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Gearing up for Lagrangian dynamics

The flywheel analogy in Maxwell's 1865 paper on electrodynamics

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Abstract James Clerk Maxwell's 1865 paper, "A Dynamical Theory of the Electromagnetic Field," is usually remembered as replacing the mechanical model that underpins his 1862 publication with abstract mathematics. Up to this point historians have considered Maxwell's usage of Lagrangian dynamics as the sole important feature that guides Maxwell's analysis of electromagnetic phenomena in his 1865 publication. This paper offers an account of the often ignored mechanical analogy that Maxwell used to guide him and his readers in the construction of his new electromagnetic equations. The mechanical system consists of a weighted flywheel geared into two independently driven crank wheels in what amounts to a mechanical differential. I will demonstrate how Maxwell made use of the analogy between his flywheel system and electromagnetic induction to ground his study of electromagnetism in clear mechanical conceptions and to structure the derivation of the equations that together are now recognized as Maxwell's equations for electrodynamics. By reconceiving specific components of his model in electromagnetic terms, while at the same time retaining many of the relations between concepts in the mechanical case, Maxwell gradually assembled increasingly generalized equations for electromotive force. Maxwell thus realized a much sought after balance between physical analogy and abstract mathematics in this, the last of his three seminal papers on electromagnetism.

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Introduction

In the introduction to his *Natural Philosophy of James Clerk Maxwell*, Peter Harman writes:

> The mechanical or dynamical worldview, which dominated the programme of physical explanation in the nineteenth century, shaped Maxwell's scientific theorising. But his attitude to mechanical explanation was complex. There was a tension in his thought between physical and mathematical models of mechanical systems; and his reflections on the relationship between mechanical representations and physical reality shaped his evolving programme of explanation. (Harman 1998, p. 4)

As historians have constructed the narrative of the maturation of Maxwell's views on electricity and magnetism, weaving it through the three seminal papers that he delivered before his *Treatise on Electricity and Magnetism* was first published in 1873, they have also necessarily constructed a narrative around this evolving tension between physical and mathematical explanation. Maxwell's use of physical analogies and models fluctuates between his three major papers. To understand the approaches Maxwell took toward demystifying electromagnetic phenomena and eventually codifying its underlying relations we must come to terms with the ways in which physical analogies and models shaped his work. A crude summary of this story runs as follows. In 1855, Maxwell used an illustrative physical analogy to fluid flow to mathematize Faraday's lines of force (Maxwell 1855). In 1862, he constructed a hypothetical physical model of the electromagnetic medium which explained and at times was even directly involved in the derivation of laws of electromagnetism (Maxwell 1862). Finally, in 1865 Maxwell dropped his prior commitments to physical analogy, instead importing the dynamical equations of Lagrange to serve as the sole aid in the development of his general equations of the field (Maxwell 1865).[1] The emphasis previous historians have placed upon the apparently abstract character of the 1865 paper sig-

[1] The picture of Maxwell's 1865 paper "A Dynamical Theory of the Electromagnetic Field" as founded exclusively on the abstract analysis of Lagrangian dynamics is commonly accompanied by quotations from two of Maxwell's letters, one before the paper's publication and one after. In the first, a letter from Maxwell to William Thomson dated October 15, 1864, Maxwell states: "I can find the velocity of transmission of electromagnetic disturbances independently of any hypothesis now and it is equal to v [the speed of light]" (Maxwell 1995, p. 180). As both Martin Goldman and Harman dutifully point out, by 1864 Maxwell had absolved himself of his early reliance on hypotheses about the electromagnetic medium as well as his impious inclusion of terms that emerge directly from his mechanical analogy (Goldman 1983, p. 155; Harman 1998, p. 113). The second quotation, from a late December 1867 letter from Maxwell to Peter Guthrie Tait, compares the 1862 and 1865 papers: "The former is built up to show that the phenomena are such as can be explained by mechanism...The latter is built on Lagrange's Dynamical Equations and is not wise about vortices" (Maxwell 1995, p. 337). Harman and Goldman both accept this statement as evidence of the purely mathematical nature of Maxwell's methodology in the 1865 paper (Goldman 1983, p. 155; Harman 1998, p. 118). Nevertheless, they both admit that despite reforming his methods, Maxwell's new conception of the electromagnetic field as a "complicated mechanism" is still explicitly based upon a physical/mechanical analogy (Maxwell 1865, p. 533). Each letter makes clear (as does the content of the paper itself) that Maxwell is no longer relying on a hypothetical model, but the second is supposed to show that Maxwell believes that the analogy in the 1865 paper is to nothing but Lagrangian dynamics.

nificantly distinguishes it from the mechanical specificity of the earlier 1862 model. As Harman puts it, in 1865 "[t]he mechanical analogy for the electromagnetic field is explicated in terms of energy relations, rather than the pictorial mechanical model contrived in 'On physical lines of force'" (Harman 1998, p. 116). There is undoubtedly truth to this claim concerning Maxwell's "Dynamical Theory," as the theory is founded on the methods of Lagrangian dynamics that apply to any system defined by the same energy and momentum relations that apply in mechanics. Nevertheless, a close look at Maxwell's application of Lagrangian dynamics yields a picture of his 1865 approach that is governed by something beyond pure dynamical abstraction. Led by the peculiarities of Maxwell's analysis, I will introduce a more harmonious account of Maxwell's explanatory approach in "A Dynamical Theory of the Electromagnetic Field," one in which Maxwell strikes a balance between his often ignored analogy to a mechanical flywheel and the more abstract mathematical analysis that is usually identified as the hallmark of the paper. This analogy between a particular differential system and electromagnetism grounds Maxwell's study of electromagnetism in clear mechanical conceptions and structures the derivation of the equations that together are now recognized as Maxwell's equations for electrodynamics.

Martin Goldman and Thomas K. Simpson both invoke Maxwell's famous bell-ringer metaphor to describe the dynamical approach of the 1865 paper (Goldman 1983, pp. 155–156; Simpson 1997, p. 372).

> In an ordinary belfry, each bell has one rope which comes down through a hole in the floor to the bellringer's room. But suppose that each rope, instead of acting on one bell, contributes to the motion of many pieces of machinery, and that the motion of each piece is determined not by the motion of one rope alone, but by that of several, and suppose, further, that all this machinery is silent and utterly unknown to the men at the ropes, who can only see as far as the holes in the floor above them. (Maxwell 1879, p. 783)

In this metaphor impressing velocities on the ropes hanging from the belfry and measuring impulses for every possible position and final velocity of the ropes is the only recourse available to the men below. Nonetheless, they still may obtain generalized Lagrange equations which describe the motion of the ropes due to any impressed force despite total ignorance of the system of bells above. The association of this sort of black-box reasoning with Maxwell's use of Lagrangian dynamics in his 1865 paper naturally discourages the discussion of any aspects of a physical/mechanical analogy above and beyond the equations of Lagrangian dynamics. If Lagrangian dynamics allows us to black-box the mechanism under investigation, then the inclusion of any sort of additional physical or mechanical analogy would be entirely superfluous. If this is Maxwell's picture of how to apply Lagrangian dynamics, we should not expect him to pair this sort of mathematical analysis with the kind of pictorial physical analogy seen in his first paper on electromagnetism, and should be even more skeptical of a pairing with a mechanical model like the one presented in his second paper. This may have become Maxwell's picture eventually, but, as we will see, it was not in 1865. The quote above comes from the first appearance of the bell-ringer analogy in print, an 1879 review by Maxwell of the second edition of Thomson and Tait's *Natural Philosophy*. The first edition of his *Treatise* was preempted by the first edition of Thomson and Tait's *Natural Philosophy* (1873) and we can find a discussion of something like a proto-bell-ringer metaphor, albeit concerning his preferred Hamiltonian form of dynamics, within the pages of Maxwell's book:

> I have applied this method [Hamiltonian/Lagrangian dynamics] so as to avoid the explicit consideration of the motion of any part of the system except the coordinates or variables, on which the motion of the whole depends. It is doubtless important that the student should be able to trace the connexion of the motion of each part of the system with that of the variables, but it is by no means necessary to do this in the process of obtaining the final equations, which are independent of the particular form of these connexions. (Maxwell 1873, Vol. 2, p.185)

Given the similar approaches found in the 1865 paper and the *Treatise* and given that the analogical scaffolding of any sort of mechanical differential (let alone the more specific flywheel system) is entirely absent from the first (and second) edition of the *Treatise*, it should not come as a surprise that the bell-ringer metaphor has been exported to describe Maxwell's method in the 1865 paper.

We should, however, not read Maxwell's affection for abstract Lagrangian dynamics suggested by the bell-ringer metaphor or for Hamiltonian dynamics from the first edition of the *Treatise* back into the 1865 paper. In fact, it seems much more likely that this black-box style of analysis was at least partly inspired by Thomson and Tait's *Natural Philosophy*, something Maxwell suggests in his 1879 review:

> The credit of breaking up the monopoly of the great masters of the spell [general theory of dynamics] and making all their charms familiar in our ears as household words, belongs in great measure to Thomson and Tait. (Maxwell 1879, p. 782)

With the bell-ringer metaphor exorcized from the narrative surrounding Maxwell's "Dynamical Theory of the Electromagnetic Field" we can reexamine whether it is appropriate to characterize Maxwell's approach in his 1865 paper as something purely "abstract and general rather than concrete and pictorial" (Siegel 1991, p. 50). Is it fair to say that Maxwell's analysis in his 1865 paper is guided only by a clever extension of Lagrangian dynamics?

In what follows I will argue that Maxwell's introduction of Lagrangian dynamics in "A Dynamical Theory of the Electromagnetic Field" is guided by an amalgam of Maxwell's approaches that preceded it. The mechanical reasoning that is so characteristic of his 1862 paper remained a central, albeit less conspicuous, aspect of his "Dynamical Theory;" however, it is the more philosophically cautious 1855 paper that most closely resembles the methodological approach that Maxwell adopted in his third major paper on electromagnetism. Indeed in his third paper Maxwell's extension of Lagrangian dynamics from mechanical interactions to electrodynamics and optics presupposes the existence of some sort of mechanical model, some "complicated mechanism"; however, unlike his second paper, this dynamical theory is not dependant on the specifics of any one model (Maxwell 1865, p. 533). We shall see that while Lagrangian dynamics offered Maxwell more than one way to structure his analysis, especially in one particular case (cf. Sect. 3.4), the approach taken by Maxwell embodies relations expressed in the dynamics of a particular mechanical system, namely the flywheel.[2] The flywheel is referenced only briefly toward the beginning of Maxwell's paper; however, we will come to recognize just how impactful the concepts and relations governing it were when transferred to electromagnetism.

Broadly speaking, in the 1865 paper Maxwell found a way to realize a successful harmony between physical analogy and abstract mathematics that he had called for in his first major paper on electromagnetism all the way back in 1855:

> The first process therefore in the effectual study of the science, must be one of simplification and reduction of the results of previous investigation to a form in which the mind can grasp them. The results of this simplification may take the form of a purely mathematical formula or of a physical hypothesis. In the first case we entirely lose sight of the phenomena to be explained; and though we may trace out the consequences of given laws, we can never obtain more extended views of the connexions of the subject. If, on the other hand, we adopt a physical hypothesis, we see the phenomena only through a medium, and are liable to that blindness to facts and rashness in assumption which a partial explanation encourages. We must therefore discover some method of investigation which allows the mind at every step to lay hold of a clear physical conception, without being committed to any theory founded on the physical science from which that conception is borrowed, so that it is neither drawn aside from the subject in pursuit of analytical subtleties, nor carried beyond the truth by a favourite hypothesis. (Maxwell 1855, pp. 155–156; quoted and discussed by Goldman 1983, pp. 140–141)

Maxwell retained much of this sense of paired physical and mathematical analyses which had brought him success in his first electromagnetic paper in "A Dynamical Theory." As Sir James Jeans notes in a posthumous essay on Maxwell's scientific methodology, "[f]rom the very beginning his [Maxwell's] mathematical ideas were not only guided but controlled by a strong sense of physical reality" (Jeans 1931, p. 96).

