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Source: *Archive for History of Exact Sciences*, Vol. 69, No. 2 (March 2015), pp. 125-151
Published by: Springer
Stable URL: https://www.jstor.org/stable/24569622

Arch. Hist. Exact Sci. (2015) 69:125–151
DOI 10.1007/s00407-014-0142-8

How Woodin changed his mind: new thoughts on the Continuum Hypothesis

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Received: 7 March 2014 / Published online: 16 December 2014

Abstract The Continuum Problem has inspired set theorists and philosophers since the days of Cantorian set theory. In the last 15 years, W. Hugh Woodin, a leading set theorist, has not only taken it upon himself to engage in this question, he has also changed his mind about the answer. This paper illustrates Woodin's solutions to the problem, starting in Sect. 3 with his 1999–2004 argument that Cantor's hypothesis about the continuum was incorrect. From 2010 onwards, Woodin presents a very different argument, an argument that Cantor's hypothesis is in fact true. This argument is still incomplete, but according to Woodin, some of the philosophical issues surrounding the Continuum Problem have been reduced to precise mathematical questions, questions that are, unlike Cantor's hypothesis, solvable from our current theory of sets.

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Communicated by: Jeremy Gray.

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1 Introduction

The Continuum Problem is the question of how many subsets there are of the natural numbers. This problem arose from the original works of Cantor. His proposed solution has become known as the Continuum Hypothesis, CH: every infinite subset of the natural numbers is either countable or can be brought into a one-to-one correspondence with the whole powerset of the natural numbers. Cantor was unable to prove this hypothesis, and today we know that it is neither provable nor disprovable from our current theory of sets; it is an independent statement. The issue became a philosophical matter and has been discussed in that way ever since.[1]

The CH is only one of the many known examples of mathematically interesting statements, which are independent from our current theory of sets. In the wake of the realisation of this, a new philosophical view on set theory arose. Assume, as so many set theorists do, some form of Platonism; that is, assume that there is a fact of the matter how the universe of sets looks like. Do the independence results show that there are in fact many different true universes, or is it the case that our theory is just not good enough yet to describe the one and only true universe fully? This is the pluralism/non-pluralism debate, explored further in Koellner (2014). This debate has obvious implications for a discussion of CH: if pluralism is true, then there are simply (true) universes where the CH is true and (true) universes where the CH fails. On a non-pluralist account, CH is either true or false in the one and only true universe of sets. As such, this debate sets a philosophical stage for what follows.[2]

W. Hugh Woodin introduces his readers to the discussion on the CH by stating

> The generally accepted axioms for set theory– but I would call these the twentieth-century choice– are the Zermelo-Fraenkel Axioms together with the Axiom of Choice, ZFC. (Woodin 2001a, p. 567).

[1] See for example Ferreirós (2007, pp. 190, 210ff, 333, 382ff), or Koellner (2013) for more on the Continuum Hypothesis.

[2] Non-platonist conceptions of the metaphysics of set theory are dismissed by Woodin as basically untenable and are not discussed in this paper. See Woodin (2009a) for Woodin's argument against non-platonism.

Here the thought that the ZFC axioms are merely our current choice and that new and better choices are possible is already visible. New axioms are necessary to resolve the Continuum Problem, but how would one find these new axioms? Woodin refers to the Axiom of Projective Determinacy, PD, (see Sect. 6) which has been, according to him, largely accepted as true.

> But the truth of $[PD]$ became only evident *after* a great deal of work. For me, the remarkable aspect of this is that it demonstrates that the discovery of mathematical truth is not a purely formal endeavour. (Woodin 2001a, p. 569).

Hence, Woodin has reason to hope that the Continuum Problem can be solved. More strongly, Woodin has reasons to believe that it can be solved on the basis of a non-pluralist account, and

> argue[s] that modulo the Ω-Conjecture [...], the generic-multiverse position [i.e. a form of pluralism] outlined above is not plausible. (Woodin 2009b, p. 17).

Woodin has recently changed his opinion on the answer to the Continuum Problem. Ten years ago, he argued for it to be false, now he believes that it will turn out to be true. This paper explores his changing positions by presenting both the old argument against CH as well as the new argument for the truth of CH. Especially for the second argument, large cardinal axioms will be of importance, so I briefly introduce them here.

2 Large cardinal axioms

A large cardinal axiom is a statement that a very big set with certain properties exists. There are many different large cardinal axioms (also referred to as *axioms of strong infinity*), all proclaiming the existence of sets with certain properties. All such sets would be cardinals (hence the name), but the existence of such cardinals cannot be proved by our theory of sets, ZFC. One of the most illuminating examples of this is the large cardinal axiom, which states that there is an inaccessible cardinal (for the definitions of any large cardinal axiom mentioned in this paper, see Sect. 5). If there is an inaccessible cardinal, then we have a model for ZFC. But having a model for ZFC is equivalent to knowing that there are no contradictions following from ZFC, i.e. knowing that ZFC is consistent. By Gödel's Second Incompleteness Theorem, we know that a theory can prove its own consistency if and only if it is inconsistent. Hence, if ZFC could prove that there is an inaccessible cardinal, then it could prove its own consistency and would hence be inconsistent.[3] Note that this does not mean that there are no inaccessible cardinals. It only means that ZFC cannot prove that such cardinals exist.

[3] Remark: it is unknown, in fact unprovable from the ZFC axioms, if ZFC is consistent. The same holds true for number theory, for example. However, experience with these axiom systems makes it reasonable to expect that these systems are in fact consistent. It is common practice in set theory to assume the consistency of ZFC, and Woodin does so as well: "There will be no discovery ever of an inconsistency in $ZF + AD$" (Woodin 2009b, p. 10) (note that the consistency of ZF implies the consistency of ZFC by Gödel's construction of L). I shall follow Woodin in this throughout this paper.

Not only can the existence of such large cardinals not be proven from ZFC, it is not even known whether the existence of such sets is even consistent with ZFC.[4] But we do know that the large cardinal axioms can be well ordered by their consistency strength.[5] Here is an (incomplete) list of large cardinal axioms, with the strongest at the top and the weakest at the bottom:

There is a Reinhardt cardinal
There is a n-huge cardinal
There is a Huge cardinal
There is an Extendible cardinal
There is a Supercompact cardinal
There is a Superstrong cardinal
There is a Woodin cardinal
There is a Measurable cardinal
There is an Inaccessible cardinal

Large cardinal axioms have an interesting feature: they can decide some of the problems, which are unsolvable from ZFC alone. An example of this to which I will come back in Sect. 4 is that the existence of a measurable cardinal implies that the universe of sets is larger than Gödel's constructible universe. Another one we saw above: ZFC+'There is an inaccessible cardinal' implies the consistency of ZFC. This feature of large cardinal axioms plus the fact that they can be well-ordered results in something very interesting. Let ϕ and ψ be two set theoretic statements which ZFC neither proves nor disproves. It is then often the case that there are two large cardinal axioms LCA_1 and LCA_2 such that $ZFC + LCA_1$ proves (or disproves) ϕ and $ZFC + LCA_2$ proves (or disproves) ψ. But LCA_1 and LCA_2 are well ordered by the large cardinal hierarchy, say, for example, LCA_1 is stronger than LCA_2. We can hence say that ϕ has a higher *degree of unsolvability* than ψ because it needs a stronger large cardinal assumption to be proven (or disproven). That is: the large cardinal axioms give us a hierarchy of degrees of unsolvability. See Koellner (2011) for further details.

Despite their usefulness, large cardinal axioms cannot decide CH. To explain this, I need to briefly discuss forcing.

A model of the language of set theory consists of a collection of sets and a relation defined on them, which models the $\in$-relation. Such a model is a model for ZFC if all the axioms of ZFC are true in the model. Cohen introduced the method of forcing, which allows the set theorist to very carefully add a specific set to a model of ZFC *whilst simultaneously making sure that the resulting structure is still a model of* ZFC. The latter part of that statement is highly non-trivial. Forcing has become one of the major tools of the set theorist. See Kunen (2006) for a good introduction to forcing, see Jech (2006) for deeper results.

4 There is one exception to this. Kunen has proved that the existence of a Reinhardt cardinal is inconsistent with ZFC. The proof relies on the Axiom of Choice, and it is unknown if the existence of a Reinhardt is inconsistent with ZF (i.e. ZFC minus the Axiom of Choice). See Jech (2006) for details.

5 Consistency strength: theory T has a higher consistency strength than theory S if the consistency of T implies the consistency of S.

