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Source: *Archive for History of Exact Sciences*, Vol. 68, No. 4 (July 2014), pp. 529-545
Published by: Springer
Stable URL: https://www.jstor.org/stable/24569618

Arch. Hist. Exact Sci. (2014) 68:529–545
DOI 10.1007/s00407-013-0131-3

On Jacobi's transformation theory of elliptic functions

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Received: 4 July 2013 / Published online: 12 October 2013

Abstract The main interpretative challenge set by the *Fundamenta Nova Theoriae Functionum Ellipticarum* lies in Jacobi's transformation theory upon which the entire theoretical edifice of the treatise depends. Unfortunately, Jacobi did not convey any indication of how he attained his general formulae for rational transformations of elliptic functions. He limited himself to providing a posteriori verification of the validity of his claims. The aim of this paper is precisely to describe the heuristic path by which in 1827 Jacobi succeeded in finding these transformation formulae. The proposed historical reconstruction will hopefully shed new light upon the emergence in Jacobi's work of the inversion process of elliptic integrals of the first kind and thus of the elliptic function sinam$u$ itself.

Contents

1 Introduction

Jacobi's *Fundamenta Nova Theoriae Functionum Ellipticarum* represents a landmark in the history of analysis; in it, a systematic treatment of the emerging theory of elliptic

Communicated by: Umberto Bottazzini.

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functions was provided for the first time. Despite its undisputed importance and long-lasting influence throughout the 19th century, this treatise, which is almost two hundred pages long, is nonetheless difficult to understand, because of both the algorithmically oriented character of the mathematical formulation of the theory and the extremely synthetic style of presentation, which, in this respect, seems to be strongly affected by its being written in Latin.

From a purely historical standpoint, the main interpretative challenge set by the *Fundamenta* lies in Jacobi's transformation theory upon which the entire theoretical edifice of the treatise heavily depends. There are two reasons for this. First, essential components of the whole of Jacobi's elliptic function theory—such as the theory of modular equations, the theory of multiplication, and the introduction of the celebrated theta functions—were presented by Jacobi as by-products of his preliminary investigations of the transformation problem for elliptic integrals of the first kind. Second, the way in which the well-known formulae for rational transformations of any order were attained was obscured by Jacobi's peculiar mode of presentation, which consisted in conveying the relevant formulae without any contextual justification whatsoever and with proofs being given only at a later stage.

The reader has the feeling that Jacobi's discoveries were the product of an act of divination or, at best, the result of a fortunate sequence of trial and error. The remarks of the Italian astronomer and mathematician, Giovanni Plana, who seemed to have shared this feeling and, at the same time, tried to reduce it by improving the readability of Jacobi's results, are a good example which is worth quoting:

> By reflecting upon the proof provided by Jacobi, one does not understand easily by what chain of ideas he could have achieved the singular form that he gives to a certain rational function of one variable which represents the basis and the point of departure of his demonstration. Chance could not produce such a hidden result.[1]

Here, Plana was referring to the first proof conveyed by Jacobi in (Jacobi 1827c), but his critique could aptly be repeated for the reformulation of the theory provided in the *Fundamenta Nova*.

Similar objections were put forth by Legendre in his report on Jacobi's theory, which was published in *Nr. 130* of the *Astronomische Nachrichten*:

> [...] it is regrettable that the author fulfills the aim which he has imposed to himself by a sort of divination, without sharing with us the secret whose conception has progressively led him to the form for $1 - y$ which is required in order to satisfy the conditions of the problem.[2]

[1] En réfléchissant sur la métaphysique de la démonstration donnée par Mr. Jacobi, on ne comprend pas facilment par quel enchaînement d'idées il a pû être conduit à la forme singulière qu'il attribue à une certaine fonction rationelle d'une seule variable, qui constitue la base et le point de départ de sa démonstration. Le hasard ne saurait enfanter un résultat aussi profondément caché (Plana 1829, 333).

[2] [...] on doit regretter que l'auteur remplisse la tâche qu'il s'est imposée par une sorte de divination, sans nous mettre dans le secret des idées dont la filiation l'a amené progressivement à la forme que doit avoir $1 - y$ pour satisfaire aux conditions du problème (Legendre 1828a, 203).

The aim of the present paper is precisely to describe and explain in full detail the heuristic process by which Jacobi obtained these transformation formulae. The proposed historical reconstruction will also shed new light upon the emergence in Jacobi's work of the inversion process of elliptic integrals of the first kind and thus of the elliptic function sinam$u$ itself.

Indeed, I believe that a close examination of Jacobi's early researches on transformation theory in the period June–December 1827 will illuminate some crucial *loci* of the *Fundamenta* and partially dissolve the aura of obscurity which surrounds them.

No historical investigation on Jacobi's theory of elliptic functions can avoid a comparison with Abel's contemporary work[3] on the same subject. However, this is not my main goal. I will limit myself to some sporadic remarks when such a comparison sheds light on Jacobi's own investigations.

2 The genesis of the theory

According to (Koenigsberger 1904, 35–36), Jacobi began his research on the transformation theory of elliptic integrals in the winter of 1826–1827. Like Abel, who seemed to have taken an interest in the subject after reading Gauss's hints about the theory of cyclotomy contained in the *Disquisitiones Arithmeticae*, Jacobi's choice is likely to have been triggered by the work of Gauss as well. In particular, it was his attentive study of a celebrated memoir (Gauss 1818) on the determination of the secular variations of the orbit of a planet that is subject to the influence of another planet that fostered his first steps into the realm of a new mathematical discipline: elliptic function theory. On that occasion, Gauss had tackled and solved the problem of reducing the elliptic differential

$$\frac{\mathrm{d}E}{\sqrt{(A-a\cos E)^2+(B-b\sin E)^2+C^2}} \tag{1}$$

to a simpler form, namely:

$$\frac{\mathrm{d}P}{\sqrt{G+G'\cos^2 P+G''\sin^2 P}}. \tag{2}$$

As Gauss proved in full detail, it turned out that such a reduction could be attained by means of a rational transformation of type

$$\begin{cases} \cos E = \dfrac{\alpha+\alpha'\cos P+\alpha''\sin P}{\gamma+\gamma'\cos P+\gamma''\sin P} \\[2ex] \sin E = \dfrac{\beta+\beta'\cos P+\beta''\sin P}{\gamma+\gamma'\cos P+\gamma''\sin P}, \end{cases} \tag{3}$$

if the 9 coefficients $\alpha, \ldots, \gamma''$ are properly chosen.