The role of physical analogy, and thus the flywheel analogy, in 1865 was roughly similar to the role of the flow analogy in his 1855 paper, both "show[ed] how to partition the space" (Wise 1977, p. 215). The flywheel provided a "natural physical partitioning" that allowed Maxwell to not only generate particular conceptual units but also a roadmap to direct his construction of a new dynamical theory of electromagnetism (Wise 1977, p. 136). As such, although the analogy to a mechanical flywheel is presented as a "dynamical illustration," its role was not limited to just clarifying

[2] Olivier Darrigol gestures to the presence of the flywheel in the 1865 paper, although he assigns it little significance (Darrigol 2000, p. 156). Francis Everitt takes the time to explain how the analogy is constructed although he does not delve much into its uses (Everitt 1975, pp. 103–105). Daniel Siegel acknowledges the status of the flywheel as a mechanical analogue for inductive circuits in his notes, citing Everitt, but given his book's focus on Maxwell's 1862 paper, he understandably does not push any deeper (Siegel 1991, p. 199). Harman and Goldman ignore its existence entirely. In a guided study of Maxwell's electromagnetic papers, Thomas K. Simpson suggests that the flywheel might be of some significance: it need "not necessarily be an inferior but quite possibly a more insightful way of grasping the principles of a connected mechanical system" (Simpson 1997, p. 367). I will advocate for something very close to this view.

physical conceptions (Maxwell 1865, p. 538).[3] Maxwell's flywheel analogy filled a critical role within his new electrodynamic theory, a role outlined by Maxwell in an unpublished essay for the Apostles Club, "Analogies: Are There Real Analogies in Nature":

> Before we can count any number of things we must pick them out of the universe, and give each of them a fictious unity by definition...The dimmed outlines of phenomenal things all merge into another unless we put on the focussing glass of theory and screw it up sometimes to one pitch of definition, and sometimes to another, so as to see down into different depths through the great millstone of the world. (Maxwell 1990, p. 377)

This illustrative analogy to a mechanical flywheel embodies that elusive "method of investigation" described by Maxwell in 1855 and not only supplied Maxwell with clear distinctions between electromagnetic phenomena by grounding them in mechanical analogues, it also suggested the unique route taken in his mathematical analysis and helped to lead him to his general equations of the electromagnetic field. The 1865 paper is then much more than an abstract application of Lagrange's methods; the electromagnetic equations it introduces are instantiations of the concepts and connections outlined by the dynamics of the flywheel, as I will now show.

This paper is structured as follows.

Section 1: covers the mechanical properties of Maxwell's flywheel, how it works and its specific components, both as an isolated mechanism and as an expanded flywheel system. The section highlights the concept of reduced momentum, which by analogy becomes a crucial feature of Maxwell's electromagnetic project. It concludes with a discussion of the example that Maxwell uses to clarify the flywheel's link to electromagnetic phenomena.

Section 2: lays out the analogy between the mechanical flywheel and electromagnetism, drawing the necessary connections between components of the flywheel and concepts in electromagnetic theory. I will consider two specific cases where circuits act on one another through the field.

Section 2.1: Demonstrates the induction of a current in a passive circuit by changes in another circuit (by alteration of its current in Sect. 2.1.1 and by setting the circuit in motion in Sect. 2.1.2). At the same time this section illustrates how the relevant concepts in electromagnetism are grounded in mechanical understanding by analogy to the flywheel and how the flywheel analogy guides Maxwell in structuring his analysis.

Section 2.2: looks at Maxwell's derivation of an equation of power and the implications his mechanical analogy has for his concept of the electromagnetic field by locating energy in the field itself. The conclusion of this section briefly parses Maxwell's justification for reversing the use of his mechanical illustration to now analyze how the field affects circuits.

[3] This is not to say that Maxwell viewed this clarifying role as unimportant. Maxwell still regarded this application of analogy as crucial when giving his "Introductory Lecture on Experimental Physics" at Cambridge in October 1871: illustrative analogy lends "vividness and relief to ideas which, when presented as mere abstract terms, are apt to fade entirely from the memory" (Maxwell 1871, p. 242).

Section 3: primarily investigates this shift to the field acting on circuits and the way in which Maxwell uses his mechanical analogy to guide his piece-by-piece construction of a generalized equation for induced electromotive force.
Section 3.1: serves primarily as an introduction to electromagnetic momentum and electromotive force, considering these concepts localized at a point in the field.[4]
Section 3.2: reconstructs Maxwell's derivation of the electromagnetic momentum of a circuit, illustrating the role of the mechanical analogy in guiding Maxwell's mathematics.
Section 3.2.1: reconstructs Maxwell's derivation of an expression for magnetic force in terms of electromagnetic momentum; this will prove useful later in Sect. 3.
Section 3.3: illustrates how the flywheel aids the construction of the first piece of a generalized equation of induced electromotive force on a circuit.
Section 3.4: reconstructs Maxwell's derivation of the final piece of this generalized equation for induced electromotive force, the induced electromotive force on a moving conductor. Again Maxwell's analogy to a mechanical flywheel is shown to direct his analysis of this particular electromagnetic phenomenon, suggesting the inclusion of terms which are ultimately unnecessary for the purposes of this derivation.
Section 4: demonstrates the deep methodological similarities between Maxwell's work in 1855 and 1865, specifically the similar role assigned to the physical analogies in both papers.
Section 5: highlights some of the ways in which the flywheel analogy and systems like it continued to be used as instructional tools by scientific and engineering professionals.
Section 6: brief concluding remarks summarizing the findings in this paper.

1 Flywheels and bevel gears

After an introductory discussion Maxwell's "A Dynamical Theory of the Electromagnetic Field"[5] moves to a general discussion of electromagnetic induction. The first part of this discussion is entitled "Electromagnetic Momentum of a Circuit," and it is here that Maxwell explicitly references a flywheel system as an analogue of the electromagnetic cases he will investigate:

> Now, if the magnetic state of the field depends on motions of the medium, a certain force must be exerted in order to increase or diminish these motions, and when the motions are excited they continue, so that the effect of the connexion between the current and the electromagnetic field surrounding it, is to endow the current with a kind of momentum, just as the connexion between the driving-

[4] What Maxwell calls electromagnetic momentum is what we would now call the vector potential **A**. It should not be confused with the modern definition of electromagnetic momentum, $\varepsilon_0(\mathbf{E} \times \mathbf{B})$, where ε_0 is the dielectric constant in vacuo and **E** and **B** are the electric and the magnetic field, respectively.

[5] Unless noted otherwise, all page references refer to the reprint of this paper in *The Scientific Papers of James Clerk Maxwell*, Volume 1. Equations marked M(x), where x is an integer or capital letter, refer to the equation number or letter within Maxwell's 1865 paper.

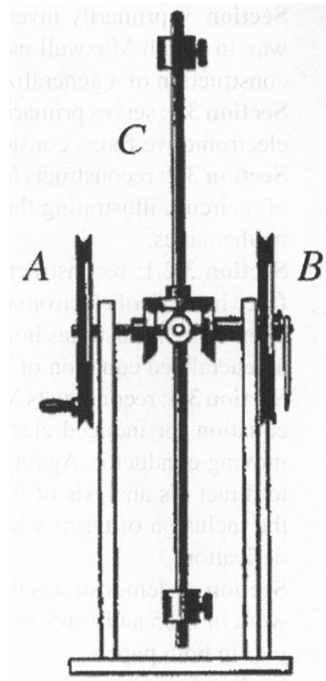

Fig. 1 Maxwell's flywheel, the labels in this image has been modified from those in the version shown in (Maxwell 1892, Vol. 2, p. 228)

point of a machine and a fly-wheel endows the driving-point with additional momentum, which may be called the momentum of the fly-wheel reduced to the driving-point. (p. 536)

There is no accompanying illustration of the flywheel system that Maxwell references in the 1865 paper; however, a sketch and description of a flywheel designed by Maxwell that is also intended to illustrate electromagnetic induction is printed in the first, unabridged edition of *The Life of James Clerk Maxwell*, and later in the third edition of Maxwell's *Treatise* (Campbell and Garnett 1882, pp. 550–554; Maxwell 1892, Vol. 2, p. 228). The later drawing from Maxwell's *Treatise* is shown in Fig. 1. This image was likely added by J.J. Thomson who took over from W.D. Niven as the editor for the *Treatise*'s third edition, carrying on despite Maxwell's death in 1879, before the release of the second edition.

The abstract of the 1865 paper as well as a deleted passage from the original manuscript, both submitted to the Royal Society in late October 1864, indicate that in addition to the flywheel, Maxwell began writing this paper with two more mechanical analogies in mind (Maxwell 1864). Another mechanical analogy could conceivably have done similar work, the flywheel analogy is after all meant to be illustrative and not a hypothetical model of reality. Maxwell was aware that "[t]he problem of determining the mechanism required to establish a given species of connexion between the motions of the parts of a system always admits of an infinite number of solutions" (Maxwell 1873, Vol. 2, p. 417). The change from "a rod acted on by two forces perpendicular to its direction" and "two horses harnessed to a carriage by the intervention of a lever so that

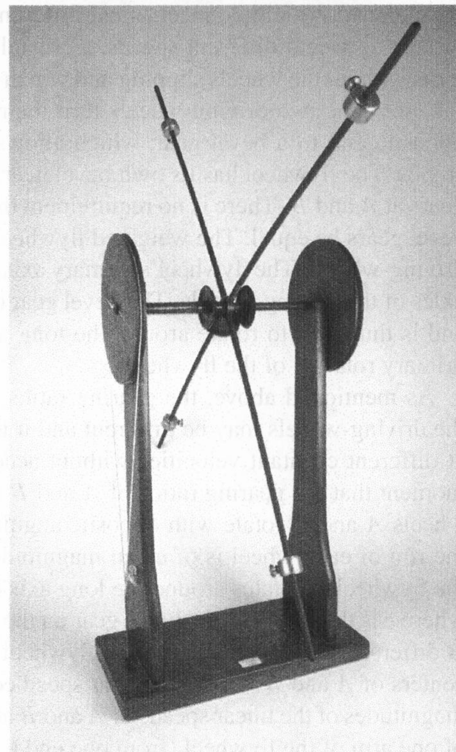

Fig. 2 Maxwell's flywheel model built for the Cavendish Laboratory (in the Whipple Museum of the History of Science, University of Cambridge Wh.2455)

each horse pulls at its own arm of the lever while the lever is attached to the carriage by its fulcrum" to the flywheel was most likely a decision based simply on the respective clarity of each analogy as "some may be more clumsy or more complex than others" (Maxwell 1995, pp. 191, 197; Maxwell 1873, Vol. 2, p. 417). The elements of the rod and horse analogies seem ill-suited to embody the components of the dynamical equations with which Maxwell begins, while these analogies taken as a whole are clumsy illustrations of Maxwell's discussion of dynamics more generally. Ultimately, Maxwell presumably saw an advantage in using only the flywheel as an analogue of electromagnetic phenomena in his 1865 paper. The perceived superiority of the flywheel analogy seems to have persisted as the flywheel was built under Maxwell's orders by Elliott Bros. in late 1876 (see Fig. 2) as a teaching aid for the newly founded Cavendish Laboratory in accordance with his design (Maxwell 2002, p. 421; Maxwell 1892, Vol. 2, p. 228). In what follows, the advantages of founding this paper on an analogy to a mechanical flywheel will become apparent.