In particular, the CH can be made true or false by forcings, which are small in the sense that they act only on the lower (infinite) levels of the cumulative hierarchy.[6] In (1967), Lévy and Solovay proved that these kinds of small forcings do not act on the level of a measurable cardinal. That is, such small forcings only affect the lower levels, whilst the higher levels, the levels of a measurable cardinal and above, are not affected by the forcing. Hence, the existence of a measurable cardinal (or any stronger large cardinal) simply has no bearing on the truth value of the CH. This is the famous Lévy-Solovay Theorem.

But even if large cardinals are not an instant-decision tool, the hierarchy of degrees of unsolvability has proven to be useful. This is at least one of the reasons why set theorists try to account for large cardinals. They are looking for arguments for the consistency of large cardinals with ZFC and frequently use large cardinals in their everyday work. This issue will be further touched upon in Sect. 4.

I now come back to the main topic of this paper: Woodin's arguments for and against CH. Since his argument against the CH was prior to his argument in favour of the statement, I shall start there.

3 The case against CH

Woodin gave three very similar yet different arguments against CH. The first one was a technical tour-de-force, presented in Woodin (1999). Woodin was then able to capture the insights of this argument in his notion of Ω-logic. The study of this strengthening of classical first-order logic has become a centre piece of Woodin's research and is explained below. Using Ω-logic, Woodin could restate the case against CH more easily. Whilst this case was already made in Woodin (1999), it was expanded upon in the papers (Woodin 2001a, b). In these papers, Woodin considered even stronger logics, Ω^*-logics, to make his point. These logics, so he realised in Woodin (2004), can be dropped by introducing a satisfiability notion for Ω-logic. The case against CH, however, always stays essentially the same. By a rational reconstruction of already existing arguments for new axioms (which have been largely accepted according to Woodin), Woodin identifies certain desirable properties of these axioms. Statements that satisfy these properties (for a certain structure) are called *good*. Woodin showed that there are good statements which could hence function as new axioms for an extension of ZFC. He did so by producing an example: the $(*)$-axiom (Star-axiom). The (boiled down) case against CH is that *all* good axioms imply $\neg CH$, and the $(*)$-axiom shows that there are axioms, which are good. According to Woodin, we should be looking to extend ZFC by good axioms, and all those extensions imply $\neg CH$. This is the case against CH.

6 The cumulative hierarchy incorporates the idea that the universe of sets is build-up from below by taking the powerset and the union operation. This results in 'levels' within the hierarchy. See also the definition of the von Neumann universe V in Sect. 6.

3.1 Exploring the universe from below

In his two 2001 papers on the Continuum Hypothesis, Woodin was encouraged by the success that the Axiom of Projective Determinacy, PD, had enjoyed.[7]

> Projective Determinacy is the *correct* axiom for the projective sets; the ZFC axioms are obviously incomplete and, moreover, incomplete in a fundamental way. (Woodin 2001a, p. 575).

This axiom had emerged from the study of a certain part of the universe of sets and had, according to Woodin, led to a canonical set theoretical theory of this sub-structure, which was previously missing. An idea of the 2001 papers was that it should therefore be possible to find a canonical theory for a larger part of the transfinite universe, and Woodin presented a candidate for such a theory.

Formal arithmetic is defined via the axioms of Peano Arithmetic, PA. As Woodin stated:

> It seems that most mathematicians do believe that arithmetic statements are either true or false. (Woodin 2001a, p. 568).

That is, PA is *empirically complete*: we do not know of any natural arithmetical statement, which is independent (i.e. neither provable nor disprovable) from PA. Of course, due to Gödel's incompleteness theorems, there are statements in the language of arithmetic that are independent from PA, and this is why Woodin talked about *natural* statements here. Statements are natural if mathematicians come across them in their endeavours. Gödel sentences and the like are artificial in that they serve to prove certain theorems, but are otherwise of no particular interest. These definitions are vague, but nonetheless it seems largely clear to the mathematical community what a natural and what an artificial statement is.

The empirical completeness of PA is supported even further by the *forcing completeness* of PA. It is an elementary fact about Cohen's method of forcing that independence from PA cannot be shown by going to a forcing extension.[8]

I need to make one small detour here. There are two major ways in which the universe of sets can be given. By far the most common and well known is the von Neumann Hierarchy, i.e. the universe of sets is given in terms of the V_α where α is an ordinal.[9]

Another way of stratifying the universe of sets is in terms of the $H(\kappa)$ for cardinals κ, where $H(\kappa)$ contains all sets whose transitive closure is less than κ.[10] It then holds

[7] The Axiom of Projective Determinacy will play only a minor part in this paper, and I shall neither give a formal definition nor an explanation of it here. The reader is referred to Jech (2006) and Kanamori (2009) for details.

[8] Woodin remarked that it is theoretically possible that a generalisation of forcing could be found, which could be used to prove independence results about PA. But not even an idea of how to come by such a generalisation, let alone the generalisation itself, is currently present.

[9] For a formal definition, see Sect. 6.

[10] A set a is *transitive* if from $b \in a$ and $c \in b$ follows that $c \in a$. Transitive sets turn out to be of particular interest to the set theorist. The transitive closure of a set x is the smallest transitive set, which contains x.

that $V_\omega = H(\omega)$. Similarly, if CH holds, then $V_{\omega+2} = H(\omega_2)$, and if the Generalised Continuum Hypothesis ($2^{\aleph_i} = \aleph_{i+1}$ for all i) holds, then these similarities are carried upwards. However, if CH fails, then $H(\omega_2)$ is less rich than $V_{\omega+2}$. On the other hand, the union of all the V_α for all ordinals α is just the same as the union of all the $H(\kappa)$ for all cardinals κ, namely the whole universe. Hence, the $H(\kappa)$ stratification is more fine grained than the von Neumann's V_α stratification. As Woodin put it:

> If CH fails, then these two structures can be very different, with the former structure [$\langle H(\omega_1), \epsilon\rangle$] possibly being fundamentally *simpler* than the latter structure. (Woodin 2001a, p. 570).

This is why Woodin worked with the $H(\kappa)$s rather than the von Neumann universe.

In terms of completeness, it is well known that one can transform any model of arithmetic into a model of set theory. Woodin did this and obtained $\langle H(\omega), \epsilon\rangle$. The theory that holds in this structure, i.e. the *structure theory* of $\langle H(\omega), \epsilon\rangle$, is ZFC with the Axiom of Infinity replaced by its negation. Just as with PA, this theory is then empirically and forcing complete for the structure. In fact, PA and this theory are logically equivalent.

Woodin then turned his attention to the structure $\langle H(\omega_1), \epsilon\rangle$, whose structure theory is second-order number theory.[11] For this structure, the theory $ZFC + PD$ is empirically complete, where PD stands for the Axiom of Projective Determinacy.

> *Projective Determinacy* settles (in the context of ZFC) the classical questions concerning the projective sets and moreover Cohen's method of forcing *cannot* be used to establish that questions of second order number theory are formally unsolvable from this axiom. (Woodin 2004, p. 3).

However, unlike PA, PD is not forcing stable, i.e. PD can be destroyed by forcing. This can be circumvented by assuming class many Woodin cardinals,[12] as will be seen in the next section. $ZFC + PD$ is hence *effectively forcing complete*.

PA and $ZFC + PD$ are hence, for their respective structures, both empirically and (effectively) forcing complete. Woodin called such theories *canonical*. But canonicalness alone did not suffice for Woodin. Recall that he is a non-pluralist platonist. If canonicalness were everything, there is to a good set theoretic theory, and then, it would be unclear why there should be only one such theory for every structure. In fact, as we will see below, in general, there are many of them. It is hence important that the axioms of the canonical theory are also *true*. The truth of PA and ZFC is usually unquestioned, and Woodin did not even mention his agreement here. The problem lies with PD. Why should PD be seen as true? Set theorists have studied this axiom extensively, and their results are seen by many to support the truth of PD. A discussion of these results would lead too far astray. The reader is referred to Woodin (2001a) for

[11] Second-order languages allow quantifications not only over variables, but also over sets of variables.

