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Historical development of Teichmüller theory

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Abstract Originally, the expression "Teichmüller theory" referred to the theory that Oswald Teichmüller developed on deformations and on moduli spaces of marked Riemann surfaces. This theory is not an isolated field in mathematics. At different stages of its development, it received strong impetuses from analysis, geometry, and algebraic topology, and it had a major impact on other fields, including low-dimensional topology, algebraic topology, hyperbolic geometry, geometric group theory, representations of discrete groups in Lie groups, symplectic geometry, topological quantum field theory, theoretical physics, and there are certainly others. Of course, the impacts on these various fields are not equally important, but in some cases (namely, low-dimensional topology, algebraic geometry, and physics) the impact was crucial. At the same time, Teichmüller theory established important connections between the fields mentioned. This, in part, is a consequence of the diversity and the richness of the structure that Teichmüller space itself carries. From a more subjective point of view, the result of pondering on these connections and applications demonstrates the unity of mathematics. The aim of this paper is to survey the origin of Teichmüller theory and the development of its early major ideas.

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1 Introduction

Many people made substantial contributions to Teichmüller theory by discovering new ideas, or formulating important problems, or proving major results. In fact, to understand the origin of this theory, one has to ponder on the origins of both Riemann surface theory and of the theory of complex functions. In this respect, we shall mention a certain number of names, but two names appear to the forefront, Riemann and Teichmüller.

There are very few mathematicians whose works had as much influence on mathematics as Riemann. Ahlfors (1953, p. 53) wrote: "The spirit of Riemann will move future generations as it has moved us." Since our purpose in this paper is not to describe the work of Riemann, but only the part of it related to Teichmüller theory, his name will be mainly attached to two fundamental objects: Riemann surfaces and the Riemann moduli space. The concept of Riemann surface arose in Riemann's attempt to associate a surface to a multivalued algebraic function defined by a polynomial equation so that the function becomes univalent, or "uniform" (this is probably the origin of the word "uniformization"). Depending on the way one can represent a Riemann surface, one can regard Riemann's moduli space from different points of view: as a space of equivalence classes of branched coverings of surfaces, as a space of birational equivalence classes of algebraic curves, as a space of equivalence classes of function fields, etc. At Riemann's time, it was not easy to make precise relations between these various points of view. Riemann wanted to understand the family of Riemann surfaces, and his moduli space arose as an attempt to study the family of equivalence classes of function fields on algebraic curves. One important step in the study of the moduli space of surfaces was to count the number of parameters needed for the space of equivalence classes of surfaces of a given topological type. Riemann made such a parameter count although he did not have a topology or a complex structure on that space, and therefore it is hard to give a precise meaning to this parameter count. In this sense, Riemann's ideas are to be considered as a program rather than as results. This was the opinion of Ahlfors (1982, vol. II, p. 207), of Klein (1882, pp. X and XI of the Preface), and of several other people. Let us quote Ahlfors (1963a, p. 4):

> Riemann's classical problem of moduli is not a problem with a single aim, but a program to obtain maximum information about a whole complex of questions which can be viewed from several different angles.

Let us also quote Bers, from a survey he wrote on the uniformization problem that was started by Riemann (Bers 1976, p. 559):

> A significant mathematical problem, like the uniformization problem which appears as No. 22 on Hilbert's list, is never solved only once. Each generation of mathematicians, as if obeying Goethe's dictum *Was du ererbt von deinen Vätern hast, erwirb es, um es zu besitzen*,[1] rethinks and reworks solutions discovered by their predecessors, and fits these solutions into the current conceptual and notational framework. Because of this, proofs of important theorems become,

[1] That which you have inherited from your fathers, earn it in order to possess it.

and if by themselves, simpler and easier as time goes by—as Ahlfors observed in his 1938 lecture on uniformization. Also, and this is more important, one discovers that solved problems present further questions.

Monastyrsky wrote about Riemann (Monastyrsky 1987, p. 52): "It is difficult to recall another example in the history of nineteenth-century mathematics when a struggle for a rigorous proof led to such productive results." A twentieth-century example is certainly that of Teichmüller. We can quote Ahlfors again, from his 1953 survey *Development of the theory of conformal mapping and Riemann surfaces through a century* (Ahlfors 1953, pp. 93 and ff.) written for a conference celebrating the hundredth anniversary of Riemann's inaugural dissertation:

> [...] Riemann's writings are full of almost cryptic messages for the future. For instance, Riemann's mapping theorem is ultimately formulated in terms that would defy any attempt of proof, even with modern methods. [...] In the premature death of Teichmüller geometric function theory, like other branches of mathematics, suffered a grievous loss [...] Nothing could be more false than to say that classical function theory has solved its problems and has therefore outlived itself. Even without the introduction of completely new ideas the classical problem of modules, vague as it is, and—to mention a more recent example—the investigation of the true role played by Teichmüller's extremal quasiconformal mappings, are questions that can keep generations busy. [...] Geometric function theory of one variable is already a highly developed branch of mathematics, and it is not one in which an easily formulated classical problem awaits its solution. On the contrary it is a field in which the formulation of essential problems is almost as important as their solution.

Today, 59 years after Ahlfors' statement, Teichmüller theory is still growing, faster than ever before. New ideas and new connections between this theory and several domains in mathematics are still emerging.

The name of Teichmüller is attached to Teichmüller space, i.e, the deformation space of marked Riemann surfaces. Teichmüller space is an orbifold covering of Riemann's moduli space. Riemann's problem of moduli was formulated precisely in terms of complex geometry by Teichmüller, who emphasized the problem of understanding the intrinsic properties of the moduli space as a variety. In particular, he realized the need of using markings to unfold the singular points. Teichmüller (1939) after introducing the deformation space that carries now his name, provided this space with a topology that is induced from a metric (the so-called "Teichmüller metric"). Teichmüller (1944) equipped this space with the structure of a complex manifold, and this gave a meaning to Riemann's parameter count. We also owe to Teichmüller the use of the theory of quasiconformal mappings and that of quadratic differentials as two major ingredients in the deformation theory of Riemann surfaces.

One special moduli space was well understood right from the beginning, and it was certainly used as a motivation; this is the moduli space of the closed surface of genus one (equivalently, the space of elliptic functions). From the point of view of uniformization, this case was completely treated by Riemann (1857) (cf. Riemann's remark, on Jacobi's Inversion Problem, p. 93 in the English *Collected Works*: "Until now the only parts of that investigation which have been worked out fully are those mentioned in Sects. 1 and 2 and the first half of Sect. 3 related to elliptic functions"). We shall recall below that likewise, Teichmüller space of the closed surface of genus one was understood and completely analyzed by Teichmüller.

Both Riemann and Teichmüller had interior visions of theories on moduli that they did not develop completely because both of them had short lives.[2] Thus, they left for other mathematicians—and not minor ones—the task of filling out the details of their theories and transforming the sketches they left into complete and rigorous results. Riemann's ideas were commented and complemented by Klein, and they were the most influential on the works of Fricke, Koebe, Weierstrass, Poincaré, Weyl, and on the whole Italian school of algebraic geometry that grew up during the early years of the twentieth century (one can mention the names of Castelnuovo, Enriques, Torelli, Severi, and others). Teichmüller's ideas were influential on all subsequent developments of complex analysis, and they were at the basis of the works of Ahlfors, Bers, Rauch, Kodaira, Spencer, and other complex analysts and geometers. It was only after the work of generations of mathematicians that it became clear that the intuitions that Riemann and Teichmüller had were completely sound. Teichmüller theory had also an important impact on modern algebraic geometers, namely, Weil, Grothendieck, Mumford, and many others. In some sense, ideas in the last paper of Teichmüller (1944) contributed to the birth of the modern theory of moduli spaces in algebraic geometry and influenced the modern formulation of algebraic geometry by Grothendieck (1960–1961). The fact that a large number of people contributed to the foundations and the development of a single mathematical theory, each person introducing an idea, or a technical tool, and sometimes a new problem, is an illustration of how mathematics generally grows.

André Weil wrote, in a commentary to a 1958 paper in his *Collected Works* (1979, vol. II, p. 545)

> One should almost write a book, or at least a beautiful article, on the history of moduli and moduli varieties; this has to be traced back to the beginning of the theory of elliptic functions.

The present article is a small contribution to that project. The period we concentrate on is roughly 1939–1959.

In what follows, we shall refer to foundational articles and surveys written by the founders themselves and others written by other authors in the field.

We are grateful to the following colleagues with whom we corresponded regarding Teichmüller's contribution: Oliver Baues, Peter Buser, Bill Fulton, Bill Goldman, Bill Harvey, Linda Keen, Chris Leininger, Enrico Leuzinger, John Ratcliffe, and Sumio Yamada. We are indebted to Bill Abikoff, to Scott Wolpert, and especially to Jeremy Gray for several corrections and for valuable suggestions on a preliminary version of this paper. Remarks and corrections by Clifford Earle were essential. We are most grateful to Norbert A'Campo for many valuable discussions and for his shared enthusiasm.

[2] Riemann, born in 1826, died at the age of 39, and Teichmüller, born in 1913, died at the age of 30.

2 The quasiconformal theory

The classical proofs of several major results in Teichmüller theory use the theory of quasiconformal mappings even though the term "quasiconformal mapping" does not always appear in the statements of these results. The results include the fact that Teichmüller space is homeomorphic to a cell, and the fact that it admits a canonical holomorphic embedding in some $\mathbb{C}^N$.