First, the structure and operation of the flywheel must be understood to make sense of the analogy Maxwell exploits to construct his equations of electromagnetism. The two driving-wheels A and B are supported by separate axles geared into the central flywheel C. Both driving-wheels A and B have a string hung over them attached to a weight acting as a friction break on each wheel. The flywheel is nothing but two rods arranged like a cross with weights on all four ends and a single gear. The setup is similar to the simple mechanical differentials found in automobiles that allow the wheels to turn at different speeds, a crucial invention that allows the driver to make turns without the wheels slipping and which is the foundation of many traction control systems. The two driving-wheels turn their respective axles, each connected on the opposite end to a bevel gear, which allows for the transmission of motion at right angles. The flywheel has its own bevel gear loosely afixed to it, geared into the bevel gears at A and B. There is no requirement that the gearing ratios of these independent bevel gears be equal. The weighted flywheel is set perpendicularly to the axles of the driving-wheels. The flywheel's primary axis of rotation is a line that passes through the axles of the driving-wheels. The bevel gear on the flywheel C is only loosely attached and is thus able to rotate around the long axis of the flywheel, perpendicular to the primary rotation of the flywheel.

As mentioned above, the gearing ratios of those bevel gears directly attached to the driving-wheels may be different and it is in this way that both wheels might spin at different constant velocities without accelerating the flywheel. Assuming for the moment that the gearing ratios of A and B are equivalent, we see that if the driving-wheels A and B rotate with opposite angular velocities such that the linear speed at the rim of each wheel is of equal magnitude, v, then the intermediary bevel gear on the flywheel C rotates around the long axis of the flywheel with an angular speed vr, where r is the radius of the bevel gear on the flywheel C. If A and B are made to rotate at different speeds, then the entire flywheel C will rotate around an axis through the centers of A and B with an angular speed equal to half of the difference between the magnitudes of the linear speeds of A and B at their peripheries multiplied by the length of one arm of the flywheel (from one end to the axis through A and B). If there is an acceleration of the velocity of driving-wheel A while B is at rest, then the flywheel C will not move at first, but its loosely attached bevel gear will communicate a motion to the wheel B in the opposite direction of that at A. As this motion at B is resisted by its attached hanging weight, this reaction by B as well as the force imparted by A will cause the flywheel C to rotate around the axis through the centers of A and B in the same direction as A and with an angular speed equal to half of the magnitude of the linear speed of A at its periphery multiplied by the length of one arm of C. As long as the driving-wheel A remains at a constant velocity, C will remain at this speed, rolling around B, which will eventually come to rest due to the resistance of the weighted string. Any acceleration of A will again drive an opposite motion in B. This is illustrated in Fig. 3. As a consequence of the weights in the system, we see a transfer of momentum, first from the accelerated driving-wheel to the flywheel, and then from the flywheel to the other driving-wheel, making this system particularly suited for the application of Lagrangian dynamics.

On the page immediately following his discussion of the "fly-wheel," Maxwell provides a Lagrangian account of a mechanical system. Although Maxwell's discussion of the flywheel is limited to the brief given at the beginning of this section, his Lagrangian analysis comfortably describes just such a system. Indeed both the fly-wheel and the abstract Lagrangian system aim to illustrate the same concept, namely reduced momentum. Maxwell's general Lagrangian system introduces two "driving-points" A and B and a connected body C. In unifying the mechanical systems involved in both the prior discussion of the "fly-wheel" and the more general description that

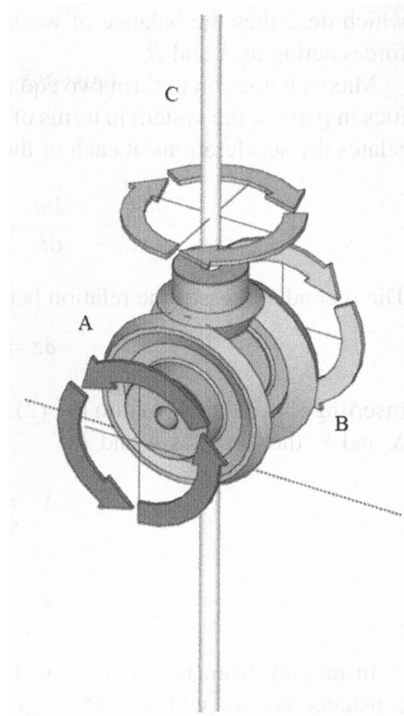

Fig. 3 Mechanical induction through the flywheel

immediately precedes Maxwell's mathematical investigation, the only leaps necessary are to equate the central flywheel connected to driving-*wheels* with the generic "body C" similarly attached to driving-*points* A and B and thus to make a jump from rotational to linear velocities. In the case of the flywheel, we take the mass of the central flywheel to be C. Maxwell sets u as the linear velocity of A, v as that of B, and w as the linear velocity of C, a motion we take to be analogous to the flywheel's primary rotation around the axis through A and B. The system is geared such that the velocity w of C is p times the velocity u of A and q times the velocity v of B. Using $\delta x, \delta y, \delta z$ for the respective simultaneous displacements (δx and δy are independent) of A, B, and C, Maxwell obtains what he calls, citing Part II, Sect. 2, §5 of Lagrange's *Analytical Mechanics*, "the general equation of dynamics" (p. 537). Lagrange's "general formula of dynamics for the motion of an arbitrary system of bodies" is represented as (Lagrange 1997, p. 186):

$$\int \left(\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\delta x + \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\delta y + \frac{\mathrm{d}^2 z}{\mathrm{d}t^2}\delta z\right) m + \int (P\delta p + Q\delta q + R\delta r + \cdots)\, m = 0. \quad (1)$$

Without parsing out Lagrange's more general dynamical equation, we can still recognize its paternity when we move back to Maxwell and his general equation of dynamics,

$$C\frac{\mathrm{d}w}{\mathrm{d}t}\delta z = X\delta x + Y\delta y, \quad (2)$$

which describes the balance of work in the system, where X and Y are the driving forces acting at A and B.

Maxwell goes on to form two equations describing the relationship of linear velocities in parts of the system in terms of what amount to gearing ratios p and q. The first relates the accelerations at each of the points A, B, and C.

$$\frac{\mathrm{d}w}{\mathrm{d}t} = p\frac{\mathrm{d}u}{\mathrm{d}t} + q\frac{\mathrm{d}v}{\mathrm{d}t}. \tag{3}$$

The second expresses the relation between their respective displacements.

$$\delta z = p\delta x + q\delta y. \tag{4}$$

Inserting Eqs. (3) and (4) into Eq. (2) and grouping δx and δy terms, Maxwell isolates X and Y, the forces at A and B.

$$\begin{aligned} X &= \frac{\mathrm{d}}{\mathrm{d}t}(Cp^2u + Cpqv), \\ & \qquad\qquad\qquad\qquad \mathrm{M}(1) \\ Y &= \frac{\mathrm{d}}{\mathrm{d}t}(Cpqu + Cq^2v). \end{aligned} \tag{5}$$

In moving from Eqs. (2)–(4) to Eq. (5), Maxwell assumes that C, p, and q are constants. As we will see, Maxwell treats the electromagnetic analogues of these quantities as if they are time dependent. That Maxwell is willing to grant this latitude to quantities that would otherwise be treated as constants in a standard Lagrangian system suggests that he may be pursuing the physical analogy to the flywheel beyond what can be strictly justified by Lagrangian dynamics.

In Eq. (5), Maxwell arrives at a key concept that he will use to motivate his discussion of the electromagnetic field, namely the quantity he calls "reduced momentum." He defines the "momentum of C referred to A" (p. 537) as the momentum of C imparted by the force at the driving-point A, given by $(Cp^2u+Cpqv)$. Similarly, the momentum of C referred to B is the momentum of C imparted by the force at the driving-point B and is given by $(Cpqu + Cq^2v)$. Reduced momentum is nothing but the momentum stemming from a particular force acting on the central body, C (or in the case of the flywheel, the central flywheel, C). Put another way, the effect of forces such as X and Y is to change the momentum of C in relation to the points A and B at which these forces are applied.

This simple system is generalized so that there may be an arbitrary number of bodies like C linked up to A and B, perhaps with different masses C and different gearing ratios p and q.[6] The quantities Cp^2, Cpq, and Cq^2 in Eq. (5) then get replaced by sums over the different values of these three parameters

[6] In his 1862 cogwheel model Maxwell had felt it necessary to insert idle wheels to overcome a similar mechanical problem, namely "coupling neighboring vortices" (Siegel 1991, p. 66). That Maxwell appeared unconcerned with the potential mechanical difficulties of coupling an arbitrary number of these systems together demonstrates his move away from modeling electromagnetic phenomena and the ether itself, instead offering only a "dynamical illustration".

$$L = \sum(Cp^2) \quad M = \sum(Cpq) \quad N = \sum(Cq^2). \tag{6}$$

Note that L and N relate only to the driving force at their respective driving-point, A or B. In the electromagnetic case, L and N refer to the shapes of the circuits and act as coefficients of self-inductance. By contrast M, which in the electromagnetic case Maxwell refers to as the coefficient of mutual inductance, relates to the driving forces at both A and B. These quantities are loosely analogous to moments of inertia insofar as they measure resistance to rotational acceleration if we think in terms of an expanded flywheel system. The momentum referred to A is

$$Lu + Mv, \tag{7}$$

while the momentum referred to B is,

$$Mu + Nv. \tag{8}$$

The forces X and Y acting on A and B to drive the expanded system can then be written as before [cf. Eq. (5)]:

$$\left.\begin{aligned} X &= \frac{\mathrm{d}}{\mathrm{d}t}(Lu + Mv), \\ Y &= \frac{\mathrm{d}}{\mathrm{d}t}(Mu + Nv). \end{aligned}\right. \qquad \mathrm{M}(2) \tag{9}$$

Resistance forces due to the weighted wires, which are also in reference to A or B and their respective velocities, take the form Ru and Sv as extra terms on the right-hand sides of these equations. The manner in which both resistance forces are incorporated in the electromagnetic case is demonstrated in Eqs. (12)–(13) at the beginning of the next section.

Although Maxwell's Lagrangian analysis is supposedly general and does not immediately reference the flywheel within his "Dynamical Theory," very nearly the same equations of Lagrangian dynamics appear 3 years later in an investigation of differentially geared governors in his paper "On Governors" (Maxwell 1868, pp. 118–120).[7] The device Maxwell describes appears remarkably similar to the differentially geared flywheel:

> In some contrivances the main shaft is connected with the governor by a wheel or system of wheels which are capable of rotation about the axis of the main shaft. These two axes may be at right angles, as in the ordinary system of differential bevel wheels; or they may be parallel, as in several contrivances adapted to clockwork. (Maxwell 1868, p. 118)

[7] Maxwell had previously described the relevance of William Siemens' differentially geared governor to electromagnetic phenomena in Part II of "On Physical Lines of Force," where he noted that elements of this governor are capable of motions similar to those of the idle wheels in his cogwheel model (Maxwell 1862, pp. 468–469).