[3] On Abel's fundamental contributions to elliptic function theory, see (Houzel 2004) and (Bottazzini and Gray 2013, Chap. I).

Jacobi soon discovered that he could provide a generalization of Gauss's theorem to the case of elliptic-type integrals of 2 variables. In his words:

> While, on several occasions, I devoted myself to that illustrious paper, I realized that the same analysis could be employed to obtain an outstanding transformation of some double integral. I believe that the result is all the more worth communicating to geometers since the theory of double integrals appears to be very much neglected.[4]

His achievements resulted in a short paper published in June 1827 in *Crelle's Journal* under the title *De singulari quadam duplicis Integralis trasformatione*, (Jacobi 1827a). Almost at the same time, under the stimulus of these researches, Jacobi began his first systematic investigations of the theory of elliptic integrals himself. The subject had recently been elevated to the rank of an autonomous and promising discipline by the indefatigable, solitary work of Adrien-Marie Legendre, who had been dealing with it for almost twenty years. Starting in the late 1780s, Legendre had provided a thorough examination of the subject that essentially consisted of the complete classification of elliptic integrals into three distinct kinds and the elaboration of suitable techniques for constructing numerical tables, thus enriching the mathematical treatment of known transcendental functions beyond the elementary cases of the exponential and trigonometric functions.

Transformation theory turned out to be essential for this general aim. For the sake of simplicity, we will limit ourselves to considering the case of the (so-called) quadratic transformations of elliptic integrals of the first kind. We will follow Legendre's discussion contained in (Legendre 1825, Chap. XVII). In Legendre's words, his motivation was the following:

> We will now see how one can, by a very simple law, form an infinite number of elliptic functions of the first kind which differ one from another both by module and amplitude and which, nonetheless, have the most remarkable property of being in constant ratios.[5]

Indeed, by denoting with $F(\phi, c)$, the elliptic integral of the first kind

$$F(\phi, c) = \int_0^{\phi} \frac{\mathrm{d}\phi}{\sqrt{1 - c^2 \sin^2 \phi}},$$

Legendre obtained the following important result:

Theorem 1 (Legendre 1825) *Let $F(\phi, c)$ and $F(\phi', c')$ two elliptic integrals with amplitudes and modules subject to the following two conditions:* $\sin(2\phi' - \phi) =$

[4] Dum egregiae illi commentationi identidem incumbebam, non fugit me, eandem fore analysin ad duplicis Integralis cujusdam insignem transformationem adhiberi posse, quam communicare cum geometris eo minus dubito, quod duplicium Integralium theoria valde jacet (Jacobi 1827a, 234).

[5] Nous allons faire voir qu'on peut, par une loi très simple, former une infinité de fonctions elliptiques de première espèce, qui diffèrent les une des autres tant par le module que par l'amplitude, mais qui ont la propriété fort remarquable d'être entre elles dans des rapports costans.

$c \sin\phi$, $c' = \frac{2\sqrt{c}}{1+c}$. *Then one has:*

$$F(c', \phi') = \frac{1+c}{2} F(c, \phi).$$

It should be observed that the relevant transformation $\sin\phi \longmapsto \sin\phi'$ is quadratic only in a broad sense, since in this case, irrationalities do appear.

Legendre exploited the existence of this transformation for the numerical computation of elliptic integrals by constructing a doubly infinite scale (*échelle*) of modules $\ldots, c^{00}, c^0, c, c', c'' \ldots$ iteratively defined by $c' = \frac{2\sqrt{c}}{1+c}$ and its backward counterpart $c^0 = \frac{1-\sqrt{1-c^2}}{1+\sqrt{1-c^2}}$.

Jacobi's first results were precisely concerned with the extension of Legendre's transformation theory. In a letter dated June 13, 1827 to H. C. Schumacher, the founder of the *Astronomische Nachrichten*, Jacobi announced the discovery of an infinite number of rational transformations for elliptic integrals of the first kind. The communication did not contain any proofs, but Jacobi was able to exhibit explicit expression for transformations of order 3 and 5 together with a sketchy discussion of related issues concerning the multiplication problem.

Reactions to Jacobi's announcement ranged from Gauss's icy indifference to Legendre's loyal acknowledgement of the scientific value of Jacobi's achievements. In a letter to Schumacher, Gauss expressed his intention to refrain from making any judgement on Jacobi's work saying that he was perplexed by the habit of publishing statements without proof or any justification whatsoever. At the same time, he wrote that he could find nothing new in Jacobi's announcement, since he was in possession of even more general results even though he had not published any.

