



Towards a Factorised Solution of the Yang-Baxter Equation with $U_q(\mathfrak{sl}_n)$ Symmetry

Benjamin Morris¹

Based on Honours thesis sup. by Prof. V. Mangazeev²

¹School of Mathematics,

University of Leeds

²Department of Theoretical Physics,

Australian National University

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Yang-Baxter Equation

The (parameter dependent) YBE on $\text{End}(V_1 \otimes V_2 \otimes V_3)$ is

$$\begin{aligned} R_{V_1, V_2}(u_1, u_2) R_{V_1, V_3}(u_1, u_3) R_{V_2, V_3}(u_2, u_3) \\ = R_{V_2, V_3}(u_2, u_3) R_{V_1, V_3}(u_1, u_3) R_{V_1, V_2}(u_1, u_2), \end{aligned}$$

$(R_{V_i, V_j}(u_i, u_j) \text{ invertible}).$

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Additive dependence $\Rightarrow R_{V_i, V_j}(u_i, u_j) = R_{V_i, V_j}(u_i - u_j)$

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Yang-Baxter Equation

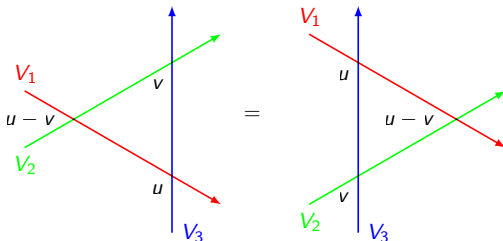
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RLL -Method

Our Goal: construct an R -matrix $\mathcal{R}(u) \in \text{End}(\mathcal{V} \otimes \mathcal{V})$,


$$\mathcal{R}_{12}(u) = \begin{array}{c} \begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \end{array} \\ \begin{array}{c} \text{---} \text{---} \text{---} \\ \downarrow \end{array} \end{array} \cdot$$

u

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
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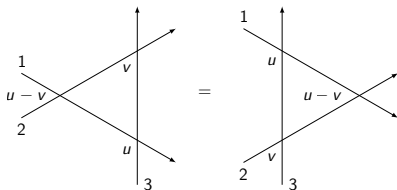
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
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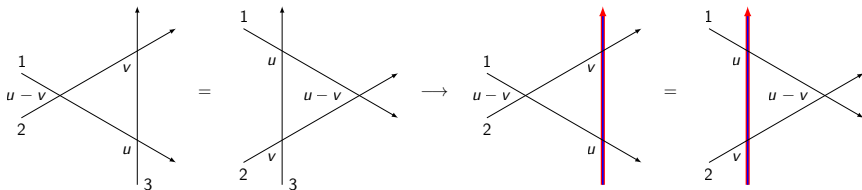
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
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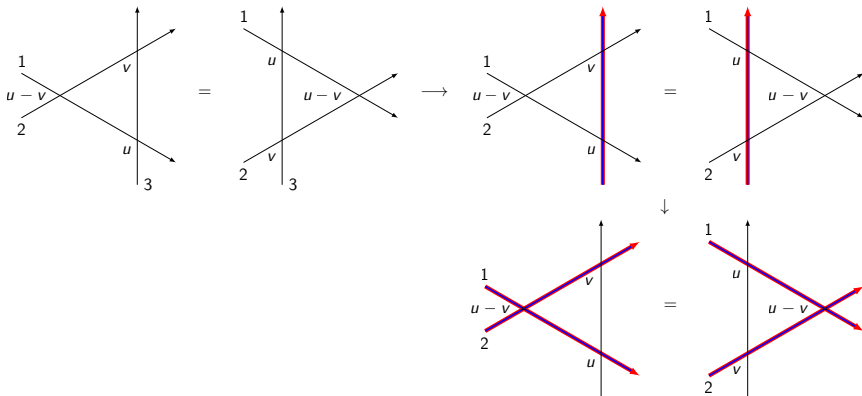
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
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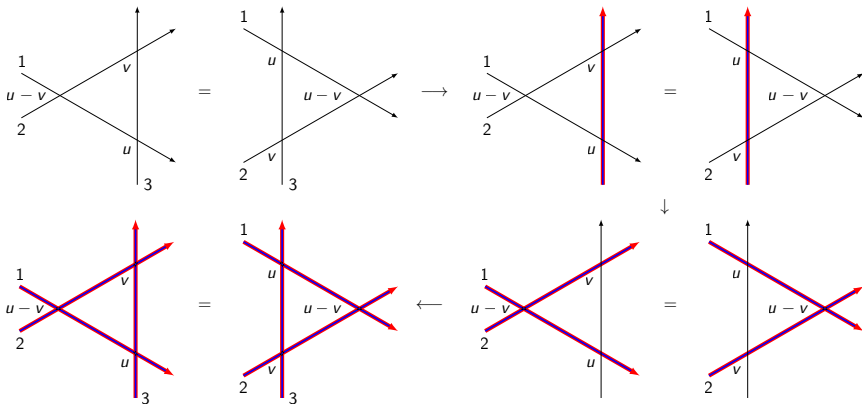
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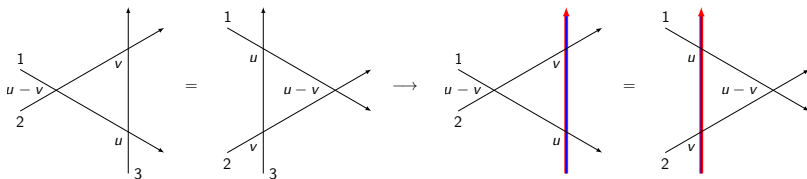
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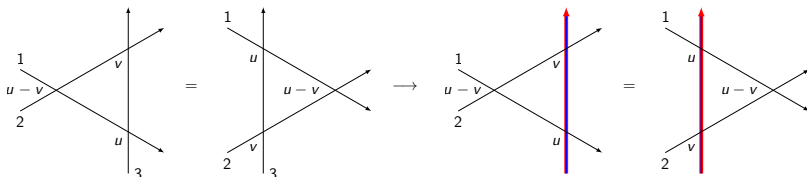
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Fundamental R -Matrix and L -operators

