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A Diagram Category for Non-Orientable Surfaces

Benjamin Morris[1]
Joint work with Dionne Ibarra[2], Gabriel Montoya-Vega[3], and Paul Martin[1] (supervisor)

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Motivation

- Enriching Skein Modules of Non-orientable surfaces

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- Construct interesting low-dim "cobordism categories" amenable to rep. th. study:
  - Linear
  - Combinatorial
  - Finite Dimensional Hom-spaces
  - More structure? (monoidal... etc)

In particular, we consider **nested** $(0, 1, 2)$ - "cobordism categories".

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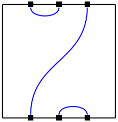
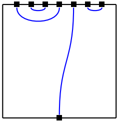
 $\in \mathrm{Hom}(3,3)$,  $\in \mathrm{Hom}(1,7)$.

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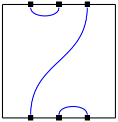
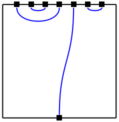
 $\in \mathrm{Hom}(3,3)$,  $\in \mathrm{Hom}(1,7)$.

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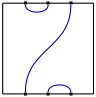
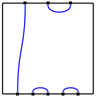
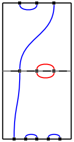
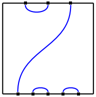
$D_2 \circ D_1 =$  $\circ$  $=$  $= \alpha$  $= \alpha\, D_2 \# D_1$

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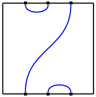
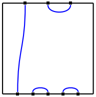
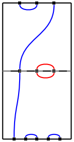
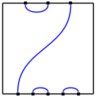
$D_2 \circ D_1 =$  $\circ$  $=$  $= \alpha$  $= \alpha \, D_2 \# D_1$

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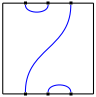
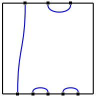
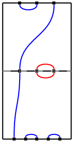
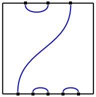
$D_2 \circ D_1 =$  $\circ$  $=$  $= \alpha$  $= \alpha\, D_2 \# D_1$

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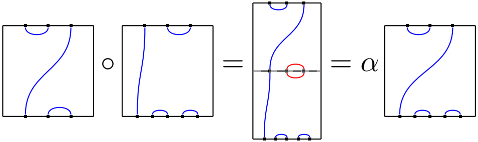
$D_2 \circ D_1 =$  $= \alpha\, D_2 \# D_1$

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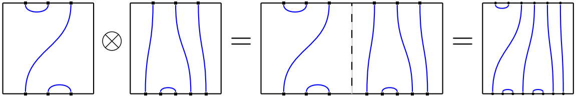
$D_2 \otimes D_1 =$ 

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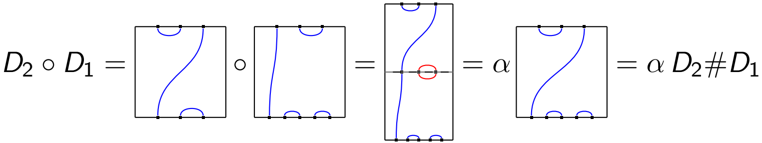

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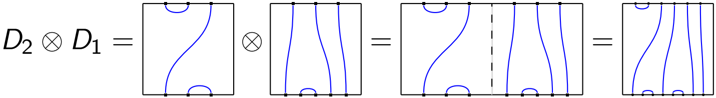

$(n_1 \otimes n_2 = n_1 + n_2)$.

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we will restrict to surface types Σ with one boundary component.

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Proceed concretely; attach "handles" to our square frame

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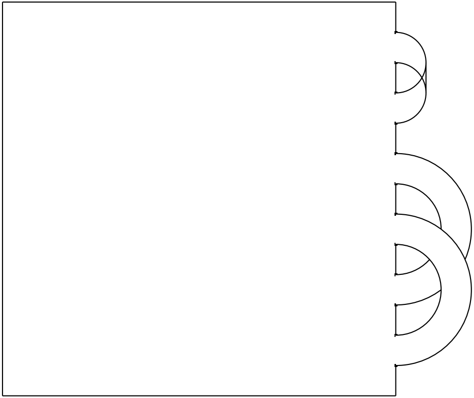

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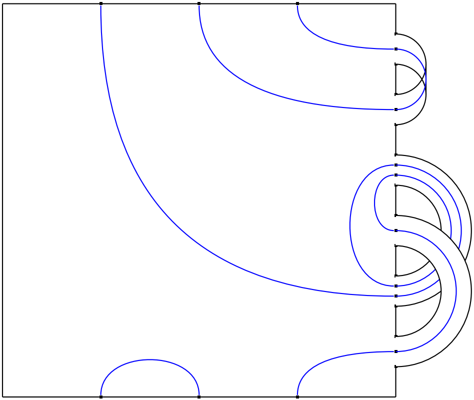

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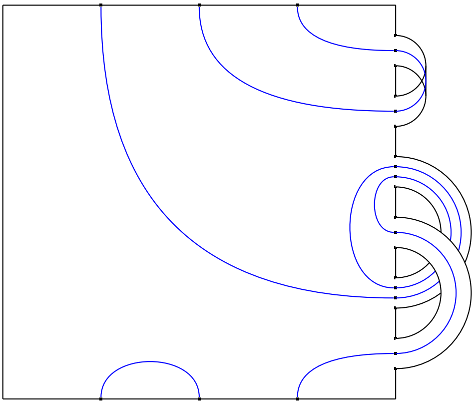

