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Complex continued fractions: early work of the brothers Adolf and Julius Hurwitz

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Abstract The two brothers Julius and Adolf Hurwitz were born in the middle of the nineteenth century in a small town near Hanover (not far from Göttingen). Already during their schooldays, the two of them became acquainted with mathematical problems and both started to study mathematics, but while the younger brother Adolf turned out to be extremely successful in his research, the elder brother and his work seem to be almost forgotten. This paper examines the lives and works of the two brothers with particular emphasis on the contributions of Julius Hurwitz, and the subsequent reception of their research. It deals with the development of an arithmetical theory for complex continued fractions by Julius and Adolf Hurwitz around 1890 and its rediscovery in the twentieth century.

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1 The road to Königsberg

Once upon a time, two brothers were born into a Jewish family in Hildesheim near Hanover: Julius on July 14, 1857, and Adolf Hurwitz on March 26, 1859.[1] There had been two more children in the Hurwitz family, namely a sister Jenny who died already in 1855 at age one, and an elder brother Max [Mosche], born August 22, 1855, who died on July 17, 1910 in Zurich (where his younger brother Adolf lived at that time). The paths of life of Julius and Adolf Hurwitz were to be both closely linked and still greatly different.

In the beginning, the education of the two brothers ran in parallel. According to their school certificates[2] both were good pupils with quite good marks, and in particular, both Julius and Adolf Hurwitz must have made an excellent impression in mathematics. It seems that Julius may have been a little more open to distractions; there is an interesting remark in one of the school documents concerning Julius visiting a tavern outside lessons. And yet the two of them spent Sunday afternoons having geometry lessons at the home of their teacher Hermann Cäsar Hannibal Schubert, the inventor of the so-called Schubert calculus in enumerative geometry. This proved to be a fruitful investment: Adolf Hurwitz's first paper (Hurwitz and Schubert 1876) is a joint work with Schubert on Chasles' theorem.[3] The teacher's driving force and his mathematical influence on the brothers were indeed fortunate since Schubert left Hildesheim for Hamburg in 1876 after only five years.[4] Although his results with Adolf Hurwitz were more obvious, Schubert decided that both pupils could have aspired to an academic career.

Unfortunately, the father of Julius and Adolf Hurwitz was sceptical about those plans, and moreover, he was not very well-off. He, Salomon Eduard Hurwitz, was a merchant, and a widower after the boys' mother, Elise Wertheimer, died[5] in 1862 when Julius was five and Adolf three years old.[6] Although the partnership of Salomon and Elise might not have been the best, the relationship of the three Hurwitz boys with their father must have been very close. Adolf Hurwitz's later wife Ida described the father as follows: "Moreover he set a high value that the young boys started smoking since he could not imagine a proper man without cigar or better a pipe."[7] When Schubert came

[1] The common Jewish surname Hurwitz (as well as Horowitz and Hurewicz) is a reference to the historically portentous small town Hořovice in Central Bohemian Region of the Czech Republic.

[2] Those certificates of the Realgymnasium Andreanum in Hildesheim can be found in the municipal archive of Hildesheim (cf. Rasche 2011, Appendix).

[3] Chasles' theorem allows one to count the number of curves satisfying certain algebraic conditions within a family of conics; it generalizes Bézout's theorem and plays a significant role in algebraic geometry. Schubert developed a calculus in order to solve counting problems of such type (1879), and Hilbert's problem number 15 asks for a rigorous justification of Schubert's enumerative calculus.

[4] See Burau (1966), Burau and Renschuch (1993) for more details on Schubert's life.

[5] Of liver malfunction according to Adolf Hurwitz's wife Hurwitz-Samuel (1984, 3), resp. kidney malfunction according to Frei (1995, 527).

[6] There is a photograph of the father with his three boys in Rowe (2007, 20).

[7] This is the authors' translation from the German original: "Ebenso legte er auch Wert darauf, dass die Knaben schon sehr frühzeitig zu rauchen begannen, da ihm ein rechter Mann ohne Cigarre oder besser noch Pfeife kaum denkbar war" (Hurwitz-Samuel 1984, 2). If not indicated differently, all translations from German to English have been made by the authors.

to Salomon Hurwitz "to convince him to let both sons choose to study mathematics,"[8] Salomon consulted a prosperous childless friend E. Edwards who offered to finance the studies of one of the sons. After questioning Schubert, finally Adolf was selected and Julius had to follow the path of his elder brother Max. Julius Hurwitz did an apprenticeship in Nordhausen, a small town in today's Thuringia, and became a bank clerk. In this profession, he worked for many years, probably first in Hamburg[9] and later in Hanover, where he and his brother Max took over the banking business of their deceased uncle Adolph Wertheimer.

At the age of 18, Adolf Hurwitz started to study at the polytechnic university in Munich[10] and attended Felix Klein's lectures. Later, he moved to Berlin,[11] where Kronecker and Weierstrass were his teachers, and in 1880, Adolf Hurwitz followed Klein to Leipzig to continue his studies.[12] In 1881, Hurwitz obtained a doctoral degree for his work on modular functions (Hurwitz 1881).[13] In a retrospective, his supervisor, Felix Klein, wrote that from his school Hurwitz and Dyck made the most important contributions to Riemann's theory of functions.[14] In fact, Klein was at the peak of his scientific career when Adolf Hurwitz did his doctorate and Klein's research benefited considerably from his pupils' investigations.

Adolf Hurwitz faced certain difficulties at the University of Leipzig with respect to his school education at the *Realgymnasium Andreanum* at Hildesheim. This type of school had been introduced in Prussia in the middle of the nineteenth century to provide an advanced education for more than the privileged youth, and it placed an emphasis on mathematics, natural sciences, and modern languages. However, some universities did not value these new institutions highly and students with such an educational background, lacking sufficient knowledge of Greek and Latin, could not obtain higher degrees (cf. Hilbert 1921, XIV; Rowe 2007, 23).

In 1882, Adolf Hurwitz returned to the University of Berlin, where Weierstrass especially was interested in his function-theoretical work and proposed him his first post-doctoral subject (cf. Hurwitz-Samuel 1984, 6). Supported by Kronecker and Klein, Hurwitz moved in the following year to the more liberal Göttingen University where he finished his Habilitation around Easter and became a *Privatdozent*.[15]

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[8] "Schubert suchte sogar den Vater auf, um ihn zu bestimmen, beide Söhne das Studium der Mathematik ergreifen zu lassen" (Hurwitz-Samuel 1984, 4).

[9] There are postcards from June 17, 1881, and September 4, 1881, from Salomon to Julius in Hamburg, courtesy of the archive of the ETH Zürich (ISIL-Code: CH-000003-X Fonds-Hurwitz-A).

[10] At that time, this institution was called *Königlich Bayerische Technische Hochschule München*; since 1970, it is the *Technische Universität München*.

[11] Since 1949 the University of Berlin has been called the *Humboldt-Universität zu Berlin*.

[12] An excerpt of a letter from Klein to Adolf Hurwitz's father explaining the situation and the prospects for his son can be found in Rowe (2007, 22).

[13] According to his later wife Hurwitz-Samuel (1984, 6), he had to borrow a tailcoat from a fellow student for his doctoral viva.

[14] "Seitdem ist das Interesse für Riemanns Funktionentheorie in immer weiteren Kreisen, auch des Auslandes, erwacht. Von meinen Schülern ist wohl besonders Hurwitz in Zürich und Dyck in München zu nennen" (Klein 1926, 276).

[15] In the German system Habilitation granted the "venia legendi," i.e., the permission to lecture as a *Privatdozent* which at that time meant to collect course fees from the students without any payment from the

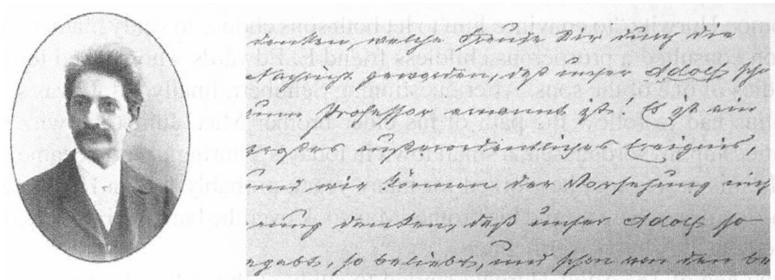

Fig. 1 *Left* portrait of Adolf Hurwitz. *Right* excerpt from Salomon Hurwitz's letter to Julius considering the new professorship of his brother

Hurwitz's friends from this highly prolific period include the famous physicist Wilhelm von Weber and the mathematician Moritz Abraham Stern, and while he had a good relationship with the first of these,[16] the second one played a more important role in Adolf's life. Born in Frankfurt in 1807, Stern, although a protégé of Gauss, faced racist obstacles in his scientific career. He only became an ordinary professor at Göttingen University in 1859, being the first unbaptized Jew to be appointed to an ordinary professorship in the rather anti-Semitic Prussia of the nineteenth century.[17]

The year 1884 brought many changes. Stern retired and started to live in Bern, where his son was a professor of history, and Stern's chair at Göttingen was filled by Klein. In the same year, Adolf Hurwitz moved to the University of Königsberg[18] at the invitation of Lindemann, another pupil of Klein, who just had proved the transcendence of π and thereby the impossibility of squaring the circle. Lindemann became aware of Hurwitz because of his *New representations of the proof of Weierstrass for the transcendence of e and π*.[19] Hurwitz obtained an extraordinary professorship (in German: *Extraordinariat*) comparable to an associate professorship in modern terms. This appointment was an overwhelming event for the whole family as a letter from his father to his brother Julius indicates:

> It is an extraordinary event, and we cannot thank destiny enough that our Adolf is so gifted, so acclaimed and is already recognized by the most important mathematicians as an excellent person (Fig. 1).[20]

Footnote 15 continued
university. For more details on exploiting scientists and the academic ladder in nineteenth century Germany see Rowe (1986).

[16] "An der lebhaften Göttinger Geselligkeit nahm H. auch sonst eifrig teil, so schwang er das Tanzbein bei dem grossen Physiker Wilh. v. Weber,..." (Hurwitz-Samuel 1984, 7).

[17] Stern was the granduncle of Anne Frank, another target of racism during the Nazi-times; see Rowe (1986, 2007) for further details about Stern's life.

[18] Nowadays Kaliningrad in Russia.

[19] *Neue Darstellung des Weierstrass'schen Beweises für die Transzendenz von e und π*, enclosed in Adolf Hurwitz's mathematical diaries: Mathematisches Tagebuch 3, 9 January 1883—ca. 1884; Hs 582:3 (Hurwitz 1972, 118–125).

