Conventions for recreational problems in Fibonacci's "Liber Abbaci"
Author(s): John Hannah
Source: *Archive for History of Exact Sciences*, Vol. 65, No. 2 (March 2011), pp. 155-180
Published by: Springer
Stable URL: https://www.jstor.org/stable/41134344
Accessed: 19-05-2020 12:25 UTC

Arch. Hist. Exact Sci. (2011) 65:155–180
DOI 10.1007/s00407-010-0074-x

Conventions for recreational problems in Fibonacci's *Liber Abbaci*

John Hannah

Received: 29 January 2010 / Published online: 3 December 2010

Abstract Fibonacci's treatment of so-called recreational problems in his *Liber Abbaci* has been interpreted as an early episode both in the history of systems of linear equations, and in the history of negative numbers. However, these problems are also interesting in their own right. We discuss some of the conventions which seem to have governed these problems. By considering certain pairs of problems, where one problem is unsolvable and its partner is solvable, we show that Fibonacci went to a significant effort to conform to these conventions. We also examine the methods which he could have used to construct his problems.

1 Introduction

Leonardo of Pisa, also known as Fibonacci, published his *Liber Abbaci* in 1202, and again in revised form in 1228. For many people his fame rests on a minor problem in Chap. 12 of *Liber Abbaci* which gives rise to the Fibonacci sequence.[1] The twelfth and longest chapter of *Liber Abbaci* has nine sections devoted mostly to various methods of solving what some people call recreational problems (Singmaster 1988) and what

[1] See Vogel (1971) and Katz (1998, Chap. 8) for a general introduction to Leonardo. His *Liber Abbaci* was published in the original Latin as Boncampogni (1857) and it has been translated into English as Sigler (2002). Leonardo's treatment of various problems, his possible sources and his own influence on later developments are discussed, for example, in Bartolozzi and Franci (1990), Docampo Rey (2009), Franci and Toti Rigatelli (1985), Høyrup (2005, 2007), Katz (1998), Rashed (1994), Sesiano (1985, 2009), Spiesser (2000), Vogel (1940, 1971).

Communicated by Menso Folkerts.

J. Hannah (✉)
Department of Mathematics and Statistics, University of Canterbury, Private bag 4800,
Christchurch 8140, New Zealand
e-mail: john.hannah@canterbury.ac.nz

other people call mathematical riddles (Høyrup 2002, pp. 364–367). As many of the problems can be interpreted as systems of linear equations, Leonardo has been seen as a pioneer in the development of methods for solving such systems (Vogel 1940). Because some of the problems involve amounts of money and are unsolvable unless you allow negative solutions (interpreted as debts), Leonardo has also been seen as an early advocate for negative numbers (Sesiano 1985). In this article, however, I want to look at these problems in their own right, trying to see them as Leonardo saw them. The most obvious effect of this change of viewpoint is that we will now be looking, not at systems of linear equations, but at problems such as:[2]

> Two men have some denari. The first says to the other:
> If you give me one of your denari, I shall have the same as you.
> The other man replies:
> And if you give me one of your denari, I shall have ten times as much as you.
> How many denari does each man have? (Boncompagni 1857, p. 190) or (Sigler 2002, p. 289)

Thus the unknowns are not usually abstract numbers, but instead concrete quantities such as "the first man's denari." The change of viewpoint also results in a change of questions. Instead of looking for the antecedents of modern ideas, such as general methods for solving systems of linear equations, or the existence of negative numbers, we shall be looking at issues which concerned Leonardo himself. We shall focus in particular on two questions: *Which kinds of questions was Leonardo allowed to ask?* and *How could he have constructed acceptable problems?*

We shall concentrate on Leonardo's treatment of three groups of problems: men giving and taking (as in the above example), men finding a purse, and men wishing to buy a horse (see Sect. 2 for examples of such problems). All three groups of problems have a long history before and after Leonardo's time (Singmaster 2004, Sect. 7.R). Leonardo contributes to this history by offering a wide range of variations (for example, increasing the number of men involved and allowing various groupings of men), by offering a variety of solution methods (such as false position and *regula recta*) and, most particularly, by exploring variations of the problems which, for one reason or another, are unsolvable. In what follows, it is Leonardo's wealth of examples which allows us to observe the conventions which seem to have applied to these families of problems, and it is his interest in unsolvable and solvable problems conforming to these conventions, which gives us the chance to see how he might have constructed his problems.

In Sect. 2, we introduce the three families of problems mentioned above, and identify the conventions which seem to have applied to such problems, especially to the more complex variations involving three or more unknowns. The most noticeable convention is that values of parameters (such as amounts of money being exchanged, or ratios of amounts of money) increase as you recite the problem, a convention which may have made it easier to remember the data in an oral environment. This convention

[2] All translations are mine. Where paraphrases are given, the reader seeking more detailed translations is referred to (Sigler 2002) which is more than adequate for that purpose.

is followed by many complex problems in many cultures, so it cannot be said to belong particularly to Leonardo, even though he seems to adhere to it more closely than many of his predecessors.

Section 3 looks at how the three families of problems fit into Leonardo's plan for Chap. 12 of *Liber Abbaci*. We shall see that Leonardo's systematic exploration of variations of these problems inevitably unearths some unsolvable problems. Earlier authors do not seem to have been interested in such unsolvable problems, so this material may actually be original to Leonardo. His main concern seems to be that his readers learn how to recognize unsolvable problems, and for this purpose he offers pairs of related problems, one unsolvable and the other solvable. To construct these problems Leonardo seems to have gone to significant effort to conform to the conventions found in Sect. 2. Although Leonardo tells his readers nothing about how to construct these problems, we show that nothing extra was needed beyond the methods taught in *Liber Abbaci*. Thus there is no need to postulate (as Sesiano (2009, p. 103) seems to do) that he had formulas for solving these problems, or that he used such formulas to choose parameter values which lead to unsolvable problems or negative values for some of the unknowns.

2 Conventions for complex problems

In this section we shall introduce three families of problems discussed by Leonardo, and then look at the general conventions which seem to govern them. At this stage all that is involved is a superficial inspection of the statements of the problems and their solutions. We postpone a deeper analysis of Leonardo's actual methods until the next section.

All the problems discussed here are variations on well-known classical puzzles. The first family of problems, about "giving and taking", occurs in the third section of Chap. 12 of *Liber Abbaci*, a section where Leonardo introduces the method of (single) false position. This group of problems deals with men taking money from one another and then making observations about the relative sizes of their holdings. Such problems have a long history, with Diophantus (third century CE) showing how to solve an abstract problem corresponding to the case of just two men; see Problem I, 15 in Heath (1910, p. 134) or Ver Eecke (1959, p. 20). Somewhat later, Mahavira (ninth century CE) offers two examples involving three men (Mahavira 1912, pp. 158–159). These problems still seem to have had some currency as challenge problems in Leonardo's time since his third example is said to have been posed to him by a teacher in Constantinople (Boncompagni 1857, p. 190) or (Sigler 2002, p. 290):

> Two men have some denari.
> If the first takes 7 denari from the second, he will have 5 times what that man has left.
> Similarly, if the second takes 5 from the first, he will have 7 times what the first has left.
> How many denari does each man have?

or, in algebraic form[3]

$$\begin{aligned} a + 7 &= 5(b - 7) \\ b + 5 &= 7(a - 5). \end{aligned}$$

We shall examine Leonardo's method of solution later, but for the moment the most interesting feature is that the solution is quite messy. Indeed, the amounts held by the two men at the start of the problem are found to be

$$a = 7\tfrac{2}{17}, \quad b = 9\tfrac{14}{17}$$

This is typical of Leonardo's examples and those of his predecessors mentioned above.

Convention 1 Data in problems are usually small integers. A consequence of this is that solutions are relatively messy (for example, non-integers).

This may seem too simple an observation to be worth making, but one important consequence is that this means such problems were generally constructed first, and the solution calculated afterwards. This contrasts, for example, with many problems in Old Babylonian texts where not only were the solutions known beforehand, but sometimes these solutions were still found "correctly" despite arithmetical errors being made during the solution process (Høyrup 2002, p. 72). Constructing problems first and accepting the solution which then arises, is probably a sign that the author is confident that his method of solution will work for any problem. Using round numbers in the problem statement may also make it easier to remember the data during the solution process, whilst the messy answer may have the advantage, in the context of challenge problems, that an opponent would be unlikely to find the correct answer simply by guessing.