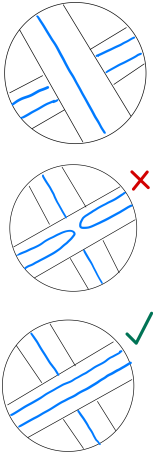

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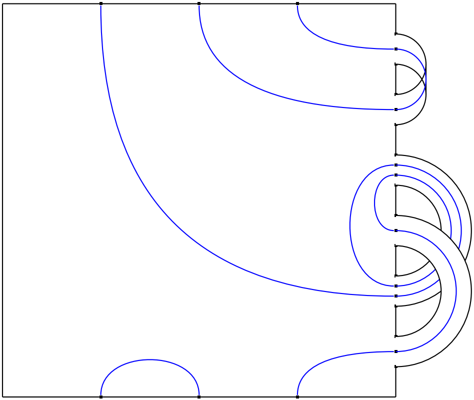

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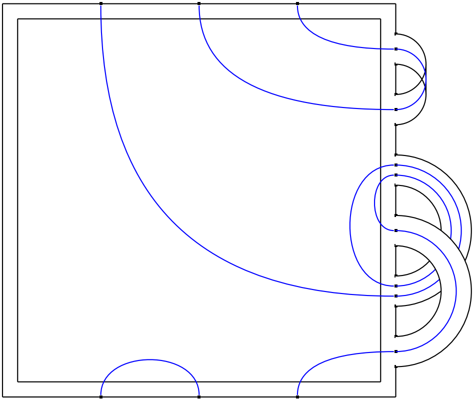

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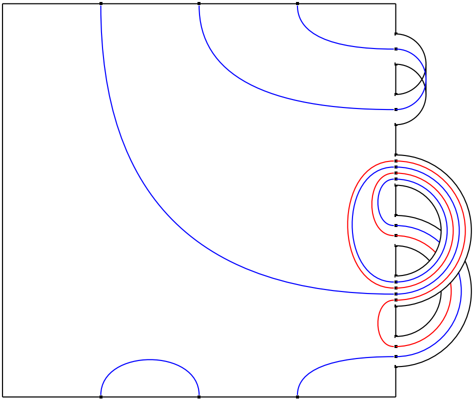
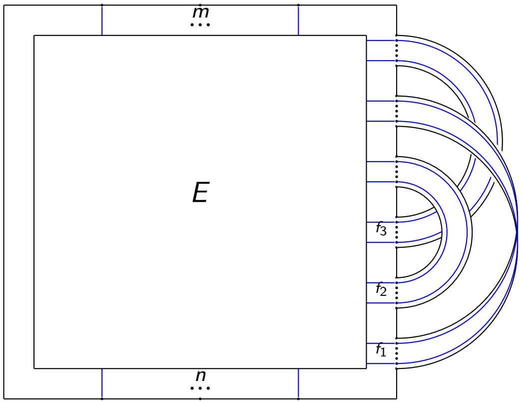
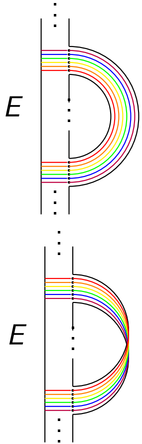


SWB diagrams

Square with bands (SWB) diagram encoded by $\Theta = (P, s, f, E)$ (type n, m)





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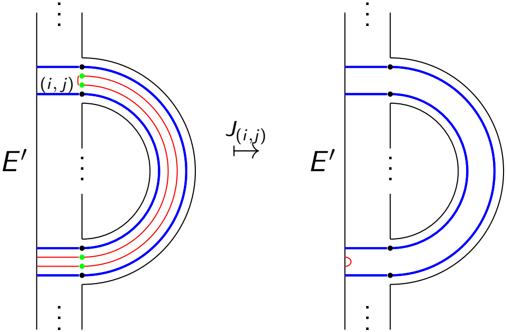

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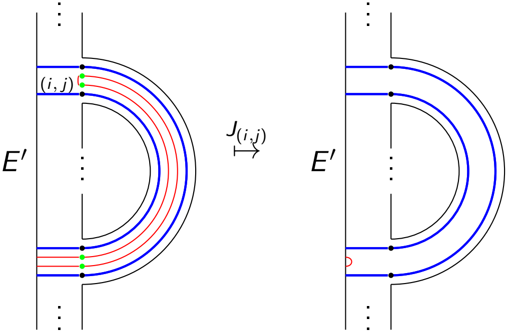

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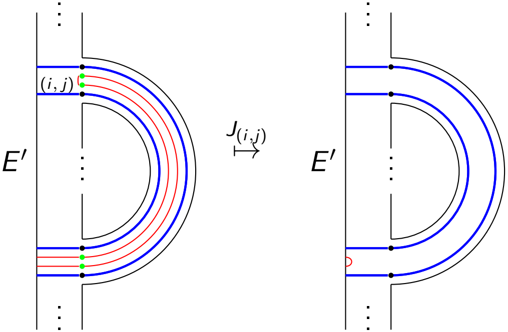

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Generate an equivalence relation with this move.

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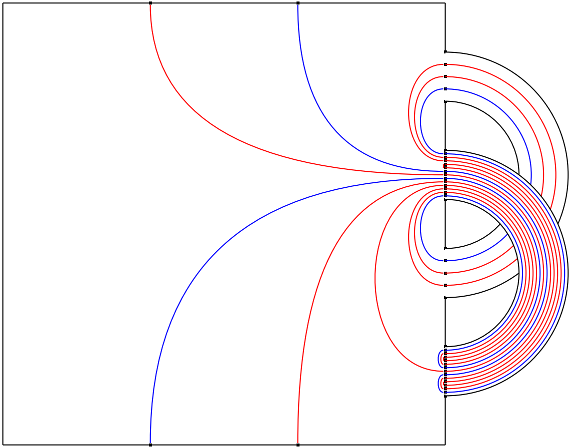

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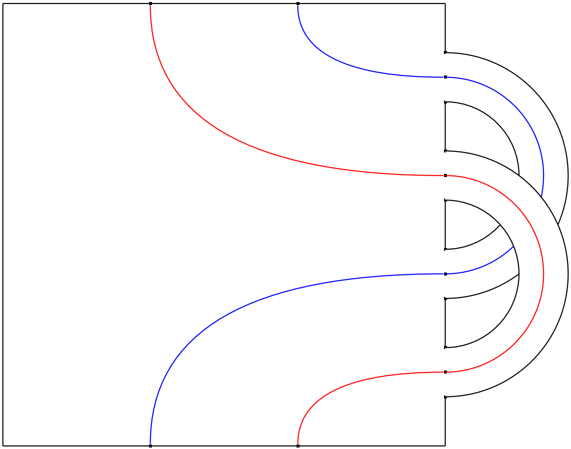

