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A new reading of Archytas' doubling of the cube and its implications

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Abstract The solution attributed to Archytas for the problem of doubling the cube is a landmark of the pre-Euclidean mathematics. This paper offers textual arguments for a new reading of the text of Archytas' solution for doubling the cube, and an approach to the solution which fits closely with the new reading. The paper also reviews modern attempts to explain the text, which are as complicated as the original, and its connections with some XVI-century mathematical results, without any documented relation to Archytas' doubling the cube.

1 Introduction

The solution attributed to Archytas (V–IV century BC) for the problem of doubling the volume of a given cube (henceforth, Archytas' solution for doubling the cube) has been unanimously praised as one of the landmarks of pre-Euclidean mathematics: «the most remarkable of all [i.e. all methods to double the volume of a cube]» (Heath 1921, p. 246, v. I); «Archytas must have had a truly divine inspiration when he found this construction» (van der Waerden 1954, p. 151); «a stunning *tour de force*» (Knorr 1986, p. 50). The configuration of geometrical elements requested by the solution is actually complex, and the reconstructions suggested by scholars « are more or less as complicated as the original».[1]

[1] See Euclide (2007, p. 83): «Sono state proposte varie ricostruzioni di come Archita possa essere giunto a concepire questo complesso intersecarsi di solidi: sono più o meno complicate come l'originale» («A number of reconstructions of the way in which Archytas made this complex intersection of solids has been suggested: they are more or less as complicated as the original»).

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This paper offers textual arguments for a new reading of the text of Archytas' solution for doubling the cube, transmitted by Eutocius (VI century AD). With this new reading, I will review modern attempts to explain or approach the text. In addition, I will offer my own approach which fits closely with the new reading.

In the first section, I present the generic problem of doubling the cube, and its reduction to the search for two mean proportionals. In the second section, the two main historical sources of Archytas' solution for doubling the cube are discussed, with particular emphasis on Eutocius' text. I will make a literal translation of the text discussing some textual issues. Then, I will analyse the structure of the text, mainly from a formal point of view. The next two sections provide some ideas about how to place this text in the context of Greek Mathematics, being aware that we have no textual evidence that supports this sort of reconstruction, except for the ones already discussed. The first of these sections attempts to reconstruct a general mathematical context for Archytas' solution for doubling the cube, including a list of mathematical requirements for carrying it out, as well as some considerations concerning the genesis of the reconstruction and its links with textual evidence. The subsequent section reviews some modern reconstructions; I highlight the fact that these modern reconstructions do not recognize some important points derived from the wording of Archytas' solution as transmitted by Eutocius; I also present some XVI century mathematical results, without any documented connection to Archytas' solution for doubling the cube, that influenced in some way a number of these reconstructions. Finally, in the conclusion, I will list the main consequences of this new reading, from the interpretation of the text to its re-evaluation.

2 Establishing the elements of the problem of doubling the cube

In this section and in some other sections, modern notation is used to describe geometrical manipulations, although this misrepresents Greek language and concepts. This anachronism does not interfere at all with the core of my arguments and therefore should not lead to any confusion.

The statement of the problem of doubling the cube is as follows: given a cube, find another cube whose volume is its double. A number of ancient geometrical solutions to this problem are based on the finding of two mean proportionals.

The concept of the mean proportional is essentially as follows: given two magnitudes, a and b, m is the mean proportional between a and b (which I call *simple* to avoid any confusion) when $\frac{a}{m} = \frac{m}{b}$. On the other hand, m_1, m_2 are two mean proportionals between a and b when $\frac{a}{m_1} = \frac{m_1}{m_2} = \frac{m_2}{b}$. In general, $a, m_1, m_2, \ldots, m_{n-1}, m_n, b$ are in *continued proportion* when $\frac{a}{m_1} = \frac{m_1}{m_2} = \cdots = \frac{m_{n-1}}{m_n} = \frac{m_n}{b}$. Therefore, if m_1 and m_2 are two mean proportionals between a and b, then a, m_1, m_2, b are in continued proportion.

The relation between doubling the cube and the finding of two mean proportionals is as follows: if $V = a^3$ is the volume of a cube of edge a, the edge of the doubled cube is the first of the two mean proportionals between a and $2a$. To see this, let m_1 and m_2 be the two mean proportionals between a and $2a$, $\frac{a}{m_1} = \frac{m_1}{m_2} = \frac{m_2}{2a}$. Therefore, $\left(\frac{a}{m_1}\right)^3 = \frac{a}{m_1} \cdot \frac{m_1}{m_2} \cdot \frac{m_2}{2a} = \frac{a}{2a}$. That is to say, $2a^3 = m_1^3$. As a consequence, m_1 is the edge of a cube that doubles the volume of a cube with edge a, and therefore, it solves the problem of doubling the volume of a given cube.

3 Textual tradition of Archytas' doubling the cube

Archytas' solution has been transmitted by Eutocius, in his commentary on Book 2 of Archimedes' *De Sphaera et Cylindro*.[2] Archytas' solution in Eutocius' text is as follows (parallel text of Heiberg's edition and my translation; the numbering is mine):

[1] Ἔστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι αἱ ΑΔ, Γ. δεῖ δὴ τῶν ΑΔ, Γ δύο μέσας ἀνάλογον εὑρεῖν. [2] γεγράφθω περὶ τὴν μείζονα τὴν ΑΔ κύκλος ὁ ΑΒΔΖ, καὶ τῇ Γ ἴση ἐνηρμόσθω ἡ ΑΒ καὶ ἐκβληθεῖσα συμπιπτέτω τῇ ἀπὸ τοῦ Δ ἐφαπτομένῃ τοῦ κύκλου κατὰ τὸ Π, παρὰ δὲ τὴν ΠΔΟ ἤχθω ἡ ΒΕΖ,	[1] Let there be two given straight lines ΑΔ, Γ. Then, one must find two mean proportionals of ΑΔ, Γ. [2] Let a circle ΑΒΔΖ be traced around the greater <straight line> ΑΔ, and let <a straight line> ΑΒ be fitted equal to Γ,[c] and, once produced, let it converge with the <straight line> from Δ tangent to the circle at Π, and let a <straight line> ΒΕΖ be drawn parallel to ΠΔΟ,
καὶ νενοήσθω ἡμικυλίνδριον ὀρθὸν ἐπὶ τοῦ ΑΒΔ ἡμικυκλίου, ἐπὶ δὲ τῆς ΑΔ ἡμικύκλιον ὀρθὸν ἐν τῷ τοῦ ἡμικυλινδρίου παραλληλογράμμῳ κείμενον.	and let a half-cylinder be conceived perpendicular to semicircle ΑΒΔ, and, on ΑΔ, a semicircle orthogonal <on semicircle ΑΒΔ>, lying in the parallelogram of the half-cylinder.
[3] τοῦτο δὴ τὸ ἡμικύκλιον περιαγόμενον ὡς ἀπὸ τοῦ Δ ἐπὶ τὸ Β μένοντος τοῦ Α πέρατος τῆς διαμέτρου τεμεῖ τὴν κυλινδρικὴν ἐπιφάνειαν ἐν τῇ περιαγωγῇ καὶ γράψει ἐν αὐτῇ γραμμήν τινα. πάλιν δέ, ἐὰν τῆς ΑΔ μενούσης τὸ ΑΠΔ τρίγωνον περιενεχθῇ τὴν ἐναντίαν τῷ ἡμικυκλίῳ κίνησιν, κωνικὴν ποιήσει ἐπιφάνειαν ἡ ΑΠ εὐθεῖα,[a] ἣ[b] δὴ περιαγομένη συμβαλεῖ τῇ κυλινδρικῇ γραμμῇ κατά τι σημεῖον.	[3] Then, this semicircle drawn around from Δ towards Β,[d] while the extreme Α of the diameter remains fixed, will cut the cylindrical surface in its being drawn around, and will trace a certain trace on it. Again, if, while ΑΔ remains fixed, triangle ΑΠΔ is led around with a motion opposite to the semicircle, straight line ΑΠ will make a conical surface, which <straight line>, then, once drawn around, will concur with the cylindrical trace at a certain point.
ἅμα δὲ καὶ τὸ Β περιγράψει ἡμικύκλιον ἐν τῇ τοῦ κώνου ἐπιφανείᾳ. [4] ἐχέτω δὴ θέσιν κατὰ τὸν τόπον τῆς συμπτώσεως τῶν γραμμῶν τὸ μὲν κινούμενον ἡμικύκλιον ὡς τὴν τοῦ ΔΚΑ, τὸ δὲ ἀντιπεριαγόμενον τρίγωνον τὴν τοῦ ΔΛΑ, τὸ δὲ τῆς εἰρημένης συμπτώσεως σημεῖον ἔστω	And at the same time, Β will also trace around a semicircle on the surface of the cone. [4] Then, on the one side, let the moved semicircle have a position at the place of convergence of the traces as the <position> of <semicircle> ΔΚΑ; on the other side, <let> the triangle drawn around in the opposite direction <have a position

[2] Archimedes (1910–1915, vol. III, 84.12–88.2). The context is a long section of the commentary in which Eutocius reviews different methods used by Greek authors of antiquity for solving the problem of doubling the cube. The heading of Archytas' solution for doubling the cube reads: «the discovery of Archytas, as reported by Eudemus». Eudemus was a disciple of Aristotle, who wrote a history of geometry. For full details about the whole textual evidence, see Euclide (2007, pp. 79–81) and Knorr (1989, pp. 100–111).

τὸ K, ἔστω δὲ καὶ τὸ διὰ τοῦ B γραφόμενον ἡμικύκλιον τὸ BMZ, κοινὴ δὲ αὐτοῦ τομὴ καὶ τοῦ BΔZA κύκλου ἔστω ἡ BZ, καὶ ἀπὸ τοῦ K ἐπὶ τὸ τοῦ BΔA ἡμικυκλίου ἐπίπεδον κάθετος ἤχθω.

as> that of ΔΛA; and let the point of the said convergence be K; also, let the semicircle traced through B be BMZ, let the common section of this <semicircle> and of circle BΔZA be BZ and let <a straight line> be drawn from K perpendicular to the plane of semicircle BΔA.

πεσεῖται δὴ ἐπὶ τὴν τοῦ κύκλου περιφέρειαν διὰ τὸ ὀρθὸν ἑστάναι τὸν κύλινδρον.

Then, it will fall on the circumference of the circle because of the fact that the cylinder stands orthogonal <on the circumference>.

πιπτέτω καὶ ἔστω ἡ KI, καὶ ἡ ἀπὸ τοῦ I ἐπὶ τὸ A ἐπιζευχθεῖσα συμβαλέτω τῇ BZ κατὰ τὸ Θ, ἡ δὲ AΛ τῷ BMZ ἡμικυκλίῳ κατὰ τὸ M, ἐπεζεύχθωσαν δὲ καὶ αἱ KΔ, MI, MΘ.

Let it fall and let it be KI, and let the <straight line> joined from I to A concur with BZ at Θ, and let AΛ <concur> with semicircle BMZ at M, and let KΔ, MI, MΘ be also joined.

[5] ἐπεὶ οὖν ἑκάτερον τῶν ΔKA, BMZ ἡμικυκλίων ὀρθόν ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ ἡ κοινὴ ἄρα αὐτῶν τομὴ ἡ MΘ πρὸς ὀρθάς ἐστι τῷ τοῦ κύκλου ἐπιπέδῳ. ὥστε καὶ πρὸς τὴν BZ ὀρθή ἐστιν ἡ MΘ.

[5] Now, since each of the semicircles ΔKA BMZ is perpendicular to the underlying plane, therefore, their common section MΘ is also at right <angles> with the plane of the circle, so that also MΘ is perpendicular to BZ.

