Planetary latitudes in medieval Islamic astronomy: an analysis of the non-Ptolemaic latitude parameter values in the Maragha and Samarqand astronomical traditions

Arch. Hist. Exact Sci. (2016) 70:513–541
DOI 10.1007/s00407-015-0172-x

CrossMark

Planetary latitudes in medieval Islamic astronomy: an analysis of the non-Ptolemaic latitude parameter values in the Maragha and Samarqand astronomical traditions

S. Mohammad Mozaffari[1]

Received: 4 July 2015 / Published online: 18 January 2016

Abstract Some variants in the materials related to the planetary latitudes, including computational procedures, underlying parameters, numerical tables, and so on, may be addressed in the corpus of the astronomical tables preserved from the medieval Islamic period (*zīj* literature), which have already been classified comprehensively by Van Dalen (Current perspectives in the history of science in East Asia. Seoul National University Press, Seoul, pp 316–329, 1999). Of these, the new values obtained for the planetary inclinations and the longitude of their ascending nodes might have something to do with actual observations in the period in question, which are the main concern of this paper. The paper is in the following sections. In the first section, Ptolemy's latitude models and their reception in Islamic astronomy are briefly reviewed. In the next section, the medieval non-Ptolemaic values for the inclinations and the longitudes of the nodal lines are introduced. The paper ends with the discussion and some concluding remarks. The derivation of the underlying inclination values from the medieval planetary latitude tables and determining the accuracy of the tables are postponed to "Appendix" in the end of the paper.

1 The Ptolemaic latitude models and their reception in medieval Islamic astronomy

In the Ptolemaic latitude theory for the superior planets in the *Almagest* (Fig. 1a), the eccentric is inclined to the ecliptic at a fixed angle i_0 which is equal to 2;30° for Saturn,

Communicated by: Noel Swerdlow.

✉ S. Mohammad Mozaffari
mozaffari@riaam.ac.ir

1 Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), P.O. Box: 55134-441, Maragha, Iran

1;30° for Jupiter, and 1° for Mars. The centre of the epicycle revolves on the eccentric in the direction of increasing longitude, reaches the northern and southern *limits*[1] at opposite locations on the ecliptic, and crosses it at the nodes. The epicycle itself is inclined to the eccentric with its perigee at the same direction as the eccentric, but at a *varying* angle i_1. i_1 is minimum and equal to i_0 when the centre of the epicycle is at the nodes and reaches its maximum when the centre of the epicycle is located at the limits: $i_{1\,\max} = 4;30°$ for Saturn, 2;30° for Jupiter, and 2;15° for Mars. The direction of these inclinations is fixed with respect to the planets' apsidal lines. $\angle ATN$ shows the longitudinal difference ω_A between the apogee A and the northern limit N, which is equal to +50° for Saturn, −20° for Jupiter, and 0° for Mars. The model in the *Handy Tables* (Fig. 1b) is as same as in the *Almagest* with the exception that i_1 is *fixed* at the same values in the *Almagest*, and that $\omega_A = +40°$ for Saturn, a value less precise than that in the *Almagest*. In the *Planetary Hypotheses* (Fig. 1c), i_1 is *fixed*, but equal to i_0, that is, the epicycle always remains parallel to the ecliptic. The inclinations are equal to i_0 in the *Almagest/Handy Tables*, with an improvement in the case of Mars where $i_0 = i_1 = 1;50°$.

The Ptolemaic model for latitude of the inferior planets is shown in Fig. 2a for Venus and Fig. 2b for Mercury. Ptolemy observed that the distance ω_A from the apogee to the northern limit of the eccentric is 0° for Venus and +180° for Mercury while they should be −1° and +196°, respectively, at his time. He also observed that the eccentric of the two planets is always inclined to the ecliptic in one direction, Venus to the north and Mercury to the south. This inclination is varied: when the centre of the epicycle is at either of the apsides, it reaches its maximum, $i_{0\,\max} = +0;10°$ for Venus and −0;45° for Mercury, but when the centre of the epicycle is ±90° from the apsidal line, $i_0 = 0°$. Consequently, Ptolemy defines a conventional position for the nodal line of the eccentric ±90° from the apsidal line. Next, when the centre of the epicycle is ±90° from the apsidal line, the diameter passing through the epicyclic apogee and perigee is inclined to the ecliptic in the line of sight at an angle i_1. This inclination is also varied, $i_{1\,\max} = 2;30°$ for Venus and 6;15° for Mercury. As the epicycle moves towards the apsidal line, it decreases until it vanishes at either of the apsides. Then, when the centre of the epicycle is in the apsidal line, the orthogonal to the diameter passing through the epicyclic apogee and perigee is slanted to the ecliptic across the line of sight at an angle i_2. The slant is also varied, $i_{2\,\max} = 3;30°$ for Venus and 7° for Mercury when the centre of the epicycle is in the apsidal line. As the epicycle moves away from the apsidal line, the slant decreases until it disappears at ±90° from the apsidal line. As shown in Fig. 2a, b, the directions of the inclination and slant are reversed for Venus and Mercury. In the *Handy Tables* and *Planetary Hypotheses*, all three components of the latitude are fixed, and the inclination and the slant of the epicycle are equal: $i_1 = i_2 = 3;30°$ for Venus and 6;30° for Mercury and $i_0 = 0;10°$ for both.[2]

[1] They are the locations in the ecliptic where a superior planet reaches its extremal northern and southern latitudes (*Almagest* XIII.1: Toomer 1998, p. 598).

[2] For the latitude models and parameters in the *Almagest*, cf. Pedersen [1974] 2010, Chapter 14; Neugebauer 1975, Vol. 1, pp. 206–226; Swerdlow and Neugebauer 1984, Chapter 6; Riddell 1978. For *Canobic Inscription*, *Planetary Hypotheses*, and *Handy Tables*, cf. Neugebauer 1975, Vol. 2, pp. 908–917, 1006–1016; Swerdlow 2005, pp. 58–68; Pedersen [1974] 2010, pp. 398–400. The structures of the latitude model

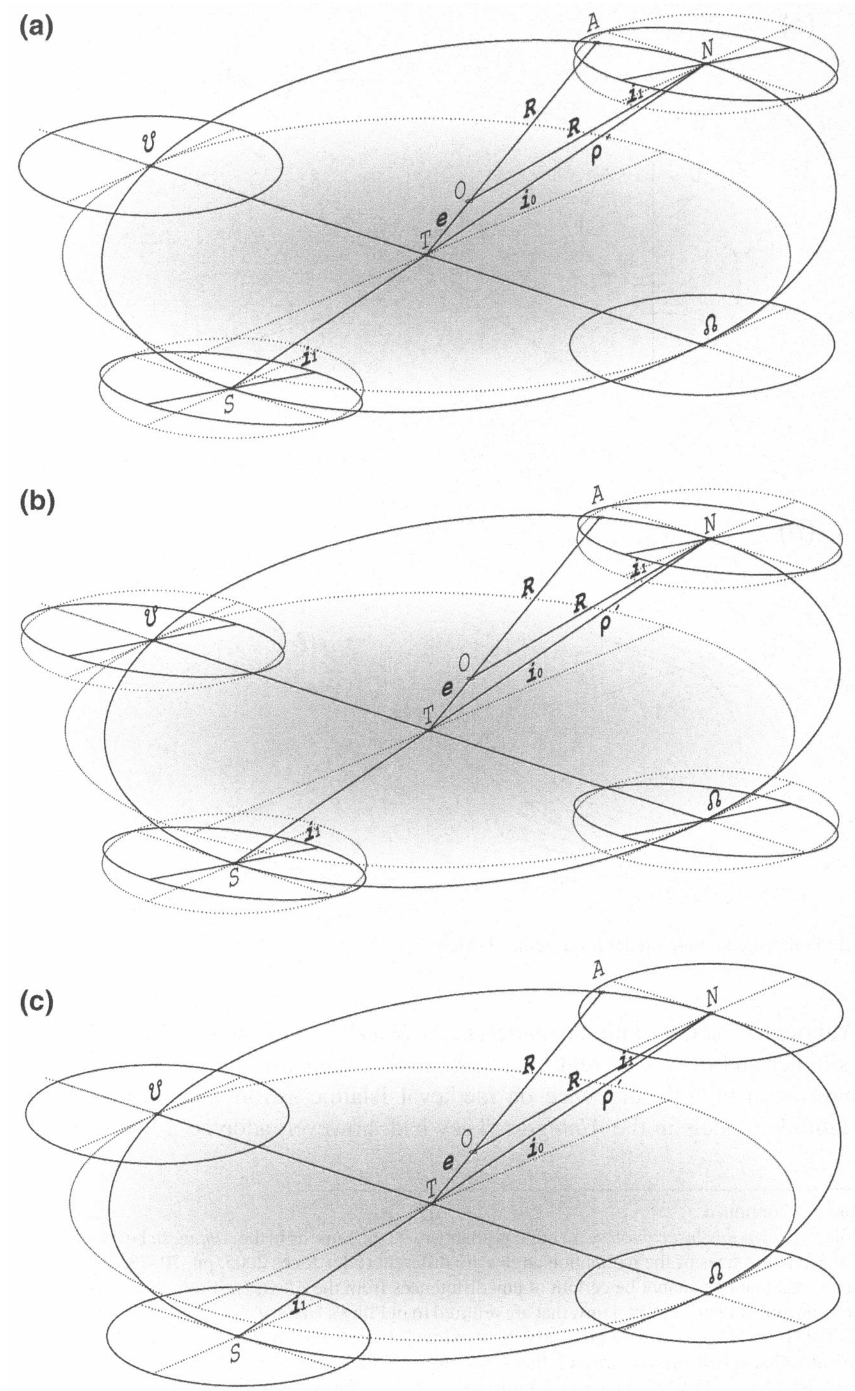

Fig. 1 Ptolemaic latitude model for the superior planets in the **a** *Almagest*, **b** *Handy Tables*, **c** *Planetary Hypotheses*

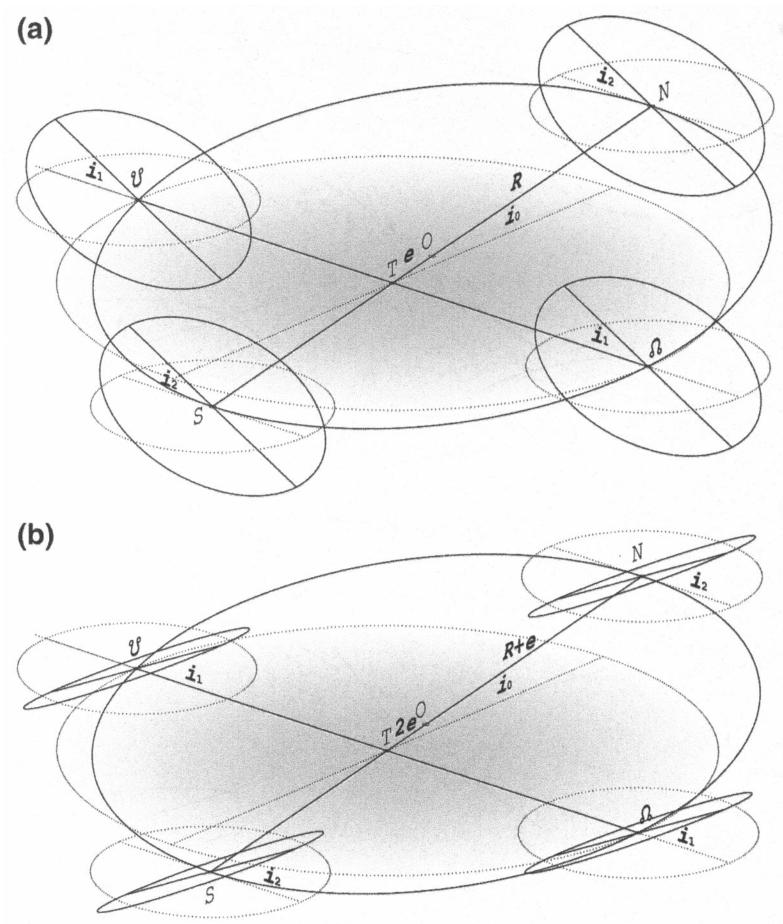

Fig. 2 Ptolemaic latitude model for **a** Venus, **b** Mercury

Although Ptolemy's latitude models in the *Handy Tables* and *Planetary Hypotheses* are simpler and more coherent than those in the *Almagest*, they nevertheless appear to have been of little influence on medieval Islamic astronomers who adhered to the latitude models in the *Almagest*. They had, however, adopted a few elements of

Footnote 2 Continued
in Ptolemy's *Canobis Inscription* seem quite probably to be the same as in the *Almagest*, but with the exception of Mars, the sizes of the inclination angles are different (after Jones 2005, pp. 70–73, 89). In fact, the text is so bad that one cannot be certain of any differences from the *Almagest*. From the planetary latitude theories presumably prior to Ptolemy, that are referred to in Pliny's *Historia Naturalis* (II.xiii.66–67: 1938–1962, Vol. 1, pp. 212–215), in which only the maximum latitudes are mentioned with reference to a 12° ecliptical/zodiacal belt , values are as follows: Moon: ±6°, Mercury: +5°/−3°, Venus: ±7°, Mars: ±2°, Jupiter: ±2 ½° or +3°/−1°, Sun and Saturn: ±1° (also, cf. Neugebauer 1975, Vol. 2, p. 782; Eastwood and Grasshoff 2003, pp. 203–6; Eastwood 2007, pp. 119–126, 136–137). The pre-Ptolemaic solar model having the latitudinal component, in which the sun reaches the maximum latitude of ±0.5°, is discussed in Jones 2000. For the planetary latitudes in Babylonian astronomy, cf. Steele 2003.

the latitude models in the *Handy Tables* and *Planetary Hypotheses*. For instance, the extremal values for the latitude of all the planets in the *Handy Tables* have been applied to the *Mumtaḥan Zīj* (see below, note 18); the maximum values for the latitudes of the inferior planets from the *Handy Tables/Planetary Hypotheses* are also mentioned in al-Battānī's *Ṣābi' Zīj* and al-Farghānī's *Book on astronomy* (although both authors do not identify their sources);[3] the table for the latitude of Venus from the *Handy Tables* is mentioned in Ibn Yūnus' *Ḥākimī Zīj* (Cairo, d. 1009) (see below). Ibn al-Shāṭir of Damascus (1304–1375/6) discussed how to adapt the latitude theory in the *Almagest* and *Planetary Hypotheses* to his own planetary model, in which two additional small circles are incorporated into the epicycle–eccentric mechanism in order that the motions of heavenly objects be produced solely by uniform circular motions.[4] The equality of the maximum values of the inclination and slant of each inferior planet, as in the *Handy Tables* and *Planetary Hypotheses*, was used in al-Kāshī's novel method of the computation of the planetary latitudes according to the *Almagest* models (see below).