We present in this section some aspects of the development of the theory of quasiconformal mappings because of its importance. We start with Grötzsch's definition and we shall review the use of this theory in Teichmüller theory. We shall also describe some major later developments and generalizations of quasiconformal mapping theory.

Before Teichmüller, work on quasiconformal mappings was done in Grötzsch (1932), Ahlfors (1935), and Lavrentieff (1935). Ahlfors gave a brief summary of the early use of quasiconformal maps (Ahlfors 1964, p. 153). According to this account, the first mention of quasiconformal mappings (by a different name) occurred in a 1928 paper by Grötzsch. Ahlfors adds, however, that the problem that Grötzsch solved there, of finding the best mapping between a rectangle and a square, was first considered as a mere curiosity, and that the full strength of quasiconformal mappings and their use in the deformation theory of Riemann surfaces was first realized by Teichmüller.[3] Let us quote Ahlfors (1964, p. 156):

> The very genesis of quasiconformal mappings was connected with the elementary extremal problem formulated by Grötzsch. Teichmüller was the first to extract a general principle: In a class of mappings it is required to find one whose maximal dilatation is a minimum. It is to be expected that the solution is unique, and that the extremal mapping is characterized by simple properties.

Likewise, Bers (1970, p. 180) writes:

> [The approach of] applying quasiconformal mappings to the "problem of moduli" was pioneered by Teichmüller 25 years ago.

In the rest of this section, after a quick review of the work of Grötzsch and of other pioneers, we shall survey the work of Teichmüller on quasiconformal mappings and their subsequent use in Teichmüller theory.

In (Grötzsch 1932), the author introduced the notion of quasiconformality of a map between surfaces as a deviation from conformality. The quasiconformal deviation (or "dilatation") of a differentiable map is defined as the supremum over all points on the surface of the ratio of the major to the minor axes of the infinitesimal ellipses that are images of infinitesimal circles. The following formula for the dilatation of a differentiable map f between two domains in the plane, at a point z, was given by Grötzsch:

$$K_z(f) = \frac{|f_z(z)| + |f_{\bar{z}}(z)|}{|f_z(z)| - |f_{\bar{z}}(z)|}.$$

[3] Teichmüller, in his main paper (Teichmüller 1939), quotes the papers of Grötzsch and of Ahlfors for the notion of quasiconformal maps.

The dilatation of f is then

$$K(f) = \sup_z K_f(z).$$

The dilatation of a map is invariant under composition by conformal mappings.

Ahlfors (1964, p. 153) writes that Grötzsch's paper was "buried in a small journal," that it first remained unnoticed, and that he himself learned of it by word of mouth in 1931 and used it only in 1936.

In 1935, in the Soviet Union, Lavrentieff (1935) published a paper on partial differential equations in which he introduced a notion of *almost-analytic function* ("fonction presque-analytique") which is equivalent to the notion of quasiconformal mapping considered by Grötzsch. Lavrentiev used these mappings in a study of the partial differential equation that controls the flow of an incompressible fluid. He proved in (Lavrentieff 1935) that if f is a sense-preserving homeomorphism between two simply connected domains D and D' of the complex plane, then for almost all x in D, one can define the eccentricity of the infinitesimal ellipses that are images by f of infinitesimal circles in D.

Ahlfors (1964, p. 153) says in his paper that it was not clear to him how the name "quasiconformal mapping" originated. In a commentary on that paper (Ahlfors 1982, vol. 1, p. 213), Ahlfors writes:

> Quasiconformal mappings had been introduced already in 1928 by Grötzch, who called them "nichtconforme Abbildungen" and almost simultaneously with my paper by A. M. Lavrentiev, who used the term "fonctions presque analytiques". I have been credited with being the first to have used the name "quasiconformal," which has become standard. The truth is that I cannot recollect having invented the name, but I have also not been able to locate it elsewhere. Little did I know at the time what an important role quasiconformal mappings would come to play in my own work.

In any case, the expression appears in print in the fundamental paper on covering surfaces (Ahlfors 1935).[4] Again, according to Ahlfors (1964, p. 153):

> In a few years it became increasingly clear that quasiconformal mappings merited a theory of their own, a view that gained its strongest support from the more prophetic than rigorous writings of Teichmüller in 1938–1940.

In (Ahlfors 1978, pp. 72–73), he makes the following remarks on the status of Teichmüller's use of quasiconformal mappings

> Quasiconformal mappings might have remained a rather obscure and peripheral object of study if it had not been for Oswald Teichmüller, an exceptionally gifted and intense young mathematician and political fanatic, who suddenly made a fascinating and unexpected discovery. At that time, many special extremal problems in quasiconformal mappings had already been solved, but these were isolated without a connecting idea. In 1939, he presented to the Prussian Academy a

[4] This is the paper for which Ahlfors earned the Fields medal.

> now famous paper which marks the rebirth of quasiconformal mappings as a new discipline which completely overshadows the rather modest beginnings of the theory. With remarkable intuition he made a synthesis of what was known and proceeded to announce a bold outline of a new program which he presents, rather dramatically, as the result of a sudden revelation that occurred to him at night. His main discovery was that the extremal problem of quasiconformal mapping, when applied to Riemann surfaces, leads automatically to an intimate connection with the holomorphic quadratic differentials on the surface. With this connection the whole theory takes on a completely different complexion: A problem concerned with non-conformal mappings turns out to have a solution which is expressed in terms of holomorphic differentials, so that in reality the problem belongs to classical function theory.

To say it briefly, Grötzsch solved the question of extremal quasiconformal mappings between quadrilaterals, and after that, Teichmüller addressed the problem of extremal quasiconformal mappings between arbitrary Riemann surfaces. Teichmüller (1939) asserted that these mappings exist, that they are affine with respect to some singular Euclidean structure on the surface, and that they are unique in their homotopy classes. He introduced the notion of quadratic differential as an ingredient in this theory, which provided an analytic setting to describe a quasiconformal mapping and the underlying Euclidean structure for the extremal map and he gave a formula relating the Beltrami coefficient of this extremal map to the quadratic differential. In that paper, Teichmüller used the notion of quasiconformality to define a metric on Teichmüller space. Starting from this work, quasiconformal mappings, used in measuring distances between conformal structures, instead of being only "non-conformal" (as Grötzsch had called them), became a basic ingredient of the theory of conformal mappings. We shall further comment on this below. According to Ahlfors (1964, p. 156), after he learned about the work by Grötzsch on extremal mappings between quadrilaterals, Teichmüller worked out the same problem for pentagons, and proved a result which according to Ahlfors, is "already a sophisticated result. [...] The problem is more complicated for hexagons, but can still be solved by the same method." Ahlfors then adds:

> Efforts to solve the [more general] problems had failed, but in a flash of genius Teichmüller came forth with a daring conjecture which he himself admitted was based on almost nothing. In the few cases that could be solved explicitly he observed that the solution was always given by a conformal mapping, followed by an affine mapping, and by another conformal mapping. Teichmüller conjectured that the same would be true in all problems. For topological reasons, however, the conformal mappings cannot be single-valued mappings, except in the simplest cases, and by heuristic reasoning (we could say by sleight of hand), he concluded that they must be determined, locally, by quadratic differentials with certain special properties. [...] After hard work Teichmüller finally managed to prove his conjecture, but it took several years to produce a reasonably neat proof. Even now, the available proofs leave much to be desired in way of directness and simplicity.

> The most remarkable feature of this result is the direct connection it establishes between quasiconformal mappings and quadratic differentials. This connection is entirely unexpected and is a sure sign that the role of quasiconformal mappings in function theory is far from superficial. Furthermore, any classically trained analyst cannot dismiss as pure coincidence that the number of linearly independent quadratic differentials is precisely equal to the number of parameters, or moduli, which according to Riemann characterize a closed Riemann surface of given genus. Teichmüller does not fail to make the most of this observation.

The theory of quasiconformal maps has since been developed, extended and applied to many different situations. The standard introduction and reference to quasiconformal maps is the book (Ahlfors 1966), the first edition of which had educated generations of mathematician. A book that emphasizes the low regularity character of quasiconformal maps is Lehto and Virtanen (1965).

3 Teichmüller space

Teichmüller space appeared for the first time in the literature in the paper by Teichmüller (1939) entitled *Extremale quasikonforme Abbildungen und quadratische Differentiale*. Before that, a space of Riemann surfaces was only implicit in Klein (1883) and in Klein and Robert (1897). A space of polygons in the hyperbolic plane that parametrize Riemann surfaces is contained in the work of Poincaré (1884), who used it in an approach to the uniformization problem known as the "continuity method"; cf. the discussion in Gray (1986), also quoted in Ratcliffe (2006). A good reference for this method and more generally for the uniformization theorem is the book (Saint-Gervais 2010).

