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On Kepler's system of conics in *Astronomiae pars optica*

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Abstract This is an attempt to explain Kepler's invention of the first "non-cone-based" system of conics, and to put it into a historical perspective.

1 Introduction

Likely in 1603, in the course of writing *Ad Vitellionem paralipomena quibus astronomiae pars optica traditur* [Supplements to Witelo by which the optical part of astronomy is explained] (1604, 2000), Johannes Kepler decided that he had to undertake a study of conic sections independently of his predecessors Apollonius and Witelo,[1] to be carried out from "a mechanical, analogical and popular" point of view.

This study resulted in a short discussion titled *De coni sectionibus* [On the sections of a cone] (Ch. IV, sect. 4, pp. 92–96), in which Kepler introduced his principle of analogy and described, for the first time in history, a "non-cone-based" system of conics in which all possible types of conic sections appear.

In Greek mathematics each conic was treated separately. Archimedes considered the ellipse, parabola and hyperbola, respectively, as the section of an acute-angled,

[1] Erazmus Ciolek Witelo (c.1230, after 1275), was author of *Perspectiva*, printed for the first time only in 1535, a treatise on optics largely based on the Latin version of *Kitab-al-manazir* (The book of optics) by the Persian Ibn al-Haytham (Alhazen). It is to Witelo's *Perspectiva* to which Kepler was referring in the title of his work. Likely Kepler used Risner's *Opticae thesaurus* (1572) containing the works of Witelo and Ibn al-Haytham.

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right-angled and obtuse-angled cone by a plane orthogonal to a line-generator of the cone itself.

Apollonius presented the ellipse, parabola, and hyperbola, as the intersection of only one (double) cone by a suitably inclined plane, but continued to consider each conic in a separate plane, although he discovered that conics of the same type have analogous properties.

For instance, Apollonius proved that for any ellipse (hyperbola) two special points exist (that, as we will see below, Kepler named "foci") such that the sum (the difference) of their distances from any point of the ellipse (hyperbola) is constant. He determined those points by a process of "application" of area, which amounted to dividing the transverse axis into pairs of segments whose product equals the square of the conjugate semi-axis (Taylor 1881, p. xliv). He also showed that at any point of the ellipse (hyperbola) the tangent makes equal angles with the lines through that point and the foci. Nevertheless, Apollonius missed any focus of the parabola, because in this case the area to be "applied" and the axis to which it was to be applied are infinite (Taylor 1883, p.14). The earliest trace of the focus of the parabola can be found in the works of Pappus, see (Taylor 1883, p.15).

This view was maintained in the Renaissance, transmitted through Commandino's translations into Latin of the works of Archimedes, Apollonius and Pappus (1558, 1566, 1588). Thus the *De coni sectionibus* may be seen as the first attempt to unify the theory of conic sections: Kepler showed, by applying his principle of analogy, that one can pass "continuously" from any conic of his system into any other conic of that system. In the same section he also gave the concept of "point at infinity", by defining the second focus of the parabola, that he called "caeco foco" (blind focus).

The principle of analogy—which is an archetype of the later *principle of continuity* used by Gaspard Monge and his disciples in the nineteenth century—has been commented by several authors,[2] and it is not among our aims to discuss this topic in depth here. What is of interest for us is the system of conics Kepler introduced by the figure here reproduced in Fig. 1.

Kepler, as we will see, imagined that one focus, a, is held fixed and the other, b, is allowed to move along the axis rf, resulting in the continuous change from one conic to another.[3]

Kepler's system of conics was first described by Taylor (1900),[4] then by Hofmann (1971), Kline (1972), and by Davis (1975). Taylor reproduced the complete Latin text of *De coni sectionibus*, but only briefly commented on it, accepting without question Kepler's statements. Hofmann wrote an equation (depending on the eccentricity) which actually does not represent the conics in Kepler's system.[5] Kline (p. 299) stressed that Kepler was the first who seemed to grasp that all conic sections, even the degenerate, i.e. those consisting of a pair of lines, are continuously derivable from each other. Davis,

[2] See for instance the detailed (Buchdhal 1972), and the more recent (Knobloch 2000).

[3] A similar figure also appeared in *Supplementum ad Archimedem, de sectonibus coni*, a section of Kepler's *Nova stereometria* (Kepler 1615), where he stated again that the centre and the second focus of the parabola are at infinity.

[4] See also the notes on this topic that Taylor inserted in his (1881) and (1883).

[5] See footnote n. 16 in Sect. 3.

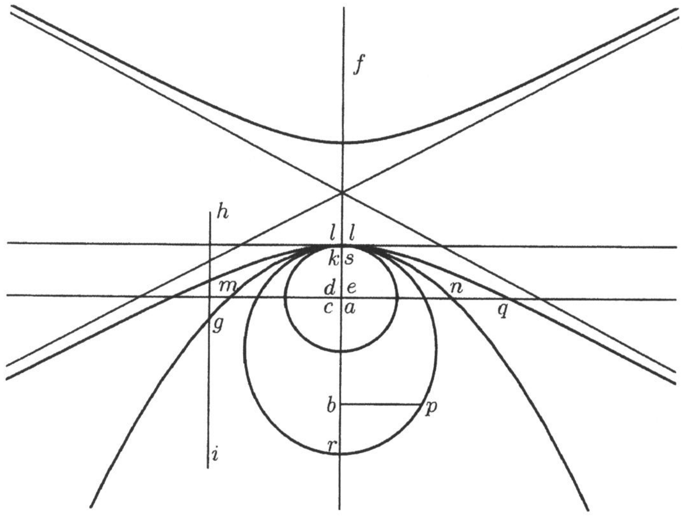


Fig. 1 Reproduction of figure at p. 94 in (Kepler 1604), with the same lettering

who based her deeper analysis of Kepler's system on the concept of eccentricity, recognized some error in his deductions. More recently, Kepler's system has been described and commented on in Field (1986, 1997) and Field and Gray (1987), which rely on Davis's analysis. Nevertheless, in none of these works is it explained how Kepler may have conceived his plane system of conics.

Here are presented two constructions of Kepler's system, for which we provide mathematical support. The first is based on the orthogonal projection of an Apollonius'-like cone-based system, as illustrated in Dürer's treatise *Underweyssung der Messung mit Zirckel und Richtsheyt* [Instruction in measurement with compass and ruler] (1525), which probably was a source of inspiration for Kepler. The second is found on the theory of shadows, and it may have been suggested to Kepler by his optical experiments.

This paper is divided into four sections, the first of which is devoted to elucidate the content of *De coni sectionibus*. In the second section, we describe the methods by which Kepler may reasonably have conceived his system of conics. In the third we give mathematical foundation to these constructions, and we provide a detailed analysis of the behaviour of conics in Kepler's system. Finally, in section four, we draw some conclusions.

2 The "De coni sectionibus"

The interpretation of Kepler's system of conics, represented in Fig. 1, cannot be attempted without analysing the content of the *De coni sectionibus*.

Astronomiae pars optica is essentially a work on optics, and Kepler's interest in conics seems to originate from burning mirrors. In fact, he introduced the subject by saying that rays from the centre of a sphere do not became parallels after reflection from its inner surface, but converge to the centre, and so, as he observed, other surfaces had to be sought which would reflect all ray from some point into parallels. He recalled that Witelo (1535, book 9 propositions 39–44),[6] had shown that the paraboloid (of revolution) was the required form. Witelo had proved—thus in part completing what was lacking in Apollonius—that the tangent at any point of the parabola makes equal angles with the parallel to the axis and the line joining that point to a suitable point of the axis (1604, p. 92, lines 1–19).

Kepler ended this introduction by writing:

> Atque id est, quod quaerebamus. Caeterùm quia difficilis est consideratio sectionum, proptera quò parum teritur, libet aliqua mecanicè analogicè et populariter de iis differere: date veniam Geometriae [And that is what we were seeking. However, because the subject consideration of [conic] sections is difficult, having been too little pursued, it is permitted to treat them somewhat mechanically, analogically and popularly. Geometers, be indulgent].

