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Al-Khwārizmī and annular solar eclipse

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1 Introduction

In the history of astronomy, theory and observation have not always moved forward hand-in-hand in many fields, as each may have been involved in its own problems. The history of prediction and observation of annular solar eclipse is one such example.[1] The annular solar eclipse occurs when the angular diameter of the moon is less than that of the sun. In such a case, if the angular distance between the center of the moon's apparent disk and that of the sun is very close, at mid-eclipse (or near to it) a part of the solar disc appears as a ring of light around the dark disc of the moon. This kind of solar eclipse occurs when the sun is at perigee and the moon is at apogee.[2]

The variation in the apparent diameters of the sun and moon had no role in Babylonian astronomy. But the Greek astronomers tried to consistently relate apparent diameters of the sun and moon with their corresponding distances from the earth based on geometrical and observational proofs. Indeed, their achievements based on

[1] It is important to know that the problem of annular solar eclipse was still being debated in seventeenth century Europe. See Stephenson (1997, p. 467).

[2] Annular solar eclipse may even not exactly fulfill the mentioned conditions.

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geometrical methods led to unsatisfactory results.[3] Ptolemy made observations by an instrument, called *Dioptra*, to obtain the exact values of the sun's apparent diameter.[4] Ptolemy compared the magnitude of two lunar eclipses and the moon's latitude to measure the moon's angular diameter. Ptolemy concluded that the apparent diameter of the sun is constant and must be the same as that of the moon at apogee.[5] This fault probably originated from his imprecise observational instrument and the intense light of the sun causing a serious problem in precisely measuring the sun's apparent diameter through direct observation. Ptolemy considered that the apparent diameter of the moon varies between 0;35,20° (at perigee) and 0;31,20° (at apogee).[6]

This determination led to his failure to predict the possibility of annular solar eclipse.[7] For ancient astronomers, observing a solar eclipse was the only way to examine the actual differences between the apparent diameters of the sun and moon in certitude. The variation in the apparent diameters of the moon and sun has a noticeable effect on the greatest value of the solar eclipse magnitude and duration. In the *Almagest*, Ptolemy shows the effect of the lunar distance from the earth in predicting solar eclipses. In his Table for solar eclipse, (VI 8), there are two separate lists of arguments of latitude. The argument of latitude indicates the ecliptic distance of the moon from the highest northern point of the moon's circle around the earth (for the

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3 We know that in Ptolemaic model, the relation between the moon's distance and its angular diameter was inconsistent, because as the moon keeps distance two times at apogee (compared to perigee), its angular diameter just drops about 11%, instead of becoming half. Therefore, it seems that Ptolemy neglected the consistency in variation of angular diameter of the moon in his lunar model. For the sun, Ptolemy's solar model permits one to suppose variation in the sun's apparent diameter, but Ptolemy has not considered it. See also Pedersen (1974, pp. 198–199).

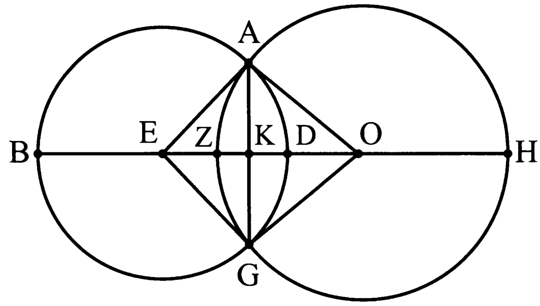
4 It seems that Hipparchus originally invented *Dioptra*, See Toomer (1998, p. 252).

5 Ibid, pp. 252–254.

6 Ibid, pp. 283–284. For the moon, the modern value lies between 0;29,22° and 0;33,31° and for the sun it lies between 0;31,28° and 0;32,32°.

7 See Pedersen (1974, p. 208). Neugebauer (1975, vol. 1, pp. 110–111) has proved that indeed Ptolemy's theory for solar distance basically permits one to conclude annular solar eclipse. Thus, it is unclear why Ptolemy neglected the variation of sun's distance in computation of its angular diameter. However, the problem of annular solar eclipse was a controversial issue in Greek astronomy. Apparently, Polemarchus (d. 404 B.C.), a younger contemporary of Eudoxus, did see an annular eclipse and this experience may have resulted in the statement found in the "Eudoxus Papyrus" that total solar eclipses are impossible. It seems that Hipparchus (190–120 B.C.), in his treatise on solar and lunar distances, probably was aware of this that sometimes angular diameter of the moon is less than that of the sun. However, this is concluded indirectly from the values given for the relative sizes of the sun and moon to the earth (see Swerdlow 1969, pp. 287–305; Ragep 1993, vol. 2, p. 461). Another annular eclipse (probably 164 A.D., Sept 4) was observed by Sosigenes. Cleomedes (sometime between mid-1st century B.C. and 400 A.D.) simply denies the reality of annular eclipses in contrast to the opinion of "some older" astronomers. Proclus (412–485 A.D.) also says that "some astronomers" report this type of eclipses, but he notes that Ptolemy's parameters exclude annular eclipses (for the references of these reports see Neugebauer (1975, vol. 2, p. 668)). Moreover, Simplicus in his commentary on Aristotle's *De caelo*, in 530 A.D. explains the possibility of annular solar eclipse. Although Bowen (2008, p. 74, see note 301) casts doubt on this subject, I see no problem to accept Simplicus' statement as evidence on annular eclipse (see also Heath 1991, pp. 69, 188; Pedersen 1974, ibid.). The main problem is that we have no information whether this statement stems from Aristotle's original work or it was simply added by Simplicus. On the other hand, based on the available evidence (see below) it is very unlikely that the notion of annular solar eclipse has been transferred from Greek to Islamic astronomy.

Fig. 1 Diagram for computing digits of solar eclipse (adopted from Toomer 1998, p. 303)



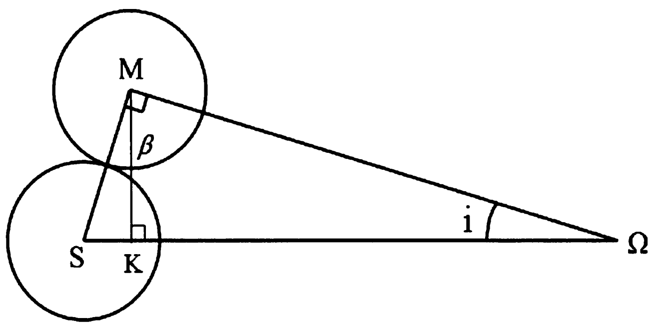


Fig. 2 The positions of the sun, *S* and the moon, *M* with respect to the lunar ascending node, *Ω*

limits of arguments of latitude, see Table 2).[8] The arguments of latitude correspond to the values of the greatest solar eclipse obscurity (in digits), for the two cases in which the moon is at apogee and at perigee. The Greek astronomers measured obscurity (or magnitude) of a solar eclipse by a unit that is called *digit*. The solar disc is divided into 12 digits.[9] In Fig. 1, E and O indicate centers of the apparent discs of the sun and moon respectively and the ratio (DZ/DB). 12 gives the value of solar eclipse magnitude in digits. For the case that the solar eclipse is total, the value of digits at mid-eclipse is obtained by: $(ZH/DB).12$.[10]

In Islamic *zīj*es (astronomical handbooks with tables), the greatest value of solar eclipse digits is normally given corresponding to mid-eclipse distance from the ascending or descending node and not from the highest northern point as Ptolemy considered (see the arc ΩM in Fig. 2). Sometimes digits are given corresponding to the moon's ecliptic latitude. The argument of latitude and the moon's ecliptic latitude can be computed from each other (see below).

[8] See Pedersen (1974, p. 201); Toomer (1998, p. 306).

[9] In Plutarch, one finds the remark that the diameter of the moon at mean distance is 12 digits. This makes only sense if taken as a definition which then must be implemented by the grading of instruments: one digit is the twelfth of the apparent diameter of the moon when at mean distance. This is in conformity with a statement by Sosigenes (175 A.D.) that a disk sometimes of 11, sometimes of 12 digits in diameter is needed to cover the moon. It is perhaps one of Ptolemy's clever innovations to define "digits" always as twelfths of the apparent diameter of sun or moon, regardless of the eccentricity (see Neugebauer 1975, vol. 2, p. 658).

