

# A Terminating $q$ -Lauricella Transformation Formula

Benjamin Morris

June 8, 2023

The purpose of this note is to give a self contained discussion and proof of a charming  $q$ -series identity discovered in a previous project of mine. We first introduce some basic notation. We will use the standard notation for the finite  $q$ -Pochhammer symbol

$$(x; q)_m = \begin{cases} (1-x)(1-qx)\dots(1-q^{m-1}x), & m > 0, \\ 1, & m = 0, \\ [(1-q^{-1}x)(1-q^{-2}x)\dots(1-q^{-m}x)]^{-1}, & m < 0, \end{cases} \quad (1)$$

aswell as the infinite  $q$ -Pochhammer symbol which is defined whenever  $|q| < 1$

$$(x; q)_\infty = \prod_{n=0}^{\infty} (1 - q^n x). \quad (2)$$

Notice that formulas (1) and (2) can be zero or singular when  $x$  is an integer power of  $q$ . Such cases require additional care as we will see. In this note we freely use identities for  $(x; q)_n$  and  $(x; q)_\infty$  from [2] Appendix I, and we will adopt the implicit base convention  $(x)_n = (x; q)_n$  and  $(x)_\infty = (x; q)_\infty$ .

We can now state the main result:

**Proposition 1.** *Fix any integer  $n \geq 1$  and non-negative integer tuples  $\mathbf{k} = (k_0, k_1, \dots, k_n) \in \mathbb{N}^{n+1}$ ,  $\mathbf{l} = (l_1, \dots, l_n) \in \mathbb{N}^n$ , and  $\mathbf{m} = (m_1, \dots, m_{n-1}) \in \mathbb{N}^{n-1}$ . Denote  $k = \sum_{j=0}^n k_j$ ,  $l = \sum_{j=1}^n l_j$ , and  $m = \sum_{j=1}^{n-1} m_j$ . Then for a complex number  $q$  with  $0 < |q| < 1$ , and arbitrary complex parameters  $y$ , and  $z$ , we have the following equality:*

$$\begin{aligned} & \frac{(y)_{l+m}}{(yz)_{l+m}} \sum_{\lambda \in \mathbb{N}^n} \sum_{\mu \in \mathbb{N}^{n-1}} \left[ \frac{(z)_{\lambda+\mu} \prod_{j=1}^n (q^{-l_j})_{\lambda_j} \prod_{j=1}^{n-1} (q^{-m_j})_{\mu_j}}{(q^{1-l-m}/y)_{\lambda+\mu} \prod_{j=1}^n (q)_{\lambda_j} \prod_{j=1}^{n-1} (q)_{\mu_j}} \right. \\ & \quad \left. \times q^{\sum_{j=1}^n \lambda_j (1+k_j + \sum_{a=1}^{j-1} (k_a - (l_a + m_a))) + \sum_{j=1}^{n-1} \mu_j (1 + \sum_{a=1}^j (k_a - (l_a + m_a)) + m_j)} \right] \\ & = z^{k_0} \frac{(y)_k}{(yz)_k} \sum_{\kappa \in \mathbb{N}^{n+1}} \left[ \frac{(z)_\kappa \prod_{j=0}^n (q^{-k_j})_{\kappa_j}}{(q^{1-k}/y)_\kappa \prod_{j=0}^n (q)_{\kappa_j}} (q^{1+k_0-k}/(yz))^{\kappa_0} q^{\sum_{j=1}^n \kappa_j (1+k_j - \sum_{a=j}^n (k_a - (l_a + m_a)))} \right], \quad (3) \end{aligned}$$

where we adopt the same labelling convention  $\kappa = (\kappa_0, \kappa_1, \dots, \kappa_n)$  for components of  $\kappa$  as per  $\mathbf{k}$ . We also use the shorthand  $\kappa = \sum_{j=0}^n \kappa_j$ ,  $\lambda = \sum_{j=1}^n \lambda_j$  and  $\mu = \sum_{j=1}^{n-1} \mu_j$ , as well as the convention  $m_n = 0$  wherever it appears.

