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Galileo's *quanti*: understanding infinitesimal magnitudes

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Abstract In *On Local Motion* in the *Two New Sciences*, Galileo distinguishes between 'time' and '*quanto* time' to justify why a variation in speed has the same properties as an interval of time. In this essay, I trace the occurrences of the word *quanto* to define its role and specific meaning. The analysis shows that *quanto* is essential to Galileo's mathematical study of infinitesimal quantities and that it is technically defined. In the light of this interpretation of the word *quanto*, Evangelista Torricelli's theory of indivisibles can be regarded as a natural development of Galileo's insights about infinitesimal magnitudes, transformed into a geometrical method for calculating the area of unlimited plane figures.

1 Introduction

Galileo was one of the most prominent mathematicians of his time in Italy, so much so that innovators such as Bonaventura Cavalieri and Evangelista Torricelli held him in high regard as a leading authority. Galileo scholars have studied his achievements in physics, astronomy, and mechanics, but few have analysed his mathematical research, and these have concentrated on his arithmetic of proportions[1] and how he applied geometry to the study of motion, adopting Archimedes' style.[2] It is notable that, for example, there is no comprehensive study on Galileo's *theory* of infinitesimal magnitudes even though he uses them in his works in mechanics. In particular, we encounter

[1] See Giusti (1986) and Giusti (1993), Palmieri (2001).

[2] See Drake (1974a, 1979), Hill (1986), Naylor (1990), Palmieri (2006), Wisan (1974).

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infinitesimal magnitudes in his description of how speed varies in accelerated motion as well as in his explanation of cohesion.

In this paper, I examine Galileo's search for a mathematical definition of infinitesimal magnitude that he needed to describe the composition and properties of ideal matter. I consider the mathematical modelling of Galileo's theory of matter from the point of view of mathematics itself without going into physics. I start with an analysis of Galileo's technical word *quanto*. According to the dictionary[3] of the *Accademia della Crusca*, the word *quanto* is related to "magnitude"[4] but the dictionary does not elaborate on its type. Stillman Drake considered the word *quanto* a physical term and translates it by "quantified parts", which means "capable of being counted" and "divided".[5] However, this is vague from a mathematical standpoint, so I shall try to clarify the meaning of this important term. When quoting Galileo, I either retain the original word (*quanto*, *quanti*) or add to Drake's English translation the original text in square brackets.[6]

2 Raising the question

Galileo's *Two New Sciences* (1638)[7] represents the culmination of 50 years of his pondering the nature of materials and local motion, and shows the way he applied mathematics to mechanics. The book has the form of a dialogue between Salviati, Simplicio, and Sagredo. These characters have three different approaches to the study of natural philosophy. The first, Salviati, represents Galileo, whose concern is primarily mathematical. The second, Simplicio, is an Aristotelian who supports the orthodox, qualitative view. The third, Sagredo, gives the impression that he occupies the middle ground but actually always agrees with Salviati. The *Two New Sciences* is divided into four *Days*. The *First Day* is devoted to consideration of the way infinity is involved in cohesion as well as in a continuous variation of speed, the *Second Day* to demonstrations of the resistance of solid bodies to fracture and their cohesion, and the *Third* and the *Fourth Days* to demonstrations concerning local motion.

Let us consider Galileo's description of variation of speed in accelerated motion presented in the treatise *On Local Motion*, at the beginning of the *Third Day*.[8] He defines equably or uniformly accelerated motion as the motion which "abandoning rest, adds on to itself equal *momenta* of swiftness in equal times".[9] In Galileo's technical language, the phrase "*momenta* of swiftness" means a sequence of intensities of speed, each acquired in a given instant of time. *Momentum* is a synonym of *degree* ("gradum")

[3] First edition 1612, now available on line at the url: http://vocabolario.sns.it/html/index.html.

[4] The original: "che ha quantità".

[5] Drake (1974a, p. xxxvi).

[6] I shall quote Galileo's original texts from Galilei (1890–1909), by giving the volume in Roman and the page in Arabic numbers.

[7] The English translation from which I shall quote is Drake (1974a).

[8] Drake (1974a, p. 190), Galilei (1890–1909, VIII, p. 190).