[10] Toomer (1998, pp. 302–305).

The value of digits should be modified by using an auxiliary table (*Almagest*, VI 8, Table 3), based on the given lunar anomaly when the moon lies between apogee and perigee.[11] The greatest value of digits in Ptolemaic model is 12^d at apogee and $12;48^d$ at perigee. This indicates that angular diameter of the moon will never be less than that of the sun,[12] thus this is consistent with the above-mentioned Ptolemaic values for the angular diameters.

The Ptolemaic theory for solar eclipse was re-examined by later astronomers from the Islamic period, particularly the part that concerns the apparent diameters of the sun and moon. Islamic astronomers achieved new values of apparent diameters based on solar eclipse or non-eclipse observations.[13] Moreover, they learned about the values of solar and lunar angular diameters that were transmitted from the Indian to Islamic astronomy (see below).

The earliest reliable record of an annular eclipse goes back to Abu'l ᶜAbbās al-Irānshahrī. According to al-Bīrūnī's explicit citation, al-Irānshahrī had observed the annularity phase of a solar eclipse in 259 A.H./873 A.D. at Neishabour (in historical texts: Nishāpūr/Nisābūr, a city in north-east of Iran).[14] The validity of this observation has been proved by modern research.[15] For post-Irānshahrī observations, in history of Islamic astronomy, there is just another known observation made by al-Wābkanwī.[16] However, this observation still awaits an independent and profound analysis.[17]

[11] For a numerical example of this computation, see ibid, pp. 656–657.

[12] Ibid, p. 306.

[13] For a brief research on various values of the apparent diameters of the sun and moon in astronomy of the Islamic period, see Ragep (1993, vol. 2, p. 460).

[14] Al-Bīrūnī (1954–1956, vol. 2, p. 632).

[15] This solar eclipse occurred on 28 July 873. See Goldstein (1985, pp. 101–102); Stephenson (1997, p. 467).

[16] See *al-Muḥaqqaq al-Sulṭānī Zīj*, ms. 'Ulumi Library (no. 68) in Yazd, Chapter (*maqāla*) 3, Book (*bāb*) 14: knowing solar eclipse based on tables (fols.130r-134v). For al-Wābkanwī's biography and works, see Rosenfeld and Ihsanoglu (2003, pp. 245–246).

[17] Indeed M. Muzaffari firstly introduced al-Wābkanwī's observation of an annular eclipse (occurred on 29 Shawwal 681 A.H./January 30, 1283 A.D.) in the following note: "Wābkanawī and the first scientific observation of an annular eclipse", *The Observatory: A Review of Astronomy*, June 2009, pp. 144–146.
However he has cited some parameters of this eclipse erroneously, because of reading or recognizing them carelessly from the manuscript. For instance, for this annular eclipse he cites the value of the angular diameter of the moon, greater than that of the sun, based on what has extracted of al-Wābkanwī's record! By referring to the *zīj*, one can find the correct mentioned values of the angular diameters of the sun and the moon as: 0; 32,59° and 0; 32,11° respectively (see also *al-Muḥaqqaq al-Sulṭānī Zīj*, ms. Ayasofya, no. 2694, fol.74v).
There are several other mistakes and deficiencies in his method of research and in the given values and computations. However, it is very important that this observation be analyzed correctly and precisely in future. It is noteworthy to say that al-Wābkanwī claims that he had computed the parameters of this eclipse in advance and he knew there would be annularity phase in it, and then he observed it in the Mughān plain (northwest of Iran). Fortunately, al-Wābkanwī has given us the algorithm of computation in details; therefore it is one of the rare cases in the Islamic astronomy, in which the author provides us with the whole numerical computation of a real solar eclipse.

Since no observational record of this phenomenon prior to Irānshahrī exist and, as a whole, because of its scarcity in the history of astronomy,[18] sometimes we are encouraged to study the problem indirectly in historical sources.

The method of such study is based on two key points which indicate the authors' presumptive information on this phenomenon. The first and more reliable one is to examine the values of digits given in solar eclipse tables. If the magnitude of solar eclipse in such tables is less than 12 (given in digit, for the case that the moon has no ecliptic latitude at mid eclipse), it would imply the authors' possible awareness of the annular eclipse.

The second point concerns the mathematical formulae that relate angular diameters of the sun and moon with their angular velocities per hour or day. Such formulae originated from Indian astronomy and were apparently transferred to the astronomy of the Islamic period through *al-Khwārizmī's Zīj*.[19] In some Islamic *zīj*es the computed results based on such formulae are shown in tabular forms.[20] If the value of the moon's angular diameter is less than that of the sun in certain limits, one may conclude the possibility of the author's awareness of annular eclipse. However, this cannot be strong evidence by itself, and if both evidences mentioned above, appear in an astronomical work together, they can lead us to a more conclusive judgment.

The aim of this article is to examine the materials presented in the *al-Khwārizmī's Zīj* with regard to the two above-mentioned techniques, considering potentially existing information on annular solar eclipse in early period of Islamic astronomy.

1.1 Annular solar eclipse: historical intricacies

A statistical study shows that about 33% of different kinds of solar eclipses are annular. Among numerous historical records of solar eclipses available to us, only a few are annular.

The scarcity of observational records of annular solar eclipses in historical texts may be justified based on several reasons. We know that annular solar eclipses are slightly more frequent than total ones, while both total and annular eclipses are less frequent than partial ones. Those observers lying at the central path of an annular eclipse can distinguish it as a brilliant ring and those who observe it out of the central path; it makes no difference with a normal partial solar eclipse, because its appearance changes to an open ring. For ancient astronomers it was a real task to compute actual course of the central path, because an error equal to one digit in computations could erroneously displace the zone of the central path 500 km far away from the accurate place.[21] Of course, the width of the central path may reach to 300 km or more in some annular eclipses.

[18] For the authentic historical records, describing the annularity phase of solar eclipses, see Stephenson (1997, pp. 62–63, 258, 394, 404, 411, 467).

[19] See Neugebauer (1962, pp. 57–59).

[20] See Suter (1997, pp. 175–180) (Tables 61–66). For the recomputed values of these tables with 6° interval, see Table 1; for another historical table, see Kūshyār ibn Labbān, *al-Jāmi^c Zīj*, ms. Fatih, no. 3418/1, fol. 86r. For its analysis and the formulae behind it, see Giahi Yazdi (2009, pp. 152–155).

[21] See Giahi Yazdi (2008, pp. 75–82).

Although, a close ring of annular eclipse may be spotted by keen observers, but it is clearly seen with naked eye, where it occurs particularly near to the local horizon because of the solar light intensity.[22] On the other hand, the annularity phase is not a long-lasting phenomenon and it takes normally less than 12 minutes.[23]

From statistical point of view, ancient astronomers had little chance to face an annular solar eclipse and say precisely, observing the annularity phase. From astronomical point of view, this problem is comparable with the scarcity of observational records of totality phase in historical texts.[24] However, the appearance of an annular eclipse would not be as splendid as a total one.

1.2 Al-Khwārizmī and his *Zīj*: historical background

Muḥammad ibn Mūsā Al-Khwārizmī was an eminent astronomer and mathematician at the court of al-Ma'mūn (Abbasid Caliph) in the ninth century.[25] He compiled a *zīj* around 230 A.H./844 A.D., which is considered as his most significant astronomical work. It was frequently referred to by later astronomers. The importance of *al-Khwārizmī's Zīj* is in that, besides the *Mumtaḥan Zīj*, it remained somehow incorporated from the early period of Islamic astronomy. *Al-Khwārizmī's Zīj* includes numerous methods from the Indian, Greek and Iranian pre-Islamic astronomy.[26] According to the historical sources, there were two editions of *al-Khwārizmī's Zīj*, but we know nothing of the differences between them.[27] Today, there is only a Latin version of the *zīj* available that had been translated by Adelard Bathi in the early twelfth century.[28] It was not translated directly from its original Arabic text, rather a recension of Andalusian astronomer al-Majrīṭī (d.1007–1008)[29] and perhaps further revised by al-Majrīṭī's pupil, Ibn al-Ṣaffār (d. 1035).[30] We can, however, get some notions of the original form of the work from extracts and commentaries made by earlier writers. Thus, from the tenth century comments of Ibn al-Muthannā, we learn that al-Khwārizmī formulated his table of sines on a base of 150 (a common Indian parameter), whereas base 60 (usually in Islamic sine tables) was employed in extant tables.[31] From the same source, we learn that the epoch of the original tables was Yazdgerd era, established

[22] It is noteworthy to say that the annularity phase of the solar eclipse of 873 (which was observed by al-Iranshahri) occurred in early morning (see Stephenson 1997, p. 467). See Footnote 15.

