

Report - Delta-Gamma hedging project

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Summary of the project

From the data provided (stock history, column B) in the file *Dataset*, we want to create the portfolio history replication of a European call with maturity on **31/12/2018**, strike **K = 6**. To do that, we are going to use a synthetic replication portfolio formed from a stock, cash and another K6.5 European call option (to neutralize the effect of gamma on the delta of the option)

Overview of the data

##	Date	Prix.action	Prix.option.strike.6	Prix.option.strike.6.5
## 1	02/10/2017	6.742	NA	NA
## 2	03/10/2017	6.955	NA	NA
## 3	04/10/2017	6.910	1.02096	0.59093
## 4	05/10/2017	7.006	1.09932	0.65885
## 5	06/10/2017	7.000	1.09306	0.65440
## 6	09/10/2017	6.950	1.04813	0.61591
## 7	10/10/2017	6.856	0.96879	0.54811
## 8	11/10/2017	6.795	0.91992	0.51253
## 9	12/10/2017	6.744	0.88119	0.49517
## 10	13/10/2017	6.561	0.74526	0.41163

I. Introduction

Let's assume here that $\sigma_{cte} = 20\%$. Let's compute now the vector of maturity thanks to the following formula $T_i = \frac{(T_{mat} - t_i)}{365}$.

