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Masters, questions and challenges in the abacus schools

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Abstract The mathematical scenario in Italy during the Late Middle Ages and the Renaissance is mainly dominated by the treatises on the abacus, which developed together with the abacus schools. In that context, between approximately the last thirty years of the fourteenth century and the first twenty years of the sixteenth century, the manuscript and printed tradition tell us of queries and challenges, barely known or totally unknown, in which the protagonists were abacus masters. We report in this work on the most significant examples and draw out interesting cues for thoughts and remarks of a scientific, historical and biographical nature. Five treatises, written in the fifteenth and sixteenth century, have been the main source of inspiration for this article: the *Trattato di praticha d'arismetricha* and the *Tractato di praticha di geometria* included in the codices Palat. 573 and Palat. 577 (c. 1460) kept in the Biblioteca Nazionale of Florence and written by an anonymous Florentine disciple of the abacist Domenico di Agostino Vaiaio; another *Trattato di praticha d'arismetrica* written by Benedetto di Antonio da Firenze in 1463 and included in the codex L.IV.21 kept in the Biblioteca Comunale of Siena; the *Tractatus mathematicus ad discipulos perusinos* written by Luca Pacioli between 1477 and 1480, manuscript Vat. Lat. 3129 of the Biblioteca Apostolica Vaticana; and Francesco Galigai's *Summa de arithmetica*, published in Florence in 1521.

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Abbreviations

ASF	Archivio di Stato, Firenze
BAV	Biblioteca Apostolica Vaticana
BCP	Biblioteca Comunale, Palermo
BCS	Biblioteca Comunale, Siena
BMLF	Biblioteca Medicea Laurenziana, Firenze
BNF	Biblioteca Nazionale, Firenze
BRF	Biblioteca Riccardiana, Firenze

1 Introduction

In the Middle Ages and the Renaissance, in Italy it was customary to pose questions to famous mathematicians, often for the purpose of testing the interlocutor's ability, sometimes during actual mathematical challenges.

Among these are the problems that Giovanni da Palermo and Master Theodore of Antioch submitted to Leonardo Pisano around 1225, during their encounters at the Court of the Emperor Frederick II, and that Fibonacci introduced and solved a little later in the *Liber quadratorum* and in the *Flos*.[1] Scholars are also well acquainted with the harsh disputes and challenges that involved Niccolò Tartaglia, Antonio Maria Fior, Girolamo Cardano, Zuanne de Tonini da Coi and Ludovico Ferrari, over the discovery of the formulae to solve third and fourth-degree equations between 1535 and 1548. We know them thanks to the book *Ars magna* (1545) by Cardano, the *Cartelli* exchanged between Tartaglia and Ferrari in 1547–1548, and the *Quesiti et inventioni diverse* (1546) by Tartaglia that collects, among others, forty-two questions with various arithmetic, algebra and geometry problems posed to the mathematician from Brescia by many of his colleagues between 1521 and 1541.[2]

In the period comprising approximately the last thirty years of the fourteenth century and the first twenty years of the sixteenth century, the manuscript and printed tradition tell us of other queries and challenges, barely known or totally unknown, which once again saw some mathematicians of that time in the vast and suggestive world of abacus schools.

The appearance of the abacus schools in Italy, institutions that were perhaps unique in the history of Europe during the Late Middle Ages and the Renaissance, began around the mid-thirteenth century. It is documented from the last twenty years of the century throughout the sixteenth century, especially in the cities with a dynamic economy; Tuscany, with Florence first, was home to some of the most important and prestigious abacus schools and masters. Most of the Italian schools of abacus were public; in that case, it was the duty of city magistrates, local bodies and, in some cases, universities to appoint the masters, often taking care of their salary, which sometimes

[1] See for instance Bortolotti 1927, 169–170; Arrighi 1970, 2004, 225–236; Galuzzi et al. 2012, 36–40, 44–68.

[2] From the vast bibliography on the subject cf. Bortolotti 1927, 171–180; Masotti 1962; Gabrieli 1986; Toscano 2009.

was paid, totally or partially, by the students' families. In Florence instead, abacus schools were private and had their premises in houses or shops owned or rented by the teachers, which for this reason were called 'botteghe d'abaco'; between the first half of the fourteenth century and the first thirty years of the sixteenth century, we can count about twenty abacus schools distributed in the four neighbourhoods of the city, but with a higher concentration in the Neighbourhood of Santa Maria Novella. It was not uncommon for two masters to cooperate in the same school.

Usually, abacus schools were the continuation of an initial cycle of studies where students learned reading and writing in Latin and in the vernacular. Their main purpose was training the students in commercial, banking, craftsmanship and artistic activities and, in general, all those activities that needed a basic mathematical culture; however, they were also attended by young men from noble families and by those who wished to continue their studies in order to later start a profession; moreover, those schools were the training camps for future abacus masters. Teaching was divided into sections called "mute"; it would last for the entire day, and it was based on written and oral exercises; masters would also assign homework. The subject followed precise programmes, partly different from school to school. It was essentially based on the Fibonacci's works, in particular the *Liber abaci* (1202, 1228) and the *Practica geometriae* (1220/21), but took inspiration from the so-called abacus treatises, a body of texts that were almost always written in the vernacular of the various regions and only very seldom in Latin. These would mainly present a selection and a synthesis compared to the monumental and complex works of Pisano, but could also propose additions and elements of novelty. Very different in their appearance, content and size, those texts reach us in about three hundred codices and in various printed works. Their authors were mainly teachers, almost invariably abacus masters, but we know of some treatises written by simple lovers of mathematics, merchants, bookkeepers or artists. The subjects treated—even though they were not always all present and, however, developed at different levels—included an introduction to the positional numbering system, indigitation, operations with integers and fractions, the 'regola del tre' and the 'falsa posizione' rules, mercantile arithmetic, practical geometry, algebra, number theory and theory of proportions; problems were often posed in the form of recreational mathematics. They were generally presented in a rhetorical way, sometimes using personal abbreviations and a particular algebraic symbolism. Some compilers would include a wide selections of works from various authors, individual problems proposed by and to other mathematicians, together with some interesting information on the history of the schools and on scholars and experts of abacus mathematics.[3]

Five treatises of the abacus tradition, four handwritten and one printed, have been the main source of inspiration in the drafting of this article:

The *Trattato di praticha d'arismetricha* included in the codex Palat. 573 (c. 1460) kept in the Biblioteca Nazionale of Florence and written by an anonymous Florentine disciple of the abacist Domenico di Agostino Cegia, best known as 'il Vaiaio'.

