set of slips). Let us summarize the discussion that has been developed since the 1950s about this vexed issue. Lao Gan mentioned the existence of the "small *dan*" and the "large *dan*" as well as the problem raised by their interpretation as early as 1950 (Lao Gan 勞幹 1950). He put forward two ideas. First, he suggested that "small *dan*" was a measuring unit used by officials, in contrast to "large *dan*," which was used by ordinary people. Secondly, he offered several examples taken from Han bamboo slips, which evidenced conversions at the time between "small *dan*" and "large *dan*." At the same time, (Yang Lien-sheng 1950) (quoted here on the basis of the reprint in Yang Lien-sheng 2006) discussed this problem and made three further points. First, in his view, "small *dan*" and "large *dan*" did not correspond to different vessels but were only nominal units used in computations, being both equal to a single actual measuring unit of capacity: the *dan*. Secondly, "small *dan*" referred to the use of *dan* when assessing unhusked grain (millet), whereas "large *dan*" designated the use of *dan* in relation to husked grain (millet). Thirdly, he pointed out the ratio of 5 to 3 as that between "large *dan*" and "small *dan*," respectively. Later authors criticized the association he made between types of grain and types of *dan*. In 1951, Utsunomiya Kiyoyoshi 宇都宮清吉 published "*Zoku Kanshi hyakkan jyuhôrei-kô sairon* 續漢志百官受奉例考再論 Resuming discussion on the Salary of Government Officials according to the *Sequel to the Treatises of (the History of Later) Han (Dynasty)*," which we quote from the reprint (Utsunomiya 1955). He dealt with this issue in a passage there (Utsunomiya 1955, 221). To begin with, he suggested that different vessels were used during the Han dynasty to measure different species and states of grains. Accordingly, the same name (*dan*) could refer to different actual measurement units. "Small *dan*" and "large *dan*" are an example thereof, but in his view, this phenomenon also affected *dou* and *sheng*. Moreover, he argued that to measure unhusked grain (millet), one would use "large *dan*," whereas for husked grain (millet), one would use the "small *dan*." Later on, Yang Lien-sheng accepted the view that there may have been two distinct measuring units. Finally, Utsunomiya put forward the hypothesis that using such vessels would enable officials to carry out conversions. Subsequently, the problem was actively debated on these bases, without reaching a conclusion generally accepted. The reader can find a summary of the debate in (Loewe 1961/1962) and (Loewe 1967, vol. 1, 330–331).

to associate *da dan* "large *dan*" and *xiao dan* "small *dan*" with two vessels embodying the *dan* for unhusked millet and that for husked millet.

Following on the first publications on the topic, the debate about these units that developed from the 1940s onward basically offered two options: Either "small *dan*" and "large *dan*" referred to two different vessels, or they designated two ways of using the same measuring unit of capacity, the relationship between *da dan* "large dan" and *xiao dan* "small dan" being established by means of conversion. Interestingly enough, this reproduces the alternative in practice that we suggested could account for the evidence provided by *The Nine Chapters*. In spite of the differences between these two views, deriving mainly from Yang Lien-sheng and Utsunomiya, they have key features in common. Both consider *dan* as a measuring unit for capacity. Moreover, both relate the ratio of 5 to 3 between *da dan* "large dan" and *xiao dan* "small dan" to the ratio between equivalent amounts of unhusked grain (millet) and husked grain (millet).

In recent decades, discussion has focused on the following issues, which have remained unanswered. First, no evidence has been found in the Han slips to confirm the existence of two different types of vessels for a *da dan* "large dan" and a *xiao dan* "small *dan*." Secondly, in the Han documents, each of these units can be used to formulate an amount in both unhusked (millet) and husked (millet) grain. The association of any of them with only one type of grain is thus problematic. Thirdly, the slips found in Juyan and Dunhuang confirm that *da dan* "large dan" and *xiao dan* "small dan" were used to express amounts for all kinds of grain. How, then, can one explain the occurrence of only two *dan* that actually relate more precisely to unhusked and husked millet?

Let us outline our way of answering these questions.

