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A forgotten solar model

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Abstract This paper analyses a kinematic model for the solar motion by Quṭb al-Dīn al-Shīrāzī, a thirteenth-century Iranian astronomer at the Marāgha observatory in northwestern Iran. The purpose of this model is to account for the continuous decrease of the obliquity of the ecliptic and the solar eccentricity since the time of Ptolemy. Shīrāzī puts forward different versions of the model in his three major cosmographical works. In the final version, in his *Tuḥfa*, the mean ecliptic is defined by an eccentric of fixed mean eccentricity and a mean obliquity fixed with respect to the celestial equator, and the center of the epicycle, which is inclined to the eccentric, moves on the eccentric with an annual period. By an additional slow motion of the sun on the epicycle, the true eccentricity of the solar deferent, defined by the annual motion of the sun, and the sun's extreme declination from the equator change, accounting for the reduction of the eccentricity and the obliquity of the ecliptic since the time of Ptolemy.

1 Introduction

A medieval astronomer working in the Ptolemaic tradition had the two choices in dealing with the differences he could ipso facto find between his own measured parameter values and those he inherited from Ptolemy or his close predecessors. As N. M. Swerdlow puts them:

[1] Swerdlow (1975, p. 50).

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> [*A*] The first was to reject all earlier observations until he could first establish a fairly accurate representation of the motions of the heavens using only extremely accurate observations from his own time. Ancient observations could then be measured against this standard to see whether they fit well enough to confirm or correct this preliminary theory, or were so far out of line that they could safely be eliminated as erroneous.
> [*B*] The alternative was to trust in the accuracy of his predecessors and develop a theory that would, in so far as possible, account for all their observations.[1]

During the medieval Islamic period (eighth–sixteenth centuries), these two distinct approaches indeed highlight two different kinds of astronomy evolved in two separated geographical domains of Islamic realm: the first can be addressed in Eastern (Middle Eastern) Islamic astronomy, but the most established steps of the second can be associated with Western Islamic astronomy (in Maghrib and Andalus) from the early eleventh century onwards, when Ibn al-Zarqālluh made the first attempts at explaining the long-term variations in the rate of precession, the obliquity of the ecliptic, and the solar eccentricity by inventing a series of historically eminent quantified models.[2] Let us call the first approach or methodology as the "Standard Establishment" and the latter as "For-All-Time Astronomy"; or "Short range observational astronomy" and "Long term theoretical astronomy", respectively, after Moesgaard.[3]

By the astronomical observations in the early Islamic period, new values for the obliquity of the ecliptic, the rate of precession, and the solar parameters (eccentricity, the longitude of apogee, and the length of solar year) were observed; the first main focus of these experimental activities was at Baghdad about 830 AD. At this primary stage, a vast amount of attempts made at justifying the fact that the solar apsidal line has moved since Ptolemy, and that similar to the planetary apsidal lines, the apogee of the sun is subject to the precession and thus sidereally fixed. Another century was needed to pass until other non-Ptolemaic values for these parameters were measured, and thereafter the medieval Middle Eastern astronomers encountered a prominent problem concerning whether these fundamental parameters are in reality constant or variable. In his vast discussion on this basic problem in the turn of the past millennium, Bīrūnī, in his *al-Qānūn al-mas'ūdī*, put forward the reasons why the method [*A*] above should be adopted; his main, and indeed intelligible, reasons were the difficulties with the observational activities and the critical sensitivity of the methods by means of which the times of occurrence of solstices and equinoxes are estimated, and hence, the length of solar year and its orbital elements are determined.[4] This work exerted a significant influence on the later astronomers in the Middle East, so that the problem no longer turned into being a central matter of busy for the later Eastern Islamic astronomers. Of the scarce attempts made there for accounting the long-term periodic or secular changes in the Ptolemaic constants, only these can be mentioned: a simple trepidation model seems to have been available in the Middle East prior to the tenth

[2] For the astronomers referred to here, see *DSB*, *NDSB*, *BEA*, EI_2, Sezgin (1978) and Rosenfeld and İhsanoğlu (2003).

[3] See Moesgaard (1989, p. 312).

[4] These are discussed at length in Mozaffari (2013a, Part 2).

century, in which the variations both in the rate of precession and in the obliquity of the ecliptic are accounted for by the rotation of the pole of the ecliptic on a small circle of radius 4° around an axis that is fixed with respect to that of the celestial equator. The reference point of this model is the summer solstice (Head of Cancer). The purely geometrical and qualitative account of this model can be found in the two tenth-century Middle Eastern treatises: Ibrāhīm b. Sinān b. Thābit b. Qurra's *Kitāb ḥarakāt al-shams* and Abū Ja'far al-Khāzin's *Zīj al-ṣafā'iḥ*.[5] This is what the astronomers working at the Maragha observatory (northwestern Iran, 1260-*c.* 1320) like Naṣīr al-Dīn al-Ṭūsī (1201–1274) and Quṭb al-Dīn al-Shīrāzī (1236–1311) discussed and discarded in his cosmographical works. Or the very curious idea that Ibn al-Amājūr maintained about the maximum value of the latitude (the inclination of the orbit) of the moon; as reported by Ibn Yūnūs,[6] Ibn al-Amājūr believed that it is not constant. Of course, no model was proposed to account for the variations observed in the inclination of the lunar orbit.

For the explanation of the variations found in the parameter values derived from the observations, which were often minor and irregular, they were reasonably attributed to observational errors or deficiencies of applied instruments. Especially, many such instances can be addressed in Bīrūnī's works, a good number of which have already been introduced and classified by O. B. Sheynin.[7] An example (not considered in those papers) is: Getting a precision of about some sixtieths of a day, or higher degrees of accuracy, in the length of the solar year is impossible; as Bīrūnī remarks,[8] this is so, because the instruments are, in principle, unable to record it thoroughly; with regard to the medieval methods of the interpolation in the solar meridian altitude within some days about the cardinal positions, i.e. the solstices and equinoxes, to estimate their times, in order to reach such a precision, one needs an instrument graduated to, e.g. every arc-second. Or Muḥyī al-Dīn al-Maghribī (d. 1283), who carried out an extensive observational programme at Maragha between 1262 and 1274, observed the eight brightest stars, then compared his derived latitudes with the surviving ancient values, and concluded that "what we found for their latitudes do not differ from the values observed by the ancients, no difference to an extent which is countable. The [inconsistencies] are due to observations, not owing to [proper] motion".[9] Another example is al-Shīrāzī's treatment of the differences in the values observed for the obliquity of the ecliptic from Hipparchus through the early Islamic period to his own time, for which he posits the idea, first in the order of priority, that the source, disorder, and irregularity of these differences may be due to the difference in the fabrication,

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[5] On the trepidation models, a vast, elaborate literature exists, of which the following works are worth reading: Comes (1996, 2001), Goldstein (1965, 1994), Hartner (1971), Mancha (1998, 2004), Mercier (1976/1977, 1996), North (1967), Samsó (1994b, 1998, pp. 93–96, 2001, pp. 169–174), Samsó and Millás (1994), Swerdlow (1975), Swerdlow and Neugebauer (1984, pp. 129–148). On Islamic astronomical tables see Kennedy (1956) and Samsó et al. (2001); these two main sources of the knowledge of Islamic astronomical tables, the so-called *zījes*, are followed by a new comprehensive survey that is currently prepared by Benno Van Dalen.

[6] King (1999, pp. 502–503). A lunar model in which a separate term/component is embedded to account for this *inequality* was not worked out until Tycho Brahe; see Swerdlow (2009, esp. pp. 35–40).

[7] See Sheynin (1973, 1992).

[8] Bīrūnī 1954–1956, Vol. 2, p. 648.

[9] Muḥyī al-Dīn, *Talkhīṣ al-majisṭī*, f. 115r.

size, graduation, and installation of the applied instruments. By irregularity, he actually means that, for example, the difference between Ptolemy's 23;51° and 23;35° as found in the observations carried out in the period of al-Ma'mūn's reign (the 'Abbāsīd caliph from 812 to 833) corresponds to a rate of decrease of 1′ in 43 Egyptian/Persian years, the interval of time between the two being 690 years, whereas Bīrūnī's value is by 2′ more than Yaḥyā b. Abī Manṣūr's 23;33°.[10]

This type of treatment seems principally to have been based upon Ptolemy's referring to the decisive effect of the discrepancies in the observational instruments,[11] which could evidentially be enhanced by the Islamic astronomers' experiences with the difficulties with astronomical instruments, regardless of their sizes or types. For example, the Banū Mūsā treated a difference of 2° they found between the motion of the solar apogee and that of Regulus (α Leo), i.e. precession, in the period between Ptolemy's and their time as "com[ing] from observational errors".[12] By such considerations, it may be said that a very primitive qualitative theory of errors in astronomical affairs had been evolved from Ptolemy's scattered allusions to them in the *Almagest* to an incidental conception and distinction of the random and systematic errors. For example, concerning Ptolemy's finding that the solar apogee is fixed by comparing his and Hipparchus's observations, Taqī al-Dīn Muḥamamd b. Ma'rūf (1525–1585), the director of the short-lived observatory at Istanbul, states that it may be because of the two sorts of the possible errors occurring in observations: the one, errors in making observations, and the other, discrepancies of the instruments, "because their situations are not investigated by repeating observations".[13] That "situation of the instruments" should be examined by "repeating observations" seems to refer, though implicitly, to a possible source of systematic errors, in the modern sense, since a defected instrument may produce such errors.