[9] Drake (1974a, p. 154). The original Latin reads "a quiete recedens, temporibus aequalibus aequalia celeritatis momenta sibi superaddit" (Galilei 1890–1909, VIII, p. 198).

as is confirmed by the phrase "the degree or the *momentum* of speed" a few lines before the definition of uniformly accelerated motion.[10]

Time and speed are continuous quantities that vary from zero to a finite value, by taking all other values in between. At one point, Simplicio says to Salviati: "But if the degrees of greater and greater tardity are infinite, it will never consume them all, and this rising heavy body will never come to rest, but will move forever while always slowing down—something that is not seen to happen".[11] His objection is founded on the hypothesis that in a *continuum*, there is an infinite number of intermediate increments or decrements ("degrees") of a certain magnitude between two of its states. Simplicio is saying that the process of deceleration of a moving body would never finish if it were true that there are an infinite number of decrements of speed that to go through before it comes to rest. Salviati replies: "This would be so, Simplicio, if the moveable were to hold itself for any time in each degree; but it merely passes there, without remaining beyond an instant".[12]

Here, Galileo is explaining the reason the final speed of a decelerating body can become zero, although it passes through infinite decrements of speed. The key point is to check the duration of each degree of speed, that is, to establish a relation between speed and time. When the degree of speed retains the same value during a small interval of time, Galileo states that it is not possible for a body in a decelerated motion, as a body thrown upward, to reach a state of rest, for if it maintains the same speed Δv_i at the beginning and end of some finite interval of time Δt_i, it could continue at the same speed through the next interval, and so on, with uniform motion *ad infinitum*. Although Galileo does not mention it, the same explanation can apply to an accelerated motion, as a falling body, for if its speed remains the same in a finite interval, it too could continue at the same speed in the next interval, and so would also move uniformly *ad infinitum*. But when the duration of each speed is an "instant", the final speed is either zero or some finite value. Thus, either the speed varies *continuously*, as we see of rising and falling bodies, or it does not vary at all. This is a characteristic Galilean argument, and the conclusion is beyond doubt. The question now is what is meant by *continuously*.

Galileo bases his argument on our everyday perception of time that a finite time interval is composed of an infinite number of time instants. However, he intends to apply the same logic to speed. In order to proceed, he needs to identify the speed equivalents of time instant and time interval. The fact that the body must pass through each speed "without remaining beyond an instant" suggests a parallel between sequences of point-like instants t_i and point-like speeds v_i. His solution then is to consider degrees of speed v_i related to t_i, for if speed v_i changes in each instant with t_i, a final finite speed is reached in a finite time interval. However, Galileo leaves some unanswered

[10] Galileo's original text is "gradum seu momentum velocitatis" (Galilei 1890–1909, VIII, p. 198, l. 10). "*Momentum* of swiftness" must not be interpreted as "mechanical moment", which points to the static effect of the heaviness of a body, a technical word which belongs to pre-modern studies on machines. Mechanical moment varies according to the distance of the body from the centre of rotation that is the length of a lever arm as in the principle of the lever. As far as the evolution of the meaning of "*momentum* of swiftness" is concerned, see Galluzzi (1979), in particular p. 364, footnote 2.

[11] Drake (1974a, p. 157), Galilei (1890–1909, VIII, p. 200).

[12] Drake (1974a, p. 157), Galilei (1890–1909, VIII, pp. 200–201).

questions: "Where is the boundary between instants t_i and time intervals Δt_i?" and "When does a decreasing Δt_i become a t_i?".

In his argument, Salviati makes the following remark: "And since in any finite time [*ogni tempo quanto*], however small, there are infinitely many instants, there are enough to correspond to the infinitely many degrees of diminished speed".[13] This is the point at which Galileo first mentions term *quanto*. The translation is correct in its meaning that *tempo quanto* is a "finite time". But what is a *quanto*? The word is here a noun meaning "quantity" or "amount", so literally *tempo quanto* means "time quantity" or "amount of time". In order to answer this question, we need to return to the part of the *First Day* concerning resistance of materials to fracture and examine Galileo's explanation of cohesion.

3 The mathematical nature of a *quanto*

During the *First Day*, Galileo explains the reasons why he believes that cohesion ("coerenza") is a direct consequence of presence of voids inside matter. He describes the role of voids, measures their presence, and explains how their interaction with matter particles gives rise to, as an example, a "continuous material" like water.[14] Galileo attempts to demonstrate that the presence of voids is enough to explain cohesion, since "although such voids are very tiny, and as a result each one is easily overpowered, still the innumerable multitude of them multiplies the resistances innumerably, so to speak".[15] It is necessary to prove geometrically "how it might be demonstrated that in a finite continuous extension it is not impossible for infinitely many voids to be found"[16] if voids are actually infinitely small. The illustration that serves his purpose is the paradox known as Aristotle's *wheel*.[17] Galileo resolves the paradox in two steps, examining two concentric, n-sided regular polygons in the first, and two concentric circles in the second.

In the first, he comes to a solution using standard geometry, while in the second, he uses a new technique of assigning to the circles properties already proved for polygons. Galileo's strategy works because a polygon can have as many sides as required, even though their number must be finite. But an indefinite number of sides is a necessary condition for the application of Eudoxus's method of exhaustion, which Galileo applies to a polygon inscribed in a circle. This allows him to inscribe polygons with an unlimited number of sides until there are so many that they look like points on a circle. On this basis, he is able to define a circle as a polygon with an *infinite* number of sides (Fig. 1).

[13] Drake (1974a, p. 157), Galilei (1890–1909, VIII, p. 201).