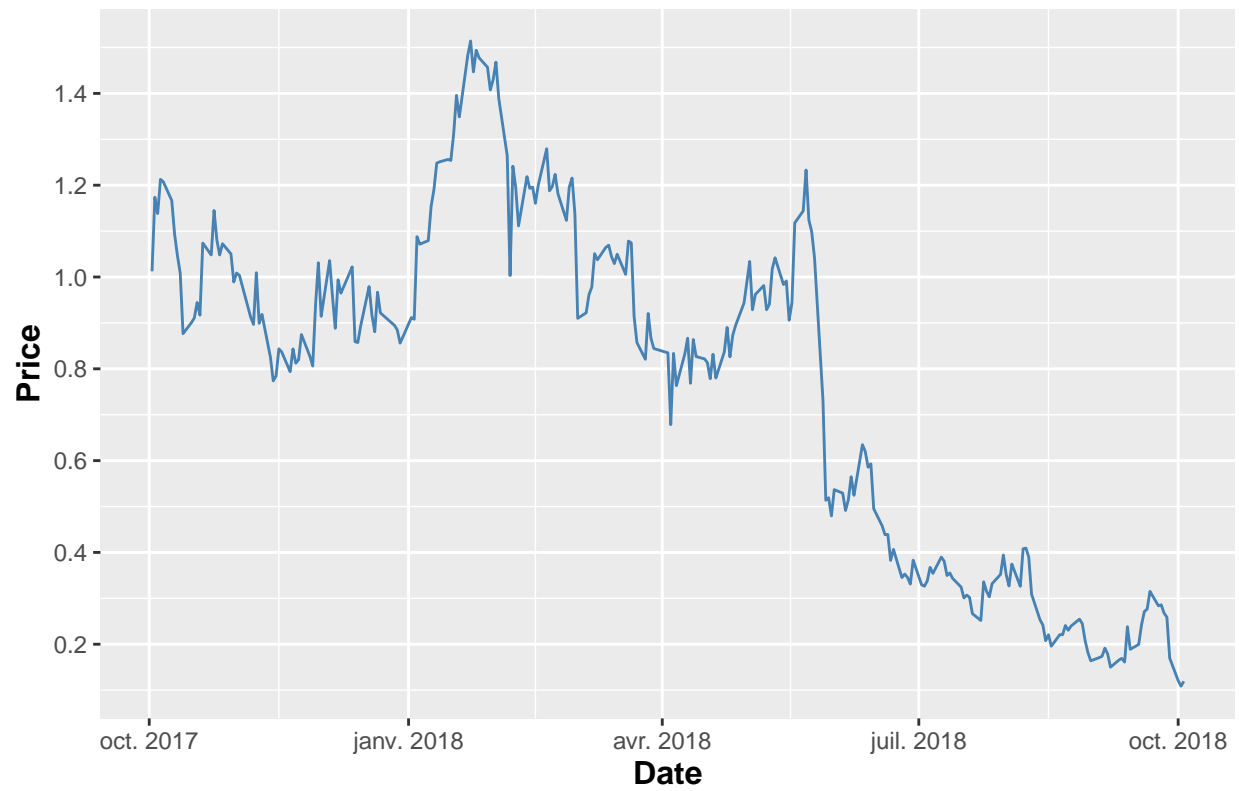
Overview of the vector of maturities

##	[1]	1.246575	1.243836	1.241096	1.238356	1.235616	1.227397	1.224658	1.221918
##	[9]	1.219178	1.216438						

Then, we are going to use the **Black-Scholes formula** to price the corresponding european option call Black-Scholes formula : $C(S, t) = S_t N(d_2) - e^{-r(T-t)} K N(d_2)$ with $r = 0$, $\sigma = 20\%$ and $K = 6$

Overview of the K6 call option value (target) over time

K6 Option value over time



Now, let's visualize the value of the second call option (strike K6.5)

K6.5 Option value over time



II. Delta-Gamma hedged portfolio with constant volatility

In this part, we are going to build a synthetic delta-Gamma hedged portfolio **formed by a stock, cash and a K6.5 call option**. The value of the portfolio **must be equal to the value of the option** to be replicated.

Overview of the principles of **delta-gamma hedging** :

	Option à Répliquer	Action	Poches euros	Option 2
Delta	δ_t	q_t^1	0	$q_t'' \delta_t''$
Gamma	Γ_t	q_t^0	0	$q_t'' \Gamma_t''$
quantité	1	q_t	q_t'	q_t''
Valeur	C_t	$q_t S_t$	q_t'	$q_t'' C_t''$

Here, **we have three parameters** : q, q' and q'' corresponding to the quantities of stock, cash and K6.5 option. We have to solve the following linear system to find the value of q, q' and q'' :

$$\begin{cases} q_t + q''\delta_t'' = \delta_t & (1) \\ q_t'' = \frac{\Gamma_t}{\Gamma_t''} & (2) \\ q_t S_t + q_t' + q_t'' C_t'' = C_t & (3) \end{cases}$$

$$\begin{cases} q_t = \delta_t - q_t''\delta_t'' & (1) \\ q_t'' = \frac{\Gamma_t}{\Gamma_t''} & (2) \\ q_t' = C_t - q_t'' C_t'' - q_t S_t & (3) \end{cases}$$

Overview of the iteration process :

t	Option	Réplication	Poche euros	Poche action	Option 2
0	C_0	$V_0 = C_0$	$C_0 - q_0 S_0 - q_0'' C_0''$	$q_0 S_0$	$q_0'' C_0''$
1	C_1	$V_1 = V_0 + q_0(S_1 - S_0) + q_0''(C_1'' - C_0'')$	$V_1 - q_1 S_1 - q_1'' C_1''$	$q_1 S_1$	$q_1'' C_1''$
2	C_2	$V_2 = V_1 + q_1(S_2 - S_1) + q_1''(C_2'' - C_1'')$	$V_2 - q_2 S_2 - q_2'' C_2''$	$q_2 S_2$	$q_2'' C_2''$

Here, $\delta_t = N(d_1(t))$ with $d_1(t) = \frac{1}{\sigma\sqrt{T_t}} \left[\ln\left(\frac{S_t}{K}\right) + (r + \frac{1}{2}\sigma^2)T_t \right]$, S_t the value of the stock and T_t the vector of maturities. Furthermore, we have $\Gamma_t = \frac{N'(d_1)}{S\sigma\sqrt{T}}$,

We can deduce the **general formula** to obtain the value of the portfolio over time :

$$V_t = V_{t-1} + q_{t-1}(S_t - S_{t-1}) + q_{t-1}''(C_t'' - C_{t-1}'')$$

Now, let's compute the value of δ_t and Γ_t over time and give an overview of them.

```
## [1] 0.7368957 0.7804467 0.7719176 0.7903820 0.7894580 0.7805751 0.7621479
## [8] 0.7496394 0.7388317 0.6968998
```

```
## [1] 0.2167716 0.1906398 0.1962999 0.1846325 0.1854742 0.1919857 0.2038638
## [8] 0.2116964 0.2183009 0.2413566
```