On the contrary, Legendre's reception was enthusiastic and very encouraging, as emerges from the well-known epistolary exchange with Jacobi that had begun on August 5, 1827. This is the date of the first letter which Jacobi sent to the unanimously recognized master of elliptic integrals in order to present the results of his early researches. He wrote the following:

> Sir, a young geometer dares to present to you some discoveries of his own in the realm of elliptic function theory to which he has been led by the restless study of your beautiful works. It is to you, Sir, that this brilliant branch of analysis owes its high degree of perfection to which it has been driven. It is only by marching along the footsteps of such a great master that geometers will be able to take it beyond the boundaries which have been imposed to this discipline thus far. Then, it is you the person to whom I feel obliged to offer what is to follow as loyal tribute of admiration and gratitude.[6]

[6] Monsieur, un jeune géomètre ose vous présenter quelques découvertes faites dans la théorie des fonctions elliptiques, auxquelles il a été conduit par l'étude assidue de vos beaux écrits. C'est à vous, Monsieur, que cette partie brillante de l'analyse doit le haut degré de perfectionnement auquel elle a été portée, et ce n'est qu'en marchant sur les vestiges d'un si grand maître, que les géomètres pourront parvenir à la pousser au delà des bornes qui lui ont été prescrites jusqu'ici. C'est donc à vous que je dois offrir ce qui suit comme un juste tribut d'admiration et de reconnaissance (Jacobi's Werke, I, 390).

Legendre informed Jacobi that he already knew of the existence of the module scale associated to the number 3 (indeed, this discovery of his was already contained in (Legendre 1825, Chap. XXXI) unknown to Jacobi); nonetheless, as he fully recognized the discovery of a third scale associated with the prime number 5 was of the uttermost importance, not to speak of the assertion that there exists a rational transformation corresponding to every prime number. At first, Legendre had to admit that he doubted the validity of such an audacious statement. However, upon reading Jacobi's more detailed letter of August 5, he convinced himself that the young geometer could actually rely upon a solid theoretical framework rather than on an unrigorous inductive methodology. For this reason, Legendre urged Jacobi to provide more analytical details shedding light on the undeniable efficiency of his techniques.

3 "Demonstratio theorematis..."

Already on November 18, 1827, Jacobi had completed a short paper, to appear in the *Astronomische Nachrichten Nr. 130*, which set out the foundations of his transformation theory of elliptic integrals, and at the same time, he provided the first proof of his ground-breaking announcements. At the start of the paper, he wrote the following:

> In the 123th number of the Astr[onomische] N[achrichten] I conveyed some properties of elliptic functions which appeared to be new and worth of the interest of geometers. The researches to which those results gave birth had advanced further and offer, if I am not mistaken, an outstanding extension of Legendre's theory. Although I cannot determine yet the time by which it will be possible to complete a treatise comprising all these investigations, I hope that geometers will appreciate that I briefly convey a fragment of these researches, namely the proof of a fundamental theorem in the transformation theory of elliptic functions.[7]

Jacobi's transformation theory consists of two essentially distinct steps: a purely algebraic one relying only on the manipulation of polynomials and the counting of arbitrary constants, and an analytical–transcendental one by means of which the explicit expression for the sought-for transformations could be attained. In this respect, the inversion of elliptic integrals and the complexification process turned out to be decisive for further investigation; in spite of that, they played no part at all in the elaboration of the preliminary algebraic component of the theory, as will be seen.

[7] Proprietates functionum ellipticarum quasdam in n. 123 Astr. N. tradidi, quae novae atque attentione geometrarum non indignae videbantur. Disquisitiones, quibus illae originem debent, exinde ulterius continuatae sunt egregiamque, ni fallor, amplificationem theoriae a Legendre datae praebent. Cum autem tempus, quo tractatui, hasce disquisitiones complectenti, finem imponere licebit, definire nondum queam, geometris non ingratum fore spero, si fragmentum harum disquisitionum, demonstrationem scilicet theorematis in doctrina de transformatione functionum ellipticarum fundamentalis, hic breviter exponam (Jacobi 1827c, 133).

Jacobi's starting point in (Jacobi 1827c) consisted of finding conditions on polynomials $U(x)$, $V(x)$ that guarantee the possibility of transforming the differential expression

$$\frac{\mathrm{d}y}{\sqrt{Y(y)}} = \frac{\mathrm{d}y}{\sqrt{(1-\alpha y)(1-\alpha' y)(1-\alpha'' y)(1-\alpha''' y)}} \tag{4}$$

into

$$\frac{\mathrm{d}x}{M\sqrt{X(x)}} = \frac{\mathrm{d}x}{M\sqrt{(1-\beta x)(1-\beta' x)(1-\beta'' x)(1-\beta''' x)}}, \tag{5}$$

where M designates a constant quantity, by means of a rational substitution $y = U(x)/V(x)$. When the expression for y is substituted in Eq. (4), as a consequence of

$$V^4 Y\left(y = \frac{U}{V}\right) = (V-\alpha U)(V-\alpha' U)(V-\alpha'' U)(V-\alpha''' U)$$

and of

$$\mathrm{d}y = \frac{1}{V^2}\left\{V\frac{\mathrm{d}U}{\mathrm{d}x} - U\frac{\mathrm{d}V}{\mathrm{d}x}\right\},$$

one obtains

$$\frac{\mathrm{d}y}{\sqrt{Y}} = \frac{\left\{V\frac{\mathrm{d}U}{\mathrm{d}x} - U\frac{\mathrm{d}V}{\mathrm{d}x}\right\}}{\sqrt{G(U,V)}},$$