In relation to the first question, it has already been argued that before the Qin dynasty was established, vessels of different capacity existed for the same measuring unit (Gao Ziqiang 高自強 1962). Now, with respect to the Han dynasty, in fact, in Juyan and Dunhuang documents of practice, we not only find evidence of *da dan* "large *dan*" and *xiao dan* "small *dan*," but the phenomenon also appears to have influenced other measuring units of capacity as well as weight: *da dou* "large *dou*" and *xiao dou* "small *dou*," *da sheng* "large *sheng*" and *xiao sheng* "small *sheng*," *da quan* 大權 "large standard weight" and *xiao quan* 小權 "small standard weight" are also used in these documents. The latter references can only relate to actual measuring standards. This supports the idea that *da dan* "large *dan*" and *xiao dan* "small *dan*" were also measuring vessels. In addition, archeologists have excavated measuring vessels of a size of the order of magnitude of *dan* and some seem to possibly embody the *da dan* "large dan."[70] Lastly, we have seen that the commentators of *The Nine Chapters* associated a measuring vessel with each sentence of the type "ONE *HU* OF UNHUSKED GRAIN (MILLET), BY REGULATION, HAS A VOLUME OF TWO *CHI* SEVEN *CUN*" (Text 16). Since we have shown above that the same type of clauses occurs in *Mathematics* and in the *Book of Mathematical Procedures*,

[70] For example, a vessel bearing an inscription from the Qin dynasty and measuring 1 *dan* 6 *dou* in Qin measuring units is mentioned in (Qiu Guangming etal. 2001, 183–184). Qiu Guangming etal. suggest that this vessel could embody the *da dan* "large *dan*."

and that they refer to the same use of the *dan* as *The Nine Chapters* bears witness for the *hu*, it is highly probable that in the former books, too, these clauses concur with the use of different vessels, depending on the grain measured. In conclusion, we have different types of evidence showing the existence of vessels associated with measuring units designated by the same name but with different volumes.

With respect to the second problem, it is relatively easy, on this basis, to explain that *da dan* "large *dan*" and *xiao dan* "small *dan*" could be used to express amounts of either unhusked or husked millet. In fact, if an official from a granary staff only had a large vessel corresponding to *da dan* "large *dan*" to measure amounts to be dispensed, he could use it not only for unhusked millet, but also for husked millet. In the latter case, he would simply have to compute, using a rule of three, the amount of "large *dan*" corresponding to the number of *dan* ("small *dan*") in husked grain that had to be dispensed. The official would then write not only the number of vessels (*da dan* "large *dan*") given in the account book, but also the value, in small *dan*, corresponding to it. In other words, whereas granaries could theoretically make use of specific vessels to dispense each type of grain, in practice actors limited themselves to using a single type of vessel and computing, using rules of three, the amount to be given with this vessel for each type of grain. Such an interpretation is supported by a passage from the *San Guo Zhi* 三國志 *Records on the three kingdoms*. The passage records the answer the king of Wei kingdom, Cao Cao, gave to someone who expressed concern that the grain stocks in the granary were insufficient. The answer reads: "可以小斛以足之。One can use the 'small *hu*' to make it enough." The document further reports that the trick was discovered by the soldiers, who bemoaned they had been cheated.[71] This piece of evidence clearly highlights the use of different vessels in the context of the granary, all called *hu*. If this statement can also be taken as supporting the hypotheses just formulated about *dan*, this derives from our conclusion that as measuring units of both capacity and value, in practice *dan* and *hu* were used in the same way.

On the same basis, we can also suggest a hypothesis for the fact that only these two types of vessels were used, even to measure other grains. We should recall that the documents of practice found in Juyan and Dunhuang were thus produced in the northwestern part of China and reflect the official distribution of grain and fodder to soldiers. On the one hand, any type of vessel could be used to dispense any type of grain, as long as it was used in relation to rules of three. The officials guaranteed fair distribution. On the other hand, given the context in the northwestern area, the grain most likely to be dispensed was millet. This explains the predominance of vessels related to millet in granaries.

These conclusions echo what we saw earlier about the complex uses of *dan*. The practice of *da dan* "large *dan*" and *xiao dan* "small *dan*" we just described is one way of bringing to light how actors implemented the notion of measuring unit of value that linked all grains with each other. They knew how to determine amounts of grain with the same value in practice, and they could do so with any available measuring vessel. Conversely, the existence of these units of value, which we have established, accounts for the existence of phenomena such as *da dan* "large *dan*"

71 (Chen Shou 陳壽 1959, 55), Chapter 1, Book of Wei. Annals of Emperor Wu.

and *xiao dan* "small *dan*," since they clarify why there were measuring units which had the same name *dan* but corresponded to different volumes.

