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Author(s): Naoki Osada
Source: *Archive for History of Exact Sciences*, Vol. 67, No. 4 (July 2013), pp. 457-476
Published by: Springer
Stable URL: https://www.jstor.org/stable/23479271
Accessed: 19-05-2020 11:58 UTC

Arch. Hist. Exact Sci. (2013) 67:457–476
DOI 10.1007/s00407-013-0117-1

Isaac Newton's 'Of Quadrature by Ordinates'

Naoki Osada

Received: 28 November 2012 / Published online: 6 April 2013

Abstract In *Of Quadrature by Ordinates* (1695), Isaac Newton tried two methods for obtaining the Newton–Cotes formulae. The first method is extrapolation and the second one is the method of undetermined coefficients using the quadrature of monomials. The first method provides n-ordinate Newton–Cotes formulae only for cases in which $n = 3, 4$ and 5. However this method provides another important formulae if the ratios of errors are corrected. It is proved that the second method is correct and provides the Newton–Cotes formulae. Present significance of each of the methods is given.

1 Introduction

Isaac Newton proposed the principle of numerical integration by polynomial interpolation in Cor of Lemma V of Book III of his *Principia* (1687).

> Hence the areas of all curves may be nearly found; for if some number of points of the curve to be squared are found, and a parabola be supposed to be drawn through those points, the area of this parabola will be nearly the same with the area of the curvilinear figure proposed to be squared: but the parabola can be always squared geometrically by methods generally known (Cajori 1947, p. 500)

Communicated by: Niccolò Guicciardini.

N. Osada (✉)
Tokyo Woman's Christian University, 2-6-1, Zempukuji,
Suginami, Tokyo 167-8585, Japan
e-mail: osada@cis.twcu.ac.jp

Numerical integration formulae based on this idea of Newton's are called of interpolatory type. As an example, Newton gave the formula for four equidistant ordinates in Scholium of Prop VI of his *Methodus Differentialis* (1711) (Newton 1711).

> If there are four ordinates at equal intervals, let A be the sum of the first and fourth, B the sum of the second and third, and R the interval between the first and fourth, then the central ordinate will be $\frac{9B-A}{16}$, and the area between the first and fourth ordinates will be $\frac{A+3B}{8}R$ (Fraser 1927).

This formula is so-called Simpson 3/8 rule. Roger Cotes published similar rules for the areas included between the curves of equidistant ordinates up to eleven ordinates in his posthumous *Harmonia Mensurarum* (1722). Therefore, numerical integration formulae by interpolation with equidistant ordinates are called the Newton–Cotes formulae or the Newton–Cotes rules.

The numerical integration formula of interpolatory type is usually applied not to the whole interval of integration, but to m equal subintervals into which the interval is divided. The full integral is approximated by the sum of the approximations to the integrals on the subintervals. This method is called the m-panel composite formula.

In the manuscript *Of Quadrature by Ordinates* (1695), which is abbreviated here to *Of Quadrature*, Newton tried two methods to obtain Newton–Cotes formulae. The first method is extrapolation and the second is the method of undetermined coefficients using the quadrature of monomials.

The first method provides n-ordinate Newton–Cotes formulae only for cases in which $n = 3, 4$ and 5. In using the first method, it is necessary to know the ratio of errors of the single Newton–Cotes formula to the corresponding composite formula, but Newton's ratios of errors were incorrect. D.T. Whiteside called this first method a "Newton's clever (but not wholly exact) approach to achieving the Cotesian formulas" (Whiteside 1976, p. xliv).

The second method has been considered as mere check of the formula. Whiteside noted "If Newton had gone on similarly to check the accuracy of the area-approximation in his preceding 'Cas. 5', he would (see note (12) above) have had a shock!" (Whiteside 1976, p. 698).

In this paper, we study Newton's two methods for obtaining the Newton–Cotes formulae and re-evaluate the manuscript *Of Quadrature*.

Since Newton used the theorem of Huygens, i.e., the first step of the Richardson extrapolation process, and the ratios of errors of the single to composite Newton–Cotes formulae, we will survey the theorem of Huygens in Sect. 2 and the error terms of the composite Newton–Cotes formulae in Sect. 3.

In Sects. 4 and 7, we will annotate *Of Quadrature* and will quote the full-text of the *Of Quadrature* from Whiteside's translation (Whiteside 1976). We will also include Newton's original Latin in footnotes. In Sect. 5, we will deal with Newton's incorrect ratios of single to composite Newton–Cotes formulae. In Sect. 6, we will describe the modern significance of Newton's numerical integration formulae by extrapolation. In Sect. 7, we will prove that the Newton–Cotes formulae are determined by the second method.

2 The theorem of Huygens

2.1 *De circuli magnitudine inventa* by C. Huygens

In *De circuli magnitudine inventa* (Huygens 1654), Christiaan Huygens proved Theorem XVI:

> Any arc less than a semicircle is greater than its subtense together with one third of the difference, by which the subtense exceeds the sine, and less than the subtense together with a quantity, which is to the said third, as four times the subtense added to the sine is to twice the subtense with three times the sine (Milne 1903)

Theorem XVI means (Rudio 1892, p. 40)

$$s' + \frac{1}{3}(s' - s) < a < s' + \frac{1}{3}(s' - s)\frac{4s' + s}{2s' + 3s}, \tag{1}$$

where a is the length of any arc $\widehat{\mathrm{AB}}$, s is the sine AM, and s' is the subtense(chord) AB in Fig. 1.