[20] "Es ist ein ordentliches Ereignis, und wir können der Vorsehung nicht genug danken, daß unser Adolf so begabt, so beliebt und schon von den bedeutendsten Mathematikern als ein hervorragender Mensch

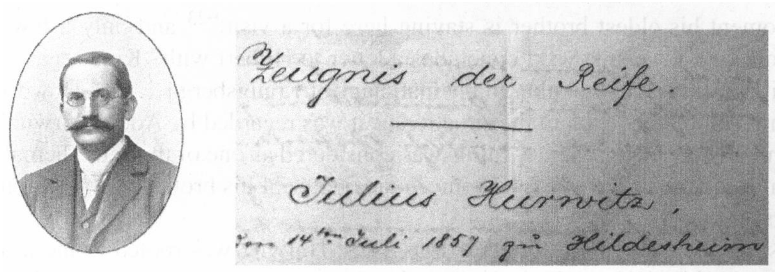

Fig. 2 *Left* portrait of Julius Hurwitz. *Right* excerpt from his *Reifezeugnis*

As the familial bonds of the Hurwitz family were always very close, one may assume that Julius Hurwitz shared his father's delight. Despite the disappointing working life in the banking business of his uncle, he never gave up his own academic plans and must have studied all the time. In retrospect, his sister-in-law, Ida Hurwitz, wrote in her biographical sketch (Hurwitz-Samuel 1984, 8):

> Since his uncle Adolph's death Max and Julius owned the bank 'Adolph M. Wertheimer's Nachf.' in Hanover. However, they felt uncomfortable in this business. Hence, first Julius left in order to return to school at the age of 33 years and finish the school examination for studying Mathematics under supervision of his younger brother.[21]

Subsequently Julius managed to get a so-called *Reifezeugnis* in Quakenbrück (not far from Bremen), which includes exams in subjects like history, geography, chemistry, and mathematics, as was necessary for studying at a university (Fig. 2).[22] Then, in 1890, at the age of 33, he finally followed his brother to Königsberg.

2 From Königsberg to Switzerland

Although the working conditions at Königsberg must have been difficult, the students were excellent, among them the young Hermann Minkowski and David Hilbert. Adolf Hurwitz became not only their guide to mathematics but also a lifelong friend of both. At that time "[Adolf] Hurwitz was at the height of his powers and he opened up whole new mathematical vistas to Hilbert who looked up to him with admiration mixed with a tinge of envy" (Rowe 2007, 25). There is a letter from Hilbert to Klein with date March 4, 1891, ending with the sentence: "[Adolf] Hurwitz send you his regards. At

Footnote 20 continued
anerkannt wird." Letter from Salomon to Julius, April 1, 1884, courtesy of the ETH library (ISIL-Code: CH-000003-X Fonds-Hurwitz-A).

21 "Seit dem Tode Onkel Adolphs waren Max u. Julius Inhaber des Bankgeschäfts 'Adolph M. Wertheimer's Nachf.' in Hannover, fühlten sich aber im Kaufmannstande nicht glücklich. So trat zuerst Julius aus, setzte sich mit 33 Jahren nochmals auf die Schulbank, um das Abiturientenexamen nach zumachen und dann bei seinem jüngeren Bruder Mathematik zu studieren" (Hurwitz-Samuel 1984, 8).

22 The certificate including several grades can still be found in the archive of Halle University (Rep. 21 Nr. 162).

the moment his eldest brother is staying here for a visit."[23] and only a few weeks later, on June 11, Minkowski concludes a letter to Hilbert with "Kind regards to the Hurwitz brothers and the other mathematicians of Königsberg [...]".[24] However, this does not completely explain the situation as it was regarded by Adolf Hurwitz's wife Ida, and it is not clear whether Julius was considered as one of those mathematicians. As a matter of fact, Julius edited some course notes of his brother Adolf's lectures at Königsberg University.[25]

Besides his mathematical dedication, Adolf Hurwitz was rooted in the academic social and cultural life of Königsberg, and in this way got to know professors' families. After years of restraint, because of the fluctuating state of his health, in summer 1892, Adolf Hurwitz married Ida Samuel,[26] a daughter of Simon Samuel, the professor of pathology, at the University of Königsberg.

More or less at the same time, Adolf Hurwitz was discussed with respect to two positions, Georg Frobenius' chair at the *Eidgenössische Polytechnische Hochschule Zürich* (in 1911 renamed in *Eidgenössische Technische Hochschule Zürich* and therefore in the sequel ETH for short), a polytechnic, and Hermann Amandus Schwarz's chair at the University of Göttingen. Adolf Hurwitz accepted a call from Zurich and remained there for the rest of his life. One reason for choosing Zurich over the more prestigious Göttingen might have been that Klein's attempts to get Hurwitz for Göttingen turned out to be time-consuming and, as time went by, less promising; we refer to Rowe (2007) for the details. However, there might have been further reasons for Adolf Hurwitz's choice. In nineteenth century, Prussian prejudices against Jews and racism were common. When the founding of the Reich was followed by a financial crash and an economic depression, the atmosphere became unfriendly. Moreover, Hurwitz's paternal friend Moritz Stern had meanwhile moved from Bern to Zurich and been made an honorary member of the local Society of Natural Scientists.[27] Adolf Hurwitz's successor at Königsberg was Hilbert who moved to Göttingen in 1895 on the promotion of Klein.[28] Certainly, Adolf did not want to leave Zurich for Göttingen.

When his brother followed the call of the ETH Zurich, Julius accompanied him; his sister-in-law Ida wrote that:

23 "[Adolf] Hurwitz lässt Sie bestens grüssen. Augenblicklich ist auch sein ältester Bruder hier bei ihm zum Besuch.", letter 63 in Frei (1985, 73); actually, Max was the eldest, and Julius the second.

24 "Grüße bestens die Gebrüder Hurwitz und die anderen Königsberger Mathematiker [...]", in Rüdenberg and Zassenhaus (1973, 45).

25 Those notes can still be found in the archives of the ETH Zürich (Hs 582: 52–53; 64–65; 96).

26 1864–1951; author of a brief and very readable account (Hurwitz-Samuel 1984) on the Hurwitz family and her husband in particular.

27 Stern died in 1894. Adolf Hurwitz and his colleague Ferdinand Rudio, a colleague from ETH and former friend from his study times at Berlin, published the collected letters from Eisenstein to Stern (Hurwitz and Rudio 1894); this had been a wish of Stern and it affirms his close relation with Adolf Hurwitz.

28 An excellent reading on this "Game of Mathematical Chairs" and the difficulties for Jewish mathematicians at that time is Rowe's (2007) article as well as the correspondence (Frei 1985) between Hilbert and Klein.

> [Also] his brother Julius soon followed him to Zurich, where he wrote his doctoral thesis for which he had received the subject from his brother.[29]

According to the curriculum vitae attached to his doctoral thesis Hurwitz (1895), Julius Hurwitz noted that he attended lectures by Stern in Zurich, which might be surprising since Stern had retired from Göttingen in 1884.[30] Even though his brother supported him, Julius Hurwitz returned to Germany for his final doctoral viva, more precisely, to the University of Halle-Wittenberg. Since the ETH was permitted to supervise dissertations only in 1909, Julius could not write his thesis under his brother's supervision.[31] One reason for his choice of Halle could have been that Schubert had received his doctorate from the University of Halle in 1870.[32] Another reason could have been his brother's acquaintance with Albert Wangerin that had begun during his studies in Berlin around 1878.

Indeed, Julius' thesis (Hurwitz 1895) at the University of Halle was officially supervised by Wangerin. Although his advisor is not widely known today, in the second half of the nineteenth century, he was a rather influential mathematician. Born 1844 in Greifenberg, after his studies at Halle under Heine, Wangerin worked as a teacher before he became a professor at the University of Berlin in 1876; there he was responsible for teaching beginners, duties which more prominent mathematicians such as Kronecker, Weierstrass, and Kummer tried to avoid. In 1882, Wangerin became professor at the University of Halle, where he remained until his retirement in 1919 and where he died in 1933. During his life, Wangerin advised the remarkable number of 53 students for their dissertation (Julius Hurwitz is number 28). The supervised topics range from calculus, in particular, differential equations, via analytic and differential geometry to mathematical physics; there are only two theses from number theory.[33]

We do not know any details about Julius Hurwitz's time in Halle. The University of Halle-Wittenberg, however, is where Georg Cantor had been appointed in 1869 on the promotion of Hermann Amandus Schwarz and where he obtained an extraordinary professorship in 1872 and became an ordinary professor in 1879 on Heine's recommendation. Cantor was prominently involved in the foundation of the *Deutsche Mathematiker Vereinigung* (DMV), and he, as well as Klein, was also involved in organizing the first International Congress for Mathematicians in Zurich in 1897, which suggests some kind of approval for his famous, or rather, controversial foundations of set theory: "[Adolf] Hurwitz openly expressed his great admiration of Cantor and proclaimed him as one by whom the theory of functions has been enriched. Jacques

29 "[Auch] sein Bruder Julius folgte ihm bald nach Zürich, wo er an der Doktorarbeit schrieb, deren Thema er von seinem Bruder erhalten hatte" (Hurwitz-Samuel 1984, 9).

30 "Viros audivi illustrissimos: [...] Turici: Franel, Geiser, Hurwitz, Rudio, A. Stadler, Stern, von Wyss, Fr. Weber" (Hurwitz 1895, 50).

31 Usually, dissertation projects were officially ratified at the University of Zurich until the ETH obtained full rights to supervise doctoral students independently in 1909; anyway, the relation between Julius and Adolf might have been too close.

32 Schubert wrote his dissertation on enumerative geometry during his studies in Berlin, however, after the death of his teacher Gustav Magnus, he decided to finish his doctorate at the University of Halle without official supervisor; see Burau and Renschuch (1993) for further details.

33 See www.mathematikuni-halle.de/history for a list.

Hadamard expressed his opinion that the notions of the theory of sets were known and indispensable instruments" [as Johnson (1972, 17) wrote]. Adolf Hurwitz first met Cantor in summer 1888 when both spent some time in a group of mathematicians around Weierstrass in the Harz Mountains,[34] and the relation between Cantor and the Hurwitz brothers seems to have been quite close. In a letter to Lemoine on July 7, 1894, Cantor wrote the post-scriptum "Many thanks for the kind acceptance of the note of my pupil Hurwitz."[35] Very likely, Cantor's pupil was Julius rather than his younger and by then well-established brother Adolf Hurwitz.

