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The natures of numbers in and around Bombelli's *L'algebra*

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Abstract The purpose of this article is to analyse the mathematical practices leading to Rafael Bombelli's *L'algebra* (1572). The context for the analysis is the Italian algebra practiced by abbacus masters and Renaissance mathematicians of the fourteenth to sixteenth centuries. We will focus here on the semiotic aspects of algebraic practices and on the organisation of knowledge. Our purpose is to show how symbols that stand for underdetermined meanings combine with shifting principles of organisation to change the character of algebra.

1 Introduction

1.1 Scope and methodology

In the year 1572 Rafael Bombelli's *L'algebra* came out in print. This book, which covers arithmetic, algebra and geometry, is best known for one major feat: the first recorded use of roots of negative numbers to find a real solution of a real problem. The purpose of this article is to understand the semiotic processes that enabled this and other, less 'heroic' achievements laid out in Bombelli's work.

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The context for my analysis of Bombelli's work is the vernacular abbacus[1] tradition spanning across two centuries, from a time where "almost all problems that are done in the abbacus way reduce to the rule of three"[2] to the Renaissance solution of the cubic and quartic equations, and from the abbacus masters, who taught elementary mathematics to merchant children, through to humanist scholars. My purpose is to track down sign practices through vernacular Italian algebra to account for the emergence of Bombelli's almost-symbolic algebra.[3]

One of the principal question concerning abbacus and Renaissance algebra revolves around the notational changes that culminated in using letters to represent unknowns and parameters, and in adopting the superscript numeral notation for exponents. These changes are recognised as having a crucial impact on mathematical development. But, as is argued in Høyrup (2009b), notation did not come in a flash of insight and did not technologically determine a superior mathematical practice. We are therefore left with the task of explaining how the slowly emerging algebraic notation was eventually adopted and disseminated.

Indeed, the history of algebraic notation is all but linear. Clever notations were introduced quite early that did not catch on (e.g. de'Mazzinghi 1967 late fourteenth century manuscript and Chuquet's 1484 *Triparty en la science des nombres*), and there is no clear correlation between the abstractness of notation and mathematical achievements (e.g. the highly evolved and just as verbose work of Cardano 1545). Furthermore, some later authors included algebra only as an independent marginal section of their treatises (e.g. Benedetto 1974) whereas some earlier authors subsumed their entire work under its aegis (e.g. Dardi 2001). So even if we believe today that algebraic notation is a good tool for expressing, processing and organising mathematical knowledge, the choice between the newer notations and the older rhetorical style was not obvious at the time.

The Algebra introduced in Italy was processed Arabic knowledge. Whilst the standard claim is that the immediate source of the abbacist tradition is Leonardo Fibonacci, Høyrup (2007) has recently argued that it was imported into Italy from a Catalan–Provençal culture, of which we no longer have any manuscript remains. Either way, this does not explain how algebra took root and evolved in the Italian scene. The most notable existing analyses of the emergence of abbacus and Renaissance algebra include Høyrup (2007, 2009a) tracking down of the sources and intellectual moti-

[1] I follow other specialists in retaining the Italian double *b* to avoid confusion with the abacus as instrument of calculation.

[2] "quaxj tutte le ragionj, che per abbacho si fanno, si riducono sotto la reghola delle tre choxe" (Paolo 1964, 153). In only slightly anachronistic terms, this is the rule that says that if a given quantity yields a second, and in the same ratio an unknown quantity yields a third, then the unknown quantity equals the given times the third divided by the second. In more contemporary terms this is the rule that says that if $a : b$ is as $x : c$, then x is $\frac{ac}{b}$.

[3] In this article I quote from fourteenth century authors Jacopo de Firenze, Maestro Dardi, Paolo dell'Abbaco and Antonio Mazzinghi, and from fifteenth century authors Piero della Francesca and Maestro Benedetto (Arrighi, the modern editor of the last four authors, erroneously attributed Benedetto's treatise to Pier Maria Calandri; the error was corrected in Van Egmond's catalogue (1980)). The authoritative survey of abbacus algebra is still that of Franci and Rigatelli (1985). Another prominent figure in this analysis will be Girolamo Cardano, whose *Ars Magna* was the main text that Bombelli sought to clarify.

vations of abbacus masters; Rose (1975) attribution of mathematical development to the humanist culture; Heeffer (2008, 2005) analysis of mathematical developments in terms of rhetorical organisation and of an epistemic shift towards model-based reasoning related to economic practices; Cifoletti (1995) reconstruction of developments in terms of imposing judicial rhetoric on mathematical texts; and in the field of didactics Radford's (2003, 1997) attempt to connect epistemico-historical analysis with the phenomenology of contemporary classrooms.

The analyses cited above shed much light on the emergence of algebra, but leave a crucial aspect of the problem unresearched. This missing aspect is the semiotic processes that reformed sign practices. When our way of doing mathematics changes, the change is hardly ever immediate and abrupt. It most often involves micro-level changes in the way we operate signs. Whether the cause of mathematical change is epistemic shift, new textual input, new cultural practices, economic activity or the internal logic of signs, change must be enabled and expressed by practical shifts in sign usage that occur inside texts. It is the fact that signs are never completely confined to any specific practice or context—the fact that signs can always be copied or practiced outside the context that is supposed to govern them—which provides signs with a motility, which, I believe, is a necessary 'energetic' condition for mathematical change.[4]

The analysis of Bombelli's own practice is in an even poorer state. Bortolotti's editorial work is careful and insightful, but at the same time entirely anachronistic, and understands Bombelli as a precursor of modern ring theory. Many accounts follow the well-known development from Dal Ferro's solution through Tartaglia, Cardano and Ferrari to Bombelli's achievements (most often in anachronistic terms), but the picture they paint is usually that of a linear progress followed by a leap of faith that is required in order to endorse roots of negative numbers. Relating this leap of faith or development to its conditions of possibility, which trace back to abbacus algebra, is, however, something that is not dealt with in contemporary literature. The one exposition I know of that takes a more detail-sensitive, non-anachronistic look at this development is La Nave and Mazur (2002) (some other aspects of Bombelli's algebraic practice are studied in the context of the immediately preceding evolution by Rivolo and Simi 1998; for a discussion of Bombelli's geometric algebra see Giusti 1992 and Wagner (forthcoming)).

In this article, therefore, rather than trace concepts, I will follow sign manipulations. I will show that at the micro-level the emergence of Renaissance algebra is not about rigid cognitive distinctions or a grand historical narrative. I will show that Renaissance algebra depends on a slow erosion of distinctions between kinds of mathematical signs, on the import of sign practices from economy into the arithmetic world, and on a non-linear emergence and repression of practices that allow transcribing some signs by other signs. This approach shifts the focus from large, vague conceptual questions such as 'how did algebra come to be' to a tractable concern with following the

[4] I do not include here a presentation of my philosophical framework; I have done this elsewhere (Wagner 2009a,b, 2010) and will return to it in future texts. Other approaches that can serve as philosophical background to this project range from the phenomenological/post structural notion of writing in Husserl's *Origin of Geometry* as reconstructed by Derrida (1989) all the way to Emily Grosholz' (2007) analytic understanding of mathematics' productive ambiguities and 'paper tools'.

evolution of aggregates of material practices with signs that we can recognise as more or less similar to what we usually call 'algebra'.

I do not claim that the pointilliste portrait of mathematical practice I include here is superior or more exhaustive than others.[5] I claim that it complements other approaches so as to enable a much more comprehensive understanding of how mathematical change occurs. Hopefully, the highly unstable portrait drawn here will help us acknowledge the unstable practices of contemporary, supposedly rigid and perfectly regimented mathematics, and the role of such practices in mathematical development.

1.2 Bombelli and *L'algebra*

Not much is known about Bombelli's life. According to Jayawardene (1965, 1963) he was born around 1526 and died no later than 1573. He was an engineer and architect involved in reclaiming marshland and building bridges. There is no record that he studied in a university, but he was obviously a learned man, so much so that a scholar from the university of Rome invited him to cooperate on a translation of the works of Diophantus.

The writing of the manuscript draft of *L'algebra*, Bombelli's only known publication, took place during a long pause in the Val di Chiana marsh reclamation, which Bortolotti (Bombelli 1929) dates to the early 1550s and Jayawardene (1965) to the late 1550s.

The manuscript (Bombelli 155?), which was uncovered by Ettore Bortolotti, is divided into five books. The first three books of the manuscript appeared with revisions in the 1572 print edition (Bombelli 1572). Bortolotti published a modern edition of the remaining two books with an introduction and comments (Bombelli 1929), which was later combined with a modern re-issue of the 1572 edition (Bombelli 1966).[6]

The first book of *L'algebra* is a treatise on arithmetic. It includes the extraction of roots up to the seventh order, and culminates in techniques for adding, subtracting, multiplying, dividing and extracting roots of sums of numbers and roots (mostly binomials, but also some longer sums).

The second book introduces the unknown (*Tanto*), and opens with what in contemporary terms would be an elementary algebra of polynomials up to and including division. Then the second book goes on to systematically present techniques for solv-

[5] In fact, Bombelli's sources include several Latin mathematicians, who rely on the classical tradition and on translations from the Arabic. Furthermore, Diophantus obviously made a strong impact on Bombelli, which resulted in some changes between the manuscript and print version. I also ignore the role of geometry in Bombelli's work (not that geometry is not important for Bombelli—on the contrary—but I defer the analysis of Bombelli's geometry to Wagner (forthcoming)). My approach is therefore not exhaustive even in terms of intellectual sources and mathematical context, and does not presume to be. For my purposes, however, the algebra of the abbacus tradition gives more than enough to work with.

[6] In referring to Books III, IV and V, I use problem number and section number (the 1929 edition is available online, so it makes more sense to use section numbers than the page numbers of the out-of-print 1966 edition). In references to Books I and II the page numbers of the 1966 edition are used, as there is no numerical sectioning. The translations from the vernacular Italian are my own. In translations I retained the original names of units, but not their shorthand notations. The exceptions are *livre* and *once* in the context of weight, which I translated as pounds and ounces.

ing quadratic, cubic and quartic equations, following the discoveries of dal Ferro, Tartaglia, Cardano and Ferrari.

The third book is a collection of recreational problems in the abbacist tradition together with problems borrowed directly from Diophantus. The problems are solved using the algebraic techniques taught in the second book.

The fourth book, which did not make it to print at the time, concerns what Bombelli calls *algebra linearia*, the reconstruction of algebra in geometric terms. It opens with some elementary Euclidean constructions, and then builds on them to geometrically reproduce the main techniques of Book II and some problems of Book III. Book V treats some more traditional geometric problems in both geometric and algebraic manner, goes on to teach some basic practical triangulation techniques, and concludes with a treatise on regular and semi-regular polyhedra. Book IV is not entirely complete. Many of the spaces left for diagrams remain empty. Book V is even less complete, and its sections do not appear in the manuscript table of contents.

There are some substantial differences between the manuscript and the print edition. Several sections that appear as marginalia in the manuscript were incorporated as text into the print edition of books I and II. Some of the geometric reconstructions of the unpublished Book IV were incorporated into the first two books. The print edition also has a much more developed discussion of roots of negative numbers and some changes in terminology and notation that will be addressed below. Book III went through some major changes. Problems stated in terms of commerce in the manuscript were removed, and many Diophantine problems were incorporated (for a full survey of these changes see Jayawardene 1973). The introduction to the print edition states an intention to produce a book that appears to be based on the manuscript Book IV, but this intention was never actualised (Bombelli died within a year of the print publication).

According to Bortolotti's introduction, *L'algebra* seems to have been well received in early modern mathematical circles. Bortolotti quotes Leibniz as stating that Bombelli was an "excellent master of the analytical art", and brings evidence of Huygens' high esteem for Bombelli as well. Jean Dieudonné, however, seems less impressed with Bombelli's achievements and renown (Dieudonné 1972; Bombelli 1929, 7–8).

1.3 Bombelli's sources

Practically all the technical achievements included in Bombelli's work had already been expounded by Cardano. The exceptions include some clever tinkering with root extraction and fine tuning techniques for solving cubics and quartics (Bombelli's achievements in reconciling algebra and geometry were not published in print at the time). But Bombelli's one undeniable major achievement is the first documented use of roots of negative numbers in order to derive a real solution of a polynomial equation with integer coefficients. He is not the first to work with roots of negative numbers, but he is the first to manipulate them extensively beyond a basic statement of their rules.

However, judging Bombelli's book through the prism of technical novelty does not do it justice. Indeed, Bombelli explicitly states in his introduction that he is representing existing knowledge. He explains that "in order to remove finally all obstacles before the speculative theoreticians and the practitioners of this science" (algebra) … "I was taken by a desire to bring it to perfect order".[7]

The earlier authors that Bombelli mentions explicitly in his list of sources (Bombelli 1966, 8–9) are the Greek Diophantus, the Arab "Maumetto di Mosè" (Al Khwarizmi), and their vaguely referenced Indian predecessors. Then he skips to authors of vernacular and Latin texts such as Leonardo Fibonacci, Oronce Finé, Heinrich Schreiber, Michael Stifel, Luca Pacioli, Girolamo Cardano, Ludovico Ferrari, Nicolò Tartaglia and "a certain Spaniard", which Bortolotti reads as Pedro Nuñez, but I suspect might be Marco Aurel (see footnote 64 below). Later in the text we can find references to Barbaro's work on Vitruvius (in the context of the doubling of the cube by Plato's school and by Archytas) and to Albrecht Dürer (in the context of a nine-gon construction). Euclid is quoted, of course, but does not play a central role. Almost all explicit quotations from Euclid are found in the first 18 sections of Book IV, where Bombelli introduces his basic geometric constructions. From the reference to Euclid's VI.12 in §18 of Book IV we can infer that Bombelli used either one of the circulating Greek editions or some edition of Zamberti's Latin translation from the Greek (The Campanus-Adelard Latin translation from an Arabic source has the Greek IV.10 as his IV.12). But in fact, Bombelli's list of sources suggests a thorough bibliographic research, and it is likely that Bombelli consulted several versions of the *Elements*, and was not committed to any one particular edition. Latin translators and commentators had already conflated arithmetic and algebra (e.g. Barlaam's commentary on Book II; but if we follow Netz 2004, we can retrace this trend to Eutocius' sixth century commentary on Archimedes), but Bombelli's reduction of Euclid's binomials to concrete sums of roots or number and root is closest to what we observe in Tartaglia's Italian translation, which is based on an integration of Campanus' and Zamberti's translations, but which takes a further step towards an arithmetic reading of Euclidean geometry (Malet 2006).