Let T_n be the circumference of n-sided inscribed regular polygon in a circle with diameter 1. The inequality (1) implies

$$T_{2n} + \frac{1}{3}(T_{2n} - T_n) < \pi < T_{2n} + \frac{1}{3}(T_{2n} - T_n)\frac{4T_{2n} + T_n}{2T_{2n} + 3T_n}. \tag{2}$$

By the Taylor expansion, we have

$$T_n = n \sin\frac{\pi}{n} = \pi\left(1 - \frac{\pi^2}{6n^2} + \frac{\pi^4}{120n^4} + O\left(\frac{1}{n^6}\right)\right). \tag{3}$$

It follows from (3) that

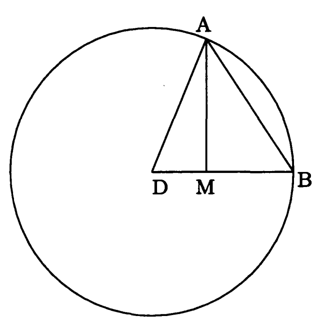


Fig. 1 Theorem XVI in *De circuli magnitudine inventa* by Christiaan Huygens

$$T_{2n} + \frac{1}{3}(T_{2n} - T_n) = \pi\left(1 - \frac{\pi^4}{480n^4} + O\left(\frac{1}{n^6}\right)\right),$$

$$T_{2n} + \frac{1}{3}(T_{2n} - T_n)\frac{4T_{2n} + T_n}{2T_{2n} + 3T_n} = \pi\left(1 + \frac{\pi^6}{22400n^6} + O\left(\frac{1}{n^8}\right)\right).$$

The lower bounds of (1) and (2) are the first step of the Richardson extrapolation which was called the theorem of Huygens by Newton. For details, see Osada (2012).

2.2 Newton's application of the theorem of Huygens

Newton referred the theorem of Huygens in the letter to Michael Dary on 22 January 1675 (Turnbull 1959, p. 333; Whiteside 1971, p. 662). In this letter, Newton applied the theorem of Huygens to a construction of the length of the arc of an ellipse. Newton wrote

> This is derived from Hugenius' Quadrature of ye Circle, and I believe approaches ye Ellipsis as near as his doth the Circle (Turnbull 1959)

Newton also mentioned the theorem of Huygens in the letter *epistola prior* to Henry Oldenburg on 13 June 1676 (Turnbull 1960, pp. 39–40; Whiteside 1971, p. 669). Let A be the chord of an arc, B the chord of half the arc. Let z be the length of the arc, and r be the radius of the circle. Then Newton derived

$$A = z - \frac{z^3}{24r^2} + \frac{z^5}{1920r^4} - \&c,$$

$$B = \frac{z}{2} - \frac{z^3}{192r^2} + \frac{z^5}{61440r^4} - \&c.$$

Newton eliminated the terms z^3/r^2 in A, B, and obtained

$$\frac{8B - A}{3} = z - \frac{z^5}{7680r^4} + \&c.$$

Newton stated:

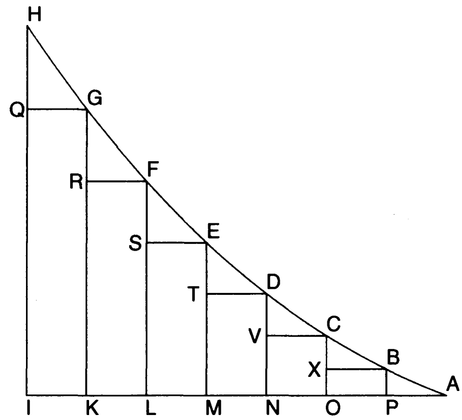
> that is, $\frac{1}{3}(8B - A) = z$, with an error of only $z^5/7680r^4$ in excess; which is the theorem of Huygens (Turnbull 1960).

3 Error terms in the Newton–Cotes formulae

3.1 The trapezoidal rule of James Gregory

James Gregory discussed numerical integration in his *Exercitationes geometricae* (1668) (Gregory 1668). As in Fig. 2, he divided the equally line segment AI into minimum parts. If the second differences of ordinates BP, CO, ..., HI are equal to Z, Gregory stated:

Fig. 2 *Fig. 8* in *Exercitationes Geometricae* by James Gregory



$$GHQ = \frac{HQ \times GQ}{2} - \frac{Z \times GQ}{12}. \tag{4}$$

By adding the square $GQIK$ to both sides of (4), we have

$$GHIK = \frac{(HI{+}GK) \times GQ}{2} - \frac{Z \times GQ}{12}, \tag{5}$$

which is the special case of the error term of the trapezoidal rule.

Whiteside (1961, p. 249) pointed out that (5) implies the Simpson rule as follows:

$$\begin{aligned} FHIL &= FGKL + GHIK \\ &= \frac{1}{2}(FL + GK)FR + \frac{1}{2}(GK + HI)GQ - \frac{1}{6}(FL - 2GK + HI)GQ \\ &= \frac{1}{3}(FL + 4GK + HI)GQ = \frac{1}{6}(FL + 4GK + HI)LI. \end{aligned}$$

3.2 Composite Newton–Cotes formulae

In the remainder of this paper, we shall use the following notation and well-known propositions and their corollaries. See, for example, Davis and Rabinowitz (1984). Let $f(x)$ be a continuous function on $[a, b]$. Put $I = \int_a^b f(x)dx$, and let n denote the number of interpolation points or that of ordinates.

Proposition 1 ($n = 2$) *Suppose $f(x)$ is of class C^2 on $[a, b]$.*

(i) *The trapezoidal rule.*

$$T_1 = \frac{b-a}{2}\,(f(a) + f(b)), \quad I = T_1 - \frac{(b-a)^3}{12} f''(\xi), \quad a < \xi < b.$$

(ii) *The m-panel composite trapezoidal rule. Let* $h = (b-a)/m, x_i = a + ih$ $(i = 0, \ldots, m)$.

$$T_m = \frac{h}{2}\left(f(x_0) + 2\sum_{i=1}^{m-1} f(x_i) + f(x_m)\right),$$
$$I = T_m - \frac{(b-a)^3}{12m^2} f''(\xi), \quad a < \xi < b.$$

Corollary 1 *Under the conditions of Proposition* 1, *the ratio of errors is as follows:*

$$T_1 - I : T_m - I \approx m^2 : 1.$$

Proposition 2 $(n = 3)$ *Suppose* $f(x)$ *is of class* C^4 *on* $[a, b]$.