[23] See Meeus (2002, pp. 116–117).

[24] According to Stephensons' survey, there is no record of total solar eclipse in Islamic *zīj*es (see Stephenson 1997, pp. 456–500). For a comprehensive analysis of the solar eclipse observations rested in Islamic *zīj*es, see Said and Stephenson (1996, pp. 259–273, 1997, pp. 29–48).

[25] For al-Khwārizmī's biography and his works, see "al-Khwārizmī" by Toomer in *Dictionary of scientific biography* (*DSB*), vol. 7, pp. 358–365; Rosenfeld and Ihsanoglu (2003, pp. 23–25).

[26] See Ibn al-Qifṭī (1903, p. 271); Van Dalen (1996, pp. 210–211).

[27] See Ibn Nadīm (1971, p. 333); Ibn ᶜbrī (1983 p. 237).

[28] For its remained manuscripts, see Carmody (1956, pp. 46–47).

[29] Ibn al-Qifṭī (1903, p. 326); Ibn Sāᶜid al-Andalūsī (1971, p. 246).

[30] See Vallicrosa and María (1947, pp. 75, 109–110). For an analysis of the remained work of Ibn al-Ṣaffār, see Castells and Samsó (2007, pp. 229–262).

[31] Goldstein (1967, p. 178).

by the last Sassanid King Yazdgerd-e III (epoch: 16 June 632) and not the lunar Hijra era (epoch: 15 July 622) which appears in al-Majrīṭī's revision.[32] There is another commentary of *al-Khwārizmī's Zīj* written by Ibn Masrūr that has not been edited and published yet.[33]

In fact, the survived version of *al-Khwārizmī's Zīj* consists of a variety of different astronomical methods that were derived from Indian, Greek and Iranian pre-Islamic astronomy. However, it is mostly based on Indian astronomy.

The *Zīj* of al-Khwārizmī, was the first of such work to reach the west. Many of al-Khwārizmī's tables reached a wide audience in the west via another work, the *Toledan Tables*, a miscellaneous assembly of astronomical tables from the work of al-Khwārizmī, al-Battānī, and al-Zarqālī which was translated into Latin, probably by Gerard of Cermona, in the late twelfth century.[34]

1.3 Annular solar eclipse: supporting evidence in the *al-Khwārizmī's Zīj*

1.3.1 Tables of lunar and solar angular velocities and apparent diameters

Tables 61–66 in the *al-Khwārizmī's Zīj* show angular velocity of the sun and moon per hour, corresponding to their apparent or angular diameters. According to the values in the Tables 61–66, col. 2–5, Prof. Neugebauer presented the following relations[35]:

$$d_s/(v_s^o/\mathrm{h}) = 13;12 \quad \text{and} \quad d_m/(v_m^o/h) = 0;58,10 \pm 0;0,2$$

where d_s and d_r are apparent diameters of the moon and sun and $v_s$and v_m are their hourly angular velocities, respectively.

Neugebauer stated that these formulae are only approximately satisfactory.[36] It can be proved that the values in the Tables 61–66 are on the basis of the formulae:

$$r_s = (5.5/20) \cdot V_s \tag{1}$$
$$r_m = (5/247) \cdot V_m \tag{2}$$

These formulae originated from Indian astronomy.[37] It should be kept in mind that the above formulae are based on solar and lunar apparent radiuses (not apparent

[32] Ibid, p. 18.

[33] For its unique copy, see King (1986, no. B37, p. 38).

[34] See also "al-Khwārizmī" in *DSB*, ibid. For the most comprehensive research on the *Toledan Tables*, see Pedersen (2002).

[35] Neugebauer (1962, pp. 105–106).

[36] Ibid.

[37] For the Indian origin of these formulae, see Brahmagupta (1970, p. 118), Suter (1997, pp. 78–80) and Neugebauer (1962, pp. 57–59). Ibn Masrūr pp. 81–82, explicitely cites these formulae from *al-Khwārizmī's Zīj* (of course multiplies them by 2 to obtain d_s and d_m), however there is no explicit mention of them in the Latin translation of *al-Khwārizmī's Zīj*. For the method underlying these formulae, see Eduardo Millas Vendrell (1963, pp. 78–79). It is important to know that Ibn Masrūr, pp. 81–82 also mentions two other formulae from *al-Khwārizmī's Zīj* concerning the computation of d_s and d_m as follows:

diameters) corresponding to their diurnal (not hourly) angular velocities. Therefore, the values of v_s and v_m appearing in Tables 61–66, should be multiplied by 24 to yield diurnal velocities (i.e., V_s and V_m). The above formulae also give the same values for the ratios that Neugebauer has mentioned. The values with interval of 6° in anomaly were recomputed based on the formulae mentioned above and the values are shown in Table 1. For the angular diameter of the sun, since the accuracy of the values for hourly motions are in second of arc, therefore each tabular value corresponds to a set of angular diameters in such a way not to exceed the range: $-4'' \leqq \Delta \leqq 4''$. For the angular diameter of the moon, there appears an approximately constant shift of $-2''$. This may imply that the formula underlying the tabular values for the moon was slightly different from the one mentioned above. However, the pattern of errors indicates that computations have been carried out well. According to the Tables 61–66, the values given for r_m, $r_s$and v_m^o/h, v_s^o/h vary in the following ranges:

$$0;15,40^\circ \le r_s \le 0;16,54^\circ \quad 0;14,38^\circ \le r_m \le 0;17,17^\circ$$
$$0;2,22^\circ \le v_s^o/h \le 0;2,34^\circ \quad 0;30,12^\circ \le v_m^o/h \le 0;35,40^\circ$$

Based on this evidence, we are able to conclude the presence of the notion of annular solar eclipse, because where the moon is at apogee, its angular diameter is less than that of the sun. Since the above formulae originated from Indian astronomy, it is very likely that these tables have been taken from the original version of *al-Khwārizmī's Zīj*.

1.3.2 Table of solar eclipse

There is a stronger evidence that supports the presence of annular solar eclipse in the *al-Khwārizmī's Zīj*, that this also interestingly concerns the values obtained above. Table 78 of *al-Khwārizmī's Zīj* provides the digits of obscurity and the value for half duration of solar eclipse (immersion) in minutes of arc, both corresponding to the argument of latitude (see Table 3). Indeed, there is no clear explanation on the rules concerning how one should use Table 78.[38] However, there is just an erroneous addition by a second hand in Chapter 35 that gives general description about it. This description, ambiguously relates the effect of the latitudinal lunar parallax with the maximal digits of obscurity appeared in Table 78 (i.e., equivalent to $12;48^d$ at perigee and $10;48^d$ at apogee). Neugebauer tried to interpret the text by relating these values to the latitudinal lunar parallax and even changed the value of $10;48^d$ to $10;44^d$ without further explanation.[39]

It should be noted that Islamic astronomers generally presented the lunar parallax in latitude as a function of zenithal distance of the highest point of the ecliptic (with

$d_s = V_s - (V_s/5 + V_s/4) \rightarrow d_s = 11V_s/20$ and $d_m = 10((V_m/10)/24.7) \rightarrow d_m = 10V_m/247$.
It is obvious that these formulae are equivalent to the formulae 1 and 2, respectively.

[38] Suter (1997, pp. 94–96) has previously declared that Chapter 35 of the *Zīj* is unclear and incomplete. He did not analyze the structure of the table, rather tried to show the role of the argument of latitude and lunar parallax in a numerical example.

[39] Neugebauer (1962, pp. 74–76).