The name "Teichmüller space" appears in a 1953–1954 survey article by Ahlfors, but it seems that the expression was introduced by Weil; see the comments by Weil in Weil's *Collected Works* (vol. II, p. 546), on his paper (Weil 1958a), where Weil writes: "[…] this led me to the decision of writing up my observations on the moduli of curves and on what I called 'Teichmüller space' […]"

Teichmüller's (1939) paper is probably his most influential one, and it contains the foundations of the theory. The paper contains ideas and few technical details. As a matter of fact, this is the case with all of Teichmüller's papers on the subject. In the introduction, Teichmüller declares that for some of his results, he gives only an idea of the proof, and that his hope is to give sufficient heuristic justification that will lead to complete proofs, and that this publication will encourage others to study these problems. The setting is very general: an orientable or nonorientable surfaces of finite topological type, with or without boundary, of genus g, with γ cross-caps, n boundary curves, h interior distinguished points, and k boundary distinguished points. Teichmüller calls this surface a *Hauptbereich*, i.e., a "principal region." The surface is equipped with an atlas the charts of which take their values in the complex plane with coordinate changes that are conformal or anti-conformal.[5] Teichmüller introduces the

[5] We note that Teichmüller includes anti-conformal maps in his setting because the surface he is considering might be nonorientable. We recall that the definition of an abstract Riemann surface is due to Weyl (1913).

notion of marked surface (where the marking is a quasiconformal map) and the notion of equivalence relation between marked surfaces. The mappings he considers preserve the set of distinguished points.

Teichmüller states that the set of equivalence classes of such Riemann surfaces (the space which is called today "Teichmüller space") is a topological manifold homeomorphic to $\mathbb{R}^{\sigma}$, with $\sigma - \rho = 6g - 6 + 3\gamma + 3n + 2h + k$, where ρ is the number of parameters of the group of holomorphic transformations of the surface. For the annulus, $\rho = 1$; for the torus, $\rho = 2$ and for a surface of higher genus, $\rho = 0$. Teichmüller's formula is valid in all cases.[6]

In this paper, Teichmüller starts by reviewing Grötzsch's work on conformal maps between rectangles. Grötzsch showed that between any two rectangles with marked vertices, a diffeomorphism which preserves the marking of the vertices and that has least dilatation is the affine map among those that preserve these marked vertices. Teichmüller gives to such a map the name *extremal quasiconformal map*. He then solves the problem of extremal quasiconformal maps for annuli, and he considers tori, showing that their deformation space is parameterized by the upper half-plane. After that, he considers general surfaces ("principal regions"). He notes that since, in general, Riemann surfaces are not conformally equivalent, one has to look at notions involving quasiconformality. He considers the problem of finding maps between two principal regions that have the least quasiconformal dilatation. He states that for such a map, the quasiconformal dilatation must be constant, i.e., it must be the same at each point. He then formulates the problem of characterizing infinitesimally extremal quasiconformal mappings. He says that for the description of the best quasiconformal maps between two surfaces, one can find vector fields on the surfaces that give the infinitesimal change of direction in the major ellipses. He makes this more precise by introducing the notion of meromorphic quadratic differentials.[7] He then defines natural local holomorphic parameters on the surface associated to quadratic differentials. These differentials establish a link between quasiconformal maps and complex structures. Indeed, Teichmüller gives a very simple local description of extremal homeomorphisms between two Riemann surfaces by means of two holomorphic quadratic differentials on these

Footnote 5 continued
Ahlfors writes in (Ahlfors 1953) about (Weyl 1913) "H. Weyl's book *Die Idee der Riemannschen Fläche* was the real eye-opener. Pursuing the ideas of Klein it brings, for the first time, a rigorous and general definition of a Riemann surface, and it marks the death of the glue-and-scissors period." Unlike the definition used by Riemann (using branched coverng of the sphere) or by Weierstrass (using power series expansions), the surfaces in Weyl's definition are not associated to specific analytic functions, and they are free from any embedding of the surface in some Euclidean space. It is a theorem that any compact Riemann surface is biholomorphically equivalent to a surface defined by an algebraic curve or to a curve defined locally by power series.

6 In the case of oriented surfaces without boundary and with no distinguished points, the dimension is $6g - 6 + \rho$. In the modern literature on the subject, one usually distinguishes between the case of surfaces of genus ≥ 2 (the dimension of the space is $6g - 6$ in the closed case) and genus 1 (the dimension is 2, which is not what the formula $6g - 6$ gives).

7 Teichmüller introduced the more general notion of a differential of order n, as an object which in the local parameters can be written as $\varphi(t)\,\mathrm{d}t^n$, where $\varphi(t)$ is a meromorphic function, satisfying an invariance transformation rule with respect to local parameter changes. For $n = 1$, these are the usual differential forms on Riemann surfaces and for $n = 2$ these are the quadratic differentials familiar to Teichmüller theorists.

surfaces, written in local parameters as $d\zeta^2$ and $d\zeta'^2$ with $\zeta = \xi + i\eta$ and $\zeta' = \xi + i\eta'$. The extremal map is then given by $\xi' = K\xi$, $\eta' = \eta$. The local dilatation of this map is constant (independent of the point). Denoting the local dilatation of an extremal map f_0 by $K(f_0)$, Teichmüller then uses the quantity $\log K(f_0)$ to define a metric on the set of equivalence classes of marked Riemann surfaces. This is the metric which was called later on the *Teichmüller metric*. In the case of genus one (elliptic curves), the space of equivalence classes of marked Riemann surfaces is parametrized by the upper half-plane and the Teichmüller metric coincides with the hyperbolic metric. Teichmüller proves that quadratic differentials provide the description of extremal infinitesimal quasiconformal mappings, and he conjectures that the converse is also true. He states several forms of this conjecture, including a global (as opposed to infinitesimal) version, stating the uniqueness of extremal mappings (i.e., that all extremal mappings are associated to quadratic differentials). He supports his conjecture by special cases which he works out in detail. He then makes a detailed discussion of quasiconformal maps on ring domains. He discusses the Grötzsch-Ahlfors method and the paper ends with a discussion on the role of quasiconformal mappings in the study of conformal mappings.

Teichmüller proved his conjecture in a later paper (Teichmüller 1943), in the form of an existence result (see the above comments by Ahlfors and the comments of Ahlfors and Bers which we reproduce below). Teichmüller's existence proof is also based on quadratic differentials, and it makes use of the continuity method. By means of this existence and uniqueness result, Teichmüller states the existence and uniqueness of geodesics between any two points in Teichmüller space and he uses this to show that the space is homeomorphic to an open ball in the vector space of holomorphic quadratic differentials, equipped with a suitable norm. He shows that this metric is a Finsler metric, i.e., the distance between two points is the infimum of the lengths of piecewise smooth paths joining them, where length of paths is measured by integrating norms of tangent vectors. The dimension count for Teichmüller space follows from its parametrization by the vector space of quadratic differentials that provides the ray structure, the dimension of this vector space being known by the theorem of Riemann–Roch.

Other results of Teichmüller that are contained in Teichmüller (1939) include the fact that the mapping class group acts properly discontinuously on Teichmüller space and that moduli space is the quotient of Teichmüller space by this action. They also include the investigation of the singular flat metric on the surface induced by a quadratic differential and the representation of Riemann surfaces by Fuchsian groups.

Teichmüller's (1939, 1943) papers were first considered by the other mathematicians as a program and not as finished papers. According to Abikoff, the fact that several ideas were sketched is only consistent with the tradition of the journal in which the article was published. Abikoff (1986) writes: "The tradition of *Deutsche Mathematik* is one of heuristic argument and contempt for formal proof. Busemann notes that Teichmüller manifested those traits early in his career but when pressed could offer a formal proof." It was after several years of hard work by several mathematicians that all the arguments in these papers were considered as being sound. In any case, Teichmüller's 1939 paper stimulated several authors to study the problems considered, and it is interesting to review some of the work done in this area.

In his ICM 1958 paper (Bers 1960c), Bers sketched a proof of Teichmüller's characterization of extremal quasiconformal mappings, of the fact that Teichmüller space is a cell, and of the fact that this space has a natural complex structure. Bers writes:

> Much of this work consists in clarifying and verifying assertions of Teichmüller whose bold ideas, though sometimes stated awkwardly and without complete proofs, influenced all recent investigators [on the classical problem of moduli], as well as the work of Kodaira and Spencer on the higher dimensional case.

Ahlfors' (1954) paper on quasiconformal mappings was motivated by Teichmüller's work. In the commentary to that paper, Ahlfors writes (*Collected papers*, Ahlfors 1982, p. 1):

> It had become increasingly evident that Teichmüller's ideas would profoundly influence analysis and especially the theory of functions of one complex variable, although nobody at that time could foresee the extent to which this would be true. The foundations of the theory were not commensurate with the loftiness of Teichmüller's vision, and I thought it was time to reexamine the basic concepts. My paper has serious shortcomings, but it has nevertheless been very influential and has led to a resurgence of interest in quasiconformal mappings and Teichmüller theory. [...] In particular, a complete proof of the uniqueness part of Teichmüller's theorem was included. Like all the other known proofs of the uniqueness it was modeled on Teichmüller's own proof, which used uniformization and the length-area method. Where Teichmüller was sketchy I tried to be more precise. In the original paper Teichmüller did not prove the existence part of his theorem, but in a following paper (Teichmüller 1943) he gave a proof based on a continuity method. I found his proof rather hard to read and although I did not doubt its validity I thought that a direct variational proof would be preferable. My attempted proof on these lines had a flaw, and even my subsequent correction does not convince me today. In any case my attempt was too complicated and did not deserve to succeed. Later, L. Bers (1960b) published a very clear version of Teichmüller's proof. The final credit belongs to R. Hamilton (1969), who gave an amazingly short and direct proof of the existence theorem. The consensus today is that the existence part is easier to prove than the uniqueness.

