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Synthetic and analytic geometries in the publications of Jakob Steiner and Julius Plücker (1827–1829)

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Abstract In their publications during the 1820s, Jakob Steiner and Julius Plücker frequently derived the same results while claiming different methods. This paper focuses on two such results in order to compare their approaches to constructing figures, calculating with symbols, and representing geometric magnitudes. Underlying the repetitive display of similar problems and theorems, Steiner and Plücker redefined synthetic and analytic methods in distinctly personal practices.

1 Introduction

One reads frequently that analytic and synthetic methods divided early nineteenth-century geometry. So Felix Klein claimed in his 1872 paper, "Vergleichende Betrachtungen über neuere geometrische Forschungen," in which he called for an end to methodological divisions in geometry. As it appeared in the English translation of 1893:

> The distinction [*Unterschied*] between modern synthesis and modern analytic geometry must no longer be regarded as essential, inasmuch as both subject matter and methods of reasoning have gradually taken a similar form [*gestaltet*]

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> in both. We choose therefore in the text as common designation of them both the term *projective geometry*. (Klein 1893, 216)

Here, we shall contest whether such a dichotomy really operated in geometry during the first half of the nineteenth century. Klein's categorization of two opposing geometries, however, aligned with the recently published biographies of the analytic geometer Julius Plücker (Clebsch 1872; Dronke 1871). In his biography, Adolf Dronke explained that Plücker had stopped research in geometry in 1847 due to opposition from the influential synthetic geometer, Jakob Steiner.

> Steiner declared that he would no longer write in Crelle's Journal, if Plücker's work still continued to appear there. Berlin was thereby ruined for him and it is easy to grasp why he submitted his scientific works mostly to foreign journals, where he knew that his contributions were at least not despised. (Dronke 1871, 11–12)[1]

In fact, Plücker did stop publishing articles in geometry in 1847, when he began a position as a professor of physics at the University of Bonn.[2] Accordingly, Plücker's research turned to experimental physics.

The perception of a long-standing dichotomy between methods and geometers persisted. During the early twentieth century, several articles written as part of Klein's *Encyklopädie der mathematischen Wissenschaften* included historical analyses of early nineteenth-century geometry. In "Gegensatz von synthetischer und analytischer Geometrie in seiner historischen Entwicklung im 19. Jahrhundert," Gino Fano assessed the opposition as a matter of difference in "style" with Plücker and Steiner positioned on opposing sides (Fano 1907). This division is apparent from the chapter titles alone: the second chapter concerned "Einsetzen der synthetischen Geometrie durch Poncelet, Möbius, Steiner, Chasles," and the following chapter "Entsprechende Entwicklung der analytischen Geometrie," featured Möbius and Plücker. Similarly, in their respective histories of elementary and projective geometries, Max Simon and Arthur Schoenflies frequently cited Plücker and Steiner. Though subscribing to different methods, the two geometers were said to have reached the same results in quick succession during the 1820s and 1830s (Simon 1906; Schoenflies 1909). Brief bibliographic references at once emphasized the importance of method and the scarcity of new results. When citing individual theorems or solutions, Steiner's contributions were labeled "geometric" or "synthetic," while Plücker's were "analytic." The pattern continues in recent general texts on the history of mathematics. The few paragraphs devoted to early nineteenth-century geometry cast Steiner as a "champion of" analytic geometry, which Steiner "loathed" (Eves 1990; Cooke 2005; Kline 1972; Bourbaki 1960; Struik 1948). When authors cite publications as evidence, they are from the

[1] "Steiner erklärte, nicht mehr in Crelle's Journal schreiben zu wollen, falls noch Arbeiten Plücker's fernerhin Aufnahme fänden. Dadurch war ihm Berlin vollständig verleidet und ist es wohl begreiflich, wie er seine wissenschaftlichen Arbeiten meist in ausländischen Journalen niederlegte, wo er wusste, dass seine Leistungen wenigstens nicht verachtet wurden."

[2] While at Bonn, Plücker would serve as Klein's advisor for his dissertation on geometry completed in 1868.

first decade of Plücker's and Steiner's publishing careers and particularly their first monographs.

Though analytic and synthetic have many possible connotations, in these historical references to the early nineteenth century, analytic geometry was understood to mean the use of coordinate equations. By contrast, synthetic geometry has been defined as figure-based (Fano 1907; Kötter 1901; Klein 1926) or as projective geometry without the use coordinate equations (Schoenflies 1909; Darboux 1904; Loria 1887).

The apparent consensus in these historical accounts implies that the descriptions were uniformly applied by Plücker, Steiner, and their contemporaries. However, early nineteenth-century geometers had argued that the descriptions of analytic and synthetic were too inconsistent to be meaningful. The Swiss mathematician and educator, Emmanuel Develey, explicitly rejected employing the terms analysis and synthesis to describe his methods because there was currently "no accord as to their distinct characters" [*n'est point encore d'accord sur le caractère distinctif*] (Develey 1812, v). Gergonne (1817d) similarly despaired of the recent confusion of the terms synthesis and analysis, which he attributed to Condillac and the metaphysicians of his school.[3]

As the editor of the *Annales*, Gergonne frequently classified articles as "analytic geometry," but never used "synthetic geometry" in this context. Instead, articles fell under elementary geometry, pure geometry, geometry of the ruler, geometry of situation, geometry of curves and surfaces, geometry, and several other alternative subject headings. These terms were not exclusive, and often the same article would have multiple such labels. The distinction between geometrical methods was in flux. When Gergonne (1814) contrasted the advantages of analytic geometry to those of pure geometry, he was corrected by Poncelet who instead divided geometry into three parts: ancient pure, modern pure, and analytic geometry (Poncelet 1817). Gergonne acquiesced (Gergonne 1817c). Geometry did not divide neatly in two.

Likewise, geometers were not simply analytic or synthetic. During their early careers, descriptions of Plücker and Steiner varied to the point of contradiction. Plücker's analytic geometry at once appeared as "restrained in calculations" (Anonymous 1827a, 173) similar to "synthesis" and lacking in new discoveries (Cournot 1828, 178), "innovative" and "fruitful" (Gergonne 1828), uniting the method of discovery—the method of analysis—with clarity and insight (Crelle cited in (Eccarius 1980), 208–211), and new "pure analytic geometry" (Plücker 1828a). During this period, Steiner's publications were considered as a reemergence of the "long-neglected" pure geometry (Cournot 1827, 299), fruitful but overly complicated (Anonymous 1828, 245), systematic and organic (Steiner), inventive and original [written by Friedrich Wilhelm Bessel in 1826 and published in Lange (1899), 18–26], particular to the author and too elementary but elegant (Steiner and Gergonne 1827, 286), complicated (Poncelet 1866, 410), merely following in the footsteps of Poncelet (Plücker 1828a), and independently derived [Crelle in Eccarius (1980)]. Not only the same geometer, but even the same publication could elicit dichotomous responses. By drawing upon

[3] Gergonne's philosophical article was an excerpt from a longer paper that has been described and analyzed by Amy Dahan Dalmedico (1986).

research articles, reviews, private correspondences, and public letters of support, this paper will show how Plücker, Steiner and their contemporaries utilized an array of attributes to assess positive and negative features of geometrical publications.

Conflicting descriptions suggest that labels of analytic and synthetic geometry, as emphasized by Klein, have obscured the respective methods of Plücker and Steiner. Rather than restricting their practices or areas of study, these geometers understood the descriptions of analytic and synthetic geometry in the context of their particular problem solving and theorem proving. We will examine two instances of intersection in their research on planar geometry: the Apollonius problem and Lamé's theorem. To focus on comparison between methods, the following episodes draw upon instances of nearly identical results. Further, re-examining publications that had been simply classified as analytic or synthetic enables a comparison between the original texts and their surrounding commentaries. Our focus on the early career of Plücker and Steiner reflects that of late nineteenth and early twentieth century historical accounts. We observe that during the first decade of their careers, neither geometer reined in his research with traditional methodological boundaries. By the end, they had each become emblematic of their respective methods. This then raises the question of when this association emerged, and how their contributions redefined both methods. In documenting how their methodological positions and practices changed over time, these findings call for a revision to the standard account of continual opposition.

We begin in Sect. 1 with Steiner's first publication, written on the subject of tangent circles in a plane. Steiner's early research included work on the so-called Apollonius problem: to find a circle tangent to three given coplanar circles. Steiner described his findings as independent, and his choice of research subject drew the attention of Joseph-Diez Gergonne, the editor of the French journal *Annales des mathématiques pures et appliquées*. Gergonne had framed his own solutions to the Apollonius problem as evidence of the elegance and simplicity of analytic geometry (Gergonne 1810, 1814, 1817a). While Gergonne (1817d) formally agreed that the choice of problem should determine the choice of method in geometry, this convention did not seem to apply to the Apollonius problem, which continued to be solved through numerous methodological approaches (Poncelet 1821; Durrande 1820). Steiner's text on circles was quickly followed by a translation and reinterpretation by Gergonne (Steiner 1826a; Steiner and Gergonne 1827). After surveying the reviews and recommendations that followed Steiner's first publications, we consider how Plücker introduced his own research on circles, including proofs of both Gergonne and Poncelet's original solutions to the Apollonius problem. Steiner and Plücker situated their publications with attention to contemporary interests, which led to reviews and repurposing of their publications. These texts contain early manifestations of the form and techniques that would characterize both geometers' publications through the 1830s. Steiner developed a new, comprehensive vocabulary to describe constructive relationships between circles, lines, and points that utilized simple algebraic calculations. Addressing the same problem, Plücker limited his calculations through carefully choosing coordinate axes to re-derive already known results. With reference to Poncelet's recent publication, Plücker adopted the newly defined ideal chords, which had not yet previously been written in coordinate equations.

The prevalence of "modern" objects and techniques will become even more apparent in Sect. 2 and the proofs by Steiner and Plücker of a theorem on conic sections first published by the French geometer Gabriel Lamé in 1817. Using Lamé's original proof as a point of reference, we will see how Steiner and Plücker recast the theorem's form and context (Steiner 1828b; Plücker 1828b). For Steiner, Lamé's theorem illustrated his ability to extend results from a circle to a general conic section via perspective, projection, and reciprocity. For Plücker, Lamé's theorem served as an application of his general theory of intersections between curves of any order. Both articles appeared in the *Annales de mathématiques pures et appliquées* and were critically appraised in the *Bulletin des sciences mathématiques, astronomiques, physiques et chimiques*. Later, the results would be incorporated in the longer manuscripts of Plücker (1828a, 1831) and Steiner (1832). The conservatism of this geometric framework, and in particular the recycling of well-known theorems and problems, invited comparisons. While developing original techniques, their work retained enough features of older publications to remain familiar to readers of the *Annales* as subsequent citations show. Their reception included debates linked to several oppositions: that of pure or synthetic and analytic, but also German and French geometers or ancient and modern geometries.

This paper will complement other reappraisals of individual contributions of Steiner and Plücker. Plücker's innovative modern geometry received new appreciation in the mid-twentieth century, when Carl Boyer featured Plücker as the protagonist of the "Golden Age of Analytic Geometry" and focused specifically on Plücker's simplification of coordinate equations in geometry (Boyer 1956). Further, Jeremy Gray has underscored Plücker's substantial contributions to the theory of duality, where "Plücker's contribution has been rather marginalised" (Gray 2010, vi). In Philippe Nabonnand's history of points and lines at infinity, Nabonnand examined Steiner's specific contributions in the development of geometry from Poncelet to von Staudt. For Steiner, "the central object of geometry is no longer the figure but becomes the notion of fundamental forms" (Nabonnand 2011a, 167).[4] This then points to a new concept of pure geometry that we will investigate with respect to Steiner's early publications. Yet as much as Steiner can be interpreted as breaking away from Euclidean geometry, his work remained committed to constructions and questions from ancient Greek geometry. With attention to this, Viktor Blasjö has attempted to rescue Steiner's *Systematische Entwicklung* from "a lasting and undeserved depreciation" by interpreting it as "a monumental unification of classical geometry" (Blasjö 2009, 21). From a more literary perspective, Anne Boyé has integrated Steiner within the Romantic movement of his time, drawing particularly on his Socratic style of pedagogy (Boyé 1999). In another vein, Wolfgang Eccarius has provided a direct comparison between the social situations of Plücker and Steiner in the late 1820s and early 1830s (Eccarius 1980). Drawing on previously unpublished letters from August Leopold Crelle, the editor of the *Journal für die reine und angewandte Mathematik* who also lobbied for salaried positions for several of his contributors, Eccarius proposed that the perceived methodological divide was greatly motivated by these economic concerns. This assessment coincides with speculations from the 1840s of personal animosity between

[4] "[...] l'objet central de la géométrie n'est plus la figure mais devient la notion de formes fondamentales."

Steiner and Plücker resulting in Steiner's distaste for analysis, rather than the other way around (Aronhold 1902, 64).

When Plücker and Steiner are directly compared, the nuances and overarching forms of their distinctive methods come to the fore. In fact, this evidence will show that the two geometers were very similar with respect to their approach to constructing figures, calculating with symbols, and representing geometric magnitudes. Moreover, both geometers claimed the same virtues of simplicity, elegance, and visualizability. Nevertheless, this was not a period of stagnant conformity. Rather the malleability of the terms analytic and synthetic allowed them to be shaped by their users, who could profess adherence to a tradition without confining their expression. Underlying the repetitive display of similar problems and theorems, Plücker and Steiner redefined analytic and synthetic methods and explored diverse paths toward greater generality and novelty.

2 Tangent circles in the plane (1826–1827)

Perhaps most emblematic of the repetition exhibited within early nineteenth-century geometry publications is the deceptively simple problem of finding a circle tangent to three given coplanar circles, known as the Apollonius problem.

2.1 Gergonne's solution to the Apollonius problem

Prior to 1826, Gergonne had published three separate articles on the Apollonius problem, with essentially the same solution but different degrees of explanation (Gergonne 1810, 1814, 1817a). His most extended version appeared in the *Annales* in 1817. Gergonne introduced his repetition by explaining he had received complaints about lacking a complete justification in his earlier version.

> I wrote for consummate scholars, and if I thought it right to be brief it appears that I made it a bit too much; several geometers, who knew of my memoir, reproached me without doubt because the thread which had guided me was not very apparent and that my calculations seemed rather to legitimize a construction found by happy accident, than to lead to discovering this construction. It appeared even that, due to my excessive brevity, many geometers had not been able to follow my methods and capture their spirit; because they return again to these two problems, about which I believed that nothing was left to say. (Gergonne 1817a, 289–290)[5]

[5] "J'écrivais pour des savans consommés, et je crus devoir être court; il paraît que je le fus un peu trop; plusieurs géomètres, qui eurent connaissance de mon mémoire, me firent le reproche, fondé sans doute, que le fil qui m'avait guidé n'y était pas assez apparent, et que mes calculs semblaient plutôt propres à légitimer une construction trouvée par un heureux hasard, qu'à faire découvrir cette construction. Il paraît même que, par suite de mon excessif laconisme, beaucoup de géomètres n'ont pu suivre mes méthodes et en saisir l'esprit; car on est revenu encore postérieurement sur ces deux problèmes, sur lesquels pourtant j'avais cru ne plus rien laisser à dire."

Against these reproaches of merely having "legitimized a graphic construction, discovered in advance," Gergonne would show that an analytic treatment led "naturally" and "absolutely inevitably" to his conclusions. The solution was "elegant," "simple," and "direct"–not "fashioned after the fact." Finally, the exposition would be entirely elementary, even though involving coordinate representation.

Gergonne began by laying out his strategy in detail. He would successively reduce the problem to determining simpler and simpler geometric objects. The desired circle could be determined by its three tangent points. Finding three tangent points reduced to finding a single tangent point. Finding that point reduced to finding a second line containing that tangent point. Finding that second line reduced to finding any two points on that line, and so on.

Gergonne then chose a set of coordinate axes with the origin at the center of one of the given circles c'' (with radius r'') and axes through the centers of the remaining two circles c (centered at (a, b) with radius r) and c' (centered at (a', b') with radius r'). If the unknown circle C was tangent to the circle c'' at the point (x, y), then, from the equation of a circle, $x^2 + y^2 = r''^2$. Similarly, the centers of each of the given circles could be expressed with respect to their distance from the center of the unknown circle and their distance from the origin.