Table 1 The values with interval of 6° in anomaly, correspond to solar and lunar hourly motions and their angular radiuses (adopted from Tables 61–66 of *al-Khwārizmī's Zīj*; see Suter 1997, pp. 175–180)

Anomaly		Sun		Moon	
		Hourly motion	Angular radius (r_s)	Hourly Motion	Angular radius (r_m)
1°	359°	2;22′	15;40′ (+3)	30;12′	14;38′ (−2)
6	354	2;22	15;40 (+3)	30;13	14;39 (−2)
12	348	2;22	15;40 (+3)	30;15	14;40 (−2)
18	342	2;23	15;41 (−3)	30;20	14;42 (−2)
24	336	2;23	15;42 (−2)	30;27	14;45 (−2)
30	330	2;23	15;44 (0)	30;34	14;49 (−2)
36	324	2;23	15;46 (+2)	30;43	14;53 (−2)
42	318	2;24	15;48 (−2)	30;52	14;58 (−2)
48	312	2;24	15;51 (+1)	31;4	15;4 (−1)
54	306	2;24	15;54 (+4)	31;16	15;9 (−2)
60	300	2;25	15;58 (+1)	31;29	15;16 (−2)
66	294	2;26	16;1/2 (−3)[a]	31;45	15;23 (−2)
72	288	2;26	16;4 (0)	32;1	15;32 (−1)
78	282	2;27	16;7 (−3)	32;24	15;41 (−3)
84	276	2;27	16;12/14 (+2)	32;42	15;51 (−2)
90	270	2;28	16;17 (0)	32;56	15;58 (−2)
96	264	2;29	16;22 (−1)	33;10	16;5 (−2)
102	258	2;29	16;26 (+3)	33;29	16;14 (−2)
108	252	2;30	16;30 (0)	33;48	16;23 (−2)
114	246	2;30	16;33 (+3)	34;6	16;32 (−2)
120	240	2;30	16;36 (+6)	34;22/12	16;40 (−2)
126	234	2;32	16;39 (−4)	34;37	16;47 (−2)
132	228	2;32	16;43 (0)	34;53	16;54 (−3)
138	222	2;32	16;47 (+4)	35;4	17;0 (−2)
144	216	2;33	16;49 (−1)	35;12	17;4 (−2)
150	210	2;33	16;50 (0)	35;20	17;7 (−3)
156	204	2;33	16;51 (+1)	35;27	17;11 (−2)
162	198	2;33	16;52 (+2)	35;33	17;14 (−2)
168	192	2;33	16;53 (+3)	35;37	17;16 (−2)
174	186	2;34	16;54 (−2)	35;39	17;17 (−2)
180	180	2;34	16;54 (−2)	35;40	17;17 (−3)

[a] Different values for the seconds of arc are cited from other manuscripts and are separated by a slash.

respect to the observer's local horizon) or the zenithal distance of the moon on altitude circle.[40] Moreover, the longitudinal or latitudinal lunar parallax cannot be considered

[40] For the various theories on lunar parallax in Islamic astronomy, see Kennedy (1983, pp. 164–184). Ptolemy's table for lunar parallax is based on lunar zenithal distance, see Toomer (1998, p. 265).

Table 2 Comparing solar eclipse limitsΩM in al-Khwārizmī's model and Ptolemy's one

Case	Al-Khwārizmī's solar eclipse limits	Ptolemy's solar eclipse limits
At apogee	$6;37° \geq \Omega M \geq 0°$ or $180° \geq \Omega M \geq 173; 23°$	$90° \geq \Omega M \geq 84°$ or $276° \geq \Omega M \geq 270°$
	$360° \geq \Omega M \geq 353°$ or $187° \geq \Omega M \geq 180°$	
At perigee	$7;11° \geq \Omega M \geq 0°$ or $180° \geq \Omega M \geq 172; 49°$	$90° \geq \Omega M \geq 83; 36°$ or $276;24° \geq \Omega M \geq 270°$
	$360° \geq \Omega M \geq 352; 30°$ or $187;30° \geq \Omega M \geq 180°$	

as a function of argument of latitude. On the other hand, it is very unusual that the author states that the latitudinal lunar parallax is computed based on digits. It is obvious that the unit of measuring the latitudinal lunar parallax is degree or minutes of arc and not digits. I will demonstrate that the values of digits in Table 78 have nothing to do with the latitudinal lunar parallax. This table shows a very similar structure compared to Table VI 8 of the *Almagest*,[41] although there are some interesting differences between them. Table 78 consists of two sections. The left section tabulates the values for the case where the moon is at apogee and the right one for where the moon is at perigee. The values given for argument of latitude are rounded off to 30 in minutes of arc. In tables of both Ptolemy and al-Khwārizmī, the limit of solar eclipse (argument of latitude) at perigee increases by increasing the angular diameters of the moon and sun.

Al-Khwārizmī's table gives two different limits for the "argument of latitude" (ΩM): one for perigee and apogee and another for east and west of each node (see Table 2).[42] Here, $\Omega M = 0°$ indicates the exact point of the ascending node. In Ptolemy's table, however, there is a single difference in limits between perigee and apogee. Since in such kind of tables, astronomers have not considered the effect of the lunar parallax in latitude in computing limits of solar eclipse, it is unexpected to see any difference between the limits given for east or west of each node in Al-Khwārizmī's table.

In Table 78, like Table VI 8 in the *Almagest*, the maximum values of obscurity (at mid-eclipse), have been tabulated in digits corresponding to the argument of latitude. For the case where the moon is at apogee, the value maximal for the digits is $10; 48^{d}$ and at perigee is $12; 44^{d}$.

With regard to the variation of r_{m}, r_{s}in the *al-Khwārizmī's Zīj*, one can obtain the greatest values of obscurity (in digits) at mid-eclipse (where $\Omega M = 0$) by the following formulae[43]:

$$\text{At apogee: } 12.(\text{Min } r_{\mathrm{m}})/(\text{Mean } r_{\mathrm{s}}) = 12.(14;38'/16;17') = 10;47^{\mathrm{d}}$$

[41] See Toomer (1998, p. 306).

[42] See also Neugebauer (1962, p. 126).

[43] Indeed, Table 3 gives this value equal to $10; 48^{d}$. It is interesting to consider that, the least possible value of a linear magnitude for an annular eclipse is 0.904 and for the total one the greatest value is 1.081 (see Meeus 2004, p. 119). Therefore, multiplying these values by 12 gives us the corresponding values in digits equal to $10; 50, 53^{d}$ and $12; 58, 19^{d}$, respectively.

At perigee: $12.(\text{Max } r_m)/(\text{Mean } r_s) = 12.(17;17'/16;17') = 12;44^d$

This shows that the author applies the same value for the mean angular diameter of the sun at both apogee and perigee, while it takes into account the variation of angular diameter of the moon at apogee and at perigee separately. According to Table 78, $\Omega M = 0$ means that the moon has to be aligned concentrically with respect to the sun at mid eclipse. Thus when the value of digits equals to 10; 47^d where $\Omega M = 0$ (at apogee), it shows that the author knew that such a case would lead to the annular solar eclipse where the sun is not totally covered by the moon, because the value of digits is less than 12 at mid eclipse. At perigee with the value of digits equal to 12; 48^d, there would be a total solar eclipse. It seems that confining the variations of the solar angular diameter to its mean value was meant to avoid extending the numbers of digits and making the table too complex.

The computational method behind the values of half eclipse durations (immersion), T in Table 78 is another important subject. The maximum value of immersion in minutes of arc at apogee and at perigee (where $\Omega M = 0$) are obtained by the following relations:

At apogee: $T = (\text{Min } r_m) + (\text{Mean } r_s) = 14;38' + 16;17' = 30;55'$
At perigee: $T = (\text{Max } r_m) + (\text{Mean } r_s) = 17;17' + 16;17' = 33;34'$

To recalculate the other values of immersion, it is necessary to primarily obtain the inclination of the lunar orbit with respect to the ecliptic i (or the maximal value of the lunar ecliptic latitude) used in al-Khwārizmī's model. In Fig. 2, we have:

$$\text{cotg}(i) = \Omega M/MS$$

(at apogee) $\text{cotg}(i) = 6;\ 37^\circ/0;\ 30,55^\circ$ and (at perigee) $\text{cotg}(i) = 7;\ 11^\circ/0;\ 33,34^\circ$ Both above formulae lead to $i = 4;\ 27^\circ$.[44] Thus by taking the values of ΩM from the table, the intermediate values of MS are obtained by the following formula:[45]

$$MS = \Omega M/\text{cotg}(i) \rightarrow MS = \Omega M \cdot \text{tg}(i) \qquad (3)$$

[44] This value is in good agreement with the maximal value given for the lunar latitude in Table 23 of the *Zīj* (equal to $i = 4;\ 30^\circ$, see Suter 1997, p. 134). Ibn Hibintā (1987, vol. 1, p. 176) also cites the same value from the original version of *al-Khwārizmī's Zīj*.