4 Second decoding of the table in *The Nine Chapters* and the "Regulation for cereals plants": "husked grains *mi*" as the key of the system

Our analysis of the "Regulation for cereal plants" (Text 7 or its revised version 7″) revealed a fact to which we now return. For the types of grain mentioned in the text, most states are associated with *specific* and *different* amounts expressed in capacity measuring units, except the first state of grain obtained by husking. This is the state called "coarsely husked grain (*li mi*) or, simply "husked grain (*mi*)," in relation to millet, and "husked grain (*mi*)" in relation to rice. Compared with other states of grain, this state represents an exception in two respects. First, in relation to this state of husked grain (*mi*), the *dan* has two meanings, as we showed above. In addition, in all regulations, husked grain (*mi*) is *constantly* associated with 1 *dan* or 10 *dou*. This fact has a very important consequence. Whereas for all states of the grain except "husked grain (*mi*)," the amount expressed in measuring units of capacity and the value of that amount expressed in value units *dan* differ, these two figures remain constantly the same for "husked grain (*mi*)." The amount corresponding to the value of one *dan* for husked grain (*mi*) has a capacity of one *dan* (where *dan* is now a capacity measuring unit). This fact holds true for any type of grain. These remarks lead us to formulate the following hypothesis, whose consequences we shall explore. "Husked grain (*mi*)" has been chosen as the basis for defining the measuring units for capacity as well as the value unit. As a result, we suggest translating *mi* as "standard husked grain." This hypothesis contradicts the common assumption that "unhusked grain (*su*)" was the basis of the system of management of grains.[72] In other words, we postulate that "standard husked grain" (*li mi* or *mi*) has played an essential role.

With this issue in mind, let us return to "Norms/divisors for unhusked and husked grains" (Text 4).

4.1 Second decoding of "Norms/divisors for unhusked and husked grains"

In contrast to the "Regulations for cereal plants," this table brings all types and states of grain in relation to each other, giving each of them the same status and associating each of them with an abstract "integer." The fact that the first nineteen problems of chapter 2 in *The Nine Chapters* all have the same pattern and deal with changing "unhusked grain (millet)" into any of the grains and states of grain gathered in the table shows how all of them were treated equally.

We have suggested above that the beginning of the table ("unhusked grain," "coarsely husked grain") referred to millet even though it was expressed in generic

[72] For expressions of this view, and related ideas, see Utsunomiya Kiyoyoshi, "*Zoku Kanshi hyakkan jyuhôrei-kô sairon*," quoted from the reprint (Utsunomiya 1955, 221–231), and more recently, (Peng Hao 2012).

terms. Indeed, millet, the most widespread crop in northern China and the most fundamental grain for theoretical purposes, would otherwise not be mentioned in the table. This leaves no doubt that it is the crop that lies behind the generic terms opening the table. This hypothesis is supported by the fact that the ratio between the various states of the grain mentioned at the beginning of the table are those associated with the sequence of states deriving from millet.

Now, we shall show that, in fact, things are more complex than the explanation so far. We can show this by comparing "Regulations for cereals plants" and "Norms/divisors for unhusked and standard husked grains" (Text 4). Let us first observe that in the table of Text 4, the grain corresponding to "paddy" has the same ratio to "coarsely husked grain (*li mi*)" as unhusked grain deriving from "paddy stalks" had in "Regulations for cereals plants" to the "husked grain (*mi*)" corresponding to it. In both cases, the ratio between either "unhusked rice" or "paddy" and "husked grain" is 2 to 1. Several important consequences can be derived from this observation. First, it indicates that "paddy" in the table has to be interpreted as designating the state of "unhusked grain." Moreover, and more importantly, "coarsely husked grain (*li mi*)" in "Norms/divisors for unhusked and standard husked grains" is *not only* the first state of husked grain associated with millet, but *also* refers to the first state of husked grain for rice. Secondly, the fact that in "Regulations for cereals plants," the same amount, in capacity as well as value, was associated with the first state of "husked grain" ["standard husked grain (*mi*)"] correlates with the fact that these two states were united in the table in *The Nine Chapters*. It thus appears that the *dispositif* presented in "Regulations for cereals plants" allowed actors to connect the series of different states of grain associated with distinct crops via the bridge constituted by "standard husked grain (*mi*)." As for the table, "standard husked grain" appears to constitute its pivot.