The meaning of these latter terms will be made clear in the course of this section. At the beginning of his study Kepler wrote (p. 92, lines 21–26):

> Coni varii sunt, rectanguli, acutanguli, obtusanguli: item coni recti seu regulares, et scaleni seu irregulars aut compressi: de quibus vide Apollonium et Eutocium in commentariis. Omnium promiscuè sectiones in quinque cadunt species. Etenim lines in superficie coni per sectione constituta aut est recta, aut circulus, aut Parabole aut Hyperbole aut Ellipsis [There are various kinds of cones, right-angled, acute-angled, obtuse-angled: the right cones are regular, and the scalene are irregular or compressed: for which see Apollonius and the commentaries by Eutocius. The [conic] sections of all these, regardless of kind, fall into five species. The line [curve] on the surface of the cone determined by a section is either a straight line, or a circle, or a Parabola, or a Hyperbola, or an ellipse].

So Kepler referred to Apollonius' work on conics with the commentaries by Eutocius, surely it was Commandino's translation (1566).[7]

Kepler continued by writing (p. 92, lines 26–30):

> Inter has lineas his est ordo causa proprietatis suae: et analogicè magis quàm Geometricè loquendo: quod à linea recta per hyperbolas infinitas in Parabolem, inde per ellipses infinitas in circulum est transitus [Among these lines [curves] there exists an order due to their properties: and speaking analogically rather than geometrically: from the straight line one passes through an infinity of hyperbolas to the parabola, and thence through an infinity of ellipses to the circle].

This suggests that, given a cone Γ (for instance a right one), of vertex V and base the circle C (see Fig. 2), Kepler looked at (part of) the system of conic sections Λ which is cut on the cone by the planes passing through a line L, lying in a plane (not through V) parallel to the plane of C, touching the cone at the point P. He considered

[6] Also (Risner 1572, pp. 398–402).

[7] Kepler in *Astronomia nova* (1609, p. 189) attested to have read this work.

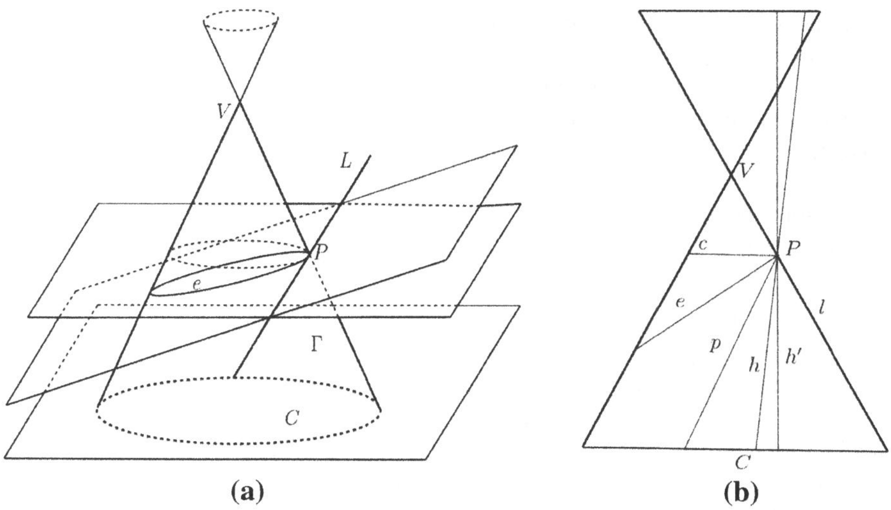


Fig. 2 **a** The cone Γ and its section e by a plane through the line L. **b** Represents the section of the cone Γ by a plane σ through its axis and orthogonal to the line L. Sections of the cone by the planes through L are identified by the traces of these planes on the plane σ labeled l, h' (vertical plane), h, p, e and c, which, respectively, correspond to the *line* (or *double-line*), *hyperbola*, *hyperbola*, *parabola*, *ellipse* and *circle*

Λ as a "continuous" system, in which each conic in it can be transformed into any other under the continuous rotation of a plane through L. It seems that Kepler was considering what in modern language is the pencil of conics cut on the cone by the planes through L.

Thus "analogy", a sort of "geometrical continuity", is the principle by which two figures, apparently dissimilar, are connected in "continuous" manner by innumerable similar intermediate forms.

Kepler went on by observing (p. 92, lines 30–32):

> Etenim omnium Hyperbolarum obtusissima est linea recta, acutissima Parabole: sic omnium Ellipsium acutissima est parabole, obtusissima Circulus [Of all hyperbolas the most obtuse is the straight line, the most acute is the parabola: likewise the most acute of all ellipses is the parabola the most obtuse is the circle],

but without saying what he meant by the terms "obtuse" and "acute" referred to the various conics.

Then, adopting a very popular style, he made some observation on "the nature" of the parabola, and of the hyperbola, that we may summarize as follows. The parabola is partly of the infinite sections and partly of the finite, being the intermediate among them, and its arms do not spread out, like that of the hyperbola, but brought near parallelism, whilst the hyperbola tends more and more to its asymptotes.

At this point Kepler went on to speak of certain noticeable points (p. 93, lines 21–29)

...quae definitionem certam habent, nomen nullum, nisi pro nominee definitionem aut proprietatem aliquam usurpes. Ab iis enim punctis rectae educatae ad contingentes sectionem, punctaque contactuum, constituunt aequales angulos iis, qui fiunt; si puncta opposita cum iisdem punctis contactuum connectatur [which have a precise definition, but no name, unless one takes for a name their definition or some property. For the straight lines drawn from these points to lines touching the section, to their points of tangency, make angles equal to those which occur if the opposite points are joined to the same points of tangency].

He called each of such "unnamed points" *focus.*[8]

Discussing the foci, Kepler introduced—without explanation—the plane system of conics he illustrated by the figure at p. 94 (see Fig. 1).

Kepler pointed out that the circle has only one focus, which coincides with its centre a; the ellipse has two b and c, equidistant from its centre, and that the more distant they are the more acute is the ellipse. He observed that in the parabola one focus d is within it, whilst the other, that he called *caeco foco* (blind focus), is imagined at infinity, and may be regarded as either without or within the parabola: "alter vel extra vel intra sectionem in axe fingendus est infinito intervallo a priore remotus" (it is to be drawn either outside or inside the section removed an infinite distance from the previous [focus]). Moreover, he noticed that:"adeò ut educta hg vel ig ex illo caeco foco in quodcunque punctum sectionis g fit axi dk parallelos", i.e. a line hg or ig from the blind focus to any point g of the parabola is parallel to the axis dk (p. 93, lines 33–35). This clearly suggests that the blind focus is located at infinite distance at both the extremities of the axis.

About the two foci of the hyperbola Kepler wrote (p. 94, lines 1–3):

In hyperbola focus externus f interno e tantò est propior quantò est Hyperbole obtusior. Et qui externus est alteri sectionum oppositarum, is alteri est internus et contra [In the hyperbola the external focus f is nearer to the internal e as the hyperbola is more obtuse. The one which is external to one of the opposite sections, is internal to the other and *vice versa*].

The meaning of the adjectives "acute" and "obtuse", attached to the various types of conic sections, seems now clear: of all ellipsis the most obtuse is the circle, being the foci coincident with the centre, whereas the most acute is the parabola, being the foci at infinite distance apart, and the hyperbola is more obtuse as its foci are closer each other. So, since the line is the most obtuse of hyperbolas, in the limiting position the distance between the foci should be the "minimum possible" The circle and the line are seen as the two extremities of the system of conics, whilst the parabola is the middle (p. 93, lines 16–17). It is also clear that the line to which Kepler refers here is no longer the line corresponding to the section l in Fig. 2b, as one has been led to think at the beginning, but the horizontal line in Fig. 1.

Summing up, it seems that Kepler conceived a plane system of conics having, in Fig. 1, a common vertex in the point $k = l = s$, a common major axis in the line fr, a

[8] This term, which means "hearth", "fire-place" and "burning-point", certainly arise from investigations on burning-mirrors. In the later *Astronomia nova*, Kepler called each of such points *puncto eccentrico*.

fixed focus in the point $a = c = d = e$, and the other free to move along *fr*. From the circle, when $b = a$, moving b down along the axis one first obtains infinite ellipses, which are more and more acute as b goes farher from a, until b has gone to infinity and the conic has become a parabola; then pushing further, b reappears, now denoted f, up on the axis *fr*, and moving it down one passes from the parabola into hyperbolas which are more and more obtuse as f approaches a.

