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Revisiting Al-Samaw'al's table of binomial coefficients: Greek inspiration, diagrammatic reasoning and mathematical induction

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Abstract In a famous passage from his al-Bāhir, al-Samaw'al proves the identity which we would now write as $(ab)^n = a^n b^n$ for the cases $n = 3, 4$. He also calculates the equivalent of the expansion of the binomial $(a + b)^n$ for the same values of n and describes the construction of what we now call the Pascal Triangle, showing the table up to its 12th row. We give a literal translation of the whole passage, along with paraphrases in more modern or symbolic form. We discuss the influence of the Euclidean tradition on al-Samaw'al's presentation, and the role that diagrams might have played in helping al-Samaw'al's readers follow his arguments, including his supposed use of an early form of mathematical induction.

1 Introduction

Al-Samaw'al ibn Yahyā al-Maghribī, born around 1130 in Baghdād, is a character of some repute in the history of mathematics. Composing a substantial work on mathematics at the prodigious age of nineteen, this son of a devout and learned Jew also famously and publicly converted to Islam from Judaism, maintained a long and distinguished career as an itinerant physician traveling throughout Iraq, Syria, Kūhistān and Ādharbayjān, and wrote on a proliferation of topics including medicine, mathematics, astronomy, religion, theories of love and erotica (Anbouba 1978).

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However, al-Samaw'al is best known for his work on mathematics: *Al-Bāhir fī al-ḥisāb* (literally: *The Splendid Book on Arithmetic*)[1] (ca. 1150). This work is important to the history of mathematics for a number of reasons. Firstly, the intellectual inspiration which al-Samaw'al owes to his predecessors is both substantial and acknowledged. Al-Samaw'al gives summaries and syntheses of a number of his predecessors in the Islamic tradition, including al-Khwārizmi and al-Karajī, and turns his attention to mathematical ideas that had been explored by mathematicians such as Qustā ibn Lūqā, al-Sijzī, al-Khayyām and Abū Kāmil (Anbouba). Furthermore, it is well established that Greek mathematical works provided fundamental inspiration for almost all mathematicians working in the Islamic tradition. Al-Samaw'al is no exception and, as we shall see below, his mathematical style was influenced by many of his Greek predecessors, including Euclid, Heron and Diophantus, whose works clearly provided much motivation for both the content and articulation of his mathematical ideas.

However, his work also contains some significant deviations from that of his Greek predecessors. In particular, al-Samaw'al epitomizes a conflation of the geometric and arithmetic approaches that had been developing in the works of many of his Islamic predecessors as they assimilated, synthesized and then built on the approaches of their Greek precursors. Notably, in various places in the *Al-Bāhir*, we can discern some decisive breaks from Greek contexts, including advancing an increasingly abstract articulation of number, algebraic styles of reasoning and a new significance for diagrams. One result of these developments is that some features retained from the Greek geometric styles of reasoning become redundant, superfluous or even unhelpful.

Al-Bāhir is a substantial work. Our study focuses on a small but significant part of it—the opening passage from fourth chapter of the second section. This passage has already attracted attention (Berggren 2003, 2007; Rashed 1972) because of its description and rendering of a table of binomial coefficients, and because of its apparent use of a mathematical mode of reasoning more frequently known today as mathematical (or complete) induction. The table of coefficients, of course, is better known as the Pascal Triangle, appearing in French mathematician Blaise Pascal's work *Traité du triangle arithmetique* (published posthumously in 1665), but it has many attested antecedents including from India, China, and Italy (Edwards 2002). The apparent use of induction has seen al-Samaw'al's insights compared to similar achievements by mathematicians in other cultures and times such as Pascal, Levi Ben Gerson, Maurolico, and Simon Stevin, among others (Rashed 1972).

The passage in question begins with five propositions in rhetorical algebra which set out rules for expanding expressions equivalent to $(ab)^n$ and $(a+b)^n$ for the cases $n = 3, 4$, followed by assertions about such expansions for higher powers. It concludes with a description of the construction of the table of binomial coefficients and gives the table up to its twelfth row.

We present here for the first time a literal English translation and detailed mathematical commentary of this passage. Then, using direct evidence from this careful textual scrutiny, we explore two issues raised by al-Samaw'al's exposition. Firstly, we consider the influence of the Greek geometric and arithmetic heritage on al-Samaw'al

[1] Although some authors have chosen the title *Al-Bāhir fī al-jabr*, the evidence of the two manuscripts we used supports our reading.

and the ways in which *Al-Bāhir* represents the transition between this source of inspiration and the new nascent modes of mathematical reckoning and practice. Secondly, we examine the diversity and role of al-Samaw'al's diagrams in his exposition and the ways in which they might have supported his mathematical arguments.

In addition, we will revisit the question of how this text is to be understood in the historical development of mathematical induction. We will offer fresh insight into recent scholarly discussion on this issue (see Acerbi 2000; Fowler 1994; Rabinovitch 1970; Rashed 1972; Unguru 1991, 1994) using al-Samaw'al's exposition to advance new perspectives on the strands of activity that may usefully be considered to fall under the purview of this mode of mathematical reasoning.

2 Text and translation

2.1 Notes on the translation

This translation is based on the relevant excerpt from an edition of *Al-Bāhir* (Ahmad and Rashed 1972, pp. 104–112) which was prepared by Ahmad and Rashed on the basis of two manuscripts. The first manuscript, numbered 2718 of Aya Sofia (116ff), was copied in 1324. The second is numbered 3155 Esat Efendi (88ff). Both are housed in the Suleymaniye library (Istanbul, Turkey).[2] This is the first time this section of *Al-Bāhir* has been translated as a whole into English, and the first time the diagrams in this section of Aya Sofia 2718 have been published and studied.[3]

For flow and clarity, we have arranged our translation and mathematical commentary proposition by proposition, and the numbered subheadings we have used to delimit the presentation are ours (not al-Samaw'al's) and are intended for the reader's convenience; the original text is continuous.

For each proposition, we have presented our mathematical analysis in two distinct parts. This dual approach is based on 'paraphrasing' the mathematical content. The

[2] There seems to be some confusion with respect to the numbering and location of these manuscripts. Ahmad and Rashed (1972, pp. 1–2) claim the Aya Sofia manuscript to be no. 2118, but in the collection in Cairo (sic!), and the other manuscript, no. 3155 Esat ef., to be in the Suleymaniye library. Anbouba (1978) states the Aya Sofia manuscript to be no. 2718 and 115ff and in Istanbul. King (1976) declares that both manuscripts are in Istanbul and were discovered by Krause (1936) and that in 1961 Anbouba summarized the work and further emphasized its importance to the scholarly community (Anbouba 1961). It may be relevant to note that only Anbouba's contribution is cited by Ahmed and Rashed, who furthermore claim responsibility for having revealed to the Suleymaniye library that ms. 3155 Esat Ef., previously classified as 'anonymous,' was a copy of al-Samaw'al's *Al-Bāhir* (Ahmad and Rashed 1972, p. 2).

[3] It is true that Rashed discusses this whole passage in Rashed (1972, pp. 3–8, 1994, pp. 63–68) as part of his investigation into al-Karajī's and al-Samaw'al's role in the history of mathematical induction (al-Samaw'al explicitly credits large portions of the contents of this passage to his predecessor al-Karajī (953–ca. 1029) and at some points his terminology seems to suggest he may even be directly quoting him). Indeed, Rashed gives full translations for propositions 1, 2 and 4 (admittedly with unacknowledged corrections to his edition) and for the construction of the binomial table. However, he gives only a modern symbolic paraphrase for proposition 3 and he omits completely the proof of proposition 5. He also omits all the diagrams and changes the order of the propositions. Berggren (2007, pp. 552–554) has translated the portion of this passage which gives the construction of the binomial triangle. A version of the diagrams from Aya Sofia 2718 was published in Ahmad and Rashed (1972), but these omitted the red line segments visible in the diagrams accompanying propositions 1, 2 and 4, leaving instead a selection of the black dots marking the end points of the lines.

first paraphrase attempts to simplify the rhetorical presentation while staying as close as possible to the original processes of exposition and reasoning (albeit using modern notation). The second (where appropriate) is an explanation of the same proposition but using modern operations and procedures and modern symbolic algebra. For example, the first paraphrase might retain the Euclidean language of ratios and operations on them, while the second interprets these in terms of fractions. Or, again, the first paraphrase might retain the original's sequential symbolic names for quantities (such as a, b, c and so on), while the second takes advantage of algebraic symbolism to identify products as ab (instead of c, say) or powers as a^3 (instead of e, say). Thus, the first paraphrase is designed to keep intact the original expressions and operations that al-Samaw'al uses, while the second, more modern paraphrase, is for ease of comprehension for those more familiar with modern mathematical procedures.

In translating this passage, we have attempted to be as literal as possible to convey the fullest impression of the original text for non-Arabic readers and also to give a sense of the distinct modes of expression original to this mathematical context. However, on occasion there was need to suggest minor modifications to the text when the mathematical context demanded it, or for the mathematical integrity and consistency of the work. These have been clearly identified and noted. Emendations to the text (as presented in the edition Ahmad and Rashed 1972) have been indicated by the use of square brackets [$\cdots$] in the English translation. When English sense requires the addition of words for fluency, or for glosses and supplementary material, we have indicated the additions by round brackets: ($\cdots$). Notes or comments which have been deemed crucial for intelligibility have been indicated in footnotes.

2.1.1 Letters as labels

Al-Samaw'al often uses Arabic letters to label the line segments (or their extremities) that he invokes in the course of his demonstrations. In Aya Sofia 2718, these letters are usually written with an over-bar to visually demarcate them in the text. We have used lower case italics to transliterate the line segments, and upper case italics when the letters indicate the extremities of line segments in order to avoid confusion. The order in which he invokes these letters shows a clear Greek influence, in the tradition of Euclid and his practice of lettering his diagrams, with features of the diagram labeled in standard order as they arise during the demonstration (Netz 1999, p. 71). Indeed, the Arabic letters which al-Samaw'al uses do not reflect the Arabic lexical ordering of them, but rather the ancient Greek lexical ordering of first α, then β, then γ, then δ, and so forth. Where exact transliterations to the Greek did not exist, the closest Arabic letter was substituted. Since this standard ordering of labels seems to contribute the flow of his diagrams (see Sect. 3.2), we have deviated from the traditional correspondence between Arabic and Latin letters (Kennedy 1991–1992) and have tried to retain this flow by using the equivalent English letters as they appear in English alphabetical order, as shown in Table 1.[4] We argue that the order in which the letters were placed

[4] Heath used a similar strategy when translating Euclid's *Elements*. Compare the diagrams in Ref. "Euclid" (1883–1916) and Heath (1956).

Table 1 Correspondence between Arabic and Latin letters used in our transliteration of the labeled diagrams in Aya Sofia 2718

ا	ب	جـ	د	ه	ز	ح	ط	ك	ل	م	ن
a	b	c	d	e	f	g	h	i	j	k	l

This preserves the alphabetic ordering of the labels

in the diagrams represents a crucial aspect of how these diagrams were intended to be read.

2.2 Translation and mathematical commentary

2.2.1 Al-Samaw'al's title and opening statement

Chapter 4 from section 2 on the geometrical demonstrations used to extract unknown numbers. There are two methods. The first method from chapter 4 section 2 consists of the arithmetical foundations.

2.2.2 Mathematical commentary

The aim of this part of *Al-Bāhir* is to offer mathematicians various techniques that might help them to solve equations (Ahmad and Rashed 1972, pp. 53–64). For example, al-Samaw'al himself later used the propositions and binomial table, that we study in this paper, to develop methods for extracting *n*th roots (Rashed 1978). The *arithmetical* methods mentioned here are being contrasted with *geometrical* methods, although as we shall see these approaches are mingled together in the demonstrations. Our extract marks the beginning of a series of propositions about numbers which are proved using techniques similar to those in the arithmetical books in Euclid's *Elements*. The shorter second part of Chapter 4 uses techniques similar to those in the geometrical books of the *Elements* (Ahmad and Rashed 1972, pp. 64–66).

2.2.3 Al-Samaw'al's proposition one

(For) any four numbers, the product of the surface[5] of the first and the second by the surface of the third and the fourth is equal to the product of the surface of the first and the third by the surface of the second and the fourth.