[3] On abacus schools and abacus mathematics, cf. Franci 1998; Ulivi 2002a, b, 2008.

The *Tractato di praticha di geometria* with the vulgarisation of Leonardo Pisano's *Liber quadratorum*, manuscript Palat. 577 (c. 1460), also kept in the Biblioteca Nazionale, attributed to the same author.

Another *Trattato di praticha d'arismetrica* written by Benedetto di Antonio da Firenze in 1463 and included in the codex L.IV.21 kept in the Biblioteca Comunale of Siena.

The *Tractatus mathematicus ad discipulos perusinos* written by Luca Pacioli between 1477 and 1480, manuscript Vat. Lat. 3129 of the Biblioteca Apostolica Vaticana.[4]

Francesco Galigai's *Summa de arithmetica*, published in Florence in 1521, with two posthumous republications in 1548 and 1552, entitled *Pratica d'arithmetica*.

2 The challenge to Maestro Giovanni di Bartolo

An intriguing passage of the *Arismetrica* by Benedetto da Firenze, one of the most interesting abacists and abacus masters of the fifteenth century, reads:[5]

> Maestro Giovanni di Bartolo inchominciò a insegnare circha 1390 e, chosì chome il suo maestro morì g[i]ovane, anchora lui g[i]ovane chominciò in questo modo. Morto il suo maestro Antonio, persuaso et aiutato da certi amici di Maestro Antonio et anchora da' suoi, benché di 19 anni fusse, gli feciono aprire la medesima schuola et favoregiandolo quant'era possibile. E per sua g[i]ovaneza pocho dagli altri che 'nsegnavano conosciuto. E benché dottissimo et chopioso di libri fusse … l'invidia che negli artefici d'un arte regnia e massime in fra quelli che insegnono al presente, in fra lloro examinato in che modo si potesse levarlo di quella voluntà, presono questa via. Chonciosiachosaché per la sua età non fusse possibile che egli potesse sapere, ragunorono, ciaschuno nella loro schuola, alchuni buoni ragioniere; e' fu nella schuola di Maestro Michele circha a 25 di varie materie, et nella schuola di Maestro Lucha circha altretanto, benché Maestro Luca pocho o niente facesse, ma Maestro Biagio suo maestro …. Et chiamato ciaschuno a ssé dissono: a noi è stato detto che un fanc[i]ullotto discepolo di Maestro Antonio à riaperto la schuola ch'egli teneva quando era in vita; e, acciò che credo che fra voi sarebbe chi meglio di lui la terrebbe, io vi fo chomandamento che ogi, quando venite alla schuola, vo' n'andiate là e pigliate le mute vostre da llui et quando vi fate insegnare mostrategli cho' vostri arghomenti che sapete, che vadi a ffare altro. A' quali ubidendo i detti discepoli andorono …. Maestro Giovanni, maravigliatosi di tanti et quali … et di diverse materie, subito stimò quel ch'era. Nientedimeno, a uno a uno chiamatogli, la materia loro che volevano mostrò … et chiarì loro in modo che stupefatti … parve loro, in quello pocho di spazio, avere più inparato che 'l resto del tempo agli altri; onde seguitando pervennono in modo che molti di loro furono per lo' propia voluntà

[4] Transcribed and published by Calzoni and Cavazzoni: cf. Pacioli 1996.

[5] Arrighi 1965, 397–398; Arrighi 2004, 156–157.

> sopinti a dire et fare villania a' loro maestri primi, solamente avendo chonpreso la intensa invidia che gli portavono ....[6]

Various archival documents on the activity of the Florentine masters suggest that the episode narrated by Benedetto be dated to around 1388–1389, when in fact Giovanni di Bartolo was still very young, about twenty-four years old. It was more than two years after the prestigious "Scuola di Santa Trinita", at that time situated in front of the church bearing the same name, had lost its most significant exponent, the skilled algebraist Antonio Mazzinghi, who had died in his thirties. After a period of inactivity, the school had been reopened by Giovanni, the modest son of a mason and a former disciple of Maestro Antonio. Some teachers from other schools of the town, all older than Giovanni and eager to take Mazzinghi's place as headmaster of the "Scuola di Santa Trinita", set a trap for the young master, sending their best pupils to follow the "mute", namely the lessons held by Giovanni, in order to confuse him with complex questions "di diverse materie", which unfortunately we do not know and which the abacist unexpectedly answered in a clear and exhaustive way. His foes, who had deceitfully challenged Giovanni with the sole intention of discrediting him in the eyes of their students, were also skilful masters: they were Michele di Gianni, who was then teaching at the "Scuola dei Santi Apostoli", Biagio di Giovanni and Luca di Matteo, both teachers at the "Scuola del Lungarno", schools that were situated a stone's throw from that of Santa Trinita, and all located in the Neighbourhood of Santa Maria Novella. Giovanni di Bartolo, who was also lecturer of astrology in the Studio Fiorentino, taught in his abacus school for many years, until his death in 1440.[7]

3 Giovanni di Bicci's questions and Massolo da Perugia's answers

The anonymous Vaiaio's disciple talks in two circumstances of one Massolo, a.k.a. Petrozzo, da Perugia, "huomo assai esperto in dette scienzie", namely in mathematical sciences.

[6] "Maestro Giovanni di Bartolo began teaching around 1390. He was young, just like his teacher Antonio when he died. After his teacher's demise, although he was only 19, some of his and Antonio's friends helped him and convinced him to open the same school. Since he was young, although he was erudite and owned many books, he was little known to the other teachers who, moved by the envy that reigns among those who perform the same activity, and most of all among those who are teaching at present, after discussing among themselves in which way they could discourage him from his intent, decided to proceed as follows. So, believing that, due to his age, Giovanni was not expert enough, each of them selected the best students from his school: about 25 from Maestro Michele's school, expert in various subjects, and almost as many from the school where Maestro Luca taught, although very little, and most of all his teacher, Maestro Biagio, taught. They said to each of them: it came to our knowledge that a young disciple of Maestro Antonio's has reopened the school that he had when he was alive, and since we think that among you there is someone better than him, today you shall go follow his lessons, showing him your ability, so that he will decide to do something else. Obedient to such orders, those disciples went to the school of Maestro Giovanni who was surprised but immediately understood the reason for so many students. However, calling them one by one, he explained to all of them the subject they requested, and he did it so well that in that short time they felt they had learned much more than they had so far from the other teachers; and many of them ended up by bad-mouthing their first teachers, since they understood the deep envy they felt for him".

