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Plimpton 322: a review and a different perspective

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1 Part I: A review

1.1 Introduction

One may reasonably question the value of yet another discussion of this famous Mesopotamian tablet. However, recent discussions by Eleanor Robson[1] and

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[1] Robson (2001, 2002a, 2002b).

To the great sorrow of his coauthors and many other colleagues, friends and relatives, John Britton died suddenly when this paper was in the last stages of preparation. He was very pleased with the final version, which he was reviewing when he passed away. In completing the article, the coauthors, with the kind assistance of Claudine Britton, made every effort to maintain the text as John had last seen it and preserve a sense of his presence throughout. We dedicate this article to his memory.
This paper came together as a result of a seminar led by Zoë Misiewicz under the supervision of Alexander Jones, held at the Institute for the Study of the Ancient World (ISAW), an affiliate of New York University (NYU), New York, NY on March 3, 2010. Part I was largely completed by Britton, but with corrections and substantial improvements contributed by his fellow authors. Part II was a collective composition .

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Jöran Friberg,[2] suggest that a dispassionate review of what is known, suspected, and conjectured about this unique text may be worthwhile. This article is divided into two part. In Part I, we survey the historical development of the current understanding of the mathematical content of the text and estimate its likely manner of calculation and unbroken extent. In Part II, we attempt to place the text in its mathematical context and present some new perspectives based on comparisons with other texts from the same period and region.

At the outset, a few words about terminology and notation may be helpful. By "Babylonian", we refer to scholar-scribes and their pupils residing in central and southern Mesopotamia and environs and using cuneiform script in their writings. Babylonian mathematical texts employ a uniquely powerful system of numerical notation, which consisted of "digits" ranging from 1 to 59 and a place-value system in which successive digits from right to left represented successively higher powers of 60. The notation did not distinguish between integers and fractions, and absolute magnitudes had to be determined from contexts.

Digits were depicted by combinations of two signs, angled wedges 𒌋 denoting tens followed by vertical wedges 𒁹 denoting units, arranged in tiers of not more than three,[3] a feature which allowed instant recognition without counting. In Old Babylonian (OB) mathematical texts, an intermediary "zero" was conventionally indicated, if at all, by omission, either of tens or units or both. This led to frequent errors, such as confusing 50,6 and 56 as we shall see, which by Late-Babylonian times (ca., −600) had led to the insertion of two small stacked angled wedges 𒑲 as a separation sign[4] to signal either a complete missing digit or the absence of tens, when a digit lacking units was followed by one lacking tens as in the example just noted.

Different conventions have been adopted for transcribing and translating Babylonian numbers. In transcriptions, we use spaces as in the Babylonian originals to reflect missing digits or tens, and transcribe digits which lack units as 10, 20, 30, ... etc. In translations, we follow Neugebauer's convention, separating digits by commas, translating missing digits by "0", and when the magnitude is known separating the integer and fractional parts of numbers with a semi-colon (";").

Division in Babylonian arithmetic was accomplished by multiplying the dividend by the reciprocal of the divisor. Consequently "regular numbers", whose reciprocals have terminating sexagesimal expressions, play a prominent role. Such numbers contain only the prime factors 2, 3, and 5 comprising 60 and consequently are expressible as $\mathrm{N} = 2^a \cdot 3^b \cdot 5^c$ (a, b, and c being integers).

Although a minor point of probably hopeless protest, dates cited here follow the "long chronology" affirmed by Huber et al. (1982), in which 1 Hammurapi = 1848 BCE, 56 years earlier than according to the "middle chronology" still widely used

[2] Friberg (2007).

[3] In formal arithmetic; less formal notation sometimes used two tiers of units for "7" and "8".

[4] Høyrup (2002, p. 15, n19) notes two texts from Susa, TMS XII and XIV, which use the same (!) separation sign to signify the absence of units (TMS XII) or tens (TMS XIV) in otherwise unambiguous numbers, in contrast to the later usage. Consequently he doubts that these should be interpreted as intermediary "zero"s.

by Assyriologists despite being quite certainly wrong.[5] Given the strength of Huber's analysis and the quality of his collaborators,[6] one might expect historians of mathematics to at least alert their readers to the existence of this issue, but for the most part dates based on the untenable "middle chronology" continue to dominate contemporary discussions of OB mathematics without qualification.

Plimpton 322 is an OB mathematical table text, of uncertain provenance. It was purchased by George Plimpton in 1922 or 1923 from the dealer Edgar Banks, and bequeathed to Columbia University in 1936, where it now resides in the Rare Book and Manuscript Library in the Butler Library (Robson, 2001, 171ff). Its name derives from its entry in Mendelsohn's catalogue of the university's cuneiform tablets, which describes it as follows:[7]

> 322. Clay tablet, left-hand edge broken away, bottom of right-hand corner, and a piece of columns 3 and 4 chipped off; fairly well preserved, dark-brown. 8.8 × 12.5 cm; on obverse 4 columns with 16 lines, reverse blank. Content: Commercial account. No date.

According to Banks,[8] the tablet's provenance was "Senkereh", ancient Larsa, and Robson notes (2001, p. 172; 2002a, p. 110) that its headings, rightmost column and landscape format are similar to characteristics of administrative tablets from Larsa written between 1878 and 1850. Goetze (Neugebauer and Sachs 1945, pp. 146–151) assigned the text provisionally to "group 1" from Larsa,[9] but noted (Neugebauer and Sachs 1945, p. 147, n353) that it "may or may not belong here". Høyrup (2002, pp. 337, 386) notes that its orthography "points to group 6 with at least as high plausiblity as group 1" and thinks the text more likely belongs with texts from the periphery, where most of the texts which utilize the Diagonal Rule[10] originate. However, two texts from the Schøyen collection (MS 3052 and MS 3971), attributed by Friberg (2007,

[5] Huber et al. (1982) showed that the "long chronology" agrees far better with both month-length data from OB reigns and data from the so-called Venus Tablet of Ammiṣaduqa than any of the other possible chronologies consistent with the Venus data, and that this agreement was better than would be expected from the best of a random set of wrong chronologies. Furthermore, both sets of data strongly and independently exclude the accepted "middle chronology", which was initially adopted without empirical support simply as a diplomatic compromise between the "long" and "short" chronologies championed by Goetze and Kammenhuber. Pottery evidence is thought to support a later chronology, which however requires rejecting the Venus data as relevant evidence. Huber (2000) has shown that all astronomical evidence—Venus data, month-length data, eclipse reports, and omens—support a single consistent chronology for the dynasties of Akkad, Ur III, and Babylon 1 with the following key dates (BCE): 1 Naram-sin = 2301; 1 Amar-sin = 2094; 1 Ammiṣaduqa = 1702. The principal evidence against this chronology is a single "floating" tree-ring analysis of a Turkish burial chamber, acknowledged to be inconsistent with a later tree ring analysis (Manning et al. 2001).

[6] A. Sachs, M. Stol, R. M. Whiting, E. Leichty, C. B. F. Walker, and G. van Driel.

[7] See Mendelsohn's catalogue page 71, in the Butler Rare Book and Manuscript Collection, Columbia University.

[8] Undated price list describing four tablets sent to Plimpton (Robson 2001, pp. 171, 203), three of which are described as coming from Senkereh and the fourth from Drehem.

[9] Citing the spelling (*sà* and *ú*) of *in-na-as-sà-hu-ú-ma*.

[10] We use the expression "Diagonal Rule" (rather than the anachronistic "Pythagorean Theorem") to designate the relation between length, width and diagonal of a rectangle, as well as the proof of this relation, as attested for example in the tablets Db 2 146 and MS 3971 #2 (see below).

449–451) to Goetze's Group 3 from Uruk (Neugebauer and Sachs 1945, p. 146), utilize the rule, undercutting this argument. Since Banks may have actually known the provenance of the text, we give his testimony particular weight. See Part II for more details.

The tablet's mathematical character was recognized by Neugebauer, who with Sachs (1945) under the heading "Pythagorean Numbers" and showed that the text concerns numbers which satisfy the relationship,

$$d^2 = \ell^2 + b^2, \tag{1}$$

where d, ℓ, and b are respectively the diagonal, length (long-side) and width (short-side) of a rectangle or right triangle. Since then, the tablet has been the object of extensive discussion and debate concerning its construction, interpretation and likely contents when complete.[11]

1.2 Physical description

The existing tablet is a large fragment, written in landscape orientation, containing four columns of numbers aligned by vertical scoring, cleanly broken at the vertical scoring on the left side of the first preserved column. Top, bottom and right edges are preserved. Columnar scoring continues over the bottom edge and onto the reverse, both of which are otherwise blank. Two lines of column headings at the top of obverse are followed by 15 rows of numbers, uniformly aligned within horizontal scorings (see photo in Appendix B).

The writing is clear and remarkably uniform (the succession of "KI"s in column IV′ are virtually indistinguishable). Column widths reflect the maximal length of the component numbers plus space for an additional (sexagesimal) "digit". For the most part vertical alignment of digits is preserved.[12] Column IV′ reflects a less formal numerical notation than the other, strictly mathematical, columns wherein "4" is depicted as 2+2 (two tiers of two verticals) and "7" and "8" as 4+3 and 4+4, respectively,[13] Thereby, emphasizing its different character. Generally, and despite the errors discussed below, the tablet reflects a high level of professional competence. It is not student work.

The dimensions are the following. Outer: 127 mm (long) × 88 mm (high) × 32 mm (thick); writing surface: 125 mm × 83 mm (the rest is edge curvature). Column-widths: I′ = 57 mm, II′ = 24 mm, III′ = 25 mm, IV′ = 19 mm. Heading height: 13 mm; 15 rows = 70 mm ∼ 4⅔ mm/row. The rows are remarkably uniform; the largest variation is between row 1 (5 mm) and row 2 (4 mm); the rest are visually uniform and slightly less than 5 mm. The writing is similarly uniform with one sexagesimal "digit" comprised of tens and units taking 7 mm or a little less pretty consistently, except where

[11] Extensive bibliographies may be found in Robson (2001, pp. 203–206) and Friberg (1981, pp. 317–318). Both omit Anagnostakis and Goldstein (1974) despite its relevance to their interpretations.

[12] Exceptions include misalignments implying empty "digits" in column I′, rows 5 and 6.

[13] "9", However is written in standard notation as 3 tiers of 3 contra Robson (2001, p. 173) but as reflected in her copy.

the initial digit is "1". "KI" at the beginning of each entry in column IV′ similarly takes 7 mm (with 1 mm extending over the scoring into the end of column III′). The total distance vertically scored but empty on lower edge and reverse equals ∼105 mm (lower edge, 22 mm; reverse, 83 mm), which is equivalent to space for an additional 22 or 23 rows of a height similar to those on the obverse.

The maximum thickness of the fragment is slightly right of the middle of col. I′, 95 mm from right edge of the writing surface and 30 mm from the break at the left edge. By symmetry roughly 65 ± 5 mm of writing surface should be missing on the left side of the tablet,[14] implying a total original length of ca., 190 mm and original dimensions of roughly $8\frac{1}{2}'' \times 3\frac{1}{2}'' \times 1\frac{1}{16}''$.

Neugebauer and Sachs (1945, pp. 39) report the presence of modern glue on the broken left edge of the tablet prior to baking and infer that the missing part was lost after its excavation. Robson (2001, p. 172) suggests that an extraneous fragment might have been attached to give an appearance of completeness, which Banks would have recognized and removed. The nature of the break, however, combined with the careful scoring and uniform writing would have made any such "amplification" immediately apparent and hardly worth the effort. More likely is that that the tablet broke, was repaired following excavation, and broke again at the weak repair with the resulting fragment(s) becoming separated prior to reaching Banks. Many tablets from Larsa were purchased around the same time by museums and collectors, and the missing fragment may well remain unrecognized among them.

1.3 Copy and transcription

Robson (2001, p.193, 2002a, p. 105) presents a clear copy, which depicts the text's contents accurately[15] in both substance and spacing. A flake of destroyed surface at the top left results in lost or damaged signs at the beginning of both lines of the heading of column I′ and obscures the initial two or three digits of the numbers in rows 1 to 4 of column I′.

A transcription of the tablet roughly consistent in spacing and alignment with the tablet (see the photo in Appendix B and Robson's copy) is presented in Table 1, where the heading of column I′ also follows Robson (2001, p. 191) restoration. Friberg (2007, p. 441) proposes to insert "a-šà" to the left of *ta-k*]*i-il-ti* in line 1 of column I′. However, there is clearly not room for this within the boundaries of column I′, nor any reason for the heading to extend into the preceding column, since ample room remains to the right to accommodate it. There has been a persistent debate whether column I′ begins

14 de Solla Price (1964, p. 1) estimates the missing fragment as "somewhat more than half as wide as that preserved", thus ca., 65–75 mm. Robson (2001, p. 193) estimates that no more than 50 mm is missing, but from the shape and thickness of the existing fragment, this is clearly too little. From the nature of the break Friberg (1981, p. 283) suggests that "as much as perhaps a third of the whole tablet is missing", which is consistent with the above estimate from curvature alone. Recently, however, he increased his estimate of the missing tablet, first to "a third or more" (Friberg 2007, p. 434), then (p. 441) to "nearly half" (ca., 100 mm), which is clearly inconsistent with the existing curvature and much less likely than his earlier estimate.

15 A trivial exception depicts "7" = 4+3 in column IV′ as "6" = 3+3. Muroi (2003, n.4) suggests there may be other errors, but gives no details, and we have found none.

Table 1 Transcription of Plimpton 322

obv	I′	II′	III′	IV′
1	*ta-k*]*i* - *il- ti ṣi - li - ip* - *tim*	íb-si₈ sag	íb-si₈ *ṣi-li-ip-tim*	mu-bi-im
2	*ša* 1 *in*] -*na-as-sà-hu-ú-ma* sag *i-il-lu-ú*			
3	1 59 15	1 59	2 49	ki 1
4	1 56 56 58 14 *56* 15	56 7	*3* *12* *1*	ki 2
5	1 55 7 41 15 33 45	1 16 41	1 50 49	ki 3
6	1 53 10 29 32 52 16	3 31 49	5 9 1	ki 4
7	1 48 54 1 40	1 5	1 37	ki 5
8	1 47 6 41 40	5 19	8 1	ki 6
9	1 43 11 56 28 26 40	38 11	59 1	ki 7
10	1 41 33 *59* 3 45	13 19	20 49	ki 8
11	1 38 33 36 36	*9* 1	12 49	ki 9
12	1 35 10 2 28 27 24 26 40	1 22 41	2 16 1	ki 10
13	1 33 45	45	1 15	ki 11
14	1 29 21 54 2 15	27 59	48 49	ki 12
15	1 27 3 45	*7* *12* *1*	4 49	ki 13
16	1 25 48 51 35 6 40	29 31	53 49	ki 14
17	1 23 13 46 40	*56*	53	ki 15

with "1", which is mysterious, since remnants of the top of the vertical wedge, "1", are visible on the tablet in lines 7 through 11 and more clearly so in lines 14 through 17, as reflected in Robson's copy and as we observed in the original.

In Table 1, numbers readable in any part are shown in bold; errors are italicized and underlined. Shaded sections approximately reflect surface damage where text is unreadable or missing. Digit spacing is as in the text, especially lines 1, 7, 8, and 13.

1.4 Errors

There are six well-known errors.[16] Two appear to be copy errors (presumably from a work tablet), to wit:

row 2, col. I′ *56* = copy error for 50, 6
row 9, col. II′ *9* = copy error for 8

The other four errors have implications for interpretation and the manner of calculation. They are:

[16] In line 7, a blank space appears in the tablet between 54 and 1, but there is no zero there; idem in line 8 between 47 and 6. In row 13, col. I′ 1;27, 0, 3,45 is correct and consistent with Robson's copy, contra Robson (2001, p. 175, Table 2) who reads 1;27, 3,45 with Neugebauer and Sachs (1945), but corrected in Neugebauer (1951, p. 37).

row 2, col.III′	<u>*3,12, 1,*</u> = copy error for 3,13; it should be 1,20,25
row 8, col I′	<u>*59*</u> = an error for 45,14 resulting from a medial zero copy error in its calculation, but with implications for the manner of calculation (cf. Anagnostakis and Goldstein, 1974).
row 13, col. II′	<u>*7,12, 1*</u> is the square of expected 2,41 or, alternatively, a copy error for 7,13.
row 15, col. II′	<u>*56*</u> is twice expected 28, or 53 is half 1,46.