Thus Kepler imagined his system as generated by the continuous movement of b along *fr*, which clearly reflects that of the planes around L and vice versa.

According to Taylor (1881, p. lix (3), (4)),[9] it is here that Kepler formulated the two concepts that later became the underlying principles of projective geometry:

(1) the two opposite points at extremities of a straight line can be regarded as only one point;
(2) parallel lines meet in a point at infinity.

We have to say that Kepler did not state these two concepts so explicitly. Of course, they could be logically deduced (by us) from Kepler's claims, but, to which extent was he aware of this? When Kepler said (see above), "The one (of the foci) which is external to one of the opposite sections, is internal to the other opposite, and *vice versa*", it seems that he thought the hyperbola as "not connected", although, according the uniqueness of the point at infinity of a straight line, the two branches of the hyperbola should meet at the points at infinity of its asymptotes.

Kepler continued his presentation of conics by formulating the second of his deduction obtained by analogy (p. 94 lines 4–6):

> Sequitur ergò per analogiam, ut in recta linea uterque focus (ita loquimur de recta, sine usu, tantum ad analogiam compledam) coincidat in ipsam rectam: sitque unus ut in circulo. [It follows by analogy, that for the straight line both foci (as we will use the term, in relation to the straight line, without precedent, only to complete the analogy) coincide on the straight line itself, and become one point as in the circle, and thus there is only one focus as in the circle]

Whilst Kepler was right in saying that when the focus f approaches the focus a, the hyperbola opens its arms until it flattens on the horizontal line (Fig. 1), he was wrong, as we will see in Sect. 3, in saying that at the limit f falls on the line and the two foci coincide.

Kepler, pushing forward the analogy (p. 94, lines 6–13, and p. 95, lines 1–2), pointed out:

> In circolo igitur focus in ipso centro est, longissimè recedens à circumferentia proxima, in Ellipsi iam minus recedit, et in parabola multò minus, tandem in recta focus minimum ab ipsa recedit, hoc est, in ipsam incidit. Sic itaque in terminis, Circulo et recta, coëunt foci, illic longissimè distat, hic planè incidit focus in lineam. In media Parabole infinito intervallo distant, in Ellipsi et in Hyperbole lateralibus bini actu foci, spatio dimenso distant; in Ellipsi alter etiam intra est, in Hyperbole alter extra. Undique sunt rationes oppositae [In the circle the focus is right at the centre, receding the farthest from the closest circumference, in the

9 See also (Davis 1975, p. 679).

ellipse it recedes less, and in the parabola much less, finally in the straight line the focus recedes the least distance, that is, it falls on it. Thus, in the limiting cases of the circle and the straight line, the foci come together in a point, which [in the first case] is the farthest from the circumference, whilst [in the second case] the focus is on the line. In the middle, the parabola, the foci are an infinite distance apart, in the ellipse and hyperbola, which are at the sides, the two foci are separated by a measured interval; in the ellipse the second focus is also inside, in the hyperbola outside. In every case the relations are opposite.].

Continuing his discussion on conics, Kepler called *chord* the segment cut by the conic on the line mn passing through the focus $a = c = d = e$ and orthogonal to the axis fr, i.e. that segment classically known as "latus rectum", and called the *sagitta* the segment between the focus $a = c = d = e$ and the closer vertex, i.e. anyone of the segments dk, es, and br where b is the second focus of the ellipse, Then he compared the lengths of the sagitta and of the chord for the five kind of sections.

In the circle the sagitta equals the half of the chord; in the ellipse half of the chord is longer than the sagitta, but the sagitta is longer than the fourth part of the chord; in the parabola "as Witelo had demonstrated"—recalled Kepler—the sagitta equals the fourth part of the chord. Then he added (p. 95, lines 11–22):

> In hyperbole *eq* plus est, quàm dupla ipsius *es*. fc. minor est sagitta *es* quam quarta chorda *eq* et semper minor, atque minor per omnes proportiones, donec evanescat in recta, ubi foco in ipsam lineam incumbente, altitudo foci seu sagitta evanescit, et simul chorda infinita efficit, coincidens fc. cum arcu suo, abusive sic dico, cum recta linea fit. Oportet enim nobis servire Voces Geometricas analogiae: plurimum namque amo analogias, fidelissimos meos magiistros, omnium naturae arcanorum conscios: in Geometria praecipue suspiciendos, dum infinitos casus interiectos intra sua extrema, mediumque, quantumvis absurdis locutionibus concludunt, totamque rei alicuius essentiam luculenter ponunt ob oculos [In the hyperbola *eq* is more than twice *es*, that is, the sagitta *es* is less than one fourth of the chord *eq*, and is ever less and less, for all ratios, until it vanishes in the straight line, where the focus falling in the line itself, the height of the focus, or the sagitta, vanishes, and at the same time the chord, which becomes infinite, coincides with its own arc, so to speak loosely, since it becomes a straight line. For it is proper for us that geometrical terms be of use for analogy, for I am very devoted to analogies, my most trustworthy teachers, aware of all the hidden secrets of nature: particularly admirable in Geometry, since they limit the infinite cases interposed between their extremes and mean, by however inconsistent ways of speaking, and set out the entire essence of anything clearly before the eyes].

Since the sagitta is supposed to be constant, equal to *ak*, it seems to be understood that it is the ratio sagitta/chord that goes to zero, when the hyperbola becomes more and more obtuse, until it is flattened on the line (the horizontal line in Fig. 1). But this may be in contrast with the subsequent claim "the focus falls… and the sagitta vanishes, and at the same time the chord becomes infinite".

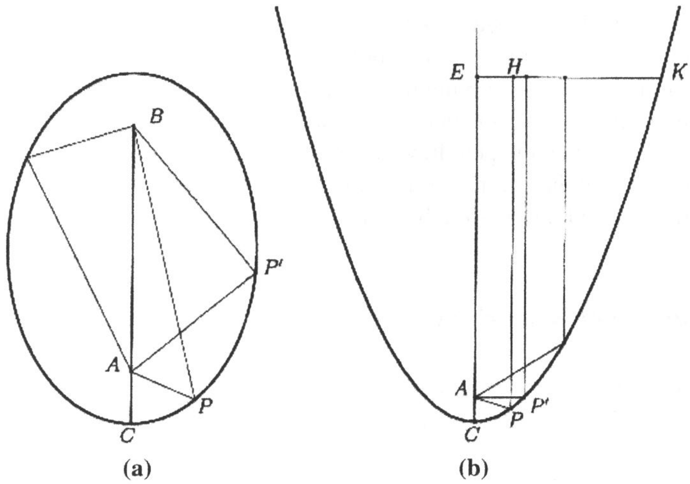


Fig. 3 **a** Construction of the ellipse: $AC + BC = AP + BP$. **b** Construction of the parabola: $EC + AC = AP + HP$

Kepler repeated the mistake about the limiting position of the focus f, and erroneously affirmed that, at the limit, "the chord coincides with its own arc". We also observe that by considering the ratio sagitta/chord, Kepler, although not directly, used the eccentricity of the conic, see formula (**) in section three where these facts will be thoroughly discussed.

From the last paragraph transcribed above it is clear that, in Kepler thought, "analogy" is the principle that leads to comprise in one definition extreme limiting forms, passing from one to another by continuous variation through an infinity of intermediate cases, and, as such, it is a tool for making new discoveries.

At the end of his discussion on conics, Kepler set out the strings–pins–pen construction of the ellipse and (a branch of) the hyperbola, as developed in Apollonius' *Conicorum*, book III, 51, 52 (Commandino 1566), and based on their definitions as those curves whose points P satisfy, respectively, the condition

$$AP + BP = AC + BC,$$
$$AP - BP = AC - BC,$$

where A and B are the foci and C is a vertex. We illustrate only the construction of the ellipse, from which, applying analogy, Kepler deduced that of the parabola.

The ends of a string, of length $AC + BC$,[10] are fixed at the foci A, B, and a pen is placed in C (see Fig. 3a). If the pen keeps the string stretched whilst it moves around the foci, it draws an ellipse.

[10] Here Kepler wrote of length "*AC* duplicata" (p. 96, line 1), i.e. $2AC$, which is inaccurate. This was already pointed out by Taylor (1900, p. 202).