Nevertheless, on the contrary, when the same values for the solar parameters and the obliquity of the ecliptic arrived at the Western Islamic lands, they gave rise to a different

[10] Shīrāzī, *Ikhtiyārāt*, f. 26v; *Tuḥfa*, f. 18r; *Nihāyat*, P1: f. 18r, P2: f. 39r. The al-Ma'mūnīc observations were made by Yaḥyā b. Abī Manṣūr (d. 830) at Baghdad and by Khālid b. Abd al-Malik al-Marwarūḍī and his team at the monastery of Murrān on a hill in the vicinity of Damascus. Shīrāzī correctly ascribes the value 23;33° to the first and associates 23;35° with the latter. Bīrūnī is more precise and mentions all the values 23;33,52°, 23;33,57°, and 23;34,27° he found in his sources concerning the observational results at Damascus from 831 to 833 AD (the true modern value at the time ~23;35,33°). Bīrūnī's own value is 23;35° derived from his observations of the extremal solar noon altitudes in the latter part of the 1010s (for the analysis of them, see Said and Stephenson 1995, esp. p. 123). As he notices, the majority of the early Islamic astronomers observed either exactly this value or the values close to it; e.g. 23;34,51° in a table in which the solar noon altitudes observed by Khālid in Damascus were written down; the Banū Mūsā at Baghdad (at Sāmarrā', they had found 23;34,30°); Sulaymān b. 'Iṣmat of Samarqand: 23;34,40° (according to Bīrūnī, Sulaymān adjusted the solar noon altitudes at the solstices by the parallax, by which he yielded 23;33,42°); al-Battānī in Raqqa (*Ṣābi' zīj*, Sect. 4: Nallino [1899–1907] 1969, Vol. 3, p. 18: from the repeatedly observations of the solar yearly extremal zenith distances, he found $z_{\min} = 12;26°$ and $z_{\max} = 59;36°$); 'Abd al-Raḥmān al-Ṣūfī (903–986) in Shiraz; and Abu 'l-Wafā' al-Būzjānī (940–997/8) and Abū Ḥāmid al-Saghānī (d. 990) at Baghdad; Bīrūnī, *al-Qānūn* IV.1: 1954–1956, Vol. 1, pp. 363–366; also, see Kennedy (1973, pp. 32–43).

[11] E.g. *Almagest* III.1: Toomer (1998, p. 134).

[12] See Neugebauer (1962, p. 267).

[13] Taqī al-Dīn, *Sidrat*, K: f. 36v. About Taqī al-Dīn's observations at Istanbul in the 1270s, see Mozaffari and Steele (2015).

kind of astronomy. The simple trepidation model mentioned above was also passed from the Middle East into Andalus where the complicated quantitative trepidation models were developed from the eleventh century on in order to account for the observed decrease in the obliquity of the ecliptic (which was thought to be a periodic variation). Ibn al-Zarqālluh (d. 1100) is perhaps the most important who accomplished, elaborated, and promoted a plan for a long-term theoretical astronomy in Andalus; he invented a solar model with a variable eccentricity[14] and some trepidation models, which appear to have also served as a point of departure for al-Biṭrūjī to work out his planetary homocentric models.[15]

By the establishment of the Maragha observatory and gathering a good number of scholars and astronomers there, the two independent observational programmes were carried out: the one by al-Maghribī and the other by the main staff of the observatory under the supervision of al-Ṭūsī. The first achieved a systematic re-measurement of the solar, lunar, and planetary orbital elements on the basis of the extensive observations which al-Maghribī documents in his *Talkhīṣ al-majisṭī*, "Compendium of the Almagest", but the second was confined to the observations of 16 fixed stars and the derivation of a new value for the size of the epicycle of Mars. Rather, the main staff of the observatory, notably, Mu'ayyad al-Dīn al-'Urḍī of Damascus and Ṭūsī, along with Shīrāzī, turned their attention to a theoretical aspect of astronomy, represented by cosmography, the tradition of Ptolemy's *Planetary Hypotheses*, in order to solve the philosophical difficulties arising from some components of Ptolemaic models such as the small circles responsible for the planetary motions in latitude as well as the separation of the centre of heavenly bodies' mean motions, i.e. the equant point, from the centre of universe. The purely geometrical models built there, which mainly use al-Ṭūsī's geometrical device known today as Ṭūsī's Couple, along with those constructed by Ibn al-Shāṭir within a century later is known today as the Maragha School, which remained the main focus of historians of astronomy throughout the past half-century concerning the activities at Maragha.[16]

[14] About this model, see Toomer (1969, 1987), Samsó (1987, 2001, pp. 207–218 and 491–492), Samsó and Millás (1994) and Calvo (1998).

[15] See Goldstein (1971, p. 10) and the other sources mentioned in note 5 above.

[16] See Ibn al-Shāṭir's models in Roberts (1957, 1966), Kennedy and Roberts (1959), Abbud (1962); Quṭb al-Dīn's models in Kennedy (1966) (these studies have been collected in Kennedy and Ghanem 1976; Kennedy 1983); al-Ṭūsī's models in his *Tadhkira*: al-Ṭūsī 1993 and Hartner (1969) that deals with Ṭūsī's lunar model; also see Hartner (1973). For al-'Urḍī's models, see esp. Saliba [1989] 1994, pp. 135–142; also, the other papers by Saliba collected in his 1994 book (hereafter, the page references are to Saliba's 1994 book). Ṭūsī and the members of the so-called Maragha School applied Ṭūsī's Couple in a more complicated and matured way to the planetary theory, and Copernicus did the same in *De Revolutionibus* III. By the translation of a cosmographical work of the Maragha circle into Greek, which did through the oral teachings of Shams al-Dīn Muḥammad al-Wābkanawī al-Bukhārī (1254?–after 1316), the most prominent astronomer of the second period of the Maragha observatory (about him, see, e.g., Mozaffari 2013b, pp. 238–242), to Gregory Chioniades, this device was entered into the Byzantine literature which were available in Padua about Copernicus's time. Neugebauer notes this and reproduces the diagram showing Ṭūsī's Couple in MS. Vatican, gr. 211 (1975, Vol. 2, p. 1035, Vol. 3, Plate IX on p. 1456; the cosmographical text in question has been edited in Paschos and Sotiroudis 1998). These opened new venues for the research on the transmission of Ṭūsī's Couple, in particular, and the Maragha models in general from the Middle East to Europe in the recent decades. It deserves noting that although the transmission had most probably occurred in reality (as early as 1956, Neugebauer notices this: 1956, p. 170; also, see his enlightening statement in 1968, p. 90;

The early Islamic astronomers had raised some questions and kept them open to be answered after more observations would be made in a long period to enable future generations to find appropriate answers to them.[17] At the core of them was the problem of secular or periodic changes in the Ptolemaic constants, and optimistically, it was expected that further continuous observations could gain new insights for solving them. Although the complicated nature of such a problem together with the methodology [*A*], mentioned in the beginning of this paper, established in Middle Eastern astronomy through about five centuries and enhanced by other observational programmes after Bīrūnī (e.g. by 'Abd al-Raḥmān al-Khāzinī, *c*. 1125, and Ibn al-Fahhād, *c*. 1175), raised a slight chance for a comprehensive programme of this kind. For example, despite the values observed for the solar eccentricity during the early Islamic period, which show a slow continuous decrease in its size, al-Khāzinī derived a greater, and in fact more erroneous, value and Ibn al-Fahhād arrived at the same value that Yaḥyā b. Abī Manṣur had measured almost three centuries and a half earlier.[18] The only parameters that clearly illustrate the probability of any change or variation in Ptolemaic constants with the passage of time were the rate of precession and the obliquity of the ecliptic.

