Report - Delta-Gamma hedging project

Benjamin DAVILA

12/12/2020

Summary of the project

From the data provided (stock history, column B) in the file *Dataset*, we want to create the portfolio history replication of a European call with maturity on **31/12/2018**, strike $\mathbf{K = 6}$. To do that, we are going to use a synthetic replication portfolio formed from a stock, cash and another K6.5 European call option (to neutralize the effect of gamma)

Overview of the data

```
##          Date Prix.action Prix.option.strike.6 Prix.option.strike.6.5
## 1  02/10/2017       6.742                   NA                     NA
## 2  03/10/2017       6.955                   NA                     NA
## 3  04/10/2017       6.910              1.02096                0.59093
## 4  05/10/2017       7.006              1.09932                0.65885
## 5  06/10/2017       7.000              1.09306                0.65440
## 6  09/10/2017       6.950              1.04813                0.61591
## 7  10/10/2017       6.856              0.96879                0.54811
## 8  11/10/2017       6.795              0.91992                0.51253
## 9  12/10/2017       6.744              0.88119                0.49517
## 10 13/10/2017       6.561              0.74526                0.41163
```

I. Introduction

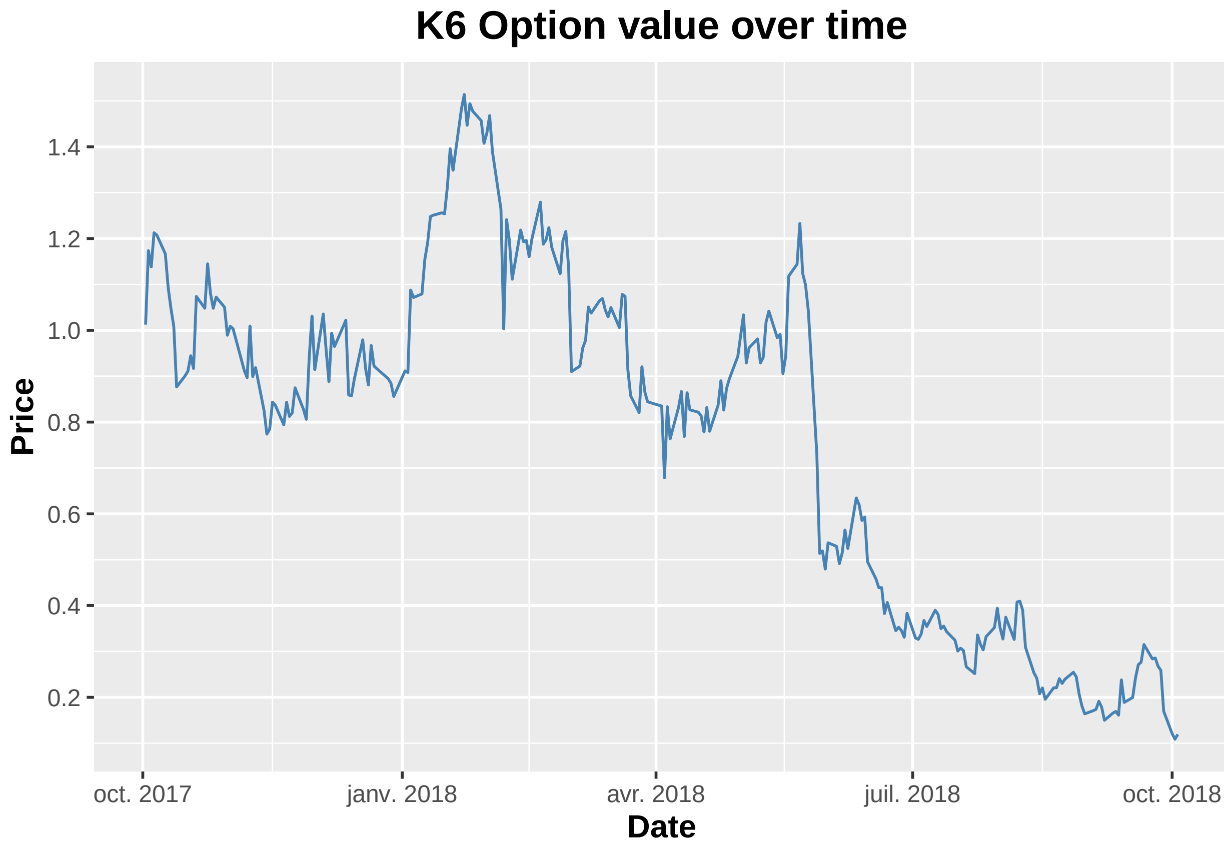
Let's assume here that $\sigma_{cte} = 20\%$. Let's compute now the vector of maturity thanks to the following formula $T_i = \frac{(T_{mat} - t_i)}{365}$.