From the equations of the given circles, Gergonne calculated an expression of a line containing the tangent point: $\frac{ax+by-r''(r''-r)}{(a^2+b^2-(r''-r)^2} = \frac{a'x+b'y-r''(r''-r')}{a'^2+b'^2-(r''-r')^2}$.

The equation of the unknown circle and the line could be combined to solve for one of the tangent points at (x, y). Rather than solving this equation (which Gergonne admitted would be complicated by radicals), he worked backward following the steps of reduction outlined above, showing that analytically simpler lines and points could instead be constructed and still lead to a uniquely determined point. Gergonne reminded the reader that there would be eight possible tangent circles satisfying the problem's conditions. These cases depended on whether the desired circle enveloped or was exterior to each of the three given circles. With respect to the analytic proof, the type of tangency corresponded to the positive or negative sign of the value of the radii.

This analytic proof concluded with a planar construction. In this construction, Gergonne differentiated two kinds of tangent lines shared by a pair of circles. The two interior tangents intersected between the two given circles, while the two exterior tangents intersected on one side of both given circles.

Given three coplanar circles c, c', c'', begin by drawing the common exterior tangents to the three circles considered pairwise and the chords of contact defined by the tangent points for each circle. The two chords of contact of circle c will meet at a point M, and their parallel chords on c' and c'' will meet at N. Similarly, the two chords of contact on c' will meet at M' with their parallel chords meeting at N', and the two chords of c'' meeting at M'' with parallels at N'' (Fig. 1).

One could then draw lines $MN, M'N', M''N''$. Line MN would meet circle c at two points, which Gergonne called t and θ. Similarly, $M'N'$ would meet c' at t' and θ', $M''N''$ will meet c'' at t'', θ''. Finally the two circles respectively drawn through t, t', t'' and $\theta, \theta', \theta''$ would be the circles sought (Fig. 2).

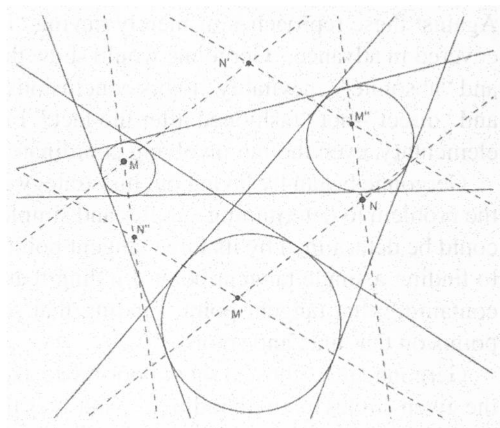

Fig. 1 Construction of tangent lines and chords of contact from Gergonne's description (Gergonne 1817a)

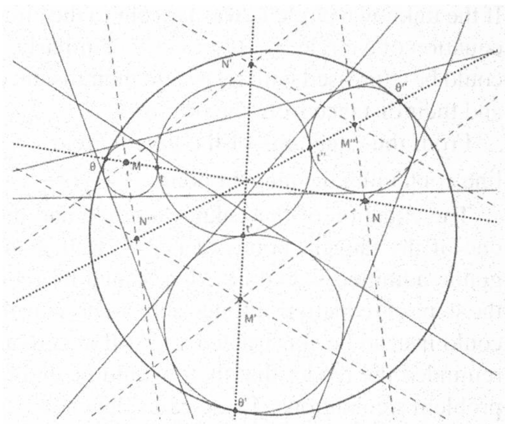

Fig. 2 Construction of intersection points and tangent circles

The remaining six tangent circles could be found by substituting combinations of exterior and interior tangents. Gergonne described his solution as the simplest yet, presenting "an entirely new face" for the application of analytic geometry in solving planar problems. Further, the construction could be applied "exactly in the same manner" to the case where the circles lay on the surface of a sphere.

Gergonne's claims for the merits of his solution stood up to those of many historically esteemed mathematicians, including Viète, Newton, Euler, and Carnot (Carnot 1803, 391). Through the nineteenth century, the Apollonius problem served as a frequent trope for geometers exhibiting their abilities and methods in geometry. By 1906 Max Simon cited around 100 different solutions that had been published in the past century (Simon 1906).

2.2 Poncelet and the Apollonius problem

Poncelet's first solution from 1811 further attests to the problem's ubiquity. Jean-Nicholas-Pierre Hachette included a statement of the problem in the *Correspondance* *sur l'École Polytechnique* the previous year as the sixth of eight problems on circles and spheres for students to solve. Hachette provided a solution to the problem by considering the given circles as great circles of three spheres and posing the three-dimensional question of finding the sphere tangent to three given spheres whose center is coplanar with the three given centers. He also cited analytic solutions from Newton in the *Arithmétique Universelle* and from Euler and Fuss in the *Mémoires de l'Académie de Pétersbourg*.

Poncelet's solution marked his first publication in geometry (Poncelet 1811). He relied upon results proved in the Hachette article, but his constructions were limited to the plane.

By contrast, Poncelet 1821 article promoted an application of his new principle of continuity.[6] The year before, Poncelet had submitted a memoir on the projective properties of conic sections to the *Académie Royale des Sciences*. This memoir had been reviewed by François Arago, Siméon-Denis Poisson, and Augustin Louis Cauchy, the last of whom wrote up a report subsequently published in Gergonne's *Annales*. While criticizing some of Poncelet's methods, Cauchy complimented Poncelet's "very elegant" solution of the problem to draw a circle tangent to three others (Poncelet and Augustin 1820, 82). Intrigued, Gergonne requested Poncelet's construction, which was subsequently published.

The constructions presented could be understood independently, but their proofs rested upon propositions and principles only fully explained in Poncelet's extensive monograph, *Traité des propriétés projectives*, which would appear the following year (Poncelet 1822). However, Poncelet used the problem to show off possible applications of his new geometric techniques such as ideal chords, an extension of the ordinary chords of conic sections to include both real and imaginary points of intersection. Steiner and Plücker would adapt these objects to their own researches in the following decade. In particular, Poncelet argued that two concentric circles would share a common ideal chord at infinity, their points of intersection being imaginary. This concept enabled Poncelet to consider simultaneously the cases of concentric circles and those having parallel shared tangent lines.

Poncelet advertised his solutions as advantageous because constructible with only a simple ruler. Poncelet further showed how to reach a "very elegant solution" with the same basis as that of Gergonne's from 1814. Poncelet assessed Gergonne's presentation favorably,

> [...] the purely algebraic path that this geometer has followed is entirely new, and appears susceptible to apply to a great number of questions reported difficult in the current state of Analysis. (Poncelet 1822, 138)[7]

Poncelet's positive comments suggest that for him, providing another proof for Gergonne's solution was not intended as a critique of the original analytic approach.

[6] For detailed analyses of Poncelet's controversial principle of continuity, see Nabonnand (2011b), Friedelmeyer (2011), Gray (2005).

[7] "[...] la marche purement algébrique qu'a suivie ce géomètre est entièrement neuve, et parait susceptible de s'appliquer à un grand nombre de questions réputées difficiles dans l'état actuel de l'Analyse."

The Apollonius problem invited multiple solutions, and the elegance of one did not preclude the advantageous qualities of another.

2.3 Steiner's tangent circles

Among Steiner's six articles that appeared in the 1826 inaugural volume of Crelle's *Journal*, his paper "Einige geometrische Betrachtungen" (Steiner 1826a) and its sequel, "Fortsetzung der geometrische Betrachtungen" (Steiner 1826c), generated attention within the French mathematical community. This was due not so much to Steiner's results, as to his development of geometric concepts that seemed valuable in further research and pedagogy.

Steiner explained that his research on tangent circles over the past three years had been motivated by three sets of problems: finding a circle tangent to three given planar circles (the Apollonius problem), the problem of inscribing three tangent circles to a triangle (the Malfatti problem), and the fifteenth theorem of the fourth book of Pappus's *Collectiones Mathematicae* (determining proportions between tangent circles inscribed in a semi-circle). Although acknowledging that each of these problems possessed well-documented recent solutions, Steiner avowed that his work was entirely independent, relying only upon theory developed by Viète and Pappus. Steiner claimed that he had but recently become aware of contemporary French publications in geometry, in particular Poncelet's *Traité*, where many of his own results had already appeared. However, Steiner also promised new findings that would further demonstrate the independence of his research. Specifically, his work contained new generalizations to a greater number of given circles, to circles intersecting at given angles rather than in tangent points, and to analogous results for second-degree curves and three-dimensional surfaces.

Though Steiner claimed that his theorems could be applied to many more problems concerning coplanar circles, the details of his solution to the Apollonius problem would appear only posthumously almost 100 years later in Steiner's anticipated book on circles, spheres, and spherical circles that he had promised as forthcoming in 1826.[8]

Instead Steiner devoted most of the text toward organizing definitions in four parts, each devoted to a particular geometric relationship, and a total of nineteen numbered sections, each beginning with a particular configuration of objects and leading to a property or theorem to be referenced by citation to that section, for example "nach (1.):" As Gergonne would observe, Steiner's exposition often delved into the very elementary. In this feature, the entire text served to reinforce Steiner's introductory remarks about the independence of his findings. Until Steiner arrived at the final part,

[8] This book was published in 1931 as *Allgemeine Theorie über das Berühren und Schneiden der Kreise und der Kugeln worunter eine grosse Anzahl neuer Untersuchungen und Sätze vorkommen in einem systematischen Entwicklungsgange dargestellt*, edited by Rudolf Fueter and Ferdinand Gonseth (Steiner 1931). As the editors explained in their introduction, during the late nineteenth century, the mathematician Fritz Bützberger had found the unpublished manuscript dating between 1823 and 1826 in a box at the Library of the Naturalist Society of Bern. The manuscript was then rediscovered by the editors, Fueter and Gonseth, among Bützberger's papers.

in which he posed and solved a series of problems, very few results were assumed as known and there was only one external citation. Steiner began from the Euclidean result (Euclid III.37) that the area of the square on a tangent segment is equal to the rectangle determined by the intersected secant and its external segment drawn from a point to a coplanar circle. He then examined proportions to determine fixed relationships between constructed points, lines, and circles, which he described in his unique vocabulary. While many of his terms had already been developed under different names in France, Steiner introduced a few new concepts, including the idea of a "circle of common power" that was specifically admired by later reviewers. Despite its absence in the article, the Apollonius problem served as an invisible motivating force for the definitions and results throughout the body of the text. This would become visible in Gergonne's later translation.

The text developed as an exploration of the relationships between coplanar geometrical objects, from two points to a point and a circle, to two circles, to three circles, to more complex planar relationships (Fig. 3). Steiner modeled his presentation as a discovery beginning from the ground up and resulting in conclusions that were surprising even when they were already known. Figures served as clarification and were never invoked as a tool of proof. Steiner presented his research without a clear map of his intended progression, and a brief summary here will motivate our understanding of his development. While Steiner never reached the Apollonius problem, we will see how his new definitions progressed toward finding tangent circles.

The first part concerned defining the power relationship between coplanar circles and contained five sections each examining a different set of geometric objects: on equations relating to perpendicular lines (1), on the power of a point with respect to a

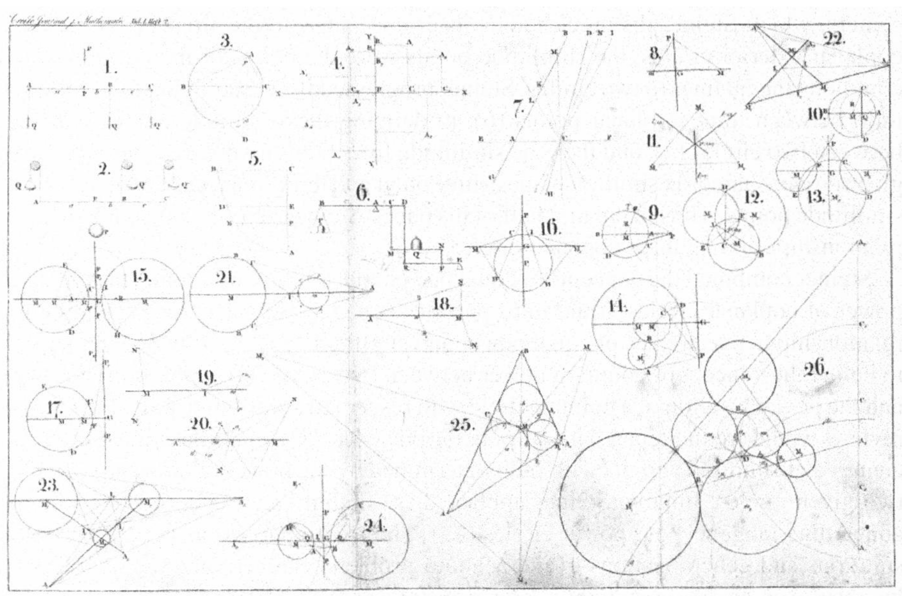

Fig. 3 Steiner's Figures (Steiner 1826a)

circle (2), on the line of equal powers between two circles (3), on the point of equal powers between three circles (4), and on power relationships between multiple and orthogonal circles (5). The concept of the *power* of coplanar circles and points derived from a well-known Euclidean relationship: when two chords of a circle intersect, then the rectangles formed by their respective segments are equal (Proposition III.35) and when a tangent and a secant line are drawn from a point to a circle, then the square on the tangent is equal to the rectangle formed by the intersected segments of the secant (Proposition III.36). Steiner denoted this product as the power of a point with respect to a circle, which could be found regardless of a coplanar points position. Extending this definition, he determined that the line containing all points of equal power to two given coplanar circles would be perpendicular to the line containing their centers. For three given coplanar circles, the lines of equal power considered pairwise all three intersected in a single point, the point of equal powers. Steiner further explored, without explicit definition or citation, the relationship of poles and polars with respect to a given circle and line or point. In particular, Steiner described how one could determine two points with respect to the line of equal powers and two given circles by choosing any point on the line of equal powers, drawing the four tangent lines to the given circles, and passing two line through each of the respective circles' two points of tangency. These newly constructed lines intersected the line containing the two given circles' centers in two fixed points (the poles of the line of equal power with respect to the respective given circles) regardless of which point on the line of equal powers was chosen. We will see that Gergonne used this construction in his exposition on the Apollonius problem.

In his second part, containing sections 6–8, Steiner continued his examination of circles, now with respect to similitude points and similitude lines. Any two coplanar circles shared two similitude points, determined by a fixed ratio between the circles' centers, which Steiner distinguished as inner and outer. In the simplest case of two coplanar exterior circles, the similitude points were the points of intersection of the common tangent lines to two circles. Steiner progressed from examining the similitude ratio between three coplanar points (6), to defining the similitude points and lines between two circles (7), and then the similitude lines between three circles (8). Three given circles shared six similitude lines, three outer and three inner, each containing two similitude points. Thus in the first half of his paper, Steiner described two independent relationships among three coplanar circles.

Steiner combined the concepts of equal power and similitude to define the common power of coplanar circles in the third part of his paper. He began by examining the relationship between lines of equal power and similitude points (9), which he applied to define the concept of common power between two circles (10), power circles (11), and the power position of a point or circle with respect to a similitude point (12). Later, reviews would credit Steiner with introducing power circles, which he defined as circles centered at similitude points with radii determined by the power relationship between two given circles. Finally, Steiner applied these definitions to a series of theorems concerning tangency and power circles (13). The fourth part of his paper concerned solutions and generalizations of the Malfatti problem. Steiner continued publishing his results on circles in a slightly later paper (Steiner 1826c), in which he explicitly addressed Pappus' Theorem, and developed numerous results pertaining to tangent circles, but did not include any further reference to the Apollonius problem.

Gergonne's long-term interest in the Apollonius problem, illustrated by his own publications and interest in publishing Poncelet's solutions, may explain why Steiner's article was quickly translated and appeared in the *Annales* under the title "Géométrie Pure. Théorie générale des contacts et des intersections des cercles" in 1827 (Steiner and Gergonne 1827). Gergonne prefaced his exposition of the "Théorie générale des contacts et des intersections des cercles" by explaining what he perceived as the advantages and flaws of Steiner's approach. Steiner had presented the elementary theories of "similitude centers, axes and planes, radical axes, planes and centers, and finally poles, polars, polar planes and conjugate polars" in order to solve "difficult" and "general" problems with "very elegant" solutions. Among Steiner's contributions, Gergonne suggested that school teachers would find "abundant resources" for "profitable" student exercises. Moreover, Gergonne's version of Steiner's "doctrine" promised to be even better than the original German text.