[14] Drake (1974a, p. 22), Galilei (1890–1909, VIII, p. 61).

[15] Drake (1974a, pp. 27–28), Galilei (1890–1909, VIII, p. 67).

[16] Drake (1974a, p. 28), Galilei (1890–1909, VIII, p. 68).

[17] The problem of the wheel (see Fig. 1) is the paradox of two concentric circles rotating around their common centre (A) and on a surface (BF). In the first case, two points (C and B) on each one of them draw two circumferences with different length around the common centre A. In the second case, the same points draw two identical straight lines (CE and BF). It was solved firstly by Jean Jacques d'Ortous de Mairan.

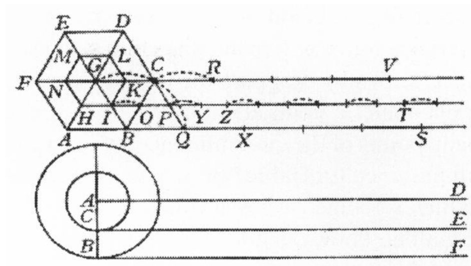


Fig. 1 The diagram exemplifying the problem of the wheel (Drake 1974a, p. 29; Galilei 1890–1909, VIII, p. 68)

Galileo's solution of the paradox is ingenious and deserves attention since it shows how he studies geometrical objects with infinitely many sides:

> I say that in polygons of one hundred thousand sides, the line passed over and measured by the perimeter of the larger—that is, by the hundred thousand sides extended [straight and] continuously—is equal to that measured by the hundred thousand sides of the smaller, but with the interposition [among these] of one hundred thousand void spaces. And just so, I shall say, in the circles (which are polygons of infinitely many sides) the line passed over by the infinitely many sides of the large circle, arranged continuously [in a straight line], is equal in length to the line passed over by the infinitely many sides of the smaller, but in the latter case with the interposition of as many voids between them.[18]

Here, we are in an exclusively mathematical domain. At the core of his solution to the paradox of the wheel is the question of how to add geometrical objects, which can be either an unlimited, infinite sequence of sides, or sides and voids, or points of tangency.

4 The grammar of the *quanti*

The *quanto* is introduced to refute the idea of Sagredo that the points on the smaller circumference can be dragged along line *CE* (see Fig. 1) over a very small segment, called the "particle" (*particella*) of the line.[19] Since the points on the line *CE* touching the smaller circumference, the "contacts" (*toccamenti*), are infinite, "the dragging along *CE* would be infinitely many; and being quantified [*quanti*], these would form an infinite line; but *CE* is finite".[20]

Salviati compares a circumference of finite length γ to a regular polygon with an infinite number of sides that are the points of tangency (*toccamenti*) of the circumference and an external line.[21] Referring to the common centre of the circles, the rotational motion of the circumference is such that every point on it corresponds to

[18] Drake (1974a, pp. 32–33), Galilei (1890–1909, VIII, p. 71).

[19] Drake (1974a, pp. 31–32), Galilei (1890–1909, VIII, p. 70).

[20] Drake (1974a, p. 32), Galilei (1890–1909, VIII, p. 71).

[21] Galileo is considering a circumference consisting of a countable infinity of points. Therefore, its length is the sum of a countable infinity of magnitudes.

a *particle* that belongs also to either line CE or BF. Looking at the motion from a reference frame external to the circles, the particle thus appears to be a very short segment, drawn by dragging the point in question along the line. At this point, we seem to encounter a contradiction: the finite length γ is composed of a *sum* of infinitely many points of the circumference, while a *sum* of the lengths of the infinite particles is infinite since for Galileo those particles are tiny and quantified (*quante*). Let us clarify Galileo's geometrical reasoning.

Galileo knows that a sum of an infinite number of some magnitudes is finite. For example, the infinitely many points on a circumference form a circumference of a finite length. However, he also knows that a sum of an infinite number of other magnitudes has to be infinite, for example, the sum of an infinite number of line segments however small, even infinitesimal, they may be. This way he obtains two infinite sets of *infinitesimal* magnitudes, such that the sum of the first set is finite, while the sum of the other is infinite.[22] He suspects that there is a fundamental difference between the elements of these two sets.

He realizes that he needs to identify a criterion for differentiating between those two sets of magnitudes, be they numbers or geometrical objects. He begins by introducing new terminology. He names *quanti* the mathematical objects, either finite or smaller than any assignable magnitude, an infinite number of which produces an infinite sum, and he calls *not-quanti* (*non-quanti*) or *indivisibles* (*indivisibili*), the mathematical objects, an infinite number of which produces a finite sum.[23] Galileo thus defines two types of *infinitesimal* magnitudes, *quanti*, however small, which may be divided, and, as we shall now write, *non-quanti* or indivisibles, which may not be divided.