For the parabola he proceeded as follows applying analogy (see p. 96, lines 8–10). He fixed a point E on the axis AC, took a string of length $EC + CA$ which pinned in A, and put the pen in A. Moving the end E of the string along the perpendicular at EK to the axis, and the pen in such a way that the floating end of the string remains parallel to the axis, then the pen draws a parabola (see Fig. 3b).

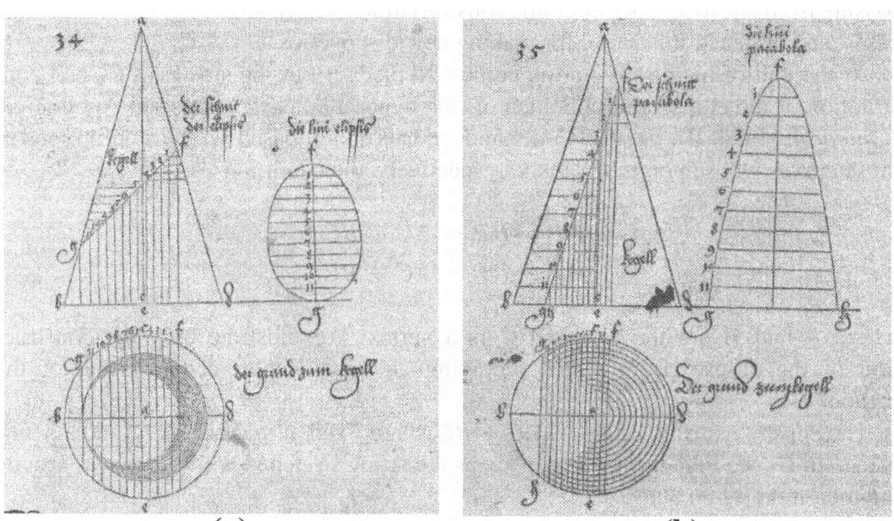
Let us remark that these constructions may give the sense to the adjective "mechanical" that Kepler used introducing his study on conics.

3 Construction of Kepler's system

It is reasonable that in order to represent the various conic sections, as they appear in Fig. 2, in a same plane, Kepler thought of an orthogonal projection into the base-plane of the cone. This hypothesis is suggested by certain figures in (Dürer 1525). In the first chapter of this book, in the course of the representation of skew curves, Dürer presented a method to draw plan and elevation of conic sections, which he illustrated by the figures 34–37 (see Figs. 4, 5).

It is clear that Dürer carried out the orthogonal projection, into the ground-plane of each type of conic section. If one projects orthogonally into the plane of C all conic sections corresponding to the letters c, e, p, h, h' and l in Fig. 2b, one gets a figure similar to the following Fig. 6, which, after a counter-clockwise rotation of 90°, is essentially the same Fig. 1 representing Kepler's system.

We do not claim for sure that Kepler was directly inspired by these figures in realizing his own system, but Dürer's treatise, which was widely spread among



(a) (b)

Fig. 4 From Dürer's (1525). **a** An elliptic section of a cone together with its orthogonal projection into the ground plane (*der Grund zum Kegel*). **b** The section of the cone by a plane parallel to a generator, so that the resulting conic section is a parabola. Also drawn is its orthogonal projection into the ground plane

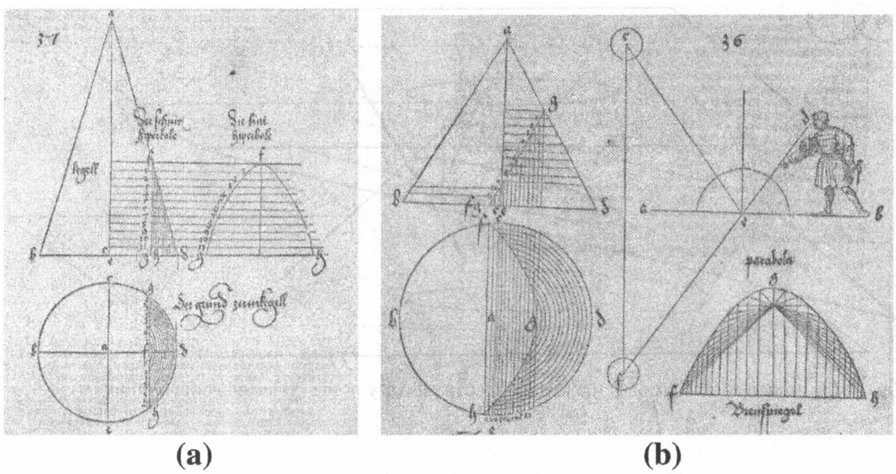
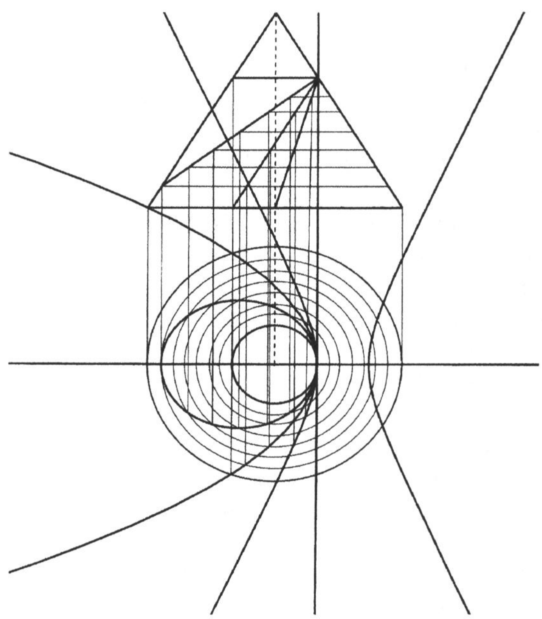


Fig. 5 From Dürer's (1525). **a** The section of the cone by a plane parallel to the axis of the cone, so that the resulting conic is a hyperbola. Also drawn is its orthogonal projection into the ground plane, in this case the line *gh*. **b** Illustration of the parabola as burning mirror (*Brennspiegel*)

Fig. 6 This figure is obtained by superimposing figures 34–37 in Dürer (1525), as sections of a single cone as in Fig. 2b. The *horizontal line* (common axis of all conics) is the projection of the line *l* in Fig. 2b, generatrix of the cone. The *vertical line* is the projection of the hyperbola corresponding to h'



German-speaking people, was certainly read by Kepler as he attested in *Harmonices mundi* (Kepler 1619).[11]

[11] The *Harmonices mundi* was planned since 1599, but not completed and published until 1619. At p. 39 of this work, Kepler explicitly referred to the wrong construction of the heptagon offered by Dürer in (1525, fig. 9), and so that he had read his treatise. We thank Aldo Brigaglia for having brought this to our attention.

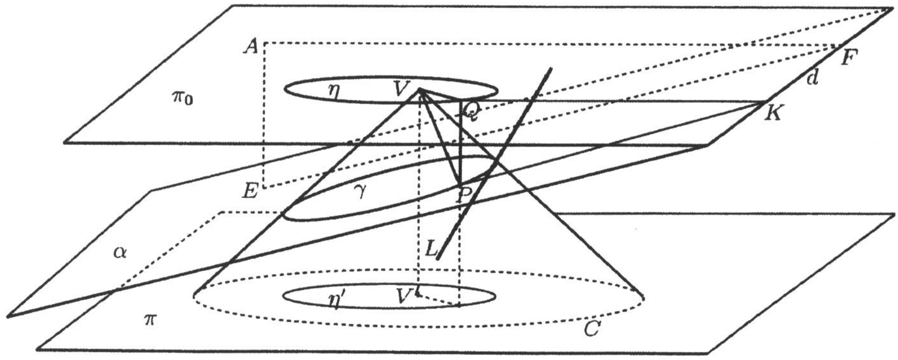


Fig. 7 Illustration for the proof of the focus-directrix property of the projected conic section

In this construction all the projected conic sections have the same horizontal axis, which also coincides with the projection of the conic section labelled l in Fig. 2b, all are tangent to the vertical line, which coincides with the projection of the hyperbola labelled h' in Fig. 2b, and, as will be shown below, all have a common focus in the point corresponding to the projection of the centre of the circle c (Fig. 2b).