> We thus suppose it will be a very useful thing for the progress of pure geometry, and consequently very agreeable to our readers, to offer here, in a concise framework, the principal points of M. Steiner's doctrine; however, without slavishly following him and permitting ourselves to deviate a bit from his path, whenever we think that it may result in some advantage with respect to clarity or brevity. We will present, in a word, these theories as we think they could and should be in elementary treatments, remembering, however, that we do not write for beginners; that is to say, in withholding, in order to abbreviate, developments which are easy for any intelligent reader to supply. (Steiner and Gergonne 1827, 287)[9]

Here was Steiner clarified, abbreviated, and overall adapted to a mathematically literate French audience.

Gergonne's choice of content appears in large part motivated toward reproving his construction for the Apollonius problem, now by purely geometric methods. As we saw above, Gergonne's original solution had been framed as evidence in favor of analytic geometry. The same construction, except for a change in the names of points, remained as simple and elegant in a purely geometric setting, thus in some ways undermining Gergonne's initial intent. Gergonne unabashedly explained his divergence from Steiner's text as an opportunity to showcase his preferred solution.

> All self love on the author's part aside, this construction, that we gave for the first time more than a dozen years ago (*Mémoires de Turin* in 1814), appears to

[9] "Nous pensons donc faire une chose très-utile pour le progrès de la géométrie pure, et conséquemment très-agréable à nos lecteurs, en offrant ici, dans un cadre resserré, les principaux points de la doctrine de M. Steiner; mais sans toutefois le suivre servilement, et en nous permettant de nous écarter un peu de sa marche, toutes les fois que nous penserons qu'il en peut résulter quelque avantage, sous le rapport de la clarté ou de la brièveté, nous exposerons, en un mot, ces théories comme nous pensons qu'elles pourraient et devraient l'être dans les traités élémentaires, en nous rappelant toutefois que nous n'écrivons pas pour des commençans; c'est-à-dire, en négligeant, pour abréger, des développemens faciles à suppléer pour tout lecteur intelligent."

us preferable to all those that have been given of the same problem before or afterward. (ibid, 310)[10]

Gergonne proved his solution using radical centers and axes (Steiner's points and lines of equal powers), similitude points and lines, and poles and polars (which Steiner had constructed and described, but left unnamed). Take the first and simplest case, to construct a fourth circle tangent to three given circles in the same manner (either all exterior or all interior points of tangency). If one drew lines through the radical center of the three given circles to the poles of their exterior similitude axis, then the intersections of these lines would determine the tangency points with the desired fourth circle. An analogous proof followed for the case where the circles were tangent in different manners.

Gergonne further connected Steiner's work in the final section, when he returned to three given coplanar circles, essentially the hypothesis of the Apollonius problem. Using Steiner's concept of common power, Gergonne derived a theorem, which he attributed to the engineer and teacher Louis Gaultier de Tours: the centers of the eight circles tangent to three given circles will be distributed pairwise on the perpendicular lines drawn from their four similitude axes to the radical center of the three circles. Gergonne thus brought Steiner's research another step toward solving the Apollonius problem, but without elucidating the details of a full proof here.

Gergonne's presentation of Steiner's article brought the idiosyncratic German article up to date with contemporary French mathematics. To begin with, the article was now in a language readable by French audiences, or at least those who read the *Annales*. Further, Gergonne had added several citations and modified Steiner's vocabulary to reflect current French usage. Most notably, rather than Steiner's points and lines of equal powers, Gergonne used the term radical centers and axes, also from the work of Gaultier. While much was added, not all of Steiner's original content survived translation. Most notably, Steiner's nineteen figures were not repeated, and with the change of notation his figures as published in Crelle's *Journal* would not have been very useful to a reader without knowledge of German. Further, the French article assumed a stronger knowledge base and skipped over Steiner's foundational exposition. Rather than progress from points to circles, Gergonne developed his research in terms of circles and polygons. Further, he employed the concept of a circle of variable size and position to extend his results to further cases. By orienting Steiner's content toward the Apollonius problem, Gergonne obscured its initial appearance as a set of tools applicable to a variety of problems. Significantly, while Steiner had emphasized that the same properties determined for a circle would likewise hold for any planar conic section, Gergonne only mentioned this generalization in the introduction and hinted at it in the conclusion.

Gergonne's conclusion, designated as a post-script, commented upon another article of Steiner (1826b) in which he had pointed out several exceptions to a theorem due to Monge. Gergonne defended Monge and dismissed the exceptions as too particular to

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[10] "Tout amour propre d'auteur à part, cette construction, que nous avons donnée pour la première fois il y a plus de douze ans (*Mémoires de Turin* pour 1814), nous paraît de beaucoup préférable à toutes celles qu'antérieurement et postérieurement on a données du même problème."

affect the theorem in its entirety. He further remarked that Steiner had perhaps failed to recognize the crucial role of generalization in his own constructions. In the work of Steiner, there was no stated mechanism for extending the result from a sphere to a plane or from a circle to a general conic section. The process was simply assumed as possible. Gergonne continued by comparing ancient and modern geometry, to the advantages of the latter.

> [...] but is it not precisely in this broader way of envisaging geometric magnitudes that the moderns are in part owed their superiority in pure geometry? Superiority that the same geometers could also contest; but is no less a patent fact for those who will not deny the obvious.[11]

Gergonne took full advantage of the dynamic capabilities of these modern concepts. Thus, Steiner's article, as interpreted by Gergonne, functioned as evidence in favor of the superiority of modern pure geometry, just as Gergonne's earlier Apollonian proofs had served him to represent the superiority of analytic geometry.

2.4 Steiner's initial reception

Soon after Gergonne's interpretation of Steiner's paper appeared it was anonymously reviewed and summarized in the *Bulletin de Férussac*. In describing Steiner's exposition, his use of new vocabulary was emphasized as a fruitful innovation in problem solving.

> Steiner's means of solution rest uniquely on the known theory of similitude axes and centers, of that of radical axes and centers, of that of poles and polars, and finally of that which the author calls *circles of common power* of two given circles, which shows all the fruitfulness and all the importance of these various theories and justifies the particular attention that several French geometers have given them already for several years. (Anonymous 1827b, 277)[12]

The *Bulletin* further pointed out how Gergonne had modified Steiner's work to include his own solution of the Apollonius problem, which Gergonne "continues to believe very preferable to all those which have been given before and after of the same problem."[13] The review continued by explaining Steiner's circle of common power, an original concept that provided "an idea of the proceedings of M. Steiner." More so than in the text of Gergonne, a background knowledge in geometric constructions

[11] "[...] mais n'est-ce pas précisément à cette manière plus large d'envisager l'étendue géométrique que les modernes sont en partie redevables de leur supériorité dans la géométrie pure? Supériorité que les mêmes géomètres pourront bien aussi leur contester; mais qui n'en demeurera pas moins un fait patent pour qui ne voudra pas se refuser à l'évidence."

[12] "Les moyens de solution de M. Steiner se tirent uniquement de la théorie connue des centres et axes de similitude, de celle des axes et centres radicaux, de celle des pôles et polaires, et enfin de celle de ce que l'auteur appelle *cercles de commune puissance* de deux cercles donnés, ce qui montre toute la fécondité et toute l'importance de ces diverses théories, et justifie l'attention particulière qui leur a été donnée par plusieurs géomètres français depuis déjà plusieurs années."

[13] "[...] persiste à croire bien préférable à toutes celles qui ont été données antérieurement et postérieurement du même problème."

(such as similitude points, ratio relations, and tangent circles) would be required to follow the review. No intermediate constructions were presented and no figures were employed.

The reviewer concluded by considering some philosophical issues concerning the relationship between analysis and geometry. In particular, he noted the simplification of both methods effected by the development of new concepts.

> Analysts, perceiving that certain quite complicated functions are reproduced frequently in their calculations, have called them exponentials, logarithms, sines, tangents, factorial derivatives, etc.; they have created abbreviated signs to designate them, and their formulas have acquired greater clarity and conciseness. And thus for certain points, certain lines and certain circles whose consideration is frequently represented in geometric speculations, it is natural to do the same with respect to them, and to call them, following their properties, similitude centers, radical centers, polars, similitude axes, radical axes, circles of common power, etc. This attention must inevitably introduce analogous simplifications in the statement of theorems and in the solution of problems which belong to the science of magnitude. (ibid, 279)[14]

Such a comment appeared to reinforce the importance given to Steiner's development of new vocabulary, such as circles of common power. Although the review had begun by praising Steiner's problem solving, the emphasis ultimately returned to his new tools and their potential for future use. In this respect, the review presented a much more accurate summary of Steiner's original mode of presentation than had Gergonne's "translation."

At this time, Steiner was working as a private teacher in Berlin and solicited his well-placed acquaintances, Karl Friedrich von Klöden, an educator and the director of the Berlin Gewerbeschule where Steiner then worked, and the mathematician and astronomer Friedrich Wilhelm Bessel to write in support of his (ultimately unsuccessful) application for funds from the ministry.[15] Here Steiner emphasized his methodological choices and declared his dedication to "geometric synthesis" [*geometrischen Synthesis*] or the "synthetic method" [*synthetische Methode*]. But while labeled as synthetic, Steiner's contemporaries recognized his method as personal and unique. Klöden began his review by pointing out the novelty as a positive attribute.

[14] "Les analystes s'étant aperçu que certaines fonctions assez compliquées se reproduisaient fréquemment dans leurs calculs, les ont appelées exponentiels, logarithmes, sinus, tangentes, dérivées factorielles, etc.; ils ont créé des signes abréviatifs pour les désigner, et leurs formules en ont acquis beaucoup de clarté et de concision. Puis donc qu'il est certains points, certaines droites et certains cercles dont la considération se représente fréquemment dans les spéculations de géométrie, il est naturel d'en user de même à leur égard, et de les appeler, suivant leurs propriétés, centres de similitude, centres radicaux, polaires, axes de similitude, axes radicaux, cercles de commune puissance, etc. Cette attention doit introduire inévitablement des simplifications analogues dans l'énoncé des théorèmes et dans la solution des problèmes qui appartiennent à la science de l'étendue."

[15] Their letters along with Steiner's applications and the ministry's response were published in Julius Lange's 1899 biography (Lange 1899, 18–26).

> The method is quite particular and has not previously been tried for these problems. (Lange 1899, 18)[16]

Similarly, Bessel described Steiner as a "an inventive and original thinker," even in comparison with the work of Poncelet (ibid, 21).[17] In particular, both Klöden and Bessel characterized recent geometry as almost exclusively confined to analytic approaches. On the one hand, they praised the speed of analytic methods that could address problems "completely inaccessible" [*völlig unzugänglich*] to synthetic methods. On the other hand, synthetic geometry had pedagogical benefits. Steiner's publications revealed the "formative power of geometry" [*bildende Kraft der Geometrie*] and formed more "coherent" and "complete" educational material than the "disjointed problems" of analytical geometry. Like Gergonne and *Bulletin* reviewers, Klöden and Bessel noted a reemergence of pure or synthetic geometry, now capable of favorable comparison against analysis.

Steiner framed his dedication to synthetic geometry as illustrative as his broader philosophical search for systematicity [*Systematizität*], organic unity [*organischen Einheit*], and intuition [*Anschauung*] based on his early education under Johann Heinrich Pestalozzi. Steiner supported a unified geometric approach, and so dismissed many contemporary results in mathematics, regardless of method, as invented haphazardly. In describing his educational development, he provided a narrative in which he studied and rejected combinatorial analysis and differential calculus. Even his first encounters with geometry textbooks had revealed, he said, an arbitrary or even empirical approach, as if the individual theorems were the aim of science and the general organic unity, remained obscured. Steiner likewise distinguished his form of synthetic geometry from that of the ancients, as more general and complete, but still employing a "rigorously genetic path" [*streng genetischen Gang*], developing increasingly complex concepts from common constructions. In these qualities of generality and completeness, Steiner observed the connection and possible contributions between his method and analytic geometry.

> The work finally will enable a rich profit for the analytic geometer, as well as an expansion of his method. (ibid, 21)[18]

While his reviewers and recommenders saw Steiner's work as synthetic geometry as an alternative to analytic geometry, Steiner viewed synthetic geometry as a much greater and autonomous achievement, whose systematicity, unity, and intuition exceeded those of *any* other area of mathematics research. In this respect, Steiner separated his work as more universal and grounded in higher principles, compared to even contemporary synthetic geometry as practiced at the École Polytechnique. We will return to the tone of these reviews and efforts at self-promotion in light of our second case study, where Steiner's results were criticized in lacking a straight-forward method.

[16] "Dabei ist die Methode durchaus eigentümlich und bisher für diese Aufgaben nicht versucht."

[17] "[...] ein erfindungsreicher und origineller Kopf [...]"

[18] "Dem analytischen Geometer endlich dürfte die Arbeit eine reiche Ausbeute, wohl gar eine Erweiterung seiner Methode gewähren."

2.5 Plücker and the Apollonius problem

Plücker was well aware of the French version of Steiner's article and the associated publicity when he composed his introduction to "Géométrie analytique. Mémoire sur les contacts et sur les intersections des cercles" (Plücker 1827). Although Plücker followed the same narrative as Steiner, from similarity to common power, Plücker's use of coordinate equations, classified by Gergonne as analytic geometry, rendered his objects very different. Further, Plücker had begun publishing and corresponding with contemporary French mathematicians in 1826, and even just one year later, he was far more in tune with contemporary research.[19] In particular, he appeared comfortable using what Poncelet had defined as ideal chords and secants in 1820, a new designation that was just beginning to be acknowledged by other geometers. Ideal objects were constructible as well-defined lines in the plane, but unlike real chords and secants, they contained imaginary points of intersection.[20] In fact, Poncelet's influence in this text is evidenced by much of Plücker's choice of vocabulary including ideals, conjugates, and inverse or direct relations as well as one of his constructions. However, in introducing his work Plücker explicitly compared it to Steiner's and strongly suggested that his own presentation was simpler and faster.

> [...] but the publicity that this geometer has given to his work does not appear to us as a sufficient motive for renouncing publishing a summary of ours, so that one will be able to better compare them, and then one will perhaps even find our path more rapid and more simple in several respects. (Plücker 1827, 29)[21]

Plücker was also familiar with Gergonne's several solutions to the Apollonius problem in 1827, which he summarized in his *Analytisch-geometrische Entwicklungen* the following year as an analytic determination of the tangency points to the desired circle lying on each of the given circles (Plücker 1828b, 102–105). In his book, Plücker credited Gergonne with simplifying the problem by finding tangency points rather than centers of the desired circles. The solution and proof Plücker provided in his *Entwicklungen* followed Gergonne much more directly than his solution published in the *Annales*. The former version was a very short exposition relying upon well-chosen coordinate axes, without any "modern" geometric terminology. In the article we consider here, Plücker would briefly cite Gergonne's solution, but his method differed substantially through the adoption of Poncelet's new geometric objects.

Plücker began directly with three circles, which he designated as $c = 0, c' = 0, c'' = 0$. Considered pairwise the real or ideal common chord, "that is, *radical axis*,"

[19] The impact of Gergonne on Plücker's first publication is discussed in Lorenat (2015).

[20] Beginning with Augustin Louis Cauchy's favorable mention of ideal objects, in his otherwise very critical review of Poncelet's research (Poncelet and Augustin 1820), Poncelet's ideal secants, chords, and points of intersection began to be discussed in the *Annales*. Often the term appeared juxtaposed to its real counterpart, such as "real or ideal" common chords, points of intersection, or tangent lines (Sturm 1826a, b; Bobillier 1827; Chasles 1828a). For historical analyses of Poncelet's ideal chords, see Gray (2005), Nabonnand (2011b), Bioesmat-Martagnon (2011).

[21] "[...] mais la publicité que ce géomètre a donnée à son travail ne nous a pas paru un motif suffisant pour renoncer à publier un sommaire du nôtre, qu'on sera peut être bien aise de lui comparer, et dont on trouvera peut-être même la marche plus rapide et plus simple à quelques égards."

for these circles would have the equations $c' - c'' = 0$, $c'' - c = 0$, $c - c' = 0$. Since these radical axes each intersected two by two at the same x and y coordinates, they all three would concur in a single point, the *radical center*.