By means of his systematic stellar observations at Maragha, Muḥyī al-Dīn deduced that the precession consists only in a uniform, continuous motion, with the rate of $1°$ in every 66 years, which is faster than the value $1°/72^y$ he had derived from his observations at Damascus and adopted it in his earlier work, the *Tāj al-azyāj*.[19] His value for the rate of precession and his belief in the continuous uniform precession in the *Tāj al-azyāj* stood against the theory of trepidation received in Western Islamic astronomy. Some astronomers that worked in the Maghrib in the fifteenth century on cast doubts on the truth of trepidation theory. In this aspect, the *Tāj al-azyāj* exerted some influences on these astronomers, because the steady uniform precessional motion posited in this work with the rate of $1°/72^y$ would lead to a good agreement with the observations carried out there from the twelfth century on.[20] Muḥyī al-Dīn's contemporaries and colleagues at the Maragha observatory such as Ṭūsī and Shīrāzī accepted the constant rate $1°/70^y$ for the precession that their near predecessors (notably, Ibn al-A'lam and Ibn Yūnus) found[21]; Shīrāzī also adds that this value is used at our time, since it too is in accordance with the "new observations", namely those made at the Maragha obser-

Footnote 16 continued
Swerdlow 1973, p. 504), nevertheless the question is still "when, where, and in what form he learned of Maragha theory" (Swerdlow and Neugebauer 1984, pp. 41–48).

17 E.g. Bīrūnī's statement concerning the solar apogee motion in *al-Qānūn* VI.8: 1954–1956, Vol. 2, p. 685; translated in Hartner and Schramm (1961, p. 218).

18 See Mozaffari (2013a, Part 1: p. 326 (Table 3, nos. 7 and 8), 330, Part 2: pp. 393, 394–395).

19 This is the best medieval approximation to the true rate $1°/71.6^y$ and also can be found in the *Barcelona Tables* (written *c*. 1381); see Dorce (2002–2003, p. 198); 2003, pp. 111, 180. It was also independently measured in Italy or France in 1306, as documented in a codex preserved in Vienna, no. 5311, f. 137r; see Goldstein (1994, pp. 193, 196–197); also, for the star tables in this manuscript, see Kunitzsch (1986).

20 About it, see Samsó (1998, pp. 94–95, 2001, pp. 170–174).

21 Ibn Yūnus, *Zīj*, L: pp. 108, 125; Caussin (1804, p. 153).

vatory.[22] The two values $1°/66^y$ and $1°/70^y$ were known in the early Islamic period, and the Muslim astronomers sometimes reached the first and sometimes arrived at the latter, but apparently never wholly considered why the first value is deduced whenever their observed longitudes are compared with Ptolemy's or Menelaus' imaginary star catalogue (i.e. Ptolemaic longitudes minus 0;25°), while the latter values is derived whenever their longitude values are compared either with their Islamic predecessors or with those of the ancient astronomers, like Hipparchus and Timocharis as recorded in *Almagest* VII.2–3. Instead, the problem became whether the speed of the precession has really changed since Ptolemy ($1°/100^y$) through the early Islamic period ($1°/66^y$) and then to the time of the turn of the eleventh century (Ibn al-A 'lam, Ibn Yūnus, and Bīrūnī: $\sim 1°/70^y$).

Also, from the extremal values observed for the solar noon/meridian altitude at Maragha in 1264, which were 76;9,30° and 29;9,30°, Muḥyī al-Dīn measured a value of 23;30° for the obliquity of the ecliptic.[23] This value is used in the other two *zījes* of the Maragha tradition, i.e. the *Īlkhānī zīj* and Wābkanawī's *Muḥaqqa zīj*, as well as in al-Kāshī's *Zīj*.[24] However, in the *Īlkhānī zīj*, al-Ṭūsī remarks that "on the basis of our observations, the obliquity of the ecliptic exceeds 23;30° by a small amount and we estimated it to be 23;30°".[25] This as compared with Ptolemy's 23;51° and the values in the range from 23;33° to 23;35° repeatedly measured by the earlier Islamic astronomers conspicuously exhibit the permanent decrease in the maximum angular distance of the ecliptic from the celestial equator.

These two problems found some echoes in the Maragha cosmographical works: Ṭūsī's *Tadhkira fī 'ilm al-hay'a* ("Memoir on the cosmography") and Shīrāzī's three major treatise, *Ikhtiyārāt-i muẓaffarī* ("Selections by Muẓaffar al-Dīn"; dedicated to Muẓaffar al-Dīn Bulāq Arsalān, d. 1305, a local ruler), *Tuḥfa al-shāhiyya fi 'l-hay'a* ("Gift to the king on astronomy"), and *Nihāyat al-idrāk fī dirāyat al-aflāk* ("Limit of Comprehension in the knowledge of celestial heavens").[26]

The variation in the rate of precession is mentioned in these works in connection with the variation of the obliquity, and both of them are initially considered in relation to the simple trepidation model of the Eastern Islamic astronomy. Ṭūsī and Shīrāzī's works give some strong impression that no one at the time could be sure of the truth of whether such secular or periodic variation in reality exists in the precession or in the

22 Al-Ṭūsī (1993, Vol. 1, pp. 123, 125); Shīrāzī, *Ikhtiyārāt*, ff. 27r, 30r; *Tuḥfa*, ff. 19r, 22v; *Nihāyat*, P1: ff. 18r, 20r, P2: ff. 39v, 43v. In his earlier wok, *Mu'īniyya* (IV.6, p. 30), Ṭūsī does not mention this value, and only refers to "$1°/100^y$ found in the time of Ptolemy and Menelaus and $1°/66^y$ observed by the moderns".

23 *Talkhīṣ* III.1: f. 31r (about this work, see Saliba [1983, 1985, 1986] 1994, pp. 163–186, 208–230; Mozaffari 2014). Muḥyī al-Dīn had already reached this figure through his observations carried out at Damascus, and this was known to the Western Islamic astronomers through the diffusion of his *Tāj al-azyāj* there (see Samsó 1998, pp. 96–97; 2001, p. 173) and also is applied to some timekeeping table by Taqī al-Dīn Muḥammad b. Ma'rūf (see King 2004/2005, Vol. 1, pp. 64, 448).

24 Al-Kāshī, *Khāqānī zīj* II.1.4: IO: f. 27r.

25 *Īlkhānī zīj*, C: p. 203, T: f. 102v, P: f. 59v, M1: f. 104v, M2: f. 89v.

26 According to Niazi (2014, pp. 85–86, 98), Shīrāzī's three works were written in the first part of the 1280s; first, *Nihāyat*, next, *Ikhtiyārāt*, and then *Tuḥfa*.

obliquity.[27] None of them extend their speculations to the variation in the precessional motion, but the change in the obliquity of the ecliptic appears more evident to them, since both Ṭūsī and Shīrāzī consider more seriously the apparent decrease in the obliquity of the ecliptic since Ptolemy's time as indicated by the observations made by their early Islamic predecessors and their own. It was this that paved the way for the simple trepidation model having been posited, of course, solely as a provisional solution which was soon discarded.[28]

Ṭusī's final solution is the employment of his own method (*wajh*) or, properly speaking, the spherical version of the so-called Tūsī's Couple.[29] Nevertheless, it appears to have been employed as a merely hypothetical consideration to highlight the power of his geometrical device for providing theoretical explanations for more problems at issue in astronomy of his time. In other words, he appears to consider the apparent permanent decrease in the obliquity of the ecliptic and the variation of the rate of precession as typical hypothetical problems for which his geometrical device could readily provide a physically justified mechanism to account.

Shīrāzī rejects the variation in the speed of precession, but appears to show some doubt about the change in the obliquity, and that the decrease in its observed values may in reality be because of the closeness of the ecliptic to the equator.[30] He then proceeds to explain the simple theory of trepidation accounting for the changes both in the speed of precession and in the obliquity in more details than Ṭūsī. His other solution is, of course, the use of the spherical version of Ṭūsī's Couple, which he specifically calls the "model/hypothesis for the declination/obliquity" (*aṣl al-mayl*), in order to account for the variation in the obliquity.[31] Meanwhile, he proposes a third alternative: as he puts it, if the value of the motion of the obliquity of the ecliptic is known precisely, then the simplest/nearest method (*aqrab wajh*) for the conception (*taṣawwur*) of its mover (*muḥarrik*) and model (*hay'a*) is that the solar model should be an eccentric-epicyclic one. This model makes the decrease in the obliquity intrinsically pertinent to that of the solar eccentricity.

This forgotten solar model appears interesting in some aspects; more notably, this is the only attempt we know from the medieval Middle Eastern astronomy to take into account the secular changes in Ptolemaic astronomy more seriously than had previously been considered. This model, which has hitherto remained unnoticed in the modern literature, is the main subject of the present paper.

This solar model is explained in all of Shīrāzī's three major treatises mentioned above, but, unfortunately, in none of them does he provide a figure, in the absence of which, it is difficult to reconstruct the model. In what follows, we summarize and

[27] Al-Ṭūsī (1993, pp. 222–223) says that his method (i.e. Ṭūsī's Couple) can be applied to accounting for the variation in the speed of the precession or in the obliquity if the truth of these two motions and their variability is ascertained. A similar remark is given by Shīrāzī prior to the explanation of his solar model.