Let us go back to the paradox of Aristotle's wheel to show the distinction of *quanti* and *not-quanti* or *non-quanti*. In the quotation, Galileo explains that, in a complete revolution, the length of the line drawn by the smaller circle is equal to the length of the line drawn by the larger, because the line drawn by the smaller includes intermediate voids. After considering polygons of 100,000 sides, he writes of circles:

> And just as the "sides" [of circles] are not *quanti* [*non son quanti*], but are infinitely many, so the interposed voids are not *quanti*, but are infinitely many; that is, for the former [line touched by the larger circle there are] infinitely many points, all filled [*tutti pieni*], and for the latter [line touched by the smaller circle], infinitely many points, part of them filled points and part voids.[24]

The sides of both circumferences are "not *quanti*", or as we say, *non-quanti*, which is a necessary condition for a finite perimeter, and they are infinite in number. In order to add voids to filled points, the voids must also be non-*quanti* and infinite in number. Galileo explains that on the line (BF) traced by the large circumference

[22] The modern technical term *infinitesimal* is used here to describe a magnitude that, as Galileo understands it, is extremely small, as small as can possibly be, but not zero in size.

[23] Galileo seems to have established a sort of relation between points and numbers as if he wanted to *measure* points in some ways. In Euclidean geometry, a point has no dimension, neither length, width, nor depth, so we can identify the dimension of a point as zero.

[24] Drake (1974a, p. 33), Galilei (1890–1909, VIII, p. 71).

there are infinitely many filled points, and on the line (*CE*) traced by the smaller circumference there are infinitely many filled and *empty* points, point-like voids. He treats voids in the same way he treats points, which is why he gives them the shape of points, an empty space inside a border. Therefore, an *empty* point is nothing but a point with a void inside. What he has done is to change Euclid's concept of a point to a new geometrical object that is either filled or empty, although he never stated it in just these terms.

Salviati, at this point, stresses a crucial fact in order to convince the reader that the above argument is correct:

> Here I want you to note how, if a line is resolved and divided into parts that are *quante* and consequently numbered [*numerate*], we cannot then arrange these into a greater extension than that which they occupied when they were continuous and joined, without the interposition of as many void [finite] spaces. But imagining the line resolved into *non-quante* parts [*parti non quante*]—that is, into its infinitely many indivisibles—we can conceive it immensely expanded without the interposition of any *quanti* void spaces, though not without infinitely many indivisible voids.[25]

The point made here is that if a line is divided into *quanti*, these *quanti* can be reassembled to form a longer line only by the imposition of finite *quanti* voids. But if the line is divided into *non-quanti*, it may be immensely expanded by the interposition of infinitely many *non-quanti* voids. And this can be extended to surfaces and solid bodies composed of infinitely many *non-quanti* atoms, which can be expanded into immense space without the interposition of *quanti* voids, but only of infinitely many *non-quanti* voids. "What is thus said of simple lines is to be understood also of surfaces and of solid bodies, considering those as composed of infinitely many *non-quanti* atoms".[26]

Referring to atoms brings the dialogue back to where it started, to the composition of water, containing an infinite number of voids, even though Galileo's concept of an atom as presented here is not material. This characteristic makes the distance between the atom of the atomists and this one insurmountable.[27] Galileo's atom consists only of space inside a three-dimensional form.

Once again Galileo offers a profound and articulated vision and, with the limited tools he has, describes an architecture that could represent a new chapter of geometry. Nevertheless, those looking for a complete mathematics of *quanti* would remain disappointed to see that Galileo limits himself to introducing some new terms and describing their application only in outline. Thus, his use of *quanti* is not a mathematical theory about infinitesimal magnitudes; rather, it is a theory expressed only verbally. But it is sufficiently detailed and consistent that it leads to powerful developments in the work of one of his followers of great talent, Evangelista Torricelli, as we shall see.

[25] Drake (1974a, p. 33), Galilei (1890–1909, VIII, pp. 71–72).

[26] Drake (1974a, p. 33), Galilei (1890–1909, VIII, p. 72).

[27] Galluzzi (2001, p. 74).

5 A new order in the infinitesimally small

In the analysis of the paradox of the Aristotle's wheel, two problems are intertwined: the analysis of the compound motion and the one-to-one relationship between the points belonging to two different circumferences. Both issues involve the concepts of continuity, infinity, and numbers:[28]

> SALV. Now let us pass to another consideration, which is that the line, and every continuum, being divisible into ever-divisibles (*sempre divisibili*), I do not see how to escape their composition from infinitely many indivisibles; for division and subdivision that can be carried on forever assumes that the parts are infinitely many. Otherwise the subdivision would come to an end. And the existence of infinitely many parts has as a consequence their being *non-quante*, since infinitely many *quante* [parts] make up an infinite extension. And thus we have the continuum composed of infinitely many indivisibles.[29]

