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Orbital motion and force in Newton's *Principia*; the equivalence of the descriptions in Propositions 1 and 6

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Abstract In Book 1 of the *Principia*, Newton presented two different descriptions of orbital motion under the action of a central force. In Prop. 1, he described this motion as a limit of the action of a sequence of periodic force impulses, while in Prop. 6, he described it by the deviation from inertial motion due to a continuous force. From the start, however, the equivalence of these two descriptions has been the subject of controversies. Perhaps the earliest one was the famous discussion from December 1704 to 1706 between Leibniz and the French mathematician Pierre Varignon. But confusion about this subject has remained up to the present time. Recently, Pourciau has rekindled these controversies in an article in this journal, by arguing that "Newton never tested the validity of the equivalency of his two descriptions because he does not see that his assumption could be questioned. And yet the validity of this unseen and untested equivalence assumption is crucial to Newton's most basic conclusions concerning one-body motion" (Pourciau in Arch Hist Exact Sci 58:283–321, 2004, 295). But several revisions of Props. 1 and 6 that Newton made after the publication in 1687 of the first edition of the *Principia* reveal that he did become concerned to provide mathematical proof for the equivalence of his seemingly different descriptions of orbital motion in these two propositions. In this article, we present the evidence that in the second and third edition of the *Principia*, Newton gave valid demonstrations of this equivalence that are encapsulated in a novel diagram discussed in Sect. 4.

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1 Introduction

In Sect. 2 of Book 1 of his *Principia*, Newton presented two different descriptions for the orbital motion of a body due to the action of an attractive central force.[1] One of these descriptions, which appears in Prop. 1, presents the orbit as uniform motion along the sides of a polygon, resulting from the action of a sequence of discrete central force *impulses* at equal time intervals. Then the orbit due to a continuous force is obtained by taking a *limit* of this polygonal motion when this time interval, and the sides of the polygon become vanishingly small. In this proposition, Newton proves that central forces lead to Kepler's area law. The other description that appears in Prop. 6 considers *locally* a short segment of an orbital curve and determines the deflection from inertial motion along the tangent of this curve due to the action of a continuous central force during a short time interval. The magnitude of this force is obtained by the *limit* when this time interval, and correspondingly the deflection, becomes vanishingly small. In Prop. 6, the proof of Kepler's area law for central forces is applied to express the time interval in geometrical form. Newton had developed a geometrical calculus, which allowed him to deal in a precise mathematical fashion with such limits of quantities which become vanishingly small, while their *ratios* approach finite values. He discussed this calculus in eleven lemmas which constitute the mathematical foundations on which the *Principia* rests.[2] In a *Scholium* following these lemmas, Newton wrote:

> In any case I have presented these lemmas before the propositions in order to avoid the tedium of working our lengthy proofs by *reductio ad absurdum* in the manner of the ancient geometers (Newton 1999, 441).

To understand these propositions, it is helpful to take a historical perspective and consider the origin of these propositions and their evolution, from Newton's first formulation in *De Motu*,[3] the earliest draft of the *Principia* that he sent to Halley in 1684, through the successive revisions which he made in the first (1687) and second (1713) editions of the *Principia*. After the publication of the first edition of the *Principia*, there is considerable evidence that will be discussed here that Newton became concerned in providing further mathematical proof for his two propositions and their equivalence in the continuum limit. The evidence for his concern is found in the changes and additions which he made in the second edition (1713) of the *Principia*, in his formulation of Props. 1 and 6 in Book 1. We will show that these modifications, which were also included in the same form in the third edition (1726) of the *Principia*, clearly indicate that Newton was well aware of the need to provide mathematical proof for the equivalence of his two description for orbital motion and force in these two propositions. In particular, in the second edition of the *Principia*, Newton gave a proof that the

[1] For this force, Newton coined the name *centripetal*, seeking the center, in contrast to Huygens' name *centrifugal*, fleeing the center (Westfall 1980, 411).

[2] In spite of their importance, these lemmas are often ignored in discussions of the *Principia*.

[3] *De motu corporum in gyrum* (On the motion of bodies) (Newton 1974, 30–74) was a nine page tract which Newton wrote, in the Fall of 1684, in response to the visit of Edmond Halley, who asked Newton if he knew the orbit for an inverse square attractive force.

measure for a central force that he obtained by taking the continuum limit of discrete impulses in Prop. 1 is precisely the same as the one which he obtained in Prop. 6 from a description of orbital motion based on calculating the deviation from 1 motion due to the action of an uninterrupted force.

In a review of the second edition of the *Principia* by an anonymous writer in the *Acta Eruditorium* (March 1713), the occurrence of changes in Props. 1 and 6 were pointed out. But the significance of these changes—to clarify these two propositions and to provide a proof of their equivalence—appears to have escaped the attention of some commentators of the *Principia* up to the present time. For example, B. Pourciau claims that "once in the *Principia* the text of both Prop. 1 and its demonstration remains virtually the same through all three editions" (Pourciau 2004, 271). But he ignored the important changes Newton made including new corollaries to this proposition and a substantial reformulation of Prop. 6 in the second edition of the *Principia*. We show that these changes provided a proof that in the continuum limit the measure of force and orbital motion in Props. 1 and 6 are equivalent

In Sect. 2, we present a brief historical background on the role of the 1679 correspondence between Newton and Hooke, and on the eighteenth-century controversy between Leibniz and Varignon on the different descriptions of force in Props. 1 and 6. In Sects. 3 and 4, we review these two propositions as they appeared in the early draft, and later revisions of *De Motu* sent to Halley in 1684, and in the first and second editions of the *Principia*, indicating the significance of changes and additions that Newton made to provide further mathematical proof to clarify the relation between these two propositions.[4] In Sect. 5, we discuss in detail the relation between the two descriptions of orbital motion and central forces in these two propositions and demonstrate, with the aid of a novel diagram, Fig. 7, that in the second and third edition of the *Principia*, Newton gave a valid mathematical proof of their equivalence. Since these two propositions remained essentially unchanged in the third edition of the *Principia*, we are able to quote from the excellent new English translation of this edition by I. B. Cohen and Ann Whitmann Newton (1999). We conclude with a brief summary in Sect. 6 and with three appendixes. Appendix 1 shows that Newton's expression for the force in terms of the curvature of an orbit, Prop. 6, Cor. 3, corresponds to the modern differential equation in polar coordinates, Appendix 2 gives a brief description of Jacob Hermann's derivation of Kepler's area law, starting from Prop. 6, instead of Prop. 1, and Appendix 3 discusses the important Lemma 11 for Prop, 6, and an extension for its application to Prop. 1.

2 Historical background

In the Fall of 1679, Newton and Robert Hooke had a long correspondence in which they discussed fundamental problems concerning orbital motion. In the first one of these letters sent by Hooke on November 24, 1679, he asked Newton for comments

[4] For other presentations, see Francois de Gant, *Force and Geometry in Newton's Principia* (de Gant 1995), Niccolo Guicciardini, *Reading the Principia* (Guicciardini 1999), and I, B. Cohen, *A Guide to Newton's Principia* (Newton 1999).

on a concept of planetary motion which Hooke had enunciated for the first time in a remarkable lecture on "Planetary Movement as a Mechanical Problem," presented at a meeting of the Royal Society of London on May 23, 1666, (Gunther 1930, 265; Nauenberg 1994a, 336; Nauenberg 2005a, 15):

> ...And particularly if you will let me know your thoughts of that compounding the celestiall motions of the planets of a *direct motion by the tangent & an attractive motion towards the central body* [italics are mine]...(Newton 1960, 297)

It is interesting to compare this early formulation of planetary motion by Hooke, with a similar formulation that Newton had recorded in his notebook in 1664,

> If a body moves progressively in some crooked [curved] line ... it may be conceived to consist of a number of straight lines. Or else, in any point of the crooked line the motion may be conceived to be on the tangent (Herivel 1965, 145)

The second part of Newton's description of orbital motion corresponds closely to Hooke's description. Evidently, he and Hooke had arrived, independently of each other, at the same formulation of orbital motion which later Newton described in a more precise mathematical form in Prop. 6: that at each moment of time, *motion* along a curve can be viewed as straight line motion along the tangent of the curve, and that, as Hooke proposed, this motion along the tangent can be "compounded" by an attractive motion toward the central body. But a few days later, Newton responded to Hooke's letter, stating that

> ...I did not before ye receipt of your letter, so much as heare (that I remember) of your Hypothesis of compounding ye celestial motion of ye Planets of a direct motion by the tang[en]t of ye curve...(Newton 1960, 81)

This response is disingenuous, because there is some evidence that Newton had read Hooke's short tract (28 pages long), published in 1674, entitled *On an Attempt to Prove the Motion of the Earth by Observation*, where Hooke had outlined his concept of planetary motion.[5] Ten years earlier, Newton had also implemented such a "compounding" of two motions mathematically, for the special case of uniform circular motion, and he derived a relation for the acceleration toward the center of the circle (Herivel 1965, 195). This relation states that the acceleration is directly proportional to the square of the velocity and inversely proportional to the radius of the circle, a relation that had been obtained earlier by Christiann Huygens.[6,7] For general orbital motion, Newton first implemented mathematically the concept

[5] Hooke's tract was also reviewed in the Philosophical Transaction of the Royal Society **101**, (1674) 11–13.