[28] Al-Ṭūsī (1993, pp. 125); Shīrāzī explains this trepidation theory in a confused way in *Ikhtiyārāt* II.4: ff. 27r–28r; *Tuḥfa* II.7: ff. 19r–20v; *Nihāyat* II.4: P1: ff. 18r–19r, P2: ff. 39r–41r; about it, also, cf. Hartner (1971, pp. 284–287).

[29] Al-Ṭūsī (1993, pp. 222–223); Ragep (1987, p. 348); about it, see Saliba and Kennedy (1991).

[30] Shīrāzī, *Ikhtiyārāt*, f. 26v; *Tuḥfa*, f. 18r; *Nihāyat*, P1: f. 18r, P2: f. 39r.

[31] Shīrāzī, *Ikhtiyārāt* II.9: ff. 87r–v; *Tuḥfa* II.8: ff. 34r–v; *Nihāyat* II.5: P1: f. 27r, P2: ff. 63r–v.

classify Shīrāzī's remarks in Sect. 2 and then present our reconstruction of this model on the basis of his account in Sect. 3.[32] Before beginning the explanation, let us first say something of Shīrāzī's contribution to astronomy and then clarify a frequently used technical term that the whole of his solar model is to account for its variation, i.e. *mayl.*

In two studies in 1979, Saliba shows that al-Shīrāzī's planetary model that became known through Kennedy's (1966) study is due to his elder colleague and contemporary, al-'Urḍī.[33] One year later, Saliba referred to Shīrāzī's criticisms of a simple concentric-epicycle model worked out by Abū 'Ubayd al-Jūzjānī (d. 1070), Avicenna's famous student.[34] A decade later, he reasonably argued that Shīrāzī's most ingenious models are for the moon and Mercury[35]; of course, he believed over two decades ago that "further studies may very well change the picture we now have of Shīrāzī's contribution to the Maragha studies". To the best of our knowledge, no further and deeper study has yet be accomplished to reject or conversely establish Shīrāzī's importance in this aspect, while the present study reveals another model of his own that seems ingenious, original, and unprecedented.

The study of Shīrāzī's astronomy has difficulties of its own: let us quote the late Kennedy and Saliba that when describing Shīrāzī's planetary models, each folio is exasperating by itself, the reason being, as the latter explains,[36] that Shīrāzī put forwards parallel models or different variants of a model side-by-side, posits and discuss the faults and discrepancies or some aspects of a part of them, and leaves the remainder to his readers to discover or decide; an exceptional attitude we shall presently encounter, too, in the account of his solar model. The other problem is that Shīrāzī, similar to many medieval scholars, does not identify his sources adequately: besides Shīrāzī's adoption of al-'Urḍī's models, he adopts al-'Urḍī's planetary order, according to which Venus is located above the sun in the geocentric view,[37] without any

[32] Shīrāzī, *Ikhtiyārāt* II.4: f. 28r; *Tuḥfa* II.7: ff. 20v–21r; *Nihāyat* II.4: P1: f. 19r, P2: f. 41v.

[33] Saliba [1979a and 1979b] 1994, pp. 114, 119–134. Saliba (1987) also demonstrates the close dependence of Shīrāzī's discussion on the height of the atmosphere on that of 'Urḍī. Also, it is noteworthy that in his non-astronomical writings, Quṭb al-Dīn shows a heavy dependence on his Islamic predecessors, often without acknowledging them; e.g. in the case of his well-known encyclopaedia, *Durrat al-Tāj*, see Pourjavady and Schmidtke (2004), in which the authors go farther to conclude that "the fact that, with the exception of portions of the section on logic, no part of the philosophical sections of *Durrat al-tāj* was originally written by Quṭb al-Dīn al-Shīrāzī, suggests that his significance as a philosopher should be reconsidered" (*ibid*, p. 320). It is noteworthy that the mathematical part (first section of *jumla* 4) of *Durrat al-tāj* seems to be a Persian translation of Muḥyī al-Dīn's al-Maghribī's *Taḥrīr al-Uṣūl* (*ibid*, p. 313), although this needs to be checked further. The astronomical part (second section of *jumla* 4) of *Durrat al-tāj* is also a translation of 'Abd al-Malik b. Muḥammad al-Shīrāzī's (d. *ca.* 596 H/1200 AD) *Talkhīṣ al-majisṭī*, as Quṭb al-Dīn himself states. About Quṭb al-Dīn's intellectual background and the manuscripts written by his own hand, see Pourjavady and Schmidtke (2007, 2009).

[34] Saliba [1980] 1994, esp. pp. 86, 89.

[35] See Saliba [1991] 1994, pp. 261–262, 265–266.

[36] Saliba [1991] 1994, p. 281.

[37] Al-'Urḍī made a drastic change in Ptolemy's order of the planets by placing Venus above the sun. He reasonably assumes the values which Ptolemy quotes from Hipparchus for the apparent radii of the planets at their *mean* distances should have been measured at their *least* distances, and as well, he takes the actual radii of the bodies of the sun, moon, and planets and the thickness of the sphere of lunar nodes into account in the computation of their distances. The latter consideration is indeed an improvement over

mention of the latter's name, after he presents a densely argumentative comparative discussion of Ptolemy's *Planetary Hypotheses* and Kūshyār b. Labbān's (tenth century) method for the measurement of the sizes and distances[38] accompanied with his critical remarks.[39] It is noteworthy that al-'Urḍī's treatise on the sizes and distances was written before his joining the Maragha observatory, and hence, it is not known whether he withdrew his opinion of putting Venus above the sun after collaborating with Ṭūsī and learning of the suspect observations of the Venus transit documented in his *Taḥrīr al-majisṭī*.[40] But it is curious that Shīrāzī poses again the

Footnote 37 continued
Ptolemy's procedure on the basis of which al-'Urḍī derives the boundaries of the lunar spheres, i.e. the limits of the convex and concave surfaces of its spheres as equal, respectively, to (maximum distance + radius of the moon + thickness of its sphere of nodes) = 64;10 + 0;17,33 + 0;2,27 = 64;30 and (minimum distance − radius of the moon) = 33;33 − 0;17,33 = 33;15,27 terrestrial radii. Then, from the Ptolemaic value of 64;10$^{t.r.}$ for the moon's greatest distance, together with committing an error in the calculation of Mercury's least distance (wrongly assumed equal to *radius of the deferent* − 3 · *eccentricity* − *radius of the epicycle*), he finds that the space between Mercury and Sun is not large enough to accommodate Venus, which was persuasive for him to place Venus above the sun. Consequently, this made the radius of the universe enlarge to 140115$^{t.r.}$, i.e. about 7 times as large as Ptolemy's (cf. Goldstein and Swerdlow 1970). Al-'Urḍī's schemata was not accepted by the later astronomers, except for Shīrāzī, but taking the thickness of the sphere of the lunar node into consideration found echoes in the later treatises; e.g. in Kāshī's *Sullam al-samā' (the stairway to the heaven)*, ff. 7r and 10r where the thickness is computed as 3;28,47$^{t.r.}$.

38 About Kūshyār, see Bagheri et al. (2010–2011).

39 Shīrāzī, *Ikhtiyārāt*, ff. 156v–175v; *Tuḥfa*, ff. 136v–153v.

40 Four Islamic reports of the Venus transit in the period of 800–1200 AD are discussed in Goldstein (1969); another report belonging to 939 AD came to light in Vaquero and Gallego (2002). As a comment upon *Almagest* IX.1, in his *Taḥrīr al-majisṭī* (P1: pp. 282–283, P2: f. 82v, P3: f. 107v), al-Ṭūsī mentions that a certain Ṣāliḥ b. Muḥammad al-Zaynabī al-Baghdādī reports in his book named *Majisṭī* the two observations of the transit of Venus made by al-Shaykh Abā 'Imrān at Baghdād and Muḥammad b. Abī Bakr al-Ḥakīm in Farsīn in the vicinity of Tūlak (all the three men are otherwise unknown); the interval of time between them was 20 years, and, in one of them, Venus was at the apogee of the epicycle (i.e. in superior conjunction) while in the other, at its perigee (i.e. in inferior conjunction). Al-Ṭūsī makes use of these reports to invalidate the idea that the two inferior planets are in the sphere of the sun as well as that the centre of their epicycles coincide with the centre of the sun's body, that is, that they rotate about the sun (for the translation of the passage in question, see Saliba [1987] 1994, p. 149). It is not known precisely whether each of the two mentioned astronomers had observed only one of the two presumed transits of Venus, or both of them had observed both of the two transits. In any case, one of the two observations is certainly incorrect simply because none of the inferior planets can transit across the solar disc in a superior conjunction, and thus, such an observation should at best be related to a large sunspot. Al-Ṭūsī mentions them after referring to Ibn Sīnā (Avicenna's) famous observation of the Venus transit, by means of which the time frame of them can be delimited: they should have been made somewhere between *ca.* 1032 (the only Venus transit that occurred during the lifetime of Ibn Sīnā) and 5 Shawwāl 644/13 February 1247 when al-Ṭūsī completed his *Taḥrīr al-majisṭī* (Saliba [1987] 1994, p. 145). In this period, the only two Venus transits took place on 22 May 1040 and 23 November 1153 (Espenak, *NASA's Six Millennium Catalog of Venus Transits*). Another note is that in the case of Venus, 5 revolution of the anomaly occurs in 8 years; so, it is a very simple matter to contend that if Venus is at the apogee of its epicycle at a given time, then 20 years later, i.e. after the two revolutions and a half through the epicyclic anomaly, it would be located at the perigee of its epicycle. Nevertheless, the consideration of the validity of the Venus transits for a medieval astronomer (as well as a modern historian of astronomy) requires a deeper scrutiny of an inferior planet's motion in longitude as well as in *latitude*. If al-Ṭūsī or al-Shīrāzī had made such a quantitative study, then it could have been known that even in the framework of Ptolemaic planetary models, and by means of applying his own parameter values, Venus transits can occur in the intervals of 105.5 or 129 years, *not* in periods of 20 years (cf. Neugebauer 1975, Vol. 1, pp. 227–229). Ptolemy himself describes qualitatively the circumstances involved in the observations of the transit of an inferior planet across the solar disc in *Planetary Hypotheses*