1.5 Contents

1.5.1 Columns

I′. The numerical contents of column I′ consist of progressively diminishing numbers,

$$\delta^2 = 1 + \beta^2 \left[= (d/\ell)^2\right] \tag{2}$$

where δ and β have terminating sexagesimal digits. As noted by de Solla Price (1964, p. 223) these numbers express the Diagonal Rule relationships among the dimensions of a series of progressively flatter rectangles or right triangles with unit length = 1, width (short side) = $\beta < 1$, and diagonal, $\delta = \sqrt{(1+\beta^2)}$.[17] They can be scaled up to give the dimensions of any specific rectangle of proportionate size by multiplying δ and β "with" (as Babylonian terminology would put it) the actual (or desired) length, ℓ, so that for a given ℓ,

$$b = \beta \cdot \ell \quad \text{and} \quad d = \delta \cdot \ell \tag{3}$$

Column I′ is thus an explicit statement of the Diagonal Rule applied to rectangles with unit length and progressively smaller short sides, starting with a rectangle very nearly square ($\beta = b/\ell = 0; 59, 30$).

The heading of this column has long been uncertain with respect to both its reading and interpretation. However, Robson (2001, 191),[18] has proposed inserting a "1" between the requisite *ša* and the missing *in-* at the beginning of line 2, and shown that what can be read of the fragmentary signs of the last word can only be consistent with *i-il-lu-ú*.[19] With this restoration the heading becomes intelligible as

[17] Neugebauer and Sachs (1945, p. 39) describe the contents as progressively diminishing ratios, $(d/\ell)^2$, which is mathematically equivalent but misses both the sense of a series of normalized rectangles with unit length and the implicit expression of the Diagonal Rule (2) by the number, its integer, and its fractional part.

[18] Citing an earlier suggestion by Bruins (1949, 1967)

[19] Following Goetze as reported by de Solla Price (1964, p. 226).

takilti ṣiliptim ša 1 *innassahuma* sag *illu*

the *takiltum* of the diagonal (from) which 1 is torn out (i.e., subtracted) and (that of) the width comes up.

We discuss *takiltum* in more detail subsequently. For now it suffices to note that if it is translated simply as "square", following Thureau-Dangin,[20] the heading accurately describes the contents of the column.

1.5.2 Columns II′ and III′

Excepting errors, these columns give the integer dimensions of the width (sag) and diagonal (*ṣiliptim*) of the rectangles whose relative proportions are implicit in column I′. Each descriptive heading (sag or *ṣiliptim*) is preceded by the qualifying term íb-si$_8$, a term frequently encountered in quadratic problem texts denoting the linear dimension or "equalside" of the completed square.[21] Thus the term may simply be a qualifier indicating that a linear dimension or line segment is meant, but it may also imply some purposeful connection with the associated square defined by this dimension. Translations as "equalside width" (II′) and "equalside diagonal" (III′) convey both possibilities. If the term only emphasizes that line segments are described, then "short-side" and "diagonal-side" would serve as well. Neugebauer's and Sachs's suggested interpretation of íb-si$_8$ as "solving-number" (Neugebauer and Sachs 1945, p. 39), seems not only removed from the semantic range of the term, but unhelpfully opaque as well. Høyrup (2002, 25ff.) discusses the relevant issues at length.

1.5.3 Column IV′

The last column successively numbers the rows as ki n meaning No. n (or nth), beginning with $n = 1$ in line 1. The heading, mu-bi-im, often translated as "its name", more accurately here means "its line" or even more simply, "line" or "item".

Robson (2001, p. 173), following Friberg,[22] remarks that the numbers in this column are written differently than in the preceding columns with "4" written as two tiers of two wedges and "7", "8" and "9" written as two tiers of 4+3, 4+4 and 5+4 vertical wedges, in contrast to the conventional three-tier notation in the preceding columns.[23] This suggests that OB notation distinguished between "mathematical" and more casual "descriptive" numbers, a distinction which disappears in Late Babylonian and Seleucid texts.

[20] Thureau-Dangin (1938, p. 226).

[21] Attinger (2008) asserts persuasively that the conventional transcription, íb-si$_8$, retained here for its familiarity, should instead be íb-sá. See also Høyrup (2002, p. 27, n45).

[22] Friberg (1981, p. 295), not cited by Robson.

[23] However, *contra* Robson (2001, p. 173), "9" is written in the normal 3-tier notation as shown in her copy (2001, p. 171).

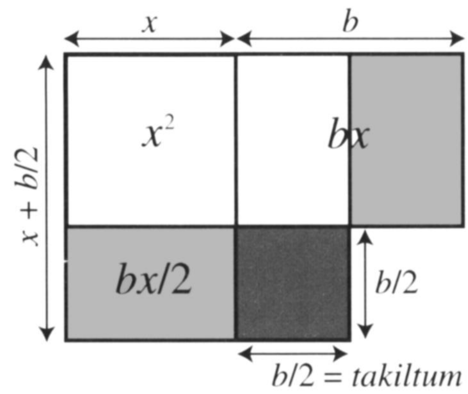

Fig. 1 Quadratic solution

1.5.4 Takiltum

The term *takiltum* is a nominative form of the verb *šutakulum* (to make hold fast together), the causal/reciprocal (Št) stem of the verb *kullum* (to hold fast),[24] which is used for multiplication in the context of rectangularization (Høyrup 2002, p. 23). Høyrup (2002) translates it as "the made hold". Muroi (2003) has collected 12 attested instances of *takiltum*, one of which is Plimpton 322. Nine of the others refer to the side of the auxiliary square which is added to a previously computed area to "complete the square", whose side—increased or decreased by the *takiltum*—is the solution to the problem.

Most of the instances occur in the context of solving a quadratic equation effectively reduced[25] to the form $x^2 \pm bx = c$, for which (in modern notation) the solution proceeds as $x = \surd\{c + (b/2)^2\} \pm b/2$, in the following steps (Fig. 1):

(1) given $x^2 \pm bx = c$, compute $b/2$
(2) multiply $b/2$ by itself to give $(b/2)^2$
(3) add $(b/2)^2$ to c, [completing $(x + b/2)^2$]
(4) obtain the square root of $\{c + (b/2)^2\}$ [26]
(5) subtract (or add) $b/2$, "your *takiltum*" [to find your answer].

In each case "takiltum" appears only in step (5) where it refers to the quantity $b/2$ multiplied by itself in step (2) to make an auxiliary square, whose area completes the square with side $x \pm b/2$. In these cases, therefore, *takiltum* refers strictly to the side

[24] A debate persists as whether *šutakulum* is a form of *akalum* (to eat) or *kullum* (to hold fast), alternatives which are indistinguishable in cuneiform writing. Thureau-Dangin, Høyrup and Robson have interpreted the verb as a form of *kullum* meaning "to make hold fast together". Neugebauer and Sachs, Muroi, and Friberg have interpreted it as a form of *akalum* meaning "to make eat one another". Høyrup (2002, p. 23), summarizes the arguments pro- and con- and argues—compellingly in our opinion—for the more sensible "make hold" interpretation. Everyone agrees that *takiltum* can only derive from *kullum*.

[25] In the more general case of $ay^2 \pm by = c$, the first step is to multiply by a, making the expression quadratic in $ay = x$, finding y in the last step as $y = x \cdot 1/a$.

[26] Typically expressed as "n^2-e *nîb*-sig", "by n^2, n is the equal side".

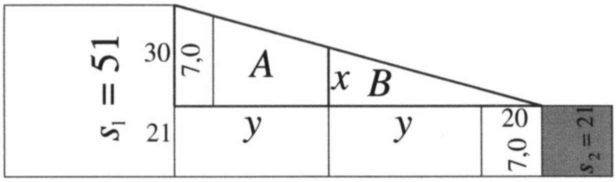

Fig. 2 VAT 8512

of the completing square, which is removed (subtracted or added) in the last step from the side of the completed square, $x \pm b/2$, to give x. Moreover, in no case is the area of the completed square, $c + (b/2)^2$ in steps (3, 4), described as other than a number.

An instructive variation on this procedure occurs in VAT 8512.[27] Here, the problem concerns a triangular field (Fig. 2) divided into two unequal parts by a transverse barrier ($pirkum = x$); one of the parts (A) has 7,0 units more area than the other (B) whose baseline, however, is 20 units longer than A's. The problem set is to find: the length of the barrier (x); the areas A and B, and the lengths of the respective baselines, y and $y + 20$. However, *takiltum* appears only in the course of first finding x, the length of the barrier, whose solution proceeds as follows.

(1) a rectangle is constructed on the 20-unit excess of the base-line of B whose area equals the 7,0 units by which the area of A exceeds that of B, making its other dimension equal to 21 units.
(2) this rectangle is extended by the length of the base-line (i.e., to length $2y + 20$), forming a trapezoid with the subject triangle, which the barrier—extended to the edge of the trapezoid (i.e., by the 21 units of height of the added base)—will bisect into equal areas.
(3) the square of length of the extended barrier is then found by the formula for bisecting trapezoids as the average of the squares of the parallel sides (s_1 and s_2), thus

$$(x + 21)^2 = 1/2\left(s_1^2 + s_2^2\right) = 1/2\left(51^2 + 21^2\right),$$

from which,
(4) $x = \sqrt{\{1/2\,(s_1^2 + s_2^2)\}} - s_2 = \sqrt{(1/2\,(51^2 + 21^2))} - 21 = 18.$

This type of problem is discussed further in Part II. See Fig. 13.

The key part of the text (lines 11–17) covering steps (3) and (4) reads as follows (Høyrup 2002, pp. 234–235):

[27] Høyrup (2002, pp. 234–238), Neugebauer (1935–1937, 341f), Thureau-Dangin (1938, pp. 101–103).

11–12. ...51 *it-ti* 51 *šu-ta-ki-il-ma* 43,21
13. 21 *ša re-eš-ka ú-ka-lu it-ti* 21
14. *šu-ta-ki-il-ma* 7,21 *a-na* 43,21 *ṣi-ib-ma* 50,42
15. 50,42 *a-na ši-na he-pé-ma* 25,21
16. íb-si$_8$ 25,21 *mi-nu-um* 39
17. *i-na* 39 21 ***ta-ki-il-tum*** *ú-s ú-uh-ma* 18

11–12. ...51 with 51 make hold: 43,21
13. 21, which your head holds, with 21
14. make hold: 7,21 to 43,21 append: 50,42
15. break 50,42 in two: 25,21
16. equal-side of 25,21, what? 39
17. from 39 tear out 21, the made hold: 18.

or, more familiarly:

51×51=43,21
21 (which you computed earlier) × **21**=7,21
7,21 + 43,21=50,42
50,42 ÷ 2=25,21
$\sqrt{25,21} = 39$
39–**21**, the *takiltum* = 18 = x

Here, we have a different formula applied, but again the construction of an auxiliary square with side $s_2 = 21$, which is not part of the subject figure, whose area is combined with that of the square on s_1 to give a quantity equal to twice the square on $x + s_2$. Halving this and finding the square-root as before yields $x + s_2$, from which s_2, parenthetically called the *takiltum*, is subtracted to obtain x. Four lines later the same dimension is subtracted from 51[28] to restore $s_1 = 30$. However, here instead of *takiltum* it is described as

21. ...21 *ša a-na ra-*[*ma-ni-šu tu-uš-ta-ki-lu i-na* 51]
...21, which you made hold itself, from 51...

Apart from offering further evidence, should any be needed, that *šutakalum* derives from *kullum*, not *akalum*, the distinct descriptions, suggest that *takiltum* implies an element that is a key step in the process of solution, and not simply a particular dimension.

In each instance where its meaning is clear *takiltum* refers to the side of an auxiliary square, which plays a key role in solving a quadratic equation. Two exceptions occur. In TMS 19 after referring to $b/2$ as described above, *takiltum* appears again referring to b, most probably as simply a scribal error. The other appears in VAT 8389 (Neugebauer 1935–1937, pp. 317–319), in a problem concerning the rents and yields of two fields. Here no square is formed, and the damaged term as restored by Neugebauer and Thureau-Dangin refers to two previously calculated results of division, effectively "yields" which are subsequently added together. Neugebauer took the term to mean something like "factor" or "coefficient", and this divergent meaning persuaded

[28] Also squared in the process and to whose square s_2^2 is added, but which is only indirectly involved in the calculation of x and never referred to as *takiltum*.

him to leave *takiltum* un-translated elsewhere. Høyrup (2002, p. 80, n106) suggests restoring *taki̱rtum* "change" for the two damaged occurrences (*ta-ki-i*[$r^?$*-tam*] and [*ta-ki-ir*$^?$]*-tam*), which arguably better fits the context of the problem. Whatever the explanation, the context is clearly anomalous and inconsistent with all of the other attestations.

In the foregoing examples, *takiltum* refers to the side of the auxiliary square but is immediately preceded by the operation of finding the side of the completed square, introduced by the term íb-sig₈ indicating that linear quantities are now involved rather than areas. It seems plausible, therefore, that *takiltum* connotes both the auxiliary square and its defining side, in the same way as *mithartum* describes both "side" and "square configuration" in BM 15285 (Høyrup 2002, pp. 60–61).[29] In this sense *takiltum* seems to have denoted an abstract square quantity, conceptually separate from the geometrical elements comprising the problem, introduced as a transitory element to enable a solution, which is removed as the solution is achieved. A translation as "number square" to connote a purely numerical quantity distinct from the geometric components of the problem, or, even more simply "square", as tentatively proposed, would seem to capture this sense of the term, while remaining consistent with Høyrup's more literal translation as "(the) made hold". Thus a more precise translation of the heading of column I′ might be

> The number square of the diagonal (from) which
> 1 is subtracted and (that of) the width comes up.

This interpretation encompasses the "square" element of previous interpretations proposed by de Solla Price (1964, p. 7), Robson (2001, p. 193), and even Friberg (2007, p. 440), although without requiring the unlikely addition of "a-šà" as Friberg proposes. It goes beyond these, however, in emphasizing an abstract numerical construct such as we find in connection with auxiliary squares used in the solution of quadratics, which is the sole context for which it is unambiguously attested. Muroi (2003, p. 262) leans in this direction and understands the term to apply to the process of completing the square, but attributes the term effectively to the whole quantity $(ax+b/2)^2$, in conflict with its consistent use in the examples he collects.

As an aside, the geographical distribution of texts containing *takiltum* parallels that of those employing the Diagonal Rule. Thus of the nine texts with consistent *takiltum* references two are from the south,[30] while seven are from the periphery.[31] By comparison, Friberg's recent (2007, p. 450) list of texts employing the Diagonal Rule includes four from the south[32] and twelve from the periphery.[33] Thus the term, which appears to lack a Sumerian equivalent, may have originated in the periphery and diffused to the center, perhaps in the company of the Diagonal Rule.

29 See also Robson's discussion of íb-sig₈ in Robson (2001, p. 173).

30 VAT 8512 and VAT 8520 from Uruk.

31 IM 53201 from Eshnunna; TMS 21, TMS 24, TMS 19, and TMS 12 from Susa; and YBC 6967 from "northern (group 5)".

32 Plimpton 322, attributed to Larsa and MS 3052, MS 3071 and YBC 8633 from Uruk.

33 BM 85194, BM85196, BM 96957, IM 55357, IM 67118, MS 3049, MS 3876, TMS 1, 2 and 19, VAT 7531, and VAT 7289.

1.6 Numerical analysis

Two questions have remained partially or inconclusively answered by previous commentators.