The proof of Teichmüller's existence theorem by Bers (1960b) is close to Teichmüller's 1943 proof and it also uses the continuity method.[8] Bers' proof uses the existence of solutions to the Beltrami equation. Hamilton's proof to which Ahlfors refers does not use the continuity method. In his paper, Ahlfors (1954, p. 4) writes:

[8] The proof is based on Teichmüller's ideas: Given a Riemann surface S, let $\mathcal{T}$ be its Teichmüller space and $Q_1(S)$ the unit ball in the vector space quadratic differentials of S (equipped with the L^1 norm, i.e., the integral of the modulus of the differential on the surface). There is a map $\pi : Q_1(S) \to \mathcal{T}(S)$ which to every quadratic ϕ of essential norm satisfying $0 < \|\phi\|_{\infty} < 1$ associates the element of Teichmüller space defined by the Beltrami equation of coefficient $\mu = k\frac{\overline{\phi}}{|\phi|}$. The new element is obtained by stretching, in the natural coordinates of ϕ, by a factor $K = \frac{1+k}{1-k}$. The image $\pi(0)$ is the structure S itself. The map π is injective and continuous. A dimension count, again by the Riemann–Roch theorem, shows that the map π is a homeomorphism.

In a systematic way the problem of extremal quasiconformal mapping was taken up by Teichmüller in a brilliant and unconventional paper (Teichmüller 1939). He formulates the general problem and, although unable to give a binding proof, is led by heuristic arguments to a highly elegant conjectured solution. The paper contains numerous fundamental applications which clearly show the importance of the problem.

In a later publication, Teichmüller (1943) has offered a proof of his main conjecture. In many respects this proof is an anticlimax when compared with the original article. It is based on the method of continuity, which of all classical methods is the least satisfactory because of its nature of a posteriori verification. It is also unduly complicated.

The main purpose of the present paper is to give a variational proof of Teichmüller's theorem. It is not our contention that the new proof is simpler than Teichmüller's, especially if the latter would be rewritten with greater conciseness. We claim for it merely the merit of greater directness. It relies heavily on real variable techniques, and can therefore hardly be classified as elementary.

One of the last papers that Bers wrote—in fact, his penultimate paper—is entitled *On Teichmüller's proof of Teichmüller's theorem* (Bers 1986). One could almost say that Bers struggled all his life to understand Teichmüller's ideas and to find a proof of that theorem that would satisfy him. In the paper Bers (1986), Bers returned to the assertion he made in his paper (Bers 1960b) that the parameters determining a marked Fuchsian group depend continuously on the parameters determining the Teichmüller Beltrami coefficient. At the same time, this gave another proof of the existence part of Teichmüller's theorem, with an argument which is again close to Teichmüller's argument. Bers (1986, p. 58) writes:

This note is a postscript to a paper (Bers 1960b) which I published many years ago and is, like that paper, essentially expository. [...] In proving a basic continuity assertion (Lemma 1 in §14C of Bers (1960b)) I made use of a property of quasiconformal mappings which belongs to the theory of quasiconformal mappings with bounded measurable Beltrami coefficients (and seems not to have been known to Beltrami). Some readers concluded that the use of that theory was indispensable for the proof of Teichmüller's theorem. This is not so, and Teichmüller's own argument is correct. This argument can be further simplified and this simplified argument will be presented here. Then we will briefly describe Teichmüller's actual argument.

We talked about Ahlfors and Bers, but there is another major figure on whom Teichmüller's papers were influential; this is André Weil. In his 1958 report at the Bourbaki seminar (Weil 1958c, p. 413), Weil writes (translated from the French):

Teichmüller proved, first heuristically, then (it is said) rigorously, that we can define a topological space, homeomorphic to an open ball of dimension $6g-6$, whose points are "naturally" in one-to-one correspondence with the classes of Teichmüller surfaces (classes with respect to the equivalence relation defined by isomorphism); Ahlfors gave another proof of the same fact. In what follows, we shall make a local study of this "Teichmüller space," and mainly, define the structure of a complex manifold (of dimension $3g - 3$), and even, of a Kähler manifold. After that, a remark by L. Bers (very simple, but based, it seems, on delicate results on partial differential equations with discontinuous coefficients) allows to recover very rapidly the principal result of Teichmüller.

In the same year, in a paper dedicated to Emil Artin, Weil (1958a, p. 413) writes:

> Perhaps the most remarkable of Teichmüller's results is the following: when provided with a rather obvious "natural" topology, the set Θ of all classes of mutually isomorphic Teichmüller surfaces is homeomorphic to an open cell of real dimension $6g-6$. This global result will neither be used nor discussed in the following pages, the chief purpose of which is to consider the local properties of Θ and to define on it a "natural" complex analytic structure, of complex dimension $3g - 3$, and a "natural" Hermitian metric.

In the same paper (Weil 1958a, p. 383), Weil describes a map from the Torelli space into the Siegel space. Combining this with the map from Teichmüller space to the Torelli space, he gets a map from Teichmüller space Θ to the Siegel space.[9] He declares that this map is holomorphic when Θ is provided with its natural complex structure. The image is an analytic subvariety of the Siegel space the points of which are all smooth except those corresponding to a hyperelliptic Riemann surface. (We shall mention below Rauch's work on that case). Weil then says

> As for $\mathcal{M}$ (moduli space) there is virtually no doubt that it can be provided with a structure of algebraic variety (non-complete of course, and with multiple points), the "variety of moduli," so the natural mapping of Θ onto $\mathcal{M}$ is holomorphic. [...] In order to justify the statements that we have made so far, we shall make use of the Kodaira–Spencer technique of variation of complex structures. This can be introduced in an elementary manner in the case of complex dimension 1, which alone concerns us here; this, in fact, had already been done by Teichmüller; but he had so mixed it up with his ideas concerning quasiconformal mappings that much of its intrinsic simplicity got lost. Perhaps the worst feature of his treatment, in the eyes of the differential geometer, is that his extremal mappings are destructive of the differentiable structure; this corresponds to the fact that his metric on Θ is almost certainly not to be defined by a ds^2, even though it is presumably a Finsler metric.

Weil (1958b, p. 91) writes:

> [...] Teichmüller's chief contribution was to define on T (Teichmüller's space) a certain topology, the "natural" one in a sense described below, and then to prove that T, with this topology, is homeomorphic to an open cell of real dimension $6g - 6$. So far, I have mainly been concerned with the local properties of the Teichmüller space and of its "natural" complex analytic structure. The definition of the latter depends upon ideas introduced by Teichmüller himself, but which

[9] The definitions of the Torelli space, the Siegel space and the map between them are recalled later on in this paper.

> do not appear to have been fully understood until Kodaira and Spencer attacked similar problems for higher dimensions.

Weil then gives an outline of these ideas. On the same page (p. 391), he describes a homeomorphism between T and a subset of $\mathbb{R}^{6g-6}$ using generators for the fundamental group and their images as matrices representing group elements of hyperbolic transformations. He then writes:

> This gives a one-to-one mapping of T onto a subset of the coordinate space $\mathbb{R}^{6g-6}$. It turns out that the latter subset is open, and that the mapping and its inverse are indefinitely differentiable (and even presumably, real-analytic) if T is provided with its "natural" differentiable structure; the proof of the latter fact is due to L. Bers.

We finally quote the comments on the AF final report that Weil wrote as an addendum to his *Collected Works* edition (Weil 1979, vol. II, p. 545) (translation from the French):

> The theory of moduli of curves, inaugurated by Riemann, has taken two decisive steps in our time, first in 1935, with the work of Siegel (*Ges. Abh. No. 20*, §13, vol. I, pp. 394–405), and then with the remarkable works of Teichmüller; it is true that some concerns were raised about the latter, but these were finally cleared up by Ahlfors in 1953 (J. d'An. Math. 3, pp. 1–58). On the other hand we realized at the end that Siegel's discovery of automorphic functions belonging to the symplectic group applied first to the moduli of Abelian varieties and then, only by extension, to those of curves, via their Jacobians and using Torelli's theorem. Thus, thanks to Siegel, we have at our disposal the first example of a theory of moduli for varieties of dimension >1. We owe it to Kodaira and Spencer (*Ann. of Math.* 67 (1958), pp. 328–566) to have discovered that progress on cohomology allows us not only to address a new aspect of the same problem but also, at least from the local point of view, to tackle the general case of complex structures on varieties.

4 On the complex structure of Teichmüller space

The existence of a natural complex structure on Teichmüller space and on the quotient moduli space is one of the major results not only in Teichmüller theory but also in the field of complex geometry. This complex structure is not easy to describe, except in the case where the Teichmüller space has complex dimension one, i.e., if the surface is the closed torus or the once-punctured torus; in these cases, the Teichmüller space is canonically biholomorphic to the upper half-plane equipped with its complex structure. It can be noted that this result is not surprising since, by the Riemann mapping theorem, there are only three distinct complex structures on simply-connected Riemann surfaces, namely, the sphere, the complex plane, and the upper half-plane, and that for the Teichmüller space of the torus, one can easily exclude the first two. We now discuss higher-dimensional Teichmüller spaces.