same order of the planets some time after al-Ṭūsī and made some efforts to justify and defend the idea by rejecting his Muslim predecessors' observational reports of the Venus transit and, instead, considering them, amazingly, as the observations of "a tiny black spot existing above the centre of the solar disc, as some people having a keen eyesight recognize it",[41] presumably without worrying about violating the Aristotelian dogma of the heavens devoid of Generation and Corruption. It also seems strange that at the beginning of his account of al-'Urḍī's schemata, Shīrāzī severely criticizes Ptolemy's procedure of computing the heavenly distances in the *Planetary Hypotheses*, at the first step of which Ptolemy rounds the least and greatest distances of the moon from the earth, 33;33 and 64;10 terrestrial radii, to 33 and 64.[42] Shīrāzī believes that Ptolemy has deliberately dropped the fractions and said that there is a reason for it, while no reason in doing so is acceptable, but Shīrāzī himself never realized that the whole of al-'Urḍī's idea that Venus is put above the sun stems from a miscomputation of Mercury's minimum distance (see, above, note 37).

In his *Nihāyat* II.8,[43] Shīrāzī attempts to give an observational proof that in the case of the three outer planets, the centre of the geocentric orbit, the deferent, on the circumference of which the epicycle centre revolves, should be located between the equant, the centre of the uniform motion in longitude, and the earth, since on the basis of it, the difference in size of the arcs of the retrograde motions of the planets can be accounted for.[44] It appears somewhat strange that an astronomer at once, on the one hand, makes attempts at constructing the alternative models to eliminate the physical/philosophical difficulties with the equant point in Ptolemaic models[45] by superimposing it on complicated systems of "cycle on epicycle, orb on orb"[46] and, on the other hand, incidentally, admits its validity in an observational aspect. Moreover, Shīrāzī's proof is indeed circular, simply because the first step in his derivation of the deferent eccentricity of the outer planets from their retrograde arcs consists in the use of the Apollonius's theorem for the determination of their stationary points; according to it, one needs to calculate the true angular

Footnote 40 continued
I: (1) the centre of the epicycle at one of the nodes and (2) the planet at that node (i.e. the longitude of that node equal to the true longitude of the planet), i.e. the planet at the (true) apogee or perigee of its epicycle. He also notes that a long time must elapse between the two successive returns of the centre of the epicycle and the planet in conjunction with the sun (Goldstein 1967, pp. 6–7, 28).

41 Shīrāzī, *Ikhtiyārāt*, f. 19v; *Tuḥfa*, f. 11v; *Nihāyat*, B: f. 27v, P1: f. 13v, P2: f. 29r. It is noteworthy that sunspots are also short-lived phenomena, and then it seems strange to maintain the existence of such an appearance for a long time; around Shīrāzī's time, there are the four reports of the observations of the large sunspots from the East Asian history dating to 15–16 September 1258, 17 February and 17 March–15 April 1276, and 31 August 1278, in which the sunspots are described as the "black spots as large as hen/goose's eggs"; see Clark and Stephenson (1978, p. 396), Yau and Stephenson (Yau and Stephenson 1988, p. 187).

42 See Goldstein (1967, p. 7).

43 Shīrāzī, *Nihāyat*, P1: ff. 37v–38r, P2: ff. 93r–94v.

44 The pertinent text is edited and translated in Gamini and Masoumi (2013).

45 Note that there is still equant motion in Shīrāzī's models produced by the eccentricity and the small epicycle.

46 The phrase is taken from Herschel (1851, p. 266).

velocity of a planet on the epicycle and that of its epicycle centre, none of which can, of course, be determined prior to an early derivation of the eccentricity of the equant.[47]

It is curious that his elder contemporary and colleague at Maragha, Muḥyī al-Dīn al-Maghribī, in his *Talkhīṣ* VII.4, makes similar remarks: that the centre of the deferent spheres carrying the epicycles of the four planets, that is, except for Mercury, is midway between the centres of the universe and the equant [circle] provides that the various observed motions of the planets agree truly with the hypothesis/model (*aṣl*) established for their motions [i.e. epicycle-eccentric]. He then adds that "Ptolemy knew it by means of the artifice (*ḥiylat*) rather than by means of proof, [but] we offer a proof for it later, God Willing".[48] The preserved incomplete manuscript of *Talkhīṣ* does not contain any such proof, and Ptolemy's iterative procedure is strictly applied to the computation of the eccentricities of the three superior planets. Nevertheless, since one of the missing two last books of the treatise is on the retrograde motions and latitudes of the planets, it is tempting to speculate that the proof (if in reality, it would have been included in the treatise) might have had something to do with the variation of the sizes of the arcs of retrograde motion of the planets, where the bisection of the eccentricity appears to account for their different sizes. This treatise appears to have been written after Muḥyī al-Dīn completed his last *zīj*, the *Adwār al-anwār*, at Maragha about the end of 1276, although some parts of it, including, especially, his extensive observations (spreading over one decade, from 1262 to 1274), the calculations embedded in it, and the parameter values derived from them, adopted in the *Adwār*, should certainly have been prepared at a time preceding the appearance of this work.[49] Due to the corruption of the only surviving MS. of the *Talkhīṣ*, we cannot, unfortunately, compare the two proofs given for Ptolemy's bisecting the equant eccentricity and examine whether Shīrāzī borrowed his proof from his elder contemporary without acknowledging him.

The term *mayl* is, indeed, a multi-purpose one in Islamic astronomy that was used for:

(a) the declination of the sun (for the planets and stars, a less specified term *bu'd*, "distance", or *bu'd al-khaf'i*, "hidden distance" was usually in use),[50]
(b) the obliquity of the ecliptic (often more specialized as *mayl al-kullī*, "total declination", *ghāyat al-mayl*, "extremal declination", etc.), or
(c) naming either the inclination (ἔγκλισις) of the planets' deferents to the ecliptic plane or that of their epicycles to their deferents that are seen in the line of sight as

[47] It is probable that Shīrāzī himself recognized the circular reasoning in his proof and consequently did not included it in his two later works, *Ikhtiyārāt* and *Tuḥfa*.

[48] Muḥyī al-Dīn, *Talkhīṣ*, f. 117v.

[49] See Mozaffari (2014, p. 71). It should be noted that the relevant section in the *Adwār* (II.6: M: ff. 19r–v, CB: ff. 18r–v) gives no information about Muḥyī al-Dīn's proof. A monograph about Muḥyī al-Dīn contribution to observational and practical astronomy at the Maragha observatory on the basis of a thorough analysis of his documented observations in the *Talkhīṣ* is in preparation by the present author.

[50] E.g. al-Battānī in *Ṣābi' zīj*, Sect. 18: Nallino [1899–1907] 1969, Vol. 3, p. 46.

well as for the inclination (according to the majority of Islamic *zījes*, the "second") component of the latitude of the inferior planets.[51]

2 Description of the model

In what follows, a comprehensive account of the solar model as found in all of Shīrāzī's three works is presented. In doing so, we strictly follow the order in which our author explains his model. His remarks are numbered for the convenience of referring to them in our reconstruction in the next section, and the variations in his account are marked with regard to the relevant treatises in order to distinguish the differences between them.

[1] The period at which the sun completes one revolution through the epicycle is equal to the period of [the variation in] the obliquity [of the ecliptic] from its maximum to minimum and then to maximum.

[2] Its [i.e., the epicycle's] equator (*minṭaqa*, lit. "belt")[52] is inclined to the path/trajectory (*madār*) of its [i.e., the sun's] epicycle centre by the maximum excess or deficit of the obliquity [of the ecliptic from its mean value].