In a footnote, Plücker attributed to Gergonne this "turn of reasoning" [*tour de raisonnement*], where common points between two coordinate equations were represented through subtracting linear equations. This use of coordinates would come to be known as "abridged notation."[22] In an extended proof using conventional coordinate equations (occupying three pages with one footnote), Plücker proved that the radical axis to a point on the circle's circumference would be a tangent line through this point.

Plücker's use of equations in this 1827 article was minimal and appeared only in the beginning. Further, his use of analysis in geometry remained like that of Gergonne in 1817, as we will see by comparison with his research the following year. While he began by describing the circles with abridged notation, after the first section the circle equations were given in their standard extended form. Further, Plücker only applied analytic geometry to find the radical axes, perpendicular lines, and concurrent points. The new geometric objects, as mentioned in the *Bulletin* review, were primarily described with respect to their positional relationship in purely geometric terms. While one could have derived analytic representations from Plücker's exposition, he did not show the equation of an orthogonal circle, a pole, or a similitude center, to take three such examples. He briefly mentioned imaginary circles as an intermediate step in obtaining a real solution, but not as a solution itself, which would have to be represented constructively.

This minimalist computation was viewed as admirable in the subsequent *Bulletin* review.

> But ordinarily it is much better, in a great number of researches, to pass alternatively from resources provided by calculation to those offered by pure geometry, and from the latter to the former, one thus avoids at once both the delays to which complicated calculations lead and the obscurity that the accumulation of great number of lemmas creates. In the first article of the issue, M. Plucker gives a very remarkable example of the application of this latter method [...] (Anonymous 1827a, 173)[23]

The anonymous reviewer suggested that too much calculation made the reader lose sight of the problem, and that Plücker's balance enabled a comprehensible presentation. We note this review implied that geometric understanding could presumably be achieved without the use of figures.

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[22] Carl Boyer traced the invention of this "abridged notation" back to Lamé's 1818 text (Boyer 1956; Lamé 1818). Plücker was among the early adopters, along with Étienne Bobillier (1798–1840), who had employed abridged notation in the article immediately preceding Plücker's. Plücker would frequently employ abridged notation in his *Analytisch-geometrische Entwicklungen* (Plücker 1828a).

[23] "Mais d'ordinaire on se trouve beaucoup mieux, dans un grand nombre de recherches, de passer alternativement des ressources que fournit le calcul, à celles qui sont offertes par la géométrie pure, et de ces dernières aux premières, on évite ainsi à la fois et les lenteurs qu'entraînent des calculs compliqués et l'obscurité que fait naître l'accumulation d'un trop grand nombre de lemmes. Dans le 1er. article de la livraison, M. Plucker donne un exemple fort remarquable de l'application de cette dernière méthode [...]."

Plücker's constructions contained numerous geometric objects, but the majority of these objects were at most enumerated and not named. With the use of coordinate representation and the new ideal objects, Plücker was able to avoid delving into most particular instances, instead covering all cases with a single construction or proof. Within Plücker's constructions the only named objects were five circles (C), (C'), (C''), (c), (c') and the two tangent points p, p', which defined the line pp'. Part of this brevity may be attributed to the fact that Plücker's publication came after those of Steiner, Poncelet, and Gergonne. Plücker could rely, even without direct citation, on the more elaborate constructions of his predecessors.

2.6 Plücker's initial reception

Situated as it was in the *Annales* and treating the very same content that Steiner had treated less than a year earlier, Plücker's article could not escape comparison with Steiner's, a fact that he acknowledged in his introduction. Yet, without Gergonne's footnotes, any acknowledgment to Steiner in the body of the text was strikingly absent. Plücker instead attributed his constructions to the earlier work of Gergonne and Poncelet. Nevertheless, this liaison between Steiner and Plücker appeared in a *Bulletin* review, which cited Steiner's treatment of the theory of circle tangency by so-called pure geometry in Crelle's *Journal*. The *Bulletin* review provided a positive assessment of Plücker's combined use of analytic and synthetic methods, which the reviewer considered much better than either method taken in isolation. Plücker, in particular, artfully avoided both complicated calculations and too many obscure lemmas.

> M. Plucker is very restrained in calculations, and all of his can, somehow, be followed by the eye; but he chooses them with great art and taste [...] (Anonymous 1827a, 173)[24]

In a review of Plücker's monograph, Cournot (who may have been the anonymous reviewer) echoed this praise of Plücker's simplicity in contrast to what he claimed to be "the often merited reproach to scholars of Plücker's country, to love complication" (Cournot 1828, 179).[25] However, Cournot also noted the drawbacks to Plücker's style of analytic geometry. While Plücker's minimalist analysis was very useful for demonstrating already known theorems, it was not the means for mathematical discovery. Because Plücker avoided all calculations that did not concern the final result, and so revealed this result, Cournot suggested that Plücker's analysis strongly resembled "la synthèse" (ibid, 178). Cournot employed the term synthesis to emphasize how Plücker's presentation worked toward a known result, and thus functioned well for proving known theorems.

Plücker's article and *Bulletin* reviews resurfaced in a letter of recommendation written for Plücker by Crelle to the Prussian culture minister, Karl vom Stein zum

[24] "M. Plucker est fort sobre de calculs, et tous les siens peuvent, en quelque sorte, être suivis de l'oeil ; mais il les choisit avec beaucoup d'art et de goût [...]."

[25] "[...] le reproche souvent mérité, que l'on fait aux savants de sons pays, d'affectionner la complication."

Altenstein in July, 1828.[26] The letter aimed to promote Plücker's recently published *Analytisch-geometrische Entwicklungen*, which had appeared earlier that year, and lends further evidence of Plücker's perceived intermediary position between analytic and synthetic methods. Moreover, in this letter Crelle provided a thorough description of the "two different methods of research in studying figures in the plane and in space." His description of the qualities of each method far exceeded the level of detail observed in either Plücker or Steiner's published work up to that point. However, as Crelle was writing a letter of recommendation, some of this detail may be attributed to simply listing Plücker's many merits. First, Crelle contrasted the opposing "synthetic, or graphic, or intuitive methods, more or less following the manner of the ancients" to the analytic method that uses calculations and is more mechanical.[27] He then outlined the advantages and disadvantages of each approach. The first method benefitted from a clear intuition about the object under investigation and mindful awareness of the operative steps of research. Thus, the synthetic method was most useful, clear, and convincing in simple cases, where the analytic method seemed too difficult. Yet, the analytic method required no additional effort to be applied generally to more complicated cases. Crelle admitted that the merits of each method had been debated, with Synthesists accusing the analytic method of "merely mechanical operations" [*bloss mechanische Operationen*] and Analysts defending "the great power and fruitfulness of analytic operations" [*die grosse Kraft und Fruchtbarkeit der analytischen Operationen*] as well as its ease. However, he insisted that both methods had their own "individual value" [*eigenthümlichen Werth*]. While the analytic method was the method of discovery, the synthetic method achieved the greatest clarity and certainty of insights For Crelle, the study of mathematics required both qualities, and both qualities had been achieved in the recent work of Plücker.

In advertising Plücker's publication record, Crelle cited Plücker's recent *Bulletin* reviews, which stated that Plücker had united both methods (Anonymous 1826, 1827a).[28] Crelle then explained that despite Plücker's claim in his preface that his monograph was purely analytic, it actually also combined both methods with success. Diplomatically, Crelle continued by correcting Plücker's accusation from the same preface that Steiner had merely been following in the footsteps of Poncelet. He countered that Steiner's work had been independently derived and was useful in its own right. Overall, Crelle suggested that the prefatory remarks expressed a tendency against synthetic methods and practitioners that was not apparent in the body of Plücker's text. Crelle concluded by listing the diverse subjects of new geometry that Plücker had included: transversals, ideal chords, polars and poles, similarity points and axes, curves of second order, osculation points, imaginary expressions, curves of higher order and other figures in space, etc. Although Plücker framed his work as analytic geometry, or even "pure analytic geometry," Crelle interpreted mixed methodological

[26] This letter is published in (Eccarius 1980) as "Document 1: *Gutachten Crelles für den Kultusminister von Altenstein über J. Plückers.*"

[27] "[...] synthetische, oder graphische, oder anschauliche Methode, mehr oder weniger nach Art der Alten."

[28] It is worth pointing out that this first review (Anonymous 1826 contained a synopsis of the article by "Pluker," which indeed was not analytic geometry, but which had been dramatically altered from Plücker's original submission by Gergonne (Plücker and Schoenflies 1904; Lorenat 2015).

tendencies in his attention to the form of the equation that would become increasingly apparent in his later works.

In both Steiner's and Plücker's approach to tangent circles, the content remained mostly elementary. Tangents and intersections among coplanar circles could be achieved through Euclidean constructions, and despite their new nomenclature, the relationships conveyed by similitude, powers, and radicals were grounded in simple proportions, collinearity, and concurrence. However, Steiner and Plücker both remarked that these same results would continue to hold should circles be replaced by any conic sections, and Plücker described the necessary projective procedure to apply circle-specific constructions to any second-order curve. This generalization extended the domain of research beyond the elementary geometry found in Euclid. Poncelet and Gergonne had likewise applied results from circles to general conic sections, but the concept was novel enough that in 1828 Chasles announced it as a new theorem.

> It follows from there, in particular, *that any circles that one likes, traced on the same plane, can always be considered as the stereographic projections of the same number of planar sections made in a second-order surface, and their centers as stereographic projections of the vertices of cones circumscribed to this surface, according to these same planar sections* (*). (Chasles 1828b, 309)[29]

Gergonne modified Chasles' claim with a footnote, signaling both the special position of Steiner and Plücker as German geometers and their publications on tangent circles at the forefront of contemporary French geometry.

> This principle does not appear to be unknown to German geometers. M. Plucker invokes it formally on page 47 of the present volume, and M. Steiner relies on it equally in the memoir of which we gave an extract on page 285 of our XVIIth volume, to transport planar constructions onto any surfaces of second order.[30]

Though Gergonne would later apologize for seeming to have trivialized Chasles' research, his remarks proved prescient in forecasting Steiner and Plücker's early adoption of new principles and practices.

3 Conic sections with four common points (1828)

In the two years following his first publication, Steiner contributed six additional articles to Crelle's *Journal* and three to the *Annales*, along with numerous posed problems in both journals. Meanwhile, Plücker had recently published the first volume of his two-part monograph, *Analytisch-geometrische Entwicklungen* and as well as

[29] "Il suit de là, en particulier, *que tant de cercles qu'on voudra, tracés sur un même plan, peuvent toujours être considères comme les projections stéréographiques d'un pareil nombre de sections planes faites dans une surface du second ordre, et leurs centres comme les projections stéréographiques des sommets des cônes circonscrits à cette surface, suivant ces mêmes sections planes* (*)."

[30] "Ce principe paraît ne pas être inconnu aux géomètres allemands. M. Plucker l'invoque formellement, à la pag. 47 du présent volume, et M. Steiner s'en appuye également dans le mémoire dont nous avons donné un extrait à la pag. 285 de notre XVII.e volume, pour transporter ses constructions planes sur des surfaces quelconques du second ordre."

a total of three articles in the *Annales* and one in the *Journal*. Both Steiner's and Plücker's articles from this period had been consistently summarized and reviewed in Ferussac's *Bulletin*, and cited in contemporary journals and monographs (Anonymous 1826, 1827a; Didiez 1828).

Solutions to geometry problems, such as the Apollonius problem, were in the form of constructions. By contrast, theorems in geometry were often independent of constructions, more theoretical, and hence better suited toward exhibiting methodological variety. First, in order to understand the original context of Lamé's theorem, we will examine his original statement and proof as it appeared in the *Annales* in 1817. Lamé's theorem states that when three or more conic sections intersect in four common points, then if one draws parallel diameters to these conics the corresponding conjugate diameters will meet in a point. Its central concept is that of conjugate diameters, which have ancient roots in Apollonius's *Conics*. In the texts considered here geometers assumed conjugate diameters as known and without need of definition. Definitions of conjugate diameters could be found in pedagogically oriented books, such as *Traité élémentaire de trigonométrie rectiligne et sphérique, et d'application de l'algèbre à la géométrie* (Lacroix 1807), in which Sylvestre François Lacroix defined the diameter conjugate to a given diameter of an ellipse as the diameter II' parallel to the tangent drawn through the endpoint L of another given diameter (Lacroix 1807, 172). To extend this construction from ellipses to every conic section, geometers often used projection or coordinate equations.

In the intervening decade between Lamé's first statement of his theorem and Steiner's proof, the whole subject of Lamé's article–the correspondence between conic sections and their conjugate diameters–remained popular in geometric articles within the *Annales*. In particular, the research of Gergonne, Brianchon, and Poncelet explored the conditions necessary to determine second-order curves and the relationships between conjugate diameters (Brianchon 1817; Brianchon and Poncelet 1820; Gergonne 1821; Poncelet 1822). For instance, in his 1822 *Traité*, Poncelet included a proof of Lamé's theorem, which he described as a direct corollary to his more general theorem on polar reciprocity found by passing a planar point to infinity.

Thus, in 1828 Steiner's choice of studying the relationship between conic sections in the plane engaged with a set of geometric questions still under lively investigation. Approximately two-thirds of the content in the nineteenth volume of the *Annales* were devoted to articles on the study of geometric curves and surfaces by at least nine different geometers (several articles were signed only as *un abonné*). However, if Steiner's subject matter was well established and well represented, his approach, as we will see, was original to him.

The following issue of the same volume of the *Annales* opened with an article of analytic geometry, "Recherches sur les courbes algébriques de tous les degrés" by "M. le docteur Plucker, professeur à l'Université de Bonn" (Plücker 1828b). Plücker's ten-page article was the first in a two-part series, the second of which explored the same questions for algebraic surfaces. In a concise introduction, Plücker explained his intention: "to give several examples of a method, by aid of which one can deduce, immediately and without any sort of calculation, a great number of general properties of curves of any degree, simply from considering the algebraic constitution of their representative equations" (Plücker 1828b, 97).[31] Plücker's development of coordinate representation without calculation characterized a new method of analytic geometry.

While both articles were about relationships between planar curves, the diverse treatments masked their parallel content except for their common reference to Lamé's theorem. If the Apollonius problem reinforced the similarities between different geometric methods in their common invocation of illustrated or virtual figures, frequently repeated constructions, and use of the same set of new geometric objects, the case of Lamé's theorem will emphasize their differences.

3.1 Lamé's theorem (1817)

Lamé had just graduated from l'École Polytechnique in 1817 when his article "Géométrie analitique. Sur les intersections des lignes et des surfaces. Extrait d'un mémoire présenté à l'Académie royale des sciences, en décembre 1816" was published in the *Annales* (Lamé 1817). As the lengthy title indicates, the article was excerpted from a longer memoir, later revised, and published in 1818 as *Examen des différentes méthodes employées pour résoudre les problèmes de géométrie* (Lamé 1818, 30–41).[32] The article's subject matter, the use of coordinate equations to solve geometry problems, was dear to Gergonne's heart and he may have requested this contribution from Lamé as he would with respect to Poncelet's memoir in 1821. This brief article set out to use rectangular coordinate equations in order to find conditions such that:

1. three first- or second-order curves [*lignes*] on the same plane concur in a point;
2. three first- or second-order surfaces in space meet on a curve;
3. four first- or second-order surfaces in space concur in a point.

The result on conjugate diameters was considered by Lamé as an "interesting theoretical and practical consequence."

For his first problem, Lamé used coordinate equations to represent three curves of second order.

$$\begin{aligned} ax^2 + 2bxy + cy^2 + 2dx + 2ey + f &= 0 \\ a'x^2 + 2b'xy + c'y^2 + 2d'x + 2e'y + f' &= 0 \\ a''x^2 + 2b''xy + c''y^2 + 2d''x + 2e''y + f'' &= 0 \end{aligned}$$

By multiplying the first two equations by the indeterminate constants m and m', respectively, and then finding their sum, he derived a single equation.

$$\begin{aligned} &(am + a'm')x^2 + 2(bm + b'm')xy + (cm + c'm')y^2 \\ &\quad + 2(dm + dm')x + 2(em + e'm')y + (fm + f'm') \end{aligned}$$

[31] "[...] donner quelques exemples d'une méthode à l'aide de laquelle on peut déduire, immédiatement et sans aucune sorte de calcul, un grand nombre de propriétés générales des courbes de tous les degrés, de la simple considération de la constitution algébrique des équations qui les représentent."

[32] Evelyne Barbin has shown how Lamé demonstrated the potential for progress and discovery in associating analysis and geometry (Barbin 2009). In particular, Barbin considers the content of Lamé's *Examen des différentes méthodes* and his development of abridged notation.

