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"Continuity and change": representing mass conservation in fluid mechanics

Alex D. D. Craik

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Abstract The evolution of the equation of mass conservation in fluid mechanics is studied. Following early hydraulic approximations, and progress by Daniel and Johann Bernoulli, its first expression as a partial differential equation was achieved by d'Alembert, and soon given definitive form by Euler. Later reworkings by Lagrange, Laplace, Poisson and others advanced the subject, but all based their derivations on the conserved mass of a moving fluid particle. Later, Duhamel and Thomson gave a simpler derivation, by considering mass flow into and out of a fixed portion of space. The later propagation of these derivations in nineteenth-century British textbooks and treatises is also examined, including Maxwell's on the kinetic theory of gases.

1 Introduction

The phrase "continuity and change" has become a cliché of historical talks and articles, but is nowhere more apt than here. This paper considers the various manifestations of the so-called "continuity equation" that describes conservation of mass and the continuity of matter in fluid flows.[1] Though several have considered the choice of words unfortunate—e.g. Thomson and Tait (1867, p. 144), Batchelor (1967, p. 74)—they

[1] This is quite distinct from the so-called "continuity principle", an expression adopted in, for example, (Mach 1919, pp. 140, 490) to denote a rather vague conceptual process of gradual adaptation that may lead to new scientific insights. Mach ascribes the origin of such reasoning to Galileo. In a letter of 1739 to Leonhard Euler, Johann Bernoulli invoked an equally vague "principle of continuity" to assert that fluid flows must vary gradually, rather than abruptly: see (Truesdell 1954a, XXXII–XXXIII). It is possible that this prompted Euler's use of the same term when he derived the equation of mass conservation.

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A. D. D. Craik (✉)
School of Mathematics and Statistics, University of St Andrews,
St Andrews KY16 9SS, Scotland, UK
e-mail: addc@st-and.ac.uk

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remain in common use. Leonhard Euler emphasized that his analysis of incompressible fluids excluded motions in which particles interpenetrate one another, and those where a portion of fluid is expanded into a larger volume. For the latter, "the continuity of the particles [is] violated" and so "individual droplets would separately perform the motion." Likewise, in his study of compressible fluids, "both the continuity and the impenetrability of the fluid impose a certain limitation ...."[2] Originally, then, "continuity" did not imply mass conservation, but rather the connectedness of a mass of fluid. Only if the fluid remains connected and particles cannot penetrate one another, is it possible to express mass conservation by treating the fluid as a continuum.

Much, of varying style and scholarship, has been written on the historical development of fluid mechanics, from earliest up to quite recent times: for instance Dugas (1955), Truesdell (1954a, 1955, 1968, 1984), Rouse and Ince (1963), Tokaty (1971), Garbrecht (1987), Maffioli (1994), Anderson (1997), Darrigol (2002, 2005), Eckert (2006), Simón Calero (2008), Darrigol and Frisch (2008), Grimberg et al. (2008). The excellent contributions of Truesdell, Rouse and Ince, Darrigol, and Darrigol and Frisch are particularly relevant, as they discuss at some length the historical development of the governing equations of motion: *Euler's equations* for "ideal" non-viscous flow, and the *Navier–Stokes equations* for viscous flow. These comprise three scalar momentum equations and the continuity equation; also, for compressible fluids, an equation of state that connects the fluid density to the pressure.[3] Though Rouse and Ince give a good account of early work on hydraulics, their treatment of more technical mathematical developments is less full. Truesdell and Darrigol perceptively discuss the development of the concepts of pressure and stress that are essential to the extension of "Newton's laws" to fluid flows. There, the acceleration of each fluid particle must equal the net force per unit mass exerted upon its surface by external pressure or stresses, together with the net force acting on its volume due to body forces. The effect of the surface forces is expressed by the gradients of pressure or stress at each point within the fluid.

For convenience, we here state the Euler equations of inviscid flow in vector form, the first being the momentum equation and the second the continuity equation that is the subject of the present paper:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{F}, \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.$$

Here, $\mathbf{v}$ denotes the velocity vector, p the pressure, ρ the density, and $\mathbf{F}$ any external body force per unit mass.

It is too often assumed by historians of science that the equations of fluid flow are little more than a routine extension of Newton's laws of particle dynamics. However, appreciation of the role of pressure (or, for viscous flows, stress) and of the importance of the "continuity equation" for long proved elusive. In particular, the continuity

[2] Abstract of Euler (1761), as translated by W. Pauls; and introduction to Euler (1757a), as translated by E. Burton and U. Frisch.

[3] Some authors reserve the name "Euler equation" or "Navier–Stokes equation" for the momentum equation (in vector form), distinguishing it from the purely kinematic "continuity equation."

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equation that expresses mass conservation provides a kinematic constraint that is an essential adjunct to "Newton's laws" when applied to a continuum, but which is absent from the individual equations that govern discrete particles as envisaged by Newton, as well as from hydrostatics.

Here, we trace the development of the expression of mass conservation in fluid mechanics, from early hydraulics through the seminal works of Daniel and Johann Bernoulli, Jean d'Alembert, Leonhard Euler, and Joseph-Louis Lagrange. The best historical accounts to date are Truesdell (1954a), Darrigol (2005) and Darrigol and Frisch (2008);[4] but even they discuss only briefly some derivations of the continuity equation, while highlighting others.[5] By placing the continuity equation centre stage in this paper, rather than as an adjunct to the momentum equations, we aim to give a full and balanced view of its various manifestations, and their relationships to one another. We also examine the later reception and reworkings by nineteenth-century French and British writers. These include a simplified derivation espoused by William Thomson and Jean Duhamel (a shift overlooked by historians, though that most familiar to today's fluid dynamicists), and James Clerk Maxwell's reformulation within the kinetic theory of gases. But we begin with some modern versions that still reflect the alternative modes of derivation used in the past.

2 Some modern accounts

We recapitulate the accounts in three well-known mid-twentieth century textbooks: Landau and Lifshitz (1959), Batchelor (1967) and Dryden et al. (1956), which of course derive from earlier sources.

Landau and Lifshitz (pp. 1–2) begin by postulating a fixed volume V_0 of space, containing a mass of fluid $\int \rho dV$, where ρ is the fluid density, $\mathbf{v}$ is the fluid velocity, t is time, and integration is over V_0:

> The mass of fluid flowing in unit time through an element **df** of the surface bounding this volume is $\rho\mathbf{v}.\mathbf{df}$; the magnitude of the vector **df** is equal to the area of the surface element, and its direction is along the [outward] normal... The total mass of fluid flowing out of the volume V_0 in unit time is therefore
>
> $$\oint \rho\mathbf{v}.\mathbf{df},$$
>
> where the integration is taken over the whole of the closed surface surrounding the volume in question.

[4] The description in Simón Calero (2008) is spoiled by many errors and misprints.

[5] For instance, Darrigol (2005) and Darrigol and Frisch (2008) describe a rather opaque and restricted version in spherical polar coordinates found in Alembert (1747), that was missed by Truesdell; but their discussion in modernized notation of later derivations by d'Alembert and Euler is less full than is given here.

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Next, the decrease per unit time in the mass of fluid in the volume V_0 can be written

$$-\frac{\partial}{\partial t}\int \rho \mathrm{d}V.$$

Equating these two expressions and transforming the surface integral to a volume integral "by Green's formula" yields

$$\int \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v})\right] \mathrm{d}V = 0.$$

(Here, differentiation under the integral sign is permitted because the volume V_0, and so each element $\mathrm{d}V$, is fixed in space.) "Since this equation must hold for any volume, the integrand must vanish, i.e.

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0. \tag{2.1}$$

This is the *equation of continuity*."

The account in Batchelor (1967, pp. 73–75) is similar, and both authors observe that the equation can be rewritten as

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0, \qquad \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}.\nabla$$

where the operator *D/Dt* denotes the *convective time derivative*, representing the time rate of change following fluid elements. Batchelor further notes that, following the changing volume $\tau(t)$ of a given mass of fluid, the above results imply that

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \int \mathbf{v}.\mathrm{d}\mathbf{S} = \int \nabla \cdot \mathbf{v} \mathrm{d}\tau$$

where $\mathrm{d}\mathbf{S}$ denotes the surface element $\mathrm{d}S$ of the moving boundary with the direction of the outward normal. In the case of an incompressible fluid with constant density ρ, one has both

$$\frac{D\rho}{Dt} = 0 \quad \text{and} \quad \nabla.\mathbf{v} = 0.$$

We shall see below that consideration of a *moving* element of fluid, rather than a fixed portion of space through which fluid flows, was key to most early derivations of the continuity equation. The latter viewpoint, though mathematically simpler, developed much later.

The derivation in Dryden et al. (1956, pp. 33–35), in a chapter by Murnaghan, is very different, using from the outset what is now usually referred to as the "Lagrangian formulation".[6] In this, individual fluid particles initially situated at points (a, b, c) in rectangular Cartesian coordinates are subsequently located at later times t at

[6] Some time before Lagrange, Euler developed such a formulation: see Sect. 7 below.

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$$x = x(a, b, c, t); \;\; y = y(a, b, c, t); \;\; z = z(a, b, c, t);$$

and their velocities are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u(x, y, z, t), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = v(x, y, z, t), \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = w(x, y, z, t).$$

On differentiating the first of these velocity components with respect to t, using the chain rule, the x-component of acceleration of fluid particles is found to be

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},$$

with similar expressions for the y and z components. In vector notation, these are just

$$\frac{D\mathbf{v}}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{v}.\nabla\right)\mathbf{v}$$

in which the convective time-derivative again appears.

Fluid initially occupying an infinitesimal rectangular volume element $\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c$ will, at later times t, occupy an infinitesimal volume $\mathrm{d}\tau$ (not usually strictly rectangular) given by the product of the initial volume and a Jacobian determinant:

$$\mathrm{d}\tau = \frac{\partial(x, y, z)}{\partial(a, b, c)}\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c.$$

Since the total mass of a portion of fluid remains constant, it follows that

$$\int \rho \mathrm{d}\tau = \int \rho\frac{\partial(x, y, z)}{\partial(a, b, c)}\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c = \int \rho_0 \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c$$

where ρ_0 denotes the initial density distribution; and, since the initial volume is arbitrary, one must have

$$\rho\frac{\partial(x, y, z)}{\partial(a, b, c)} = \rho_0. \tag{2.2}$$

This expresses the conservation of mass for each portion of fluid.

To recover the continuity equation in the form given above, set $\mathrm{d}\rho_0/\mathrm{d}t = 0$, since ρ_0 is a function of a, b, c only. This gives

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}\frac{\partial(x, y, z)}{\partial(a, b, c)} + \rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(x, y, z)}{\partial(a, b, c)}\right) = 0,$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(x, y, z)}{\partial(a, b, c)}\right) = \frac{\partial(u, y, z)}{\partial(a, b, c)} + \frac{\partial(x, v, z)}{\partial(a, b, c)} + \frac{\partial(x, y, w)}{\partial(a, b, c)}.$$

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On substituting

$$\frac{\partial u}{\partial a} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial a}, \qquad \text{etc.,}$$

the first Jacobian on the right-hand side reduces to

$$\frac{\partial(u, y, z)}{\partial(a, b, c)} = \frac{\partial u}{\partial x} \cdot \frac{\partial(x, y, z)}{\partial(a, b, c)},$$

and the others reduce similarly. It follows that

$$\left\{\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\}\frac{\partial(x, y, z)}{\partial(a, b, c)} = 0.$$

Cancelling the Jacobian yields the continuity equation, bearing in mind that $\mathrm{d}/\mathrm{d}t$ denotes the total (convective) time derivative. The result may be re-expressed, as before, in the form

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0.$$

Clearly, this "Lagrangian" derivation is more technical and less physically-intuitive than the former "Eulerian" one.