What this list of sources suppresses under a vague reference to some that came between Fibonacci and Pacioli, however, is 200 years of algebraic production carried out in the context of Italian abbacus schools. Whilst Bombelli does not consider abbacist authors worthy of being mentioned by name, he lies squarely in the path of their tradition. First, in terms of organization of knowledge, Bombelli is probably the last innovative and important author to organise his work, like leading abbacus masters (Dardi, most notably), as a long list of cases of polynomial equations, followed by a list of recreational and commercial problems (many of these problems were later replaced by Diophantine problems in the print edition, but as Jayawardene has already showed, these problems are traceable to the vernacular Italian tradition). Bombelli's terminology too comes from the Italian tradition, although, again, he tried to break this link by changing his terminology in the print edition, claiming Diophatine inspiration. Finally, Bombelli's techniques and conceptual distinctions mostly trace back to the Italian vernacular tradition. Bombelli's attempt to obscure these links fits well with the humanist attempt to erase more recent traditions in favour of reconstructing a Greek

[7] "per levare finalmente ogni impedimento alli speculativi e vaghi di questa scientia e togliere ogni scusa a' vili et inetti, mi son posto nell'animo di volere a perfetto ordine ridurla" (Bombelli 1966, 8).

affiliation, but does not change the fact that Bombelli is, in a sense, one of the last proponents of the abbacist tradition.

1.4 Bombelli's terms and notation

The name of the unknown in Italian abbacus algebra is usually *cosa* (thing), and occasionally *quantità* (quantity). Bombelli writes in his manuscript that he prefers the latter, but uses the former, because that is the received practice. In the print edition, following Diophantine inspiration, Bombelli changes the name of the unknown to *Tanto* (so much, such; I retain the Italian terms in this article in order to maintain a distance from modern practice). The second power is called *Censo* in the manuscript (Bombelli prefers *quadrata*, but again follows received practice) and *potenza* in the print edition. *Cubo* is the third power, and *Censo di Censo* or *potenza di potenza* is the fourth. Higher powers are treated and named as well, but are not relevant for this article.[8]

Bombelli's manuscript notation for powers of the unknown is a semicircle with the ordinal number of the power over the coefficient. The print edition reproduces this notation in diagrams of calculations, but, due to the limitations of print, places the semicircle next to the coefficient in the running text. So a contemporary $5x^2$ would be rendered as $5\ \overset{2}{\smile}$ in print and as $\overset{\overset{2}{\smile}}{5}$ in the manuscript. The manuscript Book II accompanies plain numbers with a $\overset{0}{\smile}$ above them, but this is almost entirely discarded in the print edition and in the other manuscript books.

The print edition also replaces the elegant script shorthand for square and cubic roots with R.q. and R.c., respectively. Bombelli uses a combination of underline and parentheses to specify the range of roots. Here too, the limitations of print made a difference (see Fig. 1), and the range of roots was bound between brackets formed by an L and a mirror-image L. Following my own print limitations, I use square brackets instead. The manuscript notation is obviously easier to follow, but after a hundred pages or so, one gets used to the print notation as well (after the first 1,000 pages one can even learn to quickly parse lengthy formulas and long calculations in the much more verbose style of abbacus masters).

Bombelli uses the shorthand *m.* for *meno* (minus) and *p.* for *più* (plus). The modern edition replaces these signs with the contemporary $-$ and $+$, but otherwise respects the notations of the 1572 edition. I follow this practice here. I revert to *m.* and *p.* only when discussing the issue of negative numbers, in order to be extra careful.

2 Undermining 'natural' distinctions

Bombelli, like many of his predecessors, distinguishes quantities (positive integers, roots, unknowns, geometric extensions) according to what he refers to as their *natura*. He describes the rules that govern the arithmetic manipulations of each kind of quantity, and attempts to exhaust the ways of compounding kinds of quantities into

[8] For the benefit of readability I use the modern terms square, cube and fourth power when referring to Bombelli's equations (but I retain the term *censo* in the context of the cossist algebra of abbacus masters).

a

e questo sarà eguale a 1 $\overset{2}{\smile}$ + R.c. 4 — R.c. 2, che levato il minor numero si haverà 1 $\overset{2}{\smile}$ eguale a R.q. LR.c. 32 $\overset{1}{\smile}$ — R.c. 16 $\overset{1}{\smile}$⅃ + R.q. LR.c. 16 + R.c. 4 + 2⅃ + R.c. 2 — R.c. 4; che agguagliato, il Tanto valerà R.q. LR.c. $\frac{1}{2}$ — R.c. $\frac{1}{4}$⅃ + R.q. LR.c. $\frac{1}{4}$ — R.c. $\frac{1}{2}$ + R.q. LR.c. 16 + R.c. 4 + 2⅃⅃.

b

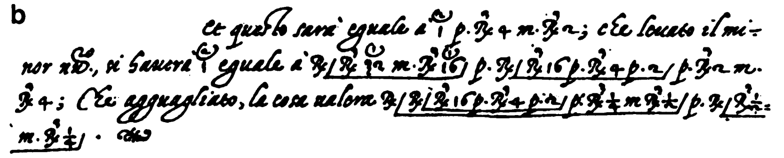

Fig. 1 Bombelli (155?, 92v, 1966, 273): manuscript and print notation

more complex combinations, culminating in the various cases of quadratic, cubic and quartic equations.

But the process I am going to describe in this section is the process of erosion of the distinction between quantities according to their natures, which resulted from economic, arithmetic and algebraic practices. As we will see, whilst 'natural' distinctions lose their footing, Bombelli attempts to articulate them more firmly without actually giving up any of the fruits of their destabilisation. But this is not in the least paradoxical. To understand Renaissance algebra one must recognise the impact of both trends: conserving distinctions and undermining them. What makes these processes non-contradictory is the fact that they operate locally, across textual micro-practices, without ever having to be confronted explicitly as opposites (and when they are confronted, they form hybrids, not conflicts). The frequent interference of these processes is the texture of Bombelli's algebra.

2.1 The natures of quantities

In Bombelli's world each quantity has a "nature" (*natura*) that imposes strict distinctions. The term 'number' (*numero*), for example, refers only to integers larger than 1 (which is not a number, but "works like the numbers").[9] A square root (*Radice quadrata*), on the other hand, is the "side of a non square number".[10] Bombelli clarifies that "the side of 20", that is, "a number, which multiplied by itself would make 20; such is impossible to find, 20 being a non square number: this side would be said to be Root of 20".[11] Therefore, roots are not numbers, and numbers are not roots. This stance radicalises Bombelli's manuscript articulation, which allowed for two kinds of root:

[9] "serve come li numeri" (Bombelli 1966, 11).

[10] "il lato di un numero non quadrato" (Bombelli 1966, 13).

[11] "come sarebbe se si havesse a pigliare il lato di 20, il che non vuol dire altro, che trovare un numero, il quale moltiplicato in se stesso faccia 20; il ch'è impossibile trovare, per essere il 20 numero non quadrato: esso lato si direbbe essere Radice 20" (Bombelli 1966, 13). The term *Lato* is not used in a consistent way throughout the text, and cannot serve as a term for unifying integer and irrational roots; indeed, on page 16 Bombelli writes that 12 has no *lato*. The term *quantità* is closer to (but is not exactly) a term encompassing

integer *Radice discreta* and non integer *Radice sorda* or *irrationale* (Bombelli 155?, 1v). For Cardano the distinction was even more violent: a number is a "true quantity", as opposed to an "irrational" (Cardano 1968, 50). Similarly, Bombelli clarifies that a binomial (*Binomio*) is a sum of two terms, only as long as they cannot be rewritten as "one quantity".[12]

Natura is a term used by Bombelli to distinguish numbers from roots, binomials from simple quantities (as well as different kinds of binomials), negatives from positives, and terms involving powers of unknowns from those that do not (e.g. Bombelli 1966, 18, 63, 65, 282). Cardano further uses the term "nature" to distinguish numbers from the "pure negative", "sophistic negative" and "entirely false"—the latter two referring to roots of negative numbers and the minuses of such roots (Cardano 1968, 220–221). Note that the distinction between quantities according to their natures relates both to well-established quantities (numbers and fractions) as well as to new and suspect constructs (roots, negatives and roots of negatives).

Homogeneity of nature is used as a valid explanation for the possibility to perform arithmetic operations. Indeed, Bombelli explains that integers such as 6 and 2 can be added "for being all of the same nature",[13] but "The subtraction of roots and numbers cannot be done except by way of minus" (e.g. $R.q.18 - 4$) "because the quantities are of different nature"[14] (at the same time, however, $R.q.24$ and $R.q.5$ can only be added with a plus sign, "even though they are similar in nature").[15]

These 'natural' distinctions are not maintained with such analytical rigour by fourteenth and fifteenth century abbacists (indeed, Bombelli's more rigorous articulations seem to be reactions against this analytic weakness), but the presence of these distinctions as principles for organising knowledge is explicit, with or without the actual use of the term 'nature'. The ontology that these distinctions establish seems to be in line with the Aristotelian mode of organising knowledge in the late Middle Ages.[16] I do not claim that abbacist mathematics was ever properly Aristotelian or systematically ontologised, but the distinction between kinds of quantities is not some superfluous frill encumbering abbacus and Renaissance algebra. As will be argued below, it is productive of algebraic development and constitutive of the organisation of abbacus and Renaissance algebraic knowledge. But this same algebra depends on the fluidity of quantities and on their convertibility just as it does on their distinctions; abbacist and Renaissance algebra is a process taking place between practices of rigid taxonomy and fluid conversion. The fact that these practices are contradictory did not preclude their productive implementation in unison, but did entail their ongoing mutual subversion. The rest of this section will describe the various ways whereby arithmetic and algebraic practices undermine the distinction between quantities, which lies at their very foundation.

all kinds of magnitudes, including unknown and geometric ones. In this article I will use 'quantity' as a general term for various kinds of numbers and magnitudes.

12 "una quantità sola" (Bombelli 1966, 65).

13 "per essere tutti di una natura" (Bombelli 1966, 63).

14 "Lo Sottrare di Radici, e numeri non si può fare se non per via del meno, per essere quantità di diversa natura" (Bombelli 1966, 24).

15 "se ben sono simili di natura" (Bombelli 1966, 65).

16 On the interaction between mathematics and philosophy in the relevant period see Høyrup (1994, Ch. 5).

2.2 Fluid articulations of quantity

The Renaissance and abbacist world of distinct kinds of quantities took place against a fluid economic practice with quantities (recall that abbacus masters taught practical arithmetic to merchant children). Benedetto's fifteenth century abbacus treatise (1974) opens with an introduction of Florentine money, weights and measures. Six leaves are required to list the various units and their subdivisions. The intricacies of this system abound. We find, for example, that a "barile of wine is larger than that of oil as that of oil is 8 quarti and that of wine is 10, that is the barile of wine is $\frac{1}{4}$ larger than the content of that of oil".[17]

When it comes to money, we learn that a "grossone is worth 20 quatrini, but at present goes for 21",[18] and whilst "the fiorino, to this money there is no fixed value, because sometimes it goes up by a few scudi and sometimes it goes down", there is "an imaginary value that's called golden scudo, which is always stable, so that the fiorino is worth 20 golden scudi and the golden scudo 12 golden denari".[19] Another imaginary monetary unit, the *fiorino di suggello*, even though as imaginary unit it is supposed to be stable, has actually been devalued with respect to the *fiorino largho*, and has mostly fallen out of use. When one exchanges 100 golden *fiorino* for gold one gets "106 or 106$\frac{1}{2}$ or 107 according to the above daily posted prices".[20] Some ten leaves are required to present the various real and imaginary unit systems of various cities and their relative exchange rate intervals, not failing to include how many days old the report of each exchange rate is. The bulk of the treatise is in fact concerned with converting units of money, merchandise and time (interest) into money. Arithmetic and algebraic recreational problems are there, according to the author, mostly "for pleasure".

In this world, where a quarter need not mean one part in four, where one uses the same unit names for different values in different contexts and places, where exchange depends on imaginary units and daily postings, and where rates are not precise but range in intervals—in such fluid world notions of quantity cannot be too rigidly distinguished. This fluidity of quantities also renders them universal enough to absorb non-quantitative entities as well. Many abbacus treatises monetarily quantified persons in partnerships (e.g. Benedetto 1974, Ch. 14; della Francesca 1970, 52; de'Mazzinghi 1967, 32), time in the contexts of labour and interest, and even units of *fatiche*—the effort involved in digging a well (Benedetto 1974, 116).

17 "Il barile da vino è maggiore che quello da olio imperò che quello da olio è 8 quarti et quello da vino è 10, coè è il $\frac{1}{4}$ maggiore quello da vino della tenuta di quello da olio" (Benedetto 1974, 33).

18 "grossone et vale 20 quattrini, ma al presente corre per 21" (Benedetto 1974, 34).

19 "Il f. nonn à a cqueste monete valuta ferma perché quando sale qualche s. et quando scende di prego, al presente vale 6 lb. 3 s. 4 d. À bene un'altra valuta immaginata che si chiama a s. a oro che sta sempre ferma imperò che'l f. vale 20 s. a oro e'l s. a oro 12 d. a oro" (Benedetto 1974, 34).