Because of the indeterminacy of m, m' this new equation, "in its generality," represented all the curves of second order passing through the intersections of the two first curves. Examining the equation, Lamé remarked that, under various relationships between the coefficients, it could belong to two different parabolas, a circle, or an infinity of ellipses and hyperbolas. The condition for the third line to have the same intersections would be met by setting each coefficient in the new equation equal to its corresponding coefficient in the third curve equation. That is,

$$am + a'm' = a'', bm + b'm' = b'', cm + c'm' = c'',$$
$$dm + d'm' = d'', em + e'm' = e'', fm + f'm' = f''.$$

Through elimination of the constants m, m' Lamé arrived at four other equations that expressed the concurrence of the three second-order curves. Moreover, the three equations containing a, a', a'', b, b', b'', and d, d', d'' could be simultaneously satisfied, signifying that the three straight lines $ax + by + d = 0$, $a'x + b'y + d' = 0$, $a''x + b''y + d'' = 0$, concurred in a point.

Each of these equations also belonged to the diameter that bisected all chords parallel to the x-axis of its corresponding second-order curve. For this result, Lamé cited an article by Bérard from the *Annales*. Then, since the direction of the x-axis could be chosen with respect to the second-order curves, Lamé concluded,

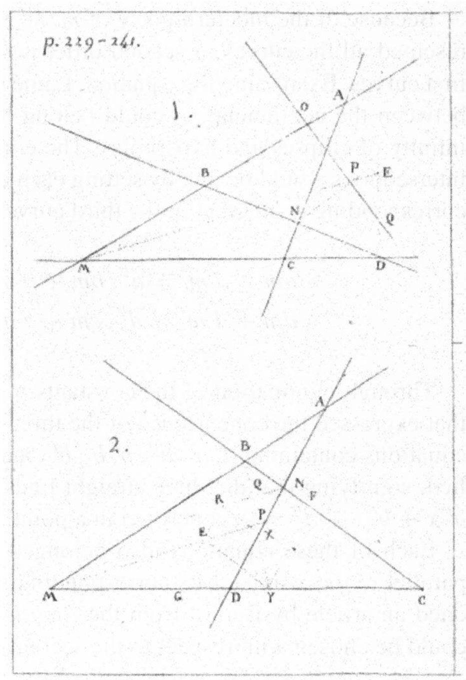
> THEOREM. If three or more conic sections have four common points; then in no matter what direction one draws parallel diameters to these conics, the corresponding conjugate diameters will concur in the same point. (Lamé 1817, 233)[33]

At this stage in the text, no figures had been employed, and all geometric objects were represented by their coordinate equations. Having derived his theorem, Lamé then applied it to determine graphically the center of a conic given five points on its perimeter and the slope of the diameters of a parabola given four points on its perimeter. These two constructive problems referenced straight edge figures, (Fig. 4). Though the problems themselves featured conic sections, the figures, which were printed at the end of the issue, contained only the given points, as was common practice for published geometrical figures at this time.

The *Annales* classified Lamé's article under the subject heading "analytic geometry," and within this article, Lamé chose coordinate equations and algebraic analysis to prove his theorems but then used these theorems to solve associated problems through constructing figures without coordinate representation. Lamé was thus able to employ the generality of algebraic expressions, without sacrificing the practice and application of graphic constructions. In fact, Lamé explicitly suggested that the application of algebra to geometry served as the method of discovery in his *Examen des différentes méthodes employées pour résoudre les problèmes de géométrie*. Following a long list of potential applications of algebraic representation of geometric loci beginning with

[33] "THÉORÈME. Si plusieurs sections coniques ont quatre points communs ; dans quelque direction qu'on leur mène des diamètres parallèles, les conjugués de ces diamètres concourront en un même point."

Fig. 4 Lamé's Figures (Lamé 1817)



the study of straight lines and extending to second-degree curves directed toward his beginning student reader, Lamé concluded,

> [...] finally in all these applications [we] must not neglect to draw attention to the constant agreement of Algebra with Geometry, an agreement that permits us to entrust calculation with the task of discovering new theorems. (Lamé 1818, 5–6)[34]

In the case of Lamé's theorem, his assessment appears to be correct, as his successors repeatedly cited his article as the source of the theorem.

3.2 Steiner proves Lamé's theorem: "Développement d'une série des théorèmes relatifs aux sections coniques" (1828)

Gergonne and *Bulletin* reviewers described Steiner's first publications as employing a method completely unique to the author. As Steiner continued writing for a French audience, his knowledge of the background literature and terminology transformed accordingly. Now writing in French, Steiner cited Carnot and Lamé (and himself).

[34] "[...] enfin ne rien négliger dans toutes ces applications pour faire remarquer l'accord constant de l'Algèbre avec la Géométrie, accord qui permet de confier au calcul le soin de découvrir de nouveaux théorèmes."

Like Steiner's first publication in the *Annales*, this paper received substantial input from Gergonne. Among Steiner's Nachlass at the Eidgenössische Technische Hochschule in Zürich a manuscript draft of Steiner's original submission is accompanied by attempts at a letter from the author to the editor (Steiner 1828e). In the most complete copy, Steiner requested that Gergonne include the following theorems in his journal. He requested that Gergonne exercise full editorial authority.

> It is only your indulgence and the hope that you will not refuse to correct in this whatever you think appropriate, which encourages me to offer my humble geometrical research in such a rough and incorrect style. (Steiner 1828a)[35]

Steiner described his difficulty with writing in French, which prevented him from communicating his method of discovering his theorems. Consequently, his style was "very aphoristic," and he hoped that Gergonne could "supply all that my work lacks in reasoning and clarity." Steiner concluded by offering to send part of his forthcoming book if Gergonne did not find his work "displeasing." Steiner's modest appeal contrasts with his confident assertions in German letters and publications. Further, Steiner recognized that his contribution lacked a "method of discovery" and thus appeared ad hoc–a feature he had criticized in analytic geometry.

The subsequent publications show that Gergonne certainly followed Steiner's directions. Steiner wrote his manuscript, entitled "Développement d'une série de théorèmes relatifs aux courbes et aux surfaces du second ordre," in three parts, each of which became a separate article appearing in the nineteenth volume (Steiner 1828b, c, d). Notably, Gergonne classified the first article as elementary geometry, the second as pure geometry, and the third as geometry of situation. Gergonne extensively changed the first and third articles through adding dual columns of writing, new theorems, and numerous citations. He limited his changes to grammar, vocabulary, and notation (Steiner's A_1 became A', for example) in the second article, "Développement d'une série des théorèmes relatifs aux sections coniques," which contained Lamé's Theorem. Gergonne also added a single additional reference: the *Annales* volume IV, page 251, an article by Gergonne himself (Steiner 1828b, 45).

"Développement d'une série des théorèmes relatifs aux sections coniques" ran to thirty pages in twenty-eight sections and included dozens of theorems and problems, which were only designated as such by the use of quotation marks around their statements (Steiner 1828d). The list-like form of Steiner's article was emphasized by its vague title, lack of preface or introduction, and extensive enumeration of small sections–some as short as a sentence, others extending several paragraphs. The same format featured in at least two of Steiner's publications of the same period, "Démonstration de quelques théorèmes" and "Einige geometrische Sätze" (Steiner 1828b, 1826b), and seemed to contradict his earlier systematic intentions or desire to show an organic unity. However, Steiner's proliferation of new articles could also be interpreted as publicizing results that would be systematically developed in his forthcoming book or as an effort to secure a permanent position (Steiner 1832).

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[35] "C'est votre indulgence seule et l'esperance, que vous ne refuserez pas d'y corriger tout ce que vous jugerez à propos, qui m'encourage de vous offrir dans un style si rude et incorrect mes faibles recherches geometriques [sic]."

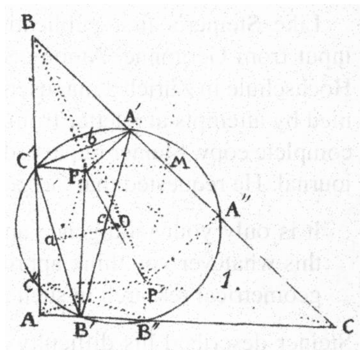


Fig. 5 Steiner's Figure 1 (Steiner 1828d)

The *Bulletin* review later summarized Steiner's article as based on a single proposition: given a triangle and three points on the lines containing the sides of this triangle such that perpendiculars raised through these points concurred in the same point, then if one drew a circumference through these points, it would intersect the given lines in three new points from which perpendiculars would also concur. Certainly, this proposition was Steiner's first result in this text, but he only explicitly referenced it in deriving his second result (whereas, for example, his fifth theorem was referenced three times throughout the text), so its role as the foundation of his research seems to be mostly positional.

As well as articles devoted to the proof of new results, Steiner's other primary output at this time were posed problems (without solutions) and theorems (without proof) offered to the readers of the *Annales* and the *Journal* (Steiner 1827a, c, d). These collections served to both secure Steiner's priority and encourage readers to practice applying his methods. Structurally, "Développement d'une série des théorèmes" was reminiscent of these catalogs in neither building from general considerations (like in Steiner's *Systematische Entwicklungen*) nor building toward some promised results (like in Steiner's "Leichter Beweis eines stereometrischen Satzes von Euler"). To introduce Steiner's style with an elementary result and for future comparison with Plücker's proof of the same proposition, we will begin with Steiner's first proposition before turning to his proof of Lamé's theorem.

Steiner began with constructing a figure (our Fig. 5), for which we will provide a series of step-by-step illustrations following Steiner's construction instructions. However, certain features of his finished Figure 1, only described later and not relevant to proving Lamé's theorem, will not be introduced.

From a point P in the plane of a triangle ABC drop perpendiculars PA', PB', PC' respectively to sides BC, CA, AB. Though not specified in the text, we follow Steiner's figure by placing P inside the triangle ABC (Fig. 6).

Then join the vertices A, B, C to P (Fig. 7).

These new segments determine fixed relationships between the parts of the right triangles.

$$\overline{BA'}^2 - \overline{CA'}^2 = \overline{BP}^2 - \overline{CP}^2,$$
$$\overline{CB'}^2 - \overline{AB'}^2 = \overline{CP}^2 - \overline{AP}^2,$$
$$\overline{AC'}^2 - \overline{BC'}^2 = \overline{AP}^2 - \overline{BP}^2.$$

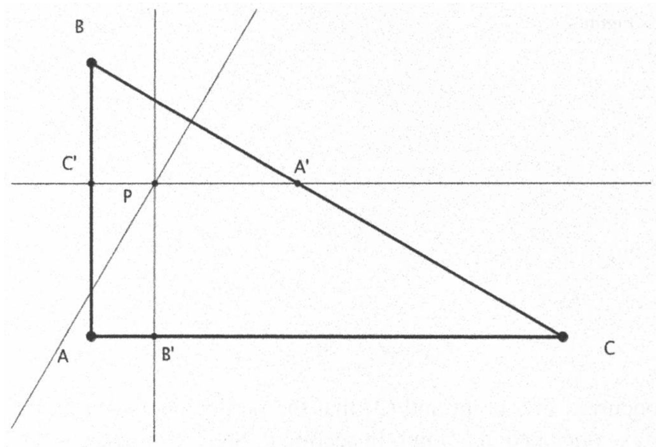


Fig. 6 Triangle ABC and point P

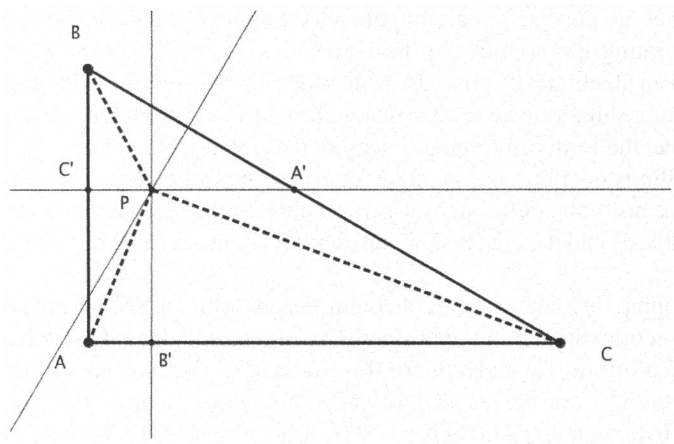


Fig. 7 Segments AP, BP, CP

By "adding, reducing and transposing" [*ajoutant, réduisant et transposant*][36] the above equations Steiner derived,

$$\overline{AB'}^2 + \overline{BC'}^2 + \overline{CA'}^2 = \overline{BA'}^2 + \overline{CB'}^2 + \overline{AC'}^2.$$

He claimed that this relationship between line segments was a necessary and sufficient condition that perpendiculars raised from the points A', B', C' on the three respective sides BC, CA, AB of a triangle ABC all concurred in the same point P. He concluded *immediately* that from this resulted (1) that the perpendicular bisectors of the sides of

[36] Gergonne frequently used the expression "adding, reducing and transposing," in the manuscript Steiner simply wrote "thus the sum is" [*donc la somme est*].

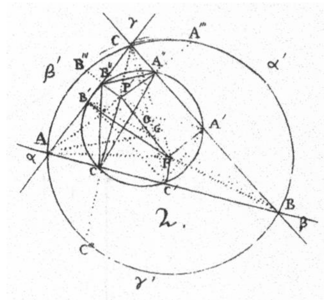


Fig. 8 Steiner's Figure 2 (Steiner 1828d)

a triangle concurred in a point and (2) that the perpendiculars from each side to its opposite vertex concurred in a point. In section 2, Steiner introduced a circumscribed circle and gradually derived further theorems on the triangles inscribed to conics.

Steiner's progression in section 1 revealed how tenuous the line between what was classified as pure and analytic geometry might be. Here Steiner began with a figure illustrating the relationship he intended to derive: the concurrence of three lines. From an algebraically computed equation, Steiner determined a criterion for the desired relationship, followed by two exemplary cases. Though Steiner's criterion was constructible, the form of presentation, as well as the use of calculations, suggests an algebraic affinity not present in the figure. Further, the sufficiency of the condition only followed because each of the steps was reversible. Figure 1 served as a representation of the hypothesis and the conclusion, but concealed the intermediary non-constructive steps.

We now jump forward to Lamé's theorem, stated toward the end of the text in section 22. In this section, Steiner initiated a new line of research. He referred explicitly only to section 6 of his article and a figure (his Figure 7) he had introduced in section 18. He thus required from his reader knowledge and memory of points only defined in section 18. In order to explain Steiner's derivation of Lamé's Theorem, we need first to determine the necessary results from section 6 required in section 22, the construction of Figure 7 in section 18, and its special case in section 19; we then shall turn to Steiner's proof per se.

Section 6 began by describing the relationships between points pictured in his Figure 2 (our Fig. 8), a circle with center P circumscribed to a triangle ABC.

Steiner had shown in section 2 that perpendiculars raised from AB, AC, BC to P bisected their respective sides at C', B', A'. In section 6, he employed this result, though without any direct citation.

Drawing $B'C'$, $C'A'$, $A'B'$, Steiner concluded that these lines were respectively parallel to the sides BC, CA, AB of the original triangle. Then, for example, if the line AP' was perpendicular to $B'C'$ it would be perpendicular to BC. So, from a result proved but not referenced from Section 1, P' would be the point where perpendiculars dropped from vertices to opposite sides concurred. Steiner designated the feet of these perpendiculars as A'', B'', C''. Then points A', B', C', A'', B'', C'' would all lie on the circumference of a circle centered at O, the midpoint of PP'. These constructive steps are shown in our Fig. 9.