[6] Huygens kept his result secret for 15 years, until finally publishing it in his *Horologium Oscillatorum* in 1674 (Huygens 1929, 253).

[7] Newton applied this relation for the acceleration to deduce from Kepler's harmonic law for planetary motions (the square of the periods of the planets is proportional to the cube of their radii) that the gravitational force on the planets dependence inversely on the square of the distance from the Sun. He also verified that

of compounding the tangential motion with an attraction or deflection toward the center of force in Theorem 3 of *De Motu*, which later became Prop. 6 of the *Principia.*

Shortly after the end of his correspondence with Hooke, Newton discovered that Kepler's area law is a general property of orbital motion for central forces. In a later recollection, he said that

> In the year 1679 in answer to a letter from Dr. Hook...I found *now*[my italics] that whatsoever was the law of the force which kept the Planets in their Orbs, the area described by the Radius drawn from them to the Sun would be proportional to the times in which they were described...(Bernard Cohen 1971, 293)

Newton's mathematical proof of the area law for "whatsoever was the law of the force...," appeared as Theorem 1 of *De Motu* and later in Prop. 1, Book I, of the *Principia.* This proof was based on his approximate description of orbital motion by "straight lines," as he had recorded it in his notebook, some twenty years earlier.

In 1688, when Leibniz first read the *Principia*, but without acknowledging it, he adopted the description of force in Prop. 1, where the deflection due to an impulse could be described by uniform motion, apparently dispensing with accelerated motion (Bertoloni Meli 1993, 80).[8] On the other hand, a contemporary of Leibniz, Pierre Varignon, preferred the description in Prop. 6 that led *ab initio* to accelerated motion. This led to a lengthy debate between them that lasted from December 1704 until 1706, aptly described as a "comedy of errors" (Bertoloni Meli 1993, 81), because these two descriptions are equivalent, but this equivalence was not clarified until 1713, when the second edition of the *Principia* appeared (see Sect. 4). Leibniz also was able to express the description of orbital motion and central force in Prop. 1 in the language of his differential calculus, and remarkably, he obtained the modern differential equation of motion in polar coordinates (Bertoloni Meli 1993, 134–137; Nauenberg 2010, 281–283). In the early part of eighteenth century, his pioneering work was followed by a number of able mathematicians in Europe, including Jacob Hermann. Johann Bernoulli, and Pierre Varignon (Nauenberg 2010, 269–300), who applied successfully Leibniz's differential calculus to the propositions in the *Principia*. But confusion and errors about the connection between

Footnote 7 continued
the ratio of the gravitational force at the surface of the earth to the gravitational force on the moon was "very nearly" proportional to the square of the ratio of the distance of the moon to the radius of the earth. However, Newton obtained the value of about 4,000, instead of 3,600, because he had used an incorrect estimate of the earth' radius (Herivel 1965, 196).

[8] There was another more subtle reason for this choice that is generally not pointed out. Leibniz had introduce the concept of first-order and higher-order differentials that were defined as differences between lower order ones. The second-order differential associated with the displacement due to impulses in Prop. 1 could be expressed as the difference of adjacent first-order differential representing the sides of the polygonal trajectory. But this connection cannot be made with the second-order deflection from tangential motion described in Prop. 6.

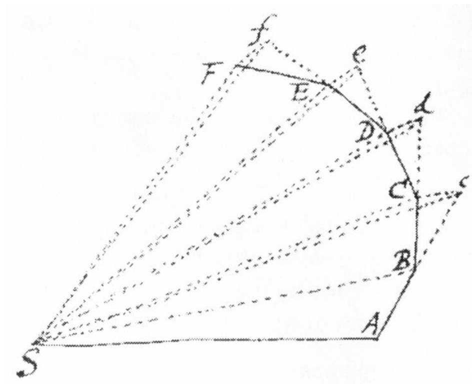

Fig. 1 Newton's diagram in *De Motu* associated with his proof of Kepler's area law. It shows the construction of a polygonal orbit, $ABCDEF$, with the motion counterclockwise under the action of a sequence of radial impulses toward a center at S

Newton's two propositions and its interpretation continues to occur up to the present time.[9,10]

3 The development of Proposition 1

In order to discuss the significance of the changes that Newton made in his formulation of Prop. 1 in the *Principia*, we review this proposition as it first appeared in early drafts and in the subsequent editions of the *Principia*. After his initial description of orbital motion in the *De Motu*, Newton became concerned in providing further mathematical proof of his fundamental propositions and considered a major revision of his book. The evidence for Newton's concern can be found by comparing the formulations of these propositions in *De Motu*, and in the first and second edition of the *Principia*.[11]

In *De Motu*, Theorem 1, Newton presented a geometrical construction for the orbit of h a body moving under the action of central impulsive forces at equal intervals of time. This construction corresponds to the description given in Prop. 1, Book 1 of the *Principia*. Referring to Fig. 1, which shows the original diagram associated with this theorem, Newton gave a proof of Kepler's area law,

[9] For example, in discussing Prop. 1, Eric Aiton wrote that "representing a curve by a sequence of first-order parabolic arcs is equivalent to representing the curve by a polygon with second-order infinitesimal sides," and that the sides of the polygon in Prop. 1 could be interpreted as second-order infinitesimals (Aiton 1989, 211). The same error also was made by Derek Whiteside (Newton 1974, 34–39, n. 19), and subsequently, it has propagated in the literature; for example, in his recent Guide to Newton's *Principia*, I.B. Cohen "warmly" recommends this erroneous analysis (Newton 1999, 115). But in Prop. 1, the correct interpretation is that these polygon sides are *first-order* infinitesimals, as is demonstrated in Sect. 4.

[10] In a recent article in this journal (Arthur 2013, 580), Richard Arthur described Leibniz's treatment of circular motion by introducing first- and second-order differentials of the radial distance, but these differentials are equal to zero.

[11] An English translation of the second edition is not available at the present time, but fortunately Props. 1 and 6 in the third edition are identically the same as in the second edition, and therefore, it is possibly to use instead the recent English translation of the third edition (Newton 1999). For an English version of the first edition, I have relied on the excellent translation of Mary Ann Rossi (Brackenbridge 1995, 235–267).

> *All orbiting bodies describe, by radii drawn to the center, areas proportional to the time*

which he generalized here for arbitrary central forces,

> Let the time be divided into equal parts, and in the first part of the time let the body by its innate force,[12] describe the straight line AB. It would then in the second part of time, were nothing to impede it, proceed directly to c, describing the line Bc equal to AB so as, when rays AS, BS, cS are drawn to the center, to make the areas ASB, BSc equal. However, when the body comes to B, let the centripetal force act in one single but mighty impulse[13] and cause the body to deflect from the straight line Bc, and proceed in the straight line BC. Parallel to BS draw cC meeting BC in C, and when the second interval of time is finished the body will be found at C. Join SC and the [area of the]triangle SBC will then, because of the parallels SB, Cc, be equal to the [area of the] triangle SBc and hence also equal to [the area of] the triangle SAB. By a similar argument, if the centripetal force acts successively at C, D, $E\ldots$, making the body in separate moments of time describe the separate straight lines CD, DE, $EF, \ldots$, the triangle SCD will be equal to the triangles SBC, SDE to SCD, SEF to SDE (and so on). In equal times, therefore, equal areas are described. Now let these triangles be infinitely small and infinite in number, such that to each individual moment of time there corresponds an individual triangle, the centripetal force acting now without interruption, and the proposition will be established. (Newton 1974, 35)

In this first draft, however, Newton did not give the slightest indication on how to implement the continuum limit by letting "these triangles be infinitely small and infinite in number."[14]

One of the first persons to have seen a copy of *De Motu*, which had been sent by Newton to Halley and registered at the Royal Society, was Robert Hooke.[15] Evidently, he recognized that Newton had implemented the concept of planetary motion that he had described to Newton in his November 24, 1679, letter, explaining the origin of the gravitational force as a sequence of periodic "pulses" acting on the planets, and directed

[12] It seems strange that Newton regarded inertial motion as motion acted on by an *innate* or inherent force. I. B. Cohen regards it "as the most puzzling of all the definitions in the *Principia*" (Newton 1999, 96–102). In definition 2 of *De Motu*, Newton states that "I call the force of a body or the force innate in a body by reason of which it endeavors to persist in its motion along a straight line," a description that he maintained also in the subsequent three editions of the *Principia*.

[13] Actually, this "mighty impulse (*impulso unico sed magno*)" is a first-order infinitesimal quantity giving rise to an instantaneous infinitesimal change δv in velocity along the direction of the impulse. At C, for example, $\delta v = Cc/\delta t$, where δt is the infinitesimal time interval between impulses, and the displacement Cc is a second-order infinitesimal quantity.

[14] In his book, *Never at rest*, Richard S. Westfall commented that "the derivation could not have survived critical examination, and Newton could not have built the *Principia* on a foundation so uncertain" (Westfall 1980, 413). But in fact, Newton did precisely that, not changing his formulation by one iota, but adding lemmas and corollaries that clarified the meaning of his proposition.