Whether Maxwell was simply examining another type of governor (Siemens designed differentially geared governors were already in use on steam-engines), or, as Otto Mayr argues, Maxwell was more thoroughly checking that the mechanical system could be trusted to illustrate electromagnetic induction in preparation for building a physical model for the Cavendish, the fact that Maxwell used nearly identical Lagrangian analysis in his 1865 paper and this later discussion of a flywheel-like device appears to bolster the claim that Maxwell conceived of the dynamical relations in his "Dynamical Theory" in terms of the flywheel (Mayr 1971, pp. 218–219).

Returning to the 1865 paper, almost as an aside, Maxwell now makes a statement that finally leads us into his discussion of electromagnetism:

> If the velocity of A be increased at the rate $\mathrm{d}u/\mathrm{d}t$, then in order to prevent B from moving a force, $\eta = \mathrm{d}/\mathrm{d}t\,(Mu)$ must be applied to it. (p. 538)

This statement is key to understanding the connection between the mechanical illustration and electromagnetism. At the risk of belaboring the point, I will unpack this quotation to clarify the analogical relationship that Maxwell unveils. So that we might better understand what Maxwell had in mind, let us investigate how these relations are reflected in the operation of the flywheel. Consider the diagram in Fig. 3. As a result of the increase in velocity at A there is an indirect force on B, $-(\mathrm{d}/\mathrm{d}t)(Mu)$, that is to be canceled by η. Acceleration of the driving-wheel A causes an acceleration of the passive wheel B in the opposite direction. The force described that would "prevent B from moving" is then just the force we would have to apply to B to stop it accelerating as a result of the acceleration of A. The stopping force at B is the opposite of the force applied to B by our acceleration of the driving-wheel A.

The force on B as a result of action at A is mediated (hence the term indirect above) by the flywheel C, which Maxwell comes to see as the embodiment of the electromagnetic field. When a force is applied at A it leads to an increase in the momentum $Lu + Mv$ of the flywheel [cf. Eq. (9)]. This increase in the flywheel's momentum is defined as the reduced momentum of A. The flywheel's momentum then decreases over time as it acts on the passive wheel B. Fig. 3 illustrates how motion is transferred from A to B through the flywheel C. The force on B takes the form of a decrease in momentum of the flywheel in time, $-(\mathrm{d}/\mathrm{d}t)(Mu)$. This process of force transferal through the flywheel is the mechanical analogue of electromagnetic induction; in the electromagnetic case, the field moderates the transferal of forces between circuits. Maxwell goes on to say that this effect on B, $-(\mathrm{d}/\mathrm{d}t)(Mu)$, is consistent with the electromotive force on a circuit which arises from the increase in strength of a nearby circuit, namely it is the induced electromotive force. Just as the flywheel's momentum decreases as it acts on B, the electromagnetic momentum of the field decreases as a result of its action on a circuit. The induced current is such that it produces a force which acts counter to the change in current at A that produced the induced electromotive force. This is Lenz's law.

Naturally there are some similarities between the flywheel analogy and the cog-wheel model presented in Maxwell's 1862 paper. Both are capable of giving a mechanical account of electromagnetic induction; however, while this feature is the primary function of the flywheel analogy, it was merely an extension of the earlier model involving secondary tangential forces of the idle wheels acting on the vortices (Siegel 1991, p. 71). Nevertheless, the mechanical analogy in the 1865 paper is significantly more general, providing an illustration of connections between the field, represented by the central body or flywheel, and objects in it, represented as driving-points, not a hypothetical mechanical model of the structure of the electromagnetic ether. Additionally, while the specific flywheel analogy deals with rotations, Maxwell's Lagrangian dynamics is concerned exclusively with linear velocities. Thus the mechanically grounded concept of electromagnetic momentum that is central to Maxwell's "Dynamical Theory" does not explicitly require considerations of rotation, in contrast to the crucial role played by torques in modeling electromagnetic induction in his 1862 paper (Siegel 1991, p. 73).

Maxwell concludes his dynamical illustration with a warning. Nevertheless, this should not be taken as a threat to the project of elevating the relative status of flywheel analogy in the historical literature that surrounds his 1865 paper.

> This dynamical illustration is to be considered merely as assisting the reader to understand what is meant in mechanics by Reduced Momentum. The facts of the induction of currents as depending on the variations of the quantity called Electromagnetic Momentum, or Electrotonic State, rest on the experiments of Faraday, Felici, &c. (p. 538)

As will be made clear in the following sections, the concept of electromagnetic momentum which is so central to Maxwell's "Dynamical Theory" is itself grounded in the mechanical concept of reduced momentum, and its role in guiding Maxwell's mathematical analysis is governed by his exploitation of analogical links to the flywheel that suggest particular derivations of certain electromagnetic expressions. The facts of electromagnetic phenomena supplied by "the experiments of Faraday, Felici, &c" are necessary for the mechanical analogy to hold; without electromagnetic phenomena similar in action to the mechanical processes there would be no analogical link through which the flywheel or even his abstract Lagrangian dynamics could affect Maxwell's investigation of electromagnetism, no "partial similarity between the laws of one science and those of another which makes each of them illustrate the other" (Maxwell 1855, p. 156). The accounts of Maxwell's mathematical analysis that follow serve to highlight the role of the mechanical foundations of his "Dynamical Theory" and the analogically grounded concept of electromagnetic momentum, a role that had previously been obscured in the literature by the choice to substitute electromagnetic momentum for the vector potential. By working within Maxwell's original formalism can we more immediately appreciate the significance of the mechanical analogy that guided his analysis.

2 Mobilizing the mechanical analogy to analyze the effects of circuits through the field

Before we proceed to examine the specific electromagnetic examples that Maxwell lays out, we need to make sure that we understand the basic connections between the mechanical flywheel and electromagnetism that constitute the analogical link. Maxwell begins anew on p. 539 with two conducting circuits A and B instead of driving-points, while keeping in mind the lessons laid out in the preceding section.

The currents x and y in A and B take the place of the velocities at the driving-points, a reasonable substitution as currents also represent a change in time. L, M, and N now represent quantities that depend on the form and relative position of the circuits, L describing the form of circuit A, N of circuit B, and M the relative position of A and B. Insofar as these quantities were originally made up of gearing ratios and the summed weights of the central flywheels in the flywheel system, together expressing the driving-wheels' moments of inertia, a strong analogy holds between their use in the mechanical and electromagnetic case. Lenz's law in the electromagnetic case is drawn directly from the construction of the mechanical example and the way motion is transferred from one driving-wheel, through the gears and flywheels to the other driving-wheel, always resulting in an acceleration opposed to that which initially drove the system. If we continue to think in terms of the flywheel analogy, the C's of our expanded mechanical system, the weighted flywheels being acted on by the driving-wheels, can now be thought of as expanding into all space and representing the field. The quantity for which Maxwell coined the term electromagnetic momentum is analogous to the reduced momentum that existed in the flywheel or system of flywheels. If the analogy holds, then electromagnetic momentum extends from the wires throughout the unbounded field. As such, in accordance with the mechanical illustration, the reduced electromagnetic momentum of A is

$$Lx + My, \tag{10}$$

and that of B is

$$Mx + Ny. \tag{11}$$

ξ, the electromotive force due to changes in A, be they changes in the strength of the current, the form of the circuit, or its relative position with respect to another circuit of current y, is given by

$$\xi = Rx + \frac{\mathrm{d}}{\mathrm{d}t}(Lx + My), \quad \mathrm{M}(4) \tag{12}$$

and the force, η, which arises from changes at B, by

$$\eta = Sy + \frac{\mathrm{d}}{\mathrm{d}t}(Mx + Ny), \quad \mathrm{M}(5) \tag{13}$$

where R and S are coefficients of resistance.

In the sections that follow we investigate Maxwell's approach to more specific cases of circuits acting on passive circuits by way of the field. It will be helpful to keep in mind not only the basic electromagnetic properties discussed above, but also the analogous mechanical operations that underpin them.

2.1 Induction of a current by another through the field[8]

It is not trivial to grasp the idea that a current in one circuit should induce a current in some other passive circuit in the field. The mechanical analogy grounds current–current interactions in clear physical concepts, illustrating the dynamical connection between components in the electromagnetic case. Through the flywheel analogy we may come to terms with Maxwell's statement that "since the two currents are in connexion with every point of the field, they will be in connexion with each other" (p. 537). Just as the driving-wheels are geared into the flywheel, so should we imagine the circuit being "geared into" the electromagnetic field.

Consider the induction of a current in the passive circuit B by that of the active circuit A (p. 540). We are told that N remains constant, which is to say that circuit B is rigid. Additionally, without some initial current driving B (B is a passive circuit), the equation describing the electromotive force due to B vanishes, i.e., $\eta = 0$. The product Ny vanishes as B does not impress a force on A through the field, nor can it initially self-induce a force. Thus, Eq. (13) reduces to

$$Sy + \frac{\mathrm{d}}{\mathrm{d}t}(Mx) = 0. \tag{14}$$

As we can see from this equation describing the process of induction of B by A, there are two distinct ways in which this phenomenon can arise, depending on which variable is held constant, M or x. If M is held constant and x is allowed to vary, then a current y in B will be induced by the variation of the current in A, while both circuits remain fixed with respect to their relative positions. If we allow M to vary and hold x constant, then a current y in B will be induced by a change in the relative position of the two circuits, the current in A remaining constant. If we think in terms of the flywheel analogy, the distinction between these two types of induction becomes immediately apparent. The latter case, where M is said to vary, is analagous to a manipulation of both gearing ratios in the flywheel system, while the former is analagous to the acceleration of the driving-wheel A. The analogy suggests an inherent physical difference between these two causes of induction. One arises from a change in the free variable of the system (velocity of a driving-wheel), while the other requires a physical alteration of the setup of the system itself (altering gearing ratios). We will follow Maxwell's lead and analyze these two possible causes for induction separately.

2.1.1 Induction of a current by the variation of another

Consider the case that the only electromotive force acting on B is due to the increase in the current in A from 0 to x. In that case M is constant, meaning the circuits will not move relative to one another, and they will not change shape, otherwise this motion would produce an additional electromotive force.

[8] Section titles closely resemble Maxwell's own titles and proceed in the order given in Maxwell's "Dynamical Theory".

As M is constant, Eq. (14) gives

$$Sy + M\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \quad \longrightarrow \quad y = -\frac{M}{S}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right). \tag{15}$$

A force arising from the change, $\mathrm{d}x/\mathrm{d}t$, in the current at A affects the field, and then acts on the current y through the coefficient of mutual inductance M that relates the relative positions of the two circuits. The electromagnetic momentum of the field is exhausted as its electromagnetic momentum is transferred to B. The analogous mechanical case has already been laid out by Maxwell in his stopping force example. If we again take refuge in the flywheel analogy, an increase in the velocity of A acts through the central flywheel to transfer momentum to B in accordance with the coefficient M. The negative sign is indicative of Lenz's law, such that the induced electromotive force produces a current which will oppose the change that produced it. Maxwell goes on to integrate Eq. (15) getting what he calls the "total induced current" (p. 540), or the total charge passing through B due to the increase in current at A from 0 to x.