Leonardo presents 16 problems on the theme of giving and taking, some involving more complicated exchanges, and others involving larger groups of men.[4] The problems involving more than two men seem to follow additional conventions, as we can see by looking at the first such problem (Boncompagni 1857, p. 198) or (Sigler 2002,

[3] In this and succeeding examples, such equations should be viewed only as a short hand notation for the problem statement. There is no intention of suggesting anything resembling an algebraic approach to the problems. One significant difference between the rhetorical version and the algebraic version in this case is that the amounts exchanged are mentioned only once in the rhetorical version, but appear twice in each equation. Thus the equations make the enunciation appear more complicated from the point of view of an oral delivery of the problem.

[4] To help the reader see the conventions discussed below, all 16 problems are listed in algebraic form in Appendix A.

p. 300).[5]

$$a + 7 = 5(b + c - 7)$$
$$b + 9 = 6(c + a - 9)$$
$$c + 11 = 7(a + b - 11)$$

Once again, we see that all the numbers are integers, but this time there is also a pattern amongst the parameters, or numbers playing the same role in each condition (or equation) of the problem. Thus the takings (7, 9, 11) and the comparison ratios (5, 6, 7) both increase as we read or recite the problem. On the other hand, the solution is still messy:

$$a = 7\tfrac{98}{263}, \quad b = 5\tfrac{206}{263}, \quad c = 4\tfrac{24}{263}.$$

This behaviour is common to all the more complex giving and taking problems in *Liber Abbaci* (see Appendix A).

Convention 2 In complex problems, the various parameters increase as you read or recite the problem.

Similar conventions can be found in Leonardo's treatment of purse problems in Sect. 4 of Chap. 12 of *Liber Abbaci.* Again, these problems have a long history, with early examples found in Iamblichus (fourth century CE) and Mahavira (Singmaster 2004). Leonardo's first problem gives the basic scenario of purse problems (Boncompagni 1857, p. 212) or (Sigler 2002, p. 317):

> Two men with denari find a purse with denari in it.
> The first says to the second, "If I were to have the denari from the purse along with those I already have, I would have three times as much as you."
> To which the other replies, "And if I were to have the denari from the purse along with my denari, I would have four times as much as you."
> How much does each man have, and how much is in the purse?

As with the giving and taking problems, I shall represent such problems in symbolic form:

$$a + p = 3b$$
$$b + p = 4a$$

but these symbolic representations should, of course, be read as stories of the above form. All 15 purse problems are given in symbolic form in Appendix B, and we can see there that Conventions 1 and 2 also apply to all of Leonardo's purse problems. This time, however, the problems are all indeterminate and so have infinitely many

[5] This example should be read as: *the first man having 7 of the others' denari, will have 5 times as much as them;* and so on. Thus the algebraic form displays more detail, and may tend to make the problem statement look more complicated, than the rhetorical form does.

solutions. Leonardo may not have a fully developed theory for systems of linear equations, but he is aware of the freedom which we associate with indeterminacy, and he uses that freedom to find integer solutions to all his problems, usually the smallest integer solutions. Thus in solving the above problem he finds that

$$a = 4, \quad b = 5, \quad c = 11.$$

It is perhaps debatable whether this solution is relatively messy compared with the initial data (as Convention 1 would suggest) but for problems involving more men the distinction is much more clear cut. For example, Leonardo's solution to the second purse problem (see Appendix B) is

$$a = 7, \quad b = 17, \quad c = 23, \quad p = 73$$

compared with initial data of 2, 3 and 4.

However, Convention 2 needs a slight modification if we are to see it applying in the group of problems considered by Leonardo in Sect. 5 of Chap. 12 of *Liber Abbaci*. These problems are variations of yet another classical problem: men buying a horse, where men (or groups of men) with money seek money from other men in order to buy a horse.[6] The second problem highlights the main features of most of these problems, (Boncompagni 1857, p. 229) or (Sigler 2002, p. 338). Symbolically, it could be represented by the following system of equations:

$$\begin{aligned} a + \tfrac{1}{3}b &= h \\ b + \tfrac{1}{4}c &= h \\ c + \tfrac{1}{5}a &= h \end{aligned}$$

or, in words, *the first man seeks one third of the denari of the second man in order to be able to buy the horse*, and so on. Notice that, because of the context of these problems, we can expect the parameter (the fractions sought, that is, $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$ in the above problem) to take values between 0 and 1. At first sight, this parameter does not appear to increase as we read through the problem. However, if we think in terms of the oral delivery of the problem, the numbers being mentioned do still increase, as in *a third, a fourth, a fifth*. Viewed in this light, these examples follow a natural extension of Convention 2. Since the fractions sought are generally expressed in terms of unit fractions, we can think of the denominators of these fractions as parameters which, in line with Convention 2, ought to increase as we read through the problems. In the few cases where fractions sought are expressed as proper fractions rather than unit fractions, Leonardo's numerators usually increase too. For example, this

[6] This group of problems has been studied in (Vogel 1940) and we refer the reader to Vogel's article for a complete listing of these problems in algebraic form.

happens in Leonardo's third horse problem (Boncompagni 1857, p. 231) or (Sigler 2002, p. 341):[7]

$$\begin{aligned} a + \tfrac{2}{3}b &= h \\ b + \tfrac{4}{7}c &= h \\ c + \tfrac{5}{9}d &= h \end{aligned}$$

Summarizing our findings then, we have one more convention.

Convention 3 In complex problems, if parameters are expected to be fractions, then they are usually expressed in terms of unit fractions, with denominators increasing as you read or recite the problem. If the parameters are expressed in terms of proper fractions, then numerators and denominators both increase as you read or recite the problem.

Lest it be thought that this last example violates Convention 1, it should be noted that Leonardo's solution to this indeterminate problem is indeed relatively messy:

$$a = 35435, \quad b = 35313, \quad c = 41412.$$

Of course, using parameters such as $\frac{2}{3}$, $\frac{4}{7}$ and $\frac{5}{9}$ may simply be an attempt to set a hard problem but, as we shall see in the next section, it is also possible that the above conventions sometimes forced Leonardo to use complicated parameters.

2.1 Discussion

We have seen that Leonardo's problems dealing with giving and taking, men finding purses, or men wishing to buy horses, almost all follow what I have called Conventions 1, 2 and 3. A glance at other ancient or mediaeval collections of problems shows that their authors usually follow the same conventions, although perhaps not as rigidly as Leonardo does. For example, when Diophantos needs to choose three or more typical values to illustrate his method for a general problem, he usually chooses according to these conventions; see, for example, Problems 16 to 25 from Book I in Heath (1910) or Ver Eecke (1959). Similarly, many of Mahavira's problems involving three or more statements use increasing (or decreasing) parameter values, although there are also several cases where the final values are out of sequence. Thus, for example, one purse problem has successive ratios 8, 9, 10, 11, but another, where fractions are taken from the purse, has the two sequences of parameter values $\frac{1}{5}$, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{3}$, and 2, 3, 5, 4 (Mahavira 1912, p. 156). Again, problems from the *Suan Shu Shu* (second century

[7] Leonardo gives just two other examples involving proper fractions. Using the numbering in Vogel (1940) they are Problem 7, which has the fractions sought $\frac{2}{3}$, $\frac{4}{7}$, $\frac{5}{11}$, $\frac{6}{13}$, $\frac{8}{19}$, and Problem 19, which has the fractions sought $\frac{2}{5}$, $\frac{3}{8}$, $\frac{4}{11}$, $\frac{6}{19}$. The only exception to these apparent conventions is Vogel's Problem 20, an unusually complicated problem posed to Leonardo by "a most skilful master" from Constantinople. Its fractions sought are expressed as sums of unit fractions which would be read as $\frac{2}{3} + \frac{1}{5}$, $\frac{2}{3} + \frac{1}{6} + \frac{1}{480}$, $\frac{2}{3} + \frac{1}{6} + \frac{1}{688}$, $\frac{2}{3} + \frac{1}{7} + \frac{1}{420}$, $\frac{2}{3} + \frac{1}{10} + \frac{1}{27} + \frac{1}{810}$.