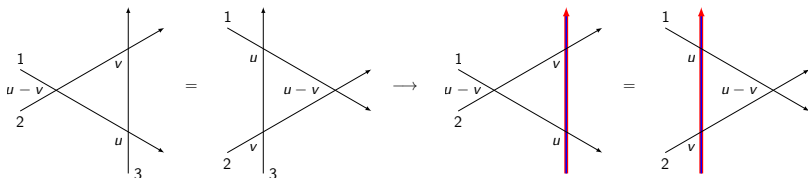


Fundamental R -Matrix and L -operators



This requires two matrices:

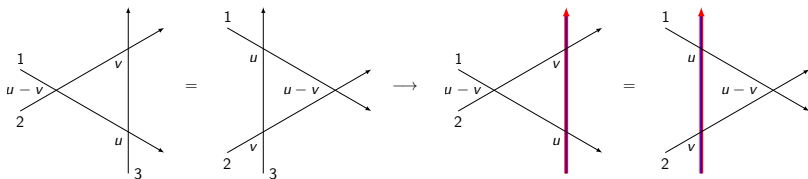
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► $R_{12}(u) = \begin{array}{c} \nearrow \\ \text{1} \\ \text{---} \\ \text{u} \\ \searrow \\ \text{2} \end{array} \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \text{ (an } n^2 \times n^2 \text{ matrix).}$

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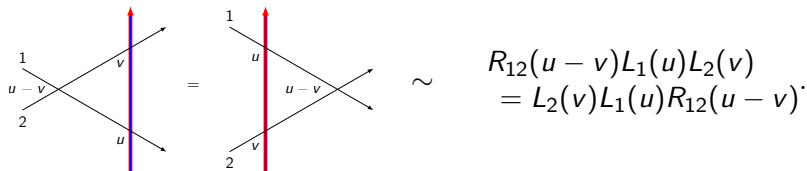


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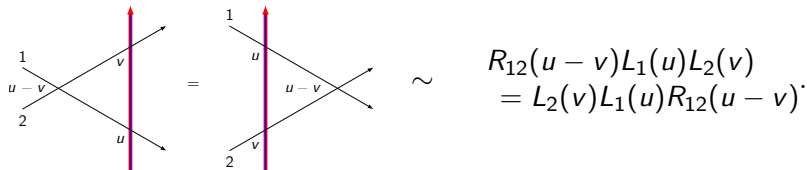
► $L_1(u) = \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ u \\ \diagdown \quad \diagup \\ 2 \end{array} \in \text{End}(\mathbb{C}^n) \otimes \mathcal{A}$, where $\mathcal{A} \subset \text{End}(\mathcal{V})$. An $n \times n$ matrix with values in \mathcal{A} .