20 "Coè s'intende che per ogni 100 f. d'oro inn oro si dia 106 o 106$\frac{1}{2}$ o 107 secondo che di per di sono posti e' sopradetti pregi" (Benedetto 1974, 34).

2.3 Economic practice and the natures of quantities

The economic practices of unit conversion and monetary exchange did not just coexist along side the arithmetic practice of distinguishing quantities according to their natures. The cohabitation of such economic and arithmetic practices in abbacus treatises (even if the materials were borrowed by authors from anterior sources) led to presenting and reconstructing these practices as related. Thus, some economic practices helped homogenise different kinds of mathematical quantities despite their supposed difference in nature.

An example for such economic practice is the rule stating that when it comes to finding a bottom line, "we reduce all the above kinds of money to one".[21] This approach is valid not only for money and measurements but also for arithmetic operations between quantities of different natures. In multiplying and dividing roots and numbers, for example, we must reduce all terms to "one nature" (Bombelli 1966, 17, 33), that is, express all quantities as roots of the same order (note the tension between Bombelli's requirement here of reducing integers to the nature of roots and his articulation above of roots of square numbers as not having the nature of roots). As a result products of the form $2\sqrt{3}$ are rare in abbacus and Renaissance mathematics, and are suppressed in favour of the form $\sqrt{4}\sqrt{3} = \sqrt{12}$. Mathematical abbacist ontology sets quantities apart according to their various natures, but bringing calculations to a simple bottom line depends on practicing quantities as convertible in ways that tend to homogenise them. I do not claim that arithmetic conversion practices are the result of economic ones (the former may have originated independently of the latter); I claim that the cohabitation of economic and arithmetic conversion practices in abbacus texts projects the fluidity of economic entities onto arithmetic ones, and undermines the practice of distinguishing quantities according to a rigid taxonomy.

To establish a clearer link between economic homognisation of units and arithmetic homogenisation of kinds of quantities, consider the following oddity. Piero della Francesca requires reducing numbers to the same *natura*—here, a common denominator—for performing addition, subtraction, division and multiplication of fractions (della Francesca 1970, 39–41).[22] At least for multiplication this practice is rather awkward.

Piero's motivation for this practice can be derived from observing the way he related economic and arithmetic practices. Let us consider the following example, where Piero asks to calculate the value of 8 ounces of silk, given that the pound is worth 5 *libre* and 3 *soldi*. To solve the problem Piero applies the rule of three: he multiplies the 8 ounces by the 5 *libre* and 3 *soldi* to obtain 41 *libre* and 4 *soldi* (these units are explicitly included in Piero's statement of the multiplied terms and of the product). Now the rule of three requires that we divide by 1 pound, but Piero explains that it "wouldn't be

[21] "Ma quando ànno a saldare la scripture riducono tutte le sopradette monete a una secondo il sopradetto hordine" (Benedetto 1974, 34).

[22] Bombelli, who multiplies and divides fractions by direct and cross multiplication, no longer requires this kind of homogeneity. But when operating with integer and fractional terms (involving powers of unknowns), Bombelli first divides the integer terms by 1 so that both terms become fractions (Bombelli 1966, 168, 176).

good, you'd better reduce to ounces, of which the pound is 12".[23] Unless the weight units are homogenised, the conversion would turn out wrong.

This rather trivial example becomes interesting when we consider how the practice of homogenising units is carried over directly to the next example in Piero's treatise, where he calculates with fractions rather than with subunits. Piero's practice of fraction denominators is literally the same as his practice with units. In the quotation below, ***meççi***, *quarti* and *octavi* (halves, quarters and eighths) are practiced as hybrids between units and fraction denominators. "You know that 44 *bracci* are worth 48 *libre* $\frac{3}{4}$, and want to know what 24 *bracci* $\frac{1}{2}$ is worth. So it is required that where there are ***meççi*** you reduce to ***meççi***", ... which "makes 49" (***meççi***) "now reduce 48 pounds $\frac{3}{4}$ to *quarti* makes 195" (*quarti*). "Multiply 49 by 195 makes 9555, which you have to divide by 44 *bracci*; but first reduce to the nature of the multiplied, which are *octavi*" (products of ***meççi*** and *quarti*), "therefore multiply 8 by 44 makes 352 and this is the divisor".[24]

The practice of converting values and units to the "same nature" in commercial contexts is presented here in full analogy to the practice of reducing to a common denominator in dividing fractions. Since homogenising units is the basis of economic calculations, and since the economic and arithmetic practices are constructed here as analogous, the abbacist practitioner projects the habit of tinkering with units onto the realm of numbers. The abbacist practitioner then becomes so habituated to the practice of tinkering with the natures of numbers for the purpose of homogenisation, that Piero recommends this practice even when it makes no practical sense, as in the above-mentioned case of multiplying fractions. The habitual tinkering with the nature of numbers makes their distinction according to these natures much less sturdy; as a result, the notion of "nature" is degraded from an immutable essential feature to something more like a denomination (Bomobelli's rearticulations of the natures of quantities mentioned above react against such degradation, but come after this degradation had already had its impact on abbacist algebra).

2.4 Economic practice and signs that carry values other than their face value

So far we saw how an economic practice promotes the homogenisation of units, denominators and the "nature"s of quantities. But this tame form of homogenisation spins away into a far more radical form of homogenisation, which pulls the carpet from under the 'natural' distinctions between quantities. The convertibility of units reaches an intensity where it is no longer always clear to which nature or denomination a

[23] "partire per 1 libra nnon e' starìa bene, convente recare ad once che la libra è 12" (della Francesca 1970, 43).

[24] "Tu sai che 44 bracci vaglano 48 Libre $\frac{3}{4}$ e voi sapere quello che vale 24 bracci $\frac{1}{2}$. Adunqua bisogna che dove sono **meççi** tu raduca a **meççi**, cioè 24 e 24 fa 48 et **meçço** ci ài fa 49; hora reca 48 Libre $\frac{3}{4}$ [a] quarti fa 195. Montiplica 49 via 195 fa 9555 li quali ài a partire per 44 bracci; ma prima reduci a la natura del montiplicato, che sono octavi, però montiplica 8 via 44 fa 352 e questo è partitore" (della Francesca 1970, 46). If this is not convincing enough, then Paolo dell'Abbaco draws an explicit analogy between dividing into fractions and converting money (Paolo 1964, 27).

given enumerator belongs. This, in turn, enables practices that view all quantities as potentially homogeneous.

This process is manifest in the following ellipsis. Piero divides 9 times $20\frac{1}{4}$ by 25. "multiply 25 by 4 makes 100, multiply 9 by $20\frac{1}{4}$ makes $\frac{729}{4}$, divide by 100 becomes $7\frac{29}{100}$".[25] Obviously, $\frac{729}{4}$ divided by 100 is not $7\frac{29}{100}$. But this is not an error. Piero practices here the sign 100 as 100 quarters, but the articulation of units is implicit—the enumerator is allowed to stand apart from the denominator, which is supposed to define its nature.[26] Such implicit articulation of the denominator (or, to use Piero's term, *natura*), far more open to interpretation than the changing values of coins and merchandise in the marketplace, makes it difficult to maintain a rigid distinction between the 100 as partaking of the nature of integer and as counter of implicit fractions.

One of the most interesting instances of such practice occurs in the tradition of counterfactual questions. Such questions as "If 4 were the half of 12, what would be the $\frac{1}{3}$ of 15?"[27] may appear perplexing, but are present across the abbacus culture. Their location in Piero's text can help clarify their context. Regardless of their origin, when they are placed, as they are, between questions concerning pricing, conversion and barter,[28] these questions are enabled by the tacit question, 4 of *what* are worth half of 12? The number is not simply standing for itself, and may take values other than its face value.

Indeed, as we will see below, in any calculation an implicit statement of units might be lurking. Not only is the distinction of quantities according to units/natures thus undermined, but even the very value of a number sign may turn our to be other than what it appears to be. The impact of this subversion of the nature and value of number signs is that even Piero, after insisting on reducing fractions to the same nature for multiplication and division, performs these operations by direct and cross multiplication with no further comment (della Francesca 1970, 41–42).[29] Where the determination of units and natures is implicit and deferred, numbers end up being operated on without the preliminary step of homogenisation. As we will see below, this implicit change of units/nature plays a role in turning numbers into parameters and variables as well.

25 "Montiplica 25 per 4 fa 100, montiplica 9 via $20\frac{1}{4}$ fa $\frac{729}{4}$, parti per 100 ne vene $7\frac{29}{100}$" (della Francesca 1970, 265). Similar ellipses occur, for example, in della Francesca (1970, 111) and in Paolo (1964, 26).

26 The resulting calculation lies midway between dividing fractions by reducing to a common denominator (quarters) and the shorthand practice of dividing fractions by cross multiplication. Indeed, by omitting the explicit statement of quarters when dividing by 100, Piero practically operates a cross multiplication of the 25 by the 4 in dividing $\frac{729}{4}$ by 25.

27 "Se 4 fusse la metà de 12, che ser la $\frac{1}{3}$ de 15?" (della Francesca 1970, 48).

28 Counterfactual questions are similarly contextualised in Benedetto (1974, 64) as well.

29 As Jens Høyrup pointed out to me, and as observed in Giusti (1991), Piero's treatise appears to be an uncritical compilation. So it would be more precise to say that the subversion of the nature and value of number signs allowed him (and the culture around him) to bind together practices that approached the multiplication and division of fractions in such very different ways.

2.5 Impracticality and the nature of quantities

The link between evolved commercial activity and algebraic or proto-algebraic practices is not unique to the Italian scene. The Arab and Indian scenes also provide such examples. To the extent that this link went further in the Italian scene than elsewhere, it may be due to the Italians' more advanced banking techniques, and, more importantly, to the fact that Italian writers of abbacus texts were not court intellectuals, but teachers of merchant youth, concerned more with credit and exchange than with high theory.

But the impact of economic practices on the distinctions between natures of quantities did not serve only to homogenise them. In many cases it was precisely economic practice that required setting positive quantities apart from negative one (interpreted as meaningless or as debt), that rendered fractions sometimes irrelevant (when one could not break actual commodities into parts, or when the fractions were too small to matter), and that made irrational roots look suspect outside geometric contexts (what is the square root of money?).[30]

But abbacus arithmetic and algebra were practiced in schools, not in the market place. The weight of recreational problems, which were of no use to merchants, was substantial in many abbacus treatises. Thus, for example, Paolo dell'Abbaco could pose a question about the treasure of a rich person composed indifferently of *bisanti* or *fiorini* (Paolo 1964, 140), Benedetto switched between *gudei* and *mori* to name those who should be tricked into abandoning ship and leave the Christians safely on board (Benedetto 1974, 143), and Jacopo could not care less if his paving question concerned a large room, a *piazza* or a house, and used all in the framework of a single problem (Høyrup 2007, 276). In fact he cared so little, that when he calculated how many houses of given side lengths could fit into a given plot of land, he did not mind that the borders of the plot would be covered with fractional houses (Høyrup 2007, 295–296). Here the distance from actual implementation of arithmetic problems works to homogenise practice with different kinds of quantities.[31]

The resulting practice is best described by the words of the *Trattato di Fioretti* concerning an irrational solution: "although this case does not result in a discrete quantity, I did not change it, because it comes out comfortably enough".[32] Inside the abbacus classroom a root is indeed just as practicable as an integer. The bottom line is that economic practices contributed, as we saw in the previous subsection, to a conception of quantities as convertible, but that this universal convertibility was further

[30] For a comment on the rarity of this last construct in the abbacus culture, and on its more frequent Arabic and Indian counterparts, see Høyrup (2007, 156–157).

[31] This impractical approach was common to most abbacists I studied. A notable exception is Paolo dell'abbaco. This author typically takes care to solve questions in ways that make practical sense. Indeed, even for the problem about the serpent that crawls up a wall during the day and slides down during the night, Paolo complements the standard solution with a solution sensitive to the fact that once the serpent reached the top, there is no longer any need to count the subsequent sliding and crawling (Paolo 1964, 152–154) (such an approach is qualified by Jens Høyrup as amongst those "so rare that they are the ones that should be taken note of" (Høyrup 2007, 92). Nevertheless, even Paolo lets the occasional slip, such as calculating interest for a fraction of a day (Paolo 1964, 63).

[32] "benchè il chaso non vengha in quantità discreta, non l'ò mutato perchè viene assaj chomodamente" (de'Mazzinghi 1967, 28).

enhanced by the impracticality of abbacus recreational problems, which, as we saw in this subsection, helped homogenise even some quantities that practical economics would have insisted on keeping apart.

3 From signs that carry values other than their face values to algebra

This section will show how the processes documented above, namely the erosion of the distinction between the natures of quantities and the practice of number signs as carrying values other than their face values, help account for the integration of algebraic practices into abbacist culture.

But I must hasten to clarify: neither algebraisation nor the destabilisation of the natures and values of number signs is a linear process, and neither 'came first'. These two processes are co-constitutive, and I am not interested here in a 'chicken and egg' question. To acknowledge the non-linearity of the evolution of algebra in the Italian context we should recall that, whilst an early fourteenth century abbacist like Dardi explicitly operated on equations (squaring both sides, dividing both sides by power terms) and only slightly later did de'Mazzinghi explicitly operate on *cosa* and *censo* binomials and on their fractions (de'Mazzinghi 1967, 21, 22, 31, 33), many fifteenth century authors were much less proficient in these techniques, or at least less willing to reproduce them. In fact, the algebra imported into the Italian vernacular was imported as an isolated method set apart from an evolved arithmetic apparatus, which, for what we would today call linear problems, was no less adequate.[33]

The challenge is therefore to understand the expansion and integration of cossist algebra into the abbacist context, leading to the eventual marginalisation of pre-cossist techniques. The initial sources of algebraisation and of the destabilisation of the natures of quantities may be independent, but their taking root is co-constitutive. This taking root is a process of a mostly (but not entirely) reinforcing interaction between the algebraic destabilisation of the natures of quantities and the enabling of algebra through homogenised quantities. And this nonlinear evolution is the reason for the persistence of both kinds of processes in such a late source as Bombelli's *L'algebra*. Even for the last proponent of the Italian abbacist tradition these processes still survive on the surface.