BCE) which involve three quantities often follow these conventions; see Problems 11, 13, 15 and 21 in Dauben (2008). And as one final example, most of Al-Karajī's problems dealing with three or more unknowns follow the above conventions; see, for example, Problems 24 to 35 in Sect. III of Woepcke (1853). Further examples can be found in the comprehensive listing of so-called recreational problems in Singmaster (2004).

However, it is still conceivable that what we have here are not really conventions, in the sense of rules which should be followed when you construct a problem. Perhaps what we have observed is simply a natural behaviour when you pose problems for people who are expected to have a general method of solution. In such cases, you seem to have complete freedom to choose values for any parameters in the problem, so why not choose the simplest numbers, such as 1, 2, 3, and so on? Choosing increasing parameter values may originally have been a response to the extra load on the hearer's memory when a complex problem is recited, as increasing (or decreasing) sequences of numbers may be easier to remember. Certainly, as Appendix A shows, this behaviour does not occur in the simpler problems dealing with only two men giving and taking. If this is the reason for the observed behaviour, then choosing messier numbers could simply represent an attempt to pose harder problems.

Although this is an appealing hypothesis, in the next section I shall argue that, at least for Leonardo, these conventions had become rules which had to be followed when you constructed a problem.

3 Conventions under strain

In this section we examine Leonardo's overall plan in the parts of his *Liber Abbaci* which deal with problems about giving and taking, or about men finding a purse, or about men wanting to buy a horse. In the natural course of following this plan, he comes across problems which, for him at least, are unsolvable. It seems that, in each case, he then looks for a solvable counterpart for the unsolvable problem. In at least one case, he goes to surprising lengths to produce a problem which conforms to the conventions we discussed in the previous section. As we follow Leonardo in this journey, we can see how he might have constructed some of these problems.

Leonardo's 16 giving and taking problems, discussed earlier, form a single coherent unit amongst the 100 or so problems which he solves in Sect. 3 of Chap. 12 of *Liber Abbaci*. Although Leonardo introduces several different methods in Sect. 3, his main thesis is that the method of (simple) false position can be used to solve many problems. To support this thesis he offers a wide variety of problems and shows how to reduce each of them to a false position calculation. For the 16 problems about giving and taking (see Appendix A) Leonardo works systematically through variations on the basic scenario, and shows that in each case the same reduction technique can be used. Perhaps one reason that he deals with so many variations, all using the same reduction, is that it turns out that some of his examples cannot be solved and so he decides to devote some time to showing his readers how to distinguish solvable problems from unsolvable ones.

To see why some problems might be unsolvable for Leonardo, let us look briefly at Problems 5 and 7 from Appendix A.[8] Problem 5 deals with two men giving and taking, but with a surplus involved in the resulting comparison (Boncompagni 1857, p. 193) or (Sigler 2002, p. 294):[9]

$$a+7=5(b-7)+12$$
$$b+5=7(a-5)+12.$$

Leonardo works in terms of a *lesser sum* which we might represent as

$$s=a+b-12$$

and he shows that

$$\left(\frac{7}{8}+\frac{5}{6}\right)s=s$$

which he says is *impossible*.[10] Leonardo does not say why this conclusion might be impossible, but he is probably referring to the Euclidean principle that *the lesser cannot be equal to the greater*. Indeed his turn of phrase (*quod est impossibile*) may be a deliberate echo of a phrase which occurs frequently in *reductio ad absurdum* proofs in the version of Euclid's *Elements* which was translated directly from the Greek, a version which Leonardo seems to have known.[11] Many of these *reductio ad absurdum* proofs reach the same kind of conclusion as Leonardo has just reached in our example: *such and such are equal; the lesser to the greater; which is impossible*.[12] By using the final part of this rather formulaic construction, Leonardo could remind his audience of the Euclidean principle encapsulated by the whole construction.[13]

From our modern viewpoint, we might say that for Leonardo all magnitudes (including this lesser sum) have to be positive numbers. Thus $s=0$ is not entertained as a possibility here. The scenario corresponding to $s=0$ could well have been excluded

[8] In modern terms, some problems are unsolvable because Leonardo is unwilling to countenance zero or negative quantities. See Sesiano (1985) for a discussion of these problems and their role in the history of negative numbers.

[9] These equations should be read as saying, *If the first man takes 7 denari from the second, he will have five times what that man has, and 12 more*, and so on.

[10] In Latin: *impossibile*. In this article I have consistently translated this word as *impossible*.

[11] Folkerts (2005, pp. 106–112) presents compelling evidence that Leonardo was at least familiar with this version of the *Elements*. Busard (1987, pp. 17–20), in his critical edition of this version, even suggests that Leonardo may have had a hand in compiling its Compendium of Books XIV and XV.

[12] For example, Proposition I, 14 says: ...*est equalis. Minor maiori. Quod est impossibile* (Busard 1987, p. 35).

[13] See Netz (1999, Chaps. 4 and 5) for a discussion of formulaic phrases in Greek mathematics. A key idea here is that formulas are naturally subject to ellipsis, and that the entire original formula can be conjured up for the reader by the mention of just a part.

on other grounds too. This value leads to the first man having just 5 denari and the second just 7. Thus, after the first exchange in the statement of the problem, the first man (with all 12 denari in his hand) would be saying, 'I have 12 more than five times what you have,' to a man who has nothing in his hand. Perhaps there is some discomfort, in this context of public challenge, with making arithmetical comparisons about amounts of money when one party actually has no money.

Leonardo's treatment of Problem 7 from Appendix A shows another way in which problems may be unsolvable. In algebraic terms the problem can be expressed as the following system (Boncompagni 1857, p. 196) or (Sigler 2002, p. 297):

$$a + 7 = 5(b - 7) + 1$$
$$b + 5 = 7(a - 5) + 15.$$

Leonardo shows that this problem is unsolvable by working in terms of a *middle sum*

$$S = a + b - 1.$$

He derives a condition which we might represent by the equation

$$\left(\frac{1}{8} + \frac{1}{6}\right) S + 9\frac{1}{4} = S$$

and, after a false position calculation, he deduces that the first man must have $4\frac{15}{17}$ denari. Leonardo says that this value is *absurd*[14] because it is less than the 5 which the first man is supposed to give the second.

If we are thinking in terms of solutions to systems of linear equations, then we might say that the obstruction to solvability this time is that Leonardo is unwilling to allow negative values (for the holdings of the first man after he has handed over 5 denari to the second man). However, if we think in terms of the original problem, the obstruction is not being able to make sense of the proposed solution in terms of the original story.[15]

Earlier writers do not seem to have been interested in unsolvable problems, so this material may be original to Leonardo. In more general settings, however, unsolvable problems must have arisen throughout the history of mathematics. Traces of some of these problems can perhaps be found in the $\delta\iota o\rho\iota\sigma\mu o\sigma$, or condition of solvability,

[14] In Latin: *inconueniens*. I have translated this word as *absurd* as this is consistent with the use of *inconueniens* as a translation of $\alpha\tau o\pi o\sigma$ in Books III to IX of the translation of the *Elements* made directly from the Greek, a text with which Leonardo seems to have been familiar (Busard 1987, pp. 17–20) and (Folkerts 2005). In *Liber Abbaci* the words *inconueniens* and *impossibile* are used as synonyms describing not only the contradictions which we might associate with inconsistent problems, but also other contradictions of Euclidean principles which we might now resolve by using negative numbers or zero. The same books of the translation of the *Elements* made directly from the Greek consistently use *impossibile* to translate $\alpha\delta\upsilon\nu\alpha\tau o\sigma$.