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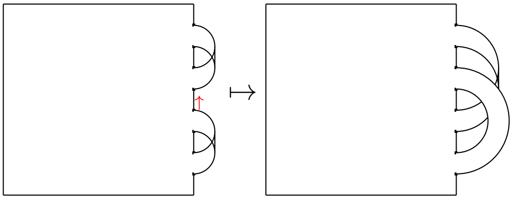

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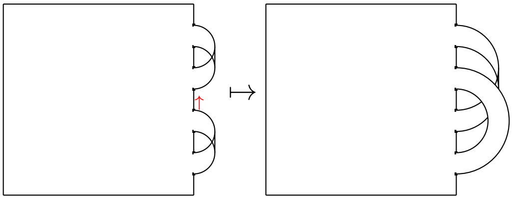

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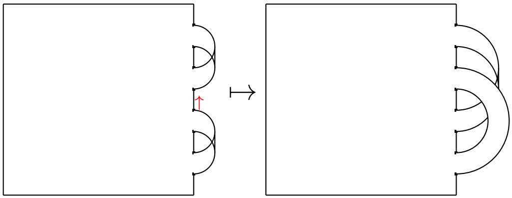

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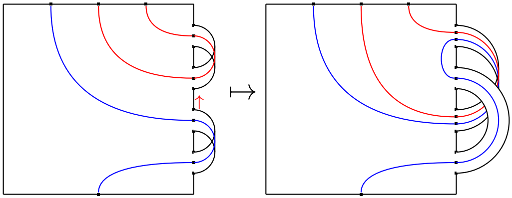

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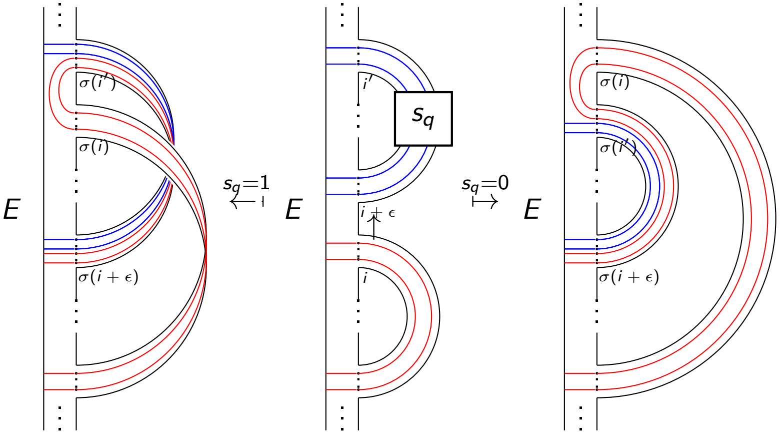

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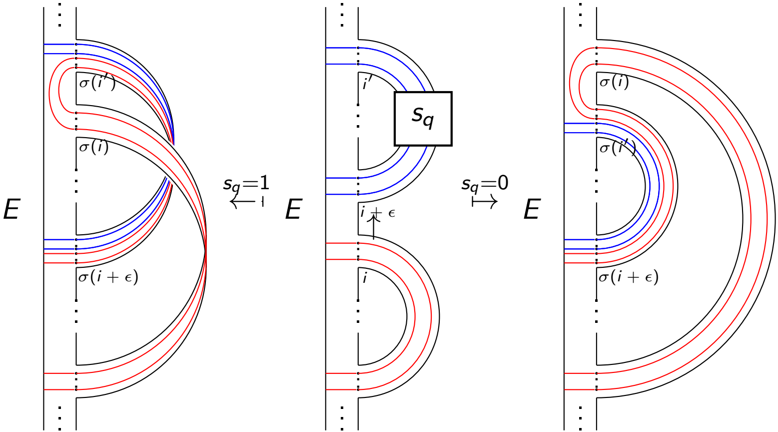

$$(P, s, f, E) \mapsto (\sigma(P), s' \circ \sigma^{-1}, f' \circ \sigma^{-1}, o(E) \cup \{\text{"new red arcs"}\})$$

SWB diagrams - Handleslide Equivalence

On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P, s) by a finite sequence of handleslides, e.g.

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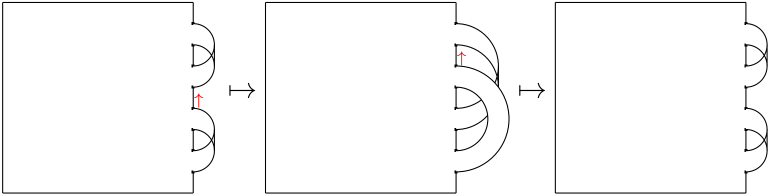

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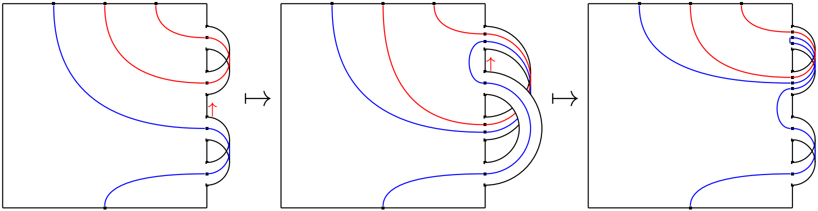

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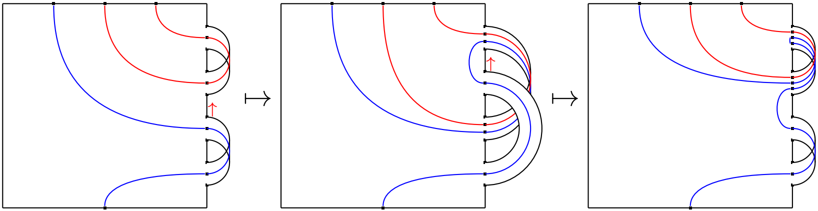

Defines an equivalence relation on **isotopy classes** of SWB diagrams - call this **Handleslide (HS) Equivalence**.

