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On the genesis of the Cartan–Kähler theory

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Abstract The theory of exterior differential systems plays a crucial role in Cartan's whole mathematical production. As he once recognized, all the germs of his subsequent work were contained there. Indeed, it provided him with powerful technical tools that turned out to be very useful in many different fields such as the theory of partial differential equations, the theory of infinite dimensional Lie groups (Lie pseudogroups) and differential geometry. Nevertheless, scarce attention has been paid to this area of historical research thus far. Although authoritative scholars have investigated the foundation of exterior differential calculus in Cartan's early papers, no specific analysis of Cartan's subsequent works laying the foundations of what nowadays is known as the Cartan–Kähler theory has been yet provided. This article represents a first attempt to remedy this unsatisfactory state of affairs by focusing on Cartan's work on Pfaffian systems at the very beginning of the past century. The analysis of Cartan's relevant papers is preceded by a description of the historical context in which such contributions were conceived. In this respect, special emphasis will be put on some works by Engel and von Weber on Pfaffian systems, which laid the basis for the subsequent geometrical developments of Cartan's theory of exterior differential systems.

1 Introduction

Historians and mathematicians are unanimous in considering Cartan's work on Pfaffian systems (what we would nowadays call exterior differential systems) as a landmark

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both for his own mathematical production and the development of twentieth century mathematics itself. The strategic role played by such systems in so many realms of mathematical research, such as the general theory of partial differential equations, Lie groups and the theory of equivalence and differential geometry, to mention only a few, is universally acknowledged.

Cartan himself was quite definite in assessing the importance of his work on total differential equations within his whole mathematical activity. Referring to the years when his attention was concentrated for the large part on developing his ideas on Pfaffian equations, he once wrote that those were years of calm and long meditation in which all the germs of his subsequent works were contained.[1]

Nevertheless, it appears that scarce attention has been paid to this area of historical research thus far. Authoritative scholars[2] have dealt with Pfaff's problem and the foundation of exterior differential calculus in Cartan's early papers; however, no specific analysis of his subsequent works laying the foundations of what is nowadays known as the Cartan–Kähler theory has yet been provided. The present article represents a first partial attempt to remedy this unsatisfactory state of affairs.[3]

2 Some technical preliminaries

This section is devoted to some general remarks on the mathematics which we are about to deal with. Historical accuracy will not be main focus of attention here; we will limit ourselves to giving the necessary information that will be helpful in understanding the discussion that follows. The interested reader can find a detailed historical account of this material in the paper by Hawkins (2005).

The main topic of our discussion will be the problem of integration of differential systems of Pfaffian equations; thus, it seems appropriate to describe briefly what a Pfaffian equation is and what it means to integrate such equations. Moreover, it is useful to emphasize a crucial separation that has to be maintained in the theory and that will be of primary importance for our purposes: the distinction between the completely integrable (*unbeschränkt integrable*) case and the not completely integrable one.

In modern terms, what nineteenth century mathematicians meant by a Pfaffian form in n variables can be identified with the local expression of a differential 1-form defined on a n-dimensional manifold. However, until 1899, when Cartan gave a symbolic definition of what he called a differential expression (*expression différentielle*), it appears that no autonomous status was attributed to it. Instead, what was considered

[1] See Appendix C in Akivis and Rosenfeld (1993):

> [...] Je garde le meilleur souvenir des quinze ans que j'ai passés en province, à Montpellier d'abord, à Lyon, et à Nancy ensuite. Ce furent des années de méditation dans la calme, et tout ce que j'ai fait plus tard est contenu en germe dans mes travaux mûrement médités de cette période.

[2] See Hawkins (2005) and Katz (1985).

[3] I would like to express my gratitude to Professor P. J. Olver for precious advice and comments on a preliminary version of this paper.

to be meaningful was the problem of its vanishing on suitable regions of space. This was interpreted as the occurrence of certain finite relations (to be determined) among the independent variables.

Thus, a Pfaffian equation in n variables is a differential relation of the following type:

$$\omega = A_1(x_1, \dots, x_n)dx_1 + \cdots + A_n(x_1, \dots, x_n)dx_n = 0. \tag{1}$$

To find integrals[4] of (1) means to determine functionally independent, finite relations among the variables $x_1, \dots, x_n$, $f_j(x_1, \dots, x_n) = 0$, $(j = 1, \dots, m)$ such that the vanishing of (1) is a consequence of the $2m$ relations $f_j = 0, df_j = \sum_{k=1}^{n} \frac{\partial f_j}{\partial x_k} dx_k = 0\,(j = 1, \dots, m)$. These integrals can be thought of geometrically as defining an integral submanifold of dimension $n - m$ given by the intersection of m hypersurfaces $f_j = 0$.[5]

During the nineteenth century, one of the main problem in the theory of Pfaffian equations was that of finding a canonical form for ω, that is, the problem of finding a suitable change of variables, $y_i = y_i(x_1, \dots, x_n)$, so that the Pfaffian expression ω could be written in such a way as to contain the minimal number of variables. Clearly, the determination of such a canonical form coincides with the determination of the minimal number of integral equivalents of $\omega = 0$ and, consequently, with the individuation of the integral varieties of maximal dimension.

The main results in this field were obtained by Frobenius (1877) in 1877 with the introduction of two notions: the bilinear covariant (*bilineare Covariante*) and the class (*Classe*) of a Pfaffian expression.
The bilinear covariant[6] of ω was defined by Frobenius as the following expression:

$$\sum_{i,j=1}^{n} a_{ij} dx_i \delta x_j, \quad \text{with} \quad a_{ij} = \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}, \tag{2}$$

where d and δ are differentials in different directions; the word *covariant* indicates the crucial property according to which if, under a change of coordinates $x'_i = \phi_i(x_k)$, $\sum A_j dx_j = \sum A'_j dx'_j$, then

$$\sum_{i,j}^{n} a_{ij} dx_i \delta x_j = \sum_{i,j}^{n} a'_{ij} dx'_i \delta x'_j.$$

4 In the classical literature, one often finds the wording *integral equivalents*.

5 The present-day definition of an integral variety of a 1-form is the same, only rephrased in different language: $i : S \hookrightarrow M$ is an integral submanifold of the equation $\omega = 0$ if, and only if, the pullback of ω, $i^*(\omega)$, vanishes identically.

6 For a detailed historical account see Hawkins (2005, §6).

An early application of this notion was Frobenius' analytical classification theorem for Pfaffian forms.[7] Indeed, he considered the matrix

$$M = [a_{ij}] \quad \text{and} \quad M' = \begin{bmatrix} a_{11} & \cdots & a_{nn} & A_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & A_n \\ -A_1 & \cdots & -A_n & 0 \end{bmatrix} \tag{3}$$

and defined the class of a Pfaffian form ω as the number $p = \frac{rk(M)+rk(M')}{2}$; this number is invariant under an arbitrary change of coordinates. He then demonstrated that p is the minimal number of independent variables in terms of which ω can be expressed. In other words, p individuates the canonical form to which ω belongs: if $p = 2r$, then $\omega = y_{r+1}dy_1 + \cdots + y_{2r}dy_r$, if $p = 2r+1$, then $\omega = dy_0 + y_{r+1}dy_1 + \cdots + y_{2r}dy_r$, under appropriate changes of coordinates.

A second application of the bilinear covariant, of which Frobenius took great advantage, was the so-called integrability theorem for systems of Pfaffian equations. A Pfaffian system of type

$$\omega_\mu = a_{\mu 1}dx_1 + \cdots + a_{\mu n}dx_n \quad (\mu = 1, \ldots, m). \tag{4}$$

was said to be completely integrable if it admits m independent integrals, that is, if it admits an integral variety of dimension $n-m$. Frobenius dealt with this special kind of Pfaffian systems en route to the proof of the analytical classification theorem for single Pfaffian equations. His main result was a characterization of complete integrability in terms of the properties of the bilinear covariants of (4):

Theorem 1 (Frobenius, 1877) *Given the system* (4) *of m linearly independent Pfaffian equations, it is completely integrable if and only if the vanishing of all its bilinear covariants is an algebraic consequence of the system itself.*

Frobenius' demonstration relied upon a result due to Clebsch which can now be interpreted as the dual counterpart of Frobenius' theorem. Indeed, Clebsch (1866) devoted his attention to a generalization of Jacobi's theory of linear partial differential equations by introducing the notion of complete (*vollständig*) integrability. A system of linear partial differential equations of type

$$A_i(f) = X_{i1}\frac{\partial f}{\partial x_1} + \cdots + X_{in}\frac{\partial f}{\partial x_n} = 0 \quad (i = 1, \ldots, r) \tag{5}$$

was said by Clebsch to be complete if all expressions $(A_i, A_j)(f) = A_i(A_j(f)) - A_j(A_i(f))$ are linear combinations (in general with non-constant coefficients) of (5). He was able to demonstrate the following:

[7] Important results in this field were obtained by Darboux almost at the same time. However, Darboux did not submit them for publication immediately. A paper by Darboux (1882) on Pfaff's problem appeared in 1882. For an analysis of Darboux' contribution and a comparison with Frobenius' approach, see Hawkins (2005, pp. 420–424).

Theorem 2 (Clebsch, 1866) *If the system (5) is complete, then it admits a system of $n - r$ functionally independent solutions $f_1, \ldots, f_{n-r}$.*

It turns out that requiring complete integrability of (4) is equivalent to the supposition that an appropriate system (actually, its dual[8]) of linear differential equations of type (5) is complete in the sense of Clebsch's definition; moreover, it should be observed that a system of integrals of (4) is also a system of solutions for (5) and vice versa.

To conclude the present introductory section, we recall that if the integrability conditions of Theorem (1) are not satisfied, then, in general, ω_μ cannot be expressed as a linear combination of the total differentials of m appropriate functions $f_j (j = 1, \ldots, m)$. If this is the case, then the system (4) is said to be not completely integrable. To be precise, one should distinguish further the case in which some (although not all) of the integrability conditions are satisfied from the case in which *none* of them is; in the former case, one speaks of incompletely integrable systems; in the latter, of non-integrable systems. Since the study of incompletely integrable systems can be traced back to the study of non-integrable ones, we will often ignore such a distinction in the following discussion.

3 The state-of-the-art in the early 1890s

Well after the publication in 1877 of the seminal work by Frobenius (1877), the problem of finding solutions of not completely integrable Pfaffian systems remained open and almost untouched. As we have just seen, Frobenius was able to give necessary and sufficient conditions that guarantee the complete integrability of a given differential system. However, except for some brief remarks,[9] no specific attention was paid by him to the more general problem of finding integral equivalents of not completely integrable systems of Pfaffian equations.

A common feeling of inadequacy in relation to the state of the theory of Pfaffian systems of this more general kind was frequently expressed by mathematicians in the early 1890s. For instance, Forsyth (1890) complained about the lack of new results in this realm of the theory and tried to indicate a path to be followed in order to achieve a satisfying generalization of the study of a single non-exact Pfaffian equation to systems of many equations. As in the case of a single equation, he said, it is desirable to have the integral equivalent of the system as general as possible and, in order to fulfill this aim, he individuated three different steps: (i) the determination of the number of equations in the integral equivalent of a non-integrable system; (ii) the deduction of some simple integral equivalent of such a system and finally, and (iii) the generalization of such an integral equivalent once it has been obtained.

According to Forsyth, some advances had been achieved only with respect to step (i) by the work of Biermann (1885), who had demonstrated that the maximal dimen-

[8] It appears that Mayer was the first one to call attention over this dual connection in Mayer (1872); see also Hawkins (2005, pp. 408–410). In this connection, Frobenius spoke of *adjungirt* or *zugehörig* system. His characterization of duality was purely algebraic, as one can see in Frobenius (1877, §13) or Hawkins (2005, pp. 411–415). Engel's interpretation in terms of infinitesimal transformations is discussed in §4 below.

[9] See Frobenius (1877, §20).

sion of the integral varieties of an unconditioned[10] Pfaffian system is given by the integer part of the ratio between the number of variables and the number of equations augmented by one, and that the rest of this division gives information about the degree of indeterminacy of the integral solutions. As far as the remaining two steps were concerned, Biermann's analysis had made it clear that the methods of integration known at that time (in particular Clebsch's so-called second method) did not permit a general solution to be obtained. Forsyth's effective synthesis of the state-of-the-art of the theory is worth quoting:

> And so the solution of the problem of obtaining the integral equivalent of a simultaneous system of unconditioned Pfaffians does not appear possible by any methods at present known which are effective for the case of a single Pfaffian. It is, in fact, one of the most general problems of the integral calculus; the discovery of its solution lies in the future.[11]

4 Engel's invariants theory of Pfaffian systems

Quite similar remarks of dissatisfaction about the state-of-the-art of the theory were expressed by Engel (1890) at the beginning of the first of two memoirs that were dedicated to the invariant theory of Pfaffian systems and were communicated by Mayer in 1889 and in 1890 to the *Sächsische Akademie der Wissenschaft* in Leipzig. Engel wrote:

> The invariant theory of a single Pfaffian equation has been completed for some time; on the contrary, as far as systems of Pfaffian equations are concerned, almost everything remains to be done.[12]

Engel's approach was deeply influenced by the work of his highly respected master Sophus Lie. Moreover, it appears that the main concern that led him to deal with such systems of total differentials equations was their application to the theory of continuous groups of transformations. Nonetheless, Engel's contributions are of considerable historical interest since they represented a source of inspiration for the later papers by von Weber and Cartan himself.