[15] Later on, Leonardo does allow negative solutions (viewed as debts) for some purse or horse-buying problems (Sesiano 1985), perhaps thinking of these later problems as more advanced material. However, none of his problems are allowed zero solutions.

which Euclid supplies for some problems in his *Elements* (Heath 1926, pp. 129–131). For example, unsolvable instances of the problem "to construct a triangle out of three straight lines equal to three given lines" are eliminated by the condition that any two of the lines taken together should be greater than the third (*Elements*, I, 22). Leonardo was certainly familiar with the *Elements* (Folkerts 2005) and so, with such Euclidean examples in mind, Leonardo may have aspired to eliminating unsolvable problems by finding similar conditions of solvability. An example where this was possible occurred in Chap. 11 of *Liber Abbaci* where he discusses alloy problems. Here he was able to offer rules for when certain problems were solvable. For example, to form an alloy of a desired richness, the reader is told to start with two alloys, one richer than the desired richness and one poorer (Boncompagni 1857, p. 151) or (Sigler 2002, p. 238). By contrast, al-Karajī gives an unsolvable problem about alloys in his collection *al-Fakhrî* and he resolves the dilemma simply by changing the desired richness to an appropriate value (Woepcke 1853, p. 79). As we shall see shortly, Leonardo was also able to demarcate the solvable cases of two simpler variations of giving and taking problems by supplying the critical cases beyond which all problems were unsolvable (Problems 5 and 7 from Appendix A). It seems likely that Leonardo was not aware of al-Karajī's work (Rashed 1994) but we shall see that Leonardo responds to other unsolvable problems in the same way that al-Karajī does, giving a modification of an unsolvable problem which renders it solvable.

The first of Leonardo's pair of four-man giving and taking problems is unsolvable and can be represented symbolically as follows (Boncompagni 1857, p. 201) or (Sigler 2002, p. 303):[16]

$$\begin{aligned}
a + b + 7 &= 3(c + d - 7)\\
b + c + 8 &= 4(d + a - 8)\\
c + d + 9 &= 5(a + b - 9)\\
d + a + 11 &= 6(b + c - 11)
\end{aligned}$$

To see why Leonardo thinks this problem is unsolvable, we shall follow a paraphrase of his explanation. Not only does this show his general method for solving such problems, but it may also shed some light on his construction of a solvable version of the same problem.

Leonardo works in terms of the *whole amount* of denari present, a quantity we might represent as

$$S = a + b + c + d.$$

The first condition in the problem, says Leonardo, implies that the first and second men, once they have the 7 denari handed over by the other two, must hold $\frac{3}{4}$ of the total amount S. Hence the third and fourth, initially, must have $\frac{1}{4}$ of S plus the 7 denari.

[16] This example should be read as: *the first and second men having 7 of the others' denari, will have three times as much as them*; and so on. Once again, the rhetorical version is not as dense with details which need to be remembered as the symbolic version is.

Similarly, using the third condition, the first and second men must initially hold $\frac{1}{6}$ of S plus the 9 denari. Combining these facts, Leonardo deduces that all four men between them must have $\frac{1}{4}+\frac{1}{6}$ of S plus 16 denari. Since he wants to use the method of false position, he rephrases this conclusion in a form which we might represent by the equation:

$$S-\left(\frac{1}{4}+\frac{1}{6}\right)S=16.$$

For this false position calculation Leonardo would normally guess that $S=24$, the least common multiple of the denominators 4 and 6. Evaluating the left-hand side of the equation would then give 14 instead of the desired 16, and so he would scale his guess to get the true value of S:

$$S=\frac{24\times 16}{14}=27\tfrac{3}{7}.$$

In the text he simply states this solution (by this stage he has already done more than fifty false position calculations in this section of *Liber Abbaci*). Proceeding in the same way with the second and fourth conditions from the problem, he derives a condition which we might represent by the equation:

$$S-\left(\frac{1}{5}+\frac{1}{7}\right)S=19.$$

Once again Leonardo simply states the solution, but his normal guess for this false position would be $S=35$. This yields 23 instead of 19, so this time the true value of S appears to be

$$S=\frac{35\times 19}{23}=28\tfrac{21}{23}.$$

He says this situation is ***absurd*** since the earlier calculation gave a different value for S.

The question naturally arises: why did Leonardo give an unsolvable problem here, and how did he find it? The situation can be clarified if, for a moment, we switch to a completely anachronistic and modern viewpoint. Giving and taking problems like this one can be written in matrix form:

$$\begin{bmatrix} 1 & 1 & -\alpha & -\alpha \\ -\beta & 1 & 1 & -\beta \\ -\gamma & -\gamma & 1 & 1 \\ 1 & -\delta & -\delta & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}=\begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix}$$

In our case, the parameter values (the ratios of what we will have to what you will have) are

$$\alpha=3,\quad \beta=4,\quad \gamma=5,\quad \delta=6$$

but, regardless of the parameter values, the coefficient matrix here has rank at most 3, and so almost all such problems are inconsistent. A glance at the complete list of giving and taking problems (Appendix A) shows that this form of problem arose naturally during Leonardo's exploration of variations on the basic problem. Hence, Leonardo did not need to search for this unsolvable problem. Once he had decided to look at this form of problem (pairs of men from amongst four men, giving and taking, with the pairs changing cyclically) it was almost inevitable that his example would be unsolvable. In particular, there is no need to postulate in this case, as Sesiano seems to do in Sesiano (2009, p. 103), that Leonardo had a general formula for solving all such problems, and that he used this formula to deliberately choose parameter values which would lead to, say, negative solutions. As we have just seen, Leonardo does have an algorithm for solving such problems but, in the absence of algebraic symbolism, it seems unlikely that he would have been able to express the solutions in terms of arbitrary parameter values. (By way of contrast, we shall see later a situation where it would have been entirely feasible for him to see the effect of choosing arbitrary parameter values.)

Immediately after the above unsolvable problem, Leonardo says that if we want to exhibit a solvable variation on this problem,[17] then we can use the example which can be expressed in symbolic form as follows (Boncompagni 1857, p. 201) or (Sigler 2002, p. 303):

$$\begin{aligned} a+b+100 &= 3(c+d-100) \\ b+c+106 &= 4(d+a-106) \\ c+d+145 &= 5(a+b-145) \\ d+a+170 &= 6(b+c-170) \end{aligned}$$

How could he have found this problem? After all, we have just seen that almost all problems with this pattern of giving and taking are unsolvable.[18] One easy way to construct a solvable problem is to start with the solution. In this case the easiest way would be to choose arbitrary amounts for the denari of each man (more than they are required to give away, of course), and then to calculate the new proportions needed to replace the proportions 3, 4, 5, 6 in the original problem. For example, keeping the same amounts being exchanged and starting with the solution $a = 5$, $b = 7$, $c = 9$ and $d = 10$ would lead to the problem:

$$\begin{aligned} a+b+7 &= \tfrac{19}{12}(c+d-7) \\ b+c+8 &= \tfrac{24}{7}(d+a-8) \\ c+d+9 &= \tfrac{28}{3}(a+b-9) \\ d+a+11 &= \tfrac{24}{5}(b+c-11) \end{aligned}$$

[17] In Latin: *nam si eam solubilem proponere uolumus ...*

[18] Notice that one modern strategy for making the problem solvable will not work here: changing one entry in the coefficient matrix destroys the structure of the story.

Unfortunately, the new parameter values (the ratios of what we will have to what you will have) no longer conform to the conventions we observed earlier. Indeed, in all Leonardo's giving and taking problems, involving more than two men, the ratios are integers which increase as you read through the problem. As we shall see shortly, Leonardo does construct some problems by starting with the solution, but it seems likely that in this case he was deterred by the resulting unsuitable values for the ratio parameter. Instead, it appears that Leonardo decided to retain the original values $(3, 4, 5, 6)$ of the ratio parameter, and to look for suitable new exchange values. This is a more intricate task, but there are several ways in which it could be done. For example, if you start with a solution, then an algebraic approach to the first "equation" could be used to show that the first taking needs to be $\frac{3}{4}S - (a + b)$ where S is total amount of money held by all four men, and there are similar expressions for the other takings. The calculations involved in this approach are certainly within Leonardo's capability, even if he lacked a symbolic algebra. However, the fact that Leonardo's first taking is 100 suggests that he actually chose this number first, rather than deriving it from a known solution.