If this was the route followed by Kepler to invent his plane system of conics, the question of how he realized that the projected conics all have a common focus in a, is a matter we can not answer with certainty. Kepler may have done accurate drawings, like those in Fig. 6, and taken into account two properties of the foci well known to him: (1) the tangent at any point of a conic make equal angles with the two lines through that point and each of the foci (a property that had extended to the parabola); (2) for the ellipse and the hyperbola the two foci have the same distance from the nearest vertex. Nevertheless the previous procedure can be slight modified in order to get a direct proof of the fact that all these conics share a common focus.

Suppose the cone Γ, defined in Sect. 1, is right-angled with C lying on the plane π. Let α be a plane cutting Γ along the conic γ, see Fig. 7. Then the orthogonal projection of γ into π is a conic having a focus in the projection V' of V into π. The proof rests on the focus-directrix property of conics: any conic is the locus of points P in the plane, such that the ratio between the distances from a fixed point (the focus) and a fixed line (the directrix) is constant e.[12]

Let Q be the orthogonal projection into the plane π_0, parallel to π and passing through V, of a point P on γ, and K its orthogonal projection on the line d intersection of α and π_0. For a point A in π_0, we denote, respectively, with E and F its orthogonal projections on α and on the line d.

[12] Kepler studied Euclid's *Elements* and the works of Archimedes, Apollonius and Pappus translated into Latin by Federico Commandino (Commandino 1558, 1566, 1588). Kepler quoted Commandino's works in *Astronomia nova*, p. 286, see also (Cronwell 1997, p. 140). In particular he knew the definition of conics through the focus-directrix property, as in Pappus' *Collectiones*, book 7, prop. 138. The constant e is the *eccentricity* of the conic.

We observe that the value e of the ratio AE/AF is constant for any choice of the point A on π_0. Since $PQ = VQ$, being $\angle QVP = \angle VPQ = \pi/4$, it follows that

$$e = \frac{PQ}{KQ} = \frac{QV}{KQ}.$$

Therefore, V is a focus of the conic η orthogonal projection of γ into π_0. As a consequence we have that V' is a focus of the conic η', orthogonal projection of γ into π.

This proof certainly did not exceed Kepler's mathematical ability, perhaps only his current interest.

We observe that the plane system of conics Ψ, which is obtained by projecting into π_0 the sections of the cone Γ with the planes through a line L, parallel to π and touching the cone in a point (see Fig. 7), is the same as Kepler's system.

Kepler may also have been induced to conceive his system of conics in another way. He devoted the first three chapters of *Ad Vitellionem paralopomena* to the study of light, of the functioning of the human eye, of the *camera obscura,* and of lunar and solar eclipses. Certainly he was involved in several optical experiments, and he may have considered the shadows cast on a plane by a sphere, under various lighting conditions. The determination of the shape of shadows was also a problem of interest in perspective, see for example (Del Monte 1600, pp. 273–275).[13]

Let S be a sphere of radius r on a horizontal plane α, touching it at the point F, and tangent at the point T to a line a orthogonal to α (see Fig. 8). Denote b the line intersection of α with the tangent plane to S at T. Suppose a point source of light L is put on the line a at a certain height h over α. The outline contour of the shadow produced by the sphere on the plane α, is determined by the intersection of the plane α with the (oblique) cone Δ with vertex in L and everywhere tangent to S. The cone Δ is tangent to S along the circle which is cut on S by the plane through T and orthogonal to the axis of Δ. Varying $h$the contour changes, as illustrated in Fig. 8. In particular: if the source of light is put at infinity, so that Δ is a cylinder, its intersection with α is the circle of centre F and radius r; if h is finite, but greater than $2r$, the intersection of Δ with α is an ellipse; if $h = 2r$, the intersection is a parabola; if $h < 2r$, but greater than r, the intersection is a (branch of a) hyperbola; finally, if $h = r$ the cone Δ degenerates in the tangent plane to S at T, and the limiting (branch of) hyperbola flattened on the line b.

If one consider Δ as a complete cone, i.e. with two slopes, one gets in the plane α a system of conics Θ, that, up to a rotation of 90° on the right, is similar to that Kepler introduced with Fig. 1.

[13] Francesco Maurolico in *Photismi de lumine et umbra*, completed in 1521, proved that any conic is the perspective image of a circle (1611, theorems XII, XIII). In this work Maurolico also studied problems concerning the *camera obscura*. It seems that Blaise Pascal, in the lost *Traité des coniques*, affirmed that all conics are the perspective image of a circle, see (Taton 1962, p. 237). We also observe that Isaac Newton, in section 29 "Genesis curvarum per umbra" of his short treatise *Enumeratio linearum tertii ordinis*, see (1704, p.157), wrote that all conics can be seen as shadows of a circle. It is interesting to notice that Taylor (1900, p. 217), wrote that such genesis may have been suggested to Newton by some of Kepler's *problemata observatoria* (Kepler 1604, pp. 201, 203).

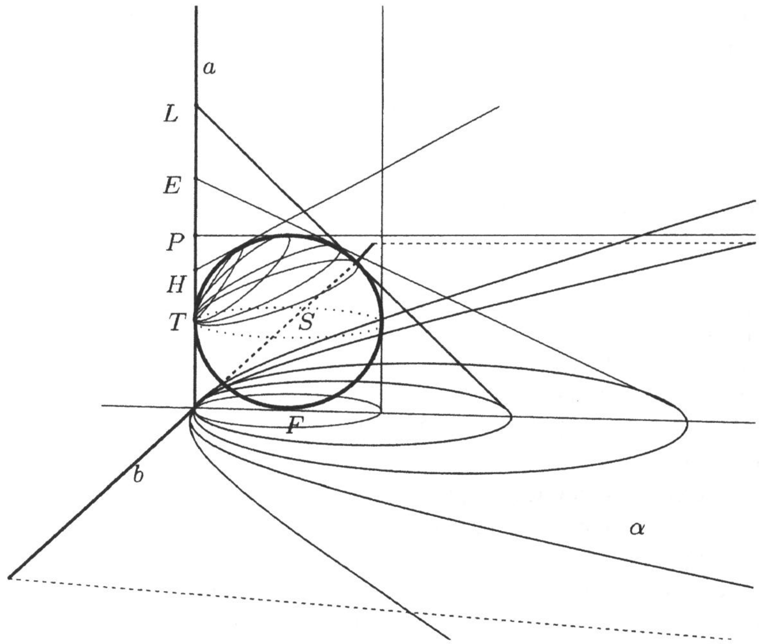


Fig. 8 In this figure the conics, intersection of various cones Δ with the plane α, are drawn in perspective. If the light source L is positioned at infinity on the line a, the sphere S produces a shadow whose contour is the circle of centre F and radius r, if it is positioned in E the contour of the shadow is an ellipse, in P is a parabola, and in H is a hyperbola

To confirm this, it is enough to show that all conics in Θ have F as a focus.

Before we do it, we like to observe that this system is no longer obtained by sectioning a single cone, but rather by sectioning different cones, whose angle at the vertex varies from 0 to π: hence we are operating more in the spirit of Archimedes than Apollonius. Moreover, it is clear that one passes (in a continuous way) from one conic to another by moving L along the line a. Allowing h to assume negative values, we get a system of conics that we still denote by Θ.

Let β be the plane of the circle along which Δ is tangent to the sphere, and denote d the intersection of β and α (see Fig. 9). To prove that F is a focus for the conic γ, it is enough to show that for any point P on γ the ratio between PF and the distance of P from d is constant.