[12] A proper class is a collection, which is too big to be a set. Classical examples of a proper classes are the universe of all sets and the collection of all the ordinal numbers. Classes are too big to be measured by standard measurements via cardinalities. A collection is said to contain *class many* elements if it contains as many elements as there are ordinals.

a brief discussion of some of these results or to Sect. 6 for a short example. Woodin answered the question about the truth of PD as follows:

> Finally the intricate connections between this axiom [PD] and large cardinal axioms provide compelling evidence that this axiom is true. For me, granting the truth of the axioms for Set Theory, the only conceivable argument against the truth of this axiom, would be its inconsistency. I also claim that, at present, the only credible basis for the belief that the axiom is consistent is the belief that the axiom is true. This state of affairs could change as the number theoretic consequences of the axiom become more fully understood. (Woodin 2004, p. 3).

Woodin hence argued that the mathematical results proven by the set theoretic community can support the truth of this axiom. Recall here also the citation given above

> the truth of [PD] became evident only after a great deal of work. For me, a remarkable aspect of this is that it demonstrates that the discovery of mathematical truth is not a purely formal endeavour. (Woodin 2001a)

That is, for Woodin, it was possible to discover true axioms via mathematical results. It was hence possible to find true canonical theories for structures for which previously no such theory was known. Woodin found this encouraging. He was now in a situation where there is a true canonical theory for $\langle H(\omega), \epsilon\rangle$ and a true canonical theory for $\langle H(\omega_1), \epsilon\rangle$. He then took the next step: $\langle H(\omega_2), \epsilon\rangle$.

In the structure $\langle H(\omega_2), \epsilon\rangle$, the Continuum Hypothesis is decided. This is because the CH can be expressed as a statement about $H(\omega_2)$. Therefore, if a true canonical theory for $\langle H(\omega_2), \epsilon\rangle$ would be found, the CH would have found an answer. This was Woodin's goal. The problem was how to come up with a canonical theory for this structure.

What Woodin was looking for was a theory for $H(\omega_2)$, which is forcing invariant, i.e. for which independence results cannot be proven via forcing. A major problem was that such a theory cannot be generated by large cardinal extensions of ZFC; see the Lévy-Solovay Theorem discussed in Sect. 2. So how to find suitable axioms that, when added to ZFC, form a theory that satisfies the request for forcing invariance?

Woodin (1999) explored the non-stationary ideal[13] on ω_1 and managed, in a technical tour de force and with the help of additional assumptions, to derive a statement $(*)$ that, when added to ZFC, resulted in a forcing invariant theory. In the same book, Woodin realised that the needed technicalities can be captured by a strengthening of classical logic to Ω-logic. This paper will be concerned only with the Ω-logic approach.

[13] A treatment of the non-stationary ideal on ω_1 is beyond the scope of this paper; see Jech (2006) for details; see Sect. 6 for a formal definition.

3.2 A brief introduction to Ω-logic

Ω-logic was first introduced by Woodin in the first edition of Woodin (1999).[14] Since then it has received further refinement by Woodin. The brief introduction to the topic that follows is based on these refinements and is meant as a short overview for the reader rather than as a description of a historical development. In the first part of this section, the ideas and concepts of Ω-logic are informally introduced. The second part then revisits these concepts and explains them in more technical detail.

3.2.1 Ideas and concepts

In formal endeavours, there are two classical ways to decide whether a certain conclusion C follows from some premisses $P_0, \ldots, P_n$. The first is to follow certain rules to construct a proof of the conclusion from those premisses. These rules of proof are captured in the provability notion, a syntactic notion, and we write $P_0, \ldots, P_n \vdash C$. The second way to decide whether the conclusion follows from the premisses is to construct a counter-example, i.e. a situation in which all the premisses are true, but the conclusion is false. Such a situation amounts to the construction of a model where all the premisses are true, but the conclusion is false. Of course, this *disproves* the conclusion. Turning this around, one obtains that the conclusion follows from the premisses if for all models in which the premisses hold, the conclusion holds as well. This is the validity notion, a semantic notion, and we write $P_0, \ldots, P_n \models C$. For the construction of Ω-logic, both the provability and the validity notion will be changed.

The method of forcing can change the truth values of some statements in the language of set theory (e.g. CH) but not of all of them (e.g. ZFC holds in every forcing extension of a ZFC model). The main idea of Ω-logic is to define a new validity notion such that all statements which are true in the ground model and whose truth value cannot be altered by forcing are considered to be valid. This makes Ω-logic a strong logic in the sense that every statement which is valid in classical (first order) logic is also valid in Ω-logic but not vice versa. For example, the consistency of ZFC, $Con(ZFC)$, is Ω-valid from ZFC ($\neg Con(ZFC)$ is not forceable from a ZFC model) but famously not classically valid from ZFC; cf. (Bagaria et al. 2006, p. 4).

With the semantic notion of Ω-validity in place, the search for an Ω-provability notion begins. In classical logic, proofs are finite. However, Ω-logic is a strong logic, and so it is reasonable to expect (or demand) that proofs are no longer finite but may be infinite. This is not unprecedented. Via Gödel numbering we are already used to regard formulae as natural numbers. Proofs are then finite sets of natural numbers. It is then a small step to allow for *infinite* sets of natural numbers as proofs. An infinite set of natural numbers is a real (i.e. an element of $\mathbb{R}$), and a collection of such infinitary proofs is hence a set of reals. In Ω-logic, a proof is such a set of reals, i.e. an Ω-proof is a subset of the reals. This makes a proof in Ω-logic big in the sense that it is uncountable, but it is also not too big in the sense that its cardinality is at most the cardinality of the continuum. Woodin then demanded of his Ω-validity notion that the proof survives

[14] Note that Woodin's Ω-logic is different from ω-logic.

the transition into forcing extensions. This is made precise by the notion of *A closed models* (see below), where A is the Ω-proof. The Ω-provability notion is hence, in a well-defined sense, invariant under forcings. This links to the considerations about Ω-validity above.

A question is if the semantic $\models_\Omega$ and the syntactic $\vdash_\Omega$ notions in Ω-logic match in the sense of Gödel's completeness theorem for first-order logic.[15] Soundness is provable.

Theorem 3.1 (Ω-soundness) *Suppose that T is a set of sentences, that ϕ is a sentence, and that $T \vdash_\Omega \phi$. Then $T \models_\Omega \phi$. (Woodin 2004, p. 7).*

For completeness on the other hand, a proof is still missing and Woodin has formulated this completeness as the Ω-conjecture (see below).

3.2.2 Technicalities

As mentioned above, the idea behind the definition of Ω-logic is to define a logic whose validity notion is forcing invariant. The new validity notion $\models_\Omega$ is defined as

> For all statements ϕ in the language of set theory: $ZFC \models_\Omega \phi$ if, for all ordinals α, in all forcing extensions in which ZFC holds at the αth level, ϕ holds at the αth level as well.

This definition can be generalised by replacing ZFC by any recursively enumerable extension T of ZFC.

$ZFC \models_\Omega \phi$ is an (at most Π_2) formula, which can be expressed in the language of set theory. It hence makes sense to ask if $ZFC \models_\Omega \phi$ holds in some forcing extension of the universe of sets V. Using this, one can now make the question for the forcing invariance of $\models_\Omega$ more precise:

> Is it the case that $ZFC \models_\Omega \phi$ holds in V if and only if $ZFC \models_\Omega \phi$ holds in every forcing extension of V?

Woodin (1999) had shown that if there is a proper class of Woodin cardinals, then the above holds.

The assumption that there is a proper class of Woodin cardinals is often made by Woodin. In fact, many results of Ω-logic rely on it. Here is an argument for this assumption. It is a very subtle one and one that was not explicitly made by Woodin in his papers. Recall that for him PD is a true axiom. PD is equivalent to the statement 'for every n there is a M_n such that M_n is a specific model in which n Woodin cardinals exist'. The truth of PD is hence a good argument for the truth of this statement. The problem is, however, that the statement is not a large cardinal axiom in the strict sense as it posits the existence of models and not sets. The weakest large cardinal assumption that implies PD is that there are infinitely many Woodin cardinals, but then there is wiggle-room in consistency strength (i.e. the reverse direction of the implication does

15 A-logic is *sound* if from the fact that there is an A-proof of the statement ϕ from some premisses T, i.e. $T \vdash_A \phi$, it follows that ϕ is also A-valid from T, i.e. $T \models_A \phi$. A-logic is *complete* if $T \models_A \phi$ implies that $T \vdash_A \phi$. From Gödel's Completeness Theorem, it follows that for classical logic, $T \models \phi$ iff $T \vdash \phi$.