[15] The Royal Astronomer, John Flamsteed, complained to Newton " I am obliged to your kind concession of ye perusall of your papers, tho I believe I shall not get a sight of them till our common friend Mr Hooke & the rest of the towne have been satisfied" (Newton 1960, 405).

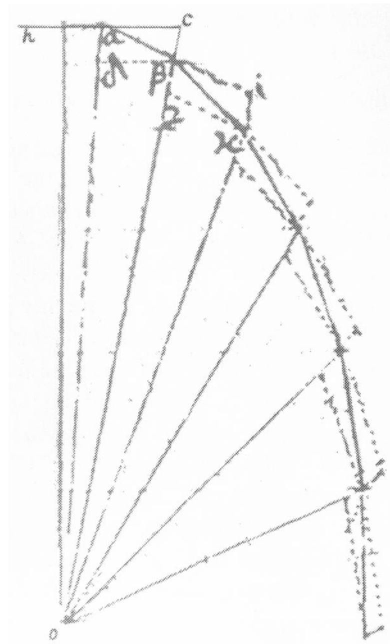

Fig. 2 *Upper right hand* part of Hooke's September 1685 diagram for a discrete elliptical orbit rotating clockwise under the action of a sequence of impulses toward O that depend linearly on the distance to this center. Some *auxiliary lines* have been deleted to show its correspondence with Newton's diagram, Fig. 1, in the *De Motu*

toward the sun (Nauenberg 2005a). He then proceeded to apply Newton's geometrical construction in a novel way, by fixing the relative magnitude of the attractive impulses which Newton had left undetermined in his description of Prop. 6. Hooke assumed that these impulses depended linearly on the distance (Hooke's law) and obtained graphically the resulting polygonal orbital motion which for this case has its vertices on an ellipse with its center at the center of the impulsive force. His result, shown if Fig. 2 where we have enlarged the upper half of his diagram, is similar to Newton's *de Motu* diagram, Fig. 1, except that the body is rotating in the opposite sense. Hooke's diagram and associated text[16] was dated September 1685, a year and a half before the appearance of the first edition of the *Principia*, but for unexplained reasons, it was left unpublished (Pugliese 1989, 181–205; Nauenberg 2005a, 12).

Newton became aware that a discussion and proof of his limits in Prop. 1 was necessary, and in an initial revision of *de Motu*, he included eleven Lemmas that describe the mathematical foundations—Newton's geometrical formulation of the calculus—on which the entire *Principia* rests. In this revision, he added an important remark at the conclusion of Prop. 1:

> ... and the proposition will then, by Corollary 4 of Lemma 3 be established Q.E.D (Newton 1974, 125).

In this Lemma, Newton discussed the area under a monotonic curve by approximating it by a sequence of rectangles or parallelograms. By evaluating both upper and lower

[16] For a full discussion of Hooke's diagram and text see (Nauenberg 1994a, 332).

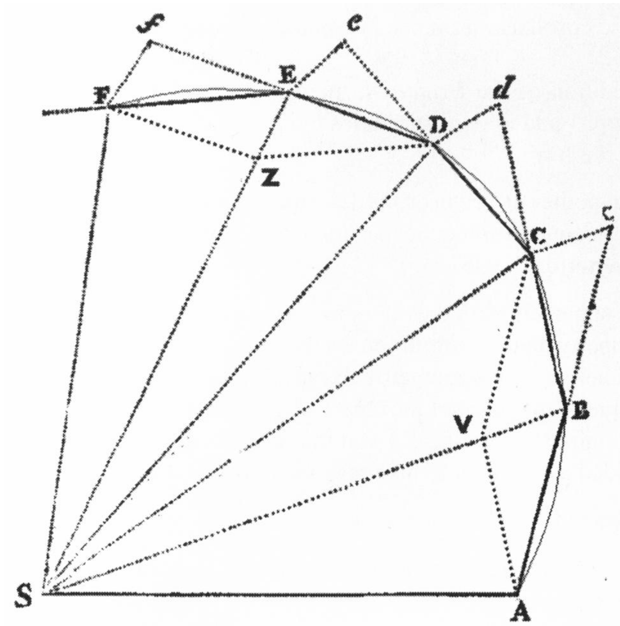

Fig. 3 Newton's diagram in Prop. 1, but with the original lines Sc, Sd, Se and Sf deleted, and a curve $\overline{ABCDEF}$ added, that does not appear in the original, containing the vertices A to F

bounds, he then gave a geometrical proof that as the width of these parallelograms become vanishingly small, the sum of their areas approaches the area under the curve. It is clear that in order to apply Cor. 4 of Lemma 3 to Prop. 1, Newton would have had in mind that the vertices A, B, C, D, E... of the triangles in his geometrical construction in Prop. 1 were on a curve that remained fixed as these "triangles be infinitely smaller and infinite in numbers." Although it was not shown in the diagram for Prop. 1, this curve corresponds to the orbital curve in the continuum limit that Newton must have assumed to be given (Nauenberg 2003, 441–456). This is also the interpretation that Leibniz took, and later also Jacob Hermann, to translate Prop. 1 into the language of Leibniz's differential calculus (Nauenberg 2010, 269–300).

In Cor. 2 of Lemma 3, Newton wrote,

> ... the rectilinear figure that is comprehended by the chords of the vanishing arcs AB, BC, CD, . .[Fig. 3] coincides ultimately with the curvilinear figure (Newton 1999, 434).

and in Cor. 3 of Lemma 3,

> And it is the same for the circumscribed rectilinear figure that is comprehended by the tangents of these same arcs (Newton 1999, 434).

In modern terms, for finite arc length, Cor. 2 gives a lower bound, and Cor. 3 an upper bound for the area under the curve, and these areas coincide in the limit of vanishing arc length. These corollaries remained essentially unchanged in all the three editions of the *Principia*.[17]

In the first edition of the *Principia*, there is an important addition in Newton's statement of Prop. 1 and in two corollaries that were added. The statement of Prop. 1 is now given in the form,

> The areas that bodies driven in orbits describe with radii having been constructed to a stationary center of force, lie in stationary planes and are proportional to the times (Brackenbridge 1995, 245).

Comparing this statement with the previous one in *de Motu*, one sees that Newton has included the property that for central forces, the orbits must lie in a plane. This property follows immediately from his geometrical construction of an orbit for impulsive forces, where the position of the plane of the orbit is determined by the *initial conditions*, i.e., the plane of the initial radial line SA and the initial displacement AB not parallel to SA. Newton added a comment to his previous description in Prop. 1:

> then these [lines] lie in the same plane

referring to the lines comprising the polygonal orbit associated with force impulses. It is evident that in the limit that this polygonal orbit becomes a continuous orbit, this orbit must also be planar, that is, lying in the same plane which is determined by the initial conditions. This constitutes a perfectly valid proof of for the planarity of orbits under the action of centripetal force. The conclusion of Prop. 1 now reads

> Now let the number of triangles be increased and the width be diminished indefinitely, and their ultimate perimeter ADF (by the fourth corollary of the third lemma) will be a curved line; hence the centripetal force in which a body is perpetually drawn away from the tangent of this curve, will act uninterruptedly; but whatever areas $SADS$ and $SAFS$ described which are always proportional to the times of the descriptions, will be proportional to the same times in this case,Q.E.D. (Brackenbridge 1995, 245)

Two added corollaries indicate the changes which occur if the force is not directed to a common center. In practice, this is the case when additional external forces are included, as is the case in the treatment of the motion of the moon and the planets discussed in Book 3, and in the case of motion of body in a gas or fluid which Newton treats as a resisting medium in Book 2 of the *Principia*.[18]

[17] For a sequence of finite periodic impulses at equal time intervals δt, the time t_n at the n th vertex of the polygonal orbit is $t_n = n\delta t$. According to Prop. 1, δt is proportional to the equal area of the triangles subtended by this orbit. Then, in the limit that δt vanishes, and n approaches infinity, $t_n \rightarrow t$, where t is the time at a given point on the orbit that is kept fixed during this limiting process. Hence, the time t at a point on the continuum orbit is proportional to the area subtended by the corresponding arc of this orbit.

[18] Corollary 1 states that

> In non resisting media if the areas are not proportional to the times, [then] the forces are not directed along the path of the radii (Brackenbridge 1995, 245).

and Corollary 2 adds,

In the second edition of the *Principia*, Newton added six new corollaries to Prop. 1, evidently intended as clarifications and further mathematical proof of this Proposition. The original 2 corollaries were moved to Prop. 2.

Cor. 1 gives an expression for the velocity v in terms of the perpendicular distance p from the center to the tangent of the orbit. This corollary implies that for central forces the angular momentum $l = vp$ is a constant of the motion.

> In nonresisting places, the velocity of a body attracted to an immobile center is inversely as the perpendicular dropped from the center to the straight line which is tangent to the orbit. For the velocities in those places A, B, C, D, and E are respectively as the bases of the equal triangles AB, BC, CD, DE, and EF, and these bases are inversely as the perpendiculars dropped to them (Newton 1999, 445).