Let be P' the intersection of the line LP with the circle of tangency, H the intersection of d with the perpendicular through P, and K the intersection with the line OF of the perpendicular through P. Moreover we denote AA' the diameter of the circle (so $A = T$ of Fig. 8), and B the intersection of the line AA' with d. We have $PH = KB$ and, since both the segments PF and PP' are tangent to the sphere S, also $PP' = PF$. Therefore

$$\frac{PF}{PH} = \frac{PP'}{KB}.$$

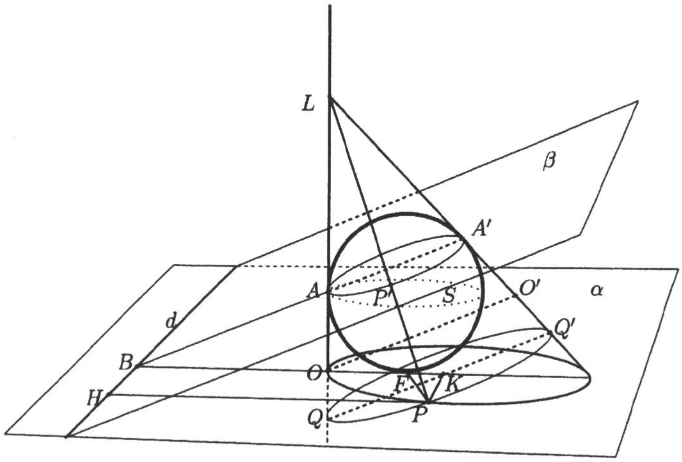


Fig. 9 Focus-directrix property for the outline contour of the shadow of a sphere

The plane passing through P and parallel to β cuts on the cone Δ a circle of diameter QQ'. We have $PP' = QA$, so by applying Thales' intercept theorem to the parallel lines BA and QQ' cut by the two orthogonal lines LQ and BK, we obtain

$$\frac{AQ}{KB} = \frac{OA}{OB},$$

and then

$$\frac{PF}{PH} = \frac{PP'}{KB} = \frac{OA}{OB},$$

Hence, since A, B and O do not depend on P, the ratio PF/PH is constant. This holds true for any position of L.

Also this proof did not exceed Kepler's mathematical ability.

4 Mathematical construction and analysis of Kepler's system

In order to explain how Kepler's system of conics really works, and to comment on the conclusions that Kepler came to concerning the limiting position of the free focus, and of the limiting value of the ratio sagitta/chord, it is convenient to use coordinate geometry.

Without loss of generality, we may consider the cone Γ with equation

$$4x^2 + 4y^2 - (z-4)^2 = 0,$$

having vertex $V = (0, 0, 4)$ and as base the circle C of equation $x^2 + y^2 = 4$ in the plane $z = 0$, and the pencil of planes

$$\mathfrak{F} : z - 2 + \lambda(2x + z - 4) = 0,$$

with axis the line L, defined by $z-2=2x+z-4=0$, tangent to Γ at the point $P=(1,0,2)$.

Let Λ be the pencil of conics cut on Γ by the planes in $\mathfrak{F}$. This pencil, under orthogonal projection into the plane of C, i.e. $(z=0)$, determines an irreducible plane system of conics, whose equation is obtained by eliminating the variable z between the equation of Γ and that of $\mathfrak{F}$:

$$\Phi : (1+\lambda)^2(x^2+y^2)-(\lambda x+1)^2=0.$$

The system Φ is obviously a quadratic system being the two-to-one projection map from Γ onto the plane. In particular this means that for a general point in the plane there pass (exactly) two conics of the system Φ. It is not hard to see that this is true for all point in the plane except the point with coordinate $(1,0)$, through which all the conics of Φ pass, and the points with coordinates $(1,y)$, with $y\neq 0$, through each of which only one passes.

All the conics in the system Φ have a vertex in the point (1,0). The other vertex is located in the point $(-1/(1+2\lambda),0)$ if $\lambda\neq -1/2$, whilst if $\lambda=-1/2$ the conic is a parabola. Moreover all the conics have a focus at the origin $a=(0,0)$, and the other at b, which, as can be easily seen by some computation, in the point

$$b=\left(\frac{2\lambda}{1+2\lambda},0\right),$$

if $\lambda\neq -1/2$, whilst if $\lambda=-1/2$ the second focus, the conic being a parabola, is at infinity.

The determinant of the general conic in Φ is equal to $-(\lambda+1)^4$, so there are only two (distinct) singular conics in the complete system $\overline{\Phi}$:[14] the one corresponding to $\lambda=-1$, i.e. the double line $(x-1)^2=0$, which has multiplicity 4, and the one corresponding to $\lambda=\infty$, i.e. the double line $y^2=0$, which has multiplicity 2. By looking for conics that are tangent to the line at infinity, we also see that in the system $\overline{\Phi}$ there are four parabolas (not necessarily distinct): for $\lambda=-1/2$, the one given by $y^2+4x-4=0$ and which we denote p; for $\lambda=\infty$, the degenerate conic given by $y^2=0$; and for $\lambda=-1$, the degenerate conic given by $(x-1)^2=0$, which we denote *dh* and has multiplicity 2.

For any $\lambda\neq -1/2$, or -1, the equation of a central conic in the system Φ can be put in the standard form:

$$\frac{\left(x-\frac{\lambda}{1+2\lambda}\right)^2}{\left(\frac{1+\lambda}{1+2\lambda}\right)^2}\pm\frac{y^2}{\left(\sqrt{\pm\left(\frac{1}{1+2\lambda}\right)}\right)^2}=1,$$

[14] This is the system obtained when one allows λ to assume the value ∞, or, which is the same, Φ is put in the projective form $\mu(z-2)+\lambda(2x+z-0)=0$.

where the sign $\pm$ has to be chosen according as $\lambda > -1/2$ (ellipses), or $\lambda < -1/2$ (hyperbolas).

Let us give the equation of some other particular conic in the system Φ.

For $\lambda = 0$ we have the circle c defined by $x^2 + y^2 = 1$, with centre in (0, 0); for $\lambda = -1/4$ we have the ellipse e defined by $8x^2 + 9y^2 + 8x - 16 = 0$, and in standard form

$$\frac{(x+1/2)^2}{(3/2)^2} + \frac{y^2}{(\sqrt{2})^2} = 1,$$

which has the second focus in $(-1, 0)$; for $\lambda = -2/3$ we have the hyperbola h defined by $-3x^2 + y^2 + 12x - 9 = 0$, which in standard form is

$$(x-2)^2 - \frac{y^2}{3} = 1,$$

and has its second focus in (4, 0).

The conics c, e, p, h, dh defined above are represented in Fig. 10, related to Fig. 2b, as are e_1, e_2, h_1, h_2, h_3 corresponding to different values of λ, with their respective second foci, which, for simplicity, we have denoted by the same letter.

We note that, up to a rotation on the left of 90°, the bold part of Fig. 10 is essentially Fig. 1.

We remark that when Kepler wrote "quod à linea recta per hyperbolas infinitas in Parabolem, inde per ellipses infinitas in circulum est transitus [we pass from the line through infinite hyperbolas to the parabola, and then from the parabola through infinite ellipsis to the circle" (p. 92, lines 26–30), he seemed to consider not the whole system Φ, but only part of it. Moreover it is not clear to which "line" Kepler is referring. If he considered only the conics corresponding to the values of $\lambda \in [-1, 0]$, the line in question is the horizontal line in Fig. 1, corresponding to the degenerate hyperbola *dh*,[15] and the projection of the hyperbola h' in Fig. 2b. In fact, for $\lambda \to -1$, the two branches of the hyperbola become closer and closer until they flattened onto the line $x = 1$. Whilst, if Kepler considered the conics corresponding to $\lambda \in (-\infty, 0]$, as it seems at the beginning of his study on conics when he referred to the actual sections of the cone, the line in question may be either the projection of the hyperbola h', i.e. *dh*, or the projection of the (double) line l of Fig. 2, i.e $y^2 = 0$.

This ambiguity may have caused Kepler's erroneous determination of the limiting position of the free focus on the degenerate hyperbola *dh*, as already noticed Davis (1975, p. 680). In fact for the limiting position of the free focus b we have that: $\lambda \to \infty$ implies $b \to (1, 0)$, and $\lambda \to -1$ implies $b \to (2, 0)$. So the free focus does not fall on the line $x - 1 = 0$ as Kepler claimed.

As is well known the type (and the shape) of a conic depends on its *eccentricity*. For a central conic, i.e. for an ellipse or a hyperbola, the eccentricity was classically defined as the value of the ratio

[15] In fact for $\lambda \to -1$, the two arms (or branches) of the hyperbola become closer and closer, until they flatten onto the line $x = 1$.