We will not give the proof of [proposition 1](#) yet, however, we can immediately note that both sides are finite sums in light of the  $(q^{-a})_\alpha$  terms in the numerators which vanish for  $\alpha > a$ . We choose to write infinite sums to make explicit the connection with  $q$ -Lauricella series. To do so succinctly, we define two auxilliary  $n$ -tuples of integers  $\mathbf{r} = (r_j)$  and  $\mathbf{p} = (p_j)$  by

$$r_j = 1 + \sum_{a=1}^j (k_a - (l_a + m_a)), \quad p_j = 1 - \sum_{a=j}^n (k_a - (l_a + m_a)), \quad (4)$$

for  $j = 1, \dots, n$  (taking  $m_n = 0$  as before). Let us also denote by  $\hat{\mathbf{r}}$  the  $(n-1)$ -tuple  $(r_1, \dots, r_{n-1})$ , and by  $\tilde{\mathbf{k}}$  the  $(n-1)$ -tuple  $(k_1, \dots, k_n)$  for convenience.

Now introduce the type  $D$ ,  $q$ -Lauricella series treated in [1]

$$\Phi_D^{(n)}[\beta; \alpha_1, \dots, \alpha_n; \gamma; q; x_1, \dots, x_n] = \sum_{\nu_1=0}^{\infty} \dots \sum_{\nu_n=0}^{\infty} \frac{(\beta)_{\nu} (\alpha_1)_{\nu_1} \dots (\alpha_n)_{\nu_n}}{(\gamma)_{\nu} (q)_{\nu_1} \dots (q)_{\nu_n}} x_1^{\nu_1} \dots x_n^{\nu_n}, \quad (5)$$

where as before we use the notation  $\nu = \sum_{j=1}^n \nu_j$ . Using (4), and (5), the equality (3) is written succinctly as  $\Theta_{\mathbf{k}, \mathbf{l}, \mathbf{m}} = \Omega_{\mathbf{k}, \mathbf{l}, \mathbf{m}}$ , where

$$\Theta_{\mathbf{k}, \mathbf{l}, \mathbf{m}} = \frac{(y)_{l+m}}{(yz)_{l+m}} \Phi_D^{(2n-1)} \left[ z; q^{-l}, q^{-m}; q^{1-l-m}/y; q; q^{r+l+(m,0)}, q^{\hat{\mathbf{r}}+\mathbf{m}} \right], \quad (6)$$

is the left hand side, and

$$\Omega_{\mathbf{k}, \mathbf{l}, \mathbf{m}} = z^{k_0} \frac{(y)_k}{(yz)_k} \Phi_D^{(n+1)} \left[ z; q^{-k}; q^{1-k}/y; q; q^{1+k_0-k}/(yz), q^{\mathbf{p}+\tilde{\mathbf{k}}} \right], \quad (7)$$

is the right hand side. Here we are using element-wise exponentiation short hand  $q^{\mathbf{x}} = (q^{x_1}, \dots, q^{x_m})$ .

A standard procedure for dealing with expressions such as (6) and (7) may be to use (4.1) from [1] to rewrite them in terms of  ${}_{m+1}\phi_m$  basic hypergeometric functions (See [2] (1.2.22)) and work with known transformation formulae thereof. This approach is not valid here. For example, applying (4.1) [1] to (6) yields

$$\Theta_{\mathbf{k}, \mathbf{l}, \mathbf{m}} \propto {}_{2n}\phi_{2n-1} \left[ \frac{q^{1-l-m}/(yz), q^{\mathbf{r}+l+(m,0)}, q^{\hat{\mathbf{r}}+\mathbf{m}}}{q^{r_1+m_1}, \dots, q^{r_{n-1}+m_{n-1}}, q^{r_n}, q^{r_1}, \dots, q^{r_{n-1}}; q, z \right], \quad (8)$$

which contains denominator arguments of the form  $q^a$  with  $a$  potentially a negative integer. In this case the  ${}_{m+1}\phi_m$  function is undefined and a similar problem occurs with the RHS (7).