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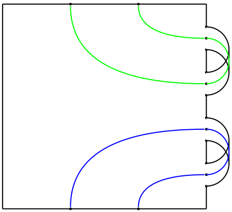
$\Theta =$



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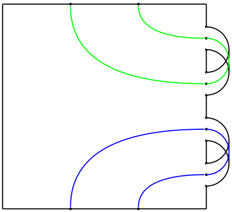
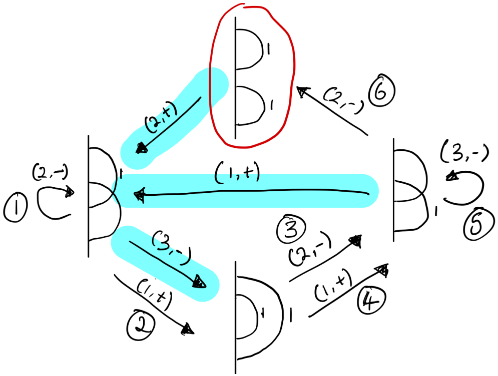
$\Theta =$ 



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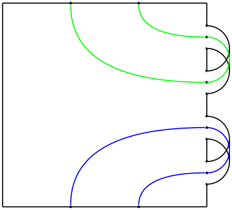
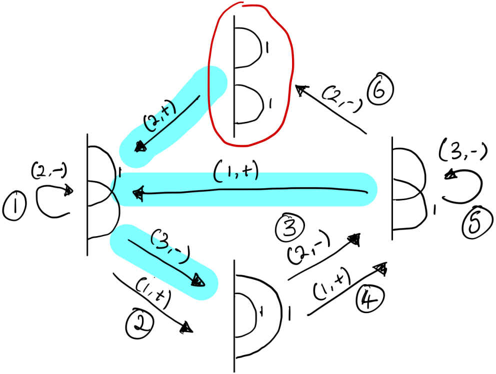
$\Theta =$ 



Associate the "reduced" sequence A_i for each edge outside the tree, e.g.

$$A_2 = (3,+) \circ (4,-) \circ (1,+) \circ (2,+)$$

SWB diagrams - Handleslide Equivalence

$A_1(\Theta) =$
$A_2(\Theta) =$
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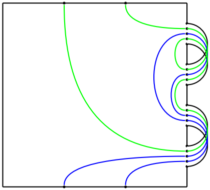

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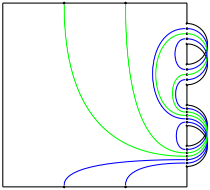

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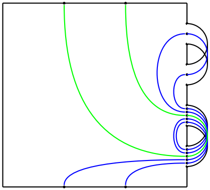

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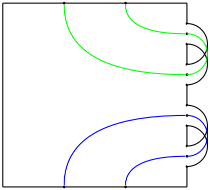

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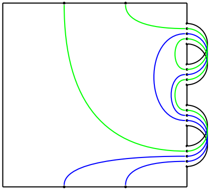

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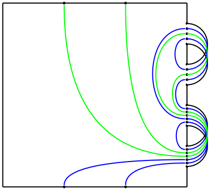

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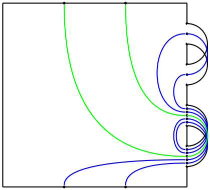

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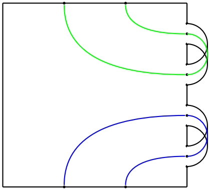

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$$\langle A_2, A_3, A_4 \mid A_3 A_2 = A_4,\ A_2 A_4 = A_4 A_2^{-1} \rangle \simeq \mathbb{Z} \rtimes \mathbb{Z}.$$

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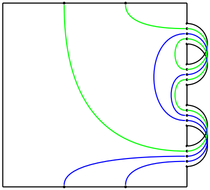,

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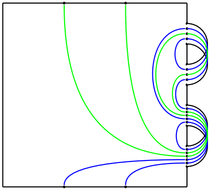,

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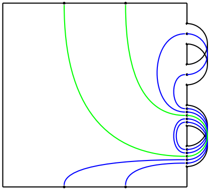,

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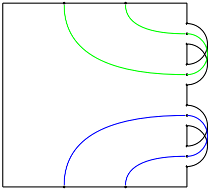.

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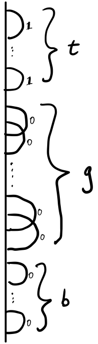
(Chord Diag. Pres. of Mapping Class Group - Bene 2009)

Handleslide Equivalence - Caravan form

FACT: Any surface (P, s) has a unique representative in the following **caravan form**:

$(P, s) \sim$ 

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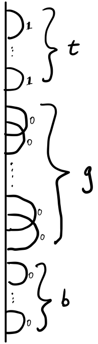
$(P, s) \sim$ 

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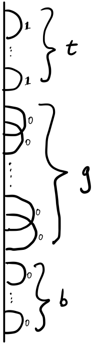