[5] Al-Samaw'al has two ways to refer to the product of two numbers. He can use the Arabic word 'مسطح' (*musaṭṭaḥ*) which literally means 'surface,' or he can use the word 'ضرب' (*ḍarb*) which means 'product.' In order to capture the difference, we preserve his choice even though the mathematical meaning is the same. The corresponding adjective in Euclid (VII, definition 16) is usually translated as 'plane' (as in 'plane numbers,' the result of multiplying two numbers together), but it is sometimes rendered as 'representing a surface.' Høyrup (2002) uses 'surface' to capture a similarly distinct notion of product in Babylonian mathematics.

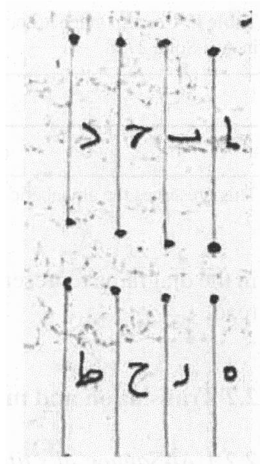

Fig. 1 Diagram accompanying proposition 1 in Aya Sofya 2718

Let us consider four numbers a, b, c, d, and let us multiply a by b to get e, and let us multiply a by c to get [f, and let us multiply b by d to get][6] g. Let us multiply c by d to get h (see the manuscript diagram in Fig. 1).

Then, I say that the product of e and h is equal to the product of f and g.
Its demonstration: When the number a is multiplied (respectively) by the two numbers b and c, then there results from the multiplication the two numbers e and f. Then, the ratio of e to f is the same as the ratio of b to c.
Moreover, when the number d is multiplied (respectively) by the two numbers b and c, then there results from the multiplication the two numbers g and h. Then, the ratio of g to h is the same as the ratio of b to c.
We know that the ratio of e to f is the same as the ratio of b to c. Then, the ratio of e to f is the same as the ratio of g to h.
Then, the surface of e and h is equal to the surface of f and g and this is what we wished to demonstrate.

2.2.4 Mathematical commentary

In symbolic terms, al-Samaw'al proves that $(ab)(cd) = (ac)(bd)$. It is tempting to think this proposition is obvious ('why not simply remove the brackets and use $bc = cb$?') and to ask why he sees a need to prove it. To some extent, this puzzle is an artifact of viewing his rhetorical algebra in symbolic terms. Indeed, by the time al-Samaw'al has named all the numbers he will need in the proof, the initial numbers a, b, c, d have lost their identity in the proposed identity $eh = fg$. But it is also not clear what his criteria were for a proposition to need a proof. For example, in the course of proving proposition 3, he feels a need to refer to *Elements II,1* to justify the rhetorical equivalent of $(a + b)x = xa + xb$ but he is happy to state without

[6] Ahmad and Rashed (1972, p. 105) fix this omission by changing a letter written by the scribe and inserting a missing phrase. Our emendation is slightly simpler, involving only a missing phrase at the start of a new line in the manuscript and no corrections of written letters.

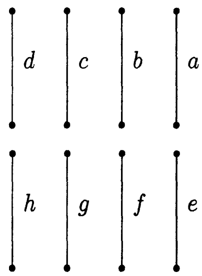

Fig. 2 Transliteration of diagram accompanying proposition 1, preserving the 'alphabetic order' of the labels

proof[7] the fact that we would now write as $(pq)r = (pr)q$. Similarly, in proposition 4 he states without proof various facts like the rhetorical equivalent of $(3a^2b)a = 3a^3b$. In any case, having decided that proposition 1 needed a proof, he uses the same techniques involving ratios that we see in *Elements VII,19* where Euclid proves that if four numbers are in proportion then the product of the first and the fourth is the same as the product of the second and the third.

First paraphrase: using modern symbols, but retaining his use of ratios. To help the reader, each step of the argument is on a separate line. Al-Samaw'al's diagram, translated in Fig. 2, may also help, as we shall see in Sect. 3.2, where we explore the meaning of such diagrams and how al-Samaw'al and his readers might have used them to help understand his proofs.

Let $a,\ b,\ c,\ d$ be four numbers.
Let $ab = e$ and $ac = f$, and let $cd = h$ and $bd = g$.
Then I say that $eg = fh$.
Demonstration Multiplying a by b and c gives e and f, respectively.
Hence[8] the ratio $e : f$ is the same as the ratio $b : c$.
Moreover, multiplying d by b and c gives g and h respectively. Hence, the ratio $g : h$ is the same as the ratio $b : c$.
Thus[9], the ratio $e : f$ is the same as the ratio $g : h$.
Hence[10] $eh = fg$ which is what we wished to demonstrate.

Second paraphrase: using symbolic algebra and replacing ratios by fractions.
I say that $(ab)(cd) = (ac)(bd)$.
Demonstration Since

$$\frac{ab}{ac} = \frac{b}{c} = \frac{bd}{cd}$$

we have

$$(ab)(cd) = (ac)(bd).$$

[7] or without referring to proposition 1, if that was how he saw it being justified.

[8] *Elements VI, 1* or *VII, 17*. Islamic mathematicians had various attitudes as to the distinction between discrete numbers and continuous magnitudes (Oaks 2011); thus, we cite both the geometric and arithmetic context from Euclid's work.

[9] *Elements V, 11*.

[10] *Elements VI, 16 or VII, 19*.

Fig. 3 Diagram accompanying proposition 2 in Aya Sofya 2718

2.2.5 Al-Samaw'al's proposition two

The surface of two sides each cubed is equal [to the cube][11] of their surface.
Let the cubic numbers be the two numbers a and b, and let their sides be c and d, and let their squares be e and f, and let c be multiplied by d to get the number g, and let a be multiplied by b to get h (see the manuscript diagram in Fig. 3).
Then, I say that the number h is equal to the cube of the number g.

Its demonstration: Indeed, it was explained in the arithmetical sections that if the square number e is multiplied by the square number f there results from this multiplication the square of the number g, the surface.
Then, if the result from this is multiplied by the surface of c and d, I mean by the number g, there results from this the cube of the number g, and it is the product of the surface of e and f and the surface of c and d.

But the result from the product of the surface of e and f and the surface of c and d is equal to the result from the product of the surface c and e and the surface [of f and][12] d, as we explained in the previous proposition.

Therefore, the result from the product of the surface of e and c and the surface of d and f is equal to the cube of the number g. But, the result from the product c and e [is the number a] and the surface of d and f is the number b.

Therefore, the surface of a and b, I mean the number h is equal to the cube of number g, and this is what we wished to demonstrate.

2.2.6 Mathematical commentary

In symbolic terms al-Samaw'al proves that $(cd)^3 = c^3d^3$.

First paraphrase: using modern symbols, but staying close to his style of reasoning.

[11] We add this for mathematical sense. Ahmad and Rashed (1972, p. 105) leave this phrase as it is, noting simply (and erroneously) in a footnote that the proposition says that $(ab)^{\frac{1}{3}} = a^{\frac{1}{3}}b^{\frac{1}{3}}$.

[12] This is originally inserted by Ahmad and Rashed (1972, p. 105).

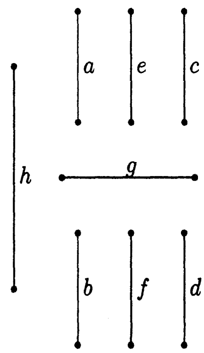

Fig. 4 Transliteration of diagram accompanying proposition 2, again preserving the 'alphabetic order' of the labels

Let a and b be cubes: $a = c^3$ and $b = d^3$.
Let $c^2 = e$ and $d^2 = f$.
Let $cd = g$ and $ab = h$ (see the manuscript diagram in Fig. 4).
Then I say that $h = g^3$.
Demonstration We already know that $ef = g^2$.
Multiplying by $cd = g$ gives $(ef)(cd) = g^3$.
By proposition 1 we have $(ef)(cd) = (ce)(fd)$
and so $(ce)(fd) = g^3$.
But $ce = a$ and $fd = b$.
Thus $ab = h = g^3$ which is what we wished to demonstrate.

Second paraphrase: using symbolic algebra and its index notation for powers.
I say that $c^3d^3 = (cd)^3$.
Demonstration We already know that $c^2d^2 = (cd)^2$.
Multiplying by cd gives $(c^2d^2)(cd) = (cd)^3$.
By proposition 1 we have $(c^2d^2)(cd) = (c^2c)(d^2d)$
and so $(c^2c)(d^2d) = (cd)^3$
But $c^2c = c^3$ and $d^2d = d^3$.
Thus $c^3d^3 = (cd)^3$ which is what we wanted.

Notice that this last version of the proof makes it much easier for one who knows about the technique to see how one might use induction to prove that $c^nd^n = (cd)^n$.

2.2.7 Al-Samaw'al's proposition three

(When) any number is divided into two parts, then its cube is equal to (the sum of) the cubes of its two parts and the product of each of its parts by the square of the other part taken three times.
Its example: If a number AB is divided at the point C (see the manuscript diagram in Fig. 5), then I say that the cube of AB is equal to the cube of AC and the cube of CB and the product of AC by the square of CB taken three times and the product of CB by the square of AC taken three times.
Its demonstration: Indeed, the square of AB is equal to the square of AC and the square of CB and the product of AC by CB taken twice.

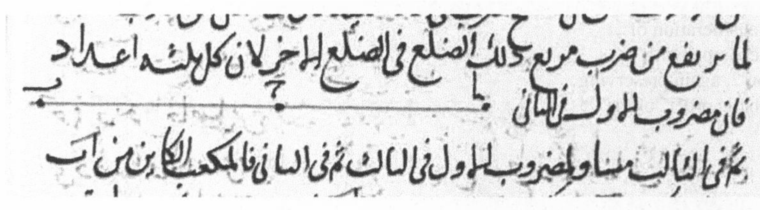

Fig. 5 Diagram accompanying proposition 3 in Aya Sofya 2718

If we multiply AB by its square, we obtain its cube. So the cube of AB is equal to the product of AB by the square of AC, the square of CB and the product of AC by CB taken twice.
The product of AB and any number is equivalent to the product of AC and CB by this number, as Euclid explained in the first proposition of Book 2.
So the product of the square of AC by AC and by CB, and the product of the square of CB by CB and by AC, and the product of double the surface encompassed by AC and CB by AC, and the product of this also by CB, is equivalent to the cube of AB.
(So) the product of the square of AC and AC, that is the cube of AC, and the product of the square of AC by CB [and the product of the square of CB and CB, that is the cube of CB],[13] and the product of the square of CB by AC and the product of double the surface that is encompassed by AC and CB by each one of AC and CB [is the cube of AB].[14]

But, for any surface, if we multiply the surface by one of its sides, then the result from this product is equal to that which arises from the product of the square of that side by the other side, since for any three numbers, the product of the first by the second and then by the third is equal to the product of the first by the third then by the second.
Thus, the existing cube[15] of AB is equal to the cube of AC and the cube of CB and the product of AC by the square of CB taken three times and the product of CB by the square of AC taken three times, and this is what we wished to demonstrate.

2.2.8 Mathematical commentary

Al-Samaw'al's AB notation and the reference to Euclid (*Elements* II,1) both suggest geometric thinking. His simple diagram in Fig. 6 is the same diagram that Heron uses in his proofs 'without a diagram' for the propositions in *Elements* II. It is possible that al-

[13] There is clearly a problem with this section of the text. We propose to emend it as we have indicated in accordance with mathematical sense. This problem is not mentioned in the edition (Ahmad and Rashed 1972), but is implicitly corrected in Rashed's symbolic paraphrases in Rashed (1972, p. 4) and (1994, pp. 64–65). See also the next footnote.

[14] This scribe appears to have omitted some terms here as well. We have corrected the text in line with the corresponding step from proposition 4, adding a missing term, the cube of CB, as well as a reminder that the whole sum comes to the cube of AB.

[15] The Arabic corresponding to this phrase is الكعب الكائن.

Fig. 6 Transliteration of diagram accompanying proposition 3. The diagram for proposition 4 is similar

Samaw'al would have been familiar with these proofs from al-Nayrizi's commentary on Euclid's *Elements* (Lo Bello 2009, p. 26).[16]

We offer a single paraphrase this time, replacing al-Samaw'al's line segments by algebraic symbols and using symbolic algebra. To stay close to his notation, we have replaced the lines AC, CB and AB by a, b and $a+b$, respectively.[17]

Paraphrase: I say that $(a+b)^3 = a^3 + b^3 + 3(ab^2 + ba^2)$.
Demonstration We already know that

$$(a+b)^2 = a^2 + b^2 + 2ab.$$

Multiplying by $a+b$ we get

$$(a+b)^3 = (a+b)(a^2 + b^2 + 2ab).$$

For any number x we have $(a+b)x = ax + bx$, as Euclid explains in *Elements* (II,1). So we get

$$a^2a + a^2b + b^2b + b^2a + (2ab)a + (2ab)b = (a+b)^3$$

or

$$a^3 + a^2b\,[\,+b^3\,] + b^2a + (2ab)a + (2ab)b\,[\,= (a+b)^3]$$

But $(xy)x = x^2y$ since for any three numbers p, q, r we have $(pq)r = (pr)q$. Thus

$$(a+b)^3 = a^3 + b^3 + 3ab^2 + 3ba^2$$

which is what we wished to demonstrate.