Fortunately we do not need the transformation rule (4.1) from [1]; we can settle for the intermediate step

$$\Phi_D^{(n)}[\beta; \alpha; \gamma; q; \mathbf{x}] = \frac{(\beta)_{\infty}}{(\gamma)_{\infty}} \sum_{a=0}^{\infty} \frac{(\gamma/\beta)_a}{(q)_a} \beta^a \prod_{j=1}^n \left( \sum_{\nu_j=1}^{\infty} \frac{(\alpha_j)_{\nu_j}}{(q)_{\nu_j}} (x_j q^a)^{\nu_j} \right). \quad (9)$$

In [1] the bracketed sums are evaluated using the infinite summation identity for  ${}_1\phi_0[\alpha; q; x]$  (II.3) [2], however, in our case these sums are terminating since  $\alpha_j$  is always a negative integer power of  $q$ . We therefore apply the finite summation identity (II.4) [2]

$${}_1\phi_0[q^{-m}; q; x] = \sum_{n=0}^{\infty} \frac{(q^{-m})_n}{(q)_n} x^n = (xq^{-m})_m. \quad (10)$$

Combining formulae (9) and (10) we obtain

$$\begin{aligned} \Theta_{\mathbf{k}, \mathbf{l}, \mathbf{m}} &= \frac{(y)_{l+m}}{(yz)_{l+m}} \frac{(z)_{\infty}}{(q^{1-l-m}/y)_{\infty}} \sum_{\lambda=0}^{\infty} \frac{(q^{1-l-m}/(yz))_{\lambda}}{(q)_{\lambda}} (q^{r_n+\lambda})_{l_n} z^{\lambda} \prod_{j=1}^{n-1} (q^{r_j+m_j+\lambda})_{l_j} (q^{r_j+\lambda})_{m_j} \\ &= \frac{(z)_{\infty}}{(q/y)_{\infty}} \frac{(y)_{l+m}}{(yz)_{l+m}} \left( \frac{(q^{1-l-m}/(yz))_{l+m}}{(q^{1-l-m}/y)_{l+m}} z^{l+m} \right) \sum_{\lambda=0}^{\infty} \frac{(q/(yz))_{\lambda-l-m}}{(q)_{\lambda}} \left( \prod_{j=1}^n (q^{r_j+\lambda})_{l_j+m_j} \right) z^{\lambda-l-m} \\ &= \frac{(z)_{\infty}}{(q/y)_{\infty}} \sum_{\lambda=0}^{\infty} \frac{(q/(yz))_{\lambda-l-m}}{(q)_{\lambda}} \left( \prod_{j=1}^n (q^{r_j+\lambda})_{l_j+m_j} \right) z^{\lambda-l-m}, \end{aligned} \quad (11)$$