τὸ ἄρα ὑπὸ τῶν BΘZ, τουτέστι τὸ ὑπὸ AΘI, ἴσον ἐστὶ τῷ ἀπὸ MΘ.

Therefore, the <rectangle contained> by BΘZ, that is, the <rectangle contained> by AΘI, is equal to the <square> on MΘ.

ὅμοιον ἄρα ἐστὶ τὸ AMI τρίγωνον ἑκατέρῳ τῶν MIΘ, MAΘ, καὶ ὀρθὴ ἡ ὑπὸ IMA. ἔστιν δὲ καὶ ἡ ὑπὸ ΔKA ὀρθή.

Therefore, triangle AMI is similar to each of <the triangles> MIΘ, MAΘ, and angle AMI <is> right. But angle ΔKA is also right.

παράλληλοι ἄρα εἰσὶν αἱ KΔ, MI, καὶ ἔσται ἀνάλογον, ὡς ἡ ΔA πρὸς AK, τουτέστιν ἡ KA πρὸς AI οὕτως ἡ IA πρὸς AM, διὰ τὴν ὁμοιότητα τῶν τριγώνων.

Therefore, KΔ, MI are parallel, and, as ΔA is to AK, that is, KA <is> to AI, so IA will be in proportion to AM, because of the similarity of the triangles.

τέσσαρες ἄρα αἱ ΔA, AK, AI, AM ἑξῆς ἀνάλογόν εἰσιν. καί ἐστιν ἡ AM ἴση τῇ Γ, ἐπεὶ καὶ τῇ AB.

Therefore, four <straight lines> ΔA, AK AI, AM are in continued proportion, and AM is equal to Γ, since it <is> also <equal to> AB.

[6] δύο ἄρα δοθεισῶν τῶν AΔ, Γ δύο μέσαι ἀνάλογον ηὕρηνται αἱ AK, AI.

[6] Therefore, two mean proportionals AK AI have been found of two given <straight lines> AΔ, Γ.

[a] We read τῆς AΠ εὐθείας in the manuscripts, but Heiberg corrected the text following the Basiliensis edition as τῇ AΠ εὐθείᾳ. ἡ εὐθεῖα is a better correction, because the verb ποιέω/to make usually has a subject, which is a mathematical object. This correction is of no consequence to the usual interpretation of the passage

[b] This relative pronoun is rather unusual in this form (the relative ὅστις is the usual form but not very frequent). A feminine article (with an implicit noun «trace») could be argued to be a good alternative but at the price of a grammatical and pragmatic issue: the phrase ἡ περιαγομένη <γραμμή>, «the <trace> drawn around» (i.e. the intersection of the conical and the semi-cylindrical surfaces), has no explicit antecedent and no previous definition

[c] AB is a chord of the circle, but the text does not say so explicitly. It is deduced easily from the fact that A and B are letters contained in the name of the circle, ABΔZ. The use of the verb ἐναρμόσθω/to fit probably points to a quotation from Euclid, *Elementa* IV.1

[d] Translations usually read «as from Δ towards B», but the use of ὡς + preposition is a typical late use that only reinforces the preposition

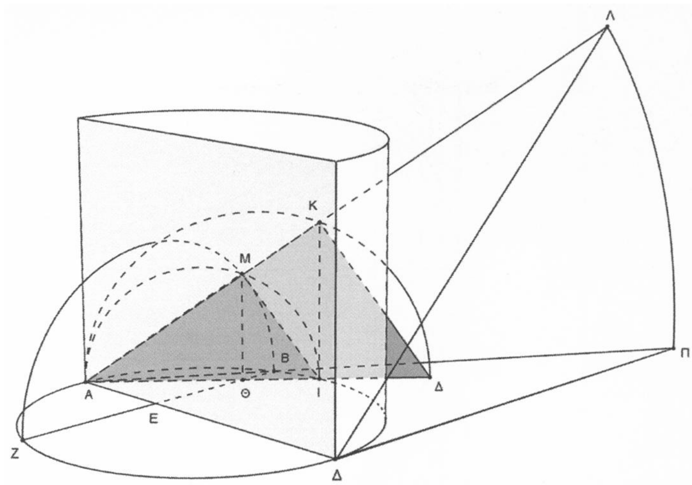


Fig. 1 Three-dimensional view of Archytas' solution for doubling the cube, following the planar representation of Fig. 2. The point O, in the prolongation of line ΠΔ, has been omitted as well as line Γ

The translation is very close to the Greek text, almost literal[3] and so it may seem a bit odd in some passages. I have also reproduced the figure as it appears in the manuscripts (see Fig. 2) and a modern reading of it (see Fig. 1).

The Arabic tradition has also transmitted a text quite close to Eutocius' text. This version appears as proposition 16 of the *Verba filiorum*, a geometric treatise by the three brothers Banû Mûsâ (Baghdad, s. IX). The older version of the text is the Latin translation of Gerard of Cremona (s. XII).[4] The structures of Eutocius and the Banû Mûsâ's texts are nearly the same, and their steps have nearly the same order, except for one small detail at the end of the solution. However, the sources used by the two texts do not seem to be exactly the same, since there are two fundamental differences in the Banû Mûsâ's text:

- The text attributes the construction to Menelaus (first century BC) and does not mention Archytas.[5]
- The stated purpose of the construction of two mean proportionals is the search for the edge of a cube, given its volume, and its use for doubling the cube is not

[3] Not only at a lexical level; I have tried to preserve grammatical categories as well (translating verbs into verbs, nouns into nouns, and so on, and even on lower grammatical levels, e.g. participles into participles). I have tried to preserve the *figurae etymologicae*, so that words with the same root are translated using the same common root: e.g. γράφω/γραμμή, to trace (verb)/trace (noun; its usual translation is «line»); ἄγω /περιάγω /ἀντιπεριάγω, to draw/to draw around/to draw around in the opposite direction. The translations of Knorr (1989, pp. 102–107) and Thomas (1951, pp. 284–289) [and Euclide (2007, pp. 82–83), in Italian] are also very close to the original text. van der Waerden (1954, pp. 150–151) is a paraphrase, and Heath (1921, vol. I, pp. 246–249) and Knorr (1986, pp. 50–52) are commentaries on the text.

[4] Edited and commented on by Clagett (1964–1984, vol. I, chapter 4, pp. 223–367). There is a detailed comparative analysis of both texts in Knorr (1989, pp. 100–110).

[5] In fact, the Latin text of Gerard of Cremona reads «Mileus» but the most likely reading from the Arabic text is «Menelaus» (Clagett 1964–1984, vol. I, p. 365, prop. 16, 12).

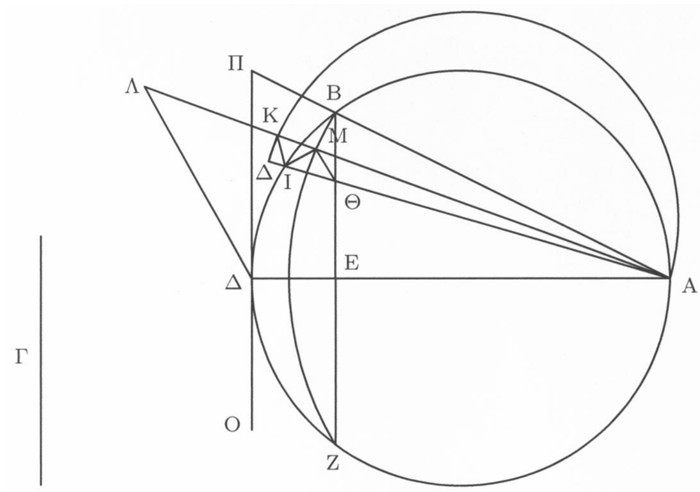


Fig. 2 Archytas' solution for doubling the cube, as shown in manuscripts

mentioned. It is a more *modern* way to enunciate the problem: more general and more *algebraic*.[6]

3.1 Parts of the text

Greek geometrical treatises transmitted from Antiquity can be divided up into *propositions*. Each proposition usually sets out a geometrical property, which is proved thereupon. The proof usually includes a constructive process (whose outcome is a geometrical construction) and a demonstrative process (whose outcome is a demonstration). Both processes are mixed throughout the proof.[7]

Some kinds of propositions are called *problems*, and we can recognize a problem either by its content or by its form. A *problem* requires the construction of some kind of

[6] In this case, the goal is to find two mean proportionals between V (whose cube root is sought) and 1. If m_1, m_2 are the two mean proportionals, then: $\frac{V}{m_1} = \frac{m_1}{m_2} = \frac{m_2}{1}$. Therefore, $\frac{V}{m_1} \cdot \frac{m_1}{m_2} \cdot \frac{m_2}{1} = \left(\frac{m_2}{1}\right)^3$. So, $V = m_2^3$, i.e. m_2 is the cube root of V, its side. The setting up of the problem as shown in the Banû Mûsâ's text is highly unlikely to be of Greek origin, because V is a volume and m_2 is a length; Greek Mathematics never build proportions with non-homogeneous magnitudes (numbers, lengths and volumes). In fact, the presence of the unity in the Arabic text allows us to suggest that the Banû Mûsâ are thinking of the numbers that are the result of the measures, and this approach is impossible from the Greek point of view. At any rate, the Banû Mûsâ only explicitly attribute the finding of two mean proportionals to Menelaus, not its use in the extraction of the «side of a cube».

[7] Proclus (s. V BC), *In Primis Euclidis*, pp. 203.1–207.25, shows the canonical division into parts of a Greek mathematical proposition (see also Netz 1999; Euclide 2007, pp. 259–312). We do not strictly follow this division, although some of the terms I use are identical to the English translation of Proclus terms. The division into parts I use is based upon the form and the content of Eutocius' text. Therefore, words such as *proof*, *demonstration*, *construction* and *statement* have the usual sense and not the technical sense in Proclus' division. Only the terms *proposition* and *problem* are used in its technical sense, and for this reason, they are written in italics in this section.

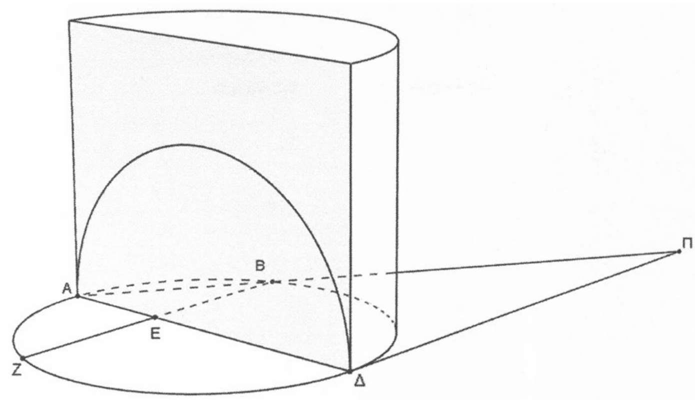


Fig. 3 *Initial Configuration* of geometrical elements as described in step [2] of the text

geometrical configuration, and the statement of the *problem* always includes a phrase of this kind: δεῖ+ infinitive, «it should be found [...]», as in our text. Therefore, Archytas' text is formally written as a *problem* and [1] could be considered the statement of the problem.

The group of the following three steps, i.e. steps [2]–[4], which I will refer to as *Archytas' Construction*,[8] builds two configurations of geometrical elements (henceforth, *Initial Configuration* and *Final Configuration*) with some common elements, and the way to transform the *Initial Configuration* into the *Final Configuration* (henceforth, *Transition*):

[2] *Initial Configuration* (see Fig. 3): there is a circle ABΔZ of a given diameter AΔ, a chord on it AB of a given length and a tangent to the circle in Δ. AB is produced to cut the tangent in Π. BEZ is a chord parallel to ΠΔ. A half-cylinder is drawn perpendicular to semicircle ABΔ, and a semicircle perpendicular to the semicircle ABΔ (henceforth, semicircle *SC*) is also drawn on the plane face of the half-cylinder.[9]
From a formal point of view, the text uses the imperative form, passive in most cases, to introduce the geometrical elements. This is a well-known characteristic of Greek mathematical style (see Acerbi 2012).

