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Towards a Factorised Solution of the Yang-Baxter Equation with $U_q(\mathfrak{sl}_n)$ Symmetry

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Based on Honours thesis sup. by Prof. V. Mangazeev[2]

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Introduction

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  - Symmetric Group Relations

Yang-Baxter Equation

The (parameter dependent) YBE on $\mathrm{End}(V_1 \otimes V_2 \otimes V_3)$ is

$$\begin{aligned} &R_{V_1,V_2}(u_1,u_2)R_{V_1,V_3}(u_1,u_3)R_{V_2,V_3}(u_2,u_3) \\ &\quad = R_{V_2,V_3}(u_2,u_3)R_{V_1,V_3}(u_1,u_3)R_{V_1,V_2}(u_1,u_2), \end{aligned}$$

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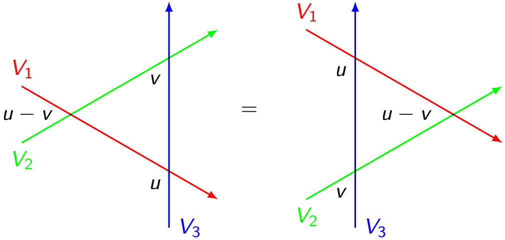
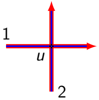


RLL-Method

Our Goal: construct an R-matrix $\mathcal{R}(u) \in \mathrm{End}(\mathcal{V} \otimes \mathcal{V})$,

$\mathcal{R}_{12}(u) =$ 

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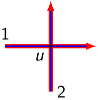
$\mathcal{R}_{12}(u) =$ 

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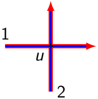
$\mathcal{R}_{12}(u) =$ 

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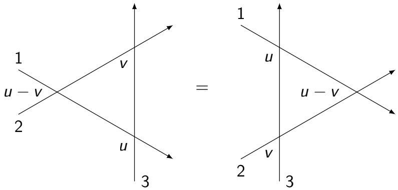

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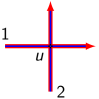
$\mathcal{R}_{12}(u) =$ 

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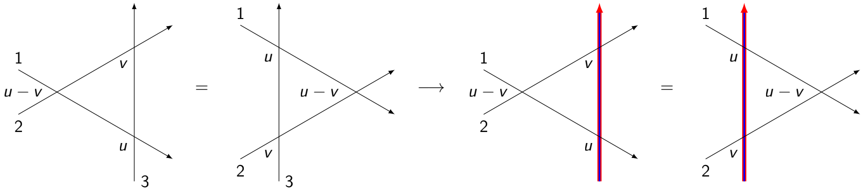

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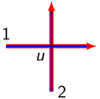
$\mathcal{R}_{12}(u) =$ 

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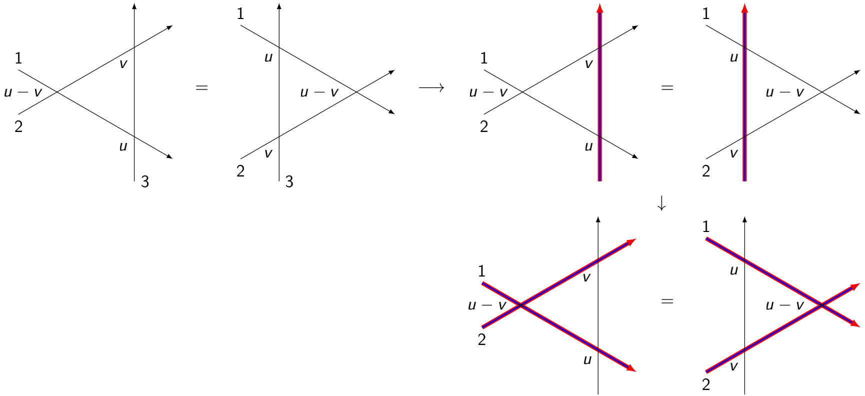

