A Terminating q-Lauricella Transformation Formula

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The purpose of this note is to give a self contained discussion and proof of a charming q-series identity discovered in a previous project of mine. We first introduce some basic notation. We will use the standard notation for the finite q-Pochhammer symbol

$$(x;q)_m = \begin{cases} (1-x)(1-qx)\dots(1-q^{m-1}x), & m>0, \\ 1, & m=0, \\ \left[(1-q^{-1}x)(1-q^{-2}x)\dots(1-q^{m}x)\right]^{-1}, & m<0, \end{cases} \tag{1}$$

aswell as the infinite q-Pochhammer symbol which is defined whenever $|q|<1$

$$(x;q)_\infty = \prod_{n=0}^{\infty}(1-q^n x). \tag{2}$$

Notice that formulas (1) and (2) can be zero or singular when x is an integer power of q. Such cases require additional care as we will see. In this note we freely use identities for $(x;q)_n$ and $(x;q)_\infty$ from [2, Appendix I], and we will adopt the implicit base convention $(x)_n = (x;q)_n$ and $(x)_\infty = (x;q)_\infty$.

We can now state the main result:

Proposition 1. *Fix any integer $n \geq 1$ and non-negative integer tuples $\boldsymbol{k} = (k_0, k_1, \dots, k_n) \in \mathbb{N}^{n+1}$, $\boldsymbol{l} = (l_1, \dots, l_n) \in \mathbb{N}^n$, and $\boldsymbol{m} = (m_1, \dots, m_{n-1}) \in \mathbb{N}^{n-1}$. Denote $k = \sum_{j=0}^{n} k_j$, $l = \sum_{j=1}^{n} l_j$, and $m = \sum_{j=1}^{n-1} m_j$. Then for a complex number q with $0<|q|<1$, and arbitrary complex parameters y, and z, we have the following equality:*

$$\begin{aligned} &\frac{(y)_{l+m}}{(yz)_{l+m}} \sum_{\boldsymbol{\lambda}\in\mathbb{N}^n} \sum_{\boldsymbol{\mu}\in\mathbb{N}^{n-1}} \left[\frac{(z)_{\lambda+\mu} \prod_{j=1}^{n}(q^{-l_j})_{\lambda_j} \prod_{j=1}^{n-1}(q^{-m_j})_{\mu_j}}{(q^{1-l-m}/y)_{\lambda+\mu} \prod_{j=1}^{n}(q)_{\lambda_j} \prod_{j=1}^{n-1}(q)_{\mu_j}} \right. \\ &\qquad \left. \times\, q^{\sum_{j=1}^{n} \lambda_j (1+k_j+\sum_{a=1}^{j-1}(k_a-(l_a+m_a))) + \sum_{j=1}^{n-1}\mu_j(1+\sum_{a=1}^{j}(k_a-(l_a+m_a))+m_j)} \right] \\ &= z^{k_0} \frac{(y)_k}{(yz)_k} \sum_{\boldsymbol{\kappa}\in\mathbb{N}^{n+1}} \left[\frac{(z)_\kappa \prod_{j=0}^{n}(q^{-k_j})_{\kappa_j}}{(q^{1-k}/y)_\kappa \prod_{j=0}^{n}(q)_{\kappa_j}} (q^{1+k_0-k}/(yz))^{\kappa_0} q^{\sum_{j=1}^{n}\kappa_j\left(1+k_j-\sum_{a=j}^{n}(k_a-(l_a+m_a))\right)} \right], \end{aligned} \tag{3}$$

where we adopt the same labelling convention $\boldsymbol{\kappa} = (\kappa_0, \kappa_1, \dots, \kappa_n)$ for components of $\boldsymbol{\kappa}$ as per $\boldsymbol{k}$. We also use the shorthand $\kappa = \sum_{j=0}^{n}\kappa_j$, $\lambda = \sum_{j=1}^{n}\lambda_j$ and $\mu = \sum_{j=1}^{n-1}\mu_j$, as well as the convention $m_n = 0$ wherever it appears.

We will not give the proof of proposition 1 yet, however, we can immediately note that both sides are finite sums in light of the $(q^{-a})_\alpha$ terms in the numerators which vanish for $\alpha > a$. We choose to write infinite sums to make explicit the connection with q-Lauricella series. To do so succintly, we define two auxilliary n-tuples of integers $\boldsymbol{r} = (r_j)$ and $\boldsymbol{p} = (p_j)$ by

$$r_j = 1 + \sum_{a=1}^{j}(k_a-(l_a+m_a)), \qquad p_j = 1 - \sum_{a=j}^{n}(k_a-(l_a+m_a)), \tag{4}$$

for $j = 1, \dots, n$ (taking $m_n = 0$ as before). Let us also denote by $\hat{\boldsymbol{r}}$ the $(n-1)$-tuple $(r_1, \dots r_{n-1})$, and by $\tilde{\boldsymbol{k}}$ the $(n-1)$-tuple $(k_1, \dots, k_n)$ for convenience.