[3] *Transition* (see Fig. 4): this step shows how to obtain the *Final Configuration* from the *Initial Configuration*. First of all, a motion that does not generate any surface is described: the semicircle *SC* rotates from its position in the *Initial Configuration*, revolving around A but remaining always perpendicular to the circle ABΔ. Then, triangle AΔΠ is revolved around AΔ and generates a cone. Derived from these two motions (the one of the semicircle *SC* and the other of the triangle AΔΠ), the *curved* traces that allow point K to be found (only mentioned as «a certain

[8] Henceforth and for clarity, I define explicitly some geometrical configurations or manipulations. I always use terms in italics and initial capital letters to denote them.

[9] This semicircle, like the other geometrical elements, is explicitly mentioned in the text (see line 13R), but does not appear in the diagram of the transmitted text, and probably for that reason scholars have ignored *SC* and do not include it in their figures.

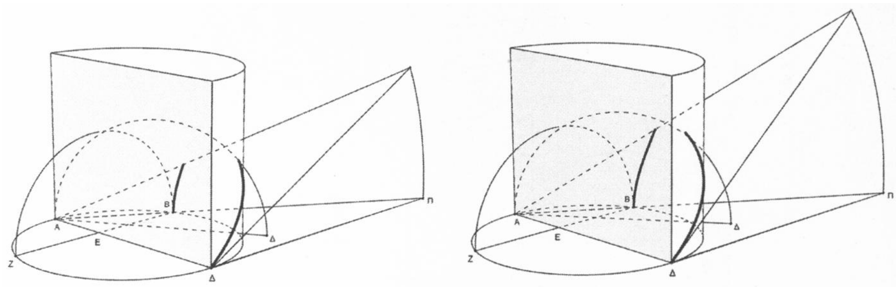


Fig. 4 Two frames (from *left* to *right*) of the *Transition* (step [3]), with the trace of the *semicircle* on the *half-cylinder* in *bold*, and also in *bold* the *trace of line* AΠ on the *half-cylinder*

point», line 28R) are introduced: the «cylindrical trace» (lines 16R–20R) and the «<straight line> drawn around» (lines 22R–26R).[10] The semicircle with diameter ZB «traced» by point B (line 29R–31R) is also introduced.

From a formal point of view, there are no imperative forms; all forms of verbs in the *Transition* are in the future. This is not a form frequently used in Greek mathematics, but it is not unusual.[11]

[4] *Final Configuration* (see Fig. 5): this configuration shows the elements as they are represented in the diagram transmitted by Eutocius' text, as the text explicitly claims in two cases.[12] The most important difference between the *Final Configuration* and the *Initial Configuration* is the position of semicircle *SC*. The exact «position» of this semicircle in the *Final Configuration* is defined as follows: «at the place of convergence of the traces» (line 33R). It is apparent that this «place» is the one mentioned a few lines before linked to a «certain point» (line 28R) in which «<the line> drawn around will concur with the cylindrical trace». This fact is confirmed a few lines after (line 38R): «let the point of the said convergence be K». The final position of the rotating triangle AΔΠ is defined in the same way.[13] Both «positions» are used to find the point of intersection, called K. Next, BZ is mentioned again to identify the intersection of *SC* and circle BΔZA. A perpendicular from K to the base semicircle ABΔ is also traced to find I. IA intersects

[10] That is to say, the rotating AΠ. An alternative reading could be the «<trace> drawn around», i.e. the intersection of the cylindrical and the conical surfaces, as discussed in note 4 (Fig. 4 also shows this trace).

[11] For example, when examining all verbal tokens in Archimedes *De Sphaera et Cylindro* (see Masià 2012, p. 204), only 5.2 % of them are future verbal tokens. Euclid and Apollonius probably use more or less the same percentage of future verbal tokens. It is usually used in the apodosis of conditional clauses like the one in lines 22R–26R.

[12] «Let the moved semicircle have a position [...] as the <position> of the <semicircle> ΔKA», line 33R–35R; «let the triangle [...] <have a position as> that of ΔΛA», lines 36R–38R.

[13] It is worth noting, again, that the diameter of the rotating semicircle *SC* is denoted in the same way in both configurations, *Initial* and *Final*. But the rotating triangle AΔΠ is denoted differently in these configurations: AΔΠ and AΔΛ. We can assume that the semicircle subsists during the motion and the rotating movement of *SC* does not generate a surface, while the line AΛ could be a line on the explicitly defined conical surface; it has different names in the two different positions: AΛ, AΠ. Then, the conical surface is also *indirectly* used to find the solution.

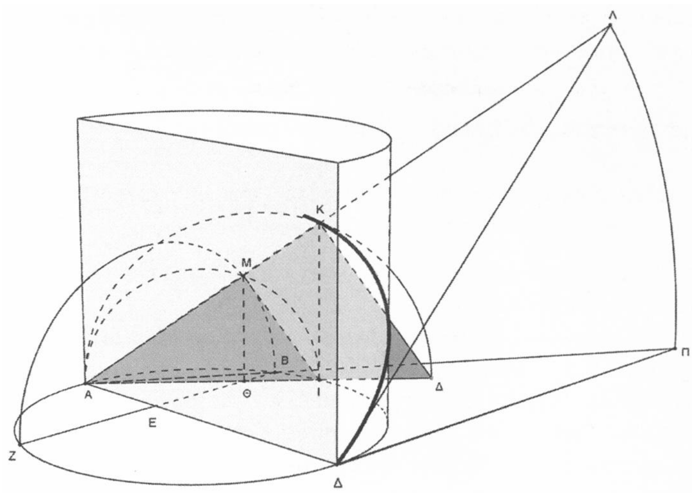


Fig. 5 *Final Configuration* of geometrical elements as described in step [4] of the text

BZ in Θ and AΛ intersects the semicircle BMZ in M. Finally, KΔ, MI, MΘ are traced.
From a formal point of view, the text returns to the imperative form to introduce the new elements, except for the future form πεσεῖται /will fall, in 41.[14]

Step [5] contains the proof that the points K, I of the triangle AKΔ at the mentioned «position» solve the problem. It is a quite regular and short proof. It begins with a usual sentence ἐπεί ... ἄρα/since ... therefore, and is introduced by οὖν/now. As usual in this part of the proposition, the number of logical particles is higher than in other parts.

Step [6] shows the conclusion, almost repeating the wording of [1], but changing the verbal form, as is usual.

After the analysis of the transmitted text, I will show in the next two sections:

- Some ideas for unfolding the process of construction and proof of Archytas' solution for doubling the cube and its mathematical requirements. These explanations follow very closely the transmitted text of Eutocius, but my goal goes further: to offer a global account of the solution, including its plausible origins and the steps whereby it was discovered. Thus, it contains considerations not mentioned in our sources; again, I only try to counterbalance some scholarly ideas or prejudices about Archytas' solution for doubling the cube, and not to show what Archytas was thinking of when he developed his solution; it is actually impossible to discover Archytas' thought processes and historiographically misleading to try to do so.

[14] The sentence with the future form is a parenthetical commentary introducing a property of the straight line from K (it will be perpendicular to some plane), justified with a postponed explanation.

- Some scholarly approaches to Archytas' solution for doubling the cube and, also, in the case of the two-dimensional reconstructions, their historical origins.

4 Some reconstructions based on the new reading

I will develop two approaches in order to provide a plausible reconstruction of Archytas' solution for doubling the cube:

1st A fully two-dimensional approach: an account of what I call *Archytas' Solution*, using easy geometrical tools.

2nd A three-dimensional approach: rewinding the motion of the elements of what I called *Archytas' Construction*, from the *Final Configuration* to the *Initial Configuration*, thereby recovering some ideas in the background of Archytas' solution for doubling the cube. The *Initial Configuration* will be the main key of the reconstruction, but also the rewriting of the problem in terms of *locus* problems.

Before I develop these two approaches, I will list all the Greek mathematical tools that my approach to the solution needs.

4.1 Mathematical tools

Only some basic technical and conceptual tools, which were available in Archytas' day, are required. They are actually reducible to the determination of the simple mean proportional between two magnitudes and some elements concerning the geometry of the sphere. More specifically, the reconstruction of Archytas' solution for doubling the cube that I propose requires knowing only:

1. One criterion for the similarity of triangles, namely the proportionality of sides: two triangles are similar if their sides are proportional.
2. The properties of a right-angled triangle inscribed in a semicircle, one of the oldest and most successful configurations of Greek geometry: if AKΔ is a triangle

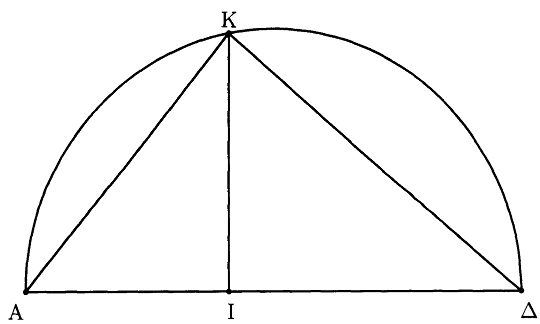


Fig. 6 *Archytas' Simple Triangle*, i.e. a *semicircle* with an inscribed right-angled triangle whose hypotenuse is the diameter and where the height of the *triangle* is KI

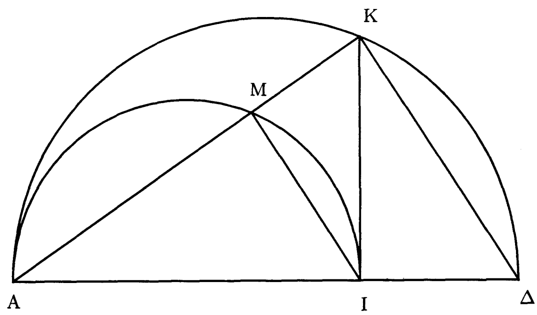


Fig. 7 *Archytas' Triangle*, i.e. two right-angled triangles inscribed in *two semicircles*, MI and ΔK parallel and KI perpendicular to AΔ. In an *Archytas' Triangle*, AK and AI are the two mean proportionals of AΔ and AM

inscribed in a semicircle and AΔ is the diameter,[15] then AKΔ is a right-angled triangle and AKΔ, AIK and KIΔ are similar triangles. Furthermore, $\frac{\mathrm{AI}}{\mathrm{IK}} = \frac{\mathrm{IK}}{\mathrm{I\Delta}}$.[16] I call this geometrical configuration *Archytas' Simple Triangle* (see Fig. 6).

3. Two properties of the sphere: (a) a great circle of the sphere[17] divides the sphere into two equal hemispheres; (b) any plane perpendicular to such a great circle intersects the sphere in two equal semicircles (and, obviously, the diameter that divides these two semicircles is a chord of the great circle).[18]

4.2 Two-dimensional approaches

As we have seen above (Sect. 2), Archytas' solution for doubling the cube comes down to the search for the two mean proportionals of two lines a and b, where $a > b$, based on the configuration of three-dimensional elements as shown in Fig. 1. The solution is derived from the triangles inscribed into the semicircle AMKΔI, as shown in two-dimensional Fig. 7.

In other words, let there be a diagram AIΔKM with right-angled triangles and semicircles AIM and AΔK (see Fig. 7, henceforth *Archytas' Triangle*). Then AI, AK are two mean proportionals of AΔ, AM, i.e. in modern notation:[19]

[15] I use the same letters A, I, K, Δ used in other figures, but the new figure only shows a general right-angled triangle inscribed into a semicircle.