2.2.9 Al-Samaw'al's proposition four

(When) any number is divided into two parts, then the square square of the divided number is equal to the square square of each one of the two parts and the product of

[16] The point of Heron's proofs seems not to be to dispense with geometric thinking itself, since the diagram is still clearly geometric. Rather, it seems that Heron wishes to transfer the bulk of the calculation to the reader's imagination, leaving just a divided line to record to essential relationships. This suits al-Samaw'al's purposes particularly well here, since he conjures up calculations beyond the reach of plane geometry, involving third and fourth powers.

[17] An intermediate paraphrase, in the style of our other first paraphrases, may be obtained by reversing these substitutions. Thus, the statement would become $ab^3 = ac^3 + cb^3 + 3(ac.cb^2 + cb.ac^2)$ and so on.

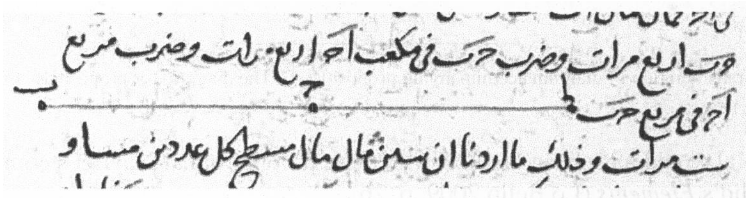

Fig. 7 Diagram accompanying proposition 4 in Aya Sofya 2718

each one of the two parts by the cube of the other part taken four times and the product of the square of one of them by the square of the other taken six times (Fig. 7).
Its example: When a number AB is divided into two parts, and they are AC and CB (see the manuscript diagram in Fig. 7), then the square square of AB is equal to the square square of AC and the square square of CB and the product of AC by the cube of CB taken four times, and the product of CB by the cube of AC taken four times, and the product of the square of AC and the square of CB taken six times.

Its demonstration: Indeed, the square square[18] of AB is the product of AB by its cube, and we explained in the previous proposition that the cube of AB is equal to the cube of AC and the cube of CB and the product of AC by the square of CB taken three times and the product of CB by the square of AC taken three times. The product of AB by any number is equal to the product of that number by AC and by CB, therefore, the product of the cube of AC by AC, which is the square square of AC, and by CB, and the product of the cube of CB by CB, which is the square square of CB, and by AC, [and the product of the surface of the square of AC by CB taken three times by AC and by CB][19] and the product of the surface of the square of CB by AC taken three times by AC and by CB, is the square square of AB.
But, three times the product of the surface of the square of AC by CB by AC [is equal to] three times the product of the cube of AC by CB. Similarly, three times the surface of the product of the square of AC by CB by CB [is equal to] three times the product of the square of AC by the square of CB. Similarly, three times the product of the surface of the square of CB by AC by AC is equal to three times the product of the square of AC by the square of CB. And three times the product of the surface of the square of CB by AC by CB is equal to three times the product of the cube of CB by AC.

Therefore, the square square of AB is equal to the square square of AC and the square square of CB, and the product of AC by the cube of CB taken four times, and the product of CB by the cube of AC taken four times, and the product of the square of AC by the square of CB taken six times, and this is what we wished to demonstrate.

[18] At this point in the proposition, al-Samaw'al switches to calling the fourth power by the 'arithmetical' term مَال مَال (*māl māl*), instead of the 'geometrical' term مربع مربع (*square square*) used in the statement of the proposition, even though he retains the term for *square* for the second power.

[19] For mathematical sense, we have inserted a term. Due to the fact that this phrase is almost identical to the last, it seems plausible there has been some omission due to scribal oversight.

2.2.10 Mathematical commentary

In this proposition, al-Samaw'al has retained the geometric notation used in proposition 3. Thus, he repeats his diagram from the previous proposition (Fig. 6) and AB once again seems to signify a line segment. On the other hand, it is possible that his switch part-way through the proof to the 'arithmetical' term *māl māl* for the fourth power is an acknowledgement that *square squares* are hardly geometric. Our paraphrase in terms of symbolic algebra uses the same conventions as in that proposition.

Paraphrase: I say that $(a+b)^4 = a^4 + b^4 + 4ab^3 + 4ba^3 + 6a^2b^2$.
Demonstration We know that

$$(a+b)^4 = (a+b)(a+b)^3$$

and we saw in proposition 3 that

$$(a+b)^3 = a^3 + b^3 + 3ab^2 + 3ba^2$$

and we know that for any number x we have $(a+b)x = xa + xb$. Therefore

$$a^3a + a^3b + b^3b + b^3a + \ [(3a^2b)a + (3a^2b)b] \ + (3b^2a)a + (3b^2a)b = (a+b)^4$$

where $a^3a = a^4$ and $b^3b = b^4$.
But $(3a^2b)a = 3a^3b$. Similarly $(3a^2b)b = 3a^2b^2$. Similarly $(3b^2a)a = 3a^2b^2$ and $(3b^2a)b = 3b^3a$. Thus

$$(a+b)^4 = a^4 + b^4 + 4ab^3 + 4ba^3 + 6a^2b^2$$

which is what we wished to demonstrate.

2.2.11 Al-Samaw'al's proposition five

The *māl māl* of the surface of any two numbers is equal to the surface of the *māl māl* of each of them.
Let the numbers be the two numbers a and b and their surface be the number c. Then, I say that the *māl māl* of c is equal to the product of the *māl māl* of a by the *māl māl* of b.
Its demonstration: a is multiplied by itself to get d and a is multiplied by d to get e and e is multiplied by a to get f, then f is *māl māl* of a.
Let b be multiplied by itself to get g and let g be multiplied by b to get h and h be multiplied by b to get i, then i is *māl māl* of b.
c is multiplied by itself to get j and j is multiplied by c to get k and multiply k by c to get l, then l is *māl māl* of c (see the manuscript diagram in Fig. 8).
Then, I say that l is equal to the surface of f and i.

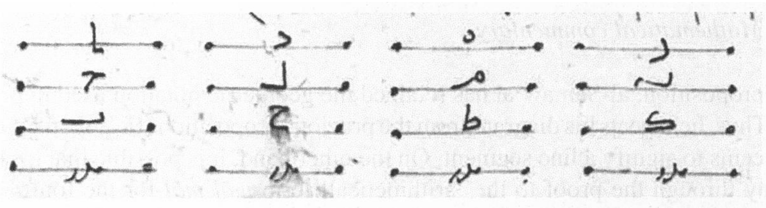

Fig. 8 Marginal diagram accompanying proposition 5 in Aya Sofya 2718 (rotated 90° clockwise). The last row of line segments is labeled by a repeated Arabic word–perhaps *qadr* meaning magnitude—whose significance is unclear to us

Therefore, because the two sides of c are the two numbers a and b, and the two sides of j are the two numbers d and g, the squares, we obtain that the ratio of the surface c to the surface j is compounded[20] of the ratio of a to d and the ratio of b to g.
But the ratio of a to d is equivalent to the ratio of d to e because the two numbers a and d multiplied by a produce d and e.

And the ratio of b to g is equivalent to the ratio of g to h because the two numbers b and g multiplied by the number b produce g and h.
[And the ratio of c to j is equivalent to the ratio of j to k because the two numbers c and j multiplied by the number c produce j and k].[21]
Therefore, the ratio of [j to k][22] is compounded of the ratio of d to e and the ratio of g to h.
But j is the surface of d and g, so k is the surface of e and h.
This is evident from the converse of proposition 5 from Book VIII of *The Elements*.[23]

But, the ratio of d to e is equivalent to the ratio of e to f and the ratio of g to h is equivalent to the ratio of h to i.
And the ratio of j to k is equivalent to the ratio of k to l.
Therefore, the ratio of k to l is compounded of the ratio of e to f and the ratio of h to i.
But when e is multiplied by h we get k, so when f is multiplied by i we get l, and this is what we wished to demonstrate.

2.2.12 Mathematical commentary

First paraphrase: using modern symbols, but retaining his argument in terms of ratios. Once again, referring to his diagram (Fig. 9) may help, as we shall see in Sect. 3.2.

Let the two numbers be a and b, and let their product be c.
Then I say that $c^4 = a^4 b^4$.

[20] مؤافه من نسبه is a standard term for compounding ratios; al-Samaw'al wishes to compound these ratios in the sense of *Elements* VIII,5.

[21] The mathematical argument is incomplete at this point. We have inserted this sentence, and modified the following ratio, so that the argument conforms to the similar argument in the next paragraph.

[22] As indicated in the previous footnote, we have replaced the text's *the ratio of c to j* by *the ratio of j to k*.

[23] *Elements* VIII,5: *Plane numbers have to one another the ratio compounded of the ratios of their sides.*

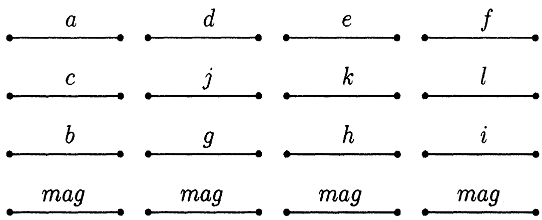

Fig. 9 Transliteration of marginal diagram accompanying proposition 5, rotated clockwise 90°, and once again preserving the 'alphabetic order' of the labels. The line segments in the bottom row have been labeled *mag* to signify the conjectural translation *magnitude*

Demonstration Let $a^2 = d$ and $ad = e$ and $ae = f$, so that $f = a^4$.
And let $b^2 = g$ and $bg = h$ and $bh = i$, so that $i = b^4$.
And let $c^2 = j$ and $cj = k$ and $ck = l$, so that $l = c^4$.
Then I say that $l = fi$.
Since $c = ab$ and $j = dg$, the ratio $c : j$ is the ratio compounded of the ratios $a : d$ and $b : g$ (*Elements* VIII,5).
But the ratio $a : d$ is equal to the ratio $d : e$ since a and d multiplied by a give d and e, respectively (*Elements* VII,17).
And the ratio $b : g$ is equal to the ratio $g : h$ since b and g multiplied by b give g and h.
[And, the ratio $c : j$ is equal to the ratio $j : k$ since c and j multiplied by c give j and k.]
Therefore, the ratio $[j : k]$ is the ratio compounded of the ratios $d : e$ and $g : h$.
But $j = dg$ so $k = eh$, by the converse to *Elements* VIII,5.[24]

But the ratio $d : e$ equals the ratio $e : f$,
and the ratio $g : h$ equals the ratio $h : i$,
and the ratio $j : k$ equals the ratio $k : l$.
Hence, the ratio $k : l$ is the ratio compounded of the ratios $e : f$ and $h : i$.
But $k = eh$ so we get $l = fi$, which is what we wished to demonstrate.

Second paraphrase: using symbolic algebra and replacing ratios by fractions.

Since $(ab)^2 = a^2b^2$, we know that

$$\frac{ab}{(ab)^2} = \frac{a}{a^2} \cdot \frac{b}{b^2}$$

But

$$\frac{a}{a^2} = \frac{a^2}{a^3} \text{ and } \frac{b}{b^2} = \frac{b^2}{b^3} \text{ and } \frac{ab}{(ab)^2} = \frac{(ab)^2}{(ab)^3}.$$

[24] In this context, *Elements* VIII,5 says that if $j = dg$ and $k = eh$ then the ratio $j : k$ is the ratio compounded of the ratios $d : e$ and $g : h$. The converse needed here would say that, if the ratio $j : k$ is the ratio compounded of the ratios $d : e$ and $g : h$ and if $j = dg$, then $k = eh$. This converse is not in the *Elements*.

Hence

$$\frac{(ab)^2}{(ab)^3} = \frac{a^2}{a^3} \cdot \frac{b^2}{b^3}.$$

But we know that $(ab)^2 = a^2b^2$ so we must have $(ab)^3 = a^3b^3$. Now

$$\frac{a^2}{a^3} = \frac{a^3}{a^4} \text{ and } \frac{b^2}{b^3} = \frac{b^3}{b^4} \text{ and } \frac{(ab)^2}{(ab)^3} = \frac{(ab)^3}{(ab)^4}.$$

Hence

$$\frac{(ab)^3}{(ab)^4} = \frac{a^3}{a^4} \cdot \frac{b^3}{b^4}.$$

But we know that $(ab)^3 = a^3b^3$ so we must have $(ab)^4 = a^4b^4$.