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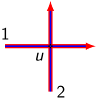
$\mathcal{R}_{12}(u) =$ 

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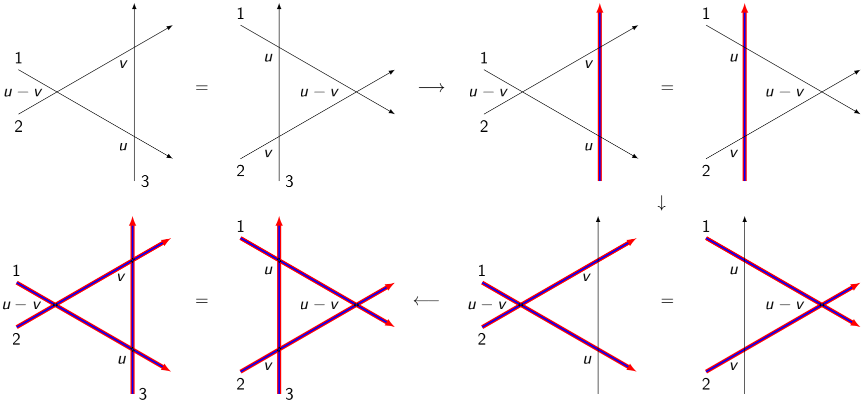
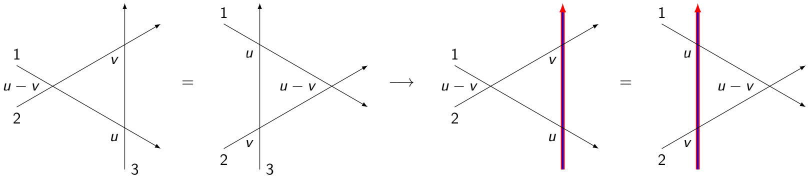


Fundamental R-Matrix and L-operators



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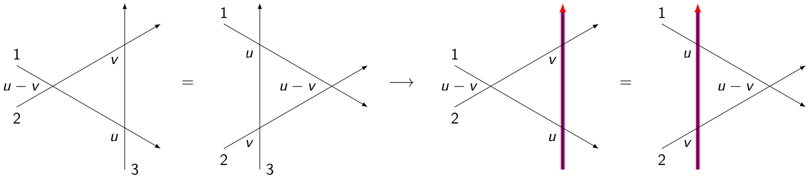

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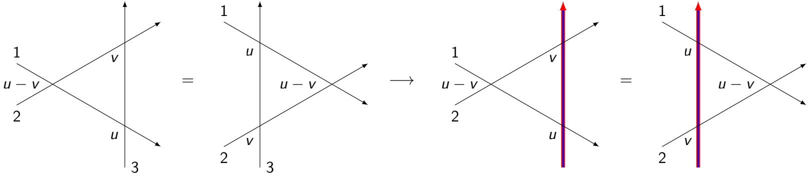

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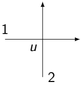
- $R_{12}(u) =$  $\in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ (an $n^2 \times n^2$ matrix).

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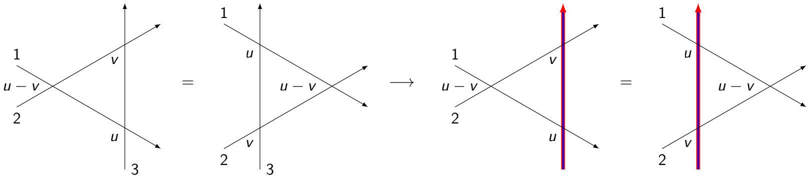

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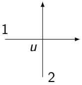
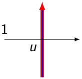
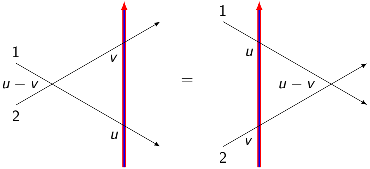
- $R_{12}(u) =$  $\in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ (an $n^2 \times n^2$ matrix).