where $G(U,V)$ indicates the homogeneous 4th order function $V^4 Y\left(y = \frac{U}{V}\right)$. Now, Jacobi supposed, if $G(U,V)$ contains a quadratic factor $(x-\gamma)^2$, then $(x-\gamma)$ divides $V\frac{\mathrm{d}U}{\mathrm{d}x} - U\frac{\mathrm{d}V}{\mathrm{d}x}$. Indeed, since U and V are supposed to contain no common factors, it is clear that $(x-\gamma)^2$ must divide one of the terms $(V-\alpha U)$, $(V-\alpha' U)$, $(V-\alpha'' U)$, $(V-\alpha''' U)$ only. Otherwise, suppose that there exist polynomials $W(x)$, $W'(x)$ such that $(V-\alpha U) = (x-\gamma)W(x)$ and $(V-\alpha' U) = (x-\gamma)W'(x)$, then it is easy to see that $U(x)$ and $V(x)$ have $(x-\gamma)$ as a common factor, contrary to the hypothesis. As a consequence of this, and of the identity $V\frac{\mathrm{d}U}{\mathrm{d}x} - U\frac{\mathrm{d}V}{\mathrm{d}x} = (V-\gamma U)\frac{\mathrm{d}U}{\mathrm{d}x} - U\frac{\mathrm{d}}{\mathrm{d}x}(V-\gamma U)$, Jacobi observed that if $G(U,V)$ contains a square factor $(x-\gamma)^2$, then $(x-\gamma)$ also divides $V\frac{\mathrm{d}U}{\mathrm{d}x} - U\frac{\mathrm{d}V}{\mathrm{d}x}$.

Thus, if $G(U,V)$, which is of degree $4n$ in the variable x (n being the maximum exponent of x in the polynomials U, V), contains $2n-2$ such square factors so that

$$G(U,V) = (1-\beta x)(1-\beta' x)(1-\beta'' x)(1-\beta''' x)T^2(x),$$

then T divides $V\frac{\mathrm{d}U}{\mathrm{d}x} - U\frac{\mathrm{d}V}{\mathrm{d}x}$. Since the degree of $V\frac{\mathrm{d}U}{\mathrm{d}x} - U\frac{\mathrm{d}V}{\mathrm{d}x}$ cannot be greater than $2n-2$, there exists a constant M such that $V\frac{\mathrm{d}U}{\mathrm{d}x} - U\frac{\mathrm{d}V}{\mathrm{d}x} = T/M$.

The following theorem, which Jacobi considered to be fundamental for the entire transformation theory,[8] was thus proved.

Theorem 2 *Let U, V, T be polynomials (*functiones rationales integras*) in the variable x such that*

$$\begin{aligned}&(V-\alpha U)(V-\alpha' U)(V-\alpha'' U)(V-\alpha''' U)\\&\quad=(1-\beta x)(1-\beta' x)(1-\beta'' x)(1-\beta''' x)T^2(x),\end{aligned}\tag{6}$$

then the expression

$$\frac{\mathrm{d}y}{\sqrt{Y(y)}}=\frac{\mathrm{d}y}{\sqrt{(1-\alpha y)(1-\alpha' y)(1-\alpha'' y)(1-\alpha''' y)}},$$

as a result of the substitution $y=U(x)/V(x)$, is transformed into

$$\frac{\mathrm{d}x}{M\sqrt{X(x)}}=\frac{\mathrm{d}x}{M\sqrt{(1-\beta x)(1-\beta' x)(1-\beta'' x)(1-\beta''' x)}},$$

where M indicates a constant quantity.[9]

Jacobi pointed out that the result could easily be applied to elliptic integrals of the first kind. Indeed, it suffices to set $\alpha=-\alpha'=1, \alpha''=-\alpha'''=\lambda$ and $\beta=-\beta'=1, \beta''=-\beta'''=k$ in order to obtain a rational substitution, which transforms $\frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-\lambda^2y^2)}}$ into $\frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}}$.

It should be remarked that this theorem does not by itself guarantee the existence of rational transformations of any order. It only asserts that if one is able to produce polynomials U, V satisfying (6), then $y=U/V$ is the desired transformation.

A rough computation of the number of unknown constants involved, which Jacobi provided in a letter to Legendre and much later in the *Fundamenta Nova*, makes it clear that the determination of the coefficients of U and V should be possible since the number of unknown coefficients is greater than the number of necessary conditions to be imposed. When reconstructing the heuristic path which had led him to the conjecture concerning the existence of transformations of any prime order, Jacobi explained that his first discovery consisted of the equation $T=V\frac{\mathrm{d}U}{\mathrm{d}x}-U\frac{\mathrm{d}V}{\mathrm{d}x}$, from which he could draw the conclusion that the deduction of transformations of any order was possible because, as he wrote to Legendre, it could be traced back to a *problème d'Analyse algébrique déterminé* (*a determined problem of algebraic Analysis*).

However, this simple argument is not sufficient to guarantee the existence of rational transformations, as Poisson explicitly observed in his report on Jacobi's *Fundamenta Nova*.

[8] Theoremate hoc fundamentum transformationis trascendentium ellipticarum continetur (The foundations of the transformation of elliptic transcendents are contained in this theorem) (Jacobi 1827c, 135).

[9] (Jacobi 1827c, 135).

Nonetheless, this enumeration of unknown quantities and conditional equations is not sufficient to guarantee a priori the possibility of equation (2); since it might happen that the conditional equations are incompatible and that one cannot satisfy them either with real values or with imaginary values of the unknowns, despite the fact that the number of the latter is greater than that of equations.[10]