Now introduce the type D, q-Lauricella series treated in [1]

$$\Phi_D^{(n)}\left[\beta; \alpha_1, \dots, \alpha_n; \gamma; q; x_1, \dots, x_n\right] = \sum_{\nu_1=0}^{\infty} \cdots \sum_{\nu_n=0}^{\infty} \frac{(\beta)_\nu\,(\alpha_1)_{\nu_1} \dots (\alpha_n)_{\nu_n}}{(\gamma)_\nu\,(q)_{\nu_1} \dots (q)_{\nu_n}} x_1^{\nu_1}\,\dots\,x_n^{\nu_n}\,, \tag{5}$$

where as before we use the notation $\nu = \sum_{j=1}^n \nu_j$. Using (4), and (5), the equality (3) is written succinctly as $\Theta_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} = \Omega_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}}$, where

$$\Theta_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} = \frac{(y)_{l+m}}{(yz)_{l+m}} \Phi_D^{(2n-1)}\left[z; q^{-\boldsymbol{l}}, q^{-\boldsymbol{m}}; q^{1-l-m}/y; q; q^{\boldsymbol{r}+\boldsymbol{l}+(\boldsymbol{m},0)}, q^{\hat{\boldsymbol{r}}+\boldsymbol{m}}\right], \tag{6}$$

is the left hand side, and

$$\Omega_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} = z^{k_0}\frac{(y)_k}{(yz)_k}\,\Phi_D^{(n+1)}\left[z; q^{-\boldsymbol{k}}; q^{1-k}/y; q; q^{1+k_0-k}/(yz), q^{\boldsymbol{p}+\tilde{\boldsymbol{k}}}\right], \tag{7}$$

is the right hand side. Here we are using element-wise exponentian short hand $q^{\boldsymbol{x}} = (q^{x_1}, \dots q^{x_m})$.

A standard proceedure for dealing with expressions such as (6) and (7) may be to use [1, (4.1)] to rewrite them in terms of ${}_{m+1}\phi_m$ basic hypergeometric functions (See [2, (1.2.22)]) and work with known transformation forumlae thereof. This approach is not valid here. For example, applying [1, (4.1)] to (6) yields

$$\Theta_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} \propto {}_{2n}\phi_{2n-1}\left[\begin{matrix} q^{1-l-m}/(yz), q^{\boldsymbol{r}+\boldsymbol{l}+(\boldsymbol{m},0)}, q^{\hat{\boldsymbol{r}}+\boldsymbol{m}} \\ q^{r_1+m_1}, \dots, q^{r_{n-1}+m_{n-1}}, q^{r_n}, q^{r_1}, \dots, q^{r_{n-1}}\end{matrix}; q, z\right], \tag{8}$$

which contains denominator arguments of the form q^a with a potentially a negative integer. In this case the ${}_{m+1}\phi_m$ function is undefined and a similar problem occurs with the RHS (7).

Fortunately we do not need the transformation rule [1, (4.1)]; we can settle for the intermediate step

$$\Phi_D^{(n)}\left[\beta; \boldsymbol{\alpha}; \gamma; q; \boldsymbol{x}\right] = \frac{(\beta)_\infty}{(\gamma)_\infty}\sum_{a=0}^{\infty}\frac{(\gamma/\beta)_a}{(q)_a}\beta^a \prod_{j=1}^{n}\left(\sum_{\nu_j=1}^{\infty}\frac{(\alpha_j)_{\nu_j}}{(q)_{\nu_j}}(x_j q^a)^{\nu_j}\right). \tag{9}$$

In [1] the bracketed sums are evaluated using the infinite summation identity for ${}_1\phi_0[\alpha; q, x]$ [2, (II.3)], however, in our case these sums are terminating since α_j is always a negative integer power of q. We therefore apply the finite summation identity [2, (II.4)]

$${}_1\phi_0[q^{-m}; q, x] = \sum_{n=0}^{\infty}\frac{(q^{-m})_n}{(q)_n}x^n = (xq^{-m})_m. \tag{10}$$