not hold). Nonetheless, the existence of infinitely many Woodin cardinals can be seen as an argument for the truth of PD. But Woodin wanted PD to hold through forcing extensions (it is a true axiom; hence, it should be true in all forcing extensions). By moving to a forcing extension, we can 'kill' Woodin cardinals, i.e. the Woodin cardinal looses its property of being a Woodin cardinal in a forcing extension. Hence, if there are only infinitely many Woodin cardinals, then we could 'kill enough of them' to reach a forcing extension where there are no longer enough of them to ensure PD. To remedy this, we need unboundedly many Woodin cardinals in the ground model. In this scenario, no matter how many Woodin cardinals we kill, there will always remain enough to ensure PD. This means that we need class many Woodin cardinals to start with.[16] Hence, we have an argument for 'there is a proper class of Woodin cardinals'.[17]

With a definition of the semantic notion $\models_\Omega$ in place, we now look for a corresponding syntactic notion $\vdash_\Omega$. As mentioned above, classically proofs are finite sequences of statements, i.e. via Gödel numbering, finite sequences of natural numbers. Infinite proofs are hence infinite sets of natural numbers, i.e. reals. The provability notion of Ω-logic relies on multiple infinite proofs, i.e. sets of reals $A \subseteq \mathbb{R}$. For technical reasons beyond the scope of this paper, it is assumed that A is universally Baire; see Sect. 6 for a definition. Such a set of proofs then proves many different formulae. However, (thanks to A being universally Baire), we can use the Wadge hierarchy (see Sect. 6) to define the length of an Ω-logic proof. It makes therefore sense to talk about the shortest Ω-proof A for ϕ. An Ω-proof hence has similarities to an ordinary proof.

The forcing invariance of $\vdash_\Omega$ translates into this picture as 'carrying over' A into the forcing extension. We are in a position where we have an Ω-proof A (which is just a subset of the reals) in some model M and we wish that A survives the transition into a forcing extension of M. This can be made precise by demanding that M is A-closed. Woodin has introduced many different ways to formalise this, all of which are equivalent; cf. (Bagaria et al. 2006, Prop. 2.9).[18]

With this in mind recall Gödel's Completeness Theorem:

A theory proves ϕ if and only if all models of said theory also model ϕ.

That is, as test-structures, *all* models are allowed. By restricting the class of test-structures, one obtains a stronger logic. A maximal case of this is allowing only a single test-structure, in which case the strong logic would decide every statement. The

[16] *For connoisseurs:* To demand unboundedly, many Woodins is to demand a process. That this process can be regarded as an object in its own right is due to a principle already present in Cantor: 'for every rule or process by means of which a collection of elements is obtained there is a set which contains exactly the elements which conform to the rule, or are obtained in the process, respectively' (Fraenkel et al. 1984, pp. 30–31). Of course today we know that the above formulation leads to inconsistencies (it allows for the Russell set) and have hence changed the word 'set' to 'class' in the above. The process of having unboundedly many Woodin cardinals is hence simply the same as having a class of Woodin cardinals.

[17] I am indebted to José Ferreirós and especially Dominik Adolf for drawing my attention to this subtlety.

[18] *For connoisseurs*: Given a universally Baire set A, an $\in$-model M of (a fragment of) ZFC. Then, M is A-closed if for all posets $\mathbb{P} \in M$ and all V generic filters $G \subseteq \mathbb{P}$:
$V[G] \models M[G] \cap A_G \in M[G]$.

above considerations present us with a different class of test-structures: the A-closed models.[19]

Woodin defined the Ω-provability relation $\vdash_\Omega$ as

> $ZFC \vdash_\Omega \phi$ if there exists a universally Baire set A such that for all A-closed, countable, transitive models M and for all ordinals α in M: if $M \cap V_\alpha \models ZFC$ then $M \cap V_\alpha \models \phi$.

Notice that the $\vdash_\Omega$ relation is defined only in case there is a model of ZFC, i.e. the consistency of ZFC is a necessary assumption for Ω-logic. Notice also that the above definition is given under the assumption that there is a proper class of Woodin cardinals. As mentioned elsewhere, this assumption is frequently made when dealing with Ω-logic. However, it is possible to define $\vdash_\Omega$ without this assumption, see Bagaria et al. (2006). Where the assumption is needed is for the forcing invariance of the $\vdash_\Omega$ relation. That is, if there is a proper class of Woodin cardinals, then $ZFC \vdash_\Omega \phi$ holds in V if and only if it holds in every forcing extension of V.

We now have a semantic and a syntactic notion. A natural question to ask is if these two coincide. Woodin (1999) showed soundness, i.e. that '$ZFC \vdash_\Omega \phi$ implies $ZFC \models_\Omega \phi$'. Completeness, i.e. the other direction of the implication, is still an open problem. Woodin has formulated this as the

> **Ω-Conjecture**: If there is a proper class of Woodin cardinals, then $ZFC \vdash_\Omega \phi$ if and only if $ZFC \models_\Omega \phi$.

3.3 Ω-Logic and the $(*)$-axiom

I now return to Woodin's search for a canonical (i.e. empirically complete and effectively forcing complete) axiomatisation of $\langle H(\omega_2), \in\rangle$. Canonical axiomatisation was, from Woodin's point of view, left as an imprecise notion. Given the above considerations about Ω-logic, Woodin saw himself in a position to make this notion more precise, namely by formalising it using the new notion of Ω-provability. His point can be put thus: an empirically complete theory is empirically complete because we cannot prove the independence of (natural) mathematical statements from this theory. Independence proofs are (mostly) done via forcing; hence, these empirically complete theories are effectively forcing complete (i.e. they are canonical). To model this situation by a proof relation in a strong logic therefore requires the new proof relation to be invariant under forcing. As seen above, $\vdash_\Omega$ is invariant under forcing (assuming a proper class of Woodin cardinals). Therefore, according to Woodin, Ω-provability can model canonicalness of theories. The following theorem is an intuition pump for this:

[19] *For connoisseurs*: There are only countably many sentences in the language of set theory, and the universally Baire sets are closed under preimages by Borel functions and countable unions. Therefore, there exists a single A_Ω such that the class of models under consideration here are the A_Ω-closed models.

Theorem 3.2 *Suppose there exists a proper class of Woodin cardinals. Then for each sentence ϕ,*

$$ZFC \vdash_\Omega \text{“}\langle H(\omega_1), \epsilon\rangle \models \phi\text{”}$$

if and only if

$$\langle H(\omega_1), \epsilon\rangle \models \phi.$$

Woodin (2001b).

We have already seen that there is a canonical theory for $\langle H(\omega_1), \epsilon\rangle$.[20] Woodin's idea was that the above theorem expresses this canonicalness.

The search for a canonical theory for $\langle H(\omega_2), \epsilon\rangle$ could hence be expressed thus:

Can there exist a sentence ψ such that for all sentences ϕ either

$$ZFC + \psi \vdash_\Omega \text{“}\langle H(\omega_2), \epsilon\rangle \models \phi\text{”}$$

or

$$ZFC + \psi \vdash_\Omega \text{“}\langle H(\omega_2), \epsilon\rangle \models \neg\phi\text{”}$$

and such that $ZFC + \psi$ is Ω-consistent? (Woodin 2001b, p. 686).

Call such ψ *good*. A good ψ is hence such a ψ that, when added to ZFC, completely determines, in Ω-logic, the structure theory of $\langle H(\omega_2), \epsilon\rangle$. Such a ψ would, granting Ω-logic, generate a canonical theory for $\langle H(\omega_2), \epsilon\rangle$. It would hence decide the CH (recall that CH becomes decided in this structure). But is there such a good ψ?

Woodin came close to constructing a good ψ. The additional assumption he needed was that there is an inaccessible cardinal, which is a limit of Woodin cardinals. Given this assumption, his $(*)$-axiom (Star-axiom) is good.[21] The definition of this axiom is technical. I shall give it here and refer the reader to Sect. 6 for a discussion of the relevant concepts.

Axiom $(*)$: There is a proper class of Woodin cardinals, and for each projective set $X \subseteq \mathbb{R}$, for each Π_2 sentence ϕ, if the theory

$$ZFC + \text{“}\langle H(\omega_2), \mathcal{I}_{NS}, X, \epsilon\rangle \models \phi\text{”}$$

is Ω-consistent, then

$$\langle H(\omega_2), \mathcal{I}_{NS}, X, \epsilon\rangle \models \phi$$

Woodin (2001b, p. 687).

[20] Keep in mind that the existence of a proper class of Woodin cardinals guarantees PD.

[21] *For connoisseurs*: The assumption is needed for Ω-consistency.