The indivisibles must not be confused with Euclid's points. For Galileo, the most basic elements of any continuous geometrical object, which is infinitely divisible by definition, are the indivisibles, otherwise known as *non-quanti*. A "divisible and *quanti*" magnitude, i.e. one that is continuous and measurable, can only be obtained by putting together an infinite number of indivisibles. If a limited number of them were used, even a very large one, the result would always be zero according to Galileo's new arithmetic laws.[30]

The concept of *quanto* is subtle. In Galileo's opinion, apart from defining a magnitude, very small but finite, it is also the smallest geometrical object that maintains the property of continuity. The indivisible, on the other hand, terminates the continuity since it cannot be further subdivided. The number of *quanti* inside continuous magnitudes is neither finite nor infinite, but exactly as large as suits the purpose: "the *quante* parts in the continuum are [...] neither finite nor infinitely many, but so many as to correspond to every specified number".[31] Further, Galileo is aware that infinity does not conform to the laws of arithmetic: "for I consider that the attributes of greater, lesser, and equal do not suit infinities, of which it cannot be said that one is greater, or less than, or equal to, another".[32] As a proof, Galileo provides a comparison of an infinite sequence of natural numbers (n) with a sequence of their squares (n^2).

The *quanto* could be an object of either very large or a very small size but never zero. The second case is what interests Galileo, since in this case *quanto* is placed among infinitesimal objects, but at a higher level than indivisibles:

[28] "We are concerned, in the problem of ARISTOTLE'S Wheel, with two matters, (1) the problem of motion, particularly the composition of motions, and (2) the point-to-point correspondence of paths of different lengths. These matters are, to be sure, only apparently independent, for both are, at least from one view-point, ultimately bound up with the problems of continuity, infinity, and the number system" (Drabkin 1950, p. 169).

[29] Drake (1974a, p. 42), Galilei (1890–1909, VIII, p. 80).

[30] Drake (1974a, pp. 38–40), Galilei (1890–1909, VIII, p. 77).

[31] Drake (1974a, p. 43), Galilei (1890–1909, VIII, p. 81).

[32] Drake (1974a, p. 40), Galilei (1890–1909, VIII, p. 78).

> SALV. Here I want to say something that will perhaps astonish you concerning the possibility of resolving a line into its infinitely many [points] by following the procedure that others use in dividing into forty, sixty, or a hundred parts; [...] Pursuing that method, anyone who believes he can find its infinitely many points is badly mistaken, for with such a procedure he will never achieve the division of the line into all its *quante* parts, even if he goes on forever; and as to its indivisibles, he would be so far from arriving at the desired end by that path that instead, he would be travelling away from it.[33]

Here, Galileo is also stating that the procedure of decomposition of a continuous magnitude into *quanti* would fail because it would have to be applied an infinite number of times. Moreover, it would not be possible to make a transition from *quanti* to indivisibles. In other words, Galileo is not able to provide a geometrical procedure that would show that *quanti* consist of indivisibles. In this sense, we can claim that there exists a discontinuity, a break, in the decomposition of a continuous magnitude into its indivisibles, the ultimate components of a geometrical continuum. Therefore, Galileo appears unable to prove the existence of indivisibles from a mathematical point of view. He seems to be creating duplicate infinitesimal objects, since *quanti* and indivisibles do not differ in their mathematical nature but only in the characteristics of the result produced by their addition. This duplication solves only the issue of summing an infinite sequence of magnitudes. Furthermore, Galileo knows that *quanti* are real mathematical objects, since they come into existence through a division of a continuous magnitude. After all, they are just another way of representing the shortest possible segments. But how can he prove that the same is true for indivisibles?

6 The proof of existence of the indivisibles

The relations between the geometrical objects, such as a point, a line, a straight line, a surface, and the space, are presented in Euclid's *Elements*. Even for Galileo "a line [...] contains infinitely many points",[34] although to separate "the points from one another and show them to you distinctly one by one on this paper"[35] is impossible. However, "it is not impossible to resolve a line into its infinitely many points, and not only that, but that this presents no greater difficulty than to distinguish its *quante* parts".[36]

The problem Galileo has to solve now is understanding if and how a point can generate a line, a line can generate a surface, and a surface can generate a solid when a line, a surface, and a solid differ from each another by one dimension. In other words, how is it possible that a point, an object without parts and zero-dimensional, can generate a line that is a one-dimensional figure? How is it possible that by summing all the elements (points), we manage to obtain a finite length that is not zero?