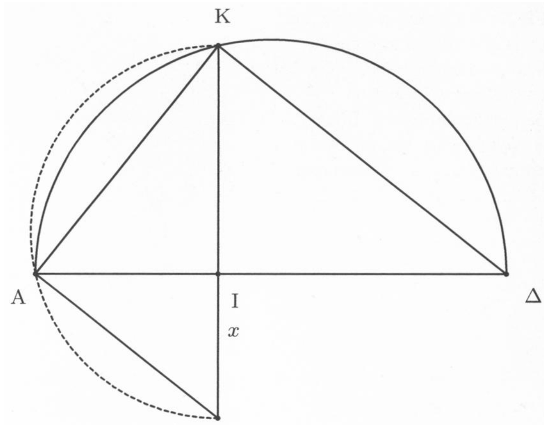
[16] Aristotle, some decades after Archytas, uses this configuration as an example in *Metaphysica* and *Analytica posteriora* (see Euclide 2007, pp. 118–119). Euclid's *Elementa* discusses the elements of this configuration in books III and VI.

[17] A great circle of a sphere is the intersection of the sphere with a plane which passes through the centre of the sphere.

[18] Solid geometry is known from very early in Greek geometry, because of its relationship with astronomy. It is also known that the division of the circle into two equal semicircles by a diameter is attributed to Thales of Miletus (s. VI BC). This result can be easily translated to the three-dimensional space: after the creation of a sphere by rotating a circle around one of its diameters, the initial circle is a great circle of the sphere, and it is apparent because of the construction that the great circle divides the sphere into two equal parts. Property 3b) could easily be derived from arguments of symmetry and the application of Thales' property of the circle.

[19] As we have said, MI is perpendicular to AK, IK to AΔ and ΔK to KA. Therefore, AΔK, AIK and AIM are similar right-angled triangles (two right-angled triangles are similar when they have a common angle in addition to the right angle). And similar triangles have proportional pairs of corresponding sides.

Fig. 8 Illustration of how to obtain two mean proportionals using the triangle AKΔI of Fig. 6 and expression 1. The outcome is equivalent to the diagram of the solution for doubling the cube attributed to Plato by Eutocius



$$\frac{\mathrm{A\Delta}}{\mathrm{AK}} = \frac{\mathrm{AK}}{\mathrm{AI}} = \frac{\mathrm{AI}}{\mathrm{AM}}$$

I will refer to the solution of the search for the two mean proportionals by means of *Archytas' Triangle* when AΔ and AM are given in length, as *Archytas' Solution*.[20] Two questions could be considered:

- How could *Archytas' Triangle* have been derived?
- How can *Archytas' Solution* be obtained? (i.e. *Archytas' Triangle* given two data)

4.2.1 How to obtain Archytas' Triangle

I show in this section some simple considerations that lead from *Archytas' Simple Triangle* (see Fig. 6) to *Archytas' Triangle* (see Fig. 7).

As mentioned, *Archytas' Simple Triangle* shows two similar triangles with a common side, the *geometric pendant* of the simple mean proportional, $\frac{\mathrm{AI}}{\mathrm{IK}} = \frac{\mathrm{IK}}{\mathrm{I\Delta}}$. There is another triangle similar to both triangles, the triangle AΔK. The similarity of the three triangles is a perfect stimulus for the search for the two mean proportionals (which requires three ratios): we can take all the possible combinations of two corresponding sides to find two mean proportionals. The fact that the three triangles are similar assures us that:

1. $\frac{\mathrm{AK}}{\mathrm{K\Delta}} = \frac{\mathrm{AI}}{\mathrm{KI}} = \frac{\mathrm{KI}}{\mathrm{I\Delta}}$
2. $\frac{\mathrm{A\Delta}}{\mathrm{AK}} = \frac{\mathrm{AK}}{\mathrm{AI}} = \frac{\mathrm{\Delta K}}{\mathrm{KI}}$
3. $\frac{\mathrm{A\Delta}}{\mathrm{K\Delta}} = \frac{\mathrm{K\Delta}}{\mathrm{I\Delta}} = \frac{\mathrm{AK}}{\mathrm{KI}}$

Each expression can be easily transformed to obtain a continued proportion and, therefore, to obtain two mean proportionals.[21] The second equality of expression 1, $\frac{\mathrm{AI}}{\mathrm{KI}} = \frac{\mathrm{KI}}{\mathrm{I\Delta}}$, could be part of a continued proportion. In contrast, the first equality of

[20] It is worth noting that Archytas' solution for doubling the cube is one way of finding in three-dimensional space *Archytas' Solution*, i.e. it is *one Archytas' Solution*.

[21] See p. 2.

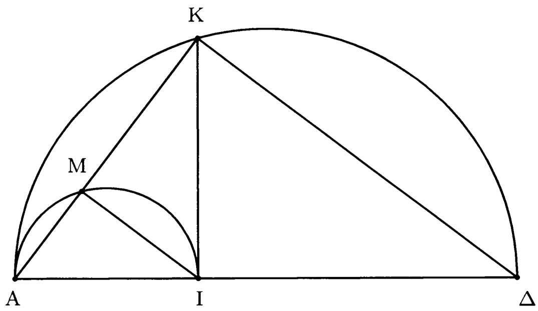


Fig. 9 Illustration of how to obtain two mean proportionals using the triangle AKΔI of Fig. 6 and expression 2. The outcome is *Archytas' Triangle*

expression 1, $\frac{\mathrm{AK}}{\mathrm{K\Delta}} = \frac{\mathrm{AI}}{\mathrm{KI}}$, should be rewritten as $\frac{x}{\mathrm{AI}} = \frac{\mathrm{AI}}{\mathrm{KI}}$, to obtain the two mean proportionals: $\frac{x}{\mathrm{AI}} = \frac{\mathrm{AI}}{\mathrm{KI}} = \frac{\mathrm{KI}}{\mathrm{I\Delta}}$ or analogously, AI and KI are the two means proportionals of x and IA. How do we find x geometrically? Taking into account the geometrical structure behind the second equality (a common side of two similar triangles and the angles between the two hypotenuses are right angles), we can deduce that the geometric structure of the first equality should be represented in the same way, as shown in Fig. 8.[22]

The first equality of expression 2 could be part of a continued proportion. In contrast, the second equality, $\frac{\mathrm{AK}}{\mathrm{AI}} = \frac{\Delta\mathrm{K}}{\mathrm{KI}}$, should be modified in this way: $\frac{\mathrm{AK}}{\mathrm{AI}} = \frac{\mathrm{AI}}{x}$. Therefore, AK and AI are the two mean proportionals of AΔ and x, because $\frac{\mathrm{A\Delta}}{\mathrm{AK}} = \frac{\mathrm{AK}}{\mathrm{AI}} = \frac{\mathrm{AI}}{x}$. The geometric structure behind the first equality is as follows: triangles AΔK, AKI; the second clipped onto the first; the shortest side of the first matches the hypotenuse of the second; the longest side of the second is perpendicular to the hypotenuse of the first. Therefore, this structure should be repeated for the triangles of the second equality, AKI, AIM[23]: the second clipped onto the first; the shortest side of the first matches the hypotenuse of the second; the longest side of the second is perpendicular to the hypotenuse of the first. In this way, we get *Archytas' Triangle* (see Fig. 9).

The third expression is equivalent to the second, because its structure uses similar elements of this kind: $\frac{\text{hypotenuse}}{\text{side}}$. The geometrical configuration of the solution will be, then, equivalent to the solution derived from expression 2.

In short, from simple geometrical considerations concerning the simple mean proportional, *Archytas' Triangle*[24] has been derived. However, *Archytas' Triangle* is not the goal (i.e. the goal is not a geometrical configuration that shows two such mean proportionals), but how to obtain an *Archytas' Triangle when* AΔ and AM are given, namely *Archytas' Solution.*

4.2.2 How to obtain Archytas' Solution

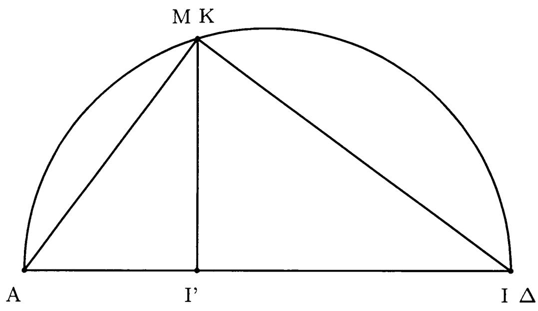
Two ways to obtain *Archytas' Solution*, both starting with an *Archytas' Simple Triangle* with two given sides, AΔ, AM (i.e. a right-angled triangle with the given sides

22 This figure turns out to be equivalent to the diagram of the solution attributed to Plato by Eutocius in the same text where Archytas' solution for doubling the cube appears (see Knorr 1986, pp. 50–66; Archimedes 1910–1915, vol. III, 56.13–58.14).

23 Assuming that IM $= x$.

24 And even the construction behind the solution attributed to Plato by Eutocius for doubling the cube.

Fig. 10 Initial position of some geometrical elements to find *Archytas' Solution* by collapsing both triangles of *Archytas' Triangle* into a unique triangle with given hypotenuse and side. It turns out to be *Archytas' Simple Triangle*. The given data are AM, AΔ



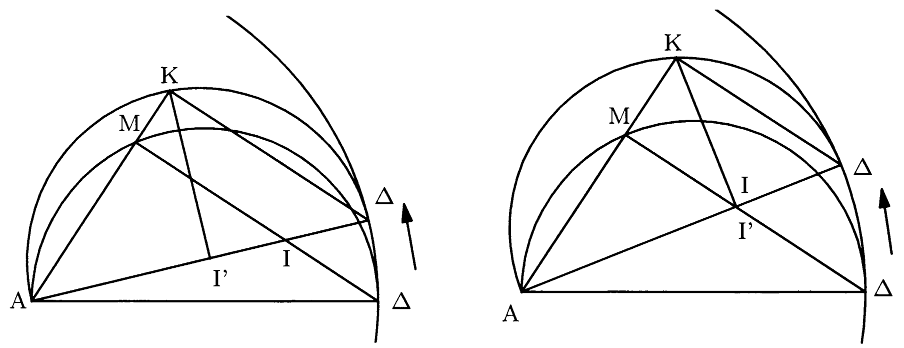


Fig. 11 On the left, a step in the rotation of one of the coincident triangles, AKΔ, of Fig. 10 in order to get an *Archytas' Triangle*, AMKΔI, on the *right*

inscribed into a semicircle) will be shown. In both approaches, during the manipulation of the *Archytas' Simple Triangle*, only one configuration of geometrical elements turns out to be an *Archytas' Triangle*, and therefore an *Archytas' Solution*.[25] We have textual evidence of one of them in Archytas' text transmitted by Eutocius, as we will see in Sect. 4.3.1.

The first solution begins with the two circles and the two triangles of *Archytas' Triangle* (see Fig. 7) collapsed into a single circle and a single triangle (see Fig. 10). It is actually an *Archytas' Simple Triangle* with given AM and AΔ, adding the triangle AIM, which is the same as AΔK. KI′ in Fig. 10 stands for KI in Fig. 7.

What changes does the figure need to undergo to become an *Archytas' Solution*? Obviously, I and I′ should match on AΔ,[26] while K should remain on AM prolonged, and KI′ and MI should remain perpendicular to AΔ and AM, respectively.

A simple way to do this is to rotate the *Archytas' Simple Triangle* AΔK around A.[27] What is going on during the rotation (see Fig. 11)?