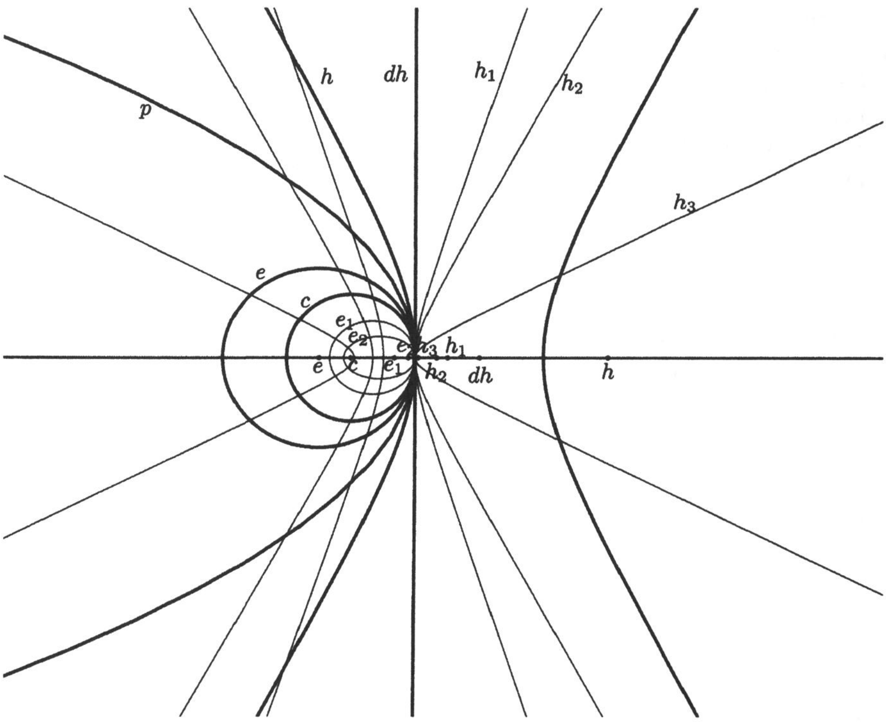


Fig. 10 In this figure conics from the system Φ are represented, those in *bold* are relative to values of $\lambda \in [-1, 0]$, the others are relative to values $\lambda \in (0, +\infty)$ (ellipses) and $\lambda \in (-\infty, -1)$ (hyperbolas). The two letters "c" denote the circle and its centre. Similarly the other pairs of equal letters denote a conic and its focus $\neq c$

$$e = \frac{FC}{CV},$$

where C is the centre of the conic, F one of its foci, and V the nearest vertex to F. Clearly we have $e = 0$ for the circle, $0 < e < 1$ for the ellipse, and $1 < e < +\infty$ for the hyperbola.[16]

When the conic is defined by the focus-directrix property, i.e. as the locus of points P whose distances from a fixed point (focus) and a fixed line (directrix) is constant, this constant is the eccentricity, that is, denoting by H the foot of P on the directrix, we have

$$e = \frac{PF}{PH}.$$

[16] We note in passing that Hofmann, see p. 336 of the reprint of his work, wrote the following equation $y^2 = 2px - (1 - e^2)x^2$, where p is a fixed number and e the eccentricity. This equation does not represent a system of semi-confocal conics. For instance, for $p = 1$ the coordinates of the foci are $(1/(1+e), 0)$ and $(1/(1-e), 0)$, and clearly both depend on the eccentricity. Let us remark that when $e \to +\infty$, the limiting position of the foci is the point $(0, 0)$; so in this system both foci coincide for the degenerate hyperbola and fall on it.

Since for the parabola we have $PF = PH$, it follows that $e = 1$.

When a non-degenerate conic is defined by the equation

$$(*) \quad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

and $B^2 - 4AC \neq 0$, i.e. it is not a parabola, the eccentricity can be expressed by the formula

$$e = \sqrt{\frac{2\sqrt{(A-C)^2 + B^2}}{\pm(A+C) + \sqrt{(A-C)^2 + B^2}}},$$

where the sign $\pm$ has to be chosen opposite to that of the determinant of the quadratic form associated to the equation (*).[17] Therefore, since the quadratic form associated to Φ has determinant $-(1+\lambda)^4$, for a conic in Φ, when $\lambda \neq -1/2$ and -1 the eccentricity is

$$e = \sqrt{\frac{\lambda^2}{(1+\lambda)^2}}.$$

It is readily seen that: $e = 0$ for $\lambda = 0$; $e = 1$ for $\lambda = -1/2$; $e = +\infty$ for $\lambda = -1$ (the degenerate hyperbola *dh*); if $\lambda \to +\infty$, then $e \to 1$; if $\lambda \to -\infty$, then $e \to 1$. The last two cases correspond to the degenerate parabola $y^2 = 0$.

In his study Kepler did not consider the eccentricity directly, but through the ratio sagitta/chord. In fact a simple computation shows that the sagitta is equal to $a(1-e)$, and the chord is equal to $a(1-e^2)$, where a is the half of the distance between the two vertices, so that

$$(**) \quad \text{sagitta/chord} = \frac{1}{2(1+e)}.$$

For the behaviour of the sagitta and of the chord (maintaining these name for the respective lengths), we observe that:

(1) the sagitta is constantly equal to 1, and the chord steadily increases from 1 to infinity, when λ goes from 0 to -1;
(2) the sagitta steadily decreases together with the chord from 1 to 0, when λ goes from 0 to $+\infty$;
(3) the sagitta steadily decreases from 1 to 0 and the chord steadily decreases from $+\infty$ to 0, when λ goes from -1 to $-\infty$.

Therefore the sagitta is not constant for all conics in the system Φ, but only for those corresponding to λ varying in the interval $[-1, 0]$, i.e. those that actually Kepler considered.

[17] See for instance (Ayoub 2003).

So, in the first case the ratio sagitta/chord goes to 0 when $\lambda \to -1$, and in the second and third case the limiting value of the ratio sagitta/chord is 1/4.

The point $(1, 0)$, which corresponds to the limiting position of the free focus when λ goes to infinity, lies on the degenerate parabola $y^2 = 0$, and, applying Kepler's analogy, we could say that both foci (0, 0) and (1, 0) lie on it, whilst the ratio sagitta/chord of 1/4 holds for every non-singular parabola.

Now we verify that both systems Ψ and Θ, introduced in the previous paragraph, are essentially the same as that of Kepler.

For the first we may suppose, without loss of generality, that the right-angled cone Γ is given by the equation

$$x^2 + y^2 - z^2 = 0,$$

and consider the pencil of planes through the line $z - 1 = 0$, $x - z = 0$, given by

$$z - 1 + \lambda(x - z) = 0.$$

By eliminating the variable z between the two equations we get

$$(1 + 2\lambda)x^2 + (1 + \lambda)^2 y^2 - 2\lambda x - 1 = 0,$$

that represents the system Ψ, which is clearly the same as Φ.

For the system Θ, we may consider the sphere S with equation

$$x^2 + y^2 + (z - 1)^2 - 1 = 0,$$

which is tangent to the plane $z = 0$ at the point (0, 0), and, changing our point of view with respect to Fig. 8, consider the plane $x = 1$ tangent to the sphere at the point $T = (1, 0, 1)$. The cone Δ, circumscribing S, has now its vertex at $L = (1, 0, h)$, and it has equation

$$(2h - h^2)(x - 1)^2 - (h - 1)^2 y^2 + 2(1 - h)(x - 1)(z - h) = 0.$$

Intersecting Δ with the plane $z = 0$, we get that the system of conics Θ is given by

$$(2h - h^2)(x - 1)^2 - (h - 1)^2 y^2 - 2h(h - 1)(x - 1) = 0.$$

In this system the circle $x^2 + y^2 = 1$ corresponds to $h = \infty$. It is readily seen that by putting $h = -1/\lambda$, $\lambda \neq 0$, in the equation above this is transformed to that of Φ.

5 Conclusions

In 1603 Kepler initiated the study of conics, to which, at least apparently, he was directed by his interests in burning mirrors, but the remark that the light-rays emitted from a light source in the focus of a parabolic mirror are reflected in the direction of the axis, is marginal compared to the other ideas that Kepler expounded in the *De coni sectionibus*. So, it seem to us that Kepler was more interested in the geometry of conics for its own sake, rather than for making use of burning mirrors.

To perform the continuous change from one conic to another in his system, Kepler imagined keeping one focus a fixed and moving the other focus b along the common axis fr to form all conics in the system (Fig. 1). When b approaches the fixed focus a, the ellipse becomes a circle, letting b go off to the infinity, the ellipse becomes a parabola, then, pushing it further, it reappears on the other side of the axis, as f, and the parabola becomes a hyperbola, and when the two foci of the hyperbola approach each other the hyperbola degenerates into a pair of (coincident) lines.