The complexity of the *Almagest* latitude models seems to have been of no concern to medieval astronomers, despite the fact that one of Ptolemy's most notable remarks on simplicity and complexity in astronomical hypotheses in the *Almagest* is stated in connection with the latitude theory.[5] This could give an impression that, since the underlying latitude models in the *Handy Tables/Planetary Hypotheses* are simpler than those in the *Almagest*, they might be preferable. But instead they followed the *Almagest* models with some attempt to remedy them in some aspects, as will be mentioned briefly below.

In the observational aspect, the Middle Eastern Islamic astronomers seem also to have possessed some persuasive observational evidence of problems with the planetary latitudes in some specific *zījes*; nevertheless, such criticisms had been directed to the adopted parameters, numerical tables, and so on, rather than to the models themselves.[6]

[3] Cf. Nallino [1899–1907] 1969, Vol. 1, p. 116, Vol. 3, p. 175; al-Farghānī XVIII, 1969 pp. 73–4, P: fol. 67r: Al-Farghānī gives the rounded values to the nearest $0;20° = \frac{1}{3}°$ for the maxima of the latitudes: $\pm 3°$ for Saturn, $\pm 2°$ for Jupiter, $+4\frac{1}{3}°$ and $-7°$ for Mars, but he specifies that for Venus it is $6\frac{1}{3}°$ in the *Almagest* and 9° in the "sources other than the *Almagest*", and gives $4\frac{1}{3}°$ for Mercury, which is indeed equal to that in the *Handy Tables/Planetary Hypotheses*. Such a mixed tradition can be found in the Andalusian section of Islamic astronomy, e.g., in the *Muqtabis Zīj* of Ibn al-Kammād (12th ct.) (cf. Chabás and Goldstein 1994, p. 32).

[4] Cf. Roberts 1966.

[5] *Almagest* XIII.2: Toomer 1998, p. 600. The two others are mentioned in III.1 and III.4 (Toomer 1998, pp. 136, 153). These passages have been rendered faithfully into Arabic (cf. *Arabic Almagest*, S: fols. 29r, 34r, 212r; PN: fols. 32v, 161v).

[6] For example, 'Alī b. Amājūr's criticisms of the planetary latitudes as computed from the *Mumtaḥan Zīj* (see below, note 18) or Wābkanawī's claim that the directions of the observed latitudes of the interior planets are opposite to what are derived from his contemporary *zījes* (Wābkanawī, T: fol. 2v, Y: fol. 3v, P: fol. 3v. About Wābkanawī, cf. Mozaffari 2013a). Of the criticisms of other kinds, it is worth noting Ibn Yūnus' severe critical remarks on Ḥabash al-Ḥāsib's (d. after 869) knowledge of the *Almagest* latitude model for the inferior planets (cf. Ḥabash, fols. 63v–65v; Debarnot 1987, p. 54) as well as his berating al-Battānī's understanding of how the direction of latitude of Mercury should be determined in the *Almagest* model (Ibn Yūnus, L: p. 4; Caussin 1804, pp. 53, 55). Factually, in the rules set forth in chapter 47 of the *Ṣābi' zīj*, the directions given for the slant component of latitude of the inferior planets are diametrically opposed to what should be taken into account according to *Almagest* XIII.6 (Toomer 1998, p. 635); corrected in Nallino [1899–1907] 1969, Vol. 3, p. 175.

In the philosophical aspect, the following are worthy of consideration: in the latitude models in the *Almagest* (XIII.2),[7] Ptolemy embedded small circles that were responsible for rotating the endpoints of the diameters of the epicycles in order to account for their variable inclinations. Some spherical versions of the Ptolemaic models were proposed in Islamic astronomy in order to remove the small circles and give a philosophically justified *physical* model for the latitudinal tilting of the epicycles: Ibn al-Haytham (965–*ca.* 1040) utilized two contacting concentric spheres with different poles whose distance is equal to the maximum inclination or slant of the epicycle, and rotating in opposite directions with the same velocity as the centre of the epicycle of the planet as seen from the earth.[8] Naṣīr al-Dīn al-Ṭūsī (1201–1274) deployed three concentric interconnected spheres, two of them producing the geometrical device known as the "Ṭūsī couple", to solve the difficulties arising from Ptolemy's small circles.[9]

In the mathematical aspect, it merits mentioning that a *geometrical* spherical version of the latitude models in the *Almagest* was proposed by Jamshīd Ghiyāth al-Dīn al-Kāshī (1380–1429); although apparently influenced by al-Ṭūsī, it was essentially intended to improve upon the *mathematics* of the models.[10] As mentioned above, it uses a distinct feature of the Ptolemaic latitude models in both the *Handy Tables* and *Planetary Hypotheses* that the extremal values of the inclination and slant of each inferior planet are equal.[11] Al-Kāshī also presented two-argument tables for deriving longitude and latitude; dispensing with lengthy computational procedures, a practitioner only needs enter them with the adjusted eccentric and epicyclic anomalies to derive the ecliptical coordinates.[12] Another example is this: in the *Almagest* models, the slant component of the latitude of Mercury from the tables should be corrected such that when the planet is in the apogeal half of the eccentric, one-tenth of the slant is subtracted, and inversely, when the planet is in the perigean part of the eccentric, the same amount is added. Muḥyī al-Dīn al-Maghribī rectified the procedure,[13] and Wābkanawī later commented upon al-Maghribī's method and improved it in the manner that the amount of the slant itself should be multiplied by the absolute value of the cosine of the planet's eccentric anomaly;[14] over one century later, Kāshī instead proposed a new interpolation function in order to calculate the slant component of the latitude of Mercury, by means of which there is no need to include this correction.[15]

[7] Toomer 1998, pp. 599–601.

[8] Cf. Mancha 1990; Ragep 1993, pp. 214–7; 2004.

[9] Cf. Ragep 1987, pp. 344–8; 1993, pp. 218–222; Saliba and Kennedy 1991.

[10] Cf. Brummelen 2006, esp. pp. 357–8.

[11] Cf. Brummelen 2006, p. 360. Al-Kāshī uses the *Almagest* maximum value of the slant of each inferior planet for its maximum inclination as well; i.e., $i_{1\,\max} = i_{2\,\max} = 3;30°$ for Venus, as is in the *Handy Tables* and *Planetary Hypotheses*, and $i_{1\,\max} = i_{2\,\max} = 7;0°$ for Mercury while the *Handy Tables* and *Planetary Hypotheses* have 6;30°. Al-Kāshī states (IO: fols. 102v, 103v, 106r) that this is what the moderns have found through new observations, which seems to be a conclusion derived from examining the latitude tables in the *zījes* of the Maragha tradition (see below, Sect. 2).

[12] Al-Kāshī, IO: fols. 142r–156r, P: pp. 136–152; in these two MSS, only the tables for the longitude of the sun, moon, Jupiter, and Venus, and those for the latitude of the inferior planets are available.

[13] Al-Maghribī, *Adwār*, CB: fol. 17v.

[14] Wābkanawī, T: fols. 55r, Y: fol. 100r, P: fol. 83v.

[15] Al-Kāshī, IO: fol. 104r; Qūshčī, pp. 340–341.

Despite all the above-mentioned refinements, the most important alternatives to Ptolemy's planetary latitude models during the medieval period were Indian models.[16] They were adopted in some *zījes* of the early ninth century such as al-Khwārizmī's *Sindhind Zīj* (*ca.* 840)[17] and Yaḥyā b. Abī Manṣūr's *Mumtaḥan Zīj* (*ca.* 832),[18] and were later passed to the Western Islamic astronomical tradition (the Maghrib and Andalus) through al-Khwārizmī's *Zīj*; e.g., in the *Toledan Tables*[19] and Ibn 'Azzūz al-Qusanṭīnī's *Muwāfiq Zīj* (d. 1354).[20] The only medieval astronomer who proposed latitude models different from Ptolemy's is apparently Levi ben Gerson (1288–1344), who supplied computational procedures and numerical tables as well.[21]

Among the equatoria invented in the late Islamic period for finding the ecliptical positions of the heavenly objects, al-Kāshī's instrument for computing the planetary latitudes may be worth mentioning.[22] However, because of the smallness of the inclinations, such a device could probably be in use only for demonstrational purposes.

2 The values for the inclinations and ω_A in medieval Islamic astronomy

In the late Islamic period, there was a great deal of systematic planetary observations to determine their underlying parameters while in the early Islamic period the investigation was of solar and lunar motions. The new values for planetary inclinations and longitudes of ascending nodes are found in the *zījes* written in association with the well-known Islamic observatories established at Maragha (*ca.* 1260–1320 AD), Beijing (*ca.* the 1270s), and Samarqand (*ca.* 1430–1449).

16 Although the Indian models are inferior to Ptolemy's, nevertheless, they have two distinct features: the eccentrics of the inferior planets coincide with the ecliptic and the epicycles of the superior planets are parallel to the ecliptic (cf. Kennedy and Ukashah 1969).

17 Cf. Neugebauer 1962, pp. 34–40; Kennedy and Ukashah 1969.

18 The *Mumtaḥan Zīj* was later revised by some astronomers working in Iraq and Damascus up to about the mid-ninth century, although the manuscripts preserved probably go back to a recension made in the tenth century (see Van Dalen 2004a, esp. p. 11; Mozaffari 2013b, pp. 328–9). For the planetary latitudes, the following elements have been adopted in this *zīj*: the Indian values for the longitudes of the ascending nodes, Ptolemy's values for the extremal latitudes as tabulated in the *Handy Tables*, and a simple sinusoidal function, seemingly influenced by the Indian latitude models (see Viladrich 1998, esp. pp. 264–6; Kennedy 1990, pp. 173–7). As Ibn Yūnūs (L: p. 100; Caussin 1804, pp. 111, 113) states, 'Alī b. Amājūr found some errors in the latitudes of the planets and their directions with respect to the ecliptic as calculated from the *Mumtaḥan Zīj*. The earliest observation Ibn Yūnus reports from the Banū Amājūr family is the conjunction of Regulus with Venus made on 10 September 885 and the latest the lunar eclipse on 4/5 November 933; cf. Ibn Yūnus, L: pp. 102, 109; Caussin 1804, pp. 123, 125, 157; concerning the solar and lunar eclipses observed by them, see Stephenson 1997, pp. 471–2, 479–482; Steele 2000, pp. 116–117.

19 Toomer 1968, pp. 8, 69–72; Van Dalen 1999, pp. 323–324.

20 Samsó 1997, p. 92; 1999, pp. 16–17.

21 Its preliminary philosophical aspects are introduced in Goldstein 2002; also, cf. Glasner 2003. The models and their parameters and technical aspects still await further research.

22 Cf. Kennedy 1951; 1960, pp. 176–180, 198–214.

Table 1 New values for the inclination of the inferior planets in the Maragha tradition

	Almagest		*Īlkhānī zīj*		al-Maghribī's *Adwār*	
	β_{max}	i_{1max}	β_{max}	i_{1max}	β_{max}	i_{1max}
Mercury	4; 4°	6;15°	[*Alm.*]	[*Alm.*]	4;35°	7; 2°
Venus	6;22	2;30	8;40°	3;25° (?)	6;40	2;37

2.1 Inclinations

In the Maragha tradition, new values were observed for the inclinations of the epicycles of the inferior planets. These are mentioned nowhere and can only be derived from the latitude tables. Al-Maghribī explains his own observations and computations made at the Maragha observatory for the purpose of determining the solar, lunar, and planetary parameters in a treatise entitled *Talkhīṣ al-majisṭī* (*Compendium of the Almagest*). Nevertheless, since the last two books, IX and X, of this treatise are unfortunately missing from its only surviving manuscript, which is in the author's handwriting (Leiden: Universiteitsbibliotheek, no. Or. 110), we do not know whether and how he derived his values for the inclinations of the inferior planets and ω_A of the superior planets (see below) from his own observations. According to the list of its contents, book IX is devoted to the planetary retrograde motions and latitudes and book X to the stereographic projection of the celestial sphere onto a plane surface tangent to the north celestial pole, which is used for the drawing of astrolabe plates.[23]

The maximum latitude of an inferior planet is due to the inclination of its epicycle and takes place when the planet is at inferior conjunction with the mean sun, i.e. at the true epicyclic anomaly of 180°. We indicate the inclination component of the latitude as $\beta_{icl}(\alpha)$ where α is the true epicyclic anomaly; $\beta_{max} = \beta_{icl}(180°)$ in Table 1.