These facts are confirmed by similar observations one can make with respect to "barley, or soy, mung, and hemp beans." Let us remind the reader that above we suggested two possible interpretations for the following statement that occurs in "Regulations for cereal plants" (text 7): "Fifteen *dou* (**unit of capacity**) of barley, or soy, mung, and hemp beans is one *dan* (**unit of value, and perhaps also capacity**)."

According to the second interpretation offered, one *dan* here would refer to the state of "husked grain," whereas "barley, or soy, mung, and hemp beans" would designate unhusked grain. Note that this interpretation fits well with the title of the table in *The Nine Chapters*: "Norms/divisors for unhusked and standard husked grains." If this is the case, then exactly the same conclusion holds as above. Namely, in the table of *The Nine Chapters*, "husked grain (*li mi*)" also designates the first state obtained through husking "barley, or soy, mung, and hemp beans." The ratio between this husked grain and unhusked "barley, or soy, mung, and hemp beans" is, in both texts, 2 to 3. This would reinforce the conclusion about the centrality of the state "standard husked grain" as a basis of the system. In all these cases, all grains would be connected at the level of the state "standard husked grain" for which the unit of capacity *dan* would merge with the unit of value *dan*. The table in *The Nine Chapters* and the "Regulations for cereal grains" would each express this fact in its own way.

If we opt for the first interpretation, "one *dan*" in this statement of "Regulations for cereal plants" would only refer to a unit of value, and this statement, by defining the unit of value for these grains, would allow the connection of barley, or soy, mung, and hemp beans to the other grains, by means of their state as "husked grain." In this last respect, "standard husked grain" would still remain the pivot of the system of grain. Note that one further step is taken in the table of *The Nine Chapters* to incorporate states of grain deriving from barley into the general system. The reader may recall that we emphasized that in "Regulations for cereal crops," there was a sentence that did not fit the overall scheme, namely, "Ten *dou* of barley is three *dou* of grits." We can see that in *The Nine Chapters,* this latter state of barley has been included in the table as "small grits of barley," associated with the number "thirteen and a half." The ratio of this state to barley is 3 to 10. Further, another state "large grits of barley" has been added to the system. In this respect, "Norms/divisors for unhusked and standard husked grains" develops and integrates further the set of grains presented in "Regulations for cereal crops," through enriching the series of states deriving from barley.

In conclusion, "standard husked grain" appears to be a state of all grains, or of the most important ones, which provides a basis for expressing all types of cereals. This central position of "standard husked grain" is reflected in an essential fact. We have mentioned Shang Yang's parallelepipedic *sheng*, designed in the fourth century B. C.E. and officially used or a long time afterward. We have also mentioned Wang Mang's bronze *hu*. We should recall that their inscriptions all testified to the relationship between capacity measuring units and volume measuring units as follows: 1 *hu* (or 1 *dan*) makes 1,620 *cun*. The numerical value chosen to make the bridge between capacity and volume is precisely the one that corresponds to "standard husked grain" in all the texts from *Mathematics* to *The Nine Chapters*. This means that "standard husked grain" was chosen as the basis for defining measuring units. Its centrality in this respect was formulated explicitly once again by Li Chunfeng, who stated in the "Monograph for pitch-pipes and the calendar" of the *History of the Sui*: "One takes the *hu* of the standard husked grain as the standard 以米斛為正." (Wei Zheng et al. 魏徵等 1973, 409). Note the crucial fact that in choosing the "standard husked grain" as the basis for connecting capacity and volume, actors chose the state of grain that allowed a vessel embodying a measuring unit of capacity simultaneously to embody the unit of value for the "standard husked grain."