33 Drake (1974a, pp. 44–45), Galilei (1890–1909, VIII, p. 82).

34 Drake (1974a, p. 44), Galilei (1890–1909, VIII, p. 81).

35 Drake (1974a, p. 53), Galilei (1890–1909, VIII, pp. 91–92).

36 Drake (1974a, p. 53), Galilei (1890–1909, VIII, p. 91).

Galileo was convinced that the answer to these questions lay in the existence of indivisible objects at the very basic level of geometrical continuum. Do they really exist? The necessity to describe the behaviour of continuous matter in a consistent way proved the existence of indivisibles and *quanti* by logical deduction. From the logical–mathematical point of view, this should have been sufficient. However, Galileo is not satisfied and he decides to go further than this. He looks for a geometrical procedure that identifies the indivisibles by construction, of which he shows examples for polygons and a circle:

> SALV. [...] when I form of this line a polygon of infinitely many sides–that is, when I bend it into the circumference of a circle–[...] such a resolution [of a line] is made into its infinitely many points [...]. The circle, which is a polygon of infinitely many sides, touches the straight line with one of its sides, which is a single point, different from all its neighbours, and therefore divided and distinguished from them no less than is one side of the polygon from its adjacent [sides]. And as the polygon, rotated on a plane, stamps out with the successive contacts of its sides a straight line equal to its perimeter, so does the circle, when rolled on a plane, describe with its infinitely many successive contacts a straight line equal to its circumference.[37]

An indivisible is, therefore, identified by the part of the circumference touched by its tangent, which Euclid calls a point but for Galileo must possess characteristics that differentiate it from Euclid's point. This could have been the origin of his idea of indivisibles.

7 Torricelli's discoveries

To arrive at his definition of indivisible, Torricelli starts from lessons of Galileo and studies of Cavalieri.[38] In particular, in his notes called *Field of Truffles*[39], he presents some simple geometrical problems (Propositions 73–80) involving indivisibles. In another set of notes, collected under the title *Against Infinite Aggregates*,[40] he uses a standard idea of indivisible that, surprisingly, leads to wrong conclusions. The samples of some of these, collected by Viviani and titled *On Indivisibles*,[41] in particular those in the section called *On the Method of Indivisibles Wrongly Applied*,[42] help us understand the issues Torricelli was grappling with.

[37] Drake (1974a, p. 53–54), Galilei (1890–1909, VIII, p. 92).

[38] For studies of Torricelli, see Bortolotti (1925, 1928, 1939), Jacoli (1875), Belloni (1987), De Gandt (1987, 1991a,b, 1992, 1995), Giusti (1989), Festa (1992), Loria (1897).

[39] I shall quote Torricelli's original texts from Torricelli (1919–1944), by giving the volume in Roman and the page in Arabic numbers. Here, Torricelli (1919–1944, I, part 2, pp. 1–43 and pp. 20–23).

[40] Torricelli (1919–1944, I, part 2, pp. 20–27). There exists another folder, which has an identical title and in which there are notes very similar (Torricelli 1919–1944, I, part 2, pp. 46–48). There are also examples involving curves.

[41] Torricelli (1919–1944, I, part 2, pp. 415–432).

[42] Torricelli (1919–1944, I, part 2, pp. 417–426).

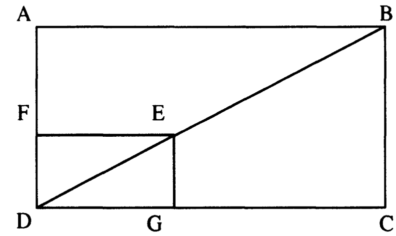


Fig. 2 Torricelli's diagram (copy of the original in Torricelli 1919–1944, I, part 2, p. 417)

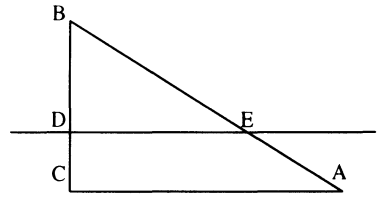
One of the examples from this section shows rectangle ABCD divided by diagonal BD (see Fig. 2). The points (for instance, E) on the diagonal are used in order to define a one-to-one relationship between two cross-sections, FE and EG, belonging to two congruent triangles, ADB and CDB. These triangles are sets or aggregates of the cross-sections and contain the same number of elements. Since every segment FE, which is an infinitesimal part of triangle ABD, is longer than the corresponding segment EG, an infinitesimal part of triangle BCD, Torricelli realizes that this would lead to a wrong conclusion that the area of triangle ABD is also larger than the area of triangle BCD.[43] Even though Torricelli does not make explicit comments in this respect, from the context we can deduce his dilemma: Under what conditions does a geometrical property that is true for each indivisible remain valid also for an aggregate of them?

Torricelli realizes that if the procedure of this example is correct from the logical–formal point of view, then the wrong conclusion must be coming from the relation between the linear cross-sections of the plane figures. There is no doubt that the number of the cross-sections is the same in both triangles since the corresponding segments are always joined at one point. Thus, the fault must be related to the way the above relation is defined, so Torricelli decides to examine this issue further. For Torricelli, differing from Galileo, the existence of a one-to-one relationship between the elements of two sets is not a sufficient guarantee of a correct result. Regarding the problem of Aristotle's wheel, Galileo had presumed a relation between points of a different inner structure made up of voids and non-voids. It looks as if Torricelli took up this idea and developed it further. His working assumption was that the indivisibles were not all the same: "[That] All indivisibles," he writes, "seem equal to one another, that is, points are equal to points, lines are equal in thickness to lines, and surfaces are equal in thickness to surfaces, is an opinion that in my judgment is not only difficult to prove, but false".[44] Coming back to the example of two concentric circles, for Torricelli, they have a one-to-one relationship because each radius of the larger circle includes a radius of the smaller circle. This radius intersects each circle in one point. As a consequence,

[43] Cavalieri's scholars have proved that the notion of a whole of lines, called aggregate, is neither to be thought of as a set of lines in the modern sense nor as a figure compounded of line-like indivisibles. See Giusti (1980), Andersen (1985), De Gandt (1991b).