For future reference, we also mention a simplified version of the continuity equation much used in engineering: that, for steady incompressible flow through a curved tube (or a "stream tube" consisting of streamlines within a flow), the volumetric rate of inflow must equal that of outflow. This is just

$$\int_{S_1} \mathbf{v} \cdot \mathrm{d}\mathbf{S} + \int_{S_2} \mathbf{v} \cdot \mathrm{d}\mathbf{S} = 0 \tag{2.3}$$

where S_1, S_2 denote the cross-sections of the tube at inflow and outflow, and the vector $\mathrm{d}\mathbf{S}$ denotes the element of surface area with direction along the outward normals of S_1 and S_2.

In what follows, we retain original notations as far as possible, to avoid anachronisms. In particular, the derivations all predate vector calculus and usually employ Cartesian coordinates. This poses a dilemma for historical writers: in Darrigol (2002, 2005), modern vector and tensor notations are freely used for conciseness and ease of understanding; but Truesdell (1954a, 1955) mostly avoids these, while introducing some modernizations of the notation of analysis. It can be argued that *any* changes of notation may lead to a distorted view of the original and sometimes conceal real difficulties that would have been "obvious" with better notation. For instance, d'Alembert typically used poor notation; and retaining this is essential for a true appreciation his work, good and bad. In contrast, Euler's notation is usually limpidly clear; but streamlining it, as done in Truesdell (1954a) by using later Jacobian notation, can

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impart a greater impression of generality than is warranted by the original.[7] Similarly, Lagrange's sometimes cavalier proliferation and manipulation of differential symbols would be concealed by attempts at modernization. Even partial and ordinary differentiation were not distinguished by the now standard use of $\partial/\partial x$ and $\mathrm{d}/\mathrm{d}x$ until quite late in the nineteenth century.

3 Hydraulic beginnings

Archimedes (c.287–212 BCE) is rightly credited with the first theoretical advances in hydrostatics in his treatise *On Floating Bodies*. But sound progress in understanding fluid motion—as distinct from incorrect philosophical speculations—developed only much later. Ingenious experiments involving hydraulics and pneumatics are described in the *Pneumatics* of Heron of Alexandria (fl. c.150 BCE). Also, Heron's discussion of the flow from a spring in his *Dioptra* contains, according to Rouse and Ince (1963, p. 21): "the earliest known expression of the relationship between cross-sectional area, velocity, volume, and time." In contrast, later Roman engineers, most notably Vitruvius and Frontinus, despite their excellence in constructing water-supply systems employing both pipes and open aqueducts, appear to have understood the underlying principles of hydraulics less well than did Heron. The measure of the water delivered to a consumer was then the cross-sectional area of the delivery pipe, rather than the actual flow rate through it.[8] Not until the Renaissance in Europe did clear improvements occur. Probably the earliest surviving accounts of "continuity" of flow in rivers (and, of course, in other open channels) are those in the manuscripts of Leonardo da Vinci (1452–1519), for instance (Rouse and Ince 1963, p. 49):

> A river in each part of its length in an equal time gives passage to an equal quantity of water, whatever the width, the depth, the slope, the roughness, the tortuosity.
> Each movement of water of equal surface width will run the swifter the smaller the depth....

(Here, the flow is taken to be steady). Though some, e.g. Macagno (1987), have claimed that Leonardo was the original discoverer of this and other results of hydraulics, others, notably (Truesdell 1968, pp. 1–83), considered that Leonardo was probably recording results already known.

Certainly, a strong Italian tradition in hydraulics developed from around this time. Its major proponents were Benedetto Castelli (c.1577–c.1644) a former pupil of Galileo, Evangelista Torricelli (1608–1647) a student of Castelli, and later Domenico Guglielmini (1655–1710) and Giovanni Poleni (1683–1761), who all wrote treatises on hydraulics.[9] Castelli's expression of the law of "continuity" for rivers is essentially that stated earlier by Leonardo, and is equivalent to (2.3):

[7] In another instance, Truesdell (1954a, p. CXXIV) claims that Lagrange was "perhaps the first author to differentiate a 3×3 determinant": but, in fact, he differentiated an algebraic expression that was only later expressed as a 3×3 determinant.

[8] See Rouse and Ince (1963, pp. 27–32) and Garbrecht (1987, pp. 23–32).

[9] See Rouse and Ince (1963), Garbrecht (1987) and Maffioli (1994).

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> Sections of the same river discharge equal quantities of water in equal times, even if the sections themselves are unequal.
>
> Given two sections of a river, the ratio of the quantity of water which passes [in a given time] the first section to that which passes the second is in proportion to the ratio of the first and second sections and to that of the first and second velocities. Given two unequal sections by which pass equal quantities of water, the sections are reciprocally proportional to the velocities.[10]

However, these writers made further advances on the important topic of efflux of liquid from the bottom of a tank or reservoir.

Also noteworthy are the hydraulic and pneumatic works of the Frenchmen Edme Mariotte (1620–1684) and Blaise Pascal (1623–1662), and the Englishmen Robert Boyle (1627–1691) and Isaac Newton (1642–1727). Newton's famous *Principia* (Newton 1687) (and later editions) contains much on fluid mechanics. Incorporating sometimes dubious empirical assumptions, his Book II treats many problems, including efflux from vessels, resistance of moving bodies, vortex motion, and gravitational oscillations of liquid in curved pipes. Truesdell's assessment (Truesdell 1954a, XII), that "Almost all of the results are original, and but few correct" is sound, if harsh: for one must admire Newton's bold attempts at mathematical modelling in contexts that still lacked sound principles.

4 Daniel and Johann Bernoulli

The next major advances were made by Daniel Bernoulli, whose *Hydrodynamica* (Bernoulli 1738) is justly famous; and by his father Johann Bernoulli, whose *Hydraulica* (Bernoulli 1742), allegedly written in 1732, was a deliberate attempt to upstage his son.[11] Though Daniel Bernoulli's *Hydrodynamica* still contained nothing on the governing differential equations of fluid flow, he successfully solved many original problems by employing the "conservation of living force" (*vis viva*). This is a principle equivalent to the conservation of mechanical energy, in which he equates the "potential ascent" and "actual descent" of all the particles of his system, subject to gravitational attraction: see e.g. Truesdell (1954a, XXIV).

Throughout, Daniel Bernoulli's treatment of mass conservation is achieved by what later became known as the "hypothesis of parallel sections." This, and the principle of *vis viva*, were earlier applied by Bernoulli to flow in canals (Bernoulli 1729). Equally applicable to flows in pipes and in open channels, this hypothesis is identical to the hydraulic approximation discussed above: it is supposed that, at any plane section perpendicular to the channel, all fluid particles flow with the same velocity perpendicular to the plane, and that this velocity is "reciprocally proportional to the magnitude of the section".[12] For flows in open channels, this approximation is restricted to steady flows,

[10] Quotation from Rouse and Ince (1963, pp. 59–60).

[11] There is an English translation of both by Carmody and Kobus (Bernoulli and Bernoulli 1968).

[12] Trans. (Truesdell 1954a, XXIV) of Bernoulli (1738, III, sect. 2).

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because the fluid depth may change in time-dependent flows; but for flows in rigid pipes, this law of continuity remains (approximately) valid also for time-dependent flows. Bernoulli exploited this fact to solve many time-dependent problems involving flow through, and oscillations within, open-ended tubes of variable cross-section. It is essential that the tube's cross-section and curvature vary sufficiently slowly for the hypothesis to yield an adequate approximation. Similar application to flow in slender stream-tubes was employed by many later writers: c.f. equation (2.3) above.

In his *Hydraulica*, Johann Bernoulli re-derived many of his son's results but obtained no new ones. Rather, the advance that he made was methodological, supplanting the global "conservation of living forces" by consideration of the forces acting upon, and the consequent acceleration of, infinitesimal slices of fluid perpendicular to the pipe or channel. Though he confused the issue by reference to a vague "eddy" (Latin *gurges*), he was the first to postulate a differential equation as furnishing the "true and genuine method"; and it was he, not Daniel, who first gave a *time-dependent* version of what is now called "Bernoulli's equation" relating pressure and flow velocity.[13] However, for mass conservation, neither father nor son improved upon the "hypothesis of parallel sections".

Meanwhile, in Scotland, Colin Maclaurin (Maclaurin 1742, sects. 537–550) attempted to reconcile the work of Daniel Bernoulli with Isaac Newton's old "cataract theory" for efflux from a hole in the bottom of a cylindrical vessel.

5 Jean le Rond d'Alembert

Like Johann Bernoulli's *Hydraulica*, Jean d'Alembert's first book on fluid mechanics (Alembert 1744) duplicated many results of Bernoulli's *Hydrodynamica*, but contained nothing new, other than recasting problems in a form consistent with his *Traité de dynamique* (Alembert 1743). In both of these works, problems of motion are reduced to equivalent statical or hydrostatical ones by invoking a fictitious "accelerative force" in place of actual acceleration, a concept that was central to d'Alembert's approach to dynamics. An admirable overview of d'Alembert's life and writings in their historical context is Hankins (1970).

Another work, *Réflexions sur la cause générale des vents* (Reflections on the general cause of winds) (Alembert 1747), won the 1746 prize competition of the Berlin Academy, but failed to make a lasting impression. In this, d'Alembert envisaged the atmosphere to be a thin layer of uniform density, and he addressed its equilibrium and motion under the influence of gravity, the attraction of Sun or Moon, and the Earth's rotation (but without the Coriolis force): in effect, he investigated atmospheric tides. This work is often mentioned as containing the first partial differential equations to be derived in mathematical physics. Among these is a version of the continuity equation in spherical polar coordinates; but this is restricted to a "shallow atmosphere", such that any change in its almost-uniform depth on the Earth's sphere is caused by gradients of the depth-averaged horizontal velocity components, and the temporal rates of change are taken as those associated with the Earth's angular velocity.

[13] See Truesdell (1954a, XXXIII).

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A careful analysis by Darrigol and Frisch (2008, pp. 1862–1863) of d'Alembert's opaque reasoning and cumbersome notation (which we forebear to reproduce) identifies his continuity equation, in modern notation, as

$$\dot{\eta} = \omega \left(\frac{\partial \eta}{\partial \theta} \cos \phi - \frac{\partial \eta}{\partial \phi} \frac{\sin \phi}{\tan \theta} \right) = - \left(\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta}{\tan \theta} + \frac{1}{\sin \theta} \frac{\partial v_\phi}{\partial \phi} \right).$$

Here, η denotes the change in local depth of the atmosphere, the dot is a time-derivative, θ and ϕ are angular coordinates of co-latitude and longitude measured relative to an Earth-axis directed towards the orbiting luminary, ω is the Earth's angular velocity, and v_θ, v_ϕ are corresponding velocity components relative to the Earth, taken as uniform over the local depth. The right-hand side is just the negative of the divergence with respect to θ and ϕ. Accordingly, this equation is close to that later derived by Laplace in his study of the tides (see below).

In 1749, d'Alembert submitted an entry for another prize competition of the Berlin Academy. This was entitled *Essai d'une nouvelle théorie de la résistance des fluides* (Essay on a new theory of the resistance of fluids), subsequently published as Alembert (1752). The Academy rejected all entries as unworthy, and urged the authors to compare their results with experiments; but the annoyed d'Alembert instead rushed into print. Leonhard Euler was almost certainly involved in the rejection. Truesdell suggests that the given reason was merely an excuse, as "the truth, much more difficult to substantiate to an author, [was] that d'Alembert's reasoning was inaccurate, tortuous, incomprehensible, and that in the illustration of his equations he had not succeeded in exhibiting a single flow" (Truesdell 1954a, LVIII).[14] Nevertheless, despite d'Alembert's frustrating shortcomings, this work contained a crucial new advance, and it is hard to believe that Euler had overlooked it.