Overview of the vector of maturities

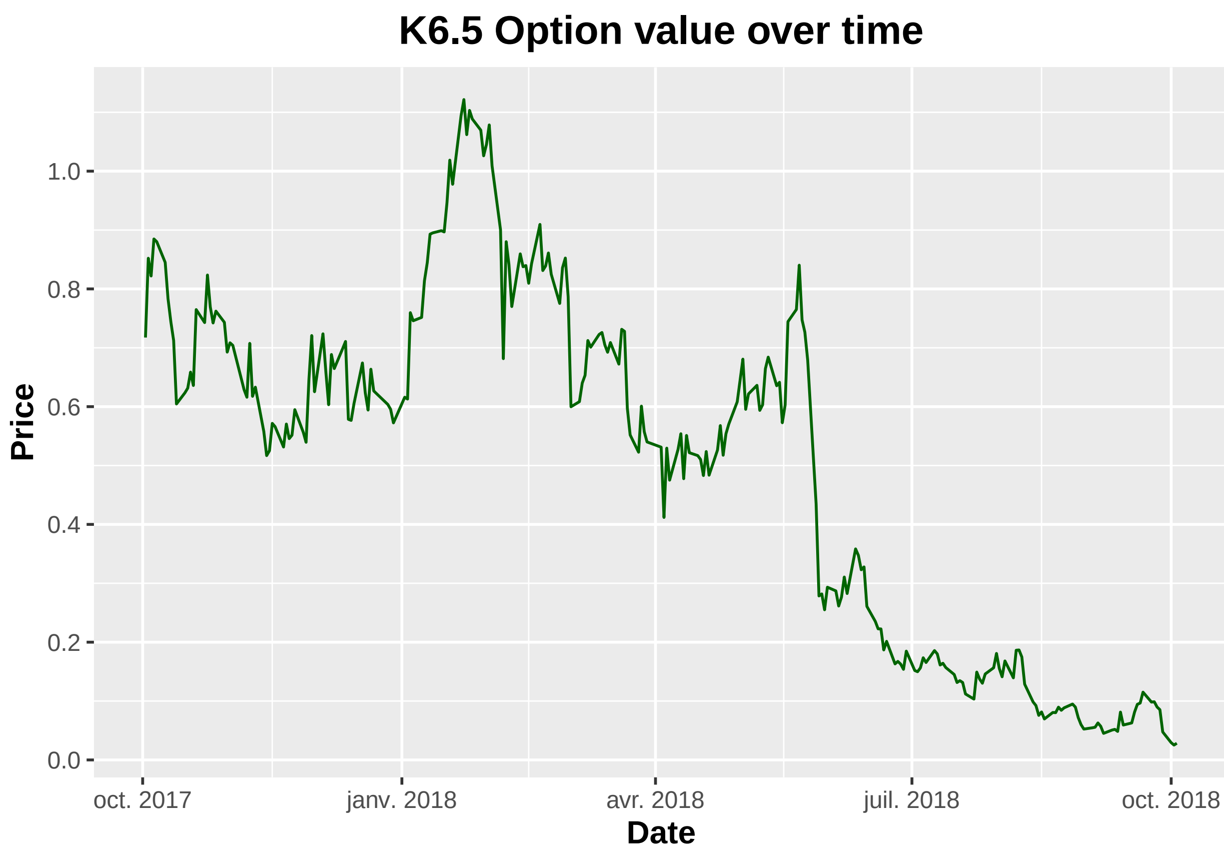
```
##  [1] 1.246575 1.243836 1.241096 1.238356 1.235616 1.227397 1.224658 1.221918
##  [9] 1.219178 1.216438
```