2.1.2 Induction of a current by the motion of the circuit

Next Maxwell investigates the induced electromotive force on B due to the two circuits A and B approaching one another. Such a motion will be represented by a change in the coefficient of mutual inductance, M. In this case, Eq. (14) gives

$$Sy + x\frac{\mathrm{d}M}{\mathrm{d}t} = 0 \quad \longrightarrow \quad y = -\frac{x}{S}\frac{\mathrm{d}M}{\mathrm{d}t}. \tag{16}$$

Again Maxwell finds that the induced electromotive force on B follows from a reduction in electromagnetic momentum in the field through its transfer to B, producing a current opposed to the change which produced it, in agreement with Lenz's law. Here the analogous mechanical effect would involve the increase in both gearing ratios which make up the M term, causing an increase in the momentum of the flywheel by the action of A, and subsequently a decrease in the momentum of the flywheel as this momentum is received by B.

The work done by our importation of Maxwell's analogy to the flywheel in both of these cases of induction is twofold. First, it clearly delineates the two cases of induction by reference to specific elements in the mechanical case, as they "resemble[ ] rather the reduced momentum of a driving-point of a machine influenced by its mechanical connexions" (p. 539). Second, it provides clear physical concepts through which we may come to terms with these electromagnetic phenomena: "L, M, N correspond to the same quantities in the dynamical illustration, except they are supposed to be capable of variation when the conductors A or B are moved" (p. 539).

2.2 Equation of work and energy

To form the equation of total work in unit time (power) done by both circuits A and B (p. 541), Maxwell proceeds analogously to a mechanical system, multiplying the electromotive forces ξ and η [from Eq. (12) and (13)] by the currents x and y, respectively (in the mechanical case, the power, P, is given by the inner product, $\mathbf{F}\cdot\mathbf{v}$)

$$\xi x + \eta y = Rx^2 + Sy^2 + x\frac{\mathrm{d}}{\mathrm{d}t}(Lx + My) + y\frac{\mathrm{d}}{\mathrm{d}t}(Mx + Ny). \qquad \mathrm{M}(8) \quad (17)$$

After dropping the terms Rx^2 and Sy^2, the energy lost as heat due to resistance, we can rewrite the right-hand side of Eq. (17) as:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(Lx^2 + 2Mxy + Ny^2\right) + \frac{1}{2}\frac{\mathrm{d}L}{\mathrm{d}t}x^2 + \frac{\mathrm{d}M}{\mathrm{d}t}xy + \frac{1}{2}\frac{\mathrm{d}N}{\mathrm{d}t}y^2. \qquad (18)$$

When L, M, and N are constant, the last three terms in Eq. (18) vanish. What is left is "the whole intrinsic energy of the currents" or the change in the energy contained within the field due to the currents x and y. Maxwell makes a prescient but guarded follow-up point, noting that as the currents are time derivatives, this energy "probably exists as actual motion, the seat of this motion being not merely the conducting circuits, but the space surrounding them." This motion and the energy is in the field, although it is "in a form imperceptible to our senses" (p. 541). While the dynamical theory will make no specific claims about the field, Maxwell's analogical reasoning has effectively physicalized empty space. In the words of one commentator, "*space* is not an empty geometrical container but a coherent, connected physical system bearing the energy of motion" (Simpson 1997, p. 312). By analogy to the flywheel, the properties of the field resemble those of a moment of inertia, a physical concept adapted to the field, but stripped of the immediate perceptibility it had in the earlier mechanical illustration.

The last three terms in Eq. (18) in which L, M, and N are variable describe the work done in unit time by the "alterations in the form and position of the conducting circuits A and B" (p. 542). Maxwell goes on to note that this impressed force must in fact be a simple mechanical force acting on a body, such as a conductor in one of the circuits, a very tangible action with electromagnetic consequences in the field. This equation then is taken to be the work done during these mechanical alterations of the circuit.[9]

Maxwell concludes this section with a justification of the dynamical project which will follow in the next section on the "General Equations of the Electromagnetic Field." If the unresisted part of an acting electromotive force generates

> a self-persistent state of the current, which we may call (from mechanical analogy) its electromagnetic momentum, and [if] this momentum depends on circumstances external to the conductor, then both induction of currents and electromagnetic attractions may be proved by mechanical reasoning. (p. 542)

Essentially Maxwell is outlining his plan of attack for the next section. If he has been justified in using the mechanical illustration and thus, as we have suggested, its accompanying flywheel analogy to investigate how electromotive forces arising

[9] Maxwell's comment on p. 542 that this mechanical force acts to maximize L, M, and N remains puzzling.

from changes in circuits affect the electromagnetic field, then such reasoning should work equally well in reverse. In the "General Equations" section, Maxwell will draw on his mechanical analogy to the flywheel to inform his investigation of induced electromotive forces, i.e., his investigation of how electromotive forces arising from changes in the electromagnetic field "due to any system of magnets or currents" affect circuits (p. 555).

3 General equations of the electromagnetic field: constructing a generalized induced electromotive force[10]

As noted at the conclusion of the preceding section, the equations that Maxwell derives in the rest of his 1865 paper are no longer from the point of view of driving-wheels or circuits, rather they are primarily equations of induction, describing the action of the electromagnetic field on circuits. The primary focus of this section is to reconstruct Maxwell's approach to deriving a fully generalized equation for the induced electromotive force on a conductor. Although there will be additional applications of the flywheel analogy within specific subsections, the most striking example of the way in which the analogy guides Maxwell's mathematics can be seen in the specific choices he makes in breaking up the analysis of induced electromotive force in a circuit. Maxwell builds the fully generalized induced electromotive force from two constituent parts, the induced electromotive force due to changes in the electromagnetic momentum of the field and the induced electromotive force due to a motion of the circuit through the field. Dividing the analysis of the electromotive force in such a way is suggested by the flywheel analogy and the distinct physical difference it draws between the effects of the acceleration of a driving-wheel (changes in currents, which define the electromagnetic momentum of the field in most examples) and changes in gearing ratios (changes in the form and/or position of a circuit). This physical distinction between two different origins of the same force in the flywheel analogy carries over into the general equations section of this paper and ultimately guides Maxwell's analysis and assembly of the fully general equation for induced electromotive force on a moving conductor. As I will show in Sect. 4, Maxwell uses the flywheel analogy in a similar manner to the flow analogy he introduced in his 1855 paper, to generate distinctions between constituent elements of the induced electromotive force that guide his mathematical analysis.

In his 1865 paper, Maxwell did not yet use the elements of vector calculus that later appears in his *Treatise on Electricity and Magnetism*. To save ourselves from rewriting each equation for each component while at the same time maintaining the ease of comparison between our reconstruction and Maxwell's text, we will write vectors by putting their three components in parentheses, (x, y, z).

[10] We will skip Maxwell's extended attempt to "bring these results within the range of experimental verification" (p. 543) and pick up again at the beginning of Part III, "General Eqations of the Electromagnetic Field," on p. 554.

3.1 Electromotive force (P, Q, R) and electromagnetic momentum (F, G, H)

Analogous to Newtonian mechanics whereby the force is equal to the time derivative of momentum,

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}, \quad \text{or} \quad (F_x, F_y, F_z) = \frac{\mathrm{d}}{\mathrm{d}t}(p_x, p_y, p_z), \tag{19}$$

the electromotive force at an arbitrary point in the field will be equal to a change in momentum, albeit with a negative sign to denote that it is an induced electromotive force which arises through a decrease in electromagnetic momentum in the field in accordance with Lenz's law.[11] Unlike what we have seen from Maxwell before, electromagnetic momentum expressed in this relation is localized at a point in the field and entirely general, insofar as it does not refer to any specific driving-points which create the field.

$$(P, Q, R) = -\frac{\partial}{\partial t}(F, G, H) \qquad \text{M(29)} \tag{20}$$

In modern notation this equation may be rewritten:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}. \tag{21}$$

where $\mathbf{A}$, now called the "vector potential," plays the role of Maxwell's electromagnetic momentum.

3.2 Electromagnetic momentum of a circuit

Returning to his concept of reduced momentum, Maxwell's mechanical analogy informs his definition of the total electromagnetic momentum referred to the circuit. The total electromagnetic momentum referred to the circuit is represented by a line integral of the electromagnetic momentum (F, G, H) over the circuit. Although it may initially appear unjustified, this expression is a natural consequence of his mechanical analogy. The circuit should be thought of as "geared into" the field at every point around its entire length:

> In the case of electric currents, the force in action is not ordinary mechanical force...but something outside the conductor, and capable of being affected by other conductors in the neighbourhood carrying currents. In this it resembles rather the reduced momentum of a driving-point of a machine as influenced by its mechanical connexions, than that of a simple moving body like a cannon ball, or water in a tube. (p. 539)

Based on the mechanical presumptions that ground Maxwell's approach, the circuits, like the analogous driving-wheels in the flywheel analogue, must be in dynamical

[11] See the end of Sects. 1 and 2 for more detailed discussions of this relation.

connection with other structures through the field, in which case "currents are in connexion with every point of the field" (p. 537) and the line integral of electromagnetic momentum around the circuit is the correct representation of the circuit's total electromagnetic momentum.

$$\oint_{\text{circuit}} (F, G, H) \cdot \mathbf{dl}. \tag{22}$$

Using Stokes' theorem, we can rewrite this as,

$$\oint_{\partial \mathbf{S}} (F, G, H) \cdot \mathbf{dl} = \iint_{\mathbf{S}} \operatorname{curl}(F, G, H) \cdot \mathbf{dS} \qquad \mathrm{M}(30) \tag{23}$$

$$\operatorname{curl}(F, G, H) = \left(\frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}, \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}, \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right), \tag{24}$$

where $\mathbf{S}$ is a surface and $\partial \mathbf{S}$ is the edge of the surface, coinciding with the circuit.

As Maxwell's quantity of total reduced electromagnetic momentum is equivalent to Faraday's "Electro-tonic State," it is ostensibly a measure of the strength of the field, or "the number of lines of magnetic force which pass through it [the circuit]" (p. 556). Thus Maxwell is able to find support for his mathematical representation of the electromagnetic momentum of the circuit not only from his flywheel analogy but also from the physical geometry he had developed in 1855.

3.2.1 Magnetic force (α, β, γ)

Maxwell's investigation of electromagnetic momentum of a circuit yielded a measure of the strength of the magnetic field passing through the circuit. The magnetic force per unit area is thus the integrand of the surface integral on the right-hand side of Eq. (23)

$$\mu\,(\alpha, \beta, \gamma) = \operatorname{curl}(F, G, H), \qquad \mathrm{M(B)} \tag{25}$$

where μ is "the ratio of magnetic induction in a given medium to that in air under an equal magnetizing force" (p. 556). In modern notation,

$$\mathbf{B} = \operatorname{curl} \mathbf{A}. \tag{26}$$

3.3 Electromotive force in a circuit

The role of the flywheel analogy in the case of induced electromotive force in a circuit mirrors its use in the case of the reduced electromagnetic momentum of the circuit. Again we should think in terms of the flywheel analogy whereby the electromagnetic momentum referred to the circuit is related to the form and relative position of the circuit (gearing ratios in the mechanical case). Maxwell founds his construction of an equation for electromotive force on the expression for the total electromagnetic momentum in the circuit, a line integral of electromagnetic momentum over the circuit, set equal to another expression of total momentum,

$$\oint_{\partial \mathbf{S}} (F, G, H) \cdot \mathbf{dl} = Lu + Mv. \quad \text{M(33)} \tag{27}$$

This other expression of total momentum, $Lu + Mv$, is just the expression for reduced momentum in the mechanical case that Maxwell derived in conjunction with the flywheel analogy. In the equation above, the total momentum referred to the driving-point A [cf. Eq. (7)] is set equal to the total electromagnetic momentum of the field referred to the circuit A [cf. Eq. (23)].