for the LHS, where again we understand  $m_n = 0$ , and

$$\begin{aligned}
\Omega_{\mathbf{k}, \mathbf{l}, \mathbf{m}} &= z^{k_0} \frac{(y)_k}{(yz)_k} \frac{(z)_\infty}{(q^{1-k}/y)_\infty} \sum_{\kappa=0}^{\infty} \frac{(q^{1-k}/(yz))_\kappa}{(q)_\kappa} (q^{1+\kappa-k}/(yz))_{k_0} \left( \prod_{j=1}^n (q^{p_j+\kappa})_{k_j} \right) z^\kappa \\
&= \left( z^{k_0} \frac{(yq^{k-k_0})_{k_0}}{(yzq^{k-k_0})_{k_0}} \frac{(q^{1-k}/(yz))_{k_0}}{(q^{1-k}/(y))_{k_0}} \right) \frac{(y)_{k-k_0}}{(yz)_{k-k_0}} \frac{(z)_\infty}{(q^{1-(k-k_0)}/y)_\infty} \sum_{\kappa=0}^{\infty} \frac{(q^{1-(k-k_0)}/(yz))_\kappa}{(q)_\kappa} \left( \prod_{j=1}^n (q^{p_j+\kappa})_{k_j} \right) z^\kappa \\
&= \frac{(y)_{k-k_0}}{(yz)_{k-k_0}} \left( \frac{(q^{1-(k-k_0)}/(yz))_{k-k_0}}{(q^{1-(k-k_0)}/(y))_{k-k_0}} z^{k-k_0} \right) \frac{(z)_\infty}{(q/y)_\infty} \sum_{\kappa=0}^{\infty} \frac{(q/(yz))_{\kappa-(k-k_0)}}{(q)_\kappa} \left( \prod_{j=1}^n (q^{p_j+\kappa})_{k_j} \right) z^{\kappa-(k-k_0)} \\
&= \frac{(z)_\infty}{(q/y)_\infty} \sum_{\kappa=0}^{\infty} \frac{(q/(yz))_{\kappa-(k-k_0)}}{(q)_\kappa} \left( \prod_{j=1}^n (q^{p_j+\kappa})_{k_j} \right) z^{\kappa-(k-k_0)}, \tag{12}
\end{aligned}$$

for the RHS. It follows from these expressions that both  $\Theta_{\mathbf{k}, \mathbf{l}, \mathbf{m}}$  and  $\Omega_{\mathbf{k}, \mathbf{l}, \mathbf{m}}$  are independent of  $k_0$ , and depend on  $l_i$  and  $m_i$ , only in the combinations  $l_i + m_i$  (both of these facts are necessary for (3) to hold). By cancelling the prefactors in (11) and (12), and relabelling  $l_i + m_i \mapsto l_i$  for  $i = 1, \dots, n-1$ ,  $k - k_0 \mapsto k = \sum_{j=1}^n k_j$  and  $q/(yz) \mapsto y$ , to better reflect the dependence, we have reduced the equality  $\Theta_{\mathbf{k}, \mathbf{l}, \mathbf{m}} = \Omega_{\mathbf{k}, \mathbf{l}, \mathbf{m}}$  to the following.

**Proposition 2.** *For any integer  $n \geq 1$ , complex parameter  $q$ , such that  $0 < |q| < 1$ , arbitrary complex parameters  $y$  and  $z$ , and non-negative integer tuples  $\mathbf{l} = (l_j) \in \mathbb{Z}_{\geq 0}^n$  and  $\mathbf{k} = (k_j) \in \mathbb{Z}_{\geq 0}^n$  ( $j = 1, \dots, n$ ), we have the following equality*

$$\sum_{\lambda=0}^{\infty} \frac{(y)_{\lambda-l}}{(q)_\lambda} (q^{r_1+\lambda})_{l_1} \dots (q^{r_n+\lambda})_{l_n} z^{\lambda-l} = \sum_{\kappa=0}^{\infty} \frac{(y)_{\kappa-k}}{(q)_\kappa} (q^{p_1+\kappa})_{k_1} \dots (q^{p_n+\kappa})_{k_n} z^{\kappa-k}, \tag{13}$$

where  $k = \sum_{j=1}^n k_j$ ,  $l = \sum_{j=1}^n l_j$ , and  $p_j$  and  $r_j$  are as per (4) (with  $(l_a + m_a) \mapsto l_a$ ).

The proof of [proposition 2](#) requires two technical lemmas.