Then, we are going to use the **Black-Scholes formula** to price the corresponding european option call Black-Scholes formula : $C(S,t) = S_t N(d_2) - e^{-r(T-t)} K N(d_2)$ with $r = 0$, $\sigma = 20\%$ and $K = 6$

Overview of the K6 call option value (target) over time



Now, let's visualize the value of the second call option (strike K6.5)



II. Delta-Gamma hedged portfolio with constant volatility

In this part, we are going to build a synthetic delta-Gamma hedged portfolio **formed by a stock, cash and a K6.5 call option**. The value of the portfolio **must be equal to the value of the option** to be replicated.

Overview of the principles of **delta-gamma hedging** :

	Option à répliquer	Action	Poche euros	Option 2
Delta	δ_t	$q_t \, 1$	0	$q''_t \delta''_t$
Gamma	Γ_t	$q_t \, 0$	0	$q''_t \Gamma''_t$
quantité	1	q_t	q'_t	q''_t
Valeur	C_t	$q_t S_t$	q'_t	$q''_t C''_t$

Here, **we have three parameters** : q, q' and q'' corresponding to the quantities of stock, cash and K6.5 option. We have to solve the following linear system to find the value of q, q' and q'' :

$$\begin{cases} q_t + q''_t \delta''_t = \delta_t & (1) \\ q''_t = \frac{\Gamma_t}{\Gamma''_t} & (2) \\ q_t S_t + q'_t + q''_t C''_t = C_t & (3) \end{cases}$$

$$\begin{cases} q_t = \delta_t - q''_t \delta''_t & (1) \\ q''_t = \frac{\Gamma_t}{\Gamma''_t} & (2) \\ q'_t = C_t - q''_t C''_t - q_t S_t & (3) \end{cases}$$

Overview of the iteration process :

t	Option	Réplication	Poche euros	Poche action	Option 2
0	C_0	$V_0 = C_0$	$C_0 - q_0 S_0 - q''_0 C''_0$	$q_0 S_0$	$q''_0 C''_0$
1	C_1	$V_1 = V_0 + q_0(S_1 - S_0) + q''(C''_1 - C''_0)$	$V_1 - q_1 S_1 - q''_1 C''_1$	$q_1 S_1$	$q''_1 C''_1$
2	C_2	$V_2 = V_1 + q_1(S_2 - S_1) + q''_1(C''_2 - C''_1)$	$V_2 - q_2 S_2 - q''_2 C''_2$	$q_2 S_2$	$q''_2 C''_2$

Here, $\delta_t = N\big(d_1(t)\big)$ with $d_1(t) = \frac{1}{\sigma\sqrt{T_t}}\Big[ln\big(\frac{S_t}{K}\big) + \big(r + \frac{1}{2}\sigma^2\big)T_t\Big]$, S_t the value of the stock and T_t the vector of maturities. Furthermore, we have $\Gamma_t = \frac{N'(d_1)}{S\sigma\sqrt{T}}$,

We can deduce the **general formula** to obtain the value of the portfolio over time :

$V_t = V_{t-1} + q_{t-1}(S_t - S_{t-1}) + q''_{t-1}(C''_t - C''_{t-1})$

Now, let's compute the value of δ_t and Γ_t over time and give an overview of them.

```
##  [1] 0.7368957 0.7804467 0.7719176 0.7903820 0.7894580 0.7805751 0.7621479
##  [8] 0.7496394 0.7388317 0.6968998
```