As a professor in the mathematical department at Halle, Cantor had to review Julius' dissertation and his signature can be found on the relevant document. On July 13, 1895, Julius Hurwitz officially finished his thesis. Besides Cantor's signature, it bears the dedication "My dear brother and distinguished teacher Herr Prof. Dr. A. Hurwitz."[36] Moreover, he expressed his gratitude to his younger brother for "the many pieces of advice with which he had supported this work."[37] Remarkably, the topic of Julius' dissertation thesis is not only close to his brother's treatment of complex continued fractions, according to the official description of his advisor Wangerin, it is even based on a published article of Adolf Hurwitz. He wrote:

> Herr J. Hurwitz examines, following a published work in *Acta Mathematica*, volume XI, by his brother, Prof. A. Hurwitz in Zürich, a certain kind of continued fraction expansion of complex numbers.[38]

In his *Jahrbuch über die Forstschritte der Mathematik* review, Hurwitz (1894a) himself wrote about his brother Julius' dissertation:

> The author examines in the present paper a certain kind of a continued fraction expansion of complex numbers from a similar point of view as the referee took as a basis for handling certain other continued fraction expansions of real and complex numbers.[39]

With a twinkling eye, this summary may be seen in the light of a rough quotation by Adolf Hurwitz saying that "A PhD dissertation is a paper by the professor written under

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[34] "Im Sommer 1888 verbrachte er [Adolf Hurwitz] auf Anregung von Prof. Mittag-Leffler in Stockholm, einige Tage in Wernigerode i/H. in interessantem mathematischen Kreis der sich um den Altmeister Prof. Weierstrass aus Berlin geschart hatte. Dort lernte er Georg Cantor und Sonja Kowalewski näher kennen" (Hurwitz-Samuel 1984, 8).

[35] "Für die freundliche Aufnahme der Notiz meines Schülers Hurwitz besten Dank.", cf. Décaillot (2011, 152).

[36] "Meinem lieben Bruder und verehrten Lehrer Herrn Prof. Dr. A. Hurwitz".

[37] "Es sei mir gestattet, meinem Lehrer, Herrn Professor A. Hurwitz, für die mannigfachen Ratschläge, mit welchen er mich bei dieser Arbeit unterstützt hat, auch an dieser Stelle meinen herzlichsten Dank auszusprechen." Rep. 21 Nr. 162, Universitätsarchiv Halle-Wittenberg, Halle.

[38] "Herr J. Hurwitz untersucht im Anschluß an eine in den Acta mathematica, Band XI, veröffentlichte Arbeit seines Bruders, des Prof. A. Hurwitz in Zürich, eine besondere Art der Kettenbruchentwickelung complexer Größen." Announcement of the disputation by A. Wangerin, Rep. 21 Nr. 162, Universitätsarchiv Halle-Wittenberg, Halle.

[39] "Der Verfasser untersucht in der vorliegenden Arbeit eine besondere Art der Kettenbruchentwickelung complexer Grössen nach ähnlichen Gesichtspunkten, wie sie der Referent der Behandlung gewisser anderer Kettenbruchentwickelungen reeller und complexer Grössen [...] zu Grunde gelegt hat."

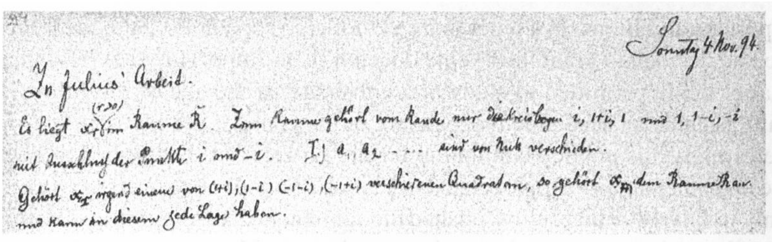

Fig. 3 Excerpt from the diary considering Julius' work an complex continued fractions (Hurwitz 1972, 94)

aggravating circumstances" (cf. Krantz 2005, 24). His mathematical diaries (Hurwitz 1972) contain several pages with explicit computations of a preliminary version of Julius' thesis with date 4 November 1894 (Fig. 3).[40]

It seems that at this early date, Adolf Hurwitz provided a few suggestions to his brother about what could be considered in the final version of his thesis. In the end, Julius' thesis must have made a good impression. In the further course of description, Wangerin emphasized that Julius Hurwitz worked diligently, efficiently, and in particular independently.[41] Besides Wangerin, the examination committee consisted of the dean Prof. Haym, Prof. Dorn (physics), and Prof. Vaihinger (philosophy). The minutes of the examination[42] record that Julius was questioned not only about mathematics, but also about Carnot's theory of heat, the philosophy of Leibniz, and other subjects. Finally, all examiners agreed on the grade *magna cum laude*.

Shortly after his disputation, Julius Hurwitz returned to Switzerland. Already in 1896, he became a member of the 'Naturforschende Gesellschaft' in Basel and in the same year he was lecturing as *Privatdocent*[43] at the University of Basel, the first university in Switzerland, founded in 1460. One of the contacts the Hurwitz brothers could have had in Basel was Karl von der Mühll who worked in Leipzig until 1889 before he was appointed a professorship in Basel. It is astonishing how quickly Julius submitted the necessary Habilitation thesis "Über die Reduction der binären quadratischen Formen mit complexen Coeffizienten und Variablen"; it must have been written in the same year 1896 although it was still far from being published. On June 2, 1896 the examination board of the department in Basel found that "the petitioner became zealously and skillfully acquainted with the field of mathematics in a short time."[44] In the following colloquium on June 17, Julius was not only questioned about number and function theory, but also had to deliver a short talk. Despite criticism of "the lack of clarity and clear mastery of presentation," the examination board decided

[40] Mathematisches Tagebuch 9, 4 April 1894–15 January 1895; Hs 582:9 (Hurwitz 1972, 99–107).

[41] "Die Durchführung der Untersuchung zeugt nicht nur von Fleiß und tüchtiger mathematischer Schulung, sondern auch von selbständigem Nachdenken." Gutachten der Dissertation von A. Wangerin, Rep. 21 Nr. 162, Universitätsarchiv Halle-Wittenberg, Halle.

[42] Gutachten der Dissertation, Rep. 21 Nr. 162, Universitätsarchiv Halle-Wittenberg, Halle.

[43] Personal file, Dozentenkartei, Universitätsarchiv F 6.2.1, Staatsarchiv Basel.

[44] "Der Petent hat sich in kurzer Zeit mit Eifer und Geschick in das Gebiet der Mathematik eingearbeitet," protocols of the sessions of the philosophical faculty, StABS, Universitätsarchiv R 3.5 Basel

to support his ambitions. In a letter to the senior civil servant Dr. Zutt, the Regent says that he had decided to grant the "Venia docendi"[45] to Julius Hurwitz.[46] So finally, six years after being permitted to study at a university, at the age of 39, Julius Hurwitz officially became an academic independent of his younger brother.

Interestingly, his public Habilitation lecture on October 27, 1896, was about "the infinite in mathematics," a topic related to Cantor's famous controversial work.[47] But it was only in July 1899 that Julius finished the summary of dissertation and Habilitation, and it took another two years before it was printed as Hurwitz (1902). In any case, it is remarkable that both Julius and Adolf Hurwitz realized their habilitation in a very short time.

Emigration turned out to be a good choice for Adolf Hurwitz too. Firstly, in Switzerland, there were fewer resentments against Jews than in Prussia. Moreover, in Zurich, there was a rich academic life with a polytechnic school, the ETH, and a university as well as a mathematical society. While in the beginning, the ETH had been a stepping stone for young researchers to obtain better positions at respectable German universities (e.g., Dedekind, H. A. Schwarz, Frobenius), the situation improved quickly thanks to the tight collaboration with the established University of Zurich. Of course, there was also a certain competition between the two institutions.[48] As with the situation in Königsberg, there were excellent students in Zurich. For instance, around 1900, Albert Einstein applied for an assistanceship with Adolf Hurwitz; however, as Einstein reported, Hurwitz must have been puzzled that a student who was never ever seen in the mathematical seminar asked for such a position; according to Einstein, for a physicist, it suffices to know and apply the elementary mathematical notions.[49] About a decade later, Albert played chamber music together with the Hurwitz family.[50] The physicist Max Born was another student of Adolf Hurwitz around 1902/03; he wrote about his teacher Hurwitz: "Once when I missed a point in a lecture I went to Hurwitz afterwards and asked for a private explanation. He invited me (...) to his house and gave us a series of private lectures on some chapters of the theory of functions of complex variables, in particular on Mittag-Leffler's theorem, which I still consider as one of the most impressive experiences of my student life. I carefully worked out the whole course, including these private appendices, and my notebook was used by Courant when he, many years later and after Hurwitz' death, published

[45] The Swiss version of the "venia legendi."

[46] "Hochgeachteter Herr, auf Antrag der philosophischen Fakultät hat E. E. Regenz in ihrer gestrigen Sitzung beschlossen, Herrn Dr. phil Julius Hurwitz die venia docendi für Mathematik zu verleihen. Herr Hurwitz hat sich durch eine Habilitationsschrift und durch ein Colloquium in der Weise ausgewiesen, daß Fakultät und Regenz überzeugt sind, daÃ in ihm unserer Universität eine tüchtige Kraft gewonnen wird." Extract from the application of the 'Venia docendi', Erziehungsakten CC 28b, Staatsarchiv Basel.

[47] Protocols of the sessions of the philosophical faculty, StABS, Universitätsarchiv R 3.5 Basel.

[48] For example, over the appointment of Minkowski; see letter 107 from Hilbert to Klein in Frei (1985, 122).

[49] "der Herr Professor darüber ein wenig verwundert gewesen sei, war doch dieser Student niemals in den mathematischen Seminaren zu sehen gewesen, da er sich mangels an Zeit nicht beteiligen konnte. (...) dass es für einen Physiker genüge, die elementaren mathematischen Begriffe zu kennen und anzuwenden, der Rest für ihn aus 'unfruchtbaren Subtilitäten' bestehe" (Fölsing 1993, 634).

[50] A nice photograph illustrating such a performance can be found in Pólya (1987, 24).

his well-known book ..." [cf. Rowe 2007, 29; the book in question is Courant and Hurwitz (1992)].

These various activities culminated in the first International Congress for Mathematicians held at Zurich in 1897. Adolf Hurwitz was not only involved in its organization, but, together with Klein, Peano, and Poincaré, he was one of the distinguished invited speakers giving a talk on recent developments in complex analysis and the impact of Cantor's set theory.[51] Julius Hurwitz also attended the congress at Zurich, but he did not attend the next one in Paris nor any other International Congress (neither did Adolf).

In 1901, Julius Hurwitz had to give up his position at the University of Basel due to illness. The protocols of the sessions of the philosophical faculty do not report what kind of illness, and it is impossible to decide whether this might be the reason why no other mathematical papers of his can be found. Four years later Julius moved to Lucerne, where he stayed until 1916 when he accompanied his companion[52] Franz Sieckmeyer to Germany. The brief note 'Deutschland, Krieg' in the file in the municipal archives of Sieckmeyer suggests that this return was not of his own free will. Referring to their last meeting, Ida Hurwitz wrote

> [...] Julius was visiting us after a long break (he had followed his companion Franz Sieckmeyer to Freiburg i/Br., where he served in a military hospital). Also Julius was suffering for many years (heart disease and arteriosclerosis), but at this last meeting, which was rather unexpected for both brothers, Adolf made a far more sickly impression.[53]

Some years before, in 1905, one of Adolf Hurwitz's kidneys had been removed; later, his second kidney ceased to work properly. Thereafter, his life became more calm and isolated than before.[54] In May 1919, Adolf Hurwitz finished his last major project, his monograph (Hurwitz 1919) on the number theory of quaternions based on his lectures at the *Königl. Gesellschaft der Wissenschaften zu Göttingen* in 1896.