(i) *The Simpson rule.*

$$S_1 = \frac{b-a}{6}(f(a) + 4f(a + (b-a)/2) + f(b)),$$
$$I = S_1 - \frac{(b-a)^5}{2880} f^{(4)}(\xi), \quad a < \xi < b.$$

(ii) *The m-panel composite Simpson rule. Let* $h = (b-a)/(2m), x_i = a + ih$ $(i = 0, \ldots, 2m)$.

$$S_m = \frac{h}{3}\left(f(x_0) + 4\sum_{i=1}^{m} f(x_{2i-1}) + 2\sum_{i=1}^{m-1} f(x_{2i}) + f(x_{2m})\right),$$
$$I = S_m - \frac{(b-a)^5}{2880m^4} f^{(4)}(\xi), \quad a < \xi < b.$$

Corollary 2 *Under the conditions of Proposition* 2, *the ratio of errors is as follows:*

$$S_1 - I : S_m - I \approx m^4 : 1.$$

Proposition 3 $(n = 4)$ *Suppose* $f(x)$ *is of class* C^4 *on* $[a, b]$.

(i) *The Simpson 3/8 rule.*

$$N_1 = \frac{b-a}{8}(f(a) + 3f(a + (b-a)/3) + 3f(a + 2(b-a)/3) + f(b)),$$
$$I = N_1 - \frac{(b-a)^5}{6480} f^{(4)}(\xi), \quad a < \xi < b.$$

(ii) *The m-panel composite Simpson 3/8 rule. Let* $h = (b-a)/(3m)$, $x_i = a + ih$ $(i = 0, \dots, 3m)$.

$$N_m = \frac{3h}{8}\left(f(x_0)+3\sum_{i=1}^{m} f(x_{3i-2})+3\sum_{i=1}^{m} f(x_{3i-1})+2\sum_{i=1}^{m-1} f(x_{3i}) + f(x_{3m})\right),$$

$$I = N_m - \frac{(b-a)^5}{6480m^4} f^{(4)}(\xi), \quad a < \xi < b.$$

Corollary 3 *Under the conditions of Proposition* 3, *the ratio of errors is as follows:*

$$N_1 - I : N_m - I \approx m^4 : 1.$$

Proposition 4 $(n = 5)$ *Suppose* $f(x)$ *is of class* C^6 *on* $[a, b]$.

(i) *The Boole rule.*

$$\begin{aligned} B_1 &= \frac{b-a}{90}(7f(a) + 32f(a+(b-a)/4) + 12f(a+(b-a)/2) \\ &\quad +32f(a+3(b-a)/4) + 7f(b)) \\ I &= B_1 - \frac{(b-a)^7}{1935360} f^{(6)}(\xi), \quad a < \xi < b. \end{aligned}$$

(ii) *The m-panel composite Boole rule. Let* $h = (b-a)/(4m), x_i = a + ih$ $(i = 0, \dots, 4m)$.

$$\begin{aligned} B_m &= \frac{2h}{45}\left(7f(x_0) + 32\sum_{i=1}^{m} f(x_{4i-3}) + 12\sum_{i=1}^{m} f(x_{4i-2}) + 32\sum_{i=1}^{m} f(x_{4i-1})\right. \\ &\quad \left. +14\sum_{i=1}^{m-1} f(x_{4i}) + 7f(x_{4m})\right), \\ I &= B_m - \frac{(b-a)^7}{1935360m^6} f^{(6)}(\xi), \quad a < \xi < b. \end{aligned}$$

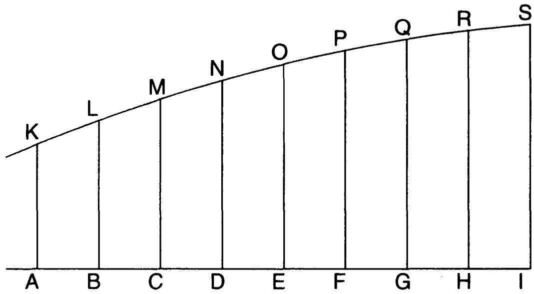
Corollary 4 *Under the conditions of Proposition 4, the ratio of errors is as follows:*

$$B_1 - I : B_m - I \approx m^6 : 1.$$

4 The first method of obtaining the Newton–Cotes formulae

In the manuscript *Of Quadrature by Ordinates*, Newton gave two methods for obtaining Newton–Cotes formulae. The first method is extrapolation and the second one is the method of undetermined coefficients using the quadrature of monomials. We study the former in this section and the latter in Sect. 7 (Fig. 3).

Fig. 3 The figure in *Of Quadrature by Ordinates* (Whiteside (1976), p. 690)



If upon the base A at equal distances be erected ordinates to any Curve AK, BL, CM &c the Curve may by y^{e} Ordinates be squared *quamproxime* as follows.[1]

Case 1 If there be given but two ordinates AK and BL, make the area $(AKLB) = \frac{1}{2}(AK + BL)AB$.

Case 2 If there be given three AK, BL and CM, say that

$$\frac{1}{2}(AK + CM)AC = \Box(AM) \tag{6}$$

and again, by Case 1,

$$\begin{aligned}\frac{1}{2}\left(\frac{1}{2}(AK + BL) + \frac{1}{2}(BL + CM)\right)AC &= \frac{1}{4}(AK + 2BL + CM)AC \\ &= \Box(AM),\end{aligned} \tag{7}$$

and that the error in the former solution is to the error in the latter as AC^2 to AB^2 or 4 to 1, and hence, the difference $\frac{1}{4}(AK - 2BL + CM)AC$ of the solutions is to the error in the latter as 3 to 1, and the error in the latter will be

$$\frac{1}{12}(AK - 2BL + CM)AC. \tag{8}$$

Take away this error and the latter solution will come to be

$$\frac{1}{6}(AK + 4BL + CM)AC = \Box(AM), \tag{9}$$

the solution required.[2]

[1] Newton did not translate the first sentence into Latin.

[2] *Cas. 1.* Si dentur duæ tantū ordinatæ AK, BL fac aream

$$AKLB = \frac{AK + BL}{2}AB$$

Newton gave the single trapezoidal rule in Case 1. In Case 2, Newton applied the single trapezoidal rule (6) and 2-panel composite trapezoidal rule (7) on $I = \text{area}(AKMC)$. Let $T_1 = \frac{1}{2}(AK + CM)AC$ and $T_2 = \frac{1}{4}(AK + 2BL + CM)AC$.