4.1 Invariant correspondences

Engel's strategy was dominated by the persistent recourse to structures invariantly connected to the given Pfaffian system. The very first example of such connected structures had been the so-called bilinear covariant of a Pfaffian expression upon which Frobenius had constantly relied in his work. Engel took up this fertile idea and

[10] An unconditioned Pfaffian system is one in which no specification of the coefficients of the Pfaffian system has been made.

[11] See Forsyth (1890, §185).

[12] 'Die Invariantentheorie einer einzelnen Pfaff'sche Gleichung ist schon lange erledigt, dagegen bleibt für die Systeme von Pfaff'schen Gleichungen fast noch Alles zu thun'.

generalized it by proposing the following definition: two differential systems (depending on the circumstances, a differential system can be a system of partial differential equations, a system of Pfaffian equations or a set of infinitesimal transformations) are said to be invariantly associated (*invariant verknüpft*) if a bijective correspondence exists between them that is preserved under arbitrary changes of coordinates. The knowledge of these connected structures, as in the case of a single Pfaffian equation or in the case of a complete system of Pfaffian equations examined by Frobenius, was considered quite useful by Engel since the study of their properties allowed him to get information, for example, on the normal form of the original Pfaffian system.

The starting point of his analysis was the observation that a reciprocal connection (*Zusammenhang*) exists between Pfaffian systems and systems of linear homogeneous partial differential equations of the first order. According to Engel, the origin of this connection stemmed from two distinct interpretations that one could ascribe to a given system of m Pfaffian equations of the following form:

$$\omega_\mu = \sum_{i=1}^{n} a_{\mu i}(x_1 \ldots, x_n)dx_i = 0 \quad (\mu = 1, \ldots, m). \tag{6}$$

One could interpret (6), in the usual way, as a system of differential equations and, correspondingly, one could undertake the task to determine all its integral equivalent equations, that is, to determine all the equations

$$\Phi_1(x_1 \ldots x_n) = 0, \ldots, \Phi_q(x_1 \ldots x_n) = 0,$$

such that the $2q$ relations

$$\Phi_1 = 0, \ldots, \Phi_q = 0, d\Phi_1 = 0, \ldots, d\Phi_q = 0,$$

imply identically $\omega_\mu \equiv 0$, $\mu = 1, \ldots, m$. On the other hand, Engel explained, one could regard the quantities $dx_1, \ldots, dx_n$ in (6) as the infinitesimal increments to which the variables $x_1, \ldots, x_n$ are subject as a consequence of the action of an infinitesimal transformation

$$X(f) = \sum_{j=1}^{n} \xi_j(x_1 \ldots x_n)\frac{\partial f}{\partial x_j}.$$

According to this interpretation, Eq. 6 define a family (*Schaar*) of infinitesimal transformations, namely the set of all infinitesimal transformations $X(f)$ that satisfy the following m relations:

$$\sum_{j=1}^{n} a_{\mu j}\xi_j = 0 \quad (\mu = 1, \ldots, m). \tag{7}$$

Since the rank of the matrix $A = \left[a_{\mu j}\right]$ is supposed to be maximal (and so equal to $m < n$), Eq. 7 admits $n - m$ linearly independent solutions $\xi_i^{(k)}$, for $i = 1, \ldots, n$ and $k = 1, \ldots, n - m$. Therefore one obtains $n - m$ linearly independent infinitesimal transformations:

$$X_k(f) = \sum_{i=1}^{n} \xi_i^{(k)}(x_1, \ldots, x_n) \frac{\partial f}{\partial x_i}, \quad (k = 1, \ldots, n - m)$$

which are the generators of the set of infinitesimal transformations that is associated to the given Pfaffian system (6). An arbitrary transformation of this set takes on the following expression:

$$W(f) = \chi_1(x_1, \ldots, x_n) X_1(f) + \cdots + \chi_{n-m}(x_1, \ldots, x_n) X_{n-m}(f), \tag{8}$$

where the χ_i $(i = 1, \ldots, n - m)$ are arbitrary functions of n variables.

Engel observed that this reciprocal correspondence between Pfaffian systems and sets of infinitesimal transformations is not only bijective but it is also preserved under arbitrary transformations of coordinates, so that it is, in fact, an example of invariant association. Finally, by setting all these transformations equal to zero, one obtains the following system of independent differential equations:

$$X_1(f) = 0, \ldots, X_{n-m}(f) = 0.$$

which is also invariantly connected with (6).

This dual connection was used by Engel to build up new auxiliary Pfaffian systems which introduce a remarkable simplification in the theory. The first of these auxiliary systems is obtained as a consequence of the action of a generic infinitesimal transformation of type (8) on the Pfaffian system (6), now supposed to be rewritten (*aufgelöst*) in the following form[13]:

$$\Delta_\mu = dx_\mu - \sum_{k=1}^{n-m} a_{m+k,\mu} dx_{m+k} = 0 \quad (\mu = 1, \ldots, m). \tag{9}$$

Correspondingly, the infinitesimal transformations which are associated with it are now written as:

$$A_{m+k}(f) = \frac{\partial f}{\partial x_{m+k}} + \sum_{\mu=1}^{m} a_{m+k,\mu} \frac{\partial f}{\partial x_\mu} \quad (k = 1, \ldots, n - m).$$

[13] Here and in what follows I adhere to Engel's original notation.

If we define $W(f)$ to be a generic transformation:

$$W(f) = \sum_{k=1}^{n-m} \chi_{m+k} A_{m+k}(f),$$

its action on (9) transforms the latter into the system[14]:

$$\Delta_\mu + \delta t(W\Delta_\mu) = 0 \quad (\mu = 1, \ldots, m)$$

which is easily demonstrated to be equivalent and invariantly connected to the following system of Pfaffian equations:

$$\Delta_1 = 0, \ldots, \Delta_m = 0, A_{m+k}\Delta_1 = 0, \ldots, A_{m+k}\Delta_m = 0 \quad (k = 1, \ldots, n-m). \tag{10}$$

Finally, a few manipulations give the following equivalent and simplified form written in terms of the coefficients of Frobenius's bilinear covariants:

$$\begin{cases} dx_\mu - \sum_{k=1}^{n-m} a_{m+k,\mu} dx_{m+k} = 0, \\ \sum_{k=1}^{n-m} \{A_{m+k} a_{m+j,\mu} - A_{m+j} a_{m+k,\mu}\} dx_{m+k} = 0 \\ (\mu = 1, \ldots, m; \quad j = 1, \ldots, n-m). \end{cases} \tag{11}$$

Engel observed that it may happen that the system (11) coincides with (9); if this is the case, then (9) is completely integrable and it admits every infinitesimal transformation (8), that is, for a generic transformation $W(f)$, $W\Delta_\mu = 0\,(\mu = 1, \ldots, m)$, are a consequence of $\Delta_\mu = 0\,(\mu = 1, \ldots, m)$.

In virtue of the dual correspondence between Pfaffian systems and sets of infinitesimal transformations, the system (11) can be considered as defining a set (*Schaar*) of infinitesimal transformations. It turns out that these transformations are precisely those transformations that leave the original Pfaffian system (9) invariant and, moreover, as Engel demonstrated that the Pfaffian system (11) is completely integrable. Using anachronistic terminology, we can call such transformations *characteristic transformations* and, correspondingly, the Pfaffian system (11) defining them a *characteristic system*.

A second differential system invariantly connected to (9) was obtained by Engel as a consequence of the following simple remark. If one considers a system of $n-m$ linear homogeneous partial differential equations of the following form:

$$C_k(f) = \sum_{i=1}^{n} \beta_{ik}(x_1, \ldots, x_n) \frac{\partial f}{\partial x_i} = 0 \quad (k = 1, \ldots, n-m), \tag{12}$$

[14] The expression $W\Delta_\mu$ is what today we would call Lie derivative of Δ_μ with respect to the vector field W. From a historical point of view, such a denomination is very appropriate. Actually, Lie was the first person to introduce it, see, for example, Lie (1888, pp. 529–530).

and the system of equations

$$\begin{cases} (C_k, C_j)(f) = C_k(C_j(f)) - C_j(C_k(f)) = 0, \\ C_k(f) = 0 \quad (k, j = 1, \dots, n - m), \end{cases} \tag{13}$$

then they are invariantly associated. Since there is an adjoint Pfaffian system associated to every system of linear homogeneous partial differential equations, it is clear that two systems of Pfaffian equations which are invariantly connected correspond to (12) and (13). As a result of this, Engel stated the following:

Theorem 3 *The system of Pfaffian equations which is dual to the system of partial differential equations*

$$\begin{cases} A_{m+k}f = \frac{\partial f}{\partial x_{m+k}} + \sum_{\mu=1}^{m} a_{m+k,\mu} \frac{\partial f}{\partial x_\mu} = 0 \\ (A_{m+k}, A_{m+j})f = \sum_{\mu=1}^{m} (A_{m+k}a_{m+j,\mu} - A_{m+j}a_{m+k,\mu}) \frac{\partial f}{\partial x_\mu} = 0 \\ (k, j = 1, \dots, n - m), \end{cases} \tag{14}$$

is invariantly connected to the Pfaffian system (9).

Other differential systems invariantly connected to the given Pfaffian system were obtained by Engel in the course of his research. However, the function they fulfilled was always the same: to deduce from them the normal form of the Pfaffian system under consideration and to develop applications in the realm of the theory of continuous groups and of the theory of contact transformations as well. To name just a few concrete examples, Engel succeeded in giving a complete invariant theory of Pfaffian systems of two equations in four independent variables; furthermore, he utilized some of his results to give a simpler treatment of the problem, already faced by Page (1888), of the classification of all imprimitive continuous transformation groups in space in four dimensions[15] and, finally, he was able to present a very clear demonstration of a theorem originally due to Bäcklund (1876) which gave a complete characterization of all contact transformations.

5 von Weber's contributions: 1898–1900

As Goursat (1922, p. 259) was once to observe, before Cartan's seminal papers Cartan (1901a,b) the first rigorous results in the field of the theory of general Pfaffian systems were obtained, along with Engel, by the young mathematician Eduard Ritter von Weber

[15] A group of r independent infinitesimal transformations in n variables is said to be *imprimitive* if it leaves a family of ∞^{n-q} q−dimensional subvarieties M_q:

$$\phi_1(x_1, \dots, x_n) = c_1, \dots, \phi_{n-q}(x_1, \dots, x_n) = c_{n-q},$$

invariant; that is, if

$$X_i(\phi_k) = \Omega_{ki}(\phi_1, \dots, \phi_{n-q}), \ i = 1, \dots, r \quad k = 1, \dots, n - q$$

where the Ω are some functions of $\phi_1, \dots, \phi_{n-q}$; see (Page, 1888, pp. 297–300).

(1870–1934) in a series of articles which laid the basis for the subsequent geometrical developments of Cartan's theory of exterior differential systems. von Weber's approach was profoundly inspired by Engel's researches. Wide use of invariantly associated differential systems, frequent application of infinitesimal characteristic transformations and consistent reference to geometrical visualization were for von Weber, as well as for Engel, the main technical and conceptual tools to which to tackle the resolution of generalized Pfaffian systems.

Nonetheless, it appears that a specific motivation guided von Weber's interest in his attempt to classify the large variety of Pfaffian systems, that is, the hope of applying Pfaffian systems to the systematic study of general system of partial differential equations already begun by Méray and Riquier. Moreover, von Weber took advantage of some of the main results of the general theory of systems of partial differential equations, namely existence theorems for the so-called passive systems[16] which he used to demonstrate the existence of integral varieties of the given Pfaffian system and the consequent possibility of writing it in a simple normal form containing a reduced number of differentials. Finally, a regular application of the theory of linear complexes and congruences in projective space has to be noted as one of the most original technical innovations introduced by von Weber into the theory.[17]

The contributions of von Weber that we are interested in are spread over a certain number of memoirs which he published between 1898 and 1900. Our attention will be mainly concentrated on von Weber (1898a), in which the notion of character of a Pfaffian system and that of derived system were introduced for the first time. Nevertheless, since it appears that some of his later developments may have played a role in influencing Cartan's geometrical approach to a generalization of the problem of Pfaff, a brief survey of von Weber (1900a,b,c) will also be given.

5.1 Character and characteristic transformations

As von Weber himself observed in the final historical remarks of von Weber (1900c, p. 609), since the introduction of the bilinear covariant by Frobenius and Darboux,

[16] An explanation of this intricate notion will be given below.