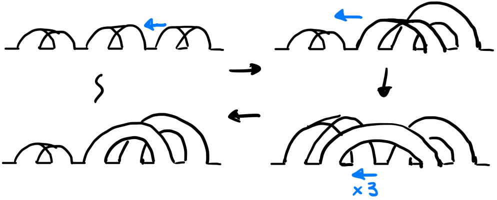

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The Category $\mathcal{SQ}$

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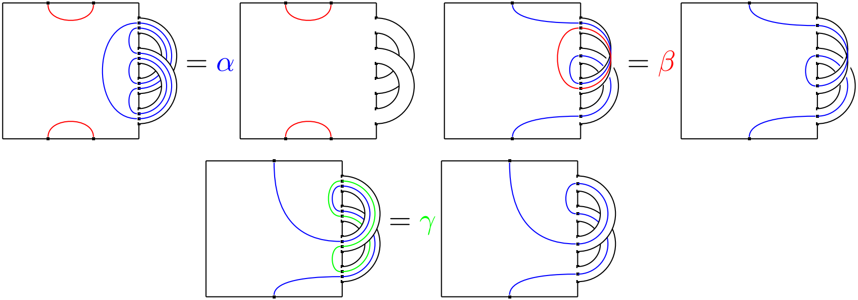

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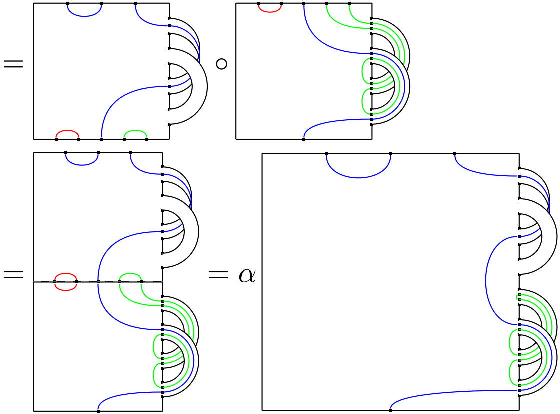

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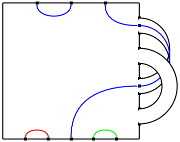
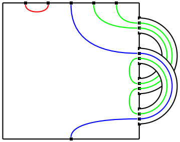
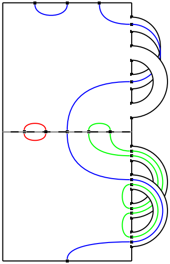
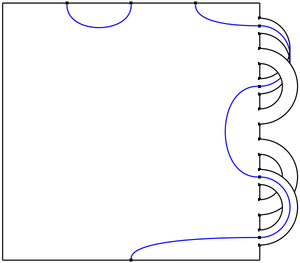
$\overline{\Theta_2} \circ \overline{\Theta_1} =$  $\circ$ 

$=$  $= \alpha\gamma$ 

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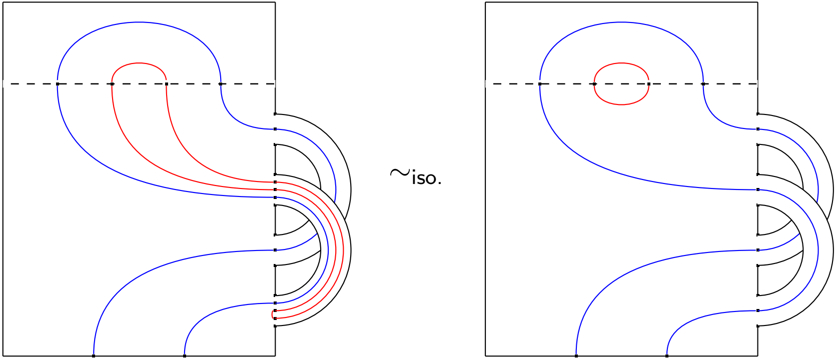

The Category $\mathcal{SQ}$ - Basic Facts

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Fact 1: For any $\Theta \in Sq(n, m)$, there exist **unique** integers l_s, l_t and l_u such that:

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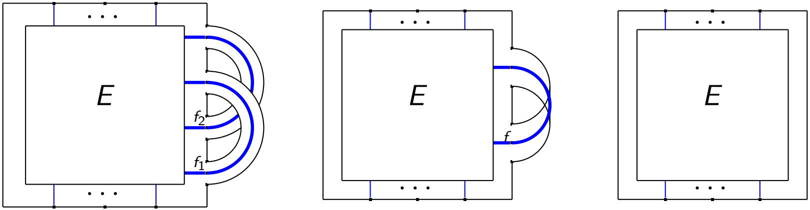

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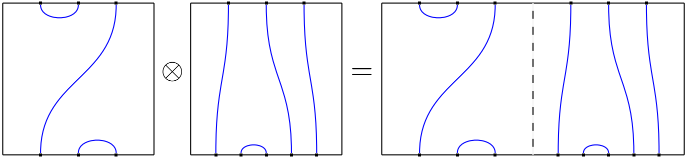

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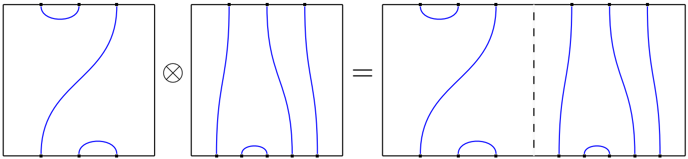

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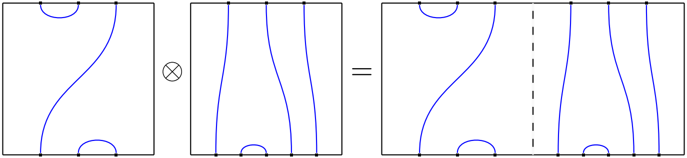

Can we extend this to a tensor product on $\mathcal{SQ}$ which has $n_1 \otimes n_2 = n_1 + n_2$ on objects. What should $\overline{\Theta} \otimes \overline{\Theta'}$ be for SWB diagrams??