Let us start with a general remark. We already recalled that any closed Riemann surface is conformally equivalent to a complex algebraic curve and, more precisely, to the set of zeros of a collection of homogeneous polynomials with complex coefficients in the projective space $\mathbb{P}^3$. Thus, one way of describing a varying complex structure on a given topological surface would be to represent this surface as the set of zeros of a collection of complex homogeneous polynomials and then to vary in a holomorphic way the coefficients of these polynomials. In principle, this gives a way to get hold on the complex structure of moduli space. Since the number of polynomials and their degrees depend on the Riemann surfaces and might change, making this idea precise enough for studying the structure, and in particular studying the singular points of moduli space, the topology of that space, or even for simply computing the dimension turns out to be very difficult.

There are other approaches to the complex structure of moduli space which are more efficient, in which one starts by proving that Teichmüller space is a complex manifold (in particular a complex space without singularities). Indeed, Teichmüller space is a branched cover of the moduli space, and it was already realized by Teichmüller himself that it is much easier to start by defining a complex structure on Teichmüller space than on moduli space. This was done in his 1944 paper "Veränderliche Riemannsche Flächen" (*Variable Riemann surfaces*) (Teichmüller 1944), his last paper on the theory. It also turns out that whereas Teichmüller space is a complex manifold, moduli space is a complex space which is not a complex manifold. By the way, it seems that the notion of complex manifold in dimension >1 was completely new at that time, and it is safe to say that Teichmüller space is historically the first interesting example of such a manifold.[10] After Teichmüller, the complex structure of Teichmüller space was studied by Rauch, Weil, Ahlfors and Bers.

In his 1944 paper, Teichmüller (1944) defined the complex structure on Teichmüller space by introducing the notion of a holomorphic family of Riemann surfaces ("analytische Schar Riemannscher Flächen"). He proved the existence of a universal object that was later called the "Teichmüller universal curve." This is a fiber space over Teichmüller space where above each point there is a Riemann surface which represents the equivalence class of that point. The existence of the structure of a complex manifold on that space gives the complex structure on Teichmüller space. We shall say more about that below. This is one way of defining this complex structure, and there are many others. Teichmüller's paper has been translated into English by Annette A'Campo-Neuen, see Teichmüller (1944); see also the commentary in A'Campo-Neuen et al. (2013).

[10] Reinhold Remmert, one of the founders of the theory of several complex variables, says the following, regarding the beginning of the theory of complex manifolds of higher dimensions (Remmert 1998, p. 225): "It seems difficult to locate the first paper where complex manifolds explicitly occur. In 1944 they appear in Teichmüller's work on "Veränderliche Riemannsche Flächen" (Teichmüller 1944, *Collected Papers*, p. 714); here we find for the first time the German expression "komplex analytische Mannigfaltigkeit." The English "complex manifold" occurs in Chern's work (Chern 1946, p. 103); he recalls the definition (by an atlas) just in passing. And in 1947 we find "variété analytique complexe" in the title of Weil (1947). Overnight complex manifolds blossomed everywhere." One may add that complex domains of higher dimension, i.e., domains of $\mathbb{C}^n$, $n \geq 2$, had been studied earlier by several people, in particular, C. L. Siegel, in the context of modular forms, or automorphic functions of many variables, but Teichmüller space was not known yet to be a complex domain. An embedding of that space in some $\mathbb{C}^N$ was found later on. See the Appendix of this paper.

Before giving an overview of the various ways in which the complex structure on Teichmüller space can be defined, let us quote some of the major mathematicians who worked on the subject.

In his paper, Ahlfors (1961a, p. 156) writes that "in recent work of several persons it has been established that the Teichmüller space of closed Riemann surfaces of genus $g > 1$ carries a natural complex analytic structure," and he refers to his own paper (Ahlfors 1960) and to (Kodaira and Spencer 1958). It is probable that since Teichmüller's paper (Teichmüller 1944) is written in the language of algebraic geometry, it did not attract Ahlfors' attention very much.

In the introduction to another 1960 paper (Ahlfors 1960), concerning the complex structure of Teichmüller space, Ahlfors observed that Teichmüller set his sights extremely high, without claiming complete success, and as a result Ahlfors found himself unable to determine how much Teichmüller had proved. He then presented his own construction of the complex structure, by means of techniques from (Teichmüller 1939), but not (Teichmüller 1944). In a commentary on the same paper, written as a note in his *Collected Papers* edition (Ahlfors 1982, vol. II), Ahlfors mentions the work of Rauch, who had by then (Rauch 1955a,b) published two notes in which he settled the problem for nonhyperelliptic surfaces. Ahlfors commented that he had by then a proof in the nonhyperelliptic case, but was delaying publication until he was able to construct the complex structure for the whole space $\mathcal{T}_g$. He then observed that "Today Rauch's and my methods would both be considered obsolete, but the variational formulae they led to are still of interest."

In the same year (1960), Grothendieck established the existence of a complex manifold structure on Teichmüller space that follows Teichmüller's algebro-geometric ideas, probably without having read Teichmüller's paper. See the commentary (A'Campo-Neuen et al. 2013) for more details.

In his 1964 survey paper on quasiconformal maps and their applications (Ahlfors 1964, p. 152), Ahlfors writes:

> Equally important, and in their ultimate consequences perhaps overwhelmingly so, were the observations that poured forth from the uncanny brain of Teichmüller. For better or for worse, his ghost has risen from an early grave to haunt a great deal of the thinking that has gone on in postwar function theory. An excellent way to acquaint the mathematical public with the trends in my special field is to relate some of the experiences of this author and fellow specialists in dealing with the heritage of Teichmüller.
> [...] Actually, the classical problem calls for a complex structure, and Teichmüller was well aware that his method did not yet solve this problem. In fact, he refutes his own method and says explicitly that it is no good for the purpose of establishing a complex structure. This prediction turned out to be wrong.
> [...] In a final effort Teichmüller produced a solution of the structure problem, by an entirely different method, but it was so cumbersome that it is doubtful whether anybody else has checked all the details. Many years later E. Rauch and this author found, independently of each other that the original method, used primarily as a convenient parametrization, does lead fairly easily to the existence of a natural complex structure on $\mathcal{T}_g$. The most significant development, however,

> is due to Bers. He found, as has been indicated, that Teichmüller's extremal problem can be dispensed with altogether, and that the most direct approach is via a closer study of the Beltrami equation. In other words, it is the full theory of quasiconformal mappings, rather than a special problem, which is the right clue to the moduli of Riemann surfaces.
> [...] It is only fair to mention, at this point, that the algebraists have also solved the problem of moduli, in some sense even more completely than the analysts. Because of the different language, it is at present difficult to compare the algebraic and analytic methods, but it would seem that both have their own advantages.

This says in particular (a fact that we already pointed out) that there are several ways for defining the complex analytic structure of Teichmüller space. We also noted that in the paper, (Teichmüller 1944) observed that Riemann's moduli space is a quotient of Teichmüller space by the action of the mapping class group. He proved that this action is holomorphic and properly discontinuous. It is known, from a result of Cartan that a holomorphic properly discontinuous action on a complex space induces a complex structure on the quotient space. There are natural holomorphic functions with respect to this structure, namely, the periods of Abelian differentials of the first kind, and in fact, one can use these functions to define the complex structure. This method is outlined at the end of (Teichmüller 1944) and it was later developed by Ahlfors and Rauch. One can recall again that the case of surfaces of genus one was known. Indeed, the fact that the shapes of complex tori represented by parallelograms in the complex plane can be specified by periods of Abelian integrals (more precisely, by the ratio ω_1/ω_2 of two periods) is a classical fact.

The fact that hyperelliptic surfaces in Teichmüller spaces of higher dimension are a source of difficulties for the definition of the complex structure using the period map was already pointed out by (Teichmüller 1944). In his papers, Rauch (1955a,b) gave prescriptions to find sets (each containing $3g-3$ elements) of periods of normal Abelian integrals of the first kind that serve as local moduli near a given nonhyperelliptic surface and analogous sets of $2g-2$ elements each serving as local coordinates near a hyperelliptic point. The work of Rauch gave the correct number locally. This difficulty was resolved by Ahlfors, who used Rauch's work and another ingredient for hyperelliptic surfaces to show the existence of a complex analytic *manifold* structure on $\mathcal{T}_g$.

There is also holomorphic embedding of Teichmüller space in a complex linear space $\mathbb{C}^N$, called the Bers embedding (see the Appendix to this paper), and we shall dwell on this below. Ahlfors (1963a,b) proved that the set of Schwarzian derivatives of univalent functions on the upper half-plane $\mathbb{H}^2$ is an open subset of the Banach space of holomorphic functions with the norm $\sup_H |\phi(z)|^2 y^2$. The Schwarzian derivative can be interpreted as a quadratic differential, and the embedding of Teichmüller space by means of the Schwarzian derivative made new connections between uniformization theory, quadratic differentials and Teichmüller theory.