One more methodological note is in order before we can proceed. I will use below the anachronistic terms 'unknown', 'parameter' and 'variable'. The unknown is fixed but its value is deducible only indirectly; the parameter is fixed within a given problem, but may vary between variations of the problem (when solving quadratics, the coeffi-

[33] This evolved apparatus contained not only the abbacists' skilled use of single and double false positioning, but also, for example, Paolo dell'Abbaco's (1964, 96–97) capacity to retrace his way from the rule for summing an arithmetic progression backwards to recovering the length of the progression from its sum, without recourse to cossist or otherwise algebraic formalisations (indeed, the consecutive division and multiplication by 2 in Paolo's text indicate that the solution was formed by retracing one's steps along the rule for summing progressions). A somewhat less reliable indication of advanced pre-cossist practice is Benedetto's solution of a system of two linear equations by forming a linear combination of the equations without recourse to explicit cossist technology (Benedetto 1974, 73). But given the relatively late date of the manuscript, this might be a recasting of a cossist technique in pre-cossist terms.

cients are parameters); the variable is practiced as taking different values within the context of a single statement (e.g. x and y in $y = x^2$, if one thinks of this equation as representing a curve). Here we are considering the interactive emergence of all three. I use the unknown–parameter–variable distinction as a narrative instrument, but I hope this section makes clear that this distinction cannot explain generative processes, only organise them post-hoc. I am interested here in the emergence of practices of reading and writing signs as standing for an underdetermined set of possible values. I am not interested in the emergence of the *concepts* of unknown, parameter and variable (concepts tend to come too late, and capture too little). We are describing here processes of becoming taking place between the poles of parameter, unknown and variable—poles that only later would be differentiated into distinct entities.

3.1 How signs that carry values other than their face values become unknowns and parameters

Practicing number signs as carrying values other than their face values is obviously related to the technique known as 'false positioning'—guessing a wrong answer, and then rescaling it to adapt to the requirements of the problem. Of course, false positioning has a genealogy separate from that of cossist methods, but it is interesting to point out the moments where one flows into the other in abbacist texts.

Piero della Francesca provides us with an example. He asks for two numbers, the square of one is 5 times the square of the other, such that the sum of their squares together with their product is 400. The solution starts like a false positioning: "Taking that the first is 1, the other should be root of 5; multiply 1 by itself makes 1, and multiply root of 5 by itself makes 5, which makes the one 5 times the other. And added together they are 6. Now 1 times root of 5 makes root of 5, put on top of 6 makes 6 and root of 5, and you want 400".[34] As this is a homogeneous problem, Piero could have gone on with (a quadratic analogue of) false positioning: divide 400 by 6 plus root of 5, take the root, and multiply the result by the original false positions 1 and root of 5. That is, for example, what Paolo dell'abbaco does in a similar situation (Paolo 1964, 120).

But even though so far the rhetoric comes from the tradition of false positioning, here is how Piero continues: "And so take that 1 be $\bar{1}$ and the other root of $\overset{\square}{5}$" (the bar is shorthand for *cosa* and the square for *censo*).[35] The problem is then turned into a quadratic equation and solved algebraically. The unknown takes here the place of the false position.

I emphasise: this is not how the unknown entered the Italian scene. The Italians did not invent the unknown by building on their false positioning techniques. The Italians got the cossist unknown from the Arabs through Latin and vernacular mediators.

[34] "Poniamo che il primo numero fusse 1, l'altro convene che sia radici de 5; montiplica 1 in sè fa 1 et montiplica radici de 5 in sè fa 5, che fa 5 tanto l'uno de l'altro. Et gionti insiemi sono 6. Hora 1 via radici de 5 fa radici de 5, poni sopra 6 fa 6 e radici de 5 e tu voi 400" (della Francesca 1970, 263).

[35] "Et però poni che 1 sia $\bar{1}$ e l'altro radici de $\overset{\square}{5}$" (della Francesca 1970, 263).

But this false-positioning-becoming-*cosa* is an event that enables the integration of abbacist algebra, and the eventual marginalisation of false positioning.

This process is a long, slow process, that goes back to earlier authors. de'Mazzinghi, for example, asks to find two numbers, whose squares sum to 100, and their product is 5 less than the square of their difference (de'Mazzinghi 1967, 30). The solution starts with an intermediary problem, where the sum of the numbers (10) is given instead of the sum of their squares. The sought numbers are modelled as 5 + *chosa* and 5 − *chosa*. Only after this is dealt with, does de'Mazzinghi return to the original question, and solves it by modelling the numbers as *chosa* plus and minus the root of some *quantità*. The number 5 was a placeholder for what is to become an unknown, when the complexity of the problem increases.

We saw above Piero's superscript bar and square signs for *cosa* and *censi*. To appreciate their place as hinges between false positioning and cossist algebra one should consider that practically any variation of their presence or absence can be found in the text. When a *cosa* is meant, sometimes there is only a bar, sometimes only the word *cosa*, sometimes both, and indeed, sometimes neither. The bar is sometimes even carried with the number where the *cosa* is no longer involved (della Francesca 1970, 92, 93, 95, 97).

We need not reconstruct these inconsistencies as errors. They are too widespread to be just that. It is more accurate to say that Piero did not practice *cosa* signs as rigidly as Bombelli later would. Piero's practice of *cosa* signs generates ambiguous positions between numbers as coefficients of unknowns, as counting some not necessarily explicit units or objects, and as false positions (note in this context the use of $\overset{c}{1}$ for a horse, where the *c* is short for *cavallo*, and the value of the horse is a sought after unknown (della Francesca 1970, 101). In Piero's practice the number and *cosa* signs are still becoming unknowns. They not simply '*are*'.

This becoming-unknown enables the process that led to Renaissance algebra. But it can also lead to blunders. For example, the equation $\overset{\square}{9}$ and $\bar{4}$ equals 44 is reduced into $\overset{\square}{1}$ and $\bar{4}/9$ *di censo* equals $4\frac{8}{9}$ (della Francesca 1970, 114). In the normalisation process the sign *censo* was carried over with the 9 into the middle term, where, from a contemporary point of view, it does not belong. This practice might reflect a treatment carried over from the practice of monetary denominations into cossist algebra; in Piero's text, *cosa* terms, like monetary denominations, were sometimes carried with their enumerators and sometimes dropped (some explanation for this practice will be suggested when we consider the carrying over of minus terms below). This did not hinder a correct solution, but is likely to have made things more difficult to follow.

But even with Bombelli's much more rigorous use of *cosa* signs, the technique of turning a false positioning into an unknown survives. Bombelli often solves problems in Book III by deriving values that partly solve a problem (that is, satisfy some requirements but not others), and then multiplying the result by an unknown in order to complete the solution of the problem (e.g. problems 90, 93, 98 and many subsequent problems in Book III). This is done partly to avoid the inconvenience of a second unknown, but since Bombelli did have and use notation for two unknowns, it cannot be just that. The fact is that for Bombelli, as a proponent of the abbacist tradition, a number carries with it the history of not simply standing just for itself. That a number is the coefficient of an implicit unknown need not be explicitly stated from the outset; this possibility is always lurking, and may emerge in due course. The numbers are not what they seem.

To strengthen this last claim, note that Bombelli sometimes practices numbers as carrying values other than their face values independently of algebraic terminology. Consider Problem 113 of the third book of *L'algebra*, which asks for three rational numbers, the product of any two plus 24 makes a square number. Bombelli begins by guessing arbitrarily, but when the third number is to be derived in accordance with the previous arbitrary choices, the result fails to be rational. He then looks at the resulting numbers, and retraces their relations to his initial arbitrary choices. This allows Bombelli to reconstruct conditions on the initial choices, which would guarantee the rationality of the final result. This practice turns the arbitrarily chosen numbers (36 and 64) into parameters to be revised, tentatively taking the place of the 'correct' numbers that will replace them—and all that without resorting to any algebraic symbolism or even a properly algebraic rhetoric. Such parametric practice with numbers runs throughout Book III, and only occasionally leads to explicit algebraic reconstructions. Recall: numbers are not what they seem.

This has to do with Bombelli's diophantine sources, but is just as dependent on the tradition of counterfactual problems, mentioned above, where numbers are practiced as if their values were other than their face values ("if 4 were half of 12"...). Whilst it is true that in Bombelli's text counterfactual questions are not brought up as such, one can find him making such statement as "and if the $7\frac{2}{5}$ were equal or greater than 12", (some operation) "would be impossible".[36]

Another manner in which numbers came to represent a general magnitude, rather than their concrete face values, is expressed in the following odd practice, adopted by as bright an algebraic mind as de'Mazzinghi's. This is the practice of refraining from simplification. de'Mazzinghi explicitly instructs not to reduce a quotient where a *cosa* and the root of a *censo* could cancel out, and explicitly insists on carrying the cubic root of 64 through a solution without replacing it by 4, making the solution appear terribly complicated, and deferring to the very end its eventual simplification to 2 (de'Mazzinghi 1967, 46–47, 52).

At first sight these practices look like a reactionary version of sticking to the natures of quantities (a root should remain a root, premature simplification would be 'unnatural'). Jens Høyrup, when he encountered these practices in the texts of Jacopo and Dardi, suspected that they were a sign of poor understanding and uncritical borrowing (Høyrup 2007, 175, 179). But even if some abbacists did borrow badly, I would like to offer a more charitable explanation for this practice. It is possible that maintaining the unsimplified form of quantities was a conscious or an unconscious tool for developing algebraic understanding. By refusing to simplify the cubic root of 64 one maintains the complicated algebraic form of the solution, which would characterise 'typical' data, rather than the simple integer result that follows from tailor made data. Since it is not reduced, the term 'cubic root of 64' can be followed through the proof without being

36 "et se il $7\frac{2}{5}$ fusse stato pari o maggiore di 12 tale divisione era impossibile" (Book IV, §81).

a

$$
\begin{array}{r}
3\ \ 2 \\
3\ \ 2 \\ \hline
6 \cdot 4 \\
9 \cdot 6 \\ \hline
9 \cdot 12 \cdot 4 \\
0\ \ 3 \cdot 2 \\ \hline
18 \cdot 24 \cdot 8 \\
27 \cdot 36 \cdot 12 \\ \hline
27 \cdot 54 \cdot 36 \cdot 8 \\
3 \cdot 2 \\ \hline
54 \cdot 108 \cdot 72 \cdot 16 \\
81 \cdot 162 \cdot 108 \cdot 24 \\ \hline
81 \cdot 216 \cdot 216 \cdot 96 \cdot 16 \\
3 \cdot 2 \\ \hline
162 \cdot 432 \cdot 432 \cdot 192 \cdot 32 \\
243 \cdot 648 \cdot 648 \cdot 288 \cdot 48 \\ \hline
243 \cdot 810 \cdot 1080 \cdot 720 \cdot 240 \cdot 32
\end{array}
$$

b

$$
\begin{array}{l}
1\overset{1}{\smile} + 2 \\
1\overset{1}{\smile} + 2 \\ \hline
1\overset{2}{\smile} + 4\overset{1}{\smile} + 4 \\
1\overset{2}{\smile} + 4\overset{1}{\smile} + 4 \\ \hline
1\overset{4}{\smile} + 8\overset{3}{\smile} + 24\overset{2}{\smile} + 32\overset{1}{\smile} + 16 \\
1\overset{1}{\smile} + 2 \\ \hline
1\overset{5}{\smile} + 10\overset{4}{\smile} + 40\overset{3}{\smile} + 80\overset{2}{\smile} + 80\overset{1}{\smile} + 32
\end{array}
$$

Fig. 2 Reproduction of Bombelli (155?, 42r, 1966, 61): number and unknown as parameters

absorbed into other numerical terms, and assumes the role of a parameter. I suspect that de'Mazzinghi was such an impressive algebraist partly because he insisted on retaining the forms of algebraic terms in the context of practices that did not have access to explicit parametric expressions such as today's $-\frac{b}{2} + \sqrt{\frac{b^2}{4} - c}$.[37]

In Bombelli's text, an impressive instance of a number becoming parameter occurs when he attempts to explain in a general way the derivation of what we today call binomial coefficients (which he needs for his method of extracting higher roots). In the manuscript Bombelli derives binomial coefficients by raising $30+2$ to the fifth power. In order to make the coefficients emerge, he does not use the usual multiplication procedure, but a clever variation that groups separately products of the anachronised form $30^i 2^{5-i}$ for each i between 0 and 5 (see Fig. 2). It is only in the print edition that 30 is replaced by the *Tanto* sign $\overset{1}{\smile}$. We clearly see that the practice of a number as *any* number, or as a formal entity, does not depend entirely on algebraic notation, and that Bombelli's unknown is not just an unknown, but also a parameter (in this example there is no question of recovering the value of the *Tanto*). The number and the unknown can serve as parameters and formal terms *avant la lettre*.

37 As for Dardi, his explicit comment that he treats "discrete roots as if they were indiscrete" ("intendendo di queste R discrete chome s'elle fusseno indiscrete" (Dardi 2001, 62)), and the eventual extraction of the root, show that his (or his source's) treatment had the pedagogical purpose of teaching the general practice through a special case, maintaining both the 'general' appearance of the solution and the ability to eventually simplify and verify it; indeed, despite Høyrup's contrary comments, Dardi occasionally does verify results following calculations with discrete roots (Dardi 2001, e.g. 40, 46).