Notice that the first half of this proof gives an alternative proof of proposition 2: $(ab)^3 = a^3b^3$.

2.2.13 Al-Samaw'al's on cases $\mathbf{n} = \mathbf{5}$ *and higher*

By the same method, it can be demonstrated that the *māl* cube of the surface of any two numbers is equal to the surface of the *māl* cube of one of them by the *māl* cube of the other, and so on in increasing order.

For a person who understands what we have done then that person can demonstrate that for any number divided into two parts the *māl* cube is equal to the *māl* cube of each of the two parts and the product of each one by the *māl māl* of the other one taken 5 times and (the product of) the square of each of them by the cube of the other taken 10 times, and so on for the next ascending terms.

2.2.14 Mathematical commentary

In symbolical terms, al-Samaw'al says that the same method (as in proposition 2 or 5, presumably) can be used to show that

$$(ab)^5 = a^5b^5.$$

Although it is not explicit, he also seems to claim that this same method will show the general result $(ab)^n = a^nb^n$ for $n = 6, 7, \ldots$.

Similarly, he says that anyone who has understood proposition 3 or 4 will also be able to show that

$$(a+b)^5 = a^5 + b^5 + 5(ab^4 + ba^4) + 10(a^2b^3 + b^2a^3)$$

as well as the corresponding results for higher powers. At this stage, it is not clear what those corresponding results are, but al-Samaw'al will tell us how to find out in the following paragraphs.

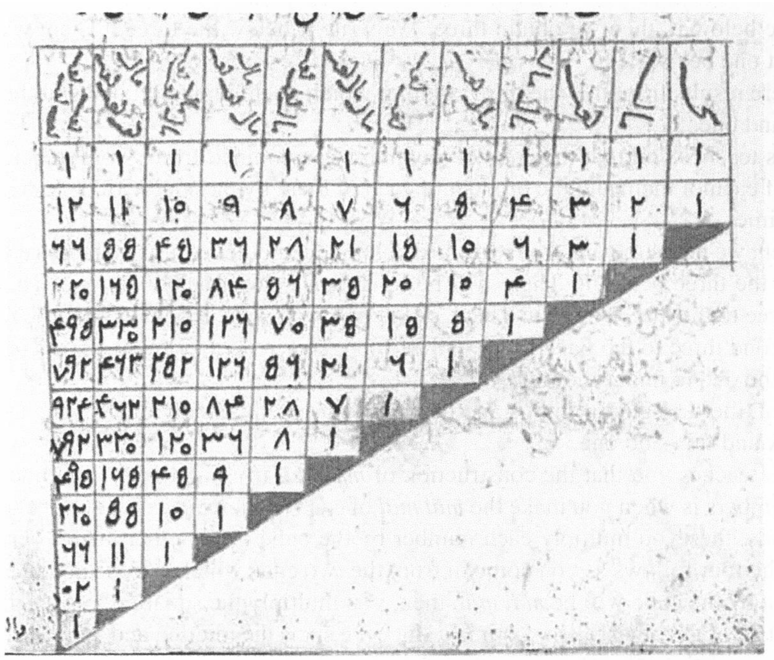

Fig. 10 Diagram accompanying statement of binomial theorem in Aya Sofya 2718

2.2.15 *Al-Samaw'al's on how to construct the table*

Let us now indicate a principle[25] for knowing the number of times that are necessary to multiply these powers[26] by each other for any number divided into two parts.
Al-Karajī says: in order to achieve that, you place on a board[27] one and one below it (see the manuscript diagram in Fig. 10).
Then, move the (first) one into another column and add the (first) one to the one (that was) below it, then there is two. Place it under it (the first one). Then, you place the last one below it.
Then, there results one and two and one.
This shows you that for any number combined from two numbers, if you multiply each of them by itself once, since the two ends are one and one, and if you multiply one of them by the other twice, since the middle term is two, there results the square of that number.
Then, we move the one from the second column to another column, and we add the one to the two. There results three and we write it under the one. We add the two to

[25] The Arabic word used here is أصلا.

[26] The Arabic word invoked here (*martaba* pl. *marātib*) means literally rank, grade, step, degree or levels, so it is likely that there is a reference at least to the ordering we now represent by powers of an algebraic symbol. See for instance Souissi (1969, p. 39–41) or Woepcke (1851, p. 6).

[27] This no doubt refers to the Arabic equivalent of a slate, a standard device for doing computations.

the one below it, then we obtain three. We write it below the three. [Then, we place the last one below it].

There results from this the third column, which is, individually: one and three and three and one.

This teaches you that the cube of any number combined from two numbers is the cube of each of them and the product of each of them by the square of the other taken three times.

Then, we move the one from the third column to another column. Then, we add the one to the three below it. There will be four. You write it below the one. Then, you add three to the three below it. There will be six. You write it below the four. You add the second three to the one. There will be four. You write it below the six. Then, you move the one to under the four.

Then, there results from this another column, which is, individually: one and four and six and four and one.

This teaches you that the construction of *māl māl* from a number combined from two numbers is when you make the *māl māl* of each of them, because of the one at the two ends, then you multiply each number by the cube of the other taken four times, since the four follows as you come in from the two ends which are one and one, since the root by the cube will be *māl māl*, then, you multiply the square of one of them by the square of the other taken six times, since the six is the middle, and since the square by the square is *māl māl*.

Then, you move the one from the fourth column into the fifth column. Then, you add the one to the four below it, and the four to six below it, the six to the four below it, and the four to the one below it. Then, you write down the results of that below the one that was moved into the aforementioned adjacent (column) and you write after this the remaining one.

We obtain from this the fifth column, its numbers: 1 and 5 and 10 and 10 and 5 and 1.

This teaches you that for any number divided into two parts, its *māl* cube is equal to the *māl* cube of each part, since the two ends are one and one, and the product of each of them by the *māl māl* of the other taken 5 times, since fives are next as you come in from the two end ones, and the product of the square of each one by the cube of the other taken 10 times, since the numbers 10 are next after the two fives. Each one from this group belongs to the type *māl* cube as the root by *māl māl* and the cube by *māl* both give *māl* cube. By this procedure, one knows the number of times for *māl* ing and cubing according to what result we want and here is its diagram (Table 2).

3 Discussion

3.1 Heritage versus development

This excerpt from *Al-Bāhir* is steeped in the tradition of classical Greek mathematics. This can be seen in its mode of exposition, its reliance on Euclidean results, its geometrical orientation, and the phrasing and grammatical conventions it often exhibits. But in many respects, it is also emblematic of the ways in which Islamic

Table 2 Table of the coefficients of the binomial expression $(a+b)^n$ with n from 1 to 12 in modern notation

x^{12}	x^{11}	x^{10}	x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x
1	1	1	1	1	1	1	1	1	1	1	1
12	11	10	9	8	7	6	5	4	3	2	1
66	55	45	36	28	21	15	10	6	3	1	
220	165	120	84	56	35	20	10	4	1		
495	330	210	126	70	35	15	5	1			
792	462	252	126	56	21	6	1				
924	462	210	84	28	7	1					
792	330	120	36	8	1						
495	165	45	9	1							
220	55	10	1								
66	11	1									
12	1										
1											

scholars were extending their results beyond this setting and propelling mathematics in entirely new directions. In this way, the work can be considered as epitomizing the transitional nature of mathematics during this period, in particular the blending of the domains of arithmetic and geometry. Indeed, given these new ambitions for mathematics, many of the imports from the original Greek setting become somewhat less useful or even incompatible, particularly those terms which have explicit geometrical meaning. Indeed, in al-Samaw'al's account, we can directly appreciate how the blending and synthesis of these concepts in new and unprecedented ways has given rise to novel mathematical insights, such as his table of binomial coefficients. In this section, we shall look at, on the one hand, al-Samaw'al's strict adherence to Euclidean conventions for demonstrations, and on the other hand, his insightful use of algebraic terms applied to geometry and arithmetic.

3.1.1 Heritage: the art of deduction

The influence of Greek geometrical practice is appreciable both explicitly and implicitly. Al-Samaw'al explicitly refers to Euclid's *Elements* only once in this extract (the reference to proposition 1 of Book II, during the demonstration of proposition 3) although it is used again without direct reference during the demonstration of his proposition 4.[28] However, al-Samaw'al's mode of expression harks back to Euclidean writing,[29] with each of his propositions following a structure similar to that which Proclus observed in his commentary on Book I of the *Elements*.[30] Thus, if we follow

[28] One other reference occurs during the demonstration of al-Samaw'al's proposition 5, where he mentions the converse of proposition 5 from Book VIII, a converse not mentioned by Euclid.

[29] This is typical of Islamic authors. For other examples which have been analyzed in similar ways, see, for instance, Oaks (2011) and the study of Abū Kāmil.

[30] We follow Heath's translation in Heath (1956, pp. 129–130). Netz gives an example of Proclus' divisions applied to a Euclidean proposition in Netz (1999, pp. 9–11).

proposition 1 as a typical example, al-Samaw'al's propositions begin with a statement of the general result in rhetorical form:

> (For) any four numbers, the product of the surface of the first and the second by the surface of the third and the fourth is equal to the product of the surface of the first and the third by the surface of the second and the fourth.

This corresponds to Proclus' *enunciation*, which is supposed to state what is given and what is sought, but which also (as Heath observes) does this in quite general terms. Next al-Samaw'al makes this general statement more concrete (this is Proclus' *setting out*) by assigning names to any quantities mentioned:

> Let us consider four numbers a, b, c, d, and let us multiply a by b to get e, and let us multiply a by c to get [f, and let us multiply b by d to get] g. Let us multiply c by d to get h.

As in the *Elements*, the names can be either single letters (as in propositions 1, 2 and 5) or paired letters representing a line segment (as in propositions 3 and 4). In propositions 3 and 4 al-Samaw'al labels this step *Its example*,[31] but there is no such indication in propositions 1, 2 and 5. Following this naming of quantities, al-Samaw'al then restates the claim of the proposition in terms of these named quantities, often opening with a characteristic phrase echoing Euclid's own usage, the use of the first person singular statement (our italics):

> *Then I say that* the product of e and g is equal to the product of f and h.

This corresponds to Proclus' *definition or specification* and Heath sees these last two steps as a way of better focusing the reader's attention (Heath 1956, p. 130). However, Netz (1999, Chapter 6) sees these first three steps as a way of stating a general result of potentially infinite scope (the enunciation) before reducing the task to proving a specific typical case (the setting out and its consequential definition).

Al-Samaw'al usually labels the next section as *Its demonstration*.[32] Here, typically, a chain of reasoning is built (Proclus' *demonstration or proof*[33]) culminating in a restatement of the original claim (Proclus' *conclusion*).

> Its demonstration: When the number a is multiplied (respectively) by the two numbers b and c, then there results from the multiplication the two numbers e and f. Then, the ratio of e to f is the same as the ratio of b to c. Moreover, when the number d is multiplied (respectively) by the two numbers b and c, then there results from the multiplication the two numbers h and g. Then, the ratio of h to g is the same as the ratio of b to c. We know that the ratio of e to f is the same

[31] Proclus' word to describe this stage is *ekthesis*. In Arabic, al-Samaw'al marks it using the word: *mithāl* (a noun that means 'example' or 'instantiation').

[32] Proclus' word to describe this section is *apodeixis*. In Arabic, al-Samaw'al marks it using the word *burhān* (a noun meaning demonstration, proof) from the verb *barhana*, to prove, demonstrate.

[33] For a discussion on the translation as *apodeixis* as 'demonstration' rather than 'proof' see Catton and Montelle (2012, p. 27ff).

as the ratio of b to c. Then, the ratio of e to f is the same as the ratio of h to g. Then, the surface[34] of e and g is equal to the surface of f and h.

Often the logical steps rely on propositions from Euclid's *Elements* but, following Euclid's own custom, this reliance is not usually explicit.[35] The completion of the demonstration is then announced in a phrase echoing Euclid's *what it was required to demonstrate*.[36]

So al-Samaw'al is in many ways emblematic of the indebtedness of the Arabic scholars to the Greek tradition of exposition and demonstration. There is no doubt that he and his predecessors were well versed in the deductive style of mathematics of Euclid from reading Arabic versions of the *Elements*.