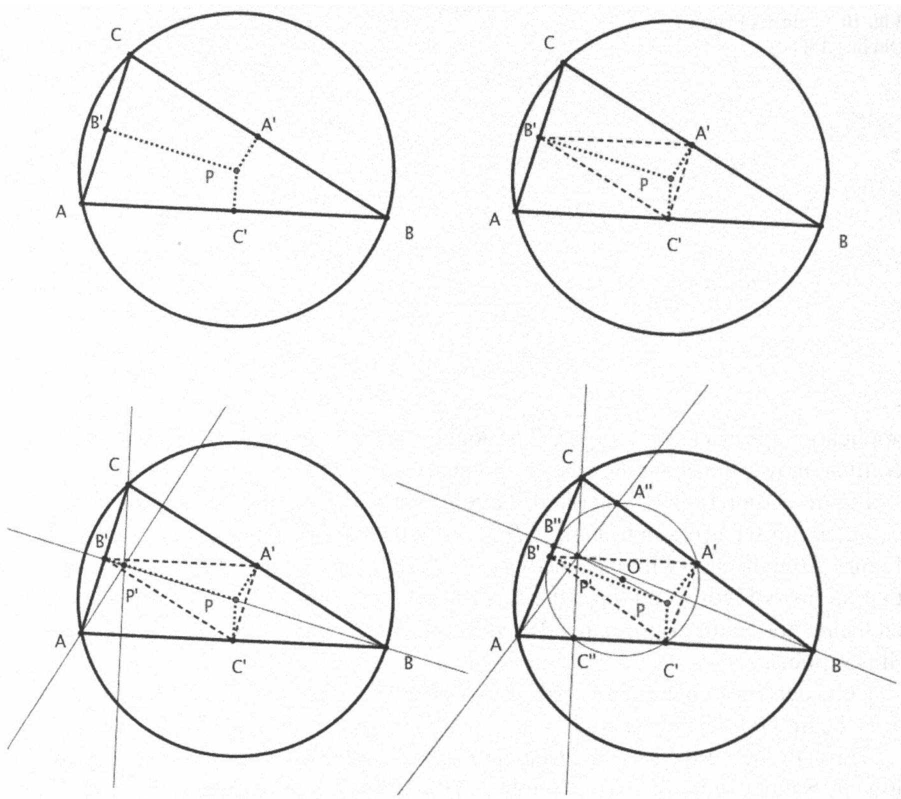


Fig. 9 Constructing Steiner's Figure 2

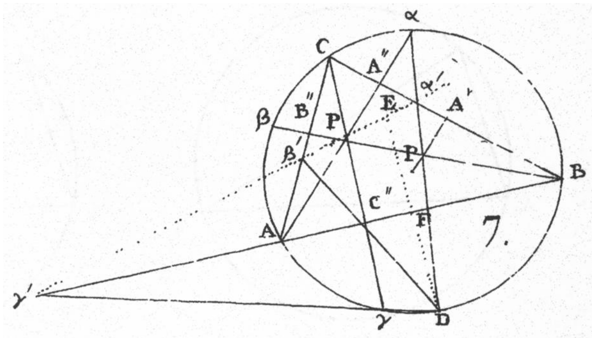
Steiner referenced Lazare Carnot (without a date or a title) in finding a fourth point G on PP' such that $GO : GP :: P'O : P'P$.[37] From this ratio P' and G were the similitude centers of the two circles centered at O and P.[38] Thus the circle centered at O also passed through the midpoints of the segments $P'A$, $P'B$, and $P'C$. This was the result that Steiner would employ in section 22.

As we have seen with his Figures 1 and 2, each of Steiner's figures carried its own, usually new, definition of points. Often these new points would have overlapping names with the points from prior figures. Since he cited Figure 7 in section 22, we must now set aside the use of point labeling from Figure 2, except in translating the

[37] This proportion reappears in many early nineteenth-century pure geometry texts. Poncelet attributed the result to Pappus and also noted Brianchon's treatment (Poncelet 1822, 12). Brianchon in turn references "l'illustre auteur de la Géométrie de position," that is, Carnot (Brianchon 1817, 6). Poncelet, Brianchon, and Steiner refer to the division as *harmonic* or *harmonic proportion*. Carnot indeed proved the existence of the fourth proportional point for any three given points in space in *De la corrélation des figures de géométrie* in 1801 (Carnot 1801, 103–125).

[38] The similitude centers between two circles were defined by precisely this ratio between circles with radii GO and $P'P$ in (Steiner 1826a).

Fig. 10 Steiner's Figure 7 (Steiner 1828d)



application of results from Figure 2 as noted. This will hopefully help avoid too much confusion over the changing roles of points.

Figure 7 (our Fig. 10) pictured a circle, but was described in section 18 as any conic circumscribed to a triangle ABC. As in the previous figures, we will construct Figure 7 step by step with intermediary illustrations along the way. However, we note that Steiner's Figure 7 pictured the construction in its entirety and incorporated other elements not featured in section 18 or 22, which we will omit from our progressive illustrations.

Consider any conic circumscribed to a given triangle ABC, as in Steiner's Figure 7 we begin with a circle (Fig. 11).

Through the vertices of this triangle and through any planar point P' (again, following Steiner's figure, placed within ABC), lines $AP'A''\alpha$, $BP'B''\beta$, $CP'C''\gamma$ respectively cut the extended triangle sides opposite the three angles in A'', B'', C'' and the conic at α, β, γ. If through any point D on the circumference, one drew lines $D\alpha$, $D\beta$, $D\gamma$ cutting respective sides BC, AC, AB in α', β', γ', then these three points would always be on a line $\alpha'\beta'\gamma'$ containing P'. This collinearity result followed when one considered $D\beta BCA\alpha D$, or any six such points, as an inscribed hexagon to the conic. Then Pascal's theorem guaranteed that the intersections of opposite sides $D\beta$ and CA, βB and $A\alpha$, BC and αD, respectively, at β', P, α', were collinear. Choosing a different hexagon and applying the same procedure would ensure the collinearity of all four points. Finally, as D moved along the circumference, the line $\alpha'\beta'\gamma'$ rotated on the point P', and (as Steiner put it) vice versa.

In section 19, Steiner considered the special case where the conic was a circle, then $P'A'' = A''\alpha$, $P'B'' = B''\beta$, and $P'C'' = C''\gamma$. While Steiner did not state whether or not the conic was a circle in Section 22, he did begin with this particular equality result, which also allied with his application of the results from section 6, proved only for a given circle not a general conic.

So, in section 22 points A'', B'', C'' were now the respective midpoints of segments $P'\alpha$, $P'\beta$, $P'\gamma$, particular cases of the points constructed in section 18. If from A'', B'', C'' one drew three lines respectively parallel to $D\alpha$, $D\beta$, $D\gamma$, where D was some point on the circumference, they would then pass through each of the midpoints of $P'\alpha'$, $P'\beta'$, $P'\gamma'$ and concur at a point D'. Then, from section 6, a conic would pass through A', B', C' midpoints of $\beta\gamma$, $\alpha\beta$, $\alpha\gamma$ (the former BC, CA, AC of

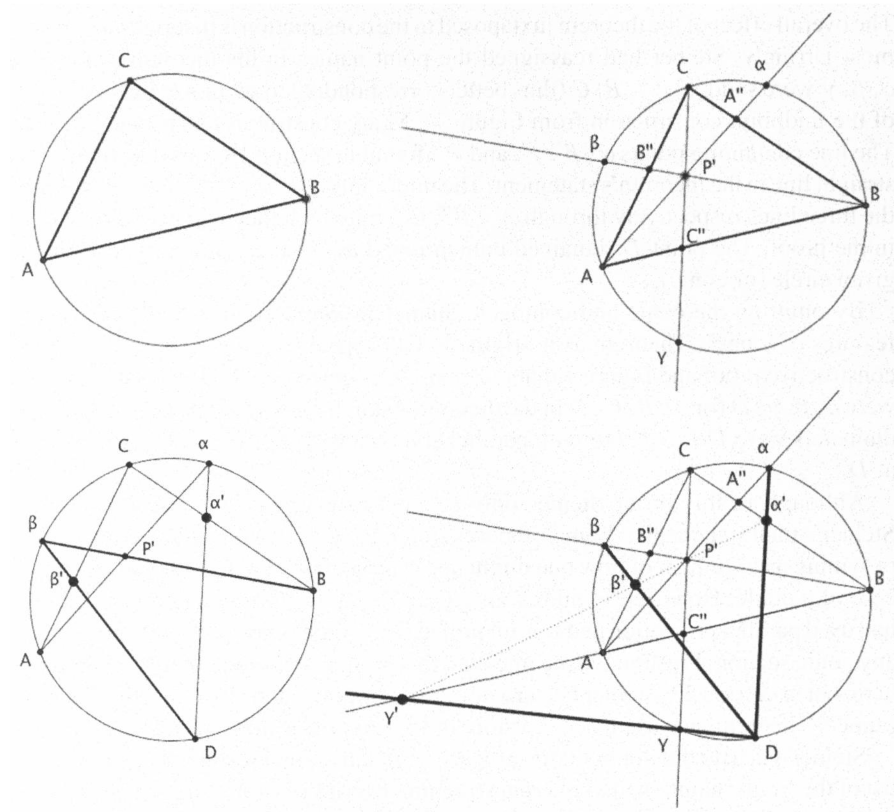


Fig. 11 Constructive steps toward Steiner's Figure 7

section 6) and A'', B'', C'' the midpoints of $P'\alpha$, $P'\beta$, $P'\gamma$ (the former $P'A$, $P'B$, $P'C$ of section 6). From the construction, the point of concurrence D' would also lie on this circumference and by Pascal's Theorem D, D', P' would be collinear. Steiner declared that from this resulted a theorem he said was due to Lamé (with no date or text cited).

> Four coplanar points A, B, C, P' determine three systems of line pairs AP' and BP' [sic, BC in manuscript], BP' and AC, CP' and AB, which intersect respectively in A'', B'', C''. If one intersects these systems by any line $\alpha'\beta'\gamma'P'$ passing through P', and if, by the points A'', B'', C'' and by the midpoints of the segments [$\alpha'P'$, $\beta'P'$, $\gamma'P$] of this line, one draws the lines $A''D'$, $B''D'$, $C''D'$, then these lines will concur in the same point D'. And the locus of this point will be a conic passing through the points A'', B'', C'' and through the midpoints of the lines BC, CA, AB AP', BP', CP', etc. (ibid, 61)[39]

[39] "Quatre points A, B, C, P' donnés sur un même plan déterminent trois systèmes de deux droites AP' et BP', BP' et AC, CP' et AB, qui se coupent respectivement en A'', B'', C''. Si l'on coupe ces systèmes par une droite quelconque $\alpha'\beta'\gamma'P'$ conduite par P', et si, par les points A'', B'', C'', et par les milieux des segmens de cette droite, on mène des droites $A''D'$, $B''D'$, $C''D'$, ces droites concourront en un même

The overall effect of the theorem juxtaposed to the construction is disorienting. Perhaps most jarringly, Steiner had reassigned the point names in his theorem's statement: α, β, γ were suddenly A, B, C (thus better corresponding to section 6, but in disregard of the midpoint construction from Figure 7). Many constructive steps were inverted. The line containing points α', β', γ', and P', found in section 18, was described as any straight line in the theorem's statement. The midpoints of $P'\alpha'$, $P'\beta'$, $P'\gamma'$ here defined the three lines respectively through A'', B'', C'', on which they had been proved to lie in the proof. The point D remained unmentioned as did the parallel relation and the given circle (or conic).

By omitting the conic and conjugate diameters, Steiner's text removed the main features of Lamé's Theorem as it originally had appeared in 1817. It was only in the constructive proof and Figure 7 that one could recognize that the four common points were α, β, γ, D (or A, B, C, D in the theorem's statement–although D is missing), the diameters were $D\alpha$, $D\beta$, $D\gamma$ with parallel conjugates $A''D'$, $B''D'$, $C''D'$ concurring at D'.

Although in this proof Steiner only seemed to examine the case of the circle, Steiner's final statement of Lamé's theorem referenced a general conic section, without providing an argument of how one might apply projection to extend the proof from the case of a circle. However, in numerous earlier examples throughout the text, Steiner had used parallel projection and central projection in order to extend circle properties to any conic section. Presumably, he intended that similar techniques could be employed here, but in section 23, Steiner began a new line of inquiry, and did not further reference Lamé's Theorem, nor use its conclusion, in the remainder of his article.

Steiner's article did not build toward any cumulative final results. Nevertheless, the list of theorems maintained a coherent structure in terms of subject matter and repeated techniques. Throughout the article, Steiner employed a common theme of progressing from a simple case to ever more general elaborations. Each new line of inquiry began with simple particular figures–most commonly a circle and straight lines–that Steiner generalized progressively into curves. He often repeated a three-step argument pattern, first a specific result for a circle or an ellipse, then via projection to a broader result for a set of conic sections, and finally via perspective to an even more general result applied to all conic sections.[40] Finding the reciprocal result was another common practice for Steiner. Sometimes he simply stated "reciprocally" or "vice versa," sometimes he wrote out the reciprocal in full in the same section, and sometimes he devoted an entirely new section to the exposition of the reciprocal. These choices do not appear motivated by the significance of the result or the length of the process. The symmetry

Footnote 39 continued
point D', et le lieu de ce point sera une conique passant par les points A'', B'', C'' et par les milieux des droites BC, CA, AB AP', BP', CP', etc."
This theorem is nearly identical between the published and unpublished versions (Steiner 1828e). Steiner, in fact, ended many theorems with "etc." or "etc. etc. etc." in his manuscript, several of which Gergonne edited out.

[40] For instance, in section 10 Steiner proved a result for an equilateral triangle circumscribed to a circle. Through parallel projection he extended the result to any triangle and the smallest possible circumscribed ellipse in section 11. Then in section 12 he applied central projection or perspective to generalize the result to any triangle and any circumscribed conic section.

of Steiner's argument patterns unified his article more than the particular contents, which ranged across objects in the geometric plane, leading to numerous results and then backtracking to different hypotheses. Although the results appeared disorganized, the repetition of generalization could serve as a guide to the reader in systematically following and applying Steiner's method. Though it did not make for easy reading, this repeated form of argument perhaps more closely resembled an imitable method of discovery than an argument directed toward a specific result. Here, no one result was placed as more important or significant than another. Some results were employed more frequently, but because Steiner proceeded from particular findings to generalizations, the more specific results appeared more often despite their limited applicability.

As in the case of section 2 and section 18, each new line of inquiry included a specified figure, where Steiner took the opportunity to wipe the slate clean and redefine previously designated points. As we have seen, this practice was not without confusion. Although Steiner included many theorems, not every proposition was illustrated. Steiner referred to figures economically, often using one figure to illustrate several constructions.

In the published version of the article, Gergonne signed his initials as the draftsman for all the engraved figures in his journal, and clearly took care in his layout. In order to

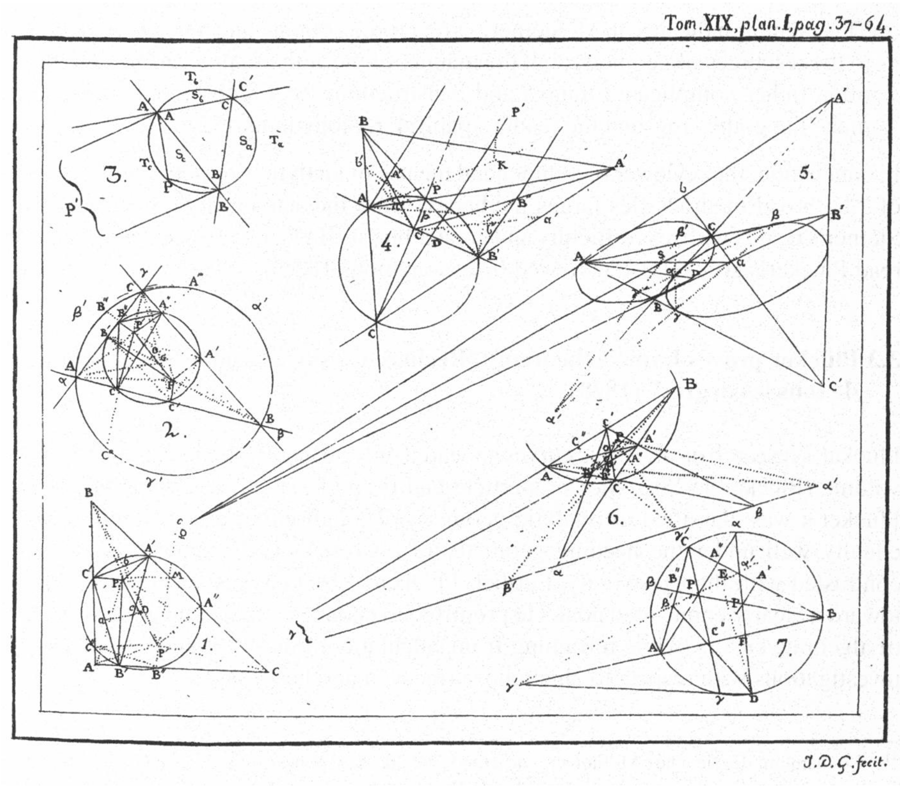


Fig. 12 Steiner's Figures (Steiner 1828d)

fit all of Steiner's figures on a single sheet (our Fig. 12), the figures were not arranged linearly, but almost in a clockwise fashion beginning with figure one in the lower left corner. In Figure 5 there were two ellipses, otherwise the conics were circles, or as in his Figure 6, partial circles. If we consider when figures were invoked, they remain linked to preliminary and more particular theorems rather than the general results derived through projection and reciprocity.