[17] The following remarks taken from the introduction to von Weber (1901) are quite enlightening. There von Weber wrote:

> If our task can be considered as a special case of the general theory of differential systems, it is also true that, from another point of view, the latter can be regarded as a special case of the former. Yet, the new auxiliary means that we will utilize in our analysis and that will be connected in many ways to the theory of differential systems, is represented by the theory of linear complexes and congruences in $(m-1)$-dimensional space as well as by the theory of families of antisymmetric bilinear forms in $2m$ variables.
> ('Unsere Aufgabe lässt sich als Specialfall der allgemeinen Theorie der Differentialsysteme auffassen, wie sich auch umgekehrt die letztere, von einem andern Standpunkt aus betrachtet, der ersteren als Specialfall einordnet. Das neue Hilfsmittel jedoch, das wir bei unseren Untersuchungen verwenden und mit der Theorie der Differentialsysteme in mannigfache Beziehung setzen werden ist die Theorie der Liniencomplexe und -Congruenzen in $m-1$-dimensionalen Raum, also der Schaaren von alternirenden Bilinearformen mit m Variabelnpaaren'.)

invariantly associated structures had played a major role in the theory of Pfaffian systems. Engel had taken great advantage of them and had succeeded in providing some new applications of Pfaffian equations especially in the classification problem of continuous groups of transformations. von Weber acknowledged the fruitfulness of this approach and tried to give it systematic basis within the context of an invariants theory of systems of Pfaffian equations.

von Weber (1898a) began his analysis by considering a system of $n - m$ Pfaffian equations in the following, resolved form[18]:

$$\nabla_s = dx_{m+s} - \sum_{i=1}^{m} a_{si} dx_i = 0, \quad (s = 1, \dots, n - m). \tag{15}$$

Engel had proved that the differential system for two independent variations dx_i and δx_i of the n variables $x_1, \dots, x_n (i = 1, \dots, n)$

$$\begin{cases} dx_{m+s} = \sum a_{si} dx_i; \quad \delta x_{m+s} = \sum a_{si} \delta x_i, \\ \sum_{k=1}^{m} \sum_{i=1}^{m} a_{iks} dx_i \delta x_k = 0, \end{cases} \quad (s = 1, \dots, n - m). \tag{16}$$

is invariantly associated with (15).[19] As a consequence of this, von Weber observed that the study of invariant quantities attached to (15) could be transferred to that of the invariants of the system (16). He defined the first of these invariants, the *character* K of the system (15), as the rank of the matrix

$$\left[\sum_{k=1}^{m} a_{iks} \lambda_k \right] \quad (i = 1, \dots, m, s = 1, \dots, n - m),$$

when $x_1, \dots, x_n$ and $\lambda_1, \dots, \lambda_m$ assume arbitrary values.

The importance of the notion of character lay in the fact that it offered a first classification criterion for the large variety of Pfaffian systems, and a measure of the degree of difficulty of the problem one has to face, as it were: the greater the character, the harder is the task to undertake. If, for example, $K = 0$ then the system (15) is completely integrable, since clearly the coefficients of the bilinear covariants vanish identically.

von Weber's attention (1898a) was almost exclusively concentrated on Pfaffian systems of character *one*, but some important results concerning characteristic transformations were obtained for the general case too. Since characteristic transformations will also play a key role in Cartan's analysis, it seems appropriate to describe their frequent use in von Weber's theory in some detail.

Already introduced by Engel, characteristic transformations are defined as the infinitesimal transformations which are dually associated with (15) and, at the same time, leave these equations invariant; that is, if we consider a generic infinitesimal

[18] Though it may appear bizarre, the ∇ notation was the one employed by von Weber.

[19] As usual, the following relations are supposed to hold: $A_i(f) = \frac{\partial f}{\partial x_i} + \sum_{s=1}^{n-m} a_{si} \frac{\partial f}{\partial x_{m+s}}$ and $a_{iks} = -a_{kis} = A_i(a_{sk}) - A_k(a_{si})$.

transformation associated with (15), $X(f) = \sum_{i=1}^{m} \xi_i A_i(f)$, this transformation is characteristic if the following identities are satisfied in virtue of Eq. 15:

$$X(\nabla_s) = 0 \quad (s = 1, \ldots, n-m). \tag{17}$$

In a more explicit form, this means:

$$X(\nabla_s) = d\xi_{m+s} - \sum_{i=1}^{m} X(a_{si})dx_i - \sum_{i=1}^{m} a_{si}d\xi_i = 0,$$

and, consequently, we have

$$d\left(\xi_{m+s} - \sum_{i=1}^{m} a_{si}\xi_i\right) + \sum_{i=1}^{m} (da_{si}\xi_i - X(a_{si})dx_i) = 0,\text{[20]}$$

and finally, since $\xi_{m+s} = \sum_{i=1}^{m} a_{si}\xi_i$ and since, as a consequence of (15), for an arbitrary function of n variables, $df = \sum_{k=1}^{m} A_k(f)dx_k$:

$$\sum_{k=1}^{m} \xi_k a_{iks} = 0 \quad (i = 1, \ldots, m;\ s = 1, \ldots, n-m). \tag{18}$$

von Weber supposed that there are h independent solutions $\vec{\xi}^{(i)} (i = 1, \ldots, h)$ of Eq. 18, so that the set of infinitesimal transformations leaving the Pfaffian system (15) invariant is generated by the following differentials operators:

$$X_i(f) = \sum_{j=1}^{m} \xi_j^{(i)} \frac{\partial f}{\partial x_j} \quad (i = 1, \ldots, h). \tag{19}$$

As already pointed out by Engel and proved by von Weber through a direct computation, the system of h differential equations $X_i(f) = 0$ is complete in the sense of Clebsch's definition.[21] von Weber acknowledged the importance of such transformations, and explained how they could be usefully employed to simplify the integration of the Pfaffian system under consideration. In particular, he proved the following:

Theorem 4 *For the Pfaffian system* (15) *to be reducible through a change of coordinates to a system of equations in $n - h$ variables, it is necessary and sufficient that it admits h (independent) characteristic infinitesimal transformations.*

[20] A modern version of this formula would read as follows: $X(\nabla_s) = d\nabla_s(X) + d(\nabla_s(X))$. Cartan is usually acknowledged as its first discoverer, see Ivey (2003, p. 339). However, this attribution appears to be not very accurate from a historical point of view. Indeed, it can be found already in Engel (1896, p. 415).

[21] In modern terms, that means that the operators $X_i (i = 1, \ldots, h)$ define an involutive distribution of tangent vector fields. See Sect. 2 above.

Indeed, if one introduces a change of coordinates in which $n - h$ variables are identified with the $n - h$ independent solutions of the complete system (19), it is easy to show that the Pfaffian system so obtained only depends upon these $n-h$ variables.[22]

5.2 Pfaffian systems of character one, I

The remaining part of the memoir (von Weber 1898a) was devoted to a thorough analysis of a very special type of Pfaffian systems, namely those whose character is equal to one. Since Cartan would take up the same topic in 1901 by reinterpreting von Weber's result in the light of his new geometrical methods based on the brand new exterior differential calculus, it appears appropriate to discuss von Weber's achievements in order to facilitate a comparison between Cartan's and von Weber's approaches.

Pfaffian systems of character one represent the simplest case one can conceive after the case of completely integrable systems. Frobenius had showed that if the system is completely integrable, then the vanishing of all its bilinear covariants is an algebraic consequence of the equations of the system itself. Instead, in the case of systems of character one the bilinear covariants reduce to a single bilinear form whose vanishing is not implied by the equations of the system itself. In other words, the following relations among the coefficients of the bilinear covariants hold: $a_{iks} = \mu_s a_{ik1} (s = 2, \ldots, n - m; i, k = 1, \ldots, m)$, where the μ_s are functions of the n variables $x_1, \ldots, x_n$. von Weber supposed that the matrix $[a_{ik1}](i, k = 1, \ldots, m)$ has rank equal to 2ν, so that he could deduce the existence of $m - 2\nu$ linearly independent characteristic transformations

$$X^{(k)} f = \sum_{i=1}^{n} \xi_i^{(k)} \frac{\partial f}{\partial x_i}, \quad k = 1, \ldots, m - 2\nu.$$

As already explained, the existence of such transformations was exploited to obtain a reduced form of the Pfaffian system under examination; indeed, by appropriate definition of new variables $y_1, \ldots, y_{\nu+1}, \ldots, y_{\nu+n-m}$, the number of differentials can be lowered to $\nu + n - m$ to give the following reduced form of (15)[23]:

$$dy_{\nu+s} = \sum_{i=1}^{\nu} \eta_{si} dy_i, \quad (s = 1, \ldots, n - m).$$

von Weber's treatment of this special type of systems was marked by the definite and profitable distinction between the case in which $2\nu = 2$ and the case in which $2\nu > 2$. Let us consider in some detail the case $2\nu > 2$. The study of such systems was carried out by exploiting the existence of the so-called derived system (das abgeleitete System von (15)) which in this case turns out to be completely integrable. Indeed,

[22] For details, see von Weber (1898a, pp. 210–211). A modern statement of this theorem can be found in Olver (1995, p. 430).

[23] It should be observed that, in general, the η_{si} are functions of the n variables $x_1, \ldots, x_n$.

von Weber considered the following system of partial differential equations (the dual counterpart of the derived system)[24]:

$$A_i(f) = 0, \quad B(f) = \frac{\partial f}{\partial x_{m+1}} + \sum_{s=2}^{n-m} \mu_s \frac{\partial f}{\partial x_{m+s}} \quad (i = 1, \ldots, m) \tag{20}$$

and demonstrated that it is complete in the sense of Clebsch. In fact, as $(A_i A_k)(f) = a_{ik1} B(f) \quad (i, k = 1, \ldots, m)$, all he had to show was that $(A_i B)(f)$ could be expressed as a linear combination (in general with non-constant coefficients) of the $A_i(f)$'s and $B(f)$. Supposing[25] $m \geq 3$, from Jacobi's identity and from $((A_i A_k) A_l) = \left(\sum_{s=1}^{n-m} a_{iks} \frac{\partial f}{\partial x_{m+s}}, A_l\right)$, it follows that

$$\Phi_{0i} a_{kl1} + \Phi_{0k} a_{li1} + \Phi_{0l} a_{ik1} = 0 \quad (i, k, l = 1, \ldots, m), \tag{21}$$

where $\Phi_{0l} \equiv -\Phi_{l0} \equiv (BA_l) - B(f) \cdot B(a_{1l})$. From this one deduces that in the $(m + 1) \times (m + 1)$ antisymmetric matrix

$$\begin{bmatrix} 0 & \Phi_{01} & \Phi_{02} & \cdots & \Phi_{0m} \\ \Phi_{10} & 0 & a_{121} & \cdots & a_{1m1} \\ \Phi_{20} & a_{211} & 0 & \cdots & a_{2m1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{m0} & a_{m11} & a_{m21} & \cdots & 0 \end{bmatrix} \tag{22}$$

all the principal minors (*Hauptunterdeterminanten*) of order four containing elements from the first column and the first row vanish. As a consequence of antisymmetry, *all* principal minors of order four vanish and consequently,[26] either the rank of $[a_{ik1}]$ is two or all Φ_{0l} vanish. Since $rk\,[a_{ik1}] > 2$, the only possibility left is that $\Phi_{0l} = -\Phi_{l0} = (BA_l) - B(f) \cdot B(a_{1l}) \equiv 0$ and so the system (20) is complete. From the complete integrability of the system (20), von Weber straightforwardly derived the complete integrability of what he called the derived Pfaffian system of (15):

$$\nabla_s - \mu_s \nabla_1 = 0 \quad (s = 2, 3, \ldots, n - m). \tag{23}$$

By indicating with

$$\begin{cases} z_{2\nu+2}(x_1, \ldots, x_n) = c_1, \\ z_{2\nu+3}(x_1, \ldots, x_n) = c_2, \\ \vdots \\ z_{2\nu+n-m}(x_1, \ldots, x_n) = c_{n-m-1}, \end{cases}$$

[24] Remember the definition of $A_i(f) = \frac{\partial f}{\partial x_i} + \sum_{s=1}^{n-m} a_{si} \frac{\partial f}{\partial x_{m+s}} \quad (i = 1, \ldots, m)$.

[25] The case $m = 2$ is indeed trivial.

[26] This implication holds in virtue of the antisymmetry. Remember that the rank of an antisymmetric matrix is always even and that it is equal to r if, and only if all the principal minors of order $r + 2$ vanish, and a non vanishing principal minor of order r exists.

its integral equivalents, he was finally able to provide a normal form for (15) given by the system

$$\begin{cases} dz_{2\nu+1} = z_{\nu+1}dz_1 + z_{\nu+2}dz_2 + \cdots + z_{2\nu}dz_\nu \\ dz_{2\nu+2} = 0, dz_{2\nu+3} = 0, \ldots, dz_{2\nu+n-m} = 0, \end{cases} \tag{24}$$

where $z_1, \ldots, z_{2\nu+1}$ are appropriate functions of $x_1, \ldots, x_n$. We will see later Cartan's reinterpretation of the notion of derived system. For the time being, it should be observed, as von Weber did, that the derived system (23) represents an example of differential structure invariantly connected to (15) in the sense of Engel's definition. Indeed von Weber's derived system (23) does coincide with the Pfaffian system introduced by Engel in Theorem 3.

5.3 Reducibility of a Pfaffian system to its normal form

Although von Weber (1898a) analysis can certainly be considered as a remarkable progress by comparison with years of relative stagnation, the results therein contained were of an unsystematic kind and often limited to very particular cases (e.g., character equal to one, as we have seen in the previous paragraph). Over the following years, von Weber tried to remedy this inconvenience and developed a more organic theory which in principle could be applied to Pfaffian systems of a general type. A crucial role, as we will see, was played by geometrical insight and by frequent reliance upon the theory of linear complexes and linear congruences in projective spaces.

As for von Weber's results in this period, the following account is mainly based upon von Weber (1900a,b). I will linger on some details, since in von Weber's papers for the first time we encounter problems, results and technical tools of great importance for the development of Cartan's geometrical theory of exterior differential systems. When discussing Cartan's papers, I will endeavour to indicate limits and relative importance of his debt to von Weber.