Fundamental R -Matrix and Universal L -operators



$$\begin{array}{c}
 \begin{array}{c} 1 \\ u-v \\ 2 \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} v \\ \text{red line} \end{array} = \begin{array}{c} 1 \\ u \\ 2 \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} \begin{array}{c} v \\ \text{red line} \end{array} \sim \\
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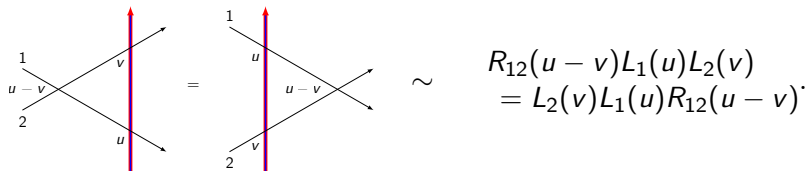
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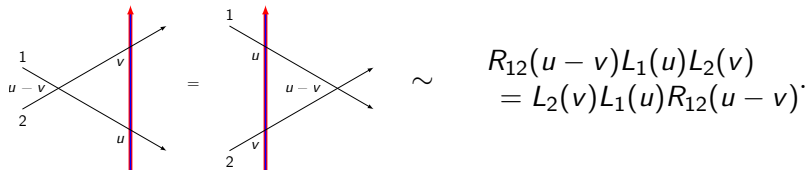


$$\begin{array}{c} \text{Diagram 1: Two lines labeled 1 and 2 cross. A vertical red line labeled } v \text{ is to the right. The crossing is labeled } u-v. \\ \text{Diagram 2: Two lines labeled 1 and 2 cross. A vertical red line labeled } v \text{ is to the left. The crossing is labeled } u-v. \end{array} = \sim \begin{array}{l} R_{12}(u-v)L_1(u)L_2(v) \\ = L_2(v)L_1(u)R_{12}(u-v) \end{array}$$

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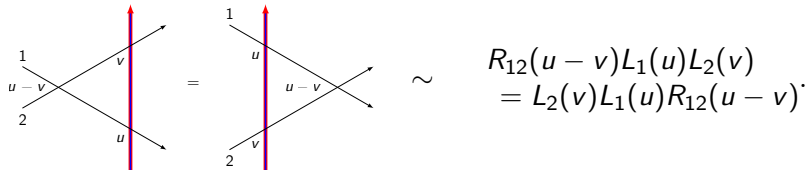
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Why YBE for R ? This is a consistency condition for associativity of \mathcal{A} .



Undeformed Case: \mathfrak{sl}_n



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The universal enveloping algebra (UEA) $\mathcal{A} = U(\mathfrak{sl}_n)$ has a fundamental R -matrix

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$$L(u) = u \cdot \text{id}_n \otimes 1_{\mathcal{A}} + \sum_{i,j=1}^n e_{ij} \otimes E_{ji},$$

where e_{ij} is the matrix unit.



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where e_{ij} is the matrix unit. Here $\{E_{ij}\}$ is the Cartan-Weyl basis for \mathfrak{sl}_n :

$$\begin{aligned} h_i &= E_{ii} - E_{i+1,i+1}, \quad \sum_i E_{ii} = 0, \quad E_{i,i+1} = e_i, \quad E_{i+1,i} = f_i, \\ [E_{ij}, E_{kl}] &= \delta_{jk} E_{il} - \delta_{ik} E_{lj}. \end{aligned}$$



Differential Representation of \mathfrak{sl}_n

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$$P_{ij} = -\partial_{ij} - \sum_{k=i+1}^n x_{ki} \cdot \partial_{kj}.$$



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[Derkachov and Manashov, 2006]



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E.g. $n = 2$ case: Taking $N_x = x\partial_x$ and $m = \rho_2 - \rho_1 + 1$,

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- ▶ It is reducible if some $m_i \in \mathbb{Z}_{\leq 0}$. It contains a finite dimensional irreducible subrep iff true for all m_i .
- ▶ It has a factorised L -operator!

$$L(\mathbf{u}) = ZD(\mathbf{u})Z^{-1} = \begin{array}{c} \uparrow \\ \text{---} u \text{---} \\ \downarrow \end{array},$$

$$\mathbf{u} = (u_i), \text{ where } u_i = u - \rho_i.$$



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$$[k_i, k_j] = 0, \quad k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{a_{ij}} f_j,$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} = \delta_{ij} [h_i]_q,$$

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$$g_i^2 g_{i\pm 1} - (q + q^{-1}) g_i g_{i\pm 1} g_i + g_{i\pm 1} g_i^2 = 0,$$

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$g_i = e_i, f_i$. The a_{ij} are components of the A_n Cartan matrix.

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The q -deformed UEA $U_q(\mathfrak{sl}_n)$: For some $q = e^h \in \mathbb{C} \setminus \{0, \pm 1\}$

- Generators: e_i, f_i , and invertible $k_i = q^{h_i}$ for $i = 1, 2, \dots, n-1$
- Relations:

$$[k_i, k_j] = 0, \quad k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{a_{ij}} f_j,$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} = \delta_{ij} [h_i]_q,$$

$$[e_i, e_j] = [f_i, f_j] = 0, \quad \text{for } |i - j| > 1,$$

$$g_i^2 g_{i\pm 1} - (q + q^{-1}) g_i g_{i\pm 1} g_i + g_{i\pm 1} g_i^2 = 0,$$

$g_i = e_i, f_i$. The a_{ij} are components of the A_n Cartan matrix.