With rather opaque arguments, and with comprehensibility further hindered by misprints and badly labelled diagrams, d'Alembert deduced the following pair of partial differential equations (Alembert 1752, sects. 43–45):

$$B' = -A - \frac{p}{z} \quad \& \quad A' = B,$$

where A, B, A', B' are defined by

$$dq = Adx + Bdz \quad \& \quad dp = A'dx + B'dz$$

as partial derivatives of q and p, respectively. Here, the velocity components of the fluid in the x and z directions are aq, ap, where a is a fixed reference velocity, the fluid being incompressible with uniform density. It appears only belatedly that x and

[14] Truesdell's frequent criticisms of d'Alembert are too extreme, in marked contrast to his near-adulation of Euler. A more charitable view of d'Alembert's style is taken by Darrigol and Frisch (2008). Nevertheless, the present writer shares some of Truesdell's reactions: d'Alembert's merits are apparent only after considerable immersion in his writings, whereas modern readers have unconsciously inherited much of Euler's analytical style.

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z denote axial and radial cylindrical polar coordinates and that d'Alembert envisages steady flow past an axisymmetric body. The first of the two equations correctly describes the equation of continuity in these coordinates. It is derived by supposing that the local velocity in any infinitesimal section varies reciprocally as the cross section, thereby applying the old hydraulic approximation to an infinitesimal portion of fluid.[15] Apart from the restricted version in spherical polars (Alembert 1747) that is mentioned above, this is the first representation of the continuity equation in differential notation. The second equation (which need not be considered here) derives from a hidden assumption and is the condition for what is now called *potential flow*.

D'Alembert proceeds to give an alternative proof of these results (Alembert 1752, sect. 48), described below, again unnecessarily imposing conditions that require the flow to be a potential one. Subsequently, he notes the equivalent results for plane flow with Cartesian coordinates x, z (Alembert 1752, sect. 73):

$$B' = -A \quad \& \quad A' = B;$$

that is,

$$\frac{\partial q}{\partial x} + \frac{\partial p}{\partial z} = 0 \quad \& \quad \frac{\partial p}{\partial x} = \frac{\partial q}{\partial z}$$

in modern notation, respectively expressing continuity and zero vorticity. Turning to "elastic fluids" in which the density δ may vary, d'Alembert later derives the continuity equation for steady axisymmetric compressible flow (Alembert 1752, sect. 116):

$$\frac{d(\delta p)}{dz} + \frac{d(\delta q)}{dx} + \frac{\delta p}{z} = 0.$$

In view of its similarity with Euler's later definitive derivations, we conclude this section by briefly describing d'Alembert's alternative proof of the continuity equation for steady axisymmetric flow (Alembert 1752, sect. 48). He envisages an infinitesimal portion of fluid initially occupying the volume of a rectangular parallelepiped. Opposite sides having corners at N, C, D, B and N', C', D', B', respectively, denote two rectangles normal to the axis of symmetry. After an infinitesimal time *dt*, the fluid particles initially at N, C, D, B move to new locations n, c, d, b; and those initially at N', C', D', B' similarly move to n', c', d', b'. Each of the six new faces are still almost rectangular. D'Alembert argues that the deviations from true rectangles due to obliquity and curvature of the sides involve infinitesimals of higher order that can be neglected. Accordingly, the new shape is also a rectangular parallelepiped at leading order.

When the axial velocity at N is *aq*, that at C is $aq + NC \cdot a(dq/dx)$, and similar reasoning applies to the other points. In this way, the new locations of every corner point after a time *dt* may be found to the appropriate order of approximation. As the volumes of the original and the new parallelepipeds must be equal because the fluid is incompressible, the calculated leading-order change must be zero. This yields his previously obtained result

[15] Euler had done similarly in the earlier (Euler and Robins 1745), but he did not there go on to derive the continuity equation: see next section.

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$$\frac{dp}{dz} + \frac{dq}{dx} + \frac{p}{z} = 0, \quad \text{or} \quad B' = -A - \frac{p}{z}.$$

D'Alembert's concept of following the motion of an initial parallelepiped of fluid is both original and sound.

Despite his frequent obscurities, d'Alembert deserves much credit for achieving these, and some other, original results, including the famous "d'Alembert paradox", that a body in a non-viscous irrotational flow experiences no net drag force.[16] D'Alembert went on to write much more on fluid mechanics in several memoirs published in his *Opuscules* (Alembert 1761–80), and, more briefly, in articles of the *Dictionnaire Encyclopaédique des Mathématiques* (Alembert 1789). The fluid dynamical memoirs in his *Opuscules* total over 300 pages of turgid meanderings, containing little that was new.[17] There are frequent references to his own previous treatises, and some to Daniel Bernoulli's *Hydrodynamica* and Johann Bernoulli's *Hydraulica*, but he conspicuously failed to learn from the clearer writings of Euler and Lagrange, and he continued to claim priority over both as the discoverer of the theory of fluid motion based on partial differential equations.[18]

Nevertheless, there was some justice in d'Alembert's claim to priority. Though not a clear expositor, he was a deep and original thinker, whose methods could have allowed more general expression of the governing equations than he achieved. The suggestion of Darrigol and Frisch (2008, p. 1865), that it "would seem appropriate to use "d'Alembert's condition" when referring to the condition of incompressibility, written as a partial differential equation" is perhaps a step too far. But they are right to emphasize that d'Alembert "was following an older tradition of mathematical physics according to which general principles, rather than general equations, were applied to specific problems."[19] Unfortunately for d'Alembert, but fortunately for mathematics, this tradition was overturned by Euler.[20]

[16] Though (Truesdell 1954a, XL) observes that Euler anticipated this result in his comments on Robins' *Gunnery* [Euler and Robins 1745], the governing equations were not known at that time. A fuller account of the paradox is in (Alembert 1761–80, 5, mém. 34). A recent authoritative discussion is (Grimberg et al. 2007).

[17] See, e.g. Truesdell (1954a, CXII-CXVIII).

[18] These memoirs in Alembert (1761–80) are at: vol. 1 (dated 1759) 4th memoir, 137–168; vol. 5 (1768) 31st, 32nd, 33rd, 34th memoirs pp. 41–138; vol. 8 (1780) 57th memoir, pp. 52–230. In claiming priority, he wrote, for example: "on peut voir ce que nous en avons dit dans les Mémoires déjà cité. On peut voir aussi les savants recherches de MM. de la Grange & Euler sur ce sujet, ... recherches fondées sur les même principes qui servent de fondement à la théorie nouvelle, générale & rigoureuse que j'ai donnée le premier du mouvement des fluides" (One can see what we have said in the Memoirs just mentioned. One can also see the scholarly researches of Messrs. de la Grange and Euler on this subject, ... researches founded upon the same principles which provide the basis for the new, general and rigorous theory that I first gave for the motion of fluids.) (Alembert 1761–80, 8: 135–136).

[19] After this paper was drafted, I was told by Olivier Darrigol of the thesis by Gérard Grimberg, *D'Alembert et les équations aux derivées partialles en hydrodynamique*, (Université Paris 7, 1998). Unfortunately, I have not been able to consult this.

[20] For more on d'Alembert's relations with other scientists, particularly his rivalry with Euler, see Hankins (1970, pp. 42–65). Hankins (pp. 63–64) also remarks on d'Alembert's style of writing, observing that Laplace considered that his mathematical works lacked clarity and that C.G.J. Jacobi later complained that "it is impossible today to choke down a single line of d'Alembert's mathematics, while most of Euler's works can be read with delight."

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6 Leonhard Euler

An interesting prelude to Euler's definitive work on fluid mechanics is his commentary on the Englishman Benjamin Robins' *New Principles of Gunnery* (Euler and Robins 1745). Without deriving the governing differential equations, he applies methods that derive from Daniel Bernoulli to calculate the force exerted on a two-dimensional solid body in uniform motion. Thereby, he anticipated the result now commonly known as "d'Alembert's paradox" that the net drag is zero: a recent account is Grimberg et al. (2008). In doing so, Euler considered flow in a thin curved "fillet" (or filament) of fluid bounded by adjacent streamlines, noting that the local velocity v along the curve and the local width δz of the fillet are such that $v\delta z$ remains constant along the fillet. This, of course, is again the well-known hydraulic approximation, but viewed in terms of local motion along streamlines within a more complex flow.

Leonhard Euler gave several derivations of the continuity equation, the two earliest long delayed in publication. His first attempt, "Recherches sur le mouvement des rivières" (Researches on the motion of rivers) (Euler 1767a), seems to have been composed in 1750 or 1751.[21] His next, the important "Principia motus fluidorum" (Principles of the motion of fluids), was written in 1752 but not published until 1761 (Euler 1761); while his "Principes généraux du mouvement des fluides" (General principles of the motion of fluids), the second of three major linked papers on fluids written during 1753–55, was published more promptly (Euler 1757a). These three papers therefore appeared in reverse order of composition.

A further derivation of the continuity equation for both incompressible and compressible fluids is contained in the second part of Euler's four-part Latin treatise on fluid mechanics (Euler 1770). This appeared during 1768–1771 in successive volumes of the proceedings of the St Petersburg Academy soon after Euler's return there in 1766.[22] The part containing the derivation of the equations of motion is entitled "Sectio secunda de principiis motus fluidorum" (Second section on the principles of the motion of fluids).

Nowhere in these papers does he mention d'Alembert's prior work. However, elsewhere, he does make a few brief remarks about d'Alembert and other contemporaries and predecessors.[23] In view of their priority dispute, a couple of these, characterized by faint praise, are worth noting.

The sequel to his "Principes généraux ...", entitled "Continuation des recherches sur la théorie du mouvement des fluids" (Euler 1757b), begins with the following paragraph (present author's translation):

> In my previous two Memoirs, having reduced all the Theory of fluids, as much their equilibrium as their motion, to two analytical equations, the consideration

[21] The dating of this and other manuscripts are those given by Enestrom and accepted by Truesdell: see e.g. Truesdell (1954a, LVIII).

[22] A German translation in 1805 by H.W. Brandes became the first separately published treatise on hydrodynamics to describe Euler's equations of motion.

[23] Just how few is evident from the remarkably brief index of names cited in vols. XII and XIII of *Euleri opera omnia* (ser. 2, XIII (1955), 375). There are just 16 references to 10 individuals, and some of these are to the editor's footnotes.

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> of these formulae may appear to be of the greatest importance, seeing that they include not only all that has already been discovered by very different, and for the most part hardly convincing methods, both about the equilibrium and motion of fluids, but also all that one can still desire in this Science. However sublime may be the researches on fluids for which we are indebted to Messrs. Bernoulli, Clairaut and d'Alembert, they flow so naturally from my two general formulae that one cannot enough admire this agreement of their profound meditations with the simplicity of the principles from which I have drawn my two equations, and to which I have been led immediately by the first principles of Mechanics.[24]

A few years later, in a Latin paper on the resistance of fluids (Euler 1763), he mentions d'Alembert's *Essai*:

> The most celebrated d'Alembert has commented on these things concerning the resistance of fluids in an extraordinary [Lat. *peculiari*] Tract: they more than usually demonstrate, rather than remove, the greatest difficulty in investigating the true resistance.[25]

(a) *"Recherches sur le mouvement des rivières"* (Euler 1767a)

The writing in 1750–51 of his "Recherches sur le mouvement des rivières" (Euler 1767a) predates d'Alembert's *Essai* by a couple of years. That d'Alembert saw this work in manuscript (though he does not identify its author) is clear from a statement at the end of the *Essai* (Alembert 1752, p. 189). He is at pains to point out that:

> I had found the principles on which my method is based by the end of 1749 … more than a year before the memoir in question fell into my hands …. It would not even be impossible that the method presented in my work was unknown to the author of the memoir of which I speak, and did not aid him in his researches on the flow of rivers." (Translation by Truesdell 1954a, LVIII retaining d'Alembert's errors of "unknown" for "known" and "did not aid" for "aided" that reverse his intended meaning).