Despite the elementary nature of the figures, they contained recent geometric innovations. Specifically, both Steiner's Figure 3 and his Figure 4 indicated an infinite point P' as the intersection of parallel lines. As in the figures of Poncelet in his *Traité*, the point at infinity was suggested as being off the page (Poncelet 1822). In generalizing his theorems, Steiner referred to cases where the points of intersection passed to infinity or were situated at infinity. These procedures of projection, perspective, and reciprocity maintained an analogy to constructive results yet extended beyond their constructive limitations.

The aphoristic style that Steiner perceived in his introductory letter to Gergonne was observed and criticized by his readers. An anonymous reviewer in the *Bulletin* summarized Steiner 1828d as containing too many theorems even to list within the bounds of the review. The reviewer continued that while Steiner's fruitfulness might be impressive, his complicated presentation required extraordinary patience and fortitude even for a well-disciplined reader.

> This memoir indicates, in M. Steiner, a great force of mind, and much habituation to the resources of geometry; but the manner of the author's proceeding, and the often rather complicated figures and constructions, to which he has recourse, make the going slow and the reading painful. (Anonymous 1828, 245)[41]

In conclusion, the reviewer recommended the use of analytic geometry, or some other of "the broad methods [les larges méthodes] from the school of Monge," to render Steiner's results with less difficulty and greater brevity. In fact, the reviewer had already read Plücker's article [also reviewed in (Anonymous 1828)], which did exactly that.

3.3 Plücker proves Lamé's theorem: "Recherches sur les courbes algébriques de tous les degrés" (1828)

Plücker's research on algebraic curves appeared only 30 pages later in the same *Annales* volume (Plücker 1828b). Steiner's article had been classified as pure geometry and Plücker's was classified as analytic geometry. Though either article might have fit equally well under the heading geometry of curves, their disparate classifications contrasted with their proximity. Further, Plücker's research was succinct and driven toward a clearly enunciated goal. He promised to deduce a great number of properties of algebraic curves, while refraining from calculation. Plücker would continue these investigations with respect to algebraic surfaces in another essay.

41 "Ce mémoire décèle, chez M. Steiner, une grand force de tête, et beaucoup d'habitude des ressources de la géométrie; mais la manière de procéder de l'auteur, et les figures et constructions, souvent assez compliquées, auxquelles il a recours, en rendent la marche lente et la lecture pénible."

The article comprised three sections. The first motivated the research with a review of Cramer's paradox and a discussion of well-known curve properties. Plücker cited Gabriel Cramer as the first geometer to have noticed, in his *Introduction à l'analyse des courbes algébriques* (1750), that a planar curve of degree m was completely determined by $\frac{m+1}{1} \cdot \frac{m+2}{2} - 1$ points, but two planar curves of this degree could intersect in up to m^2 points. When $m > 2$, there was an apparent paradox, since different curves could share as many or more points than the number required to completely determine one of them. For instance, when $m = 3$ two planar curves could intersect in 9 points, which should determine a single curve of degree 3. Following Cramer, Plücker explained this seemingly "surprising" result by remarking that when completely determining a planar curve one always assumed the given points were chosen "by chance and contain no particular common relation." Turning to his own research in the theory of osculating curves and its geometric interpretation, Plücker had met several similar theorems that initially appear rather "singular," but were "very fruitful with beautiful corollaries." Though these theorems had appeared elsewhere (Plücker did not say where precisely), he would reproduce them here in a more developed manner followed by some of their applications.[42]

The second section contained Plücker's two main theorems and their proofs, these were the only theorems labeled and enumerated in the text. Plücker began with an analytic representation of the properties discussed in Section I, showing that if one represented two curves of any degree $m > 2$ as

$$M = 0, M' = 0,$$

then the equation of the same degree

$$\mu M + M' = 0,$$

in which μ is a constant indeterminate coefficient, would express an infinity of other curves of degree m passing through the m^2 points of intersection of M, M'. But if one found an arbitrary new point on one of the curves, one could find μ from the resulting linear equation, and so the curve containing the new point would be completely determined. Plücker continued with this line of reasoning in the case where there were $\frac{m+1}{1} \cdot \frac{m+2}{2} - 2$ given points (one fewer than to completely determine a conic). He then invoked the principle of duality, which extended the result from curves of a given degree and fixed points to corresponding curves of a given class and fixed lines. Following these dual results, Plücker demonstrated his Theorem I in parallel columns.[43]

[42] Plücker would return to Cramer's Paradox in the second volume of his *Analytisch-geometrische Entwicklungen*, published in 1831 (Plücker 1831, 242).

[43] In all theorems in this article, Plücker consistently described both given and deduced points and lines as "fixed," perhaps in order to fix the general equation of the second-order curve.
"THÉORÈME I. Toutes les courbes du $m.^{ieme}$ degré qui passent par les $\frac{m+1}{1} \cdot \frac{m+2}{2} - 2$ mêmes points fixes, se coupent en outre aux $m^2 - (\frac{m+1}{1} \cdot \frac{m+2}{2} - 2)$, autres mêmes points fixes."
"THÉORÈME I. Toutes les courbes du $m.^{ieme}$ classe qui passent par les $\frac{m+1}{1} \cdot \frac{m+2}{2} - 2$ mêmes droites fixes, se coupent en outre aux $m^2 - (\frac{m+1}{1} \cdot \frac{m+2}{2} - 2)$, autres mêmes droites fixes."

THEOREM I. All the curves of mth degree that pass through the same $\frac{m+1}{1} \cdot \frac{m+2}{2} - 2$ fixed points will also meet in $m^2 - (\frac{m+1}{1} \cdot \frac{m+2}{2} - 2)$ other fixed points

THEOREM I. All the curves of mth class that are tangent to the same $\frac{m+1}{1} \cdot \frac{m+2}{2} - 2$ fixed lines, are also tangent to $m^2 - (\frac{m+1}{1} \cdot \frac{m+2}{2} - 2)$ other fixed lines

As an example, and still in parallel columns, Plücker detailed the case of third-degree curves passing through the same eight points and thus meeting in a ninth fixed point. The parallel dual case treated third-degree curves tangent to the same eight lines and thus tangent to a ninth fixed line. Similarly, fourth-degree curves passing through thirteen points would meet in an additional three fixed points, and so on. Returning to a single column of text, Plücker noted that some of the points could be of higher order and so not necessarily unique.

Plücker then continued to consider the situation when the coefficients of the representative equations were not indeterminate, but instead subject to certain conditions, which would serve to lower the number of necessary fixed points for the theorem's hypothesis. Plücker summarized these results in Theorem II, presented in only one column.

> Given n coefficients of the general equation of mth degree in two variables, or given n linear equations among all or part of these coefficients; then all the curves represented by the general equation, so modified and passing by the same $\frac{m+1}{1} \cdot \frac{m+2}{2} - (n+2)$ fixed points, will meet in the same $m^2 - \frac{m+1}{1} \cdot \frac{m+2}{2} + (n+2)$ other fixed points. (ibid, 101)[44]

Plücker concluded his second section with a reminder that the theorem required the inequality, $n > \frac{m+1}{1} \cdot \frac{m+2}{2} - 1$.

In the third and final section, Plücker demonstrated the fruitfulness of Section II, by showing assorted examples of easily deducible curve properties, including Lamé's Theorem. In particular, Plücker focused on consequences of Theorem II, which could be applied to second-degree curves. In this case, given n coefficients or n linear equations containing the coefficients, then all the curves passing through $4 - n$ fixed points would have n additional fixed points in common.

Plücker considered a general equation of a second-degree curve in two indeterminate variables with six coefficients.

$$Ax^2 + By^2 + Cxy + 2Dx + 2Ey + F = 0. \tag{1}$$

Then if F was supposed known, any combination of four coefficients, equations containing coefficients, or points on the curve (e.g., one coefficient and three points) would give an infinite number of curves passing through the same four points. This relationship formed the basis for all Plücker's subsequent investigations in this article.

[44] "Etant donnés n coefficiens de l'équation générale du $m.^{ieme}$ degré à deux indéterminées, ou encore étant données n équations linéaires entre tous ou partie de ces coefficiens ; toutes les courbes représentées par l'équation générale, ainsi modifièe et passant par les $\frac{m+1}{1} \cdot \frac{m+2}{2} - (n+2)$ mêmes points fixes donnes, se couperont en outre aux $m^2 - \frac{m+1}{1} \cdot \frac{m+2}{2} + (n+2)$ autres mêmes points fixes."

Plücker began with the special case of the equilateral hyperbola. In rectangular coordinates, the equation representing an equilateral hyperbola would have coefficients such that $A + B = 0$. Such an equation reduced the number of necessary givens in equation (1). Drawing from this condition, Plücker determined that all equilateral hyperbolas sharing three given points would meet in a fourth point. Since systems of two perpendicular lines could be considered as hyperbolas, Plücker applied the property to show that the three heights of a triangle concur in the same point.[45]

Plücker returned to the general conic. From (1), Plücker concluded that knowing either ratio $\frac{C}{A}$ or $\frac{C}{B}$ would give two conjugate diameters of the curve, where one would be parallel to one of the coordinate axes. Thus all conics with two conjugate diameters parallel to two fixed lines and passing through three fixed points would meet in a fourth fixed point, and conversely.

In particular, the equation of a diameter whose conjugate was parallel to the x-axis would be $By + Cx + E = 0$. So if the ratio $\frac{E}{B}$ were given, then one would know the intersection point of the given diameter with the y-axis. That is, if the y-axis met the curve, then the intersection point would be the midpoint of the chord intercepted. Further, given any coordinate point (a, b) on the diameter's extension, one could conclude that $Bb + Ca + E = 0$. Similarly, if one were given another point (a', b'), now on the extension of the diameter whose conjugate was parallel to the y-axis, then $Aa' + Cb' + D = 0$.

Continuing this investigation with a given line of the form $\alpha x + \beta y + \gamma = 0$, Plücker found that the equation of the diameter whose conjugate was parallel to this given line would be $\alpha(By + Cx + E) = \beta(Ax + Cy + D)$, a linear equation relating A, B, C, D, E. One could then be given one, two, three, or four equations of the same form containing A, B, C, D, E. In the last case, and supposing one of the coefficients as given, the curves would be completely determined except for the last term F. From these considerations, Plücker deduced "immediately" [*sur-le-champ*] that all conics passing through three given points and in which the conjugates of the diameters that were parallel to the same fixed line concurred in the same fixed point, would meet at a fourth point. Further, if any number of conics passed through the same four points, the conjugates of their diameters parallel to the same fixed line would concur in a fixed point.

The latter result, in which we recognize Lamé's theorem, Plücker indeed attributed to Lamé, giving a precise citation of the *Annales* VII, page 229. He suggested that the theorem could be "completed" as follows.

> If the line to which the diameters are parallel turns on one of its points, the point of concurrence of the diameters describes a conic, the geometric locus of the centers of all conics through the four given points (*).

[45] Though not relevant to the remainder of his argument, Plücker's unexpected triangle result provided a striking contrast to Steiner's proof of the same result, merely 70 pages earlier in the same journal.

> If two conics intercept, on the same given line, chords whose midpoints coincide; then the same thing will occur for all the conics which, passing by the four points of intersection of these two, intercept the given line. (ibid, 105)[46]

Finally, Plücker generalized the result for conics passing through $4-n$ given points, which would then have conjugate parallel diameters meeting in n points. Plücker explained that if $n = 4$, then the conics would be similar and concentric such that the given points of intersection would pass to infinity. Plücker's consideration of this special case shows an attention to Poncelet's innovations in the *Traité des propriétés projectives* in which he established the common points at infinity among similar, concentric conics.

Plücker used coordinate equations without digressing into computation. He presented the necessary equations representing conjugate parallel diameters as known, from there the reader only needed to count the number of coefficients and the number of givens in order to reach the desired result. The entire article remained within the system of rectangular coordinate equations. The conic section was represented exactly by (1), and Plücker seemed to use the two terms of reference (conic and equation) interchangeably in his proofs. Specific conics, such as the equilateral hyperbola, were designated with respect to their coordinate relations in (1). Plücker's only mention of constructing the determined conics was to point out that the construction of a fourth point shared by two conics would be very easy and one could then find all the points one desired. Nowhere did he give a constructive procedure and there were no problem solutions, only theorems and proofs. That this was geometry, and not algebra, was emphasized in the names of things, equations represented geometric objects. While equations and not figures were the form of representation, the objects remained conic sections, diameters, parallel lines, hyperbolas, and points of intersection. Just as Steiner's figures were nowhere to be seen in his theorem statements, so, too, Plücker made no mention of his coordinate equations and coefficients when stating theorems.

Though Lamé and Plücker both used coordinate equations to represent geometric objects, we begin to see how the label "analytic geometry" carried dissonant connotations in the context of the two different publications over ten years apart. Lamé would introduce the use of abridged notation in 1818, but in 1817 he relied upon standard coordinate equations to represent curves and surfaces. In Lamé's exposition, the equations became increasingly complex and numerous. While his paper did not include a great deal of calculation, this was because Lamé left the elimination of variables to the reader and simply showed the end result. In part relying upon the technology developed by Lamé, Plücker could use symmetry and the best choice of coordinate axes to simplify his equations. Plücker further avoided calculation by emphasizing that the curves could be determined, without showing how one might use a given equa-

[46] "Si la droite, à laquelle les diamètres sont parallèles, tourne sur l'un quelconque des points de sa direction, le point de concours des conjugués de ces diamètres décrira une conique, lieu géométrique des centres de toutes les coniques passant par les quatre points donnés (*)."
"Si deux coniques sont telles qu'elles interceptent, sur une même droite donnée, des cordes dont les milieux coïncident ; la même chose aura lieu pour toutes les coniques qui, passant par les quatre points d'intersection de ces deux là, couperont la droite donnée (*)."
The asterisk will be explained in Sect. 3.4.

tion to find the exact coefficients. These specific, computationally derived, results were not Plücker's aim. With these techniques, his form of analytic geometry without calculation appeared more frequently in this article than it had in his earlier research.

As pointed out in Cournot's 1828 review, Plücker often progressed toward already known results, rather than making new discoveries (Cournot 1828, 178). However, his approach to these known results incorporated recent developments that had not yet been represented through coordinate equations. Gergonne had introduced dual columns that showed corresponding geometric results side-by-side in 1824, but they had been used almost exclusively in non-coordinate geometry since that time. In this article, Plücker used dual columns to state his theorems and showed how to apply polar reciprocity in his final example, where he provided an equation that represented when one point with coordinates (a, b) lay on the polar of another point (a', b') with respect to the second-order curve (1).

$$Aaa' + Bbb' + C(ab' + ba') + D(a + a') + B(b + b') + F = 0$$

Plücker deduced from the symmetry of equation that the result was reciprocal—the point (a', b') lay on the polar of the point (a, b).

Plücker further incorporated new geometric practices with his use of common points at infinity shared by any concentric conics. Poncelet had originally motivated this choice of terminology to provide an interpretation within "pure geometry" of the imaginary points from "analytic geometry," where one could in general represent points of intersection for any two second-degree equations. Plücker did not provide a coordinate representation for these points, although the step of sending points or lines to infinity was not strictly constructive either. Within research articles, it was becoming a common procedure, even in articles that Gergonne designated as "elementary geometry" [such as in Steiner (1828c)]. Plücker demonstrated a strong familiarity with contemporary research, and an eagerness to expand the domain of analytic geometry. With the new geometric objects developed by Poncelet and the new abridged notation, the practice of using coordinate equations to solve geometry problems was evolving. His review in the *Bulletin* (two pages after Steiner's) credited Plücker with demonstrating, without any sort of calculation, a multitude of properties of second-order lines (Anonymous 1828).