[45] Indeed, this formula is based on the plane trigonometry, and it gives approximate results. Since in Fig. 2, the sides of the spherical triangle are small in degrees, ancient astronomers generally dealt with it as a plane triangle (See Pedersen 1974, pp. 227–228). However, it should be noted that the difference is negligible in most cases. The modern exact formula is based on the spherical trigonometry. By using the Napier formulae, we have:
$\sin(\Omega M) = \text{tg}(MS) \cdot \text{tg}(90 - i) \rightarrow \text{tg}(MS) = \sin(\Omega M) \cdot \text{tg}(i)$. See Smart (1999, pp. 14–15).
The values recomputed of T (see formula 7) based on the values of MS derived from the Napier formula, are shown in Table 3. Comparing these values with those derived from formulae 3 and 7 indicate it is unlikely that an exact spherical formula(e), similar to the above one, underlie the tabular values (see Table 3). On the other hand, historical evidence from the *Toledan Tables* support the idea that the formula is based on the plan trigonometry rather than the exact method.

Table 3 Table 78 of *al-Khwārizmī's Zīj* (Suter 1997, p. 193)

Moon is at its greatest distance (apogee)					Moon is at its least distance (perigee)				
Arguments of latitude (ΩM)		Digits (d)	Minutes of immersion (T)		Arguments of Latitude (ΩM)		Digits (d)	Minutes of immersion (T)	
			Plan trig.	Nap. Rule				Plan trig.	Nap. Rule
6; 37°	173; 23°	0 (0)	0′(0)	(0)	7;11°	172;49°	0 (0)	0′(0)	(0)
6;30	173;30	0;11 (−1)	5;30 (−23)	(−42)	7;0	173;0	0;17 (−3)	7;56 (+19)	(−2)
6;0	174;0	1;5 (+4)	13;7 (+3)	(−4)	6;30	173;30	1;9 (−4)	14;11 (−9)	(−17)
5;30	174;30	1;55 (+5)	17;10 (−3)	(−6)	6;0	174;0	2;0 (−6)	18;32 (+3)	(−2)
5;0	175;0	2;45 (+6)	20;10 (−6)	(−8)	5;30	174;30	2;53 (−6)	21;37 (0)	(−2)
4;30	175;30	3;37 (+9)	22;41 (0)	(−1)	5;0	175;0	3;47 (−6)	24;2 (−5)	(−6)
4;0	176;0	4;29 (+12)	24;41 (+3)	(+2)	4;30	175;30	4;27 (−19)[a]	26;12 (+2)	(+1)
3;30	176;30	5;21 (+16)	26;15 (0)	(0)	4;0	176;0	5;28 (−11)	27;13 (−40)	(−41)
3;0	177;0	6;13 (+19)	27;21 (−13)	(−13)	3;30	176;30	6;20 (−12)	29;17 (−2)	(−2)
2;30	177;30	7;6 (+23)	28;39 (+1)	(+1)	3;0	177;0	7;12 (−13)	30;19 (−11)	(−11)
2;0	178;0	7;57 (+25)	29;28 (0)	(0)	2;30	177;30	8;5 (−13)	31;31 (+3)	(+3)
1;30	178;30	8;48 (+27)	30;7 (0)	(0)	2;0	178;0	8;56 (−15)	32;15 (+1)	(+1)
1;0	179;0	9;39 (+29)	30;34 (0)	(0)	1;30	178;30	9;47 (−17)	32;49 (0)	(0)
0;30	179;30	10;32 (+33)	30;51 (+1)	(+1)	1;0	179;0	10;38 (−20)	33;15 (+1)	(+1)
0;0	180;0	10;48 (0)	30;55 (0)	(+1)	0;30	179;30	11;30 (−21)	33;30 (+1)	(+1)
359;30	180;30	10;32 (+33)	30;51 (+1)	(+1)	0	180;0	12;44 (0)	33;34 (0)	(0)
359;0	181;0	9;39 (+29)	30;34 (0)	(0)	359;30	180;30	11;30 (−21)	33;30 (+1)	(+1)
358;30	181;30	8;48 (+27)	30;7 (0)	(0)	359;0	181;0	10;38 (−20)	33;15 (+1)	(+1)
358;0	182;0	7;57 (+25)	29;28 (0)	(0)	358;30	181;30	9;47 (−17)	32;49 (0)	(0)
357;30	182;30	7;6 (+23)	28;39 (+1)	(+1)	358;0	182;0	8;56 (−15)	32;15 (+1)	(+1)
357;0	183;0	6;13 (+19)	27;21 (−13)	(−13)	357;30	182;30	8;5 (−13)	31;31 (+3)	(+3)
356;30	183;30	5;21 (+16)	26;15 (0)	(0)	357;0	183;0	7;12 (−13)	30;19 (−11)	(−11)
356;0	184;0	4;29 (+12)	24;41 (+3)	(+2)	356;30	183;30	6;20 (−12)	29;17 (−2)	(−2)
355;30	184;30	3;37 (+9)	22;41 (0)	(−1)	356;0	184;0	5;28 (−11)	27;13 (−40)	(−41)
355;0	185;0	2;45 (+6)	20;10 (−6)	(−8)	355;30	184;30	4;27 (−19)	26;12 (+2)	(+1)
354;30	185;30	1;55 (+5)	17;10 (−3)	(−6)	355;0	185;0	3;47 (−6)	24;2 (−5)	(−6)
354;0	186;0	1;5 (+4)	13;7 (+3)	(−4)	354;30	185;30	2;53 (−6)	21;37 (0)	(−2)
353;30	186;30	0;11 (−1)	5;30 (−23)	(−42)	354;0	186;0	2;0 (−6)	18;32 (+3)	(−2)
353;0	187;0	0 (0)	0 (0)	(0)	353;30	186;30	1;9 (−4)	14;11 (−9)	(−17)
					353;0	187;0	0;17 (−3)	7;56 (+19)	(−2)
					352;30	187;30	0 (0)	0 (0)	(0)

[a] Suter (1997, p. 193) mentions the value of $4;37^{d}$ from another manuscript in footnote. It seems that this value to be correct, because the difference of this value with our computation is −9, therefore it lies well between previous value of difference: −6 and the succeeding one: −11. Moreover, the value of $4;37^{d}$ is in agreement with the computation that was carried out based on the "coefficient method" (see Table 4). On the other hand, Pedersen (2002, pt. 4, p. 1461) cites this value from the *Toledan Tables*, in which a similar table appears based on *al-Khwārizmī's Zīj* (see below).

Now the values of digits d corresponding to ΩM are computed by means of the following formulae:

$$\text{at apogee:}\quad d = (1 - (MS/0;30,55)) \cdot 10;48 \tag{4}$$

$$\text{at perigee:}\quad d = (1 - (MS/0;33,34)) \cdot 12;44 \tag{5}$$

In Ibn Muthanā's commentary on *al-Khwārizmī's Zīj*, the following formula appears for computing the values of immersion in hours and minutes[46]:

$$T_{\mathrm{h}} = 24\sqrt{(r_{\mathrm{s}} + r_{\mathrm{m}})^2 - MS^2}/(V_{\mathrm{m}} - V_{\mathrm{s}}) \tag{6}$$

where V_{m} and V_{s} are daily angular velocities of the moon and sun, respectively.