In this paper, Euler considers steady motion in two spatial dimensions, taking x horizontally downstream and y vertically. He gives the first expression of the equations of motion in material coordinates. Particles initially situated at heights z above the bed at some chosen section of the river have coordinates at later times t denoted by x and y, where

$$dx = Pdt + Qdz \quad \text{and} \quad dy = Rdt + Sdz.$$

Equating the cross-derivatives gives

$$\frac{dP}{dz} = \frac{dQ}{dt} \quad \text{and} \quad \frac{dR}{dz} = \frac{dS}{dt}.$$

[24] It is interesting that Euler refers to his "two" equations, for continuity and momentum, though the latter are three in number along coordinate directions: a proto-vectorial viewpoint!

[25] Author's translation of *Eulerii opera omnia* ser.2, XII, p. 219.

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Choosing points O and O' initially at z and $z+dz$, with respective horizontal and vertical velocity components $P = m$, $R = -n$ at O and $P = m+dm$, $R = -n-dn$ at O', Euler envisages that, after a short time $d\tau$, O and O' respectively move to o, and o', and he shows that, at leading order, the area (he says "masse') $OO'oo'$ is equal to $mdzd\tau$.

More generally, after any time t, O is at M and O' at M', with respective coordinates x, y and $x + Qdz$, $y + Sdz$; and at time $t + d\tau$, they are at m and m' with respective coordinates

$$x + Pd\tau, \quad y + Rd\tau \qquad \text{and} \qquad x + Pd\tau + Qdz, \quad y + Rd\tau + Sdz.$$

At leading order, the area $MM'mm'$ is found to be $PSdzd\tau - QRdzd\tau$, which by conservation of mass must equal $mdzd\tau$. Accordingly, the equation of continuity is

$$PS - QR = m, \quad \text{or} \quad \frac{\partial x}{\partial t}\frac{\partial y}{\partial z} - \frac{\partial x}{\partial z}\frac{\partial y}{\partial t} = \frac{\partial(x, y)}{\partial(t, z)} = m \tag{6.1}$$

in modern notation.[26]

Euler himself says that this condition expresses the "continuity of the fluid" ("la continuité du fluide") (Euler 1767a, sect. 12)—an early occurrence of the word "continuity" to mean mass-conservation. This result is a precursor of the equation of continuity expressed in so-called "Lagrangian coordinates", in which a Jacobian of the transformation of space coordinates appears: see (2.2) above.[27] But Euler's version is here restricted to steady, incompressible, and two-dimensional flow; and time t, rather than a second material coordinate, appears in the Jacobian.[28] But nowhere in this paper does there appear the now familiar continuity equation expressed in so-called "Eulerian variables", which, as seen above, d'Alembert derived in special cases.

(b) *Principia motus fluidorum* (Euler 1761)

Euler's "Principia motus fluidorum" (Euler 1761) was written in 1752, the same year in which d'Alembert's *Essai* appeared in print and about two years after Euler had seen it in manuscript. Here is the first of Euler's derivations of what we now know as the *Euler equations* of incompressible hydrodynamics. This paper is summarized and commented on in Truesdell (1954a, LXII–LXXV).[29] There is also a brief online summary, focusing on Euler's derivation of the continuity equation, by Sandifer (2008), and another account (unfortunately marred by many misprints) in Simón Calero (2008). Truesdell observed that Euler's treatment is the first to clearly separate kinematics

[26] Carelessly, Euler here uses m in two senses, to denote both the initial horizontal velocity component and the position of a point. That in (6.1) is the horizontal velocity.

[27] Euler later returned to the equations in material coordinates in Euler (1762, 1766), around the same time as did Lagrange: see Sect. 7 below.

[28] The two-dimensional incompressible equivalent of (2.2), with material coordinates a, z, is formally obtained by setting $da = mdt$ in (6.1).

[29] There are now two full English translations: one by Enlin Pan in the online Euler Archive E258 at http://www.math.dartmouth.edu/~euler/, the other by Walter Pauls in *Physica D* 237 (2008): 1840–1854.

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from dynamics: Euler's first section concerns the continuity equation and his second the equations connecting pressure and acceleration.

Euler's derivation of the continuity equation shows similarities with d'Alembert's *Essai*, but it is much clearer and more general. He begins with two-dimensional flows, introducing the now standard notation of u, v for velocity components in the Cartesian x- and y-directions, and their partial derivatives defined by

$$du = Ldx + ldy, \quad dv = Mdx + mdy.$$

He considers a small triangular element of fluid with vertices initially at l, m, n, where $lm = dx$ and $ln = dy$. Taking (u, v) as the velocity components of the point l, it follows that those of m and n are, respectively, $(u + Ldx,\ v + Mdx)$, and $(u + ldy,\ v + mdy)$.[30] Accordingly, after an infinitesimal time *dt*, the points l, m, n are moved to the points p, q, r by respective distances parallel to the coordinate directions of $(udt,\ vdt)$, $\{(u + Ldx)dt,\ (v + Mdx)dt\}$ and $\{(u + ldy)dt,\ (v + mdy)dt\}$. Because the initial triangle *lmn* is infinitesimally small, "its sides cannot receive curvature from the motion" over the time *dt*; accordingly, *pqr* is also a triangle. The area of *lmn* is just $\frac{1}{2}dxdy$, but that of *pqr* requires calculation by considering the areas of three associated trapezia. He finds the area of *pqr* to be

$$\frac{1}{2}dxdy(1 + Ldt)(1 + mdt) - \frac{1}{2}Mldxdydt^2,$$

which must equal the area *lmn*. Therefore,

$$Ldt + mdt + Lmdt^2 - Mldt^2 = 0$$

correct to second order. Cancelling *dt* and ignoring remaining infinitesimals yields the desired condition

$$L + m = 0, \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

in modern notation.

Euler then proceeds to consider motion in three dimensions, in which the velocity at an arbitrary point λ with Cartesian coordinates x, y, z has velocity components u, v, w. The various partial derivatives of u, v, w with respect to x, y, z are defined by

$$du = Ldx + ldy + \lambda dz, \quad dv = Mdx + mdy + \mu dz, \quad dw = Ndx + ndy + \nu dz.$$

Neighbouring points μ, ν, o are chosen with respective coordinates $(x + dx,\ y,\ z)$, $(x,\ y + dy,\ z)$ and $(x,\ y,\ z + dz)$: here, Euler uses λ, μ and ν in two senses, but there is no risk of confusion. These four points define a triangular pyramid with volume

[30] Euler does not use such brackets.

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$\frac{1}{6}dxdydz$. The respective velocity components of the four points are (u, v, w), $(u + Ldx,\ v + Mdx,\ w + Ndx)$, $(u + ldy,\ v + mdy,\ w + ndy)$ and $(u + \lambda z,\ v + \mu dz,\ w + \nu dz)$.

In a short time interval dt, the points λ, μ, ν, o are carried with these velocities to the points π, ϕ, ρ, σ, the coordinates of which are readily found. To calculate the volume of the new pyramid defined by these points, Euler first has to determine the volumes of four associated prisms, at length finding that the pyramidal volume is

$$\tfrac{1}{6}dxdydz[1 + (L + m + \nu)dt + (Lm - Ml + L\nu + m\nu - n\mu - N\lambda)dt^2$$
$$+(Lm\nu - Ml\nu - Ln\mu + Mn\lambda - Nm\lambda + Nl\mu)dt^3].$$

Here he assumes, as before, that the sides cannot become curved. As the two pyramidal volumes must be equal, it follows by ignoring higher-order differentials that

$$L + m + \nu = 0, \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

in modern notation.[31]

It might seem that Euler has here derived the continuity equation only for *steady* flows. But he goes on to observe (sect. 38) that, for unsteady flows, it is only necessary to write the differentials as

$$du = Ldx + ldy + \lambda dz + \mathsf{L}dt, \quad dv = Mdx + mdy + \mu dz + \mathsf{M}dt,$$
$$dw = Ndx + ndy + \nu dz + \mathsf{N}dt$$

and "it will still always hold that $L + m + \nu = 0$."

(c) *Principes généraux du mouvement des fluids* (Euler 1757a)

Written a few years after the *Principia motus fluidorum*, but published before it, this was the second of three classic papers written in French on the equilibrium and motion of fluids.[32] Here, Euler's notation and exposition have matured, and this paper can be read with ease and pleasure by any modern fluid-dynamicist.

For the continuity equation, the main differences are that Euler here considers fluid with variable density; and, instead of a triangular pyramid, he takes his initial element of fluid to be a rectangular parallelepiped with sides *dx, dy, dz*. Representing velocity

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[31] In his summary of this paper, Truesdell expresses the above results in suggestive but anachronistic Jacobian notation, and he observes that Euler's higher-order terms in dt^2 and dt^3 are not generally correct (Truesdell 1954a, LXIII–LXIV, LXXI–LXXII) as they omit terms involving second spatial derivatives of the velocity. In fact, such additional terms are indeed negligible compared with those retained provided the spatial infinitesimals *dx, dy, dz* are taken to be an order of magnitude smaller than the displacements *udt, vdt, wdt*: but this was surely not in Euler's mind. In any case, only the leading-order terms matter, as d'Alembert had earlier realized.

[32] An English translation by T.E. Burton, adapted by U. Frisch, is in *Physica D* **237** (2008), 1825–1839.

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components by u, v, w as above, he determines the velocities at each corner of the parallelepiped and hence their subsequent positions after an infinitesimal time *dt*. Unlike in his *Principia*, he retains only leading-order terms from the outset: this ensures that the shape of the new element is "infinitely little different from a rectangular parallelepiped"—as d'Alembert had earlier realized. With this simplification, it is easily found that the new volume is

$$dxdydz\left(1+dt\left(\frac{du}{dx}\right)+dt\left(\frac{dv}{dy}\right)+dt\left(\frac{dw}{dz}\right)\right).$$

By way of further justification, he adds: "If one still had any doubt of the justness of this conclusion, one would only have to read my Latin piece: *Principia motus fluidorum*, where I have calculated the volume without neglecting anything" (sect. 15)—however, this would not appear in print for another four years.

If the initial fluid density within the element is q, this changes over the interval *dt* to

$$q+dt\left(\frac{dq}{dt}\right)+udt\left(\frac{dq}{dx}\right)+vdt\left(\frac{dq}{dy}\right)+wdt\left(\frac{dq}{dz}\right),$$

because the leading-order displacement of the element has components *udt, vdt, wdt*. (This is just $q + dt(Dq/Dt)$, where Dq/Dt is the "convective derivative" of q following the motion.) Euler then briefly observes that, "since the density is inversely proportional to the volume, this quantity will be to q as *dxdydz* to

$$dxdydz\left(1+dt\left(\frac{du}{dx}\right)+dt\left(\frac{dv}{dy}\right)+dt\left(\frac{dw}{dz}\right)\right)\text{''}.$$

Consequently,

$$\left(\frac{dq}{dt}\right)+u\left(\frac{dq}{dx}\right)+v\left(\frac{dq}{dy}\right)+w\left(\frac{dq}{dz}\right)+q\left(\frac{du}{dx}\right)+q\left(\frac{dv}{dy}\right)+q\left(\frac{dw}{dz}\right)=0,$$

or

$$\left(\frac{dq}{dt}\right)+\left(\frac{d\cdot qu}{dx}\right)+\left(\frac{d\cdot qv}{dy}\right)+\left(\frac{d\cdot qw}{dz}\right)=0,$$

which reduces to the previous result for incompressible flow when q is constant.