To conclude, we confirm here what we have begun to understand in relation to "Regulation for cereal crops," namely, that "standard husked grain" was central to the system of grains. It was central in three respects. First, it was the basis for the system of value measuring units *hu* (or *dan*). Secondly, it was the basis for the definition of the measuring unit of capacity. Thirdly, it was the state that allowed different types of grain to be related to each other. "Norms/divisors for unhusked and standard husked grains" express all these facts.

4.2 Second decoding of "Regulations for cereal crops"

Let us return to "Regulations for cereal crops" for the last time. We have already seen that this text listed series of states deriving from basic crops and associated with each of them (except two exceptions) the measure in capacity expressing the amount having the value 1 *dan*. Moreover, we have shown that this 1 *dan* in Text 7 occurred in general in association with the state "standard husked grain" and in relation to the expression of the value, bringing the various sequences of grain into relation with each other. In this respect, the use of the "Regulations for cereal crops" is comparable to the use of "Norms/divisors for unhusked and standard husked grains."

However, there is a huge difference between the two, namely, the part played by weight measuring units. Weight measuring units never occur in *The Nine Chapters* in relation to grains. By contrast, they play a role in "Regulations for cereal crops." We see that the two sequences of states of grain deriving from millet and paddy stalks both start from the plant just cut, with stalks and leaves. Measuring units for weight are used to assess their amount. Moreover, in Text 7, the two sequences start from the same amount of plants, whether it is millet or paddy, namely, the amount corresponding to a weight of 1 *dan*. How can we interpret this striking echo between the equal weights from which the two series start and the fact that these amounts in weight correspond precisely to 1 *dan*, this time as a measuring unit for capacity, for the coarsely husked grain?

We suggest two elements for the answer to this question.

First, the largest measuring unit for capacity *dan,* which has the same name as the largest measuring unit for weight, derived from the latter precisely on the basis of the transformation that leads from the plant just cut to standard husked grain. In "Regulations for cereal crops," this would give us a key way in which the definition of capacity relies on the definition of weight on the basis of the management of grains. The fact that grains provided essential elements in the definition of measuring units is also evidenced in the "Monograph on pitch-pipes and the calendar" of the *History of the Han.*[73] It is interesting that in the case of "Regulations for cereal crops," the establishment of measuring units for capacity on the basis of weight is shown to occur for both millet and paddy.

Second, through the state of standard husked grain, the measuring unit for value *dan* receives its foundation by relying on the largest measuring unit for weight: The amount of standard husked grain having the value 1 *dan* is the one deriving from an amount of plants weighing 1 *dan*. In this respect, this relation gave the foundation of *dan* as an absolute measuring unit for value through "standard husked grain." It is by the derivation from this basic measuring unit for value that the value units for the other states of grain could be obtained. This is precisely the meaning of "Regulations for cereal crops" and "Norms/divisors for unhusked and standard husked grains."

Such an explanation reveals how the measuring unit *dan* for both capacity and value originates from the measuring unit of weight bearing the same name, *dan*. It

[73] See above, note 17.

also provides clues as to what was at stake in the reform carried out by Wang Mang, when he simply substituted *hu* for *dan* as the name of the largest measuring unit for capacity and for value. In doing so, he severed the essential link that had existed between weight and capacity–value measuring units. One can thus speculate that he was discarding weight as the basis for the definition of other measuring units and was promoting another way of defining value and capacity. This is perhaps the construction of the system of measuring units on the basis of grains that is presented precisely in the "Monograph on pitch-pipes and the calendar" of the *History of the Han*.[74] Moreover, perhaps such a transformation occurred together with the promotion of a single grain, namely millet, as the basic grain, a fact reflected in both *The Nine Chapters* and the "Monograph on pitch-pipes and the calendar" of the *History of the Han*. Could the split introduced between weight and capacity be related to a change in the management of grains? This is not impossible. In fact, *The Nine Chapters* contains no instance of weight measuring units used in relation to grain. By contrast, we have seen that *Mathematics* provided standard clauses to relate weight and capacity for various plants (Text 9). On the other hand, throughout the time period studied, the definition and measure of value remained closely related to capacity measurement: The name of the key measuring unit for value remained that of the measuring unit for capacity and the measurement could be done by vessels.

5 Conclusions

The above arguments have led to several conclusions that we summarize in three main points before suggesting some hypotheses on the actual deployment of the various kinds of measuring units that, as we have seen, could be used to assess grains.