For this reason, the actual existence of transformations of any order could be attained only by producing polynomial functions satisfying the hypothesis of theorem (2). Already in the above-mentioned letter on August 5 to Schumacher, Jacobi had communicated, without proof, the general form for the pth order transformation (p being an odd number) equation by means of the following trigonometric expression:

$$\tan\left(\frac{\pi}{4}-\frac{\psi}{2}\right)=\tan\left(\frac{\pi}{4}\pm\frac{\phi}{2}\right)\frac{\tan\frac{\alpha_1-\phi}{2}}{\tan\frac{\alpha_1+\phi}{2}}\cdots\frac{\tan\frac{\alpha_{p-2}\pm\phi}{2}}{\tan\frac{\alpha_{p-2}\mp\phi}{2}}, \tag{7}$$

where α_m is so chosen as to satisfy the relation $F(k,\alpha_m)=\frac{m}{p}K$, and it is agreed that the upper sign is to be taken when p is of the form $4n+1$ and the lower sign when it is of the form $4n-1$. The first rigorous proof of (7) was provided by Jacobi only in December 1827 when, after reading Abel's memoir, he explicitly introduced the inverse function sinamξ to denote the sine of the angle (*amplitudo*) ϕ that satisfies the relation

$$\xi=\int_0^{\phi}\frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}}.$$

It should be observed that already in (7) Jacobi implicitly made use of the inverse function. Indeed, the coefficients α_m are defined precisely by the relations $\sin\alpha_m = \mathrm{sinam}\frac{m}{p}K$, where K, the so-called complete integral, is defined to be equal to $\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}}$. Nonetheless, it is still unclear whether Jacobi had the idea of inverting elliptic integrals before getting to know Abel's results. One can safely say that Abel's emphasis on the necessity of dealing with inverse functions from the very beginning of his investigation has no match in Jacobi's early researches. While in Abel's *Recherches* (Abel 1827, 1828) the inversion process appeared as a preliminary step to highlight essential properties of elliptic functions such as double periodicity, Jacobi was led to the inversion of elliptic integrals in a somehow indirect way through his detailed study of transformation theory. We will see later on how the coefficients sinam $\frac{m}{p}K$ emerged in a quite fortuitous form along a very interesting heuristic path.

How can one evaluate the impact played by Abel's work on Jacobi's discovery of the existence of rational transformations of any order as proven in (Jacobi 1827c)? Despite the fact that the proof provided in (Jacobi 1827c) made no use of either

[10] Mais cette énumération des inconnues et des équations de condition ne suffit pas pour établir a priori la possibilité de l'équation (2); car il pourrait arriver que les équations de condition fussent incompatibles, et qu'on n'y pût satisfaire, ni par de valeurs réelles, ni par des valeurs imaginaires des inconnues, quoique le nombre de celles-ci fût plus grand que celui des équations (Poisson 1831, 87). By Eq. (2) Poisson referred to Eq. (6) above.

complexification or of double periodicity so fundamental for Abel's *Recherches*, the introduction of the function sinam$u$ turned out to be essential. In particular, Jacobi exploited the single periodicity $\operatorname{sinam}(u+4K) = \operatorname{sinam} u$ in order to deduce algebraic properties required for the application of Theorem (2). It should be remarked that in his previous investigations, Jacobi had merely limited himself to considering specific values of the function sinam$u$, namely those corresponding to rational multiples of the complete integral K. It was only in (Jacobi 1827c), after reading Abel, that the abstract notion of the inverse function was given autonomous status and, consequently, was made the central object of his analysis.

The structure of the proof provided by Jacobi was quite simple in principle. He rewrote the transformation (7) in the following form:[11]

$$y = \frac{U}{V} = \frac{x}{M} \frac{\prod_{m=1}^{\frac{p-1}{2}} \left\{ 1 - \frac{x^2}{\sin^2 \operatorname{am} \frac{2mK}{p}} \right\}}{\prod_{m=1}^{\frac{p-1}{2}} \left\{ 1 - k^2 x^2 \sin^2 \operatorname{am} \frac{2mK}{p} \right\}} \tag{8}$$

and he verified *a posteriori* that the polynomials U and V satisfy the hypotheses of Theorem (2).

In order to do that, Jacobi relied upon an outstanding symmetry property of the differential equation

$$\frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-\lambda^2 y^2)}} = \frac{\mathrm{d}x}{M\sqrt{(1-x^2)(1-k^2x^2)}}, \tag{9}$$

according to which the Eq. (9) remains invariant upon simultaneous substitution of x by $\frac{1}{kx}$ and of y by $\frac{1}{\lambda y}$. As a consequence of this, Jacobi succeeded in providing explicit expression for the quantities $1 \pm y$, $1 \pm \lambda y$ from which he could draw the conclusion that there exists a polynomial function $T(x)$ such that

$$(V-U)(V+U)(V-\lambda U)(V+\lambda U) = (1-x^2)(1-k^2x^2)T^2(x), \tag{10}$$

as was required by the purely algebraic statement of Theorem (2).

It should be remarked, as in (Legendre 1828a), that such a computational trick, which Legendre called double substitution principle, played a decisive role in the proof, since a direct verification of the formula (8) would otherwise be forbidding. In this respect, Legendre wrote the following:

> The difficulty which one experiences here is such that there would not be any hope of attaining the general demonstration, had not Jacobi found a very ingenious way of avoiding the substitution of the formulae into the differential equation

[11] It is by no means trivial to prove that the two expressions are indeed equivalent. A careful verification of this fact was provided in (Legendre 1828b, 5–6).

which consists in exploiting a particular property of this equation which is shared by its integrals too.[12]

We now tackle the question of how Jacobi could have come to his conjecture. Once again, the already mentioned letter to Legendre (April 1828) turns out to be very enlightening. Jacobi wrote the following:

> You wanted me to have given the chain of ideas which led me to my theorems. However, the path which I pursued is not susceptible to geometrical rigor. Once the result was found, one could replace it with another one which could be achieved in a purely rigorous way. Then, it is for you only, Sir, that I add what follows:
> The first thing which I found (in March 1827) was the equation $T = \frac{V\mathrm{d}U}{\mathrm{d}x} - \frac{U\mathrm{d}V}{\mathrm{d}x}$; from which I recognized that, for any number n, the determination was a known problem of algebraic analysis, the number of arbitrary constants being always equal to the number of conditions. By means of undetermined coefficients, I was able to produce the transformations of order 3 and 5 [...] By a fortunate groping, I noticed in both these cases the complementary transformation giving rise to the multiplication formula. At that moment I wrote my first letter to Schumacher, the method being general and verified by some examples. Afterward, by a closer examination of the two transformations $z = \frac{ay+by^3}{1+c^2y^2}$, $y = \frac{a'x+b'x^3}{1+c'x^2}$ written in the form which I conveyed in my first letter, I discovered that by putting $x = \mathrm{sinam}\frac{2K}{3}$, z must vanish and, since, in the mentioned form, $\frac{b}{a}$ was positive, I concluded that y also vanished. In this manner, I found by induction the factorization; this being confirmed by some examples, I conveyed the general theorem in my second letter to Schumacher.[13]