(1) Where do the numbers come from and what principle governed their selection and organization?
(2) How were the numbers computed and what does this imply about the author's approach to the exercise and understanding of the relationships involved?

Where do the numbers come from? Neugebauer and Sachs (1945, pp. 38–41) concluded that the tablet was constructed by first selecting regular numbers r and s (in Friberg's notation) so that

$$d/\ell = 1/2(r/s + s/r) \tag{4a}$$

declines more or less steadily at the rate of roughly 0;1 per row. Pythagorean numbers—i.e., triples of integers satisfying the Diagonal Rule (1)—were thought to have then been formed with these values of r and s by the number-theoretical formula

$$d = r^2 + s^2, b = r^2 - s^2, \ell = 2rs. \tag{4b}$$

This conclusion was emphasized in Neugebauer's *Exact Sciences in Antiquity*[34] and became regarded as the number-theoretical essence of the authors' understanding of the text, eclipsing the earlier conclusion[35] that the numbers in the text arose from the *selection* of pairs of regular numbers, r and s, whose *ratios*, combined as in (4), resulted in desired values of d/ℓ meeting the essentially subjective criterion of a steady rate of decrease. No constraint was suggested for r and s beyond the suggestion that they were taken from combined multiplication tables, which of course included standard tables of reciprocals.

Subsequent commentators, notably Bruins (1949, 1955, 1957), Schmidt (1980), Voils (per Buck, 1980) and most recently Robson (2001), argued that the more likely starting point for the construction of the text was the ratios, r/s, and their reciprocals, rather than their components, a possibility acknowledged but rejected by Neugebauer and Sachs, who believed that "only the simple numbers r and s" could have been the "point of departure". In the alternative view, the starting point for the construction of the text consisted of pairs of reciprocals extracted from lists of extended reciprocals subject to some limitation on the number of places (i.e., sexagesimal "digits") comprising either or both. Robson (2001) champions this interpretation but also acknowledges its shortcomings, namely that among OB mathematical texts no such tables of extended reciprocals are known, and further that if r/s is restricted to 4 places and s/r to 3 (a more severe limitation than exhibited in the text), then missing from the text are entries corresponding to three such reciprocal pairs, namely:[36]

[34] Neugebauer (1951, p. 38).

[35] Neugebauer and Sachs (1945, p. 41).

[36] Robson (2001, p. 197, Table 8).

No	r	s	r/s	s/r	$(d/\ell)^2 = I'$
4a	4,48	2,5	2;18,14,24	0;26,2,30	1;51,27,6,59,24,9
8a	2,15	1,4	2;6,33,45	0;28,6,40	1;40,6,47,17,32,36,15
11a	2,5	1,4	1;57,11,15	0;30,43,12	1;31,9,9,25,42,2,15

all of which result in values of $(d/\ell)^2$ with fewer "digits" than that in row 10 of the text.[37]

These critiques (other than Bruins' which came earlier) ignore the more plausible answer proposed by de Solla Price (1964) and affirmed by Friberg (1981), namely that the contents of the text arise naturally from the set of *all* distinct ratios of regular integers, r/s, where $1 < r/s < \sqrt{2}+1 \cong 2;24,51,10 \cong 2;25$ and s is a regular sexagesimal "digit" (i.e., $1 \leq s < 1,0$).[38] The right side of the first condition is a consequence of considering only rectangles (or corresponding right triangles) where $b < \ell$ and therefore $b/\ell = 1/2(r/s - s/r) < 1$. Since $d/\ell = 1/2(r/s + s/r) < \sqrt{2}$, $(d+b)/\ell = r/s < \sqrt{2}+1 \cong 2;25$, and the left side follows from $0 < b/\ell = 1/2(r/s - s/r)$ which implies $1 < r/s$.

There turn out to be just 38 reduced combinations of r and s which satisfy the above conditions, whose first 15 ratios, arranged in descending order, lead to precisely the examples found in the text. All of which was understood and explained in detail by de Solla Price, who observed, contra Neugebauer and Sachs, that "...it follows that the regularity of decrease of [r/s] or in column [I′] is simply the result of considerable uniformity in the density of such regular numbers among the integers. (de Solla Price 1964, p. 3)

de Solla Price made another significant observation, namely that the tablet, scored as it is over the lower edge of the obverse and on the reverse, appears unfinished and that the space remaining is just sufficient to accommodate entries corresponding to the remaining 23 ratios of r and s.[39] See the photo of the reverse in Appendix B. Careful measurement supports this estimate, and the close fit[40] implies that the dimensioning of the tablet was done with this objective in mind. Thus to the question: where do the numbers come from and what principal governed their selection, the answer seems unambiguous. The tablet represents the unfinished result of an investigation of *all* ratios, r/s, of regular numbers, where $1 < r/s < 2;25$ and $1 \leq s < 1,0$, arranged in descending order of r/s.

Table 2 shows the values of r for each value of s, which lead to unique reduced ratios with $1 < r/s < 2;25$. Table 3 shows the resulting values of r/s arranged in descending order. In a sense, the "point of departure" for the construction of the text was indeed the simple regular numbers r and s, as suggested by Neugebauer and Sachs, but with s limited to sexagesimal digits, and r constrained only by the condition that

[37] Joyce (1995) follows a similar line of argument, but caps p at 2, 5, reducing the number of missing entries to one (11a).

[38] de Solla Price (1964, p. 221) observes that the restriction of s to sexagesimal "digits" is demonstrated by the omission of entries for r/s = 125/64 (2,5/1,4) and 135/64 (2,15/1,4) which "would have occurred between lines 8 and 9 and 11 and 12, respectively".

[39] Friberg (1981, 2007) embraces de Solla Price's conjecture; Robson (2001, 2002a, 2002b) ignores it.

[40] The likely arrangement would seem to be: 4 rows on the lower edge; 17 rows on the reverse; and 2 rows curving over on to the back side of the top edge.

Table 2 Regular numbers r and s, such that r and s have no common factors, and $1 \leq s < 1,0$ and $1 < r/s < 2;25$

s	r				s	r				
1	2				18	25				
2	3				20	27				
3	4	5			24	25				
4	5	9			25	27	32	36	48	54
5	6	8	9	12	27	32	40	50	64	
8	9	15			32	45	75			
9	10	16	20		40	81				
12	25				45	64				
15	16	32			50	81				
16	25	27			54	125				

Table 3 $s, r, r/s$ and s/r arranged in descending order of r/s for all regular numbers, r and s, with no common factors, $1 \leq s < 1,0$ and r/s <2;25. Shown in bold are the elements underlying row n in Plimpton 322

txt	r/s = in descending order				txt	r/s = in descending order			
No	s	r	r/s = α	s/r = 1/α	No	s	r	r/s = α	s/r = 1/α
1	**5**	**12**	**2 24**	**25**	20	5	8	1 36	37 30
2	**27**	**64**	**2 22 13 20**	**25 18 45**	21	16	25	1 33 45	38 24
3	**32**	**75**	**2 20 37 30**	**25 36**	22	2	3	1 30	40
4	**54**	**125**	**2 18 53 20**	**25 55 12**	23	27	40	1 28 53 20	40 30
5	**4**	**9**	**2 15**	**26 40**	24	25	36	1 26 24	41 40
6	**9**	**20**	**2 13 20**	**27 0**	25	45	64	1 25 20	42 11 15
7	**25**	**54**	**2 9 36**	**27 46 40**	26	32	45	1 24 22 30	42 40
8	**15**	**32**	**2 8**	**28 7 30**	27	18	25	1 23 20	43 12
9	**12**	**25**	**2 5**	**28 48**	28	20	27	1 21	44 26 40
10	**40**	**81**	**2 1 30**	**29 37 46 40**	29	3	4	1 20	45
11	**1**	**2**	**2**	**30**	30	25	32	1 16 48	46 52 30
12	**25**	**48**	**1 55 12**	**31 15**	31	4	5	1 15	48
13	**8**	**15**	**1 52 30**	**32 0**	32	5	6	1 12	50
14	**27**	**50**	**1 51 6 40**	**32 24**	33	27	32	1 11 6 40	50 37 30
15	**5**	**9**	**1 48**	**33 20**	34	8	9	1 7 30	53 20
16	9	16	1 46 40	33 45	35	9	10	1 6 40	54
17	16	27	1 41 15	35 33 20	36	25	27	1 4 48	55 33 20
18	3	5	1 40	36	37	15	16	1 4	56 15
19	50	81	1 37 12	37 2 13 20	38	24	25	1 2 30	57 36

r/s be greater than 1 and less than $\sqrt{2}+1$. At issue remains whether r and s continued to play any further role in the construction of the text, as Neugebauer and Sachs *inter alia* affirmed, or whether their role was limited to defining the set of reciprocal ratios which comprised its basic raw material. For the answer, we need to look at the likely manner of its computation.

How were the numerical entries computed? Two distinct approaches have dominated the discussion of this question. Neugebauer and Sachs (1945)—and especially Neugebauer (1951) —followed by Gillings (1953), Aaboe (1964), and de Solla Price (1964), assumed use of the number-theoretical generating function (4b above) whereby

$$\begin{aligned}\delta^2\left(\mathrm{I}'\right) &= (d/\ell)^2 = \left\{\left(r^2+s^2\right)/2rs\right\}^2 \\ b\left(\mathrm{II}'\right) &= r^2 - s^2, \\ d\left(\mathrm{III}'\right) &= r^2 + s^2,\end{aligned} \tag{5a}$$

and thus $\ell = 2rs$.

An alternate procedure was proposed by Bruins, initially in two papers (Bruins 1949, 1955), whose shortcomings[41] included gross mistakes, limited accessibility, and an undisguised animosity towards Neugebauer that impeded serious consideration of his views. He may have realized this, for a subsequent paper in The Mathematical Gazette (Bruins, 1957)[42] presents a succinct analysis of the text in a straightforward, comprehensible style, unencumbered by hyperbole or hints of personal animus. While certain of his proposals are incorrect,[43] his principal contribution was to emphasize the reciprocal formulation of the problem, whereby

$$\begin{aligned}\delta^2\left(\mathrm{I}'\right) &= \{1/2\,(r/s+s/r)\}^2 = 1/4\left\{(r/s)^2+(s/r)^2\right\}+1/2, \\ \beta^2\left(\mathrm{I}'\right) &= \{1/2(r/s-s/r)\}^2 = \delta^2\left(\mathrm{I}'\right)-1, \\ b\left(\mathrm{II}'\right) &= (r/s-s/r)\cdot rs = 2\beta\cdot rs, \text{ and} \\ d\left(\mathrm{III}'\right) &= (r/s+s/r)\cdot rs = 2\delta\cdot rs,\end{aligned} \tag{5b}$$

and to have proposed explanations for each of the errors encountered, which were arguably more "natural" than those advanced by advocates of the number-theoretical formulation.

Apart from utilizing the reciprocal formulation (5b), a central feature of Bruins' approach was a method for computing b and d, which skirted direct multiplication by rs implied by (5b) above. Instead, Bruins proposed that this was accomplished by removing common regular factors from β and δ by repeated multiplications by the reciprocals of regular factors in the last "digit" of the resulting number, and continuing this procedure until no regular factors remained in the last digit of at least one of the pair. Friberg (2007, p. 24) calls this the "trailing part algorithm", and it is essentially a variant of the "Technique" described by Sachs (1947)[44] for finding the reciprocal of any regular number. Done correctly (i.e., by the same factors for b and d) it yields the same result as $1/2(r/s \pm s/r)\cdot 2rs$.

Bruins' approach was reiterated with some variation of detail (and considerable variation in clarity) by Schmidt (1980), Voils,[45] Buck (1980), Friberg (1981, 2007) and Robson (2001, 2002a), none of whom mentioned his unique explanation of the

41 Described in Robson (2001, p. 185).

42 Curiously mentioned by neither Friberg (1981) nor Robson (2001).

43 Notably his belief that column I′ contained only β^2 and lacked a left-most "1", and his claim that the text reflected all ratios of regular numbers between 2;24 and 1;48 inclusive, as noted by Robson (2001, pp. 185–186).

44 See also VAT 6505 (Neugebauer 1935–1937, 270ff.) and Proust (2011).

45 In an unpublished paper destined for *Historia Mathematica* as described by Buck (1980).

error in row 13 (see below). Its application to Plimpton 322 is shown in Table 4. Row 11 is once again exceptional, since there we should find $b = 3, d = 5$, were the procedure strictly applied. However, as noted, the canonical form of this relationship is 45:1,0:1,15, so this discrepancy is without significance. Apart from avoiding multiplication by numbers outside the standard multiplication tables (e.g., 57 in row 2) and by 2-digit numbers (e.g., 57,36 and 1,48 in rows 2 and 10), it is hard to see much benefit from this procedure over direct multiplication by rs. However, as we shall see from the analysis of errors, it does appear to have been the procedure followed.

We can appraise the relative likelihood of the two techniques by examining how each might have led to the four non-trivial errors in the text.

I′(8): 1,41,33,***59***,3,45 instead of 1,41,33,***45,14***,3,45. Robson glosses over this error, characterizing it[46] as a "simple arithmetical error: two places added together" and lumping it with the copy errors as an artifact without computational consequence of transferring calculations from the work-tablet or copy board to finished tablet. However, as Anagnostakis and Goldstein (1974) showed in a note overlooked by both Friberg and Robson,[47] the error results naturally from ignoring a medial zero in the course of calculating this entry. Specifically, they showed that if I′(8) was calculated as

$$\begin{aligned}\delta^2 &= \{1/2\,(r/s + s/r)\}^2 = 1/4\{(r/s)^2 + (s/r)^2 + 2r/s \cdot s/r\} \\ &= 1/4\left\{(r/s)^2 + (s/r)^2\right\} + 1/2\,(r = 32, s = 15)\,, \\ &= 1/4\left\{(2;8)^2 + (0;28,7,30)^2\right\} + 0;30 && \text{(i)} \\ &= 1/4\left\{4;33,4 + 0;13,11,\underline{\mathbf{0}},56,15\right\} + 0;30 && \text{(ii)} \\ &= 1/4(4;46,15,\underline{\mathbf{0}},56,15) + 0;30 && \text{(iii)} \\ &= 1;41,33,45,14,3,45, && \text{(iv)}\end{aligned}$$

but if the medial zero in ($\boldsymbol{s/r^2}$) is ignored in either step (ii) or step (iii) of the calculation,

$$\begin{aligned}\delta^2 &= 1/4 \cdot (4;46,15,56,15) + 0;30 && \text{(iii)} \\ \delta^2 &= 1;41,33,\mathbf{59},3,45 && \text{(iv)}\end{aligned}$$

results instead, as we find.