[7] For more detailed information on the aforesaid schools and masters, see Ulivi 2004a, 2013.

In the tenth part of the *Arismetricha* contained in Palat. 573, dedicated to algebra, he narrates that around 1397 Massolo sent a letter to Giovanni di Bicci de' Medici, in response to some questions he had posed on the solution of equations.[8] The text of the letter was also the copy of a reply to another letter that the Perugino had received from a certain Maestro Diamante, who taught in Venice in 1396, and begins like this:

> Chonprendo vi paia gran fatto che cubi, censi et chose si possono aguagliare al numero, chonsiderato l'opinione di tutti e' passati è suta non essere possibile; io ve dicho che onne adeguagliamento è possibile a definire et, se 'l modo non fusse molto lungho et dificile, in questa ve 'l manderia; bene che, quello nne richiedete, vi mando.[9]

It is unclear whether the Umbrian mathematician knew formulas for solving third-degree equations, and if so, which of them;[10] from what he writes later, it seems that he wished to refer to the possibility of constructing any typology of algebraic equation by assuming that it has a given root. So, for instance, "la radice chubica dello 84, chollo abattimento di 5 chose, è 4; perché 5 chose sono 20 che, abattutte del'84, rimanghono 64, la chui radicie chubicha è 4".[11] Since in those times the 'cosa' meant the variable x, it means that a solution of the equation $84 - 5x = x^3$ is $x = 4$.

We find another reference to Massolo in the introduction to the vernacular version of Fibonacci's *Liber quadratorum*, which is part of Palat. 577.[12] Here we can read that, on an unspecified date, the Perugino sent nothing less than a "treatise" containing "10 ragioni a Giovanni de' Bicci de' Medici, le quali, il detto Giovanni, per lo tempo passato, gli aveva chieste per darle ad alchuni valenti che erano a quel tempo dimostratori, cioè tenevano in questa città schuola".[13] Those problems concerned square numbers and, as the author of Palat. 577 pointed out, "Nientedimeno Lionardo Pisano chiaro dimostra e' numeri quadrati avere certe nature per le quali l'asolutione delle quistioni sopra quelle trovate prestamente s'ànno".[14] One in particular contained the answer to

8 Arrighi 1967a, 401–402; Arrighi 2004, 165–166.

9 "I understand that to you it seems a great fact that cubi, censi and cose can be equalised to the number, since in the past everybody thought it impossible; I am telling you that every equalisation can be defined and, if the procedure wasn't very long and difficult, I would explain it to you in this letter; however, I am sending you what you have requested".

10 In the fourteenth century, four treatises were known reporting formulas for the solution of third-degree equations that were completely wrong: Paolo Gherardi's *Libro di ragioni*, written in Montpellier (1328: BNF, Magl. XI, 87), the anonymous *Trattato dell'alcibra amuchabile* (c. 1365: BRF, Ricc. 2263), the treatise *Aritmetica e geometria* by Gilio da Siena (1384: BCS, L.IX.28), the anonymous *Libro merchatantesche* (1398: BCP, 2QqE13). The *Aliabraa argibra,* written in Pisa by some Maestro Dardi in 1344, found in different codices, and the aforementioned *Libro merchatantesche* propose formulas for complete third- and fourth-degree equations that are exact, but only within specific problems. The same formulas can be found in many texts of the fifteenth century. On this subject, see Franci 2013.

11 "The cube root of 84, decreased by 5 cose, is 4; because 5 cose are 20 that, subtracted from 84, gives 64, whose cubic root is 4".

12 Arrighi 1967c, 774; Arrighi 2004, 208.

13 "Ten problems to Giovanni de' Bicci de' Medici that said Giovanni sent to him in the past to pass them on to some expert mathematicians, who at that time had schools in this city".

14 "No less Lionardo Pisano clearly demonstrates that square numbers have certain properties, by which one can immediately find the solutions to the questions asked on them".

the question: "trovare uno numero quadrato che agunto, o vero trattone, uno numero rimangha quadrato".[15]

It is the well-known problem that translates into two undetermined equations

$$x^2 + N = y^2$$
$$x^2 - N = z^2$$

where the numbers N and x^2, integer or rational, are called *congruous* and *congruent*, respectively.

The question, already present in Diophantus's *Aritmetica* and in some works written by Arab mathematicians of the tenth to eleventh century, is one of the more complex among those tackled by Fibonacci.

Moving from the question posed to him by Giovanni da Palermo with $N = 5$, Pisano dealt with the problem more generally in the *Liber quadratorum* and then in the *Flos* for that specific case only. The version of the Book of Squares included in Palat. 577 and also another vernacular version of Leonardo's treatise that Benedetto da Firenze included in the *Arismetrica* of codex L.IV.21 contain significant original contributions to the solution of the congruous–congruent numbers problem.[16]

Besides the mathematical aspects, the interest of the tales reported in the codices Palat. 573 and Palat. 577 on Massolo consists firstly in the fact that the questions were posed to him by a figure of primary importance in the history of Florence, Giovanni de' Medici, son of Averardo, called Bicci, and father to Cosimo il Vecchio. Precisely in 1397, the year in which he received Perugino's answer on the equations, Giovanni moved the headquarters of the Bank of Rome that he had shared with his uncle Vieri to Florence, triggering the quick economic and political rise of the Medici family. The Vaiaio's anonymous disciple reports that Giovanni, undoubtedly trained in an abacus school, and a keen promoter of works of art and architecture, also had mathematical interests and close relations with the best teachers of the Florentine schools; he probably met the Umbrian mathematician in person.

A significant note concerns Massolo himself, about whom we have no biographical information, although we can suppose he was one of the abacus masters whose public activity with the Studio di Perugia is known from 1389:[17] we cannot rule out that he is the "magister geometricus" mentioned in the Statutes of the Conservatori della Moneta dated 1396.[18]

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[15] "Find a square number such that, whether you add or subtract a number, remains a square number".

[16] See Franci 1984.

[17] Cf. Ermini 1971, 185; Ulivi 2002b, 125–126.