In the *Īlkhānī Zīj*, the maximum tabular value for the latitude of Venus is 8;40°,[24] corresponding to $i_{1\,max} \approx 3;25°$. This is close to the extremal value 8;52° as derived from the *Handy Tables* and *Planetary Hypotheses* in which $i_{1\,max} = 3;30°$.[25] Nevertheless, the table itself suffers from some difficulties as explained in the "Appendix", so that it is actually impossible to assign one single value to $i_{1\,max}$ from which the table has been computed. It is noteworthy that Ibn Yūnūs in his *Ḥākimī Zīj* has an extra table for the latitude of Venus where $\beta_{icl}(0°) = 1;29°$ and $\beta_{icl}(180°) = 8;52°$.[26] With Ibn Yūnus' non-Ptolemaic $e \approx 1;3$ for the eccentricity of Venus (corresponding to the maximum equation of centre $= 2;0,30°$)[27] and $r \approx 43;28$ for the radius of the epicycle (corresponding to the maximum epicyclic equation $= 46;25°$ at mean distance),[28]

23 Al-Maghribī, *Talkhīṣ*, fol. 2r; Mozaffari 2014, pp. 68–71.

24 Al-Ṭūsī, C: p. 128, T: fol. 76r, M: fol. 77v, P: fol. 44r. Also, see note 30.

25 Neugebauer 1975, Vol. 2, pp. 1011–16.

26 Ibn Yūnus, O: fols. 101r–103v.

27 Ibn Yūnus maintained the Indian-originated idea adopted in early Islamic astronomy that Venus' eccentricity is equal to the sun's and their apsidal lines coincide (cf. Ibn Yūnus, L: p. 121; Caussin 1804, p. 221).

28 Ibn Yūnus, L: p. 121; Caussin 1804, p. 221. Also, see "Appendix".

it can be found that both latitude values result from the rounded $i_{1\,\max} = 3;30°$. It seems that Ibn Yūnus obtained this table either directly or *via a medium* from the *Handy Tables*. Note that a mixed tradition of the *Almagest-Handy Tables* for the latitudes of the inferior planets already existed in earlier Islamic works (see note 3 above). The values 3;25° or 3;30° are in better agreement with the correct inclination of the planet than 2;30° in the *Almagest* and other Islamic *zījes*.

In al-Maghribī's *Adwār al-anwār*,[29] the maximum latitude of Venus is 6;40°, which corresponds to $i_{1\,\max} \approx 2;37°$. In his first *zīj*, the *Tāj al-azyāj*, written at Damascus in 1258 before his joining Maragha, he has the value 8;30° for the maximum latitude of Venus,[30] which corresponds to $i_{1\,\max} \approx 3;21°$. His value for the maximum latitude of Mercury in the *Adwār* is 4;35°, corresponding to $i_{1\,\max} \approx 7;2°$.[31] This is approximately equal to the Ptolemaic maximum value for the slant of Mercury, i.e., $i_{2\,\max} = 7;0°$. Since al-Maghribī's table of the slant is identical to the table in *Almagest* XIII.5,[32] it is straightforwardly deduced that al-Maghribī took the maximum inclination of Mercury equal to its maximum slant, as in the *Handy Tables/Planetary Hypotheses*, although in the *Handy Tables* and *Planetary Hypotheses* Ptolemy takes in the inclination as 6;30°, not 7;0°. It should be noted that the values near 7;0° are in excellent agreement with the true value of the inclination of the planet.[33] These values are summarized in Table 1.

It appears to have been generally accepted after the Maragha observatory that the value 3;30° for the inclination of Venus was a recent improvement upon the parameters of the *Almagest*. For instance, Ibn al-Shāṭir refers to it as "what the moderns have amended [in the *Almagest*]".[34] Also, one may conclude that in each of the two independent observational programs carried out at the Maragha observatory, i.e., by al-Maghribī and by the main staff of the observatory, who were engaged in preparing the *Īlkhānī Zīj*, it was found that the maximum values of the inclination and slant of one of the inferior planets are equal, although neither maintained this for the other inferior planet. This can be considered a partial rediscovery of Ptolemy's latitude model in the *Handy Tables* and *Planetary Hypotheses*. Over a century after the Maragha observatory, al-Kāshī appears to have achieved the same result that in the new observations made by recent predecessors and modern scholars both the inclination and the slant of Venus and Mercury are, respectively, equal to 3;30° and 7;0°.[35] He was familiar with both the *zījes* of the Maragha tradition, so that he was involved in the task of

[29] Al-Maghribī, *Adwār*, CB: fol. 87v, M: fol. 89v; Wābkanawī, T: fol. 163v.

[30] Dorce 2003, p. 218. It is noteworthy that some close values 8;35° and 8;36° are found in the Western Islamic, Hebrew, and Spanish astronomical tables; e.g., the *Muqtabis zīj* of Ibn al-Kammād (Chabás and Goldstein 1994, p. 32), the *Alfonsine Tables of Toledo*, and the canons to the tables of Judah ben Asher II of Burgos (d. 1391) (Chabás and Goldstein 2003, pp. 164–165). No relation between them and the *Tāj al-azyāj* or *Īlkhānī zīj* seems to exist.

[31] A close value 4;38° is found in the *Alfonsine Tables of Toledo* and in the canons to the tables of Judah ben Asher II (Chabás and Goldstein 2003, p. 164). No relation between it and al-Maghribī seems to exist, however.

[32] Toomer 1998, p. 634.

[33] Also, cf. Swerdlow 2005, pp. 63, 68.

[34] Roberts 1966, p. 216; the addition in brackets is ours.

[35] Al-Kāshī, IO: fols. 102v, 103v, 106r.

Table 2 New values obtained for the inclinations of the superior planets at the Samarqand observatory

	$\beta_N(90°)$	$\beta_S(90°)$	i_0	$\beta_N(180°)$	$\beta_S(180°)$	i_{1max}
Saturn	+2;29°	−2;29°	2;30° [*Alm.*]	+3;14°	−3;18°	6; 0°
Jupiter	+1;28	−1;29	1;30 [*Alm.*]	+2; 8	−2;16	2;45
Mars	+1; 9	−1; 4	1;20	+4;32	−7;15	2; 7

revising the *Īlkhānī Zīj*, and hence named his own work *Zīj-i Khāqānī dar takmīl-i Zīj-i Īlkhānī* (*Khāqānī Zīj*; *the improvement of the Īlkhānī Zīj*); and in his discussion of the latitude theory, he explicitly refers to al-Maghribī and his *Zīj al-kabīr* (*Great Zīj*),[36] an alternative name for al-Maghribī's *Adwār al-anwār*.[37]

The tables of the latitude of Venus in Ibn al-Shāṭir's *Jadīd Zīj*,[38] Ulugh Beg's *Sulṭānī Zīj*[39] and al-Kāshī's *Khāqānī Zīj*[40] are identical to the corresponding table in the *Īlkhānī Zīj*, as are the tables of the latitude of Mercury aside from the correction for distance in computing the slant.[41] This illustrates the influence that the formal work of the Maragha astronomy exerted on later Middle Eastern astronomers and that the inclinations of the inferior planets from the Maragha observatory appeared trustworthy to them. Perhaps for this reason, the astronomers at the Samarqand observatory turned to measuring the inclinations of the superior planets. The values derived for the inclinations of the superior planets from the critical entries in the northern and southern latitude tables in Ulugh Beg's *Sulṭānī Zīj*[42] are as listed in Table 2. The critical entries are the latitudes for the true anomaly of 90° and of 180°; the first is used to determine the inclination i_0 of the eccentric and the latter to derive the maximum inclination $i_{1\,max}$ of the epicycle. The subscripts 'N' and 'S' denote, respectively, the northern and southern latitudes. Note that our derivation of the inclinations from the extremal latitudes as explained in "Appendix" are based on the new values observed for the eccentricities and the radii of the epicycles of the superior planets at the Samarqand

[36] Al-Kāshī, IO: fol. 104r.

[37] This title can be found in other sources as well; e.g., al-Kamālī, fols. 230v and 231r.

[38] Ibn al-Shāṭir, O: fol. 56r, K: fol. 71r; the table is *Almagest*-type, namely the entries in are for each 6° of the argument in the range from 0° to 90°/270° to 360° and for each 3° of the argument in the interval from 90° to 270°. This table can also be found in Muḥammad al-Ṭabīb al-Muhtadī al-Mūṣilī's commentary on Aḥamad b. Ghulām Allāh's *al-Lum'a fī ḥall al-kawākib al-sab'a* (fol. 58r), which is based upon Ibn al-Shāṭir's *Jadīd zīj*. (Muḥammad al-Ṭabīb appended this commentary to another treatise of his own on the sine quadrant, *Risāla fī al-rub' al-mujayyab*, and then called the two altogether as *al-Jam' al-mufīd*).

[39] Ulugh Beg, P1: 146v, P2: fol. 163v.

[40] Al-Kāshī, IO: fol. 139v–140r, P: p. 124.

[41] *Īlkhānī zīj*, C: p. 137, T: fol. 83v, M: fol. 83r, P: fol. 47v; al-Kāshī, IO: fol. 140v–141r, P: p. 125–126; Ulugh Beg, P1: fol. 149v, P2: fol. 166v–167r. In the *Īlkhānī zīj*, the values of the slant of Mercury are tabulated as $11/10\beta_{S1}$ and $9/10\beta_{S1}$ in order to eliminate the additional step for correcting this latitude by adding/subtracting one-tenth of the value obtained from the table in the *Almagest*. In Ulugh Beg's and al-Kāshī's *zījes*, only the entries related to $11/10\beta_{S1}$ can be found, which is because of the new interpolation function proposed by al-Kāshī.

[42] Ulugh Beg, P1: fols. 137v, 140v, 143v; P2: fols. 153v, 156v, 160r. The latitudes of the superior planets in al-Kāshī's *Khāqānī zīj* as well as the values he mentions for their inclinations are Ptolemaic (IO: fols. 100v, 139v, P: p. 131).

Table 3 Difference ω_A between the longitudes of the apogees and the northern limits of the superior planets in the *zījes* of the Maragha and Samarqand observatories

	Ptolemy [*Alm.*][(1)]	Mod.	Al-Maghribī[(2)]	Wābkanawī[(3)]	Mod.	Ulugh Beg[(4)]	Mod.
Saturn	+53° /+50°	+49°	+60°	+60°	+61°	+60°	+63°
Jupiter	−19 / −20	−11	−10	−10	− 4	− 8	− 3
Mars	− 4.5/ 0	− 8	+ 6	−12	+ 4	+ 4	+ 6

(1) *Almagest* XIII.1 (Toomer 1998, p. 598). For Saturn: +27° in the *Canobic Inscription* and +40° in the *Handy Tables* and *Planetary Hypotheses*; *cf.* Neugebauer 1975, Vol. 2, pp. 910, 916, 1010; Goldstein 1967, pp. 20–25; Jones 2005, pp. 74–75, 90.
(2) Al-Maghribī, *Adwār al-anwār* II.5.2: CB: fol. 17v, M: fol. 18v. Tables of the latitudes: CB: fol. 87v, M: fol. 89v (also, Wābkanawī, T: fol. 163v). In his commentary on the *Īlkhānī zīj*, al-Nīshābūrī (P1: fol. 117v, P2: fols. 141r–v) gives the differences in longitude between the apogee and the ascending node of each superior planet, according to "the new observations, as Muḥyī al-Dīn has mentioned in his *zīj*", as Saturn 150°, Jupiter 80°, and Mars 96°.
(3) Wābkanawī, *Zīj* III.5.3: T: fol. 54r, P: fol. 82v, Y: fols. 98v-99r.
(4) Ulugh Beg, *Sulṭanī zīj*, Tables of the latitudes: P1: fols. 137v, 140v, 143v, P2: fols. 153v, 156v, 160r. The tables of the minutes of proportion give $\cos(\kappa + 60)$ for Saturn, $\cos(\kappa - 8)$ for Jupiter, and $\cos(\kappa + 4)$ for Mars. No further corrections are mentioned in the related explanatory text in III.4 (P1: fols. 107v-108r, P2: fols. 119r-v). Consequently, the values of ω_A for the Saturn, Jupiter, and Mars should be, respectively, +60°, −8°, and +4°. (It is worthwhile that also in al-Ṭūsī's *Īlkhānī zīj*, the entries in the tables of the minutes of proportion for Saturn and Jupiter are displaced by the Ptolemaic values for $-\omega_A$ (C: pp. 101, 110, 119, P: fols. 35r–v, 37v, 39v, T: fols. 54v, 62r, M: fols. 62r, 66v, 72r). But we are told in the canons of this *zīj* (II.3: C: p. 42, P: fol. 15r, T: fol. 21v, M: fol. 27r) to add the values 7° and 12°, respectively, to the adjusted eccentric anomaly of Saturn and Jupiter prior to entering the tables. These are *only* due to the asymmetrical tables of the equations of the epicyclic anomaly of the two planets in this *zīj*, the entries of which are displaced and always additive by adding, respectively, 7° and 12° to those in the corresponding *Almagest* table, and thus these amounts should have been subtracted from the mean eccentric anomalies before tabulating. Consequently, after *adjusting* the mean eccentric anomaly by the equation of centre, the result is still smaller than the *true* eccentric anomaly by 7° and 12°, respectively.) These are in agreement with the values 150°, 82°, and 94° which ʻAlī b. Muḥamamd Qūshčī (*ca.* 1402–74), one of the Samarqand astronomers contributing to the preparation of the *Sulṭānī zīj* (his name is explicitly mentioned in the prologue of the *Sulṭānī zīj* (P1: fol. 1v, P2: 1r)), gives in his *Commentary on Zīj of Ulugh Beg* (pp. 332–333) for the differences in longitude between the apogees and the ascending nodes of these planets in the *Sulṭanī zīj*

observatory, which are summarized in the "Appendix" (Tables 8, 9). This explains why the values of i_0 for Saturn and Jupiter are Ptolemy's while the corresponding latitudes are non-Ptolemaic.