In his paper, Ahlfors (1961a) defined a fiber space over Teichmüller space where the fiber above each point is a Riemann surface representing the point, and he described a complex analytic structure on that fiber space. This is the same fiber space that Teichmüller (1944) considered and which we already mentioned, but Ahlfors' methods were completely different. Let us quote Ahlfors from his ICM talk (Ahlfors 1963a, p. 4):

> The present lecture reports on the results obtained by an analytic method which is based on the geometric idea of quasiconformal mapping. This analytic treatment has its roots in the work of O. Teichmüller. [...] Teichmüller surmised, but did not establish, a link between moduli and quasiconformal mappings. The results on moduli that he achieved later were based on a different method which did not have the simplicity of his original approach.
> Teichmüller's work was too sketchy to serve as a satisfactory theory. During the 1950s, his results were confirmed and reorganized by H. Rauch, L. Bers, and the speaker. Through these efforts it was firmly established that the space of moduli has a natural complex analytic structure. As the work progressed, strong connections became apparent with Beltrami's classical theory of conformal mapping of surfaces with a Riemannian metric. Much more delicate existence theorems were needed, but it was found that they had already been proved by C. Morrey for different purposes. Ideas of I. N. Vekua and Boyarskii led to important simplifications of these existence proofs.

The proofs by Vekua and Boyarskii referred to by Ahlfors make use of the theory of Calderon and Zygmund on singular integrals. A sketch of the proof is given in (Ahlfors 1964).

Kodaira and Spencer, motivated by Riemann's and by Teichmüller's counts also tried to count the number of moduli, and we shall mention their work below.

In what follows, we review in some detail several approaches to the definition of the complex structure of Teichmüller space.

4.1 Quasiconformal maps and the Beltrami equation

In the approach to the complex structure using the Beltrami equation, Teichmüller space is considered as a space of equivalence classes of Beltrami differentials on a fixed Riemann surface S. To make things more precise, we review some definitions.

We recall that a *Beltrami differential* μ on S is an invariant object which is written in any holomorphic coordinate chart U with holomorphic local coordinate z as $\mu(z)\mathrm{d}\overline{z}/\mathrm{d}z$, where μ is an essentially bounded measurable function on U. The invariance of μ means that in any other holomorphic local coordinate $w = w(z)$, the Beltrami differential μ is written as $\nu(w)\mathrm{d}\overline{w}/\mathrm{d}w$, where

$$\nu(w(z)) = \mu(z)\overline{\frac{\mathrm{d}w}{\mathrm{d}z}}/\frac{\mathrm{d}w}{\mathrm{d}z}.$$

For such a Beltrami differential μ, the real-valued function $|\mu|$ which a priori is defined only in local coordinates, is well defined on the surface S. The differential μ is said to be *essentially bounded* if the essential supremum of $|\mu|$ on S is <1.

Given an essentially bounded Beltrami differential μ on S, one associates to it the partial differential equation

$$f_{\bar{z}} = \mu f_z$$

which is called the *Beltrami equation* with coefficient μ, in which $f_z = \frac{1}{2}(f_x - if_y)$ and $f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$. The invariance property of μ insures that the equation $f_{\bar{z}} = \mu f_z$, which a priori is defined only in coordinate charts, is well defined on the surface S.

One searches for quasiconformal mappings $f : S \to S$ which are solutions of the Beltrami equation. The existence theory of such solutions is a difficult chapter in analysis. When a solution f exists, then the Beltrami differential μ is called the *complex dilatation* of f. If $\mu \equiv 0$, then a solution to the Beltrami equation is a conformal map since the equation $f_{\bar{z}} = 0$ is equivalent to the classical Cauchy–Riemann equation.

There is a notion of equivalence between Beltrami differentials, defined in such a way that equivalent conformal structures on S are represented by equivalent Beltrami differentials.

Given a quasiconformal mapping $f : S \to S$ satisfying the Beltrami equation, a new conformal structure on S is obtained by composing the coordinate charts of the original structure with maps induced from the quasiconformal homeomorphism f. The basepoint of Teichmüller space (i.e., the equivalence class of the surface S) corresponds to the (equivalence class of the) Beltrami differential $\mu \equiv 0$.

The space of essentially bounded Beltrami differentials on S is the unit norm of the complex vector space of measurable functions equipped with its natural structure of a Banach complex space. The complex analytic structure of Teichmüller space is defined using this complex Banach space structure. This is Ahlfors' approach, and it is still the most common way of defining the complex analytic structure of Teichmüller space. Though in some sense it is awkward that a complex structure on a finite-dimensional space has to be defined through a complex structure on an infinite-dimensional space, the spaces involved are essentially finite dimensional since, as we recalled, different Beltrami differentials can define the same complex structure, and their equivalence classes give the Teichmüller space. Picking out and identifying the complex structure (or representatives) might not be obvious, and this infinite complex dimensional Banach space provides the convenience and flexibility to go around this difficulty. One must also add that this method also works for infinite-dimensional Teichmüller spaces.

For a concise survey on the development of the complex structure of Teichmúller space using the Beltrami equation, we refer the reader to Earle (2010).

4.2 The Ahlfors–Bers approach using holomorphic families

We already recalled that in his paper, Teichmüller (1944) provided a description of the complex structure of Teichmüller space using holomorphic families of marked Riemann surfaces. Another approach using families was developed later on by Ahlfors (1961a) and by Bers (1958), using solutions of the Beltrami equation with the Beltrami coefficient depending on a parameter. In this approach, the complex structure on Teichmüller space is characterized by the fact that if μ is a Beltrami coefficient defining a normalized complex structure obtained by solving the Beltrami equation $f_{\bar{z}} = \mu f_z$, and if the Beltrami coefficient $\mu = \mu_u(z)$ depends continuously (respectively, differentiably and holomorphically) on a variable u, then the map $u \mapsto [\mu_u]$ is continuous (respectively, differentiable and holomorphic). The Ahlfors–Bers theory gives a precise version of this result, in which a family of Riemann surfaces is obtained by means of Beltrami coefficients $\mu(z) = \mu_u(z)$ that depend on a complex parameter u. The complex structure on the fiber space is defined locally using the projection on the Riemann surface factor. The theory of the solution of the Beltrami equation is also used, i.e., the fact that when one starts with a measurable function $\mu : \mathbb{C} \to \mathbb{C}$ whose essential supremum $\|\mu\|$ is < 1, the Beltrami equation $f_{\bar{z}} = \mu f_z$ has a unique solution f^μ that is a homeomorphism fixing 0, 1 and ∞, and that for any point z in the complex plane, the map $\mu \mapsto f^\mu(z)$ is holomorphic from the unit ball of $L^\infty(\mathbb{C})$ to $\mathbb{C}$. This is the so-called "measurable Riemann mapping theorem" of Ahlfors and Bers.

The development of the theory of holomorphic families of Riemann surfaces is surveyed in the recent paper (Earle and Marden 2012).

4.3 The complex structure defined by periods of Abelian differentials

As we already recalled, the complex structure of Teichmüller space can also be defined using periods of Abelian differentials, and this is related to the fact that this complex structure is expected to be natural in the sense that natural maps from Teichmüller space to some known complex manifolds should be holomorphic. We also noted that the idea of a construction of a complex structure on Teichmüller space $\mathcal{T}_g$ using periods is contained in (Teichmüller 1944). Periods of Abelian differentials define maps from Teichmüller space to the Siegel upper half-space, which is a Hermitian symmetric space H_g, and it was tempting to use this map to define a complex structure on the Teichmüller space $\mathcal{T}_g$. Though this map is not injective, it factors through an injective map from the Torelli space Tor^g to H_g. This map was shown by Rauch to be a local embedding away from hyperelliptic Riemann surfaces and hence defines a complex structure on $\mathcal{T}_g$ outside the locus of hyperelliptic Riemann surfaces. The complex structure was extended to the whole Teichmüller space by Ahlfors.

We recall now more precisely some of these notions and constructions.

The Siegel upper half-space H_g of degree g (which is also called the Siegel generalized upper half-plane) is defined by

$$H_g = \{X + iY \mid X, Y \text{ are real } g \times g \text{ symmetric matrices }, Y > 0\}.$$

When $g = 1$, this is the usual upper half-plane.

It is known that the symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ acts holomorphically and transitively on the Siegel upper half-space, and that the stabilizer of iI_g is isomorphic to $U(g)$. This implies that H_g can be written as

$$H_g = \mathrm{Sp}(2g, \mathbb{R})/U(g),$$

and it is a Hermitian symmetric space of noncompact type, i.e., a symmetric space of noncompact type with an invariant complex structure.

We now recall the definition of the period map of Riemann surfaces. Let $S = S_g$ be a closed Riemann surface of genus g and $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ be a symplectic basis for the first homology group $H_1(S, \mathbb{Z})$, where the word symplectic means here that the intersection numbers of the cycles satisfy $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$ if $i \neq j$ and $\alpha_i \cdot \beta_j = \delta_{ij}$. With such a choice, given an Abelian differential w of the first kind on S (i.e., a holomorphic 1-form on S), its periods are $A_j = \int_{\alpha_j} w$ and $B_j = \int_{\beta_j} w$. Consider now a basis $w_1, \ldots w_g$ of Abelian differentials of the first kind satisfying $\int_{\alpha_j} w_i = \delta_{ij}$ for all i and j. The matrix $(Z_{ij}) = (\int_{\beta_j} w_i)$ is called the *period* of S with respect to the symplectic basis $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$. The Riemann bilinear relations imply that $(Z_{ij}) \in H_g$. Once a symplectic basis of the homology of a base Riemann surface S_0 in $\mathcal{T}_g$ is fixed, it gives a choice of a symplectic basis for the homology of every marked Riemann surface in $\mathcal{T}_g$. This defines a map

$$\mathcal{T}_g \to H_g, \quad S \mapsto (Z_{ij}).$$

The complex structure that we aim for on Teichmüller space should satisfy the condition that each such function is continuous and holomorphic.