[18] Two more questions included in Palat. 573 and in L.IV.21 concern abacus masters from out of Florence and their relations with Florentine abacus schools. The two problems, both solved algebraically, have no particular mathematical significance; therefore, we will provide only some notes of a historical nature. The manuscript Palatino's problem was posed by Filippo de' Folli da Pisa to Maestro Giovanni di Bartolo in 1399 (Arrighi 1967a, 404; Arrighi 2004, 166–167). This problem is the only problem we know by Maestro Filippo. The Folli, from an illustrious family from Pisa, taught at least in 1398–1399, cooperating with his junior Iacopo di Maestro Tommaso, one of the five abacists active in Pisa in the fourteenth century who belonged to the lineage of the Bonagi dell'Abaco, perhaps the same of Fibonacci's (Ulivi 2011, 258–271). The question of the manuscript L.IV.21 was sent to Florence by a master from L'Aquila in 1445, even if

4 Marco Trevigiano's question

In 1372, one Marco Trevigiano sent an interesting question to the Florentine masters. Perhaps this Marco, supposedly originally from Treviso, was an abacus master in that town, where, however, we know nothing about the teaching of the abacus that, conversely, is documented in nearby Venice from the early fourteenth century, nearly always in private schools.[19] The question can be placed in recreational mathematics. It concerned the Earth's circumference, of which it reported the measure, in Arab miles, obtained by the Arab astronomer al-Farghani, a.k.a. Alfraganus, who lived in the first half of the ninth century at the time of the Abbasid Caliph al-Ma'mun, a measure that also Dante mentions in the *Convivio* (Cv. III, V, 11). It also required knowledge of the formula from the sum of the first n natural numbers and that of their cubes. For this reason right in the second chapter of the fifth part of the *Trattato di praticha d'arismetricha* of Palat. 573, "dove s'asolve chasi per natura e propietà di numeri", Vaiaio's anonymous disciple proposed that problem:[20]

> Credesi, per gli astrologhi, la Terra girare 20400; dove adunque 2 huomini si partono, et l'uno va inverso levante et l'altro inverso ponente, et muovonsi da uno punto solamente. Et l'uno va, il primo dì un miglio, e il sechondo 2 miglia, e il terzo dì 3 miglia, e chosì ogni dì il primo va più un miglio che 'l dì passato. E il sechondo huomo va, il primo dì uno miglio, il sechondo dì 8 miglia, et chosì ogni dì va il seguente numero chubo, cioè il terzo dì 27 miglia et il quarto dì 64 miglia. Adimandasi in quanti dì si schonterranno insieme. Questa quistione mandò a Ffirenze Marcho trevigiano nel 1372. E, benché varii oppinioni sieno, io tengho questo ....[21]

The solution, described in the usual rhetorical language, and here presented in our symbolism, is the following:

Indicating by n the required days, with $P_1(n)$ and $P_2(n)$ the routes of the two men, then

Footnote 18 continued
it is included by Benedetto da Firenze in the "chasi exenplari alla regola dell'algebra di Maestro Biagio"; said Biagio, called "il vecchio", had died around 1340, after being master and collaborator of the renowned Paolo dell'Abaco, probably in the "Bottega di Santa Trinita" (Arrighi 1965, 377; Arrighi 2004, 136; M° Biagio 1983, 116–119). As of today, the problem of the anonymous master from L'Aquila is the only statement that provides evidence of the existence of abacus schools in the capital city of Abruzzo, at least in the mid-fifteenth century.

[19] Bertanza and Dalla Santa 1907; Ulivi 2002b, 131–132; Ulivi 2008, 407.

[20] BNF, Palat. 573, cc. 138r-138v; Arrighi 1967a, 400 and 421; Arrighi 2004, 165 and 181.

[21] "Astrologists believe that the Earth has a circumference of 20400. Where two men start from the same point, and one goes east, while the other goes west. And on the first day, one travels one mile, the second day 2 miles, and the third day 3 miles, and so every day the first man has travelled one mile more than the previous day. And the second man travels one mile on the first day, 8 miles on the second, and so every day he travels the next cube number, that is 27 miles on the third day and 64 miles on the fourth. The question is: in how many days will they meet? Marco Trevigiano sent this question to Florence in 1372. And, although there are various methods, I follow this one ...".

$$P_1(n) = 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$$
$$P_2(n) = 1 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{1}{2}n(n+1)\right]^2$$

Therefore, it must be

$$P(n) = P_1(n) + P_2(n) = \frac{1}{2}n(n+1)\left[1 + \frac{1}{2}n(n+1)\right] = 20400$$

The author of the treatise now proceeded with a sort of interpolation.

He observed that $P(16) = 18632$ and $P(17) = 23561$; therefore, $n = 16 + f$ where

$$f = \frac{20400 - P(16)}{P(17) - P(16)} = \frac{1768}{4930} \quad \text{whence} \quad n = 16 + \frac{1768}{4930}$$

This problem has been discussed many times by historians, not in relation to Marco Trevigiano's question included in Palat. 573 but to three important mathematicians of the fifteenth and sixteenth century, namely Luca Pacioli, Girolamo Cardano and Niccolò Tartaglia, who reported it in three of their works, which can be included, despite at a very high level, in the body of abacus treatises.[22]

Pacioli, a Franciscan friar born in Borgo Sansepolcro, proposed it in the first part of the *Summa de arithmetica, geometria, proportioni et proportionalita*, published in 1494. His solution is based on the algebraic method.[23] He writes:

> Fa così, poni che li si congionga in $1co.$ de dì, adonca serà andato lo primo in questo tempo $\frac{1}{2}ce.\,e\,\frac{1}{2}co.$ E lo secondo sirà andato in questo tempo $\frac{1}{4}ce.ce.\,e\,\frac{1}{2}cu.\,e\,\frac{1}{4}ce.\ \ldots$.[24]

Therefore, it takes as variable $x = 1$ cosa, the number of days to be determined.[25] With the previous notation, since

$$P_1(x) = \frac{1}{2}x^2 + \frac{1}{2}x$$
$$P_2(x) = \frac{1}{4}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2$$

by summing, he gets

$$\frac{1}{4}x^4 + \frac{1}{2}x^3 + \frac{3}{4}x^2 + \frac{1}{2}x = 20400$$

22 See, for example, Loria 1982, 280; Gavagna 1999, 302–304; Gavagna 2007, 126–128; Giusti 2007, 172–173.

23 *Summa*, I: distinctio secunda, tractatus quintus, 38r and 44r.