- Notation: $[x]_q = (q^x - q^{-x}) / (q - q^{-1})$



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and a universal L -operator [Jimbo, 1986]

$$L(u) = q^u L^+ + q^{-u} L^- \in \text{End}(\mathbb{C}^n) \otimes U_q(\mathfrak{sl}_n),$$

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Now specialise:

Is there an analogous class of representations for $U_q(\mathfrak{sl}_n)$? How about a factorised L -operator?



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\mathfrak{sl}_n : differential representation $\leftrightarrow U_q(\mathfrak{sl}_n)$: “ q -difference” representation:



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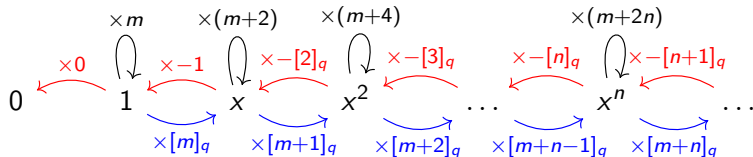
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$n = 2$ case: Just one variable $x_{21} = x$

$$f = -D_x, \quad e = x[m + N_x]_q, \quad h = 2N_x + m,$$





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$$E_{ij}^{(n)} = -\rho_{n+1-i} - \sum_{j=1}^{i-1} N_{ij} + \sum_{j=i+1}^n (N_{ji} + 1),$$

$$f_i^{(n)} = -D_{i+1,i} q^{\sum_{j=1}^{i-1} (N_{ij} - N_{i+1,j})} - \sum_{j=1}^{i-1} x_{ij} D_{i+1,j} q^{\sum_{k=1}^{j-1} (N_{ik} - N_{i+1,k})},$$

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$$= x_{i+1,i} \left[m_i + N_{i+1,i} + \sum_{j=i+2}^n (N_{ji} - N_{j,i+1}) \right]_q + q^{-m_i} \sum_{j=i+2}^n x_{ji} D_{j,i+1} q^{\sum_{k=j}^n (N_{k,i+1} - N_{k,i})} \\ - q^{m_i + 2N_{i+1,i}} \sum_{j=1}^{i-1} x_{i+1,j} D_{ij} q^{\sum_{k=i+2}^n (N_{ki} - N_{k,i+1}) + \sum_{k=j+1}^{i-1} (N_{i+1,k} - N_{i,k})},$$



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[Awata, Noumi, and Odake, 1994]



Factorised L -operator?



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$$\underline{\mathfrak{sl}_n}: L(\mathbf{u}) = ZD(\mathbf{u})Z^{-1}$$

$$Z = \begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ x_{31} & x_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n1} & x_{n2} & \dots & x_{n,n-1} & 1 \end{pmatrix}, \quad D(\mathbf{u}) = \begin{pmatrix} u_n & P_{21} & P_{31} & \dots & P_{n1} \\ & u_{n-1} & P_{32} & \dots & P_{n2} \\ & & \ddots & \ddots & \vdots \\ & & & u_2 & P_{n,n-1} \\ & & & & u_1 \end{pmatrix},$$



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$$\underline{U_q(\mathfrak{sl}_n)}: \text{Postulate } L(\mathbf{u}) = Z_1(\mathbf{u})D(\mathbf{u})Z_2(\mathbf{u})^{-1}$$

$$D(\mathbf{u}) = \begin{pmatrix} [u_n]_q q^{b_1} & P_{21} & \dots & P_{n1} \\ & \ddots & \ddots & \vdots \\ & & [u_2]_q q^{b_{n-1}} & P_{n,n-1} \\ & & & [u_1]_q q^{b_n} \end{pmatrix},$$

$$P_{ij} = -D_{ij}q^{b_{ij}} - \sum_{k=i+1}^n x_{ki} D_{kj} q^{b_{ijk}}, \quad Z_i(\mathbf{u}) = \begin{pmatrix} 1 & & & & \\ x_{21} q^{a_{21}^{(i)}} & 1 & & & \\ \vdots & \ddots & \ddots & \ddots & \\ x_{n1} q^{a_{n1}^{(i)}} & \dots & x_{n,n-1} q^{a_{n,n-1}^{(i)}} & 1 \end{pmatrix},$$