On June 2/3, 1919, Julius returned to Lucerne, where he checked into the Hotel des Alpes located directly on the famous Lake Lucerne. Somedays later, he wanted to leave for Lugano[55] but although his notice of departure can be found in the city's records, it never came to pass.

[51] Adolf Hurwitz had previously supported the forerunner of the ICM at Chicago's World's Columbian Exposition in 1893 by submitting a contribution *in absentia*; see Lehto (1998, 5), resp. www.mathunion.org/ICM/.

[52] In German: Gesellschafter.

[53] "[...] war Julius nach langer Pause wieder einmal bei uns zu Besuch (er war seinem Gesellschafter Franz Sieckmeyer nach Freiburg i/Br. gefolgt, wo dieser Lazarettdienst leistete). Auch Julius war seit Jahren sehr leidend (Herzleiden und Arterienverkalkung), auch machte bei diesem letzten Zusammensein, auf welches beide Brüder wohl kaum mehr gerechnet hatten, Adolf den weitaus kränklicheren Eindruck." (Hurwitz-Samuel 1984, 13)

[54] Pólya (1987, 25) wrote "His health was not too good so when we walked it had to be on level ground, not always easy in the hilly part of Zürich, and if we went uphill, we walked very slowly." Pólya at that time was around 30 years old whereas Adolf in the mid-fifties.

[55] Alphabetische Ausländerkontrolle, F8/7:10, Stadtarchiv Luzern.

> On 15 June in Lucerne, during one of the frequent heart attacks he endured, Julius suddenly closed his eyes forever at the age of 61. [Adolf] H. accepted this news, which was delivered him with the greatest caution, with complete sincerity, praising the destiny of his beloved brother, who had overcome everything now, and longing for the same for himself.[56]

The health of Adolf Hurwitz deteriorated more slowly. As late as October 28, 1919, he ran a seminar at home where his family was "listening at the door admiring the control and clarity with which he knew how to talk."[57] Somedays later, on November 18, 1919, he died in Zurich of kidney failure at 60 years of age. Life expectancy was just around 54 years at that time. All the three brothers, Max, Julius, and Adolf Hurwitz, were buried next to each other in an family grave at the Sihlfeld cemetery in the center of Zurich.[58]

Adolf Hurwitz's collected papers (Hurwitz 1932) were edited by Pólya and appeared in 1932.[59] The ETH holds the complete rack of 31 mathematical diaries (Hurwitz 1972) of Adolf Hurwitz ranging from 1882 until 1919. During his scientific life, he supervised altogether at least 23 doctoral students,[60] There is much more to say about Adolf Hurwitz's life and research.[61] For instance, the detailed obituary W.H.Y. (1922) published by the London Mathematical Society makes a good reading; Adolf Hurwitz had been an honorary member of this society since 1913.[62] Curiously, the author of the obituary (W.H.Y. 1922) signed just with his intitials W.H.Y., so we may only guess that it was William Henry Young, the president of the London Mathematical Society from 1922 to 1924. When dealing with Adolf Hurwitz's work on continued fractions, W.H.Y. wrote

> His papers on continued fractions and on the approximate representation of irrational numbers are also very original, as well as curious. (W.H.Y. 1922, Li)

[56] "Am 15. Juni schloss Julius in Luzern bei einem der häufigen Herzanfälle, die er erlitt, ganz plötzlich seine Augen für immer. [Adolf] H. nahm die Nachricht, die ihm mit größter Vorsicht beigebracht wurde, voll Ergebung auf, das Geschick des Bruders preisend, der nun alles überstanden habe, und für sich selber das Gleiche ersehnend." (Hurwitz-Samuel 1984, 13)

[57] "[wir,] an der Tür Lauschenden bewunderten die Beherrschtheit und Klarheit, mit der er vorzutragen vermochte." (Hurwitz-Samuel 1984, 14)

[58] The family grave in field D had number 81201; the grave was removed in 2000.

[59] In Pólya (1987, 25), George Pólya wrote: "My connection with Hurwitz was deeper and my debt to him greater than to any other colleague." It was indeed on Adolf Hurwitz's invitation that Pólya was offered an appointment as *Privatdozent* at Zurich.

[60] According to the Mathematics Genealogy Project http://genealogy.math, all within the period 1896–1919; however, in his collected works (Hurwitz 1932, 754, vol. II) there are just 21 listed. Among his pupils one can find the later professors Gustave du Pasquier at the Université de Neuchâtel, Eugène Chatelain, Alfred Kienast, Émile Marchand, and Ernst Meissner, all at ETH Zurich, as well as Kerim Erim, who obtained his doctorate at the University of Erlangen-Nuremberg and later became a professor at the University of Istanbul.

[61] Very good accounts on Adolf Hurwitz's life and work are given by his wife Hurwitz-Samuel (1984) and Frei (1995).

[62] It is interesting to notice that he was also an honorary member of the mathematical societies of Hamburg and Kharkov, and a corresponding member of the Academia di Lincei at Rome (which is rather different from the image of a couch potato that one could have in view of his absence from International Congresses outside Zurich).

But he did not explain what he meant by *curious*.

In the following section, we shall give a short account of Adolf Hurwitz's work on continued fractions in general, and complex continued fractions in particular. The largely forgotten work of Julius Hurwitz on complex continued fractions will be discussed in the subsequent section.

3 Complex continued fractions according to Adolf Hurwitz

Continued fractions of rational and real irrational numbers have been studied for centuries. Of course, continued fractions are closely related to the Euclidean algorithm. For instance, it is said that Pythagoras' pupil Hippasos may have discovered the incommensurability of the diagonal of a square, i.e., the irrationality of $\sqrt{2}$, by means of a geometric version of the Euclidean algorithm giving the regular continued fraction expansion[63]

$$\sqrt{2} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots .$$

In the sixth century, the Indian mathematician Āryabhaṭa used finite continued fractions to solve linear diophantine equations (cf. Brezinski 1991, 29). In the sixteenth century, the Italian mathematicians Bombelli and Cataldi gave first examples of infinite continued fractions [as for, e.g., $\sqrt{13}$; see Brezinski (1991, 61)]. To mention another historical benchmark, the first continued fraction for π was found by the English school, when Lord Brouncker gave the semi-regular expansion

$$\frac{4}{\pi} = \frac{1}{1} + \frac{1^2}{2} + \frac{3^2}{2} + \frac{5^2}{2} + \frac{7^2}{2} + \cdots$$

in response to Wallis' infinite product (Dutka 1982, 116).[64] The first systematic theory of regular continued fractions is often attributed to the Dutch astronomer, physicist, and mathematician Christiaan Huygens, motivated by his work on a gear driven model of the solar system. Later Euler, Lambert, Lagrange, Legendre, Galois, and Gauss made significant contributions to the further development of the theory of continued fractions. More information about the history of continued fractions can be found in Brezinski (1991). The first classic and yet standard reference for continued fractions is the treatise '*Die Lehre von den Kettenbrüchen*' (Perron 1913) by Oskar Perron (and we refer to this source for definitions, details, and fundamental facts). This wonderful monograph covers both the arithmetic and approximation theory of continued fractions, and the third edition from 1954 includes a brief introduction to Adolf Hurwitz's approach to extending the theory of continued fractions from real numbers to complex numbers.

[63] For details about the various appearances of continued fractions in ancient Greek mathematics we refer to Fowler (1990).

[64] See Stedall's (2000) articles for more details.

Even before his time in Königsberg, Adolf Hurwitz had deep interest in continued fractions, as the following entry in his first mathematical diary (Hurwitz 1972, 16) shows:

> The theory of continued fractions having units as numerators and algebraic integers as denominators play a crucial role in the theory of functions as well as in number theory. In the case of numbers $a+bi$ and $a+b\rho$ ($i=\sqrt{-1}$, $\rho=\frac{-1+i\sqrt{3}}{2}$) there is in principle no difficulty in developing a corresponding theory in view of the possibility of Euclid's method for determining the greatest common divisor. Nevertheless, a careful and thorough foundation of such a theory appears to be of great value, for example, for solving diophantine equations of second degree for the corresponding number systems. This is a good doctoral thesis for a young and ambitious mathematician.[65]

This entry bears no date; however, it cannot have been made later than April 13, 1883 (as one may deduce from a later entry with this date); probably, the entry is from 1882. It took a few years that young Adolf Hurwitz returned to continued fractions during his time at Königsberg around 1886/1887.[66]

As a matter of fact, Adolf Hurwitz picked up this idea a few years later by extending the continued fraction expansion to the nearest integer for real numbers to complex numbers. Given a real number $x \in [-\frac{1}{2}, \frac{1}{2})$, its continued fraction to the nearest integer is of the form

$$x = \frac{\epsilon_1}{a_1} + \frac{\epsilon_2}{a_2} + \frac{\epsilon_3}{a_3} + \cdots,$$

where $\epsilon_n = \pm 1$, $a_n \in \mathbb{N}$, and $\epsilon_n + a_n \geq 2$.[67] It is not difficult to see that a real number x has a finite nearest integer continued fraction if, and only if, x is rational. Probably Minnigerode (1873) was the first person to consider continued fractions to the nearest

[65] "Die Theorie der Kettenbrüche, deren Zähler durchgehends Einheiten, deren Nenner ganze algebr. Zahlen sind spielen sowohl im funktionen- wie zahlentheoretischer Hinsicht eine bemerkenswerte Rolle. Im Falle der Zahlen $a+bi$ und $a+b\rho$ ($i=\sqrt{-1}$, $\rho=\frac{-1+i\sqrt{3}}{2}$) ist, wegen der Möglichkeit des Euklid. Verfahrens zur Bestimmung des größten gemeins. Theilers, die Entwickl. der betreffenden Theorie ohne prinzipielle Schwierigkeiten. Nichts desto weniger scheint eine sorgfältige und gründliche Durchführung derselben von großem Werte, z.B. für die Lösung Diophant. Gleichungen des zweiten Grades für die betr. Zahlengebiete. Dieses ist eine gute Doctor-Arbeit für einen jüngeren strebsamen Mathematiker." (Hurwitz 1972, 16), Mathematisches Tagebuch 1, 25.04.1882–09.04.1884; Signatur Hs. 582:1.

[66] According to his mathematical diary no. 5, February 1886–March 1888, pp. 49–69; Hs 582:5 (Hurwitz 1972, 49–69). Unfortunately, there is no date attached, but a later entry has date May 1, 1887.