[25] As we will see in Sect. 5.1, there are a number of historical and historiographical two-dimensional approaches to the solution of the two mean proportionals. All of them generate a curve: for each curve, every point on it generates an *Archytas' Triangle* and only one of them is *Archytas' Solution*. Therefore, these approaches are methodologically far from the *Archytas' Solutions* that I am ready to explain.

[26] In an *Archytas' Solution* KI′ and MI should intersect in $I = I'$.

[27] It is evident that the triangle that solves the problem should have a lesser angle ΔAM. When making this rotation, the angle is reduced. It should be remembered that the lengths of AM, AΔ are given and this motion keeps AM and the length of AΔ the same.

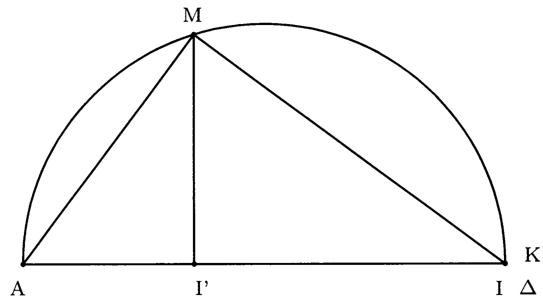


Fig. 12 *Initial Configuration* to find *Archytas' Solution* by collapsing the triangle AKΔ of *Archytas' Triangle* into a single segment AΔ (i.e. K = Δ). The given data are AM, AΔ

- The line AΔ rotates around A, but MI should remain perpendicular to AK. Then, the new I is the intersection of the rotated AΔ and the fixed MΔ.
- The semicircle AKΔ also rotates around A, but K should remain on AM prolonged. Then, the new K is the intersection of the rotated semicircle AKΔ and the initial AM prolonged.
- The new I′ is found on the rotated AΔ and on the new KI′, which is perpendicular to AΔ.

The rotation should stop when I and I′ match, as shown in Fig. 11, right.[28]

Another possibility for obtaining *Archytas' Solution* is even simpler: the starting point is, again, *Archytas' Simple Triangle* with given data AΔ and AM (see Fig. 12). But in this case, K and I coincide with Δ: the semicircle AMI contains the right-angled triangle AMI with given side AM; the triangle AΔK is collapsed into a given line AΔ.

If we move I towards A, the new K is found drawing the perpendicular from I to AΔ and cutting off the semicircle AMΔ.[29] Thus, we obtain the triangle AKΔ. Then, the semicircle of diameter IA could be drawn. The new AM is found in this semicircle, because it has a given length. Thus we obtain the triangle AMI (see Fig. 13). *Archytas' Solution* is found when A, M, K are aligned obtaining *Archytas' Triangle*.[30] We will see in Sect. 4.3.1 that this configuration could be derived easily from the transmitted text of Archytas' solution for doubling the cube.[31]

4.3 Three-dimensional approaches

Archytas' Construction, as mentioned in Sect. 3.1, is a group of three steps (*Initial Configuration*, *Transition* and *Final Configuration*) needed to obtain Archytas'

[28] See this interactive diagram: http://www.geogebratube.org/student/m119505. The use of these interactive diagrams is merely illustrative and facilitates the reading of our arguments; they do not introduce any supplementary (visual) argument.

[29] Or, alternatively, K can be moved around the circumference and the new I is found drawing the perpendicular from K to AΔ, and cutting off te line AΔ.

[30] See this interactive diagram: http://tube.geogebra.org/student/m208427.

[31] And it is also in the background of Becker's curve (see Sect. 5.1).

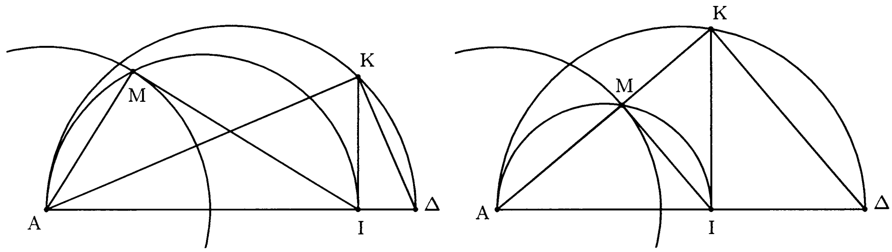


Fig. 13 Steps in the transition (*left*) from Fig. 12 to *Archytas' Triangle* (*right*). I moves from Δ towards A

solution for doubling the cube. I present two approaches to the proof, the first of them fitting the wording of the transmitted text and the three steps just mentioned; for this reason I call it a *Historical Reconstruction*. The three diagrams are different from the diagrams drawn from the literal reading of the transmitted text, because they are deduced, first, by rewinding the motion from the *Final Configuration* backwards and, then, moving forward again from the new *Initial Configuration*. In any case, I maintain the names of *Initial Configuration*, *Transition* and *Final Configuration* for the steps of the reconstruction.

4.3.1 A historical reconstruction

As said before, the *Initial Configuration* of the reconstruction is obtained by rewinding the motion of the semicircle AKΔ from the *Final Configuration* (see Fig. 5)[32] to the *Initial Configuration*: after the rewinding, triangles AMI and AKΔ are collapsed into a triangle and a line, and following the explicit traces of K, I, M, it is easy to see that both points K and I coincide with Δ, and M is the intersection of both semicircles. The new *Initial Configuration*, then, shows two intersecting semicircles that are perpendicular to each other, AMΔ, BMZ, and also perpendicular to a circle, ABΔ, and points Δ, K, I coincide (see Fig. 14). I have appended a sphere that circumscribes the circle and semicircles.[33] Therefore, the original configuration could be an *Archytas'*

[32] The line AΠ and the cone are not shown in the figure to simplify the diagram. In fact, the cone is implicitly shown by the generatrix AM and the circle BMZ.

[33] The sphere is not in the transmitted text and is actually unnecessary for the technical details of the demonstration shown in it (the entire explanation I will develop does not need the sphere). But one of our goals is to find some ideas that may be behind the solution. It seems apparent that the starting point (temporally and logically) of the demonstration is this new *Initial Configuration*: the three intersecting and perpendicular (semi)circles. From a historical point of view, we cannot go any further. But we can try to explain where this *Initial Configuration* comes from. An explanation is that an *Archytas' Triangle* with given sides could be circumscribed into a sphere, in the search for an *Archytas' Solution*. Then, the sphere is only useful for my justification of the *Initial Configuration*, but not in the development of the rationale. And this justification is based on this argument: it is unlikely that whoever invented Archytas' solution for doubling the cube did not *see* the sphere in this new *Initial Configuration*; in addition, as a seminal idea, it seems to be easier to imagine an *Archytas' Triangle* inscribed into a sphere than the three (semi)circles mentioned. I postulate the sphere with the inscribed *Archytas' Triangle* to be previous to the figure with the intersection of circle and semicircles. That is, of course, a matter of opinion. But, from this point on, the sphere is irrelevant and could easily be removed from the description of what is happening with *Archytas' construction* and the demonstration (although it is more difficult to explain why the *inventor*

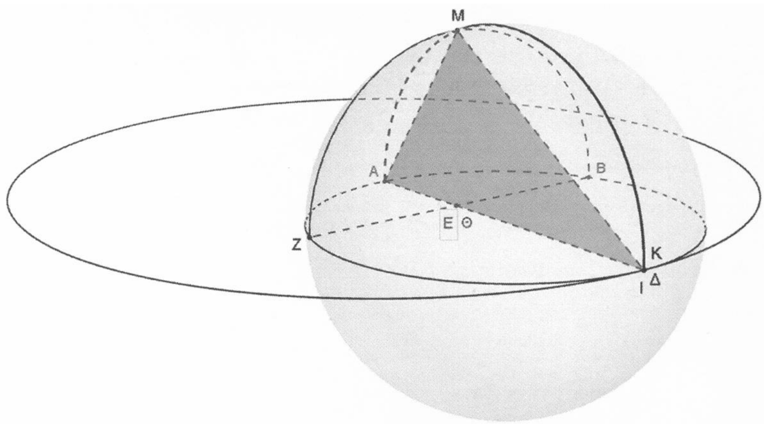


Fig. 14 *Initial Configuration* once rewound from the *Final Configuration* as described in the transmitted text

Simple Triangle inscribed into a sphere, with AΔ the diameter and AM a side and perpendicular to a great circle of the sphere.

Once a plausible *Initial Configuration* has been recovered from the backwards motion, we can come up with some clues to explain how to derive *Archytas' Construction* from this right-angled triangle inscribed into a sphere. Whatever the motion, M has two restrictions: AM is a given length and MI must be perpendicular to AM. These conditions have some obvious geometric translations in terms of *locus* problems: the first condition is satisfied when M is on the surface of a sphere of centre A and radius AM; the second condition, when the triangle AMI is perpendicular to the great circle and its base is a chord of the great circle and M is on the surface of the sphere. In summary, A, M, I must be on the sphere: M on a semicircle ZMB on the sphere perpendicular to the hypotenuse of the triangle; A and I on the great circle ABZ.

On the other hand, neither Δ nor K have to be on the hemisphere. Therefore, *Archytas' Simple Triangle* may rotate freely on the great circle, with centre A and radius AΔ, with the triangle perpendicular to this circle.[34] At every moment of the rotational motion of *Archytas' Simple Triangle* (see Fig. 15), there is another semicircle AMI in the rotating semicircle AKΔ. K must be on the rotating semicircle and KI must be perpendicular to AΔ: Fig. 13 shows two steps of this two-dimensional configuration in the plane of the semicircle AKΔ; it is precisely the 2nd two-dimensional reconstruction as described in Sect. 4.2.2. The figure shows two right-angled triangles, AΔK and AMI; KI is perpendicular to AΔ and tangent to the semicircle AMI. While Δ moves around A, I moves from Δ towards A until K, M, A are aligned; at this precise moment,

Footnote 33 continued
of the demonstration performs certain motions). Finally, the large circle of centre A and radius AΔ is added to help the reader to link Figs. 14, 15 and 16.

[34] The rotational motion is suggested by the restriction on I, which must be on the great circle. It is worth noting that the triangle must stay perpendicular to the great circle during the entire process.

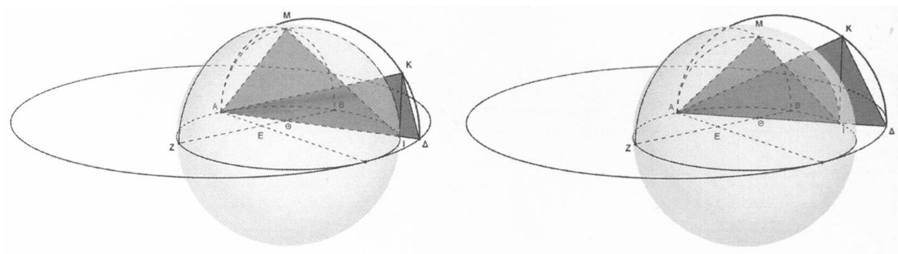


Fig. 15 Two frames of the *Transition*

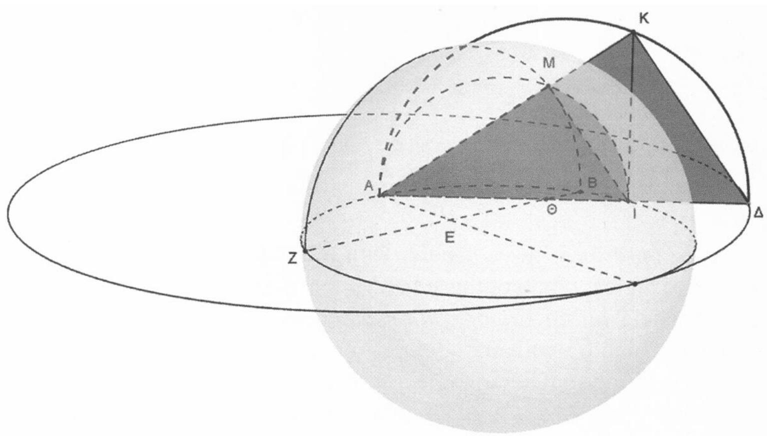


Fig. 16 *Final Configuration*, when AMK are aligned

Archytas' Triangle is shown (see Fig. 16). The sequence of the changes inside the circle AKΔ is implicit, but we can reconstruct the motion following the traces of K and M explicitly mentioned in lines 27R and 29R.[35]

If we continue the process of translating the restrictions into the language of *loci*, the condition «IK must be perpendicular to AΔ» is translated as «I, K are on the surface of a (semi)cylinder of base the great circle of the sphere». And, finally, the last condition «A, M, K must be aligned» is translated as «AK is a line on the cone defined by AM».[36]

In short, the problem is solved with a simple sequence: first, inscribing *Archytas' Simple Triangle* into a sphere, with hypotenuse, the diameter AΔ, and a side, AM (almost the *Initial Configuration* of *Archytas' Construction*); second, rotating the triangle on a great circle that contains AΔ, holding the point A. This sequence fully solves the problem when points A, M, K are aligned; at this precise moment, *Archytas' Solution* is shown. That is to say, there is no need to have in mind, from the very beginning, all the elements of *Archytas' Construction* to solve the problem; in fact,

[35] See this interactive diagram: http://tube.geogebra.org/student/m210965.