Two choices of symplectic bases of $H_1(S, \mathbb{Z})$ are related by elements in $\mathrm{Sp}(2g, \mathbb{Z})$, and their corresponding periods lie in an $\mathrm{Sp}(2g, \mathbb{Z})$-orbit in H_g. Therefore, we obtain a well-defined map

$$J : \mathcal{M}^g = \mathrm{Mod}(S_0)\backslash\mathcal{T}_g \to \mathrm{Sp}(2g, \mathbb{Z})\backslash H_g.$$

The locally symmetric space $\mathrm{Sp}(2g, \mathbb{Z})\backslash H_g$ is called the Siegel modular variety and it is the moduli space of principally polarized abelian varieties of dimension g. For each Riemann surface S, $J(S)$ is the Jacobian variety of S. Because of this, the map J is also called the Jacobian map. Reference for the period map and abelian varieties include (Griffiths 1970; Griffiths and Harris 1994).

In equipping Teichmüller space $\mathcal{T}_g$ with a complex structure by pulling back the natural complex structure of the Siegel upper half-space, one needs to know that the period map is a local immersion, and this was proved by Rauch at points of $\mathcal{T}_g$ away from the locus of hyperelliptic Riemann surfaces. His method was to compute differentials of the period via the so-called Rauch's variational formula. Rauch (1955a, p. 43) says that Riemann's $3g - 3$ count using Riemann–Roch and Euler's characteristic for the number of parameters that are involved in the construction of branched coverings of the sphere is "manifestly too vague a specification; one would like numerical moduli—a set of numbers associated with each surface whose equality would guarantee conformal equivalence between two surfaces." He then describes a set of periods of Abelian integrals π_{ij} and declares that his result solves "a reasonable and long-standing conjecture saying that a suitable subset of $3g - 3$ of the π_{ij} is a set of numerical moduli." He then says that the "key and unifying element" would be "the application of the concept of quadratic differential to conformal mappings by Teichmüller." In a second paper, Rauch (1955b) published an extension of his results to surfaces with boundary. It seems that some of Rauch's result had some gaps since Gerstenhaber produced Riemann surfaces that have the same set of periods specified by Rauch and which are not conformally equivalent.[11]

In his paper, Ahlfors (1960) also worked out the complex structure of Teichmüller space using holomorphic families and the period map, completing the work of Rauch.

We already mentioned that the period map $\mathcal{T}_g \to H_g$ is not injective and factors through the Torelli space of a surface. It may be useful to recall that the Torelli space Tor_g of a closed surface of genus g is a space of equivalence classes of "marked" Riemann surfaces (f, S), where the marking here keeps track only of the action of the homeomorphism f on the first homology group of the surface. Alternatively, one can consider an element of the Torelli space as a Riemann surface equipped with a basis for its first homology group. Note that in the classical marking of Riemann surfaces, instead of specifying a homeomorphism f between the base surface and the varying Riemann surface S, one can as well specify a basis for the first homotopy group of S. In the case where the surface is the torus, the Teichmüller and the Torelli spaces coincide since the fundamental group of the surface is commutative and thus the first homotopy group and the first homology group coincide.

In the same way, as the Riemann moduli space is a quotient of the Teichmüller space by the mapping class group $\mathrm{Mod}_g = \mathrm{Mod}_g(S)$, the Torelli space is a quotient of Teichmüller space by the Torelli group $\mathcal{I}_g = \mathcal{I}_g(S)$, i.e., the group of mapping classes that induce the identity on the first homology group of the surface. The Torelli group has no torsion. Indeed, a torsion element of the mapping class group acts as an isometry for a certain hyperbolic metric on the surface, and such an isometry, if it is not the identity, cannot act trivially on homology (Farkas and Kra 1992, Sect. V.3). The Torelli group acts without fixed point on Teichmüller space, and the Torelli space, which is the quotient space of this action, is equipped with a complex manifold quotient structure induced from that of Teichmüller space. The moduli space $\mathcal{M}_g$ is the quotient of the Torelli group by the quotient group $\mathrm{Mod}_g/\mathcal{I}_g \cong \mathrm{Sp}(2g, \mathbb{Z})$. The mapping class group has torsion, and the moduli space is an orbifold quotient space.

André Weil, in his 1958 Bourbaki seminar (Weil 1958c) and in his paper (Weil 1958a), used the terminology *Teichmüller surface* for the object called today a *marked Riemann surface*. He also introduced the terminology *Torelli surface*. This choice of names is coherent because it reflects clearly the sequence of maps $\mathcal{T}_g \to \mathrm{Tor}_g \to \mathcal{M}_g$ between the Teichmüller, the Torelli and the Riemann spaces.

A Torelli surface is considered as *reinforced*, i.e., it carries more structure; the adjective is used by Weil (1958a) in the same sense in which a Teichmüller surface is a reinforced Riemann surface.

It is clear that the period map $\mathcal{T}_g \to H_g$ is invariant under the action of the Torelli group $\mathcal{I}_g$, and hence it descends to a period map

$$\mathrm{Tor}_g = \mathcal{I}\backslash\mathcal{T}_g \to H_g.$$

Under the action of the Siegel modular group $\mathrm{Sp}(2g, \mathbb{Z})$, we recover the Jacobian map

$$J : \mathcal{M}_g = \mathrm{Sp}(2g, \mathbb{Z})\backslash\mathrm{Tor}_g \to \mathrm{Sp}(2g, \mathbb{Z})\backslash H_g.$$

[11] See Gerstenhaber's review of Rauch's papers in Zentralblatt (Zbl 0067.30502).

Let us now quote Rauch (1962) from his paper in which he derives the complex structure on moduli space from that on the Torelli space (p. 390):

> The set of conformal equivalence classes of compact Riemann surfaces (henceforth: classes) of fixed genus $g \geq 2$ can be endowed with the structure of a normal analytic space M^g of complex dimension $3g - 3$. This structure is derived from the structure of a complex analytic manifold, the Torelli space $\mathfrak{I}_g$, by identification under the action of a properly discontinuous group of analytic automorphisms, the reduced mapping class (Siegel modular) group G^g. In this description, $\mathfrak{I}_g$ appears as a branched covering of M^g. $P \in \delta^g$, the branch locus, if and only if it is a fixed point of a nontrivial finite subgroup $G^g(P)$ of G^g, in which case P lies over a class (an H-class) in M^g containing a surface S (H-surface) admitting at least one nontrivial conformal automorphism and $G^g(P)$ is isomorphic with $H(S)$, the automorphism group of S.

The local Torelli theorem amounts to the injectivity of the differential of the period map at nonhyperelliptic curves, see Kodaira (1964, Theorem 17), where this result is attributed to Andreotti and Weil.

4.4 Teichmüller space as a bounded complex domain

This is an important part of Teichmüller theory, but the presentation is too technical, compared to the rest of the paper, and we have made it an Appendix. The Bers embedding, which we describe there, embeds the Teichmüller space $\mathcal{T}$ of a closed surface of genus g as a bounded domain in $\mathbb{C}^{3g-3}$, and it can be used to define the complex structure of that space. Again, quadratic differentials play here a prominent role. In developing this theory, Bers considered a quadratic differential as a Schwarzian derivative of the conformal extension of quasiconformal mappings, obtained using the Beltrami equation. Ahlfors writes in his 1978 ICM paper (Ahlfors 1978, p. 79):

> The Bers mapping is not concerned with extremal quasiconformal mappings, and it is rather curious that one again ends up with holomorphic quadratic differentials. The Bers model has a Kählerian structure obtained from an invariant metric, the Petersson–Weil metric, on the space of quadratic differentials. The relation between the Petersson–Weil metric and the Teichmüller metric had not been fully explored and is still rather mystifying.

4.5 Kodaira–Spencer theory

In 1957, Kodaira and Spencer developed a theory of deformations of compact higher-dimensional complex manifolds using a notion of universal family (Kodaira and Spencer 1958). They gave examples of such families, and the general existence result of universal families was solved in Kuranishi (1962, 1965). In this approach, one defines an *almost complex structure* on a manifold M to be a complex vector bundle structure on the real tangent bundle of M. This is given by an endomorphism J satisfying $J^2 = -\mathrm{Id}$. A famous theorem of Newlander–Nirenberg gives a sufficient condition for an almost-complex manifold to be a complex manifold. The condition is a kind of commutation condition of complex vector fields with respect to the operator J, and it is called *Frobenius complex integrability*. In this theory, the space that plays a major role is a cohomology group $H^1(M, \Theta)$, where Θ is the sheaf of germs of sections of the holomorphic tangent bundle (or, equivalently, holomorphic vector fields) of the manifold. A deformation of a complex manifold given by an atlas is then defined by moving the charts one relative to another, and the infinitesimal variation of the structure is described as a variation of the collection of chart coordinate changes. Such a variation gives rise to a holomorphic vector field on each component of the intersection of two charts and this collection of holomorphic vector fields is shown to satisfy cocycle conditions which arise from the compatibility relations satisfied by the holomorphic coordinate changes; therefore, they define elements of the first cohomology group $H^1(S, \Theta)$. Using Serre duality and a theorem of Dolbeault, the space $H^1(S, \Theta)$ is identified with the tangent space $T_0(\mathcal{T}(R))$ of the Teichmüller space $\mathcal{T}(S)$ at the basepoint. This makes the relation with our theory. In conclusion, Kodaira–Spencer theory makes the space of infinitesimal deformations equal to the cohomology space $H^1(S, \Theta)$.