At the beginning, von Weber (1900a) singled out the main problem in the theory of general Pfaffian equations as the following question (indeed, a genuine generalization of the problem of Pfaff for a single total differential equation):

Problem 1 *What are the necessary and sufficient conditions for the system* (15) *to be reducible to the following (normal) form*

$$\sum_{s=1}^{\tau} F_{sh} df_s = 0 \quad (h = 1, \ldots, n-m), \tag{25}$$

containing only τ *differentials, where* $f_1, \ldots, f_\tau$ *are independent functions of* $x_1, \ldots, x_n$ *and* τ *indicates an integer not smaller than* $n - m$ *and not greater than* $n - 2$?[27]

[27] The reason for these limitations is easily explained: if $\tau = n - 1$, then one is brought back to the problem of determining 1-dimensional integral manifolds of (15); if, on the other hand, $\tau = n - m$, this means that the system (15) is completely integrable and so Frobenius' theory can be applied.

As already observed by Frobenius,[28] if the Pfaffian system (15) admits a normal form of type (25), then the system of τ equations

$$dx_{m+h}=\sum_{i=1}^{m} a_{ih}dx_i;\ \ df_1=0,\ldots,df_\rho=0\ \ (h=1,\ldots,n-m;\ \rho=\tau-n+m) \tag{26}$$

is completely integrable. Thus, a necessary and sufficient condition for the existence of the normal form (25) is that the $n-m$ bilinear forms

$$\Omega(dx,\delta x)=\sum_{i=1}^{m}\sum_{k=1}^{m} a_{iks}dx_i\delta_k x\quad (s=1,\ldots,n-m) \tag{27}$$

vanish as a consequence of the following relations:

$$\sum_{i=1}^{m} A_i(f_k)dx_i=0,\quad \sum_{i=1}^{m} A_i(f_k)\delta x_i=0\quad (k=1,\ldots,\rho)^{29} \tag{28}$$

von Weber observed that the same problem can be considered from a different, more geometrical perspective whose usefulness Cartan would thoroughly examine in his work.

If the Pfaffian system (15) can be rewritten in the normal form (25), then the equations

$$f_1(x_1,\ldots,x_n)=c_1,\ f_2(x_1,\ldots,x_n)=c_2,\ldots,\ f_\tau(x_1,\ldots,x_n)=c_\tau \tag{29}$$

represent (for arbitrary values of the c_i's) an integral equivalent of (15), that is, an $(n-\tau)$-dimensional integral variety, which von Weber indicated with $M_{n-\tau}$. If Eq. 29 are replaced by their equivalent parametric expression

$$x_i=\phi_i(u_1,\ldots,u_\nu)\quad (i=1,\ldots,n;\ \nu=n-\tau), \tag{30}$$

[28] See Frobenius (1877, §20).

[29] This is a good point at which the following important observation should be made: in the classical literature no conceptual and notational distinction was made between basis elements of what we would nowadays call cotangent space and the components of tangent vectors. If one cannot resist the temptation to restore such a distinction, it should be observed that in formulas (27) and (28) the dx_i's and the δx_i's have to be regarded as components of tangent vectors whereas in formula (26) the dx_i's and df_k's are indeed to be considered as elements of a cotangent space. Clearly, this comes as no surprise in so far as the theory lacks a formal definition of what a differential form is. Somehow more surprising is the discovery that even Cartan's theory was affected by this 'flaw'.

then these functions are solutions of a first order differential system, to be indicated with the symbol S_ν, which takes on the following form[30]:

$$\begin{cases} \frac{\partial x_{m+h}}{\partial u_r} = \sum_{i=1}^m a_{ih} \frac{\partial x_i}{\partial u_r} \quad (r = 1, \dots, \nu; h = 1, \dots, n-m), \\ \sum_{i=1}^m \sum_{k=1}^m a_{ikh} \frac{\partial x_i}{\partial u_r} \frac{\partial x_k}{\partial u_s} = 0 \quad (r, s = 1, \dots, \nu; h = 1, \dots, n-m). \end{cases} \tag{31}$$

Thus, if the differential system S_ν is such that there is a $M_{n-\tau}$ integral variety through every point $(x_1^0, x_2^0, \dots, x_n^0)$ of a certain domain of the whole variety M_n, then the Pfaffian system admits the normal form (25). In this way, problem (1) was traced back to the analysis of the conditions guaranteeing the existence of solutions of the differential system S_ν.

It was at this very point that von Weber used the theory of the so-called general systems of partial differential equations developed by Méray and Riquier among others.[31] As we will see below, a crucial role was played by the notion of passivity (or involution) which assures, under regularity conditions, the existence of integrals of the system itself.

If one adds to the system S_ν all the equations that can be obtained from S_ν through repeated derivations (finite in numbers), one of the following two cases must occur: either a contradiction, that is, a relation among the variables $x_1, \dots, x_n$ only is produced, or a differential system is obtained that can be put in the so-called *canonical passive form* Σ by solving it with respect to certain partial derivatives.[32] If this is the case and if ν constants $u_1^0, \dots, u_\nu^0$ are arbitrarily chosen, the system S_ν admits a

[30] The second group of equations can be easily deduced from the first one simply by differentiating with respect to $u_1, \dots, u_\nu$ and by remembering that $\frac{\partial^2 x_i}{\partial u_r \partial u_s} = \frac{\partial^2 x_i}{\partial u_s \partial u_r}$. For a detailed verification of this very simple statement, see, e.g., Amaldi (1942, pp. 110–112).

[31] For a historical account of the theory, see, e.g., Riquier's remarks in the preface to Ref. Riquier (1910).

[32] I will not insist on detail. For further details, see Riquier's papers (Riquier 1893) and his treatise (Riquier 1910) or von Weber's encyclopaedia article (von Weber 1900d) on partial differential equations. Here I limit myself to the following, few remarks. The expression *canonical form* means a system (to be indicated with Σ) of equations of type:

$$\frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_\nu} x_i}{\partial u_1^{\alpha_1} \partial u_2^{\alpha_2} \cdots \partial u_\nu^{\alpha_\nu}} = \phi_{i,\alpha_1,\dots,\alpha_\nu}\left(x_1, \dots, x_\nu, \dots, \frac{\partial^{\beta_1+\cdots+\beta_\nu} x_k}{\partial u_1^{\beta_1} \cdots \partial u_\nu^{\beta_\nu}}\right) \tag{32}$$

which satisfies the following requirements: (i) No derivative occurring in the left side of (32) is contained in the right side. (ii) for every derivative $\frac{\partial^{\beta_1+\cdots+\beta_\nu} x_k}{\partial u_1^{\beta_1} \cdots \partial u_\nu^{\beta_\nu}}$ contained in the expression for $\phi_{i,\alpha_1,\dots,\alpha_\nu}$, is $\sum_{j=1}^\nu \beta_j \le \sum_{j=1}^\nu \alpha_j$; if, in particular, $\sum_{j=1}^\nu \beta_j = \sum_{j=1}^\nu \alpha_j$ then $k \le i$; if $k = i$ then the first non-vanishing number in the series of differences $\beta_1 - \alpha_1, \beta_2 - \alpha_2, \dots$ is required to be positive.
On the other hand, passivity coincides with the following hypothesis: if the partial derivatives contained in the left side of the equations in Σ and all the other deduced from them through repeated derivations with respect to $u_1, \dots, u_\nu$ are called *principal*, and the remaining ones, along with the dependent variables $x_1, \dots, x_n$, are called *parametric* quantities, then it is required that from Σ and from equations deduced by Σ through differentiation, every principal derivative is required to be expressed in terms of the parametric quantities in a unique way.

unique solution $x_k(u_1, \ldots, u_\nu)(k = 1, \ldots, n)$ with the property that the parametric quantities valued in $u_1^0, \ldots, u_\nu^0$ assume the initial values

$$x_1^0, \ldots, x_n^0, \ldots, \left(\frac{\partial^{i+k+\cdots+l} x_h}{\partial u_1^i \partial u_2^k \cdots \partial u_\nu^l}\right)_0 \cdots,$$

provided that the right-hand sides of equations composing Σ are sufficiently regular, and also provided that the n power series (the sum being extended to all parametric derivatives):

$$\sum \left(\frac{\partial^{i+k+\cdots+l} x_k}{\partial u_1^i \cdots u_\nu^l}\right)_{u_i = u_i^0} (u_1 - u_1^0)^i \cdots (u_\nu - u_\nu^0)^1 \tag{33}$$

converge in a certain region of C^n. In this way, the admissibility of a normal form of type (25) and the consequent existence of $n - \tau$-dimensional integral varieties of the Pfaffian system was characterized by von Weber in terms of the possibility to write the differential system S_ν in a canonical passive (involutive) form. Moreover, as he observed, to establish whether such an eventuality was fulfilled or not, reduced, at least in principle, to a simple procedure consisting of differentiations and eliminations to be operated on S_ν.

At this point, von Weber made an interesting observation that was to assume a role of outstanding importance in Cartan's theory. Motivated by the possibility of developing fruitful applications in the realm of the theory of general partial differential equations, he introduced further hypotheses which guarantee the existence of integral varieties of increasing dimensions. Indeed, he explained, if one supposes that the system S_ν can be put in a canonical passive form simply by solving it with respect to certain derivatives $\frac{\partial x_i}{\partial u_\nu}$ and, furthermore, if one supposes that from S_ν and from equations obtained from it by differentiation and elimination no relation can be deduced among the variables $x_i, \frac{\partial x_i}{\partial u_1}, \ldots, \frac{\partial x_i}{\partial u_\nu}, \quad (i = 1, \ldots, n)$, already contained in $S_{\nu-1}$, and so on for the systems $S_{\nu-1}, \ldots, S_1$, then every 1-dimensional integral variety of (15) M_1 belongs at least to one 2-dimensional integral variety M_2, etc. and, finally, every $\nu - 1$-dimensional integral variety $M_{\nu-1}$ belongs at least to one ν-dimensional integral variety M_ν.

Thus, von Weber arrived at the statement of what I will refer to as the second problem of his theory of Pfaffian systems.

Problem 2 *What are the necessary and sufficient conditions that have to be satisfied so that every 1-dimensional integral variety M_1 of (15) belongs at least to one 2-dimensional integral variety M_2, etc. and so that every $\nu - 1$-dimensional integral variety $M_{\nu-1}$ belongs at least to one ν-dimensional integral variety M_ν?*

Clearly, both for Problems 1 and 2, the aim was that of expressing such conditions by means of algebraic and differential relations among the coefficients $a_{si}(s = 1, \ldots, n - m;\ i = 1, \ldots, m)$ only. For instance, it turns out that a necessary condition for the system (15) to possess ν-dimensional integral varieties M_ν (Problem 1) is the existence of a system of linearly independent functions $\eta_i^{(s)}$, $(i = 1, \ldots, m; s = 1, \ldots, \nu)$ satisfying the following bilinear equations:

$$\sum_{i=1}^{m} \sum_{k=1}^{m} a_{ikh} \eta_i^{(r)} \eta_k^{(s)} = 0, \quad (r, s = 1, \ldots, \nu; h = 1, \ldots, n-m). \tag{34}$$

A detailed account of von Weber's relevant achievements would go well beyond our present purposes; I will limit myself to a few observations. His analysis was mainly based on the theory of bilinear forms and, more precisely, on the classification of linear complexes and linear congruences in $(m-1)$-dimensional[33] projective spaces. This should come as no surprise, since the coefficients of the bilinear covariants, a_{iks}, $(i, k = 1, \ldots, m; s = 1, \ldots, n-m)$ were interpreted by von Weber geometrically as defining a system of linear complexes, equal in number to the character K of (15):

$$\sum_{i=1}^{m} \sum_{k=1}^{m} a_{iks} \eta_i \xi_k = 0, \quad (s = 1, \ldots, K). \tag{35}$$

In this geometrical context, von Weber identified the characteristic transformations $X_i(f) = \sum_{j=1}^{m} \xi_j^{(i)} \frac{\partial f}{\partial x_j}$ $(i = 1, \ldots, h)$ as those for which the components ξ_j represent the singular points of the congruence consisting of all the straight lines belonging to complexes (35). von Weber was able to give a detailed analysis of Pfaffian systems with $m = 3, 4, 5, 6$ but ultimately, despite his hopes,[34] his approach did not succeed in providing a general theory of unlimited validity. Nevertheless, he should be acknowledged for opening a new phase in the studies of general Pfaffian systems by assessing some of the problems to be considered relevant, and also for introducing useful technical tools which were destined to outlive his theory and to be reinterpreted in the light of Cartan's exterior differential calculus.

6 The foundations of the exterior differential calculus

After the completion of his doctoral thesis (Cartan 1894), where he gave a rigorous and complete treatment of the classification of finite-dimensional, semi-simple, complex Lie algebras already considered by Killing, Cartan devoted himself for some years to applications of the theoretical results contained therein. The theory of partial differential equations appears to have been one of the main fields of his interest. This emerges quite clearly, for example, from the reading of the dense memoir Cartan (1896) devoted to the theory of those systems of partial differential equations whose

[33] The number m, that is the difference between the number of variables and the number of equations of which the system (15) consists, was called by von Weber *die Stufe* and it was considered by him, along with the *character*, as a measure of the difficulties one has to face in tackling the study of a given system of Pfaffian equations. See von Weber (1901, p. 387).