This is the continuity equation in full generality in Cartesian coordinates: c.f. (2.1). Euler's account is masterly: his argument is succinct and his clear notation is virtually that adopted today.

(d) *Sectio secunda de principiis motus fluidorum* (Euler 1770)

In his later Latin treatise, Euler derived his equations of motion afresh. Perhaps feeling that the above brief derivation of the continuity equation was insufficiently rigorous,

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he surprisingly reverts to considering a pyramidal volume as in his *Principia motus fluidorum.* Though in different notation, he proceeds much as before, this time finding that the volume of his (almost) pyramidal element after time dt is

$$\tfrac{1}{6}dxdydz\left(1+dt\left(\frac{du}{dx}\right)\right)\left(1+dt\left(\frac{dv}{dy}\right)\right)$$
$$\times\left(1+dt\left(\frac{dw}{dz}\right)\right)\sqrt{(1-\lambda\lambda-\mu\mu-\nu\nu+2\lambda\mu\nu)}.$$

Here, λ, μ and ν are the cosines of the angles made by the three line elements joining the new vertices of his pyramid. As each angle is close to 90°, these cosines are first-order infinitesimals and the term under the square root may be replaced by unity with an error of the second order. Correct to first order, the volume is therefore

$$\tfrac{1}{6}dxdydz\left(1+dt\left(\frac{du}{dx}\right)+dt\left(\frac{dv}{dy}\right)+dt\left(\frac{dw}{dz}\right)\right)$$

as before.

If the initial density of the element is q and that after time dt is q', it follows that

$$\frac{q'-q}{q'dt}=\frac{q'-q}{qdt}=-\left(\frac{du}{dx}\right)-\left(\frac{dv}{dy}\right)-\left(\frac{dw}{dz}\right),$$

where

$$q'=q+udt\left(\frac{dq}{dx}\right)+vdt\left(\frac{dq}{dy}\right)+wdt\left(\frac{dq}{dz}\right)+dt\left(\frac{dq}{dt}\right),$$

since the displacements in the x, y, z directions during the time dt are udt, vdt, wdt. The final continuity equation immediately follows, as stated above.

Later in this paper, Euler gives a clear derivation of the equations of motion in material coordinates, employing a method that differs from that of Lagrange (1760–61). Denoting the instantaneous coordinates of fluid particles at times t by x, y, z, and those at the initial time $t = 0$ by X, Y, Z, he finds the continuity equation for compressible flow in the form[33]

$$\left(\frac{d\cdot Kq}{dt}\right)=0,$$

[33] This is in *Euleri opera omnia* ser.2, XIII, pp. 138, 142.

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where q is density and[34]

$$K = \left(\frac{dx}{dX}\right)\left(\frac{dy}{dY}\right)\left(\frac{dz}{dZ}\right) + \left(\frac{dz}{dX}\right)\left(\frac{dx}{dY}\right)\left(\frac{dy}{dZ}\right) + \left(\frac{dy}{dX}\right)\left(\frac{dz}{dY}\right)\left(\frac{dx}{dZ}\right)$$
$$- \left(\frac{dx}{dX}\right)\left(\frac{dz}{dY}\right)\left(\frac{dy}{dZ}\right) - \left(\frac{dz}{dX}\right)\left(\frac{dy}{dY}\right)\left(\frac{dx}{dZ}\right) - \left(\frac{dy}{dX}\right)\left(\frac{dx}{dY}\right)\left(\frac{dz}{dZ}\right).$$

(e) *Flow in tubes*

Elsewhere, Euler gave approximate forms of the continuity equation for flow in curved tubes of variable circular cross section. In Euler (1757b)[35] he derives

$$rrqv = ff\phi\omega - \int rr\mathrm{d}s\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right).$$

Here, r denotes the radius of the tube, q the density, and v the velocity of the fluid, all varying with axial distance s along the tube; f, φ, ω are the respective values of r, q, v at some chosen location; and the integral is taken along the tube between the two locations. The connection with the hydraulic approximation of Sect. 3 is clear.

In a later paper (Euler 1767b)[36] he gives the corresponding differential form,

$$\left(\frac{\mathrm{d}\cdot qrrv}{\mathrm{d}s}\right) + rr\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right) = 0.$$

The extension to flow in non-axisymmetric tubes is obvious: one only needs to replace r^2 by A/π, where A is the local cross-sectional area (compare equation (2.3)). Such versions of the continuity equation in "streamline coordinates" occur frequently in later literature and remain commonplace in engineering textbooks.

7 Joseph-Louis Lagrange

Lagrange's writings on the governing equations of fluid mechanics are threefold: (i) the concluding sections XL–LII of a long paper devoted to applying his recently-developed calculus of variations (Lagrange 1760–61); (ii) his "Mémoire sur la théorie du mouvement des fluides" (Lagrange 1781); and (iii) the final sections X-XII of the second volume of his treatise *Mécanique analytique* (Lagrange 1788/1853–55). The last is mainly an improved recapitulation of his earlier work.

His 1781 paper begins with a fulsome tribute to d'Alembert, who "has reduced to analytical equations the true laws of movement of fluids." He states that a great number of subsequent researches have appeared in the *Opuscules* of d'Alembert, and

[34] Euler's K is the same as Lagrange's θ (see Sect. 7) and both are equal to the Jacobian $\frac{\partial(x,y,z)}{\partial(X,Y,Z)}$ of the coordinate transformation, as in (2.2).

[35] At *Euleri opera omnia* ser. 2, XII, p. 101.

[36] At *Euleri opera omnia* ser. 2, XII, p. 251.

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in the journals of the Academies of Berlin and of St Petersburg, but he fails to mention Euler by name anywhere in this paper. He then proceeds to derive the equations in a manner little different from Euler's, apart from some changes of notation.

For the continuity equation, he considers an infinitesimal parallelepiped with sides of initial length $\delta x, \delta y, \delta z$, and he calculates its position and volume after a time dt, neglecting "infinitely small quantities of third order". He duly arrives at the new volume

$$\delta x \delta y \delta z \left(1 + \frac{dp}{dx}dt + \frac{dq}{dy}dt + \frac{dr}{dz}dt\right),$$

where p, q, r denote the three velocity components in Cartesians x, y, z; while the new density has changed from Δ to

$$\Delta + \frac{d\Delta}{dt}dt + \frac{d\Delta}{dx}pdt + \frac{d\Delta}{dy}qdt + \frac{d\Delta}{dz}rdt.$$

As mass is conserved, the product of these expressions must equal the original mass $\Delta\delta x\delta y\delta z$.The continuity equation results in the form

$$\frac{d(\Delta p)}{dx} + \frac{d(\Delta q)}{dy} + \frac{d(\Delta r)}{dz} + \frac{d\Delta}{dt} = 0,$$

and Lagrange notes the reduction for incompressible flows.

Though Lagrange later tries to find some new solutions of the governing equations—with very limited success—his derivation of these equations is totally unoriginal, and his lack of acknowledgment of Euler's prior work seems churlish.

Nevertheless, his earlier paper (Lagrange 1760–61) contains fresh insights. In his section XL, he expresses the equations of motion for incompressible flow as a *variational principle*; and in his section XLIV, he gives his first derivation of the equations of motion in terms of the initial locations X, Y, Z of fluid particles and the elapsed time t. In the latter, the instantaneous coordinates x, y, z of fluid particles are regarded as functions of their initial locations X, Y, Z and the elapsed time t. In sections XLVIII–L he extends the variational formulation to "elastic fluids" of variable density; and in section LI he briefly considers how these equations may be recast in terms of initial locations X, Y, Z and time t.

Despite being a pioneer of variational calculus, Euler showed no interest in the variational formulation for fluid mechanics. However, around this time, he and Lagrange corresponded about casting the equations in terms of material coordinates X, Y, Z, and time t. A letter of January 1760 from Euler, published by Lagrange (Euler 1762) and cited by him in Lagrange (1760–61), further supports Euler's priority in the use of such "Lagrangian coordinates." He had first considered these (in restricted fashion) in 1750–51 in his "Recherches sur le mouvement des rivières" (Euler 1767a), a paper that had still not appeared in print. Then, in Euler (1766, 1770), he published derivations of the equations of motion in material coordinates that differ from that of Lagrange

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(1760–61).[37] But, whoever was first, Lagrange's account was the one that became best known, and led to the misnomers of "Lagrangian" and "Eulerian" coordinates.[38]

All that is worthwhile in Lagrange's article (Lagrange 1760–61) is described again in his *Mécanique analytique* (Lagrange 1788/1853–55), with better notation and the removal of much analytical meandering. Accordingly, we discuss this later version, with references mainly to the more accessible third edition.

Lagrange's Section X of volume 2 "on the principles of hydrodynamics" (Lagrange 1788/1853–55, 2: 243–249) is a brief historical survey, mentioning the contributions of Newton, Varignon, Torricelli, Daniel and Johann Bernoulli, Maclaurin, Clairaut and, especially, d'Alembert. He praises d'Alembert's *Essai* (Alembert 1752) as having first established "the rigorous equations of the movement of fluids, whether incompressible, or compressible or elastic, equations that belong to the class of those called *of partial differences* ..." Incredibly, in his first edition, Lagrange here makes no mention of Euler. But, in later editions, the following passage was inserted:

> But these equations still did not have all the generality or simplicity to which they were susceptible. It is to Euler that one owes the first general formulae for the movement of fluids, founded on the laws of their equilibrium, and presented with the simple and luminous notation of partial differences.

He proceeds almost immediately (Sect. XI) to state his variational formulation for incompressible fluids:

$$S\left[\left(\frac{d^2x}{dt^2}+\mathrm{X}\right)\delta x+\left(\frac{d^2y}{dt^2}+\mathrm{Y}\right)\delta y+\left(\frac{d^2z}{dt^2}+\mathrm{Z}\right)\delta z\right]\mathrm{D}m+S\lambda\delta\mathrm{L}=0. \quad (7.1)$$

Here, S is an integral taken over all the mass of fluid, Dm an element of mass, X, Y, Z are now the Cartesian components of the external "accelerative force" acting on the element, and x, y, z the position coordinates of the element. (In Lagrange's definition, (X, Y, Z) are the components of the force per unit mass acting on the element along the *negative* coordinate directions. The first integral derives from the variation of the "Lagrangian" $L = T - V$, where T and V are the kinetic and potential energy, respectively.) The incompressibility condition is $\mathrm{L} = \mathrm{D}x\mathrm{D}y\mathrm{D}z$ − const., expressing the constancy of volume of any infinitesimal parallelepiped of fluid, and λ is an "undetermined quantity" (nowadays called a "Lagrange multiplier"). The quantities $\delta x, \delta y, \delta z, \delta\mathrm{L}$ denote arbitrary small variations of x, y, z, and L. This formulation resembles those used in previous sections of the treatise for rigid bodies (without the incompressibility constraint) and for hydrostatics (without the acceleration terms), and it is in accord with Lagrange's objective to provide a general theory for all of mechanics.