Newton used Corollary 1 which follows from Gregory's (5). Suppose $Z = CM - 2BL + AK = DN - 2CM + BL = EO - 2DN + CM$, we have $4Z = EO - 2CM + AK$. Then

$$T_1 - I = \frac{1}{2}(AK+CM)AC - \square(AM) = \frac{1}{12}(EO - 2CM + AK) \cdot AC = \frac{1}{3}Z \cdot AC.$$

On the other hand,

$$T_2 - I = \frac{1}{4}(AK + 2BL + CM)AC - \square(AM) = \frac{1}{12}Z(AB + BC) = \frac{1}{12}Z \cdot AC.$$

Therefore,

$$T_1 - I : T_2 - I = 4 : 1. \tag{10}$$

For $m > 2$, we can derive similarly that the error in the single trapezoidal rule is to the error in the m-panel composite trapezoidal rule as m^2 to 1.

Newton derived $T_1 - T_2 : T_2 - I = 3 : 1$ from (10). Since $T_1 - T_2 = \frac{1}{4}(AK - 2BL + CM)AC$, Newton obtained

$$T_2 - I = \frac{1}{12}(AK - 2BL + CM)AC.$$

Therefore,

$$T_2 - \frac{1}{12}(AK - 2BL + CM)AC = \frac{1}{6}(AK + 4BL + CM)AC,$$

which is the Simpson rule.

Newton derived $S_1 = \frac{1}{6}(AK + 4BL + CM)AC$ from

$$S_1 = T_2 - \frac{1}{3}(T_1 - T_2), \tag{11}$$

Footnote 2 continued

Cas. 2. Si dentur tres AK, BL, CM, dic $\frac{AK + CM}{2}AC = \square AM$, et rursus $\frac{AK + BL}{4} + \frac{BL + CM}{4}$ in $AC = AK + 2BL + CM$ in $\frac{1}{4}AC = \square AM$ (per Cas. 1) et errorem solutionis prioris esse ad errorem solutionis posterioris ut AC^q ad AB^q seu 4 ad 1 adeoque solutionum differentiam $\frac{AK - 2BL + CM}{4}AC$ esse ad errorem posterioris ut 3 ad 1, et error posterioris erit $\frac{AK - 2BL + CM}{12}AC$. Aufer hunc errorem et solutio posterior evadet

$$\frac{AK + 4BL + CM}{6}AC = \square AM. \text{ Solutio quæsita.}$$

which is the theorem of Huygens, i.e., the first step of the Richardson extrapolation process.

Newton continued:

Case 3 If there be given for ordinates AK, BL, CM and DN, say that

$$\frac{1}{2}(AK + DN)AD = \square(AN) \tag{12}$$

likewise, that

$$\frac{1}{3}\left(\frac{1}{2}(AK + BL) + \frac{1}{2}(BL + CM) + \frac{1}{2}(CM + DN)\right)AD,$$

that is,

$$\frac{1}{6}(AK + 2BL + 2CM + DN)AD = \square(AN). \tag{13}$$

The errors in the solutions will be as AD^2 to AB^2 or 9 to 1, and hence the difference in the errors—which is the difference $\frac{1}{6}(2AK - 2BL - 2CM + 2DN)AD$ in the solutions—will be to the error in the latter as 8 to 1. Take away this error and the latter will remain as[3]

$$\frac{1}{8}(AK + 3BL + 3CM + DN)AD = \square(AN). \tag{14}$$

Newton applied the 3-panel composite trapezoidal rule on I = area($AKND$). Let $T_1 = \frac{1}{2}(AK + DN)AD$ and $T_3 = \frac{1}{6}(AK + 2BL + 2CM + DN)AD$. By $T_1 - I : T_3 - I = 3^2 : 1$, Newton derived $T_1 - T_3 : T_3 - I = 8 : 1$. Using $T_1 - T_3 = \frac{1}{3}(AK - BL - CM + DN)AD$, Newton obtained

$$T_3 - \frac{T_1 - T_3}{8} = \frac{1}{8}(AK + 3BL + 3CM + DN)AD, \tag{15}$$

and he gave (15) in his *Methodus Differentialis*. Newton's derivation

[3] *Cas. 3.* Si dentur 4 Ordinatæ AK, BL, CM, DN: dic $\frac{AK+DN}{2}AD = \square AN$. Item $\frac{AK+BL}{6} + \frac{BL+CM}{6} + \frac{CM+DN}{6}$ in AD (id est $\frac{AK+2BL+2CM+DN}{6}AD) = \square AN$. Et solutionū errores erunt ut AD^q ad AB^q seu 9 ad 1 adeoque errorum differentia (quæ est solutionū differentia $\frac{2AK-2BL-2CM+2DN}{8}AD$) erit ad errorem posterioris ut 8 ad 1. Aufer hunc errorem et posterior manebit

$$\frac{AK + 3BL + 3CM + DN}{8}AD = \square AN.$$

$$N_1 = T_3 - \frac{T_1 - T_3}{8} \tag{16}$$

is the theorem of Huygens.