2.2 The values of ω_A

In Ptolemaic astronomy, the apsidal and nodal lines of the planets are sidereally fixed. Indian models have completely different features: the apsidal lines of the planets move in the direction of increasing longitude, but their nodal lines in the retrograde direction, like that of the moon, at unequal rates.[43] The majority of (especially, early) Islamic *zījes* have Ptolemy's values for ω_A.[44] The non-Ptolemaic values for ω_A from the Maragha

[43] E.g., *Súrya Siddhánta* I.43–44: pp. 29–30.

[44] E.g., the *Zīj* of Ḥabash (fol. 66r); al-Battānī's *Ṣābi' zīj* (Nallino, [1899–1907] 1969, Vol. 2, pp. 140–141); Bīrūnī's *al-Qānūn al- masʻū dī* X.10 (1954–1956: Vol. 3, p. 1323); Ibn Yūnus, O: fols. 78r–79r; al-Kāshī, IO: fol. 101v (cf. Kennedy 1951, p. 19). The Ptolemaic values were also prevalent in the Latin sources insofar as the

and Samarqand traditions are listed in Table 3 (the true modern values at the times are indicated in the columns headed Mod.). These values are either explicitly mentioned in the canons to the tables or can be extracted from the tables or columns for "the minutes of proportion". The minutes of proportion c_5 is a simple cosine function of the argument of latitude $\omega_P = \kappa + \omega_A$ where κ is the true eccentric anomaly of the planet. In the *Almagest*, the user has to add the values of ω_A to κ and then enter the table with the result. This is also the case with both al-Maghribī's and Wābkanawī's *zījes*, but in the *Sulṭānī Zīj* the entries are shifted by $-\omega_A$ with respect to the corresponding entries in the *Almagest* tables. As a result, the procedure is shortened by one step.

3 Discussion and conclusion

Unlike the detailed examples in the *Almagest* for determining the parameters of the planets' motions in longitude, Ptolemy did not mention specific observations for planetary latitudes, only what appear to be approximate extreme values, which gave rise to some medieval objections. [45] Consequently, the medieval astronomers had no access to specialized methods and techniques required for making observations to determine the latitude parameters.[46] The required observations for the derivation of the inclinations and the values of ω_A should essentially be made at some specific conditions which are very rarely possible to satisfy: in order to measure the inclinations of the eccentric and the epicycle of a superior planet, it should be observed when it is *both* close to a limit and at opposition to (or as near as possible to conjunction with) the mean sun. This can take place once in about 30 years for Saturn, 12 years for Jupiter, and 15 or 17 years for Mars. The longitude of the ascending node can be determined when the planet crosses the ecliptic, in this case, when the centre of the epicycle is at a node, whose longitude is thus equal to the mean longitude of the planet.[47] As mentioned briefly in Sect. 1, early Islamic astronomers employed Indian methods or the ungrounded (if not meaningless) mixed versions of the Hindu and Ptolemaic latitude theories, as well as having difficulties with understanding the latitude models in the *Almagest*. Among the surviving early Islamic *zījes*, presumably al-Battānī's is one of the first works in which the *Almagest* latitude theory appeared in an understandable and correct abridged form (though, having a minor fault; see above, note 6), i.e., at the turn of the tenth century. These perhaps explain the reason it took a relatively long period until the first attempts were made to measure the inclinations of the planets (at the Maragha observatory in the mid-thirteenth century). Also, for the same reason, the

Footnote 44 Continued
materials reflected in the secondary literature indicate; e.g., Copernicus' *De Revolutionibus* (Swerdlow and Neugebauer 1984, p. 498), Bianchini (mid-fifteenth century; cf. Goldstein and Chabás 2004, p. 460; Chabás and Goldstein 2009, p. 95). The values of ω_A are not properly defined in the *Alfonsine Tables of Toledo* (Chabás and Goldstein 2003, p. 163).

[45] E.g., Goldstein 2002, p. 27.

[46] We can address some alternative strategies developed in Islamic astronomy to facilitate obtaining specific parameters, to remove anticipated difficulties, or to secure desired results. But these alternative methods themselves are based upon already-existing ones, mostly proposed by Ptolemy himself (e.g., the two alternative methods for measuring the solar eccentricity, cf. Mozaffari 2013b, pp. 320–324).

[47] For the difficulties with the latitude observations, cf. Swerdlow 2005, pp. 47, 49.

derivation of the latitude parameters from the observational data can be considered the acme of medieval observational astronomy.

Concerning the Eastern branch of Islamic astronomy, a partial improvement upon Ptolemy's values for the inclinations of the inferior planets in the *Almagest* was made at the Maragha observatory. This can, however, be considered a rediscovery of Ptolemy's latitude models of the inferior planets both in the *Handy Tables* and *Planetary Hypotheses*, of which the Maragha astronomers were quite probably aware. Of course, al-Maghribī's value 7;2° for the inclination of Mercury is in excellent agreement with the modern data, and thus clearly better than Ptolemy's values in the *Almagest* (6;15°) and in the *Handy Tables/Planetary Hypotheses* (6;30°). As the contents of al-Maghribī's *Talkhīṣ al-majisṭī* reveal, there is no doubt that he had made the actual observations at the Maragha observatory, as well as having the technical and mathematical skills to derive the parameters from the observational data. In the beginning of 1262, the observations were begun in the Maragha observatory (al-Maghribī's first documented observation in the *Talkhīṣ* is the lunar eclipse on 7 March 1262) and in April–May 1276, he completed the canons of his last *zīj*, *Adwār al-anwār*, and then engaged in preparing its tables.[48]

Ulugh Beg's planetary parameters (Table 4; "Appendix", Tables 8, 9) can explicitly indicate systematic observations at Samarqand, although nothing about them appears to have been documented. Unlike the improvement made at the Maragha observatory in the case of the inferior planets, the values the Samarqand observers determined for the inclinations of the superior planets are either equal to or larger than the corresponding values in the *Almagest* while they should be very close (if not essentially equal) to each other. Accordingly, it may be that the attempts made in the Samarqand observatory resulted in a deterioration in the *Almagest 's* underlying values for the inclinations of the superior planets, while these had already been improved significantly in Ptolemy's other two works. We are not told how and when the Samarqand observers reached their own values for the extremal latitudes of the superior planets; in his *Commentary on Zīj of Ulugh Beg*, Qūshčī only mentions the maximum and minimum latitudes at the limits, expressing "we found" them, but without giving any more information.[49] The true values of the maximum latitudes of the three planets at oppositions, the zodiacal signs where they occurred in the period of 1430–49 when the Samarqand observatory was active, and their differences from the extremal values in the *Sulṭānī Zīj* (Table 2) are given below. It is noteworthy that these errors are comparable with those found in the stellar latitudes in Ulugh Beg's star catalogue incorporated into his *zīj*.[50]

Mars:	–6;38°	Aqr	–37′
	+4;30	Cnc	+ 2
Jupiter:	–1;40	Psc	–36
	+1;36	Vir	+32
Saturn:	–2;49	Psc	–29
	+2; 7	Leo	+67

48 Mozaffari 2014, p. 71.

49 Qūshčī, p. 329. When mentioning the extremal latitudes of the inferior planets, his statement changes to "they determined" (pp. 336, 340).

50 Cf. Shevchenko 1990; Krisciunas 1994, esp. p. 273; Verbunt and van Gent 2012, esp. pp. 7–8.

Table 4 Longitude of the apogee of the superior planets in the *zījes* of the Maragha and Samarqand traditions

	Ptolemy (1)		Al-Maghribī (2)		Ulugh Beg (3)	
Saturn	233.0+1°	[236.5]	258.5°	[258.5]	250.0+ 7°	[261.5]
Jupiter	161.0+1	[160.5]	178.0	[179.0]	167.5+12	[181.5]
Mars	115.5+1	[116.5]	137.0	[137.5]	142.0	[140.5]

(1) For 8 July 136, 7/8 October 137, and 27/28 May 139, respectively, for the three planets. (2) For 27 February 1273, 12 August 1274, and 26 February 1271, respectively, for the three planets. These are the dates when Ptolemy (Toomer 1998, pp. 484–498, 507–519, 525–537) and al-Maghribī (*Talkhīṣ*, fols. 123r–123v, 128r–129r, 132v–133r) observed the last of their own trios of the oppositions of a superior planet to the mean sun required for determining its eccentricity and the direction of its apsidal line in the *Almagest*. The addition of one degree to Ptolemy's values is due to his well-known error in determining the moment of the vernal equinox by more than one day, making a displacement in the position he determined for the vernal equinox by a little over one degree. Consequently, it is required to take this error into account by adding one degree to Ptolemy's values in order to fix his longitudes in a correct tropical frame of reference. (3) The date considered for the *Sulṭānī zīj* is the epoch of this *zīj*, 4 July 1437. The additional values are due to the fact that the longitudes of the apogees of Saturn and Jupiter have decreased by the values k_2 in the tables of the equations of anomaly and then tabulated (cf. "Appendix")

About the values of ω_A, it should be noted that observing no new value in the case of the inferior planets cannot be so strange because unlike the superior planets, the directions of the nodal lines of the eccentrics of the inferior planets in Ptolemaic astronomy are an intrinsic convention of the model that they are ±90° from the apsidal lines. It is not imaginable that an observer could recognize that the two are not perpendicular to each other when the centre of the epicycle is at either of the apsides, but make angles as small as about 1° and 16° to the perpendicular. In particular for Mercury, it should be considered that Ptolemy's longitude of the apogee of the planet, 191° in 135 AD, is over −30° from the true value of about 223°. The same error can also be found in the majority of the Islamic *zījes* with the exception of those based upon Indian parameters (e.g., al-Khwārizmī's).[51]

The two interrelated problems in the values of ω_A for the superior planets in Table 3 are as follows:

(a) These might initially be interpreted as indicating that the medieval astronomers were aware of the change in the relative directions of the apsidal and nodal lines. Then, an inevitable consequence of the continuous increase in ω_A with respect to Ptolemy's values would be that *the apsidal and nodal lines of the superior planets do not share the same motions*, in contrast to the fundamental assumption in Ptolemaic astronomy that they are sidereally fixed. Nevertheless, such a conclusion was never posed in the Eastern (Middle Eastern) Islamic astronomy, where permitting any secular change in the Ptolemaic constants or fundamental assumptions and hypotheses was considered very unlikely. Accordingly, any new value, variation, or change found in a parameter was undoubtedly attributed to faulty observations, defects and discrepancies in instruments, and/or untrustworthy ancient data. However, a diametrically opposed point of view was adopted in the Western Islamic astronomy, where it was allowed to consider proper motions and/or models to account for the observationally found

[51] This surprising subject will be discussed by the present author elsewhere.

secular changes.[52] For instance, about 1075, Ibn al-Zarqālluh found that the solar apogee had the proper motion of 1° in 279 Julian years in the direction of increasing longitude, which is then added to the constant rate of precession 1°/66y. Several Andalusian astronomers applied this proper motion to the apogees of the planets as well.[53] Nevertheless, whether this proper motion was thought to be the case for the nodal lines of the planets, or whether other such proper motions were proposed for them, must await a thorough inspection of the materials related to the planetary latitudes in the Western Islamic *zījes*.

(b) How were they obtained? These medieval values perhaps were the result of actual observations. Needless to say, the values of ω_A are dependent upon the values adopted for the longitudes of the apogees. By means of the two, one can also examine to what extent the values obtained for the longitudes of the nodes/limits were accurate.

The longitudes of the apogees of the superior planets in the Maragha and Samarqand astronomical traditions are listed in Table 4.[54] The values in brackets are the true values computed for the times. We round both the historical and modern values to the nearest 0.5° which is sufficient for our purpose.

As seen in Table 4, al-Maghribī has more precise values for the longitudes of the apogees of the superior planets than those applied to the *Sulṭānī Zīj*. As regards Tables 3 and 4, one can deduce that the values obtained by al-Maghribī for the longitudes of the nodes/limits of the three planets are about +1°, +5°, −2.5° in error, respectively. Wābkanawī's value for ω_A of Mars increases the error in the longitude of the nodes/limits of the planet to +15.5°. The corresponding errors in the *Sulṭānī Zīj* are −1.5°, +3°, and +3.5°.

As we shall see below, the possibility exists that al-Maghribī derived his values of ω_A from actual observations as accurate as the others documented in his treatise *Talkhīṣ al-majisṭī*. Following him, Wābkanawī mentioned in his *zīj* some observations performed during his 40-year career, which were directed to reconciling theory and observation.[55] Let us first consider whether and how the astronomers at Maragha and Samarqand were able to obtain their own values for ω_A from observations in the periods of the activities in the two observatories.

In the period of al-Maghribī's observations at Maragha (1262–76), he had one opportunity in the case of Jupiter and Saturn to find that Ptolemy's value of ω_A for each planet is invalid at the time and another to determine a correct value. In this period, the mean longitude of Saturn changed from about 21° to about 197°, the longitude of its perihelion was about 79°, and its ascending node was at the longitude of about 107°. From the three observations of Saturn made on 25 October 1263, 9 December 1266, and 27 February 1273 when the planet crossed the meridian of Maragha, by means

[52] The present author has already discussed these diverse attitudes in more depth in the case of the solar parameters (cf. Mozaffari 2013b).

[53] Cf. Samsó and Millás 1998, p. 269.

[54] Note that Wābkanawī converted al-Maghribī's values to the epoch of his *zīj*, 13 March 1266, by adopting the precessional rate of 1° in 66 Persian years (of 365 days unvarying).