- $L_1(u) =$  $\in \mathrm{End}(\mathbb{C}^n) \otimes \mathcal{A}$, where $\mathcal{A} \subset \mathrm{End}(\mathcal{V})$. An $n \times n$ matrix with values in $\mathcal{A}$.

Fundamental R-Matrix and Universal L-operators



$$\sim \quad \begin{aligned} &R_{12}(u-v)L_1(u)L_2(v) \\ &= L_2(v)L_1(u)R_{12}(u-v) \end{aligned}.$$

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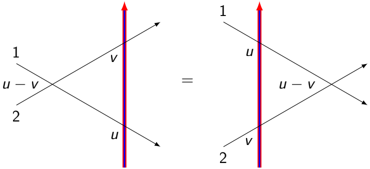

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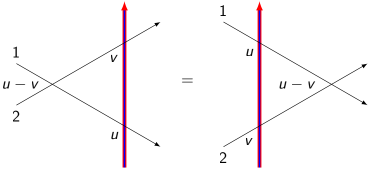

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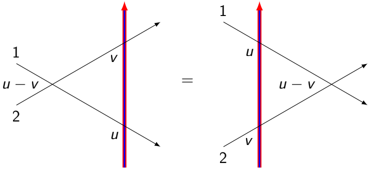

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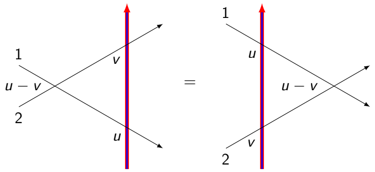

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Why YBE for R? This is a consistency condition for associativity of $\mathcal{A}$.

Undeformed Case: $\mathfrak{sl}_n$

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The universal enveloping algebra (UEA) $\mathcal{A} = U(\mathfrak{sl}_n)$ has a fundamental R-matrix

$$R_{12}(u) = u \cdot \mathrm{id}_{n^2} + P_{12} : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n,$$

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where, P_{12} is the flip $P_{12}(x_1 \otimes x_2) = x_2 \otimes x_1$, and a universal L-operator

$$L(u) = u \cdot \mathrm{id}_n \otimes 1_{\mathcal{A}} + \sum_{i,j=1}^{n} e_{ij} \otimes E_{ji},$$

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where e_{ij} is the matrix unit. Here $\{E_{ij}\}$ is the Cartan-Weyl basis for $\mathfrak{sl}_n$:

$$h_i = E_{ii} - E_{i+1,i+1}, \quad \textstyle\sum_i E_{ii} = 0, \quad E_{i,i+1} = e_i, \quad E_{i+1,i} = f_i,$$
$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{ik} E_{lj}.$$

Differential Representation of $\mathfrak{sl}_n$

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[Derkachov and Manashov, 2006]

Differential Representation of $\mathfrak{sl}_n$

E.g. $n = 2$ case: Taking $N_x = x\partial x$ and $m = \rho_2 - \rho_1 + 1$,

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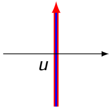
- It is reducible if some $m_i \in \mathbb{Z}_{\leq 0}$. It contains a finite dimensional irreducible subrep iff true for all m_i.
- It has a factorised L-operator!

$$L(\boldsymbol{u}) = ZD(\boldsymbol{u})Z^{-1} =$$



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$\boldsymbol{u} = (u_i)$, where $u_i = u - \rho_i$.

q-Deformed Case: $U_q(\mathfrak{sl}_n)$

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The q-deformed UEA $U_q(\mathfrak{sl}_n)$: For some $q = e^h \in \mathbb{C} \setminus \{0, \pm 1\}$

- Generators: e_i, f_i, and invertible $k_i = q^{h_i}$ for $i = 1, 2, \ldots, n-1$

q-Deformed Case: $U_q(\mathfrak{sl}_n)$

The q-deformed UEA $U_q(\mathfrak{sl}_n)$: For some $q = e^h \in \mathbb{C} \setminus \{0, \pm 1\}$