As will be shown in the appendix, Jacobi was able to obtain the expression for third-order transformations by purely algebraic means. The first transformation, which he

[12] La difficulté qui se présente ici est d'une telle nature qu'il ne resteroit guère d'espoir de parvenir à la démonstration générale, si Mr. Jacobi n'eût trouvé un moyen très ingenieux d'éviter la substitution à faire dans l'équation différentielle, et d'y suppler par une propriété particulière de cette équation qui doit être commune aux intégrales qui la représentent (Legendre 1828a, 204).

[13] Vous auriez voulu que j'eusse donné la chaîne des idées qui m'a conduit à mes théorèmes. Cependant la route que j'ai suivi n'est pas susceptible de rigueur géométrique. La chose étant trouvée, on pourra y substituer une autre sur laquelle on aurait pu y parvenir rigoureusement. C'est ne donc que pour vous, Monsieur, que j'ajoute le suivant: La prémière chose que j'avais trouvée (dans le mars 1827), c'était l'équation $T = \frac{V\mathrm{d}U}{\mathrm{d}x} - \frac{U\mathrm{d}V}{\mathrm{d}x}$; de là je reconnus que, pour un nombre n quelconque, la détermination était un problème d'Analyse algébrique déterminé, le nombre des constantes arbitraires égalant toujour celui des conditions. Au moyen des coefficients indéterminés, je formai les transformations relatives aux nombres 3 et 5 [...] Par un tâtonnement heureux, je remarquais dans ceux deux cas l'autre transformation complémentaire pour la multiplication. Là j'écrivis ma prémière lettre à M. Schumacher, la méthode étant générale et vérifiée par des examples. Depuis, examinant plus de proche les deux substitutions $z = \frac{ay+by^3}{1+c^2y^2}$, $y = \frac{a'x+b'x^3}{1+c'x^2}$ sous la forme présentée dans ma première lettre, je vis qu'etant mis $x = \mathrm{sinam}\frac{2K}{3}$, z devra s'évanouir, et comme, dans ladite forme, $\frac{b}{a}$ était positif, j'en conclus que y devra s'évanouir aussi. De cette manière je trouvai par induction la résolution en facteurs, laquelle étant confirmée par des examples, je donnai le théorème général dans ma seconde lettre à Schumacher (Jacobi's Werke I, 415–416).

referred to in the above letter to Legendre as $z = \frac{ay+by^3}{1+c^2y^2}$, was written in (Jacobi 1827b, p. 34) in the following trigonometric form:[14]

$$\sin\psi = \frac{\sin\phi\left[ac + \left(\frac{a-c}{2}\right)^2 \sin^2\phi\right]}{c^2 + \left(\frac{a-c}{2}\right)\left(\frac{a+3c}{2}\right)\sin^2\phi}. \tag{11}$$

The second transformation $y = \frac{a'x+b'x^3}{1+c'x^2}$ took on the following form:

$$\sin\phi = \frac{\sin\theta\left[-3ac + \left(\frac{a+3c}{2}\right)^2 \sin^2\theta\right]}{a^2 - 3\left(\frac{a-c}{2}\right)\left(\frac{a+3c}{2}\right)\sin^2\theta}. \tag{12}$$

We are now in the position to fully understand Jacobi's observation. As a consequence of the relation

$$\frac{\mathrm{d}\psi}{\sqrt{1-\kappa\sin^2\psi}} = \frac{3\mathrm{d}\theta}{\sqrt{1-\kappa\sin^2\theta}} \quad \text{where } \kappa = \left(\frac{a-c}{2c}\right)\left(\frac{a+3c}{2a}\right)^3, \tag{13}$$

it is clear that z ($= \sin\psi$), when considered as a function of x ($= \sin\theta$), vanishes when $x = \mathrm{sinam}\frac{2K}{3}$. Indeed, if $F(\psi, \sqrt{\kappa}) = 3F\left(\mathrm{am}\frac{2K}{3}, \sqrt{\kappa}\right) = 2K$, then, since

$$\int_0^{\pi/2} \frac{\mathrm{d}\psi}{\sqrt{1-\kappa\sin^2\psi}} = \int_{\pi/2}^{\pi} \frac{\mathrm{d}\psi}{\sqrt{1-\kappa\sin^2\psi}},$$

$\psi = \pi$ and $z\left(x = \mathrm{sinam}\frac{2K}{3}\right) = 0$. Now, in order to conclude that $y\left(\mathrm{sinam}\frac{2K}{3}\right) = 0$, we have to prove that the product ac in (11) is positive. Indeed if, by absurdum $y\left(\mathrm{sinam}\frac{2K}{3}\right) \neq 0$, we would have

$$ac + \left(\frac{a-c}{2}\right)^2\left[y\left(\sin\mathrm{am}\frac{2K}{3}\right)\right]^2 = 0,$$

which contradicts the requirement that $ac > 0$.