[3a] [In the *Nihāyat* and *Tuḥfa*:] The path of the centre of the epicycle actually describes the ecliptic.

[3b] [In the *Ikhtiyārāt*:] The path of the centre of the epicycle lies in the plane of the ecliptic.

[4] [Only in the *Tuḥfa*:] Or, its [i.e., the epicycle's] equator/belt is perpendicular to the path/trajectory of the centre [of the epicycle], so that the diameter of the path of the centre of the sun's body through the epicycle from the north to the south and vice versa becomes equal to the sum of the chords of the arcs of the [maximum] excess and deficit [of the obliquity of the ecliptic from its mean value].

[5] Thus, the sun becomes closer to or further from the path of the epicycle centre, and so the ecliptic (since the path of the centre of the epicycle lies in the plane of the ecliptic), by the amount of the inclination (*mayl*) of the equator/belt of the epicycle.

[6] Therefore, the sun becomes closer to or further from the celestial equator, and the obliquity increases or decreases, because the sun is not always moving in the plane of the ecliptic, but is inclined to it in either of the two directions

51 *Almagest* XIII.3: Toomer (1998, pp. 601–602). This equivalence is according to Hunayn-Thābit's translation which constituted the standards in the Arabic astronomical terminology: Arabic *Almagest*, S: ff. 212v–213r, PN: ff. 162r–v. For the components of Ptolemy's planetary latitude models in the *Almagest*, see Pedersen (1974, pp. 358–359, 369–370), where ἔγκλισις is rendered into "deviation"; Neugebauer (1975, Vol. 1, pp. 209, 214), Swerdlow (2005, pp. 51–52).

52 Since our author deals with these matters from a cosmographical (*hay'a*) point of view (as in Ptolemy's *Planetary Hypotheses*), the orbital components such as epicycle and eccentric are always treated as spheres/orbs. Accordingly, instead of simply using epicycle or eccentric as the geometrical instruments/devices (as in the *Almagest*) in order to describe the compound motions of a heavenly object, he indicates this by referring to the equator/belt of the epicycle or eccentric spheres, that is, the great circle 90° distant from the poles of these spheres. In order not to confuse it with the celestial equator, we add "belt" immediately after "equator" in these cases.

[i.e., towards north or south], except for when it is located in the two points intersecting the equators of its epicycle and eccentric;

[7] [In this situation], the [eccentric] circle that the centre of the sun's body describes is equal to the equator/belt of the eccentric; in other situations: if the sun is at the apogee of the epicycle or in vicinity of it, the circle that the centre of the sun's body describes is greater than the equator of the eccentric, and if the sun is at the perigee of the epicycle or in its neighborhood, the circle that the centre of the sun's body describes is less than the equator of the eccentric.

[8] From this, it necessitates that the amount of eccentricity varies, because it is a constant thing (Persian: *yik čīz*, Arabic: *shay' wāḥid*) that is once taken in proportion to a greater value and at another time, to a lesser value. In Ptolemy's time, the sun was at the perigee of the epicycle, and for this reason, his value for the eccentricity is larger than what is derived from the observations of the moderns.

[9a] [In the *Nihāyāt* and *Ikhtiyārāt*:] if we put the ecliptic in the plane of the path of the sun's body, then although this necessitates that the sun would always be/move in the plane of the ecliptic, it also necessitates that the equator/belt of the eight sphere is not always in the plane of the ecliptic, because the situation of the ecliptic changes while the situation of the equator/belt of the eight sphere does not, because there is no mover to move the equator/belt of the eight sphere in latitude.

[9b] [In the *Nihāyat*:] We prefer the second, because the observations indicting that the sun moves in the plane of the ecliptic is more correct and closer to the truth than the observations denoting that the equator/belt of the eight sphere lies in the plane of the ecliptic. This is revealed to whoever uses the book of the *Almagest* and investigates the observations mentioned in it.

[9c] First, if we put the ecliptic in the plane of the path of the centre of the sun's body, then the sun would still move in the plane of the ecliptic, but this necessitates that the ecliptic does not become a great circle, since the centre of the sun's body describes the circles parallel to that described by the centre of the epicycle that passes through the centre of the cosmos, and thus since they do not pass through it, they are not the great circles. Second, this also necessitates that the celestial equator does not lie in the midway/midpoint of the arc that is between the greatest [noon/meridian] altitude of the sun towards the north and its least altitude [noon/meridian] towards the south. Therefore, it is not only the most preferable, but also essential to make/put the circle described by the centre of the epicycle as the ecliptic, not that described by the centre of the sun's body.

3 Reconstruction of the model

The model consists of an inclined epicycle, the small circles with diameter *ab* in Fig. 1, the centre of which rotates on an eccentric, the circle of diameter c_1c_2. The sun rotates through the epicycle very slowly; the period of its rotation, as stated in [1],[53] is

[53] The initial remark [1] is somewhat corrupted in the *Ikhtiyārāt* and *Nihāyat*, but complete in the *Tuḥfa*.

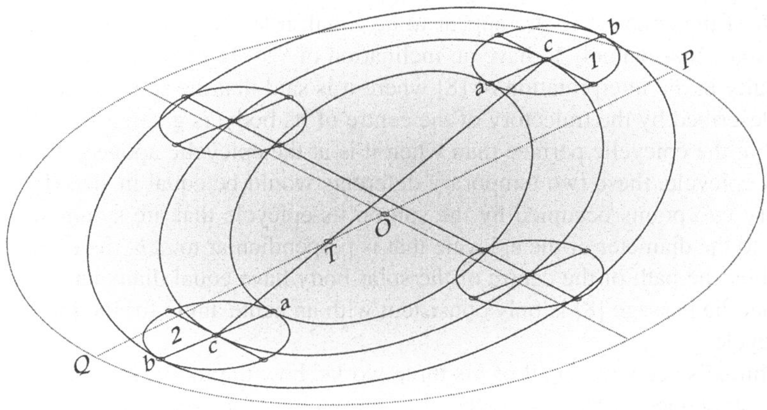

Fig. 1 Reconstruction of Shīrāzī's solar model

equal to the period of the variation in the obliquity of the ecliptic. On the basis of the quantities Shīrāzī refers to, the bounds/limits of the obliquity are not greater than 24° and not yet reaching a value less than 23;30° according to the observations carried out at Maragha in Shīrāzī's time. By comparing the value 23;35° obtained in al-Ma'mūnic observational programme at Baghdad/Damascus about 830 AD and Ptolemaic value 23;51° in the 690 Egyptian/Persian years intervening between the two, Shīrāzī finds the approximate rate 1′ in every 43 E/P years. These are the only parameter values Shīrāzī mentions in connection with this solar model. It is, of course, not known why he did not compare Ptolemy's value with the value 23;30° derived from the Maragha observations, which also constitutes his lower limit; by taking it into consideration, he could find a longer time interval, which is one of the Ptolemaic norms for establishing the rates of motions, and thus could arrive at a more precise value for the rate of the decrease in the obliquity of the ecliptic (about 1′ in each 53 years).[54]

By the solar motion through the epicycle, it also departs from the ecliptic or approaches towards it [5], and hence, its vertical angular distance from the celestial equator, the highlighted circle of diameter *PQ* in Fig. 1, varies as well [6].

The angle *acT* of the inclination of the epicycle from the eccentric c_1c_2, we are told in [2], is equal to the amplitude of the variation in the obliquity of the ecliptic (angle *aTc* and angle *bTc*) from its mean value (angle *cTP* = angle *cTQ*); this, of course, cannot be the case, simply because the inclination can be derived only from the amplitude of the variation in the ecliptic and two of the lengths *Tc*, *Ta*, and the radius *ac* of the epicycle. Our author seemingly recognize this difficulty and later in the *Tuḥfa* [4], says that the epicycle is perpendicular to the path of its centre; in this case, the size of the epicycle can be calculated from the amplitude of the variation of the obliquity in terms of the arbitrary units assigned to either the length of *Tc* or of *Ta*. In addition, since the amplitude of the variation in the obliquity towards the north and south of

[54] Note that Muḥyī al-Dīn's observation of the obliquity (above, note 23) was carried out in the year 633 Yazdigird, and the interval between it and Yaḥyā b. Abī Manṣūr's observations (200 Y) is about 433 E/P years. This added to Shīrāzī's value 690 years between the latter and Ptolemy yields 1123 years; thus, (23;51,20°–23;30°)/1123 ≈ 0;0,1,8° per year or 1′ in each 53 E/P years.

the path of the epicycle centre appear to be equal, it is necessary for the epicycle to be perpendicular, namely to have an inclination of 90°, to c_1c_2. But, this introduces a difficulty in the interpretation of [8] where it is said that the solar deferent, i.e. the circle described by the trajectory of the centre of its body, is greater when the sun is located in the epicyclic perigee than when it is at the epicyclic apogee, while with a vertical epicycle, these two temporary deferents would be equal in size (In general, for every two points occupied by the sun on its epicycle that are symmetrical with respect to the diameter of the epicycle that is perpendicular to *acb*, the two deferents marked by the path of the centre of the solar body have equal diameters.). It is then clear that the passage [8] is only consistent with an acute angle for the inclination of the epicycle.