Leonardo's attempt to solve the original problem may give us a hint as to how he found the new exchange values and the corresponding solution. As we saw earlier, in his investigation of the original problem, Leonardo found that the total amount S had to satisfy the two conditions

$$S - \left(\tfrac{1}{4} + \tfrac{1}{6}\right) S = \text{sum of first and third takings}$$

and

$$S - \left(\tfrac{1}{5} + \tfrac{1}{7}\right) S = \text{sum of second and fourth takings.}$$

Since he wanted both conditions to be true, his usual guess for a false position would be $S = 420$, the least common multiple of 4, 5, 6, 7. Using this value of S in the above two calculations shows that he needed to have

$$\text{sum of first and third takings} = 245$$

and

$$\text{sum of second and fourth takings} = 276.$$

From the many combinations which satisfy these two conditions, Leonardo chose $245 = 100 + 145$ and $276 = 106 + 170$, and then his solution method yielded the values

$$a = 100, \quad b = 115, \quad c = 115, \quad d = 90.$$

It seems reasonable to assume that the especially round numbers here (the first taking being 100, and the first man's initial holding being 100) represent free choices by Leonardo. If he was guided by the proposed conventions, then the construction process was reasonably straightforward, even if he was not guided by a formula for solving all such problems. Once he had chosen the first taking to be 100, the third taking had to be 145, and then the second taking had to lie between these by Convention 2. Choosing 106 had the advantage of making the first and fourth men's share total to a relatively round number (since then $a + d = \frac{1}{5}S + 106 = 190$). Of course, since Leonardo was constructing a problem, rather than solving an already given problem, the false position guess $S = 420$ did not need to be scaled, but as a consequence some of the parameters in the problem look unusually large.

Leonardo does not actually tell us how he constructed this problem, but he does solve it, using false position calculations similar to those just mentioned. In the course of doing this, he observes that there are many solutions. For example, in his solution, there are 215 denari which can be *divided at will* between the first and second men (Boncompagni 1857, p. 201) or (Sigler 2002, p. 303). In modern terms, this is what we would expect: as the coefficient matrix of the system of linear equations has rank at most 3, once we make the system consistent, there will be an infinite number of solutions. Leonardo does not have the modern theory of linear equations at his disposal, of course, but his methods are still powerful enough to reveal the indeterminacy of his solvable problem.

To see a case where Leonardo probably did construct a problem by starting with a solution, we look at another instance where he considers a pair of problems, one unsolvable and one solvable. This occurs when he discusses pairs of men (from amongst four men) all wanting to buy a horse. He says that he offers these two examples so that the reader will know better how to distinguish the solvable from the unsolvable (Boncompagni 1857, p. 251) or (Sigler 2002, p. 365).[19] Once again the unsolvable example comes first:[20]

$$\begin{aligned} a + b + \tfrac{1}{2}(c + d) &= h \\ b + c + \tfrac{1}{3}(d + a) &= h \\ c + d + \tfrac{1}{4}(a + b) &= h \\ d + a + \tfrac{1}{5}(b + c) &= h \end{aligned}$$

Leonardo tries two different methods of solving this problem.[21] The first method, which has been used in several previous problems, works in terms of what Leonardo calls the residue, the total amount of money left to the other men after a group has acquired the price of the horse. When the calculations are done in full, this method uses a false position calculation similar to the one above except that, because the problem is undetermined, the guess is simply a convenient value for the main unknown and no scaling step is needed. However, by this stage, Leonardo has reduced the method to a recipe, so he simply tells the reader that using the earlier method gives the combined

[19] In Latin: *quare ponamus unam questionem insolubilem et aliam solubilem de IIII^{or} hominibus ut habeas melius notitiam cognoscendi solubiles ab insolubilibus.*

[20] *The first and the second men seek half the denari of the third and fourth in order to be able to buy the horse,* and so on.

[21] Vogel (1940, p. 233) mentions only the second method.

holdings of the four men as 73 bezants, and the residue as 24 bezants. Hence the price of the horse is $73 - 24 = 49$ bezants. Using the first condition in the problem, he then shows that the first and second men must have $49 - 24 = 25$ bezants, whilst the third condition shows that the third and fourth must have $49 - \frac{1}{3}24 = 41$ bezants. But this is *impossible*, he says, because they ought to have $73 - 25 = 48$ bezants. In Leonardo's second method, he considers the first and second men as one man, and the third and fourth as another man. Using the first and third conditions from the problem, he shows that the price of the horse must be $\frac{7}{10}$ of the combined holdings of both new men (that is, of all four original men). In a similar way, though, the second and fourth conditions of the problem show that the price of the horse must be $\frac{7}{11}$ of the combined holdings of all four original men. This, he says, is *absurd* and so the problem is unsolvable. This could be taken as a refusal to allow zero solutions, but it is easy to understand how an audience might disapprove of a solution to a horse-buying problem, if the solution called for the horse to be free and for the combined holdings of all the men to be zero.

As with the unsolvable giving and taking problem, this problem was easy to find. First, as can be seen from the problems listed in Appendix C, the form of the problem arose naturally as Leonardo systematically explored variations on the classic problem of "men wishing to buy a horse". It is worth noting, however, that he presents a solvable five-man problem first, before giving the unsolvable four-man problem. This break from the natural sequence of increasing complication (see Appendices A and B, for example) suggests that Leonardo wished to establish first that his methods did still work for this variation of horse-buying problems. Secondly, if we take the modern perspective once again, horse-buying problems involving pairs of men from amongst four men can be written in matrix form:

$$\begin{bmatrix} 1 & 1 & \alpha & \alpha \\ \beta & 1 & 1 & \beta \\ \gamma & \gamma & 1 & 1 \\ 1 & \delta & \delta & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = h \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

In our case, the parameter values (the fractions that we need from what you hold) are

$$\alpha = \tfrac{1}{2}, \quad \beta = \tfrac{1}{3}, \quad \gamma = \tfrac{1}{4}, \quad \delta = \tfrac{1}{5}$$

but, regardless of the parameter values, the coefficient matrix still has rank at most 3 and so, if $h \neq 0$, almost all such problems are inconsistent. Thus, once again, Leonardo did not need to search for this unsolvable problem, or to use a formula to choose suitable values for the parameters. In a sense, he probably stumbled on the problem.

As Leonardo promised, he immediately gives a solvable problem of the same type (Boncompagni 1857, p. 252) or (Sigler 2002, p. 366):

$$\begin{aligned} a + b + \tfrac{1}{2}(c + d) &= h \\ b + c + \tfrac{3}{7}(d + a) &= h \\ c + d + \tfrac{3}{11}(a + b) &= h \\ d + a + \tfrac{5}{13}(b + c) &= h \end{aligned}$$

In Leonardo's solution the four men have 5, 6, 7 and 9 bezants (and so the price of the horse is 19 bezants). So it seems likely that he constructed this problem by starting from his solution. This time substituting the values from a simple solution has resulted in proper fractions which obey the earlier conventions for values of parameters which are expected to take fractional values. Further evidence that Leonardo proceeded this way lies in the fact that this is the "first" problem which satisfies these conventions. More precisely, if you construct a four-man problem starting from a solution of the form $a = n$, $b = n + 1$, $c = n + 2$ and $d = n + 4$ (so that $a + b + \frac{1}{2}(c + d)$ is a whole number) then $n = 5$ is the first case which gives parameter values conforming to the above conventions. In other words, in a systematic search for a solvable problem, this is the first suitable example which Leonardo would have found. Amongst the smaller values of n, the cases $n = 1,\ 2$ lead to scenarios where two of the men can already buy the horse, whilst the cases $n = 3,\ 4$ gives sequences of parameter values $\frac{1}{2}, \frac{2}{5}, \frac{1}{7}, \frac{1}{3}$, and $\frac{1}{2}, \frac{5}{12}, \frac{2}{9}, \frac{4}{11}$, respectively, which do not conform to the above conventions.