- Generators: e_i, f_i, and invertible $k_i = q^{h_i}$ for $i = 1, 2, \ldots, n-1$
- Relations:

$$[k_i, k_j] = 0, \quad k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{a_{ij}} f_i,$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} = \delta_{ij} [h_i]_q,$$

$$[e_i, e_j] = [f_i, f_j] = 0, \quad \text{for } |i - j| > 1,$$

$$g_i^2 g_{i\pm 1} - (q + q^{-1}) g_i g_{i\pm 1} g_i + g_{i\pm 1} g_i^2 = 0,$$

$g_i = e_i, f_i$.

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- Notation: $[x]_q = (q^x - q^{-x})/(q - q^{-1})$

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$$R(u) = q^u R + q^{-u} R^{-1} \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n),$$

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Now specialise:

Is there an analogous class of representations for $U_q(\mathfrak{sl}_n)$? How about a factorised L-operator?

q-Difference Representation of $U_q(\mathfrak{sl}_n)$

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$q^{\alpha + \sum \alpha_{ij} N_{ij}} f(x_{21}, \dots, x_{n,n-1}) = q^{\alpha} f(q^{\alpha_{21}} x_{21}, \dots, q^{\alpha_{n,n-1}} x_{n,n-1})$

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<u>$n = 2$ case:</u> Just one variable $x_{21} = x$

$$f = -D_x, \quad e = x[m+N_x]_q, \quad h = 2N_x + m,$$

$$\begin{tikzcd}
0 & 1 \arrow[l, bend right, "\times 0"'] \arrow[loop above, "\times m"] \arrow[r, bend right, "\times [m]_q"'] & x \arrow[l, bend right, "\times -1"'] \arrow[loop above, "\times (m+2)"] \arrow[r, bend right, "\times [m+1]_q"'] & x^2 \arrow[l, bend right, "\times -[2]_q"'] \arrow[loop above, "\times (m+4)"] \arrow[r, bend right, "\times [m+2]_q"'] & \cdots \arrow[l, bend right, "\times -[3]_q"'] \arrow[r, bend right, "\times [m+n-1]_q"'] & x^n \arrow[l, bend right, "\times -[n]_q"'] \arrow[loop above, "\times (m+2n)"] \arrow[r, bend right, "\times [m+n]_q"'] & \cdots \arrow[l, bend right, "\times -[n+1]_q"']
\end{tikzcd}$$

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- For $\boldsymbol{\rho} \in \mathbb{C}^n$, there is an analogous representation $\mathcal{V}_{\boldsymbol{\rho}}$ of $U_q(\mathfrak{sl}_n)$ [Dobrev, Truini, and Biedenharn, 1994].

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- An Explicit formula: $m_i = \rho_{n-i} - \rho_{n+1-i} + 1$

$$E_{ii}^{(n)} = -\rho_{n+1-i} - \sum_{j=1}^{i-1} N_{ij} + \sum_{j=i+1}^{n} (N_{ji}+1)\,,$$

$$f_i^{(n)} = -D_{i+1,i}\, q^{\sum_{j=1}^{i-1}(N_{ij}-N_{i+1,j})} - \sum_{j=1}^{i-1} x_{ij} D_{i+1,j}\, q^{\sum_{k=1}^{j-1}(N_{ik}-N_{i+1,k})}\,,$$

$$e_i^{(n)} = \begin{array}{l} x_{i+1,i}\Big[\, m_i + N_{i+1,i} + \sum_{j=i+2}^{n}(N_{ji}-N_{j,i+1})\Big]_q + q^{-m_i} \sum_{j=i+2}^{n} x_{ji} D_{j,i+1}\, q^{\sum_{k=j}^{n}(N_{k,i+1}-N_{k,i})} \\ \qquad - q^{m_i+2N_{i+1,i}} \sum_{j=1}^{i-1} x_{i+1,j} D_{ij}\, q^{\sum_{k=i+2}^{n}(N_{ki}-N_{k,i+1}) + \sum_{k=j+1}^{i-1}(N_{i+1,k}-N_{i,k})} \end{array},$$