Bruins (1957, p. 27), who believed column I′ to comprise only β^2, proposes a similar explanation from calculating β^2 alone, to wit:

$$\begin{aligned}\beta^2 &= 0;49,56,15^2 = (0;50 - 0;0,3,45)^2 && \text{(i)} \\ &= 0;41,40 + 0;\mathbf{0},\mathbf{0},\mathbf{0},14,3,45 - 0;0,6,15) && \text{(ii)} \\ &= 0;41,33,45,14,3,45, && \text{(iii)}\end{aligned}$$

but, if the second term is telescoped by a sexagesimal place to 0;0,0,14,3,45, the result is

$$\begin{aligned}\beta^2 &= 0;41,40 + 0;\mathbf{0},\mathbf{0},14,3,45 - 0;0,6,15 && \text{(ii)} \\ &= 0;41,33,\mathbf{59},3,45 && \text{(iii)}\end{aligned}$$

46 Robson (2001, p. 175, Table 2).

47 Friberg (1981), Robson (2001). No reference to it appears in either text or bibliography of either paper.

Table 4 Computation of columns II′(b) and III′(d) by the trailing part algorithm

row	s	r	r/s	s/r	½(r/s - s/r)	x	½(r/s + s/r)	x
1	5	12	2 24	25	59 30 1 **59**	2	1 24 30 2 **49**	2
2	27	1, 4	2 22 13 20	25 18 45	58 27 17 30 1 56 54 35 23 22 55 4 40 35 56 **7**	2 12 12 12	1 23 46 2 30 2 47 32 5 33 30 25 6 42 5 1 20 25 *16 5* *3* ***13*** 3 *12 1*	2 12 12 12 12 *12* <~
3	32	1,15	2 20 37 30	25 36	57 30 45 3 50 3 1 16 **41**	4 20	1 23 6 45 5 32 27 1 50 **49**	4 20
4	54	2,5	2 18 53 20	25 55 12	56 29 4 14 7 16 3 31 **49**	15 15	1 22 24 16 20 36 4 5 9 **1**	15 15
5	4	9	2 15	26 40	54 10 5 25 1 5	6 12 !	1 20 50 8 5 1 **37**	6 12
6	9	20	2 13 20	27	53 10 5 **19**	6	1 20 10 8 **1**	6
7	25	54	2 9 36	27 46 40	50 54 40 2 32 44 38 **11**	3 15	18 41 20 3 56 4 56 **1**	3 15
8	15	32	2 8	28 7 30	49 56 15 3 19 45 13 **19**	4 4	1 18 3 45 5 12 15 20 **49**	4 4 0
9	12	25	2 5	28 48	48 6 8 **1**	10	1 16 54 12 **49**	10
10	40	1,21	2 1 30	29 37 46 40	45 56 6 40 2 17 48 20 6 53 25 1 22 **41**	3 3 12	1 15 33 53 20 3 46 41 40 11 20 5 2 16 **1**	3 3 12
11	1	2	2	30	45	!	1 15	!
12	25	48	1 55 12	40	41 58 30 1 23 57 27 **59**	2 20	1 13 13 30 2 46 27 48 **49**	2 20
13	8	15	1 52 30	32	40 15 2 **41**	4	1 12 15 4 **49**	4
14	27	50	1 51 6 40	32 24	39 21 20 1 58 4 29 **31**	3 15	1 11 45 20 3 35 16 53 **49**	3 15
15	5	9	1 48	33 20	37 20 1 52 *56*	3 *30*	1 10 40 3 32 **53**	3 15

as we find. Bruins claims that this binomial computation of β^2 is simpler than squaring the semi-sum of the reciprocals, at least for 12 of the 14 relevant entries. If so, column I′ could have easily been calculated as $\delta^2 = \beta^2 + 1$, instead of directly. Although seemingly more prone to similar errors than the computational procedure described by Anagnostakis and Goldstein, Bruins' conjecture is supported by the error in II′ (13), discussed below, where b^2 appears in place of b.

While at first glance this appears to compel a computational path based on reciprocals, an opportunity to make a similar error arises from Neugebauer's and Sachs's procedure (5a). Properly computed this entails the following steps

$$\begin{aligned}\delta^2 &= \left(r^2+s^2\right)^2 / \left(4r^2s^2\right) \\ &= \left(32^2+15^2\right)^2 \cdot (0;15) \cdot (0;1,52,30)^2 \cdot (0;4)^2 && \text{(i)} \\ &= (20,49)^2 \cdot (0;15) \cdot (0;0,3,30,56,15) \cdot (0;0,16) && \text{(ii)} \\ &= (7,13,2\mathbf{0},\mathbf{1}) \cdot (0;0,3,30,56,15) \cdot (0;0,4) && \text{(iii)} \\ &= 1;41,33,45,14,3,45. && \text{(iv)}\end{aligned}$$

However if 7,13,20, 1 is mistaken as 7,13,21 (as in I′(2)), the result is also

$$\begin{aligned}\delta^2 &= (7,13,\mathbf{21}) \cdot (0;0,3,30,56,15) \cdot (0;0,4) && \text{(iii)} \\ &= 1;41,33,\underline{\mathbf{59}},3,45 && \text{(iv)}\end{aligned}$$

Thus, there are several paths, all entailing misplaced or telescoped sexagesimal digits, which can have led to the error in I′(8). Whichever the path, however, the error clearly arises in the course of computation, and is not simply an artifact of some copying process. Since it did not affect the correct calculation of b or d in the two subsequent columns, it shows that *columns II′ and III′ were not computed from column I′*.

We are left with three errors in columns II′(b) and III′(d),

III′(2):	$d = 3, 12, 1$ instead of 1,20,25
II′(13):	$b = 7, 12, 1$ instead of 2,41
II, III′(15):	$b = 56$, instead of 28, or $d = 53$ instead of 1,46

and two computational paths to choose from:

$$b = r^2 - s^2, \quad d = r^2 + s^2, \tag{6a}$$

and

$$b = 1/2\,(r/s - s/r) \cdot k_b, \quad d = 1/2\,(r/s + s/r) \cdot k_d. \tag{6b}$$

III′(2):d = 3,12,1 (or 3,13) for 1,20,25. This has been widely recognized to have resulted from two separate errors, one of which Bruins proposed[48] was simply a copy error of 13 as 12,1. By (6a),

$$d = 1,4^2 + 27^2 = 2^{12} + 3^6 = 1,8,16 + 12,9 = 1,20,25,$$

which is so straightforward it is difficult to see a plausible path to either result. Nor have the proponents of this approach been able to propose one.

48 Bruins (1957, p. 27) followed by Friberg (1981, 2007) and Robson (2001, 2002a, 2002b).

By (6b) the procedure is shown in Table 4, row 2. For b, three successive eliminations of the factor 5 (i.e., multiplications by 12) lead to the non-regular result 56, 7, and the factoring of d should have stopped there. However, it appears that the factoring of d continued without regard to b, until it reached the non-regular number 3,13 after two more steps. Thus by this approach, the error is a combination of copy error from 13 to 12,1 plus an uncoordinated application of the trailing part algorithm, both of which seem perfectly plausible.

II′(13) :$b = 7, 12, 1$ (or 7,13), instead of 2,41. Neugebauer and Sachs note (1945, p. 38, n107) that the number is the square of the correct number, and virtually all subsequent commentators have considered it such, without offering a compelling rationale for why it should be.[49] Exceptionally, Bruins (1957, p. 27) proposed that rather than the square of 2,41, the number is a copy error for 7,13 similar to that apparently affecting III′(2), and that instead of $(r/s - s/r) =$ 1,20,<u>30</u>, the author miscopied $(r/s + s/r)$ as 2,24,<u>*20*</u> leading to multiplication by 3 instead of 2 in the trailing part algorithm, and thus to 7,13. This, however, effectively assumes three errors, at least one too many to be plausible. A simpler explanation, favoring the reciprocal algorithm, is that column I′ was computed as $\beta^2 + 1$, and that β^2 was mistakenly used to obtain b by the trailing part algorithm.

II′, III′(15) :$b = 56$ for 28 or $d = 53$ for 1,46. The alternatives are:

$$b = r^2 - s^2 = 1,21 - 25 = 56, d = r^2 + s^2 = 1,21 + 25 = 1,46, \tag{7a}$$

and

$$b = 1/2(r/s - s/r) \cdot 2rs, d = 1/2(r/s + s/r) \cdot 2rs \tag{7b}$$

The presence of the factor 2 in (7b) makes the error more easily explained as arising from multiplying $1/2(r/s + s/r)$ by rs instead of $2rs$, than from (7a), where the factor 2 does not appear. More plausibly, applying the trailing part algorithm, but in computing b either stopping a step short or removing only the factor 2 in the last step, instead of 4 as in computing d, leads to the numbers we find, and suggests that the error was in II′ (15), which should have been 28 instead of 56.[50] Details are in Table 4 for row 15.

In sum, the number-theoretical algorithm (7a) offers a plausible explanation for the error in I′ (8) but fails to explain any of the errors in columns II′ and III′. In contrast the reciprocal procedure (7a), utilizing the trailing part algorithm, offers explanations of all four significant errors, based on one or more of the following plausible errors:

- medial zero or place value error: [I′ (8)]
- copy or reading errors: 12, 1 for 13 [III′ (2)]
- disparate factorization or factorization steps [III′ (2), II′ (15)]
- use of β^2 from computation of I′ for β [II′ (13)]

[49] Friberg (1981) speculates from the relative paucity of errors in column I′ that it was checked by computing $b^2 \cdot (1/\ell^2)$ and that b^2 was simply written down by mistake in place of b. However, column I′ appears no more accurate than II′ or III′, and the error in I′(8) argues against his explanation.

[50] As in Robson (2001, p. 175, Table 2).

On balance, there is no contest; the reciprocal procedure yields plausible explanations for *all but one* of the text's non-trivial errors, while the number-theoretical algorithm fails to offer likely, let alone compelling, explanations for *any* of the errors in columns II′ and III′. Thus we may confidently exclude the number-theoretical formulation (5a) as the basis for the text's computation.

1.7 Missing columns

Any attempt to estimate the missing contents of the tablet is necessarily speculative. However, by paying close attention to the physical characteristics of the surviving tablet, the possibilities can be reduced. Furthermore, since it is possible, even likely, that at least part of the missing fragment is buried in some collection, perhaps mischaracterized as was Plimpton 322, it seems worthwhile to hazard a guess as to the possible contents, in the hope of possibly aiding its future identification.

At the outset we noted that by symmetry from the thickest part of the tablet it appears that between 60 mm and 70 mm of writing surface is missing to the left, and that each column of complete sexagesimal "digits" (i.e., containing both tens and units) takes up ~7 mm or a little less horizontally. Consequently the missing section should have had room for 8½ to 10 columns of full sexagesimal "digits", and perhaps up to 10½ if the missing width is slightly greater than our estimate.[51] This limitation rules out the possibility that the complete tablet contained all the numbers entailed in its production, each of which requires space for the number of digits shown below.[52] (Here digits lacking 10s are treated as requiring ½ digit; those lacking units are treated as whole digits, consistent with preserved text.)

A glance at Table 5 shows that the missing fragment could have contained either β and δ or r/s and s/r, but not both pairs, contra Friberg's most recent (2007) proposal. The latter pair could have been accompanied by either r and s or by $2rs$ for a total of 10 digits taking up 70 mm of width, but combining β and δ with either r and s or $2rs$, would add up to 11 digits requiring 77 mm, exceeding the likely maximal extent of the missing fragment. In the other direction including only the pair of reciprocal ratios as proposed by Robson (2001) would seem to take up too little space. Thus the likely alternatives appear to be either the pair β and δ, or r/s and s/r plus either r and s or $2rs$.

Of these, it seems far more likely that the missing fragment contained β and δ on several grounds. First, the likely method of computation of columns II′ and III′ requires both and appears to have been unaffected by errors in column I′. Furthermore, since at least part of the tablet concerns the dimensions of rectangles of unit length, as reflected in column I′, beginning the tablet with β would have amounted to a statement of the relative flatness of the rectangles in each line. Finally, the inclusion

51 de Solla Price, who had a good eye for such things, estimated "more than a third" of the tablet to be missing. This would seem consistent with slightly more than my estimate, perhaps 70–75 mm, which would allow room for 10 or 101/2 digits but probably not 11.

52 In the existing text, the initial "1" in column I′ evidently takes up less than a full digit's space, while all three columns leave room for units following a last digit which contains only tens.

Table 5 Possible elements and width of missing columns

Element [digits]	Combined	
	digits	width (mm)
s [1], r [1½]	2½	17.5
r/s [3½], s/r [4]	7½	52.5
ℓ: 2rs	2½	17.5
β: ½(r/s – s/r)	4	28
δ: ½(r/s + s/r)	4½	31.5

of β and δ followed in column I′ by the diagonal triple, $\delta^2 = 1 + \beta^2$, would seem to follow a natural order, all of whose elements concern normalized rectangles of unit length.

The alternative—including the reciprocal ratios plus either their component digits or $2rs$ —suffers in comparison, first by omitting β and δ, needed for the computation of columns II′ and III′ but also because the ratios are at least a step removed from the remaining contents of the tablet. Furthermore, while one might expect columns II′ and III′ to have been accompanied by the corresponding lengths equal to $2rs$, the natural place for this to have appeared would have been between columns I′ and II′. Placing $2rs$ before column I′ would have made no logical sense, and making it the first column would have introduced the appearance of random order conflicting with both convention and the text's natural order. Finally, since neither r nor s seem to play any direct role in the remainder of the tablet, there seems no motivation for including them.

On balance, we think it can be assumed with fair confidence that the missing fragment contained β and δ as suggested in Friberg (1981), probably unaccompanied by s, r, or $2rs$ as displayed in Table 6.

1.8 Purpose

Friberg and Robson both regard Plimpton 322 as a pedagogical "teacher's aid" intended to serve as a source of parameters and checkable intermediate results for sets of "solvable" problems—"solvable" meaning solutions comprised of numbers with finite sexagesimal expressions, called "semi-regular" by Friberg. Less clear is exactly what sorts of problems were the object of the exercise. Robson (2001, p. 201) summarizes the dilemma succinctly as follows:

> ...But we are still left with a decision to make: was this teachers' list a compendium of suitable sets of right triangles (Friberg) or reciprocal pairs (Voils/Buck)? If we go with Friberg, we have to explain why Column I is in the table and why there is no (extant) column for l. If on the other hand we follow Voils/Buck, the problem is to explain why Columns II and III contain values which have been scaled up by the factor l—and why the word "diagonal" crops up in the heading of Columns I and III.

Table 6 Plimpton 322. Conjectured reconstruction of the contemplated completed tablet

β	δ	$\delta^2 = 1+\beta^2$	b	d	N°	
[-I']	[0']	I'	II'	III'	IV'	
sag	*ṣi-li-ip-tum*	*ta-k]i-il-ti ṣi-li-ip - tim* *šá 1 in]-na-as-šà-hu-ú-ma sag i-il-lu -ú*	íb.si₈ sag	íb.si₈ *ṣi-li-ip-tim*	mu.bi.im	
59 30	1 24 30	1 59 0 15	1 59	2 49	ki 1	
58 27 17 30	1 23 46 2 30	1 56 56 58 14 *50* *6* 15	56 7	*1* *20* *25*	ki 2	
57 30 45	1 23 6 45	1 55 7 41 15 33 45	1 16 41	1 50 49	ki 3	
56 29 4	1 22 24 16	1 53 10 29 32 52 16	3 31 49	5 9 1	ki 4	
54 10	1 20 50	1 48 54 1 40	1 5	1 37	ki 5	
53 10	1 20 10	1 47 6 41 40	5 19	8 1	ki 6	obv
50 54 40	1 18 41 20	1 43 11 56 28 26 40	38 11	59 1	ki 7	
49 56 15	1 18 3 45	1 41 33 *45* *14* 3 45	13 19	20 49	ki 8	
48 6	1 16 54	1 38 33 36 36	*8* 1	12 49	ki 9	
45 56 6 40	1 15 33 53 20	1 35 10 2 28 27 24 26 40	1 22	2 16 1	ki 10	
45	1 15	1 33 45	45	1 15	ki 11	
41 58 30	1 13 13 30	1 29 21 54 2 15	27 59	48 49	ki 12	
40 15	1 12 15	1 27 0 3 45	*2* *41*	4 49	ki 13	
39 21 20	1 11 45 20	1 25 48 51 35 6 40	29 31	53 49	ki 14	
37 20	1 10 40	1 23 13 46 40	*28*	53	ki 15	
36 27 30	1 10 12 30	1 22 9 12 36 15	2 55	5 37	ki 16	
32 50 50	1 8 24 10	1 17 58 56 24 1 40	7 53	16 25	ki 17	lo.e
32	1 8	1 17 4	8	17	ki 18	
30 4 53 20	1 7 7 6 40	1 15 4 53 43 54 4 26 40	1 7 41	2 31 1	ki 19	
29 15	1 6 45	1 14 15 33 45	39	1 29	ki 20	
27 40 30	1 6 4 30	1 12 45 54 20 15	6 9	14 41	ki 21	
25	1 5	1 10 25	5	13	ki 22	
24 11 40	1 4 41 40	1 9 45 22 16 6 40	14 31	38 49	ki 23	
22 22	1 4 2	1 8 20 16 4	11 11	32 1	ki 24	
21 34 22 30	1 3 45 37 30	1 7 45 23 26 38 26 15	34 31	1 42 1	ki 25	
20 51 15	1 3 31 15	1 7 14 53 46 33 45	16 41	50 49	ki 26	rev
20 4	1 3 16	1 6 42 40 16	5 1	15 49	ki 27	
18 16 40	1 2 43 20	1 5 34 4 37 46 40	5 29	18 49	ki 28	
17 30	1 2 30	1 5 6 15	7	25	ki 29	
14 57 45	1 1 50 15	1 3 43 52 35 3 45	6 39	27 29	ki 30	
13 30	1 1 30	1 3 2 15	9	41	ki 31	
11	1 1	1 2 1	11	1 1	ki 32	
10 14 35	1 0 52 5	1 1 44 55 12 40 25	4 55	29 13	ki 33	
7 5	1 0 25	1 0 50 10 25	17	2 25	ki 34	
6 20	1 0 20	1 0 40 6 40	19	3 1	ki 35	
4 37 20	1 0 10 40	1 0 21 21 53 46 40	52	11 17	ki 36	
3 52 30	1 0 7 30	1 0 15 0 56 15	31	8 1	ki 37	u.e.
2 27	1 0 3	1 0 6 0 9	49	20 1	ki 38	

Robson notes that neither Friberg (1981) nor Buck (1980) satisfactorily resolves the issue and takes a contextual run at it before conceding that the manner in which the text was computed (i.e., by reciprocal sums and differences) does not necessarily reflect the problems it was designed to set, which, she concludes, edging towards Friberg (1981), must be "some sort of right triangle problems" (Robson 2001, p. 202).