The bases of the equal triangles correspond to $v\delta t$, and the perpendiculars that we label p give the differential area of each of these triangles, $\delta A = vpdt$, that by Prop. 1 is proportional to δt, i.e., vp is a constant.

Cor. 2 gives further evidence that in Prop. 1, Newton had in mind that the vertices $A, B, C\ldots$ lie on a fixed curve,

> If chords AB and BC of two arcs successively described by the same body in equal times in non-resisting spaces are completed into the parallelogram $ABCV$, and the diagonal BV (in the position that it ultimately has when those arcs are decreased indefinitely) is produced in both directions, it will pass through the center of forces(Newton 1999, 445)

Evidently Newton had in mind that the sides $AB, BC, CD\ldots$, of the polygon in his geometrical construction for Prop. 1, Fig. 1, are the chords of successive arcs $\overline{AB}, \overline{BC}, \overline{CD}$ ..., of a curve, but he did not draw this curve in his diagram. This curve is the orbit of the body in the continuum limit (Nauenberg 2003, 441–456) and justifies Newton's application of Lemma III, Cor. 4, to describe this continuum limit in Prop. 1.[19]

Footnote 18 continued

> In all mediums, if the description of areas is accelerated, the forces do not tend towards the point where the radii meet but deviate forward [or in consequentia] from it. (Newton 1999, 447)

Cohen and Koyre (1971, 91) have pointed out that with some further revisions, Corollary 1 of the first editions plus the addition in the MS Errata became in the second edition the Corollary 1 to Prop. 2 (see also footnotes aa, bb, cc in I. B. Cohen, Guide to Newton's *Principia* (Newton 1999, p. 447)) which reads

> In nonresisting spaces or mediums, if the areas are not proportional to the times, the forces do not tend toward the point where the radii meet, but deviate forward [or in consequentia] from it, that is, in the direction toward which the motion takes place provided that the description of the areas is accelerated; but if it is retarded, they deviate backward [or in antecedentia, i.e., in a direction contrary to that in which the motion takes place] (Newton 1999, 447).

19 Here, Newton has set the stage to obtain a measure of the force in the continuum limit. Since according to Newton's construction in Fig. 1, Cc has been taken parallel to SB, it follows that for finite chords AB, BC the diagonal BV is along the line SB. But in this corollary, Newton states that BV will "pass through the center of forces" in the continuum limit, while in his diagram, this is also true for finite time intervals. Evidently, Newton must have been aware that when AB and BC are finite chords of two arcs described in

In Cor. 3, Newton states that

> If chords AB, BC and DE, EF of arcs described in equal times in nonresisting spaces are completed into parallelograms $ABCV$ and $DEFZ$, then the forces at B and E are to each other in the ultimate ratio of the diagonals BV and EZ when the arcs are decreased indefinitely. For the motions BC and EF of the body are (by corol. 1 of the laws) compounded by the motions Bc, BV and Ef, EZ; but in the proof of this proposition BV and EZ, equal to Cc and Ff, where generated by the impulse of the centripetal force at B and E, and thus are proportional to these impulses (Newton 1999, 445)

In this corollary, Newton shows that the diagonals BV and EZ are equal to the displacements Cc and Ff, which are proportional to the impulsive forces acting at two distinct points B and E of the orbit. He then asserts, without giving a proof, that in the continuum limit when BV and EZ become vanishingly small, their ratio BV/EZ remains finite and approaches a finite value. This limit corresponds to the ratio of the continuous force acting at B and at E, because these impulses act at equal time intervals as will be discussed further in connection with Prop. 6 in the next section. In this proposition, Newton gave a proof that in the continuum limit, the force at B is the limit of the ratio $BV/\delta t^2$ of the displacement BV divided by the square of the time interval δt elapsed between impulses. Likewise, since the force at E is obtained by the ratio $EZ/\delta t^2$, the ratio of these two forces is BV/EZ, and the existence of this ratio in the continuum limit is therefore ensured.[20]

In Cor. 4, Newton added,

> The forces by which any bodies in nonresisting spaces are drawn back from rectilinear motions and are deflected into curved orbits are to one another as those sagittas of arcs described in equal times which converge to the center of forces and bisect the chords when the arcs are decreased indefinitely. For these sagittas are halves of the the diagonals with which we dealt in corol. 3

Footnote 19 continued

"equal time" intervals, the location of C is not generally the same as the location obtained by his geometric construction with finite chords, where equal time refers to time interval time elapsed between successive inertial motion along the chord, instead of orbital motion along the arc, see Fig. 3. In Newton's geometrical construction for Prop. 1, equal times correspond to equal areas of the triangles bounded by the chords and the radii. But for continuous orbital motion and forces, which is the limit of vanishingly small triangles, equal times for finite arcs of the orbit is associated with equal areas of the sectors bounded by these arcs and the radial lines. Thus, for finite arcs, $\overline{AB}$ and $\overline{BC}$, given A and B the position of C obtained by the requirement that the areas of sectors ASB and BSC be the same, will in general differ from the position C obtained from Newton's geometrical construction described in Prop. 1, where equal areas refer to the areas of the finite triangles ASB and BSC. In Cor. 2, Lemma 3, however, Newton indicated that in the limit that "the width of the triangle becomes vanishingly small and the number goes to infinity" the sum of the areas of the triangles bounded by chords, approach the area of the sum of the sectors bounded by the corresponding arcs.

[20] In Prop. 1, Newtow did not prove that the displacements Cc, Ff, etc., are second-order quantities and, therefore, that the ratios Cc/dt^2, Ff/dt^2 are finite in the limit that these differentials vanish. This proof was given in Prop. 6 by applying Lemma 10, in the first edition, and lemma 11 in the second one, assuming that the curvature is finite.

In this corollary Newton introduced the *sagitta* (arrow) of an arc which is a line segment between the midpoint on the arc and a point on its chord which approaches the middle of the chord in the continuum limit.[21] Newton relates impulsive displacements, such as BV and EZ discussed in the previous corollary, Cor. 3, with the sagittas of a pair of adjacent arcs. For example, the chord of the two arcs $\overline{AB}$ and $\overline{BC}$ adjacent to the point B is described by the line AC, which is the diagonal of the parallelogram $ABCV$ introduced in Cor. 1. Hence, half the line BV is then, by Newton's *definition*, the sagitta of the arc $\overline{AC}$,

Cor. 5 states,

> And therefore these forces are to the force of gravity as the these sagittas are to the sagittas, perpendicular to the horizon, of the parabolic arcs that projectiles describe in the same time

In this corollary, Newton illustrated his description of force in the continuum limit for an orbit under the action of gravity near the surface of the earth. In this case, the force is approximately a constant directed perpendicularly toward the surface which Newton refers here as the horizon. Galileo had demonstrated that in this case, the orbit is a parabola, and Newton also found that finite arcs of this orbit are "parabolic arcs."

Since the motion along the sides of the polygonal orbit under the action of force impulses is inertial motion, the velocity on each side is constant. Its magnitude v_i is given by the ratio of the first-order differential length δl_i of each side (labeled here by the subscript i) divided by the periodic first-order differential time interval δt between pulses, i.e., $v_i = \delta l_i/\delta t$ which approaches a finite value in the limit that δt, and correspondingly δl_i, vanishes. Each impulse gives an *instantaneous* change in velocity δv_i directed toward the center of force, but in Prop. 1, Newton did not give a proof that its magnitude must be a first-order differential proportional to this time interval, i.e., $\delta v_i = a_i \delta t$, where a_i is a measure of the magnitude of the impulse that in the continuum limit becomes the acceleration imparted by the force. Hence, $\delta^2 l_i = \delta v_i \delta t$ is the magnitude of the displacement due to an impulse at the ith vertex, transversed at the uniform differential velocity δv_i; a second-order differential such that $a_i = \delta^2 l_i/\delta t^2$ is in accordance with the definition of accelerative force in Prop. 6.[22]

Early on, by applying Prop. 1, both Leibniz and Hermann obtained an expression for the force (acceleration) in terms of differentials. For example, Hermann's diagram, shown in Fig. 4, clearly corresponds to a segment of Newton's Prop. 1 diagram, Fig. 3, but with the motion directed clockwise along the curve $ABCD$ around the center of

[21] Newton aptly chose the Latin word *sagitta* for *arrow* to describe the various displacements of the midpoint of an arc from its chord. Andrew Motte, however, who in 1728 translated the *Principia* into English, gave it the mathematical expression *versed sine*, which applies only to the special case that the sagitta is normal to the chord.

[22] A proof that the deviations $\delta^2 l_i$ from inertial motion due to impulses are second-order differentials, requires the assumption that the arcs of the underlying orbital curve (left out of Newton's diagram for Prop. 1 but included in Fig. 3) that *determines* the magnitude of these deviations have finite curvature. Then, a proof can be obtained by a straightforward extension of Lemma 11, considering the deviation from inertial motion from an extension of the chord, instead of the tangent, of an arc (see Appendix 3). For uniform circular motion, which has constant curvature, such a proof was carried out by Leibniz (Bertoloni Meli 1993, 80–81).