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$$e_i^{(n)} = \begin{matrix} x_{i+1,i}\Big[\, m_i + N_{i+1,i} + \sum_{j=i+2}^{n}(N_{ji}-N_{j,i+1}) \,\Big]_q + q^{-m_i} \sum_{j=i+2}^{n} x_{ji} D_{j,i+1}\, q^{\sum_{k=j}^{n}(N_{k,i+1}-N_{k,i})} \\ - q^{m_i+2N_{i+1,i}} \sum_{j=1}^{i-1} x_{i+1,j} D_{ij}\, q^{\sum_{k=i+2}^{n}(N_{ki}-N_{k,i+1}) + \sum_{k=j+1}^{i-1}(N_{i+1,k}-N_{i,k})} \end{matrix}\,,$$

[Awata, Noumi, and Odake, 1994]

Factorised L-operator?

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$\underline{\mathfrak{sl}_n:}$ $L(\boldsymbol{u}) = ZD(\boldsymbol{u})Z^{-1}$

$$Z = \begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ x_{31} & x_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n1} & x_{n2} & \dots & x_{n,n-1} & 1 \end{pmatrix}, \quad D(\boldsymbol{u}) = \begin{pmatrix} u_n & P_{21} & P_{31} & \dots & P_{n1} \\ & u_{n-1} & P_{32} & \dots & P_{n2} \\ & & \ddots & \ddots & \vdots \\ & & & u_2 & P_{n,n-1} \\ & & & & u_1 \end{pmatrix},$$

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$\underline{U_q(\mathfrak{sl}_n)}$: Postulate $L(\boldsymbol{u}) = Z_1(\boldsymbol{u})D(\boldsymbol{u})Z_2(\boldsymbol{u})^{-1}$

$$D(\boldsymbol{u}) = \begin{pmatrix} [u_n]_q q^{b_1} & P_{21} & \dots & P_{n1} \\ & \ddots & \ddots & \vdots \\ & & [u_2]_q q^{b_{n-1}} & P_{n,n-1} \\ & & & [u_1]_q q^{b_n} \end{pmatrix},$$

$$P_{ij} = -D_{ij} q^{b_{ij}} - \sum_{k=i+1}^{n} x_{ki} D_{kj} q^{b_{ijk}}, \quad Z_i(\boldsymbol{u}) = \begin{pmatrix} 1 & & & \\ x_{21} q^{a^{(i)}_{21}} & 1 & & \\ \vdots & \ddots & \ddots & \\ x_{n1} q^{a^{(i)}_{n1}} & \dots & x_{n,n-1} q^{a^{(i)}_{n,n-1}} & 1 \end{pmatrix},$$

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n=2: Yes [Derkachov, Karakhanyan, and Kirschner, 2007]

$$L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ q^{u_1 - N_x} x & 1 \end{pmatrix} \begin{pmatrix} [u_2]_q q^{-N_x - 1} & -D_x q^{N_x} \\ 0 & [u_1]_q q^{N_x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q^{u_2 - N_x} x & 1 \end{pmatrix}.$$

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n=3: Yes [Valinevich et al., 2008], $L(u_1,u_2,u_3)=Z_1 D Z_2^{-1}$ with

$$D=\begin{pmatrix}[u_3]_q q^{-N_{21}+N_{31}} & (D_{21}+x_{32}D_{31}q^{N_{31}-N_{32}-1})q^{N_{21}+N_{31}} & D_{31}q^{N_{31}}\\ 0 & [u_2]_q q^{N_{21}-N_{32}} & D_{32}q^{u_2-N_{31}+N_{32}}\\ 0 & 0 & [u_1]_q q^{N_{32}+N_{31}}\end{pmatrix},$$