24 "Do as I say; assume that they meet in a number of days equal to $1co.$; hence, in such time, the first would have travelled $\frac{1}{2}ce.$ and $\frac{1}{2}co.$ And in that time, the second would have travelled $\frac{1}{4}ce.ce.$ and $\frac{1}{2}cu.$ and $\frac{1}{4}ce.\ \ldots$".

25 On the symbolism used by Pacioli, see also the next question.

that is

$$x^4 + 2x^3 + 3x^2 + 2x = 81600$$

that, adding one to both members, becomes

$$(x^2 + x + 1)^2 = 81601$$
$$x^2 + x = \sqrt{81601} - 1$$

hence, the solution

$$x = \sqrt{\sqrt{81601} - \frac{3}{4}} - \frac{1}{2}$$

Many years later, both the mathematician Pavese Cardano in the last chapter of its *Practica arithmetice* of 1539, entitled "De erroribus Fratris Luce", and Tartaglia in the Second Part of the *General trattato di numeri et misure,* published in 1556, and still among the "Errori" of the friar from Borgo Sansepolcro, highlighted the incorrectness of Pacioli's algebraic method; indeed, this would have led to an irrational result, while the very procedure of the solution assumed it to be a natural number. Therefore, they both proposed an alternative solution by interpolation, already described in the codex Palat. 573, which leads anyway to a non-integer result. In the Sixth Part of the *General trattato,* published posthumously in 1560, Tartaglia re-proposed the problem in algebraic form, but replaced the value 20400 with 29412 so as to obtain the equation

$$(x^2 + x + 1)^2 = 117649$$

whose solution is $x = 18$.[26]

Pacioli, Cardano and Tartaglia made no reference whatsoever to the far more ancient origins of the question, which, however, must have been subject of discussions in the Italian abacus schools for quite a while, also judging from the allusion to the knowledge of "varii oppinioni" on the subject made by the author of the palatine manuscript written around 1460.

The mathematician from Borgo Sansepolcro, in addition to holding various positions as a lecturer of mathematics at university level, received various assignments as an abacus master in the Studio di Perugia. Tartaglia had been abacus master at least in the schools of Verona and then public lecturer in Venice and Brescia.

As for Cardano, we know that he taught mathematics in Milan in the Scuole Piattine, but we have no accurate information about what he taught. However, various considerations lead us to assume that he had contacts with the abacus schools. There are

[26] The German mathematician Michael Stifel, in his *Arithmetica integra* of 1544 (306r-307v), re-examining the problem from Cardano's work, solved it algebraically, like Pacioli and Tartaglia, but with 44310 instead of 20400 so as to obtain the solution $x = 20$. Soon after, observing that $P_2(x)$ is the square of $P_1(x)$, without developing the two expressions, solved the $P_1(x) + P_2(x) = 44310$ by changing the variable $(x^2 + x)/2 = y$, thus avoiding the fourth-degree equation and finding x through two second-degree equations.

significant and repeated references in his works to a certain Gabriele Aratori or degli Aratori da Caravaggio "arithmeticus optimus", whom Cardano recalls as "arithmeticam Mediolani publice docente" and for whom he also prepared a horoscope, knowing his exact date of birth, 31 December 1480. This connection offers us the chance to report an interesting hypothesis about the Aratori discussed by art historians. According to Calvesi, that Gabriele Aratori could be a forefather of Michelangelo Merisi, "il Caravaggio", whose mother was Lucia di Giovan Giacomo Aratori.[27] Considering the mathematical interests of the Aratori cited by Cardano, we believe that this hypothesis is supported by a document of 1525 in which one Gabriele of Luigi Aratori appears as witness of a notary deed of Giovan Antonio di Stefano Aratori who, according to Berra, can be reasonably considered the maternal great-grandfather of Caravaggio, and by other various documents in which the maternal grandfather of the great painter, Giovan Giacomo, is identified as "agrimensore",[28] that is, as a surveyor who had to measure real estate properties, an activity that required mathematical competence and for a long time had been carried out by expert abacists, often by actual abacus masters.

5 A question posed by Giovanni del Sodo

Luca Pacioli received his first assignments as an abacus master from the University of Perugia between 1477 and 1480. In those years, he composed his second abacus treatise,[29] dedicated to his students, the *Tractatus mathematicus ad discipulos perusinos*. The work is divided into sixteen parts followed by a "Tariffa mercantesca" that precedes the last group of mathematical problems. The "Capituli de algebra" occupy the sixteenth part, albeit incomplete due to the absence of the first twenty-five papers, where we assume that Pacioli had also described the peculiar algebraic symbolism that he used in his treatise and that we can reconstruct by reading his problems. Towards the end of the chapters on algebra and among the problems following the "Tariffa", Pacioli reported four questions pointing out, or writing notes in the margin, that he had sent the solutions to Florence. In one, reported below, he explicitly mentioned the name of the master who proposed it to him in 1480, Giovanni del Sodo dei Sodi;[30] Frate Luca had also sent questions to the Florentine master, but without getting any reply, at least until then:

> Trovame 3 numeri proportionali ch' el quadrato del terzo sie uguale a la summa di quadrati degli altri doi, e multiplichare el primo numero nel secondo facia 10; dimando che fia ciascun numero per sé. Sapi questa esser bona domanda, e a dì

[27] Calvesi 1992, 18–19; Calvesi 2000, 40.

[28] Cf. Berra 2005, 125–133, 342. Oddly, it is precisely Berra, in producing these documents and only three more notary deeds where some Gabriele Aratori is mentioned, who believes "invece evidente" that the Gabriele mentioned by Cardano "non fosse in alcun modo parente del nonno materno del Caravaggio perché il suo nome non compare mai, neppure marginalmente, associato a quello dei familiari del pittore".

[29] The first, gone missing, dates back to 1470 and was dedicated to the children of the merchant Antonio Rompiasi, of whom Luca was the tutor during his first stay in Venice.

[30] BAV, Vat. Lat. 3129, 359v-360r; Cavazzoni 1998, 594–595; Derenzini 1998, 187, 189. Note that the passage is crossed out with a stroke of the pen.