But there are other, different contexts, where numbers come to be practiced as formal terms. For instance, one of the equations that Bombelli solves leads to a rather long expression that involves the term $R.c.\left[2 \pm R.q.\frac{107}{27}\right]$. In order to verify that his solution is correct, Bombelli substitutes this solution into the original equation. The resulting expression is a monstrous applications of all sorts of arithmetic operations (multiplication, squaring and additions) to $R.c.\left[2 \pm R.q.\frac{107}{27}\right]$. But instead of calculating the result of these arithmetic manipulations, Bombelli's text simply states that they are applied to $R.c.\left[2 \pm R.q.\frac{107}{27}\right]$, as if this term were an unknown (Bombelli 1966, 256). Eventually these complicated manipulated terms cancel out, and Bombelli's solution of the equation is proven correct. But for our purpose it is important to observe that a specific known number can be subjected to practices carried over from the treatment of unknowns. This further binds numbers and unknowns in a manner that could anachronistically be interpreted as a first step towards an abstract formal algebra. Given this practice, the formal treatment of mathematical signs, such as roots of negative numbers, as if they were unknowns that may eventually cancel out, appears much more plausible.

But I believe that the most impressive example of practicing a mathematical sign as carrying a value other than its face value occurs where even a sign standing for a non-existent quantity is subjected to arithmetic manipulation. This happens, for instance, where Bombelli considers the equation of $1 \overset{3}{\smile} +165$ to $9 \overset{2}{\smile} +9 \overset{1}{\smile}$, which cannot be solved "because that which one seeks is either impossible or the derivation of the equation was badly done".[38] Such equations are solved by applying a linear change of variable to derive a simpler auxiliary cubic equation. In this case too, Bombelli derives the auxiliary equation. Knowing full well that this auxiliary equation does not have a positive solution, he writes: "the value of the *Tanto*" (of the auxiliary equation) "being found (if it could be), if we would add 3"... "the sum would be the value of the *Tanto*" of the original equation, namely an equation already qualified as "impossible".[39] We see here that even non-existing quantities could be other than they actually are(not). With this practice in one's arsenal, the hypothetic practice of roots of negative numbers may seem not all that startling.

3.2 Becoming variable

Things can get tricky where the same sign is used for different unknowns in a single line of text. For example, in solving a cubic equation Bombelli reaches the point where the cube of $\overset{1}{\smile} -2$ equals $12 \overset{1}{\smile} +120$. He then conducts what we would today call a change of variable. He explains that the cube of $\overset{1}{\smile} -2$ "could be said to be $1 \overset{3}{\smile}$, such that the number that composes it is 2 less than it was before. And this cube is equal $12 \overset{1}{\smile} +120$. But because the $12 \overset{1}{\smile}$ are worth 2 less each than they were worth before, it is necessary that whatever one takes from the Tanti, one gives it to the number, so

[38] "perchè quello che si cerca o è cosa impossibile overo fu fatta male la positione" (Bombelli 1966, 261).

[39] "che trovata la valuta del Tanto (potendo), se li aggiongerà 3, che valeva più avanti la trasmutatione e la somma sarà la valuta del Tanto avanti detta trasmutatione" (Bombelli 1966, 262).

that adding 24 to 120 one would have $\overset{3}{\smile}$ equals 12 $\overset{1}{\smile}$ +144".[40] In a somewhat more modern manner we would say that if $\overset{1}{\smile}$ −2 is replaced by $\overset{1}{\smile}$ on the left hand side, then $\overset{1}{\smile}$ on the right hand side must be replaced by $\overset{1}{\smile}$ +2, turning 12 $\overset{1}{\smile}$ +120 into 12 $\overset{1}{\smile}$ +144.

What is interesting here is that the *Tanto* means different things in a single sentence. This plurality is partly controlled by the temporalisation of the *Tanto* (referring to it as the *Tanto* before and after the change). But in the quoted sentence the *Tanto* goes from the new value to the old one and back again (if we wish the expressions to make sense), whilst the syntax of the sentence does not quite respect this temporalisation. If we are to respect the mathematical and linguistic practice, we should acknowledge that the *Tanto* in the middle term 12 $\overset{1}{\smile}$ means more than one thing (the old *Tanto* in referring back to 12 $\overset{1}{\smile}$ +120, the new *Tanto* in being worth 2 less than before). The unknown does not just stand for itself. Rather than a single unknown value, or two different unknowns under the same sign, what we have here is a motion of becoming. In the middle of the sentence a single unknown sign is changing its meaning as we read. When we get there it still has one value, but when we move on, it already has another. The unknown is becoming a variable. This is indeed confusing for the uninitiated, but this confusion survives in modern day change of variables that uses x both in the substituted and the substituting variable ('substitute $x + 2$ for x'), as any calculus teacher can testify.

Another expression of the process of turning unknowns into variables occurs in indeterminate problems that require integer solutions (Diophantine problems). This occasionally gives rise not to an equation, but to quadratic inequalities (e.g. problems 110, 200, 220–222 of book III). In this context unknowns range, rather than stand fixed. But even though in the treatment of quadratic inequalities there are explicit ranges, and the arbitrariness of choice is obvious, Bombelli includes no explicit talk of the motion of the value of the unknown. If we want proper variables, we have to pass through Cardano.

When Cardano explains why some cubic equations must have solutions, he uses language such as "decreasing the estimated value of the thing" or "increasing the cube and thus increasing the value of the thing" (Cardano 1968, 22).[41] The analysis is vague, and the conclusions sometimes inaccurate, but there are explicit calculations of the range of values that a power term expression can take, and a motion of decreasing or increasing coefficients (Cardano 1968, 70, 134, 225, 248). Such language is hard to track in Bombelli's text, but his unexplained observations concerning the existence of solutions of some cubic and quartic equations together with his excellent command of Cardano's work do testify to a tacit practice of unknowns as variables.

[40] "si potrà dire essere 1 $\overset{3}{\smile}$ che il numero che lo compone è 2 meno che non era prima. E questo cubo è eguale a 12 $\overset{1}{\smile}$ +120. Ma perchè li 12 $\overset{1}{\smile}$ vagliono 2 men l'uno, che non valevano prima, bisogna quello che si toglie loro nelli Tanti darglielo nel numero, che aggionto a 120 24 si haverà 1 $\overset{3}{\smile}$ eguale a 12 $\overset{1}{\smile}$ +144" (Bombelli 1966, 235).

[41] The quoted translation uses the anachronistic x. The original has the Latin *res* for the Italian *cosa*. To avoid anachronism I replace x by 'thing'.

We can, with some risk, track in Bombelli's *L'algebra* at least one example of treating the unknown as a variable. When considering equations of the form 'fourth power, cube, square and number equal *Tanti*', Bombelli provides a sufficient condition for the non-existence of a positive solution. If the equation's number is greater or equal than the coefficient of the *Tanto*, and if the sum of the coefficients of the fourth, third and second powers is greater than the coefficient of the *Tanto*, then the equation has no positive solution. This rule comes with no explanation, but the following reconstruction appears plausible (for the benefit of the reader, I bring it in anachronistic notation). Let the equation be $ax^4 + bx^3 + cx^2 + d = ex$. Bombelli's condition is $d \geq e$ and $a + b + c > e$. If $x \geq 1$, then the left hand side is greater than $(a + b + c)x$, which, according to the condition, is greater than ex. If $x < 1$, then the left hand side is greater than d, which, according to the condition, is greater than e, which is greater than ex. Either way an equality cannot hold. If this was indeed Bombelli's implicit argument, we can see here an analytic practice of equations as functions of a dynamic variable.

3.3 Closing the circle: how algebra helped undermine the distinction between quantities according to their natures

As stated above, neither algebraic practice nor the destabilization of the natures and values of number signs 'came first'. Their emergence depends on (mostly) mutually reinforcing interaction. Since above I focused on how the instability of number signs helped promote the homogenisation of quantities that algebraic practice depends on, here I will show how algebraic practices helped further destabilise the distinction between quantities according to their natures.

It appears that the distinction between the natures of quantities always had a context dependent dimension. Piero della Francesca explains: "When numbers are multiplied by themselves, then these numbers are called roots and these products are called squares or *censi*. And when the numbers are not in relation to roots or squares, then they are called simple numbers. So according to this definition every number is sometimes root, or square, or simple number".[42] Of course this is a much looser distinction than the one presented by Bombelli. But in Bombelli's evolved algebraic context, distinguishing quantities according to their natures has an even more pronounced context-sensitive streak.

For example, when Bombelli asks to equate $\overset{1}{\smile}$ +12 to $R.q.300$, he states that "in that operation the R.q. are like number, not having the sign of a power".[43] In the manuscript, where quantities involving integers and roots, but not a power of the *cosa*, are all crowned with a homogenising $\overset{0}{\smile}$, Bombelli further adds that "they're

42 "Quando i numeri se montiplicano in sè, alora quelli numeri se dicono radici et quelli producti se dicono quadrati o vero censi. Et quando e' numeri non ànno respecto a le radici o vero quadrati, alora se dicono numeri semplici. Adunqua secondo questa definitione omne numero è alcuna volta radici, o vero quadrato, o vero numero semplici" (della Francesca 1970, 75).

43 "in questa operatione le R.q. sono come numero non havendo segno di dignità" (Bombelli 1966, 184).

rendered with the sign of the number".[44] Relative to power terms, roots and numbers gain homogeneity.

The algebraic context also serves to relativise the positive/negative distinction. When discussing the roots of one kind of cubic equation, Bombelli brings up a transformation that turns a negative root of one equation into a positive root of another. The following sentence illuminates the transition: "one will have $1\ \overset{3}{\smile} +16$ equals $12\ \overset{1}{\smile}$, which equated, the *Tanto* will equal 2, and this is minus, as in $1\ \overset{3}{\smile}$ equals $12\ \overset{1}{\smile} +16$ the true value is 4 and the false is -2".[45] The contracted Italian syntax is perfect for expressing the intermediary position of 2 between the two equations and the relativity of its sign: it is the 'same' 2 which is positive with respect to the former equation and negative with respect to its latter transformation (but this does not contradict, in the context of this practice, the "false"ness of the negative solution).[46]

In fact, algebraic technique makes it difficult for Bombelli even to adhere to the distinction between equations according to their 'normal' forms (where all coefficients are positive, and each power of the unknown appears only on one side of the equation). When applying Cardano's trick for solving irreducible cubic equations, for example, one places the number term of the equation on the same side as the *Tanto*, which generates equations where the number term is negative, but which Bombelli nevertheless practices as regular (Bombelli 1966, e.g. 267). In the context of completing quartics to squares the preparatory regrouping of terms led to regularising forms, where one side consisted of a single negative number (Bombelli 1966, e.g. 280). Further non-standard equations, such as power terms and number equal zero, arose as auxiliary forms of standard ones (Bombelli 1966, 254). In one such case Bombelli explains that "the number changes and one has $\overset{3}{\smile}$ and $\overset{1}{\smile}$ equal minus number, so the Tanto equals minus, and serves as such".[47]

A similar indeterminacy of sign occurs in the context of dividing by a negative term. First, Bombelli declares that he "has never encountered that division by minus could occur".[48] Embarrassingly enough, some pages later occurs a division by -41. Bombelli states that although it did happen, it can always be avoided. However, in the context of power terms, Bombelli explicitly acknowledges the possibility of dividing by a minus (Bombelli 1966, 101, 162). We see here that as one loses control over the value of a term through the intervention of unknowns, the sign becomes a less stable ground for defining the natures of numbers and regimenting arithmetic operations.

The drift continues when subtracting power terms. In the manuscript Bombelli states that he will call the subtracted quantity "minor" and the quantity from which it

44 "e si vede, che si fa loro il segno del numero" (Bombelli 155?, 61r).

45 "si haverà $1\ \overset{3}{\smile} +16$ eguale a $12\ \overset{1}{\smile}$, che agguagliato il Tanto valerà 2, e questo è meno, però di $1\ \overset{3}{\smile}$ eguale a $12\ \overset{1}{\smile} +16$ la vera valuta è 4 e la falsa è -2" (Bombelli 1966, 246).

46 See Cardano (1968, 154) for a similar effect.

47 "l'agguagliamento viene sempre a $\overset{3}{\smile}$ e $\overset{1}{\smile}$ eguale a numero, overo a $\overset{3}{\smile}$, $\overset{1}{\smile}$ e numero eguale a 0, che in quel caso si muta il numero e si ha $\overset{3}{\smile}$ e $\overset{1}{\smile}$ eguale a - numero, che il Tanto vale meno, che tanto serve" (Bombelli 1966, 292).

48 "io (per quanto ho operato) mai ho conosciuto, che possa accadere partire per meno" (Bombelli 1966, 63).

is subtracted "major", which turns major and minor from substantial to positional terms (Bombelli 155?, 58v). This statement did not find its way into the print edition, but the terminology itself did (Bombelli 1966, e.g. 180). Similarly, Bombelli's directive to order summed elements in descending order breaks down with no explicit comment as power terms come into the picture (Bombelli 1966, 18, 163, 167). Furthermore, two distinct methods for summing and subtracting roots collapse into a single method in the context of roots of negative numbers, as one can not clearly decide which root is larger (Bombelli 1966, 19, 21, 147).

Things become even more obscure when extracting roots of complicated terms. Bombelli's techniques for finding the cubic root of a binomial involve some guesswork. It therefore becomes less than obvious how to determine the nature of cubic roots of binomials. For example, when equating the innocent looking $1 \overset{4}{\smile} +16 \overset{1}{\smile}$ to 48, Bombelli derives the monstrous solution $R.q.$[of $R.q.[R.c.[4608 + R.q.4456448] + R.c.[4608 - R.q.4456448] + 16] + R.c.[R.q.68 + 2] - R.c.[R.q.68 - 2]]$. He then states that he assumes that "the binomial and the trinomial should have side" (root), "because the *Tanto* has to be worth 2. But such side as yet I could not find".[49] The correct determination of the nature of the cubic roots of the binomial and the trinomial depends on indirect evidence, and remains undecided, in the hope that (as Cardano had put it in a slightly different context) "these irrational quantities serving as numbers can be reduced to numbers" (Cardano 1968, 246). The nature of quantities becomes here a deferred object.