**Lemma 1.** *With  $\mathbf{k}, \mathbf{l}, k, l$ , and  $p_j$  as per [proposition 2](#), suppose that  $k \geq l$  and define  $\Delta = k - l \geq 0$ . Then for any integer  $0 \leq \kappa \leq \Delta - 1$  we have*

$$(q^{p_1+\kappa})_{k_1} \dots (q^{p_n+\kappa})_{k_n} = 0. \tag{14}$$

*Proof.* It suffices to show that for any  $0 \leq \kappa \leq \Delta - 1$ , there exists  $j$  such that  $p_j + \kappa \leq 0$  and  $p_j + \kappa + k_j > 0$ , that is,  $\{0, 1, \dots, \Delta - 1\} \subset U$  where  $U = \bigcup_{j=1}^n (-(p_j + k_j), -p_j]$ . Since  $-(p_j + k_j) = -(p_{j+1} + l_j) \leq -p_{j+1}$  (for  $j = 1, \dots, n-1$ ), it follows that  $U$  is an overlapping union of intervals giving

$$U = \left( \min_{j=1}^n (-(p_j + k_j)), \max_{j=1}^n (-p_j) \right] := (m, p].$$

Now note that  $p \geq -p_1 = \Delta - 1$  and  $m \leq -(p_n + k_n) = -(1 + l_n) < 0$  so we are done.  $\square$

**Lemma 2.** *With  $\mathbf{k}, \mathbf{l}, k, l$ , and  $r_j$  as per [proposition 2](#), suppose that  $k \geq l$  and define  $\Delta = k - l \geq 0$ . Then for any integer  $\lambda \geq 0$  we have*

$$(q^{r_1+\lambda})_{k_2} \dots (q^{r_{n-1}+\lambda})_{k_n} = 0, \quad \Leftrightarrow \quad (q^{r_1+\lambda})_{l_1} \dots (q^{r_{n-1}+\lambda})_{l_{n-1}} = 0, \quad \Leftrightarrow \quad \lambda \leq r := \max_{j=1}^{n-1} (-r_j). \tag{15}$$

Furthermore, for any integer  $\lambda > r$  we have

$$\frac{(q^{1+\lambda})_{k_1} (q^{r_1+\lambda})_{k_2} \dots (q^{r_{n-1}+\lambda})_{k_n}}{(q^{r_1+\lambda})_{l_1} \dots (q^{r_n+\lambda})_{l_n}} = \frac{(q)_{\lambda+\Delta}}{(q)_\lambda}. \tag{16}$$

*Proof.* For the first claim define  $K_0 = \{\lambda \in \mathbb{Z}_{\geq 0} \mid (q^{r_1+\lambda})_{k_2} \dots (q^{r_{n-1}+\lambda})_{k_n} = 0\}$  and similarly define  $L_0 = \{\lambda \in \mathbb{Z}_{\geq 0} \mid (q^{r_1+\lambda})_{l_1} \dots (q^{r_{n-1}+\lambda})_{l_{n-1}} = 0\}$ . Clearly  $\lambda \in K_0$ , if and only if there is some  $1 \leq j \leq n-1$  such that  $k_{j+1} > -(r_j + \lambda) \geq 0$ , giving  $L_0 = \left( \bigcup_{j=1}^{n-1} (-(r_j + k_{j+1}), -r_j] \right) \cap \mathbb{Z}_{\geq 0}$ . Similarly, one can find that  $L_0 = \left( \bigcup_{j=1}^{n-1} (-(r_j + l_j), -r_j] \right) \cap \mathbb{Z}_{\geq 0}$ . These unions of intervals are overlapping since  $-(r_j + k_{j+1}) = -(r_{j+1} + l_{j+1}) \leq -r_{j+1}$  and  $-(r_j + l_j) = -(r_{j-1} + k_j) \leq -r_{j-1}$ , for  $j = 1, \dots, n-2$  and  $j = 2, \dots, n-1$  respectively, therefore

$$\begin{aligned} \bigcup_{j=1}^{n-1} (-(r_j + k_{j+1}), -r_j] &= \left( \min_{j=1}^{n-1} (-(r_j + k_{j+1})), \max_{j=1}^{n-1} (-r_j) \right] := (m_1, r], \\ \bigcup_{j=1}^{n-1} (-(r_j + l_j), -r_j] &= \left( \min_{j=1}^{n-1} (-(r_j + l_j)), \max_{j=1}^{n-1} (-r_j) \right] := (m_2, r]. \end{aligned}$$

Now the fact that  $m_1 \leq -(r_{n-1} + k_n) = -(1 + \Delta + l_n) < 0$ , and  $m_2 \leq -(r_1 + l_1) = -(1 + k_1) < 0$ , gives  $K_0 = \{0, 1, \dots, r\} = L_0$ , which proves the first claim.