To compute the values of immersion in minutes of arc, the above formula should be modified as follows:

$$T = \sqrt{(r_{\mathrm{s}} + r_{\mathrm{m}})^2 - MS^2} \tag{7}$$

Therefore, it is plausible to suppose that the values of solar eclipse table were basically computed based on the formula 7. Of course, one should enter r_{m} (min.) and r_{s} (mean) (at apogee) and r_{m} (max.) and r_{s} (mean) (at perigee) (For the values, see above).
The following relations also give the values of immersion in minutes of arc based on digits:

$$\text{At apogee:}\quad T = 30;55\sqrt{d(21;36 - d)}/10;48 \tag{8}$$

$$\text{At perigee:}\quad T = 33;34\sqrt{d(25;28 - d)}/12;44 \tag{9}$$

It should be noted that the formulae 8 and 9 are equivalent to the formula 7, all of which resulting in similar values for T.
The recomputed values of T, based on the formula 7, are shown in Table 3. The differences between the recomputed values and the tabular values may have originated from the errors of copyists or from original computations. It is possible to distinguish between them in some extent, based on some evidence.

For instance, the divergence in minutes of immersion corresponding to $\Omega M = 6;30°$ and $\Omega M = 3°$ (at apogee) and for $\Omega M = 7°$ and $\Omega M = 4°$ (at perigee) may be justified as errors of copyists. Particularly for the case of $\Omega M = 4°$ the error is -40 indicating that the correct value is 27;53′ rather than 27;13′. In the *Abjad* numerical system, if the original computed minutes of that tabular value would be "53 = نج", it could erroneously be read "13 = يج" which the latter value appears in Table 3.

This conjecture was confirmed by considering the values given in the solar eclipse table that appeared in the *Toledan Tables*. Toomer proved that this table was taken from the *al-Khwārizmī's Zīj*.[47] The interesting fact is that, for the case $\Omega M = 4°$ (at perigee) the *Toledan Tables* give the correct value of 27;53′ which is similar to

46 See Goldstein (1967, pp.138, 139, 241).

47 See Toomer (1968, pp. 86–87).

our computation. On the other hand, for the case $\Omega M = 5;30°$ (at perigee) it gives an incorrect value equal to $21;17'$ (instead of $21;37'$ that appears in *al-Khwārizmī's Zīj*), therefore there is a miscopying of 37 = لز as 17 = يز. Of course, there are other serious errors in minutes of immersion corresponding to $\Omega M = 6;30°$ (at apogee) and $\Omega M = 7°$ (at perigee), that they cannot be justified easily.[48]

However, it is odd that the recomputed values for digits (based on the formulae 4 and 5) have enormous differences with the tabular ones, whereas employing the values computed for digits in formulae 8 and 9 yields good results for minutes of immersion T. Since the errors in digits follow two systematic patterns, thus it seems that they were computed based on the other two formulae. The errors for the values of digits at apogee follow a dwindling pattern and at perigee follow a cumulative one (see Table 3). A simple analysis shows that the ratio of each tabular value with its corresponding recomputed value is 1.055 at apogee and 0.970 at perigee.[49] Therefore, the formulae 4 and 5 need the following slight modifications (for the recomputed results see Table 4, column for coefficients):

$$\text{At apogee:}\quad d = 1.055(1 - (MS/0;30,55)) \cdot 10;48 \tag{10}$$

$$\text{At perigee:}\quad d = 0.970(1 - (MS/0;33,34)) \cdot 12;44 \tag{11}$$

The main issue is to define the role of the mentioned coefficients in computations. It is not justifiable to consider that these coefficients were originally used in the formulae. However, if we multiply the mentioned coefficients by the maximum digits, they will result in the following formulae:

$$\text{At apogee:}\quad d = (1 - (MS/0;\,30,55)) \cdot 11;23,38 \tag{12}$$

$$\text{At perigee:}\quad d = (1 - (MS/0;\,33,34)) \cdot 12;21,4 \tag{13}$$

This idea that the values for digits were computed based on the different values of maximum digits seems to be peculiar at first. But the interesting fact is that the maximum values for digits (in the formulae 12 and 13) are very approximate compared to those we find in the *al-Ṣābi' Zīj* of al-Battānī (as 11;23,30 and 12;33,0).[50] However, if we replace the above maximal values for digits with values that appear in the *al-Ṣābi'Zīj*, it is necessary to replace the denominators (the maximal value of immersion for each case) with those values that appear in the *al-Ṣābi' Zīj* as well[51]:

$$\text{At apogee:}\quad d = (1 - (MS/0;31,0)) \cdot 11;23,30 \tag{14}$$

$$\text{At perigee:}\quad d = (1 - (MS/0;34,0)) \cdot 12;33,0 \tag{15}$$

[48] Cf. ibid. Pedersen (2002, pt. 4, p. 1461) also cites the values as $27;52'$ and $27;53'$ of two different manuscripts of the *Toledan Tables*. Of course there are other discrepancies between the values given in the *Toledan Tables* and *al-Khwārizmī's Zīj* that do not help us in interpreting the difference between the tabular and computed values of *al-Khwārizmī's Zīj*.

[49] It is impossible to define the exact value of these ratios, because there are small differences from one to another parameter.

[50] See Nallino (1969, pt. 2, p. 91).

[51] For these values, see ibid.

Table 4 The recomputed digits based on coefficient method and al-Battānī's values; correspond to the argument of latitudes

At apogee				At perigee				
Argument of latitude (ΩM)		Digits (d) coefficient	Digits (Battānī i =4;27°)	Argument of latitude (ΩM)		Digits (d) coefficient	Digits (Battānī i =4;27°)	Digits (Battānī i =4;32°)
6;37°	173;23°	0 (0)	(0)	7;11°	172;4°	0 (0)	0	(0)
6;30	173;30	0;11 (−2)	(−3)	7;0	173;0	0;17 (−2)	(−12)	(+1)
6;0	174;0	1;5 (+1)	(−1)	6;30	173;30	1;9 (−2)	(−12)	(+1)
5;30	174;30	1;55 (−1)	(−2)	6;0	174;0	2;0 (−3)	(−13)	(−1)
5;0	175;0	2;45 (−2)	(−4)	5;30	174;30	2;53 (−1)	(−11)	(−1)
4;30	175;30	3;37 (−2)	(−3)	5;0	175;0	3;47 (+1)	(−9)	(+1)
4;0	176;0	4;29 (−2)	(−3)	4;30	175;30	4;27 (−10)	(−21)	(−12)
3;30	176;30	5;21 (−1)	(−2)	4;0	176;0	5;28 (−1)	(−11)	(−4)
3;0	177;0	6;13 (−1)	(−2)	3;30	176;30	6;20 (0)	(−11)	(−4)
2;30	177;30	7;6 (0)	(0)	3;0	177;0	7;12 (0)	(−11)	(−5)
2;0	178;0	7;57 (0)	(−1)	2;30	177;30	8;5 (+2)	(−9)	(−5)
1;30	178;30	8;48 (−1)	(−1)	2;0	178;0	8;56 (+1)	(−10)	(−6)
1;0	179;0	9;39 (−1)	(−2)	1;30	178;30	9;47 (+1)	(−11)	(−8)
0;30	179;30	10;32 (0)	(0)	1;0	179;0	10;38 (0)	(−12)	(−10)
0;0	180;0	10;48	11;23,30	0;30	179;30	11;30 (0)	(−11)	(−10)
		(11;23,38)		0	180;0	12;44 (12;21,4)	(12;33)	(12;33)

It is also important to recognize by which value of i computations of MS were originally carried out. The computed values of MS in the formula 14 (based on $i = 4; 27°$) lead to good results for digits at apogee, but at perigee, with the formula 15, there appears a shift in range of −11 to −13, approximately (see Table 4). This approximate constant shift indicates that the value $i = 4; 27°$ could be applied in computation. However, it is possible that the underlying values have been transformed from *al-Ṣābi' Zīj* based on a careless computation or adoption.[52] This conjecture may be confirmed by considering that in the *al-Ṣābi' Zīj*, the values of digits are given corresponding to the lunar latitude, β (not argument of latitude, as it appears in the *al-Khwārizmī's Zīj*, see Fig. 2). The values recomputed based on the formulae 10, 11 (coefficients) and 14, 15 (al-Battānī's values with $i = 4; 27°$ and $i = 4; 32°$) are shown in Table 4.