[55] Cf. Mozaffari 2013a, esp. pp. 239–240; see also his remark concerning the observed latitudes of the inferior planets in note 6 above.

of Ptolemy's iterative algorithmic procedure, al-Maghribī obtained the longitude of the eccentric apogee of the planet as 258.5° (Table 4), which, of course, could be taken for the whole period of his observations at Maragha. Then, with Ptolemy's value $\omega_A = +50°$, the latitude of the planet should have been zero when its mean longitude was about $\lambda_\Omega = \lambda_A - \omega_A - 90° = 258.5° - 50° - 90° = 118.5°$; this took place, according to modern theories, about mid-December 1269, and according to al-Maghribī's *Adwār*, about mid-November 1269. About this time, the planet culminated at Maragha during nights and its latitude was about $+0;43°$,[56] which is in the range of the accuracy of the instruments of the Maragha observatory.[57] It is thus probable that if al-Maghribī had made systematic observations in this time interval, he could achieve the result that the planet had a sizable latitude while, adopting Ptolemy's $\omega_A = +50°$, its epicycle centre should have been located very near the ascending node, and thus its latitude should have been about zero.

About mid-May to mid-December 1268, for some period before the last visibility of Saturn on 11 June and after its first visibility on 18 July, the longitude of the planet changed from about 99° to 113° and its latitude from about $-0;10°$ to $+0;10°$ and reached zero on 14 September. From observations of the planet near the horizon as accurate as those recorded in his *Talkhīṣ al-majisṭī*, al-Maghribī might then conclude that the latitude of the planet had reached zero somewhere in this period. His tables give about 100° to 107° for the mean longitude of Saturn in this time interval.[58] This could lead to the result that ω_A for Saturn should be about 61.5°–68.5°. Therefore, it is probable that al-Maghribī's denial of Ptolemy's $\omega_A = +50°$ and his own rounded $+60°$ were found by observation.

In the similar situation for Jupiter, with the Ptolemaic $\omega_A = -20°$, the planet should have crossed the ecliptic when its mean longitude was equal to $\lambda_\Omega = \lambda_A - \omega_A - 90° = 178° + 20° - 90° = 108°$. This took place according to both the modern ephemeris and al-Maghribī's *Adwār* about 9–21 April 1267 while the latitude of the planet was about $+0;26°$. Unlike Saturn, the culmination of Jupiter in this interval occurred during daylight, but an observer had time enough to observe the planet after sunset and find that it had a slight latitude, while according to Ptolemy its latitude should be about zero.

In the period of the activities at the Samarqand observatory (1430–49), the longitude of Saturn changed from about 277° to 174°, its ascending node was at about 109°, and its perihelion/perigee at about 82°. The Samarqand observers could thus observe the

[56] Al-Maghribī's *Adwār* gives a value $\sim +0;40°$.

[57] As I have shown elsewhere, al-Maghribī's mean errors in the measurements of meridian altitudes by the central quadrant of the Maragha observatory are (regardless of sign) about 0;3° for the sun, 0;6° for the eight bright stars, and 0;5° for the planets. The mean errors in the latitudes of the stars and planets he computed from the observed meridian altitudes by the trigonometrical rules of spherical astronomy are, respectively, about 0;6° and 0;10°. In particular for Saturn, in the three above-mentioned observations, the errors in the meridian altitude of the planet are, respectively, $-0;9°$, $-0;7°$, and $+0;6°$, which resulted the errors of $-0;15°$, $-0;8°$, and $+0;4°$ in the latitude of the planet as calculated by al-Maghribī. The mean error in the latitudes of the 16 stars recorded in the non-Ptolemaic star table of the *Īlkhānī Zīj* is about 0;15° (they were probably made with the aid of the armillary sphere of the observatory).

[58] Al-Maghribī's *Adwār* gives the planet's longitude as ~99° to ~113° and its latitude as $-0;13°$ to $+0;6°$.

planet when crossing its nodes, southern limit, and perigee, and this 20-year period was sufficient for Jupiter and Mars to be observed at all the critical points.

To sum up, this paper has established that new values for the latitude parameters, inclinations, and directions of the lines of the limits/nodes were observed at the Maragha and Samarqand observatories in late medieval Islamic astronomy, and has analysed their accuracy. However, in the absence of further documented information, it has not settled the question of how these values were determined, although some ways for deriving them from observations were proposed.

Acknowledgments I would like to thank Prof. Benno Van Dalen (Germany) for his valuable helps; especially, I owe my information of the planetary latitudes in Ibn Yūnus' *Ḥākimī Zīj* to him, who kindly supplied me with his informative transcription and translation of the related chapter in that *zīj*. Of course, only I should be held responsible for any error. This work was financially supported by the Research Institute for Astronomy and Astrophysics of Maragha (RIAAM).

Appendix: Planetary inclinations and latitude tables

In order to derive the inclinations from the planetary extremal latitudes and parameter values adopted in the *zījes* of the Maragha and Samarqand traditions, we make use of both modern and Ptolemy's methods. The first consists in solving trigonometrical equations resulted from the configuration of the ecliptic, eccentric, and epicycle at the northern or southern limit (in the case of the superior planets) and at the nodes (for the inferior planets). The latter is to extract the inclinations from the maximum and minimum latitudes by interpolation in the epicyclic equation tables, an approximate method explained in *Almagest* XIII.3.[59]

Maragha: inferior planets

Figure 3 shows the configuration of the eccentric and epicycle of Venus when the centre of the epicycle is at either of the nodes of the eccentric, i.e., $\pm 90°$ from the apsidal line; in this position, the inclination of the epicycle is in the line of sight. The radius of the eccentric $OC = R = 60$, the eccentricity $TO = e$, the distance from the earth to the centre of the epicycle $TC = (R^2 - e^2)^{1/2}$, and the radius of the epicycle $CP = r$. We let the true epicyclic anomaly $\alpha = \angle ACP$, so that $PP' = QQ' = r \sin \alpha$, $PQ = P'Q' = r \cos \alpha \sin i_{1\,\max}$, and $CQ' = r \cos \alpha \cos i_{1\,\max}$. For Mercury, the direction of the inclination of the epicycle is reversed, as the epicyclic apogee A is inclined to the north; also, $TO = 2e$. The latitude $\beta = \angle PTQ$ is found from the below formula (negative sign for Venus and positive sign for Mercury):

$$\sin \beta_{\text{icl}}(\alpha) = \pm \frac{PQ}{\sqrt{(TC + CQ')^2 + QQ'^2 + PQ^2}}. \tag{1}$$

59 Toomer 1998, pp. 601–605. Ptolemy's method is adequately explained in Pedersen [1974] 2010, pp. 361–365, Neugebauer 1975, Vol. 1, pp. 209–216, and Swerdlow 2005, pp. 43–46. So, there is no need for repeating it in detail here.

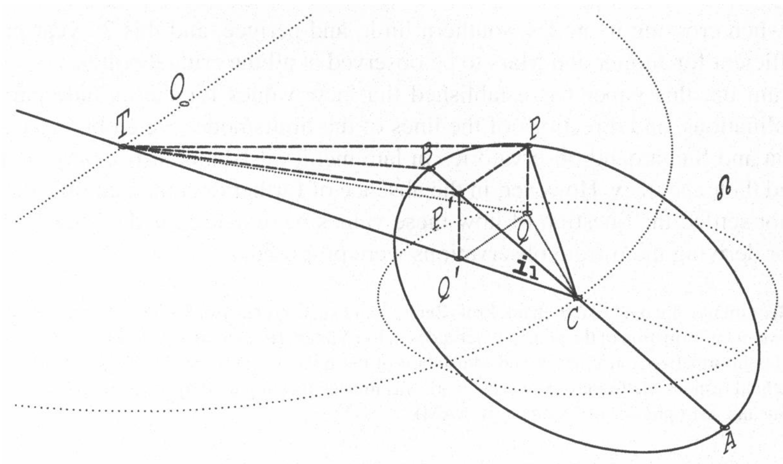

Fig. 3 Configuration of the epicycle and the eccentric of Venus when the centre of the epicycle is at the ascending node of the eccentric, ±90° from the apsidal line

If $\alpha = 180°$, for which the latitude of the inferior planets reaches its maximum value, all that is necessary is to solve triangle TBC in which $\angle BTC = \beta_{max}$:

$$\frac{TC}{\sin(\angle TBC)} = \frac{CB}{\sin(\angle CTB)} \text{ or } \frac{TC}{\sin(\beta_{max} + i_{1\,max})} = \frac{r}{\sin(\beta_{max})} \qquad (2)$$

and so, $i_{1\,max} = (\beta_{max} + i_{1\,max}) - \beta_{max}$. Therefore, $i_{1\,max}$ can be conveniently found from TC, r, and β_{max}.

The underlying planetary parameters in the *zījes* of the Maragha tradition are as follows: the eccentricity of Venus in the *Īlkhānī Zīj* is $e \approx 1;2$ where $R = 60$, computed from the maximum tabular value $q_{max} = 1;59°$ for the equation of centre of the planet,[60] in al-Maghribī's *Adwār al-anwār* $e \approx 1;3$, from $q_{max} = 2;0°$.[61] The value adopted for the eccentricity of Mercury, $e = 3;0$, and the radii of the epicycles of both the inferior planets in these two *zījes* are Ptolemaic, Venus: 43; 10, Mercury: 22;30.

Due to Ptolemy's complicated model for Mercury's motion in longitude of a movable eccentric, in Fig. 3: $TC \neq (R^2 - e^2)^{1/2}$, but with Ptolemy's value $e = 3;0$, $TC \approx 56;40$.[62] Then, with al-Maghribī's value $\beta_{max} = \beta_{icl}(180°) = 4;35°$ for the maximum latitude of Mercury, one can compute $i_{1\,max} \approx 7;2°$ from (2).

Our historical method, i.e., Ptolemy's solution, of which al-Maghribī quite probably made use, is an analogy between the extremal latitudes and inclination and the epicyclic equations and true epicyclic anomalies, respectively, in the sense that the extremal latitudes $\beta_{icl}(0)$ (= $\angle ATC$ in Fig. 3; line AT not drawn) and $\beta_{icl}(180°)$ (= $\angle BTC$) can

60 Al-Ṭūsī, C: pp. 125–6; T: fols. 73v–75r; P: fols. 42v–43r; M: fols. 76r–77r.

61 The equation tables of the inferior planets in al-Maghribī's *Adwār*, M: fols. 87v–89r, CB: fols. 85v–87r; also preserved in Wābkanawī, T: fols. 160v–163r and Kamālī, fols. 248v–251r.

62 Neugebauer 1975, Vol. 1, p. 221; Toomer 1998, p. 609.

Table 5 Al-Maghribī's table of the inclination of Mercury $i_{1\,\max} = 7;2°$

	Tab.	Re-c.		Tab.	Re-c.
6°	2; 0°	1;59°	96°	0;17°	0;17°
12	1;58	1;58	102	0;35	0;35
18	1;56	1;55	108	0;54	0;54
24	1;53	1;51	114	1;14	1;14
30	1;49	1;47	120	1;35	1;36
36	1;42	1;41	126	1;57	1;58
42	1;34	1;34	132	2;21	2;21
48	1;26	1;26	138	2;44	2;44
54	1;17	1;17	144	3; 7	3; 8
60	1; 7	1; 7	150	3;29	3;30
66	0;55	0;56	156	3;50	3;51
72	0;43	0;44	162	4; 8	4; 9
78	0;29	0;30	168	4;20	4;23
84	0;15	0;16	174	4;30	4;32
90	0	0	180	4;35	4;35

144: MS. M: 3;6

be taken as the epicyclic equations in the correction tables for longitude corresponding, respectively, to the true epicyclic anomalies corresponding to the anomalies $\pm i_{1\,\max}$ and $180° \pm i_{1\,\max}$ (in Fig. 3, assume that the epicycle is perpendicular to the ecliptic). When the centre of the epicycle of an inferior planet is at either of the nodes, i.e., $\pm 90°$ from the apsidal line, its true eccentric anomaly, $\angle OTC$ in Fig. 3, is $\kappa = \pm 90°$, which corresponds to the mean eccentric anomaly $\bar{\kappa} \approx \pm 93°$. At this position, from al-Maghribī's table of the equations of Mercury, the following values for the epicyclic equation p can be derived:[63]

$$5;12° \text{ for } \alpha = 180° - i_{1\,\max} = 172° \text{ and}$$
$$4;34° \text{ for } \alpha = 180° - i_{1\,\max} = 173°.$$

Al-Maghribī's $\beta_{\max} = 4;35°$ falls between these two values. So, with taking it as the epicyclic equation, the linear interpolation between $p = 5;12°$ and $p = 4;34°$ results in $\alpha = 180° - i_{1\,\max} \approx 172;58°$, and thus $i_{1\,\max} \approx 7;2°$, which is in best agreement with our earlier derivation.

Table 5 shows that a table computed from this value is in good agreement with al-Maghribī's tabular entries.[64]

In the case of Venus, using formula (2) with al-Maghribī's value $\beta_{\max} = \beta_{\text{icl}}(180°) = 6;40°$ results in $i_{1\,\max} = 2;37°$ (note that because the eccentricity of Venus is small, $TC \approx 60$). But this value cannot be derived from al-Maghribī's table

[63] The components of the epicyclic equation in the *Almagest* (about them, see Neugebauer 1975, Vol.1, pp. 183–186) with the tabular values in al-Maghribī's *Adwār* are given below.

$\bar{\kappa}$	$c_8(\bar{\kappa})$	$\alpha = 180° - i_{1\,\max}$	$c_6(\alpha)$	$c_7(\alpha)$	$p = c_6 + c_8 \cdot c_7$
93°	0;43°	172°	4;45°	0;38°	5;12°
		173	4;10	0;33	4;34

[64] The agreement is maintained using the rounded value $i_{1\,\max} \approx 7;0°$ as well.