[34] See von Weber (1901, p. 388).

solutions depend only upon arbitrary constants, and such that they admit a continuous group of transformations.

Cartan's work (1899) on Pfaffian forms, and more specifically on Pfaff's problem, was part of this interest. Indeed, as Lie had shown, the integration of partial differential equations and the integration of Pfaffian forms were considered as equivalent formulations of the same problem. As Katz (1985) and Hawkins (2005) have already given a full, authoritative account of the material contained in Cartan (1899), I will limit myself to recall the main notions, which will be useful in the rest of our discussion.

Cartan organized his treatment in a deductive way by first presenting a full set of definitions and conventions. He started by giving a *symbolic definition* of what a differential expression in n variables is; this was defined as a homogenous expression built up by means of a finite number of additions and multiplications of the n differentials $dx_1, \ldots, dx_n$ as well as of certain coefficients which are functions of $x_1, \ldots, x_n$. In this way, a Pfaffian expression was defined as a differential expression of degree one such as $A_1 dx_1 + \cdots + A_n dx_n$; a differential form of degree two was given, for example, by $A_1 dx_2 \wedge dx_1 + A_2 dx_3 \wedge dx_2$.[35]

A very important notion of Cartan's new calculus was the exterior multiplication of two differential expressions.[36] Cartan himself observed that he had realized back in 1896 that the formulas for the change of variables in multiple integrals could be easily derived by submitting the differentials under the integration sign to appropriate laws of calculation, which coincide with Grassmann's exterior calculus. By developing such an intuition, in 1899 he was able to present convincing arguments to justify such rules, which relied upon the idea of the *value* of a differential form.

To this end, Cartan considered a differential expression ω of degree h and then supposed that the n variables involved are functions of h arbitrary parameters $(\alpha_1, \ldots, \alpha_h)$. Indicating by $(\beta_1, \ldots, \beta_h)$ one of the $h!$ permutations of the parameters $\alpha_1, \ldots, \alpha_h$, Cartan associated to it the value that ω assumes when the differentials occupying the ith $(i = 1, \ldots, h)$ position are replaced by the corresponding derivative of x with respect to β_i. By attributing to such a quantity the sign + or − depending on the parity of the permutation considered, and then by summing over all $h!$ permutations, Cartan finally obtained what he called the *value* of the differential expression. For example, the value of the differential form $A_1 dx_2 \wedge dx_1 + A_2 dx_3 \wedge dx_2$ is:

$$A_1 \frac{\partial x_2}{\partial \alpha_1} \frac{\partial x_1}{\partial \alpha_2} + A_2 \frac{\partial x_3}{\partial \alpha_1} \frac{\partial x_3}{\partial \alpha_2} - A_1 \frac{\partial x_2}{\partial \alpha_2} \frac{\partial x_1}{\partial \alpha_1} - A_2 \frac{\partial x_3}{\partial \alpha_2} \frac{\partial x_2}{\partial \alpha_1}.$$

At this point, Cartan defined two differential expressions of degree h to be equivalent if their value is the same independently of the choice of parameters $\alpha_1, \ldots, \alpha_h$. In this way, he was able to establish Grassmann's well-known multiplication rules, which were to be interpreted, in Cartan's view, as equalities between equivalence classes of exterior differential forms. For example, one has $dx_1 \wedge dx_2 = -dx_2 \wedge dx_1$ or

[35] Cartan did not employ the wedge product symbol $\wedge$.

[36] One can find the germs of this crucial notion already in some of Poincaré's work on integral invariants. See Katz (1985, p. 322) and Olver (2000, p. 69) for the relevant bibliography.

$dx_4 \wedge dx_4 = 0$, as is easy to see on calculating the values of the differential expressions appearing in the equations.

A second crucial novelty of Cartan's theory was the exterior derivative of a given Pfaffian expression,[37] which he explicitly connected with Frobenius' and Darboux's notion of bilinear covariant. Cartan's definition reads as follows. Given a Pfaffian form like $A_1 dx_1 + \cdots + A_n dx_n$, its derived expression is the form of degree two:

$$\omega' = dA_1 \wedge dx_1 + \cdots + dA_n \wedge dx_n.$$ [38]

The invariant character of such a derivative is then established by relying on the notion of *value* on observing that if $\bar{\omega}$ indicates the expression of ω with respect to a new set of coordinates $y_i(\vec{x})$ then the differential forms of degree two $\overline{\omega'}$ and $(\bar{\omega})'$ are equivalent in the sense specified above.

On the basis of this new calculus, Cartan was not only able to reformulate all the known results in the theory of Pfaffian equations, including Frobenius' analytical classification theorem,[39] but he also to succeed in obtaining remarkable new results about the resolution of systems that consist of a single Pfaffian equation and a certain number of finite relations.[40] These were particularly relevant for the theory of partial differential equations of first order.

7 Cartan's theory of general Pfaffian systems

After laying the foundations of his new exterior differential calculus, Cartan devoted himself to the study of integrable systems of Pfaffian equations that are not completely. As with von Weber, and in accordance with the motivations driving Cartan (1899), it appears that the main reason for this was the many applications of Pfaffian expressions to the theory of partial differential equations.

Cartan (1901a) began his analysis by recalling Biermann's[41] efforts to determine the maximal dimension and the degree of indeterminacy of integral varieties of unconditioned Pfaffian systems. In this respect, Cartan complained about the lack of rigorous (not generic) and systematic results, and emphasized the urgency of providing the theory with solid theoretical grounds in order to remedy this unsatisfactory state of affairs.

As observed by Hawkins (2005, p. 430), a key role was played by the notion of the bilinear covariant which, as we have seen, Cartan interpreted as the first exterior derivative of a Pfaffian form. However, one should not forget that the use of such a notion was not Cartan's alone since, as we have seen, it was quite frequently employed by other mathematicians. Instead, what characterizes Cartan's approach with respect

[37] The definition was generalized to include derivatives of differential forms of degree greater than one in Cartan (1901b, p. 243).

[38] The notation $d\omega$ was introduced by Kähler (1934, p. 6).

[39] See §2 above.

[40] See Cartan (1899, Chapter V).

[41] See §3 above.

to his contemporaries was the ubiquitous recourse to geometrical insight, and the foundational role of his exterior differential calculus which introduced considerable simplifications in the theory.

7.1 Geometrical representation

It seems that Lie had been the first one to attribute a geometrical interpretation to the system of Pfaffian equations (15) in the context of his synthetic approach to differential equations. Engel took it from Lie and profitably applied it to his researches in Engel (1890) by writing:

> We can attribute to the system (15) also an illustrative representation. Indeed, through the equations (15), to every point of the space $x_1, \dots, x_n$ the corresponding plane bundle of directions: $dx_1 : dx_2 : \cdots : dx_n$, is associated. If in every point of the space the corresponding bundle of directions is considered then one obtains a figure which is the exact image of the system (15).[42]

von Weber himself took great advantage of such a geometrical representation and even widened it by introducing for the first time (in September 1900, see von Weber (1900b)) the notion of an *element* of a Pfaffian system as the set of directions tangent to the integral variety of the system itself.

Clearly inspired by this long-standing tradition, Cartan opened his analysis in Cartan (1901a) by emphasizing the importance of geometrical representations in the problems he was about to deal with. Let us consider a system of Pfaffian equations of the following type:

$$\begin{cases} \omega_1 = a_{11}dx_1 + \cdots + a_{1r}dx_r = 0 \\ \omega_2 = a_{21}dx_1 + \cdots + a_{2r}dx_r = 0 \\ \vdots \\ \omega_s = a_{s1}dx_1 + \cdots + a_{sr}dx_r = 0. \end{cases} \tag{36}$$

Cartan supposed that n out of the r variables should be regarded as independent, so that the remaining $r - n$ could be expressed as functions of them. In this way, a n-dimensional variety M_n of the r-dimensional total manifold[43] was defined. Then, Cartan observed, the system (36) can be thought of geometrically as prescribing the conditions that have to be satisfied by the differentials $dx_1, \dots, dx_r$ when one considers an arbitrary displacement on M_n. Furthermore, since the differentials dx_i can be assimilated[44] to the direction parameters of the tangent lines to the variety M_n,

[42] 'Man kann mit dem Systeme (15) auch eine anschauliche Vorstellung verbinden. Durch die Gleichungen (15) wird nämlich jedem Punkte des Raumes $x_1, \dots, x_n$ ein ebenes Bündel von ∞^{m-1} Fortschreitungsrichtungen: $dx_1 : dx_2 : \dots : dx_n$ zugeordnet. Denkt man sich in jedem Punkte des Raumes das zugehörige Bündel von Fortschreitungsrichtungen, so erhält man eine Figur, welche das genaue geometrische Bild des Systems (15) ist'.

[43] I designate with the expression *total manifold* the set of all the r-uples $(x_1, \dots, x_r)$.

[44] Again, no distinction between base elements of cotangent spaces and components of tangent vectors was made.

the system (36) can be interpreted as saying that, as a consequence of (36), these tangent lines belong to a certain linear variety (*multiplicité plane*) which depends upon the point considered. Hence, the problem of finding n-dimensional integral varieties of (36) was traced back to the following:

> We associate to every point of the space a linear variety passing through this point; then we determine an n-dimensional variety M_n, such that in its every point all the tangents to the variety belong to the linear variety corresponding to this point.[45]

After setting the problem in precise geometrical terms, Cartan defined a linear element (*élément linéaire*) to be the set consisting of a point and a straight line passing through the point, which can be denoted $(\vec{x}, d\vec{x})$. Then, he specified a linear element to be integral if the differentials $dx_1, \ldots, dx_n$, regarded as direction parameters of the tangent line of the linear element, satisfy the systems of linear equations obtained from (36) after evaluating the coefficients a_{ij}, $(i = 1, \ldots, s;\ j = 1, \ldots, r)$ at $\vec{x}$. He finally arrived at the following proposition which characterized the integral varieties of (36).

Proposition 1 *For a variety to be integral is necessary and sufficient that every linear element of it be integral.*

The notion of linear element was then generalized by Cartan to a greater number of dimensions. Indeed, a p-dimensional element was defined to be the couple consisting of a point and a linear variety passing through the point. Cartan indicated it with the symbol E_p. It was clear that every p-dimensional element E_p of an integral variety M_n (necessarily $p \leq n$) consists of *integral* linear elements; however, as already observed by von Weber,[46] E_p has to satisfy further conditions which, in general,[47] are not algebraically implied by (36); they are the relations obtained by requiring the vanishing of all bilinear expressions

$$\begin{cases} \omega_1' = \sum_{i,k=1}^{r} \left(\frac{\partial a_{1i}}{\partial x_k} - \frac{\partial a_{1k}}{\partial x_i}\right)(dx_i\delta x_k - dx_k\delta x_i), \\ \vdots \\ \omega_s' = \sum_{i,k=1}^{r} \left(\frac{\partial a_{si}}{\partial x_k} - \frac{\partial a_{sk}}{\partial x_i}\right)(dx_i\delta x_k - dx_k\delta x_i), \end{cases} \tag{37}$$

[45] *'A chaque point de l'espace on fait correspondre une multiplicité plane passant per ce point; déterminer une multiplicité à n dimensions M_n, telle qu'en chacun des ses points toutes les tangentes à cette multiplicité soient situées dans la multiplicité plane correspondente à ce point'.* Cartan's formulation of the integration problem can be translated as follows. Consider the exterior ideal $\mathcal{I}$, simply generated by the 1-forms $\omega_1, \ldots, \omega_s$. To find integral submanifolds of $\mathcal{I}$ is equivalent to the problem of determining integral submanifolds of the distribution of vector fields which is dual to $\mathcal{I}$. If we indicate this distribution with $\mathcal{V} = \{v_1, \ldots, v_{r-s}\}$, the vector space $\mathcal{V}\mid_x$ is precisely the linear variety to which Cartan referred. Indeed, for a submanifold M_n to be an integral variety of $\mathcal{V}$ is necessary and sufficient that $TM_n \mid_x \subset \mathcal{V}\mid_x$, $\forall x \in M_n$.

[46] Here I am referring to von Weber's remarks according to which one has to consider in the differential system S_v, together with equations of type $\frac{\partial x_{m+h}}{\partial u_r} = \sum a_{ih}\frac{\partial x_i}{\partial u_r}$, also the equations obtained by them through differentiation.

[47] Unless the system (36) is completely integrable.

where $d\vec{x}$ and $\delta\vec{x}$ are arbitrary integral linear elements belonging to E_p. Thus, Cartan arrived at the crucial definition of integral element of more than one dimension:

> An element consisting of linear integral elements is said to be an integral element of dimension 2, 3, ... if every two linear integral elements satisfy (37).[48]

Then he defined two integral linear elements $d\vec{x}$, $\delta\vec{x}$ to be *associated* or in *involution* if all of the bilinear expressions (37) vanish. As a consequence of this, the definition of integral element could be rephrased as follows:

> A linear integral element of dimension 2, 3, ... is an element consisting of linear integral elements which are associated pairwise.[49]

Once more, Cartan found it useful to emphasize his geometrical approach by observing that if the quantities $dx_i\delta x_k - dx_k\delta x_i$ are regarded as Plücker's coordinates of a straight line, then the bilinear relations (37) can be interpreted as defining linear complexes in projective spaces; in this connection, he often insisted upon the possibility of yielding a thorough classification of all Pfaffian systems on the basis of the classification of linear complexes themselves.