> 4 aprile 1480 me fo mandata da Firenza da Maestro Giovani Sodi per le mani de Giovan Giacopo or[a]fo de Peroscia e facemmoli resposta aprobatissima e ancho a cert'altre, e mandammoli a l'incontro altre domande de le qual finora non abiam resposta etc.[31]

Indicating by a, b, c the quantities to be determined, which are assumed to be in continued proportion, the problem translates, using the modern symbolism, into the relations

$$\begin{aligned} a : b &= b : c \\ a^2 + b^2 &= c^2 \\ ab &= 10 \end{aligned}$$

The same problem is solved by Leonardo Pisano in the fifteenth chapter of the *Liber abaci*.[32] Like Fibonacci, Pacioli assumed as variable $a = x$.

From the first and third equation, he obtained $b = \frac{10}{x}$ and $c = \frac{100}{x^3}$, and finally, substituting in the second equation, he obtained

$$x^2 + \frac{100}{x^2} = \frac{10000}{x^6}$$

and then

$$\begin{aligned} x^{10} + 100x^6 &= 10000x^2 \\ x^9 + 100x^5 &= 10000x \\ x^8 + 100x^4 &= 10000 \end{aligned}$$

Again, as in the *Liber abaci*, where, however, the deduction of the last equation is more immediate, the solution is obtained by reducing the eight-degree trinomial equation to one of second degree in the variable x^4.

The problem is quite significant because in the entire *Tractatus* it is the one with the equations of highest degree, allowing us to identify the symbols used by Pacioli for the variable and the ensuing powers up to the tenth, with the sole exclusion of the seventh.

The previous three equations read, respectively

$$\begin{aligned} &100^{\square\triangle}\ \textit{più}\ 1^{\square\oslash}\ \textit{equale a}\ 10000^{\square} \\ &100^{\oslash}\ \textit{più}\ 1^{\triangle\triangle}\ \textit{equale a}\ 10000^{co} \\ &100^{\square\square}\ \textit{più}\ 1^{\square\square\square}\ \textit{equale a}\ 10000 \end{aligned}$$

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[31] "Find 3 proportional numbers so that the square of the third is equal to the sum of the squares of the other two, and if you multiply the first number by the second the result is 10; I am asking what the value of each number is. Mind you, this is a good question sent to me on the 4th April 1480 from Florence by Maestro Giovanni Sodi at the hand of Giovan Giacomo, goldsmith in Perugia, and we gave him a very exact answer, and to other questions too, and then we sent him, in our turn, other questions that so far have not been answered, etc.".

[32] Leonardo Pisano 1857, 1862, I, 447–448.

So, Pacioli used the basic symbols *co*, □, △, ⊘ to indicate the first, second, third and fifth power of the variable. The other powers are obtained by arranging those symbols according to the multiplication principle.[33]

Although with some adjustments, the *Tractatus ad discipulos perusinos* was used almost entirely by Pacioli in the realisation of his bigger, almost monumental, printed work, the *Summa*, without using in it algebraic symbolism of the *Tractatus*, but instead more rhetorical and traditional notations. As we have already seen in the previous question of Marco Trevigiano: $1\,co.(1\text{ cosa}) = x$, $1\,ce.(1\text{ censo}) = x^2$, $1\,cu.(1\text{ cubo}) = x^3$, $1\,ce.ce.(1\text{ censo di censo}) = x^4$; moreover, $1\,p^\circ.r^\circ.(1\text{ primo relato}) = x^5$, and so on.

In particular, the problem proposed by Giovanni dei Sodi, but without any reference to the Florentine mathematician, appears in the first part of the *Summa*, among those on numbers in continued proportion.[34] So, for instance, the previous equation of tenth degree is written in the form: $10000\,ce.\ equali\ a\ 100\ ce.cu.\ \tilde{p}.\ 1\,ce.p^\circ.r^\circ.$

Some treatises of the fifteenth and sixteenth century have points in common with the peculiar symbolism of the *Tractatus*.

The observation appears almost obvious when comparing Pacioli's text to the *Trattato d'abaco* by Piero della Francesca, written in the seventies of the fifteenth century. In Piero's book, Luca's illustrious friend and fellow citizen, here too in an irregular way and with the coexistence of abbreviations in rhetorical style, we find the same symbols used by Pacioli for the second, third and fourth power of the variable, placed above the relevant coefficient; the variable is identified by a dash placed on the relevant coefficient, and the triangle that identifies the third power is often replaced by a *c*, initial of cube.[35]

Far more evident, although apparently undetected until today, is the correspondence between the *Tractatus* symbolism and the symbolism described in some short "Capitoli d'arcibra" situated at the end of the Florentine codex Ash. 353 in the Biblioteca Medicea Laurenziana, a codex that Van Egmond has dated around 1470 based on the filigree analysis.[36]

In a table of this codex, we find the notations used by Maestro Luca up to the sixth and for the eighth power,[37] with a symbol also for the seventh power of the variable, the "sechondo relato", a small circle cut by two diameters (Fig. 1), maybe even present in one of the lost pages of Pacioli's manuscript.

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[33] Actually, in the manuscript, the author proceeds in a rather erratic way, placing them as exponents or above the relevant coefficient.

[34] *Summa*, I: distinctio sexta, tractatus sextus, 93r.

[35] Piero della Francesca 2012, I, Testo e note, XIX, L-LI. We notice that in the vernacular version of Piero della Francesca's *Libellus quinque corporum regularium*, which is part of the *Divina proportione*, published in 1509, Pacioli again used a symbolic notation limited to the first, second and fourth power, but different from both that of the *Tractatus* and that used by Piero in the original text: cf. Piero della Francesca 1995, I, Testo e note, XXXII and XXXIV. At the end of the second part of the *Summa*, in the "Particularis tractatus circa corpora regularia et ordinaria", although taken from the geometrical part of Piero's *Trattato d'abaco*, Luca obviously follows the symbolism that is used in his entire text.

[36] The "Capitoli d'arcibra" occupy the folios 124r-127r of the manuscript and are preceded by a treatise by Matteo di Niccolò Cerretani, written with a different handwriting: cf. Van Egmond 1980, 78.

[37] BMLF, Ash. 353, 124v.