Combining formulae (9) and (10) we obtain

$$\begin{aligned}
\Theta_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} &= \frac{(y)_{l+m}}{(yz)_{l+m}}\frac{(z)_\infty}{(q^{1-l-m}/y)_\infty}\sum_{\lambda=0}^{\infty}\frac{(q^{1-l-m}/(yz))_\lambda}{(q)_\lambda}(q^{r_n+\lambda})_{l_n} z^\lambda \prod_{j=1}^{n-1}(q^{r_j+m_j+\lambda})_{l_j}(q^{r_j+\lambda})_{m_j} \\
&= \frac{(z)_\infty}{(q/y)_\infty}\frac{(y)_{l+m}}{(yz)_{l+m}}\left(\frac{(q^{1-l-m}/(yz))_{l+m}}{(q^{1-l-m}/y)_{l+m}}z^{l+m}\right)\sum_{\lambda=0}^{\infty}\frac{(q/(yz))_{\lambda-l-m}}{(q)_\lambda}\left(\prod_{j=1}^{n}(q^{r_j+\lambda})_{l_j+m_j}\right)z^{\lambda-l-m} \\
&= \frac{(z)_\infty}{(q/y)_\infty}\sum_{\lambda=0}^{\infty}\frac{(q/(yz))_{\lambda-l-m}}{(q)_\lambda}\left(\prod_{j=1}^{n}(q^{r_j+\lambda})_{l_j+m_j}\right)z^{\lambda-l-m},
\end{aligned} \tag{11}$$

for the LHS, where again we understand $m_n = 0$, and

$$\begin{aligned}
\Omega_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} &= z^{k_0} \frac{(y)_k}{(yz)_k} \frac{(z)_\infty}{(q^{1-k}/y)_\infty} \sum_{\kappa=0}^{\infty} \frac{(q^{1-k}/(yz))_\kappa}{(q)_\kappa} (q^{1+\kappa-k}/(yz))_{k_0} \left(\prod_{j=1}^{n} (q^{p_j+\kappa})_{k_j} \right) z^\kappa \\
&= \left(z^{k_0} \frac{(yq^{k-k_0})_{k_0}}{(yzq^{k-k_0})_{k_0}} \frac{(q^{1-k}/(yz))_{k_0}}{(q^{1-k}/(y))_{k_0}} \right) \frac{(y)_{k-k_0}}{(yz)_{k-k_0}} \frac{(z)_\infty}{(q^{1-(k-k_0)}/y)_\infty} \sum_{\kappa=0}^{\infty} \frac{(q^{1-(k-k_0)}/(yz))_\kappa}{(q)_\kappa} \left(\prod_{j=1}^{n} (q^{p_j+\kappa})_{k_j} \right) z^\kappa \\
&= \frac{(y)_{k-k_0}}{(yz)_{k-k_0}} \left(\frac{(q^{1-(k-k_0)}/(yz))_{k-k_0}}{(q^{1-(k-k_0)}/(y))_{k-k_0}} z^{k-k_0} \right) \frac{(z)_\infty}{(q/y)_\infty} \sum_{\kappa=0}^{\infty} \frac{(q/(yz))_{\kappa-(k-k_0)}}{(q)_\kappa} \left(\prod_{j=1}^{n} (q^{p_j+\kappa})_{k_j} \right) z^{\kappa-(k-k_0)} \\
&= \frac{(z)_\infty}{(q/y)_\infty} \sum_{\kappa=0}^{\infty} \frac{(q/(yz))_{\kappa-(k-k_0)}}{(q)_\kappa} \left(\prod_{j=1}^{n} (q^{p_j+\kappa})_{k_j} \right) z^{\kappa-(k-k_0)}, \qquad (12)
\end{aligned}$$

for the RHS. It follows from these expressions that both $\Theta_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}}$ and $\Omega_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}}$ are independent of k_0, and depend on l_i and m_i, only in the combinations l_i+m_i (both of these facts are necessary for (3) to hold). By cancelling the prefactors in (11) and (12), and relabelling $l_i + m_i \mapsto l_i$ for $i = 1, \ldots, n-1$, $k - k_0 \mapsto k = \sum_{j=1}^{n} k_j$ and $q/(yz) \mapsto y$, to better reflect the dependence, we have reduced the equality $\Theta_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} = \Omega_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}}$ to the following.