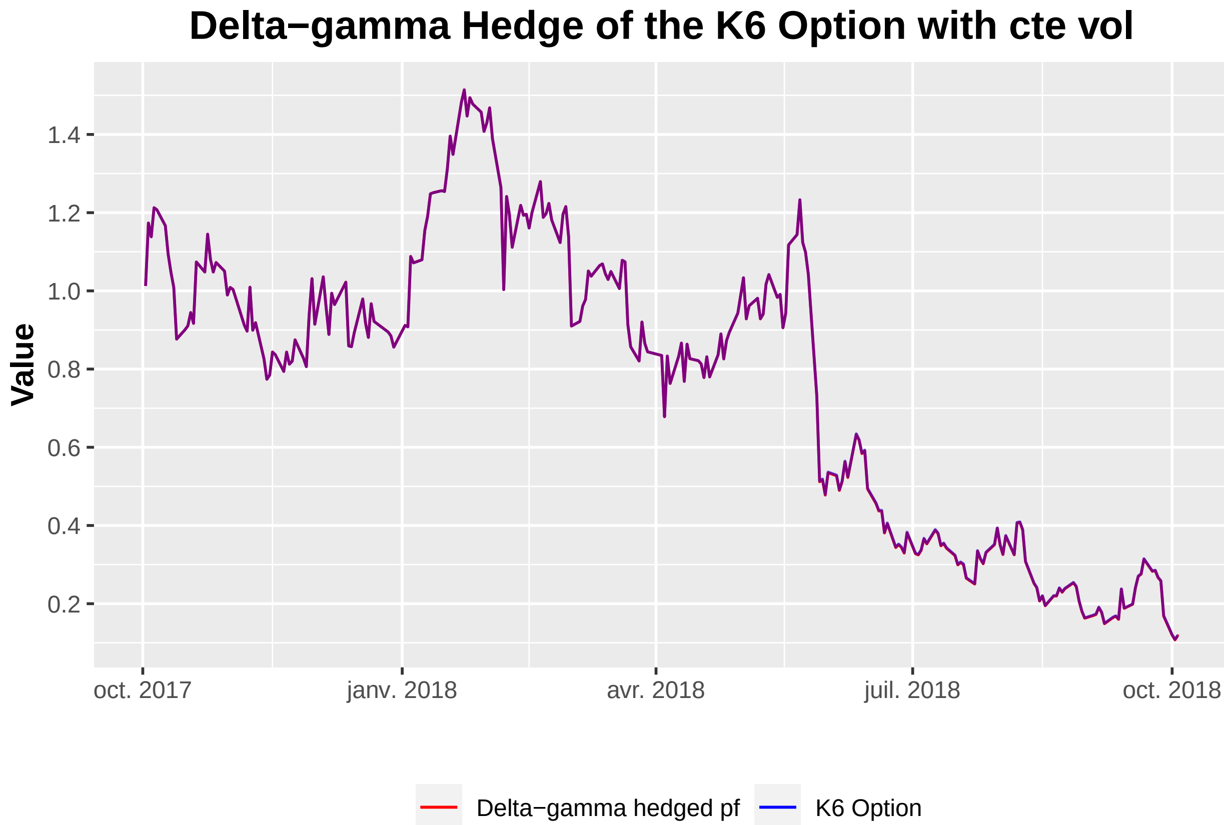
```
##  [1] 0.2167716 0.1906398 0.1962999 0.1846325 0.1854742 0.1919857 0.2038638
##  [8] 0.2116964 0.2183009 0.2413566
```

We do the same thing for δ'_t and Γ'_t over time.

```
##  [1] 0.6084767 0.6608759 0.6502269 0.6729618 0.6716604 0.6601580 0.6374242
##  [8] 0.6222151 0.6092211 0.5606582
```

```
##  [1] 0.2551340 0.2359551 0.2405229 0.2314111 0.2322393 0.2378938 0.2471499
##  [8] 0.2530031 0.2577701 0.2724619
```

Here we are, we are going to use the previous iteration method mentioned above to build the portfolio history replication of the K6 option.



Interpretation : As we can see on the chart above, the two times series are almost equal over time. We can deduce that the hedge method is really efficient with a constant volatility.