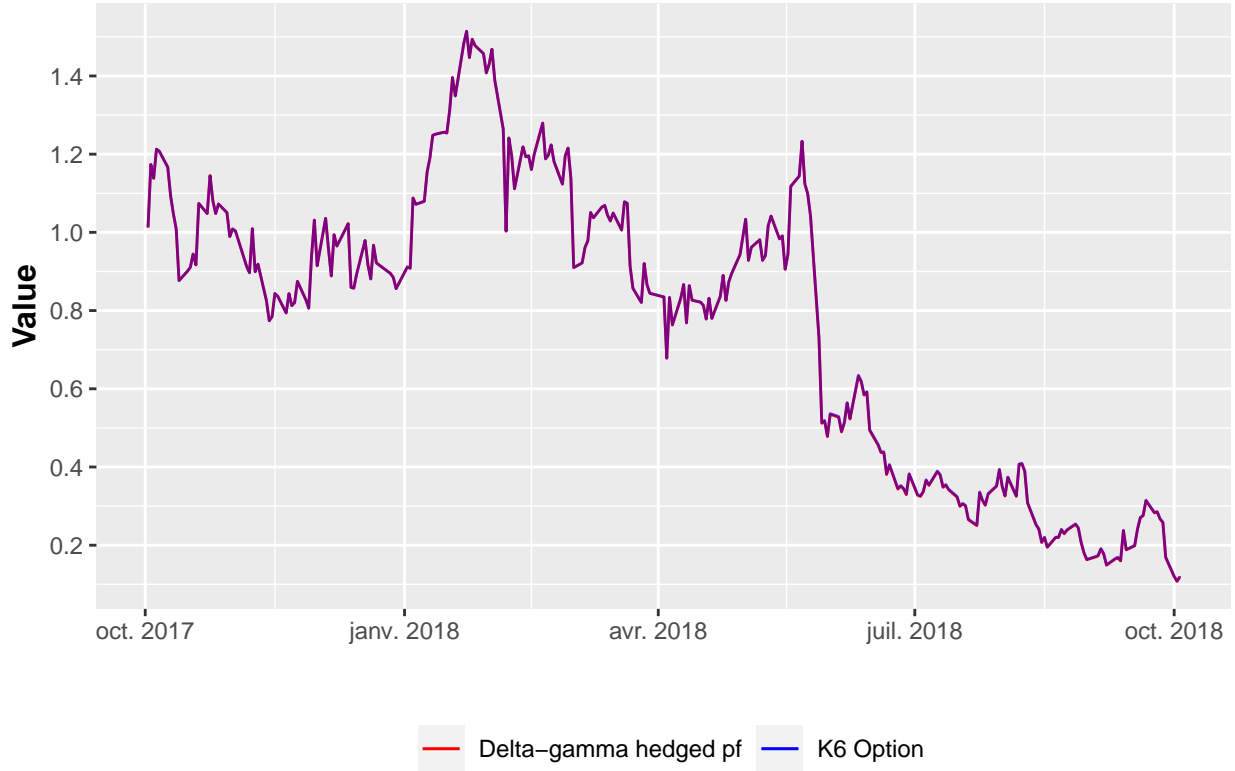
We do the same thing for δ_t' and Γ_t' over time.

```
## [1] 0.6084767 0.6608759 0.6502269 0.6729618 0.6716604 0.6601580 0.6374242
## [8] 0.6222151 0.6092211 0.5606582

## [1] 0.2551340 0.2359551 0.2405229 0.2314111 0.2322393 0.2378938 0.2471499
## [8] 0.2530031 0.2577701 0.2724619
```

Here we are, we are going to use the previous iteration method mentioned above to build the portfolio history replication of the K6 option.

Delta–gamma Hedge of the K6 Option with cte vol



Interpretation : As we can see on the chart above, the two times series are almost equal over time. We can deduce that the hedge method is really efficient with a constant volatility.