5.1 Capacity, weight, value, and the nature of early Chinese mathematical writings

Mathematical writings provide evidence that allowed us to establish that *dan*, in the Qin and Western Han dynasty, and *hu* in the period of Wang Mang and the Eastern Han dynasty, had a key feature in common. They were both the highest-order measuring units for capacity as well as measuring units for value. This last feature seems to have been overlooked in previous historical accounts of the systems of measuring units. Let us emphasize the importance for us, in facilitating understanding of this fact, of having the context of use of these measuring units that mathematical books provide. Without the books, this feature would have probably remained unremarked. One feature, however, appeared to differ between *dan* and *hu*; namely, *dan* was also the name of the highest-order measuring unit for

[74] At the beginning of the "Monograph on pitch-pipes and the calendar" of the *History of the Han*, Ban Gu, its author, makes clear that he has used texts by the scholar who helped Wang Mang to implement his reforms, namely, Liu Xin 劉歆 (Ban Gu etal. 1962, 955).

weight, whereas *hu* had no relation with the weight system. Does the fact that both *dan* and *hu* were used as measuring units for value explain why officials could be ranked by means of the number of *dan* they received as salary? We know that such officials could be paid in any type of grain. Is the *dan* in occurrences referring to officials in terms of expressions such as "six hundred *dan*" or "two thousand *dan*" to be interpreted as a unit of value?[75] This question awaits further study. In any event, the fact that in mathematical sources the unit of value *dan* could be used, regardless of the type of grain, appears to confirm that this concept of value must have been absolute.

If we believe the evidence provided by mathematical sources, *dan* or *hu* seem to have been used as measuring units for value especially in close connection to granaries. In this context, we have suggested that from the time when *Mathematics* was compiled until the time when *The Nine Chapters* was completed, the term *rong* 容 designated the assessment of the content of granaries in measuring units of value.

Mathematical sources thus provide evidence about key features of the use of measuring units in relation to grain. Note that this holds true for documents that were handed down as well as for documents provided by archeology.

Seen from another perspective, these facts show how closely the sources documenting mathematical practice in early China were related to the bureaucracy. Let us recall the features demonstrating the close relationship of mathematical documents with the activities of officials, especially with respect to grains, which was our main purpose in this article. First, we remember that the *Book of Mathematical Procedures* has a section in common with the chapter "Statutes on granaries" from the Qin edicts and statutes and that many of its procedures show continuity with the data provided in this section. This can be correlated with the fact that, like other mathematical manuscripts, the *Book of Mathematical Procedures* was found in a tomb that also contained a set of statutes and edicts. Secondly, *The Nine Chapters* provides a table that appears to define units of values, in that it yields numbers whose ratios are the same as the ratios between the value units *hu* associated, the text says, by "regulations *cheng* 程," with various types of grain. One may assume that the content of this table reflects a statute of the same type as that contained in the "Statutes for granaries" and quoted by the *Book of Mathematical Procedures*. In any event, when Li Chunfeng wrote a technical monograph in the seventh century on the history of measuring units for the official histories of, respectively, the Jin and the Sui dynasty, he used the evidence provided by *The Nine Chapters* and Liu Hui's commentary on measuring units in relation to grains as if they were unquestionably reliable. This shows indirectly that he understood that these documents reflected official practices in these respects.

In addition, other features of mathematical sources indicate a close connection to officials' activities. We have seen how problems and data contained in at least some of these documents, perhaps particularly in some of the chapters of *The Nine Chapters,* reflected actual practices, and more specifically, practices related to edicts

[75] See Utsunomiya, "*Zoku Kanshi hyakkan jyuhôrei-kô sairon,*" published in (Utsunomiya Kiyoyoshi 1955, 221–231) and (Bielenstein 1980, 4–5).

and statutes. For instance, problems related to granaries, to the distribution of grain, and to the stockpiling of grain appeared to have counterparts in actual practice (Peng Hao forthcoming). These remarks not only cast light on some characteristics of these books, but also invite historians to take the evidence provided by these problems and data more seriously as historical evidence.