[67] In modern mathematical language this convergent expansion is obtained by iteration of the mapping $x \mapsto Tx := T(x) := \frac{\epsilon}{x} - \lfloor \frac{\epsilon}{x} \rfloor$ for $x \neq 0$ and $T0 = 0$, where $\lfloor z \rfloor$ denotes the integer part of z (which is unique provided $z - \frac{1}{2} \notin \mathbb{Z}$); here, the subsequent partial quotients are given by $a_n := \lfloor T^{n-1}|x| \rfloor$ and the sign $\epsilon_n = \epsilon$ equals the sign of $T^{n-1}|x|$. The first iteration leads to

$$x = \frac{\epsilon_1}{\lfloor \frac{\epsilon_1}{x} \rfloor + T|x|} = \frac{\epsilon_1}{a_1} + \frac{\epsilon_2}{a_2 + T^2|x|}$$

and so forth.

rational integer as an alternative method for solving Pell's equation; Roberts (1884) developed a related approach independently about a decade later.

Hurwitz (1888) extended Minnigerode's approach by considering complex numbers $z = x+iy$ instead of real numbers and replacing rational integers by Gaussian integers; here and in the sequel, the imaginary unit $i = \sqrt{-1}$ denotes the square root of -1 in the complex upper half-plane. It is not difficult to see that this yields a continued fraction expansion with partial quotients in the ring $\mathbb{Z}[i]$. By analogy to the real situation, a complex number has a finite continued fraction to the nearest Gaussian integer if, and only if, it is a rational Gaussian number. The proof relies on a variation of the Euclidean algorithm in $\mathbb{Z}[i]$. However, Adolf Hurwitz was considering a far more general situation.

Let S be any set of complex numbers such that (i) sum, difference, and product of any two elements in S belong to S, (ii) any finite domain of the complex plane contains only finitely many points from S (from which it already follows that besides zero there is no point in the open unit disk inside S), and, finally, (iii) $1 \in S$. Starting from some complex number x, Adolf Hurwitz built up the following chain of equations:

$$x = a_0 + \frac{1}{x_1}, \quad x_1 = a_1 + \frac{1}{x_2}, \ldots, \quad x_n = a_n + \frac{1}{x_{n+1}},$$

where $a_n \in S$ and none of the x_j is assumed to vanish. This leads to a continued fraction expansion

$$x = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{1}{x_{n+1}},$$

which one can continue ad infinitum if all $x_j \neq 0$. Supposing further that (iv) the nth convergent $\frac{p_n}{q_n} := a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$ differs from z by a quantity less than a fixed constant multiple of $\frac{1}{q_n^2}$, Hurwitz (1888) obtained the following

Theorem *Both the infinite continued fraction*

$$x = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \cdots$$

and the sequence of convergents $\frac{p_n}{q_n}$ converge with limit x (which cannot be an element of S). Moreover, if x is the solution of a quadratic equation with coefficients from S, then the sequence of the x_n takes only finitely many values.

His proof of the above theorem is more or less straightforward. We give a sketch of his reasoning. Firstly, by induction,

$$x = \frac{p_n x_{n+1} + p_{n-1}}{q_n x_{n+1} + q_{n-1}} = \frac{p_n}{q_n} + \frac{(-1)^{n-1}}{q_n^2(x_{n+1} + q_{n-1}/q_n)},$$

and convergence follows from assumption (iv) (respectively, showing $|q_n| > |q_{n-1}|$). To go on, Hurwitz considered for a quadratic irrational the corresponding irreducible

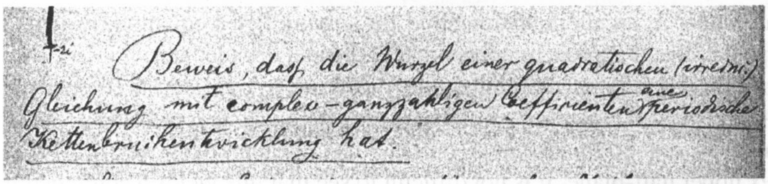

Fig. 4 Original version of the mentioned theorem in Adolf Hurwitz's mathematical diaries no. 5 (Hurwitz 1972, 52), HS 582:5

quadratic equation $Ax^2 + Bx + C = 0$ and deduced another irreducible quadratic equation for x_{n+1} with the same discriminant $D = B^2 - 4AC$ and coefficients bounded by an explicit quantity depending on $|\sqrt{D}|$ such that only finitely many values for x_{n+1} are possible. If in addition the numbers a_n are taken according a certain rule, e.g., as nearest Gaussian integer, then the sequence of the x_n and, henceforth, the sequence of partial quotients a_n are eventually periodic (Fig. 4). For regular continued fractions of real numbers, this is a celebrated theorem of Lagrange (1770, 136); the same reasoning holds for continued fractions to the nearest integer, and even in the case of complex numbers x when the partial quotients are taken to be the nearest Gaussian integers. In his proof, Adolf Hurwitz gives a reference to Hermite (1885).

Concerning Adolf Hurwitz's assumptions on the "system" S [which is what he called a set of numbers satisfying conditions (i)–(iii)], it should be mentioned that S is in fact a ring with the additional assumption that it does not contain any accumulation point. Actually, Adolf Hurwitz wrote in a footnote that "A theory of such number systems has been developed in the well-known works of Kronecker and Dedekind, particularly in the case of algebraic numbers. Cf. in particular Supplement XI to Dirichlet's lectures on number theory. Third edition."[68] Dedekind's supplements to Dirichlet's (1879) lectures deal with the arithmetic in algebraic number fields and gave birth to such fruitful concepts as rings, ideals, and modules; however, the notion of a ring came only in the 1890s with Hilbert's work on his *Zahlbericht* (cf. Dieudonné and Guérindon 1985, 116). In 1894, Adolf Hurwitz (1894c) published a note on principal ideals (probably influenced by Hilbert's work on the *Zahlbericht*). Outside algebraic number theory rings and their substructures became widely accepted only with Emmy Noether's modern algebra. Being a discrete additive subgroup, also a two-dimensional lattice in the complex plane serves well for a system S as considered by Adolf Hurwitz provided that 1 is a lattice point. The notion of a lattice had been established in the important work of Hermann Minkowski in the 1890s, whose *Geometry of Numbers* is based on lattices; previously Eisenstein, Liouville, Jacobi, and Weierstrass had used lattices when dealing with elliptic functions.

As already mentioned, in Hurwitz (1888, 197) Adolf Hurwitz applied his general theorem to the case when S is a ring of Gaussian integers $\mathbb{Z}[i]$. He furthermore wrote

68 "Eine Theorie solcher Zahlsysteme ist in den bekannten Arbeiten von Kronecker und Dedekind, vorzugsweise für den Fall algebraischer Zahlen entwickelt. Vgl. insbesondere das XI. Supplement zu Dirichlet's *Vorlesungen über Zahlentheorie*. Dritte Auflage" (Hurwitz 1888, 187).

> Apart from the here considered continued fraction expansion in the range of complex numbers $m + ni$ there exist more for which the above theorem holds, however, I do not want to go on with this here.[69]

Another continued fraction expansion with partial quotients in the set of Gaussian integers was studied by Adolf Hurwitz's brother Julius as the main topic of his dissertation; we shall investigate this in detail in the following section. Moreover, Adolf Hurwitz mentioned that his approach could be used to build up a *complex* theory of the Pell equation $t^2 - Du^2 = 1$ where D is a given number and solutions t and u are to be complex integers. This topic also is not unrelated to what had been considered by Julius. Finally, in Hurwitz (1888, 197) Adolf Hurwitz studied the ring of Eisenstein integers $\mathbb{Z}[\rho]$ associated with a primitive cube root of unity $\rho = \frac{1}{2}(-1 + i\sqrt{3})$ and pointed out that his continued fraction expansion to the nearest Eisenstein integer is different from what one would obtain with Bachmann's Euclidean algorithm for $\mathbb{Q}(\rho)$ in Bachman (1872, 189). Further applications are possible and were considered by his contemporaries (as we shall indicate in the final section).

Klein's influence is apparent in much of Adolf Hurwitz's work; however, Hurwitz's research on continued fractions and related topics from diophantine approximation seems to have independent roots. Both of his teachers, Schubert and Klein, included continued fractions in some of their writings, namely the textbooks Schubert (1902) and Klein (1908), but these sources are addressed to beginners in mathematics and do not indicate any deeper relation to this topic.

What might have been the motivation for Adolf Hurwitz to start a new direction of research and investigate continued fractions? His friend from Göttingen, Moritz Stern, had received a doctorate on the theory of continued fractions Stern (1829), being the first candidate examined by Gauss (cf. Rowe 1986, 429). For several years, after the work of Euler and Lambert and the investigations of the French school around Lagrange and Legendre, Stern had been the leading expert in this subject; he published quite a few papers on continued fractions in Crelle's *Journal*, and his *Lehrbuch der Algebraischen Analysis* (Stern 1860) contains a whole chapter on this topic.

In fact, continued fractions were a major topic of investigation in the eighteenth and nineteenth centuries. The well-known contributions of Euler, Lagrange, and others focussed on arithmetical questions as, for example, solving Pell's equation or periodic expansions, and Gauss' research marks the beginning of the metrical theory of continued fractions. First results in the nineteenth century highlight analytic questions about convergence and divergence, e.g., the Seidel–Stern convergence theorem due to Seidel (1846) and (independently) Stern (1848). Building on previous work of Stern (1860), Stolz (1885) investigated periodic continued fractions with complex entries with respect to convergence; this is now known as the Stern–Stolz divergence theorem (for details we refer to Lorentzen and Waadeland (1992)). The very first complex

[69] "Ausser der hier betrachteten giebt es übrigens noch andere Kettenbruchentwicklungen im Gebiete der complexen Zahlen $m + ni$, für welche der obige Satz ebenfalls gilt, worauf ich indessen an dieser Stelle nicht eingehen will."

continued fractions can be found implicitly in the general approach of Jacobi; see Jacobi (1868), published in 1868 17 years after Jacobi's death.

Adolf Hurwitz's contribution (Hurwitz 1888) on complex continued fractions was submitted on November 29, 1887, to the journal *Acta Mathematica* and published there in March 1888. It seems that this paper is the first to study complex continued fractions in a systematic way. At this time, there was another study of this topic by the Italian mathematician Michelangeli (1887), and in a letter (Bianchi 1959, 99) to his friend Bianchi from January 12, 1891, Adolf Hurwitz asked about Michelangeli's (1887) work on continued fractions. However, Hurwitz's results seem to go far beyond Michelangeli.[70] We cannot be sure whether Adolf Hurwitz was aware of previous results such as those of Jacobi or Stolz mentioned above. We may only speculate that he could have got a rigorous introduction to *real* continued fractions from his friend Stern; any research on *complex* continued fractions might have been unknown to him. During his studies in Munich, Adolf Hurwitz attended courses by Seydel and Pringsheim, both well-known for their contributions to the theory of continued fractions; however, their research went in a different direction at that time, although that does not exclude the possibility that continued fractions were a topic of their courses and interests.[71] It is a fact that Hurwitz did not cite any work on continued fractions apart from Minnigerode's (Minnigerode 1873) paper, even not Lagrange's or Galois' theorems on periodic continued fraction expansions, which might be from ignorance or *curious* (to use the words of W.H.Y.). However, this canniness in citing other mathematician's works is quite common in the nineteenth century (and applies to Stern and Stolz as well). Summing up, Adolf Hurwitz's point of view is rather different from that of Jacobi or Stolz—namely, arithmetical, not analytical—and it led to a revival of the arithmetical theory of continued fractions. His approach generalizes Minnigerode's approach from real to complex numbers, and there certain phenomena arise with complex continued fractions that do not occur in the real case.