3.4 Gergonne's footnote, citation, and text

The asterisk in Plücker's declared completion of Lamé's theorem (Sect. 3.3) referred the reader to a footnote by Gergonne who succinctly stated that "this is precisely what was demonstrated on page 106 of the preceding volume." Gergonne's note was perhaps not *precisely* Plücker's statement, but certainly a special case. There, Gergonne had given a proof that the conjugates of the parallel diameters of all the ellipses circumscribed to the same quadrilateral concur in a point and this point was constantly on the perimeter of the hyperbola which was the locus of centers of the ellipses (Gergonne 1827). The first part of the result was attributed to Lamé, although a different text than Plücker's citation. As reference in an accompanying footnote, Gergonne pointed to "a very elegant proof of this proposition, as well as many other interesting things, in a little work of Lamé, *Examen des différentes méthodes, etc.*" (Gergonne 1827, 106).[47]

Gergonne, although proving the extended version of this theorem, attributed the original statement to an article by Steiner in Crelle's *Journal* from 1827 written in response to a question posed by the geometer Étienne Bobillier in the *Annales* on finding an ellipse closest to a circle given certain constraining conditions (Steiner 1827b).[48] Despite admiring what he described as Steiner's elegant theorem, Gergonne expressed dissatisfaction with the proof, claiming that it too often relied upon what Steiner described as "generally known" results, while they were not well known at all. Gergonne did not attempt to reconstruct Steiner's proof, instead taking the material in his own direction. While Steiner's proof had been constructive and figure-based, Gergonne's proof relied upon calculations with coordinate equations and did not include descriptions of figures. Accordingly, Gergonne did not consider this a translation or abridgment of Steiner, but claimed it as an independent result. Like Plücker, Gergonne began with the same general form of a conic, Eq. (1). Unlike Plücker, he then introduced many other variables to serve in algebraic, trigonometric, and differential calculations, which were eventually simplified to reach the desired conclusions. The proof only applied to ellipses and resulting hyperbola loci, not a general second-order curve. Further, Gergonne made no comment on the particularity of his version nor whether one might be able to extend the results to other conic sections.

Readers of the *Annales*, and certainly Steiner among them, might have known that Steiner had already given another version of Lamé's Theorem in an earlier context. On the one hand, Lamé's theorem and its proof was known, as the common citation in all three articles demonstrates. On the other hand, Steiner's extreme brevity (his article runs less than two full pages in Crelle's *Journal*–Gergonne's analytic proof of the same results goes for ten), certainly demanded a great deal of background knowledge on the part of his reader. However, we suggest that Gergonne's proof in the *Annales* was also motivated by an opportunity to advertise his prowess in analytic geometry, as he had done earlier with the Apollonius problem. Steiner himself made no reference to this earlier result when returning to Lamé's theorem the following year, and as we have seen his statement of the theorem took quite a different form without conics or conjugate diameters in this later version, and it is only through Gergonne's reference that we are able to identify the two results as following from the same source, further attesting to the popularity of proving Lamé's theorem.

3.5 Conclusions

That Lamé's theorem was repeatedly re-proven was not a statement against the validity of Lamé's original proof. Then why prove Lamé's theorem again and again? Certainly,

[47] "[...] une démonstration fort élégante de cette proposition, ainsi que beaucoup d'autres choses intéressantes, dans un petit ouvrage de M. LAMÉ, ayant pour titre : *Examen des différentes méthodes, etc.*"

[48] This example of cross-pollination between journals shows the overlapping audience between the publications from Berlin and Montpellier in the late 1820s.

none of these texts aimed at a comprehensive or systematic survey of properties of conic sections. Steiner's collection of theorems had little overarching design. Similarly, Plücker described his own applications as a chance assortment of possible results. Both geometers hinted at selecting examples from a vast wealth of other similar results. With Lamé's theorem, either geometer might have simply cited the well-known result, but chose instead to use the opportunity to show off their particular methods. Yet, neither geometer attempted a self-contained exposition. Steiner pointed the reader vaguely to Carnot for the rationale behind finding a point of determinate ratio. Even less explicitly, Plücker introduced several equations with the preface "one knows," while his use of coordinate equations assumed a familiarity with the second-degree equation for conic sections, a topic common in textbooks on analytic geometry or algebra applied to geometry.[49]

Although Plücker and later Steiner promoted new methods over new results, the resulting theorem statements in their articles obscured the derivation and innovation. Alongside the association of projective techniques with modern geometry, they described their methods as "elementary," thus further connecting their texts, if not to Euclid, then at least to Apollonius. Certainly, the problems and theorems about planar geometric objects seemed to adhere to this tradition. Steiner and Plücker stated their theorems without revealing their respective methods. In his proofs and figures, Steiner defined each point and line with respect to a given conic section. However, this conic section remained unmentioned in the subsequent theorem. Since three non-collinear points uniquely determined a circle, there was no need to explicitly describe it. Plücker stated that the representation of geometric objects with coordinate equations was well known, but no coordinate equations appeared in his theorems. Instead, "all conics" implied the general second-degree equation in two variables with six coefficients that Plücker had introduced earlier in his article.[50] For both geometers, the method could not be inferred from the individual results.

In 1832 Steiner published his first book, *Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von Einander, mit Berücksichtigung der Arbeiten alter und neuer Geometer über Porismen, Projections-Methoden, Geometrie der Lage, Transversalen, Dualität und Reciprocität, etc.*, where he developed a public formal statement of his personal geometric method (Steiner 1832). However, even in this context, Steiner did not exclude or denigrate other geometric methods. To the contrary, in prefacing a list of posed problems and theorems for his readers, he offered the option of employing and practice his method or of following another method instead (Steiner 1832, 439). In this later text, Steiner differentiated a specific type of pure intuition in geometry. In particular, he claimed that stereometry and three-dimensional geometry could be better understood without means to make them graspable by the senses [*Versinnlichungsmittel*], such as figures or equations. Instead, one could correctly perceive these constructions [*Gebilde*] only through the use of

49 To take one example among many, in Lacroix's popular *Traité élémentaire de trigonométrie rectiligne et sphérique, et d'application de l'algèbre à la géométrie* he demonstrated that $Ay^2 + Bxy + Cx^2 + Dy + Ex = F$ is the general form of the equation for all conic sections, drawing upon results proved in elementary geometry and algebra (Lacroix 1807, 154–172).

50 In these articles, neither geometer addressed the degenerate conic composed of two straight lines.

inner imagination [*innere Vorstellungskraft*] (ibid, 306). This statement from Steiner marks a departure from the kind of geometry employed in his planar solution to the Apollonius problem and proof of Lamé's Theorem, in which he relied on figures to introduce constructions and support textual explanations.

Steiner's desire for systematicity, unity, and intuition was not immediately apparent in these shorter articles, many of which Steiner presented as excerpts from his longer work. Steiner cited these virtues in his correspondence and unpublished work from 1826 onward [excerpted in Lange (1899)], but these claims did not manifest publicly until his first monograph.[51] Like his article on the Apollonius problem, Steiner's monograph began with definitions built up from relationships between coplanar points and lines. This feature was absent in many of his other articles in the *Journal* and the *Annales*. Steiner's earliest publications were in part aimed at establishing priority and independence in his research domain. In the context of circle relations and the Apollonius problem, Steiner was able to achieve both a thorough, systematic exposition in an article format. However, not only Steiner, but also Gergonne (1817b) and Poncelet (1817), often previewed their newest research on conic sections by merely listing results or providing solutions without proof. While his specific results could appear disjointed and even unexpected, Steiner suggested a uniform and followable form of geometry in his overarching pattern of proofs, constructions, and definitions. In his proofs, Steiner began with what he wanted to show and worked backward to initial conditions, a strategy that could be used to extend a particular result to further cases. Similarly, in his research on tangent circles, Steiner methodically extended his definitions from points, to two circles, to sets of circles in the plane, in space, or on the surface of a sphere. As discussed above, Steiner's argument structure was perhaps more imitable in his mode of generalization through projection, perspective, reciprocity, and points at infinity. In this way, Steiner demonstrated an applicable method of discovery and achieved some systematic unity at this structural level that was sometimes obscured by the proliferation of new results.

Plücker's methodological position emerged more powerfully in his *Analytisch-geometrische Entwicklungen* (Plücker 1828a). In the preface, he described his method as "new" and "pure analytic geometry" such as that of Gaspard Monge. Plücker drew attention to his method in order to differentiate his results from those recently announced by Steiner in the first two volumes of Crelle's *Journal* and pointed out that Steiner had merely stated his findings, often without proof, so seemed to be

[51] Steiner's lifelong dedication to these higher principles is attested by Jacobi's 1845 letter of recommendation for Steiner to receive a full professorship, in which Jacobi summarized Steiner's contributions over the past twenty years. Jacobi described *Systematische Entwicklung* as a holistic and exemplary text for all of mathematics.

> Thus he revealed the *Organismus* through which diverse manifestations in space are interconnected, he has not only promoted geometric synthesis, but also established a model of a perfect method and execution for all other branches of mathematics. (Jahnke 1903, 278)
>
> Indem er so den Organismus aufdeckte, durch welchen die verschiedenartigsten Erscheinungen in der Raumwelt miteinander verbunden sind, hat er nicht bloss die geometrische Synthese gefördert, sondern auch für alle anderen Zweige der Mathematik ein Muster einer vollkommen Methode und Durchführung aufgestellt.

"following in the footsteps of Poncelet." Yet, as suggested by Crelle's letter of recommendation described above, Plücker's preface was not necessarily indicative of his monograph's contents, the first volume of which Plücker had written before reading the work of either Poncelet or Steiner. Moreover, we suspect that Plücker may have directly belittled Steiner's recent publications as derivative in order to further contrast the independence of his own work.

In Plücker's form of analytic geometry examined here, he began with a general initial theorem pertaining to curves of any degree. He then proceeded to focus on the more particular case of second-degree curves. His arguments were clearly introduced and his use of coordinate equations was straightforward, but did not suggest applicability beyond variations on the same theme of four given conditions to determine a set of conic sections because the number of variables was so crucial to the argument. For Plücker the use of coordinate equations to represent geometric objects opened a path toward simplification for analytic geometry, a contribution that in his view outweighed discovering particular theorems or problems. Plücker remained vocally committed to his analytic geometry throughout his career. In a letter to Poncelet from 1835, he explained his recent findings on the relationship between curves of any order and their duals:

> I have still not decided if I will print the general results that I have reached all together or in a memoir, in German or in French. Regardless, I still must make it analytic geometry. (Plücker 1835a)[52]

His subsequent publication of these results in 1835 (summarized in French in Crelle's *Journal* and proved in German in his monograph *System der analytischen Geometrie*) employed homogeneous coordinates, exhibiting a form of analytic geometry only introduced five years prior and thus beyond the scope of Monge's publications (Plücker 1834, 1835b).

By the end of their careers, Steiner and Plücker had become emblematic of the competing methods of analysis and synthesis. Contrary to these stereotypes, neither exhibited an orthodoxy of method in these early articles. Further, as responses to their publications show, the line between pure and analytic geometry was but faintly drawn and could be characterized neither in terms of calculation nor by the presence of figures. Both Steiner and Plücker used some calculation and both geometers provided constructions based in figure manipulation (in Plücker's monograph and his articles in Crelle's *Journal*, there are illustrated figures too). The tools of figures and calculation were at times complementary and certainly served different purposes, one could not replace the other. For Steiner and Plücker, a proof using algebraic computation of coordinate equations without translation into figures would not qualify as a geometric solution. Analogously, while the common (ideal) chord between two non-intersecting conics could be easily portrayed in a linear equation, drawing such an object defied common sense (and in part explains why the less figurative term "radical axis" was often preferable). As we can see by comparison with the contemporary geometries of

52 "Je ne suis pas encore décidé si je ferai imprimer les résultats généraux auxquels je suis parvenu, le tout ensemble ou par mémoire, en allemand ou en français. Malgré moi je dois encore faire de la géométrie analytique."

Gergonne, Poncelet, and Lamé, as well as the wider reception and attempted classifications of their work, it does not make sense to speak only of synthetic geometry for Steiner and analytic for Plücker. Their ways of doing geometry were peculiar to them.

Within constructive, planar geometry, evolving techniques allowed addressing the same problem from numerous different approaches. Geometers emphasized the differences between their methods in part to compensate for the similarity of their results when describing their work, or when their work was described by others. In 1839, Crelle wrote a letter of support for Plücker to the ministry of culture, in which various styles of synthetic geometry were lumped together in order to differentiate Plücker's contributions.

> The analytical opposed, or rather next to its ruling synthetic method, by which Poncelet, Mobius, etc. and especially Steiner have accomplished so much that is so admirable, can succeed as well in all results, however the further development of mathematics occurs almost more with the improvement of method than the results; because the methods are the tools to promote still further new results; and to the improvement of the analytic method in geometry, in my opinion, Herr Plücker has in turn acquired meaningful and significant worth by his earlier and now the present writings. (Ernst 1933, 31)[53]

From a different perspective, we have seen how French mathematicians associated Steiner and Plücker as German geometers. As Steiner was of Swiss origin and Germany as a nation had yet to exist, this designation appears to be derived from their common language. Similarly, Crelle's *Journal*, published in Berlin, was considered a German publication. The association with German scholars and "complication," as mentioned in Cournot's review of Plücker (Sect. 2.6), may have also had its origins in linguistic differences more than mathematical ones. As "foreigners," the French reception of both geometers emerged through the influential filter of Gergonne and his journal. Gergonne appeared to respect and value the contributions of Steiner and Plücker, but, as comparisons between manuscripts and publications reveal, he exercised strong editorial intervention. Whether neutral or critical, this label imposed by their French colleagues may have further prompted Steiner and Plücker to distance themselves from each other.

This same muting of idiosyncrasies between Poncelet and Steiner as synthesists or between Gergonne and Plücker as analysts, each of whom saw the other's work as very different, could be perceived as an apparent dichotomy. The image of geometry that appeared in the late nineteenth century portrayed their predecessors as separated and ultimately calcified by a spurious methodological divide. In support of this image, this next generation of geometers, especially those who wrote biographies and histories

[53] "Auch die der analytischen gegenüberstehende, oder vielmehr neben ihr bestehende synthetische Methode, mit welcher Poncelet, Möbius usw. und besonders Steiner so Vieles und Bewunderungswertes geleistet haben, mag zu allen Resultaten ebenfalls gelangen können, indessen kommt es bei der weiteren Entwicklung der Mathematik fast noch mehr als auf die Resultate auf die Vervollkommnung der Methoden an; denn die Methoden sind die Werkzeuge, um immer noch weitere neue Resultate zutage zu fördern; und um der Vervollkommnung der analytischen Methode in der Geometrie hat sich nach meiner überzeugung Herr Plücker seinerseits schon durch seine früheren Schriften und jetzt wieder durch die vorliegende, ein wesentliches und bedeutendes Verdienst erworben."

like Alfred Clebsch, Gaston Darboux, Felix Klein, and Gino Loria, cited the frequent methodological claims made by their predecessors. Yet, as we have seen, the methodological claims did not always coincide with the qualities of the particular geometer or of the recurring content.

Just as the choice of a problem or theorem did not limit what geometric method could be applied, so the designation of one or another geometric method did not limit the author's particular approach. Rather than static and opposing methods, Steiner and Plücker respectively allied their work to evolving practices that changed in response to their contributions.

While Klein pointed to the opposition between Steiner and Plücker as mirroring the opposition between synthetic and analytic methods, his personal description of each geometer belied his classifications of methods that we considered in our introduction. Klein provided a nuanced overview of Plücker's geometric method dominated by his skillful "internal intuition" [*inneren Anschauung*] (Klein 1926, 122). To Klein, Plücker appeared as visually focused on the "true geometric image of forms" [*die wahre geometrische Gestalt des Gebildes*], rather than calculations (ibid, 125). In summarizing Steiner's contributions, Klein noted his systematic treatment of geometry and peculiar "art of instruction." As an example, Klein relayed how Steiner's devotion to the Socratic method extended so far as to not include figures in his geometry lessons. Instead,

> [...] the active thinking of the listener should produce such a clear picture in his mind that he could dispense with the sensory evidence. (ibid, 128)[54]

So, while Plücker, the analytic geometer, emphasized the visual and intuitive properties of geometry, Steiner, the synthetic geometer, excelled at systematizing but often avoided graphical representations.

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[54] "[...] das lebendige Mitdenken des Hörers sollte ein so deutliches Bild in seiner Vorstellung erzeugen, dass er das sinnlich Angeschaute entbehren könnte."

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