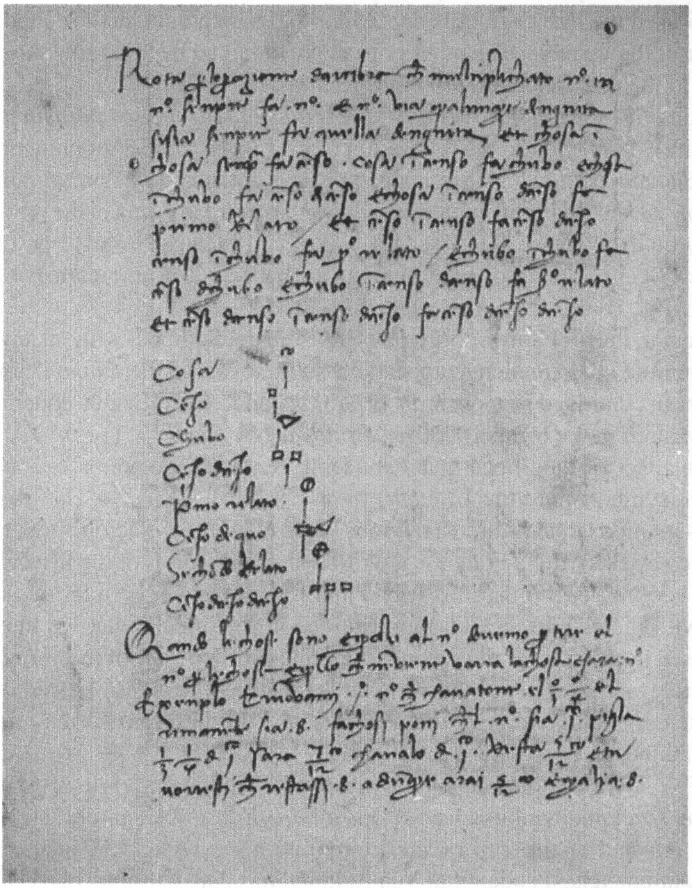

Fig. 1 BMLF: Ash. 353, 124v. Reproduced with permission of MiBACT. Further reproduction by any means is prohibited

We do not know who is the author of the "Capitoli" contained in the Laurentian manuscript, which seem to precede Pacioli's text some years. We know that that symbolism, already in the eighties, was known at least by Giovanni dei Sodi and probably by those attending his school. Giovanni, defined in a Book of Memories of the late fifteenth century as "el migliore maestro d'abacho da Firenze", taught for more than twenty years in the Neighbourhood of Santa Maria Novella. He had among his students Francesco Galigai and maybe even Raffaello Canacci, future abacus masters in the Neighbourhoods of San Giovanni and Santo Spirito.[38] Both in the second edition of Canacci's *Ragionamenti d'algebra* (c. 1495), contained in the codex Palat. 567 of the Biblioteca Nazionale of Florence, and in Galigai's *Summa de arithmetica*, published in 1521, we find the same algebraic symbolism that Maestro Francesco says he took from an *Arcibra*, unknown to us, of his "preceptore" Giovanni del Sodo, which reaches

[38] On Raffaello Canacci, Giovanni dei Sodi and Francesco Galigai cf. Ulivi 2004b, 125–134, 162–166; Ulivi 2012.

the thirteenth power of the variable. Perhaps Giovanni, drawing inspiration from the problems he received from Pacioli, had created his personal symbolism that actually corresponds to that of the *Tractatus* only for the first, second and fourth power, while for the others he used a composition of squares and rectangles that in the written equation are situated on the right of the relevant coefficient.[39]

6 Maestro Agnolo del Carmine's questions

The question submitted by Giovanni dei Sodi to Luca Pacioli introduces us to an interesting subject in the abacus body of treatises, namely continued proportion, which is the subject of this final portion of our work.

In abacus treatises, the problems, especially those relevant to mercantile arithmetic, are often solved by applying the 'regola del tre' and the 'falsa posizione' rules, which use proportionality, but they are mainly presented in an exclusively practical form, bypassing the concept of proportion. However, quite a few texts submit theoretical problems about proportional quantities, applying specific identities that characterise them. The best treatises, starting from the *Liber abaci*, also dedicate considerable space, sometimes even entire chapters, to the theory of proportions and to a systematic exposition of the relevant rules and properties with ensuing applications. This is the case with Pacioli, starting from the *Tractatus ad discipulos perusinos*, but mostly in the sixth distinction of the first part of the *Summa*, where he listed, among other things, thirteen "regole" and fifteen "chiavi" relevant to three, four or five quantities in continued proportion.[40]

Francesco Galigai also dealt widely with this subject in his *Summa de arithmetica*, reporting in the third book a group of "casi sottili" posed to him by a certain Maestro Agnolo del Carmine "excessivo geometro, et le risposte da me factogli e absolutogli con regole et modi aptissimi, come apieno si vedra".[41] There are six problems, one of which is repeated twice; they are solved with very little use of algebra, mainly through other methods and particular relations. We will present the three problems that we consider the most significant.

The first question reads:

> Quando vvuoi dividere 11 in tre parte continue proportionali, per sapere ciascuna parte per sé. Questa mi propose Maestro Agnolo dal Carmine non mi dicendo in che proportione se la volessi, la composi nella doppia proportione.[42]

It is an undetermined problem to which Galigai found a solution using the simple 'falsa posizione'. After calling the three parts a, b and c, he assumed they are 1, 2 and 4, respectively, whose sum is 7; then, he obtained a from the proportion

[39] *Summa de arithmetica*, 71r-71v. Cf. Franci and Toti Rigatelli 1985, 68.

[40] *Summa*, I: distinctio sexta, tractatus sextus, 86v-89v.

[41] *Summa de arithmetica*, 22r, 25r-27r.

[42] "When you wish to divide 11 in three continued proportional parts, find each part. This is the question that Maestro Agnolo dal Carmine posed to me, without telling me in which proportion he wanted it; I solved it in the double proportion".

$$1{:}7 = a{:}11 \quad \text{hence } a = \frac{11}{7}, b = \frac{22}{7}, c = \frac{44}{7}$$

Taking 10 as number to be divided, Fibonacci had already solved the problem using the same method in the third paragraph of the twelfth chapter of the *Liber abaci*[43]; then we find it, for instance, in the twelfth part of Luca Pacioli's *Tractatus*, with the title "Divisioni e partimenti di numeri".[44]

Agnolo del Carmine's second and fourth questions differ only in the numerical data. The first formulation is the following:

> Fammi di 14 tre parte continue proportionali che multiplicato ciascuna contro all'altre 2, et gli avvenimenti giunti insieme faccino 112; domando le dette quantità.[45]

In this case, it must follow that

$$a + b + c = 14$$
$$a{:}b = b{:}c$$
$$a(b + c) + b(a + c) + c(b + d) = 112$$

Galigai first obtained b using the identity:

(*) $2(a + b + c)b = a(b + c) + b(a + c) + c(a + b)$ which is valid if $b^2 = ac$, that is if the three quantities are, as in this case, in continued proportion.