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$n=2$: Yes [Derkachov, Karakhanyan, and Kirschner, 2007]

$$L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ q^{u_1 - N_x} & 1 \end{pmatrix} \begin{pmatrix} [u_2]_q q^{-N_x - 1} & -D_x q^{N_x} \\ 0 & [u_1]_q q^{N_x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q^{u_2 - N_x} & 1 \end{pmatrix}.$$

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$n=3$: Yes [Valinevich et al., 2008], $L(u_1, u_2, u_3) = Z_1 D Z_2^{-1}$ with

$$D = \begin{pmatrix} [u_3]_q q^{-N_{21} + N_{31}} & (D_{21} + x_{32} D_{31} q^{N_{31} - N_{32} - 1}) q^{N_{21} + N_{31}} & D_{31} q^{N_{31}} \\ 0 & [u_2]_q q^{N_{21} - N_{32}} & D_{32} q^{u_2 - N_{31} + N_{32}} \\ 0 & 0 & [u_1]_q q^{N_{32} + N_{31}} \end{pmatrix},$$

$$Z_1 = \begin{pmatrix} 1 & 0 & 0 \\ q^{u_2 - N_{31} + N_{32} - N_{21}} x_{21} & 1 & 0 \\ q^{-u_1 - N_{31} + N_{32}} x_{31} & q^{u_1 - u_2 - N_{32}} x_{32} & 1 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 1 & 0 & 0 \\ q^{c_{21}} x_{21} & 1 & 0 \\ q^{c_{31}} x_{31} & q^{c_{32}} x_{32} & 1 \end{pmatrix},$$

$$c_{21} = u_3 - N_{21}, \quad c_{31} = -u_3 - N_{31} - N_{21} - 1, \quad c_{32} = N_{21} + N_{31} - N_{32}.$$



Factorised L -operator?

$n=4$:



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“Controlled deformation” breaks - We have “pure quantum phenomena” in the Cartan-Weyl elements:

$$E_{42} = [f_3, f_2]_q = -D_{42}q^{N_{21}-N_{32}-N_{41}-1} - x_{21}D_{41}q^{-(1+N_{31})} \\ + (q - q^{-1})x_{31}D_{41}D_{32}q^{N_{21}-N_{31}-1}.$$

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Such terms cannot arise from our ansatz.



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$n=4$: A modified factorisation $L(\mathbf{u}) = Z_1(\mathbf{u})D(\mathbf{u})Z_2(\mathbf{u})^{-1}$



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Factorised L -operator?

General n : Order of highest term in $(q - q^{-1})$

$$\mathcal{O}(L^+(\mathbf{u})) \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ & & 0 & 0 & 1 & 2 & 2 & 2 \\ & & & 0 & 0 & 1 & 2 & 3 \\ & & & & 0 & 0 & 1 & 2 \\ & & & & & 0 & 0 & 1 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{pmatrix}$$

So the factorisation must involve higher order terms in $(q - q^{-1})$.
Potentially can be factored further into elementary row/column matrices which are only first order.

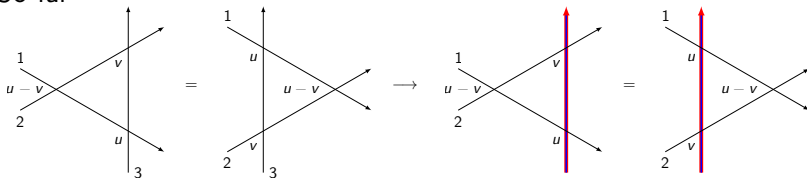


Parameter Permutations and YBE



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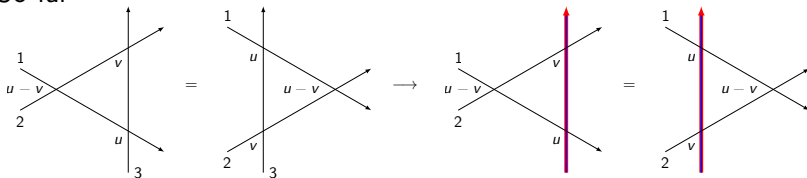
So far






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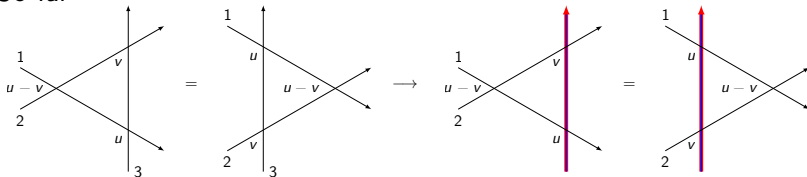
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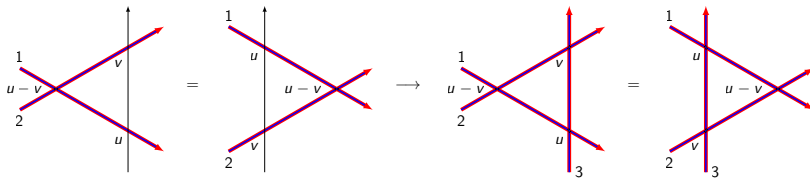
Now we're seeking $\mathcal{R}(u) =$