We should do some computations to understand the relation between al-Khwārizmī's tabular values and those appearing in the *al-Ṣābi' Zīj*. In Fig. 2, $KM = \beta$ which is the lunar latitude. So we have:

[52] Of course my research showed that al-Battānī's values with $i = 4; 32°$ produce good results as well, particularly in the first half of the table. But in the second half it produces a systematic increasing pattern of errors that makes it difficult to accept this parameter of i underlying the tabular values at perigee. On the other hand, it is unjustifiable to suppose that different values for i were independently used at apogee and at perigee.

$$\sin(i) = \beta/\Omega M \qquad (16)$$

By considering $i = 4; 27°$ and the values for β, taken from *al-Ṣabi' Zīj* (see Table 5), the corresponding values of ΩM are computed by the above formula. Now, based on each computed values for ΩM, we take its corresponding value of digits from al- Khwārizmī's table[53] and modify it with its computed difference for the nearest value of digits at both apogee and perigee (for the values of differences see Table 4, column 3, section of apogee and perigee).[54] The obtained results are very close to those appearing for solar eclipse digits in the *al-Ṣabi' Zīj* (see Table 5). As a whole, this analysis indicates the values of immersion were probably computed based on a relatively correct formula, whereas the values for digits were taken from another table or they were computed based on a method irrelevant to the computational procedure of the minutes of immersion as it was discussed above. If the value for digits have been taken from al-Battānī's *Zīj*, then it is plausible to suppose that these values stem from al-Majrīṭī's manipulation, particularly we know that al-Majrīṭī provided a summary of *al-Ṣābi' Zīj*.[55]

This raises several questions:

1- Did al-Khwārizmī's original *Zīj* include eclipse digits or not?
2- Can we suppose that al-Majrīṭī himself awkwardly juxtaposed the values of digits and arguments of latitude without doing precise computation?
3- Did al-Majrīṭī find serious errors in al-Khwārizmī's digits or he simply preferred the tabular values of *al-Ṣābi' Zīj*?

Since, we may assume that the sections pertaining to the values for digits were inseparable parts of each solar eclipse table, it is very unlikely that there were no eclipse digits in al-Khwārizmī's original *zīj*, having in mind that Muslim astronomers were familiar with the structure of such tables through *Almagest*. It is possible that al-Majrīṭī preferred those digits in the *al-Ṣābi' Zīj* but made serious mistakes in their adoption and transfer. This conjecture may be supported by the fact that the tabular values for digits in the *al-Ṣābi'Zīj* correspond to the values for lunar latitude and not for argument of latitude, as seen in the *al-Khwārizmī's Zīj*.

2 Concluding remarks

For a long period of time, historians of astronomy have simply introduced al-Battānī (d. 137 A.H./929 A.D.) as the pioneer in Islamic astronomy (or even in the history of astronomy) who knew the possibility of annular solar eclipse.[56] I have not dealt

[53] Since the values of ΩM are not the same as those appear in Table 3, in most cases, it is necessary to interpolate the corresponding values for digits, based on the tabular values.

[54] If the value of difference is negative, it should be added to the obtained value of digit. For instance, for $\Omega M = 4; 59°$ (at perigee), the corresponding value for digit is equal to $3; 46^d$ (see Table 3). By adding +9 (the nearest value of difference), it yields $3; 55^d$ which is very close to 4^d that appears in al-Battānī's table.

[55] See Ibn al-Qifṭī (1903, p. 326).

[56] See "Eastern Arabic astronomy between the eighth and the eleventh centuries" by Morelon in *Encyclopedia of the History of Arabic Science*, vol. 1, p. 47; see "al-Battānī" by Hartner in *DSB*, vol. 1, p. 511.

Table 5 Al-Battānī's solar eclipse table (Nallino, 1899-1907, pt. 2, p. 91)

Moon and sun both at perigee				Moon and sun both at apogee			
Lunar latitude, β	Rec. ΩM	Digits d		Lunar latitude, β	Rec. ΩM	Digits d	
Tab.		Tab.	Rec.	Tab.		Tab.	Rec
34;0′	7;18°	0	(0)	31;00′	6;40°	0	(0)
31;18	6;43	1	(+1)	28;18	6;5	1	(+3)
28;35	6;08	2	(+1)	25;35	5;30	2	(+3)
25;53	5;34	3	(+3)	22;53	4;55	3	(+2)
23;10	4;59	4	(+5)	20;10	4;20	4	(+3)
20;28	4;24	5	(−1)	17;28	3;45	5	(+2)
17;45	3;49	6	(+2)	14;45	3;10	6	(+2)
15;3	3;14	7	(+1)	12;3	2;35	7	(+3)
12;20	2;39	8	(+2)	9;20	2;0	8	(+2)
9;38	2;4	9	(+1)	6;38	1;25	9	(+2)
6;55	1;29	10	(0)	3;55	0;50	10	(+2)
4;13	0;54	11	(0)	1;13	0;16	11	(+21)[a]
1;30	0;19	12	(+3)	0	0	11;23,30	(0)
0	0	12;33	(0)				

[a] It should be noted that according to the rule mentioned in Footnote 54, the value recomputed of $\Omega M = 0;16°$ lies nearer to the ΩM=0; 30° in al-Khwārizmī's table (see Table 4). Its difference with al-Battānī's digit is zero, whereas the difference for succeeding entry is (11;23,30^{d}–10; 48^{d}=0;35,30^{d}). Thus, based on the mentioned rule, the difference obtained equal to zero may not be meaningful. However in view of the fact that for most cases the value of difference between each couple nearside recomputed al-Battānī's digits is on scale of 0;1 or 0;2, therefore such interpolation scheme does not produce serious problem except for $d = 11$.

with this issue in this article.[57] But based on this research, there is no doubt that the notion of annular solar eclipse took roots from the early period of Islamic astronomy.[58] Obviously our knowledge about *al-Khwārizmī's Zīj* mostly originates from the recension of al-Majrīṭi and two commentaries written by Ibn al-Muthannā and Ibn Masrūr. However, it is difficult to separate materials based on al-Khwārizmī's own *Zīj* from those originating from the later additions. As for the table of solar eclipse, it can be said that it is a mixture of materials from al-Khwārizmī's original *Zīj* and later addition probably from *al-Ṣābi' Zīj*. It seems that the values for immersion (in Table 78) are derived from the earliest version of *al-Khwārizmī's Zīj* and not from later additions (see above). Based on the parameters of the angular diameters of the sun and moon, used in Table 78, it can be concluded that they originated from Indian astronomy.

This means that the concept of annular eclipse was probably known in early Islamic astronomy based on knowledge transfer rather than direct observation of such a phenomenon by al-Khwārizmī or his contemporaries. This hypothesis may be confirmed by considering the fact that al-Bīrūnī by citing al-Irānshahrī's report on annular eclipse, stated that the concept of annular eclipse could be concluded from Indian astronomy,

[57] I have dealt with the problem of al-Battānī's contribution to annular solar eclipse in a chapter of my doctoral dissertation. See Giahi Yazdi (2009, pp. 132–140).

[58] To support this hypothesis, we can also refer to the Debarnot's research, in which she introduced a trace of the notion of annular eclipse in the *Ḥabash's Zīj* (see Debarnot 1987, p. 50). For a detailed analysis, see also Giahi Yazdi (2009, pp. 96–98).

but that its authenticity would be proved by observation.[59] This implies that Islamic astronomers knew about the possibility of this phenomenon before al-Bīrūnī's era, probably through translations of the Indian astronomical texts into Arabic during the early stage of the translation movement in the Abbasid era of eighth and ninth centuries. However, it seems that in the absence of observational evidence, there is no clear explanation of such a phenomenon in the works prior to al-Bīrūnī.

The novelty in al-Khwārizmī's approach seems to be employment of the Indian values of angular diameters in a solar eclipse table. Al-Khwārizmī demonstrated effectiveness of his method clearly by computing non-Ptolemaic values for solar eclipse magnitudes, which it led to the possibility of annular eclipse. However, his theories were apparently not immune from manipulations throughout history.

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References

Al-Bīrūnī, Muḥammad ibn Aḥmad, 1954–1956. *al-Qānūn al-mas^{c}ūdī*, 3 Vols. Hyderabad: Osmania Oriental Publications Bureau.

Bowen, A.C. 2008. Simplicius' commentary on Aristotle, De caelo 2.10–12: An annotated translation. Part 2. *SCIAMVS* 9:25–131.