Given all those practices, where the nature of quantities becomes dependent on context and on future determination, one can be much more open to tentative work with suspect quantities of obscure nature. So much so, apparently, that Bombelli was led to tolerate roots of negative numbers in solving cubic equations, hoping that their combinations will eventually turn out to be of the nature of numbers, as they occasionally did.

4 The benefits of distinguishing quantities according to their natures

According to a standard historical narrative, setting continuous (geometric) quantities apart from discrete (arithmetic) quantities, and maintaining distinctions between integers, fractions, irrationals and negatives inhibited the evolution of mathematics. This narrative is not entirely unfounded, but does depend on selective vision. In order to prevent the impression that distinguishing quantities according to their natures had a strictly inhibiting effect on abbacist and Renaissance algebra, and that the history of algebra is the history of an emancipation from this inhibition, I point out in this section some of the productive aspects of sticking to 'natural' distinctions in the context of abbacist and Renaissance algebra.

First, this distinction motivated attempts to simplify complicated quantities, an effort that Cardano qualified as "the greatest thing to which the perfection of human intellect, or rather, human imagination, can come" (Cardano 1968, 246). Pursuing the

[49] "tengo che il Binomio et il Trinomio habbia lato, perchè il Tanto habbia da valere 2. Ma tal lato per ancora non ho potuto ritrovare" (Bombelli 1966, 271).

nature of quantities, researching such questions as whether a binomial can be reduced to a 'single' quantity, or whether the root of a binomial can be expressed as another binomial, made many algebraic observations possible, including, of course, Bombelli's solution of cubic equations by reducing sums of cubic roots of binomials with roots of negatives to plain integers.

Indeed, the attempt to reduce to a single nature the products of arithmetic combinations of quantities of different natures sometimes appears to be taken too far. de'Mazzinghi, for example, following Paolo's directive to "always in multiplication make it similar against similar"[50] (that is, write products of number and root as products of roots), expresses the product of the root of 5 and a *cosa* as the root of the product of 5 and a *censo*. This obfuscates such expressions as *cosa* plus root of 5 *censi*, and obscures the fact that this sum is in fact a linear term (de'Mazzinghi 1967, 41). The problem is further exacerbated when abbacists such as Dardi, de'Mazzinghi and Piero often square equations whose coefficients include roots so as to turn all coefficients into integers, even at the cost of raising the degree of the equation (e.g. de'Mazzinghi 1967, 40; della Francesca 1970, 120, 164). This practice seems to be linked to the above practice of homogenisation, which discourages the mixing of roots and numbers.[51] But this practice of squaring equations in order to get rid of roots in coefficients had a positive side effect: it helped algebraists recognise quartic equations where both sides are, or can be completed to, squares. Indeed, Bombelli demonstrates the capacity to easily gage opportunities for completing quartic and even eighth degree equations to squares (e.g. problem 255 of Book III). This skill must have been a prerequisite for eventually coming up with Ferrari's solution of the quartic by, precisely, finding out how to transform it into an equation between squared quadratic terms.

But the most impressive results of taking seriously the distinction between quantities according to their natures in the later stages of Renaissance algebra has to do with researching the natures of solutions of equations. Cardano explicitly researched the relations between problems and the possible forms (or, for him, "nature") of solutions. He derived non-trivial knowledge of which equations can yield which forms of solutions, and of relations between coefficients and solutions (Cardano 1968, e.g. 48–49, 169, 176).

Bombelli also follows up explicitly on this issue. Whilst analysing the solutions of cubics, he explains, amongst other similar observations, why the solution of an equation of the form 'cube equals *Tanti* and number' cannot be a binomial with the number greater than the root (Bombelli 1966, 245). Indeed, Bombelli asks whether $2 + R.q.2$ could solve such an equation. Raising this number to the third power yields $20 + R.q.392$. Now one considers separately the number and the root. To balance the root on the right hand side one has to multiply the prospective solution $2 + R.q.2$ by 14. But then one gets $28 + R.q.392$ on the rights hand side, and adding a number cannot make the right hand side equal to the $20 + R.q.392$ on the left. Bombelli then states, without argument, that this holds generally for this kind of equation.[52] The point is

[50] "senpre nel multjprichare fae che xxia simjlj chontro a xximjlj" (Paolo 1964, 63).

[51] A more sinister interpretation would be that abbacus masters attempted to increase the complexity of problems artificially in order to impress clients (Høyrup 2009a).

[52] In fact, this kind of reasoning goes back to Fibonacci's *Flos* (Fibonacci 1862, 227–234).

that this argument depends on a separate treatment of number and root, ignoring the possibility that roots may equal numbers.[53] Such arguments served Bombelli to rule out various kinds of quantities as solutions of cubic equations, which eventually led him to state "that (according to my judgement) I take it to be impossible to find such a general rule",[54] that is, a general rule for solving cubics that bypasses the use of roots of negative numbers.

This manoeuvre shows that exploring relations between equations and the natures of their solutions was a method practiced to find rules for solving equations. This in turn confers plausibility on the conjecture that the solution of the cubic equation was discovered by exploring various prospective kinds of solutions, until eventually stumbling on the sum of cubic roots of a binomial and its conjugate.[55] As in Bombelli's example above, one could start by trying to construct a cubic whose solution is $R.c.[2 + R.q.3] + R.c.[2 - R.q.3]$. Raising this term to the third power yields $4 + 3[R.c.[2 + R.q.3] + R.c.[2 - R.q.3]]$. Now consider separately the number and the sum of roots. To balance the sum of roots on the left hand side, one must multiply the prospective solution by 3 on the right. To balance the number 4 on the left one must add 4 to the right hand side. The prospective solution therefore solves the equation 'Cube equals 3 *Tanti* and 4'.

After having considered several such examples, one can try to derive general observations. Keeping to the ontological distinction between the root and the number terms allows to note that the number on the right hand side (the above 4) must be twice the integer under the cubic root (the above 2). From there the way is not long to finding the relation between the coefficient of the *Tanto* and the terms under the cubic root, and to formulating a rule for solving such cubic equations. Note, however, that such a rule remains valid even when any of the roots involved is reducible to a quantity of a 'simpler' nature, undoing the distinction on which the above argument depends.

This derivation of the solution of cubic equations is of course purely conjectural; Cardano's and Bombelli's explicit arguments mentioned above, however, show that this kind of reasoning belonged to the arsenal of sixteenth century Italian algebraists. So even if this suggestion for the route to the initial discovery of the solution of the cubic is false, the analysis of relations between solutions and equations did rely on the distinction of quantities according to their natures, and did contribute to algebraic understanding, as demonstrated by Bombelli's argument above.

[53] I do not claim that the analysis is faulty, given Bombelli's definition of root as a quantity that is not a number, and of a binomial as not reducible to a 'single' quantity. I claim that this distinction helps Bombelli construct good arguments.

[54] "Sì che (quanto al mio giuditio) tengo impossibile ritrovarsi tal regola generale" (Bombelli 1966, 245).

[55] Tartaglia claimed that his discovery was geometric, but he was keen on keeping his methods secret, and his inspiration may have had nothing to do with Cardano's dissected geometric cubes. Indeed, expressions of the form $\sqrt{a+\sqrt{b}} - \sqrt{a-\sqrt{b}}$ figure in Tartaglia's translation of Euclid (Tartaglia 1543, Book X, problems 9,10) and in his solution of bi-quadratic equations (Tartaglia 1560, 10r,13v). Earlier encounters with such expressions may have inspired research into their cubic root analogues by Tartaglia or dal Ferro.

5 Negatives (and their roots)

The discussion of negative numbers in the mediaeval and early modern context is tricky. It is easy to find the term *meno*[56] outside the context of subtraction, but one must also bear in mind that even as late as the nineteenth century negative numbers were held as suspect by some authors. The line between minus as operation (subtraction) and as modifier (changing the *natura* of a number, and so closer to the minus sign in the modern sense) is not always clear, even if one is modest enough to simply attempt a reconstruction of the practice, rather than the concept. Here I will document abbacist and Renaissance practices with *meno*, and attempt to understand their status by appealing to the distinction between quantities according to their natures.

5.1 Away from substraction

Operations with and on isolated *meno* terms can be found rather early in the abbacus tradition. de'Mazzinghi, for example, asks to "divide a *chosa meno* in 2 parts such that one multiplied by 3 and the other by 5 make *meno* 4 *chose*".[57] A somewhat less abstract and more common treatment of an isolated *meno* occurs in such statements as "multiply *meno* root of 2 by *meno* root of 2 makes 2 più".[58] But note that in both cases the *meno* terms are extracted from a binomial, where they are subtracted from a positive term.

However, the fact that *meno* terms are extracted from subtraction operations does not mean that they are reducible to them. When Cardano writes that "to subtract 4 from 12 is the same as to add 4 m̃. to 12" (Cardano 1968, 134),[59] it is clear that 4 m̃. does not unquestionably mean the subtraction of 4, as otherwise this explanation would have been redundant, and adding a *meno* (literally, adding a subtraction, or adding less) would have been a very odd way of putting things. This kind of rhetoric suggests that the *meno* has already gained some modifier status, which Cardano wants to erase and reduce back to that of a subtraction operation. This situation attests to the problematic status of the term *meno*, which we will try to elucidate in this section.

Likewise, a formulation such as "add in m., so to speak" (Cardano 1968, 248) reinforces the irreducibility of adding in *meno* to subtraction, as otherwise explicit subtraction would have been used. This statement arises when Cardano starts with adding for the purpose of solving a problem; then, given a variation of the problem, Cardano needs to replace addition with subtraction. Adding in *meno* is the intermediary figure that Cardano uses to enable a smooth transition and a unified presentation, but the semiotic price is to grant *meno* the role of a modifier. The same difference is

[56] In this section I will use the term *meno* for minus and *più* for plus (or their shorthand notations) when referring directly to abbacus and Renaissance texts, in order to keep a sense of estrangement that may help us avoid jumping to conclusions.

[57] "Ora ti resta a dividere una chosa meno in 2 parti che, l'una multiplichata per 3 e l'altra per 5, faccino meno 4 chose" (de'Mazzinghi 1967, 21).

[58] "montiplica meno radici de 2 via meno radici de 2 fa 2 più" (della Francesca 1970, 80).

[59] I use here Cardano's m̃ for *minus* rather than the English translation's minus sign.

apparent in Bombelli's statement that "to sum p. 16 with m. 8 is as if I had 16, and was indebted 8, so that the debt being paid I would be left with 8 scudi".[60] Again, some rhetorical distance is maintained between adding a *meno* or debt and simply subtracting.

5.2 Accepting and rejecting the *meno*

Before we attempt to further understand the status of this *meno* that is on its way from a subtraction operation to becoming a modifier of the nature of numbers, let us try to understand to what extent *meno* was accepted or rejected in practice. The typical narrative suggests that at least as far as the sixteenth century negative quantities were rejected essentially, but accepted opportunistically, and never served as end result. This description may more or less work for the abbacists, but the situation in Bombelli's and Cardano's work is more intricate.

On the side of endorsement, on top of many negative interim results, we find Bombelli giving $R.q.8-4$ as an example of subtraction with no explanation or qualification. Later, he allows "*meno* $R.c.[R.q.1452-38]$" as a final result of an example (Bombelli 1966, 24, 129). On the other hand, whilst trying to find the cubic root of a complicated term, running into an intermediary subtraction with a negative result, Bombelli has no qualms with the apodeictic statement that "it can't be subtracted".[61] And whilst in the manuscript introduction for Book III Bombelli explains that "where I say subtract such from such, and that which is to be subtracted is bigger, then it does not follow that the rule is not good, because that which remains will be *meno*",[62] Bombelli is almost exclusively interested in positive answers to the questions of Book III, and in underdetermined questions, when a solution fails to be positive, he typically revises the algebraic model or reviews his intermediary numerical choices to guarantee positive final results (e.g. problems 17 and 16 of Book III respectively).

Cardano too has a complex attitude. He explains that positive solutions are more appropriate than negative ones (Cardano 1968, 29), and then goes on to note negative solutions or ignore them without any obvious consistent motivation (Cardano 1968, e.g. 115, 127, 148, 151; in the context of the equations of Book II Bombelli also occasionally includes and occasionally neglects negative solutions). Eventually Cardano presents problems where negative numbers are incorporated even into the data, but in the same context appears the sentence "So progresses arithmetic subtlety the end of which, as is said, is as refined as it is useless" (Cardano 1968, 220). Note that the objection here is pragmatic, not ontological.[63]

[60] "a sommar p. 16 con m. 8 è come se io havessi 16, e ne havessi debito 8, che pagato il debito mi restarebbono scudi 8" (Bombelli 1966, 62–63). I use here p. for *più* and m. for *meno* as in the sixteenth century print edition, rather than the + and − of the modern edition.

[61] "non si può cavare" (Bombelli 1966, 141).

[62] "Et ove dico cavasi il tale del tale, et che sia maggiore quello che si ha da cavare: però non resta, che la regola non sia buona: perchè quello che resterà sarà meno" (Bombelli 1966, 316).

[63] In later texts Cardano's objections to negatives seem more ontological, cf. Tanner (1980, 166).

Roots of negative numbers suffer a similar ambiguity. When Bombelli first presents them in Book I, he does it so casually that he does not even give them a separate section heading. But in the context of quadratics a solution involving a root of a negative number is presented on a par with the other "sophistic method" of simply replacing the impossible subtraction with an addition (Bombelli 1966, 201).[64] A few pages later, in the context of the bi-quadratic without real solutions, solutions involving roots of negatives are not mentioned at all, not even under the mark sophistry. It is simply claimed that "one cannot solve" this case "because it concerns the impossible".[65]

In the cubic case, on the other hand, a solution involving roots of negatives is first endorsed in the manuscript on pragmatic grounds, because a real solution can be derived from it. But then another solution with roots of negative numbers is included, even though Bombelli cannot rewrite it as a real solution (Bombelli 155?, 72v, 76v). In the print edition solutions of cubic equations involving roots of negatives are further justified by the fact that Bombelli "found their demonstration in lines", that is, he constructed a geometric model of the equation and its solution.[66] Nevertheless, no argument is included which would allow Bombelli to justify the claim that his expressions involving roots of negatives are actually related to the geometrically constructed solution (more on that in Wagner (forthcoming)). Later, in the context of quartics, Bombelli refrains from solutions involving roots of negative numbers.