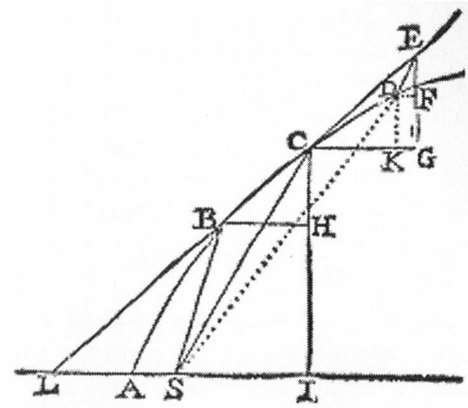

Fig. 4 Hermann's version of Newton's diagram for Prop. 1 (see Fig. 3), but with the motion along the arc $\overline{ABCD}$ directed clockwise

force at S. The impulse at C is represented here by the short line ED, taken parallel to the radial line SC, where the vertex E is obtained by extending the chord BC to CE, setting it equal in length to BC, and D is the intersection of this line with the curve. Then, Hermann introduced Cartesian coordinates by taking the horizontal line LASI as the x axis, and he obtained the displacement ED as a second-order differential,

$$ED = \mathrm{dd}x\sqrt{x^2 + y^2}/x, \tag{1}$$

where $\mathrm{d}x = BH$, $\mathrm{d}x' = CG$, and $\mathrm{dd}x = \mathrm{d}x' - \mathrm{d}x = KG$.[23] Assuming the validity of Kepler's area law,[24] he set the time interval $\mathrm{d}t$ equal to twice the area of the triangles $BSC = CSD$, i.e., $\mathrm{d}t = x\mathrm{d}y - y\mathrm{d}x$,[25] and applying Prop. 6, he obtained a differential expression for the force,[26]

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[23] The first-order differentials $\mathrm{d}x$, $\mathrm{d}x'$ are not equal, but defined to be transversed during equal time intervals $\mathrm{d}t$.

[24] At the time, Hermann, Varignon and others, failed to recognize that Prop. 1 demonstrates the validity of Kepler's area law. In Cartesian differential coordinates, it takes the form $\mathrm{d}(x\mathrm{d}y - y\mathrm{d}x) = 0$. A one line proof is obtained by also expressing the displacement ED in terms of the second-order differential $\mathrm{dd}y$, which implies that $\mathrm{dd}y = (y/x)\mathrm{dd}x$ (Nauenberg 2010, 277). But neither Hermann nor his contemporaries noticed this relation, and it took him another 6 years before he developed a proof, based on Prop. 6 instead of Prop. 1, of this fundamental theorem (see Appendix 2). He published it in his main work, *Phoronomia*, and claimed that it was the first valid proof of Prop. 1 (Guicciardini 1999, 211–215).

[25] In polar coordinate r, θ, the area $\mathrm{d}A$ of a differential triangle with a vertex at the center of coordinates, base of length r, and height $r\mathrm{d}\theta$ is

$$\mathrm{d}A = (1/2)r^2\mathrm{d}\theta. \tag{2}$$

In Cartesian coordinates, $r = \sqrt{x^2 + y^2}$, and $\tan\theta = y/x$. Hence $\mathrm{d}(\tan\theta) = \mathrm{d}\theta/\cos^2\theta = (x\mathrm{d}y - y\mathrm{d}x)/x^2$, and substituting $r^2\cos^2\theta$ for x^2, obtain

$$\mathrm{d}A = (1/2)(x\mathrm{d}y - y\mathrm{d}x) \tag{3}$$

[26] According to N. Guicciardini, (Guicciardini 1999, 207), Hermann obtain this expression by "focusing" on Prop. 6. But it is important to recognize that, like Leibniz, he first applied Prop. 1 to obtain a relation for the force in terms of the displacement from the extension of the chord of an arc of the orbital curve, rather than from its tangent as described in Prop. 6.

$$F \propto \frac{ED}{dt^2} = \frac{ddx\sqrt{x^2+y^2}}{x(xdy - ydx)^2}, \tag{4}$$

that he solved for the inverse square force, $F = 1/(x^2 + y^2)$, demonstrating that the orbit is an ellipse (Nauenberg 2010, 272–276).

4 The development of Proposition 6, Book 1

In *De Motu*, Theorem 3, which later became Prop. 6 in the *Principia*, Newton gave an expression for a continuous centripetal force acting on a body moving on a curve that, according to Prop. 1, must satisfy Kepler's area law. Referring to the diagram associated with this proposition, Fig. 5, he wrote,

> If a body [located at the point] P in orbiting around the center S shall describe any curved line APQ, and if the straight line PR touches that curve at any point P and to this tangent from any other point Q of the curve there be drawn QR parallel to the distance SP, and if QT be let fall perpendicular to this distance SP: I assert that the centripetal force[27] is reciprocally as the solid $SP^2 \times QT^2/QR$, provided that the ultimate quantity of that solid when the points P and Q come to coincide is always taken (Newton 1974, 41),

In the first edition of the *Principia*, this theorem became Prop. 6, theorem 5, and except for minor clarifying alterations in language it remained essentially unchanged. The only significant change was that Newton now cited Lemma 10 to justify his assertion that the measure of central force is proportional $QR/\delta t^2$, where δt is the time interval elapsed for the body to travel from P to Q, that by Prop. 1 is proportional to $SP \times QT$.[28] In Lemma 10, Newton demonstrated that,

> Spaces which a body describes at the urging of any standard force are, at the very start of motion, in the double ratio of the times. (Brackenbridge 1995, 241)

This lemma is a generalization for the case, first considered by Galileo, of the acceleration of falling bodies under a constant gravitational force. Then the arc QP is a segment of a parabola (Fig. 5).[29]

The extended statement of Prop. 6 now reads:

[27] Newton considered here the *accelerative* centripetal force or simply the acceleration, as opposed to the *motive* centripetal force that appears in the *Principia*, Definition 6, as the conventional definition – force = mass times acceleration. But this distinction is often ignored by commentators of Prop. 6, although it is evident that this proposition has nothing whatsoever to do with mass.

[28] The constant of proportionality is the inverse of the angular momentum l for unit mass, $l = vSY$. where v is the velocity, and SY is the component of SP perpendicular to the direction of the velocity, i.e., $\delta t = (1/l)QT \times SP$ (see Cor. 1 of Prop. 1). For Hermann's proof that l is a constant which is based only on Prop. 6 see Appendix 2.

[29] For circular motion, it was first applied by Huygens in 1659 to derive the central acceleration a for a body moving with uniform velocity v in a circular orbit of radius ρ. Huygens approximated the arc QP of the circle by the arc of the parabola $QR = (1/2\rho)RP^2$ with vertex at P. Then setting $RP = v\delta t$, and $QR = (1/2)a\delta t^2$, he found $a = v^2/\rho$. In Prop. 6, the factor 1/2 is missing.

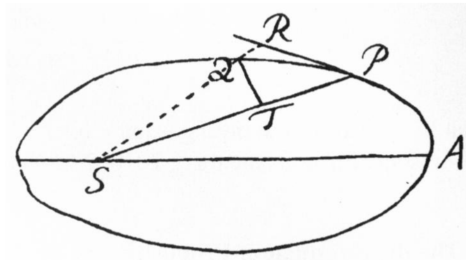


Fig. 5 Newton's diagram for Theorem 3 in *De Motu*, which later became Prop. 6 in the *Principia*. In this diagram, QR is drawn as an extension of SQ

> In the indefinetely small figure $QRPT$, Fig. 1, the nascent line segment QR is, given the time [interval], as the centripetal force (by law 2), and given the force, as the square of the time (by Lemma 10) (Brackenbridge 1995, 252).

In other words, $QR \propto F\delta t^2$, where F is the force (acceleration) acting during the time interval δt. Actually, $QR = (1/2)a\delta t^2$, where a is the acceleration, but the factor 1/2 is missing in all the three editions of the *Principia*, where Newton confined his expressions to proportionality relations. But this factor 1/2 is important to relate, in the continuum limit, the expression for accelerative force in Prop. 1 which is equivalent to $F = 2QR/\delta t^2$ (see Sect. 5).