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Parameter Permutations and YBE

So far



Now we're seeking $\mathcal{R}(u) = \text{crossing} \in \text{End}(\mathcal{V}_\rho \otimes \mathcal{V}_\sigma)$





Parameter Permutations and YBE

For $\check{\mathcal{R}}(u) := P \circ \mathcal{R}(u) \in \text{End}(\mathcal{V}_\rho \otimes \mathcal{V}_\sigma)$



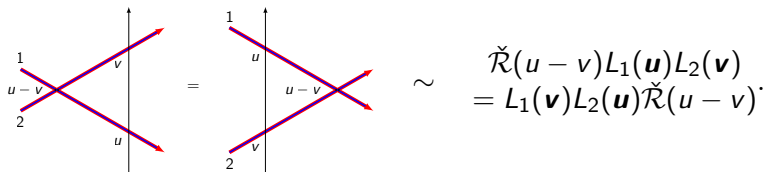
Parameter Permutations and YBE

For $\check{\mathcal{R}}(u) := P \circ \mathcal{R}(u) \in \text{End}(\mathcal{V}_\rho \otimes \mathcal{V}_\sigma)$ the defining RLL -relation is

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} \sim \check{\mathcal{R}}(u-v) L_1(\mathbf{u}) L_2(\mathbf{v}) = L_1(\mathbf{v}) L_2(\mathbf{u}) \check{\mathcal{R}}(u-v).$$

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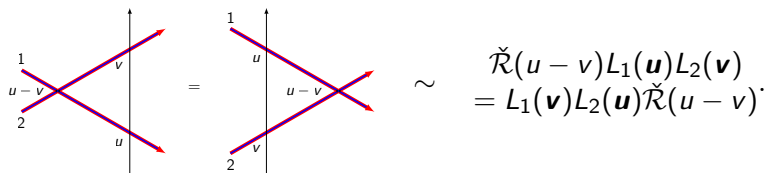


$$\begin{array}{c} 1 \\ u-v \\ 2 \end{array} \begin{array}{c} v \\ u \end{array} = \begin{array}{c} 1 \\ u \\ 2 \end{array} \begin{array}{c} u-v \\ v \end{array} \sim \check{\mathcal{R}}(u-v)L_1(\mathbf{u})L_2(\mathbf{v}) \\
 = L_1(\mathbf{v})L_2(\mathbf{u})\check{\mathcal{R}}(u-v).$$

$\check{\mathcal{R}}$ realises the permutation $(\mathbf{u}, \mathbf{v}) \mapsto (\mathbf{v}, \mathbf{u}) \in \text{Perm}(\mathbf{u}, \mathbf{v}) \simeq S_{2n}$.

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IDEA: Factorise $\check{\mathcal{R}}(u-v)$ in terms of elementary transposition operators $\mathcal{S}_i \in \text{End}(\mathcal{V}_\rho \otimes \mathcal{V}_\sigma)$

$$\mathcal{S}_i L_{12}(\mathbf{u}, \mathbf{v}) = L_{12}(\mathcal{S}_i(\mathbf{u}, \mathbf{v}))\mathcal{S}_i, \quad (L_{12}(\mathbf{u}, \mathbf{v}) = L_1(\mathbf{u})L_2(\mathbf{v}))$$

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Parameter Permutations and YBE

IDEA: Factorise $\check{\mathcal{R}}(u - v)$ in terms of elementary transposition operators $\mathcal{S}_i \in \text{End}(\mathcal{V}_\rho \otimes \mathcal{V}_\sigma)$

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Simplification: Can just find $n - 1$ -“intertwining” operators

$\mathcal{T}_i \in \text{End}(\mathcal{V}_\rho)$:

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and a single “exchange” operator:

$$\mathcal{S}_n(\mathbf{u}, \mathbf{v}) L_{12}(\mathbf{u}, \mathbf{v}) = \mathcal{S}_n(\mathbf{u}, \mathbf{v}) L_{12}(u_1, \dots, u_{n-1}, v_1, u_n, v_2, \dots, v_n).$$



Parameter Permutations and YBE



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$$\check{\mathcal{R}}_{12}(v-w)\check{\mathcal{R}}_{23}(u-w)\check{\mathcal{R}}_{12}(u-v) = \check{\mathcal{R}}_{23}(u-v)\check{\mathcal{R}}_{12}(u-w)\check{\mathcal{R}}_{23}(v-w).$$