This axiom is provably good in the above sense. Furthermore, the following important result can be proven:

Theorem 3.3 *The* $(*)$*-axiom implies* $2^{\aleph_0} = \aleph_2$,

i.e. the $(*)$-axiom implies that the CH is false. This amounts to a case against CH. Recall that $(*)$ is good. By the representation of canonicalness via the $\vdash_\Omega$ relation, this means that $ZFC + (*)$ is a canonical theory, i.e. $ZFC + (*)$ is effectively forcing complete and empirically complete. But this is what Woodin had individuated as criteria that the theories PA (or its set theoretic representation) and $ZFC + PD$ satisfy. $ZFC + (*)$ satisfies these criteria also and hence could be seen as suitable or even as true.

The above argument is weak. There might be other axioms that, when added to ZFC, also satisfy these criteria. Whilst true, this is not troublesome. Here is the argument. Consider, for any good axiom ψ, the theory T_ψ which contains all those ϕ such that $H(\omega_2) \models \phi$ is Ω-valid from $ZFC + \psi$. That is: $T_\psi = \{\phi \mid ZFC + \psi \models_\Omega$ "$H(\omega_2) \models \phi$"$\}$. Woodin had shown that if there is a proper class of Woodin cardinals and the Ω-conjecture holds, then $\neg CH$ is an element of T_ψ for all good ψ. This means that as long as we add a good axiom to ZFC we will get a structure theory for $H(\omega_2)$ which includes the negation of the CH, i.e. which says that CH is false. In short: all good extensions of ZFC imply that CH is false.

There is, however, another problem. By a result of 2009 by Koellner and Woodin, the good extensions of ZFC are manifold. They managed to construct out of a good extension of ZFC, T_{ψ_0}, another good extension of ZFC, T_{ψ_1}, such that $T_{\psi_0} \neq T_{\psi_1}$. So even though the theories all agree that the CH is false, they disagree elsewhere. But then: which theory should we accept? This question is important, especially in the light of a non-pluralist understanding of the metaphysics of set theory.

Woodin has given no clear answer to this question. In Woodin (2001b), he presented $T_{(*)}$ as the most promising candidate. Nonetheless, he concluded his paper with

> So, is the Continuum Hypothesis solvable? Perhaps I am not completely confident the 'solution' I have sketched is the solution, but it is for me convincing evidence that there is a solution. (Woodin 2001b, p. 690).

A possible reading of this quote is that Woodin made a case against CH, not a case for the $(*)$-axiom. The $(*)$-axiom was only used to show that there are good axioms. This conflicts with his focus on the $(*)$-axiom in the paper. Throughout the paper, it seems that the $(*)$-axiom is presented as the axiom candidate we should accept. Accepting this axiom would not only show that the CH is false (as every good axiom would) but it would settle the question of the power of the continuum: under the $(*)$-axiom $2^{\aleph_0} = \aleph_2$.

Woodin (2004) only discussed the $(*)$-axiom. The whole confusion that there are other good axioms were left out. This pushed the $(*)$-axiom again as *the* good axiom we should accept.

Koellner (2013) showed that $T_{(*)}$ is in a well-defined sense maximal amongst the T_ψ (for good ψ). Therefore, we should accept the $(*)$-axiom. This seems reasonable, but an argument why we should accept maximal theories is not given. In fact, $T_{(*)}$ is maximal for an enrichment of $H(\omega_2)$ by the non-stationary ideal (on ω_1) and a subset of the reals of $L(\mathbb{R})$.[22] Why is maximality for this structure desirable? This question is not answered in Koellner's paper.

3.4 Summary

Woodin had considered fragments of the universe of sets V. The first fragment, $H(\omega)$, is (equivalent to) the structure of (first order) number theory. ZFC with the Axiom of Infinity replaced by its negation is a canonical (i.e. good) theory for this structure.

The second fragment is $H(\omega_1)$. The canonical theory for this structure is $ZFC + PD$. This theory was found only by much work, work which has not been discussed in this paper.

The third fragment of V is $H(\omega_2)$. It is in this fragment that the CH becomes decided. Via Ω-logic (plus the assumption that there is a proper class of Woodin cardinals; that there is an inaccessible cardinal which is a limit of Woodin cardinals; and that the Ω-conjecture holds), Woodin could show that there is a canonical theory for this structure: $ZFC + (*)$. This theory implies $2^{\aleph_0} = \aleph_2$ (i.e. $\neg CH$). What is more, *every* canonical theory for this structure implies $\neg CH$.

Is this process of gradually working ourselves upwards the infinite ladder of the fragments of V satisfying? Could it not be that the process stops at some point? And is it not off-putting that we have analysed only such a minuscule part of the transfinite universe? Woodin disagrees:

> I see no reason why this [process] should stop here and I am not unduly discouraged by the fact that these structures are negligible initial segments of the universe of sets. (Woodin 2004, closing words).

4 The case for CH

In 2010, Woodin's tone changes. Where he was earlier ready to analyse the set theoretic universe fragment by fragment, he now presents an argument for a form of extension of ZFC based on the whole universe. He moved from a local to a global argument. This change came about, according to Woodin, because of new mathematical results that he was able to prove.

4.1 The fate of the Ω-conjecture

The results in Sect. 3 rely on the Ω-conjecture. Without it, Ω-logic would be far less compelling, which would in turn weaken Woodin's results. So the question becomes: How to prove the Ω-conjecture?

As a first observation it has to be noted that if the Ω-conjecture holds in V, then it holds in every forcing extension of V. This is a very strong result, which indicates

22 See Sect. 6 for a discussion of $L(\mathbb{R})$.

that the Ω-conjecture might be disprovable. After all, we would only have to find a forcing extension of V where the conjecture does not hold in order to disprove it.

> Thus it is not unreasonable to expect both that the Ω-Conjecture has an answer and further if that answer is that it is false, then the Ω-Conjecture be refuted from some large cardinal hypothesis. (Woodin 2010b, p. 4).[23]

But it turns out that the level of such a large cardinal in the large cardinal hierarchy would have to be rather big; it would have to be beyond the level accounted for by a certain type of inner model. This will be explained in more detail below. But if the Ω-conjecture would be true, then there would be no such critical level. This question, is there a critical level or not, is Woodin's motivation to start his project in inner model theory, the results of which can be found in large parts in Woodin (2010b) and Woodin (2011).

As it turns out there is evidence (but no formal proof) that there is no such critical level and that the Ω-conjecture is in fact true. But in the course of collecting the evidence for this, something else happened. Woodin was able to show that if an inner model could be found which accommodates supercompact cardinals, then, contrary to what holds for smaller large cardinals, this inner model would accommodate all other large cardinals (consistent with ZFC). This remarkable result (or more precisely: the work surrounding it) then gives rise to a new kind of axiom, the axiom that V is Ultimate-L. In turn, this axiom would decide the independent statements, statements such as the CH. The aim of this section is to make what has been said thus far more precise.

4.2 Inner model theory

Gödel had shown the consistency of the Axiom of Choice, AC, with ZF (the axioms of ZFC without AC) by a construction of a model for ZF. Gödel assumed that the ZF axioms are consistent and constructed the so-called *constructible universe*, which is classically denoted by L; (Gödel 1938, 1939, 1940). In this model, AC holds. Hence, assuming the consistency of ZF, there is no inconsistency in $ZF + AC$ (i.e. ZFC). This model, L, gives a very well understood structure, and its structure theory is effectively complete in a sense similar to the effective completeness of $ZFC + PD$ for $H(\omega_1)$. So why not accept $V = L$, i.e. the axiom which states that the universe of sets is in fact the constructible universe, as correct? $ZFC + V = L$ is a well understood theory which resolves most set theoretical questions. For example, it proves the CH. However, L does not accommodate large cardinals. This was shown by Scott:

Theorem 4.1 *Suppose there is a measurable cardinal. Then $V \neq L$; Scott (1961).*

Woodin is known to add in his presentations after Scott's theorem the following "(meta) corollary" [e.g. in Woodin (2010c)].

Corollary 4.2 $V \neq L$

[23] This argument can be strengthened by regarding the corresponding claim for the non-trivial Ω-satisfiability of the Ω-conjecture, see (Woodin 2010b, p. 4).

Woodin expresses here his convictions that there are measurable cardinals. Measurable cardinals are relatively small large cardinals (i.e. they are low in the hierarchy). And if L cannot accommodate these small large cardinals, it cannot accommodate any larger large cardinals. So, if $V = L$, then there are no large cardinals at the level of a measurable cardinal or above. This is Woodin's case against $V = L$, and it is a case which he shares with a majority of the set theoretical community.