Table 6 Tables of the epicyclic equation of Venus in the *Īlkhānī zīj*, al-Maghribī's *Adwār*, and Ibn Yūnus' *Ḥākimī zīj*

1	2	3	4	5	6	7
	Almagest	*Īlkhānī zīj*	al-Maghribī	Re-c. (r = 43;10)	Ibn Yūnus	Re-c. (r = 43;28)
1°	—	0;26°	0;26°	0;25	0;25°	0;25°
2	—	0;51	0;51	0;50	0;51	0;50
3	—	1;16	1;16	1;15	1;16	1;16
...	...	...	...	...	...	...
6	2;31	2;31	2;31	2;31	2;32	2;31
...	...	...	...	...	...	...
12	5; 1	5; 1	5; 1	5; 1	5; 4	5; 2
...	...	...	...	...	...	...
30	12;30	12;30	12;31	12;30	12;37	12;33
...	...	...	...	...	...	...
...	...	...	...	...	...	...
135	45;59	45;59	45;59	46; 0	46;25	46;24
136	—	45;59	46; 0	46; 0	46;25	46;25
...	...	...	...	...	...	...
170	—	23;12	23;12	23;12	23;25	23;42
171	21;15	21;15	21;17	21;15	21;27	21;43
172	—	19;11	19;16	19;12	19;22	19;38
173	—	17; 2	17;10	17; 3	17;12	17;27
174	14;47	14;47	14;59	14;48	14;54	15; 9
175	—	12;27	12;42	12;29	12;34	12;47
176	—	10; 4	10;20	10; 5	10;10	10;20
177	7;38	7;38	7;53	7;37	7;42	7;48
178	—	5; 9	5;20 (1)	5; 6	5;12	5;14
179	—	2;35	2;42	2;34	2;37	2;38
180	0	0	0	0	0	0

(1) MS. M: 5;22

of the epicyclic equation of Venus according to Ptolemy's procedure. As mentioned earlier, both in the *Īlkhānī Zīj* and in al-Maghribī's *Adwār*, the radius of the epicycle of Venus is Ptolemaic, and as shown in Table 6, the tables of the epicyclic equation of the planet at mean distance in both *zījes* are equivalent to the corresponding one in the *Almagest*. Nevertheless, al-Maghribī has different values for the epicyclic equation for arguments 171°–179° (enclosed by the dashed lines in Table 6). I cannot explain why these entries are different, larger than the *Almagest* entries. As mentioned earlier, Ibn Yūnus has a greater value for the radius of the epicycle of Venus than Ptolemy ($r \approx 43;28$; see Table 6).[65] But, al-Maghribī's equations for arguments 171°–179° can in no way be related to Ibn Yūnus's.

With Ptolemy's procedure, the maximum latitude $\beta_{\max} = \beta_{\text{icl}}(180°) = 6;40°$ taken as the epicyclic equation at mean distance corresponds to an epicyclic anomaly $\alpha = 180° - i_{1\,\max} = 177;28,38°$, and thus $i_{1\,\max} = 2;31,22°$ which is nearly equal to the Ptolemaic 2;30°. It can be assumed that 6;40° is estimated from the observed latitudes of Venus near inferior conjunction with the sun, although estimating latitude

[65] Ibn Yūnus, L: pp. 188–190.

in this location is extremely difficult and al-Maghribī's error, like Ptolemy's is close to $-2°$, but it cannot be determined which of the two values for $i_{1\,\max}$ al-Maghribī actually derived from it. However, since the tabular entries for the inclination of Venus in his *Adwār* are in better agreement with computation from $i_{1\,\max} = 2;37°$ than from $i_{1\,\max} = 2;30°$ (see below), it seems to be safe to conclude that it is the first value that, in reality, underlies al-Maghribī's table in the *Adwār*.

In Table 7, Cols. 2 and 3, respectively, indicate the *Almagest*[66] and al-Maghribī's table of the inclination of Venus in the *Adwār*. Cols. 4 and 5 give the recomputed values from the two values for $i_{1\,\max}$. Neither of the two sets of the re-computed latitudes is as consistent with the tabular entries as we have seen earlier in the case of al-Maghribī's table of the inclination of Mercury (Table 5). The table was probably computed according to the following procedure: the entries for the arguments 0°–96°) are nearly identical to the corresponding entries in the *Almagest*, except for the slightly improved increments of 0; 1° for the first five entries; this seems reasonable, since a small change in $i_{1\,\max}$ from 2;30° to 2;37° causes no critical change in the latitudes in this range (at most, 0;3°). But most of the entries in the latter part of the table appear to have been derived from multiplying the corresponding entries in the *Almagest* table by 6;40/6;22 (see Col. 6). It is noteworthy that applying similar procedures to computing the planetary equation tables can be found in Islamic astronomy, more remarkably, in al-Maghribī's *Tāj al-azyāj*[67] as well as in the *Īlkhānī Zīj*.[68]

In the *Tāj al-azyāj*, al-Maghribī has for Venus $\beta_{\max} = \beta_{\mathrm{icl}}(180°) = 8;30°$. The table of the planet's epicyclic equation in this *zīj* does not have those strange entries found in the *Adwār*, but is identical to the *Almagest/Handy Tables*, and so to the *Īlkhānī Zīj* (Table 6). Both formula (2) and interpolation in the anomalistic equation table of Venus result in $i_{1\,\max} \approx 3;21°$. Al-Maghribī's values for the inclination and the recomputed values are shown in Table 7 (Cols. 7 and 8).

The entries in the latitude tables of the inferior planets in the *Īlkhānī Zīj* are given for each integer degree of the argument. Col. 9 in Table 7 shows the entries in the table of the inclination of Venus in the *Īlkhānī Zīj*. Both solving the trigonometrical equation (2) with $\beta_{\max} = \beta_{\mathrm{icl}}(180°) = 8;40°$ and interpolation in the table of the epicyclic equation of Venus results in $i_{1\,\max} \approx 3;25°$. But the first half of the table, in the region of the arguments 0°–90°, is identical to the corresponding table in the *Almagest* computed from $i_{1\,\max} = 2;30°$. Moreover, re-computation of the table from $i_{1\,\max} = 3;25°$ (Col. 10) shows that the results are by no means in agreement with the entries in the latter part of the table. In MS. P (fol. 44r) of the *Īlkhānī Zīj*, an unknown commentator has restored the latter half of the table to the Ptolemaic numbers and thus reduced the maximum latitude of the planet to 6;22°. We are told that this modification was done on the basis of the corresponding table in al-Maghribī's *Zīj*, while, as we have

[66] *Almagest* XIII.5: Toomer 1998, p. 633.

[67] See Dorce 2002–3, pp. 202–204; 2003, pp. 127–137.

[68] E.g., the table of the epicyclic equation of Mars in the *Īlkhānī zīj* (C: p. 116, P: fols. 38v–39r, M: fols. 70v–71v) has the maximum value 42;12°, corresponding to $r \approx 40;18$. Al-Kāshī (IO: fol. 99r, 112r) notices that this value is different from Ptolemy and that the entries of the table are derived from multiplying the entries of the *Almagest* table (with the maximum value 41;9°; cf. Toomer 1998, p. 551) by 42;12/41;9. Spot checks show that this is indeed the case.

Table 7 Tables of the inclination β_{icl} of Venus in the *Almagest*, al-Maghribī's *Tāj*, and *Adwār* the *Īlkhānī Zīj*

1	2	3	4	5	6	7	8	9	10
	Almagest	al-M. *Adwār*	Re-c. 2;37°	$i_{1\max}$ = 2;30°	[*Alm.*] $\times\frac{6;40}{6;22}$	Al-M. *Tāj*	Re-c. $i_{1\max}$ = 3;21°	*Īlkh. zīj*	Re-c. $i_{1\max}$ = 3;25°
6°	1; 2°	1; 3°	1; 5°	1; 3°	1; 5°	1;24°	1;24°	1; 2°	1;26°
12	1; 1	1; 2	1; 5	1; 2	1; 4	1;24	1;23	1; 2	1;25
18	1; 0	1; 1	1; 3	1; 0	1; 3	1;22	1;21	1; 1	1;24
24	0;59	1; 0	1; 1	0;59	1; 2	1;20	1;19	0;59	1;23
30	0;57	0;58	0;59	0;56	1; 0	1;17	1;15	0;57	1;20
36	0;55	0;55	0;56	0;53	0;58	1;13	1;11	0;55	1;17
42	0;51	0;51	0;52	0;50	0;53	1; 9	1; 7	0;51	1;13
48	0;46	0;46	0;48	0;46	0;48	1; 5	1; 1	0;47 (1)	1; 8
54	0;41	0;41	0;43	0;41	0;43	1; 0	0;55	0;41 (2)	0; 3
60	0;35	0;35	0;38	0;36	0;37	0;52	0;48	0;35	0;56
66	0;29	0;29	0;32	0;30	0;30	0;43	0;41	0;29	0;49
72	0;23	0;23	0;25	0;24	0;24	0;32	0;32	0;23	0;41
78	0;16	0;16	0;17	0;17	0;17	0;21	0;22	0;16 (3)	0;33
84	0; 8	0; 8	0; 9	0; 9	0; 8	0;11	0;12	0; 8 (4)	0;23
90	0	0	0	0	0	0	0	0	0;12
93	0; 5	——		0; 5		——		0; 7	0; 6
96	0;10	0;10	0;10	0;10	0;10	0;13	0;13	0;14	0;13
99	0;15	——		0;15		——		0;22	0;20
102	0;20	0;21	0;21	0;20	0;21	0;25	0;27	0;29	0;28
105	0;26	——		0;26		——		0;37	0;36
108	0;32	0;33	0;34	0;32	0;34	0;43	0;43	0;45	0;44
111	0;38	——		0;39		——		0;54	0;53
114	0;44	0;46	0;48	0;45	0;46	0;58	1; 1	1; 2	1; 2
117	0;50	——		0;53		——		1;11	1;12
120	0;59	1; 2	1; 3	1; 0	1; 2	1;19	1;21	1;20	1;22
123	1; 8	——		1; 9		——		1;30	1;34
126	1;18	1;21	1;21	1;17	1;22	1;45	1;44	1;43	1;46
129	1;28	——		1;27		——		1;57	1;58
132	1;38	1;48	1;41	1;37	1;43	2;11	2;10	2;12	2;12
135	1;48	——		1;48		——		2;28	2;27
138	1;59	2; 8	2; 5	2; 0	2; 5	2;40	2;40	2;44 (5)	2;43
141	2;11	——		2;13		——		3; 6	3; 1
144	2;23	2;38	2;33	2;27	2;30	3;14	3;16	3;30	3;20
147	2;43	——		2;42		——		3;58	3;41
150	3; 3	3;13	3; 7	2;59	3;12	4; 5	4; 0	4;27 (6)	4; 4
153	3;23	——		3;18		——		4;54	4;30
156	3;44	3;55	3;48	3;38	3;55	5; 0	4;52	5;22	4;58
159	4; 5	——		4; 1		——		5;50	5;28
162	4;26	4;39	4;37	4;25	4;39	5;58	5;54	6;20	6; 1
165	4;49	——		4;51		——		6;53	6;36
168	5;13	5;29	5;31	5;17	5;28	6;56	7; 3	7;27 (7)	7;11
171	5;36	——		5;41		——		7;57 (8)	7;44
174	5;52	6; 9	6;19	6; 3	6; 9	7;44	8; 4	8;20	8;13
177	6; 7	——		6;17		——		8;35	8;32
180	6;22	6;40	6;40	6;22	6;40	8;30	8;30	8;40	8;40

(1) M: 0;46 (2) M: 0;40 (3) C: 0;17, M: 0;15 (4) M: 0;7 (5) C: 2;45 (6) M, T: 4;26 (7) M, T: 7;26 (8) M, T: 7;56

Table 8 Equations of centre and eccentricities in Ulugh Beg's *Sulṭānī Zīj*

1	2	3	4	5	6	7	8
	Max	Arg(s)	Min	Arg(s)	k_1	q_{max}	e
Saturn	13;37,45°	260°	0;22,15°	86°	7°	6;37,45°	3;28,30 ≈ 3;29
Jupiter	11;18,39	261–2	0;41,21	86–7	6	5;18,39	2;46,58 ≈ 2;47
Mars	23;50,48	252	0; 9,12	84	12	11;50,48	6;13,30 ≈ 6;14
Venus	3;39,19	267	0;20,41	89	2	1;39,19	0;52
Mercury	7; 2	259–263	0;58	89–93	4	3; 2	3; 0 [*Alm.*]

Sulṭānī zīj, P1: fols. 135r, 138r, 141r, 144r, 147r; P2: fols. 152r, 155r, 158r, 161v, 165r

seen above, his *Tāj* and *Adwār* have, respectively, the values 8;30° and 6;40° for the maximum latitude of Venus. It is worth noting that in his commentary on the *Īlkhānī Zīj*, al-Nīshābūrī briefly refers to this problem in the table of the latitude of Venus as a subtle point, but he offers no solution.[69] Since the table is intrinsically contradictory, I cannot speculate about the reasoning behind it. However, it was apparently reasonable enough for Ibn al-Shāṭir and the Samarqand astronomers to quote it in their own *zījes*.

Samarqand: superior planets

The new planetary parameters obtained at the Samarqand observatory, as deduced from Ulugh Beg's *Sulṭānī zīj*, are summarized in Tables 8 for the equation of centre and 5 for the equation of anomaly. In both, Cols. 2 and 4 show, respectively, the maximum and minimum equations tabulated in this *zīj* and Cols. 3 and 5 show the arguments, mean eccentric or true epicyclic anomalies, for which the extremal values are given.