Meanwhile, Friberg, who concluded in (1981, p. 300)

> that it may have been the intention of the author of the tablet to find the front and diagonal of *all* rational right triangles with flank (i.e., length) = 1 under the sole condition for practical reasons that the parameter $t = s/r$ …must be a regular sexagesimal number such that, for instance, $s < 1, 0$.

simply noting (1981, p. 303) as a secondary matter,

> the interesting possibility of a close connections between the "Babylonian triangle parameter equations" and the treatment of some types of quadratic equations in Babylonian mathematics

tacks back in (2007, p. 433), and describes the text in the title of Appendix 8 as "Plimpton 322: a Table of Parameters for igi-igi.bi Problems." Here he is clearly influenced by the discovery of two previously unknown texts, MS 3971 and MS 3052, which contain six similar problems: given *igi* = r/s and thus implicitly *igibi* = s/r, find the short side β as $\surd\{1/2(r/s + s/r)^2 - 1\}$. As Friberg notes, solutions to the 5 problems in MS 3971 comprise lines 18, 22, 29, 32, and 37 in the completed extension of Plimpton 322 shown in Table 6, while MS 3052 concerns the simplest of ratios: r/s = *igi* = 2, *igibi* = 0;30, β = 0;45, corresponding to line # 11 in the actual text.

Significantly, all six of these problems assume a normalized rectangle with unit length, $\lambda = 1$, and all employ a round-about solution path starting with $r/s = a$ and computing $1/a, \delta = 1/2(a + 1/a), \delta^2, \beta^2 = \delta^2 - 1$, and finally $\beta = \surd\{\delta^2 - 1\}$, largely reversing the construction of column I′ in Plimpton 322 and eschewing the shortcut of $\beta = 1/2(a - 1/a)$ implicit in the underlying algorithm. Since the procedure works for any regular $a < 2; 25$, it is hard to see what Plimpton 322 would contribute, apart from the (strangely indirect) intermediate calculations and answers. Finally, as Robson notes, what is to be made of columns II′ and III′ in the absence of their widely varying third dimension, ℓ=2*rs*?

Thus, despite their sometimes spirited differences Friberg and Robson arrive by different paths at a largely similar view that Plimpton 322 was intended to serve as a teachers' aide linking reciprocal relationships to the Diagonal Rule, without ever quite pinpointing what problems it would have uniquely helped construct, or what mathematical purpose it might have served. Otherwise they differ in emphasis and perspective with Robson minimizing the tablet's mathematical character in favor of its more prosaic pedagogical attributes, and Friberg—embracing de Solla Price's conjecture that it was intended to encompass a complete set of entries—emphasizing its mathematical attributes.

1.9 Summary

- Plimpton 322 concerns sets of diagonal triples, β, 1, δ, for normalized rectangles of unit length computed from ratios r/s of regular numbers as

$$\beta = 1/2(r/s - s/r), \delta = 1/2(r/s + s/r).$$

- From the scoring on the lower edge and reverse and measurements of the unwritten surface area it appears the its author intended to include entries for all 38 unique ratios, r/s satisfying the conditions

$$1 < r/s < 1 + \surd 2, \text{ and } 1 \leq s < 1,0.$$

- It is highly likely that the missing fragment to the left contained only β and δ.
- Column I′ reflects the fundamental diagonal relationship, $\delta^2 = 1 + \beta^2$, in a single number for each entry, probably computed in the same fashion. The term *takilti* in its heading means "square" but with emphasis on its abstract numerical rather than concrete geometrical nature.

- Columns II′ and III′, reflecting the integer values for width and diagonal corresponding to each β, were computed by successive applications of the "trailing part algorithm" to β and δ.
- The four non-trivial errors can be explained as due to
  - medial zero or place value error: [I′ (8)]
  - copy or reading error: 12, 1 for 13 [III′ (2)], or
  - disparate factorization steps [III′ (2), II′ (15)]
  - use of β^2 from computation of I′ for β [II′ (13)].
- The writing, layout, and construction of the tablet reflect a high degree of skill on the part of its author.

2 Part II. A Different Perspective

Many of the difficulties encountered by Robson and to some extent Friberg in describing the purpose of Plimpton 322, arise from their shared assumption that the tablet was created to serve a restricted pedagogical purpose. The following discussion abandons this assumption and considers the tablet's place as a *mathematical* artifact in its *historical* and *mathematical* context. We draw heavily upon the new material from the Schøyen Collection, published with illuminating commentary by Friberg (2007), as well as upon the geometrical perspectives advanced by Høyrup (2002) and earlier papers.

We find the tablet intimately connected with the discovery of the algorithm for finding triples of finite sexagesimals satisfying the Diagonal Rule, therefore closer to a problem text than has hitherto been proposed, and substantively "deeper" than a mere pedagogical aid. In the following commentary, we have avoided, as much as possible, using the term "triples", since there is no reference to triples in the texts under consideration, including P 322, but to pairs (width and diagonal). These two numbers define completely the rectangles which are the object of the problem, that is, rectangles with length, width and diagonal represented by finite sexagesimal numbers. We will call such rectangles "finite sexagesimal rectangles". (In the spirit of Old Babylonian mathematics, we refer to the associated rectangles, instead of the right triangles.)

Our understanding of the historical and mathematical context is derived from a quite rich collection of cuneiform mathematical tablets dating from approximately the same period, that is, Old Babylonian, and coming from the same geographical region (according to a provisional assessment of both provenance and date). These texts reflect diverse scribal traditions. The Old Babylonian period covers several centuries: an early formative period (Isin-Larsa) ending with the conquest of Larsa by Hammurabi; an overlapping period encompassing the reigns of Hammurabi (1848–1806) and Samsu-iluna (1805–1768) up to the abandonment of the southernmost cities (Ur, Uruk, Larsa) by 1795 and of Nippur and Isin by around 1776; and a late period during which scribes fled to the north after the destruction of the southern cities and which extended past the end of the first dynasty of Babylon (1651) into the period of Kassite rule. With respect to the provenance of cuneiform mathematical texts, there are several regions: the south—Larsa, Uruk, Ur, comprising the post-Ur III "core" in Høyrup's terminology, the center—Nippur, Isin; the northern region of the

Mesopotamian plain —Kiš and Sippar; the middle Euphrates, Mari and the kingdom of Ešnunna; and finally Susa to the east. The last three are often called "peripheral".

Common linguistic features, first analyzed by Goetze, Neugebauer and Sachs (1945, pp. 146–151), and more recently and extensively by Høyrup (2002, pp. 315–361) have led to the recognition of several groups of texts, some of which can be correlated with varying degrees of certainty with distinct locations. Thus using Goetze's groupings as expanded by Høyrup, texts comprising groups 7 and 8 are known to come from Ešnunna and Susa, respectively. They are also among the earliest and the latest of Old Babylonian mathematical texts, the former being dated to around 1840 to 1830 and the latter thought to be late from paleographic criteria (Høyrup 2002, p. 318). Texts from group 6 are probably from Sippar (Høyrup 2002, p. 332). The few texts comprising group 5 are not specifically located but reflect both northern and southern characteristics and are believed to come from the periphery after the abandonment of at least the southernmost cities in 1795. Texts from groups 1 to 4 are all from the south, with group 1 attributed "in all probability" (Goetze, Neugebauer and Sachs (1945), p. 148) to Larsa, and groups 3 and 4 attributed to Uruk.

Temporally, texts from Ešnunna are from before ca., 1830, from Ur, Larsa and Uruk before 1795, those from Sippar and group 5 probably after 1795, and those from Susa probably later but before the end of the first dynasty of Babylon in 1651. A separate group of "series texts" is also thought to be late from paleographic and substantive considerations.

2.1 Provenance

Since evidence of mathematical activity at the southernmost cities disappears after 1795, the historical context of the tablet is intimately related to its provenance. Høyrup (2002, p. 386) has questioned the likelihood of a Larsa provenance, noting,

> Goetze ascribes Plimpton 322 very tentatively to group 1 saying that 'it may or may not belong here' [Neugebauer and Sachs (1945), pp. 147, 353]. His only argument is the spelling [*in-*]*na-as-sà-hu-ú-ma*, which may indeed just as well (if not better) point to group 6.

Similarities of vocabulary—***elum*** (comes up) for results, ***nahasu*** for subtract, ***šutak-ulum*** for rectangularization and íb-si$_8$ as a nominal rather than verbal form point perhaps even more strongly to group 5, which Høyrup (Høyrup 2002, p. 332) finds closely related to the body of texts from Ešnunna.

Høyrup's other reason for doubting a Larsa provenance was the greater frequency of problems invoking the Diagonal Rule in texts from the periphery rather than from the south, suggesting that P 322 fits better with the former than the latter, as well as a later date than would be plausible for texts from the south. Friberg's discovery of texts in the Schøyen Collection apparently from the south which involve the Diagonal Rule, weakens but hardly refutes this argument. A stronger case for a late date, moreover, can be made from the quasi absence of evidence of finite sexagesimal rectangles other than the classical triple 45:1:1,15, presumably known from deep antiquity.

Thus, before considering the text's mathematical context, we shall examine in greater detail the evidence bearing on whether it originated in the south, implying a pre-1795 date, and specifically from Larsa. The chief evidence for a southern origin, if not specifically Larsa, is found in MS 3971, published in Friberg (2007, pp. 245–254 and especially #3, pp. 252–253). It describes five problems of determining the diagonals and widths of rectangles of implicitly unit length from the sums and differences of reciprocals. As noted by Friberg (2007, p. 444) the five problems correspond to line numbers 37, 18, 22, 29, and 32, from the unfinished section of P 322. Apart from BM 85200+VAT 6599 #16 which also reflects the other classictriple, 5:12:13, corresponding to (unfinished) line 22, and seemingly known from Sargonic times, the examples from MS 3971 are the only Old Babylonian evidence of finite sexagesimal rectangles (other than the classic two) attested or implied in P 322. Significantly, they also imply that someone completed the unfinished exercise contemplated by P 322's author. Friberg places MS 3971 firmly in Goetze's group 3 from Uruk, based on an analysis of similarities among Sumerian elements. While this may merit confirmation from further analysis, it strongly suggests a temporal proximity between P 322 and MS 3971, which would rule out a late date for P 322, and makes a southern provenance more likely than not.

A less compelling, but still significant, bit of evidence favoring an earlier over a later date is the correct, extremely precise, and very fine handwriting of the tablet's author (seen clearly in the photograph) which reflects none of the cursive writing found in later "series" or Susa texts. While not indicative of even regional origins, it does suggest an association of the author with the scribal traditions of a major center.

The foregoing evidence suggests on balance that P 322 came from the south more probably than not, and—based on orthography and terminology—from somewhere other than Uruk. As noted in Part I the strongest evidence for a Larsa provenance is Banks's specific statement that the tablet was found at Senkereh. In an undated prospectus, probably written around 1923, describing four tablets and their prices, Banks described the tablet (No. 3) to Plimpton as follows

> No. 3 $10.00 Found at Senkereh. A very large burned tablet with one edge broken away, but with the inscription practically complete. It is a mathematical tablet, and the column at the right contains the numerals 1 to 15. The numbers in the columns at the left are very large, and it seems to me that they are the cubes or squares of the numbers in the column on the right. It was used as a mathematical text book (sic). The date is about 2250 B.C." (i.e., during the reign of Hammurabi as thought then). (Plimpton file, Rare Book and Manuscript Library, Columbia University)

Tablets Nos. 1 and 2 were also described as from Senkereh, while No. 4, an Ur III accounting of sheep and goats, was found at Drehem. Other tablets offered to Plimpton around this time were all individually provenanced, with most from Senkereh, some from Ur, one from Drehem and one (P348) from Jokha (ancient Umma). Thus all of the tablets offered by Banks to Plimpton came from locations in the south with the northernmost coming from Drehem. A similar pattern of origins characterized the much larger volume of tablets sold by Banks to collectors and institutions across the continent both before and shortly after World War I, the vast majority of which came from the south.

The pattern of Banks's tablet sales correlates with his travels and connections within Iraq. After obtaining his Ph.D. at Breslau under Delitzsch in 1897, Banks spent several years in Bagdad endeavoring to obtain permision to excavate first Ur, which was denied, and then to Bismaya (ancient Adab), which was eventually granted in 1903 to the University of Chicago. The university then appointed him "Field Director of the Babylonian section of the Oriental Exploration Fund" and an "instructor in Turkish and Semitic languages". While serving as field director, he made many acquaintances among the "dealers and manufacturers" of antiquities, as he describes them in his 1912 account of the expedition to Bismaya. On his travels to and from Bismaya he passed through Hillah and Nippur, and late in 1904 he made a trip south via the Euphrates to visit the main southern sites of Uruk, Larsa and Ur. Of Larsa he wrote (Bismaya 418) "Of all the ruins in Babylonia Senkera (sic) is the most promising, and this the illicit Arab diggers have long known." Suspected by Turkish authorities of playing loose with the antiquities laws, he resigned his position with the expedition at the end of 1904 and left Iraq. In 1912 he returned to Bagdad and purchased the remaining 11,000 or so of a huge trove of illicitly excavated Ur III tablets mostly from Drehem and Jokha, which he sold widely to US institutions and collectors in subsequent years.

In a 1937 letter to J.G. Phelps Stokes, to whom he had recently sold a Nebuchadnezzar cylinder, Banks described his source of tablets as follows.

Dear Mr. Stokes,

> I thank you for your fine letter and for the enclosed check in full payment for the Nebuchadnezzar cylinder. I can give you very little further information regarding the cylinder. I obtained it in Bagdad some years ago from a native friend who has supplied me with such antiquities as he can get from the natives who find them. These native have been trained, some of them by myself, to search among the ruins and to take to Bagdad whatever they find. In this way I built up my collections during the past forty years or more. In the earlier days under the Turks [i.e. pre WW I], all of the Babylonian objects which reached this country came in this manner, and it was about the only way to save them from the Turkish Museum, where they were dumped into the basement and left to perish. However, the conditions now with an archaeological school and museum in Bagdad are much better...(Plimpton file, Rare Book and Manuscript Library, Columbia University)

P 322 probably followed the same route, which would imply that the tablet passed through the hands of at least two other persons besides Banks—his agent in Baghdad and the local excavator. The latter would almost certainly be associated with a single location known to the agent, so the most likely source of provenance error would lie with the agent's record keeping. It also seems likely that it would have been the Baghdad agent who glued the missing fragment to the tablet, which subsequently re-broke.