But there was also a back-door entry for negative terms into Bombelli's text, in the context of cubic equations. There Bombelli's method (following dal Ferro, Tartaglia and Cardano) is a change of variable. He adds or subtracts a number from the *Tanto*, plugs the changed variable into the cubic, solves the resulting simplified cubic, and then recovers the original *Tanto*. But given this method, a negative solution of the auxiliary cubic could yield a positive solution of the original cubic. Initially Bombelli includes in such cases an extra step of transforming the auxiliary cubic so that instead of having to add a negative solution, one would subtract a positive solution. But later on Bombelli apparently cannot be bothered with the intermediary step, and simply works with the negative auxiliary solutions (Bombelli 1966, 255–257). As a byproduct of this economy of attrition some negative solutions of the *original* cubic occasionally enter the habitus of the mathematician as well (Bombelli 1966, e.g. 259, 263).[67]

[64] This method might be explained by the work of Iberian author Marco Aurel. In his *Despertador de ingenios, Libro Primero de Arithmetica Algebratica* (1552) he offers as solution to the irreducible case of $x^2 + c = bx$ the negative counterpart of Bombelli's "sophistic" suggestion, namely $-\frac{b}{2} - \sqrt{\frac{b^2}{4} + c}$ (his notation is of course not modern). If one postulates that the product of two negatives is negative (a postulate to be discussed below), one indeed obtains a correct solution. (I derive this observation from a talk by Fàtima Romero Vallhonesta delivered in the PASR conference in Ghent on August 27, 2009). This fact raises the possibility that the "certain Spaniard" referred to by Bombelli was Aurel, rather than Pedro Nuñez, as suggested by Bortolotti.

[65] "tal capitolo non si potrà agguagliare per trattarsi dell'impossibile" (Bombelli 1966, 207).

[66] "ho trovato la sua dimostratione in linee" (Bombelli 1966, 133). The demonstration is included in Bombelli (1966, 226, 228).

[67] A similar situation occurs when solving quartics. There too there is an initial solution technique (completing the quartic to an equation between squares) that might involve negative auxiliary terms, as well as a bypass technique that replaces the addition of negative auxiliary terms by the subtraction of positive ones

It is interesting to compare these last practices to the corresponding practices in the manuscript. In the first of the corresponding manuscript examples the auxiliary negative solution is (possibly unconsciously) glossed over by simply not carrying the solution through to its end result—the necessary steps are only indicated. The next corresponding example is worked through and includes the intermediary step of transforming the auxiliary cubic to avoid a negative solution (Bombelli 155?, 82v, 84r). But in the print edition, where both solutions are carried through, and both examples are recognised as involving negative auxiliary solutions, the intermediary step is included in the first example and omitted from the second. It is the repetition that brings about the ellipsis of the negative-bypass steps, and in turn habituates the mathematician to practice negative solutions. Here, as in many other cases, repetition yields difference.

5.3 So what is *meno*?

We see, therefore, that the terms of endorsing or rejecting suspect quantities cannot be properly explained by a distinction between opportunistic endorsement and ontological rejection, but change along with the context. Bombelli is obviously highly capable of dealing with negative numbers, but prefers to bypass them, unless it involves too much effort. what can explain Bombelli's attitude? A detailed look at his practice will be helpful.

Note that even in the context of Bombelli's highly rigorous use of algebraic signs, we can still find expressions that attest to a practice of *meno* that is not reducible to either subtraction or modifying the nature of a term. When equating $1 \overset{4}{\smile}$ with $R.q.192 \overset{3}{\smile} +12$, Bombelli transforms the left hand side of the equation into the square of $1 \overset{2}{\smile} -R.q.48 \overset{1}{\smile} -1 \overset{1}{\smile}$ *di numero* (the last term here is an unknown number: the $1 \overset{1}{\smile}$ indicates that this is one time an unknown, and the *di numero* indicates that this unknown stands for the number, or free coefficient, of the quadratic term). After Bombelli derives the conditions that would yield a felicitous choice of unknown for the purpose of solving the quartic, he solves these conditions and states that the sought unknown is "6, and this is the value of the *meno* $1 \overset{1}{\smile}$ *di numero*".[68] But, in fact, Bombelli's transformed left hand side of the equation is the square of $1 \overset{2}{\smile} -R.q.48 \overset{1}{\smile} -6$. So today we would rather say that 6 is the value of the $1 \overset{1}{\smile}$ *di numero* itself, not of its minus. A similar effect occurs when Bombelli writes: "6, from which one subtracts the cube of m. $R.c.7$, remains m. 1".[69] Again, we would say

Footnote 67 continued
(Bombelli 1966, e.g. 282–283). But later we find Bombelli using the initial technique even where it involves negative terms (Bombelli 1966, e.g. 288), and later still we can find him stating that for a certain kind of quartic equation one can only use the bypass technique, but immediately adding that "It is also possible to" use the initial technique, "but the *Tanto* would be meno" ("E solo si può fare la positione di $-1 \overset{1}{\smile}$ di mumero. Si potrebbe anco ponere $+1 \overset{1}{\smile}$ di numero, ma il Tanto valerebbe meno") (Bombelli 1966, 299).

68 "Il Tanto valerà 6, e questo è la valuta del meno $1 \overset{1}{\smile}$ di numero" (Bombelli 1966, 279).

69 "6, del quale si cavi il cubato di m. $R.c.7$ restarà m. 1" (Bombelli 1966, 114). A similar formulation appears on page 115 as well.

that the cube of the negative number is added, not subtracted, and the explicit rules set by Bombelli (1966, 63–64) agree with this modern reformulation.

It is tempting to reconstruct such expressions either as indications of the anachronism of imposing our syntax on that of Bombelli, or simply as errors. But these supposedly aberrant formulations are in tension not only with modern ones, but also with other formulations routinely employed by Bombelli, and even if they were errors, one should try to explain their origin, as they are not too rare to matter.[70]

It seems here that quantities carry their sign with them farther than they 'should' (this recalls Piero's carrying of *cosa* terms with their coefficients, even where they should have been left behind; see the quotation from della Francesca (1970, 114) in Sect. 3.1). Similar effects occur with units of measurement and, as the appendix shows, with root signs. I believe that there are two practices that maintain these phenomena.

The first practice applies *meno* signs (as well as units, *cosa* terms and root signs) not strictly as operations or modifiers, but also as indexes or tags. In order to show that one is speaking about 'that' *Tanto* or 'this' *R.c.*7, one refers to it with the minus sign that originally accompanied it. It is important to acknowledge this indexical use of mathematical modifiers (which is easier to demonstrate in the context of root signs), as it is an often unacknowledged aspect of mathematical practice (it is much more common in oral delivery than in contemporary written math).

The second practice that enables holding on to the sign (and, again, units, *cosa* terms and roots) is more relevant for this article. We must recall that quantities were distinguished according to their natures. It was therefore sometimes hard to leave behind the signs that marked these natures, and separate them from the quantities that they qualified.

To support this last explanation, we should recall a lineage of sign rules that do not coincide with our own. Piero della Francesca, for example, asks to find the height of a ladder, such that when it leans against a wall of the same height as the ladder, it reaches 2 *braccia* below the top of the wall, and its foot is 6 *braccia* away from the wall. The height of the ladder is modelled as $\overline{1}$ (one *cosa*), and so we get a right angle triangle with sides 6 and $\overline{1} - 2$ and hypotenuse $\overline{1}$. Piero then writes: "multiply $\overline{1}$ *meno* 2 by $\overline{1}$ *meno* 2 makes $\overset{\square}{1}$ *censo* and 4 *meno* $\overline{4}$ *cose*, then multiply $\overline{1}$ by $\overline{1}$ makes $\overset{\square}{1}$" (that is 1 *censo*) "and 6 *meno* by 6 *meno* makes 36 *meno*. You have $\overset{\square}{1}$ and 4 *numero meno* $\overline{4}$ *cose* equal to $\overset{\square}{1}$ *meno* 36".[71] The point to note here is how the product of differences matches modern standards, as opposed to the product of the isolated negative ("6 *meno*") with itself, which produces a negative result. Along with this deviant practice we can list Marco Aurel's deviant articulation of the square of a negative as negative (see footnote 64 above) and Cardano's attempt to explain why the product of negatives

[70] In fact the last formulation quoted, from page 114, was changed from the manuscript, where it had been in line with modern practice, whereas the parallel form on page 115 had appeared in this non-modern form already in the manuscript. This suggests that Bombelli's choice was deliberate.

[71] "montiplica $\overline{1}$ meno 2 via $\overline{1}$ meno 2 fa $\overset{\square}{1}$ censo e 4 meno $\overline{4}$ cose, poi montiplica $\overline{1}$ via $\overline{1}$ fa $\overset{\square}{1}$ et 6 meno via 6 meno fa 36 meno. Tu ài $\overset{\square}{1}$ e 4 numero meno $\overline{4}$ cose equale ad $\overset{\square}{1}$ meno 36" (della Francesca 1970, 108).

is negative in his *De Regula Aliza* (Cardano 1663, 398–400).[72] Bombelli himself, in his manuscript, claims that "*meno* times *meno* is *più* when it is accompanied by the *più*, but by itself alone is *meno*",[73] but the promised demonstration is lacking, a marginal note states that "this is not the case",[74] and the statement is omitted from the print edition.

Apparently, it was hard to let go of the sign marking the nature of a quantity even when arithmetical manipulations were supposed to leave it behind. This interpretation is further confirmed by Cardano's reference to a quantity's own and alien nature, and its inability to go outside its own nature, as motivating his aberrant rules. (Tanner 1980, 166).

But this latter explanation does not suppress the practice of signs as indexes. Both practices operate in the text, and there is no point trying to 'decide' between them. At the same time, the powerful stream of practices documented above made the nature of quantities ever less stable, and rendered it easier to leave the marks of nature behind. My conclusion is that the hesitant but increasingly spread practice with negative entities is the result not of a conscious effort to review the ontology of quantities, but the impact of, to recapitulate the factors reviewed so far, economic practices, algebraic practices and attrition, which rendered the taxonomy of quantities a less stable and less practicable endeavour in the context of abbacist and Renaissance algebra. The conflict apparent in Bombelli's practice of negative numbers reflects his double commitment to keeping the natures of numbers constant and distinct (do not mix up negative and positive numbers, do not trust negative ("false") numbers), as well as to practical considerations where the stability of such distinctions is undermined.

6 The organisation of knowledge

Bombelli's stated objective was to "remove finally all obstacles before the speculative theoreticians and practitioners of this science" (algebra), "and take away any excuse from the unworthy and inapt", who use the obscurity of existing presentations to justify their lack of understanding.[75] Nevertheless, the results, as professed in the manuscript introduction, "for the beginners are brief, and for those who already understand are long".[76]

Bombelli's guiding principle is an economy of clarity and brevity negotiated in the context of a heterogeneous readership. This renders Bombelli a rather pragmatic writer. In a characteristically modest tone he acknowledges that "it could be that today I may teach a rule, which would please more than the others given in the past, and

[72] I refer to the 1663 edition rather than the original 1570 edition because the former is more accessible.

[73] "meno, via meno fa puì" [sic] "qua[n]do è accompagnato col più; ma per se solo fa meno" (Bombelli 155?, 11r). This is repeated slightly differently in Bombelli (155?, 51v), and it is unclear whether this has anything to do with the rules for multiplying roots of negative numbers.

[74] "non fa l'effetto".

[75] "per levare finalmente ogni impedimento alli speculativi e vaghi di questa scientia e togliere ogni scusa a' vili et inetti, mi son posto nell'animo di volere a perfetto ordine ridurla" (Bombelli 1966, 8).

[76] "Et per questo rispetto, quanto a li principianti sono stato breve; et quanto a gl'intendenti sono stato lungo" (Bombelli 1966, 316).

then another may come, and may find a more practicable and easy rule, and so that new rule would now be accepted, and mine rejected".[77] However, trying to economise his presentation was not always consistent with the organisation of knowledge around the distinction between quantities according to their natures.

A simple example of a failed attempt to organise knowledge around such distinctions occurs in Book III—a collection of questions to be solved by the algebraic means developed in Book II. To start with, Bombelli tries to be careful concerning what kind of solution is required. If the solution is an integer or a fraction, Bombelli asks for numbers in the question. When a root is the answer he asks for it explicitly (e.g. problem 24). And when quadratic questions first appear he asks for "numbers or quantities", as quadratic equations can yield both (problem 46). But by problem 144 cubic roots in the solution are referred to as number, and by problem 187 Bombelli asks for "two numbers", even though the result is irrational (this is repeated e.g. in problems 247 and 250). Holding on to the 'natural' distinctions between quantities becomes a tiresome burden to maintain, and midway through the third Book Bombelli no longer has enough energy to spare. Attrition eats away at the distinction between quantities according to their natures.

Such processes are manifest earlier in the abbacist tradition. Dardi's treatise (2001), for instance, has attracted attention for its proliferation of kinds of equations resulting from a separate treatment of equations with integer and with root coefficients (194 kinds, not including the four cases of special cubics and quartics, and not exhausting all cases that Dardi could have treated). Whilst this proliferating treatment made sense in mid 14th century, when the combination of roots and power terms was sufficiently new and odd to sustain it, it later became experienced as repetitive and tedious to the point where some of it appeared too tiresome to retain. Later authors borrowing from Dardi's work, such as Piero della Francesca, settled for sampling only a few of the cases that he had treated.