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These operators should define an action of S_{2n} , i.e.,

$$s_{i_j} \dots s_{i_2} s_{i_1} \mapsto \mathcal{S}_{i_j}(s_{i_{j-1}} \dots s_{i_1}(\mathbf{u}, \mathbf{v})) \dots \mathcal{S}_{i_2}(s_{i_1}(\mathbf{u}, \mathbf{v})) \mathcal{S}_{i_1}(\mathbf{u}, \mathbf{v}),$$

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YBE then follows from equivalence of the decompositions in $\text{Perm}(\mathbf{u}, \mathbf{v}, \mathbf{w})$

$$(\mathbf{u}, \mathbf{v}, \mathbf{w}) \xrightarrow{\check{\mathcal{R}}_{12}} (\mathbf{v}, \mathbf{u}, \mathbf{w}) \xrightarrow{\check{\mathcal{R}}_{23}} (\mathbf{v}, \mathbf{w}, \mathbf{u}) \xrightarrow{\check{\mathcal{R}}_{12}} (\mathbf{w}, \mathbf{v}, \mathbf{u}),$$

$$(\mathbf{u}, \mathbf{v}, \mathbf{w}) \xrightarrow{\check{\mathcal{R}}_{23}} (\mathbf{u}, \mathbf{w}, \mathbf{v}) \xrightarrow{\check{\mathcal{R}}_{12}} (\mathbf{w}, \mathbf{u}, \mathbf{v}) \xrightarrow{\check{\mathcal{R}}_{23}} (\mathbf{w}, \mathbf{v}, \mathbf{u}).$$



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$U_q(\mathfrak{sl}_2)$ Case: [Derkachov, Karakhanyan, and Kirschner, 2007]

$U_q(\mathfrak{sl}_3)$ Case: [Valinevich et al., 2008]



q -Deformed Case

Proposition

The intertwiners for the $U_q(\mathfrak{sl}_n)$ ($|q| < 1$) L -operator are given by

$$\mathcal{T}_{n-i}^{(n)}(\alpha) = \left(\Lambda_{n-i}^{(n)} \right)^\alpha \frac{e_{q^2}(q^{2(N_{i+1,i}+1)} \mathbf{x}_{n-i}^{(n)})}{e_{q^2}(q^{2(N_{i+1,i}+1-\alpha)} \mathbf{x}_{n-i}^{(n)})},$$

$$e_{q^2}(\mathbf{Z}) = ((\mathbf{Z}; q^2)_\infty)^{-1} = [(1 - \mathbf{Z})(1 - q^2 \mathbf{Z})(1 - q^{2 \cdot 2} \mathbf{Z}) \dots]^{-1},$$

$$\frac{e_{q^2}(\mathbf{Z})}{e_{q^2}(q^{-\alpha} \mathbf{Z})} = \sum_{j=0}^{\infty} \frac{(q^{-\alpha}; q)_j}{(q; q)_j} \mathbf{Z}^j, \quad \Lambda_{n-i}^{(n)} = (x_{i+1,i})^{-1} q^{\beta_i}$$

where $\alpha = u_{n-i} - u_{n+1-i}$, and

$$\mathbf{x}_{n-i}^{(n)} = 1 + x_{i+1,i} \sum_{j=i+2}^n \frac{x_{j,i+1}}{x_{ji}} (q^{N_{ij}} - q^{-N_{ij}}) q^{\gamma_i}.$$



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Obtained using an approach from [Valinevich et al., 2008].



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Proof.

The only non-trivial relation is the braid relation

$$\mathcal{T}_i(\alpha)\mathcal{T}_{i+1}(\alpha + \beta)\mathcal{T}_i(\beta) = \mathcal{T}_{i+1}(\beta)\mathcal{T}_i(\alpha + \beta)\mathcal{T}_{i+1}(\alpha).$$



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After a series expansion it is reduced to a family of (terminating) q -series identity relating rank $i + 1$ and rank $2i - 1$ q -Lauricella series. □



q -Series Identity



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(Type D) q -Lauricella Function: q -Lauricella functions are a family of multivariable hypergeometric series:

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$$\begin{aligned} & \Phi_D^{(n)}[b; a_1, \dots, a_n; c; q; x_1, \dots, x_n] \\ &= \sum_{m_1=0}^{\infty} \dots \sum_{m_n=0}^{\infty} \frac{(b; q)_M (a_1; q)_{m_1} \dots (a_n; q)_{m_n}}{(c; q)_M (q; q)_{m_1} \dots (q; q)_{m_n}} x_1^{m_1} \dots x_n^{m_n}, \quad (\star) \end{aligned}$$

where $M = \sum_{i=1}^n m_i$ and

$$(x; q)_m = (1-x)(1-qx) \dots (1-q^{m-1}x).$$



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where $M = \sum_{i=1}^n m_i$ and

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[Andrews, 1972] gives a general transformation formula allowing us to rewrite (\star) in terms of a ${}_{n+1}\phi_n$ hypergeometric series.