[37] A work with the same title as (Euler 1766) was presented to the Berlin Academy in December 1759: see the *Euler Archive Online*, E306, citing C.G.J. Jacobi, http://www.math.dartmouth.edu/~euler/ (accessed 30 May 2012). In Lagrange (1760–61), the derivation of the continuity equation in material coordinates starts from his already-established equation in the usual Cartesian coordinates, whereas Euler derives it from first principles, considering the evolution of an initial infinitesimal tetrahedron of fluid.

[38] See (Truesdell 1954a, CXXn; 1954b, 30-31n) for a full discussion of the question of priority.

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Lagrange's incompressibility condition $\mathrm{L} = \mathrm{D}x\mathrm{D}y\mathrm{D}z -$ constant, where $\Delta \mathrm{D}x\mathrm{D}y\mathrm{D}z$ is the mass $\mathrm{D}m$ of a parallelepiped with constant density Δ, has the variation

$$\delta L = \delta(\mathrm{D}x\mathrm{D}y\mathrm{D}z) = \mathrm{D}x\mathrm{D}y\mathrm{D}z\left(\frac{\mathrm{D}\delta x}{\mathrm{D}x} + \frac{\mathrm{D}\delta y}{\mathrm{D}y} + \frac{\mathrm{D}\delta z}{\mathrm{D}z}\right).$$

Substituting this into $\mathrm{S}\lambda\delta L$ and integrating by parts produces

$$S\lambda\delta\mathrm{L} = -S\left(\frac{\mathrm{D}\lambda}{\mathrm{D}x}\delta x + \frac{\mathrm{D}\lambda}{\mathrm{D}y}\delta y + \frac{\mathrm{D}\lambda}{\mathrm{D}z}\delta z\right)\mathrm{D}x\mathrm{D}y\mathrm{D}z + \text{three surface integrals},$$

the three latter integrals being supposed to vanish at the boundary of the fluid. Separating the independent variations δx, δy, δz then yields the three dynamical equations

$$\Delta\left(\frac{d^2x}{dt^2} + \mathrm{X}\right) - \frac{\mathrm{D}\lambda}{\mathrm{D}x} = 0, \qquad \Delta\left(\frac{d^2y}{dt^2} + \mathrm{Y}\right) - \frac{\mathrm{D}\lambda}{\mathrm{D}y} = 0,$$
$$\Delta\left(\frac{d^2z}{dt^2} + \mathrm{Z}\right) - \frac{\mathrm{D}\lambda}{\mathrm{D}z} = 0.$$

Next, Lagrange derives from these equations the corresponding equations of motion in material form that are satisfied by x, y, z as functions of initial position a, b, c and time t. (These need not be stated here.) Later (art. 10), he shows that the acceleration components can also be written as

$$\frac{d^2x}{dt^2} = \frac{dp}{dt} + p\frac{dp}{dx} + q\frac{dp}{dy} + r\frac{dp}{dz}, \qquad \frac{d^2y}{dt^2} = \frac{dq}{dt} + p\frac{dq}{dx} + q\frac{dq}{dy} + r\frac{dq}{dz},$$
$$\frac{d^2z}{dt^2} = \frac{dr}{dt} + p\frac{dr}{dx} + q\frac{dr}{dy} + r\frac{dr}{dz}.$$

Thereby result Euler's three momentum equations, with velocity components p, q, r and with the "undetermined quantity" $-\lambda$ identified as the pressure.

The appearance of the pressure in this way, without any consideration of its physical role, and its association with the constraint of mass conservation as an undetermined multiplier (with which it has no obvious physical connection), still seem rather mysterious: a bit of "mathematical magic" without physical interpretation. This no doubt pleased Lagrange, whose aim was to reduce all of mechanics to analysis, and who used no diagrams, and a minimum of physical discussion, in his treatise.

Returning to the incompressibility condition, replacing δx, δy, δz by *dx, dy, dz*, and dividing by *dt* yields (art. 7)

$$\frac{\mathrm{D}\cdot\frac{dx}{dt}}{\mathrm{D}x} + \frac{\mathrm{D}\cdot\frac{dy}{dt}}{\mathrm{D}y} + \frac{\mathrm{D}\cdot\frac{dz}{dt}}{\mathrm{D}z} = 0.$$

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As *dx/dt, dy/dt, dz/dt* are just the velocity components p, q, r, this is the now familiar continuity equation for incompressible fluids; but one can hardly approve Lagrange's cavalier handling of δ's, d's, and D's.

Lagrange then derives from this equation the corresponding one in material coordinates. Suffice to say that this, together with his derivation of the momentum equations in material form, is a considerable analytical achievement. After considerable manipulation (arts. 5-7), it turns out that the above equation is equivalent to $d\theta/dt = 0$, where

$$\theta = \frac{dx}{da}\frac{dy}{db}\frac{dz}{dc} - \frac{dx}{db}\frac{dy}{da}\frac{dz}{dc} + \frac{dx}{db}\frac{dy}{dc}\frac{dz}{da} - \frac{dx}{dc}\frac{dy}{db}\frac{dz}{da} + \frac{dx}{dc}\frac{dy}{da}\frac{dz}{db} - \frac{dx}{da}\frac{dy}{dc}\frac{dz}{db}.$$

Further, since $x = a$, $y = b$ and $z = c$ at time $t = 0$, $\theta = 1$.[39]

In his later section XII, he derives the corresponding equations for "compressible and elastic fluids" where Δ is the variable density. Mass conservation is expressed as

$$d.(\Delta \mathrm{D}x\mathrm{D}y\mathrm{D}z) = 0, \tag{7.2}$$

and logarithmic differentiation yields

$$\frac{d\Delta}{\Delta} + \frac{\mathrm{D}dx}{\mathrm{D}x} + \frac{\mathrm{D}dy}{\mathrm{D}y} + \frac{\mathrm{D}dz}{\mathrm{D}z} = 0.$$

Replacing *dx, dy, dz* by *pdt, qdt, rdt*, changing D into d, substituting for $d\Delta$ its "complete value"

$$\left(\frac{d\Delta}{dt} + \frac{d\Delta}{dx}p + \frac{d\Delta}{dy}q + \frac{d\Delta}{dz}r\right) dt,$$

and then dividing by *dt*, he arrives at the continuity equation (2.1),

$$\frac{d\Delta}{dt} + \frac{d\cdot(\Delta p)}{dx} + \frac{d\cdot(\Delta q)}{dy} + \frac{d\cdot(\Delta r)}{dz} = 0.$$

(Again, one is struck by Lagrange's rather informal manipulations—but he knew the answer in advance.)

He also derives the continuity and momentum equations for compressible fluid in material coordinates. The former is $\Delta\theta = H$, in place of $\theta = 1$ for incompressible fluid, where $\mathrm{H}(a, b, c)$ is the value of the density Δ at $t = 0$, and θ is as stated above: this is precisely equation (2.2).

8 Transmission and modification

Book 1 of Pierre Simon Laplace's influential *Mécanique céleste* provided an elementary treatise of mechanics, serving as introduction for his many more specialized later

[39] In modern Jacobian notation, this is $\frac{\partial(x,y,z)}{\partial(a,b,c)} = 1$. It is the incompressible version of the result derived in Euler (1770): see Sect. 6(d).

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Books that eventually filled five volumes (Laplace 1799–1825). The final chapter VIII of Book I is devoted to "The motion of fluids". Rather like Lagrange, he begins by expressing the relation between force and acceleration of a fluid particle in the variational form

$$\delta V - \frac{\delta p}{\rho} = \delta x \cdot \left(\frac{ddx}{dt^2}\right) + \delta y \cdot \left(\frac{ddy}{dt^2}\right) + \delta z \cdot \left(\frac{ddz}{dt^2}\right)$$

where external force components P, Q, R (per unit mass) satisfy $P \cdot \delta x + Q \cdot \delta y + R \cdot \delta z = \delta V$ and ρ is the density. But he has already identified p as the pressure in his previous section on hydrostatics: it is no longer an undetermined multiplier associated with the continuity condition.

Adopting material coordinates, he then derives the continuity equation as

$$\rho\beta = (\rho),$$

(equivalent to Lagrange's $\Delta\theta = \mathrm{H}$ above) where (ρ) denotes the initial density of the particle, ρ that at time t, and β the Jacobian given by θ above. He then obtains the equations of motion in now-so-called "Eulerian form," in notation close to Euler's rather than Lagrange's, with velocity components represented by u, v, w. Later, he goes on to re-express these equations of motion in spherical polar coordinates, needed for his study of water waves and tides on the Earth's surface. In the latter, he was the first to incorporate the so-called "Coriolis force" due to the Earth's rotation.

Siméon Denis Poisson derived the hydrodynamic equations in the last Book VI of his popular *Traité de mécanique* (Poisson 1811). His account and notation are almost the same as Euler's and so need no description. For the continuity equation, he considers the subsequent motion of an infinitesimal parallelepiped as described above.

It was through the work of Lagrange, Laplace, and Poisson that British scientists first began to study theoretical fluid mechanics and to cast aside old-fashioned Newtonian prejudices. Rather improbably, among the first was Thomas Young (a brilliant physicist, but reactionary mathematician) who gave an anonymous and rather eccentric reworking of Book 1 of Laplace's *Mécanique céleste* (Laplace and Young 1821).[40] He replaced large sections of Laplace's writings with his own, inserted many diagrams, and in his section "Of the motions of fluids", quoted many pages from Poisson's *Traité de mécanique* (Poisson 1811) that he considered clearer than Laplace and "reduced to more elementary principles." Other, more faithful, translations of Laplace's Book 1 were by the Cambridge-educated John Toplis (Laplace 1814) and the Dublin-educated Henry Harte (Laplace 1822). Harte also translated Poisson's *Traité de Mécanique* (Poisson 1842). In his introduction, Toplis favourably quotes the judgment of S.F. Lacroix that "the Mechanique Analytique and the Mechanique Celeste are the true sources from which a complete and methodical knowledge of all the properties of the equilibrium and of the motion of bodies either solid or fluid ... can be obtained" (Laplace 1814, Preface v (footnote)). It was no coincidence that Toplis's translation was contemporaneous with the activities of the Cambridge Analytical Society that

[40] See also Craik (2010).

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promoted the continental calculus by translating Lacroix's *Traité élémentaire* (Lacroix 1816). A freer translation and condensation of much more of Laplace's *Mécanique céleste* is Mary Somerville's *The mechanism of the heavens* (Somerville and Laplace 1831). Her Chapter VII on the motion of fluids contains derivations of the equations of motion close to those of Laplace and Poisson. A more faithful translation and copious commentary of all but the final volume of the *Mécanique céleste* was later given by the American Nathaniel Bowditch (Laplace and Bowditch 1829). In contrast with the attention to Laplace, there was no English translation of Lagrange's *Mécanique analytique* until 1997 (Lagrange 1997).

An inauspicious contribution was made by Peter Barlow in his long article 'Hydrodynamics' for the London-based *Encyclopaedia Metropolitana* (Barlow 1840). After much about hydrostatics, efflux from vessels, and practical hydraulics, he at last comes to the equations of motion (pp. 275–278). Following Poisson's "Traité de Mechanique", but choosing to re-express the analysis in outdated Newtonian 'dot' notation, Barlow derives the continuity equation as

$$\frac{\dot{d}}{\dot{t}} + \frac{d\dot{u}}{\dot{x}} + \frac{d\dot{v}}{\dot{y}} + \frac{d\dot{w}}{\dot{z}} = 0$$

where d is density and u, v, w the components of velocity. He concludes with the common view that these fluid equations are intractable: "unfortunately, their general integration cannot be obtained by any means at present known" (p. 278).