Banks appears to have been conscientious in passing on the information available to him regarding the provenance of all the tablets that he sold and open about acknowledging uncertainties of provenance where he was aware of them. His reported contacts, experience and travels were restricted to Baghdad and areas to its south, and there is no evidence that he ever visited Ešnunna or the middle Euphrates north of Hillah. Tablets sold by him, both before and after WWI, came largely from the deep south, the principal exception being NB tablets from Babylon, acquired while Babylon's excavation by Koldewey was ongoing. Banks was certainly familiar with both the process and people engaged in the illegal antiquities trade and specifically with both the promise and vulnerability of Larsa for pillage. Banks's notoriety notwithstanding, there seems little reason to doubt his assertion that P 322 was found at Larsa, apart from the possibility that he or his agent in Baghdad mis-recorded its provenance.

Thus while a "middle" peripheral site such as Sippar cannot be completely ruled out, a substantial preponderance of evidence supports the conclusion that P 322 was indeed found at Larsa. This suggests that the tablet was probably slightly later than tablets from Ešnunna in the north and either slightly earlier or contemporaneous with tablets from Uruk such as MS 3971.

2.2 Mathematical context

The following discussion is based on examples from other tablets which may be reasonably considered as from the same general area and the same period. Theses related texts are mainly the ones published and commented by Friberg in *A Remarkable Collection of Babylonian Mathematical Texts, Manuscripts in the Schøyen Collection Cuneiform Texts I* (2007). We argue that P 322 presents a coherent *mathematical* project, which is to generate a series of finite sexagesimal rectangles from a basic algorithm, or, in other terms, that P 322 is a problem text including a complete solution. To a large extent, this approach revives Neugebauer's original interpretation as modified by Bruins' introduction of reciprocals and other subsequent improvements.

In Høyrup (2002 and earlier) ground-breaking papers, Høyrup showed that much of Old Babylonian mathematics rested upon a geometrical foundation variously characterized as "naive" or "cut and paste" geometry (Høyrup) or "metrical algebra" (Friberg), involving a series of procedures based on diagrams that may have been sketched on a sandy floor with a pointer. Babylonian mathematics appears to have developed around the practical needs and knowledge of surveyors at an early date, and subsequently melded with the arithmetic of accountancy by the Ur III period, if not earlier. Whatever its precise chronology, it preceded P 322, and some of the basic constructions, (Figs. 3, 4, 5 below) could have formed the background for its context.

The notation in this part is chosen to be consistent with part I:

- For a rectangle of length 1 (as in column I′ of P 322) the width will be denoted by β and the diagonal by δ
- For an arbitrary rectangle the width will be denoted by b, the length by a and the diagonal by d.

The simplest "sand-stick" construction is shown in Fig. 3. It shows a relation between two concentric squares:

Figure 4 shows the Old Babylonian representation of the Diagonal Rule:

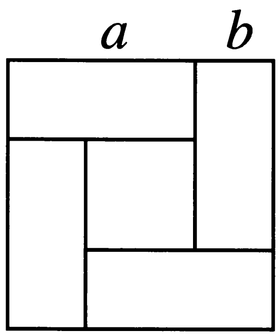

Fig. 3 $(a + b)^2 = (a - b)^2 + 4ab$ (8)

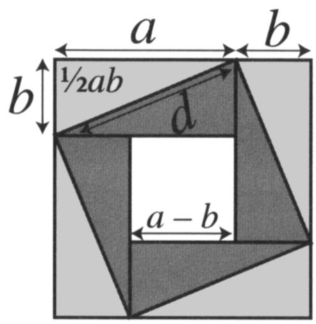

Fig. 4 $d^2 = (a + b)^2 - 2ab = a^2 + b^2$ (9)

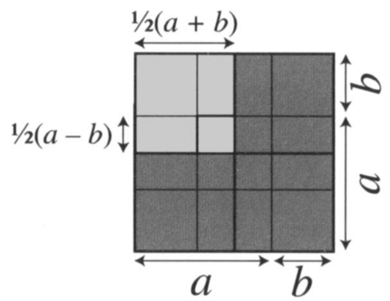

Fig. 5 Standard solution of quadratic problems

Figure 5 illustrates another important relation from Old Babylonian mathematics. It gives a geometric representation of the method of completing the square starting with a rectangle of sides a and b.

This figure illustrates the standard solution of quadratic problems such as: given $ab = \mathrm{A}$ and $a - b = c$, find a and b. The solution proceeds by completing the square on

$$\begin{aligned} a - (1/2)c = b + (1/2)c = 1/2(a+b), \\ (1/2(a+b))^2 = ab + (1/2(a-b))^2. \end{aligned} \tag{10}$$

Hence,

$$1/2(a+b) = \surd\left(\mathrm{A} + ((1/2)c)^2\right) \tag{11}$$

$$a = \surd\left\{\mathrm{A} + ((1/2)c)^2\right\} + (1/2)c. \tag{11a}$$

$$b = \surd\left\{A + ((1/2)c)^2\right\} - (1/2)c. \tag{11b}$$

The mathematical contribution of P 322 is making a connection between the Diagonal Rule, as shown in Fig. 4 and expressed in Eq. 9 and the procedure of completing the square, as shown in Fig. 5 and expressed in Eq. 10.

Figures 4 and 5 are used by Høyrup (2002, pp. 257–261) to interpret the solution path and subsequent proof of the problem presented in Db_2-146, a tablet from Ešnunna, dating to around 1830, and thus probably (although not certainly) slightly earlier than P 322.

The translation and transliteration of $\mathrm{Db}_2 - 146$ after Baqir (1962, pl. 2–3) and Høyrup (2002), pp. 258–259, with some corrections, is as follows.[53]

Obverse

1. *šum-ma ṣí-li-ip-ta-a-am i-ša-lu-ka*
2. *um-ma šu-ú-ma* 1,15 *ṣí-li-ip-tum* 45 a-šà
3. *ši-di ù* sag-ki *ki-ma-a-ṣí at-ta* [*i*]*-na e-pé-ši-ka*
4. 1,15 *ṣí-li-ip-ta-ka mé-he-er-šu i-di-i-ma*
5. *šu-ta-ki-il-šu-nu-ti-i-ma* 1,33,45 *i-li*
6. 1,33,45 šu KU.Ú.ZU/BA?
7. 45 a-šà-*ka a-na ši-na e-bil-ma* 1,30 *i-li*
8. *i-na* 1,33,45 *hu-ru-úṣ-ma* 1,33,45^{sic}*ša-pí-il-tum*
9. ib-sí 3,45 *le-qe-e-ma* 15 *i-li mu-ta-su*
10. 7,30 *i-li a-na* 7,30 *i-ši-i-ma* 56,16^{sic}*i-li.*
11. 56,15 šu-*ka* 45 a-šà-*ka e-li* šu-*ka*
12. 45,56,15 *i-li* ib-si 45,56,15 *le-qe-ma*
13. 52,30 *i-li* 52,30 *mé-he-er-šu i-di-i-ma*
14. 7,30 *ša tu-uš-ta-ki-lu a-na iš-te-en*
15. *ṣí-ib-ma i-na iš-te-en*
16. *hu-ru-úṣ*1 uš-*ka* 45 sag-ki *šum-ma* 1 *uš*

[53] According to the handcopy by Baqir (1962, pl. 2), Baqir and/or Høyrup's transliteration should be corrected as follows. In li. 3: -[*i*]-, not -*i*–; li. 4: *mé*-, not *me*-; li. 6: Ú, not U; li. 7: -*bil*-, not -*bi-il*-; li 10; 16^{sic}, not 15; li. 17: ⌈45 <sag>-ki⌉, not 45 sag-ki; li. 19: -[*eš*]-⌈*ka*⌉, not -*eš-ka*-; li. 22: -*qe*-, *not* -[*qe*]-; li. 23:-<*ta*>-, not -[*ta*]-.

17. ⌈45 <sag>-ki⌉ a-šà *ù ṣí-li-ip-ti ki-ma-ṣí*
18. [*at-ta i-na e*] -⌈*pé*⌉-*ši-ka ši-da šu-ta-ki-il-ma*
19. [1 *i-li* ...] *re*-[*eš*]-⌈*ka*⌉*li-ki-il*

Reverse

20. ⌈x x x⌉-*ma* 45 sag-ki *šu-ta-ki-il-ma*
21. 33,45 *i-li a-na ši-di-ka ṣí-ib-ma*
22. 1,33,45 *i-li* ib-si 1,33,45 *le-qe-ma*
23. 1,15 *i-li* 1,15 *ṣí-li-ip-*<*ta*>*-ka* uš-*ka*
24. *a-na* sag-ki *i-ši* 45 a-šà-*ka*
25. *ki-a-am ne-pé-šum*

Obverse

1. If, about a (rectangle with) diagonal, (somebody) asks you
2. thus, 1,15 the diagonal, 45 the surface;
3. length and width corresponding to what? You, by your proceeding,
4. 1,15, your diagonal, its counterpart lay down:
5. make them hold: 1,33,45 comes up,
6. 1,33,45 may (?) your (?) hand hold (?)
7. 45 your surface to two bring: 1,30 comes up.
8. From 1,33,45 cut off: 3,45$^{!}$ the remainder.
9. The equalside of 3,45 take: 15 comes up. Its half-part,
10. 7,30 comes up, to 7,30 raise: 56,15 comes up
11. 56,15 your hand. 45 your surface over your hand,
12. 45,56,15 comes up. The equalside of 45,56,15 take:
13. 52,30 comes up, 52,30 its counterpart lay down,
14. 7,30 which you have made hold to one
15. append: from one
16. cut off. 1 your length, 45 the width. If 1 the length,
17. 45 the width, the surface and the diagonal corresponding to what?
18. (You by your) making, the length make hold:
19. (1 comes up ...) may your head hold.

Reverse

20. [...]: 45, the width, make hold:
21. 33,45 comes up. To your length append:
22. 1,33,45 comes up. The equalside of 1,33,45 take:
23. 1,15 comes up. 1,15 your diagonal. Your length
24. to the width raise, 45 your surface.
25. Thus the procedure.

The problem presented is: given the diagonal, d, and area, ab, of a rectangle, what are its dimensions? The solution first forms the square on the diagonal as in Figure 4, then subtracts twice the area ($2ab$) to obtain the square on $a - b$, and thence $a - b$, equivalent to c in the example above. It then proceeds to find a and b by quadratic completion, Eq. 10, and finishes by proving that the results yield the diagonal by

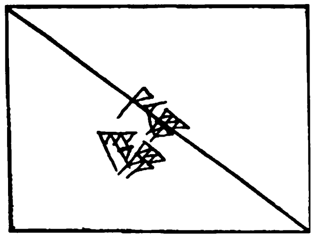

Fig. 6 Diagram drawn at the end of the text of Db2-146. Handcopy by Baqir (1962, pl. 2)

the Diagonal Rule (lines 16–25), showing unequivocally, as Høyrup notes, that the Diagonal Rule was known in Ešnunna by this date.

Another example of quadratic completion, YBC 6967, is cited by Robson (2001, pp. 183–185) as illustrating the connection between P 322 and *igi*/*igibi* problems. Here **the difference** in the above example is described as the excess of the *igibi* over the *igi*, i.e., *igibi*–*igi* = 7, and the solution method is once again based on Eq. 10. The right side of (10) is calculated in line 9 below. The solution assumes that the product, *igibi* × *igi* = 60, leading to *igibi* = 12, *igi* = 5. The problem as defined, replaces conventional length and width with the slightly more abstract *igibi* and*igi*, thereby establishing the area of the subject rectangle indirectly, but otherwise has no bearing on the Diagonal Rule since 60 is not a square number (Fig. 7).

YBC 6967 (Transliteration and translation after Høyrup 2002, pp. 55–57)
Obverse

1. [igi]-⌈bi⌉*e-li* igi 7 *i-ter*
2. [igi] *ù* igi-bi *mi-nu-um*
3. ⌈*at-ta*⌉7 *ša* igi-bi
4. ugu igi *i-te-ru*
5. *a-na ši-na he-pé-ma* 3,30
6. 3,30 *it-ti* 3,30
7. *šu-ta-ki-il-ma* 12,15
8. *a-na* 12,15 *ša i-li-kum*
9. [1 a-šà]$^{⌈la⌉-am}$*ṣí-ib-ma* 1,12,15
10. [íb-sig$_8$ 1],12,15 *mi-nu-um* 8,30

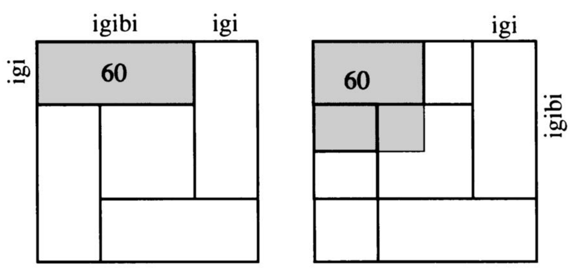

Fig. 7 Solution of YBC 6967

11. [8,30 *ù*] 8,30 *me-he-er-šu i-di-ma*

Reverse

1. 3,30 *ta-ki-il-tam*
2. *i-na iš-te-en ú-su-uh*
3. *a-na iš-te-en ṣí-ib*
4. *iš-te-en 12 ša-nu-um* 5
5. 12 igi-bi 5 *i-gu-um*

Obverse

1. The *igibi* over the *igi*, 7 it goes beyond,
2. *igi* and *igibi* what?
3. You, 7 which the *igibi*
4. over the *igi* goes beyond
5. to two break: 3,30;
6. 3,30 together with 3,30
7. make hold:12,15.
8. to 12,15 which comes up for you
9. [1 the surface] append: 1,12,15
10. [The equalside of 1],12,15 what? 8,30.
11. [8,30 and] 8,30, its counterpart, lay down.

Reverse

1. 3,30, the made hold,
2. from one tear out,
3. to one append.
4. The first is 12, the second is 5
5. 12 is the *igibi*, 5 is the *igi*.

We now examine in more detail the evidence provided by tablets from Schøyen collection which are conceptually closely linked to P 322. Consider first problems #2 and #3 of MS 3971 which provide a key to the understanding of P 322.

MS 3971 #2 (Friberg 2007, p. 251, with notations following the conventions of this article as well as some slightly modifications[54])

1. 1,15 *ṣi-li-ip-tum* 45 a-šà
2. uš *ù* sag en.nam
3. 1,15 du_7-du_7 1,33,45 in-sì

[54] li. 1: Friberg's transliteration reads: 15 *ṣi-i[l-i]p-tum*. According to the copy by Al-Rawi (Friberg 2007, 249) and the "conform transliteration" by Friberg (*ibid*, 248), the reading of all the signs are clear and therefore there is no need of bracket in this word; moreover, the second sign is *li*, not *il* (note that this orthography is the same as in P 322).

li. 4: Friberg's transliteration reads e-ta[b], but /tab/ is clear on the tablet (*ibid*).

li. 5: Friberg's transliteration reads d[ah], but /dah/ is clear on the tablet (*ibid*).

li. 12: in-si_3 is written under sag.