The fact that this problem is in some sense the smallest suitable example, raises the issue of whether Leonardo consciously sought out what Thomaidis calls "first admissible cases" (Thomaidis 2005, pp. 604–605). This certainly seems to be the case for Problems 5 and 7 in Appendix A, for example. The parameter values 12 and 15 (respectively) are the first values which yield unsolvable problems for giving and taking problems with that structure, and Leonardo does comment on this fact when discussing Problem 5.[22] As Thomaidis makes clear, Diophantos showed an implicit interest in this issue (in the sense that he sometimes uses the first admissible case to illustrate how to solve a problem) and Leonardo's approximate contemporary, Maximus Planudes, showed an explicit interest when commenting on Diophantos' work. In the present context, the question is whether Leonardo was interested in demarcating the boundary between solvable and unsolvable problems. Certainly his treatment of all these unsolvable problems suggests an interest, but it may be that a full investigation was beyond his capability. For example, in his treatment of Problem 9 in Appendix A (which involves both a surplus $+6$ and a deficit -8) he mentions that, although his example is solvable, some such problems are not solvable (Boncompagni 1857, p. 198) or (Sigler 2002, p. 300). However his chosen example is not a critical example this time, which suggests that he did not know the critical values. Of course, if he had a formula for the solution of such problems, it should have been a straightforward matter to provide such a critical example.

Finally we look at a situation where it appears that Leonardo was unable to find a problem satisfying the expected conventions. As before, the problems concerned focus on the solvability issue, this time dealing with four men, pairs of whom take a

[22] Boncompagni and Franci (1857, p. 193) or Sigler (2002, 294).

purse (Boncompagni 1857, p. 227) or (Sigler 2002, p. 336):[23]

$$\begin{aligned} a+b+p &= 2(c+d) \\ b+c+p &= 3(d+a) \\ c+d+p &= 4(a+b) \\ d+a+p &= 5(b+c) \end{aligned}$$

Once again it worth seeing why Leonardo thinks this problem is unsolvable. As before, he works in terms of the *whole amount* of money present:

$$S = a+b+c+d+p.$$

Using a similar argument to the one we have already seen, he uses the first and third conditions to show that the combined holdings of the four men are

$$\left(\tfrac{1}{3}+\tfrac{1}{5}\right)S, \quad \text{or} \quad \tfrac{32}{60}S$$

whilst the other two conditions show that the combined holdings are

$$\left(\tfrac{1}{4}+\tfrac{1}{6}\right)S, \quad \text{or} \quad \tfrac{25}{60}S.$$

He calls this situation *absurd*. Once again it is easy to understand how an audience might disapprove of a solution to a money problem where the total amount of money present is zero.

As with the earlier unsolvable examples, it seems likely that Leonardo stumbled upon this example, as opposed to deliberately constructing it. Indeed, as Appendix B shows, the form of the problem arose naturally during Leonardo's systematic exploration of variations on the classic problem "men finding a purse". Furthermore, if we once again take the modern viewpoint, purse problems like this one can be written in matrix form:

$$\begin{bmatrix} 1 & 1 & -\alpha & -\alpha \\ -\beta & 1 & 1 & -\beta \\ -\gamma & -\gamma & 1 & 1 \\ 1 & -\delta & -\delta & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = -p \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

In our case, the parameter values (ratios comparing what we will have with what you already have) are

$$\alpha = 2, \quad \beta = 3, \quad \gamma = 4, \quad \delta = 5.$$

But, regardless of the parameter values, the coefficient matrix still has rank at most 3 and so, if $p \neq 0$, almost all such problems are inconsistent.

[23] *The first and the second men, with the purse, have twice the denari of the third and fourth,* and so on.

From what we have seen with the preceding problems, we might expect Leonardo to follow this unsolvable problem with a solvable one having a similar structure. Once again, this requires care since almost any other choice of proportions α, β, γ, δ instead of 2, 3, 4, 5 will produce the same absurdity which we have just seen. It is possible that Leonardo tried to construct a solvable problem by starting with a solution. However, his calculations above show that a solvable problem needs to have

$$\frac{1}{\alpha+1}+\frac{1}{\gamma+1}=\frac{1}{\beta+1}+\frac{1}{\delta+1}. \tag{1}$$

Furthermore, in this case, the relationship between the original parameter values $\alpha, \beta, \gamma, \delta$ and the condition which they need to satisfy is simple enough that even someone without symbolic algebra could reasonably be expected to observe the pattern. Indeed, Leonardo's habit of representing fractions as sums of unit fractions means that he would see, on the calculating surface in front of him, a condition for solvability quite similar to the above algebraic condition. Hence, it is entirely conceivable that Leonardo realized that no solvable problem could satisfy the convention for purse problems, namely that:

$$\alpha < \beta < \gamma < \delta.$$

In any case, Leonardo makes no comment this time about offering a solvable example of the same type. As we might expect, his next problem is indeed a solvable one, but this time it involves an extra man (Boncompagni 1857, p. 227) or (Sigler 2002, p. 336):

$$\begin{aligned}
a+b+p &= 2(c+d+e)\\
b+c+p &= 3(d+e+a)\\
c+d+p &= 4(e+a+b)\\
d+e+p &= 5(a+b+c)\\
e+a+p &= 6(b+c+d)
\end{aligned}$$

As can be seen from a glance at the corresponding matrix formulations, from the modern point of view the unsolvable purse problem is essentially the same as the previous unsolvable horse-buying problem. However, from Leonardo's point of view, there may have been a subtle difference. The unsolvable purse problem is followed by a solvable problem involving an extra man, but the unsolvable horse-buying problem is followed by a solvable problem with the same number of men. As we have seen, Leonardo could not give a solvable purse problem for four men without abandoning his apparent conventions for the wording of such problems. For purse problems, this meant having increasing integer parameter values as you read through the statement of the problem. However, in the case of the horse-buying problem, his conventions did allow him to present a solvable four-man problem. Horse-buying problems involved the transfer of some fraction of a man's (or a group's) money, and his convention for fraction parameter values did allow him to satisfy condition (1).

4 Conclusions

We have seen that the so-called recreational problems in Leonardo's *Liber Abbaci* seem to follow various conventions. Thus Leonardo shows a marked preference for problems which have relatively simple data (small integers or unit fractions) and relatively complicated solutions (non-integers or relatively large integers). Such problems were almost certainly constructed first, with the solution calculated afterwards. This probably means that the poser of the problem possessed, and expected any potential solver to possess, a general method of solution. Simple data may also have been easier to remember during the solution process, whilst a complicated answer may have discouraged a potential solver from seeking the correct answer simply by guessing. All these features would make such problems useful, not just in the classroom, but also in the context of challenge problems posed for one another by professional mathematicians (for want of a better title).

Another convention appears in complex problems (those involving three or more unknowns). Here we see that the various parameters usually increase as you read through the problem. In particular, if parameters were expected to be fractions, then they were usually expressed in terms of unit fractions, with increasing denominators. Similarly, proper fractions usually had increasing numerators and denominators. Increasing parameter values may originally have been a response to the extra load on the hearer's memory when a complex problem was recited, as increasing (or decreasing) sequences of numbers are probably easier to remember.

A glance at other ancient or mediaeval collections of problems shows that their authors usually followed these same conventions, although perhaps not quite as rigidly as Leonardo did (see the extensive lists in Singmaster (2004), for example). We have seen that Leonardo's own attachment to these conventions was tested when there was a potential conflict between the conventions and his overall plan for Chap. 12 of *Liber Abbaci*. This plan involved showing that various solution methods (such as false position and *regula recta*) could be used to solve many problems. As he worked systematically through variations on some basic scenarios, such as men giving and taking, or men finding a purse, or men wishing to buy a horse, he stumbled on examples which were not solvable. Problems 14 from Appendices A and B, and Problem 22 from Appendix C, could all arise in this way, since almost all problems with the same structure are also unsolvable. In particular, there is no need to postulate, at least not for these examples, that Leonardo had a general formula for solving all such problems, and that he used this formula to deliberately choose parameter values which would lead, say, to negative solutions. It has to be admitted that Problems 5 and 7 from Appendix A do seem to have been sought out deliberately, as they represent critical cases on the border between solvability and unsolvability. But these examples are small enough that a patient incremental search could have unearthed them.