The others following all had Cambridge affiliations. George Biddell Airy's long encyclopaedia article "Tides and Waves" (Airy 1841), important in other respects (see, e.g. (Craik 2004)), follows Laplace's formulation of the continuity equation, but restricts attention to incompressible flows in two dimensions and to tidal motions on a sphere. John Henry Pratt's derivation for three-dimensional compressible flows in *The mathematical principles of mechanical philosophy* (Pratt 1836, pp. 574–577) also follows Laplace and Poisson. The same is true of the Cambridge textbooks of William Hallowes Miller (1850, 4th edn., 62–63) and Thomas Webster (1836, pp. 137–139), the latter of whom cites both Poisson and "the published papers of Professor Challis" in his preface. More antiquated was the collection of problems on hydrostatics and hydrodynamics published by William Walton (1847), that no doubt reflected much of the teaching and examining at Cambridge at that time. His hydrodynamical problems all concern efflux from and oscillations in pipes, employing the method of parallel sections: nowhere do the general equations of motion appear.

James Challis, Cambridge's Plumian Professor, wrote several papers on fluid mechanics, all of them misguided. In Challis (1851, p. 32), he follows Lagrange in deriving the continuity equation by setting to zero the variation of the total mass $\mathrm{S}(\rho \mathrm{D}x\mathrm{D}y\mathrm{D}z)$ where S denotes integration, see (7.2) above. But Challis was convinced that the established equations of motion were insufficient and that a further condition had to be applied, namely that "the direction of motion is everywhere normal to a system of continuous surfaces". An increasingly testy exchange took place with G.G. Stokes, by then Cambridge's Lucasian Professor, who pointed out Challis's error (Stokes 1851).

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9 A new derivation

It might seem that there is little more to be said about the continuity equation, but this is not so. A major methodological puzzle remains. Why were all the above derivations of the continuity equation based on a *moving fluid element*, relating its initial volume to that after a time *dt*? As we have seen, this was first done by d'Alembert, refined and generalized by Euler, and accepted and recapitulated by Lagrange, Laplace, and Poisson. It was perhaps natural that the first derivations should have focused on a moving particle of fluid, as this was necessary for expressing Newton's laws of momentum. But the latter equations require consideration of just a *single* typical moment of time, whereas the corresponding derivation of the continuity equation involves *two* moments of time, with calculation of the new position of an element of fluid after an infinitesimal time *dt*, in order to find its new volume. But this is an unnecessarily complicated way of arriving at the result: it follows more naturally from considering *a fixed portion of space* through which the fluid flows. The latter method involves just a *single* typical moment of time, like the momentum equations, and is the true "Eulerian" derivation, though unknown to Euler. This improvement was adopted by the next generation of scientists in Britain. In the remainder of this section, we outline their reception of the tradition and their reworking of it.

When the recently graduated William Thomson (later to become Baron Kelvin) took over the editor's duties of the *Cambridge Mathematical Journal*, he enlarged its scope and changed its name to the *Cambridge and Dublin Mathematical Journal.* For it, he and his friend George Gabriel Stokes together wrote six "Notes on hydrodynamics" with the aim of countering the lack of suitable textbooks. The first note was "On the equation of continuity" (Thomson 1847). Here, at last, is the simple derivation based on fluid entering and leaving a fixed volume of space. He begins:

> The following proof of the Equation of Continuity is simpler than that which is generally given in treatises on Hydrodynamics .... Thus, instead of considering a portion of the moving fluid and the varying space which the particles composing it occupy at successive instants, as in the ordinary proof, we imagine a space S fixed in the interior of the fluid, and we consider the fluid which flows into this space, across part of the bounding surface, and that which flows out of it, across the remainder in a given interval of time.

Considering a small parallelepiped with edges α, β, γ parallel to the Cartesian coordinate axes, the quantity of fluid lost in time *dt* is

$$-\frac{d\rho}{dt}dt \cdot \alpha\beta\gamma$$

where ρ is density. Also, "the excess of the quantity of fluid which leaves the parallelepiped across one of the faces $\beta\gamma$ above that which enters it across the other" is

$$\frac{d(\rho u)}{dx} \cdot \alpha\beta\gamma \cdot dt.$$

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Adding to this the corresponding quantities associated with the other two pairs of sides, it readily follows that

$$\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d(\rho w)}{dz}+\frac{d\rho}{dt}=0,$$

where u, v, w are the components of velocity. He then goes on to apply the result to an arbitrary finite space S, involving integrals over its volume.

Thomson does not claim his demonstration as original, remarking in a footnote that: "The proof in the text has been frequently given in lectures in Cambridge, and elsewhere, and it is likely to occur to anyone reading Fourier's Theory of Heat; but I am not aware that it has been hitherto published in any work except Duhamel's *Cours de Mécanique* (Deuxième Partie: Paris 1847)." Thus, he gives credit for first publication to Duhamel (1845–46) shortly before his own note appeared; and it seems that Duhamel at the École polytechnique in Paris and others in Cambridge had previously given this demonstration in lectures.[41] But who in Cambridge would have given this in lectures? It does not appear in any of the books mentioned above, and official lectures in Cambridge were in any case then haphazard and ill-attended. Another possible individual is Thomson's and Stokes' private tutor, William Hopkins; but the notes on hydrostatics and hydrodynamics copied by Stokes and Thomson from Hopkins' booklets contain nothing on Euler's equations (Stokes 1838–40; Thomson 1842–44).

What of Thomson's claim that "it is likely to occur to anyone reading Fourier's Theory of Heat" (Fourier 1822)? Fourier derived the heat conduction equation as follows, here given in the later English translation by Alexander Freeman (Fourier 1878, pp. 101–102). Considering a "prismatic molecule enclosed between six planes at right angles" having opposite corners at x, y, z and $x+dx$, $y+dy$, $z+dz$:

> The quantity of heat which during the instant dt passes into the molecule across the first rectangle $dydz$ perpendicular to x, is
>
> $$-K\,dydz\frac{dv}{dx}dt,$$
>
> and that which escapes in the same time from the molecule, through the opposite face, ... is
>
> $$-K\,dydz\left(\frac{dv}{dx}\right)dt-K\,dydzd\left(\frac{dv}{dx}\right)dt,$$
>
> the differential being taken with respect to x only.

Here, v denotes temperature and K the "conductability" of the material. Considering the other two pairs of opposite faces gives corresponding results. The net heat gained by the molecule is therefore

[41] Duhamel's demonstration is at pp. 263–266 of his second volume. It differs from that given above only by the use of dx, dy, dz in place of α, β, γ.

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$$K dy dz d\left(\frac{dv}{dx}\right) dt + K dz dx d\left(\frac{dv}{dy}\right) dt + K dx dy d\left(\frac{dv}{dz}\right) dt$$

or

$$K dx dy dz \left\{\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right\} dt,$$

and this must equal $(CDdxdydz)dv$ where D is density, C is the heat capacity, and dv the small change in temperature of the molecule in the time dt. The heat conduction equation

$$\frac{dv}{dt} = \frac{K}{CD}\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right)$$

follows.

With Thomson's versatility and capacity for seeing analogies between different areas of physics, it was no doubt evident to him that a corresponding derivation could be made of the continuity equation of fluid mechanics. But one must doubt whether it would have occurred to "anyone" reading Fourier's *Theory of Heat*. Though the matter seems simple nowadays, it had escaped d'Alembert, Euler, Lagrange, Laplace, Poisson *et al.*

10 Consolidation

Later British treatises and textbooks propagated the equations of hydrodynamics to future generations of scholars and students. We here consider only the main ones.

A full account of the continuity equation was given in Thomson and Peter Guthrie Tait's *Treatise on Natural Philosophy* (Thomson and Tait 1867, 1: 144–149) in their long preliminary section on "Kinematics". Noting that the term "equation of continuity" is "unhappily chosen"—a sentiment echoed in (Batchelor 1967, p. 74)—, they proceed to the formulation in terms of material coordinates x, y, z that depend on the initial values a, b, c and time t. Much as was done by Lagrange, they arrive at the result

$$\rho \begin{vmatrix} \frac{dx}{da}, \frac{dy}{da}, \frac{dz}{da} \\ \frac{dx}{db}, \frac{dy}{db}, \frac{dz}{db} \\ \frac{dx}{dc}, \frac{dy}{dc}, \frac{dz}{dc} \end{vmatrix} = \rho_0$$

where ρ is the density at time t and ρ_0 that at the initial time.[42]

[42] This is just equation (2.2) and Lagrange's result $\Delta\theta = H$.

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To find the corresponding form in terms of velocity components, they then choose a, b, c as the coordinates at the initial time $t - dt$ and x, y, z those at the slightly later time t. Thus,

$$x - a = \frac{dx}{dt}dt = udt, y - b = \frac{dy}{dt}dt = vdt, z - c = \frac{dz}{dt}dt = wdt,$$

where u, v, w are the velocity components at time t. Substitution into the determinant yields the value

$$1 + \left(\frac{du}{da} + \frac{dv}{db} + \frac{dw}{dc}\right)dt,$$

on ignoring all higher powers of *dt*. "The corresponding ratio of variation of density is

$$1 + \frac{D\rho}{\rho}$$

if $D\rho$ denote the differential of ρ ... as it moves from the position (a, b, c) to (x, y, z) ...". The continuity equation readily follows as

$$\frac{1}{\rho}\left(\frac{d\rho}{dt} + u\frac{d\rho}{da} + v\frac{d\rho}{db} + w\frac{d\rho}{dc}\right) + \frac{du}{da} + \frac{dv}{db} + \frac{dw}{dc} = 0.$$

Their shortcut gives a derivation far simpler, though less general, than that in Dryden et al. (1956, pp. 33–35) described in Sect. 2.

Thomson and Tait then move on to the more direct derivation, considering flow into and out of an infinitesimal parallelepiped, as done in Thomson (1847). They do not give the equivalent derivation involving calculation of the volume of a moving particle of fluid, already implicit in their previous demonstration.

William H. Besant (Besant 1859, pp. 125–127) gives a derivation similar to that of Duhamel (1845–46) and Thomson (1847). In his third edition (Besant 1877, pp. 168–171), Besant expands this account to derive the continuity equation in material coordinates, much as in Thomson and Tait (1867), explicitly citing Lagrange's *Mécanique analytique*.

Horace Lamb's treatise (Lamb 1879, pp. 5–6) first gives a rather cursory derivation of the continuity equation by considering a moving fluid particle, initially a parallelepiped, but without detailed calculation of the disturbed volume, and he uses a now confusing notation $\frac{\partial\rho}{\partial t}$ to denote the total (convective) derivative. But a few pages later (p. 9) he gives a better version, observing that if "Q denote the measure, estimated per unit volume, of any quantity connected with the properties of fluid" its rate of increase in a fixed rectangular space *dxdydz* is

$$\frac{dQ}{dt}dxdydz$$

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and "the total gain of Q due to the flow across the boundary" is

$$-\left(\frac{d \cdot Qu}{dx} + \frac{d \cdot Qv}{dy} + \frac{d \cdot Qw}{dz}\right) dxdydz.$$

If Q is the density, the continuity equation is recovered.

In the first edition of his full treatise *Hydrodynamics* (Lamb 1895, pp. 4–7), Lamb gave a more satisfactory derivation for a moving fluid particle, citing G.G. Stokes as the originator of the notation *D/Dt* to denote differentiation following the motion. He then gives "Another, and now more usual method" fixing attention on an element of space *dxdydz* as in Thomson (1847). Then, in pages 14–15, he gives the "Lagrangian equations" of motion in material coordinates, employing the Jacobian notation in the continuity equation.