3.1.2 Developments for algebraic powers applied to arithmetic and geometry

On the other hand, al-Samaw'al's treatment of algebraic products and powers is part of a story of continual revision and development, with his Arabic predecessors transforming and building on ideas inherited from Euclid and Diophantus.

Arabic mathematicians developed a notion of number in various ways from Greek conceptions, sometimes conflating the domains of geometry and arithmetic in both nuanced and direct manners. For instance, al-Khwārizmī (b. ca. 780) used what Høyrup (2002, p. 412) would call naive, or cut-and-paste, geometrical demonstrations to validate his algebraic procedures (Oaks 2011). Later, Thābit ibn Qurra (b. 836) would use Euclidean geometry to justify al-Khwārizmī's algebraic rules (Høyrup 2002, p. 412). On the other hand, Abu Kāmil (b. ca. 850) used both geometrical and arithmetical 'proofs' alongside one another (Oaks 2011). Typically, this dual approach offered geometrical demonstrations, in the style of al-Khwārizmī, for specific calculations (such as $\sqrt{9} \cdot \sqrt{4} = \sqrt{9 \cdot 4}$) but gave arithmetical demonstrations, in the style of Euclid's material on number theory, for general results (such as $\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$). Later still, al-Khayyām (b. 1048) was dedicated to the notion that number could only be a positive integer, and held firm to the Aristotelian conviction that quantity was either discrete or continuous. Consequently, for him, because algebra allowed both fractions and irrational roots, it belonged *in toto* to the domain of geometry (Oaks 2011, p. 62).

As we shall see, al-Samaw'al also retains remnants of Greek conceptions while adopting and extending the more developed notions of his direct predecessors. This is most noticeable in al-Samaw'al's conflation of numerical and geometrical magnitudes. To some extent, Euclid himself began this process with his definitions of plane and solid numbers and their sides in *Elements* VII. We see a similar usage by al-Samaw'al in proposition 5 above, where he says that *the two sides of* c *are the two numbers* a *and* b and then goes on to talk of *the surface* c (formed from a and b). But al-Samaw'al goes

[34] Notice that al-Samaw'al uses the word *surface* here as a synonym for the word *product* which appeared in the definition or specification earlier in the demonstration.

[35] For example, in this case, al-Samaw'al has used either *Elements* VI,1 or VII,17 to derive the equivalence of the ratios e to f and b to c. He has also used *Elements* V,11 and either VI,16 or VII,19.

[36] For example, see Heath (1956, p. 248).

Table 3 Al-Samaw'al's terminology for geometrical and algebraic terms applied to arithmetic

Arabic	English	Symbolic
ḍil', *shay'*, or *jadhr*	side, thing, root	x
māl, *murabba'*, or *majdhūr*	*māl*, square, radicand	x^2
muka''ab	cube	x^3
māl māl, or *muraba' muraba'*	square–square	x^4
māl muka'ab	square-cube	x^5

further. For instance, he uses two distinct terms to express the concept 'product.' The first is '*ḍarb*' which is from the verb 'to multiply', usually used in a general context, particularly arithmetic. The second, however, has geometric origins; '*musaṭṭaḥ*' which he uses to express the product literally translates as 'surface.' Despite these different origins, for al-Samaw'al these words are synonyms, as we see by the substitution of one for the other in the setting out and final conclusion of proposition 1 above. On the one hand, all such calculations yield numbers, as we see in proposition 2 where he multiplies *by the surface of* c *and* d, *I mean by the number* g. But on the other hand, the geometric meaning allows him to quote the geometric *Elements* II,1 in an arithmetic context like proposition 3.

Why keep two different words for what had become the same concept? Having two words for multiplication may have been helpful in following the rhetoric of the many repeated multiplications in these propositions. For example, in proposition 1, the careful use of both words in *the product of the surface of the first and the second by the surface of the third and the fourth* gives a structure to the statement that might not be so apparent if all the multiplications were simply *products*. A similar usage in proposition 4: *the product of the surface of the square of* AC *by* CB *by* AC, makes it clear that the choice of word was not driven by any geometric imagery. This use of language shows how transitional this way of mathematical thinking was, as al-Samaw'al and his contemporaries shifted to an orientation where the ties to geometry were increasingly less useful.

Al-Samaw'al's algebra is purely rhetorical and, in particular, he has no symbolism to denote the algebraic powers which appear in these propositions. For the reader's convenience, Table 3 summarizes al-Samaw'al's terminology for geometrical and algebraic terms applied to arithmetic, along with their English and modern symbolic equivalents.

When it comes to these higher algebraic powers, it is well known that Arabic mathematicians introduced new terminology[37] which had a greater degree of generality than ever before. The names of the simplest powers of an unknown quantity (number or length) have come down to us from al-Khwārizmī (2009, p. 96): the unknown quantities themselves are called *root*, *side* or *thing*, while an unknown quantity multiplied by itself is called *māl*[38] (an arithmetical term) or *square* (its geometric equivalent).

[37] See for instance, Berggren (2003, pp. 102–103) or (2007, pp. 542–544).

[38] In Arabic مال. See the column headers in Fig. 10.

Later writers expanded this vocabulary. Abū Kāmil (2012, pp. 444, 454, 478) uses *cube*[39] (a thing multiplied by a square), *square–square* (a square multiplied by itself), *cube–cube* (a cube multiplied by itself) and *square–square–square–square* (a square–square multiplied by itself). There does not, at this stage, seem to be any systematic generation of these powers, in that names seem to grow by concatenation as products appear in algebraic calculations, typically expansions of squared binomials. Laws involving products of these powers are certainly known, presumably from adding up the number of factors involved. For example, Abū Kāmil knows that multiplying *cube* by *square* gives the same as multiplying *square–square* by *thing* (Abū Kāmil 2012, p. 454), but this product does not seem to have a standard name. Such standard names seem to appear with the translation of Diophantus into Arabic. Thus, al-Karajī adopts Diophantus's name *square-cube* for the product just mentioned, but whereas Diophantus's powers were generated from the products mentioned in their names Diophantus of Alexandria (1893, pp. 2–5) (as with Abū Kāmil's names above), al-Karajī also defines his powers by the successive addition of another single factor *thing* (Woepcke 1982, p. 48). With this idea, he extends Diophantus's list (which went up to what we would now call the sixth power) so that it includes powers up to the ninth power, while retaining Diophantus's naming strategy of combining *square*'s and *cube*'s. Al-Samaw'al follows this tradition, and the top row of his table of binomial coefficients (Fig. 10) shows names that go up to the twelfth power. However, it is important to note that, with this naming system, the recursive relationship between successive powers is not visible numerically (as we might now see in the numerical index of an x^n, for example). Indeed, this relationship is effectively obscured by having the powers expressed as the appropriate combinations of *square* and *cube*. In the next section, we shall see how al-Samaw'al removes this obscurity with the help of diagrams and tabular arrays.

3.2 The role of diagrams

There are two different traditions of diagrams represented in the text. The first, with each number represented as a lettered line segment, is seen in propositions 1, 2 and 5 (see Figs. 1, 3, 8). This is arguably the most common in the Arabic tradition of diagrams and is in essence similar to Euclidean practice. In the second group of diagrams, an initial sum is represented as a partitioned and lettered line segment, but all other numbers involved in the proposition are omitted. These simple diagrams are seen in propositions 3 and 4 (see Figs. 5, 7). Such diagrams can be traced back to Heron's proofs 'without a diagram' in his commentary on book II of the *Elements*, which al-Samaw'al was possibly familiar with, as we said earlier, through al-Nayrizi's commentary (Lo Bello 2009). Al-Samaw'al's use of diagrams in this section thus is another example of the way in which he both follows and adapts earlier Greek usage (in particular Euclid's use of diagrams in the arithmetical books (VII to IX) of the *Elements*

[39] In Arabic مكعب. See the column headers in Fig. 10.

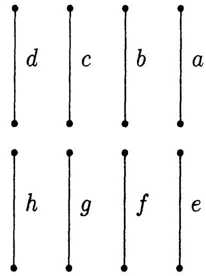

Fig. 11 Transliteration of diagram accompanying proposition 1. The *top row* represents four given numbers, while the *bottom row* represents the relevant pairwise products of these numbers

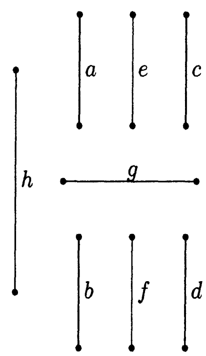

Fig. 12 Transliteration of diagram accompanying proposition 2. Powers of c and d, increasing from right to left, appear in the *top* and *bottom rows*, respectively

which he would have seen in Arabic translation).[40] Like Euclid, al-Samaw'al makes no comment about his diagrams and, in both cases, the diagrams appear at first sight to be an unstructured display of the numbers discussed in the relevant proposition. However, we shall see that some features of al-Samaw'al's diagrams reflect important relationships between these numbers and that similar structures can be found in some Euclidean diagrams. In the absence of an abstract symbolic notation for the unknown and its various powers, the alignment and 'flow' of such arrays function as a shorthand for the processes of reasoning, offering a comprehensive view of the mathematical relations of the proposition.

The diagram accompanying proposition 1 consists of two rows of lettered line segments (Fig. 11). The letters correspond to the various numbers discussed in the proposition, and as in Euclidean practice (Netz 1999, p. 71), they are named in alphabetic order as they arise during the demonstration. Thus, four arbitrary numbers, a, b, c and d, are listed in order in the top row, while the relevant products of pairs of these numbers, $e = ab$, $f = ac$, $g = bd$ and $h = cd$, are listed in the bottom row.[41] It is tempting to imagine a teacher drawing and labeling each line as each new number is encountered during the demonstration.

The diagram associated with proposition 2 seems to have been constructed in the same way (Fig. 12). Thus, the proposition claims that $(cd)^3 = c^3d^3$ and the diagram

[40] Heath (1956), and Euclid (1883–1916) before him, are not reliable witnesses to the manuscript tradition regarding diagrams. For the *Elements*, we shall rely on the witness of the Tehran manuscripts of the Elements in Arabic, as well as the more readily accessible Bodleian manuscript. For *Al-Bāhir*, we refer to the Aya Sofya 2718 manuscript since this is the only manuscript with completed diagrams.

[41] Just as in Arabic writing, these diagrams are usually read from right to left.

shows the numbers constructed as we go through the demonstration. Again, alphabetic order indicates the order in which the numbers are named or constructed. Thus, al-Samaw'al begins by naming the cubes ($a = c^3$ and $b = d^3$) and their corresponding sides (c and d). He then constructs the corresponding squares ($e = c^2$ and $f = d^2$) and finally constructs the products of the two sides ($g = cd$) and of the two cubes ($h = ab$).

However, this diagram seems to be not just recording these quantities but also highlighting the key relationships between them. Thus, the top and bottom rows list the increasing powers of the basic quantities c and d, and the columns seem to group them according to their degrees. For example, linear quantities (c and d) are shown in the first (right-most) column, quadratic quantities ($e = c^2$ and $f = d^2$) in the second column (from the right) and cubic quantities ($a = c^3$ and $b = d^3$) in the third column. The products ($g = cd$ and $h = ab$) are drawn at a level between the two rows, perhaps symbolizing a mixture, and in the same degree-based order, but their exact placement (with h further to the left and the line for g drawn horizontally instead of vertically) suggests a looser connection to the other components of the diagram. Thus, this diagram is carrying some of the information that nowadays we would record in the algebraic symbolism c^2, d^2, c^3, d^3, cd and $(cd)^3$.

By contrast, al-Samaw'al's diagrams for propositions 3 and 4 (the binomial expansions for $n = 3$ and $n = 4$) consist of a single line segment AB representing the number that has been divided into two parts AC and CB (Fig. 6). Such diagrams would probably have been familiar from Heron's commentary on book II of the *Elements* (Lo Bello 2009). Perhaps understandably there is no attempt to extend Euclid's partitioned square which illustrates his demonstration of the case $n = 2$ (*Elements* II, 4). The extension of algebra to higher powers has made such a diagram either difficult (for proposition 3) or impossible (for proposition 4) to draw. By using a divided line to record essential relationships, al-Samaw'al can transfer the bulk of the calculations (applications of what we would call the distributive law) to the reader's imagination, thus helping the reader through calculations which reach beyond plane geometry, involving third and fourth powers.