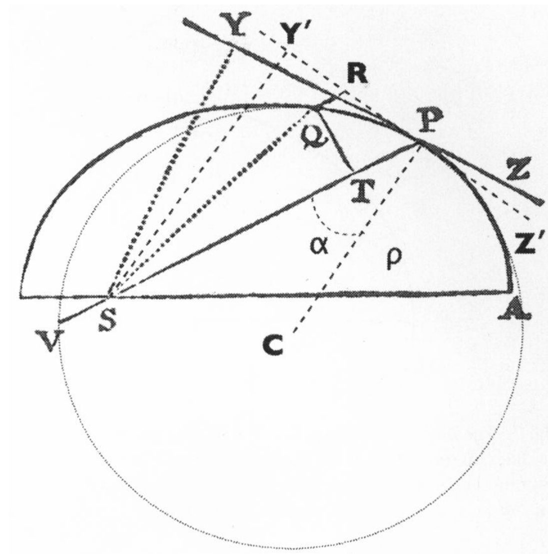
In the second edition of the *Principia*, there appeared a drastic alteration in the formulation of Prop. 6. The statement of this proposition that appeared in the first edition is now relegated to a corollary, and the new statement of Prop. 6 reads as follow:

> If in a nonresisting space a body revolves in any orbit about an immobile center and describes any just-nascent arc in a minimally small time, and if the sagitta of the arc is understood to be drawn so as to bisect the chord and, when produced, to pass through he center of forces, the centripetal force in the middle of the arc will be as the sagitta directly and as the square of the time inversely For the sagitta in a given time [interval] is as the force (by Prop. 1 and Cor. 4), and if the time [interval] is increased in any ratio, then -because the arc is increased in the same ratio - the sagitta is increased in that ratio squared (by Lemma 11, Corols. 2 and 3), and therefore is as the force once and the time squared [the product of the force and the square of the time interval].[30] Take away from both sides the squared ratio of the time, and the force will become as the sagitta directly and as the time squared inversely Q.E.D. (Newton 1999, 453)

Newton defined the *sagitta* (arrow) associated with a point P on a small arc of a curve by the segment PX of the radial line SP intersecting the chord of this arc at its midpoint X. But he did not show this arc and the corresponding sagitta in his diagram, Fig. 6,

[30] In the first edition of the *Principia* Newton referred to Lemma 10 for this relation, but he did not invoke it in the second edition, referring instead to Lemma 11. In this lemma, Newton gave a proof that $QR = QP^2/2\rho$, when the force is directed along the radius of curvature ρ. For the case of a circle, his proof corresponds to Euclid's Prop. 36 in Book 3. Since $QP = v\delta t$, in this case $QR = a\delta t^2$, which is the content of Lemma 10.

Fig. 6 Diagram for Prop. 6 in the second and third edition of the *Principia*, with a section of the circle of curvature at P and center at C superimposed. The *dashed line* PC is the radius ρ of this circle, with the chord $PV = 2\rho cos(\alpha)$ along the radial line SP. The *dashed line* $Z'PY'$ is the tangent line at P, which was drawn incorrectly in the original diagram as the *black line* ZPY



associated with this proposition.[31] As the paragraph following this new formulation of Prop. 6 shows, Newton introduced the sagitta for the specific purpose of relating the definition of a continuous central force in Prop. 6 to the definition of force in Prop. 1, where it is given in terms impulses during equal time intervals in the limit that these intervals vanish. For any finite arc, however, only one of Newton's two requirements for the sagitta can be met simultaneously, unless the line "produced to pass through the center of force" is normal to the tangent of the curve at P. But this problem disappears in the limit that the arc length becomes vanishingly small.[32]

Newton followed his new statement of Prop. 6 with a new corollary, Cor. 1, which consisted of the formulation of Prop. 6 in the first edition of the *Principia*. In Cor. 1, Newton also added the statement that,

> the displacement QR is equal to the sagitta of an arc which is twice the length of arc QP, with P being in the middle. (Newton 1999, 454)

According to this description, the *sagitta* is nearly equal to QR, where QR is taken parallel to SP, and the middle of the extended arc is now the geometrical middle, because in this case the corresponding chord must be nearly parallel to the tangent line ZPR (see Fig. 7). The *sagitta*, which is also taken parallel to SP, does not bisect

[31] We have added the sagitta PX to a corrected version of this diagram in Fig. 7.

[32] In his new statement of Prop. 6, Newton *defined* the centripetal force at a given point P of the orbit as the limit of the ratio $sagitta/\delta t^2$, where the *sagitta* was first introduced in Cor. 4 of the version of Prop. 1 in the second edition of the *Principia*. In Prop. 6, Newton describes the *saggita* for a small arc of the orbit, with P located on the middle of the arc, as a line from P which bisects the cord of the arc. A subtle point here is what Newton meant by the middle of the arc where the centripetal force is defined. The answer is obtained by turning to Prop. 1, Cor. 4 where the arc describing the *sagitta* is defined by two adjacent arcs transversed in *equal time* intervals. Hence, the middle of the arc is not the geometrical middle, which is obtained by dividing an arc into two arcs with the same length.

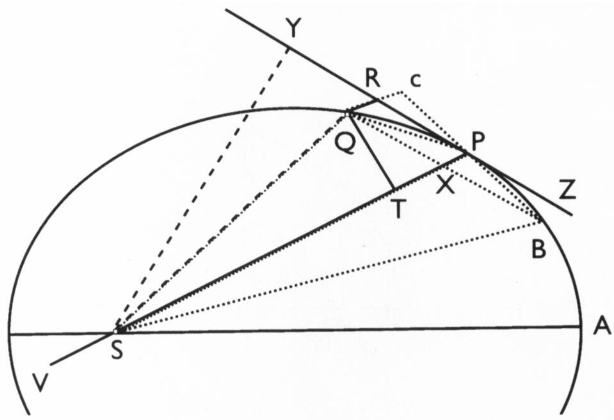


Fig. 7 The diagram for Prop. 6 in the second and third edition of the *Principia*, Fig. 6, with a section of the diagram for Prop. 1, Fig. 3, superimposed on it in *dotted lines*. The cord BQ, not drawn in the original diagram, of the arc $\overline{BPQ}$, is approximately parallel to the tangent line ZPY at P and intersects the radial line SP at X, describing the *sagitta* $PX \approx QR$

the chord as announced in the statement of this proposition, but as we will show in the next section, it does in the limit as Q approaches P.

In addition, Newton included an important new corollary, Cor. 3. based on the geometrical concept of the local radius of curvature of a general curve, that he, and independently Huygens, had developed earlier (Brackenridge and Nauenberg 2002). In his 1679 correspondence with Hooke (see Introduction), there is evidence that he applied this concept early on to obtain the approximate shape of orbits under the action of general central forces (Nauenberg 1994b, 222–225). In a preliminary corollary, Cor. 2, he applied the relation $SY/SP = QT/QP$, where SY is a line from the center of force at S normal to the tangent line ZPR intersecting it at Y, to obtain that "the centripetal force is inversely as the solid $SY^2 \times QP^2/QR$."

Cor. 3 states,

> If the orbit APQ either is a circle or touches a circle concentrically or cuts it concentrically - that is, if it makes with the circle an angle of contact or of section which is the least possible - and has the same curvature and the same radius of curvature at point P, and if the circle has a chord going from the body through the center of forces, then the centripetal force will be inversely as the solid $SY^2 \times PV$. For PV is equal to QP^2/QR.[33]

In Newton's diagram for Prop. 6, the circle of curvature and its radius ρ was not drawn. In Fig. 6, we show this circle with center at C by a dotted line and its radius of curvature $PC = \rho$ by a dashed line from P to the center of curvature C (Brackenbridge 1995,

[33] Around 1705, the expression for the central force in terms of the radius of curvature was obtained independently by the mathematician Abraham De Moivre, who then showed it to Newton. But Newton replied that he had already obtained a similar formula (Guicciardini 1999, 226).

115).[34,35] Then the chord of this circle "going from the body [at P] through the center of forces" is $PV = 2\rho cos(\alpha)$, where α is the angle between ρ and the radial line SP. The relation $PV = QP^2/QR$ follows from a generalization of Lemma 11, where Newton shows that $QR = QP^2/2\rho$ for the special case that $\alpha = 0$.[36]

This corollary, which was absent in the first edition of the *Principia*, [37] is extremely important, because it justifies, for curves with a finite radius of curvature ρ, the existence of a limit for the ratio QR/QP^2 when $QP \to 0$. Moreover, it gives an expression for the force in terms of finite instead of infinitesimal quantities. In particular, expressing the radius of curvature in polar coordinates, it reduces to the modern differential equation for the orbit in these coordinates [Nauenberg 2005b, 345 (see Appendix 7)].

5 Proof of the equivalence of Propositions 1 and 6 in the continuum limit

In this section, we show that in the second and third edition of the *Principia*, Newton gave a mathematical proof that established, in the *continuum limit*, the equivalency of the two apparently distinct descriptions in Props. 1 and 6 of orbital motion under the action of a centripetal force. In Prop. 1, the force is a *periodic* impulse that acts *instantaneously*, while in Prop. 6, the force acts *continuously* in time. But in this continuum limit, these two mathematical descriptions give the same result.

To demonstrate this equivalence, we combine in Fig. 7 the diagram associated with Prop. 6 in the second and third edition of the *Principia*, Fig. 6, with a segment of the diagram associated with Prop. 1, Fig. 3. This segment is shown in Fig. 7 by the dotted lines, but the labels for the vertices in Fig. 3, and those in this figure are different.[38] In particular, the arc $\overline{ABC}$, which does not appear in the Prop. 1 diagram, but is mentioned in Cor. 2 of this proposition, corresponds to the arc $\overline{BPQ}$.[39]

According to Prop. 6, given any point P on the curve $\overline{APQ}$, Fig. 7, the deviation from inertial motion along the tangent line ZPR to this curve at P, due to a *continuous* force acting along the arc $\overline{QP}$, is represented by the short line QR, *parallel* to SP, from a nearby point Q, intersecting this tangent line at R. Correspondingly, according

34 After Lemma 11, Newton devotes a Scholium to discuss curves for which the radius of curvature become infinite at certain points.