In order to prove this, we observe that the transformation (11) satisfies the following differential relation

[14] With a slight change in notation. A minor misprint contained in the first formula of *Théorème I* (Jacobi 1827b, 34) should be pointed out. The square outside the square bracket must be shifted to the term $\sin\phi$. A corrected version of the formula is contained in (Jacobi's Werke, I, 32).

$$\frac{d\psi}{\sqrt{a^3c - \left(\frac{a-c}{2}\right)\left(\frac{a+3c}{2}\right)^3 \sin^2\psi}} = \frac{d\phi}{\sqrt{ac^3 - \left(\frac{a-c}{2}\right)^3\left(\frac{a+3c}{2}\right)\sin^2\phi}} \tag{14}$$

If we insist, as Jacobi did, that the modules of the two elliptic integrals lie between -1 and 1, we see that $1 < \frac{a}{c} < 3$ and thus that $ac > 0$, which is what was to be proved. In this respect, it is interesting to note that the limitation of Jacobi's analysis to the real domain, though provisional, turned out to be providential at this stage of the discovery process.

By pursuing the heuristic path here described, Jacobi thus deduced the correct factorization of the numerator of (7) in the order 3 case. Indeed, as a consequence of the oddness of the function (12), the existence of the other zeros, $-\mathrm{sinam}\frac{2K}{3}$ and 0, follows immediately. As for the generalization of the formula to any prime number p, at least until November 1827, it remained a fortunate and audacious conjecture, despite the fact that Jacobi had venture claimed it already in August of that year. As he confessed to Legendre, the proof of the theorem was attained only at a later stage.[15] In this respect, it is interesting to emphasize that Jacobi's confidence in the efficiency of his techniques extended well beyond his capability of providing rigorous proofs. Of course, as with the whole of mathematical knowledge, standards and rules for public communication of mathematical results are not immutable entities; as such, they too are subject to developments imposed by history. But at the same time, Gauss's above-mentioned sharp reproach of the bad habit of publishing statements of theorems without contextual proof is a sign of a changing attitude toward standards of rigor and prerequisites for publication. Finally, I suggest that Jacobi's audacity could be explained by his faith in the purely algebraic method that consists in the counting of arbitrary constants and conditional relations that might have induced him to underestimate caveats like those pointed out by Poisson in his *Rapport*.

4 Further developments

On September 9, 1828, Jacobi wrote to Legendre that his researches of the last 18 months, in which all of his energy and attention had been absorbed, night and day, by the study of elliptic functions, would be gathered into a *petit ouvrage* under the title: *Fundamenta Nova Theoriae Functionum Ellipticarum*. Despite Jacobi's modest estimate, the result was by no means *petit*; on the contrary, the treatise stood out for his thoroughness and systematic character, which assured him a long-lasting influence upon investigations to come.

As for transformation theory, to which the first part of the *Fundamenta Nova* was devoted, Jacobi did not limit himself to presenting his previous results again. On the contrary, he conveyed a reformulation of the entire theory which provided a powerful extension of the version contained in (Jacobi 1827c).

[15] (Jacobi's Werke, I, 416).

In this respect, to be sure, the reading of Abel's works played an essential role in shaping Jacobi's reformulation of his previous ideas, as the epistolary exchange with Legendre witnesses. The centrality of the inversion of elliptic integrals, the recourse to the complexification of the argument of the function sinam$u$, and the focus on double periodicity should be regarded as a direct manifestation of Abel's decisive influence. This is not to underestimate Jacobi's contributions to elliptic function theory. On the contrary, the historians' problem consists in placing his contributions in the proper context. While, despite their undisputed ingenuity and sagacity, Jacobi's early investigations were more in the spirit of Legendre's work, the *Fundamenta Nova* represented a departure from that tradition. Although Jacobi maintained his own priorities and peculiarities, he was able to enrich them with the stimuli which he had long been absorbing from Abel.

Not only was transformation theory endowed with a brand new depth, but also its far reaching consequences were carefully investigated. As for transformation theory, the introduction of complex arguments for the function sinam$u$—*via* what Jacobi called *théorème fondamental de Abel*, i.e., the equation $\mathrm{sinam}(i\xi, k) = i\tan(\xi, k')$ (with $k'^2 = 1 - k^2$)—played a dominant role. It allowed Jacobi to extend the number of possible rational transformations of a given order; it turned out that there exist $p+1$ of them if p is the order of the transformation.

Perhaps, the more striking consequence of this new systematization consisted in the well-known development of elliptic functions sinam$u$ and cosam$u$ into ratios of infinite products. These achievements directly stemmed from transformation theory. Indeed, Jacobi succeeded in obtaining a special type of transformation, which he called *prima transformatio supplementaria*, that allowed him to express the function $\mathrm{sinam}(nu, k)$ in terms of $\mathrm{sinam}\left(\frac{u}{M}, \lambda\right)$. By replacing u with $\frac{u}{n}$ and taking the limit for $n \to \infty$, he obtained the following expansion for sinam$u$:

$$\mathrm{sinam}u = \frac{2Ky}{\pi} \frac{\prod_{n=1}^{\infty}\left(1 - \frac{y^2}{\sin^2 \frac{niK'}{2K}}\right)}{\prod_{n=1}^{\infty}\left(1 - \frac{y^2}{\sin^2 \frac{(2n-1)iK'}{K}}\right)} \tag{15}$$

where $y = \sin\frac{\pi u}{2K}$ and $K = \int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}}$, $K' = \int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1-(1-k^2)^2\sin^2\phi}}$.

This and similar formulae were exploited to obtain the well-known infinite product developments of the module k and the complete integral K with respect to

$$q = \exp -\frac{K}{K'}.$$

A crucial step was finally represented by the introduction of the theta function

$$\Theta(u) = \Theta(0)\exp\int_0^u Z(t)\mathrm{d}t, \tag{16}$$

where $Z(t)$ was defined in terms of the elliptic integrals of first and second kind, which was to play a completely anew foundational role in his later formulation of the theory.