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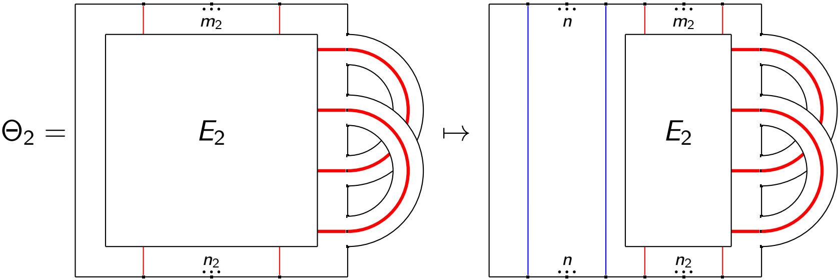
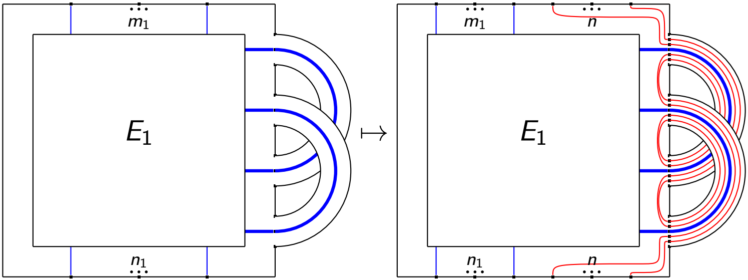


$\overset{(?)}{:=} \mathrm{id}_n \otimes \Theta_2$

The Category $\mathcal{SQ}$ - Tensor Product

Indirect answer: Step 2 - Put the identity diagram on the right:

$\Theta_1 =$



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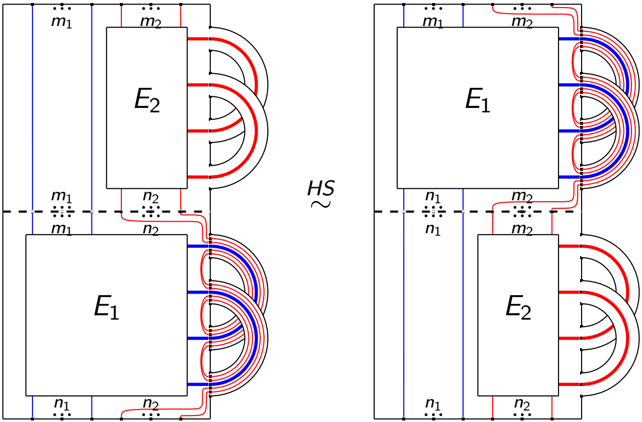
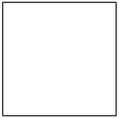
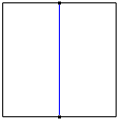
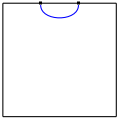
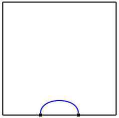
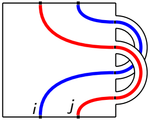
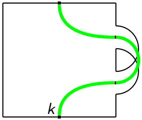


Monoidal Generating Set?

Conjecture: The following is a monoidal generating set for $\mathcal{SQ}(\alpha, \beta, \gamma)$:

$\mathrm{id}_0 =$ , $\mathrm{id}_1 =$ , $U =$ ,

$V =$ , $\mathbb{T}_{i,j} =$ 

, $\mathbb{M}_k =$ 

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$$\begin{tikzcd} & \mathcal{SQ} \arrow[r, "\_/Ker"] & UT(?) \\ Ker \arrow[ur, hook] \arrow[r, hook] & \mathcal{SQ}^+ \arrow[u, hook] \arrow[r, two heads, "F \simeq \_/Ker"] & T \arrow[u, hook, dashed, "?"] \end{tikzcd}.$$

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F a full and essentially surjective, monoidal functor, and T a target monoidal $\mathbb{K}$-linear category with f.d. hom spaces. Call F a **finitising functor**.

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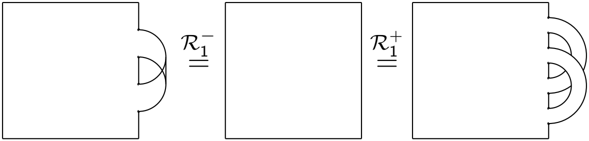

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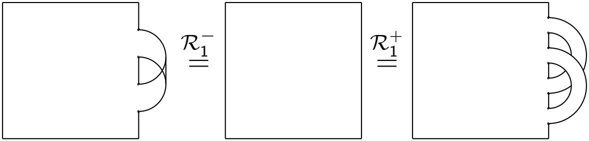

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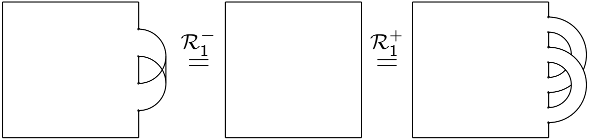

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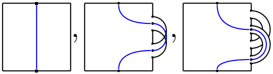
- $\mathrm{Hom}(1,1) = \mathbb{K}\Big\{$  $\Big\} \simeq \mathbb{K}\langle a \mid a^3 = a\rangle$,

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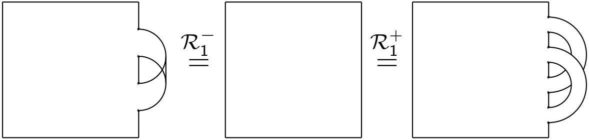
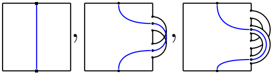


Conclusion: $\mathcal{SQ}/\mathcal{R}_1$ is a $\mathbb{K}$-linear category with finite dimensional hom-spaces:

- $\mathrm{Hom}(1,1) = \mathbb{K}\Big\{$  $\Big\} \simeq \mathbb{K}\langle a \mid a^3 = a\rangle$,
- $\dim(\mathrm{Hom}(2,2)) \geq 23$ *

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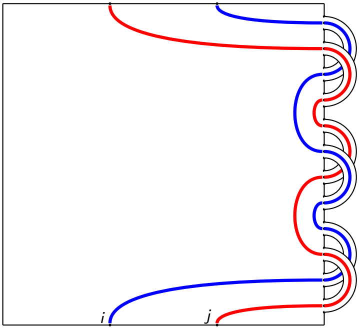
Example in $\mathcal{SQ}/\mathcal{R}_1$