[36] It is worth noting, as mentioned before (see step [4] and Sect. 3.1), that the text explicitly underlines the fact that the last step of the process reads: A, M, K should be aligned.

the *Final Configuration* of the construction is the outcome of imposing the geometrical restrictions derived from the motion from *Archytas' Simple Triangle* to *Archytas' Triangle*.[37] The only object we have to have in mind from the very beginning is an *Archytas' Simple Triangle* inscribed into a (hemi)sphere, and this figure is *almost* what we can find in the *Initial Configuration* of *Archytas' Construction*.

4.3.2 A sequence of discovery

The fact that an *Archytas' Simple Triangle* appears in the (new) *Initial Configuration* leads us to think that the implicit goal of Archytas' solution for doubling the cube was to find an *Archytas' Solution* from an *Archytas' Simple Triangle* with given sides. Then, we can assume that *Archytas' Triangle* was known to be the solution, perhaps by planar means (in Sect. 4.2.2, I have given simple explanations without any documental support).

Having in mind this main goal, a simple way to order some ideas in a temporal sequence of discovery of Archytas' solution for doubling the cube could be as follows:

1. Discovery of *Archytas' Triangle* as a solution the two mean proportionals problem (for example, as shown in Sect. 4.2.1).
2. Planar manipulation of *Archytas' Simple Triangle* to obtain *Archytas' Solution*: it is only necessary to move the two triangles inside the semicircle, beginning with *Archytas' Simple Triangle* and moving I from its initial position to find *Archytas' Solution* (as shown in Sect. 4.2.2, 2nd approach).
3. The *bright idea* of placing *Archytas' Simple Triangle* inscribed into a (hemi)sphere.
4. Rotation of *Archytas' Simple Triangle* which produces the same two-dimensional sequence (see step 2) that drives into *Archytas' Solution*.
5. Translating the condition of the planar solution (A, M, K must be aligned) into the language of *loci*: the cylindrical trace should intersect the rotating generatrix of the cone.

We only have no textual evidence of steps 1 and 3. Step 3 does not need to have textual evidence, and even no logical explanation, precisely because it is a *bright idea*; the use of *bright ideas* does not leave witnesses outside the personal notes or drawings

[37] In other words, the entire process is a sequence of *geometrization* of all restrictions (in the form of a sphere, a cone, a cylinder and a semicircle), that is to say, restrictions are *reduced* geometrically:

- «AM and AΔ are given in length» and «AMI is a right-angled triangle» is equivalent to the fact that the triangle is inscribed into a sphere of radius AΔ, perpendicular to one great circle, and AM is a *side* (i.e. *generatrix*) of a right cone.
- «AΔK is a right-angled triangle» is equivalent to the fact that it is inscribed on the semicircle AΔK, in the *Initial Configuration*.
- «IK is perpendicular to AΔ» is equivalent to the fact that IK is on the semicylinder.
- Finally, «A, M, K are aligned» is equivalent to the fact that K is on the cone. This is the last step of the process, achieved by rotating the semicircle AKΔ.

This kind of process of *geometrization* of all restrictions is usual in so-called *locus* problems (see Euclide 2007, pp. 83–85). It is worth noting that the sphere disappears from the demonstration because it is actually unnecessary once the other conditions have been satisfied.

of the mathematician.[38] It must be stressed that this reconstruction minimizes the *bright idea* component of the solution. Some reflections about step 1 are developed in Sect. 4.2.1.

In the end, it is all about delimiting the extent of the *bright idea*. Scholars seem to have decided, implicitly or explicitly, that the *bright idea* is Archytas' solution as a whole.[39] Besides being a poor hermeneutical approach, it has a serious flaw: it overlooks the *Initial Configuration*, almost explicitly mentioned in the text, and its evolution.[40] In other words, if *Archytas' Construction* was revealed to the mathematician as a whole, what is the role of the *Initial Configuration* and its evolution? Could they appear by chance, and the mathematician not be aware of them before he had the *bright idea*? If this is true, he was aware of a highly complex three-dimensional configuration without being aware of some simple two-dimensional figures (whose elements are explicitly mentioned in the text).

In any case, all these considerations go further than the limits of the text, because we have no sources to support some of them. Therefore, the reconstruction or approach must be regarded *cum granus salis*;[41] in fact, my only goal with this approaches is to counterbalance some modern approaches to the issue (see Sect. 5), and not to try to discover Archytas' thoughts.

4.3.3 Another reconstruction

In this short section, I present another reconstruction of Archytas' solution for doubling the cube, very close to the historical one. There are two goals of this reconstruction: first, to show the fact that the *Initial Configuration* of *Archytas' Construction* is a productive configuration; second, to highlight the difference between historical and non-historical reconstructions, even in reconstructions which are very similar.

The *Initial Configuration* is the same, but with the point K displaced from Δ to M (see Fig. 17). Of course, it is not a *historical reconstruction* because, after rotating the semicircle, the trace of K does not coincide with the trace described in Archytas' text: now, K moves on the surface of the cone when the semicircle AKΔ is rotated, because it is the intersection of this semicircle with the line AM. In addition, the evolution of the figures on the plane AKΔ is the one discussed in the first approach of Sect. 4.2.2 (see Figs. 10, 11)

The *Final Configuration* of this reconstruction is the same *Final Configuration* (see Fig. 16). The only difference, again, is the trace of K and, therefore, the evolution

38 It is worth noting that there are no such texts (private notes or drawings by mathematicians) in the Ancient Greek mathematical corpus. There are not even any *original* mathematical texts, but only later editions made by scholars centuries after the mathematician's death.

39 With differing degrees: van der Waerden is perhaps the most explicit, while in Euclide (2007) this suggestion is only implicit. The general wonder that produces Archytas' solution for doubling the cube is due, mainly, to this implicit decision.

40 Namely *Archytas' Simple Triangle* of the *Initial Configuration*, and the evolution of the two triangles inscribed into the semicircle.

41 As Burkert remarks (1972, p. 303): «Still and all, in the history of science logical necessity and historical sequence are not always identical».

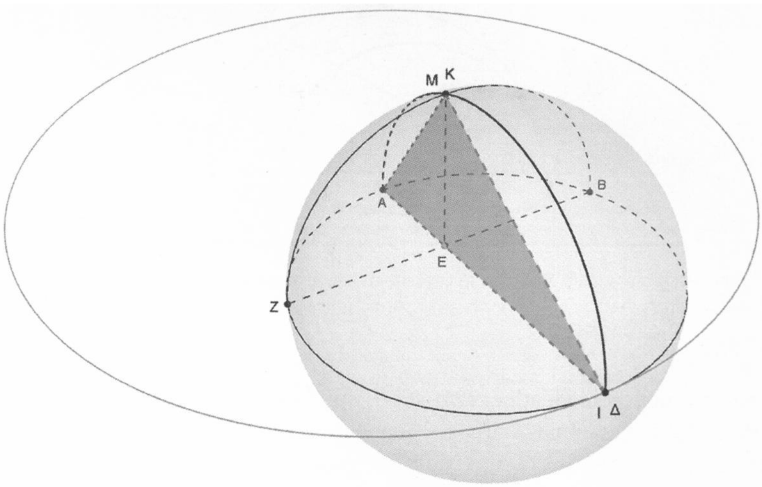


Fig. 17 *Initial Configuration* of another reconstruction. In this case, K is the same point M. The *Final Configuration* will be the same as that of the historical reconstruction, only the *Transition* is different (and the trace of K is not as described in the transmitted text)

of the figure into the semicircle AΔK: the motion stops when KI is perpendicular to AΔ.[42]

5 Some modern reconstructions

Some scholars have attempted to explain or reconstruct Archytas' solution for doubling the cube.[43] All of these reconstructions are difficult, and a number of them are also confusing and/or even wrong at some point.

The translations of Eutocius' text, in which the reconstructions are supported, are not homogeneous: the translation in van der Waerden (1954, pp. 150–152) of Archytas' solution for doubling the cube is in fact a paraphrase of Eutocius' text: there are the quite literal translations of Euclide (2007, pp. 82–83) and Knorr (1989, pp. 102–107) and, in some passages (Thomas 1951, pp. 284–289). All translations, except the ones of Acerbi and Knorr, differ in one point from Eutocius' text: the letter Δ in Eutocius' text designates two different points, both in the diagram and the text, while the translations denote the point in the rotated semicircle differently.[44]

42 See this interactive diagram: http://www.geogebratube.org/student/m210935.

43 A number of them mention the fact that Archytas' solution is developed in a *synthetic* way, and the reconstruction is *analytical*. The discussion about *analysis/synthesis* in Greek mathematics is a well-known subject (for a complete discussion on this subject, see Euclide 2007, section III B.1, pp. 439–518), but it is avoided in this article because my arguments stand apart from this issue.

44 Van der Waerden (1954, p. 150, n. 2) use of Δ' is «for greater clarity». Diels, the editor of Eutocius' text, also corrected the text, denoting the rotated point as Δ'. This is a «inutile» correction (Euclide 2007, p. 162, n. 258).

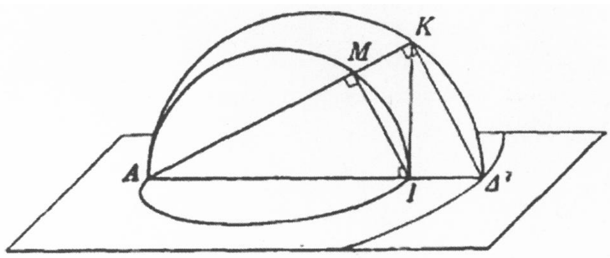


Fig. 18 Figure used by van der Waerden to explain his reconstruction of Archytas' solution for doubling the cube. It is apparent that the circles in the plane are erroneously concentric, and not tangent

The reconstruction in Heath (1921, vol. I, pp. 246–249) uses, as is usual in Heath's reconstructions, the «language of analytical geometry». Obviously, this modern reading is not useful for our discussion or for the evaluation of the discovery of Archytas, since it has nothing to do with the demonstration process as reported in Eutocius' text.[45]

In contrast, in van der Waerden (1954, pp. 150–152) *Archytas' Construction* is explained from the text itself, searching for «the figure which Archytas saw before his mind's eye [...] the right triangle AKΔ′» (p. 151).[46] The explanation is a bit confusing and even erroneous in one point: «the semicircle AMI will then describe a hemisphere whose diameter is AΔ»; this sentence is only true in the *Initial Configuration* of the semicircle, when I coincides with Δ.[47] The diagram accompanying the *analysis* contributes to the confusion (van der Waerden 1954, p. 151, Fig. 50) (see Fig. 18), since the segments of circle that contain I and Δ′, respectively, should be tangent, and not concentric, as is apparent from the diagram.