Newton continued:

Case 4. If there be given five ordinates, say (by Case 2) that

$$\frac{1}{6}(AK + 4CM + EO)AE = \Box(AO), \tag{17}$$

likewise that

$$\frac{1}{2}\left(\frac{1}{6}(AK + 4BL + CM)AC + \frac{1}{6}(CM + 4DN + EO)\right) AE = \Box(AO),$$

and that the errors are as AE^2 to AB^2 or 16 to 1; then, since the difference in the errors is $\frac{1}{12}(AK - 4BL + 6CM - 4DN + EO)AE$, the error in the lesser will be $\frac{1}{180}(AK - 4BL + 6CM - 4DN + EO)AE$, and when this is taken away there will remain[4]

$$\frac{1}{90}(7AK + 32BL + 12CM + 32DN + 7EO)AE = \Box(AO). \tag{18}$$

In Case 4, Newton applied the single and 2-panel Simpson rule on $I = \text{area}(AKOE)$. Let $S_1 = \frac{1}{6}(AK + 4CM + EO)AE$ and $S_2 = \frac{1}{12}(AK + 4BL + 2CM + 4DN + EO)AE$. Newton wrote $S_1 - I : S_2 - I = AE^2 : AB^2 = 16 : 1$, which is correct as good luck. The exact ratio is $S_1 - I : S_2 - I = 2^4 : 1^4 = 16 : 1$ by Corollary 2. Similarly to Case 3, from $S_1 - S_2 : S_2 - I = 15 : 1$ and

[4] *Cas. 4.* Si dentur 5 Ordinatæ, dic (per Cas.2)

$$\frac{AK + 4CM + EO}{6} AE = \Box AO.$$

Item $\dfrac{AK + 4BL + CM}{12} + \dfrac{CM + 4DN + EO}{12}$ in $AE = \Box AO$ et errores esse ut AE^q ad AB^q seu 16 ad $1_{[,]}$ et cum errorum differentia sit

$$\frac{AK - 4BL + 6CM - 4DN + EO}{12} AE$$

error minoris erit $\dfrac{AK - 4BL + 6CM - 4DN + EO}{180} AE$ quem aufer et manebit

$$\frac{7AK + 32BL + 12CM + 32DN + 7EO}{90} AE = \Box AO.$$

$$S_1 - S_2 = \frac{1}{12}(AK - 4BL + 6CM - 4DN + EO)AE,$$

Newton derived

$$S_2 - \frac{S_1 - S_2}{15} = \frac{1}{90}(7AK + 32BL + 12CM + 32DN + 7EO)AE,$$

which is the Boole rule. Newton's derivation

$$B_1 = S_2 - \frac{S_1 - S_2}{15} \tag{19}$$

is the theorem of Huygens.

Newton continued:

Case 5 In the same way, if there be seven ordinates there will come to be[5]

$$\frac{1}{280}(17AK + 54BL + 51CM + 36DN + 51EO + 54FP + 17GQ)AG = \square(AQ). \tag{20}$$

Case 6. While if there be given nine there will come[6]

$$\begin{aligned}&\frac{1}{5670}(217AK + 1024BL + 352CM + 1024DN + 436EO\\&\quad + 1024FP + 352GQ + 1024HR + 217IS)AI = \square(AS).\end{aligned} \tag{21}$$

The formula (20) is derived by

$$N_2 - \frac{1}{6^2 - 1}(N_1 - N_2), \tag{22}$$

but $N_1 - I : N_2 - I = 36 : 1$ is incorrect. Newton derived the incorrect formula (20) from the mistaken ratio $N_2 - N_1 : N_2 - I = 35 : 1$.

5 *Cas. 5.* Eodem modo si dentur 7 Ordinatæ, fiet

$$\frac{17AK + 54BL + 51CM + 36DN + 51EO + 54FP + 17GQ}{280}AG = \square AQ.$$

6 *Cas. 6.* Et si dentur 9, fiet

$$\frac{\begin{array}{c}217AK + 1024BL + 352CM + 1024DN + 436EO\\ +1024FP + 352GQ + 1024HR + 217IS\end{array}}{5670}AI = \square AS.$$

By Corollary 3, the exact ratio is $N_1 - I : N_2 - I = 2^4 : 1 = 16 : 1$. Corrected extrapolation is

$$\begin{aligned} &N_2 - \frac{1}{15}(N_1 - N_2) \\ &= \frac{1}{120}(7AK + 24BL + 21CM + 16DN + 21EO + 24FP + 7GQ)AG, \end{aligned} \tag{23}$$

which was pointed out by Whiteside (Whiteside (1976), p. 695). As Whiteside noted the rule (23) is distinct from the seven ordinates Newton–Cotes formula.

By Corollary 4, $B_1 - I : B_2 - I = 2^6 : 1^6 = 64 : 1$. In this case Newton's ratio is perhaps $B_1 - I : B_2 - I = AI^2 : AB^2 = 64 : 1$, which is correct but only by good luck. From $B_2 - B_1 : B_2 - I = 63 : 1$, Newton derived

$$\begin{aligned} R_1 &= B_2 - \frac{1}{63}(B_1 - B_2) \\ &= \frac{1}{5670}(217AK + 1024BL + 352CM + 1024DN + 436EO \\ &\quad + 1024FP + 352GQ + 1024HR + 217IS)AI, \end{aligned} \tag{24}$$

which will be mentioned in Sect. 6.

Newton concluded:

> These are quadratures of the parabola which passes through the end-points of all the ordinates.[7]

This sentence makes it apparent that Newton mistakenly thought that the numerical integration formulae obtained by extrapolation coincide with Newton–Cotes formulae.

5 Incorrect ratio of errors by Newton

Newton used incorrect ratios of errors in the single to composite Newton–Cotes formulae in *Of Quadrature* as follows:

Case 2 $T_1 - \square(AM) : T_2 - \square(AM) = AC^2 : AB^2 = 4 : 1$.
Case 3 $T_1 - \square(AN) : T_3 - \square(AN) = AD^2 : AB^2 = 9 : 1$.
Case 4 $S_1 - \square(AO) : S_2 - \square(AO) = AE^2 : AB^2 = 16 : 1$.
Case 5 $N_1 - \square(AQ) : N_2 - \square(AQ) = AG^2 : AB^2 = 36 : 1$.
Case 6 $B_1 - \square(AS) : B_2 - \square(AS) = AI^2 : AB^2 = 64 : 1$.

Let Q_m be m-panel composite Newton–Cotes formula of n ordinates for $I = \int_a^b f(x)\mathrm{d}x$. Newton seemed to believe the following incorrect ratio of error in the single to composite Newton–Cotes formula:

$$Q_1 - I : Q_m - I \approx (n-1)^2 m^2 : 1. \qquad \text{(incorrect form)}$$

[7] Hæ sunt quadraturæ Parabolæ quæ per terminos Ordinatarum omniū transit.