III. Delta-Gamma hedged portfolio with non-constant volatility

III.I Calibration

First, we are going to calibrate the implied volatilities for the two options with the price history. As a reminder, implied volatility is the parameter that allows to adjust the observed price and the price stipulated by the model $C^{obs}(t, T, K) = C^{BS}(t, T, K, \Sigma^{impl})$. To find the parameter σ , we need to solve an **linear optimization problem**. To do that, we are going to use two principles algorithms :

1. Newton-Raphson's algorithm

To sum up, we are going to initialize a choosen value of σ_{init} and use the following iteration process :

$$\sigma_{i+1} = \sigma_i - \frac{C_{market} - C^{BS}(\sigma)}{\frac{\partial C^{BS}}{\partial \sigma}} \text{ with } \frac{\partial C^{BS}}{\partial \sigma} = S\sqrt{T}N'(d_1) = vega.$$

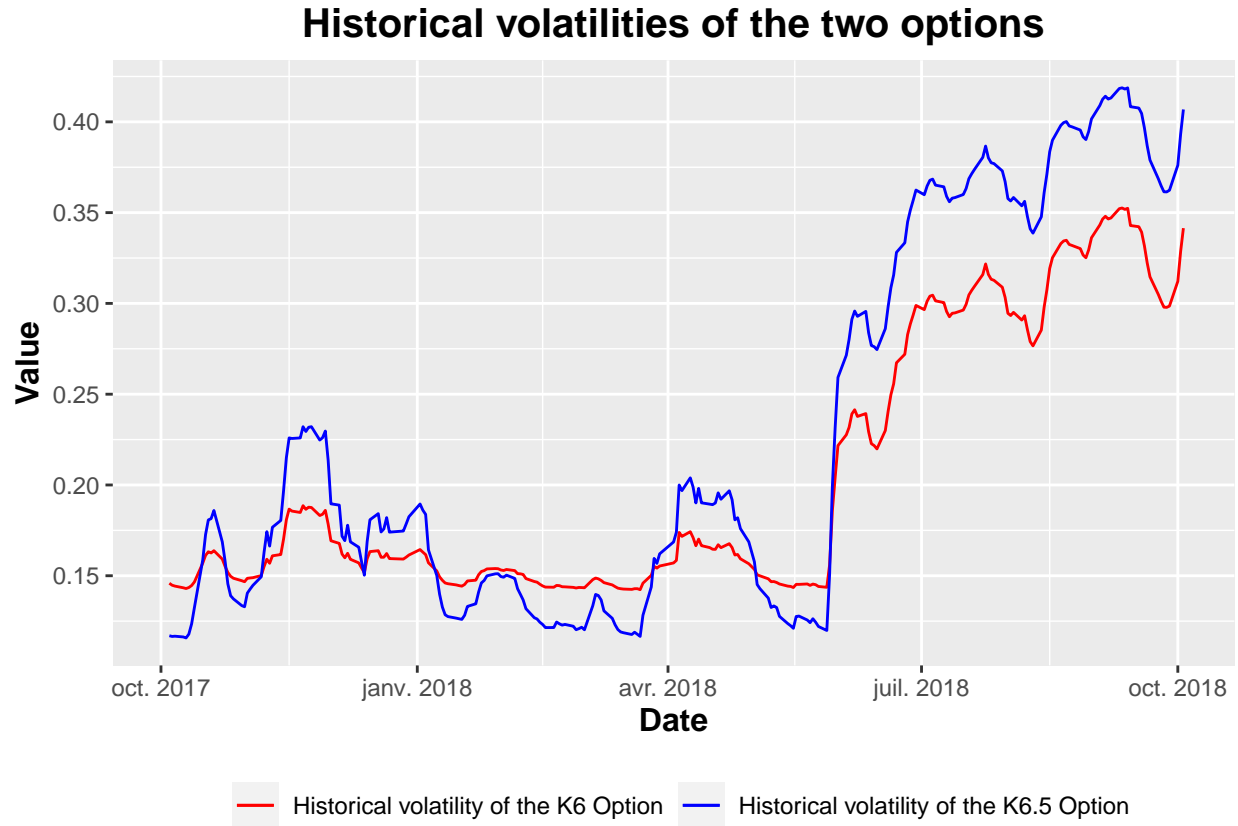
2. Bisection method

To sum up, we are going to initialize the algorithm then follow an iterative process in these three steps :