Whilst Dardi did realise that some rules were "common to discrete and indiscrete roots",[78] and that "number and root of number is of similar substance",[79] he nevertheless held on to his distinction setting discrete and indiscrete roots apart, either ideologically "because the intellect cannot understand nor comprehend"[80] the latter, or pragmatically because they "carry greater difficulty" ... "as many indiscrete roots can't be added in one enunciation with some numbers or roots".[81] Between Dardi and his successors operates a delicate economy of valuing differences on the one hand, whilst on the other giving them up due to attrition and to the similar treatment of the supposedly different kinds.

[77] "potria essere, che hoggi io insegnassi una regola, la quale piacerebbe più dell'altre date per il passato, e poi venisse un altro, e ne trovasse una più vaga, e facile, e così sarebbe all'hora quella accettata, e la mia confutata" (Bombelli 1966, 38).

[78] "aviallo fatto per ragione che lla reghula sia chomuna a R di numero discreta o indiscreta" (Dardi 2001, 46).

[79] "niente di meno numero e R di numero è d'una simile sustantia" (Dardi 2001, 275).

[80] "perché l'ontelletto no lla può intendere né conprendere" (Dardi 2001, 39).

[81] "porta maggiore dificultà sicome tu àj veduto nell'amaestramento dinanti che molte R indiscrete non si può giungere in una vocie chon anchunj numerj overo R" (Dardi 2001, 284).

When we reach Cardano, despite his strong interest in the natures of quantities, he states that an equation with irrational coefficients is to be solved "in the same manner as for rational numbers" (Cardano 1968, 189), and for Bombelli this fact does not even deserve mentioning. Note the self refuting process: Dardi's commitment to the root-number distinction gave rise to his exhaustingly repetitive catalogue—a repetitiveness which, in turn, rendered the number-root distinction less rigid *in practice*. Insistently repeating the articulation of a difference between roots and numbers, even where it made little practical difference, eventually rendered this difference too banal to merit ongoing attention. The excessive repetition of a difference has led to its dissolution.

Another aspect of failed classification concerns roots embedded into other roots, which emerged as cubics and quartics were solved. These entities seemed to Cardano too obscure to deserve proper attention in terms of classifying their natures. Whilst he did pay attention to the different natures of possible solutions of equations with integer coefficients, he stated that solving an equation with irrational rather than integer coefficients "leads to an incongruous quantity" (Cardano 1968, 259). The daunting interminability of the project of articulating the natures of emergent compound quantities made its ontological framework less viable.

Bombelli made a more earnest attempt at classifying the possible natures of solutions of the different kinds of polynomial equation. In the section entitled "discussion of the six cases above" (Bombelli 1966, 244) the first two cases of the cubic ('cube and *Tanti* equal number' and 'cube equals *Tanti* and number') are the basic cases to which all the others are reduced or compared. For the first equation it is summarily stated that the solution can be an integer or a sum of cubic roots. For the reducible cases of the second equation (where the method of dal Ferro and Tartaglia works) there is a similar statement on the possible natures of the solutions, as well as a list of possibilities that Bombelli ruled out as solutions for the irreducible case (see the end of Sect. 4). But this is also a point where Bombelli puts the whole project of charting the natures of solutions into question. Try as he may exploring cases and transforming the problem, some cubics that can be solved geometrically and with roots of negatives could not be solved by any 'traditional' quantity that Bombelli had in store. He therefore concludes "that (according to my judgement) I take it to be impossible to find such a general rule".[82]

Here the project of charting the natures of solutions hit an obstacle that for Bombelli was insurmountable. But the obstacle is not just the 'objective' problem of the expressive power of Bombelli's language. There is another dimension to the infeasibility of this project. Indeed, for all but the first two cases of the cubic Bombelli settles for vague remarks and observations concerning the nature of solutions. In fact, whilst the manuscript version of the "discussion" section handled all thirteen cases of the cubic (saying precious little about the latter, more complicated cases), the print edition comments only on the first six cases. The reason is not only the fact that Bombelli was uneasy with solutions involving roots of negatives; it is also that exhausting the information becomes a tedious task, whose results have little value in a world of a destabilised distinction between the natures of quantities. Even in cases where a full

[82] "Sì che (quanto al mio giuditio) tengo impossibile ritrovarsi tal regola generale" (Bombelli 1966, 245).

taxonomy of solutions was possible, such as 'cube equals square and number', none was attempted, and the discussion revolves instead around the reduction of this case to previous cases.

The information concerning equations becomes more and more haphazard as the text progresses. There is a less and less systematic treatment of the natures of solutions, solvability conditions, the number of solutions, transformation to previous cases, solutions with roots of negatives and negative solutions (including comments as odd as "This Case will rarely have more than one true solution and one false", where no negative solution is possible, and where this knowledge was easily accessible to Bombelli).[83] There is a clear impression that Bombelli is getting out of breath.

Bombelli's conclusion of the text is highly revealing. "I am of the opinion that I have left many unsatisfied in these last Sections" ... "and having wanted to put all cases" ... "would have made more a volume of collected Civil Code than a brief epilogue" ... "which was always farthest from my nature, being a most avid scholar of brevity" ... "I won't hold back now from saying this: that these Sections are a Chaos, and infinitely many steps and things occur there, which are impossible to teach in their entirety" ... "and these Sections have so many distentions" ... "that it's a deep chasm" (perhaps more figuratively: have so many protruding land-heads ... that it is a deep sea).[84]

7 Conclusion

Bombelli's project ends in rumbling discontent. With Bombelli, the abbacist organisation of knowledge is obviously at a point of crisis. It is therefore fitting to consider Bombelli as the last proponent of abbacist algebra, even though he was obviously no abbacus master. The erosion of the distinction between the natures of quantities, the emergence of algebraic entities such as parameters and variables, and the practically unfeasible organisation of knowledge under unbounded lists of types of equations and rules for solving them—all combined together to mark the structuring principles of *L'algebra* as a dead end.

But these processes had to combine to produce Bombelli's sense of helplessness. Had the 'natural' distinctions not been undermined, the mere infeasibility of the project need not have been considered a crisis. Indeed, the taxonomies of natural history are unboundedly large, but taxonomic zoology and botany still survive; the modern

83 "e questo Capitolo rare volte haverà più d'una valuta vera ed una falsa" (Bombelli 1966, 247).

84 "Son di opinione che a molti non haverò sodisfatto in questi ultimi Capitoli dove intervengono le potenze di potenze (per essere stato breve) ma questi Capitoli sono tali che chi intende bene uno di essi li intenderà tutti, et havendo voluto mettere tutti li casi che potevano intravenire, nelle loro agguagliationi, si saria fatto più tosto un volume d'un corpo di Testi civili, che un breve epilogo di Capitoli di potenze, Tanti e numero, il che fu sempre lontanissimo dalla natura mia, per essere studiosissimo della brevità. Però me ne sono passato con brevità, parendomi che sia bastato a chiarire bene li sei Capitoli primi ... Non restarò già hora di dir questo, che questi Capitoli sono un Caos, et infiniti passi e cose vi occorrono, le quali non si possono insegnar tutte, delle quali ne darrò qualche saggio; e li prudenti ne potranno trovare dell'altre, ma gli huomini rozzi e ancora mediocri non ci si affatichino che getteranno il tempo, perchè sono cose difficilissime; e questi Capitoli hanno tanti capi (come ho detto di spora) ch'è un pelago profondo" (Bombelli 1966, 311–312).

project of classifying finite simple groups was even more monstrous than the charting of all cases of quartics, but was nevertheless brought to an end; to be less anachronistic, Dardi's project of classifying equations according to the rationality of their coefficients had not been experienced as a crisis, even though it was just as unfeasible and messy, involving combinations of reductions and actual solutions not always correctly exhausted. If, on the other hand, the project had happened to be more containable, then the unstable distinctions underlying it need not have put it into question, as in the case of the distinction between six kinds of quadratic equations, which lasted well beyond the expiry date of the principles that had shaped it. But the semiotic micro-processes that destabilised the distinction between the natures of numbers, compounded by the failure of abbacist organisation of knowledge to contain what it had set out to order, resulted in Bombelli's experience of "Chaos" and "deep chasm".

The next steps in the 'grand narrative' of mathematics belong to Viète and Descartes. Often, the transition to this next step is characterised as depending on the invention of better symbolism—the representations of multiple parameters and variables by letters, and of exponents by superscript numbers. But this explanation underplays the existence of variants of exponential notations before the end of the fifteenth century, the representation of parameters by terms such as *numero*, *quantità* and shorthand symbols, the increasing practice of numbers as parameters, and the diagrammatic representations of solution techniques (such as the one in Fig. 3, which shows how to solve a cubic equation), which were expressive enough for purposes much more intricate than usually acknowledged today.

And so, if we reject the premises that Cartesian algebraic notation was a spark of genius that technologically determined mathematical progress, we must see it as a tool whose origins are older than we usually recognise, and whose widespread

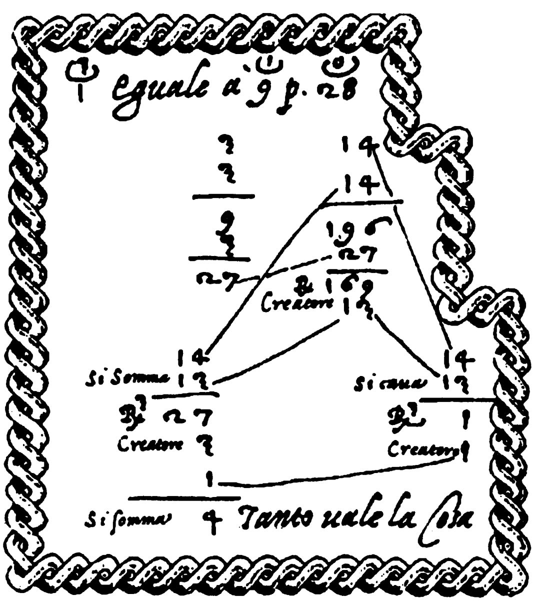

Fig. 3 Bombelli (155?, 71v): a computational diagram for solving a cubic equation

endorsement is an issue that requires explanation. I hope that the interactions between semiotic micro-processes that destabilised the distinction between quantities and equations according to their natures, the practice of numbers and other symbols as proto parameters and variables, and the problems of the abbacist organisation of knowledge, help us understand the non-linear and not necessarily conscious evolution of mathematical knowledge and notation culminating in symbolic algebra.

But the main message of this article is in its fine-grain detail: over and over again we find that mathematical signs stand for more than themselves, and for much more than the regimented roles assigned them by authors and philosophers of mathematics.

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Appendix: The status of roots

Even a cursory look at the use of root signs in abbacist texts shows that they are subject to multiple standards. In Jacopo's text one finds the expression "Now multiply" ... "4, which is cube root, against the cube root of 3375".[85] The cubic root is practiced as an operation applied to 3375, whereas for 4 it is practiced as a modifier marking its kind or nature.

Piero, a century and a half later, writes "And wanting to divide" one square root by another "one divides root by root" and the answer "will be the root of that number which comes out".[86] For a contemporary ear it sounds odd that one would apply a further root to the division of root by root. In fact, in the first half of this sentence the root actually points to the number under the root. This usage is in line with contemporary phrases such as '4 children have 12 bonbons. Divide the bonbons by the children, and the answer will be that each child has as many bonbons as the outcome'; first we divide the *numbers* qualified by 'bonbons' and 'children', then the result is given its qualifier, 'bonbons'. The root is in fact sometimes practiced like a designation or index pointing to the number without actually modifying it. To make this point even more explicit note that Maestro Dardi often uses the phrase "the number called root" for pointing to numbers under root signs.[87]

I must be very clear here. I do not claim that abbacists necessarily conceived of roots differently than we do, whatever 'conceiving' may mean. I do not claim that their practice of roots led to results that today we would consider wrong. In Piero's case, for example, the rules for solving equations of the form power (or root of power) equals root of power are inconsistent in regards to what portion of the text is covered by the term 'root' (della Francesca 1970, 88–91); in concrete examples, however, his results are almost always correct (or at least can be read as such with a reasonable

[85] "Ora multipricha la radice chubica, cioè 4, che è radice chubicha, contra ala radice chubicha {contra ala radice chubicha} de 3375" (Høyrup 2007, 326).

[86] "E volendo partire l'una per l'altra, se parte radici per radici et serà radici de quello numero che ne verà" (della Francesca 1970, 76).

[87] "lo numero chimamto R" (Dardi 2001, e.g. 52).

interpretation of the range of root signs). My claim is that the textual practice of the term 'root' combines practices deriving from indices, units and modifiers.

In Bombelli's text too various practices of the term 'root' survive, which treat root and number as convertible units. Bombelli asks to add numbers "as if the one and the other were roots" instead of adding their roots[88]; he writes "Having multiplied 2 by *R.q.*3 (as if each were number)" for multiplying 2 by 3[89]; he uses the phrase "difference between *R.q.*128 and 8 as number" for the difference between 128 and 64[90]; and concerning the issue of whether pairs such as *R.q.*50 and *R.q.*2 should be said to have a ratio as number to number or as square to square, Bombelli is ambiguous, and uses both.[91]

I include these observations to show how mathematical modifiers were rhetorically practiced also as indexes or units. This obviously has to do with the pragmatics of teaching and rules, where an abbacus master would refer to a number by the sign standing next to it. But it also has to do with the conception of modifiers as expressing the nature of quantities, and therefore, like units, as expressions that should be carried with the number, even when it is the unmodified number that is operated on. Acknowledging these phenomena in the context of roots helps understand corresponding practices in the context of *meno* and *cosa* signs.

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88 "come l'uno e l'altro fosse R.q." (Bombelli 1966, 87).

89 "Moltiplicato 2 via *R.q.*3 (come se ciascuno fosse numero)" (Bombelli 1966, 120).

90 "differenza ch'è da *R.q.*128 a 8 numero" (Bombelli 1966, 121).

91 The manuscript seems to prefer the latter. The print edition seems to prefer the former, but acknowledges that the latter is a more common practice (Bombelli 1966, 64, 73, 124; 155?, 12v).

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