The exact ratio is

$$Q_1 - I : Q_m - I \approx \begin{cases} m^{n+1} : 1. & \text{if } n \text{ is odd,} \\ m^n : 1. & \text{if } n \text{ is even.} \end{cases}$$

See Davis and Rabinowitz (1984, p. 79) for the proof. For $n > 2$, the couples (n, m) for which Newton's ratios are correct by good luck are $(n, m) = (3, 2), (4, 3)$, and $(5, 2)$. The only incorrect case that Newton used is $(n, m) = (4, 2)$, i.e. $N_1 - I : N_2 - I = 36 : 1$.

Since "With Gregory's *Exercitationes* Newton was well familiar" (Whiteside (1976), p. 693), Newton knew the ratio of errors in the single to composite trapezoidal rule. Newton might generalize this case, i.e., $n = 2$. Whiteside noted "Newton in sequel seems to adopt the argument *per analogiam* that, where $n + 1$ equidistant ordinates are given, the corresponding ratio of the errors is $n^2 : 1$, but this will fail him in 'Cas. 5'" (Whiteside (1976), p. 692).

6 Rediscovery of Newton's integration formulae by extrapolation

After 1900, Newton's numerical integration formulae by extrapolation have been rediscovered.

Sheppard (1900) gave

$$S_{m/2} = \frac{4T_m - T_{m/2}}{3}, \tag{25}$$

for cases in which m is even. When $m = 2$, (25) coincides with Newton's (11). For cases in which m is a multiple of 3, Sheppard also gave

$$N_{m/3} = \frac{9T_m - T_{m/3}}{8}. \tag{26}$$

When $m = 3$, (26) coincides with Newton's (16).

(Milne-Thomson, 1933, p. 199) gave an exercise "By eliminating the error term in h^4 between two Simpson formulae of $2p + 1$ and $4p + 1$ ordinates, obtain the Newton–Cotes' formula for $n = 5$." This exercise for $p = 1$ coincides with Newton's (19).

Romberg (1955) proposed the Romberg integration which is an application of Richardson extrapolation process to $T_1, T_2, T_4, T_8, \ldots$, i.e.,

$$S_m = \frac{4T_{2m} - T_m}{3}, \quad m = 1, 2, 4, \ldots,$$
$$B_m = \frac{16S_{2m} - S_m}{15}, \quad m = 1, 2, 4, \ldots,$$
$$R_m = \frac{64B_{2m} - B_m}{63}, \quad m = 1, 2, 4, \ldots.$$

This process is computed from top to bottom and from left to right in the following triangular array:

$$\begin{array}{llll} T_1 & & & \\ T_2 & S_1 & & \\ T_4 & S_2 & B_1 & \\ T_8 & S_4 & B_2 & R_1 \\ \vdots & & & \ddots \end{array}$$

Newton's 9-ordinate rule (24) is appeared in the fourth column. We remark that the fourth, fifth,... columns are not composite Newton–Cotes formulae. See Ralston and Rabinowitz (1978, p. 124).

7 The second method of obtaining the Newton–Cotes formulae

Newton proposed a method of determining the values of coefficients in Newton–Cotes formulae in the last part of *Of Quadrature*. Newton stated:

> These quadratures may also be ascertained in this manner. Let a be the sum of the first and last ordinates, b the sum of the second and next to the last ones, c that of the third and next but one to last, d the sum of the fourth from the beginning and the fourth from the end, e the sum of the fifth from the beginning and the fifth from the end,..., m the middle ordinate and A the abscissa, and let there be $\frac{za+yb+xc+vd+te+sm}{2z+2y+2x+2v+2t+s}A = \square^{\text{re}}$ required. Then in place of the ordinates write first an equal number of the opening terms of this series of square numbers $0, 1, 4, 9, 16, 25, \ldots$; next, as many first terms of this one of cubes $0, 1, 8, 27, 64, 125, \ldots$; thereafter the same number of this one of fourth powers $0, 1, 16, 81, 256, \ldots$, so proceeding (if need be) to as many series, less one, as there are unknown quantities $z, y, x, v, \ldots$, and in place of the quadrature required write the quadrature of the parabola to which these ordinates accord. There will ensure equations from which, when they are collated, $z, y, x, \ldots$ will be determined.[8]
>
> For instance, if there be four ordinates, I set $a = 0 + 9, b = 1 + 4, A = 3$ and the quadrature $= 9$. In this way there comes to be $\frac{9z+5y}{2z+2y} \times 3 = 9$ and

[8] Investigantur etiam hæ quadraturæ in hunc modum. Sit a summa Ordinatæ primæ et ultimæ, b summa secundæ et penultimæ[,] c tertiæ & antepenultimæ[,] d summa 4^{tae} a principio et quartæ a fine[,] [e] summa quintæ a principio et quintæ a fine &c[,] m Ordinata media et Abscissa A et sit

$$\frac{za + yb + xc + vd + te + sm}{2z + 2y + 2x + 2v + 2t + s}A = \square^{\text{æ}}\text{quæsitæ},$$

et pro ordinatis primo scribantur termini totidem primi hujus seriei numerorū quadratorum 0.1.4, 9.16.25 &c[,] dein termini totidem primi hujus cubicorum 0.1.8.27.64.125 &c[,] dein totidem hujus quadrato-quadraticorū 0.1.16.81.256.&c pergendo (si opus est) ad tot series una dempta quot sunt incognitæ quantitates z, y, x, v &c, et pro quadratura quæsita scribe quadraturam Parabolæ cui hæ ordinatæ congruunt. Et provenient æquationes ex quibus collatis determinabuntur z, y, x &c.