The *inner model programme* seeks L-like structures, i.e. models which are as well understood as L, and which can accommodate large cardinals. This programme is based on work of Lévy (1957, 1960) and Hajnal (1956, 1960). These mathematicians presented independently from each other the generalisations $L[E]$ and $L(E)$ of L, respectively.[24] Scott's theorem cited above can be said to have started the inner model programme; (Mitchell 2012, p. 5). Solovay, working on measurable cardinals at the time, then initiated work on $L[E]$ as analogous to L. One of the main results of this direction of study is that $L[E]$ can be constructed in such a fashion that it contains a measurable cardinal. However, Solovay also showed that this $L[E]$ only contains a single measurable cardinal and cannot contain any large cardinal, which is higher in the large cardinal hierarchy than a measurable cardinal; in $L[E]$, a version of Scott's theorem holds. $L[E]$ is an inner model, a concept introduced by Shepherdson who had largely proof-theoretic aims in mind; (Shepherdson 1951). Today the term 'inner model' is reserved for a special case: A proper class M is an inner model if and only if M is a transitive ϵ-model of ZF (Kanamori 2009, p. 33).

Powell (1974) introduced the concept of extenders in a general context. Extenders are functions derived from elementary embeddings, where elementary embeddings are functions between models, which preserve truth. Mitchell (1979) then used extenders in large cardinal theory and his work was further extended by Dodd (1982). The idea was that large cardinals can be defined in terms of elementary embeddings from which extenders can be derived. These can then be used in the construction of Solovay's $L[E]$ for example. By allowing extender sequences, Mitchell, Jensen and Dodd were able to construct L-like models for strong and superstrong cardinals. There is not enough room here to fully develop the history of the inner model programme. It was influenced by work on determinacy and by its brother-programme, the core model programme. See Kanamori (2009) for the early stages of this history, see Mitchell (2012) for more recent developments. Today the best results, according to Woodin, are the Mitchell-Steel extender models, presented in Mitchell and Steel (1994), which can accommodate Woodin cardinals.

To understand the heuristic of the inner model programme, it will be helpful to sketch the construction of Solovay's $L[E]$ in the modern extender-terminology.

Assume we are looking for a model which accommodates a measurable cardinal. Just as Gödel assumed the consistency of ZF for the construction of L, we now

[24] In the ordinary construction of L, we use the definable powerset operation $\mathcal{P}_{Def}$ for successor steps: $L_{\alpha+1} = \mathcal{P}_{Def}(L_\alpha)$, where $L_{\alpha+1}$ contains all those subsets of L_α that are definable from a finite number of elements of L_α by a formula relativised to L_α. For the construction of $L[E]$, not only the elements of L_α are used as defining parameters, but also elements of E. The construction of $L(E)$ proceeds just like the construction of L but rather than starting form the empty set $\emptyset$ the construction is started from the set E. See Sect. 6 for formal definitions. The differences between $L[E]$ and $L(E)$ are most clearly seen by observing that, in general, $E \in L(E)$ but not $E \in L[E]$.

assume the consistency of $ZFC+$'There is a measurable cardinal'.[25] The idea is to use Lévy's model $L[E]$. Let E be the extender given to us by the elementary embedding, which defines the measurable cardinal. Then, $L[E]$ is the model one obtains when constructing L whilst using the information coded in E. In this model, $ZFC+$'There is a measurable cardinal' holds. This was the first goal. But furthermore this model is, similar to L, 'simple' and can be very well analysed; it is L-like.

As stated above, this L-likeness even goes so far that there is a corresponding result to Scott's theorem in $L[E]$: $L[E]$ cannot accommodate any large cardinals above the level of a measurable cardinal. That is why there has never been a proposal for a new axiom of the form $V = L[E]$. So it is up to the inner model programme again to find L-like structures, which could accommodate even stronger axioms. However, it turns out that this is not possible with a single extender. That is, the example given above is the only case in which an L-like structure can be obtained by one extender. The problem lies in the derivation of the extender E from the elementary embedding. Recall that the elementary embedding is closely linked to the large cardinal under consideration. An extender then codes some of the information about this embedding, but not all of it. The problem is that for stronger large cardinals, a single extender does not code enough information about the embedding, too much gets lost.

There is an obvious remedy to this—allow for sequences of extenders—but it comes with a problem. The more information is fed into the construction of the new L-like structure, the less L-like it is going to be. The new structure will be more and more complicated and less and less well analysable. It is this trade-off between reaching larger large cardinals and the loss of L-likeness that the inner model programme has to balance.

Mitchell, Jensen and Dodd allowed for sequences of extenders, which changed the heuristic as follows: target the large cardinal axiom ϕ and assume $ZFC + \phi$ to be consistent. Hence, there is an elementary embedding. From this embedding, one can generate a sequence of extenders. Add the information coded in this sequence to the construction of L to obtain a model for $ZFC + \phi$. This method has proven to be very fruitful in the sense that models which can accommodate large cardinals which are stronger than measurable cardinals have been constructed. However, a version of Scott's theorem holds again in all these models. Hence, every model constructed in this way is limiting, it can only accommodate large cardinals up to the targeted large cardinal axiom, but no larger one. So it is up to the inner model programme to find yet another model to reach higher and higher levels of large cardinal axioms. As mentioned above, Mitchell-Steel extender models can accommodate Woodin cardinals and, assuming an iteration hypothesis, they extend it to a level just below that of a supercompact.[26]

[25] Notice one key difference in the assumptions between Gödel's construction of a model and how the inner model programme resolves the issue. Gödel did not assume anything about AC or CH, yet his resulting model believed both statements. Hence, he had proven consistency of AC and CH with ZF. The inner model programme on the other hand assumes the consistency of ZFC plus whatever large cardinal axiom is under scrutiny and then tries to find a model for it. That is, the inner model programme does not prove consistency of ZFC plus the large cardinal axiom because consistency is already set as a premiss.

[26] The reader might find it helpful to refer back to Sect. 2 and the (incomplete) list of large cardinal axioms given there to 'see the picture'.

Recent work of Woodin has led to a remarkable result (Woodin 2010b). Woodin starts by assuming that there is an inner model, which can accommodate a supercompact cardinal. It has to be noted that this is a strong assumption. No such model has been found thus far (in the sense in which a model which can accommodate a measurable cardinal has been found). The surprising result is that if such an inner model could be found, then it would accommodate *all* large cardinals (that are consistent with ZFC).

> [T]his is a paradigm shift in the whole conception of inner models. (Woodin 2009a, p. 21).

Where the structures thus far produced by the inner model programme stop somewhere near the middle of the large cardinal hierarchy (due to versions of Scott's theorem), this new model would go all the way up to the top. Furthermore, this inner model would be 'close to V' in a well-defined sense. This makes this model the, as Woodin calls it, *Ultimate L*. It is L-like, can accommodate all large cardinal axioms (consistent with ZFC) and is 'close to V'. Furthermore, the axiom V = Ultimate L could resolve the undecidability problems. This is discussed in more detail below.

To recapitulate. The Inner Model Programme seeks L-like structures that can accommodate large cardinal axioms. Each such structure comes with a version of Scott's theorem, resulting in the fact that the targeted large cardinal can be accommodated, but no stronger one. This changes at the level of a supercompact cardinal. However, this level has not been reached yet.

In fact, the level of a supercompact cannot be reached by the current methods of the Inner Model Programme. The point is that up until now *extender models* have been used by the Inner Model Programme. However, no (Mitchell-Steel) extender model can accommodate supercompact cardinals, as Woodin has shown (Woodin 2010b). But this does not mean that supercompact cardinals cannot be accommodated in any inner model, it only means that the methods currently used are insufficient for Woodin's aim. Woodin changes the methods and presents *strategic extender models*. A definition and explanation of these kinds of models lie beyond the scope of this paper. Instead I shall discuss how Woodin came to individualise them.