Al-Samaw'al's diagram for proposition 5 (which, in modern terms, asserts that $(ab)^4 = a^4b^4$) has a similar (but not identical) structure[42] to the one we saw for the related proposition 2 (which says that $(cd)^3 = c^3d^3$). Again the numbers are labeled alphabetically in the order in which they arise in al-Samaw'al's exposition. Thus, al-Samaw'al begins by naming the numbers (a and b) and their product $c = ab$. He then constructs the relevant powers of a (namely, $d = a^2$, $e = ad = a^3$ and $f = ea = a^4$), and of b (namely, $g = b^2$, $h = gb = b^3$ and $i = hb = b^4$), and of c (namely, $j = c^2$, $k = jc = c^3$ and $l = kc = c^4$). Notice that, by naming these numbers in this order, and then building this order into his diagram, he encodes the relationships between successive powers, and then displays these relationships in a way which helps the reader to 'see' the ordering which we would now encode using algebraic symbolism

[42] This time the whole diagram, letters as well as lines, has been rotated 90° to fit in the margin of the manuscript. Unlike the diagram for proposition 2, all the lines are horizontal, allowing the structure of the $c = ab$ row to follow that of the a and b rows. The blank space for this diagram in the Esat Efendi 3155 manuscript is a similar shape and so was presumably intended to hold a similarly rotated diagram.

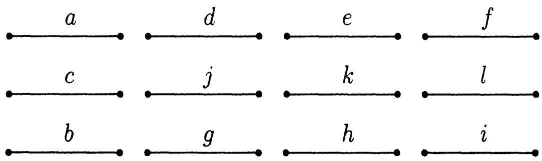

Fig. 13 Transliteration of marginal diagram accompanying proposition 5, rotated clockwise 90°. Powers of a, b and $c = ab$ appear in the *top*, *bottom* and *middle rows*, respectively. The seemingly irrelevant bottom row of Fig. 9 has been omitted

and our index notation for powers. This time though, unlike the situation we saw for proposition 2, the diagram is read from left to right (as we follow increasing powers of a, b and c).[43] Furthermore, the structure of the diagram is much tighter, with entries in each column all corresponding to the same power of a, b or c. Thus, the first and third rows list the increasing powers of the basic quantities a and b, while the second row lists the increasing powers of their product $c = ab$.

Diagrams like Figs. 12 and 13, where a two-dimensional array reflects a mathematical structure within an arithmetical proposition, have antecedents in the manuscript tradition of Euclid's *Elements*.[44] We shall content ourselves here with looking at two propositions (VII,27 and VIII,2) where the reader is offered such structured diagrams. As witnesses to the manuscript tradition, we shall use the Tehran *Elements* (xxxx) (as evidence of the Arabic tradition) and the Bodleian *Elements* (xxxx) (from the Greek tradition but more readily accessible than the Tehran manuscripts). Firstly, consider the diagram accompanying proposition VII,27: if two numbers are prime to one another, then so are their squares, and so are their cubes. Figures 14 and 15 show the relevant images from the above manuscripts, and Fig. 16 gives clearer drawings of each diagram.

The setting out of this proposition says[45]

> Let α, β be two numbers prime to one another, let α by multiplying itself make γ and by multiplying γ make δ, and let β by multiplying itself make ϵ, and by multiplying ϵ make ζ.

Thus, the proposition deals with numbers α, β and their squares and cubes. Euclid constructs the powers of α first, $\gamma = \alpha^2$ and $\delta = \alpha^3$, and then of β, with $\epsilon = \beta^2$ and $\zeta = \beta^3$. The relationship between these powers is displayed clearly in the Bodleian image. In Fig. 16 (*right*), these powers are displayed down the left and right columns

[43] This change of direction does not seem to be caused by the rotated orientation of the diagram, since the letters (and so the actual page) have been rotated too. Perhaps this demonstration, and its diagram, comes from a different source. This may help explain why al-Samaw'al offered two different demonstrations that $(cd)^3 = c^3d^3$ in propositions 2 and 5.

[44] This may come as a surprise for readers brought up on Heath's translation of Euclid's *Elements* (Heath 1956), or indeed for those familiar with Heiberg's critical edition (Euclid 1883–1916), but neither author seems to have been concerned to reproduce the original diagrams. For a discussion of why and how modern editions do not contain accurate diagrams see Saito and Sidoli (2012) or De Young (2005).

[45] For the reader's convenience, we use Heath's translation but use the letters from the Bodleian manuscript. This matches the original Greek lettering from Heiberg's edition, except that it uses lower case rather upper case letters.

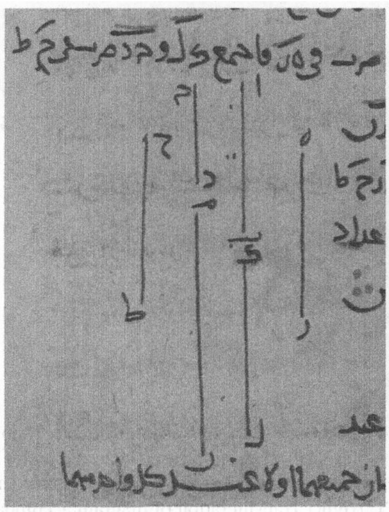

Fig. 14 Diagram from the Tehran Euclid's *Elements* VII, 27

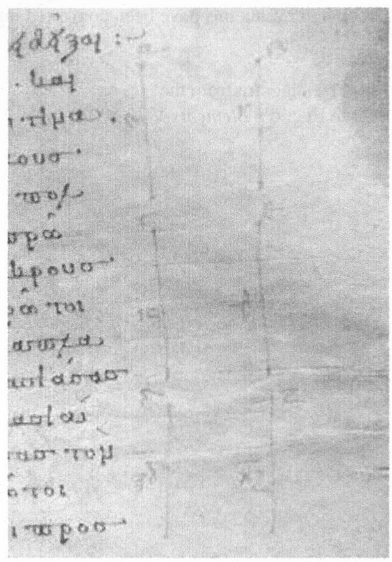

Fig. 15 Diagram from the Bodleian version of Euclid's *Elements* VII, 27 in Greek

(respectively) and the relationships are further strengthened by numerical examples (although these are almost certainly added by the scribal tradition since they are not referred to in the text). The diagram in the Tehran image, Fig. 16 (*left*), has a similar but looser structure, with the vertical flow broken somewhat by the squares being displaced outwards, as if the Bodleian diagram had been squashed vertically. In both cases, however, the structure of Fig. 16 closely parallels that in the first and third rows of Fig. 13, although this times the line segments are vertical and increasing powers go down the page.

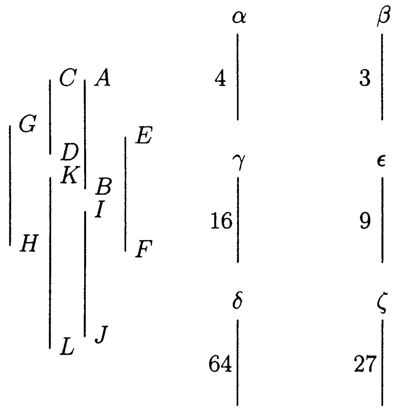

Fig. 16 *(left)* Diagram accompanying proposition VII, 27 in the Tehran *Elements*. The lines AB and CD represent given co-prime numbers, EF and GH represent their respective squares, and IJ and KL represent their respective cubes. The original lettering has been transliterated in accordance with the scheme in Table 1. (right) Transliteration of diagram accompanying *Elements* VII, 27 in the Bodleian. Powers of α and β appear in the *left* and *right columns*, respectively. The diagram appears twice in the manuscript, once in the margin of f.137r and then within a space set aside in the text at the end of the demonstration on f.137v. Greek letter numerals have been converted into modern Arabic numerals

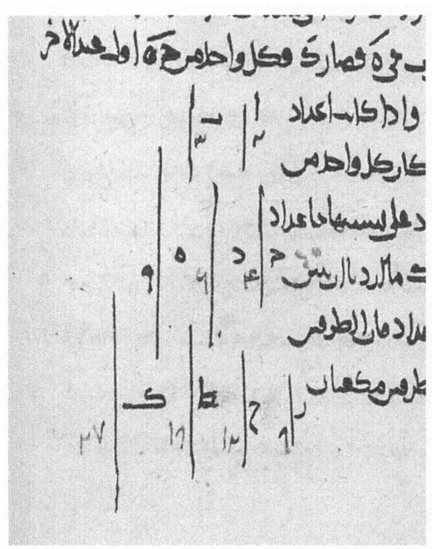

Fig. 17 Diagram from the Tehran Euclid's *Elements* VIII, 2

Our second example comes from *Elements* VIII,2 (to find numbers in continued proportion, as many as may be prescribed, and the least that are in a given ratio). Figures 17 and 18 show the relevant images from the above manuscripts. This time the diagrams are essentially the same and Fig. 19 gives a clearer drawing of the Bodleian diagram.

In this proposition, Euclid starts with two numbers α, β that are the least in the given ratio. He then constructs three numbers in the continued proportion $\alpha : \beta$, in modern terms the numbers

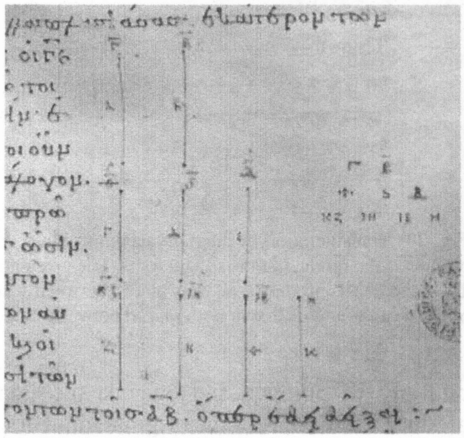

Fig. 18 Diagram from the Bodleian version of Euclid's *Elements* VIII, 2 in Greek

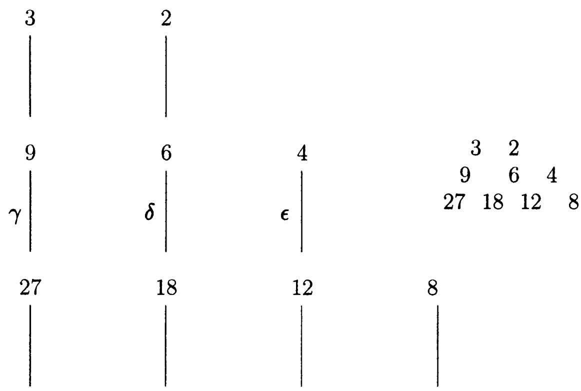

Fig. 19 Transliteration of diagram accompanying *Elements* VIII, 2 in the Bodleian Euclid (1883–1916, f.145r). The numerical example has been repeated in a small array to the right of the main diagram

$$\gamma = \alpha^2,\ \delta = \alpha\beta \text{ and } \epsilon = \beta^2.$$

Using these numbers, he then constructs four numbers in the same continued proportion, namely

$$\zeta = \alpha\gamma = \alpha^3,\ \eta = \alpha\delta = \alpha^2\beta,\ \theta = \alpha\epsilon = \alpha\beta^2,\ \text{and } \kappa = \beta\epsilon = \beta^3.$$

We see in Fig. 19 that the diagram records some of this structure, with the given numbers α, β in the first row, the three numbers γ, δ, ϵ in continued proportion in the second row, and the four numbers ζ, η, θ, κ, in continued proportion in the third row. Again numerical examples beside each line help to reinforce this structure. On the surface at least, this diagram has similar themes to Fig. 12 with powers and products displayed, but this time the rows of Fig. 19 represent separate sequences of numbers in continued proportion rather than a conscious attempt to represent degree structure.

Similar diagrams, highlighting two-dimensional relationships between numbers, were also used by al-Samaw'al's more immediate predecessors. For example, Fig. 20 shows a diagram used by Abū Kāmil in his Euclidean-style demonstration that if you

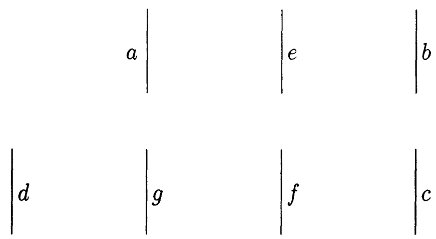

Fig. 20 Transliteration of diagram accompanying Abū Kāmil's demonstration that the square root of a product is the product of the square roots. The given numbers are a and b, and their respective square roots are d and c. The product $e = ab$ is placed between a and b, and its square root $f = \sqrt{ab}$, and the product of the roots $g = \sqrt{a}\sqrt{b}$, are both placed between the roots c and d. (Adapted from Oaks 2011)

multiply two numbers and then take the square root, then you get the same thing as the product of the roots of the original numbers (Oaks 2011). Once again, the structure of the diagram reflects some of the mathematical structure of the demonstration. Thus, (following our convention for lettering) the given numbers b and a appear in the first row, and their roots $c = \sqrt{b}$ and $d = \sqrt{a}$ appear in the second row. The product $e = ab$ appears between b and a, as it did in our earlier diagrams for propositions 2 and 5 (see Figs. 12, 13), while its root $f = \sqrt{ab}$ and the product of the roots $g = \sqrt{a}\sqrt{b}$ appear in the second row between the roots c and d.