Euclidean practice suggests that the ideal response to unsolvable problems is to state necessary and sufficient conditions which guarantee solvability, as in *Elements, I, 22 and VI, 28*. Leonardo's critical examples in Problems 5 and 7 from Appendix A serve this purpose for those simple variations of giving and taking problems. Furthermore he reaches towards this goal when he remarks, before Problem 22 from Appendix C, that some problems may be unsolvable if the number of men is even and two or more men take from the others (Boncompagni 1857, p. 251) or (Sigler 2002, p. 365). But it seems likely that Leonardo could not supply necessary and sufficient conditions for the solvability of any of the complex variations on classical problems discussed in this article. Indeed, even recognizing that a problem was unsolvable was a non-trivial calculation. Perhaps because unsolvable problems could arise accidentally, just by exploring natural variations on standard problems, Leonardo instead devoted some time to showing his readers how to distinguish between the solvable and unsolvable cases. To do this he needed to construct solvable examples sufficiently like the unsolvable ones, as with the pairs of Problems 14 and 15 from Appendix A and Problems 22 and 23 from Appendix C. In the latter case, the solvable problem was relatively easy to construct from a simple solution, although it seems that Leonardo ignored two "smaller" examples in order to make sure that his example fitted the above conventions. But the construction of Problem 15 from Appendix A was significantly more intricate. Here, if you start from a solution and retain the most convenient parameter values (the amounts being exchanged) a straightforward calculation gives the values for the other parameter (the ratio between people's holdings after the exchanges) but unfortunately these values typically violate the conventions. Instead, Leonardo decided to retain the values of the ratio parameter, and seems to have used some careful juggling of information to calculate both suitable values of the other parameter and a solution to go with them. This suggests that Leonardo took considerable care to make sure his examples fitted the conventions, as if the conventions were rules which had to be followed when you constructed a problem.

Owing to the existence of unsolvable problems, Leonardo needed to add a second focus to his basic plan. Not only did he need to show his readers how to solve problems, but also how to recognize unsolvable problems. Where possible, he seems to have sought out solvable variations of each unsolvable problem. But Leonardo makes no mention of how one might deliberately construct an unsolvable problem, and instead focuses on distinguishing between solvable and unsolvable problems. In other words, these sections of *Liber Abbaci* could equip the reader to solve challenge problems (using Leonardo's various methods of solution), and to set challenge problems (simply by choosing their own favourite parameter values in any of these typical problems discussed by Leonardo) and finally to avoid setting unsolvable challenge problems (because he has shown his readers how to recognize the unsolvable cases). It seems that his readers did not need to know how to construct an unsolvable problem (perhaps it was considered absurd or even unfair to set unsolvable problems). Indeed, as we have seen, it is possible that Leonardo himself did not know how to construct an unsolvable problem, unless it was by executing patient incremental searches or by stumbling upon them in the course of a systematic examination of families of problems. But Leonardo did know how to construct solvable problems under difficult circumstances, and although he withheld this knowledge from his readers, we have seen that he could have constructed these problems using just the techniques which he taught in *Liber Abbaci*.

Appendices

Appendix A: giving and taking problems from Liber Abbaci

Listed below are the 16 problems on giving and taking from Sect. 3 of Chap. 12 of *Liber Abbaci*. Section 3 consists of almost 100 problems, and its main purpose is to explain the method of (single) false position. To help the modern reader understand the relationships between the various problems, I have presented the problems in symbolic form. To see the problems as Leonardo would have seen them you would, of course, need to read them as stories about men giving and taking denari. For examples of such readings, see the footnotes earlier in the article. For Leonardo, Problems 5, 7 and 14 are unsolvable.

1. $a + 1 = b - 1$
   $b + 1 = 10(a - 1)$
   1(a). $a + 1 = b - 1$
   $b + 1 = 12(a - 1)$
2. $a + 7 = 5(b - 7)$
   $b + 5 = 7(a - 5)$
3. $a + 6 = 5\frac{1}{4}(b - 6)$
   $b + 4 = 7\frac{2}{3}(a - 4)$
4. $a + 7 = 5(b - 7) + 1$
   $b + 5 = 7(a - 5) + 1$
5. $a + 7 = 5(b - 7) + 12$
   $b + 5 = 7(a - 5) + 12$
6. $a + 7 = 5(b - 7) + 1$
   $b + 5 = 7(a - 5) + 2$
7. $a + 7 = 5(b - 7) + 1$
   $b + 5 = 7(a - 5) + 15$
8. $a + 7 = 5(b - 7) - 1$
   $b + 5 = 7(a - 5) - 3$
9. $a + 7 = 5(b - 7) + 6$
   $b + 5 = 7(a - 5) - 8$
10. $a + 7 = 5(b + c - 7)$
    $b + 9 = 6(c + a - 9)$
    $c + 11 = 7(a + b - 11)$
11. $a + 7 = 5(b + c - 7) + 1$
    $b + 9 = 6(c + a - 9) + 1$
    $c + 11 = 7(a + b - 11) + 1$
12. $a + 7 = 5(b + c - 7) + 1$
    $b + 9 = 6(c + a - 9) + 2$
    $c + 11 = 7(a + b - 11) + 3$
13. $a + b + 7 = 5(c - 7)$
    $b + c + 9 = 6(a - 9)$
    $c + a + 11 = 7(b - 11)$

14. $a+b+7=3(c+d-7)$
    $b+c+8=4(d+a-8)$
    $c+d+9=5(a+b-9)$
    $d+a+11=6(b+c-11)$
15. $a+b+100=3(c+d-100)$
    $b+c+106=4(d+a-106)$
    $c+d+145=5(a+b-145)$
    $d+a+170=6(b+c-170)$
16. $a+b+c+7=2(d+e-7)$
    $b+c+d+8=3(e+a-8)$
    $c+d+e+9=4(a+b-9)$
    $d+e+a+10=5(b+c-10)$
    $e+a+b+11=6(c+d-11)$

Appendix B: purse problems from Liber Abbaci

Listed below are the 15 problems about finding a purse from Sect. 4 of Chap. 12 of *Liber Abbaci*. As with the giving and taking problems in Appendix A, these problems should be read as stories. For examples of such readings, see the footnotes earlier in the article. Some of the problems are followed by subproblems where, for example, values for the contents of the purse (p) are specified. For Leonardo, Problem 14 is unsolvable.

1. $a+p=3b$
   $b+p=4a$
2. $a+p=2(b+c)$
   $b+p=3(c+a)$
   $c+p=4(a+b)$
3. $a+p=3(b+c+d)$
   $b+p=4(c+d+a)$
   $c+p=5(d+a+b)$
   $d+p=6(a+b+c)$
4. $a+p=2\frac{1}{2}(b+c+d+e)$
   $b+p=3\frac{1}{3}(c+d+e+a)$
   $c+p=4\frac{1}{4}(d+e+a+b)$
   $d+p=5\frac{1}{5}(e+a+b+c)$
   $e+p=6\frac{1}{6}(a+b+c+d)$
5. $a+p=2b$
   $b+p=3c$
   $c+p=4a$
6. $a+p=2\frac{1}{2}b$
   $b+p=3\frac{1}{3}c$
   $c+p=4\frac{1}{4}a$
7. $a+p=2b$
   $b+p=3c$

$c + p = 4d$
$d + p = 5a$

8. $a + p = 2b$
$b + (p + 3) = 3c$
$c + (p + 7) = 4d$
$d + (p + 13) = 5a$
9. $a + p = 2b$
$b + (p + 13) = 3a$
10. $a + p = 2(b + c)$
$b + (p + 10) = 3(c + a)$
$c + (p + 10 + 13) = 4(a + b)$
11. $a + p = 2(b + c + d)$
$b + (p + 10) = 3(c + d + a)$
$c + (p + 10 + 13) = 4(d + a + b)$
$d + (p + 10 + 13 + 19) = 5(a + b + c)$
12. $a + b + p = 2c$
$b + c + p = 3a$
$c + a + p = 4b$
13. $a + b + p = 2c$
$b + c + p = 3d$
$c + d + p = 4a$
$d + a + p = 5b$
14. $a + b + p = 2(c + d)$
$b + c + p = 3(d + a)$
$c + d + p = 4(a + b)$
$d + a + p = 5(b + c)$
15. $a + b + p = 2(c + d + e)$
$b + c + p = 3(d + e + a)$
$c + d + p = 4(e + a + b)$
$d + e + p = 5(a + b + c)$
$e + a + p = 6(b + c + d)$

Appendix C: horse-buying problems from Liber Abbaci

Listed below are five problems about men wishing to buy a horse from Sect. 5 of Chap. 12 of *Liber Abbaci*. The numbering of the problems comes from Vogel (1940) and the full collection of horse-buying problems is listed and analysed there. The five problems listed below make up Vogel's Group 6, dealing with situations where subgroups of the men (for example, pairs) seek a fraction of what all the other men hold in order to have enough money to buy the horse. The case involving pairs of men doing the buying, from a group of three, was covered in an earlier group of problems (Vogel's Group 4) under the guise of one man giving to the rest. As usual, these problems should be read as stories. For examples of such readings, see the footnotes earlier in the article. For Leonardo, Problem 22 is unsolvable.