The description is based on the observation that «in Archytas' diagram, everything is in motion; he thinks kinematically». This view may seem founded in the language of the text itself, which includes some terms that denote/connote motion, mainly rotational motions. However, this kinematic appearance does not take into account the terminology of Greek geometry: the use of terms such as ἄγω, φέρω, γράφω (that I have translated always as «draw, lead, trace») and words derived from these verbs, such as περιάγω, ἀντιπεριάγω, περιφέρω, περιγράφω, ... (that I have translated always as «draw around, draw around in the opposite direction, lead around, trace around»,...), are systematically used by Greek geometers, and their use is very homogeneous, even in our text. Van der Waerden seems to consider the second group of terms a kinematic lexicon, but not the first group, although all of them have the same root. Therefore, the issue is not whether Archytas' text has a kinematic char-

45 I mention this reconstruction only as an outstanding example of a modern reading in the early twentieth century, but useless for understanding Archytas' solution for doubling the cube.

46 Van der Waerden even seems convinced of catching on to Archytas' intuition: «Archytas hit upon the following idea» (p. 152). As mentioned, van der Waerden and Knorr denote as Δ′ one of the two Δ of the transmitted text and diagram.

47 Attempting to understand what the scholar is trying to explain and leaving out the error, we could agree that the semicircle AMI will describe *half* a hemisphere, but not a complete hemisphere.

acter, but whether this character is different from Greek geometry. This lexicon is as kinematic as the lexicon of Greek geometry,[48] because it is almost the same lexicon.[49]

The key to the actual kinematic or static character of Greek geometry (and even of Greek mathematics), in its deepest sense, is not the lexicon, but the grammar: the systematic use of perfective forms[50] in the construction of geometrical objects verbs shows that Greek geometry is mainly static, interested only in the result of the motion, not in its process. Archytas' text shows this same character. Only step [3][51] shows a different character, not because of the lexicon, but because of the grammar: the future verbal forms are the main forms in this step. But this characteristic is not so unusual in Greek mathematics in similar contexts, although perhaps the proportion of future verbal forms may be less.

That said, two of the three alleged surfaces of the Archytas' solution for doubling the cube are never explicitly used, but only the motions of the objects that generate these surfaces: the conical surface, defined, as usual in Greek mathematics, as a rotating motion of a triangle, is never explicitly used; the *toric* surface is not even mentioned: in this case, the text only mentions the rotation of a semicircle.[52]

Wilbur Knorr provides two explanations for Archytas' solution for doubling the cube: the first is described by Knorr as an *analysis* of the construction (Knorr 1986, pp. 51–52) and the second, less important for our goal, is an attempt to design a mechanical device to solve the problem according to *Archytas' Construction* (Knorr 1989, pp. 109–110). The first one is based on the search for *loci*, and not on the relentless motion of some geometric elements: «[...] one can seek the common intersection of the *loci* associated with the points K, I, and M».

Knorr's argument attempts to «approach» the «arrangement» of *Archytas' Triangle* «available to Archytas» to make «clearer the pattern of thought». In other words: the reader might understand that Knorr will show how the three-dimensional construction will appear from *Archytas' Triangle*. It starts with a fully two-dimensional approach. But then it swaps abruptly from that two-dimensional approach to a three-dimensional approach.[53] The link, if any, between the two explana-

48 If we decide to qualify the verb περιάγω /«draw around» as a kinematic verb, the verb ἄγω /«draw» must be qualified as a kinematic verb too. But this last verb is one of the most common verbs in Greek geometry, used to draw lines. Therefore, we also have to characterize all Greek geometry as kinematic. In fact, I would agree that the lexicon of Greek geometry is kinematic in a very elemental way: it is built with words that describe a drawing activity, which, of course, is a kinematic activity.

49 In fact, only one *kinematic* root is not usually found in Greek geometry, with two occurrences: the noun κίνησις/«movement», and the participle κινούμενον/«moved». It is worth noting that there are also two explicitly *static* words in the text, unusual in Greek geometry (except that the first one is quite common in Euclid, *Data*: Def. 4, 6, 8, 13, 14, 15, Prop. 25–41): θέσις, τόπος/«position, place», repeated several times, but sometimes implicit (see beginning of section [4] of the translation).

50 Underlining the *resultative* aspect of the verbal action.

51 And some passages of [4] in which certain elements of step [3] are mentioned.

52 In addition, following the reading proposed in note 4, both rotating elements (semicircle and triangle) are not used to find the solution, but instead their traces on the cylindrical surfaces, and these traces are *static* lines. With the usual reading, only the triangle is a moving element in the solution.

53 The two-dimensional approach and its origin are broadly discussed in Sect. 5.1.

tions is unknown, and we do not know why Knorr abandons the two-dimensional approach.

The three-dimensional exposition (qualified as a «clearer underlying pattern of thought» and a different «approach available to Archytas») needs all of the geometrical elements of *Archytas' Construction* (*Archytas' Triangle* with the cone and the cylinder). In other words, it must assume that the entire construction appeared fully formed in Archytas' mind, because «Archytas' approach» needs all the elements of the construction; that is not only an extreme assumption but it cannot explain anything more than the «synthetic expository», because it does not simplify anything in either the construction or the rationale. The innovation is in the reformulation of some arguments in the language of *loci*; for example, «the length AM can be kept equal to AB by setting M on the trace of B as its semicircle rotates on its axis $A\Delta$» is the parallel formulation in terms of *locus* problems of these two facts: $Z\Theta \cdot \Theta B = M\Theta^2$ and ZB is perpendicular to $A\Delta$. But this reformulation hardly «clarifies the underlying pattern of thought», nor is it a different «approach available to Archytas». It is the same approach as the «synthetic exposition mode» but using different language, the language of *loci*.

Acerbi (Euclide 2007, pp. 83–84) deepens Knorr's line of argument using *loci* on surfaces. The goal of his explanation is «to search for understanding of how to figure out the demonstration» and the relation between *Archytas' Triangle* and the three-dimensional construction around it.[54] He systematically translates the geometrical restrictions into the language of *loci* on surfaces but, again, this translation does not help very much to simplify the proof: the explanation requires all the elements of *Archytas' Construction* and all the arguments of the demonstration working together; it isolates neither clusters of geometrical elements nor groups of arguments that can run independently.

In fact, there are two main elements that obscure all the reconstructions we have reviewed of Archytas' solution for doubling the cube:

1. The condition which ensures that the final construction is correct is always expressed in the original form of Eutocius' text. This condition links the lines $A\Theta I$, $B\Theta Z$ and their point of intersection, M, as follows: $A\Theta \cdot \Theta I = BZ \cdot Z\Theta = \Theta M^2$. Taking into account that A, B, I, Z are on the same circle, this condition (in the context of Greek geometry) is *virtually* equivalent to the fact that M is on the surface of a hemisphere with base on circle ABIZ.[55]
2. More importantly, there is a common methodological shortfall in all the reconstructions: *Archytas' Triangle*, $AMK\Delta I\Theta$, appears at the beginning of all considerations and its (possible) origins are not discussed by scholars; it seems that either the

[54] «Sono state proposte varie ricostruzioni di come Archita possa essere giunto a concepire questo complesso intersecarsi di solidi: sono più o meno complicate come l'originale. Cercando anche noi di capire come funziona la dimostrazione [...]» (Euclide 2007, p. 83).

[55] As we have shown before, only van der Waerden seems aware of this fact, even though the error in the text and in the diagram linked to the solution suggests that his intuition is not accurate.

origin cannot be discussed, or the triangle appeared in the mind of the mathematician already complete, like Athena from the head of Zeus.

My approach has discussed both elements and has tried to link them with a plausible rationale. In the next section, I will look at some historical and historiographical approaches to the second element, usually in contexts that have no direct relation with Archytas' solution for doubling the cube.

5.1 Two-dimensional approaches to the resolution of the two mean proportionals problem

Knorr's reconstruction begins with a two-dimensional approach to Archytas' solution for doubling the cube, which is abandoned and replaced with a purely three-dimensional approach. As I have mentioned before, there is no apparent link between the two approaches in Knorr's text and, in fact, the two-dimensional approach has a very ancient origin in a group of resolutions of the two mean proportionals problem using curved lines, called *proportionatrix* (*prima* and *secunda*), oval, *folium* or *multiplicatrix*, with no plausible historical relation to Archytas' solution for doubling the cube.[56] A technical link between this group of resolutions and Archytas' solution could be suggested but without any basis in all cases, except one.

These two-dimensional solutions, although they have no historical or technical relation to Archytas' solution, have what we will call an Archytas' solution *flavour*. But it is only a mirage; I will briefly review some technical, historical and historiographical details of the issue to show the differences. In summary, I will show now that there is no technical or direct historical relation between these curves and Archytas' solution for doubling the cube.[57] In fact, the original historiographical link between these two-dimensional approaches and Archytas' solution is quite entangled and, in the end, the link has been obscured.

In recent historiographical works, these curves appeared in Becker (1966, p. 79) as opposed to Tannery's proposal to Eudoxus' solution to the two mean proportionals (Tannery 1912–1915, "Sur les solutions du problème de Délos par Archytas et par Eudoxe. Divination d'une solution perdue", pp. 53–61). Eudoxus' solution is mentioned by the same Eutocius' text that contains Archytas' solution for doubling the cube, but it is only mentioned in passing: «since he [Eudoxus] says in the introduction that he discovered it *by means of a curved trace* (διὰ καμπύλη γραμμή). However, in the solution, in addition to not using curved lines, he even finds a discrete proportion and then uses it as if it were continuous» (Archimedes 1910–1915, v. III, 56.3–8). With this single mention, Tannery attempts to reconstruct the «curved line» and, later on, Becker attempts his own curve (see Becker 1966, p. 79). Becker achieves an «elegant and straightforward» curve that «directly ties in with the planimetrical core piece of

[56] Or, if there were some relation, it disappeared before the XVI century.

[57] Only Becker's curve has an indirect relation through an alleged Eudoxus' solution for doubling the cube.

solution of Archytas ».[58] In addition, he claims that «already Father Villapaudo (*sic*)[59] used it as "Duplicatrix" (*sic*)».[60]

A reference to Becker's work appears in a note to Knorr's two-dimensional reconstruction (Knorr 1986, p. 88, n. 6): «Becker (*op. cit.* p. 79) presents this curve via a somewhat modified construction and cites its use by Villapaudo (*sic*) as reported by Viviani (1647); see also G. Loria. *Ebene Kurven*, 1902, p. 317. Kepler's use of the same curve is noted by W. Breidenbach, *Das delischt Problem*, Stuttgart, 1953, pp. 31 f». In fact, Loria seems to be the original source of all recent scholarly work on the issue, in the section «Multiplicatrix and mediatrix curves».[61]

The sequence of facts described by Loria is erroneous in one point, as Kepler's *Astronomia Nova*[62] is later than Villalpandos' book about the Solomon temple.[63] The correct sequence is as follows: Villalpando probably invented the *proportionatrix* (*prima* and *secunda*) to calculate two mean proportionals (Prado and Villalpando 1596–1604, Problema X and XI, pp. 289–290). Later on, Kepler uses an oval (known as Kepler's folium), similar to the one Villalpando used, to describe a possible model of the path of the planet Mars (Kepler 1609). Viviani invents a *new way*, as he claims, to trace Villalpando's *proportionatrix secunda*.[64] Loria and Becker derived their definitions of this curve from Viviani's text, or from posterior texts (of Seidel or Longchamps).[65] None of them mentions Archytas' solution for doubling the cube as the initial inspiration for the curve; rather it is a new fully two-dimensional approach to solve the problem of the two mean proportionals.