In surveying the complex structure of Teichmüller space, it is natural to talk about its Kähler structure. This is what we do in the next and final section.

4.6 The Kähler structure and the Weil–Petersson metric

A Kähler metric is a Hermitian metric on a manifold satisfying an integrability condition which says roughly that the holonomy group of its Riemannian part is contained in the unitary group $\mathrm{U}(n)$ associated to the underlying complex structure. A Kähler structure on a manifold is a kind of unifying structure since it involves at the same time a Riemannian structure, a symplectic structure and a complex structure. When a certain compatibility condition is satisfied, the three structures together determine a Kähler structure. In any case, Kähler geometry is strongly related to the complex structure of a manifold, and the fact that a metric on a complex manifold is Kähler has many important consequences in algebraic geometry. For example, the Hodge decomposition of cohomology groups is crucial in the transcendental approach of algebraic geometry. Kähler manifolds are also in some sense generalizations of projective varieties.

It seems that it is André Weil who first included the study of Teichmüller space in the setting of Kähler geometry. He introduced in (Weil 1958a) a Riemannian metric on Teichmüller space and he conjectured that it is Kähler. This metric is defined by a version of an inner product which had been previously defined by Hans Petersson in the context of automorphic forms (hence the name "Weil–Petersson metric"). Weil concluded his paper (Weil 1958a) with the following (p. 389):

> This raises the most interesting problems of the whole theory: is this a Kähler metric? has it an everywhere negative curvature? is the space Θ, provided with its complex structure and with this metric, a homogeneous space? It would seem premature even to hazard any guess about the answers to these questions.

Weil comments on that paper, in his *Collected Works* (Weil 1979, vol. II, p. 546):

> I was asked for a contribution to a volume of articles to be offered to Emil Artin in March 1958 for his sixtieth birthday; this led me to the decision of writing up my observations, even incomplete, on the moduli of curves and on what I called "Teichmüller space." Soon later, I noticed that on more than one point I was matching Ahlfors and Bers; these continued their research, and soon after they overtook me. As for the questions that I asked at the end of (Weil 1958a), they showed that the response is positive for the first two, and negative for the third.

Ahlfors (1961a), in his paper, proved that the metric introduced by Weil on Teichmüller space is Kähler. In the same paper, he wrote, as a footnote: "According to an oral communication the fact has been known to Weil, but his proof has not been published." Besides proving that the Weil–Petersson metric is Kähler, Ahlfors showed that its holomorphic sectional and Ricci curvatures are negative. In his commentary on his papers (Ahlfors 1961a,b) that he made for his *Collected papers* edition, vol. II (p. 155), Ahlfors wrote:

> Actually, it was André Weil's idea to make Teichmüller space a Riemannian space by using the Petersson bilinear product. [...] Weil knew that the Petersson metric was Kählerian, but had not published the proof. This turned out to be an almost immediate consequence of the calculation in Ahlfors (1961a,b) I showed through hard work that the metric has negative Ricci and sectional curvatures. When I reread the paper I was reminded of Dr. Johnson's remark about the dancing poodle; it was not very good, but remarkable that it could be done at all.

Tremendous developments were done later on Teichmüller theory. In particular, in the 1970, a spectacular new point of view was brought by Thurston. We cannot expand that here, since the aim of the article is not to report on the later developments, but only on the foundation of Teichmüller theory.

Appendix (Teichmüller space as a bounded complex domain)

There are several known embeddings of Teichmüller space as a bounded domain in a complex vector space, such that the complex structure on Teichmüller space is induced from this embedding.

In his paper (Bers 1960a, with the *corrigendum* published in 1961), Bers showed that the Teichmüller space of a surface of finite type (g, n) has a canonical holomorphic embedding as a bounded domain in $\mathbb{C}^N$, with $N = 3g - 3 + n$. In 1964, Bers and Ehrenpreis showed that any finite-dimensional Teichmüller space can be embedded as a domain of holomorphy in some $\mathbb{C}^N$ (Bers and Ehrenpreis 1964). (We recall that a domain of holomorphy is a subset of a complex space $\mathbb{C}^N$, which is maximal in the sense that there exists a holomorphic function on this set which cannot be extended to a larger set. This notion is only interesting for higher-dimensional complex variable theory since in dimension one every subset of $\mathbb{C}$ is a domain of holomorphy.) The Bers embedding is defined in terms of Kleinian groups, and in that theory, quasiconformal mappings and fine properties of their Schwarzian derivatives also play a central role. The embedding is based on Bers' *simultaneous uniformization theorem.* To describe this, we briefly review some elements of Kleinian group theory.

A *Kleinian group* is a torsion-free discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$. Such a group acts by fractional linear transformations on the Riemann sphere S^2, identified with the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. This action extends to hyperbolic 3-space, seen as the extended upper half-space $\{(x_1, x_2, x_3) \mid x_1, x_2 \in \mathbb{R}, x_3 \geq 0\}$ with boundary the extended complex plane $\overline{\mathbb{C}}$. We can also view the hyperbolic 3-space as the unit ball in 3-dimensional Euclidean space whose boundary is the unit sphere S^2.

Given a Kleinian group Γ acting on $\mathbb{H}^3$, and given a point x in $\mathbb{H}^3$, the limit set $\Lambda = \Lambda_\Gamma$ of Γ is the intersection $\overline{\Gamma x} \cap S^2$, and it does not depend on the choice of x. It is a closed invariant subset of S^2. The group Γ acts properly discontinuously on the complement $S^2 \setminus \Lambda = \Omega$, called the *domain of discontinuity* of Γ.

A *quasiconformal deformation* of a Kleinian group Γ is a quasiconformal homeomorphism $h : S^2 \to S^2$ such that for any g in Γ there is an associated element g^h of $\mathrm{PSL}(2,\mathbb{C})$ defined by $g^h = h \circ g \circ h^{-1}$. The set of such transformations g^h forms a group called the *deformed group* $G^h = hgh^{-1}$, and the map $g \mapsto g^h$ is an isomorphism between the group G and the deformed group G^h.

A quasiconformal deformation of a Kleinian group G with limit set Λ can be described by means of a Beltrami coefficient μ for G, i.e., a Beltrami coefficient μ on S^2 satisfying $\mu|\Lambda = 0$ and the equivariance condition

$$\mu(g(z))\,\overline{g'(z)}/g'(z) = \mu(z) \quad \forall g \in G.$$

Let $f^\mu : S^2 \to S^2$ be a normalized solution of the Beltrami equation with coefficient μ. Then, for every g in G, $f^\mu \circ g : S^2 \times S^2$ is again μ-conformal, therefore, there is a Möbius transformation g_1 such that $f^\mu \circ g = g_1 \circ f^\mu$. In this way, the deformed group $G^\mu = f^\mu G (f^\mu)^{-1}$ is again a Kleinian group. This group is considered as a *quasiconformal deformation of the group* G, and the map

$$G \to G^\mu$$

defined by

$$g \mapsto f^\mu \circ g \circ (f^\mu)^{-1}$$

is a *quasiconformal isomorphism between* G *and* G^μ.

We have $\Omega(G^\mu) = f^\mu(\Omega(g))$ and $\Lambda(G^\mu) = f^\mu(\Lambda(g))$. Furthermore, the restricted map

$$f^\mu : \Omega(G) \to \Omega(G^\mu)$$

induces a quasiconformal mapping between the (not necessarily connected) surfaces $\Omega(G)/G$ and $\Omega(G^\mu)/G^\mu$.

In the case where G is Fuchsian, G^μ is said to be *quasi-Fuchsian*. It fixes the circle $f^\mu(\overline{\mathbb{R}})$. If the Beltrami coefficient μ satisfies the condition

$$\mu(\overline{z}) = \overline{\mu(z)},$$

then G^μ is again Fuchsian.

A *quasi-Fuchsian group* can also be defined as a Kleinian group the limit set Λ of which is a Jordan curve. The quotient by the group action of the two connected components of $S^2 \setminus \Lambda$ consists in two Riemann surfaces. There is a preferred marking between these two surfaces, and hence between such a surface and a base Riemann surface. Such a pair of Riemann surfaces can therefore be considered as an element of the product of two copies of Teichmüller space. Thus, the same quasi-Fuchsian group Γ uniformizes simultaneously (and exactly) two Riemann surfaces, $S_1 = \Omega_1/\Gamma$ and $S_2 = \Omega_2/\Gamma$. This is Bers' (1960a) *simultaneous uniformization*. Conversely, any element in $\mathcal{T} \times \mathcal{T}$ determines a unique Kleinian group up to conjugation. Bers showed that given any two Riemann surfaces of the same finite type and given any orientation-reversing homeomorphism between them, there is a quasi-Fuchsian group that simultaneously uniformizes the pair. The space of quasi-Fuchsian groups of a given type is homeomorphic to a product $\mathcal{T} \times \mathcal{T}$ of Teichmüller spaces of a given surface. Fixing an element S in the first factor and taking it as a basepoint, one gets an embedding of the second factor Teichmüller space $\mathcal{T} = \mathcal{T}(S)$ into the space of Kleinian groups, and this embedding is called a *Bers embedding*.

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