For the second claim, we now know that  $\lambda > r$  means that  $\frac{(q^{r_j+\lambda})_{k_{j+1}}}{(q^{r_j+\lambda})_{l_j}} = (q^{r_j+l_j+\lambda})_{k_{j+1}-l_j}$  is non-zero and non-singular. Therefore, (16) becomes

$$\begin{aligned} & (q^{1+\lambda})_{k_1} (q^{r_1+l_1+\lambda})_{k_2-l_1} \dots (q^{r_{n-1}+l_{n-1}+\lambda})_{k_n-l_{n-1}} [(q^{1+\Delta+\lambda})_{l_n}]^{-1} \\ &= (q^{1+\lambda})_{k_1+k_2-l_1} (q^{r_2+l_2+\lambda})_{k_3-l_2} \dots (q^{r_{n-1}+l_{n-1}+\lambda})_{k_n-l_{n-1}} [(q^{1+\Delta+\lambda})_{l_n}]^{-1} \\ &\vdots \\ &= (q^{1+\lambda})_{\Delta+l_n} [(q^{1+\Delta+\lambda})_{l_n}]^{-1} = (q^{1+\lambda})_{\Delta} = (q)_{\Delta+\lambda} [(q)_{\lambda}]^{-1}, \end{aligned}$$

as desired, where we iterate the rule  $r_j + l_j = 1 + k_1 + \sum_{a=1}^{j-1} (k_{a+1} - l_a)$  to collapse the product.  $\square$

Now we can prove [proposition 2](#) and hence [proposition 1](#).

*Proof of [Proposition 2](#).* Since  $(k_j, l_j) \mapsto (l_{n+1-j}, k_{n+1-j})$  is a symmetry of (13) we can assume WLOG that  $k \geq l$  and define  $\Delta = k - l \geq 0$ . By [lemma 1](#) we can shift the summation of the RHS of (13) to  $\kappa \mapsto \lambda + \Delta$  with  $\lambda \geq 0$ . Making use of the relations  $p_{j+1} + \Delta = r_j$  for  $j = 1, \dots, n-1$ , and  $p_1 = 1 - \Delta$ , we have

$$\begin{aligned} \sum_{\kappa=0}^{\infty} \frac{(y)_{\kappa-k}}{(q)_{\kappa}} (q^{p_1+\kappa})_{k_1} \dots (q^{p_n+\kappa})_{k_n} z^{\kappa-k} &= \sum_{\lambda=0}^{\infty} \frac{(y)_{\lambda-l}}{(q)_{\lambda+\Delta}} (q^{1+\lambda})_{k_1} (q^{p_1+\lambda})_{k_2} \dots (q^{p_{n-1}+\lambda})_{k_n} z^{\lambda-l} \\ &= \sum_{\lambda=0}^{\infty} \frac{(y)_{\lambda-l}}{(q)_{\lambda}} (q^{r_1+\lambda})_{l_1} \dots (q^{r_n+\lambda})_{l_n} z^{\lambda-l}, \end{aligned} \tag{17}$$

where the last equality follows from [lemma 2](#).  $\square$

## References

- [1] G. E. Andrews. Summations and transformations for basic appell series. *Journal of the London Mathematical Society*, s2-4(4):618–622, 1972.
- [2] G. Gasper and M. Rahman. *Basic Hypergeometric Series*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2 edition, 2004.