Thus, he found $b = \frac{112}{2\times 14} = 4$.

Then, he determined $a = 2$ and $c = 8$ from the conditions $a + c = 10, ac = 16$, like solutions of the equation $x^2 - 10x + 16 = 0$.

Despite with different numerical data, this problem appeared various times in the abacus treatises from the fourteenth century. Antonio Mazzinghi solved it like Galigai in his *Trattato di fioretti* (c. 1370),[46] work of which Benedetto da Firenze presented a wide selection in his *Praticha d'arismetrica.* Luca Pacioli did the same in his *Summa*, where the relation (*) constitutes its fourteenth "chiave".[47] Other authors solved the problem only with algebra, like Maestro Giovanni di Bartolo in his "miracholose ragioni" reported in the codex Palat. 573.[48]

The third question posed by Maestro Agnolo is the following

[43] Leonardo Pisano 1857, 1862, I, 181; Giusti 2002, 91.

[44] Cf. Pacioli 1996, 384–385 and 393. On the problems of chapter "Divisioni e partimenti di numeri" cf. Heeffer 2010.

[45] "Divide 14 in three continuously proportional parts so that, by multiplying each of them by the sum of the other two, the summed up products give 112; I am asking said quantities". In the second formulation the numbers 14 and 112 are replaced by 20 and 160 respectively.

[46] Mazzinghi 1967, 15–16; Franci 1988, 245.

[47] *Summa*, I, 89v and 91r.

[48] BNF, Palat. 573, 477r-477v; Arrighi 1967b, 22–23.

> Truova 4 quantità continue proportionali che la somma della prima e quarta sia 18, et la somma della seconda e terza sia 12; domando quanto sarà ciascuna per sé solo.[49]

Thus, it must be

$$a{:}b = b{:}c = c{:}d$$
$$a + d = 18$$
$$b + c = 12$$

Here Galigai used the identity

$$bc = \frac{(b+c)^3}{3(b+c)+(a+d)} \text{ valid because } ac = b^2 \text{ and } bd = c^2$$

He then obtained b and c as solutions of the second-degree equation

$$x^2 - (b+c)x + \frac{(b+c)^3}{3(b+c)+(a+d)} = 0$$

with the formula

$$(^{**})x = \frac{b+c}{2} \pm \sqrt{\left(\frac{b+c}{2}\right)^2 - \frac{(b+c)^3}{3(b+c)+(a+d)}}$$

In the specific case:

$bc = \frac{12^3}{3\times12+18} = 32$, the previous equation is $x^2 - 12x + 32 = 0$; hence, $b = 4, c = 8$, wherefrom $a = 2, d = 16$.

Also in this case, the problem is solved in the same way in earlier abacus treatises. We find it with different numerical values in the codex Val. Lat. 4826 (c. 1450) that contains Iacopo da Firenze's *Tractatus algorismi* written in 1307, and in the work *Una raccolta di ragioni* by Filippo Calandri of the Sienese codex L.VI.45 (c. 1495).[50] With the same data, it is included in Pacioli's *Summa*, where (**) constitutes the eleventh "regola" for calculating the second and third of four quantities in continued proportion.[51] Later Maestro Luca proposed another solution to the problem with different data and using a different property of the numbers in proportion.[52] Another solution is described by an anonymous author in the codex Magl. XI.120 (c. 1400) of the Biblioteca Nazionale of Florence.[53]

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49 "Find 4 continuously proportional quantities so that the sum of the first and fourth is 18, and the sum of the second and third is 12; I am asking how much will each be".

50 Calandri 1982, 32–33; Høyrup 2007, 116–118, 327–328.

51 *Summa*, I, 87v; Franci 1990, 25; Høyrup 2007, 117–119.

52 *Summa*, I, 96v.

53 BNF, Magl. XI.120, 5v-6r.

As we said, when Francesco Galigai introduced the questions of Maestro Agnolo del Carmine in his book, he called him "excessivo geometro". And in other texts, the same master is described as an expert theologian and rich in mathematical competence, especially geometrical, with reference to the Florentine convent of Santa Maria del Carmine.[54] Maestro Agnolo or Angelo, whose real name was Bicci, was indeed a friar of that convent. According to the cadastral data, he was born around 1456[55] from the old tanner Andrea di Antonio Catastini and his second wife Monna Andrea, daughter of Bicci di Lorenzo and sister of Neri di Bicci, renowned Florentine painters of the fifteenth century. In his interesting *Ricordanze*, an actual shop and family diary, Neri mentions his nephew many times. First, when his widowed sister, wearing rags and holding her child by the hand, was forced to seek refuge from her brother after her stepson excluded her from her dead husband's house; then, when Bicci, still very young, soon after her mother's second marriage with a notary, was sent to the friars of the convent Carmine, where he was consecrated and celebrated his first Mass in 1476. A few years later, he obtained a degree in theology, gave a lecture of logic in the Studio Fiorentino and became a provincial Padre for twenty years. He died on 7 July 1529.[56] In the inventory of his movable assets left at the convent, among his many books we find three works of astronomy, an edition of Euclide's *Elementi* and "Magister Lucas de Burgo Sancti Sepulcri geometria volgare", perhaps Pacioli's *Summa* or the transcription of only the geometrical part of the treatise.[57] While there are no doubts about Frate Angelo's mathematical interests, as of today no document enables us to establish whether Catastini can be added to the already long list of those who taught the abacus in Florence.[58]

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54 Cf. for instance Richa 1989, 10, 92; BNF, Magl. XXV.398, 95.

55 His name does not appear in the *Baptismal Records* of the Opera di Santa Maria del Fiore in Florence.

56 On Angelo Catastini and his family cf. Negri 1973, 42; Neri di Bicci 1976, ad vocem; Giovannini and Vitolo 1981, 88, 96–98; Verde 1973–2010, 6, ad vocem. ASF, Catasto 66 (year 1427), 179v; Catasto 652 (year 1447), 19r-19v; Catasto 793 (year 1458), 297r-298r.

57 ASF, Corporazioni religiose soppresse dal governo francese 113, 20, 152r-152v.

58 Ulivi 2002a, 196–198.

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