[58] But Becker, strangely, attributes the discovery to Eudoxus (an alleged disciple or follower of Archytas). Would it not be easier to attribute or link this curve to Archytas, if the curve is found in the «core» of his solution? In any case, Archytas' solution does not contain a reference to a curve of this kind, even indirectly, although the construction of the curve is certainly very close to some elements of *Archytas' Construction*. Note that nobody has associated the curved traces on the cylinder (of the semicircle and, perhaps, of the triangle) explicitly mentioned in Archytas' text with the «curved trace» linked to Eudoxus.

[59] Knorr, Becker and Loria misspell the name of the Spanish Jesuit Juan Bautista Villalpando, mathematician and architect, who wrote, with another Jesuit, Jerónimo Prado, a commentary on the prophet Ezekiel and the Solomon temple, *In Ezechielem explanationes et apparatus urbis, ac templi Hierosolymitani commentariis et imaginibus illustratus* which includes some chapters devoted to proportional lines (Prado and Villalpando 1596–1604). Probably, Loria only consulted Viviani's book where Villalpando is mentioned, misspelling the name of Villalpando, Becker copied Loria and Knorr both Becker and Loria. Villalpando would seem to have not been consulted by these scholars.

[60] The curve mentioned is Villalpando's *proportionatrix secunda*, although Becker offers a slightly different definition; he calls it "Duplicatrix" because it is the name that Loria (the source of Becker) uses for these kinds of curves.

[61] Loria (1902, p. 317). It is worth noting that the point of view of Loria's book is completely algebraic, and not geometric. There is no mention of Archytas' solution for doubling the cube.

[62] I have not found Kepler's mention of this curve; the Loria citation «*Astronomia Nova* (Prag, 1609) S. 337» in Loria (1902, p. 317, n. 1) is not correct, as p. 337 of Kepler's book does not contain the oval; it is actually the last page of the treatise.

[63] *Astronomia Nova* was published in 1609 and Villalpando's third and last volume of his work was published in 1604.

[64] «Vi ò nello stesso tempo osservato che questa curva può segnarsi in altro modo senza bisogno di quel secondo mezzo cerchio» (Viviani 1674, p. 279).

[65] Only in Becker's text the context is the discussion of Eudoxus' *curve* mentioned by Eutocius. As I have said, he is answering Tannery's proposal for Eudoxus' curve, both are without any textual support.

Therefore, Villalpando's *proportionatrix prima* and *secunda* are, as far as I know, the most ancient description of curves of this kind. The curves of Viviani, Becker and Knorr are equivalent to the *proportionatrix secunda* but defined in a different way, and Knorr's definition is even the same as Viviani's definition; the only difference is that Knorr cuts off this curve with a circle to obtain Archytas' solution. Becker's curve is the only one associated with the implicit process developed in Archytas' solution for the problem of doubling the volume of the cube, as Becker himself highlights: «Eudoxos could also have used another construction, which is elegant and straightforward. Besides, *it directly ties in with the planimetrical core piece* of solution of Archytas».[66] All the rest have no direct historical or technical relation with Archytas' solution for doubling the cube.

5.1.1 Proportionatrix secunda and Becker's curve

Villalpando's *proportionatrix secunda* is built as shown in Fig. 19: let there be AG the diameter of a semicircle, D the centre of the semicircle and DG the diameter of a second semicircle. For each point F of the greater semicircle, we can define a point K of the *proportionatrix secunda* as follows: FG cuts the smallest circle in H; I is the intersection of AG with the perpendicular to AG that passes through H; the circle of centre I and radius IG cuts FG in K. Point K generates the *proportionatrix secunda* when F moves around the greater semicircle.

Now, if we define L as the intersection of the circle GK with the line AG and we unite AF, LK, the triangle ALGKF is an *Archytas' Triangle*, i.e. for each K of the *proportionatrix secunda* there is an *Archytas' Triangle*, and, therefore, the curve can be used to find two mean proportionals between lines a, b, with $a > b$, i.e. to find an *Archytas' Solution*, in this way: if we set AG $= a$ and draw the *proportionatrix secunda*, we can cut it with a circle of centre G and radius b, and obtain K, with GK $= b$. F is obtained by lengthening GK and cutting it with the semicircle of diameter AG; the resulting *Archytas' Triangle* solves the problem.[67]

Becker's curve is built as shown in Fig. 20: let there be AG the diameter of a semicircle. For each point F on the semicircle, we can define a point K of Becker's curve as follows: L is the intersection of AG with the perpendicular to AG passing through F; FA cuts the semicircle of diameter AL in K. Point K generates Becker's curve when F moves on the semicircle, and the figure ALGFK is an *Archytas' Triangle*, i.e. for each K of Becker's curve there is an *Archytas' Triangle*, and, therefore, the curve can be used to find an *Archytas' Solution* in a way equivalent to that of the *proportionatrix secunda* just mentioned.

[66] Becker (1966, p. 79) (italics are mine). Becker, as we have said, also associates this curve with Villalpando's *Proportionatrix secunda*: «The curve is a beautiful oval, is consequently completely contained within the finite. It is symmetrical around the axis AD. From the 17th c. onwards, this curve is well known. Already Father Villapaudo (*sic*) used it as "Duplicatrix" (*sic*)» (Becker 1966, *loc.cit.*). Although this is a clever piece of intuition by Becker, which seems not to have been understood by Knorr (he actually uses Viviani's curve, and not Becker's), he does not explain exactly what the «direct tie» consists of. In fact, the curve is only indirectly tied with «the planimetrical core piece of solution of Archytas», as we will see.

[67] Villalpando's actual use of this curve to find the two mean proportionals between two assigned straight lines is a bit different, but is still equivalent to the one just explained.

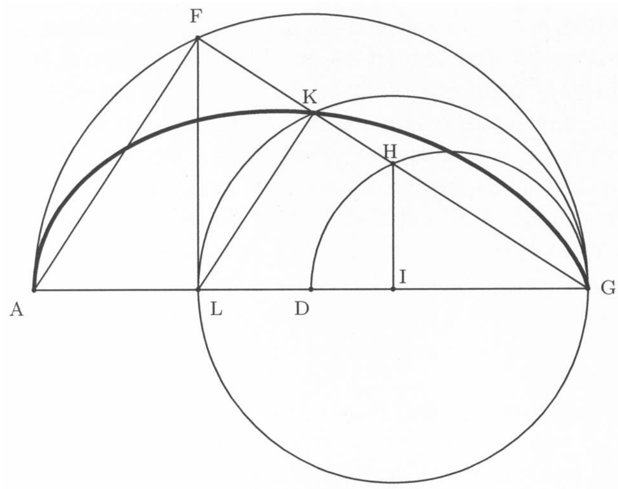


Fig. 19 Villalpando's *proportionatrix secunda* (in *bold*) defined by point K

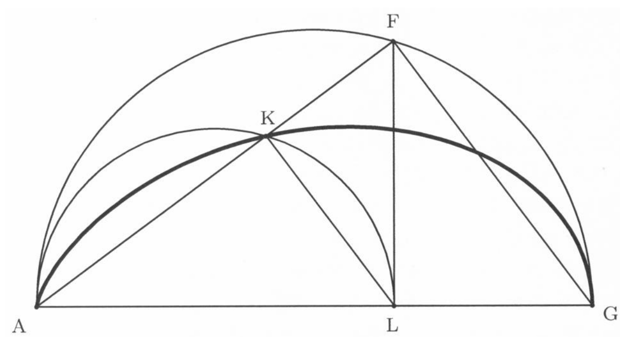


Fig. 20 Becker's curve (in *bold*), defined by point K

Becker's curve is equivalent to Viviani's (and Knorr's) curve, but its definition is slightly different. That definition is the one most like the *Archytas' Construction*, and Becker was aware of this fact: the curve «directly ties in with the planimetrical core piece of solution of Archytas» as is easy to see comparing the elements of the process of construction of Becker's curve (see Fig. 20) and those of the triangles AΔK and AIM (see Figs. 13, 15, 16).

In short, the curves of Loria, Becker and Knorr have no technical relation with the motion in the semicircle AKΔ during the *Transition* in *Archytas' Construction*, nor any historical relation, except for the fact that all curves and motions are defined into a semicircle and their goal is to build two mean proportionals. In fact, every point of each curve mentioned in this section can be used to build two mean proportionals, but the process inside the rotating semicircle of *Archytas' Construction* only shows a unique two mean proportional. The way of building Becker's curve is the closest to *Archytas' Solution*, but the point that defines his curve is never defined or even mentioned in Eutocius' text, and neither is its trace, of course: it is the intersection point of the semicircle AIM and KA (see Fig. 13), in other words, the trace of the intersection of AK with the sphere (see Figs. 15, 16).[68]

6 Conclusion

I have presented a translation very close to the original Greek text and, in a number of points, a new reading of Archytas' solution for doubling the cube transmitted by Eutocius, mainly unfolding the structure behind the text and the steps of *Archytas' Construction*. This reading has helped us to find some simple and original ideas behind the incredibly clever solution provided by Archytas and to discard some traditional ideas transmitted by scholars, namely:

- The solution is found in three explicit steps, that I call *Initial Configuration*, *Transition* and *Final Configuration*. Scholars have focussed their attention on the *Final Configuration*, but have given less importance to the *Transition* and the traces explicitly made in this step of the construction, as well as ignoring the *Initial Configuration*.
- The solution of the problem is never couched in terms of the intersection of three surfaces; this is only an anachronistic interpretation of the text. The elements of the solution are a trace on the cylindrical surface and a rotating triangle.[69] There is no *toric* surface, but only a rotational movement of a certain semicircle that intersects the cylindrical surface to make a trace; there is a conical surface, but it never seems to be explicitly used, only perhaps implicitly.

These ideas have helped us to unfold a new approach to Archytas' solution for doubling the cube. The key part of my approach takes the steps of *Archytas' Construction* further: I rewind the motion from the *Final Configuration* to a certain *original* configuration, more complete than the *Initial Configuration* (see Fig. 14). This *original* configuration, made of an *Archytas' simple triangle* inscribed into a hemisphere, can be claimed to be a logically previous step to Archytas' solution for doubling the cube. This *original* configuration gives us a way to divide up Archytas' solution for doubling the cube into some simple steps, almost all derived easily from *Archytas' Construction*. The goal of this approach is not to try to show Archytas' thoughts, but to provide some arguments that counterbalance scholars' global approaches and even to reassess in some way the aura that surrounds this solution.[70] For this reason, I have also presented a review of some modern approaches to Archytas' solution for doubling the cube in the light of

[68] A definition and an interactive visualization of all curves (*proportionatrix prima*, *proportionatrix secunda*, Viviani's (and Knorr's) approach and Becker's curve) and my *reconstructions* discussed before are shown in http://tube.geogebra.org/book/title/id/791183. All curves, except the *proportionatrix prima*, are in fact the same (as Becker points out, it is $r = a\cos^3\theta$) and *Archytas' Triangle* could be shown easily in the final diagram of each one. Tannery's curve (Tannery 1912–1915, "Sur les solutions du problème de Délos par Archytas et par Eudoxe. Divination d'une solution perdue", pp. 53–61) is shown in circle ABΔ and not in semicircle AKΔ because the point he is searching for is I.

[69] But following an alternative reading presented in note 4, these elements could be two cylindrical traces that meet at one point.

[70] In fact, no Ancient source places distinction on it.

the new reading and of some XVI century mathematical results concerning the search for two mean proportionals that have influenced the historiography of the problem.

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