Sample calculation in $\mathcal{SQ}/\mathcal{R}_1$:

$\mathbb{T}_{i,j} \circ \mathbb{T}_{j,i} \circ \mathbb{T}_{i,j} =$ 

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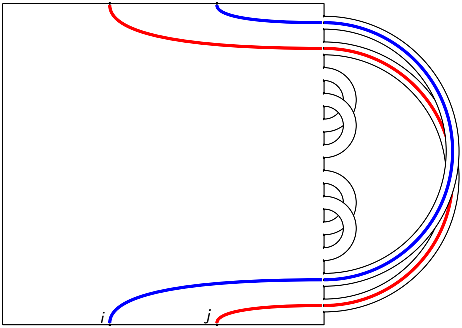
$\mathbb{T}_{i,j} \circ \mathbb{T}_{j,i} \circ \mathbb{T}_{i,j} =$ 

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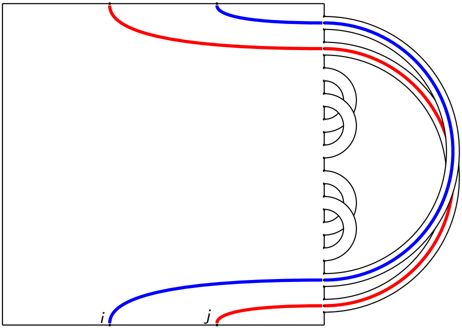
$\mathbb{T}_{i,j} \circ \mathbb{T}_{j,i} \circ \mathbb{T}_{i,j} =$  $= \mathbb{T}_{i,j} \mod \mathcal{R}_1$

The Infinity problem

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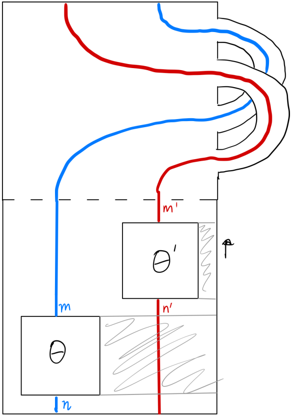

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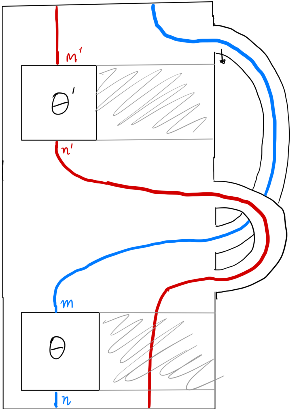

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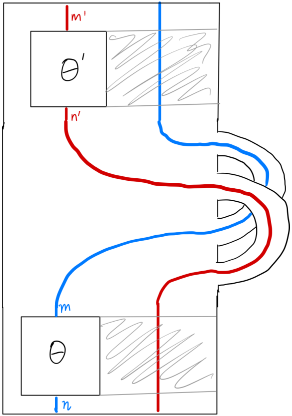

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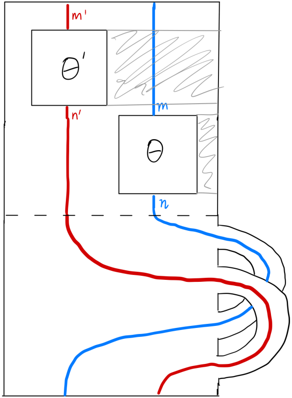

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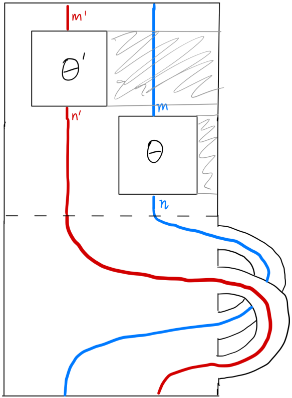

$= (\Theta' \otimes \Theta) \circ \mathbb{T}_{n',n}$

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Any other candidates for quotients? Note that the $\mathbb{T}_{i,j}$ are components of a braiding $\mathbb{T} : \otimes \to \otimes^{\mathrm{op}}$:

Can check the $\mathbb{T}_{i,j}$ satisfy the categorical YBE:

$$
\begin{aligned}
&(\mathbb{T}_{j,k} \otimes \mathrm{id}_i) \circ (\mathrm{id}_j \otimes \mathbb{T}_{i,k}) \circ (\mathbb{T}_{i,j} \otimes \mathrm{id}_k) \\
&= (\mathrm{id}_k \otimes \mathbb{T}_{i,j}) \circ (\mathbb{T}_{i,k} \otimes \mathrm{id}_j) \circ (\mathrm{id}_i \otimes \mathbb{T}_{j,k})
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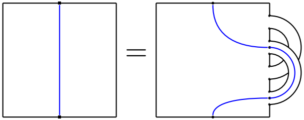
However, $\mathcal{SQ}$ is NOT a braided mon.cat. The smallest such quotient is obtained by imposing the relation $\mathcal{R}_2$ (as well as $\mathcal{R}_1^+$ *):



NOTE: This implies $\alpha = \gamma$. * not necessary if α invertible.

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The Infinity problem

Consider the functor $St := \_/(\mathcal{R}_1^+ + \mathcal{R}_2) := \_/\mathcal{R}^+$:

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\end{tikzcd} .
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$(\mathcal{SQ}/\mathcal{R}^+)/\mathcal{R}_1^- := \mathcal{SQ}/\mathcal{R} \simeq^* dBr(\alpha,\beta)$.

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\end{tikzcd}.$$

$(\mathcal{SQ}/\mathcal{R}^+)/\mathcal{R}_1^- := \mathcal{SQ}/\mathcal{R} \simeq^* \mathit{dBr}(\alpha,\beta)$.