HOD stands for *Hereditary Ordinal Definable* and the class it refers to contains all those sets which are definable from ordinal parameters.[27] It was Gödel who first proposed the concept of ordinal definability in a talk in 1946, cf. Gödel (1965). The theory of this class was developed independently by Vopěnka et al. (1968) and Myhill and Scott (1967). HOD is a model for ZFC, but, unlike L, HOD is not absolute. That means that depending on in which model HOD is calculated, HOD changes. For example, if calculated in L, then $HOD = L$, but if calculated in V (and assuming a technical assumption), then $HOD \neq L$. This makes HOD very difficult to study and we know very little about this structure (compared to our knowledge about L). But it also makes it a very interesting structure to study since (in general) there are no versions of Scott's theorem in HOD. That means that HOD can accommodate very strong large cardinal axioms, depending on where HOD is calculated.

[27] See Sect. 6 for a formal definition.

In the late 1950s, there was a renewed interest in infinite games amongst Polish mathematicians. Mycielski and Steinhaus proposed in their paper ‘A mathematical axiom contradicting the axiom of choice’ of 1962 the Axiom of Determinacy, AD, which states that every game is determined, i.e. has a winning strategy. This axiom, though powerful, contradicts the Axiom of Choice, as Mycielski's and Steinhaus' title shows. The authors accepted AC but noted that there might be subuniverses of the universe of sets

> which reflect some physical intuitions which are not fulfilled by the classical sets (e.g. paradoxical decompositions of the sphere are eliminated by [AD]) (Mycielski and Steinhaus 1962)

The natural possibility for such a universe is $L(\mathbb{R})$, as Solovay pointed out (see also $L(E)$), and in the late 1960s, he conjectured that AD holds in $L(\mathbb{R})$ relative to the existence of a supercompact cardinal (Solovay 1969). $L(\mathbb{R})$ became the focus of serious study and is today a very well understood structure. This makes it natural to look at HOD calculated in $L(\mathbb{R})$, written as $HOD^{L(\mathbb{R})}$. What happens is summed up in the following theorem.

Theorem 4.3 (Woodin) *Assume there is a proper class of Woodin cardinals. Then $HOD^{L(\mathbb{R})}$ is not a Mitchell-Steel extender model. Woodin (2010a, p. 16), emphasis in original.*

Woodin asks the relevant question:

> But then what is $HOD^{L(\mathbb{R})}$? It belongs to a different, previously unknown, class of extender models, these are the *strategic extender models*. (Woodin 2010a, p. 16), emphasis in original.[28]

In what way do these strategic extender models help us? Can they accommodate supercompact cardinals? We do not know yet. But they have a different feature which is of major importance:

> The structure and theory of strategic extender models will be fully revealed by the inner models $HOD^{L(A,\mathbb{R})}$, where A is universally Baire. (Woodin 2010a, p. 16).

This means that by studying these structures $HOD^{L(A,\mathbb{R})}$ we might reveal that there is an inner model which can accommodate supercompact and hence all large cardinals that are consistent with ZFC. The

> understanding (and even discovering) [of] large cardinal axioms would have to depend on *structural* considerations of “Ultimate-L”. (Woodin 2009b, p. 32).

Hence the philosophical debate on large cardinals is given *mathematical traction* by Woodin's results.[29]

[28] *For connoisseurs*: the concept of a *strategic extender model* is an evolution of Jensen's and Dodd's concept of a *mouse*. I am indebted to Dominik Adolf for pointing this out to me.

[29] This term was coined in Koellner (2013).

4.3 Mathematical traction

There is more to the mathematical traction than mentioned above. But before discussing this, it will be useful to retrace the steps that have led us to where we are.

It all started with the Ω-conjecture. A way to disprove it would be via large cardinals. This led to a study of large cardinals and in particular to structures, which can accommodate these large cardinals. As it turns out, the classical approaches by the Inner Model Programme do not suffice for our purposes. Hence, Woodin introduces a new approach via strategic extender models. If these models can accommodate a supercompact cardinal, then they can accommodate all large cardinals (whose existence is consistent with ZFC). This is a remarkable and completely surprising result. If such a model could be found, it would reflect back to the Ω-conjecture and serve as an argument for (but not a proof of) the correctness of the conjecture. Let us, with Woodin, call such a strategic extender model L_S^Ω. Whether such an L_S^Ω exists will, according to Woodin, be revealed by studying the strategic extender models of the form $HOD^{L(A,\mathbb{R})}$, where A is universally Baire.

Recall that Woodin has rejected $V = L$ on the basis that this axiom is too limiting, it does not allow for all large cardinals that are consistent with ZFC. But what about $V = L_S^\Omega$? This axiom would not be limiting in this way because, as mentioned above, a model which can accommodate a supercompact cardinal can accommodate all large cardinals consistent with ZFC. Interestingly, this axiom can, in the light of the existence of a supercompact cardinal, be formulated thus:

Definition 4.4 (*The axiom* $V = L_S^\Omega$) There is a proper class of Woodin cardinals. Then for each Π_2 sentence ϕ which holds in V there is a universally Baire set A such that

$$HOD^{L(A,\mathbb{R})} \cap V_\Theta \models \phi$$

where $\Theta = \Theta^{L(A,\mathbb{R})}$, with $\Theta^{L(A,\mathbb{R})}$ being the supremum of all the ordinals α such that there is a surjection $\pi : \mathbb{R} \to \alpha$ such that $\pi \in L(A, \mathbb{R})$.

This axiom states that V is Ultimate-L. With this axiom, the questions about truths in V reduce to questions about models of the form $HOD^{L(A,\mathbb{R})}$. But these structures are accessible to study by our current means: $L(A, \mathbb{R})$ is L-like, which makes HOD calculated in this structure L-like as well (keeping in mind that L-likeness is left as a vague notion).

Here is the main result of this section:

Theorem 4.5 *The axiom $V = L_S^\Omega$ implies the CH (Woodin 2010a).*

But this result fades into the background when considering the full fruitfulness of the axiom: $V = L_S^\Omega$ is generalisable. One could allow for more or different ϕ and instead of a single universally Baire set A a whole sequence Γ of them might be considered, i.e. $L(\Gamma, \mathbb{R})$ instead of $L(A, \mathbb{R})$. Together with these generalisations, $V = L_S^\Omega$ would, so Woodin believes (but there is no formal proof), reduce all undecidable statements of set theory to questions about large cardinals. As Woodin expresses it, it would

> banish the spectre of undecidability as demonstrated by Cohen's method of forcing. (Woodin 2010a), closing words.

This is mathematical traction. The philosophical pluralism/non-pluralism debate mentioned in Sect. 1 could be settled on the basis of answers to precise mathematical questions. These are questions about structures that we already know and which we can define, questions about the $HOD^{L(A,\mathbb{R})}$. These questions about $HOD^{L(A,\mathbb{R})}$ are not independent from ZFC; independence would cease to be an issue.

And if $V = L_S^{\Omega}$ should turn out to be true, then the CH would be true as well. But here a problem slips in. $V = L_S^{\Omega}$ is not the only axiom for 'V is Ultimate-L'. If there is one such Ultimate-L, then there are many. And another one, such as $L_{(*)}^{\Omega}$, for example, would imply that the CH is false. But this no longer matters too much. *All* axioms that state that V is Ultimate-L resolve all independence questions (at least so Woodin hopes). And all of them involve the study of already known structures, i.e. they all reduce independence questions to precise mathematical questions, which have solutions. This is what makes $V =$ Ultimate-L interesting; the question about CH has become secondary.

At a conference in Bristol in 2013, Woodin implied that he believes that L_S^{Ω} will be the correct Ultimate-L. This would mean that CH is true, and this also means that Woodin has changed his believe about the truth of CH.

Many questions remain; precise mathematical questions. They are mathematically difficult questions to be sure, but we have no reason to believe that they do not have an answer. And Woodin has started to try and solve them.

Acknowledgments I would like to thank Brendan Larvor, Dominik Adolf, Jeremy Grey and José Ferreiós for their helpful comments.

5 Appendix A: Large cardinal axioms

This appendix gives definitions of all the large cardinal axioms mentioned in Sect. 2. All of the definitions are minor reformulations of the definitions given in Jech (2006) and Kanamori (2009). The latter is considered to be a classical text on the historical development of large cardinal axioms. A good and short introduction to the topic is Honzik (2013).

An *elementary embedding*, j, is a truth-preserving function between two models. The *critical point* of j, $crit(j)$, is the smallest ordinal α such that $\alpha \neq j(\alpha)$. A simple proof by transfinite induction shows that $crit(j) < j(crit(j))$, and it can be shown that $crit(j)$ is always a cardinal. The identity-embedding is the trivial elementary embedding and in all the definitions that follow j is non-trivial.

Definition 5.1 A *Reinhardt cardinal* is a $\kappa = crit(j)$ for some elementary embedding $j : V \to V$.