$$Z_1=\begin{pmatrix}1 & 0 & 0\\ q^{u_2-N_{31}+N_{32}-N_{21}}x_{21} & 1 & 0\\ q^{-u_1-N_{31}+N_{32}}x_{31} & q^{u_1-u_2-N_{32}}x_{32} & 1\end{pmatrix},\quad Z_2=\begin{pmatrix}1 & 0 & 0\\ q^{c_{21}}x_{21} & 1 & 0\\ q^{c_{31}}x_{31} & q^{c_{32}}x_{32} & 1\end{pmatrix},$$

$$c_{21}=u_3-N_{21},\ c_{31}=-u_3-N_{31}-N_{21}-1,\ c_{32}=N_{21}+N_{31}-N_{32}.$$

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"Controlled deformation" breaks - We have "pure quantum phenomena" in the Cartan-Weyl elements:

$$E_{42} = [f_3, f_2]_q = -D_{42}q^{N_{21}-N_{32}-N_{41}-1} - x_{21}D_{41}q^{-(1+N_{31})} + (q-q^{-1})x_{31}D_{41}D_{32}q^{N_{21}-N_{31}-1}.$$

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A similar term appears in the E_{24} Cartan-Weyl element.

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Such terms cannot arise from our ansatz.

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n=4: A modified factorisation $L(\boldsymbol{u}) = Z_1(\boldsymbol{u})D(\boldsymbol{u})Z_2(\boldsymbol{u})^{-1}$

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<u>n=4:</u> A modified factorisation $L(\boldsymbol{u}) = Z_1(\boldsymbol{u})D(\boldsymbol{u})Z_2(\boldsymbol{u})^{-1}$

$$Z_1 = \begin{pmatrix} 1 & & & \\ x_{21}q^{a_{21}} & 1 & & \\ x_{31}q^{a_{31}} & x_{32}q^{a_{32}} & 1 & \\ x_{41}q^{a_{41}} & x_{42}q^{a_{42}} & x_{43}q^{a_{43}} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & & & \\ \begin{matrix} x_{21}q^{a_{21}} \\ -(q-q^{-1})x_{31}D_{32}q^{a_{321}} \end{matrix} & 1 & & \\ x_{31}q^{a_{31}} & x_{32}q^{a_{32}} & 1 & \\ x_{41}q^{a_{41}} & x_{42}q^{a_{42}} & x_{43}q^{a_{43}} & 1 \end{pmatrix},$$

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Factorised L-operator?

<u>General n:</u> Order of highest term in $(q - q^{-1})$

$$\mathcal{O}(L^+(\boldsymbol{u})) \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ & & 0 & 0 & 1 & 2 & 2 & 2 \\ & & & 0 & 0 & 1 & 2 & 3 \\ & & & & 0 & 0 & 1 & 2 \\ & & & & & 0 & 0 & 1 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{pmatrix}$$

So the factorisation must involve higher order terms in $(q - q^{-1})$.

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So the factorisation must involve higher order terms in $(q - q^{-1})$.
Potentially can be factored further into elementary row/column matrices which are only first order.

Parameter Permutations and YBE

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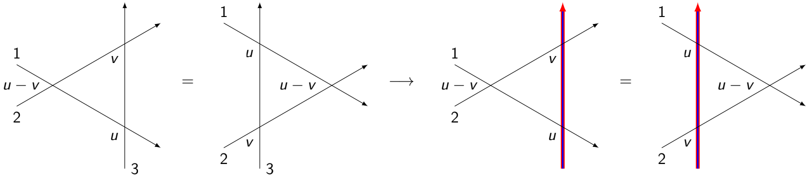
Parameter Permutations and YBE

