Answer: For the semi supervised M-step we maximise over $\theta = \phi, \mu, \Sigma$ while keeping Q fixed the expression $ELBO(Q,\theta) + l_{\text{sup}}(\theta) = \sum_{i=1}^{n} \sum_{j=1}^{k} Q_i(z_i)_j \log \frac{p(x_i,z_i;\phi,\mu,\Sigma)}{Q_i(z_i)_j} + \alpha \sum_{i=1}^{\tilde{n}} \log p(x_i, z_i; \phi, \mu, \Sigma)$ So having pre-computed in E-step the fixed $Q_i(z_i)_j = w_j^i$, we can differentiate in turn for each of ϕ, μ, Σ. Rewriting the whole expression to be maximised as

$$\sum_{i=1}^{n} \sum_{j=1}^{k} w_j^i \log \frac{\phi_j \; N(x_i; \mu_j, \Sigma_j)}{w_j^i} + \alpha \sum_{i=1}^{\tilde{n}} \log \left(\phi_{\tilde{z}_i} \; N(\tilde{x}_i | \tilde{z}_i; \mu, \Sigma)\right)$$

mu

We get

$$\begin{aligned} \frac{\partial}{\partial \mu_l} &= \sum_{i=1}^{n} w_l^i \frac{\partial}{\partial \mu_l} \log N(x_i; \mu_j, \Sigma_j) + \alpha \sum_{i=1}^{\tilde{n}} \frac{\partial}{\partial \mu_l} \log N(\tilde{x}_i | \tilde{z}_i; \mu, \Sigma) \\ &= \sum_{i=1}^{n} w_l^i \Sigma_l^{-1}(x_i - \mu_l) + \alpha \sum_{\tilde{z}_i = l} \Sigma_l^{-1}(\tilde{x}_i - \mu_l) \end{aligned}$$

Setting this to 0 we get

$$\mu_l = \frac{\sum_{i=1}^{n} w_l^i x_i + \alpha \sum_{\tilde{z}_i = l} \tilde{x}_i}{\sum_{i=1}^{n} w_l^i + \alpha \#\{\tilde{z}_i = l\}}$$

phi

Similarly for ϕ we have

$$\frac{\partial}{\partial \phi_l} = \sum_{i=1}^{n} w_l^i (1/\phi_l) + \alpha \sum_{\tilde{z}_i = l} (1/\phi_l) = (1/\phi_l) \sum_{i=1}^{n} w_l^i + \alpha \#\{\tilde{z}_i = l\}$$

with constraint that $\sum_l \phi_l = 1$, so we apply the Lagrangian multipliers method:

$$\begin{aligned} \mathcal{L}(\phi, \lambda) &= \sum_{l=1}^{k} \sum_{i=1}^{n} w_l^i \log \phi_l + \alpha \#\{\tilde{z}_i = l\} \phi_l \; + \; \lambda(\sum_l \phi_l - 1) \\ \frac{\partial \mathcal{L}}{\partial \phi_l} &= (1/\phi_l) \sum_{i=1}^{n} w_l^i + \alpha \#\{\tilde{z}_i = l\} \; + \; \lambda = 0 \end{aligned}$$

this applies for each $l = 1, ..., k$, with λ constant, so we surmise that

$$\phi_l = C \sum_{i=1}^{n} w_l^i + \alpha \#\{\tilde{z}_i = l\} \quad \text{for some constant } C \text{, again constant across each } l$$

$$\text{so applying constraint } \sum_l \phi_l = 1 \text{ we get} \quad \phi_l = \frac{\sum_{i=1}^{n} w_l^i + \alpha \#\{\tilde{z}_i = l\}}{n + \alpha \tilde{n}}$$

Sigma

I cannot be bothered to write this all out again, you get the pattern, so we end up with

$$\Sigma_l = \frac{\sum_{i=1}^{n} w_l^i (x_i - \mu_l)(x_i - \mu_l)^T \;+\; \alpha \sum_{\tilde{z}_i = l} (\tilde{x}_i - \mu_l)(\tilde{x}_i - \mu_l)^T}{\sum_{i=1}^{n} w_l^i \;+\; \alpha \# \{\tilde{z}_i = l\}}$$