Therefore, beyond the fundamental idea of "point at infinity" that he implicitly defined as meeting point of a set of parallel lines, by introducing the concept of the blind focus of the parabola, and of the straight line as "a closed line", like a circle, by means of its point at infinity, he showed a completely new way to look at conics by putting them in a continuous system, in which all conics of the system share some common properties.

Kepler used analogy in almost all fields of his researches, but it seems that in it he trusted the most when applied to mathematics: reliable enough to justify the introduction of fanciful notions like the "foci of a (double) line" or "point at infinity" (Field and Gray 1987). He made analogy a tool for discovering new results.

Kepler's work could not escape the attention of the mathematicians of his time.

In 1625, Henry Briggs wrote a letter to Kepler in which he stated a theorem that clearly shows he had comprehended and accepted Kepler's way of treating at parallels as lines to, or from, a point at infinity in one direction or its opposite, see (Taylor 1900, pp. 203–204). Certainly many others, who have left no record, learned the new ideas set out in the *De coni sectionibus* in its author's lifetime.

In 1636, 6 years after Kepler's death, Marin Mersenne, in his *Harmonie universelle*, after describing the optical and acoustical properties of conics, wrote: "L'adiouste seulement icy une figure pour expliquer de certaines analogies qui se rencontrent dans toutes les sections dont nous avons parlé"[I only add here a figure for explaining certain analogies that one encounter in every sections we have treated], and reproduced Kepler's Fig. 1, but without quoting him (Mersenne 1636, p. 62). Girad Desargues —who was acquainted with Mersenne—certainly had Mersenne's work in his hands, and the likely also had Kepler's work.[18] As already noted in (Field 1997), we can not exclude that Desargues may have been influenced by Kepler in his geometrical studies.

One century and a half after the publication of *Ad Vitellionem paralipomena*, Roger J. Boscovich gave an extended discussion of the principle of analogy (*continuitatis lege*) in *Sectionum conicarum elementa nova*, in the third volume of his *Elementorum universae matheseos* (1757), see (Taylor 1881, pp. lxxiii–lxxvii).

It seems to us that for these reasons Kepler's way of considering conics assumes a particular importance for the history of Geometry. His ideas were much more rel-

[18] Desargues in the *Brouillon project* (1639) adopted the term "foyer", used for the first time by Kepler. Taylor (1900, p. 205) wrote: "Desargues must have learned directly or indirectly from the work in which Kepler propounded his new theory of these points, first called by him Foci (foyers)".

evant than commonly known, with them he anticipated the concepts of "point at infinity", "transformation of curves" and "continuous family of curves", concepts that have a fundamental role in the development of geometry in the nineteenth century. Unfortunately his fruitful ideas were soon forgotten, perhaps obscured by his great achievements in astronomy, and it was not until Poncelet that they resurfaced and were completely developed in his treatise on the projective properties of the figures (1822).

References

Ayoub, A.B. 2003. The eccentricity of a conic section. *The College Mathematics Journal* 34(2): 116–121.

Boscovich, R.J. 1757. *Elementorum universae matheseos*, vol. III. Venetiis, apud A. Perlini.

Buchdhal, G. 1972. Methodological aspects of Kepler's theory of refraction. *Studies in History and Philosophy of Science* 3: 265–298.

Commandino, F. 1558. *Archimedes opera non nulla a Federico Commandino urbinate nuper in latinum conversae et commentaries illustrata.* Venetiis: apud P. Manuntium.

Commandino, F. 1566. *Apollonii pergaei conicorum libri quattruor, una cum Pappi alexandrini lemmatibus et commentaries Eutocii ascalonitae....* Bononiae, A. Bennatii.

Commandino, F. 1588. *Pappi alexandrini mathematicae collectiones a Federico Commandino urbinate in latinum conversae et commentariis illustratae.* Pisauri, apud H. Concordiam.

Cronwell, P.R. 1997. *Polyedra.* Cambridge: Cambridge University Press.

Davis, A.E.L. 1975. Systems of conics in Kepler's work. *Vistas in Astronomy* 18: 673–685.

Del Monte, G. 1600. *Perspectiva libri sex.* Pisauri: apud H. Concordiam.

Desargues, G. 1639. *Brouillon proiect d'une atteinte aux evenemens des rencontres du Cone avec un Plane*, Paris, also in *L'Oeuvre mathématique de G. Desargues*, ed. R. Taton. 1951. Paris: Presses Universitaires de France.

Dürer, A. 1525. *Underweyssung der Messung mit Zirckel und Richtsheyt.* Norimbergae: J. Petreius.

Field, J.V. 1986. Two mathematical inventions in Kepler's Ad Vitellionem paralipomea. *Studies in History and Philosophy of Science Part A* 17: 449–468.

Field, J.V., and J.J. Gray. 1987. *The geometrical work of Girard Desargues.* New York: Springer.

Field, J.V. 1997. *The invention of infinity: Mathematics and art in the Renaissance.* Oxford: Oxford University Press.

Hofmann, J.E. 1971. Über einige fachliche Beiträge Keplers zur Mathematik, in Internationales Keplersymposium Weil der Stadt 1971, eds. F. Kraff Meyer and B. Sticker (1973), 1–84; we quote reprint in J.E. Hofmann (1990) *Ausgewählte Schriften*, ed. C.J. Scriba, vol. 2, 327–350. Hildesheim: Olms Verlag.

Kepler, J. 1604. *Ad Vitellionem paralipomea quibus astronomiae pars optica traditur* eds. C. Marnium Francofurti and H.I. Aubrii; also in *Gesammelte Werke* II, ed. M. Caspar. Munich 1937.

Kepler, J. 1609. *Astronomia nova.. .seu physica coelestium, tradita commentariis de motus stellae martis.* Prague; also in *Gesammelte Werke*, vol. III, ed. M. Caspar. Munich 1937.

Kepler, J. 1615. *Nova stereometria dolorium vinariorum.* Lincii: J. Plancus; also in *Gesammelte Werke*, vol. IX, ed. Franz Hammer. Munich 1955.

Kepler, J. 1619. *Harmonices mundi libri V.* Lincii: Tampachii.

Kepler, J. 2000. *Optics: Paralipomena to Witelo, & Optical Part of Astronomy*, Complete English translation, by W. H. Donahue, of Kepler's *Ad Vitellionem paralipomea quibus astronomiae pars optica traditur*, Frankfurt, 1604. Santa Fe: Green Lion Press.

Kline, M. 1972. *Mathematical thought from ancient to modern times.* Oxford: Oxford University Press.

Knobloch, E. 2000. Analogy and the growth of the mathematical knowledge. In *The growth of mathematical knowledge*, ed. E. Grosholz, and H. Breger, 295–314. New York: Springer.

Maurolico, F. 1611. *Photismi de lumine et umbra ad perspectivam, et radiorum incidentiam facientes.* Neapolis: T. Longi.

Mersenne, M. 1636. *Harmonie universelle contenant la théorie et la pratique de la musique.* Paris: S. Cramoisy.

Newton, I. 1704. *Opticks.* London: Smith and Walford.

Poncelet, J.-V. 1822. *Traité de propriétés projectives des figures.* Paris: Bachelier.

Risner, F. 1572. *Opticae thesaurus: Alhazeni Arabis libri septem, nunc primùm editi. Eiusdem liber De Crepusculs et Nubium Ascensionibus. Item Vitellionis Thuringopoloni libri* X. Basileae: Episcopius.

Taton, R. 1962. L'œuvre de Pascal en géométrie projective. *Revue d'Histoire des Sciences et de leur app* 15: 197–262.

Taylor, C. 1881. *An introduction to ancient and modern geometry of conics*. Cambridge: Deighton Bell and Co.

Taylor, C. 1883. On the history of geometrical continuity. *Proceedings of the Cambridge Philosophical Society* IV: 14–17.

Taylor, C. 1900. The geometry of Kepler and Newton. *Transaction of the Cambridge Philosophical Society* VIII: 197–219.

Witelo, E.C. 1535. *Perspectivam*. Norimbergae: J. Petreius.