Note that Maxwell is mixing electromagnetic and mechanical terms in this equation. He forgoes including the equivalent electromagnetic quantities, choosing the velocities u and v instead of the currents x and y. We thus see that in this first step toward constructing a generalized equation of electromotive force, Maxwell grounds his analysis in the relations obtained in his mechanical analogy.

It follows that ξ, the complete induced electromotive force on A arising from both self and mutual induction [the opposite of Eq. (9) in the mechanical analogue and the opposite of Eq. (12) without resistance forces in the electromagnetic case], is given in terms of the total momentum by:

$$\xi = -\frac{\mathrm{d}}{\mathrm{d}t}(Lu + Mv), \quad \text{M(34)} \tag{28}$$

where the right-hand side is again expressed as a mechanically derived quantity. Using Eq. (27), we can also write the right-hand side in explicitly electromagnetic terms,

$$-\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\partial \mathbf{S}} (F, G, H) \cdot \mathbf{dl}. \tag{29}$$

With the help of Stokes' theorem, the line integral above can be rewritten as

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbf{S}} \operatorname{curl}(F, G, H) \cdot \mathbf{dS} &= -\iint_{\mathbf{S}} \frac{\partial}{\partial t} \operatorname{curl}(F, G, H) \cdot \mathbf{dS} \\ &= -\iint_{\mathbf{S}} \operatorname{curl} \frac{\partial}{\partial t} (F, G, H) \cdot \mathbf{dS}. \end{aligned} \tag{30}$$

The circuit should still be thought of as "geared into" the field at every point around its entire length justifying the use of another line integral to describe the total induced electromotive force "in a circuit." Thus, the left-hand side of Eq. (28) becomes

$$\oint_{\partial \mathbf{S}} (P, Q, R) \cdot \mathbf{dl}. \quad \text{M(32)} \tag{31}$$

Using Stokes' theorem again, we can rewrite this integral as

$$\iint_{\mathbf{S}} \operatorname{curl}(P, Q, R) \cdot \mathbf{dS}. \tag{32}$$

Substituting expressions (32) and (30) for the left and right-hand sides of Eq. (28), respectively, we find:

$$\iint_{\mathbf{S}} \operatorname{curl}(P, Q, R) \cdot d\mathbf{S} = -\iint_{\mathbf{S}} \operatorname{curl} \frac{\partial}{\partial t} (F, G, H) \cdot d\mathbf{S}. \tag{33}$$

It follows from this equation that (P, Q, R) should be equal to $-\frac{\partial}{\partial t}(F, G, H)$ *modulo* a term $\nabla\psi$, since $\operatorname{curl}(\nabla\psi) = 0$ for any ψ. Therefore,

$$(P, Q, R) = -\frac{\partial}{\partial t}(F, G, H) - \nabla\psi, \qquad \text{M(35)} \tag{34}$$

or in modern terms

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi. \tag{35}$$

And thus we have recovered Maxwell's "indeterminate" electric potential, the gradient of ψ. While ψ cannot effect a current in a circuit, it is supposed to indicate "the existence of a force urging the electricity to or from certain definite points in the field" (p. 558). Factoring in circuits and currents and looking at the induced electromotive force around the length of the circuit A, as opposed to dealing with electromotive force at some arbitrary point in the field, changes Eq. (21), $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$, to Eq. (35), $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi$ (where φ is the electric potential).

3.4 Electromotive force on a moving conductor

Finally, to complete his project of constructing a generalized equation of induced electromotive force on an element in the field, Maxwell considers the effect of changes to the form and position of a circuit on the electromagnetic momentum of that circuit. Maxwell aims to find the induced electromotive force on a moving conductor by introducing correction terms for the expression of induced electromotive force on a rigid and stationary circuit given in Eq. (34). It may appear odd that Maxwell was driven to complicate this demonstration of the electromotive force on a moving conductor by also considering a change in form of the circuit. A modern course on electricity and magnetism and even the reconstruction of this paper in Thomas Simpson's guided study involve only the motion of a rigid circuit through the field or deal exclusively with the moving conductor and the field. Simpson's modernized derivation of $\mathbf{v} \times \mathbf{B}$ describes it as merely as "the expression for the electromotive force $\mathbf{E}$ induced in a conductor moving in a field of magnetic flux $\mathbf{B}$ with a velocity $\mathbf{v}$" (Simpson 1997, p. 380). Why divide this analysis of a moving conductor up into elements "due to the motion of [the] conductor" and others "due to the lengthening of [the] circuit" (p. 559), and why begin with the total electromagnetic momentum of the circuit at all? Would it not be much simpler to just start with a rigid circuit moving through the field and then analyze the force on it by the field, or perhaps analyze the force on the moving conductor directly?

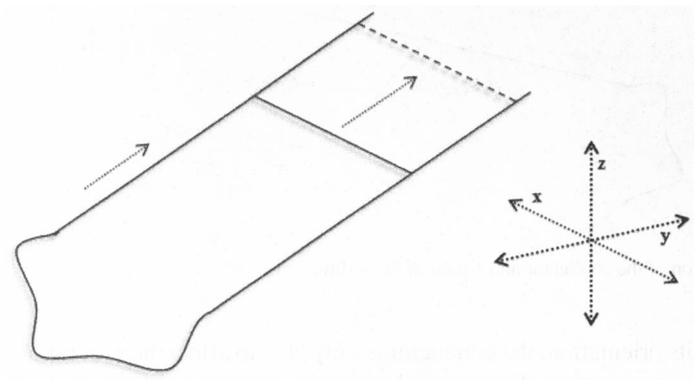

Fig. 4 Total motion of the conductor and circuit

To understand Maxwell's reasoning for complicating this problem, we must take a closer look at the system Maxwell investigates and the specific question he seeks to answer. A conductor which makes up the circuit and the circuit itself are allowed to move, putting not only the entire circuit in motion through the field, but lengthening the circuit as well (as pictured in Fig. 4).

> Let a short straight conductor of length a, parallel to the axis of x, move with a velocity whose components are $dx/dt, dy/dt, dz/dt$, and let its extremities slide along two parallel conductors with a velocity ds/dt. Let us find the alteration of the electromagnetic momentum of the circuit of which this arrangement forms a part. (p. 558)

The motion of the short straight conductor, $\mathrm{d}x/\mathrm{d}t$, $\mathrm{d}y/\mathrm{d}t$, $\mathrm{d}z/\mathrm{d}t$, is an absolute motion in all directions simultaneously. The wires that make up the whole circuit are also in motion; however, Maxwell only provides the relative velocity of the conductor with regard to the circuit, $\mathrm{d}s/\mathrm{d}t$. Thus, the whole circuit not only possesses some absolute motion, it also expands in all directions as the conductor rolls along its wires, due to the difference between the circuit's motion and the motion of the short straight conductor. Initially at least, Maxwell is purely concerned with deriving the change in the total electromagnetic momentum [cf. Eq. (23)] of the circuit as a result of the absolute motion of the circuit, only later will he use this expression to determine the electromotive force on a moving conductor.

From Maxwell's description of electromotive force, in this case the P-component, "represent[ing] the difference of potential per unit of length in a conductor placed in the direction of x at the given point" (p. 555), we gather that a straight conductor parallel to an axis may only change the component of electromagnetic momentum corresponding to the axis to which it lies parallel. Using this, we consider a few special cases.

First, as shown in Fig. 5, we examine the case of the short straight conductor moving along the axis of y with velocity $\mathrm{d}y/\mathrm{d}t$. The rest of the circuit is also moving in the y-direction and the relative velocity of the short straight conductor with regard to the circuit is $\mathrm{d}s/\mathrm{d}t$.

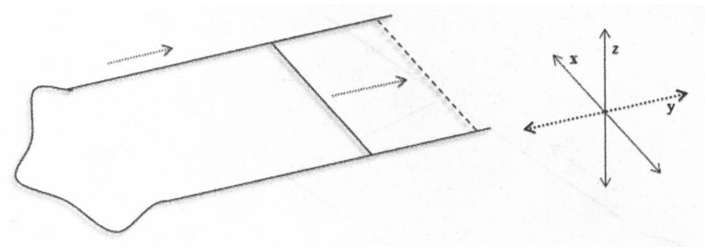

Fig. 5 Motion of the conductor and circuit in the y-direction

Due to its orientation, the conductor is only able to affect the x-component of electromagnetic momentum, F. The conductor will produce a change in electromagnetic momentum that corresponds to its changing position on the y axis.

$$a\frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \tag{36}$$

The relative motion of the conductor along the y axis with regard to the circuit, $\mathrm{d}s/\mathrm{d}t$, will naturally cause an expansion of the circuit. The wires of the circuit which lie parallel to the y axis will lengthen, $\mathrm{d}y/\mathrm{d}s$, altering the y-component of electromagnetic momentum G, per unit length in x,[12] resulting in the expression:

$$a\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\partial G}{\partial x}\frac{\mathrm{d}y}{\mathrm{d}s} = a\frac{\partial G}{\partial x}\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{37}$$

Maxwell is interested in the change in electromagnetic momentum of the circuit as a result of the absolute motion of the circuit through the field. To transform these changes in electromagnetic momentum due to the motion of the conductor and due to the change in configuration of the circuit into the expression for the motion circuit as a whole, we must subtract Eq. (37) from Eq. (36), the terms which result from the relative motion of the conductor with regard to the circuit from those which stem from the absolute motion of the conductor. This gives the change in electromagnetic momentum due to the total absolute change in position of the circuit:

$$a\frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} - a\frac{\partial G}{\partial x}\frac{\mathrm{d}y}{\mathrm{d}t} = a\left(\frac{\partial F}{\partial y} - \frac{\partial G}{\partial x}\right)\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{38}$$

The factor in parentheses is just minus the z-component of curl(F, G, H), which by Eq. (25) is equal to the z-component of the magnetic field, $\mu\,(\alpha, \beta, \gamma)$; hence, the right-hand side of Eq. (38) can be written as

[12] Here we are looking at the electromagnetic momentum of a circuit (not a single conductor as before) from a perspective such that the circuit is defined as the boundary of the area $\mathrm{d}y\mathrm{d}x$. As the circuit lengthens in the y-direction, the area increases per unit length in x. As such, the number of lines of magnetic force that pass through the circuit will also increase per unit length in x. If we remember that the number of these lines is a measure of electromagnetic momentum, the fact that the expansion is in the y-direction entails a change in G, the y-component of electromagnetic momentum, per unit length in x (cf. p. 15).

$$-a\mu\gamma\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{39}$$

Now consider the case of motion in the z-direction. To imagine this we may simply switch the labeling of the axes in Fig. 5 to obtain Fig. 6 below.

Working similarly with this orthogonal case of motion and expansion along the z axis, we find

$$a\frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}t} - a\frac{\mathrm{d}H}{\mathrm{d}x}\frac{\mathrm{d}z}{\mathrm{d}t} = a\left(\frac{\mathrm{d}F}{\mathrm{d}z} - \frac{\mathrm{d}H}{\mathrm{d}x}\right)\frac{\mathrm{d}z}{\mathrm{d}t} = a\mu\beta\frac{\mathrm{d}z}{\mathrm{d}t}, \tag{40}$$

where in the last step we used that the expression in parentheses is the y-component of curl(F, G, H), which by Eq. (25) is equal to $\mu\beta$.