The derivations of the continuity equation in (Bassett 1888, pp. 7–9) are compact, broadly similar to those of Thomson and Tait (1867), and also including its form in spherical polars (first given by Laplace). But, rather than considering the net mass flux into or out of a parallelepiped, Basset instead invokes "a particular case of Green's Theorem" (Green 1828) that

$$\iiint \left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) dxdydz = \iint (l\xi + m\eta + n\zeta) dS$$

where ξ, η, ζ are any functions of x, y, z, and l, m, n are the direction cosines of each point of the surface S drawn outwards.[43] Basset later gives a brief derivation of the equations of motion by the Principle of Least Action (Bassett 1888, pp. 32–33). In this, the requirement of mass conservation occurs as an integral constraint, multiplied by an undetermined function that turns out to equal the fluid pressure. This is as first given by Lagrange, as in (7.1) onwards, but here more carefully argued. According to Basset it is "as Mr Larmor has shown": but no reference is given, and Joseph Larmor does not give this derivation in Larmor (1884), that seemed a likely source. There seems to be no account of this variational derivation in any other nineteenth-century British textbook.

11 Extension to kinetic theory of gases

All of the derivations so far discussed employ the *continuum hypothesis* of a fluid, as is still normally assumed in fluid mechanics. However, it is far from obvious, a priori, that the continuum hypothesis is valid for liquids and gases, for they are known to be composed of discrete molecules. The random motions of molecules of gases are considerable; and those of liquids, though less so, are detectable as Brownian motion. An early attempt to incorporate random molecular motions into a theory of fluids is Section 10 of Daniel Bernoulli's *Hydrodynamica* (Bernoulli 1738). There, Bernoulli describes pressure as due to the impact of particles of air on the wall of the containing

[43] This result, usually known as the *Divergence Theorem*, is commonly attributed to C.F. Gauss: see Crowe (1985).

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vessel. This and later developments are surveyed in Truesdell (1968, VI) and Brush (1976). For present purposes, the key advance was made by James Clerk Maxwell in his paper "On the Dynamical Theory of Gases" (Maxwell 1867) following an earlier attempt in 1860, see also the clear account of this and subsequent work in Jeans (1940, Ch. IX).

In Maxwell's own words

> Molecular theories suppose that all bodies, even when they appear to our senses homogeneous, consist of a multitude of particles .... In fluids the molecules are supposed to be constantly moving into new relative positions, so that the same molecule may travel from one part of the fluid to any other part.
> If we consider an element of volume which always moves with the velocities u, v, w, we shall find that it does not always consist of the same molecules, because molecules are continually passing through its boundary. We cannot therefore treat it as a mass moving with the velocity u, v, w, as is done in hydrodynamics (Maxwell 1867, pp. 49–50, 68).

Instead, each molecule has velocity components $u+\xi,\ v+\eta,\ w+\zeta$; and the number dN of such molecules, per unit volume, having ξ, η, ζ with values lying between ξ_1 and $\xi_1 + d\xi_1$, η_1 and $\eta_1 + d\eta_1$, and ζ_1 and $\zeta_1 + d\zeta_1$, is given by

$$dN = f(\xi_1, \eta_1, \zeta_1) d\xi_1 d\eta_1 d\zeta_1,$$

where f is some distribution function. (Maxwell also allows for gases composed of molecules of two types with respective number densities N_1, N_2 and distributions f_1, f_2.)

For an equilibrium state under collisions, Maxwell argued that

$$f(\xi, \eta, \zeta) = \frac{N}{\alpha^3 \pi^{\frac{3}{2}}} e^{-\frac{\xi^2+\eta^2+\zeta^2}{\alpha^2}},$$

where α is a known constant, inversely proportional to the absolute temperature of the gas. This is now known as the *Maxwellian distribution*, and was later established with greater rigour.

Maxwell then considers a plane of unit area moving perpendicular to the x-axis with velocity u', observing that $(u+\xi-u')dN$ molecules will pass from the negative to the positive side of the plane in unit time, for each ξ. Denoting by Q "any property belonging to the molecule, such as its mass, momentum, *vis viva* &c. which it carries with it across the plane", then "the quantity of Q transferred across the plane in the positive direction in unit time is

$$\int (u-u'+\xi) Q dN, \qquad \text{or} \ldots \qquad (u-u')\overline{Q}N + \overline{\xi Q}N\text{''},$$

where $\overline{Q}N \equiv \int Q dN$ and $\overline{\xi Q}N \equiv \int \xi Q dN$. He in turn takes Q to denote (i) the mass M of each molecule, (ii) the momentum $M(u+\xi)$ in the x-direction of each molecule, and (iii) the kinetic energy of each molecule.

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He then considers the rate of variation of Q in an element of volume with sides dx, dy, dz (p. 71), finding that "by the ordinary investigation of the increase or diminution of matter in an element of volume as contained in treatises on Hydrodynamics",

$$\begin{aligned}\frac{\partial \overline{Q}N}{\partial t} &= \frac{\delta \overline{Q}}{\delta t}N - \frac{d}{dx}\left\{(u-u')\overline{Q}N + \overline{\xi Q}N\right\} \\ &\quad - \frac{d}{dy}\left\{(v-v')\overline{Q}N + \overline{\eta Q}N\right\} - \frac{d}{dz}\left\{(w-w')\overline{Q}N + \overline{\zeta Q}N\right\}.\end{aligned}$$

Here, "we employ the symbol δ to denote the variation of Q due to actions of the first kind on the individual molecules [i.e. that due to their mutual action or to external forces], and the symbol ∂ to denote the actual variation of Q in an element moving with the mean velocity of the system ...." The remaining terms express the changes due to molecules passing into and out of the element across the three pairs of parallel sides. Next, "If we perform the differentiations and then make $u' = u$, $v' = v$, $w' = w$, ... the equation becomes

$$\frac{\partial \overline{Q}N}{\partial t} + \overline{Q}N\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + \frac{d}{dx}(\overline{\xi Q}N) + \frac{d}{dy}(\overline{\eta Q}N) + \frac{d}{dz}(\overline{\zeta Q}N) = \frac{\delta Q}{\delta t}N.$$

(This last process, holding u', v', w' fixed while differentiating u, v, w, and then setting $u' = u$, $v' = v$, $w' = w$, seems decidedly dubious, though it yields correct results.)

On choosing suitable values for Q, both the continuity equation and the momentum equations may be derived. Setting $Q = M$, the constant mass of a molecule, and writing the density $MN = \rho$, it follows that

$$\frac{\partial \rho}{\partial t} + \rho\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0,$$

the usual continuity equation (2.1) (remembering that Maxwell's $\partial/\partial t$ denotes the convective derivative). Also, putting $Q = M(u + \xi)$, the x-component of momentum of a molecule, the equation yields

$$\rho\frac{\partial u}{\partial t} + \frac{d}{dx}(\rho\overline{\xi^2}) + \frac{d}{dy}(\rho\overline{\xi\eta}) + \frac{d}{dz}(\rho\overline{\xi\zeta}) = X\rho,$$

where X denotes an external force per unit volume in the x-direction. (In fact, Maxwell's equation includes another term, here ignored, as he considers a gas consisting of two species of molecules.) Corresponding equations for the y- and z-components of momentum of course follow. The terms in d/dx, d/dy and d/dz in these three equations are interpreted as gradients of normal and tangential stresses. Fluid dynamists will notice their close resemblance to the gradients of the "turbulent Reynolds stresses" first proposed by Reynolds (1895) for the equations of mean velocity in turbulent flows.

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When the distribution function is Maxwellian, these stresses reduce to a pure pressure and so yield the equations of inviscid hydrodynamics. Under other hypotheses, these stress gradients turn out to yield terms identical to those of the *Navier–Stokes equations* of viscous fluid mechanics. The latter had been formulated by Stokes (1845), and earlier versions were obtained by others under more restrictive hypotheses: see Darrigol (2002). How Maxwell, and the later workers who extended his work, achieved this correspondence is interesting and impressive; but this need not concern us here, as the continuity equation is not affected. Suffice to say that the problem is a *non-equilibrium* one, where the Maxwellian distribution is modified by the varying motion of the fluid. It was no doubt reassuring that the continuity equation is unaltered from that based on the continuum hypothesis. But the main achievement of the kinetic theory of gases was that, by providing a *molecular* explanation of the *continuum* concepts of pressure and viscosity, the full equations of fluid mechanics were validated for real gases. In similar manner, it provided an explanation of heat conduction and, when two species of molecules are present, of diffusion of concentration: see, e.g. (Jeans 1940).

12 Conclusion

The derivation of the equation of mass conservation, the so-called continuity equation, passed through several manifestations. The first hydraulic approximations led to the "doctrine of parallel sections" exploited by Daniel Bernoulli and later by Lagrange and Laplace in situations where the full equations remained intractable (notably for shallow-water waves and tides). The derivations by d'Alembert, restricted to axisymmetric and two-dimensional steady flows (but capable of generalization), prepared the way for Euler's definitive versions for unsteady flows in three (Cartesian) dimensions. Later accounts by Lagrange, Laplace, and Poisson were largely based upon that of Euler (usually without acknowledgment—just as Euler had rarely acknowledged d'Alembert); but Lagrange also developed the derivation in terms of material coordinates (previously addressed by Euler), and a variational expression of the full equations of motion employing the Principle of Least Action.

In retrospect, it is surprising that derivations of the continuity equation were so long based upon calculating the shape and volume of a fluid element at a time $t + dt$, given its shape as an infinitesimal parallelepiped or tetrahedron at time t. The first published direct derivations, considering the net flow of mass into a fixed infinitesimal volume of space, were those of Duhamel and Thomson, the latter of whom considered it "likely to occur to anyone reading Fourier's Theory of Heat." Later British treatises and textbooks incorporated the new version, while also giving variants of the older ones.

Though usually derived and used in Cartesian form, some early derivations of the continuity equation were given in other coordinate systems, namely d'Alembert's restricted versions in spherical polar and cylindrical polar coordinates, and Laplace's in spherical polar coordinates for his investigation of tides on a spherical earth. With increased proficiency in the calculus of partial derivatives, it became an easy matter to re-express the equations of motion in other coordinate systems. Also, the later

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development of quaternion, vector, and tensor notations allowed more compact formulations, as in the introductory section of this paper: see also Crowe (1985), Aris (1962), and Truesdell (1954b).

Yet the full equations of hydrodynamics and, a fortiori, those of viscous fluid dynamics long defied solution in cases of practical interest. Nearly all the exact solutions of Euler's equations obtained before 1900 were for irrotational flows, for which the troublesome nonlinear terms disappear to yield Laplace's equation. Even then, solutions so obtained were often unrealistic, on account of the neglect of viscous effects. Indeed, the troublesome d'Alembert paradox, that a finite body moving through inviscid fluid experiences no drag force, led to an understandably hostile reaction from engineers: see Darrigol (2005) and Eckert (2006) for the eventual reconciliation. The one big early success was the theory of water waves and tides, and there were advances in the theories of vortex motion and propagation of sound. Eventually, however, new mathematical methods were discovered. Conformal transformations yielded solutions for inviscid flows past streamlined bodies and accurately predicted aerodynamic lift; and the development of the boundary-layer approximation for slightly viscous flows, and the low-Reynolds-number approximation for highly viscous ones, led to crucial advances. Gradually, new areas of application, ranging from meteorology to the swimming of micro-organisms, were successfully explored. However, relatively few studies have employed the equations of motion in material coordinates: rather, particle paths have usually been reconstructed from the "Eulerian" velocities. With the advent of high-speed computing, a further revolution has occurred. Now, even the full Navier–Stokes equations hold few terrors: off-the-peg computer packages are readily available, and their results are at least sometimes believable.

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