So far



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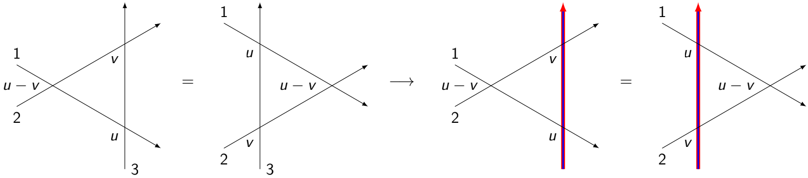
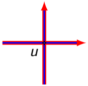


Now we're seeking $\mathcal{R}(u) =$ 

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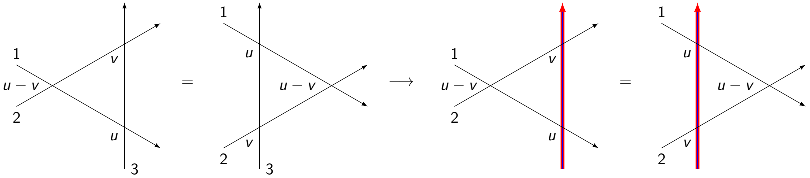
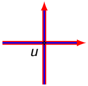
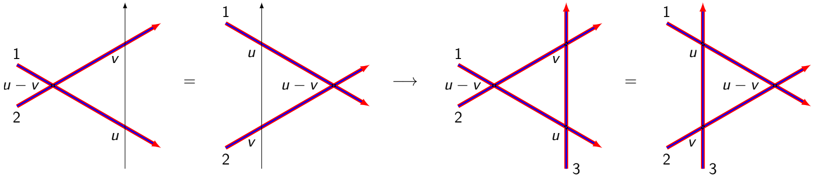


Now we're seeking $\mathcal{R}(u) =$  $\in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\sigma}})$



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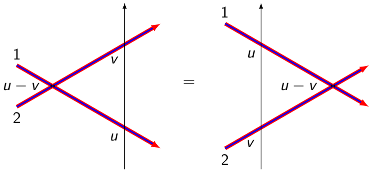

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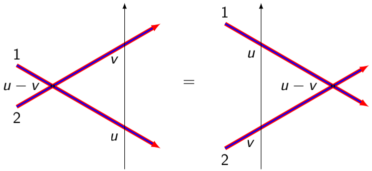

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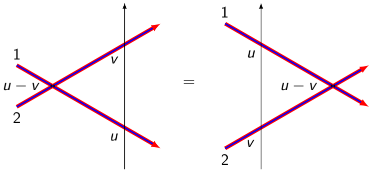

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IDEA: Factorise $\check{\mathcal{R}}(u-v)$ in terms of elementary transposition operators $\mathcal{S}_i \in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\sigma}})$

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and a single "exchange" operator:

$$\mathcal{S}_n(\boldsymbol{u},\boldsymbol{v})L_{12}(\boldsymbol{u},\boldsymbol{v}) = \mathcal{S}_n(\boldsymbol{u},\boldsymbol{v})L_{12}(u_1,\ldots,u_{n-1},v_1,u_n,v_2,\ldots,v_n).$$

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These operators should define an action of S_{2n}, *i.e.*,

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YBE then follows from equivalence of the decompositions in $\mathrm{Perm}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$

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$U_q(\mathfrak{sl}_2)$ Case: [Derkachov, Karakhanyan, and Kirschner, 2007]

$U_q(\mathfrak{sl}_3)$ Case: [Valinevich et al., 2008]

q-Deformed Case

Proposition

The intertwiners for the $U_q(\mathfrak{sl}_n)$ ($|q|< 1$) L-operator are given by

$$\mathcal{T}^{(n)}_{n-i}(\alpha) = \left(\Lambda^{(n)}_{n-i}\right)^{\alpha} \frac{e_{q^2}(q^{2(N_{i+1,i}+1)}\boldsymbol{X}^{(n)}_{n-i})}{e_{q^2}(q^{2(N_{i+1,i}+1-\alpha)}\boldsymbol{X}^{(n)}_{n-i})},$$