Proposition 2. *For any integer $n \geq 1$, complex parameter q, such that $0 < |q| < 1$, arbitrary complex parameters y and z, and non-negative integer tuples $\boldsymbol{l} = (l_j) \in \mathbb{Z}^n_{\geq 0}$ and $\boldsymbol{k} = (k_j) \in \mathbb{Z}^n_{\geq 0}$ $(j = 1, \ldots, n)$, we have the following equality*

$$\sum_{\lambda=0}^{\infty} \frac{(y)_{\lambda-l}}{(q)_\lambda} (q^{r_1+\lambda})_{l_1} \ldots (q^{r_n+\lambda})_{l_n} z^{\lambda-l} = \sum_{\kappa=0}^{\infty} \frac{(y)_{\kappa-k}}{(q)_\kappa} (q^{p_1+\kappa})_{k_1} \ldots (q^{p_n+\kappa})_{k_n} z^{\kappa-k}, \qquad (13)$$

where $k = \sum_{j=1}^{n} k_j$, $l = \sum_{j=1}^{n} l_j$, and p_j and r_j are as per (4) (with $(l_a + m_a) \mapsto l_a$).

By comparison of the tail of both sides in (13) ($\lambda > l$ and $\kappa > k$ for the LHS and RHS respectively) with the series ${}_1\phi_0[y; q, z]$ we have absolute convergence. The proof of proposition 2 requires two technical lemmas.

Lemma 1. *With $\boldsymbol{k}, \boldsymbol{l}, k, l,$ and p_j as per proposition 2, suppose that $k \geq l$ and define $\Delta = k - l \geq 0$. Then for any integer $0 \leq \kappa \leq \Delta - 1$ we have*

$$(q^{p_1+\kappa})_{k_1} \ldots (q^{p_n+\kappa})_{k_n} = 0. \qquad (14)$$

Proof. It suffices to show that for any $0 \leq \kappa \leq \Delta - 1$, there exists j such that $p_j + \kappa \leq 0$ and $p_j + \kappa + k_j > 0$, that is, $\{0, 1, \ldots, \Delta - 1\} \subset U$ where $U = \bigcup_{j=1}^{n} (-(p_j + k_j), -p_j]$. Since $-(p_j + k_j) = -(p_{j+1} + l_j) \leq -p_{j+1}$ (for $j = 1, \ldots, n-1$), it follows that U is an overlapping union of intervals giving

$$U = \left(\min_{j=1}^{n} (-(p_j + k_j)), \max_{j=1}^{n} (-p_j) \right] := (m, p].$$

Now note that $p \geq -p_1 = \Delta - 1$ and $m \leq -(p_n + k_n) = -(1 + l_n) < 0$ so we are done. □

Lemma 2. *With $\boldsymbol{k}, \boldsymbol{l}, k, l,$ and r_j as per proposition 2, suppose that $k \geq l$ and define $\Delta = k - l \geq 0$. Then for any integer $\lambda \geq 0$ we have*

$$(q^{r_1+\lambda})_{k_2} \ldots (q^{r_{n-1}+\lambda})_{k_n} = 0, \quad \Leftrightarrow \quad (q^{r_1+\lambda})_{l_1} \ldots (q^{r_{n-1}+\lambda})_{l_{n-1}} = 0, \quad \Leftrightarrow \quad \lambda \leq r := \max_{j=1}^{n-1} (-r_j). \qquad (15)$$

Furthermore, for any integer $\lambda > r$ we have

$$\frac{(q^{1+\lambda})_{k_1} (q^{r_1+\lambda})_{k_2} \ldots (q^{r_{n-1}+\lambda})_{k_n}}{(q^{r_1+\lambda})_{l_1} \ldots (q^{r_n+\lambda})_{l_n}} = \frac{(q)_{\lambda+\Delta}}{(q)_\lambda}. \qquad (16)$$

Proof. For the first claim define $K_0 = \{\lambda \in \mathbb{Z}_{\geq 0} \mid (q^{r_1+\lambda})_{k_2} \ldots (q^{r_{n-1}+\lambda})_{k_n} = 0\}$ and similarly define $L_0 = \{\lambda \in \mathbb{Z}_{\geq 0} \mid (q^{r_1+\lambda})_{l_1} \ldots (q^{r_{n-1}+\lambda})_{l_{n-1}} = 0\}$. Clearly $\lambda \in K_0$, if and only if there is some $1 \leq j \leq n-1$ such that $k_{j+1} > -(r_j + \lambda) \geq 0$, giving $L_0 = \left(\bigcup_{j=1}^{n-1}(-(r_j + k_{j+1}), -r_j]\right) \cap Z_{\geq 0}$. Similarly, one can find that $L_0 = \left(\bigcup_{j=1}^{n-1}(-(r_j + l_j), -r_j]\right) \cap Z_{\geq 0}$. These unions of intervals are overlapping since $-(r_j + k_{j+1}) = -(r_{j+1} + l_{j+1}) \leq -r_{j+1}$ and $-(r_j + l_j) = -(r_{j-1} + k_j) \leq -r_{j-1}$, for $j = 1, \ldots n-2$ and $j = 2, \ldots n-1$ respectively, therefore