These facts seem to suggest that for the local officials who used them at the time, the mathematical documents excavated recently were considered, at least partly, as auxiliary books to the administrative documents for the issue of edicts and statutes. Do the theoretical sections in these books enable practical action, or are they developments spurred by other motivations? The answer to this question should be determined by closer analysis. We leave it open here.

5.2 The central role played by standard husked grain

In contrast to previous conceptions with respect to grains, we were led to suggest that the central role in the management of grains was played by *mi* "husked grain," which we also suggested interpreting as "standard husked grain." This central role is evident in the relation between this state of grain and the definition of *dan* as a unit for value. In Qin and Western Han China, we have seen that this state for different types of grain was the basis for the definition of a *dan* that was the measuring unit for value. The same conclusion appears to have held true for the *hu* during Wang Mang's time period and the Eastern Han dynasty. These units of value were each closely related to the two texts (texts 4 and 7) that we have studied. Each in its way, these two texts, headed, respectively, "Regulations for cereal crops" and "Norms/divisors for unhusked and standard husked grains," represent two modes of mutually interrelating all types of grain and different states of these grains through the state of "standard husked grain." Given the connection established between different types of grain through "standard husked grain," the *dispositifs* described by these two texts provided a definition of value that was absolute for "standard husked grain." The relative concept of value for the other states specifically related to each grain derived from this value.

The centrality of "standard husked grain" and its role in a statute inserted in the "Statutes on granaries" suggest a hypothesis on the origin of *dan* as a measuring unit for capacity and value. In other words, *dan* as a measure of capacity derived from *dan* as a measure of weight through the administrative definition of "standard husked grain." The basic relation was established by the fact that a weight of 1 *dan* of harvested grain corresponded to a capacity of 1 *dan* of "standard husked grain." Accordingly, grains appear to have played a central role in the definition of the system of measuring units. This evokes the part played by grain in the "Monograph on pitch-pipes and the calendar" included at the end of the first century in the *History of the Han* and probably written by the scholar who defined the *hu* in bronze for Wang Mang. Even though the uses of grain in the two contexts were different, the parallel seems to support our hypothesis about the origin of *dan* from a unit of weight.

5.3 Vessels

Further, the central part played by "standard husked grain" is also reflected by the fact that it offered an absolute basis for the definition of a standard measuring unit for capacity, called either *dan* or *hu*, in relation to the volume of 1,620 *cun*. This connection between capacity and volume based on the definition of the value unit 1 *dan* for "standard husked grain" is embodied in two official vessels whose use was enacted by successive governments in China: the parallelepipedic *sheng* designed by Shang Yang 商鞅 in 344 B.C.E. and the bronze *hu* promoted by Wang Mang 王莽. Their inscriptions are unambiguous on this question. In the first half of the seventh century, Li Chunfeng explicitly stated the connection with "standard husked grain" when he wrote, "One takes the *hu* of the standard husked grain as the standard 以米斛為正." In short, the generic *hu* and probably the generic *dan* were vessels connected to "standard husked grain" and consequently defined, for this state of the grain, its measure both in terms of capacity and value units. In other words, "standard husked grain" was the only state of grain for which a single measurement vessel was to be used. For all the other states, the measurement of the capacity and the measurement of the value required different vessels. The vessels attached to "standard husked grain" were, we suggest, those to which the "*xiao dan* 小石 small *dan*" used in Juyan and Dunhuang refers. By contrast, another vessel, attached to "unhusked millet" and specifically measuring its value, seems to have been used in parallel to measure all types of grain. This use of a second vessel accounts for the introduction in the documents of the other measuring unit, "*da dan* 大石 large *dan*." This is our suggestion for a solution to the appearance in these sources of two measuring units for capacity associated with the name *dan*. Of these two units, one, the "small *dan*," had an absolute meaning. This is probably connected to the fact that the use of the single term *dan* in these texts also refers to the standard measuring unit *dan*, that is, to "small *dan*." By contrast, the unit "large *dan*" was only defined relatively to the latter.

Altogether, the documents studied seem to bear witness to a remarkable continuity in the system of the management of grains, despite differences to be found in mathematical texts and that await further study.