21. $(a+b)+\frac{1}{2}(c+d+e)=h$
 $(b+c)+\frac{1}{3}(d+e+a)=h$
 $(c+d)+\frac{1}{4}(e+a+b)=h$
 $(d+e)+\frac{1}{5}(a+b+c)=h$
 $(e+a)+\frac{1}{6}(b+c+d)=h$
22. $(a+b)+\frac{1}{2}(c+d)=h$
 $(b+c)+\frac{1}{3}(d+a)=h$
 $(c+d)+\frac{1}{4}(a+b)=h$
 $(d+a)+\frac{1}{5}(b+c)=h$
23. $(a+b)+\frac{1}{2}(c+d)=h$
 $(b+c)+\frac{3}{7}(d+a)=h$
 $(c+d)+\frac{3}{11}(a+b)=h$
 $(d+a)+\frac{5}{13}(b+c)=h$
24. $(a+b+c)+\frac{1}{2}(d+e+f+g)=h$
 $(b+c+d)+\frac{1}{3}(e+f+g+a)=h$
 $(c+d+e)+\frac{1}{4}(f+g+a+b)=h$
 $(d+e+f)+\frac{1}{5}(g+a+b+c)=h$
 $(e+f+g)+\frac{1}{6}(a+b+c+d)=h$
 $(f+g+a)+\frac{1}{7}(b+c+d+e)=h$
 $(g+a+b)+\frac{1}{8}(c+d+e+f)=h$
25. $(a+b+c)+\frac{1}{3}(d+e+f+g)=h$
 $(b+c+d)+\frac{1}{4}(e+f+g+a)=h$
 $(c+d+e)+\frac{1}{5}(f+g+a+b)=h$
 $(d+e+f)+\frac{1}{6}(g+a+b+c)=h$
 $(e+f+g)+\frac{1}{7}(a+b+c+d)=h$
 $(f+g+a)+\frac{1}{8}(b+c+d+e)=h$
 $(g+a+b)+\frac{1}{9}(c+d+e+f)=h$

References

Bartolozzi, M., and R. Franci. 1990. La teoria delle proporzioni nella matematica dell'abaco da Leonardo Pisano a Luca Pacioli. *Bollettino di Storia della Scienze Matematiche* 10: 3–28.

Boncompagni, B. 1857. *Liber Abbaci*. In: *Scritti di Leonardo Pisano*. Rome: Tipografia delle Scienze Mathematiche e Fisiche.

Busard, H.L.L. 1987. *The Mediaeval Latin translation of Euclid's "Elements": made directly from the Greek*. Stuttgart: Franz Steiner Verlag.

Dauben, J.W. 2008. *Suan Shu Shu* a book on numbers and computations. *Archive for History of Exact Sciences* 62: 91–178.

Docampo Rey, J. 2009. A new source for medieval mathematics in the Iberian Peninsula: The commercial arithmetic in Ms 10116 (Biblioteca Nacional, Madrid). *Revue d'Histoire des Mathématiques* 15: 123–177.

Folkerts, M. 2005. Leonardo Fibonacci's knowledge of Euclid's Elements and of other mathematical texts. *Bollettino di Storia della Scienze Matematiche* 24 (2004), no. 1, Leonardo Fibonacci. Matematica e societa nel Mediterraneo nel secolo XIII. vol. II: 93–113.

Franci, R., and L. Toti Rigatelli. 1985. Towards a history of algebra from Leonardo of Pisa to Luca Pacioli. *Janus* 72: 17–82.

Heath, T.L. 1910. *Diophantus of Alexandria: A study in the history of Greek algebra*. Cambridge: Cambridge University Press.

Heath, T.L. 1926. *The thirteen books of Euclid's Elements*, 2nd ed. Cambridge: Cambridge University Press. Reprint: Dover Publications, New York, 1956.

Høyrup, J. 2002. *Lengths, widths, surfaces. A portrait of old Babylonian algebra and its kin*. New York: Springer-Verlag.

Høyrup, J., 2005. Leonardo Fibonacci and *Abbaco* culture. A proposal to invert the roles. *Revue d'Histoire des Mathématiques* 11: 23–56.

Høyrup, J. 2007. *Jacopo da Firenze's Tractatus algorismi and early Italian abbacus culture*. Science networks. Historical studies, vol. 34. Basel: Birkhäuser Verlag.

Katz, V. 1998. *A history of mathematics*, 2nd ed. Reading: Addison-Wesley.

Mahavira, 1912. *Ganita Sara Sangraha* (trans: Rangacarya, M.). Madras: Government Press.

Netz, R. 1999. *The shaping of deduction in Greek mathematics. A study in cognitive history*. Cambridge: Cambridge University Press.

Rashed, R. 1994. Fibonacci et les mathématiques arabes. In *Le scienze alla corte di Federico II*. Micrologus 2: 145–160.

Sesiano, J. 1985. The appearance of negative solutions in mediaeval mathematics. *Archive for History of Exact Sciences* 32(2): 105–150.

Sesiano, J. 2009. *An introduction to the history of algebra*. Mathematical World, vol. 27. Providence: American Mathematical Society.

Sigler, L.E. 2002. *Fibonacci's Liber Abaci*. New York: Springer.

Singmaster, D. 1988. Some early sources in recreational mathematics. In *Mathematics from manuscript to print, 1300–1600 (Oxford, 1984)*, ed. C. Hay, 195–208. Oxford: Oxford University Press.

Singmaster, D. 2004. *Sources in recreational mathematics: An annotated bibliography*. Source files available at http://www.g4g4.com/MyCD5/SOURCES/singmaterial.htm. Accessed 21 Jan 2010.

Spiesser, M. 2000. Problèmes linéaires dans le Compendy de la praticque des nombres de Barthélemy de Romans et Mathieu Préhoude (1471). *Historia Mathematica* 27: 362–383.

Thomaidis, Y. 2005. A framework for defining the generality of Diophantos' methods in "Arithmetica". *Archive for History of Exact Sciences* 59: 591–640.

Ver Eecke, P. 1959. *Diophante d'Alexandrie: Les six livres arithmétiques et le livre des nombres polygones*. Paris: Albert Blanchard.

Vogel, K. 1940. Zur Geschichte der linearen Gleichungen mit mehreren Unbekannten. *Deutsche Mathematik* 5: 217–240. Available as a reprint in: Vogel, K. 1988. *Kleinere Schriften zur Geschichte der Mathematik*. Boethius: Texte und Abhandlungen zur Geschichte der Exakten Wissenschaften, vol. 20, 1–2. Stuttgart: Franz Steiner Verlag, pp. 281–304.

Vogel, K. 1971. *Fibonacci*. In *Dictionary of scientific biography, vol. IV*, ed. C.C. Gillespie, 604–613. New York: Scribner's.

Woepcke, F. 1853. *Extrait du Fakhrî, traité d'algèbre par Aboû Bekr Mohammed Ben Alhaçan Alkarkhî*. Reprint: Hildesheim: Georg Olms Verlag, 1982.