We briefly mention further work of Adolf Hurwitz on continued fractions. In Hurwitz (1889), he introduced a new type of semi-regular continued fractions[72] and studied them with respect to equivalent numbers and quadratic forms. These so-called singular continued fractions are a mixture of the regular continued fraction and the continued fraction to the nearest integer.[73]

[70] At least if the summery of Vivanti in the *Jahrbuch über die Fortschritte der Mathematik* (Vivanti 2005) provides an appropriate picture of Michelangeli (1887). Unfortunately, the authors were not able to find any copy of Michelangeli's work.

[71] "Neben Klein, dessen Vorlesungen über seine Forschungen im Gebiet der Modulfunktionen ihn in hohem Masse fesselten, hörte er bei Gustav Bauer, Seydel, Pringsheim, Brill und Beetz. Bauer u. Pringsheim trat er persönlich näher..." (Hurwitz-Samuel 1984, 6); probably, *Seydel* is misspelled here.

[72] In German: 'Kettenbrüche zweiter Art'.

[73] To explain that we recall Nakada's α-continued fraction from Nakada (1981). Given a fixed real number $\alpha \in [\frac{1}{2}, 1]$, the α-continued fraction of a real number $x \in I_\alpha := [\alpha - 1, \alpha)$ is a convergent finite or infinite semi-regular continued fraction of the form $x = \frac{\epsilon_1}{a_1} + \frac{\epsilon_2}{a_2} + \cdots + \frac{\epsilon_n}{a_n} + \cdots$, where the partial quotients a_n are positive integers and the $\epsilon_n = \pm 1$ are signs determined by iterations of the transform T_α on $[\alpha - 1, \alpha)$ given by $T_\alpha(0) = 0$ and $T_\alpha(x) := \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor$ otherwise. For $\alpha = 1$ Nakada's α-continued fraction

In Hurwitz (1894b), Adolf Hurwitz showed that for any irrational real number x there is an infinite sequence of rational numbers $\frac{p_n}{q_n}$ such that their distance to x is strictly less than $1/(\sqrt{5}q_n^2)$, and this bound is best possible; this improves upon a previous result due to Hermite (1885) and is the starting point for the investigations on the Markov spectrum.[74] His method of proof is based on the regular continued fraction expansion, and the result can be found in diophantine textbooks under the keyword "Hurwitz's approximation theorem" (although often with a proof using the Farey sequence, avoiding continued fractions). There are some refinements of this result, e.g., Borel's (1903) work. Hurwitz's (1894b) paper provides a link between the Farey sequence and continued fractions of irrationals with an application to the Pell equation. Finally, in Hurwitz (1896), he generalized some classical results due to Euler and Lambert on e and related values to more general continued fractions with partial quotients which form an arithmetic progression.

This impressive list of publications shows that continued fractions played a central role in Adolf Hurwitz's investigations. This is confirmed by Hilbert's (1921, 163) obituary where he wrote that Hurwitz had an affection for the theory of *arithmetic* continued fractions (in contrast to questions about convergence when considered as functions of the partial quotients). It is interesting to notice that Hilbert stresses among Adolf Hurwitz's various results on continued fractions the paper in which he realized the generalization from real to complex numbers.[75]

In texts about Adolf Hurwitz's life and work, it is often mentioned that he benefited greatly from his teacher Felix Klein. It is well-known that Klein himself had a very high opinion of the triangle Hurwitz, Hilbert, and Minkowski at Königsberg; in his treatise on the development of mathematics in the nineteenth century, Klein (1926) attributed the description "aphorist" to Hurwitz and considered him as a "problem solver" writing "complete works," whereas Minkowski was a theory builder who found new links between a "geometrical view" and "number theoretical problems."[76] This is

Footnote 73 continued
expansion is nothing but the regular continued fraction, for $\alpha = \frac{1}{2}$ one obtains the continued fraction to the nearest integer, and for $\alpha = \frac{\sqrt{5}-1}{2}$ it is the singular continued fraction due to Adolf Hurwitz (1889).

[74] See Cassels' (1957) monograph for further information.

[75] "Ein mit Vorliebe von Hurwitz behandeltes Thema war die Theorie der arithmetischen Kettenbrüche. In seiner Arbeit 'Über die Entwicklung komplexer Größen in Kettenbrüche' ging er dabei über den bisher allein berücksichtigten Bereich der reellen Zahlen hinaus und stellte einen allgemeinen Satz über die Periodizität der Kettenbruchentwicklung relativ quadratischer Irrationalitäten auf, der auf die Kettenbruchentwicklungen in den Körpern der dritten und vierten Einheitswurzeln eine interessante Anwendung findet" (Hilbert 1921, 163).

[76] "Und glücklicherweise findet sich um 1885 für fast wieder ein Jahrzehnt, eben auch wieder in Königsberg, ein Dreibund junger Forscher zusammen, welche diese Tendenz in neuer Weise in die Tat umsetzen und damit denjenigen Standpunkt schaffen, von dem aus die Neuzeit, wenn sie es vermag, weiterzugehen hat. Es sind dies Hurwitz, Hilbert und Minkowski. (...) und so möchte ich über Hurwitz und Minkowski hier vorweg ein paar Worte sagen, welche deren Arbeitsweise charakterisieren sollen. Man hat Hurwitz einen Aphoristiker genannt. In voller Beherrschung der in Betracht kommenden Disziplinen sucht er sich hier und dort ein wichtiges Problem heraus, das er jeweils um ein bedeutendes Stück fördert. Jede seiner Arbeiten steht für sich und ist ein abgeschlossenes Werk. (...) Minkowskis hier in Betracht kommende Arbeiten beruhen zumeist auf der Verbindung durchsichtiger geometrischer Anschauung mit zahlentheoretischen Problemen. (...) Ich selbst habe mich seinerzeit darauf beschränkt, gewisse schon bekannte Grundlagen

the picture of a *frog* and a *bird* according to Dyson's classification of mathematicians' characters (Dyson 2009). As is well-known, Minkowski remained in contact with both Hurwitz and Hilbert after their common time at Königsberg, with the first of them during his time in Zurich from 1896 until 1902 and with the latter at Göttingen from 1902 until his untimely death in 1909. It seems that Minkowski and Hilbert were closer than the other vertex of this unequal triangle.

4 Complex continued fractions according to Julius Hurwitz

Adolf's support is emphasized on the very first page of Julius' dissertation (Hurwitz 1895), where he wrote that "the thesis follows in aim and method two publications due to Herr. A. Hurwitz to whom I owe the encouragement for this investigation."[77] The two mentioned publications are Hurwitz (1888, 1889). Interestingly, it is his elder brother Adolf Hurwitz who wrote a review in *Jahrbuch über die Forstschritte der Mathematik* (Hurwitz 1894a) about his brother's doctorate starting as follows:

> The complex plane may be tiled by straight lines $x + y = v,\ x - y = v$, where v ranges through all positive and negative odd integers, into infinitely many squares. The centers of the squares are complex integers divisible by $1 + i$. For an arbitrary complex number x, one may develop the chain of equations
>
> $$x = a - \frac{1}{x_1}, \quad x_1 = a_1 - \frac{1}{x_2}, \ldots, \quad x_n = a_n - \frac{1}{x_{n+1}}, \ldots \tag{1}$$
>
> following the rule that in general a_i is the center of the square which contains x_i. In the case when x_i lies on the boundary of a square, some further rule has to be applied which we ignore here for the sake of brevity. For x the chain of equations (1) leads to a continued fraction expansion $x = (a, a_1, \ldots, a_n, x_{n+1})$ further investigation of which is the topic of this work.[78]

Footnote 76 continued
geometrisch klarzustellen, während Minkowski Neues zu finden unternahm. Diese Untersuchungen zeigen deutlich, daß Geometrie und Zahlentheorie keineswegs einander ausschließen, sofern man sich in der Geometrie nur entschließt, diskontinuierliche Objekte zu betrachten" (Klein 1926, 326).

77 "Die Arbeit schliesst sich, nach Ziel und Methode, eng an die nachstehend genannten zwei Abhandlungen des Herrn A. Hurwitz an, dem ich auch die Anregung zu dieser Untersuchung verdanke."

78 "Die complexe Zahlenebene werde durch die Geraden $x + y = v, x - y = v$, wo alle positiven und negativen ungeraden ganzen Zahlen durchläuft, in unendlich viele Quadrate eingeteilt. Die Mittelpunkte dieser Quadrate werden durch die durch $1 + i$ teilbaren ganzen complexen Zahlen besetzt. Wenn nun x eine beliebige complexe Zahl ist, so bilde man die Gleichungskette:

$$x = a - \frac{1}{x_1}, \quad x_1 = a_1 - \frac{1}{x_2}, \ldots, \quad x_n = a_n - \frac{1}{x_{n+1}}, \ldots \tag{1}$$

nach der Massgabe, dass allgemein a_i den Mittelpunkt desjenigen Quadrates bezeichnet, in welches der Punkt x_i hineinfällt. Dabei sind noch bezüglich des Falles, wo x_i auf den Rand eines Quadrates fällt, besondere Festsetzungen getroffen, die wir der Kürze halber übergehen. Durch die Gleichungskette (1) wird nun für x eine bestimmte Kettenbruchentwickelung $x = (a, a_1, \ldots, a_n, x_{n+1})$ gegeben, deren nähere Untersuchung der Gegenstand der Arbeit ist."