$$e_{q^2}(\boldsymbol{Z}) = ((\boldsymbol{Z};q^2)_\infty)^{-1} = \left[(1-\boldsymbol{Z})(1-q^2\boldsymbol{Z})(1-q^{2\cdot 2}\boldsymbol{Z})\dots\right]^{-1},$$

$$\frac{e_{q^2}(\boldsymbol{Z})}{e_{q^2}(q^{-\alpha}\boldsymbol{Z})} = \sum_{j=0}^{\infty} \frac{(q^{-\alpha};q)_j}{(q;q)_j}\boldsymbol{Z}^j, \qquad \Lambda^{(n)}_{n-i} = (x_{i+1,i})^{-1}q^{\beta_i}$$

where $\alpha = u_{n-i} - u_{n+1-i}$, and

$$\boldsymbol{X}^{(n)}_{n-i} = 1 + x_{i+1,i}\sum_{j=i+2}^{n}\frac{x_{j,i+1}}{x_{ji}}(q^{N_{ij}} - q^{-N_{ij}})q^{\gamma_i}.$$

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Obtained using an approach from [Valinevich et al., 2008].

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The only non-trivial relation is the braid relation

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After a series expansion it is reduced to a family of (terminating) q-series identity relating rank $i+1$ and rank $2i-1$ q-Lauricella series. $\square$

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where $M = \sum_{i=1}^{n} m_i$ and

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[Andrews, 1972] gives a general transformation formula allowing us to rewrite $(\star)$ in terms of a ${}_{n+1}\phi_n$ hypergeometric series.

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For $n \geq 1$ and non-negative integer tuples

$$\boldsymbol{k} = (k_0, \ldots, k_n) = (k_0, \tilde{\boldsymbol{k}}), \quad \boldsymbol{l} = (l_1, \ldots, l_n), \quad \boldsymbol{m} = (m_1, \ldots, m_{n-1}),$$

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The identity we need is the equality $\Theta_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} = \Omega_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}}$

$$\Theta_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} = \frac{(\xi; q)_{L+M}}{(\xi\,\zeta; q)_{L+M}} \Phi_D^{(2n-1)}\big[\,\zeta; q^{-\boldsymbol{l}}, q^{-\boldsymbol{m}}\,; q^{1-L-M}/\xi; \, q^{\boldsymbol{r}+\boldsymbol{l}+(\boldsymbol{m},0)}, q^{(r_i,\hat{r}_n)+\boldsymbol{m}}\,\big],$$

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for arbitrary complex parameters ξ, ζ.

Exchange Operator

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Now we can reduce the defining relation to

$$\begin{aligned} &Z_2^{(x,\tilde{\boldsymbol{u}})}(v_1)\left[(D^{(x,\tilde{\boldsymbol{u}})})^{-1}\,\mathcal{S}_n\,D^{(x,\tilde{\boldsymbol{u}})}\right]\left(Z_2^{(x,\tilde{\boldsymbol{u}})}(u_n)\right)^{-1} \\ &\qquad = Z_1^{(y,\tilde{\boldsymbol{v}})}(u_n)\left[D^{(y,\tilde{\boldsymbol{v}})}\,\mathcal{S}_n\,(D^{(y,\tilde{\boldsymbol{v}})})^{-1}\right]\left(Z_1^{(y,\tilde{\boldsymbol{v}})}(v_1)\right)^{-1}, \end{aligned}$$

if $\mathcal{S}_n^{(x,y)}$ commutes (element wise) with $Z_1^{(x)}$ and $Z_2^{(y)}$.

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Recall in the $n \geq 4$ case the postulated ansatz for the factorisation was inconsistent - the outer most factors will now have q-difference terms.

This seems to represent a serious obstruction to constructing the exchange operator - unclear whether to expect a multiplication operator (by shifted variables) to work or not

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- We explain how the failure of the factorisation property for the $U_q(\mathfrak{sl}_4)$ L-operator represents an obstruction to constructing the missing "exchange" operator.

Thank You!

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Questions?

References

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