In Shīrāzī's accounts in all of his three works, how the model behaves is not sufficiently explained. Since the model is to account for the variation in the obliquity, and the inclination of the epicycle is taken as fixed [2], it should safely be assumed that the epicycle keeps its positions fixed at all times; namely its surface at various positions on the eccentric c_1c_2 remains parallel to itself, and therefore, the model basically resembles Ptolemy's model for the latitude of the superior planets in the *Handy Tables* and *Planetary Hypotheses*, in both of which the inclination of the epicycle is fixed, although inclined to the ecliptic in the *Handy Tables* and parallel to the ecliptic in the *Planetary Hypotheses*. Also, it can be understood from [7] that when the sun is located at the epicyclic perigee a, the eccentric described by the centre of its body has the least diameter (a_1a_2), and, on the contrary, when the sun is at the epicyclic apogee b, the eccentric so described reaches its largest size (b_1b_2). In [8], the minimum size of the eccentric deferent is associated with about Ptolemy's time when the obliquity was close to its maximum value and, on the contrary, the maximum diameter of the deferent should naturally pertain to the minimum limit observed for the obliquity at Shīrāzī's time. To put these together straightforwardly leads to the model illustrated in Fig. 1, which appears to be the most reasonable reconstruction of what our author had in mind.

The circle c_1c_2 is simply called the path of the centre of the epicycle, the role of which in the model is multifold: it lies in the plane of the ecliptic, as said in [3b], or the ecliptic itself is formed by it, as remarked in [3a]. It also defines the solar eccentric deferent when the sun is located at either of the points intersecting it and the epicycle; in this way, it can be taken as the representative of a *mean* eccentric. The trajectories of the centre of the solar body are the temporary eccentrics, continuously connected with one another to form a perplexing spiral path through which the sun moves around the earth.

The difficulties arise from the passage [9a] in which Shīrāzī exhibits his suspicion about the above-mentioned underlying assumptions of the model. From the remarks in [9a], the two versions of this model arise. Despite his initial statements in [3], our author appears now undecided about which does define, or lie in the plane of, the ecliptic: the path of the centre of the solar epicycle or the path of the centre of the solar body. In both the *Nihāyat* and *Ikhtiyārāt*, he considers these two versions as equivalent and so leaves the matter to his reader to choose which of the two is preferable; however, in the *Nihāyat*, he adds the passage [9b] which clearly contradicts all of his earlier statements in [3] and [9a]: that the path of the centre of the solar body determines the ecliptic, the

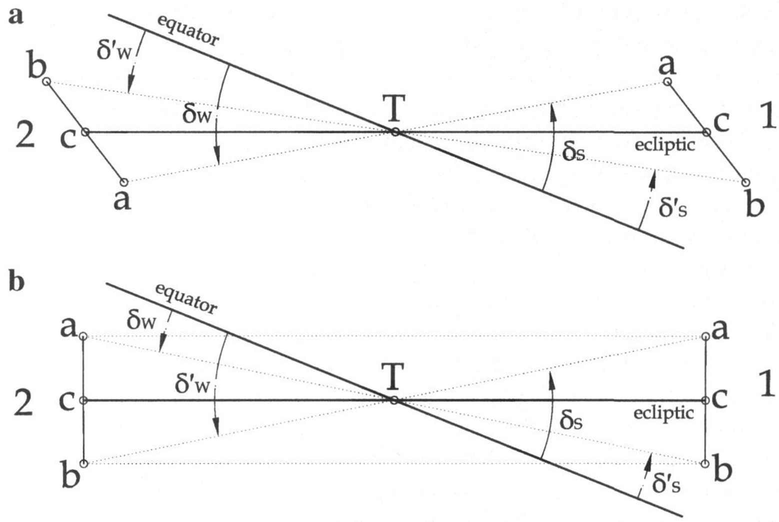

Fig. 2 **a** δ is the solar annual greatest declination at the perigee of the epicycle and δ', at the apogee of the epicycle. Subscript S denotes the summer and W the winter. In this model, the solar yearly maximum and minimum declinations, around the summer and winter solstices, respectively, are equal: $\delta_S = \delta_W$ and $\delta'_S = \delta'_W$. **b** The epicycle does not remain fixed, in spite of the model in (**a**): $\delta_S \neq \delta_W$ and $\delta'_S \neq \delta'_W$

obliquity of which varies, and so the circle of diameter c_1c_2 is simply taken as laying in the plane of the equator/belt of the eight sphere (besides being the sun's mean eccentric and the mean ecliptic as well), which is permanently stable and unmoving, because, from a physical point of view, there is no mover to move it. But, by his additional statements in the *Tuḥfa* [9c], he refrains from this second version and returns to his primary assumption in [3]: that the trajectory of the centre of the solar epicycle defines or lies in the plane of the ecliptic, and thus, this is no longer the obliquity of the ecliptic that varies, but the sun is implicitly given a slight latitude which smoothly and very slowly changes, and, as a result, the solar maximum declination varies, while the ecliptic is fixed in the surface c_1c_2. Nevertheless, greater difficulties of the other sort still arise from the two reasons Shīrāzī posits in [9c] for his change of mind in favour of a fixed ecliptic. As we have shown earlier, in order to enable the model to account for the variation either in the maximum declination of the sun or in the obliquity of the ecliptic, the epicycle should remain fixed everywhere on the mean eccentric, because, as shown simply from a lateral view in Fig. 2a, only by this can the symmetrical, continuous change in the declination of the sun during a solar year between the two equal limits (one positive, i.e. towards the north pole, and another negative, i.e. towards the south pole) be maintained, and consequently, the altitude of the highest point of the celestial equator in a given horizon (i.e. the co-latitude of a location) would be equal to the arithmetic mean of the sun's yearly greatest and least meridian/noon altitudes, which take place around the summer and winter solstices, respectively. In this model, the path of the centre of the sun's body, e.g. the circles with diameters a_1a_2 and b_1b_2, do not in principle belong to the same sphere, and therefore, one cannot speak of them as being the great or small circles on the surface of the celestial sphere, in spite of what Shīrāzī posits as his first reason in [9c]. This reason can, at first sight, be the case only if we assume that the epicycle does not keep its position fixed with respect to the path of the epicycle centre. This is shown in Fig. 2b: the trajectories a_1a_2 and b_1b_2 of the centre of the sun's body are parallel to the path c_1c_2 of the centre of the sun's epicycle. Accordingly, if the ecliptic is identical to the circle described by the path of the centre of the solar body, the ecliptic would no longer cross the equator at T, the earth's centre, and thus, the sun's yearly maximum northern declination would not be equal to its maximum southern declination, and therefore, the celestial equator would not lie midway between the greatest solar meridian altitude towards the north from the horizon and its least meridian altitude towards the south. Shīrāzī seems to have totally lost the main purpose of the model, because, regardless of which of the two paths marks the ecliptic, the model as illustrated in Fig. 2b can in no way maintain the symmetry of the annual increase-and-decrease in the solar declination. This gives an impression of Shīrāzī's confusion with this model, which we have already seen in his positing different variants of the model and freely passing from one assumption, premise, or consequence to another, without presenting a schematic view of the model and adequately setting out the necessary premises and logical sequences leading to some of the statements in all of his three major treatises.

At any rate, if (a) we put Shīrāzī's hesitation in [9a] aside, (b) neglect his reasons in [9c] which, although correct, indicate a variant of the model, by means of which neither the long-term variation in the obliquity of the ecliptic nor in the maximum declination of the sun can be accounted for, and (c) accept the general stemma deduced and drawn by K. Niazi for the order in time of Shīrāzī's writing of his main cosmographical works, i.e. *Nihāyat* → *Ikhtiyārāt* → *Tuḥfa*,[55] then it can be said that we confront a qualitative model, according to which the obliquity of the ecliptic is constant, but both the maximum declination and eccentricity of the sun vary periodically with the passage of time, which is accounted for by mounting an inclined epicycle on the solar eccentric. It should be noted that if this is in reality the case, then the term "*mayl*" in all of the above passages should be rendered into the "maximum declination of the sun".