35 We have superimposed this circle on Newton's diagram as it appeared in the third edition of the *Principia*. The original tangent line ZPY at P had been drawn incorrectly, and an improved line is shown as a dashed line $Z'PR'$ (The center C that is obtained by the intersection with the line perpendicular to ZPY at P gives a grossly exaggerated radius of curvature which does not contain the chord PV).

36 For uniform motion with velocity v on a circle of radius ρ, by 1669 Newton already had shown that the central acceleration $a = v^2/\rho$. Since according to Prop. 1, $v \propto 1/SY$, then for $\alpha = 0$, $1/a \propto SY^2 \times PV \propto 2\rho/v^2$.

37 But already in Prop. 28, Book3, in the first edition of the *Principia*, Newton applied his curvature relation to obtain the shape of the lunar orbit due to the perturbation of the sun.

38 The vertices A, B, C, c in Fig. 3, become B, P, Q, c, respectively, in Fig. 7.

39 Cor. 2 starts with,

> If chords AB and BC of two arcs successively described by the same body in equal times in nonresisting spaces ...

to Prop. 1, the deviation Qc, also parallel to SP, due to a radial *impulse* at P, is obtained by extending the chord PB from a certain point B on the curve to a point c such that $Pc = PB$.[40] For a continuous force, the motion along QR is assumed to be uniform accelerated motion from rest, and by Lemma 10, the acceleration is $a = 2QR/\delta t^2$, while an impulse at P generates an *instantaneous* infinitesimal change in velocity $\delta v = Qc/\delta t$ along Qc. Hence, Qc is transversed with a constant but infinitesimal velocity, as Leibniz had insisted in his lengthy exchange with Varignon (see Introduction). But in the continuum limit, it corresponds to a radial acceleration $a' = \delta v/\delta t = Qc/\delta t^2$. Hence, for the demonstration of equivalence, i.e., $a' = a$, it is sufficient to show that in the limit that Q approaches P, $Qc/QR = 2$. Such a proof is straight forward when the arc $\overline{BPQ}$ is the arc of a circle, which in this limit is the curvature circle.[41,42,43]

6 Summary

After the publication of the first edition of his *Principia* in 1687, Newton started to consider major revisions of his work in order to clarify and justify further some of the fundamental propositions that supported his demonstrations (Brackenbridge 1995, 141). In this paper, we discuss only those changes and additions that he made for this purpose in propositions 1 and 6. In Prop. 1, Newton gave a proof for the area law and the planarity property of orbital motion under the action of central forces, and in Prop. 6, he derived a relation to calculate these forces for a given orbital curve of finite curvature satisfying the area law. Newton's proofs in these two propositions were based on two different descriptions of orbital motion. In Prop. 1, he described the orbit in the continuum limit of polygonal motion due to periodic central force *impulses*, while in Prop. 6, he attributed the orbit to the action of an uninterrupted central force which continually deflected it from inertial straight line motion along the local tangent of this orbit. While in the first edition of his *Principia*, Newton did not relate these two different descriptions of orbital motion, the changes and additions which appeared in the second edition indicate that he was well aware of the need to demonstrate their mathematical equivalence. Indeed, in five new corollaries added to Prop. 1, Newton demonstrated that the force measure in the continuum limit of forces

[40] To obtain the point c, first extend the line QR and then find its intersection with the extension of the chord BP of an arc $\overline{BP}$ such that $Pc = BP$.

[41] Such a proof is obtained by replacing the tangent line in Lemma 11 with the extension of a chord, as shown in Fig. 7 (see Appendix 3). In Cor. 4 of Prop. 1, Newton asserted that $PX = (1/2)Qc$, which is evident from his diagram for Prop. 1.

[42] In the literature, it is customary to represent Newton's second law of motion in two forms: (1) F $\propto \delta v$ for an impulsive force and (2) F $\propto \delta v/\delta t$ for a continuous force (Newton 1999, 116). But in Prop. 1, we have shown that for 1) the correct expression is $\delta I \propto \delta v$, where δI is an infinitesimal *first-order impulse*, and that $F = \delta I/\delta t \propto \delta v/\delta t$. In the continuum limit, this is the same expression for force as in 2).

[43] According to Michel Blay, "it is nevertheless difficult to account for the exact relation between 'the force acting at once with a great impulse', and the centripetal force acting 'uninterrupted'" (Blay 2001, 233). This quote is repeated in Cohen's guide to Newton's *Principia* (Newton 1999, 71,Note 73), but the supposed *difficulty* is resolved by the realization that the impulse is not "great," as Newton misleadingly wrote in Prop. 1, but it is a first-order infinitesimal impulse $\delta I = F\delta t$, where F is the force in the continuum limit.

impulses corresponded to the force measure defined in Prop. 6. This proposition was also substantially extended, with its original formulation becoming a corollary, and with the addition of three important new corollaries. The new opening statement of Prop. 6 expresses the deflection from inertial motion in terms of the sagitta of a small arc of the orbital curve, instead of the equivalent displacement from the tangent at a point on the curve, given in the first edition. This change was evidently introduced to show the equivalence of the definition of force in this proposition and in corollaries 3 and 4 of Prop. 1.

Since the appearance of the first edition (1687) of the *Principia*, however, the equivalence of the descriptions in these two propositions has been the subject of controversies. Perhaps the earliest one was the exchange from December 1704 to 1706 between Leibniz and Varignon (Bertoloni Meli 1993, 81), but confusion about this subject has remained up to the present time. Recently, for example, B. Pourciau has claimed that Newton took the equivalence of his two descriptions for orbital motion for granted, either as an "axiom" or as "an article of faith" that does not require any mathematical proof (Pourciau 2004, 284; Nauenberg 2012, 931). But as has been shown in Sects. 3 and 4, there is ample evidence that Newton was concerned to provide mathematical proof for the equivalence of these two descriptions. This evidence is found in the changes for this purpose that Newton made to Props. 1 and 6 in the second edition of the *Principia*, which remained unchanged in the third one.

The description in Prop. 1 of central forces by impulses is in accordance with Newton's formulation of the second law of motion, in all the three editions of the *Principia*,

> A change in motion [mass times velocity] is proportional to the motive force and takes place along the straight line in which that force is impressed

But the fact that such impulses δI, and the corresponding velocity changes δv, are both first-order infinitesimals was not spelled out in the *Principia*, and this neglect has been a source of continuous misunderstanding (see footnote 9). This problem is clarified in Sects. 3 and 4, where we demonstrate that in the continuum limit, $\delta I/\delta t$ is equal to the *accelerative* force F described in Prop. 6.[44]

Although Prop. 1 and Prop. 6 are equivalent, these two propositions lead to different mathematical expressions for central forces. In the *Principia*, Newton primarily applied the continuum expression in Prop. 6, to deduce the force acting on bodies moving on various orbital curves, while Leibniz and Hermann translated the impulse formulation in Prop. 1 in terms of first- and second-order differentials, leading to equations that could be solved for the orbital curve, given the radial dependence of the force.[45] The reason for not basing this translation on Prop. 6 was that in this proposition, the deflection QR for the measure of the central force is taken from the tangent

[44] In definition 7, Newton wrote that

> the accelerative quantity of centripetal force is the measure of this force that is proportional to the velocity that is generated in a given time (Newton 1999, 407).

[45] In this sense, Newton's and Leibniz's approach were not "equivalent in practice," as has been emphasized by N. Guicciardini (Guicciardini 1999, 250–255).

line at a fixed point P of a curve. But such a tangent cannot be expressed directly in terms of the differentials dx and dy associated with a curve in Cartesian coordinates x and y, while these differentials are directly related to the sides of the polygonal orbit in Prop. 1, when these sides are viewed as the chords of a limiting orbital curve. Then, the difference between the differentials for adjacent chords is a second-order differential, $ddx = dx' - dx$, and $ddy = dy' - dy$, that could be directly related to the action of the force impulse, giving rise to a differential equation of motion (Nauenberg 2010, 272–276).[46]

In some advanced applications, Newton also resorted to the description of forces by the periodic impulses introduced in Prop. 1. For example, in Corollaries 3 and 4 of Prop. 17, he outlined the effects of external perturbations to the elliptical motion due to an inverse square force, by a periodic sequence of impulses that *instantaneously* changed the parameters (major axis, eccentricity, and orientation) of the ellipse. In a manuscript in the Portsmouth collection of his mathematical papers (Newton 1969, 508–538), he also showed that in the continuum limit, this perturbation gave rise to differential equations for the time evolution of these parameters that now are attributed to Euler and Lagrange (Nauenberg 2000, 167–194).

To conclude, we quote Newton's own assessment of the equivalence of Prop. 1 and Prop. 6. When dealing with the effects of air resistance, in connection with Prop. 10, Book 2 (Nauenberg 2011, 567–587), he remarked in a *Scholium* that remained unpublished:

> "It would have been permissible to imagine that the projectile were to proceed in the chords, GH, HI, IK of the arcs, and be disturbed through the force of gravity and the force of resistance at the sole points G, H, I and K, exactly as in Proposition I of the first book a body was disturbed through an intermittent centripetal force; and then the chords to be infinitely diminishes so that the forces were rendered continuous. And the solution of the problem by this means would have turned out very easy" (Newton 1981, 366).[47]

Acknowledgments I would like to thank Niccolò Guicciardini for many interesting comments on several of the topics covered here.