Corollary: Suppose $F : \mathit{SQ}^+(\alpha,\beta,\gamma) \twoheadrightarrow T$ is a f.f. with $F \circ \mathbb{T}$ a braiding in a BMC. Then $\alpha = \gamma$, and F factors through $\mathcal{SQ}^+(\alpha,\beta,\alpha)/\mathcal{R}^+ \simeq \mathit{VTL}(\alpha) \simeq \mathit{Br}(\alpha)$.

Non-orientable extension of TL?

$TL(\alpha)$ is a BMC. Assume a f.f. $F : \mathcal{SQ}(\alpha, \beta, \alpha) \to TL(\alpha)$ sends $\mathbb{T}$ to the braiding in TL.

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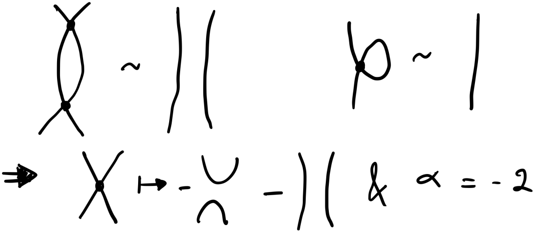

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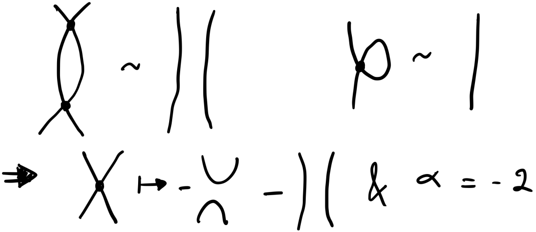

If $2 \neq 0 \in \mathbb{K}$, this quotient on $UVTL(-2, \beta)$ is more severe than hoped...

Thank You!

Questions?

A Categorical Interpretation of the $\mathbb{M}_k$

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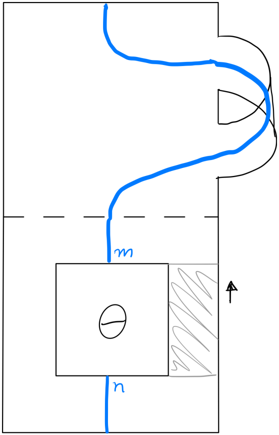
$\mathbb{M}_m \circ \Theta =$ 

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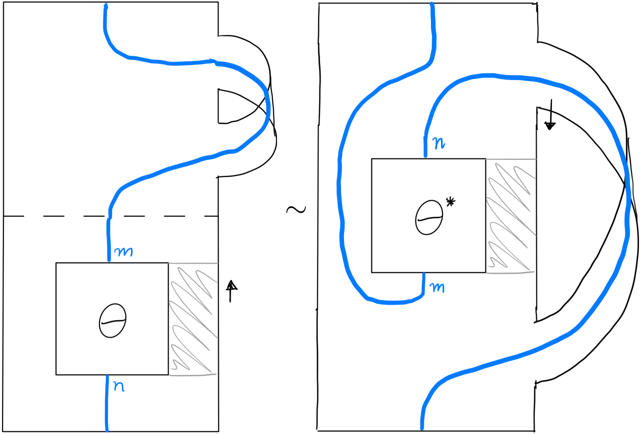
$\mathbb{M}_m \circ \Theta =$ 

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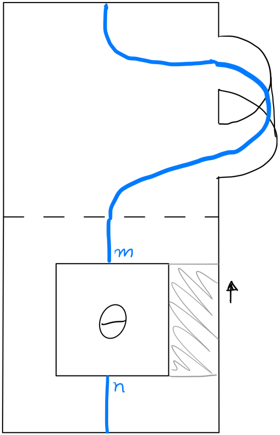
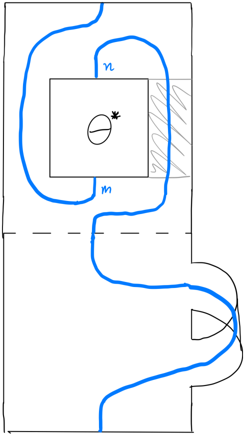
$\mathbb{M}_m \circ \Theta =$  $\sim$ 

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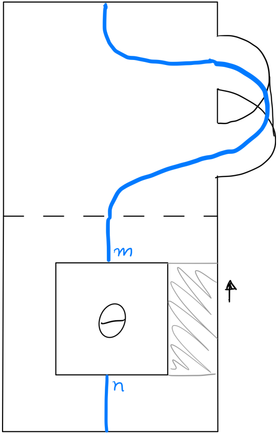
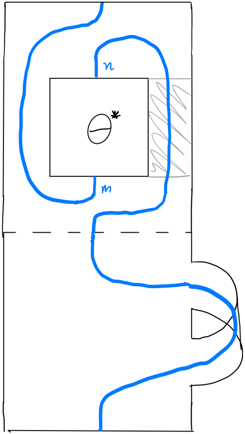
$\mathbb{M}_m \circ \Theta =$  $\sim$  $:= F(\Theta) \circ \mathbb{M}_n$

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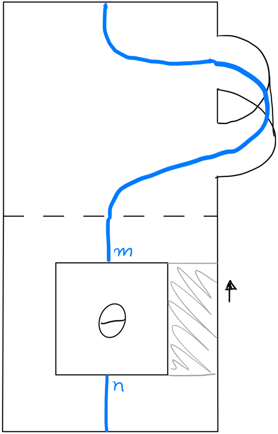
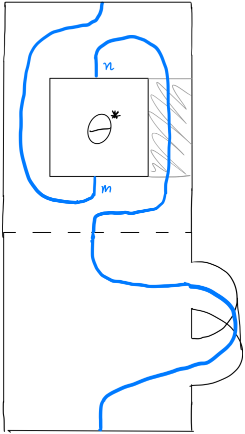
$\mathbb{M}_m \circ \Theta =$  $\sim$  $:= F(\Theta) \circ \mathbb{M}_n$

where $F^2 = \mathrm{id}$, $F \circ (\Theta \otimes \Theta') = F(\Theta') \otimes F(\Theta)$.