thence $y = 3z$. Therefore in place of y I write $3z$ and the equation $\frac{za+yb}{2z+2y}A = \square$ becomes $\frac{1}{8}(a+3b)A = \square$, as in the third case.[9]
Again, if there be five ordinates, I set $a = 0+16$, $b = 1+9$, $c = 4$, $A = 4$ and $\square = \frac{64}{3}$, and thus there comes to be $\frac{16z+10y+4s}{2z+2y+s} \times 4 = \frac{64}{3}$, that is, $8z = y + 2s$. Again I set $a = 0+64$, $b = 1+27$, $c = 8$, $A = 4$ and $\square = 64$, and in this way there comes $\frac{64z+28y+8s}{2z+2y+s} \times 4 = 64$, or $8z = y+2s$ as above. I therefore put [$a = 0+256$, $b = 1+81$, $c = 16$, $A = 4$ and the quadrature $= \frac{1024}{5}$. And in this way there comes to be $\frac{256z+82y+16s}{2z+2y+s} \times 4 = \frac{1024}{5}$ or $384z - 88s = 51y = 408z - 102s$, that is, $14s = 24z$ or $7s = 12z$ and hence $7y = 32z$. Accordingly, in place of s and y I write $\frac{12}{7}z$ and $\frac{32}{7}z$ respectively, and the equation $\frac{za+yb+sm}{2z+2y+s}A = \square$ becomes $\frac{1}{90}(7a+32b+12m)[A] = \square$, as in the fourth case.] [10]

Newton proposed the second method under two assumptions.

1. The coefficients of the Newton–Cotes formula are symmetric about the coefficient of the the central ordinate.
2. The n-ordinate Newton–Cotes formula is exact for $1, x^2, \dots, x^{n-1}$. When n is odd, this is exact for $1, x^2, \dots, x^n$

The first assumption is correct. See, Davis and Rabinowitz (1984, pp. 77–78) for the proof. The second assumption follows from Proposition 2(i), 3(i), and 4(i), for $n = 3, 4$, and 5, respectively. For the cases in which n is greater than 5, see Davis and Rabinowitz (1984, pp. 79).

Let n be a fixed odd number greater than 2. Let $x_i = x_0 + ih$ $(i = 0, \dots, n-1)$ be equidistant abscissae y_i $(i = 0, \dots, n-1)$ the corresponding ordinates. By the

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9 Ut si Ordinatæ sint quatuor, pono $a = 0+9$ & $b = 1+4$. $A = 3$ et quadraturam= 9. Sic fit $\frac{9z+5y}{2z+2y} \times 3 = 9$. et inde $y = 3z$. Igitur pro y scribo $3z$ et æquatio $\frac{za+yb}{2z+2y}A = \square$ fit $\frac{a+3b}{8}A = \square$. ut in casu tertio.

10 Rursus si ordinatæ sint quinque pono $a = 0+16$, $b = 1+9$, $c = 4$. $A = 4$, $\square = \frac{64}{3}$. Et sic fit $\frac{16z+10y+4s}{2z+2y+s} \times 4 = \frac{64}{3}$ seu $8z = y+2s$. Rursus pono

$$a = 0+64,\ b = 1+27,\ c = 8,\ A = 4 \text{et} \square = 64.$$

et sic fit $\frac{64z+28y+8s}{2z+2y+s} \times 4 = 64$. seu $8z = y+2s$ ut supra. Pono igitur [$a = 0+256$, $b = 1+81$, $c = 16$, $A = 4$ et $\square = \frac{1024}{5}$. Et sic fit

$$\frac{256z+82y+16s}{2z+2y+s} \times 4 = \frac{1024}{5}$$

seu $384z - 88s = 51y = 408z - 102s$, hoc est $14s = 24z$ sive $7s = 12z$ adeoque $7y = 32z$. Igitur pro s et y scribo $\frac{12}{7}z$ et $\frac{32}{7}z$ respective et æquatio

$$\frac{za+yb+sm}{2z+2y+s}A = \square \text{ fit } \frac{7a+32b+12m}{90}|A| = \square$$

ut in casu quarto.]

assumptions, Newton put the n ordinates Newton–Cotes formula

$$R\frac{\sum_{i=0}^{l-1} c_i(y_i + y_{n-i-1}) + c_l y_l}{2\sum_{i=0}^{l-1} c_i + c_l y_l}, \tag{27}$$

where c_i $(i = 0, \dots, l)$ are constants and $l = (n-1)/2$. The second assumption implies

$$\begin{aligned} \frac{\sum_{i=0}^{l-1} c_i(i^k + (n-i-1)^k) + c_l l^k}{2\sum_{i=0}^{l-1} c_i + c_l}(n-1) &= \int_0^{n-1} x^k \mathrm{d}x \\ &= \frac{(n-1)^{k+1}}{k+1}, \quad k = 2, \dots, l+2. \end{aligned} \tag{28}$$

We remark that $l + 2 \leqq n$ if $n \geqq 3$. By solving the simultaneous equations (28), Newton represented c_i $(i = 1, \dots, (n-1)/2)$ in c_0. By substituting c_i, $(i > 0)$ to (27), Newton reduced the fraction and obtained the quadrature formula.

We now prove that the second method provides the n-ordinate Newton–Cotes formula. Let

$$\boldsymbol{x}_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n, \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ \frac{1}{3}(n-1)^2 \\ \frac{1}{4}(n-1)^3 \\ \vdots \\ \frac{1}{n+1}(n-1)^n \end{pmatrix} \in \mathbb{R}^n,$$

and

$$\boldsymbol{x}_i = \begin{pmatrix} 1 \\ i^2 \\ \vdots \\ i^n \end{pmatrix} \in \mathbb{R}^n, \quad i = 1, \dots, n-1.$$

Since the matrix $(\boldsymbol{x}_0, \boldsymbol{x}1, \cdots, \boldsymbol{x}_{n-1})$ is nonsingular, there is the unique solution $(d_0, \dots, d_{n-1})$ of the equation

$$d_0\boldsymbol{x}_0 + \cdots + d_{n-1}\boldsymbol{x}_{n-1} = \boldsymbol{b}. \tag{29}$$

The equation (29) is equivalent to

$$\sum_{i=0}^{n-1} d_i = 1,$$
$$\sum_{i=1}^{n-1} d_i i^k = \frac{1}{k+1}(n-1)^k, \quad k = 2, \dots, n.$$

By Proposition 5 which we will cite in the end of this section,

$$R \sum_{i=0}^{n-1} d_i y_i$$

is the n-ordinate Newton–Cotes formula. By the uniqueness and the symmetry of the coefficients, we have

$$\begin{aligned} &d_0 \neq 0, \\ &d_i = d_{n-1-i}, \quad i = 0, \dots, l-1. \end{aligned}$$