III. Delta-Gamma hedged portfolio with non-constant volatility

III.I Calibration

First, we are going to calibrate the implied volatilities for the two options with the price history. As a reminder, implied volatility is the parameter that allows to adjust the observed price and the price stipulated by the model $C^{obs}(t,T,K) = C^{BS}(t,T,K,\Sigma^{impl})$. To find the parameter σ, we need to solve an **linear optimization problem**. To do that, we are going to use two principles algorithms :

1. Newton-Raphson's algorithm

To sum up, we are going to initialize a choosen value of σ_{init} and use the following iteration process :

$\sigma_{i+1} = \sigma_i - \frac{C_{market} - C^{BS}(\sigma)}{\frac{\partial C^{BS}}{\partial \sigma}}$ with $\frac{\partial C^{BS}}{\partial \sigma} = S\sqrt{T}N'(d_1) = vega.$

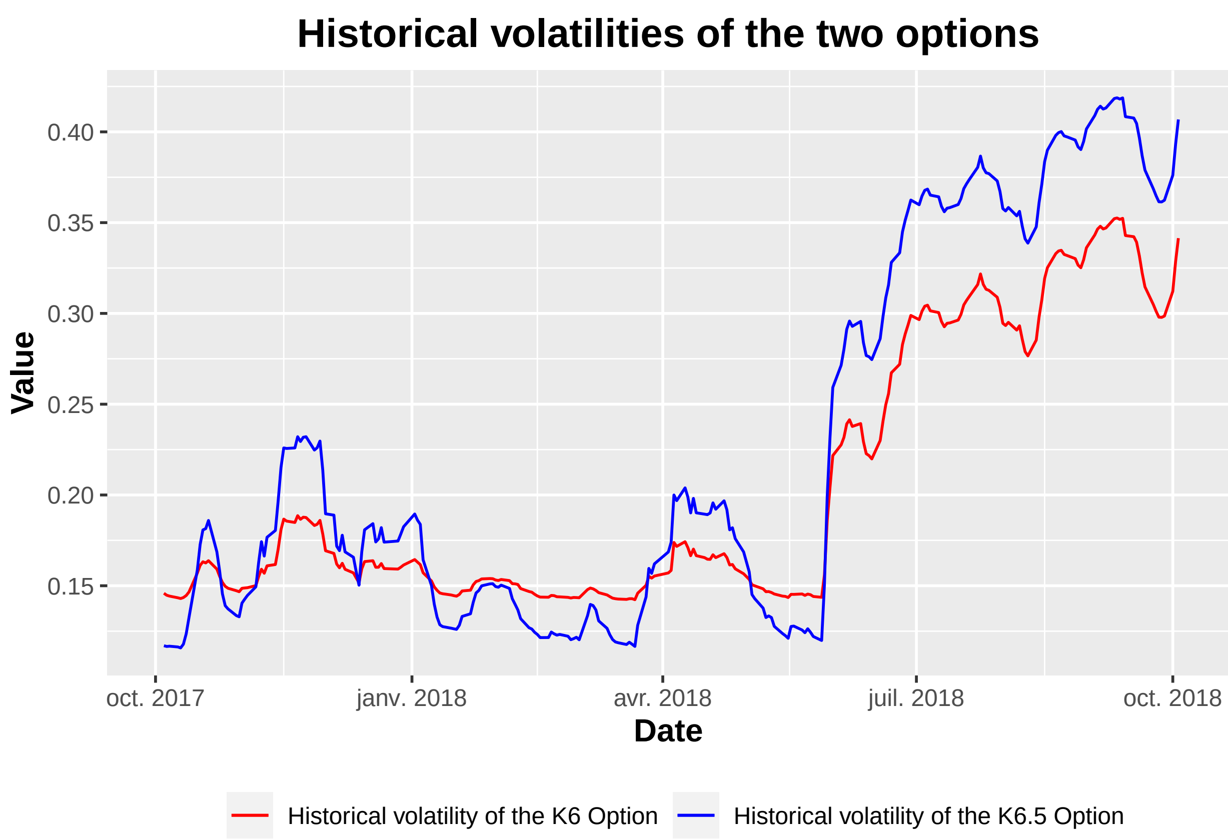
2. Bisection method

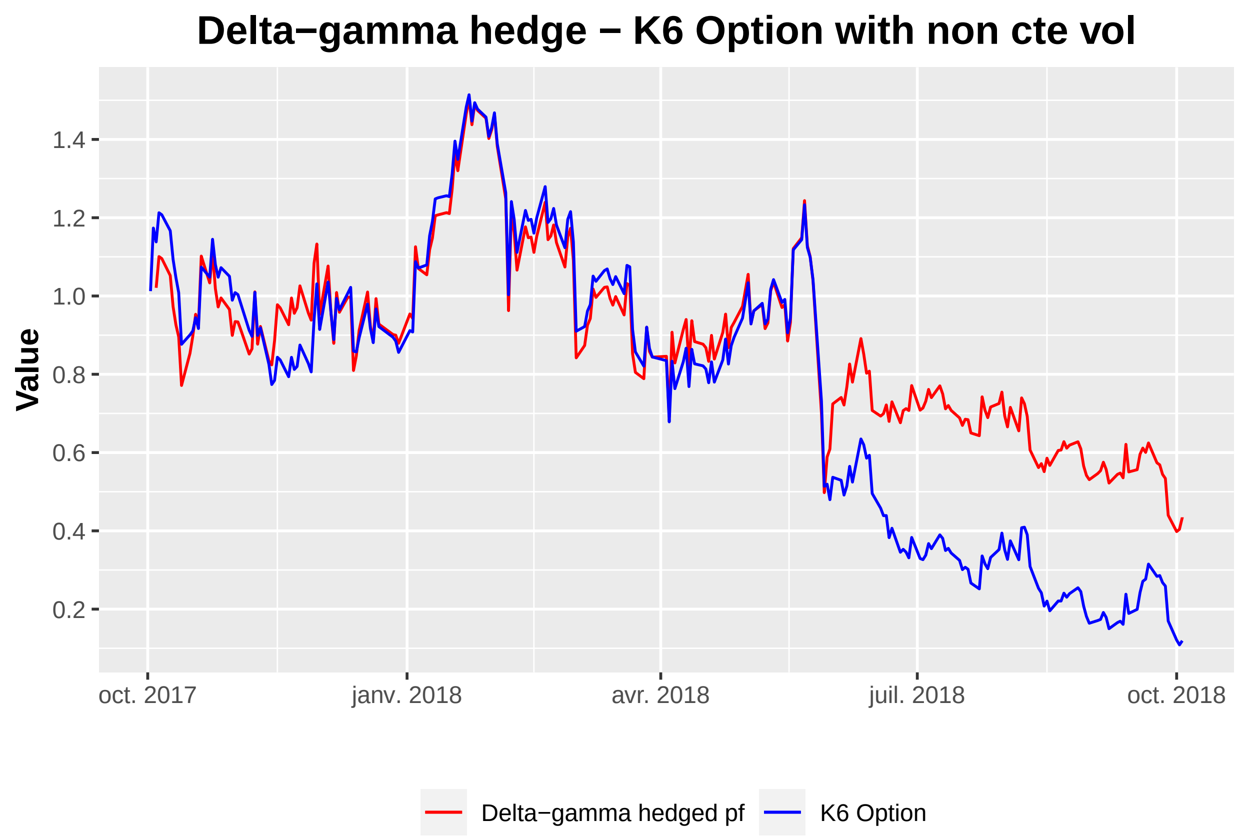
To sum up, we are going to initialize the algorithm then follow an iterative process in these three steps :

- **Initialization** : $\sigma_0^a = \sigma^a$ and $\sigma_0^b = \sigma^b$ such that $f(\sigma^a)f(\sigma^b) < 0$.
- **Itération** : $f(\sigma_i^a)f(\sigma_i^b) < 0.$. Let $\sigma_i^c = (\sigma_i^a + \sigma_i^b)/2$. So
- If $f(\sigma_i^c) = 0$: end of the algorithm, **the solution is found**.
- If $f(\sigma^c)f(\sigma^a) < 0$: $\sigma_{i+1}^a = \sigma_i^a$ and $\sigma_{i+1}^b = \sigma_i^c$

- If $f(\sigma^c)f(\sigma^a) > 0 :\ \sigma^a_{i+1} = \sigma^c_i$ and $\sigma^b_{i+1} = \sigma^b_i$ with $f(\sigma^\beta_i) = C_{market} - C^{BS}(\sigma^\beta_i)$

We used the previous algorithms to determine the implied volatilities of the K6 and K6.5 call option. The results are presented on the chart below.





Interpretation : We can see on the chart above that the replication is not perfect, especially from June, 2018. This is not very surprising because in this section we considered a **non-constant volatility** for the two options because this corresponds to a **more realistic situation**.