Finally, we note that the triangle (or Pascal Triangle) can be considered to be a diagram too.[46] This diagram presents the reader with a dilemma, since it needs to be read in two different ways. Each column is constructed from top to bottom using data from the previous column but, to be used as al-Samaw'al intended, each column needs to be read from the outside ends inwards. Indeed, al-Samaw'al's statements for propositions 3 and 4 both follow the latter order[47]

$$(a+b)^3 = a^3 + b^3 + 3(ab^2 + ba^2)$$

and

$$(a+b)^4 = a^4 + b^4 + 4(ab^3 + ba^3) + 6a^2b^2$$

and it requires a special effort or insight to 'unfold' these expressions and see the columns 1, 3, 3, 1 and 1, 4, 6, 4, 1 given in the table. Of course, these orderings look obvious to us, at least partly because of our algebraic symbolism. In pre-symbolic times, it might not have been so obvious, with the rhetorical fluency of (the equivalent of) $a^4 + b^4 + 4(ab^3 + ba^3) + 6a^2b^2$ providing a counter-current to the influence of degree diagrams like the one discussed earlier. These competing views of the binomial

[46] See Dörfler (2004) where Dörfler argues for a broader meaning for the term 'diagram' and discusses the role of these more general 'inscriptions' in mathematical reasoning.

[47] Our modern algebraic symbolism is a little misleading here. For example, it forces us to identify the individual parts a and b when the rhetorical form finds a way of mentioning the symmetric roles of each part without naming either of them: *(When) any number is divided into two parts, then its cube is equal to the sum of the cubes of its two parts and the product of each of its parts by the square of the other part taken three times.*

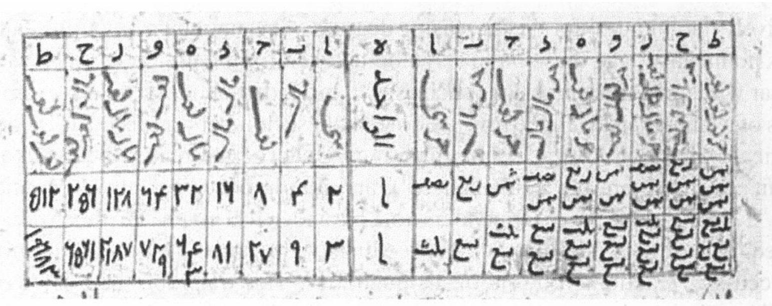

Fig. 21 Al-Samaw'al's table of powers in Aya Sofya 2718

Table 4 Al-Samaw'al's table of powers (Ahmad and Rashed 1972, p. 21)

7	6	5	4	3	2	1	0	1	2	3	4	5	6	7
māl	cube	*māl*	*māl*	cube	*māl*	*thing*	unit	part	part	part	part	part	part	part
māl	cube	cube	*māl*					*thing*	*māl*	cube	*māl*	*māl*	cube	*māl*
cube											*māl*	cube	cube	*māl*
														cube
128	64	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{4}\frac{1}{4}$	$\frac{1}{4}\frac{1}{8}$	$\frac{1}{8}\frac{1}{8}$	$\frac{1}{4}\frac{1}{4}\frac{1}{8}$
2187	729	243	81	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{3}\frac{1}{9}$	$\frac{1}{9}\frac{1}{9}$	$\frac{1}{3}\frac{1}{9}\frac{1}{9}$	$\frac{1}{9}\frac{1}{9}\frac{1}{9}$	$\frac{1}{3}\frac{1}{9}\frac{1}{9}\frac{1}{9}$

We have retained his rhetorical names for the powers. Thus, *thing* is what we would now call x, *māl* is x^2, *cube* is x^3 and so on. Reciprocals are indicated by *part*. We have also retained the factorizations in his numerical examples. Notice that in the case $x = 2$ these factorizations mirror the names of the powers. To save space, we have omitted the two outermost columns at each end of the table, corresponding to the 8th and 9th powers

expansions make al-Karajī's achievement all the more impressive. To construct the triangle, he not only had to notice that the column entries arose from calculations like $3 + 1 = 4$ (done twice) and $3 + 3 = 6$, done in no particular order, but also had to observe that following some variation of degree order (for example, making the rows of the triangle correspond to the degree of the second term) made those calculations follow a simple pattern.

As we have seen, al-Samaw'al's rhetorical rendering of his quantities (square, cube, square–square, and so on; see Table 3) does not easily indicate the various mathematical relationships between them (there is no 'obvious' mathematical relationship between the words 'cube' and 'square–square,' for instance). However, he does show an innovative attempt to present his quantities in a manner which demonstrates more clearly the mathematical relations between his objects. This is not by symbolic representation, but rather by diagrammatic arrays in which mathematical relations are embodied via the relative spatial positioning of their contents. This can be seen most distinctly in his diagram in the first chapter of the first section of *Al-Bāhir*, in which he presents a diagrammatic array of his unknown quantities, arranged in successive 'powers' (see Fig. 21; Table 4).

Along the top row are numbers, this time in *abjad* form: '0' is in the middle, and to the left and to the right, the numbers increase by one (up to 9). These are effectively the 'indices' for the increasing degrees of his unknown quantities, although he has no means to tag these on to a symbolic token for the unknown (such as we are familiar with today in expressions such as x^n). Instead, their relations are exhibited by means of a tabular array. In the row underneath these numbers are listed al-Samaw'al's rhetorical expressions for each of his quantities. Under zero is placed the word for 'one' or 'unit' (to represent our equivalent x^0), and then proceeding leftwards, under the glyph '1,' the Arabic *shay* (equivalent to x^1), under the glyph '2,' the Arabic word *māl* (equivalent to x^2) and so on. Proceedings rightwards, he lists his terminology for successive fractional powers of his quantities. This diagram, then, is directed at capturing relations between his quantities spatially. The relations are captured not by symbolic markers, but rather by relative positioning. Furthermore, one reasons not via numerical-symbolic manipulations, but rather diagrammatically.[48] As one moves to the left or right in the diagram, one detects and establishes relationships between the quantities, as al-Samaw'al himself explains to his readers when describing how to multiply these powers (Berggren 2003, p. 114).

This principle can also be applied to his diagrams in our passages in question. This can be seen most prominently in the table of binomial coefficients itself, the spatial arrangement and alignment of whose entries contain multiple mathematical relationships and can be 'read' in various directions. However, this principle is also invoked in the diagrams accompanying his propositions. Letters which symbolize the various quantities in al-Samaw'al's propositions have been placed in a diagrammatic array which preserves mathematical relations. For instance, in the diagram accompanying proposition 5 (Fig. 13) along the top line, we have presented the letters a, d, e, f, which, as the text reveals, are successive powers of a. This is the same in the second and third row, but rather with successive powers of c and b, respectively. In addition, reading vertically down the first column, we have a, c, b. However, c is also the product of a and b, so that the essence of the proposition (i.e., that $c^4 = (ab)^4$) is captured visually by the respective arrangements of the three rows. Therefore, the reasoning set out in the text is captured and can be followed diagrammatically in the array.[49]

Thus, al-Samaw'al offers a rendering of quantities and their relations which is in many senses more mathematically descriptive than the rhetorical means he invokes, but quite different to the symbolic representations of unknown quantities which are familiar to us. He represents relationships by spatial arrangement on the page, presenting quantities and their mutual relations diagrammatically so that they can be understood as one moves vertically or horizontally within the diagram. In the next section, we shall consider how these structured diagrams might have helped al-Samaw'al's readers to follow the rhetoric of his demonstrations.[50]

48 Within a century of so, Ibn al-Bannā' was offering a rule for calculating the 'exponent' of a term and observing that you add exponents when you multiply different terms together Ibn al-Bannā' (1969). We would like to thank the referee for pointing out this reference to us.

49 It is true that none of the above-mentioned authors (Euclid, or his Bodleian or Tehran copyists, or Abū Kāmil, or al-Samaw'al himself) makes any comment about this use of their diagrams. However, the consistency of practice across time and cultural differences makes it unlikely that the observed structure is accidental.

50 Of course, the structured diagrams discussed above may also have helped Euclid's and Abū Kāmil's readers (Oaks 2011) too.

3.3 Al-Samaw'al and mathematical induction revisited

After carefully setting out the five propositions, al-Samaw'al then goes on to describe the procedure for the construction of his table of binomial coefficients. But before he does, he makes some telling comments about generalizing the essence of these propositions for higher powers (see Sect. 2.2.13). This passage has been deemed especially significant (see Rashed 1972, 1994, pp. 63–68) because of the appearance here of the so-called 'Pascal Triangle' in a distinctly different and notably earlier context, but also for the appearance of a mode of reasoning closely related to mathematical induction—a style of proof which has traditionally been viewed as being first formalized by Blaise Pascal in 1654 (Acerbi 2000, p.57), five centuries after al-Samaw'al.

The search for antecedents to mathematical induction has led to much argument in the scholarly literature, with possible attestations being reported ever earlier. Thus, Vacca thought he recognized induction in the work of Maurolico in the sixteenth century (Vacca 1909; Bussey 1917; Ernest 1982) and Rabinovitch (1970) found it in the work of Levi ben Gerson in the fourteenth century. On the other hand, Freudenthal (1953) reasserted the primacy of Pascal and classified earlier methods as archaic precursors of the form of mathematical induction used by Pascal. Rashed (1972) extended Freudenthal's classification to take into account yet another precursor of induction that he found in this present extract from al-Samaw'al's *Al-Bāhir* in the twelfth century. Finally, several writers have debated the possible occurrence of induction in the ancient mathematical corpus, including in Euclid and Plato (Itard 1961; Fowler 1994; Unguru 1991, 1994; Acerbi 2000).

These disputes show that the quest to identify the first to articulate this mode of reasoning is somewhat futile, particularly given that many early attempts were bound to the context and conventions of their own cultures of inquiry in which the rigors of formal logic, symbolic styles of expression and abstract conceptions of number were either irrelevant or far from purview. Indeed, as Acerbi (2000, p. 58) succinctly put it, the issue became to some extent historiographical: '...every single scholar sets up his own reading of the principle of [complete induction], and on this basis he is able to affirm or to deny that specific proofs constitute well formed examples of it.' Acerbi then rather delightfully introduced his own contributions on the issue as 'adding to the confusion,' rather than trying to settle the matter. More broadly his comments touch upon a wider systemic issue in histories of mathematics: investigations which prioritize locating the *first* instance of an important mathematical principle, rule or concept are likely to encounter difficulty defending their resulting position.[51] In light of this, then, and taking our cue from Acerbi, we seek to explore what al-Samaw'al's aim in this section of *Al-Bāhir* was and the extent to which his mode of reasoning has affinities with the more formal process of mathematical induction.

Indeed, there has been a proliferation of ways in which to identify mathematical induction in historical sources. After much scholarly discussion, this type of reasoning is generally realized as a spectrum of techniques which include at one end the method

[51] For the historiographical implications surrounding the issues of priority see, for instance, Clark and Montelle (2012).

of generalizing examples and *incomplete induction*[52] through to formal mathematical induction at the other. Rashed (1994), building on the work of Freudenthal (1953), highlights three precursors of mathematical induction which he labels R_1, R_2 and R_3. Two of these come from Freudenthal (see Rashed 1994, p. 73). Thus, Rashed's R_2 is Freudenthal's *quasi-general* method of proof, where the proof would be valid for any positive integer n but is, in fact, given only for one specific value of n. Similarly Rashed's R_3 is Freudenthal's *regression*, where the same argument is repeated for each successively smaller integer until the smallest case is reached (again this method typically starts with a specific integer, but the argument is thought of as applying more generally). Rashed expands this catalog by introducing R_1, a method which deals with specific values of n but attempts to prove the transition n to $n+1$ in a uniform way that does not depend on n (Rashed 1994, p. 76). Rashed argues that al-Samaw'al uses this last form of reasoning in the section of *Al-Bāhir* which we have been discussing, and in this section, we wish to examine this claim.