At first glance, Julius' expansion could be mistaken as a specification of the general complex continued fraction investigated by his younger brother. The key difference is a modification of the set of possible partial quotients. While Adolf Hurwitz allowed all Gaussian integers, Julius restricted this set to all multiples of $1+i$ which is—in modern language—the ideal generated by $\alpha := 1+i$, i.e., $(\alpha) = (1+i)\mathbb{Z}[i]$, within the ring $\mathbb{Z}[i]$. It should be noticed that 1 is not an element of this ideal; hence, condition iii) of Adolf Hurwitz's setting for his "system" S of partial quotients is not fulfilled. Geometrically speaking, Julius' set of partial quotients is a sublattice $(1+i)\mathbb{Z}[i]$ of the lattice formed by the Gaussian integers. This leads to a tiling of the complex plane and enables consequently the definition of a "nearest" partial quotient $a_n \in (1+i)\mathbb{Z}[i]$ to each complex number $x \in \mathbb{C}$. Proceeding in a similar way as in the case of Adolf Hurwitz's case one finds

$$x_0 = a_0 - \frac{1}{x_1} = a_0 - \frac{1}{a_1 - \frac{1}{x_2}} = \cdots = a_0 - \frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3 - \frac{1}{x_4}}$$

and so on and so forth with partial quotients $a_n \equiv 0 \bmod 1+i$; here, we have replaced a by a_0 and x by x_0 in (1) in order to follow Julius' notation closely.[79]

A natural question is *which complex numbers have a finite continued fraction expansion?* Of course, the answer may depend on the type of continued fraction expansion, and indeed, there are differences between the continued fraction proposed by Adolf Hurwitz and the one investigated by Julius. For Adolf Hurwitz's continued fractions, the expansion for a complex number x terminates if, and only if, $x \in \mathbb{Q}(i)$, as follows from the analogue of the Euclidean algorithm for the ring of integers $\mathbb{Z}[i]$. However, the situation in the case of Julius Hurwitz-continued fractions is different:

$$\frac{1+8i}{5+7i} = \frac{1}{1-i} + \frac{1}{-2i} + \frac{1}{-4},$$
$$\frac{18}{95} = \frac{1}{6} + \frac{1}{-2} + \frac{1}{2} + \frac{1}{-2} + \frac{1}{-2} + \frac{1}{2},$$

whereas

$$\frac{1+7i}{5+7i} = \frac{1}{1-i} + \frac{1}{-2i} + \frac{1}{3+i} + \frac{1}{0} + \cdots,$$
$$\frac{17}{95} = \frac{1}{6} + \frac{1}{-2} + \frac{1}{-2} + \frac{1}{-2} + \frac{1}{-0} + \cdots;$$

79 Actually, each iteration is determined by the transform T which is defined by $T(0) = 0$ and $T(x) := \frac{1}{x} - [\frac{1}{x}]$ otherwise, in analogy with the approach of his younger brother; here the bracket $[x]$ is given by $[x] := \lfloor u + \frac{1}{2} \rfloor \alpha + \lfloor v + \frac{1}{2} \rfloor \overline{\alpha}$ for $x = u\alpha + v\overline{\alpha}$ with $\alpha = 1+i$, and T maps the square $X := \{x = u\alpha + v\overline{\alpha} : -\frac{1}{2} \leq u, v < \frac{1}{2}\}$ onto X. What turns up is an expansion of a unique complex continued fraction $x = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3 + T^3 x}$. This is modern notation and is different from what can be found in Julius' dissertation Hurwitz (1895).

here the dots indicate that the continued fraction expansion does not terminate. These examples are related to the case of iterates of z coming from the complementary set $1 + (1+i)\mathbb{Z}[i]$ of partial quotients; here, we observe

$$i = \frac{1}{1-i} + \frac{1}{0} + \frac{1}{0} + \cdots$$

which shows that there is no convergence in such cases. Generalizing from these examples, Julius Hurwitz arrived at the following

Theorem *The continued fraction expansion of the first kind of a complex rational number finishes with a partial quotient divisible by* $1+i$ *if, and only if, after canceling all common factors, neither the numerator nor the denominator are divisible by* $1+i$.[80]

The formulation of this theorem might be confusing. As a matter of fact, *the Julius Hurwitz-continued fraction for a complex number* x *with all partial quotients from* $(1+i)\mathbb{Z}[i]$ *is finite if, and only if,* $x = \frac{a}{b}$ *with coprime* $a, b \in \mathbb{Z}[i]$ *satisfying either* $a \equiv 1, b \equiv 0 \bmod \alpha$ *or* $a \equiv 0, b \equiv 1 \bmod \alpha$. In addition, one may allow a final partial quotient from the complement of $(1+i)\mathbb{Z}[i]$ in order to have a finite expansion for all complex rationals. With the notion of a continued fraction of the first kind, we mean the expansion given above; in a later part of his thesis, Julius also considers the analogue of continued fractions of the second type as introduced by his brother.

Here, we essentially reproduce Julius' proof of his theorem. Obviously, all finite expansions with partial quotients from $(1+i)\mathbb{Z}[i]$ lead to complex rationals. In order to prove the converse implication, one may expand a given Gaussian rational number $x_0 = \frac{m+ni}{r+si} = \frac{\mu}{\mu_1}$ such that

$$\frac{\mu_1}{x_1} = \mu_1 a_0 - \mu =: \mu_2 \in \mathbb{Z}[i];$$

since $|x_1| > 1$, it follows that $|\mu_1| > |\mu_2|$. This leads to $x_1 = \frac{\mu_1}{\mu_2} = a_1 - \frac{1}{x_2}$ and again

$$\frac{\mu_2}{x_2} = \mu_2 a_1 - \mu_1 =: \mu_3 \in \mathbb{Z}[i].$$

Going on we obtain a sequence of Gaussian integers μ_n such that any subsequent pair satisfies $|\mu_{n+1}| > |\mu_{n+2}|$. Consequently, the norms of the μ_n form a strictly monotonic decreasing sequence of positive integers; hence, there exists an index m such that $\mu_{m+1} \neq 0 = \mu_m$. This goes along with $\frac{\mu_m}{\mu_{m+1}} = x_m = a_m$. If μ and μ_1 are coprime in the Euclidean ring $\mathbb{Z}[i]$, then the Euclidean algorithm provides

$$\begin{aligned} \mu &= a_0\mu_1 - \mu_2, \\ \mu_n &= a_n\mu_{n+1} - \mu_{n+2} \qquad \text{for} \quad n = 1, \ldots, m, \\ \mu_m &= a_m\mu_{m+1}, \end{aligned}$$

[80] "Die Kettenbruch-Entwicklung erster Art einer complexen rationalen Zahl endigt dann und nur dann mit einem nicht durch $1+i$ teilbaren Teilnenner, wenn, nach Forthebung gemeinsamer Faktoren, weder der Zähler noch der Nenner der Zahl durch $1+i$ teilbar sind" (Hurwitz 1895, 27/28; Hurwitz 1902, 246).

where μ_{m+1} is a unit. If $a_m \equiv 1 \bmod 1+i$, then it follows by induction that the same congruence holds for all predecessors μ_n. However, if $a_n \equiv 0 \bmod 1+i$, then the same reasoning implies $\mu_n \equiv 0 \bmod 1+i$.[81]

As often happens, the complex viewpoint provides new insights about the real case. Julius Hurwitz's continued fraction applied to real numbers leads to partial quotients from $2\mathbb{Z}$, and indeed, these continued fraction expansions coincide with the continued fraction with even partial quotients which was studied first by Schweiger (1982).[82]

Another result of Julius' is the analogue of his brother's theorem on characterizing all eventually periodic continued fractions as roots of irreducible quadratic equations having Gaussian integers as coefficients. Another analogy to Adolf Hurwitz's paper is what kind of partial quotients can occur. For instance, if a partial quotient equals $a_n = 1+i$, then the next partial quotient a_{n+1} is different from $2, 1-i, -2i$. A similar problem was already considered by Adolf Hurwitz for his continued fractions to the nearest Gaussian integers. Julius' doctorate also contains results about those *admissible* sequences. Adolf Hurwitz stated in his *Jahrbuch über die Forstschritte der Mathematik* review (Hurwitz 1894a) that the partial quotients "$a, a_1, \ldots, a_n, x_{n+1}$ fulfill certain constraints. (...). It is then proved that the expansion of any complex quantity converges, that it is finite, resp. periodic, if the quantity is a rational complex number, resp. satisfies a quadratic equation with integer complex coefficients. There is a relation of the investigated continued fraction expansion with another in which the tiling of the complex plane in the above mentioned squares is replaced by domains bounded by circular arcs. For this second expansion analogous results are proved as

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81 A short algebraic proof goes as follows. The set of Gaussian integers is a disjoint union of the principal ideal $(\alpha) = (1+i)\mathbb{Z}[i]$ and $1+(\alpha)$. Denoting the Julius Hurwitz-continued fraction of a complex number x by $x = [a_0, a_1, \ldots, a_n, \ldots]$ the numerators p_n and denominators q_n of its convergents $\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$ satisfy the following recursion formulae:

$$\begin{cases} p_{-1} = \alpha, \ p_0 = a_0, \quad \text{and} \quad p_n = a_n p_{n-1} + p_{n-2}, \\ q_{-1} = 0, \ q_0 = \alpha, \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2}. \end{cases}$$

The proof is analogous to the one for regular continued fractions, and only the initial values differ. To continue, we rewrite the recursion formulae in terms of 2×2-matrices as

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \quad (n \in \mathbb{N}_0).$$

After reducing the convergents $\frac{p_n}{q_n}$ with respect to common powers of α in the numerator and denominator, it follows from the recurrence formuale that

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n \bmod \alpha,$$

where the congruence is with respect to each entry. Hence, comparing the first columns on both sides, we have shown that each finite Julius Hurwitz-continued fraction is of the form predicted by the theorem. To show the converse, one may use Fermat's descent method.

82 In Perron (1913, 186, vol. I), 3rd ed., Perron writes "Eine andere Vorschrift für die Wahl der Teilnenner stammt von J. Hurwitz (...); sie bildet das Analogon zu den halbregelmäßigen Kettenbrüchen mit geraden Teilnennern." Actually, Julius Hurwitz-continued fractions of real numbers are exactly continued fractions with even partial quotients.

for the first."[83] These continued fractions of the second kind are the singular continued fractions developed in Hurwitz (1889).

Building on his continued fraction expansion to the nearest lattice point in $(1+i)\mathbb{Z}[i]$ from his dissertation, in his Habilitation thesis, Julius Hurwitz developed a method for the reduction of quadratic forms with complex coefficients and variables; this was published as Hurwitz (1902) in the same renown journal *Acta Mathematica* as his younger brother's first paper (Hurwitz 1888). Julius had this very application in mind even during the preparation of his dissertation. In fact, Wangerin wrote in his report about Julius Hurwitz's dissertation that

> The author believes that the investigated continued fraction expansion may serve as basis for a theory of quadratic forms with complex variables and complex coefficients.[84]

His approach is quite similar to the one for real quadratic forms with positive determinants. He solved the problem of determining whether two given forms with equal determinants are equivalent, and of finding all substitutions which transform one form to any of its equivalent forms. Both questions were already solved by Dirichlet (1842) by a different method.