Finally, if we follow these same steps but for a motion of the conductor parallel to x and an expansion of the circuit along the x axis, there will be no change in the electromagnetic momentum of the circuit:

$$a\frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} - a\frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = 0. \tag{41}$$

To find the total change in electromagnetic momentum in the case of a simultaneous motion of the conductor parallel to the x axis in all three directions we add up the terms from Eqs. (39) and (40)

$$-a\mu\gamma\frac{\mathrm{d}y}{\mathrm{d}t} + a\mu\beta\frac{\mathrm{d}z}{\mathrm{d}t} = a\mu\left(\beta\frac{\mathrm{d}z}{\mathrm{d}t} - \gamma\frac{\mathrm{d}y}{\mathrm{d}t}\right). \tag{42}$$

As Maxwell has been interested in determining the induced electromotive force (P, Q, R), or the decrease over time of the electromagnetic momentum of the field which is to be transferred to the circuit due to its motion [cf. Eq. (20)], he must use the opposite of Eq. (42), the total change in the electromagnetic momentum of the circuit, to form P. Although Maxwell began this investigation looking at changes to the electromagnetic momentum of the circuit due to the motion of the circuit, by definition the components of electromotive force (P, Q, R) only affect conductors lying parallel to the relevant axes, and thus P is the electromotive force on the moving conductor parallel to the x axis. Additionally, as (P, Q, R) is measured in unit length, we find that the x-component of the electromotive force on the conductor is given by

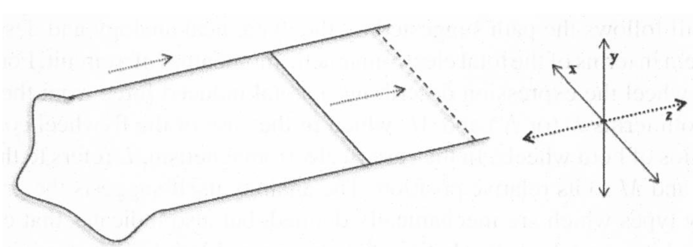

Fig. 6 Motion of the conductor and circuit in the z-direction

$$P = \mu\gamma\frac{\mathrm{d}y}{\mathrm{d}t} - \mu\beta\frac{\mathrm{d}z}{\mathrm{d}t}. \quad \text{M(36)} \tag{43}$$

This is the x-component of the cross-product of the velocity $(\dot{x}, \dot{y}, \dot{z})$ and the **B**-field $\mu(\alpha, \beta, \gamma)$.

In the cases of the conductor laying parallel to the y or z axes minor variations yield results for Q and R. Eq. (43) is a correction to Eq. (34) to account for the effects due to the motion of a conductor. The completed generalized equation of induced electromotive force (P, Q, R) is then

$$\begin{aligned} P &= \mu\left(\gamma\frac{\mathrm{d}y}{\mathrm{d}t} - \beta\frac{\mathrm{d}z}{\mathrm{d}t}\right) - \frac{\partial F}{\partial t} - \frac{\partial\psi}{\partial x} \\ Q &= \mu\left(\alpha\frac{\mathrm{d}z}{\mathrm{d}t} - \gamma\frac{\mathrm{d}x}{\mathrm{d}t}\right) - \frac{\partial G}{\partial t} - \frac{\partial\psi}{\partial y} \qquad \text{M(D)} \\ R &= \mu\left(\beta\frac{\mathrm{d}x}{\mathrm{d}t} - \alpha\frac{\mathrm{d}y}{\mathrm{d}t}\right) - \frac{\partial H}{\partial t} - \frac{\partial\psi}{\partial z}. \end{aligned} \tag{44}$$

In terms we recognize it is

$$\mathbf{F} = \mathbf{v} \times \mathbf{B} - \frac{\partial\mathbf{A}}{\partial t} - \nabla\varphi. \tag{45}$$

Substituting in the relation between **B** and **A** that Maxwell established in his discussion of magnetic force, we may rewrite the equation (Darrigol 2000, p. 160):

$$\mathbf{F} = \mathbf{v} \times (\mathrm{curl}\mathbf{A}) - \frac{\partial\mathbf{A}}{\partial t} - \nabla\varphi. \tag{46}$$

Returning to the question with which we began this discussion, why then did Maxwell complicate this derivation by changing the shape of the circuit and beginning with the total electromagnetic momentum of the circuit if he ends up with the same equations for induced electromotive force as the much simpler alternatives? The answer to our question is that the analogical foundation of Maxwell's 1865 paper suggests this more convoluted analytical route. The concept of electromagnetic momentum is at the heart of the dynamical analogy and although the line integral of electromagnetic momentum around the circuit has replaced the more evidently mechanical quantity of reduced/electromagnetic momentum, $Lu + Mv$, Maxwell's analysis still follows the path suggested by the dynamical analogy and describes the initial system in terms of the total electromagnetic momentum of a circuit. For a passive or driving-wheel the expression describing its total induced force from the flywheel contains both terms L (or N) and M, which in the case of the flywheel contains the gearing ratios of both wheels. In the case of electromagnetism, L refers to the form of the circuit and M to its relative position. The analogy itself suggests the grouping of analysis by types which are mechanically defined, but also indicates that changes in M necessarily impact L or N. Having already covered induced electromotive forces due to changes in the electromagnetic momentum of the field (in the mechanical case changes in the velocity of driving-wheels), the final step is to cover induced electromotive forces due to motion of a circuit through the field (changes in gearing ratios). In deriving the effect of the field on a moving circuit, the analogical foundations of this paper require an analysis of a change in form of the circuit in motion so we can remove its effect on the electromagnetic momentum of the circuit and isolate the change due to the absolute motion of the circuit itself. After a careful setup and analysis, Maxwell is able to remove the change in electromagnetic momentum due the alteration of the circuit's configuration and isolate the change due to the motion of the circuit (we get the same result starting with a moving rigid circuit); nonetheless, it is the flywheel analogy that suggests this roundabout approach to deriving the generalized equation for the induced electromotive force on a moving conductor.

4 Similar methods: 1855 and 1865

In the introduction to this paper I quoted a paragraph from Maxwell's "On Faraday's Lines of Force" demonstrating the balance he hoped to find between physical analogy and abstract mathematics. I have shown how "A Dynamical Theory of the Electromagnetic Field" achieves a similar balance by demonstrating the significance of Maxwell's analogy to a mechanical flywheel as a complement to his abstract mathematics, the latter having already been widely explored in the literature. If such a balance exists in Maxwell's 1865 paper, the origin of this quote suggests that there is likely a strong methodological similarity between his 1855 and 1865 papers.

Maxwell's "On Faraday's Lines of Force" achieved what Maxwell later described as a translation of "what I considered to be Faraday's ideas into a mathematical form" (Maxwell 1873, Vol. 1, pp. ix–x). In William Thomson's work on potential theory, Maxwell found a fitting mathematical expression to translate Faraday's lines of force carving up space into a symbolic form, one which would preserve the idea of "continuous action in a medium" inherent in Faraday's geometrical concept and which would form the foundation of Maxwell's similarly geometric theory of the electromagnetic field (Harman 1998, p. 72). Maxwell would also borrow from the formal analogy that Thomson had created between the equations of electrostatics and heat flow, although he replaced heat with an imaginary incompressible fluid in part because by that time heat was no longer considered to be a substance. Norton Wise, Peter Harman, Francis Everitt, Olivier Darrigol, and the late Martin Goldman all give an account of the ways in which Maxwell makes this analogy "useful in exciting appropriate mathematical ideas" (Maxwell 1855, p. 157). Nevertheless, Wise's account of Maxwell's 1855 paper is not only the most comprehensive, it is also the only history to unequivocally characterize the geometry and analogies within that paper as something more than abstract mathematical analogy.

Wise first establishes two related uses of the flow analogy in Maxwell's first paper. Initially it serves to provide a physical distinction between electric and magnetic quantities and intensities beyond the purely geometrical distinction that Maxwell had derived from his interpretation of Faraday. This distinction is inherent to the flow analogy in the form of flux and pressure gradient; however, he must imagine that the fluid is incompressible (to preserve conservation of flux akin to Faraday's continuous lines of force) and that it flows through a resistive medium (to obtain a pressure gradient). As Wise makes clear, "[t]he constant [of the medium of propagation] provides, at least in principle, an observational basis for the earlier purely geometrical distinction" (Wise 1977, p. 188). What was once a "purely geometrical distinction between 'polarization along a line' and 'polarization over a surface'," was transformed by the introduction of this constant into "a highly physical one" (Wise 1977, pp. 188). The added "constant pressure surfaces separated by unit pressure difference" not only completes the translation of countable geometrical elements into physical ones, "[b]ut this time the elements are explicitly 'unit cells' distinguished by the fact that 'the work spent in overcoming resistance is...unity in every cell in every unit of time.' Here is another physical aspect of the fluid analogy; it serves conceptually to distribute the 'potential of the system,' analogized to the total work per unit time done by the fluid; over all space" (Wise 1977, p 188).

Additionally, Wise describes a third use of the flow analogy, working in tandem with Faraday's image of the "mutually embracing curves" of magnetic and electric lines,[13] that directs Maxwell's construction of his own version of Faraday's electrotonic state. The symmetry expressed by the "embracing curves" is clarified and bolstered by the flow analogy, insofar as both curves, current and magnetic lines, can be represented as fluxes (Wise 1977, p. 207). In search of the exact relation suggested by this symmetry, between electric intensity in induction and the change in magnetic quantity, to be described by the electrotonic state, Maxwell's approach reveals the guiding force of the flow analogy in his analysis: "the fundamental equation for continuous tubes of flow, *Divergence* $\mathbf{a}[\text{quantity/flux}] = 0$, is what guarantees the possibility of the new [electrotonic] state" (Wise 1977, pp. 203–207). Faraday's Electro-tonic State would return in 1862 and, as we have seen, even more spectacularly in 1865, reconstituted as electromagnetic momentum. In each case it was birthed from Maxwell's exploitation of physical analogy.

While the use of physical analogy proved to be a fruitful approach in "On Faraday's Lines of Force," Maxwell remained extremely cautious of overextending the relationship throughout the paper. The accompanying flow analogy was intended as purely illustrative,

> [i]t is merely a collection of imaginary properties which may be employed for establishing certain theorems in pure mathematics in a way more intelligible to many minds and more applicable to physical problems than that in which algebraic symbols alone are used. (Maxwell 1855, p. 160)

He reminded his readers that he was creating a physicalized geometry based on Faraday's ideas and informed by the flow analogy, but that he was by no means supplying a physical hypothesis.