[44] Torricelli (1919–1944, I, part 2, p. 320). Translation of Curtis Wilson in De Gandt (1995, p. 188). I shall quote English translations of Torricelli's texts by referring to De Gandt (1995). When not specified, the translation is mine.

Fig. 3 Right-angle ABC cut by straight line *r* in D and E



the two circumferences have the same number of points. However, the circumferences have different lengths, proportional to their radii.

Starting with these assumptions, for Torricelli, the natural consequence must be that the corresponding point-like objects are not zero-dimensional, but instead have at least a length. This length is far too short to be treated by the traditional geometrical procedures, but is fundamental when working at the level of infinitesimals. Referring to the concentric circles, every radius intersecting both the circumferences cuts out two arcs of a very short length, but whose ratio is precisely defined, equal to the ratio of the two radii. Using this ratio enables us to explain the reason their lengths differ even though they are made up of an identical number of infinitesimal elements. Could this example be generalized into a working geometrical procedure? If yes, under what conditions?

In Torricelli's work, two prototypical situations appear which reappear in other problems of plane geometry. The first involves right-angle triangle ABC (see Fig. 3):

> If through the triangle ABC, having its side AB greater than its side BC, we imagine to draw all the infinite straight lines parallel to the base AC [*r* is one of them], the points [D] marked by the line segment [*r*] on the straight line AB will be as many as those [E] on BC. Thus, one point [D] on that is to one point [E] of this as the whole line [BC] to the whole line [BA].[45]

On the basis of what we have just seen, Torricelli believes that indivisibles are not all indistinguishable and identical to each other. Their infinitesimal dimension is reflected on a macroscopic scale in the length of the segment they form, allowing calculation of a well-defined ratio. The indivisibles of Torricelli, thus, must be the result of a decomposition of a geometrical, continuous object like, in the language of Galileo, *quanti*.

Let us consider another situation presented in Fig. 4, which shows two parallelograms (rectangle ABEC and parallelogram ABDF) that have base AB in common, same height EB, but the two remaining sides different. From Euclid 1.35, the parallelograms have the same area. From each point H on AB, we can trace segments HG and HI, which are parallel to the second side, respectively, of the rectangle and of the parallelogram. As a consequence, all segments HG, potentially infinitely many, taken together as aggregate would be equal to the aggregate of all of the equally numerous segments HI taken together. This is paradoxical since each HG has a length shorter than the corresponding HI. The only possible explanation for Torricelli is to consider the indivisibles of the plane figure, HG and HI, having two dimensions which must

[45] Torricelli (1919–1944, I, Part 2, pp. 320–321).

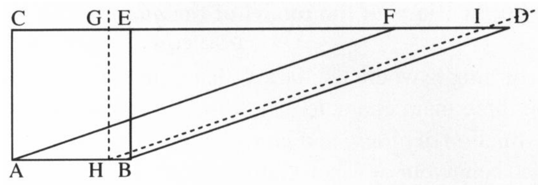


Fig. 4 Torricelli's diagram of two parallelograms with an identical area (copy slightly modified of the original in Torricelli 1919–1944, I, part 2, p. 321)

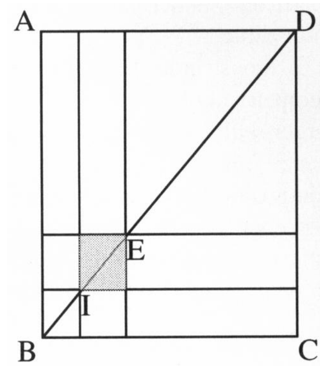


Fig. 5 Infinitesimal rectangle EI is a point-like indivisible of parallelogram ABCD (copy of the original in Torricelli 1919–1944, I, part 2, p. 322)

be inversely proportional to each other: the longer the length of HG, the shorter its width. In this way, their aggregates can have equal areas.