Brahmagupta, 1970. *The Khandakhādyaka of Brahmagupta*, 2 Vols. (edited and translated by Bina Chatterjee, with the commentary of Bhattotpala). New Delhi: Viva Books.

Carmody, F.J. 1956. *Arabic and astrological sciences in Latin Translation*. London: McGraw-Hill Book Company, Inc.

Castells, M., and J. Samsó. 2007. Seven Chapters of Ibn al-Ṣaffār's Lost *Zīj*. *Archives Internationales d'Histoire des Sciences* 45:229–261. Reprinted in Julio Samso, *Astronomy and Astrology in Al-Andalus and the Maghrib*. Aldershot: Ashgate Publishing.

Debarnot, M.T. 1987. The Zīj of Ḥabash al-Ḥāsib: A survey of ms. Stanbul Yeni Cami 784/2. In *From deferent to equant: A volume of studies in the history of science in the ancient and medieval near east in honor of E. S. Kennedy*, ed. D. A. King and G. Saliba. New York: New York Academy of Sciences.

Dictionary of scientific biography (DSB), 14 Vols and 2 Supp. Vols., ed. Charles Coulston Gillispie, New York, 1970–1980.

Eduardo Millas Vendrell, S.I. 1963. *El Comentario de Ibn al-Mutannā a las Tablas Astronomicas de al-Jwārizmī*. Madrid: CSIC.

Encyclopedia of the History of Arabic Science, 3 Vols., ed. R. Rashed, London/New York, 1996.

Giahi Yazdi, H.-R. 2008. Solar eclipses in medieval islamic civilization: A note on social and cultural aspects. *Tārīkh-e'Elm: Iranian Journal for the History of Science* 6:75–82.

Giahi Yazdi, H.-R. 2009. *La Théorie des Éclipses Solaires Chez des Savants de l'Est de l'Empire Musulman (IX^{e}-Début du XI^{e} Siècle): Contribution à l'Étude de la Phase Islamique de l'Astronomie*. Thèse de Doctorat, Université de Lille 1, France (non-publiée).

Goldstein, B.R. 1967. *Ibn Muthannā's commentary on the astronomical tables of al-Khwārizmī: Two Hebrew versions*, edited and translated, with an astronomical commentary. New Haven: Yale University Press.

Goldstein, B.R. 1985. Medieval observations of solar and lunar eclipses. *Archives Internationales d'Histoire des Sciences*, 29:101–156 (1979). Reprinted in *Theory and observation in ancient and medieval astronomy* (XVII). London: Variorum Reprints.

[59] See Footnote 14. It seems that al-Bīrūnī's statement refers to the theories of angular diameters of the sun and moon in Indian astronomy; thereby, the possibility of annular eclipse can be concluded. However, I have not found any astronomical text from Indian astronomy referring to annular solar eclipse observation or its possibility explicitly.

Heath, T.L. 1991. *Greek astronomy*. London: Dent. Originally published in 1932. Reprinted in New York.

Ibn Hibintā. 1987. *al-Mughnī fī aḥkām al-nujūm*, Facsimile edition, 2 Vols. Frankfurt: Institute for the History of Arabic-Islamic Science.

Ibn al-Nadīm *al-Fihrist*, 1350 H.S/1971. ed. R. Tajaddud. Tehran: Marwi.

Ibn al-Qifṭī. 1903. *Tārikh al-ḥukamā'*. J. Lippert, ed. Leipzig: Dieterich.

Ibn ᶜIbrī. 1403/1983. *Tārikh mukhtaṣar al-duwal*. Beirut: Dar al-Mashriq.

Ibn Ṣāᶜ id al-Andalusī, 1376 H.S/1997. *al-Taᶜrīf bi ṭabaqāt al-'umam*, ed. Gh.R. Jamshid Nezhad awwal. Tehran: Hijrat.

Kennedy, E.S. 1956. A survey of islamic astronomical tables *Transaction of the American Philosophical Society, N.S.* 46(pt 2):123–177.

Kennedy, E.S. 1983. Parallax theory in islamic astronomy. *Isis* 47:33–53. Originally published in 1956. Reprinted in *Studies in the Islamic Exact Sciences*, E.S., Kennedy ed., 164–184. Beirut: American University of Beirut.

King, D. 1986. *A Survey of the scientific manuscripts in the Egyptian National Library*. Winona Lake, IN; Cairo: American Research Center in Egypt.

Meeus, J. 2002. *More mathematical astronomy morsels*. Richmond: Willmann-Bell, Inc.

Meeus, J. 2004. *Mathematical astronomy morsels, III*. Richmond: Willmann-Bell, Inc.

Nallino, C.A. 1969. *Al-Battānī sive Al-Batenii opus astronomicum*, 3 Vols. Milan: Publicazioni del Reale Osservatorio di Brera in Milano, XL, 1899–1907. Reprinted in Frankfurt.

Neugebauer, O. 1962. *The astronomical tables of al-Khwārizmī* (translation with commentaries of the Latin version edited by H. Suter). Kommission hos Munksgaard: København

Neugebauer, O. 1975. *A history of ancient mathematical astronomy*, 3 Vols, Berlin: Springer-Verlag.

Pedersen, O. 1974. *A survey of the Almagest*. Odense University Press: Odense.

Pedersen, F.S. 2002. *The toledan tables: A review of the manuscripts and the textual versions*, 4 Pts. Copenhagen: Kongelige Danske Videnskabernes Selskab.

Ragep, F.J. 1993. *Naṣīr al-Dīn al-Ṭūsī's memoir on astronomy (al-Tadhkira fī ᶜilm al-hay'a)*, 2 Vols. New York: Springer-Verlag.

Rosenfeld, B.A., and E. Ihsanoglu, 2003. *Mathematicians, astronomers, and other scholars of islamic civilization and their works (7th–19th c.)* Istanbul: IRCICA.

Said, S.S., and F.R. Stephenson, 1996. Solar and lunar eclipse measurements by medieval muslim astronomers I: Background. *Journal for the History of Astronomy*, 27:259–273.

Said, S.S., and F.R. Stephenson, 1997. Solar and lunar eclipse measurements by medieval muslim astronomers II: Observations. *Journal for the History of Astronomy* 28:29–48.

Smart, W.M., 1999. *Textbook on spherical astronomy*, 6th edn. Revised by R.M. Green, 1977. Cambridge: Cambridge University Press (transferred to digital printing 1999).

Stephenson, F.R., 1997. *Historical eclipses and earth's rotation*. Cambridge University Press: Cambridge.

Suter, H. 1997. *Die Astronomischen Taflen des Muḥammed ibn Mūsā Al-Khwārizmī*. Kobenhaven, 1914. Repr. in *Islamic Mathematics and Astronomy*, ed. Fuat Sezgin, vol. 7, Frankfurt.

Swerdlow, N.M. 1969. Hipparchus on the distance of the sun. *Centaurus* 14:287–305

Toomer, G.J. 1968. *A survey of the Toledan tables*. *Osiris* 15:5–174.

Toomer, G.J. 1998. *Ptolemy's Almagest*. Princeton: Princeton University Press.

Vallicrosa, M. and J. María, 1947. *El Libro de los Fundamentos de las Tables Astronómicas de R. Abraham ibn ᶜEzra*. Madrid: Fundacion Pastor de Estudios Clasicos.

Van Dalen., B. 1996. Al-Khwārizmī's astronomical tables revisited: Analysis of the equation of time. *From Baghdad to Barcelona: Studies in the Islamic exact sciences in honor of Prof. Juan Vernet*, ed. J. Casulleras and J. Samsó, Vol. 1, 195–252. Barcelona: Universitat de Barcelona.

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ERRATUM

Erratum to: Al-Khwārizmī and annular solar eclipse

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Unfortunately only after online first article publication some imperfections were noticed, the article text should had read the following:

At Footnote 22: al-Irānshahrī
At Footnote 27: Ibn ᶜIbrī

Under Heading **2 Concluding remarks**: For a long period of time, historians of astronomy have simply introduced al-Battānī (d. 317 A.H./929 A.D.) as the pioneer in Islamic astronomy (or even in the history of astronomy) who knew the possibility of annular solar eclipse.[56]

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