In Jacobi's skillful hands, elliptic function theory and its plethora of new formulae attained complete autonomy. The *Fundamenta Nova* somehow marked the dawn of an independent discipline that, along with many fruitful applications in the realm of number theory and projective geometry, was to acquire a decisive role in the development of nineteenth-century analysis.

5 Appendix: algebraic derivation of third-order transformation

The first published version of the algebraic deduction of the existence of third-order transformations is contained in Jacobi's *Fundamenta Nova*. Such a delayed appearance somehow betrayed the actual historical development of the theory. Nonetheless, at the same time, the fact that Jacobi decided to include it in his first systematic presentation of the dawning theory elliptic functions is highly indicative of the strategic role played by algebraic considerations in the discovery path pursued by Jacobi. Even more so, since the algebraic treatment of transformations of order 3 (and 5) was presented as a useful exemplification of an extensive preliminary, algebraic investigation on transformation theory.

Before discussing the particular case of order 3, it is first necessary to carry out some general consideration concerning transformations of odd order $p = 2m + 1$. The case of interest will then represent the easiest possible eventuality ($m = 1$).

By following Jacobi, we will consider a general rational transformation of the following form:

$$y = \frac{U}{V} = \frac{x\left(a + a'x^2 + a''x^4 + \cdots + a^{(m)}x^{2m}\right)}{1 + b'x^2 + b''x^4 + \cdots + b^{(m)}x^{2m}}. \tag{17}$$

In order to attain the desired transformation, we have to require the following equations to be true:

$$\begin{aligned} V + U &= (1+x)A^2(x), && (18)\\ V - U &= (1-x)B^2(x), && (19)\\ V + \lambda U &= (1+kx)C^2(x) && (20)\\ V - \lambda U &= (1-kx)D^2(x). && (21) \end{aligned}$$

As for the counting of conditional relations and arbitrary coefficients, to which Jacobi referred in his April 1828 letter to Legendre, it seems useful to recall what Jacobi wrote

> Here too, one has to satisfy equations (1), (3) [i.e. (18), (20)] only, from which, by changing x in $-x$, the other two follow directly. In order for $V + U$ and $V + \lambda U$ to satisfy these equations it is necessary that each of them contains m linear factors pairwise equal; to this end, $2m$ conditional equations must be satisfied to which one has to add that $V + U$ contains the factor $(1+x)$. Therefrom it follows

> that the number of conditional equations is $2m + 1$ which coincides with the number of undetermined quantities $a, a', a'', \ldots, a^{(m)}; b', b'', \ldots, b^{(m)}$. Hence, the problem is determined in this case too.[16]

At this point, Jacobi introduced a lemma which highly simplifies the task. Referring to equations (18) and (20) above, he states the following:

> I affirm that if the functions U, V are determined in such a way that, by replacing x with $\frac{1}{kx}$, $y = \frac{U}{V}$ is transformed in $\frac{1}{\lambda y} = \frac{V}{\lambda U}$, then any of the two equations directly follows from the other.[17]

Thus, if one puts $U = xF(x^2)$, $V = \phi(x^2)$ and one supposes, according to the previous lemma, that $F(x^2) = \sqrt{\frac{k^{2m+1}}{\lambda}} x^{2m}\phi\left(\frac{1}{k^2x^2}\right)$, then the only condition to be imposed reduces to the requirement that the quantity

$$\frac{\phi(x^2) + \sqrt{\frac{k^{2m+1}}{\lambda}} x^{2m}\phi\left(\frac{1}{k^2x^2}\right)}{1+x} = \frac{V+U}{1+x} \tag{22}$$

is a square factor.

Now, in order to obtain the expression for the 3rd-order transformation, let us suppose $m = 1$. Thus, we have $U = x(a + a'x^2)$ and $V = 1 + b'x^2$. In addition, by following the prescription of the above lemma, we require that $a = \sqrt{\frac{k}{\lambda}}\frac{b'}{k}$ and $a' = \sqrt{\frac{k^3}{\lambda}}$.

As a consequence of this, if we assume A^2 to be of the form $(1+\alpha x)^2$ it is sufficient to require that $V + U = (1 + x)A^2$. We thus find the following:

$$a = 1 + 2\alpha, \quad a' = \alpha^2, \quad b' = \alpha(2+\alpha), \quad \text{where} \quad \alpha = \sqrt[4]{\frac{k^3}{\lambda}}. \tag{23}$$

Jacobi thus easily obtained the transformation sought for which, despite a different choice of parametrization, coincided with the one conveyed in the first letter to Schumacher (Jacobi 1827b, 34):

$$y = \frac{(1+2\alpha)x + \alpha^2x^3}{1+\alpha(2+\alpha)x^2}. \tag{24}$$

16 Hic quoque solummodo aequationibus (1), (3) [i.e. (18), (20)] satisfaciendum erit, quippe e quibus mutando x in $-x$ duae reliquae sponte manant. Ut illis satisfiat, et $V + U$ et $V + \lambda U$ singulae m vicibus duos inter se aequales habeant factores lineares necesse est, quem in finem $2m$ aequationibus conditionalibus satisfaciendum erit, quibus una accedit, ut insuper $V + U$ nanciscatur factorem $(1 + x)$. Hinc numerum aequationum conditionalium esse videmus $2m + 1$, qui et ipse est numerus indeterminatarum $a, a', a'', \ldots, a^{(m)}; b', b'', \ldots, b^{(m)}$. Unde et hoc casu determinatum est problema (Jacobi 1829, §10).

17 Iam dico, si quidem ita functiones U, V determinentur, ut, loco x posito $\frac{1}{kx}$, abeat $y = \frac{U}{V}$ in $\frac{1}{\lambda y} = \frac{V}{\lambda U}$, aequationes illas alteram ex altera sponte sequi (Jacobi 1829, §12).

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