The tables of the equation of centre in the *Almagest* are symmetrical and additive for half the table and subtractive for the other half, but in the *Sulṭānī Zīj*, they are displaced and always additive; the values $k_1 = 1/2(\text{Max} + \text{Min})$ of the displacements and $q_{\text{max}} = 1/2(\text{Max} - \text{Min})$ of the maximum equations of centre (note that $k_1 > q_{\text{max}}$), and the corresponding eccentricities $e = R \cdot \tan 1/2\, q_{\text{max}}$, where the radius of the eccentric $R = 60$, are given in the last three columns of Table 8.

The tables of the equation of the epicyclic anomaly are also always additive, but they use a different type of the displacement explained as follows: for all the planets, the first half of the table, for arguments 0°–180°, corresponds to the equation p_{A} of the epicyclic anomaly at greatest distance ($R + e$; for Mercury: $R + 3e$), when the centre of the epicycle is located at the apogee, but the latter half of the table, for arguments 180°–360°, to the equation p_{Π} of the epicyclic anomaly at the least distance ($R - e$, excepting Mercury), when the centre of the epicycle is located at the perigee. In the case of Saturn and Jupiter, all the entries are increased by an amount k_2. For Mars and the two inferior planets, the first half of the table gives p_{A} while the entries in the latter part of the table were increased by 360°, i.e., $360° - p_{\Pi}$.[70] The values k_2 are

69 Al-Nīshābūrī, P1: fol. 119r, P2: fol. 143v.

70 It is noteworthy that the method used in the *Sulṭānī zīj* for the displacement of the entries in the tables of the equation of centre of all the planets and the displacement method for the tables of the equation of anomaly of Saturn and Jupiter are those invented by Kushyār b. Labbān in the latter part of the tenth century (see Brummelen 1998); they are also employed in the *Īlkhānī zīj* (see above, note 45). The method of 360° displacements appeared as early as the tables of the equation of centre of all the planets in Ibn al-Fahhād's

shown in Col. 6. Cols. 7 and 8 display the maximum values of the epicyclic equation, respectively, at apogee and perigee (for Saturn and Jupiter: $p_{A,max} = \text{Max} - k_2$ and $p_{\Pi,max} = k_2 - \text{Min}$). The last column gives the corresponding epicyclic radii (for all the planets, except Mercury, $r = \sin(p_{A,max}) \cdot (R+e) = \sin(p_{\Pi,max}) \cdot (R-e)$). In the *Sulṭānī zīj*, there is a table for the differences $p_\Pi - p_A$ and two interpolation tables for each planet, one for the epicyclic anomaly 0°–180° and another for the epicyclic anomaly 180°–360°. The entries of the equation tables are carried to two sexagesimal places, except Mercury to one place, and are given for each integer degree of the argument (for Mars, from 150° to 210°, the arguments are in steps of 0;20°). Their reconstruction on the basis of the values derived above is in good agreement with the tabular entries.

It merits noting that in his *Commentary on Zīj of Ulugh Beg*, Qūshčī explains the layout of the equation and mean motion tables in this *zīj*. The values he mentions for the maximum equations agree with what can be extracted from the tables, with the exception of very slight differences in q_{max} of Saturn (6;37, 4***6***°) and Jupiter (5;18, ***42***°). The parameters we deduced from the tables are in agreement with those he gives. As Qūshčī remarks, for all the planets, the mean eccentric anomalies are reduced and, inversely, the mean epicyclic anomalies are increased by the above-mentioned values for k_1 and are then tabulated; and for Saturn and Jupiter, the longitudes of the apogees are decreased by the above-mentioned values for k_2.[71]

Figure 4 shows the configuration of the epicycle and eccentric, inclined to the ecliptic, of a superior planet when the centre of the epicycle is at the northern limit. In order to derive the inclinations from the extremal latitudes in the *Sulṭānī zīj*, we should first know the distance $TC = \rho'$ between the earth T and the centre C of the epicycle of a superior planet at the northern limit, which can be computed from the eccentricity (Table 8) and the value adopted for ω_A as mentioned earlier in Table 3. Where $R = 60$:

Planet	ρ' northern	ρ' southern
Saturn	61;57,29	58;29,44
Jupiter	62;45,23	57;15, 4
Mars	66;13, 8	53;47,26

In Fig. 4, the distance $TC = \rho'$ and the radius of the epicycle $CP = r$. We again let the true epicyclic anomaly $\alpha = \angle ACP$, and $\angle CTN' = i_0$, so that $PP' = QQ' = r \sin\alpha$, $PQ = P'Q' = r \cos\alpha \sin i_{1max}$, and $CQ' = r \cos\alpha \cos i_{1max}$, and now $QN = Q'N' = (TC + CQ') \sin i_0$. The latitude $\beta = \angle PTN$ at either limit can be computed from:

Footnote 70 Continued
'Alā'īzīj (*ca.* 1172) (*cf.* Kamālī, fols. 48v–49r). A simple version of the 360° displacements in the tables of the equation of anomaly is adopted (to the best of my knowledge, for the first time) in the *Īlkhānī zīj*, of which, it can be said, the method employed in the *Sulṭānī zīj* is a more developed version. For a brief survey of the methods of the displacements, see Dalen 2004b, pp. 841–843.

71 Qūshčī, pp. 273–274, 320–324, 371.

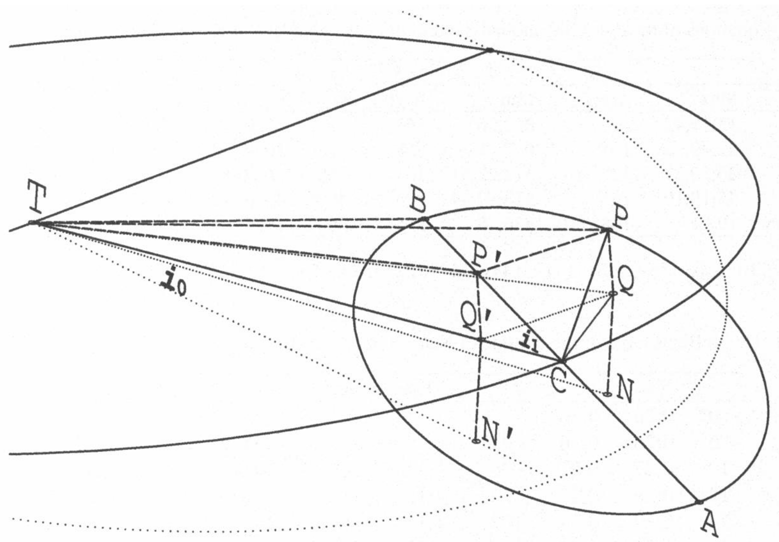

Fig. 4 Configuration of the epicycle and the eccentric of the superior planets with respect to the ecliptic when the centre of the epicycle is at the northern limit

$$\sin|\beta(\alpha)| = \frac{(TC + CQ')\sin i_0 - PQ\cos i_0}{\sqrt{(TC + CQ')^2 + QQ'^2 + PQ^2}} = \frac{r\cos\alpha\sin(i_0 - i_{1\,\max}) + \rho'\sin i_0}{\sqrt{(\rho' + CQ)'^2 + QQ'^2 + PQ^2}}. \tag{3}$$

First, in order to derive i_0, we let $\alpha = 90°$ in (3); thus, we have:

$$\sin i_0 = \sqrt{1 + \frac{r^2}{\rho'^2}} \cdot \sin\left(\left|\beta_{\text{icl}}(90°)\right|\right) \tag{4}$$

Then, in order to derive i_1, we can solve a long trigonometric equation resulting from (3) for $\alpha = 180°$, after substituting i_0 computed from (4), a given value for the northern or southern latitude for this epicyclic anomaly ($\beta_{\max} = \angle BTN'$), and r and ρ' in (3). Instead, similar to what we have done earlier in the case of the inferior planets, a simpler solution is to solve triangle *TCB* in which $\angle BTC = \beta_{\max} - i_0$:

$$\frac{TC}{\sin(\angle TBC)} = \frac{CB}{\sin(\angle CTB)} \text{ or } \frac{\rho'}{\sin(\beta_{\max} - i_0 + i_{1\,\max})} = \frac{r}{\sin(\beta_{\max} - i_0)} \tag{5}$$

and thus, $i_{1\,\max} = (\beta_{\max} - i_0 + i_{1\,\max}) - (\beta_{\max} - i_0)$.

Our formulae (4) and (5), applying the planetary parameters and the tabular latitudes in Ulugh Beg's *zīj* (as mentioned in Tables 2, 8, 9), result in the inclinations in Table 2. Using Ptolemy's solution, in which the extremal latitudes at the limits are taken as the epicyclic equations and the inclinations as the true epicyclic anomalies, as we

Table 9 Equations of the epicyclic anomaly and radii of the epicycles in Ulugh Beg's *Sulṭānī Zīj*

1	2	3	4	5	6	7	8	9
	Max	Arg(s)	Min	Arg(s)	k_2	$p_{A,\max}$	$p_{\Pi,\max}$	r
Saturn	13;11,42°	96°	0; 2,26°	263°	7°	6;11,42°	6;57,34°	6;51
Jupiter	22;49, 2	101	0; 7, 1	258	12	10;49, 2	11;52,59	11;47
Mars	36;50,57	127	312;22,51	101	360	36;50,57	47;37, 9	39;43
Venus	45;10,10	155	313; 7,44	223	360	45;10,10	46;52,16	43;10 [*Alm.*]
Mercury	19; 1	108–110	336; 8	245–8	360	19; 1	23;52	22;30 [*Alm.*]

Sulṭānī zīj, P1: fols. 135v, 138v, 141v, 144v, 147v; P2: 152v, 155v, 158v-159r, 162r-v, 165v

Table 10 Ulugh Beg's table of the latitude of Mars at the southern limit $i_0 = 1;20°$, $i_{1\,\max} = 2;7°$

	Tab.	Re-c.		Tab.	Re-c.		Tab.	Re-c.
0°	0;26°	0;26°						
6	0;26	0;26	93°	1; 7°	1; 8°	138°	2;38°	2;38°
12	0;27	0;27	96	1;11	1;11	141	2;49	2;50
18	0;28	0;27	99	1;15	1;15	144	3; 2	3; 2
24	0;29	0;28	102	1;19	1;18	147	3;16	3;17
30	0;30	0;30	105	1;23	1;23	150	3;32	3;33
36	0;32	0;31	108	1;27	1;27	153	3;50	3;51
42	0;34	0;33	111	1;32	1;32	156	4;11	4;12
48	0;36	0;36	114	1;37	1;37	159	4;34	4;35
54	0;38	0;38	117	1;42	1;42	162	5; 1	5; 1
60	0;41	0;41	120	1;48	1;48	165	5;29	5;29
66	0;44	0;45	123	1;55	1;55	168	5;58	6;58
72	0;48	0;49	126	2; 2	2; 2	171	6;27	6;27
78	0;53	0;53	129	2;10	2;10	174	6;50	6;52
84	0;58	0;59	132	2;19	2;18	177	7; 6	7;10
90	1; 4	1; 4	135	2;28	2;28	180	7;15	7;16

have explained for the derivation of $i_{1\,\max}$ of the inferior planets, results in the same inclinations.

The tables in the *Sulṭānī Zīj* were generally calculated carefully, showing only minor deviations in the cases that have been examined,[72] which is also true of the latitude tables at the northern and southern limits. Re-computation of the latitudes from the values derived for the inclinations is in good agreement with the tables in the *Sulṭānī Zīj* with no difference exceeding 0;1° in the case of Saturn, 0;3° for Jupiter, and 0;4° for Mars. Table 10 is a specimen for Mars at the southern limit, which shows the greatest differences.

References

Arabic Almagest, Ḥunayn b. Isḥāq and Thābit b. Qurra (trs.), MSS. S: Iran, Sipahsālār Library, no. 594 (copied in 480 H/1087-8 AD), PN: USA, Rare Book and Manuscript Library of University of Pennsylvania, no. LJS 268 (written in an Arabic Maghribī/Andalusian script at Spain in 783 H/1381 AD; some folios between 37v and 38r, corresponding to Almagest III.3-IV.9, are omitted).

[72] For example, the table of the equation of centre of the moon; cf. Mozaffari 2014, p. 106.

al-Bīrūnī, Abū al-Rayḥān, 1954–1956, *al-Qānūn al-mas'ūdī* (*Mas'ūdic canons*), 3 vols. Hyderabad: Osmania Bureau.

Brummelen, G.V. 1998. Mathematical methods in the tables of planetary motion in Kūshyār ibn Labbān's *Jāmi' Zīj*. *Historia Mathematica* 25: 265–280.

Brummelen, G.V. 2006. Taking latitude with Ptolemy: Jamshīd al-Kāshī's novel geometric model of the motions of the inferior planets. *Archive for History of Exact Sciences* 60: 353–377.

Caussin de Perceval, J.-J.-A. 1804. Le livre de la grande table hakémite, Observée par le Sheikh, ..., ebn Iounis. *Notices et Extraits des Manuscrits de la Bibliothèque nationale* 7: 16–240.

Chabás, J., and B.R. Goldstein. 1994. Andalusian astronomy: al-Zīj al-Muqtabis of Ibn al-Kammād. *Archive for History of Exact Sciences* 48: 1–41.

Chabás, J. and Goldstein, B. R., 2003, *The Alfonsine Tables of Toledo*, Archimedes; New studies in the history and philosophy of science and technology, vol. 8. Dordrecht: Springer.

Chabás, J., and B.R. Goldstein. 2009. *The astronomical tables of Giovanni Bianchini*. Leiden: Brill.

Debarnot, M.-T. 1987. The Zīj of Ḥabash al-Ḥāsib: A Survey of MS Istanbul Yeni Cami 784/2. In *From deferent to Equant: A volume of studies on the history Dof science of the ancient and medieval near east in honor of E. S. Kennedy*, ed. G. Saliba and D. A. King. Annals of the New York Academy of Sciences, vol. 500, 35–69.