It is somehow surprising that Cartan never mentioned von Weber's valuable researches in this field,[50] in particular von Weber's frequent recourse to the theory of linear complexes and bilinear forms. It is true that they had been published only a year before the publication of Cartan's (1901a) epoch-making article, and so he might not have had the opportunity of studying them carefully or even of reading them at all. Nevertheless, it is of considerable historical interest that many ideas fully developed by Cartan could already be found in von Weber's work.

We will see below that even for the crucial notion of a Pfaffian system in involution (closely related to Problem 2 of von Weber's theory) no mention of von Weber's analysis was made by Cartan. Once more, it is probable that Cartan was not aware of von Weber's work, and that he introduced involutive systems by borrowing them directly from the theory of general systems of partial differential equations as developed by Méray, Riquier and Delassus (see below).

48 'Appelons élément intégral à 2, 3, ... dimensions un élément formé d'éléments linéaires intégraux et tel, de plus, que deux quelconques d'entre eux satisfassent au system (37)'. In modern terms, Cartan's definition of integral element of (36) can be translated as follows: a p-dimensional integral element of (36) is a subspace $E_p \subset TM_r\,|_x$ such that:

$$\langle\omega_j; v_i\rangle = 0 \quad\wedge\quad \langle d\omega_j; v_i, v_k\rangle = 0 \quad \forall v_i, v_k \in E_p, \quad (j = 1, \ldots, s).$$

49 'Un élément integral à 2, 3, ... dimensions est un élément formé d'éléments linéaires intégraux associés deux à deux'.

50 Among von Weber's works considered here, von Weber (1898a) was the only one explicitly cited by Cartan (1901a,b).

7.2 Cauchy's first theorem

What Cartan called *Cauchy's first problem* was the following:

> Given a p-dimensional integral variety M_p of a system of total differentials equations, to determine a $(p+1)$-dimensional integral variety M_{p+1} passing through M_p.[51]

Its relevance can hardly be overestimated. Indeed, it turns out that the problem of integration of (36) can be solved by a step-by-step procedure consisting of determining integral varieties of increasing dimensions. Clearly, a necessary condition for the existence of a $(p+1)$-dimensional integral variety M_{p+1} passing through M_p is that every p-dimensional integral element E_p of M_p is contained at least in one $(p+1)$-dimensional integral element E_{p+1}. However this condition is not sufficient. To guarantee the existence of such integral varieties, one has to require more, i.e., that at least one $(p+1)$-dimensional E_{p+1} integral element passes through *every* p-dimensional integral element of the space.

Before moving on to the resolution of Cauchy's first problem, Cartan observed, it is useful to make a few geometrical remarks on the structure of the integral elements E_{p+1}. If E_p is supposed to be generated by p linearly independent vectors, i.e., $E_p = \langle e_1, \dots, e_p \rangle$, then an integral element $E_{p+1} \supset E_p$ can be defined by adding a linear element e to E_p, which is linearly independent of E_p; clearly, one requires both that e is integral and that e is in involution with e_i, $\forall i = 1, \dots, p$.

The $(p+1)$-dimensional element E_{p+1} containing E_p depends on $r-p$ homogeneous parameters,[52] and the equations which express that E_{p+1} is integral are linear in these parameters. Let us suppose that these equations reduce in number to $r-p-u-1$ (with $u \geq 0$), then at least one integral element E_{p+1} passes through every integral element E_p. In particular if $u = 0$ then such a E_{p+1} is unique. In general, these integral elements build up an infinite family depending on u arbitrary constants. It may happen that in particular cases the degree of indeterminacy is greater than u; in this eventuality, Cartan said that E_p is a *singular* element, otherwise E_p was said to be regular.[53] The notion of singularity was easily transferred to integral varieties by defining them to be singular when all their integral elements are singular in the specified sense.

At this point Cartan was ready to solve Cauchy's first problem by proving what he called Cauchy's first theorem. This is a crucial point in the whole theory of exterior differential systems; indeed, it can be considered as the gist of what is nowadays known as Cartan–Kähler theorem, since one can find in it the proof of the inductive step that is crucial for assessing the existence of varieties that 'integrate' integral elements. As

[51] 'Étant donnée une multiplicité intégrale à p dimensions M_p d'un système d'equations aux différentielles totales, faire passer par M_p une multiplicité intégrale à $p+1$ dimensions M_{p+1}'.

[52] The numbers of effective parameters is $r-p-1$; in general, every q-dimensional element E_q containing a p-dimensional element E_p depends on $(q-p)(r-q)$ effective parameters. See Goursat's (1922, §81) proof.

[53] A clearer explanation of Cartan's notion of regularity will be given in the next section in terms of the so-called characteristic integers.

it is well-known, the validity of the theorem is limited to the class of analytic Pfaffian systems. For this reason we will suppose, as Cartan explicitly did, that the coefficients of (36) are analytic functions of x.

Theorem 5 (Cauchy's first theorem) *Suppose that every p-dimensional, regular, integral element passes through at least one $(p+1)$-dimensional integral element E_{p+1}; then, given a non-singular p-dimensional integral variety M_p of* (36), *a $(p+1)$-dimensional integral variety M_{p+1} exists which passes through M_p. More precisely, if ∞^u $(p+1)$-dimensional integral elements passes through every regular, integral element E_p, then many integral varieties M_{p+1} exist which depend upon u arbitrary functions.*

In order to offer a general idea of how it works, it will be enough to limit ourselves to the following remarks.
Cartan's starting point was to translate the geometrical content of the statement into an analytic form. To this end, he considered on M_p a p-dimensional regular integral element E_p^0 that he supposed to have its centre at a fixed point $P^0 = (x_1^0, \ldots, x_r^0)$, of M_p. In an appropriate open subset containing P^0, M_p can be represented by $r-p$ analytic functions expressing, for instance, the variables $x_{p+1}, \ldots, x_r$ in terms of $x_1, \ldots, x_p$. Furthermore, the linear equations (with constant coefficients) defining the integral element E_p^0 are solvable with respect to the differentials $dx_{p+1}, \ldots, dx_r$. Then, if a $(p+1)$-dimensional integral element E_{p+1}^0 passing through E_p^0 is considered, the $r-p-1$ equations defining it are solvable with respect to the $r-p-1$ differentials $dx_{p+2}, \ldots, dx_r$, say. Thus, a $(p+1)$-dimensional integral variety M_{p+1} admitting E_{p+1}^0 will be expressed, in an appropriate neighbourhood of P^0, by $r-p-1$ analytic functions, $x_{p+2}, \ldots, x_r$ of $x_1, \ldots, x_{p+1}$.

At this point, Cartan introduced some simplification in the notation: x_{p+1} was replaced by x and $x_{p+2}, \ldots, x_r$ by $z_1, \ldots, z_m$ (clearly, $m = r-p-1$). As a consequence of this, the equations for M_p (in a neighbourhood of P^0) can be written as follows:

$$\begin{cases} x = \phi(x_1, \ldots, x_p), \\ z_1 = \phi_1(x_1, \ldots, x_p) \\ \vdots \\ z_m = \phi_m(x_1, \ldots, x_p), \end{cases} \tag{38}$$

while the variety M_{p+1} can be expressed (in a neighbourhood of P^0) by:

$$z_j = z_j(x, x_1, \ldots, x_{p+1}), \quad (j = 1, \ldots, m = r-p-1). \tag{39}$$

If x is replaced by $x - \phi(x_1, \ldots, x_p)$ then (38) can be rewritten as

$$\begin{cases} x = 0, \\ z_j = \phi_j(x_1, \ldots, x_p), \quad (j = 1, \ldots, m). \end{cases}$$

Consequently, the condition that M_p passes through M_{p+1} requires that:

$$z_j(0, x_1, \dots, x_p) = \phi_j(x_1, \dots, x_p) \quad (j = 1, \dots, m).$$

Now, the problem consists of writing down the differential equations for the $r - p - 1$ unknown functions that have to be satisfied for M_{p+1} to be an integral variety of (36). To this end, it is first necessary to consider the generic $(p + 1)$-dimensional element (having its centre in a neighbourhood of P^0) consisting of the linear elements that one obtains when infinitesimal increments are attributed to every single variable $x, x_1, \dots, x_p$. Their direction parameters are given by the following table:

$$\begin{array}{ll} e: & \frac{dx}{1} = \frac{dx_1}{0} = \cdots = \frac{dx_p}{0} = \frac{dz_1}{\frac{\partial z_1}{\partial x}} = \cdots = \frac{dz_m}{\frac{\partial z_m}{\partial x}}, \\ e_1: & \frac{dx}{0} = \frac{dx_1}{1} = \cdots = \frac{dx_p}{0} = \frac{dz_1}{\frac{\partial z_1}{\partial x_1}} = \cdots = \frac{dz_m}{\frac{\partial z_m}{\partial x_1}}, \\ \vdots & \qquad\qquad\qquad \vdots \\ e_p: & \frac{dx}{0} = \frac{dx_1}{0} = \cdots = \frac{dx_p}{1} = \frac{dz_1}{\frac{\partial z_1}{\partial x_p}} = \cdots = \frac{dz_m}{\frac{\partial z_m}{\partial x_p}}. \end{array} \tag{40}$$

Then one requires that this $(p+1)$-dimensional element is indeed an integral element; that is, one requires (i) that every linear element is integral and (ii) that the linear integral elements are in pairwise involution. Cartan observed that it is useful to separate such a system of differential equations into two groups: the first one (I) assures that the element $E_p = \langle e_1, \dots, e_p \rangle$ is integral, the second one (II) that the linear element e is integral and in involution with E_p. It is easily seen that (I) does not contain the derivatives $\frac{\partial z_1}{\partial x}, \dots, \frac{\partial z_m}{\partial x}$, while the second system of equations (II) is linear with respect to such derivatives. At this point, the regularity assumptions come into play: essentially, the fact that the indeterminacy degree of E_{p+1} is minimal guarantees the possibility of rewriting the system (II) in a Cauchy–Kovalevskaya form where $m - u$ derivatives are expressed in terms of an equal number of analytic functions:

$$\begin{cases} \frac{\partial z_1}{\partial x} = \Phi_1 \left(x, x_i, z_k, \frac{\partial z_k}{\partial x_j}, \frac{\partial z_{m-u+1}}{\partial x}, \dots, \frac{\partial z_m}{\partial x} \right) \\ \vdots \\ \frac{\partial z_{m-u}}{\partial x} = \Phi_{m-u} \left(x, x_i, z_k, \frac{\partial z_k}{\partial x_j}, \frac{\partial z_{m-u+1}}{\partial x}, \dots, \frac{\partial z_m}{\partial x} \right). \end{cases} \tag{41}$$

The theory of the systems of partial differential equations of this particular kind guarantees the existence of holomorphic solutions in a neighbourhood of P^0 depending on u arbitrary[54] functions $z_{m-u+1}, \dots, z_m$, such that for $x = 0$ they reduce to $\phi_1, \dots, \phi_m$.

Cartan's final step consisted in proving that the solution so obtained actually represents a $(p+1)$-dimensional integral variety of the system (36), that is, it satisfies both systems (I) and (II). First, he proved that every solution of the Cauchy–Kovalevskaya

54 Clearly, for $x = 0$ they have to reduce to $\phi_{m-u+1}, \dots, \phi_m$.

system satisfies systems (I) and (II) for $x = 0$. Then he proved that every solution satisfying (I) and (II) for a generic x also satisfies (I) and (II) for every infinitesimally close point $x + \delta x$, showing in such a way that the theorem is true for every point within an appropriate neighbourhood of P^0.

As we have seen, von Weber had been the first person to set the problem of the integration of general Pfaffian systems in terms of the existence of integral varieties of increasing dimension. Yet, he was unable to provide a systematic analysis of the conditions whose determination was the object of Problem 2 of his theory. On his part, Cartan considered the determination of chains of integral varieties of increasing dimension as the gist of the integration procedure of general Pfaffian systems. To this end, he introduced the following definition of Pfaffian system in involution. A Pfaffian system (36) was said to be in involution if at least one 2-dimensional integral variety M_2 passes through each integral curve M_1, at least one 3-dimensional integral variety M_3 passes through each 2-dimensional integral variety M_2, etc. and, finally, at least one g-dimensional integral variety M_g passes through each $(g-1)$-dimensional integral variety M_{g-1}.[55] Now, from Cauchy's first theorem, it follows that a necessary and sufficient condition for the system (36) to be in involution is that every regular 1-dimensional integral element E_1 belongs at least to one 2-dimensional integral element E_2, every regular 2-dimensional integral element E_2 belongs at least to one 3-dimensional integral element, etc. and, finally, every regular $(g-1)$-dimensional integral element E_{g-1} belongs at least to one g-dimensional integral element E_g. In this way, Cartan was able to provide a complete answer to von Weber's Problem 2 by obtaining at the same time an answer to Problem 1 since the determination of the number g coincides with the determination of integral (regular) varieties of maximal dimension.