7 Appendix 1: Prop. 6, Cor. 3, on curvature and force in polar coordinates

In 1671, Newton obtained the radius of curvature ρ of a general curve in coordinates that are equivalent to polar coordinates r, θ. Written in the language of Leibniz, in terms of the first-order differentials $dr, d\theta$ and the second-order differential ddr, it takes the form (Newton 1969, 169–173; Nauenberg 2005b, 345),

[46] Varignon, Johann Bernoulli, de Moivre, Keill and Cotes, were able to translate Prop. 6 into differential calculus after discovering that the second-order displacement from the tangent could be expressed in terms of the radius of curvature of the orbit, before Newton included this connection in the second edition of the *Principia*, in Cor. 3 of this proposition (Guicciardini 1999, 223, 220).

[47] I am indebted to N. Guicciardini for calling my attention to this interesting Scholium.

$$\frac{1}{\rho \cos^3(\alpha)} = \frac{2\mathrm{d}r^2 - r\mathrm{dd}r + r^2\mathrm{d}\theta^2}{r^3\mathrm{d}\theta^2}, \tag{5}$$

where

$$\cos(\alpha) = \frac{r\mathrm{d}\theta}{\sqrt{r^2\mathrm{d}\theta^2 + \mathrm{d}r^2}}, \tag{6}$$

and α is the angle between the radial line and the normal to the tangent of the curve at r, θ, shown in Fig. 6.

Since $SY = r\cos(\alpha)$, it follows from Prop. 6, Cor. 3, that[48]

$$F \propto \frac{1}{SY^2 \times PV} = \frac{1}{2r^2\rho\cos^3(\alpha)}, \tag{7}$$

where F is the central force (acceleration). Substituting Eq. 5 leads to the modern equation of classical mechanics,

$$r^2 F \propto \left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + 1\right)\frac{1}{r}, \tag{8}$$

apart from the constant of proportionality. Setting $\mathrm{d}t = (1/2)(SP \times QT)/l$, where l is the constant angular momentum for unit mass, one finds that this constant is l^2.

By expressing the second-order displacement QR and the first-order line QT (see Fig. 6) in polar coordinates, this equation can also be obtained from Newton's basic expression for F in Prop. 6,

$$r^2 F \propto \frac{QR}{QT^2}. \tag{9}$$

But such a derivation hides the important role of curvature in Newton's dynamics.

8 Appendix 2: Hermann's derivation of Kepler's area law

It is of historical interest that in 1716, Jacob Hermann obtained a derivation of the generalized Kepler area law that Newton had given for an impulsive central force in Prop. 1 of the *Principia*, by starting, instead, with his expression for a continuous central force in Prop. 6. For this purpose, he applied the relation for this force in terms of the radius of curvature of the trajectory.[49] Referring to the diagram, Fig. 8, associated with this proposition, he set the differential time interval $\delta t = PQ/v$, where v is the velocity at P, and then demonstrated that the ratio $l = SP \times QT/\delta t$ is a constant, corresponding to the angular momentum (for unit mass) $l = v \times SY$. In essence, Hermann's proof was a consistency check, filling a gap in Newton's demonstration

[48] For similar derivations by Bernoulli and by Varignon see reference (Nauenberg 2010, 285–289).

[49] First published in the 1713 (second) edition of the *Principia* in Prop. 6 Cor. 3.

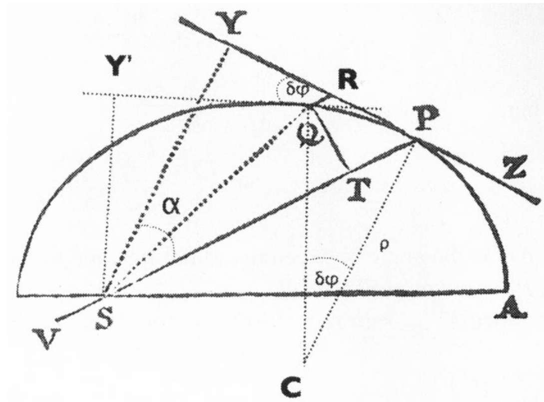


Fig. 8 The diagram for Prop. 6 in the second and third edition of the *Principia*, with the addition of the line QY' tangent at Q, the normal SY' to this line, and the radius of curvature ρ at P with center at C

of the equivalence of Prop. 6 with Prop. 1 that was lacking in the first edition of the *Principia*.[50] For completeness, we present a brief derivation of Hermann's proof in somewhat different form.[51]

Referring to Fig. 8, which corresponds to Newton's diagram for Prop. 6, Fig. 6, but with the addition of a line QY' tangent at Q, and the normal SY' to this line, let $p = SY$, $p' = SY'$, $\delta p = SY' - SY$, $q = PY$ and $\delta s = PQ$. Then

$$\delta p = -q\delta\phi, \tag{10}$$

where $\delta\phi$ is the angle between the tangential lines QY and QY' and also between the corresponding perpendicular lines PC at P, and QC at Q, which intersect a C. Hence, $PC = QC = \rho$, the curvature radius of the arc PQ and $\delta s = \rho\delta s$. We have

$$\delta v = F_T\delta t \tag{11}$$

where F_T is the tangential component of the force. Applying the expression for this component in terms of the radius of curvature ρ,

$$F_T = \frac{v^2}{\rho}\frac{q}{p}, \tag{12}$$

and substituting $\delta t = \delta s/v = \rho\delta\phi/v$ in Eq. 11,

$$\mathrm{d}v = \frac{vq}{p}\delta\phi. \tag{13}$$

[50] Hermann claimed that he was the first to give a proof of the area law (Guicciardini 1999, 216), which is valid provided it is qualified that his proof was based on Prop. 6. instead of Prop. 1.

[51] Hermann's original derivation is given in reference (Guicciardini 1999, 211–215).

Finally, substituting this relation for $\delta\phi$ in terms of δv in Eq. 10,

$$\delta p = -\frac{p}{v}\delta v \tag{14}$$

which implies that the angular momentum $l = pv$ is a constant of the motion.

To understand why the radius of curvature ρ does not appear in Eq. 13, we express the second-order displacement QR from inertial motion in the form

$$QR = \frac{\delta s \delta\phi}{2\cos\alpha} \tag{15}$$

where α is the angle between the lines SP and SY. Then, by Prop. 6,

$$F_T = \frac{2QR\sin\alpha}{\delta^2 t}, \tag{16}$$

and

$$\delta v = F_T \delta t = \frac{\delta s \delta\phi}{\delta t}\tan\alpha = v\delta\phi\frac{q}{p} \tag{17}$$

which corresponds to Eq. 13.

9 Appendix 3: An extension of Lemma 11

In this appendix, we discuss Lemma 11 for Prop. 6 and an extension for its application to the continuum limit in Prop. 1. In Fig. 9, associated with this lemma, Newton has

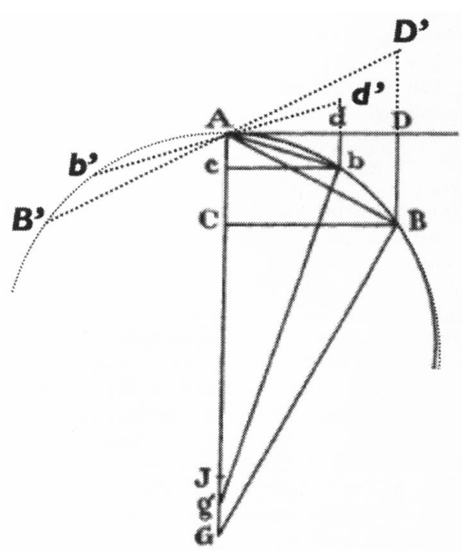


Fig. 9 Newtons diagram for Lemma 11, with an extended arc $\overline{B'b'A}$, and lines $B'AD'$, $b'Ad'$ superimposed in *dotted lines*

drawn the tangent AD at a point A of an arc $\overline{AB}$ of a curve and the perpendicular AG to this tangent. The line DB is the perpendicular from B intersecting the tangent at D, AB is the chord of the arc, and BG is a perpendicular to AB intersecting AG at G. The vertices d, b and g and corresponding lines are similar to those associated with D, B and G. If $\overline{AB}$ is the arc of a circle with radius $AJ/2$, then $G = g = J$ and

$$DB = \frac{AB^2}{AG}, \quad db = \frac{Ab^2}{Ag} \tag{18}$$

For a general arc $\overline{AB}$, in the limit that B and b both approach A, BG and bg intersect at J, where $AJ/2$ is the radius of curvature of the arc at A.

In the absence of the tangent line AD, as is the case in Prop. 1, Newton's proof is readily generalized by assuming that A is a point approximately in the middle of an arc $\overline{B'AB}$ and $b'Ab$. Let $AD' = AB$ be the extension of the chord AB' and Ad' the corresponding extension of the chord Ab'. Then, if $\overline{B'AB}$ is the arc of a circle,

$$D'B = 2\frac{AB^2}{AG}, \quad d'b = 2\frac{Ab^2}{Ag}, \tag{19}$$

Likewise, for a general arc $\overline{B'AB}$ the proof is the same as the previous one.

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