$$\bigcup_{j=1}^{n-1}(-(r_j + k_{j+1}), -r_j] = \left(\min_{j=1}^{n-1}(-(r_j + k_{j+1})), \max_{j=1}^{n-1}(-r_j)\right] := (m_1, r],$$

$$\bigcup_{j=1}^{n-1}(-(r_j + l_j), -r_j] = \left(\min_{j=1}^{n-1}(-(r_j + l_j)), \max_{j=1}^{n-1}(-r_j)\right] := (m_2, r].$$

Now the fact that $m_1 \leq -(r_{n-1} + k_n) = -(1 + \Delta + l_n) < 0$, and $m_2 \leq -(r_1 + l_1) = -(1 + k_1) < 0$, gives $K_0 = \{0, 1, \ldots, r\} = L_0$, which proves the first claim.

For the second claim, we now know that $\lambda > r$ means that $\frac{(q^{r_j+\lambda})_{k_{j+1}}}{(q^{r_j+\lambda})_{l_j}} = (q^{r_j+l_j+\lambda})_{k_{j+1}-l_j}$ is non-zero and non-singular. Therefore, the LHS of (16) becomes

$$\begin{aligned}
&(q^{1+\lambda})_{k_1}(q^{r_1+l_1+\lambda})_{k_2-l_1} \cdots (q^{r_{n-1}+l_{n-1}+\lambda})_{k_n-l_{n-1}} \left[(q^{1+\Delta+\lambda})_{l_n}\right]^{-1} \\
=&(q^{1+\lambda})_{k_1+k_2-l_1}(q^{r_2+l_2+\lambda})_{k_3-l_2} \cdots (q^{r_{n-1}+l_{n-1}+\lambda})_{k_n-l_{n-1}} \left[(q^{1+\Delta+\lambda})_{l_n}\right]^{-1} \\
&\vdots \\
=&(q^{1+\lambda})_{\Delta+l_n} \left[(q^{1+\Delta+\lambda})_{l_n}\right]^{-1} = (q^{1+\lambda})_{\Delta} = (q)_{\Delta+\lambda} \left[(q)_{\lambda}\right]^{-1},
\end{aligned}$$

as desired, where we iterate the rule [2, (I.17)] with $r_j + l_j = 1 + k_1 + \sum_{a=1}^{j-1}(k_{a+1} - l_a)$ to collapse the product in the numerator. $\square$

Now we can prove proposition 2 and hence proposition 1.

Proof of Proposition 2. Since $(k_j, l_j) \mapsto (l_{n+1-j}, k_{n+1-j})$ is a symmetry of (13) we can assume WLOG that $k \geq l$ and define $\Delta = k - l \geq 0$. By lemma 1 we can shift the summation of the RHS of (13) to $\kappa \mapsto \lambda + \Delta$ with $\lambda \geq 0$. Making use of the relations $p_{j+1} + \Delta = r_j$ for $j = 1, \ldots n-1$, and $p_1 = 1 - \Delta$, we have

$$\begin{aligned}
\sum_{\kappa=0}^{\infty} \frac{(y)_{\kappa-k}}{(q)_{\kappa}}(q^{p_1+\kappa})_{k_1} \cdots (q^{p_n+\kappa})_{k_n}\ z^{\kappa-k} &= \sum_{\lambda=0}^{\infty} \frac{(y)_{\lambda-l}}{(q)_{\lambda+\Delta}}(q^{1+\lambda})_{k_1}(q^{r_1+\lambda})_{k_2} \cdots (q^{r_{n-1}+\lambda})_{k_n}\ z^{\lambda-l} \\
&= \sum_{\lambda=0}^{\infty} \frac{(y)_{\lambda-l}}{(q)_{\lambda}}(q^{r_1+\lambda})_{l_1} \cdots (q^{r_n+\lambda})_{l_n} z^{\lambda-l},
\end{aligned} \tag{17}$$

where the last equality follows from lemma 2. $\square$

References

[1] G. E. Andrews. Summations and transformations for basic appell series. *Journal of the London Mathematical Society*, s2-4(4):618–622, 1972.

[2] G. Gasper and M. Rahman. *Basic Hypergeometric Series*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2 edition, 2004.