5 Reception in the twentieth century

Continued fractions are always linked with the name of Perron. Oskar Perron was born on May 7, 1880 in Frankenthal (near Mannheim), and died in Munich February 22, 1975. He studied in Munich and obtained his doctorate there in 1902; his dissertation was about the rotation of a rigid body and was supervised by Lindemann (who was already responsible for Adolf Hurwitz's appointment in Königsberg). In his post-doctoral research, however, Perron became interested in the work of Pringsheim, another professor at Munich with an expertise in complex analysis (and father of Katharina Pringsheim, later Katia Mann, the wife of Thomas Mann). At that time, Pringsheim (1900) was investigating the Stern–Stolz criterion (Stolz 1885) for convergence of periodic complex continued fractions. Perron did his Habilitation in 1907 on a related question, namely Jacobi's general continued fraction algorithm (Jacobi

[83] "Denkt man sich die durch $1+i$ teilbaren ganzen complexen Zahlen $a, a_1, \ldots, a_n$ und die complexe Grösse x_{n+1} willkürlich gewählt, so wird der Kettenbruch $(a, a_1, \ldots, a_n, x_{n+1})$ durch seine Einrichtung eine bestimmte complexe Grösse x ergeben. Dieser Kettenbruch braucht aber nicht notwendig mit der nach obigem Gesetze erfolgten Entwickelung von x übereinzustimmen. Vielmehr müssen, damit dies der Fall sei, $a, a_1, \ldots, a_n, x_{n+1}$ gewisse Bedingungen erfüllen. Diese stellt der Verfasser zunächst auf. Sodann wird der Nachweis geführt, dass die Entwickelung jeder complexen Grösse convergirt, dass sie abbricht, resp. periodisch wird, wenn die Grösse eine complexe rationale Zahl ist, resp. einer quadratischen Gleichung mit ganzzahlig complexen Coefficienten genügt. Mit der untersuchten Kettenbruchentwickelung steht nun ferner eine andere in genauestem Zusammenhange, bei welcher an Stelle der Einteilung der complexen Zahlenebene in die oben genannten Quadrate eine solche in Gebiete tritt, die von Kreisstücken begrenzt sind. Für diese zweite Entwickelung werden die analogen Sätze wie für die erste bewiesen."

[84] "Der Verfasser glaubt, dass die von ihm genauer erforschte Art der Kettenbruchentwicklung als Grundlage für die Theorie der quadratischen Formen mit complexen Variablen und complexen Koeffizienten dienen könne" (Archive of Halle University, Rep. 21 Nr. 162).

1868); this topic led Perron and later Frobenius to the discovery of the famous Perron–Frobenius theorem for matrices with non-negative entries.[85]

The standard reference in the theory of continued fractions is Perron's monograph Perron (1913), but neither the first edition from 1913 nor the second edition from 1929 mentions Adolf or Julius' Hurwitz's work on complex continued fractions. However, the third edition from 1954 contains a whole section on continued fractions in imaginary quadratic number fields (§46). Here, Perron gives a brief introduction to Adolf Hurwitz's (1888) work. A proof is given that this expansion is finite if, and only if, the number in question is a Gaussian rational. Some attention is also paid to approximation properties as well as to admissible sequences, and periodic complex continued fractions. It is a natural question to ask for what other fields than $\mathbb{Q}(i)$ one can obtain similar results. It seems that Dickson (1927) was the first to investigate in which quadratic fields $\mathbb{Q}(\sqrt{D})$ an analogue of the Euclidean algorithm is possible. He proved that for imaginary quadratic fields there exists a Euclidean algorithm if, and only if, $D = -1, -2, -3, -7, -11$; however, his proof for real quadratic fields turned out to be false and was corrected by Perron (1932). With regard to this, Lunz (1937) considered in his dissertation (supervised by Perron) the field $\mathbb{Q}(\sqrt{-2})$. Already in this case, fundamental questions, such as the growth of the denominators of the convergents in absolute value, seem to be more difficult to answer than in the Gaussian number field. Further studies were made by Arwin (1926, 1928) for several other imaginary quadratic fields. Gintner (1936) proved in her doctoral thesis at the University of Vienna in 1936 that in non-Euclidean imaginary quadratic number fields one can find examples where the corresponding continued fraction expansion does not converge, e.g.,

$$z = \tfrac{1}{2}\sqrt{-d} \quad \text{if } d \not\equiv 3 \bmod 4, \quad z = \tfrac{2d+1}{2d}\sqrt{-d} \quad \text{if } d \equiv 3 \bmod 4.$$

Moreover, she studied diophantine approximation in imaginary quadratic fields not only with continued fractions but also using Minkowski's geometry of numbers. Further results along these lines were found by her thesis advisor (Hofreiter 1938). Summing up, *a continued fraction expansion to the nearest integer is possible if, and only if, the order of the imaginary quadratic field is Euclidean.*

In 1912, Mathews considered binary quadratic forms with complex coefficients; he stressed that his approach differs from Julius Hurwitz's method in his Habilitation thesis, published as Hurwitz (1902). Mathews avoided considering condensation points; his reasoning shows immediately that the number of reduced forms is finite. Moreover, "the roots of a reduced form are expressible as pure recurrent chain-fractions appears as a corollary, instead of being a definition".[86] Ford (1918, 23) wrote in a footnote: "The continued fractions involving complex integers have been little studied. Only one of such fraction has, so far as I know, appeared in the literature. See [Adolf] Hurwitz, *Acta Mathematica*, vol. 11 (1887), pp. 187–220; Auric, *Journal de mathématiques*, 5th

[85] See Hawkins (2008) for a very informative study of this topic.

[86] See Mathews (1912, 329). The old-fashioned, direct translation *chain fraction* from the German *Kettenbruch* is outdated. The English notion *continued fraction* has been used since around 1900 the first articles on this topic in English had been published.

ser., vol. 8 (1902), pp. 387–431. "Ford was mostly interested in extending Hermite's approach for rational approximations to complex numbers rather than in Adolf Hurwitz's treatise of complex continued fractions. Auric (1902) gave further applications of Adolf Hurwitz's continued fractions. In 1927, Stein used in her dissertation Julius Hurwitz-continued fractions in order to compute units in quadratic extensions of number fields. This line of investigation was proposed by her supervisor Helmut Hasse. The divisibility of the denominators of the convergents by $1+i$ is here an essential tool. The research monograph of Koksma (1936) mentions both the contributions of Adolf Hurwitz and the dissertation (Hurwitz 1895) of Julius. Dickson's (1923) encyclopedia on the history of number theory briefly quotes Julius' publication (Hurwitz 1902) and the many works of Adolf Hurwitz. Whereas we can find several references to the work of the Hurwitz brothers before the big changes the world had to face when the Nazis came to power in Germany and set half of the world on flame, there are nearly no citations after World War II. Brezinski (1991, 270) mentions Julius Hurwitz in a list of contributors to the theory of continued fractions but refers to Koksma (1936) for details.

In the twentieth century, continued fractions were studied for many different reasons. Whereas in the nineteenth century much attention was given to convergence criteria and diophantine approximation, new lines of investigation after World War II were the ergodic theory of continued fractions and the consideration of a continued fraction expansion as a product of linear fractional transformations. The latter approach is related to the modular group, while the former has roots in an old problem on the statistics of the partial quotients of the regular continued fraction expansion posed by Gauss.

The first sentence of Ito and Tanaka (1981, 153) illustrates the different approaches for research on continued fractions: "The simple continued-fraction expansion of real numbers is an important concept in the theory of numbers. And the continued-fraction expansion defined by Hurwitz is also important because it is the expansion by the nearest integers. These two continued-fraction expansions give rise to many interesting problems not only in the theory of numbers but also in ergodic theory." Here, Adolf Hurwitz's continued fraction is meant. There is previous work by Kaneiwa et al. (1975, 1976) which already contains a certain similarity to Julius' continued fraction. If ρ denotes the cube root of unity in the upper half-plane, a bracket is defined by

$$[z] := \lfloor u \rfloor \rho + \lfloor v \rfloor \overline{\rho} \qquad \text{for} \quad z = u\rho + v\overline{\rho}$$

with real u and v. Given a complex number z, iterations of the transform $z \mapsto \frac{1}{z} - [\frac{1}{z}]$ yield a continued fraction in a similar way as in the previous section for Julius' expansion. Their intention was similar: Admissible sequences are discussed and diophantine properties investigated, and a Lagrange-type theorem for general quadratic extensions is included in Kaneiwa et al. (1976). In a subsequent paper, Shiokawa (1976) introduced tools from ergodic theory in order to show that T is ergodic which allows one to deduce several metrical results about this complex continued fraction expansion. In none of these papers is Julius' work is mentioned; it seemed that his insights had fallen into oblivion.

In 1985, about one century after Julius' doctorate, Shigeru Tanaka published the identical continued fraction transformation a second time. He refers only to the work of Adolf Hurwitz, and it is rather unlikely that he was familiar with Julius' thesis, which was only published in German and is hardly accessible [if one does not know that a short version is contained as first part in Hurwitz (1902)].[87] Furthermore, his approach is very different. While Julius' point of view is quite geometrical, Tanaka's motivation was to investigate ergodic properties of the continued fraction. He succeeded in determining a natural extension of the transformation T and thereby constructing an invariant measure with respect to which the continued fraction map T is ergodic. With this property, Tanaka was, among other things, able to transfer results from Shiokawa's ergodic theoretical approach to Adolf Hurwitz's continued fraction to the theory of Julius' complex continued fraction. In that sense Tanaka's work built on Shiokawa's in a similar way to the way that Julius was continuing Adolf Hurwitz's approach.

Recently, complex continued fractions have been studied in a rather different context. Nearly nothing is known about the regular continued fraction expansion of real algebraic irrationals of degree strictly larger than two. For instance, it is an open question whether the sequence of partial quotients of such a real algebraic irrational is bounded or not; the same problem is also unanswered for other real continued fractions. It follows from Adolf Hurwitz's work that the situation for complex algebraic irrationals is quite different: Complex irrationals satisfying an irreducible quadratic equation with coefficients from $\mathbb{Z}[i]$ have a periodic, henceforth bounded sequence of partial quotients (extending Lagrange's celebrated theorem). However, Hensley (2006) discovered a far more surprising phenomenon: There exist complex algebraic irrationals having a bounded but not eventually periodic sequence of partial quotients in Adolf's continued fraction expansion; an example is $z = \sqrt{2}-1+i(\sqrt{5}-2)$ which is a solution of the irreducible biquadratic equation

$$Z^4 + (4+8i)Z^3 - (12-24i)Z^2 - (32-16i)Z + 24 = 0.$$

Bosma and Gruenewald (2012) proved the existence of complex algebraic numbers of arbitrary even degree having a continued fraction expansion with bounded partial quotients (being non-periodic for degree larger than two over the Gaussian number field).

What makes these approaches to implementing continued fraction algorithms for complex numbers interesting for current research in number theory is that there are many open questions concerning algebraic and ergodic features of complex continued fraction expansions. Another approach to diophantine approximation of complex numbers via continued fractions is due to Schmidt (1975, 1982); his papers include references to the work of Julius and Adolf Hurwitz. Schmidt's type of continued fraction is superior if the quality of approximation is paramount, and it allows the use of tools from ergodic theory too, at the price that his continued fraction expansions lack the simplicity of the continued fraction expansions found by the Hurwitz brothers. Further approaches are due to Cassels, Ford, Gintner, LeVeque, Mahler, Nakada, and others; their discussion is beyond the scope of our article.

[87] Perron's (1913) monograph is not yet translated into English!

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