5.4 What hypotheses can we make on the management of grains based on mathematical and administrative texts?

At the end of this investigation, we saw that many distinct types of measuring units were used in the past in relation to the management of grains, that is, measuring units of length, volume, capacity, weight, and value. This naturally provokes questions as to the contexts in which each of these sets of measuring units was chosen and the explanation for the choice of different types of measuring units. In what follows, we suggest hypotheses inspired for the most part by the evidence provided by mathematical sources. In presenting them, we shall make clear on which kind of evidence we rely and where we add assumptions.

- Clearly, officials used measuring units of capacity when they determined (and expressed) the tax to be paid in grain for a certain kind of cropland. Evidence for this is provided by many problems included in mathematical manuscripts, such as the following, which is found in the *Book of Mathematical Procedures*:

(19) "租吳(誤)券 田一畝租之十步一斗，凡租二石四斗。今誤券二石五斗，欲益耎其步數, 問益耎幾何。(...)/96/

When levying taxes, erroneously inscribing on the certificate. A cropland (having an area of) one *mu* (NB: one *mu* is a unit specific to area and represents 240 *bu*) is taxed one *dou* for ten *bu*, which adds up to a total tax of two *dan* four *dou*. Suppose that, having erroneously inscribed one *dan* five *dou* on the certificate, one wants to vary the quantity of *bu* corresponding (to the tax of one *dou*), one asks how much it should vary (...)/96/".[76]

Such problems show how officials established a relationship between the area of the cropland and the tax to be paid. That at this stage, officials used measuring units of capacity seems to indicate that taxpayers also used capacity measuring units to assess the amount of grain to be paid to the taxation office. At this stage of the management of grains, what accounts for the use of measuring units of capacity is the *simplicity of the measure.*

– On the basis of the evidence provided by *Mathematics* (text 12) that at some point conversions between measuring units of capacity and measuring units for weight were used, we assume that tax officials used weight measures to check the grain brought to pay tax. What may account for this practice is the *accuracy of such a measure*. Note that it would be surprising for accurate scales to be widespread outside the relevant offices of the bureaucracy. This hypothesis raises two questions: (1) Why is it that only in *Mathematics* we find evidence of such conversion between weights and capacities for grain? The fact is that whereas the *Book of Mathematical Procedures* still mentions the use of a measuring unit of weight in relation to grain, such mentions are completely absent from *The Nine Chapters*. (2) How does this relate to the reform carried out by Wang Mang and to the separation between measuring units for weight and for capacity? This question remains unanswered.

– It is clear that in some cases, for instance when dispensing daily rations of grain, officials working in granaries made use of measuring vessels. We provided evidence for this above. Such a method was certainly practical for rations. However, paying salaries with grain would be a rather lengthy process if it involved using such vessels to measure amounts of the order of magnitude of dozens, or even hundreds and thousands of *dan*. This is precisely the situation dealt with in Text 18 from the *Book of Mathematical Procedures* and text 15 from *The Nine Chapters*. This leads us

[76] Peng Hao (2001, 83). *Mathematics* also provides evidence that measuring units of weight are used to express amounts to be paid for tax. However, this does not occur in relation to grains, but in relation to rather coarse stuff. See, for instance, Zhu Hanmin and Chen Songchang (gen. ed.) (2011, 41). This use of weight measuring units can be compared to their use to measure amounts of cereal crops that were just harvested, as evidenced in Text 7 ("Regulation on cereal crops"). It must be distinguished from the use of weight measuring units explained below, in relation to grains.

to suggest that when paying significant amounts of grain, officials working in granaries could use the method described in these two texts: make a heap of grain, measure its dimension by means of measuring units of length, compute its volume, and use the conversion between volume and units of value (Texts 9, 13 (*Mathematics*), 18 (*Book of Mathematical Procedures*), 15, 16 (*The Nine Chapters*)) to determine the number of measuring units of value a given heap of grain contained. These are the cases, we suggest, for which measuring units for length, volume, and value were used. The reason for using such practices and such units would be the *speed of the measure*.

– Perhaps the content supervision of granaries, i.e., how many measuring units of value they contained (*rong*), also required making use of the relationship between volumes and value measuring units. This is, perhaps, what Text 14 (*Mathematics*) tells us. One may assume that similar practices were implemented to check the capacity of vessels.

All these cases reveal the key part played by mathematical documents in addressing issues important for history in general. Naturally, many questions remain open, and we intend to return to them in the future.

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