4 Final remarks

By the final version of the model in his *Tuḥfa*, Shīrāzī appears to display his firm adherence to the already-established methodology of the Middle Eastern Islamic astronomy by assuming/believing that both the solar eccentricity and the obliquity of the ecliptic are constant. The change in the first is due to the oscillating-in-size deferent of the sun that is temporarily defined by the trajectory of the centre of its body, which itself revolves on the epicycle; in fact, the eccentricity, a constant in Ptolemaic astronomy, remains constant in this model, by interpreting this parameter/concept as "a length", not "a ratio". In this model, it is the dimension of the geocentric orbit, i.e. the diameter of the deferent, of the sun that oscillates with a long period equal to that of the variation in the maximum declination of the sun, and so, by the change of its radius, the ratio

[55] See above, note 26.

of the length of eccentricity to the radius of the deferent varies as well. And the variation in the second is in fact because of the change in the solar maximum declination that is the inevitable consequence of the sun's very slow motion through its inclined epicycle. In the other words, the problem is no longer the variation in the obliquity of the ecliptic, but in the maximum declination of the sun. In fact, the problem of the decrease in the obliquity of the ecliptic is reduced to implicitly assuming a slight latitude for the sun.

Shīrāzī does not derive the underlying parameters of the model and so does not quantify it. Both the simple trepidation theory and this solar model remained (one can perhaps even say that both Ṭūsī and Shīrāzī kept them) in a theoretical level in the cosmographic works of the *hay'a* genre, without achieving any application in the mathematical and practical astronomy. Thus, the entirety of these considerations about the decrease in the obliquity of the ecliptic or in the maximum declination of the sun in Middle Eastern medieval astronomy was confined merely to a qualitative hypothesis. Shīrāzī, of course, remarks, perhaps as an excuse, that the size of the sun's motion in latitude or in maximum declination (namely the amplitude of the inclination of its deferent either from the ecliptic or from the celestial equator) has not been precisely ascertained, but only its bounds/limits are known: not reaching greater than 24° and not being less than 23;30°. From this situation, one can generally argue that the Maragha astronomers had never heard of the existence of such hypotheses and the elaborately constructed quantitative models in the Western Islamic astronomy during almost three centuries before them.[56]

It remains to point out that in the *Tuḥfa* and *Nihāyat*, Shīrāzī says that the use of Ṭūsī's Couple is generally better and more useful, by which it is the change of the obliquity that is accounted for, not of the maximum declination of the sun, as in the final version of the first model, as presented above.

The discovery that the obliquity of the ecliptic in reality decreases seems to have received no attention in Eastern Islamic astronomy until Taqī al-Dīn's observations at Istanbul in the 1570s, perhaps because the values observed in the intermediate period are either greater than the value 23;30° observed at Maragha (e.g. Ibn al-

[56] Cf., also, Comes (2001, esp. pp. 316–318), where she made some attempts at exhibiting some traces of Ibn al-Zarqālluh's trepidation model in Ṭūsī's discussions in his *Tadhkira*. Although some exchanges might have been existed between Castile and Iran (cf. Comes 2004), it seems that there is slight chance of demonstrating the implication of the Andalusian trepidation models for the Maragha cosmographical works; two out of the three issues of similarity the late M. Comes enumerates in his 2001 study, i.e. the hypothesis of the proper motion of the solar apogee and the maximum bound of 24° for the obliquity of the ecliptic (originated in Hindu astronomy, which is, of course, different from Ibn al-Zarqālluh's maximum limit of 23;53°) were already available in the works of Ṭūsī's Middle Eastern predecessors; Comes's third issue is al-Ṭūsī's adoption of the vernal equinox as the reference point of the trepidation; this is correct, but it can be simply contended that al-Ṭūsī points to it only when he briefly describes the simple zigzag trepidation model, and does not identify any reference point when presenting his summary of the simple trepidation model of the Eastern Islamic astronomy. It should also be noted that his discussion on the trepidation in the *Tadhkira* is so abridged that can not give any insight to a probable transmission of the elements embedded in the model; but, Shīrāzī's account of the model, in which the summer solstice (Head of Cancer) is evidently specified as the reference point, does not leave any room for doubt that the Maragha astronomers only describe the simple trepidation model of Eastern Islamic origin. Moreover, Shīrāzī's solar model with an oscillating-in-size deferent is substantially different from Ibn al-Zarqālluh's solar model with the variable eccentricity.

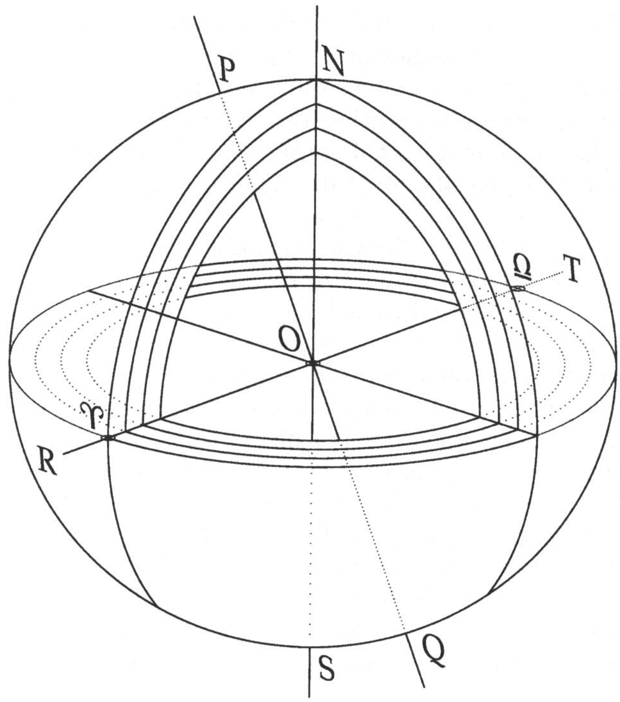

Fig. 3 Taqī al-Dīn's simple model for accounting for the variation of the obliquity

Shāṭir's 23;31°) or nearly equal to it (e.g. 23;30,17° in Ulugh Beg's *Zīj* as observed by the Samarqand astronomers).[57] The hypothesis reappeared in Taqī al-Dīn's *Sidrat* II.4.7: his observed value 23;28,54,8°[58] appears to have convinced him to assert that the trustworthy, sound observations since Ptolemy's time testify that the obliquity decreases successively. He derives the motion of the ecliptic per solar year and per day, and mentions that after about 91,000 years, the ecliptic coincides with the equator, some consequences of which he briefly explains for various geographical latitudes. His derived yearly and daily motions, as can be read from MS. K of this work, are equal, respectively, to 0;0,0,55,34,44,13° and 0;0,0,0,9,14,16°, which are not, of course, in precise agreement with each other by taking his own value for the length of solar year into account (365;14,38,34,17,8,34,17... days); he does not explain how he has obtained these figures, but it should be noted that the value 23;28,54,8° taken as the upper limit yields an annual motion of about $55'''44^{iv}$.[59] He later crossed out the above-mentioned figures in the MS. K, the reason for which seems to be that he reasonably was not certain about the minimum value of the obliquity, because he was not certain

[57] Ulugh Beg, *Sulṭānī zīj* II.4, P1: f. 19r; P2: f. 12v; Qūshčī, N: p. 118, P: p. 69, PN: f. 101v.

[58] Measured from his observed values for the extremal noon altitudes of the sun at Istanbul in 1577: 72;30,8,29° on 11 June 1577 and 25;32,20,14° on 11 December 1577 (Taqī al-Dīn, *Sidrat*, K: f. 17v on the right margin). See Mozaffari and Steele (Mozaffari and Steele 2015, pp. 350, 352).

[59] Taqī al-Dīn, *Sidrat*, K: ff. 18r–v.

about the additional sphere that should contain the ecliptic and be responsible for its upward and downward motion; as he states, on the upper margin of f. 18v,

> We are unable to know its configuration (*hay'a*) and position (*waḍ'*) of its poles with respect to the ecliptic until we know the extremal value of this decrease [in the obliquity], which is not facilitated, except in some thousands of years. Nevertheless, on the basis of the hypotheses of this technique, I say: that sphere/orb contains the ecliptic, its poles are the two equinoctial points, and the plane of its equator (*minṭaqa*, lit. "belt") lies in the plane of the circle passing through the poles [of the celestial equator and ecliptic]. It moves from the north towards the south to make the ecliptic coincident with the celestial equator, and then the two are removed from each other and the ecliptic approaches the south pole; thereafter, the motion becomes from the south towards the north.

The model he puts forward in order to account for this upward and downward motion of the ecliptic is substantially much simpler than both models proposed by the Maragha astronomers: he only adds an additional sphere (the middle sphere with the axis *RT* in Fig. 3) whose poles are the equinoxes, and whose equator lies in the plane of the great circle passing through the poles of the celestial equator (so-called the sphere of Atlas, the outer/upper sphere with the axis *NS*) and the ecliptic (the inner/lower sphere with the axis *PQ*). It is located above the sphere of the ecliptic, and its slow periodic oscillatory motion around the axis passing through the equinoxes simply accounts for the change in the obliquity.

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