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For $n \geq 1$ and non-negative integer tuples

$$\mathbf{k} = (k_0, \dots, k_n) = (k_0, \tilde{\mathbf{k}}), \quad \mathbf{l} = (l_1, \dots, l_n), \quad \mathbf{m} = (m_1, \dots, m_{n-1}),$$

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$$r_i = 1 + \sum_{a=1}^i (k_a - (l_a + m_a)), \quad p_i = 1 - \sum_{a=i}^n (k_a - (l_a + m_a)).$$



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The identity we need is the equality $\Theta_{\mathbf{k}, \mathbf{l}, \mathbf{m}} = \Omega_{\mathbf{k}, \mathbf{l}, \mathbf{m}}$

$$\Theta_{\mathbf{k}, \mathbf{l}, \mathbf{m}} = \frac{(\xi; q)_{L+M}}{(\xi \zeta; q)_{L+M}} \Phi_D^{(2n-1)} \left[\zeta; q^{-\mathbf{l}}, q^{-\mathbf{m}}; q^{1-L-M}/\xi; q^{\mathbf{r}+\mathbf{l}+(\mathbf{m}, 0)}, q^{(\mathbf{r}_i, \hat{r}_n)+\mathbf{m}} \right],$$

$$\Omega_{\mathbf{k}, \mathbf{l}, \mathbf{m}} = \zeta^{k_0} \frac{(\xi; q)_K}{(\xi \zeta; q)_K} \Phi_D^{(n+1)} \left[\zeta; q^{-\mathbf{k}}; q^{1-K}/\xi; q^{1+k_0-K}/(\xi \zeta), q^{\mathbf{p}+\tilde{\mathbf{k}}} \right],$$

for arbitrary complex parameters ξ, ζ .



Exchange Operator



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The defining relation for the exchange operator S_n is

$$S_n L_1(u_n) L_2(v_1) = L_1(v_1) L_2(u_n) S_n.$$



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The defining relation for the exchange operator S_n is

$$S_n L_1(\textcolor{red}{u}_n) L_2(\textcolor{green}{v}_1) = L_1(\textcolor{green}{v}_1) L_2(\textcolor{red}{u}_n) S_n.$$

Recall the (postulated) factorisation for $L(\mathbf{u})$. This can be put into the form:

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Now we can reduce the defining relation to

$$\begin{aligned} Z_2^{(x, \tilde{\mathbf{u}})}(\mathbf{v}_1) \left[(D^{(x, \tilde{\mathbf{u}})})^{-1} S_n D^{(x, \tilde{\mathbf{u}})} \right] \left(Z_2^{(x, \tilde{\mathbf{u}})}(\mathbf{u}_n) \right)^{-1} \\ = Z_1^{(y, \tilde{\mathbf{v}})}(\mathbf{u}_n) \left[D^{(y, \tilde{\mathbf{v}})} S_n (D^{(y, \tilde{\mathbf{v}})})^{-1} \right] \left(Z_1^{(y, \tilde{\mathbf{v}})}(\mathbf{v}_1) \right)^{-1}, \end{aligned}$$

if $S_n^{(x, y)}$ commutes (element wise) with $Z_1^{(x)}$ and $Z_2^{(y)}$.



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Recall in the $n \geq 4$ case the postulated ansatz for the factorisation was inconsistent - the outer most factors will now have q -difference terms.



Exchange Operator

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Recall in the $n \geq 4$ case the postulated ansatz for the factorisation was inconsistent - the outer most factors will now have q -difference terms.

This seems to represent a serious obstruction to constructing the exchange operator - unclear whether to expect a multiplication operator (by shifted variables) to work or not



Summary



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- We introduced the RLL -method as a means for obtaining solutions to the YBE in the class of differential (q -difference) representations of \mathfrak{sl}_n ($U_q(\mathfrak{sl}_n)$). A key feature here is a factorisation property of the L -operators.



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- ▶ We described explicitly all but one of the transposition operators in the $U_q(\mathfrak{sl}_n)$ case, and prove they obey the necessary symmetric group relations.
- ▶ We explain how the failure of the factorisation property for the $U_q(\mathfrak{sl}_4)$ L -operator represents an obstruction to constructing the missing “exchange” operator.



Thank You!










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Questions?



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