Definition 5.2 An *n-huge cardinal* is a cardinal κ such that there exists an elementary embedding $j : V \to M$ with critical point κ such that $M^{j^n(\kappa)} \subset M$.

Definition 5.3 A *huge cardinal* is a cardinal κ such that there is an elementary embedding $j : V \to M$ with critical point κ such that $M^{j(\kappa)} \subset M$.

Definition 5.4 An *extendible cardinal* κ is such that for every $\alpha > \kappa$ there is an ordinal β and an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ.

Definition 5.5 A *supercompact cardinal* is an uncountable cardinal κ such that for every A with cardinality greater or equal to κ there exists a normal measure on $P_\kappa(A) = \{X \subset A \mid |X| < \kappa\}$.

Definition 5.6 A *superstrong cardinal* is a cardinal κ such that there exists a elementary embedding $j : V \to M$ with critical point κ such that $V_{j(\kappa)} \subset M$.

Definition 5.7 A *Woodin cardinal* is a cardinal δ such that for all $A \subset V_\delta$ there are arbitrarily large $\kappa < \delta$ such that for all $\lambda < \delta$ there exist an elementary embedding $f : V \to M$ with critical point κ such that $j(\kappa) > \lambda$, $V_\lambda \subset M$ and $A \cap V_\lambda = j(A) \cap V_\lambda$.

Definition 5.8 A *measurable cardinal* is an uncountable cardinal κ such that there exists a κ-complete non-principal ultra-filter on κ.

Definition 5.9 A (strongly) *inaccessible cardinal* κ is such that κ is uncountable, regular and for every $\lambda < \kappa$, $2^\lambda < \kappa$.

6 Appendix B: Definitions and results

This appendix collects technical definitions and results left out in the main body of this article.

6.1 Results

The *von Neumann hierarchy* is formally defined as follows, where V is the universe of sets and the V_α are levels within this universe.

Definition 6.1 $V = \bigcup_{\alpha \in ON} V_\alpha$, where

$V_0 = \emptyset$
$V_{\alpha+1} = \mathcal{P}(V_\alpha)$
$V_\alpha = \bigcup\{V_\beta \mid \beta < \alpha\}$ where α is a limit ordinal

The non-stationary ideal on ω_1, $\mathcal{I}_{NS}$, is defined as follows:

Definition 6.2 $\mathcal{I}_{NS}$ is the σ-ideal of all sets $A \subseteq \omega_1$ such that $\omega_1 \setminus A$ contains a closed unbounded set. A set $S \subseteq \omega_1$ is stationary if for each closed unbounded set $C \subseteq \omega_1$, $S \cap C \neq \emptyset$. A set $S \subseteq \omega_1$ is co-stationary if the complement of S is stationary (Woodin 2001b).

6.1.1 Universally baire sets

Definition 6.3 A set A in a compact Hausdorff space Ω has the *property of Baire* if there is an open set $O \subseteq \Omega$ such that the symmetric difference $O \triangle A$ is meager. A *meager* set is a union of countably many nowhere dense sets. A *nowhere dense* set is a set whose closure has empty interior.

Definition 6.4 A set $A \subseteq \mathbb{R}^n$ is called *universally Baire* if for every continuous function

$$F : \Omega \to \mathbb{R}^n,$$

where Ω is a compact Hausdorff space, the preimage of A by F has the property of Baire.

Lemma 6.5
- *Every Borel set is universally Baire.*
- *The universally Baire sets form a σ-algebra, which is closed under preimages of Borel functions.*
- *The universally Baire sets are Lebesgue measurable.*

Definition 6.6 (*Wadge Hierarchy*) $A <_W B$ iff $A = f^{-1}(B)$ for some continuous $f :^{\omega} \omega \to^{\omega} \omega$ (Kanamori 2009).

In the presents of the Axiom of Determinacy, the Wadge hierarchy is a well order of the universally Baire sets, as Martin has shown in an unpublished article in 1973 called 'The Wadge Degrees are well ordered'. In the absence of this axiom, the following theorem holds. Hereby is $\mathbb{K}$ the Cantor set and a set $A \subseteq \mathbb{K}$ is said to be *strongly reducible to* $B \subseteq \mathbb{K}$ if there is a continuous function $g : \mathbb{K} \to \mathbb{K}$ such that $A = f^{-1}(B)$ and for all $x, y \in \mathbb{K}$, $|f(x) - f(y)| \leq (1/2)\,|x - y|$.

Theorem 6.7 *Suppose that $(A_k : k \in \mathbb{N})$ is a sequence of subsets of $\mathbb{K}$ such that for all $k \in \mathbb{N}$ both A_{k+1} and $\mathbb{K} \setminus A_{k+1}$ are strongly reducible to A_k. Then there exists a continuous function $g : \mathbb{K} \to \mathbb{K}$ such that $g^{-1}(A_1)$ does not have the property of Baire (Woodin 2001b).*

Because the universally Baire sets are closed under preimages of Borel functions, the above theorem shows well-foundedness of $<_W$ on the universally Baire sets.

6.1.2 Projective determinacy

As mentioned in the article, too much would have to be said to give an argument as to why the Axiom of Projective Determinacy, PD, should be accepted. I will here only give a very brief sample case as presented in Woodin (2001a): the Banach–Tarski Paradox.

Given the unit sphere in a three-dimensional space, there is a finite partition of the sphere into pieces which, after moving them around without changing their size, can be put together again to obtain two copies of the unit sphere. This is the Banach–Tarski Paradox. As is well known, it is an implication of the Axiom of Choice, AC. Hence, if one wants to keep AC, one may only hope to put constrains on the type of partitions in which the sphere may be divided. PD implies that these pieces may not be projective sets.

The projective sets are generalisations of the Borel sets.

Definition 6.8 (*Luzin*) A set $X \subseteq \mathbb{R}^n$ is a *projective set* if for some integer k it can be generated from a closed subset of $\mathbb{R}^{n+k}$ in finitely many steps, applying the basic operations of taking projections and complements (Woodin 2001a).

The sets encountered in everyday mathematics are all projective sets. This may instil the idea that the projective sets are somehow 'reasonable'. Hence, with PD, any partition of the unit sphere into reasonable pieces will not allow for the paradox. This is seen by some as argument for the truth of PD. See Woodin (2001a) for further details on PD and the Banach–Tarski–Paradox.

As mentioned in the above, $ZFC + PD$ is not a forcing complete theory:

Theorem 6.9 *PD is not forcing stable, i.e. there is a forcing which destroys PD.*

Proof (sketch) Let A be a set of ordinals, which codes all of $\mathbb{R}$. If $A^\sharp$ would exist in $L[A]$, then there would be a Reinhardt cardinal in $L[A]$. This contradicts AC. Hence, $A^\sharp$ does not exist in $L[A]$. Now force with $Col(\omega, A)$ to collapse A onto ω. Since $A^\sharp$ is effectively a definable subset of A and Col-forcing is homogeneous, $A^\sharp$ does not exist in the forcing extension either. Since A is countable in the extension, A is effectively a real in the extension. By "$\Pi^1_1(x)$ determinacy iff $x^\sharp$ exists" (proven by Harrington 1978) it follows that $\Pi^1_1(x)$ determinacy, and hence PD, fails in the extension.[30] □

6.2 Definitions

The following two definitions formally define what I have called 'defining L by using the information coded in E' and 'adding A to L', respectively.

Definition 6.10 Let E be a set. Then,

1. $L_0[E] = \emptyset$
2. $L_{\alpha+1} = \mathscr{P}_{Def}(Z)$, where

$$Z = L_\alpha[E] \cup \{E \cap L_\alpha[E]\}$$

and $\mathscr{P}_{Def}(Z)$ refers to the definable powerset of Z.

3. $L_\alpha = \bigcup\{L_\beta \mid \beta, \alpha\}$ for α limit ordinal.

and $L[E]$ is the class of all sets a such that $a \in L_\alpha[E]$ for some α.

Definition 6.11 Suppose that A is a transitive set. Then,

1. $L_0(A) = A$
2. (Successor Case) $L_{\alpha+1}(A) = \mathscr{P}_{Def}(L_\alpha(A))$
3. (Limit Case) $L_\alpha(A) = \bigcup\{L_\beta(A) \mid \beta < \alpha\}$

and $L(A)$ is the class of all sets a such that $a \in L_\alpha(A)$ for some α.

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30 I am indebted to Dominik Adolf for this proof.

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