4. 45 a-šà *a-na* 2 e-tab 1,30
5. 1,30 *a-na* 1,33,45 dah 3,3,45
6. 3,3,45-e 1,4[5 íb]-si$_8$
7. 1/2 1,45 gaz 5[2,30 in-sì]
8. 52,30 du$_7$-du$_7$ 45,5[6],15 in-sì
9. 45 a-šà *i-na* 45,56,15 zi
10. 56,15 in-sì 56,15,7,30 íb-si$_8$
11. 7,30 *a-na* 52,30 dah 1 uš in-sì
12. 7,30 *i-na* 52,30 zi 45 sag / in-sì

1. 1,15 the diagonal, 45 the surface.
2. The length and the width are what?
3. 1,15 square 1,33,45 it gives.
4. 45, the surface, to 2 you repeat, 1,30.
5. 1,30 to 1,33,45 join, 3,3,45.
6. 3,3,45 makes 1,45 equalsided.
7. 1/2 of 1,45 break, 52,[30 it gives.]
8. 52,30 square, 45,56,15 it gives.
9. 45, the surface, from 45,56,15 tear off,
10. 56,15 it gives. 56,15 <makes> 7,30 equalsided.
11. 7,30 to 52,30 join, 1, the length, it gives.
12. 7,30 from 52,30 tear off, 45, the width, it gives.

The problem is to find the length (a) and the width (b) of a rectangle given the diagonal and the area. The solution procedure is explained as follows:
Line 3: an "oblique" square is constructed on the diagonal (see Fig. 4).
Lines 4–5: the four lateral 1/2 rectangles (total area 2 times 45 equals 1,30) are added to this oblique square (area 1,33,45), providing the area of the large square (3,3,45).
Line 6: the side of the large square is obtained by taking square root ("3,3,45 makes 1,45 equalsided"). This provides $a + b$. The scribe then computes 1/2 of this, $(a + b)/2$, because he is considering the upper left quarter of Fig. 5 for the particular case ab = A = 45.
Line 7–10: in order to calculate $(a - b)/2$, the scribe subtracts from the area the gnomon (the grey area in Fig. 5) which equals the area of the initial rectangle and calculates $((a + b)/2)^2 - ab = ((a - b)/2)^2$, Eq. 10 above.
Lines 11–12: a and b are calculated by adding and subtracting $(a + b)/2$ and $(a - b)/2$.

There are several things worth noting: First, for the calculation of $(a - b)/2$, the scribe could calculate the area of the small square by subtracting the four lateral 1/2 rectangles from the oblique square (as in Db2-146), but he prefers to focus on the gnomon. Second, in this example the half-sum and half-difference relate the length and the width, whereas in P 322, half-sum and half-difference relate the width and the diagonal. Third, line 6 involves finding a square root, and the result will not always be a finite sexagesimal. For a particular problem with $d^2 + 2A$ a perfect square, i.e., the square of a finite sexagesimal (in line 6, $\surd(d^2 + 2A) = 1{,}45$), the solution procedure gives $a + b = \surd(d^2 + 2A)$, see Eq. 11 above. If $(a + b)/2)^2 - A$ is also a perfect square, $1/2(a - b) = \surd((a + b)/2)^2 - A = \surd\ ((1/2(a + b))^2 - ab)$ (lines 10,11, $(a - b)/2 = 7{,}30$), then the length a and width b are both finite sexagesimals. This is

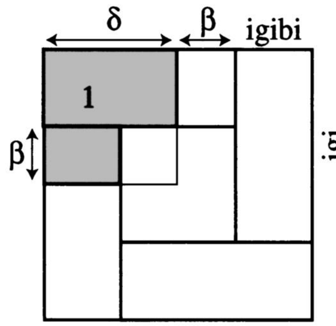

Fig. 8 Solution of MS 3971 #3

fine for a single problem, but what if the problem is to generate a list of examples as in MS 3971 #3?

MS 3971 #3 (Friberg 2007, pp. 252–254) gives five examples of how to compute δ and β from pairs of reciprocals, all of which are among the 23 entries which would have completed P 322. Removing any doubt as to the object of the exercise, the section is introduced as "*aššum* 5 *ṣilipatum amarika*—in order for you to see (i.e., construct) 5 diagonals—and concludes with the summary emphasis "5 *ṣilipatum*".

MS 3971 #3 (Friberg 2007, pp. 252–253, with notations following the conventions of this article as well as some slightly modifications[55])

3a	1–2.	*aš-šum* 5 *ṣi-il-pa-tum* / *a-ma-ri-ka*
	3.	[1,04] igi *ù* igi-bi 5[6],15[!]
	5–8.	[…]
3b	1.	ki 2 [1,40 igi 36 igi-bi]
	2.	1,40 *ù* 3[6 gar-gar 2,16 in-sì]
	3.	1/2 2,16 gaz [1,8 in-si]
	4.	1,08 du_7-du_7 [1,17,4 in-sì]
	5.	1 *i-na* 1,17,4 ⌈zi⌉
	6.	17,4 in-sì 17,4-e [32 íb-si_8]
	7.	32 sag in-sì

55 3a, li. 1-2: Friberg's transliteration reads: *ṣ;i-il-p*[*a-tum*]. According to the copy by Al-Rawi and the "conform transliteration" by Friberg, signs are clear, thus there is no need of bracket in this word. Idem for *ka* in *a-ma-ri-ka*.

3a, li. 3: the line begins probably by ki 1. The entry with ki 1 is consistent with the following sections, and there is room for these signs. Otherwise, the restoration of the pair 1,4 / 56,15 is uncertain and fits poorly in the following series of *igi* / *igibi*.

3b, li. 2: according to the other sections (and according to the Friberg's translation), the line ends with /in-sì/; Friberg omitted to restore this sequence in his transliteration.

3b, li. 5: Friberg repeats twice "17 04 in-sì". Since this sequence appears in the beginning of line 6, there is no reason that it appears in the end of the line 5, in the damaged part.

3c, li. 4: Friberg's transliteration reads <1 *i-na* 1 10 25 zi>. The tablet is damaged, and there is room for this sequence, which could be restored.

3d, li. 4: typo: insertion of "333" in the transliteration.

3c 1. ki 3 1,30 igi 40 igi-bi
2. 1,30 *ù* 40 gar-gar 2,10 in-sì
3. [1/2] 2,10 gaz 1,5 in-sì
4. 1,5 du$_7$-du$_7$ 1,10,25 [1 *i-na* 1,10,25 zi] 10,25 in-sì
5. 10,[25-e] 25 sag ki 3
3d 1. ki 4 1,20 igi 45 igi-bi 1,20 *ù* 45
2. gar-gar 2,5 1/2 2,5 gaz 1,2,30 [in-sì]
3. 1,2,30 du$_7$-du$_7$ 1,5,6,15
4. 1 *a-na* uš zi 5,6,15 in-sì
5. 5,6,15-e 17,30 íb-si$_8$
6. 17,30 sag *ṣi* – ⌈*li-ip*⌉-*ti* ki 4
3e 1. ki 5 1,12 igi 50 igi-bi 1,12 *ù* 50 [gar-gar]
2. 2,2 1/2 2,2 gaz 1,1
3. 1,1 du$_7$-du$_7$ 1,2,1
4. 1 *i-na* 1,2,1 zi 2,1in-sì
5. 2,1-e 11 íb-si$_8$ 11 sag ki 5
ṣi-il-pa-tum

As noted by Friberg, the term *ṣiliptum* means both the diagonal and the rectangle. The algorithm applied in these five examples is based on the relation

$$(1/2(igi + igibi))^2 - 1 = (1/2(igi - igibi))^2.$$

Thus if $\delta = 1/2(igi + igibi)$, and $A = 1$, then $\delta^2 - -$A is also a perfect square, and $\beta = 1/2(igi - igibi)$ (Fig. 8).

This is precisely the algorithm that explains the first existing column of P 322.

3a 1–2 In order for you to see five diagonals:
3 1,4 (is) the *igi*, and the *igibi* 56,15
5–8 [...]
3b 1 The 2nd (example). 1,40 the *igi*, 36 the *igibi*.
2 1,40 and 36 heap, 2,16 it gives.
3 1/2 of 2,16 break, 1,8 it gives.
4 1,8 square, 1,17,4 it gives.
5 1 from 1,17,4 tear off,
6 17,4 it gives. 17,4 makes 32 equalsided.
7 32, the width, it gives.
3c 1 The 3rd. 1,30 the *igi*, 40 the *igibi*.
2 1,30 and 40 heap, 2,10 it gives.
3 1/2 of 2,10 break, 1,5 it gives.
4 1,5 square, 1,10,25.
1 from 1,10,25 tear off, 10,25 it gives.
5 10,25 makes <25 equalsided>. 25, the 3rd width.
3d 1 The 4th. 1,20 the *igi*, 45 the *igibi*. 1,30 and 45
2 heap, 2,5 it gives. 1/2 of 2,5 break, 1,2,30 it gives.
3 1,2,30 square, 1,5,6,15.
4 1 from the length tear off, 5,6,15 it gives.

5 5,6,15 makes 17,30 equalsided.
6 17,30, the width of the 4th diagonal.
3e 1 The 5th. 1,12 the *igi*, 50 the *igibi*.
2 1,12 and 50 heap, 2,2 it gives. 1/2 of 2,2 break, 1,1.
3 1,1 square, 1,2,1.
4 1 from 1,2,1 tear off, 2,1 it gives.
5 2,1 makes 11equalsided. 11, the 5th width.
5 diagonals.

This relation is a particular example of Fig. 5, and Eq. 10, with b replaced by $1/a$, and the area ab of the gnomon in the upper left corner equal to 1. The difference between the squares is not geometrically a square, but rather the shaded gnomon with area 1. Indeed it is the property of the *number* 1, that the result of its multiplication by itself is still 1, that makes Eq. 5 a general algorithm for finding sets of numbers which obey the Diagonal Rule:

$$(1/2(a+b))^2 - (1/2(a-b))^2 = (1/2(a+1/a))^2 - (1/2(a-1/a))^2 = 1 = 1^2 \tag{12}$$

The significance of this step can hardly be overstated, for it represents a distinct departure from the purely geometric demonstrations which preceded it towards a relationship depending on the properties of *numbers*, and which was, in fact, not demonstrable by traditional techniques. The essential insight, that $1^2 = 1$, which also reflects a different application of terminology and understanding of reciprocation than is found in YBC 6967, appears explicitly in line 7 of MS 3052 #2, with the consequence that $\delta^2 - \beta^2 = 1$ is a relation involving a triple of square numbers.

MS 3052 #2 (Friberg 2007, p. 275)

1. [igi] 2 uš sag *ù ṣi-li-ip-tum* / en-nam
2. [za]-e ak-da-zu-dè
3. [2] igi *ù igi-bé-e* en-nam (?)
4. 2 igi *ù* 30 ⌈*igi-bé*⌉ – *e* gar
5. 2 igi *ù* 30 [igi-bi gar(?)]-gar-*ma*
6. 2,30 in-[sì 1/2 2,30 gaz-*ma* 1,15 in]-⌈sì⌉1,15 *ṣi-li-ip-tum*
7. 1,[15 du$_7$-du$_7$-*ma* 1,33,45 in-sì] 1 uš du$_7$-du$_7$-*ma* 1 in-sì
8. 1 [*i-na* 1,33,45 zi-*ma*] 33,45 in-sì
9. 33,[45-e 15 íb-si$_8$ 15] ⌈sag⌉ in-sì

1. The *igi* (is) 2. Length, width, and diagonal, what?
2. You, with your doing:
3. [2] the *igi* and the *igibi*, set.
4. 2 the *igi*, and 30, the *igibi*, set.
5. 2, the *igi*, and 30, the *igibi*, heap,
6. then 2,30 it gives. 1/2 of 2,30 break, 1,15 it gives. 1,15 is the diagonal.
7. 1,15 (let) butt (itself), then 1,33,45 it gives. 1, the length, square, then 1 it gives.
8. 1 from 1,33,45 tear off, then 33,45 it gives.
9. 33,45 makes 15 equalsided. 15, the width, it gives.

The problem defines a in expression (8) as the "igi" = 2 and asks for the length, width and diagonal of the rectangle this is assumed to define. The text then steps painstakingly through the solution, first calculating $1/a = 0;30$, then $a + 1/a = 2;30$, then $1/2(a + 1/a) = 1;15$, which it declares to be the diagonal, δ, revealing an implicit assumption that the length ostensibly sought is equal to 1. At this point one might expect the calculation of the width as $\beta = 1/2(a - 1/a)$, but instead the text follows the diagonal algorithm and computes the width as

$$\beta = \sqrt{\left(\delta^2 - 1^2\right)}.$$

carefully remarking in line 7, that "1, the length, multiplied by itself, gives 1" (1 uš du$_7$. du$_7$-*ma* 1 in-sì). This 1, i.e., 1^2, is then subtracted from δ^2 to give β^2, thence β, which corrected for typos[56] is found to be 0;45 (see again Fig. 10).

MS 3052 #2 misstates its problem and assumes, rather than finds, a rectangle of unit length. Nevertheless, its careful explanation that $1 \times 1 = 1$, suggests the central role which this observation played in the discovery of the diagonal algorithm. The fact that the solution follows the path of the Diagonal Rule, rather than simply assuming the diagonal algorithm, also suggests that the latter was a novelty, to be used sparingly, and that following the Diagonal Rule to a solution was regarded as a sounder procedure than relying entirely on the diagonal algorithm.

MS 3052 #2 departs from the conventions reflected in YBC 6967, most obviously in assuming that product *igi*×*igibi* = 1, rather than 60, but also and more subtly in associating *igi* with the length (the longer dimension) and *igibi* with the width (the smaller dimension). In YBC 6967 the opposite convention is employed, and *igi* is associated with the shorter of the implicit dimensions and *igibi* with the longer one. Whether this is significant or happenstance is unclear, but it is interesting that it appears in association with a different basis for reciprocation, the discovery of the diagonal algorithm, and a move away from geometrical towards more purely numerical analysis.

In MS 3971 #3 the area δ^2 is found, and then the side β is computed as the square root of $\delta^2 - 1$.

The similarities between MS 3971 #3 and P 322 are striking:

- Both are underdetermined problems. While the MS tablet gives five samples, P 322 gives the complete list of solutions in a given set of numbers (see in part I the conditions on r and s).
- In both texts, the samples are numbered by the index "ki + N".

This suggests that the problem solved in P 322 is a kind of generalization of the problem solved in MS 3971 #3. The problem of P 322, which could be formulated by something like "In order for you to see *all* the diagonals". Moreover, the indexing of samples by the "ki + N" system is a unique feature, not attested in other mathemati-

[56] Correcting 15 to 45 in line 9. Regrettably, Friberg's valuable volume has been wretchedly copy-edited, leaving it infected with a distressing number of errors, of which inexcusably omitting column I′ from the "transliteration" of Plimpton 322 on page 434 is only one example of many.

cal texts,[57] and thus could reveal that authors of P 322 and MS 3971 were sharing a common scribal culture.

2.3 A new approach to P 322

First, let us consider the heading of column I′.

> The number square [*takilti*] of the diagonal [*ṣiliptim*](δ) from which 1 is subtracted and (that of) the width [sag](β) comes up.

The terms "diagonal" and "width" leave no doubt that the *subject* of column I′ is a series of rectangles whose diagonals and widths, denoted (column IV′) by δ and β, are related as described in the heading of column I′ as:

$$\delta^2 = 1 + \beta^2.$$

The relation described in the heading could be illustrated in Fig. 9:

In fact, the text does not say explicitly that the 1 that is subtracted is a square. Nor does it say that the result after subtracting is a square, rather, "sag *illu*", that is, "the width comes up." It seems that the term "sag" actually is being used for the square with side β, which would explain the appearance of the term "íb-si$_8$" in columns II′ and III′ which refer to linear dimensions. Or it may be that *takilti* is being used to refer to both the sag and *ṣiliptim*. Another important point is, as explained in Part I, that the use of the term *"takiltum* " suggests that the square with side β,the *takiltum,* is included in the square with side δ as in Fig. 10 below, with a *gnomon* of area 1 in between. In any case, the appearance of the term *takiltum* is significant, because it reflects the configuration illustrated in Fig. 10, the square of side β is contained in the square of side δ which is the configuration appearing in the method of "completing the square".