Take any nonzero number c_0. Put

$$c_i = \frac{c_0}{d_0} d_i, \quad i = 1, \dots, n-1.$$

Then for $k = 2, \dots, n$,

$$\frac{\sum_{i=1}^{n-1} c_i i^k}{\sum_{i=0}^{n-1} c_i} = \frac{\sum_{i=1}^{n-1} d_i i^k}{\sum_{i=0}^{n-1} d_i} = \frac{1}{k+1}(n-1)^k,$$

and $c_i = c_{n-i-1}$ $(i = 0, \dots, l-1)$. By Proposition 5,

$$R \frac{\sum_{i=0}^{n} c_i y_i}{\sum_{i=0}^{n-1} c_i}$$

is the n-ordinate Newton–Cotes formula. Therefore, the second method provides the n-ordinate Newton–Cotes formula.

When n is even, Newton did not explain in general, but he gave the four ordinates rule, i.e., the Simpson 3/8 rule. In this example, we can see the second method. Starting from

$$R \frac{\sum_{i=0}^{l-1} c_i (y_i + y_{n-i-1})}{2 \sum_{i=0}^{l-1} c_i}$$

with $l = n/2$ and solving

$$\frac{\sum_{i=0}^{l-1} c_i(i^k + (n-i-1)^k)}{2\sum_{i=0}^{l-1} c_i} = \frac{(n-1)^k}{k+1}, \quad k = 2, \ldots, l+1.$$

For the rest of these, the determination of the coefficients is similar to the odd case.

Although Newton wrote "These quadratures may also be ascertained in this manner", the Newton–Cotes formulae can be determined by the second method.

In the twentieth century, Newton's idea has been formulated as the following Proposition 5 which is used for determining the integration formulae of interpolatory type, such as Gaussian integration.

Proposition 5 *Let $x_0, \ldots, x_{n-1}$ be fixed and distinct points lying between a and b prescribed in advance. The two ways of determining $w_0, \ldots, w_{n-1}$,* (i) *and* (ii) *yield the same numbers.*

(i) *Interpolate to the function $f(x)$ at the points $x_0, \ldots, x_{n-1}$ by a polynomial of degree at most $n-1$. Then integrate the interpolation polynomial, and express the result in the form*

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} w_i f(x_i).$$

(ii) *Select the constants $w_0, \ldots, w_{n-1}$ so that*

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} w_i f(x_i),$$

for $f(x) = 1, x, \ldots, x^{n-1}$.

Proof See Davis and Rabinowitz (1984, p. 74). □

8 Conclusion

In *Of Quadrature by Ordinates*, Newton tried two methods for obtaining the Newton–Cotes formulae. The first method is extrapolation and the second one is the method of undetermined coefficients using the quadrature of monomials.

The first method provides n-ordinate Newton–Cotes formulae only for cases in which $n = 3, 4$ and 5. However, this method provides another important formulae if the ratios of errors are corrected. Newton's numerical quadrature by extrapolation, i.e., the first method, is not systematic but can be said to be a germ of the Romberg integration.

The second method is proposed under two assumptions. We proved that two assumptions are correct and that the second method provides the Newton–Cotes formulae. In the twentieth century, the second method has been formulated and used for determining the integration formulae of interpolatory type, such as Gaussian integration.

References

Cajori, Florian. 1947. *Sir Isaac Newton's Mathematical Principles: Of Natural Philosophy and his system of the world*. Berkeley: University of California Press.

Davis, Philip, and Rabinowitz, Philip. 1984. *Methods of numerical integration*, 2nd ed. Orlando: Academic Press.

Fraser, Duncan C. 1927. *Newton's Interpolation Formulas*. London: Reprinted in Bibliolife.

Gregory, James. 1668. Exercitationes geometricae. http://books.google.co.uk/.

Huygens, Christian. 1654. De Circuli Magnitudine Inventa. Lugd. Batavia. http://books.google.com/.

Milne, R.M. 1903. Extension of Huygens' approximation to a circle arc. *Mathematical Gazette* 2: 309–311.

Milne-Thomson, L.M. 1933. *The calculus of finite differences*. New York: Macmillan.

Newton, Isaac. 1744. Methodus Differentialis 1711. In *Isaaci Newtoni, Equitis Aurati, Opuscula Mathematica, philosophica et philologica, Collegit partímque Latinè vertit ac recensuit Joh*, Castillioneus Jurisconsultus. Lausannæ & Geneva: Apud Marcum-Michaelem Bousquet.

Osada, Naoki. 2012. The early history of convergence acceleration methods. *Numerical Algorithms* 60: 205–221. doi:10.1007/s11075-012-9539-0.

Ralston, Anthony, and Rabinowitz, Philip. 1978. *A first course in numerical analysis*, 2nd ed. New York: McGraw-Hill.

Romberg, Werner. 1955. Vereinfachte numerishe Integration. *Det Kongelige Norske Videnskabers Selskabs Forhandlinger, Bind* 28: 30–36.

Rudio, Ferdinand. 1892. *Archimedes, Huygens, Lambert, Legendre*. Teubner: Vier Abhandlungen Über die Kreismessung.

Sheppard, W.F. 1900. Some quadrature formulae. *Proc. London Math. Soc.* 32: 258–277.

Turnbull, Herbert Westren. 1959. *The correspondences of Isaac Newton*, vol. 1. Cambridge: Cambridge University Press.

Turnbull, Herbert Westren. 1960. *The correspondences of Isaac Newton*, vol. 2. Cambridge: Cambridge University Press.

Whiteside, Derek T. 1961. Patterns of mathematical thought in the later seventeenth century. *Archive Hist. Exact Sci.* 1: 179–388.

Whiteside, Derek T. 1971. *The Mathematical Papers of Isaac Newton*, vol. IV. Cambridge: Cambridge University Press.

Whiteside, Derek T. 1976. *The Mathematical Papers of Isaac Newton*, vol. VII. Cambridge: Cambridge University Press.