Some remarks on the historical origins of the notion of involutive systems of Pfaffian equations are in order here. To my knowledge, von Weber was the first to draw attention on the relation between the general theory of canonical passive (involutive) systems of partial differential equations and the existence of chains of integral varieties of (36) of increasing dimensions. Thus, it would be natural to trace back Cartan's notion of involution to von Weber's works on general Pfaffian systems, namely to von Weber (1900a). However, I cannot produce any evidence that Cartan derived from von Weber the inspiration for his researches on this point. On the contrary, Cartan's (1931, pp. 28–29) recollections seem to support the possibility that he developed such a notion independently of von Weber. In fact, he recognized that his own notion of involution was analogous to the one introduced by Méray and Riquier, among others, in the context of the theory of general systems of partial differential equations. Finally, it seems that an important influence on Cartan was exerted by the work of Delassus (1896), namely, in which, for the first time, the study of solutions of general systems of partial differential equations was traced back to the study of Cauchy–Kovalevskaya systems. In particular, Delassus had shown that the whole integration procedure could be reduced to successive integrations of Cauchy–Kovalevskaya systems in an increasing

55 The number g was called by Cartan the *genre* of the system (36); it is defined as the maximal dimension of regular integral varieties of (36). In the next section, we will see how the existence of such an integer can be deduced from the so-called characteristic integers of (36).

number of independent variables. From this point of view, Cartan's achievements can be seen as a translation and a development of Delassus' results in the geometric, coordinates-independent language of exterior differential forms.

7.3 Genre and characters

Cauchy's first theorem highlights the need to proceed toward a detailed geometrical analysis of the properties of integral elements. In particular, as we have just seen, Cartan was interested in studying the conditions guaranteeing the existence of integral elements of increasing dimensions. His procedure can be summarized as follows: let us consider a p-dimensional integral element E_p of a certain point x; we can think of it as being generated by p linearly independent linear integral elements reciprocally in involution: $E_p = \langle e_1, \ldots, e_p \rangle$. To construct an integral element E_{p+1} containing E_p, as we already know, one has to add to E_p an integral linear element e independent of E_p and in involution with e_i, $(i = 1, \ldots, p)$.[56] If, as Cartan would do in Cartan (1901b), we introduce the linear variety[57] $H(E_p)$, the *polar element*[58] of E_p, of all integral linear elements in involution with E_p, we can rephrase the preceding sentence by saying that $E_{p+1} = \langle E_p, e \rangle$, with $e \in H(E_p)$ and $e \notin E_p$.

In order to characterize the structure of such polar elements as well as to obtain information on the degree of indeterminacy of integral elements, Cartan introduced a sequence of integers, which we will indicate with $\tilde{r}_i$, to be defined as follows:

$$\dim H(E_p) = \tilde{r}_{p+1} + p + 1.$$

Geometrically, this means that the polar element $H(E_p)$ is generated by the p base vectors of E_p and by $\tilde{r}_{p+1} + 1$ linear integral elements, $e_0, e_1, \ldots, e_{\tilde{r}_{p+1}}$. As a consequence of this, the $(p+1)$-dimensional integral elements E_{p+1} passing through E_p depend on $\tilde{r}_{p+1}$ parameters or, as Cartan expressed himself, $\infty^{\tilde{r}_{p+1}}$ integral elements E_{p+1} pass through E_p.

Although Cartan was not very explicit, it is clear that the coefficients $\tilde{r}_p$ depend not only on the point x to which the E_ps belong but also on the choice of the basis of the tangent space at x to the integral variety (indeed, even on the ordering of such a basis).[59] Thus, to be rigorous, he should have written $\tilde{r}_p(x; e_1, \ldots, e_{p-1})$ instead of $\tilde{r}_p$. In fact, it appears that the coefficients r_p effectively introduced by Cartan, the so-called *characteristic integers*, should be interpreted as the minimum values of $\tilde{r}_p(x; e_1, \ldots, e_{p-1})$ when x varies over M_r and the e_i's $(i = 1, \ldots, p-1)$ vary over

[56] Remember that e is in involution with $E_p = \langle e_1, \ldots, e_p \rangle$ if, and only if, $d\omega_k(e, e_i) = 0$, $(k = 1, \ldots, s; i = 1, \ldots, p)$.

[57] The fact that $H(E_p)$ is a linear variety is a consequence of the bilinearity of (37). It should be observed that, although consisting of linear integral elements, in general, $H(E_p)$ is *not* an integral element. The reason for this is that two linear integral elements in involution with a third are not necessarily in involution between themselves. See Cartan (1901a, p. 250).

[58] To my knowledge, this denomination was first introduced by Cartan when discussing Pfaffian systems of character one in Cartan (1901b, §18).

[59] See Olver (1995, pp. 450–454).

$TM_r \mid_x$:

$$r_p = Min\{\tilde{r}_p(x, e_1, \ldots, e_{p-1}) \mid x \in M_r,\ e_i \in TM_r \mid_x\}.$$

This lack of notational precision was justified, in Cartan's view, by the necessity of focusing his analysis[60] on the so-called *regular integral elements*, i.e., on those integral elements for which $\tilde{r}_p = r_p$. More precisely, according to Cartan's definition, a p-dimensional integral element E_p is regular if, and only if, $\tilde{r}_{p+1} = r_{p+1}$,[61] or, in another words, when its polar element has minimal dimension. As a consequence of this, in what follows we will limit our attention to *non-singular* (i.e., regular) integral elements, as Cartan did, and for this reason we will ignore the distinction between r_p and $\tilde{r}_p$.

From these preliminary remarks, Cartan moved on to demonstrate certain arithmetical relations among the characteristic integers, which turn out to be very useful for the following discussion.

A first result is that the integers r_p decrease when the index p increases. Indeed, let us consider a regular integral element E_p and a regular integral element E_{p-1} contained therein; since every linear integral element in involution with E_p is, a fortiori, in involution with $E_{p-1} \subset E_p$, we have $H(E_p) \subset H(E_{p-1})$ and thus, $r_p \geq r_{p+1} + 1$. From this, it follows that the succession of integers $\{r_p\}$ is decreasing, and that an integer g exists such that $r_{g+1} = -1$. Therefore, the polar space $H(E_g)$ does coincide with E_g (supposed to be regular) and no $(g+1)$-dimensional integral element E_{g+1} passes through E_g. The integer g Cartan called the *genre* of the differential system (36).

Another chain of inequalities gives information on the differences among three consecutive characteristic integers:

$$r_p - r_{p+1} \geq r_{p+1} - r_{p+2} \quad (p \leq g-2).$$

As with the preceding inequality, its demonstration relies on geometric considerations concerning the polar spaces $H(E_{p-1})$, $H(E_p)$ and $H(E_{p+1})$.

From this and from $r_{g-1} - r_g - 1 \geq r_g$, Cartan eventually deduced the following fundamental chain of inequalities:

$$r - r_1 - 1 \geq r_1 - r_2 - 1 \geq \cdots \geq r_{g-1} - r_g - 1 \geq r_g. \tag{42}$$

The numbers occurring in such inequalities assume a great importance in the theory. Like the characteristic integers r_p, they are invariants of the system (36) with respect

60 As we will see below, characteristic elements are a major exception.

61 It should be observed that this is different from the notion of regularity as given, for example, by Olver (1995, p. 456).

to arbitrary changes of coordinates,[62] and provide a useful tool for the classification of general Pfaffian systems. Cartan indicated them with $s_1, \ldots, s_g$, namely[63]:

$$\begin{cases} s_1 = r_1 - r_2 - 1 \\ \vdots \\ s_{g-1} = r_{g-1} - r_g - 1 \\ s_g = r_g. \end{cases} \tag{43}$$

Cartan observed that the first of these integers, s_1, had already been introduced by von Weber under the denomination of the *character* of the system (36). By generalizing such a notion to the subsequent integers, Cartan spoke of second, third, etc. characters, respectively. As we have seen, von Weber had introduced the character s_1 in a purely algebraic manner (except for the subsequent interpretation in terms of linear complexes) as the number of linearly independent relations built up with the bilinear covariants of the Pfaffian system. It is easy to demonstrate that Cartan's definition coincides with that of von Weber's. Indeed, it is sufficient to observe that, according to Cartan's definition, s_1 is the number of linearly independent equations which one has to add to (36) in order to obtain the polar element of a (regular) linear integral element E_1.

The relevance of such integers was clarified by the possibility of determining the most general (regular) integral variety M_g of (36) by repeated application of Cauchy's first theorem. More precisely, Cartan was able to yield a full characterization of the indeterminacy degree of the solutions of (36) by demonstrating the following

Theorem 6 (Cauchy's second theorem) *Given a Pfaffian system of s linearly independent equations in r variables, of genre g and with characters $s_1, \ldots, s_g$. Then, the r variables can be divided into $g + 2$ groups:*

$$\begin{array}{cccc} x_1, & x_2, & \ldots, & x_g; \\ z_1, & z_2, & \ldots, & z_s; \\ z_1^{(1)}, & z_2^{(1)}, & \ldots, & z_{s_1}^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ z_1^{(g)}, & z_2^{(g)}, & \ldots, & z_{s_g}^{(g)}. \end{array}$$

such that on the most general integral variety M_g the variables $x_1, x_2, \ldots, x_g$ can be regarded as independent and, in a neighbourhood of a regular point $(x_1^0, \ldots, x_g^0)$, M_g is determined by the following initial conditions: on M_g, $z_1^{(g)}, z_2^{(g)}, \ldots, z_{s_g}^{(g)}$ reduce to s_g arbitrary functions of $x_1, \ldots, x_g$; for $x_g = x_g^0$, the $z_1^{(g-1)}, \ldots, z_{s_{g-1}}^{(g-1)}$ reduce to s_{g-1} arbitrary functions of $x_1, \ldots, x_{g-1}$; etc.; for $x_g = x_g^0, \ldots, x_2 = x_2^0$ the

62 Such an invariance property is essentially due to the covariance of the exterior derivative. See Cartan (1901b, pp. 236–237).

63 The first integer $r - r_1 - 1$ is ignored since it is easily demonstrated to be equal to s, the number of linear independent Pfaffian equations of the system. s was sometimes called the 0th character of (36), for example, in Amaldi (1942).

$z_1^{(1)}, z_2^{(1)}, \ldots, z_{s_1}^{(1)}$ reduce to s_1 functions of x_1; finally, for $x_g = x_g^0, \ldots, x_2 = x_2^0, x_1 = x_1^0$, the $z_1, \ldots, z_g$ reduce to s arbitrary constants.

As Cartan observed, the theorem includes the results already obtained by Biermann[64] in the case of unconditioned Pfaffian systems. Clearly, Cartan's achievements were far more general, rigorous and complete. Furthermore, the characterization of the indeterminacy of the integral solutions agreed with analogous results obtained by Delassus (1896) in his researches on general systems of partial differential equations. Nevertheless, Cartan claimed that his new approach through exterior differential forms was superior because it had the advantage of being independent of a particular choice of coordinates.

7.4 Characteristic elements

Thus far, we have dealt with regular integral elements. However, singular integral elements play an important role, too. Following Engel, von Weber had introduced characteristic transformations as those infinitesimal transformations which are dual to the Pfaffian equations of the system and leave the system invariant. On his part, Cartan introduced what he called *characteristic elements* by observing that in some cases the differential equations of the Cauchy–Kovalevskaya system determining the integral variety M_{p+1} passing through a given integral variety M_p assume a particularly simple form which greatly simplifies their integration. By referring to the notation of §7.2, Cartan considered the eventuality in which such equations do not depend upon the derivatives $\frac{\partial z_i}{\partial x_k}$, $(i = 1, \ldots, r - p - 1; k = 1, \ldots, p)$, that is the case in which the Cauchy–Kovalevskaya system is independent of the linear integral elements $e_1, \ldots, e_p$ generating the integral element E_p. As a consequence of this, the partial derivatives $\frac{\partial z_1}{\partial x}, \ldots, \frac{\partial z_{r-p-1}}{\partial x}$ define a linear integral element e which depends only on the point considered and is in involution with all the integral elements E_p passing through this point. If this is the case, at every point of the space[65] a linear integral element exists that is in involution with *every* linear integral element drawn from this point. Such linear integral elements were called *characteristic* by Cartan.[66] Clearly, a linear integral characteristic element is also *singular*, since ∞^{r_1-1} integral elements E_2 pass through it.

As von Weber had already observed, the importance of characteristic elements[67] lay in the possibility of exploiting their existence to simplify the integration of the Pfaffian system under examination. Indeed, after defining the characteristic Pfaffian system as the system of total differential equation determining characteristic elements, Cartan was able to reformulate von Weber's Theorem 4 in the following way:

64 See §3 above.

65 Actually, we should limit ourselves to some open subset of the space.

66 The denomination stemmed from the theory of partial differential equations, namely from the theory of Cauchy's characteristics. The connection between Cartan's characteristic elements and Cauchy's characteristics is explained very clearly in Amaldi (1942, pp. 180–182).

67 As Engel had done, he actually spoke of transformations leaving the Pfaffian system invariant.

Theorem 7 *The minimal number of variables upon which, by means of a change of variables, the coefficients and the differentials of a given Pfaffian system can depend is equal to the number of linear independent equations of the characteristic system; these variables are given by the integration of such a system.*

As for von Weber's analysis, a crucial point in Cartan's treatment was the fact that the characteristic system is completely integrable. However, whereas von Weber had established such a property by having recourse to what we called characteristic transformations, and by relying upon Clebsch's theorem on complete systems of linear partial differential equations, Cartan deliberately avoided such expedients and managed to demonstrate the complete integrability of the characteristic system using differential forms only. Within a few months, he was able to offer two different proofs. The first one (Cartan 1901a, pp. 302–305) consisted of a step-by-step procedure which relied on the basic property according to which a Pfaffian system of $r - 1$ equations in r variables is necessarily completely integrable. The second one (Cartan 1901b, pp. 248–249) was presented instead as a more direct application of the symbolic calculus with exterior differential forms that he himself had developed in Cartan (1899).