We find then that in both 1855 and 1865 Maxwell utilized very similar methods. The physical analogies in both papers provide clear conceptual foundations for otherwise unclear electromagnetic phenomena and guided Maxwell's mathematical analysis. In 1855, the flow analogy (and the accompanying physical geometry) grounds the

[13] For discussion of Maxwell's use of Faraday's "mutually embracing curves" see (Wise 1979).

electromagnetic concepts of quantity and intensity in the more accessible concepts of fluxes and pressure gradients, while in 1865 the analogy to a mechanical flywheel does similar work to ground electromagnetic concepts in mechanics, for example relating coefficients of self and mutual induction to moments of inertia. In both papers the analogies suggest fundamental reconsiderations of space. The flow analogy distributes the "potential of the system" across empty space, and the flywheel does the same for the "intrinsic energy of the currents." These two analogies help him distinguish particular electromagnetic phenomena by analogy to the corresponding physical case, providing a "natural physical partitioning," a systematic framework within which Maxwell's mathematical analysis could operate (Wise 1977, p. 136). These elements defined through the flow and flywheel analogies then suggest particular routes for Maxwell's mathematical analysis that allow for the construction of the electrotonic state in 1855 and a generalized equation for induced electromotive force in 1865 respectively. Finally, in neither case does Maxwell go so far as to suggest that these analogies might be hypothetical models of reality, rather they are both regarded as illustrative, despite the crucial role each physical analogy played in the development of Maxwell's thoughts on electromagnetism.

Before we move on, we should take note of a subtle difference between Maxwell's 1855 flow analogy and the flywheel analogy in his "Dynamical Theory." Neither analogy suggests any particular microscopic construction of the electromagnetic ether; however, they do make different claims about the nature of electromagnetic systems. The earlier flow analogy is able to draw upon two distinct analogues, fluid flow and heat flow, and insofar as the latter sufficiently expresses the quantities and relations Maxwell exports to electromagnetism, there is no motivation to adopt a mechanically grounded interpretation of electromagnetism (as might be suggested by the fluid flow analogy alone) (Wise 1979, p. 1311). The direct transcription of dynamical quantities and relations from a differential system (of which the flywheel is just one example) to electromagnetism in Maxwell's "Dynamical Theory" lacks the agnosticism of the flow analogy. In 1865 there is, broadly speaking, only one analogue, dynamics. Electromagnetism is just another dynamical system, containing the same mechanical quantities of energy and momentum as the flywheel. As such, the 1865 analogy represents a special subset of physical analogy, it is a "dynamical analogy," as it necessarily involves dynamics on at least one side of the analogy, but it also borders upon "dynamical explanation," as electromagnetic energy and momentum do more than correspond to their mechanical counterparts, they are identical (Turner 1956, pp. 36–37).

In this way, although the methodologies guiding these two papers are very similar and strike a similar balance between physical analogy and abstract analysis, they operate at different levels of physical theory. The theory built upon the physical geometry and flow analogy in "Faraday's Lines" is more general than his later theory built upon dynamical analogy (and at times dynamical explanation) as the former makes no claims regarding the nature of electromagnetic systems. To borrow a phrase from Maxwell's Apostles Club essay, he has dialed up the "focusing glass of theory…to one pitch of definition, and [later] to another" (Maxwell 1990, p. 377).

5 The flywheel as an "Experiment of Illustration"

Maxwell's flywheel analogy continued to be used as a mechanical illustration of electromagnetic induction by theoreticians across Europe as well as electrical engineers in England, "throw[ing] light upon some scientific idea so that the student may be enabled to grasp it" (Maxwell 1871, p. 242). In 1890 Lord Rayleigh would reimagine Maxwell's flywheel, already in use at the Cavendish, as a system of weights and pulleys because he lacked access to differential gears (Rayleigh 1890, p. 434). Although Ludwig Boltzmann knew of the existence of Maxwell's flywheel through Rayleigh, he admitted, "unfortunately, I am not familiar with [it]" (Boltzmann 1891, p. 45). Thus when imagining his own system to illustrate electromagnetic induction, Boltzmann based his design on Rayleigh's "very simple apparatus" (Boltzmann 1891, p. 45). Nevertheless, Boltzmann's device remains tied to Maxwell's flywheel not only through their shared purpose, but also through a shared mechanical lineage joined by Rayleigh. Boltzmann not only included an illustration of a significantly more complicated apparatus or *Bizykel* in his 1891 "Lectures on Maxwell's Theory of Electricity and Light," he also had a physical model built, one which he held very dear and often brought with him on trips (Boltzmann 1891, p. 21, Fig. 15; Eckert 2001). Boltzmann's flywheel is shown in Fig. 7.

The relation of the components and operation of this flywheel to electromagnetic phenomena mirrors those described by Rayleigh and can be translated into the differential gear analogy exploited by Maxwell. The added complications were inspired by a major flaw Boltzmann had made note of in Rayleigh's design (one common to Maxwell's flywheel), namely that "the parameters can not be changed in it which for

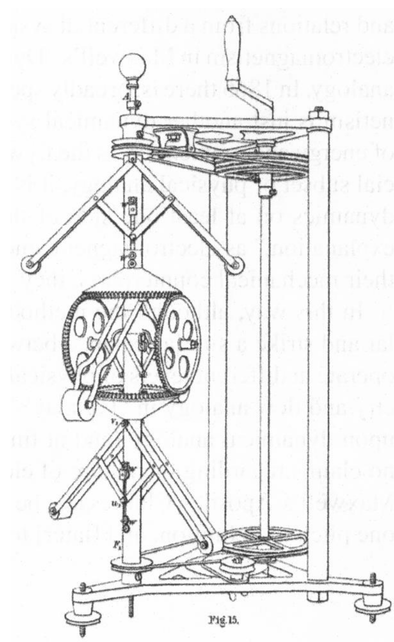

Fig. 7 Boltzmann's flywheel, (Boltzmann 1891, Fig. 15; reprinted and discussed in Eckert 2001)

us is rather essential" (Boltzmann 1891, p. 45). Boltzmann's additions only deepened the connection to electromagnetism, allowing Boltzmann to alter the terms L, M, and N, thus providing a powerful mechanical demonstration of the effects of moving and morphing circuits (Boltzmann 1891, pp. 27–28, 45). Boltzmann also lent the model out to Arnold Sommerfeld and an image of the central mechanism of a differential gear flywheel made an appearance in Sommerfeld's textbook on mechanics (Sommerfeld 1952, p. 255; Eckert 2001). There, it was again distinguished as an illustrative analogy through which students could conceptualize electromagnetic induction, although by this time Sommerfeld already thought the flywheel was "much more complicated than Maxwell's theory which it was intended to illustrate" and therefore was better served as "an exercise on the differential of an automobile" (Sommerfeld 1952, p. 225).

Although Maxwell was judicious in his use of an analogy between mechanics and electromagnetism in the *Treatise*, certain English electrical engineers shared Maxwell's opinion that "[i]t is difficult, however, for the mind which has once recognized the analogy between phenomena of self-induction and those of the motion of material bodies, to abandon altogether the help of this analogy, or to admit that it is entirely superficial and misleading" (Maxwell 1873, Vol. 2, p. 181; quoted and discussed by Gooday 2004, p. 180). One of those engineers, John Hopkinson, appealed explicitly to Maxwell's flywheel analogy in his lecture "On Some Points in Electric Lighting," delivered in 1883 to the Institution of Civil Engineers (Hopkinson 1883). In trying to give a theoretical account of the possibilities for alternating current machines, Hopkinson used the analogy to illustrate that electrical circuits acted as if they had inertia (Hopkinson 1883, p. 60; discussed by Gooday 2004, pp. 186–188). In this way, the flywheel analogy remained an important teaching tool until nearly the end of the 19th century, entrenched by the underlying connections between the construction and operation of the mechanical device and Maxwell's formulation of his equations for electrodynamics.

6 Conclusion

In this study of Maxwell's 1865 paper "A Dynamical Theory of the Electromagnetic Field," I set out to demonstrate the significance of an analogy to a mechanical differential and the flywheel analogy in particular to Maxwell's approach in this culmination of his three major papers on electromagnetism. The received wisdom concerning this paper has emphasized the Lagrangian dynamics that Maxwell exploits to build his theory of electromagnetism, often entirely ignoring the complementary contributions of the analogy to a flywheel. Given then that Maxwell leaned heavily on a mechanical analogy while constructing his 1865 paper, the prior work done to understand Maxwell's use of Lagrangian dynamics as well as his usage of analogy in general is necessarily incomplete. I have attempted to fill this gap in the literature and uncover the ways in which Maxwell's flywheel analogy worked alongside his abstract mathematics, particularly in those instances where the underdeveloped physics of electromagnetism was ill-equipped to direct his analysis. The flywheel analogy was shown to elucidate electromagnetic phenomena by providing clear physical concepts in which more complicated electromagnetic analogues could be grounded. At times the analogy suggested, or at the least justified mathematical operations which proved crucial in Maxwell's journey toward building up a complete and fully generalized electromotive force. More importantly the flywheel analogy defined and generated distinctions in the mechanical case and by analogy in electromagnetism, guiding Maxwell in his grouping of particular phenomena and suggesting a particular route for his further analysis. As such, the gap between Maxwell's 1862 hypothetical model of the electromagnetic medium and his 1865 publication has shrunk considerably. They no longer represent the extremes of possible uses of physical analogy. The 1862 paper remains where it was, with Maxwell overcommitted to a hypothetical model of the ether; however, the 1865 paper can no longer be described as driven entirely by an extension of Lagrangian dynamics. Instead the 1865 paper is characterized by a complementary relationship between a pictorial mechanical analogy and an extension of Lagrangian dynamics, moving it away from the purely mathematical extreme and closer in approach to "On Physical Lines of Force."

Including the flywheel analogy in the story of Maxwell's maturing thoughts on electromagnetism allows us to note a striking similarity between Maxwell's use of illustrative analogies in his 1855 paper "On Faraday's Lines of Force" and in 1865's "A Dynamical Theory of the Electromagnetic Field." The flow analogy's role in 1855, as reconstructed by Wise parallels that of the flywheel, insofar as in both cases they not only provide clear physical conceptions, but also guide Maxwell's mathematical analysis. Nevertheless, the impulse toward mechanics inspired by the dynamical analogy highlights the greater generality of Maxwell's 1855 approach and suggests an interpretation of his "Dynamical Theory" as a compromise between the mechanical model of 1862 and the noncomittal flow analogy of 1855.

In light of the importance of the flywheel analogy, Maxwell's approach in "A Dynamical Theory of the Electromagnetic Field" is best seen not as a theory built purely on an analogy to Lagrangian dynamics, but rather as one that achieved a cooperative balance between physical analogy and abstract mathematics. The 1865 paper is an "effectual study of science" in the sense that Maxwell proposed in the introduction to "On Faraday's Lines of Force." The success of Maxwell's "Dynamical Theory" is rooted in his ability to retain "sight of the phenomena to be explained" by grounding them in clear physical conceptions drawn from his mechanical analogy and to "obtain more extended views of the connexions of the subject" by using this mechanical analogy to guide his mathematical analysis. Similarly, the status of the flywheel analogy, not as a "physical hypothesis" but an illustrative analogy, guards Maxwell from "that blindness to facts and rashness in assumption which a partial explanation encourages." By 1865 Maxwell had again arrived at this methodological harmony, "which allows the mind at every step to lay hold of a clear physical conception, without being committed to any theory founded on the physical science from which that conception is borrowed, so that it is neither drawn aside from the subject in pursuit of analytical subtleties, nor carried beyond the truth by a favorite hypothesis" (Maxwell 1855, pp. 155–156). As historians have pointed out, in 1865 Maxwell rid himself of commitments to any hypotheses about the physical makeup of the field; however, this does not mean that he made the opposite move and chose to deal exclusively in the language of abstract mathematics. Maxwell makes liberal use of a strong, as he puts it, "illustrative" mechanical analogy to supply himself with clear physical conceptions and to fully extend the reach of his "Dynamical Theory."

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