A derivation of the algorithm underlying the examples in MS 3971 #3 and MS 3052 #2 can be given in the spirit of Old Babylonian mathematics by cutting and pasting the "leg" of the gnomon in Fig. 10 producing a rectangle of area 1 and sides $\delta + \beta$ and $\delta - \beta$, as in Fig. 11.

Thus

$$1 = \delta^2 - \beta^2 = (\delta + \beta)(\delta - \beta)$$

and the dimensions are reciprocal numbers *igi* and *igibi*:

$$\delta + \beta = igi \quad \text{and} \quad \delta - \beta = igibi$$

The initial values β and δ can be then easily found by adding and subtracting the two equations,

$$\delta = 1/2(igi + igibi) \text{ and } \beta = 1/2(igi - igibi).$$

With the examples of MS 3971 #3 as background, we offer our interpretation of the problem underlying P 322, which may be stated:

[57] A comparable system is attested in some omen texts (Glassner 2009).

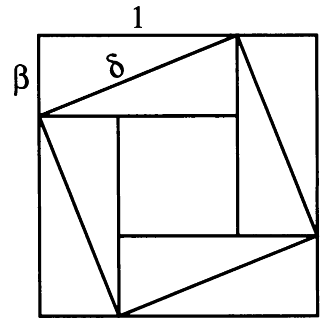

Fig. 9 Representation of the heading of column I′

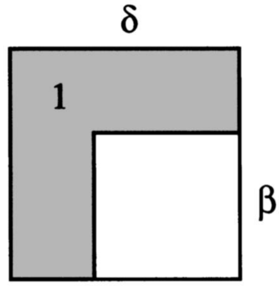

Fig. 10 Other representation of the heading of column I′

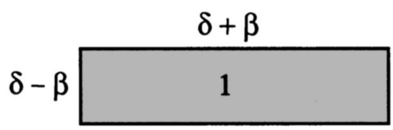

Fig. 11 Derivated rectangle of area 1

"Make the list of all the rectangles with length (uš) equal to 1 and width (sag) and diagonal (*ṣiliptum*) finite sexagesimal numbers, (column −1 and 0) and represent the dimensions, in reduced form, without common sexagesimal factor (columns II′ and III′)" Something like this statement could have been noted in the heading of one of the lost columns. In this case, the heading of column I′ may be interpreted as a verification of col. −1 and 0, as suggested by tablet Db_2-146 (lines 16–25).

Robson and Friberg consider the purpose of P 322 to be a "teacher's aid" for the setting of problems related either to the diagonal of a normalized rectangle or for rectangular-linear systems of equations where either the sum or the difference of *igi* and *igibi* is known. To the contrary, we consider the text itself as presenting a challenging mathematical problem and its solution, comparable to the example given in M 3971 #3, but complete.

Let us come back to some problems:

The absence of a column for the length of the normalized rectangle in the preserved part of P 322 should be explained. The length of third side is not listed because the purpose of columns II′ and III′ was to find the relationship between the dimensions, β (sag) and δ (*ṣiliptum*) in reduced form.

In terms of floating sexagesimal notation the half sum or difference of reciprocals would be a product of two factors

$$1/2(r/s + s/r) = \left(r^2 + s^2\right) / (2rs) = \left(r^2 + s^2\right) \left(\overline{2rs}\right)$$

where $\overline{2rs}$ represents the reciprocal of 2rs. Columns II′ and III′ represent the result of factoring out the common factor $(\overline{2rs})$ in

$$\beta = \left(r^2 - s^2\right) \left(\overline{2rs}\right)$$

and

$$\delta = \left(r^2 + s^2\right) \left(\overline{2rs}\right).$$

This is the same as what we would regard as expressing the ratio β/δ in reduced form $\beta/\delta = n/d$. Columns II′ and III′ may be regarded as completing a conceptual process not generally reducible to the single regular number, n/d, since with only two exceptions in Table 7, d is an irregular number. Moreover, the length of the "primitive" rectangle (l) is easily found as the reciprocal of common divisor of β and δ, that is, the product of the factors which appear in the right sub-column when applying the trailing part algorithm to β and δ (see Table 6). For example, for the data given in the table in Part I, above:

In ki 2, the length of the "primitive" rectangle (l) is $2 \times 12 \times 12 \times 12$ (that is 57,36).
In ki 15, the length of the "primitive" rectangle (l) is $3 \times 30 \times 30$ (that is 45).

Thus, the length of the "primitive" rectangle is automatically provided by the application of the trailing part algorithm to β and δ. Friberg's explanation of columns II′ and III′ involves finding the "factor-reduced core" of the terms in column I′, that is δ^2, which goes hand in hand with his conclusion that "it was never the intention of the author of P 322 to reduce his series of *normalized* diagonal triples (with a length equal to 1) to a corresponding series of *primitive* diagonal triples (with the width, the length and the diagonal integers without common factors)." The difference between his approach and the method of the trailing part algorithm applied to β and δ in parallel, as demonstrated in Part I, is seen clearly in lines 2 and 15, where Friberg's reduction is carried out to a different number of steps for β^2 and δ^2. Thus, the entry in column III′ line 2 (read as 3,13) and the entry in column III′ line 15 (56) are not considered as errors, even though they do not give the correct pair of terms in a finite sexagesimal rectangle.

Another problem is the hypothesis that the pairs β and δ derive from a set of *igi* and *igibi* pairs, r/s and s/r. In this connection it is noteworthy that problems C20 – C33 in the series text YBC 4668 (Neugebauer 1935–1937, pp. 430–431) involve ratios of lengths to widths, their inverses and their sums and differences, closely analogous to r/s, s/r, $r/s + s/r$, $r/s - s/r$, accompanied by the technical term and formula, "igi-te-en *šá* sag uš-šè", "relation of width towards length", which proves to be the ratio of length divided by width (uš/sag). This shows that ratios similar to those underlying P 322 were sufficiently remarkable to have a technical term describing them.

Interestingly, the parameters of the problem set, uš = 30, sag = 20, are not components of a finite sexagesimal rectangle; nor is there any evidence in the text of awareness of the diagonal algorithm. Nevertheless, the product of such ratios and their inverses is clearly equal to 1, and it is not hard to imagine the discovery of the diagonal algorithm emerging from work with such problems.

Finally, the appearance in MS 3971 #3 of 5 examples from the unfinished extension of P 322 suggests that a completed version of the tablet was known to the author of MS 3971. Remarkably, this text and MS 3052 are also the only evidence for knowledge of the underlying diagonal algorithm apart from P 322 itself.

Seen in this light P 322 appears to be an elucidation, and perhaps even celebration, of the discovery of the bridge between the Diagonal Rule and the method of completing the square. As such, the tablet and its composition seem a much more deeply mathematical undertaking than simply constructing a pedagogical aid—something already suggested by the discovery that the scribe contemplated encompassing a complete set of reasonably defined arguments.[58] The historical relation between the discovery and the text is not presently knowable. That they may have been contemporaneous is suggested by the lack of evidence, apart from MS 3052 and 3971, of either awareness or use of it until much later. Whatever the case, P 322 *is mathematics* as Neugebauer recognized, not exactly as Neugebauer understood it, but real mathematics focused on a discovery which significantly expanded the capabilities of the art. As such it appears far more deeply rooted than simply a utilitarian exercise in pedagogical convenience.

This is not to argue that P 322 served no utilitarian purpose. Its headings clearly reflect a focus on the dimensional relationships of a set of rectangles intended to comprise *all* rectangles of unit length for which $\boldsymbol{\beta} < 1$ and constructed from ratios of regular numbers to regular sexagesimal digits. The resulting distribution of the normalized rectangles preserved in the text is illustrated to scale in Fig. 12, whose components are readily scalable by any factor, regular or otherwise, further reason for assuming the presence of $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ in the missing fragment.

The link between the finite sexagesimal rectangles and problems involving partition of triangle and trapezoid[59] was underlined by Vaiman (1961, pp. 192–206). Figure 13 illustrates the relationship between the Diagonal Rule as conceptualized in P 322 and the bisection of the trapezoid (the light gray area) is equal to the dark gray one:

In fact, Trapezoid Rule appears to have been known much earlier, as evidenced by IM 58045 (Friberg, 1990, p. 541), which dates from the Sargonic reign of Šar-kali-šari and describes the bisection of a trapezoid with sides 7 and 17, implying knowledge

[58] Extending s to 2 places, i.e., considering all reduced ratios for $1 < r/s < 1;24,51,10$ and $1 \leq s < 1, 0, 0$, yields few entries outside the range of Plimpton 322 if completed. These include the couple r = 52,5, s = 21,36 and β = 0;59,53,46,54,24 at the upper end of the range and at the lower end: the couple r = 4,10, s = 4,3 and β = 0;1,42,15, 6,40; the couple r = 2,8, s = 2,5 and β = 0;1,25,20,15; the couple r = 52,5, s = 51,12 and β = 0;1,1,34,57,40,30; the couple r = 1, 21, s = 1,20 and β = 0;0,44,42,30; and finally the couple r = 34, 8, s = 33,45 and β = 0; 0,40,39,33,26,15. Reflecting continuity and extended application of the algorithm the couple r = 1, 21, s = 1,2 *is* attested in the Seleucid text from Uruk AO 6484 rev. 10-27 (Neugebauer 1935–1937, pp. 98-99; Friberg, 2007, pp. 444–447), where the problem set is to find *igi (r/s)* and *igibi* (s/r) given their sum as δ, δ^2, $\beta^2 = \delta^2 - 1$, β, $\beta + \delta$ = igi, $\beta - \delta$ = *igibi*.

[59] These problems played an essential role in cuneiform mathematics since the third millennium, See for exemple the Old Akkadian tablet from Nippur IM 58045 (CDLI: P216852) and the analysis by Friberg of Ur III field plans (CDLJ 2009:3).

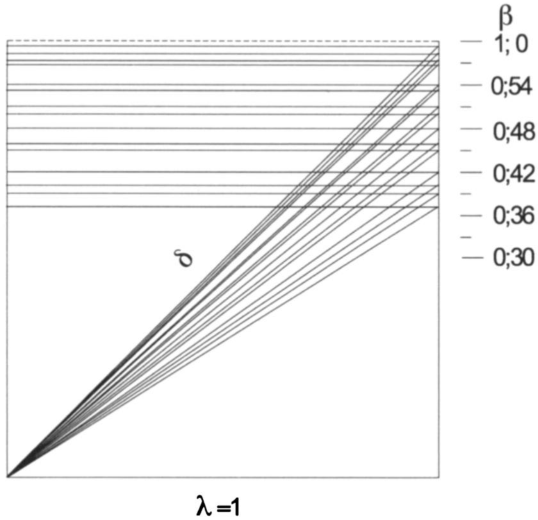

Fig. 12 The 15 normalized rectangles of P 322

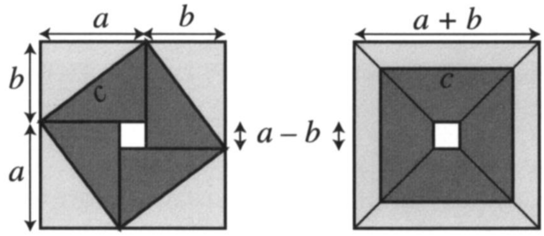

Fig. 13 Bisection of the trapezoid

of the trapezoidal triple, 7:13:17. Certainly, the 3:4:5 triple, conventionally expressed unreduced as 45:1,0:1,15, and its (reduced) trapezoidal equivalent, 1:5:7, were also known by this time, and it seems likely that all of the relationships described above may date from at least the Sargonic period, perhaps explaining the later reference noted above to the "Akkadian method".

As a consequence, the triples $1+\beta$, δ, and $1-\beta$ associated with each line of P 322 could be considered as giving a bisection of the trapezoids with parallel sides of length 1 and β. Therefore it would have been a simple exercise to compute the parameters for trapezoidal bisection problems from the tablet P 322. Perhaps the line numbering in column IV′ contemplated linkage with a second tablet with these parameters, the potential details of which appear in Appendix A. That the author may have had other practical purposes in mind seems perfectly possible but also wholly speculative. What seems less speculative is that—its occasional errors not withstanding—P 322 reflects a high degree of mathematical sensibility which distinguishes both it and its author from the typical standard of everyday Old Babylonian mathematical practice.

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3 Appendix A: Trapezoidal triples

Table 7 Trapezoidal triples (μ, δ, ν) and normalized trapezoidal triples (m, c, n) corresponding to the 38 lines of reconstructed P322

μ = 1+β	δ	ν=1−β	m	c	n	line
1 59 30	1 24 30	0 0 30	3 59	2 49	1	1
1 58 27 17 30	1 23 46 2 30	0 1 32 42 30	1 53 43	1 20 25	1 29	2
1 57 30 45	1 23 6 45	0 2 29 15	2 36 41	1 50 49	3 19	3
1 56 29 4	1 22 24 16	0 3 30 56	7 16 49	5 9 1	13 11	4
1 54 10	1 20 50	0 5 50	2 17	1 37	7	5
1 53 10	1 20 10	0 6 50	11 19	8 1	41	6
1 50 54 40	1 18 41 20	0 9 5 20	1 23 11	59 1	6 49	7
1 49 56 15	1 18 3 45	0 10 3 45	29 19	20 49	2 41	8
1 48 6	1 16 54	0 11 54	18 1	12 49	1 59	9
1 45 56 6 40	1 15 33 53 20	0 14 3 53 20	3 10 41	2 16 1	25 19	10
1 45	1 15	0 15	7	5	1	11
1 41 58 30	1 13 13 30	0 18 1 30	1 17 59	48 49	12 1	12
1 40 15	1 12 15	0 19 45	6 41	4 49	1 19	13
1 39 21 20	1 11 45 20	0 20 38 40	1 14 31	53 49	15 29	14
1 37 20	1 10 40	0 22 40	1 13	53	17	15
1 36 27 30	1 10 12 30	0 23 32 30	7 43	5 37	1 53	16
1 32 50 50	1 8 24 10	0 27 9 10	22 17	16 25	6 31	17
1 32	1 8	0 28	23	17	7	18
1 30 4 53 20	1 7 7 6 40	0 29 55 6 40	3 22 41	2 31 1	1 7 19	19
1 29 15	1 6 45	0 30 45	1 59	1 29	41	20
1 27 40 30	1 6 4 30	0 32 19 30	19 29	14 41	7 11	21
1 25	1 5	0 35	17	13	7	22
1 24 11 40	1 4 41 40	0 35 48 20	50 31	38 49	21 29	23
1 22 22	1 4 2	0 37 38	41 11	32 1	18 49	24
1 21 34 22 30	1 3 45 37 30	0 38 25 37 30	2 10 31	1 42 1	1 1 29	25
1 20 51 15	1 3 31 15	0 39 8 45	1 4 41	50 49	31 19	26
1 20 4	1 3 16	0 39 56	20 1	15 49	9 59	27
1 18 16 40	1 2 43 20	0 41 43 20	23 29	18 49	12 31	28
1 17 30	1 2 30	0 42 30	31	25	17	29
1 14 57 45	1 1 50 15	0 45 2 15	33 19	27 29	20 1	30
1 13 30	1 1 30	0 46 30	49	41	31	31
1 11	1 1	0 49	1 11	1 1	49	32
1 10 14 35	1 0 52 5	0 49 45 25	33 43	29 13	23 53	33
1 7 5	1 0 25	0 52 55	2 41	2 25	2 7	34
1 6 20	1 0 20	0 53 40	3 19	3 1	2 41	35
1 4 37 20	1 0 10 40	0 55 22 40	12 7	11 17	10 23	36
1 3 52 30	1 0 7 30	0 56 7 30	8 31	8 1	7 29	37
1 2 27	1 0 3	0 57 33	20 49	20 1	19 11	38

4 Appendix B: Photo of the tablet

Plimpton Cuneiform 322, Rare Book and Manuscript Library, Columbia University, Courtesy Jane Siegel, photo C. Proust (see Figs. 14, 15).

Fig. 14 Plimpton 322, obverse

Fig. 15 Plimpton 322, reverse

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