It should be noticed that Cartan's refusal to utilize infinitesimal transformations was by no means casual. I suggest that his need to avoid any recourse to them was due to the project of developing an approach to continuous 'Lie groups' purely in terms of Pfaffian forms without any use of infinitesimal transformations, which in his view[68] did not represent an appropriate technical tool for dealing with the structural theory of infinite dimensional continuous groups of transformations.

7.5 Pfaffian systems of character one, II

The present paragraph is devoted to a discussion of Cartan's analysis of this special type of Pfaffian systems as given in Cartan (1901b) with the aim of drawing a comparison between von Weber's and Cartan's approaches. Whereas von Weber's treatment was almost entirely based on analytical and algebraic considerations, Cartan carried out his analysis in geometrical terms, relying heavily on the new properties of his exterior differential calculus, as he had done in his approach to characteristic elements. At the beginning of his discussion, Cartan stated that his analysis of Pfaffian systems of character one did not lead to any new results with respect to von Weber's (1898a) paper. Nevertheless, he set out to reinterpret von Weber's achievements in order to yield a concrete application of the principles of his theory.

Let us begin with Cartan's deduction of the so-called *derived* system of (36). Contrary to von Weber's analysis, which had recourse to infinitesimal transformations (indeed, the dual counterpart of differential expressions) to deduce via Theorem 3 the existence of the system (23) invariantly connected to the Pfaffian system under examination, Cartan first defined the notion of congruence between two differential forms,[69] and then he introduced the derived system in the following way.

[68] See, for instance, Cartan (1904, Chap. 2).

[69] If Ω and Π designate two differential forms with the same degree, and $\omega_1, \ldots, \omega_p$ designate p homogeneous differential forms with degree less or, at most, equal to that of Ω (and Π), then Cartan defined

He considered a Pfaffian system of s independent equations, $\omega_i = 0$, $(i = 1, \ldots, s)$, in r variables and he introduced $r - s$ Pfaffian forms $\bar{\omega}_j$, $(j = 1, \ldots, r - s)$ such that $\{\omega_i, \bar{\omega}_j\}$ $(i = 1, \ldots, s;\ j = 1, \ldots, r - s)$ are n independent Pfaffian forms.[70] As a consequence of this, the s bilinear covariants of the system can be written as

$$\omega_i' \equiv \sum_{j,k=1}^{n-s} A_{ijk} \bar{\omega}_j \wedge \bar{\omega}_k = \Omega_i, \quad (\operatorname{mod} \omega_1, \omega_2, \ldots, \omega_s), \quad (i = 1, \ldots, s). \tag{44}$$

Now, in general, the differential forms Ω_i are not independent; if, for example[71]:

$$l_1\Omega_1 + l_2\Omega_2 + \cdots + l_s\Omega_s = 0,$$

then one has:

$$(l_1\omega_1 + l_2\omega_2 + \cdots + l_s\omega_s)' \equiv 0, \quad (\operatorname{mod} \omega_1, \omega_2, \ldots, \omega_s). \tag{45}$$

In such a way, Cartan demonstrated that appropriate linear combinations of the Pfaffian equations of the system (36) exist such that every couple of integral elements of (36) is in involution with respect to them. He then considered all the equations of type $l_1\omega_1 + l_2\omega_2 + \cdots + l_s\omega_s = 0$, and built up what he called the *derived system* of (36). It is clear that Cartan's definition was a generalization of von Weber's, which was limited to systems of character one. Furthermore, it is worth emphasizing that the introduction of derived systems was brought about by Cartan purely in terms of exterior differential forms without any recourse to operations with vector fields (infinitesimal transformations, in his wording). To this end, an important role may have been played by the remark, already implicit in Engel's and von Weber's work, that exterior differentiation could be considered in a certain sense as the dual counterpart of the Lie-bracketing operation between two infinitesimal transformations.[72]

Specializing his discussion to systems of character one, in accordance with von Weber's results Cartan was able to show that in this case the derived system of (36) consists of the $s - 1$ equations:

$$\omega_i - l_i\omega_1 = 0 \quad (i = 2, \ldots, s).$$

and, consequently, that the Eq. 36 could be chosen in such a way that:

$$\omega_2' \equiv \omega_3' \equiv \cdots \equiv \omega_s' \equiv 0 \quad (\operatorname{mod} \omega_1, \ldots, \omega_s).$$

Footnote 69 continued
Ω and Π to be congruent module $\omega_1, \ldots, \omega_p$ if p differential forms $\chi_1, \ldots, \chi_p$ exist such that $\Omega = \Pi + \omega_1 \wedge \chi_1 + \cdots + \omega_p \wedge \chi_p$.

70 In modern terms, one can say that $\{\omega_i, \bar{\omega}_j\}$ $(i = 1, \ldots, s;\ j = 1, \ldots, r - s)$ define a *coframe*.

71 Here, and in what follows, l_i $(i = 1, \ldots, s)$ indicate s arbitrary functions of $x_1, \ldots, x_r$.

72 For a detailed discussion of the notion of derived system with special emphasis on duality, see Stormark (2000, pp. 24–26).

Before turning to a detailed study of the derived system and exploiting its properties to integrate the Pfaffian system under examination, it is necessary, Cartan observed, to examine carefully the geometric properties of the (unique) linear complex associated to (36). The first problem to be solved is the determination of the maximal dimension of (regular) integral elements and consequently the maximal dimension of (regular) integral varieties. In the light of von Weber's results, one may expect that characteristic elements play a role of strategic importance and, indeed, it turns out that this is the case also for Cartan's treatment of the subject.

Cartan started by considering the linear variety of *all* linear integral elements of (36); he indicated it with H_ρ, where $\rho = r - s$ designates its dimension. He supposed that σ is the dimension of the greatest characteristic element ϵ_σ and then considered a linear integral element $E_1 \notin \epsilon_\sigma$; since E_1 is supposed to be regular and the character of (36) is assumed to be equal to 1, its polar element $H_{\rho-1}$ is a linear variety of dimension $\rho - 1$. Now, Cartan observed with respect to the linear integral elements of $H_{\rho-1}$ a characteristic element $\epsilon_{\sigma+1}$ of dimension $\sigma + 1$ exists such that $\epsilon_{\sigma+1} = \langle \epsilon_\sigma, E_1 \rangle$.[73] It turns out, as Cartan demonstrated in full detail, that $\epsilon_{\sigma+1}$ is the greatest characteristic element with respect to $H_{\rho-1}$. At this point, he considered another linear integral element E'_1 not belonging to $\epsilon_{\sigma+1}$; thus, the linear elements of $H_{\rho-1}$ in involution with E'_1 generate a linear variety $H_{\rho-2}$ whose greatest characteristic element $\epsilon_{\sigma+2}$ can be described as the linear variety $\langle \epsilon_\sigma, E_1, E'_1 \rangle$. Iterating the same process an appropriate number of times, one finally arrives at a (necessarily integral) element[74] $H_{\rho-\nu}$ which coincides with its characteristic element $\epsilon_{\sigma+\nu}$. The number ν is obtained by equating the dimension of $H_{r-\nu}$ with that of $\epsilon_{\sigma+\nu}$; thus:

$$\rho - \sigma = 2\nu.$$

Consequently, since $H_{\rho-\nu}$ is one of the integral elements of (36) of maximal dimension, the *genre* of the Pfaffian system is $\rho - \nu$.[75] Now, from the previous geometrical construction of maximal integral elements it follows that the bilinear covariant ω'_1 must be expressed in terms of 2ν independent Pfaffian forms, so that:

$$\omega'_1 = \bar{\omega}_1 \wedge \bar{\omega}_{\nu+1} + \cdots \bar{\omega}_\nu \wedge \bar{\omega}_{2\nu} \pmod{\omega_1, \ldots, \omega_s}. \tag{46}$$

This formula is the starting point for the subsequent analytical study of integral varieties of systems of character one.

To emphasize the novelty of Cartan's technical tools with respect to those utilized by von Weber, let us consider his proof of the theorem according to which if $\nu > 1$ then the derived system of (36) is completely integrable. Cartan supposed that the Pfaffian forms of (36) are chosen in such a way that its derived system can be written as:

73 The fact that $\epsilon_{\sigma+1}$ is characteristic with respect to $H_{\rho-1}$ means that $e \in \epsilon_{\sigma+1}$ if, and only if, $\omega_i(e) = 0 (i = 1, \ldots, s)$ and $\omega'_1(e, e') = 0 \; \forall e' \in H_{\rho-1}$. The statement that $\langle \epsilon_\sigma, E_1 \rangle$ is characteristic with respect to $H_{\rho-1}$ should now be clear if one recalls the definition of polar element.

74 Cartan indicated with h what here is indicated with ν. The change in notation is aimed at facilitating the comparison with von Weber (1898a).

75 This is in accordance with von Weber's normal form (24) for the case in which $\nu > 1$.

$$\omega_2 = \omega_3 = \cdots = \omega_s = 0.$$

From Cartan's definition of derived system, it follows that:

$$\omega_2' \equiv \omega_3' \equiv \cdots = \omega_s' \equiv 0 \pmod{\omega_1, \ldots, \omega_s}. \tag{47}$$

As a consequence of $\omega_2' \equiv 0$ (mod$\omega_1, \ldots, \omega_s$) one has:

$$\omega_2' \equiv \omega_1 \wedge \chi, \pmod{\omega_2, \ldots, \omega_s}, \tag{48}$$

where χ is a form of degree one which depends upon $\omega_1, \bar{\omega}_1, \ldots, \bar{\omega}_{2\nu}$.[76] By calculating the derivative of the last congruence above, one obtains

$$\omega_1' \wedge \chi - \chi' \wedge \omega_1 \equiv 0 \pmod{\omega_2, \ldots, \omega_s; \omega_1', \ldots, \omega_s'}$$

and, consequently, $\omega_1' \wedge \chi \equiv 0$, (mod $\omega_1, \ldots, \omega_s$). This is equivalent to:

$$\omega_1' = \chi \wedge \pi + \omega_1 \wedge \pi_1 + \cdots + \omega_s \wedge \pi_s,$$

for appropriate forms $\pi, \pi_1, \ldots, \pi_s$ of degree one. Now, unless $\chi \equiv 0$ (mod ω_1), the integral elements of (36) that satisfy $\chi = 0$ and $\pi = 0$ would be characteristic elements, and consequently the number of equations of the characteristic system would be $s + 2$. However, in such an eventuality, we would have $\nu = 1$ which contradicts the hypothesis of the theorem to be proved. Thus, the only possibility left is that $\chi \equiv 0$ (mod ω_1). As result of this, from (48) we obtain: $\omega_2' \equiv 0$ (mod $\omega_2, \ldots, \omega_s$) and, after repeating the same reasoning for $\omega_3, \ldots, \omega_s$, we eventually deduce that

$$\omega_2' \equiv \omega_2' \equiv \cdots \equiv \omega_s' \equiv 0 \pmod{\omega_2, \ldots, \omega_s},$$

which, according to Cartan's reformulation of Frobenius' theorem,[77] implies the complete integrability of the derived system of (36).

8 Concluding remarks

Contrary to prevalent opinion, far from being the achievement of an isolated mathematical genius Cartan's theory of exterior differential systems (later on generalized by Kähler (1934) to differential systems of any degree) was deeply rooted in the historical context of the late nineteenth century theory of partial differential equations. Indeed, as we have seen, his achievements were placed at the intersection of two closely related strands of research: the theory of not completely integrable systems of Pfaffian equations as developed by Engel and von Weber, and the theory of general systems of

[76] This is due to the fact that, according to (46), the characteristic element of maximal dimension is individuated by: $\bar{\omega}_i = 0$ $(i = 1, \ldots, 2\nu)$.

[77] See Cartan (1901b, p. 247) and Hawkins (2005, p. 429).

partial differential equations that was the main focus of attention of Méray, Riquier and Delassus among others. Cartan's great merit was to reinterpret them systematically in a new and powerful geometrical language whose central core was represented by his exterior differential calculus. At the same time, the very emphasis placed by him on the language of exterior differential forms may be indicated as the main cause for the undeserved lack of attention that for some years characterized the response of the mathematical community towards his achievements in this field. In this connection, it is interesting to remark that still in 1924 Vessiot, while praising the beauty of Cartan's integration theory, felt the necessity to translate it into its dual counterpart by replacing exterior Pfaffian forms with the notion of *faisceau* of infinitesimal transformations.

As Kähler suggested in the introduction to his masterpiece (Kähler 1934) in a really effective and historically accurate way, such a double historical origin was reflected in the twofold virtue of the theory: on one hand, with its emphasis on exterior forms it provided Cartan with the necessary tools for subsequent applications to geometry (namely, the method of moving frames) as well as to the theory of infinite continuous Lie groups. On the other hand, it offered a deeper insight into the machinery (*Mechanik*) of partial differential equations.

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