Dorce, C. 2002–3. The *Tāj al-azyāj* of Muḥyī al-Dīn al-Maghribī (d. 1283): Methods of computation. *Suhayl* 3: 193–212.

Dorce, C., 2003, *El Tāy al-azyāy de Muḥyī al-Dīn al-Maghribī*, In: Anuari de Filologia, vol. 25, Secció B, Número 5, Barcelona: University of Barcelona.

Eastwood, B.S. 2007. *Ordering the heavens; Roman astronomy and cosmology in the Carolingian renaissance*. Leiden: Brill.

Eastwood, B., and G. Grasshoff. 2003. Planetary diagrams—Descriptions, models, theories; From Carolingian deployments to Copernican debates. In *The power of images in early modern science*, ed. W. Lefèvre, et al., 197–226. New York: Springer.

al-Farghānī, Aḥmad b. Muḥammad b. Kathīr. 1669. *Kitab fi 'l-ḥarakāt al-samāwiyaa wa jawāmi' al-'ilm al-nujūm*, Arabic edition and Latin translation by Jacobus Golius: *Muhammedis Fil. Ketiri Ferganensis, qui vulgo Alfraganvs dicitur, Elementa Astronomica Arabicè et Latinè. Cum Notis ad res exoticas sive Orientales, quae in iis occurunt, Opera Jacobi Golii. Amstelodami.* MS. P: Princeton University Library, Islamic Manuscripts, Garrett no. 135L, retrieved from http://arks.princeton.edu/ark:/88435/x920fw91b.

Glasner, R. 2003. Gersonides unusual position on position. *Centaurus* 45: 249–263.

Goldstein, B.R. 1967. The Arabic version of Ptolemy's Planetary Hypotheses. *Transactions of the American Philosophical Society* 57: 3–55.

Goldstein, B.R. 2002. Levi ben Gerson's preliminary remarks for a theory of planetary latitudes. *Aleph* 2: 15–30.

Goldstein, B.R., and J. Chabás. 2004. Ptolemy, Bianchini, and Copernicus: Tables for planetary latitudes. *Archive for History of Exact Sciences* 58: 453–473.

Ḥabash al-Ḥāsib, *The zij of Ḥabash al-Ḥāsib*, MS. Berlin, Ahlwardt 5750 (formerly Wetzstein I 90).

Ibn al-Shāṭir, 'Alā' al-Dīn Abu 'l-Ḥasan 'Alī b. Ibrāhīm b. Muḥammad al-Muṭa''im al-Anṣārī, *al-Zīj al-Jadīd*, MSS. K: Istanbul, Kandilli Observatory, no. 238; O: Oxford, Bodleian Library, no. Seld. A inf 30.

Ibn Yūnus, 'Alī b. 'Abd al-Raḥmān b. Aḥmad, *Zīj al-kabīr al-Ḥākimī*, MSS. L: Leiden, no. Or. 143, O: Oxford Bodleian Library, no. Hunt 331.

Jones, A. 2000. Studies in the astronomy of the Roman Period IV: Solar tables based on a non-Hipparchian model. *Centaurus* 42: 77–88.

Jones, A. 2005. Ptolemy's Canobic Inscription and Heliodorus' observation reports. *SCIAMVS* 6: 53–97.

al-Kamālī, Muḥammad b. Abī 'Abd-Allāh Sanjar (*Sayf-i munajjim*), *Zīj-i Ashrafī*, MS. Paris, Bibliothèque Nationale, no. 1488.

al-Kāshī, Jamshīd Ghiyāth al-Dīn, Khāqānī zīj, MSS. IO: London: India Office, no. 430, P: Iran: Parliament Library, no. 6198.

Kennedy, E.S., 1951. An Islamic computer for planetary altitudes. *Journal of the American Oriental Society* 71: 13–21. **(Reprinted in Kennedy et al. 1983, 463–471)**.

Kennedy, E.S. 1960. *The planetary equatorium of Jamshīd Ghiyāth al-Dīn al-Kāshī*. Princeton: Princeton University Press.

Kennedy, E.S. 1990. Two topics from an astrological manuscript: Sindhind and planetary latitudes. *Zeitschrift für Geschichte der Arabisch-Islamischen Wissenschaften* 6:167–178 **(Reprinted in Kennedy 1998, Trace VI)**.

Kennedy, E.S. 1998. *Astronomy and astrology in the medieval Islamic world*. Aldershot: Ashgate-Variorum.

Kennedy, E.S. and Ukashah, W. 1969. Al-Khwārizmī's planetary latitude tables. *Centaurus* 14: 86–96. **(Reprinted in Kennedy et al. 1983, pp. 125–135)**.

Kennedy, E.S. 1983. Colleagues, and former students. In: Studies in the Islamic exact sciences. Beirut: American University of Beirut.

Krisciunas, K. 1994. A more complete analysis of the errors in Ulugh Beg's star catalogue. *Journal for the History of Astronomy* 24: 269–280.

al-Maghribī, Mūḥyī al-Dīn, *Adwār al-anwār*, MSS. M: Iran, Mashhad, Holy Shrine Library, no. 332; CB: Ireland, Dublin, Chester Beatty, no. 3665.

al-Maghribī, Mūḥyī al-Dīn, *Talkhīṣ al-majisṭī*, MS. Leiden: Universiteitsbibliotheek, no. Or. 110.

Mancha, J.L. 1990. Ibn al-Haytham's homocentric epicycles in Latin astronomical texts of the XIVth and XVth centuries. *Centaurus* 33: 70–89.

Mozaffari, S.M. 2013a. Wābkanawī's prediction and calculations of the annular solar eclipse of 30 January 1283. *Historia Mathematica* 40: 235–261.

Mozaffari, S.M., 2013b. Limitations of methods: The accuracy of the values measured for the Earth's/Sun's orbital elements in the Middle East, A.D. 800 and 1500. *Journal for the History of Astronomy*. 44: Part 1: issue 3, pp. 313–336, Part 2: issue 4, pp. 389–411.

Mozaffari, S.M. 2014. Muḥyī al-Dīn al-Maghribī's lunar measurements at the Maragha observatory. *Archive for History of Exact Sciences* 68: 67–120.

Muḥammad al-Ṭabīb al-Muhtadī al-Mūṣilī, *al-Jam' al-mufīd*, MS. Iran: Parliament Library, no. 16238/3.

Nallino, C.A., ed. [1899–1907] 1969. *Al-Battani sive Albatenii Opus Astronomicum*. Publicazioni del Reale osservatorio di Brera in Milano. n. XL, pte. I–III, Mediolani Insubrum, Milan. The Reprint of Nallino's edition: Minerva, Frankfurt, 1969.

Neugebauer, O. 1962. *The astronomical tables of Al-Khwārizmī*, In: Hist. Filos. Skr. Dan. Vid. Selsk. 4, no. 2, Copenhagen.

Neugebauer, O. 1975. *A history of ancient mathematical astronomy*, 3 vols. Heidelberg: Springer.

al-Nīshābūrī, Niẓām al-Dīn A'raj, *Kashf al-ḥaqā'iq-i* Zīj-i Ilkhānī (*Opening of Truths of the* Ilkhānī zīj, a Commentary on al-Ṭūsī's Ilkhānī zīj), MSS. P1: Iran, Parliament Library, no. 1210, P2: Iran, Parliament Library, no. 1426.

Pedersen, O. 1974. *A survey of Almagest*, Odense: Odense University Press, 1974. With annotation and new commentary by A. Jones, New York: Springer, 2010.

Pliny, 1938–1962, *Natural History*, English translation (trans: Rackham, H.), 10 vols. Cambridge, MA–London: Harvard University Press/William Heinemann.

Qūshčī, 'Alī b. Muḥammad, *Sharḥ-i Zīj-i Ulugh Beg* (*Commentary on Zīj of Ulugh Beg*), MS. Iran: National Library, no. 20127–5.

Ragep, F.J. 1987. The two versions of the Ṭūsī Couple. In *From Deferent to Equant: A volume of studies on the history of science of the ancient and medieval Near East in Honor of ES Kennedy (Annals of the New York Academy of Sciences)*, ed. G. Saliba and D.A. King, vol. 500, 329–356.

Ragep, F. J., 1993, *Naṣīr al-Dīn al-Ṭūsī's* Memoir on Astronomy (*al-Tadhkira fī 'ilm al-hay'a*), 2 vols. New York: Springer.

Ragep, F.J. 2004. Ibn al-Haytham and Eudoxus: The revival of homocentric modeling in Islam. In *Studies in the History of the Exact Sciences in Honour of David Pingree*, ed. Ch. Burnett, J.P. Hogendijk, K. Plofker, and M. Yano, 786–809. Leiden-Boston: Brill.

Riddell, R.C. 1978. The latitudes of Venus and Mercury in the Almagest. *Archive for History of Exact Sciences* 19: 95–111.

Roberts, V. 1966. The planetary theory of Ibn al-Shāṭir: Latitudes of the planets. *Isis* 57: 208–219. **(Reprinted in Kennedy 1983, pp. 72–83)**.

Saliba, G., and Kennedy, E.S. 1991. The spherical case of the Ṭūsī couple. *Arabic Science and Philosophy* 1: 285–291. **(Reprinted in Kennedy 1998, Trace VI)**.

Samsó, J. 1997. Andalusian astronomy in 14th century Fez: *al-Zīj al-Muwāfiq* of Ibn 'Azzūz al-Qusanṭīnī", *Zeitschrift für Geschichte der Arabisch-Islamischen Wissenschaften* 11: 73–110. **(Reprinted in Samsó 2007, Trace IX)**.

Samsó, J. 1999. Horoscopes and history: Ibn 'Azzūz and his retrospective horoscopes related to the battle of El Salado (1340). In: *Between demonstration and imagination; Essays in the history of science* *and philosophy presented to John D. North*, ed. L. Nauta and A. Vanderjagt, 101–124. Leiden: Brill. **(Reprinted in Samsó 2007, Trace X)**.

Samsó, J. 2007. *Astronomy and astrology in al-Andalus and the Maghrib*. Aldershot: Ashgate.

Samsó, J. and Millás, E. 1998. The computation of planetary longitudes in the zīj of Ibn al-Banna. *Arabic Science and Philosophy* 8: 259–286. **(Reprinted in Samsó 2007, Trace VIII)**.

Shevchenko, M. 1990. An analysis of errors in the star catalogues of Ptolemy and Ulugh Beg. *Journal for the History of Astronomy* 21: 187–201.

Stephenson, F.R. 1997. *Historical Eclipses and Earth's Rotation*. Cambridge: Cambridge University Press.

Steele, J.M. 2000. *Observations and predictions of eclipse times by early astronomers*. Dordrecht: Kluwer. reprinted by Springer.

Steele, J.M. 2003. Planetary latitudes in Babylonian mathematical astronomy. *Journal for the History of Astronomy* 34: 269–289.

Súrya Siddhánta: [1860] 1997. *The Súrya Siddhánta: A textbook of Hindu astronomy*, ed. P. Gangooly and E. Burgess (tr.), Delhi: Motilal Banarsidass.D

Swerdlow, N.M. 2005. Ptolemy's theories of the latitude of the planets in the Almagest, handy tables, and planetary hypotheses. In *Wrong for the right reasons*, ed. J.Z. Buchwald, and A. Franklin, 41–71. Dordrecht: Springer.

Swerdlow, N.M., and O. Neugebauer. 1984. *Mathematical astronomy in Copernicus's De revolutionibus*. New York: Springer.

Toomer, G.J. 1968. A survey of the Toledan Tables. *Osiris* 5–174.

Toomer, G.J. (ed.). 1998. *Ptolemy's Almagest*. Princeton: Princeton University Press.

al-Ṭūsī, Naṣīr al-Dīn, *Ilkhānī Zīj*, MSS. C: University of California, Caro Minasian Collection, no. 1462; T: University of Tehran, Ḥikmat Collection, no. 165 + Suppl. P: Iran, Parliament 6517 (Remark: The latter is not actually a separate MS, but contains 31 folios missing from MS. T. The chapters and tables in MS. T are badly out of order, presumably owing to the folios having been bound in disorder), P: Iran, Parliament Library, no. 181, M: Iran, Mashhad, Holy Shrine Library, no. 5332a.

Ulugh Beg. *Sulṭānī Zīj*, MSS. P1: Iran, Parliament Library, no. 72; P2: Iran, Parliament Library, no. 6027.

Van Dalen, B. 1999. Tables of planetary latitude in the *Huihui li* (II). In *Current perspectives in the history of science in East Asia*, ed. Yung Sik Kim and Francesca Bray, 316–329. Seoul: Seoul National University Press.

Van Dalen, B. 2004a. A second manuscript of the Mumtaḥan Zīj. *Suhayl* 4: 9–44.

Van Dalen, B., 2004b. The *Zīj-i Naṣirī* by Maḥmūd ibn Umar: The earliest Indian Zij and its relation to the *'Alā'ī Zīj*. In *Studies in the history of the exact sciences in honour of David Pingree*, ed. Charles Burnett et al., 825–862. Leiden: Brill.

Verbunt, F., and R.H. van Gent. 2012. The star catalogues of Ptolemaios and Ulugh Beg; Machine-readable versions and comparison with the modern Hipparcos Catalogue. *Astronomy & Astrophysics* 544: A31.

Viladrich, M. 1988. The planetary latitude tables in the Mumtaḥan Zīj. *Journal for the History of Astronomy* 19: 257–268.

al-Wābkanawī al-Bukhārī, Shams al-Dīn Muḥammad, *Zīj-i muhaqqaq-i sulṭānī*, MSS. T: Turkey, Aya Sophia Library, No. 2694; Y: Iran, Yazd, Library of 'Ulūmī, no. 546, its microfilm is available in the University of Tehran central library, no. 2546; P: Iran, Library of Parliament, no. 6435.