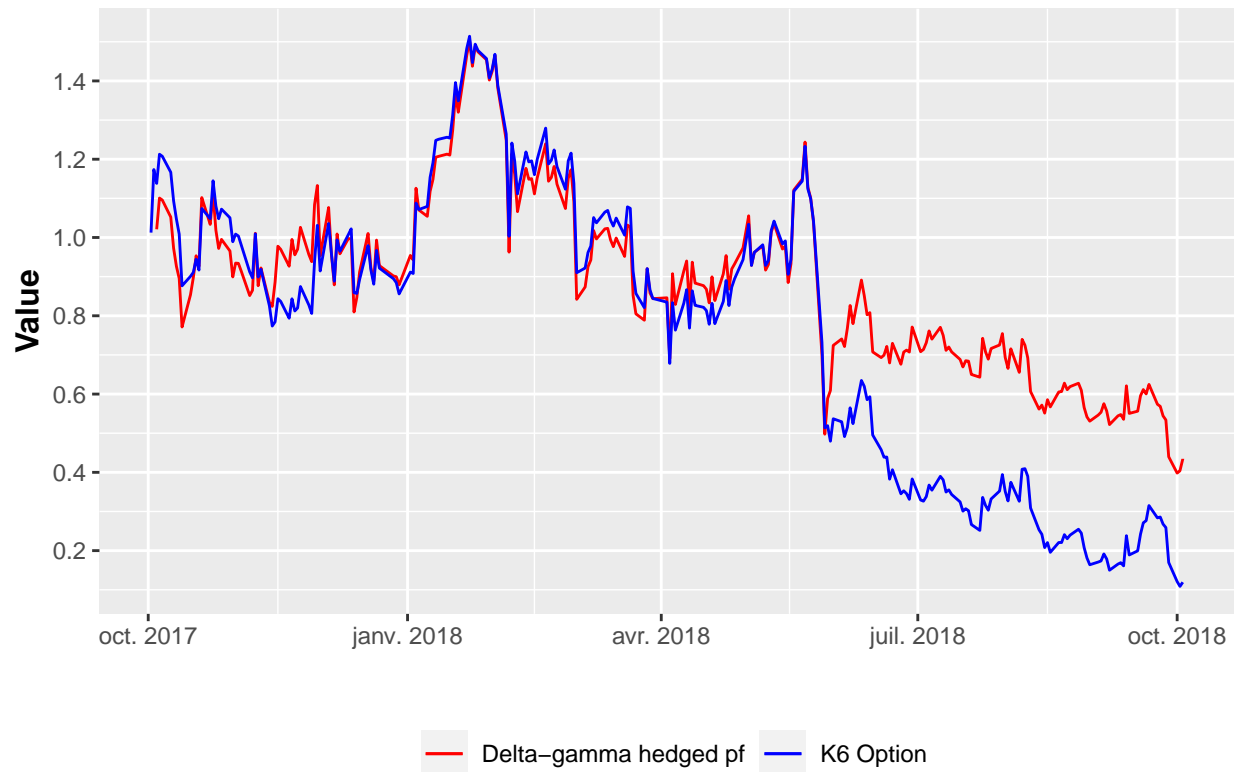
- **Initialization :** $\sigma_0^a = \sigma^a$ and $\sigma_0^b = \sigma^b$ such that $f(\sigma^a)f(\sigma^b) < 0$.
- **Itération :** $f(\sigma_i^a)f(\sigma_i^b) < 0$. Let $\sigma_i^c = (\sigma_i^a + \sigma_i^b)/2$. So
- If $f(\sigma_i^c) = 0$: end of the algorithm, **the solution is found**.
- If $f(\sigma_i^c)f(\sigma_i^a) < 0$: $\sigma_{i+1}^a = \sigma_i^a$ and $\sigma_{i+1}^b = \sigma_i^c$

- If $f(\sigma^c)f(\sigma^a) > 0$: $\sigma_{i+1}^a = \sigma_i^c$ and $\sigma_{i+1}^b = \sigma_i^b$ with $f(\sigma_i^\beta) = C_{market} - C^{BS}(\sigma_i^\beta)$

We used the previous algorithms to determine the implied volatilities of the K6 and K6.5 call option. The results are presented on the chart below.



Delta–gamma hedge – K6 Option with non cte vol



Interpretation : We can see on the chart above that the replication is not perfect, especially from June, 2018. This is not very surprising because in this section we considered a **non-constant volatility** for the two options because this corresponds to a **more realistic situation**.