For Torricelli, the indivisible must always have the same number of dimensions as the geometrical object to which it belongs and whose properties it reproduces on infinitesimal scale. Therefore, when working with plane figures, the indivisible of a line is a point-like rectangle (see EI in Fig. 5), while the indivisible of a plane figure is a line-like rectangle. In case of solids, the indivisible of a line is a point-like parallelepiped, the indivisible of a plane figure is a line-like parallelepiped, and the indivisible of a solid is a surface-like parallelepiped. In this sense, for instance, a linear indivisible is not a line, but a three-dimensional object which has two infinitesimal dimensions out of three. In this way, it becomes possible to use the relationship between the linear dimension and the second dimension (infinitesimal but not zero) to calculate the area of a figure, assumed to be the sum of an infinite number of linear indivisibles. The capabilities of these new indivisibles are applicable also to a completely new domain, such as that of curved plane figures, of surfaces of solids of revolution and of solid figures.

With these assumptions, Torricelli applies the rigour of Euclid's geometry on an infinitesimal scale in order to define a proper theory of indivisibles. His definition gives the indivisibles some properties of Galileo's *quanti*, since they are continuous objects and have finite dimensions. At the same time, they differ from them, since a sum of infinitely many indivisibles is not necessarily infinite. In this way, Torricelli simultaneously eliminates Galileo's distinction between *quanti* and indivisibles, and provides the basis of a primordial infinitesimal calculus. His new mathematical instrument was so powerful that by applying it, Torricelli became the first mathematician who correctly calculated the area of an equilateral hyperbola, which is a plane figure that is not closed.

8 Applications of the model of the *quanti*

Returning to where we started, that is, to the beginning of the *Two New Sciences*, where the three main characters are discussing the cohesion of bodies, we can see why the distinction of *quanti* and *non-quanti* is a geometrical model suitable for describing the fluid behaviour of water. Galileo asserts that "the minimum [particles] into which water seems to be resolved [...] are quite different from *quanti*-like and divisible minimum [particles], and I cannot find any other difference here besides that of their being indivisible".[46]

It looks, indeed, as if there exists a hierarchy in the order of magnitude of the geometrical objects, which begins with continuous and finite-dimension objects on a larger scale, and goes down to *quanti* and to *non-quanti* or indivisibles on a smaller scale. This order is also present in nature: "an experience that tends in the direction of composition of infinitely many indivisibles into physical materials".[47] This composition goes from the level of bodies, to particles, and down to atoms. A similar hierarchy is also noticeable in the structure of temporal continuum from measurable time, to *momenta*, and down to instants:

> If we were to apply similar reasoning to the case of circles, we should have to say that where the sides of any polygon are contained within some number, the sides of any circle are infinitely many; the former are *quanti* and divisible, the latter *non-quanti* and indivisible; either end of each side of the revolving polygon stays fixed for a time [...], whereas in circles the delays of the ends of their infinitely many sides are momentary, because an instant in a finite time is a point in a line that contains infinitely many [points].[48]

However, the study of how this model of *quanti* was applied in Galileo's mechanics lies beyond the scope of this essay.[49]

9 Conclusion

The linguistic analysis provided here shows that Galileo gave a technical meaning to the word *quanto*. In trying to identify its role and technical meaning, I have attempted to clarify Galileo's view of the structure of the continuum in geometry. On the one hand, *quanto* is a mathematical object that may be divided into indefinitely many parts at which point it could be described as an infinitesimal magnitude, continuous by nature. A sum of infinitely many infinitesimal *quanti* is always infinite, and therefore, Galileo cannot assume that the *quanto* is the ultimate constituent of a line. On the other hand, Galileo knows that neither a finite nor an infinite number of Euclidean points creates a line. Thus, he finds it reasonable to assume the existence of indivisibles as

[46] Drake (1974a, p. 48), Galilei (1890–1909, VIII, p. 86).

[47] Drake (1974a, p. 60), Galilei (1890–1909, VIII, p. 99).

[48] Drake (1974a, p. 56), Galilei (1890–1909, VIII, p. 95).

[49] Numerous scholars have studied Galileo's physical atomism. Here is a selection of the major works: Baldini (1977), Redondi (1985), Murdoch (2001), Shea (2001), Galluzzi (2001), Biener (2004).

the ultimate constituents of geometric objects, be they a line or a *quanto*. Galileo's indivisibles are so small as to be unquantifiable, *non-quanti*, and point-like. Infinitely many indivisibles create a line, and a sum of infinitely many indivisibles is finite. They can be described as the limit of a set of *quanti*, and this limit is not continuous. While this solution was good enough for natural philosophy, it was not for mathematics since the level of idealization of the infinitely small that he reached was insufficient to provide a proper arithmetic for them.

Torricelli agreed with Galileo's view on a philosophical level. By trying to fill the mathematical gap between *quanti* and indivisibles, he created a new infinitesimal object, which he also named indivisible. His notion of indivisible, however, has the characteristics of being divisible like a continuous magnitude or *quanto*, as well as being at the base of geometrical objects. Only in this way was he able to explain how a point can generate a line or a line can generate a surface. By giving a size to his indivisibles, he was also able to develop a mathematical procedure adequate to identify either the area of a surface or the volume of a solid figure of the most general shape.

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