If we express his propositions in algebraic form (to help modern readers appreciate the overall structure) then al-Samaw'al proves the following results:

1. $(ab)(cd) = (ac)(bd)$
2. $(cd)^3 = c^3d^3$
3. $(a+b)^3 = a^3 + b^3 + 3(ab^2 + ba^2)$
4. $(a+b)^4 = a^4 + b^4 + 4(ab^3 + ba^3) + 6a^2b^2$
5. $(ab)^4 = a^4b^4$ He then states the fifth-power equivalents of the last two propositions:
6. $(cd)^5 = c^5d^5$
7. $(a+b)^5 = a^5 + b^5 + 5(ab^4 + ba^4) + 10(a^2b^3 + b^2a^3)$

saying that they can be proved, using the same methods, by anyone who has understood those earlier demonstrations. In each case, the assertion concludes with a phrase like *'and so on,'* or *'in increasing order'*. The meaning of this phrase is reasonably clear in the case of proposition 5, but for proposition 4 al-Samaw'al needs to indicate how those 'coefficients' (3, then 4 and 6, then 5 and 10) can be calculated. The ensuing description shows how to construct and use the entries in his triangle, but it does not give any further explanation as to why these are the right numbers to use.

It seems likely that al-Samaw'al saw this section of material as a coherent unit, so the order of the material may hold clues to his way of thinking. In particular, the way he interleaves the two themes, $(ab)^n = a^nb^n$ and the binomial expansion of $(a+b)^n$, suggests that he may have been at least as interested in the general methods of demonstration as in the results themselves.

Al-Samaw'al's demonstrations of propositions 2 through to 4 all deal with a transition from a known case of a particular n to the next one $n+1$. proposition 5 is slightly different in that it proceeds from a known case $n = 2$ through a case $n = 3$ (which he has already proven a different way in proposition 2) and on to the case $n = 4$. This might show that he was interested not just in the particular instance covered in the

[52] In the terminology of Rashed (1994, p. 80) and Acerbi (2000, p. 60, footnote 13). This seems to be the induction of the philosophers, where from a few cases you hope that you can draw a general conclusion. Rashed and Acerbi refer to mathematical induction as complete induction.

proposition, but rather its connection to other instances. Al-Samaw'al's assertions that the $n = 5$ case of proposition 5 can be proved *'by the same method'*, and that this can be done *'and so on in increasing order'*—and similar assertions about the $n = 5$ case of proposition 4—mean that he could see some similarity between the way the case $n = 4$ was proved (from the case $n = 3$ perhaps) and the way to prove those higher cases.

However, there are some steps that al-Samaw'al remains silent on which makes us less sure of the emphasis of this passage. For instance, he does not make a similar assertion after proposition 2, and he makes no comment about achieving another demonstration of proposition 2 while proving proposition 5, and indeed, he makes no comment about the similarity between the step $n = 2 \rightarrow n = 3$ and the step $n = 3 \rightarrow n = 4$ within the demonstration of proposition 5. This may indicate that there is some room for doubt about his intent.

Furthermore, while the general idea of multiplying the expansion for $(a + b)^{n-1}$ through by $(a + b)$ and expanding is clear enough, it is not clear in what sense al-Samaw'al has achieved a proof of the induction step for the binomial theorem. First and foremost, al-Samaw'al lacks a way to express the mechanics of recognizing like terms and gathering them in the general case. Even for proposition 4 (the expansion of $(a + b)^4$) just recognizing all the like terms, which we can accomplish in a single line

but $(3a^2b)a = 3a^3b,\quad (3a^2b)b = 3a^2b^2,\quad (3b^2a)a = 3a^2b^2$ and $(3b^2a)b = 3b^3a,$

takes him half a dozen lines of rhetoric.

Both of al-Samaw'al's demonstrations for $(ab)^3 = a^3b^3$ (in his demonstrations of propositions 2 and 5) can be adapted to construct a fully general induction step. But the lack of an algebraic notation for expressing the general statement seems to render this step out of reach for al-Samaw'al. However, al-Samaw'al may have achieved this in an alternative fashion. That is, his accompanying diagram, and the way in which the reader is guided through the array, offer a visual representation of both the basic mathematical relationships and the method of progress from the case $n = 2$ to the case $n = 3$. Furthermore, the diagram does this in a way that allows the perceptive reader to 'see' what the general induction step would look like and how it would proceed. We explore this next.

3.3.1 Mathematical induction via diagrammatic reasoning?

Algebraic symbolism can be a powerful aid to creative thinking in mathematics. In the absence of such symbolism, diagrams can sometimes play a similar role (see for example Grosholz 2007; Dörfler 2004; Netz 1999; Macbeth 2010). Without any symbolic algebra, it might seem a tall order for al-Samaw'al's readers to *'understand what we have done'* and see what the so-called *'same method'* might be (we urge the reader to read our literal translations in Sect. 2.2 without the help of the paraphrases). However, let us look a little closer at the diagrams accompanying the demonstrations of propositions 2 and 5, and particularly at how these diagrams might interact with the demonstrations.

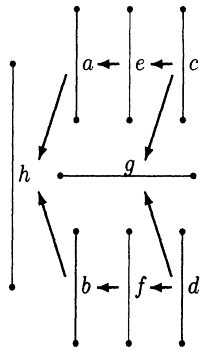

Fig. 22 Relationships between quantities in proposition 2

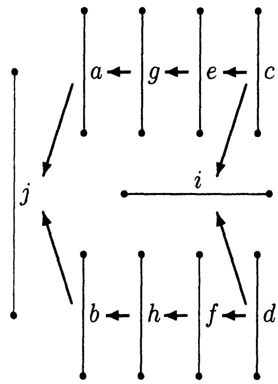

Fig. 23 Possible diagram for proving the $n = 4$ case of proposition 2 with key relationships indicated

We begin with the diagram for proposition 2. The defining relationships for the quantities in the diagram could be represented by the arrows in Fig. 22.

Thus, the arrows along the two rows represent the generation of successive powers of c and d, while the diagonal arrows represent the construction of the products $g = cd$ and $h = ab$. At the start of the demonstration, al-Samaw'al reminds us of a known fact, another relationship between some the quantities in the diagram: $ef = g^2$. His demonstration is basically a process of getting from this relationship to the next case, namely $ab = g^3$. To function as an induction step, or even as an example of Rashed's R_1 mode of reasoning (Rashed 1994, p. 76), this process needs to be general, in the sense that it is independent of the actual case under discussion. What is the process? It consists of three steps:

1. multiply both quantities (here ef and g^2) by cd or (equivalently) g
2. use proposition 1 to rearrange the resulting equality (giving $(ec)(df) = g^3$)
3. recognize the bracketed terms as the appropriate powers of c and d.

We claim that the diagram helps the reader to make this final identification step and to see what would be needed to use the *'same method'* to prove the next case. Thus, to show that $(cd)^4 = c^4d^4$ the reader could use a diagram as in Fig. 23, where quantities would be defined in a similar order to the ones in Fig. 12 so that e and f are the squares of c and d, and g and h are the cubes, while $i = cd$ and $j = ab$.

But the fact that al-Samaw'al actually offers second demonstration as part of proposition 5 may indicate that it was not so easy to see how the demonstration of proposition

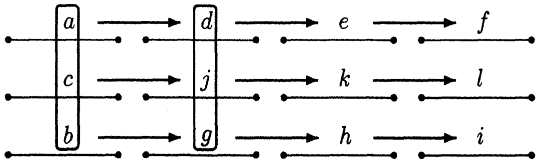

Fig. 24 Basic relationships in the diagram accompanying proposition 5

2 generalizes to the higher powers. Perhaps the first two steps above were harder to see in the diagram.

On the other hand, as we shall now see, his demonstration of proposition 5 can be carried out more or less entirely within its diagram. As with the diagram for proposition 2, the ordered labeling of the diagram imposes a structure on the diagram which reflects some of the mathematical structure of the setting out. Thus, multiplying a and b gives c, which we can think of as a vertical relationship within the first column of the diagram. Next the powers of these three numbers are given by d, e, f and g, h, i and j, k, l, respectively, a list which imposes a horizontal flow along the rows of the diagram, with powers increasing as we move from left to right (See Fig. 24).

The demonstration begins by recalling two relationships $c = ab$ and $j = dg$ that tell us about the structure of each of the first two columns of the diagram. Again, if this demonstration is to function as an induction step, even at the level of Rashed's R_1, then the process of going from $j = dg$ (or $(ab)^2 = a^2b^2$) to $k = eh$ and onwards to $l = fi$ needs to be in some sense independent of whether it started at $j = dg$ or $k = eh$ and so on. As with the demonstration of proposition 2, we can follow the process in the diagram. Thus, the first step uses *Elements* VIII,5 to express the ratio $c : j$ (the first step along the middle row) as the compound of the ratios $a : d$ and $b : g$ (the first steps along the two outer rows). But multiplication by a, b and c carries us along the upper, lower and middle rows (respectively) in sequences which are in continued proportion. Hence, $j : k$ (the second step along the middle row) must be the compound of the ratios $d : e$ and $g : h$ (the second steps along the two outer rows). But we know that $j = dg$, so the converse of *Elements* VIII,5 tells us that $k = eh$. This ought to complete the induction step, but al-Samaw'al repeats the process to show that $l = fi$. Perhaps he wanted to show that the same process can indeed be used and that it just corresponds to moving over one column in the diagram. At any rate, it is now clear how the rule can be extended to higher powers, and it is the structure of the diagram which makes this clear.[53]

3.4 Concluding remarks

The passage we have considered here from al-Samaw'al's *Al-Bāhir* has been singled out by past studies because of its inclusion of a table of binomial coefficients and the related mathematical reasoning which has similarities to mathematical induction. However, through a careful, complete, and historically sensitive examination of the whole passage in its entirety, we have revealed that there is more significance to

[53] We would like to emphasize that we are not claiming that al-Samaw'al is using the form of mathematical induction that was first formulated by Pascal, but rather that his diagrams represent a way in which he could convey to his readers the similarity of all the proofs $n = k \rightarrow n = k + 1$.

al-Samaw'al's exposition than being a more-or-less casual expression or informal anticipation of mathematical induction.

Previous accounts of this passage have focused solely on al-Samaw'al's text, ignoring the accompanying diagrams and sometimes relying on modern paraphrases to help the reader follow al-Samaw'al's arguments. Our literal translation of the whole passage brings out the rhetorical nature of al-Samaw'al's exposition and highlights the very real difficulties faced by his readers as they attempted not just to follow the arguments but also, as bidden by al-Samaw'al, to extend the same arguments to more general situations. Looking through modern eyes, as in our paraphrases, it is easy for us to see embryonic arguments using mathematical induction which would achieve the generalizations claimed by al-Samaw'al, but looking solely at the text, it is difficult to see how even al-Samaw'al himself could have done this.

We argue that key to al-Samaw'al's line of reasoning are his structured diagrams, which have evolved from the Euclidean tradition, but which are used with a slightly different intent in this context. We have seen that both al-Samaw'al's presentation and the content of his propositions are also clearly inspired by Greek mathematics. But we have also revealed how al-Samaw'al's mathematical ambitions have outgrown the Euclidean geometric context, now contemplating four-dimensional, five-dimensional or even higher dimensional products. Despite this though, al-Samaw'al remains committed to retaining some features of this mode of exposition (such as referring to products as 'surfaces,' or using propositions about parallelogram areas to justify higher dimensional calculations) even though, at first sight, they seem no longer useful or maybe even unhelpful to him.

In particular, we have advanced the notion that al-Samaw'al's diagrams are not just intended to be representative of a specific mathematical relation, but also prescriptive of a process of reasoning. Although al-Samaw'al's text includes a method closer to that of generalizing examples than a formal account of inductive reasoning, we argue that his diagrams more immediately concern the latter. The active movement which the diagrams compel the reader to engage in, in both a horizontal and vertical direction, may be emblematic of a process he wants to model for extending his results to higher powers in a manner similar to an inductive step, albeit achieved diagrammatically. Alphabetic ordering, alignment and mutual arrangement of elements in a diagrammatic array all seem directed to this aim. Indeed, where notational symbolism could not reveal the inherent relationships going from one 'power' of the unknown to the next, relative positioning in the diagram could. Therefore, reading and interpreting both text and diagram as an integrated whole in this way gives us new insight, a more complete picture of al-Samaw'al's intentions and a better understanding of how rhetorical mathematics could make progress in the absence of the power of algebraic symbolism. We also see a more complex picture of the development of mathematical induction, not a story of linear progress but more of a tapestry of ideas, with diagrammatic reasoning appearing as a hitherto unnoticed thread, enriching a story that can be traced from Plato to Pascal, and beyond.

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