Sujet de Travaux Dirigés / Pratiques - TP MRF - IMA203

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Bayesian analysis for image classification

Objective of the session :

In this PW we will perform the binary classification of a grayscale image "Iobservee.png" (image of the observations, realization y of the field Y) using a Markovian model.
In this ideal case, we are given the ideal solution x (binary image "IoriginalBW.png"), realization of the field of classes X, which will be used to evaluate the quality of the solution $\hat{x}$ that we will obtain. (NB : In practice usually, we don't have access to x).
You have to fill by hand-writing the printed version of the practical work (this document) and upload the filled jupyter notebook on e-campus.

This report should be given on the 9th of december. You can do it in pair (2 students), put both names on the document. The filled notebook should be also uplodaded on e-campus for the 9th of december.

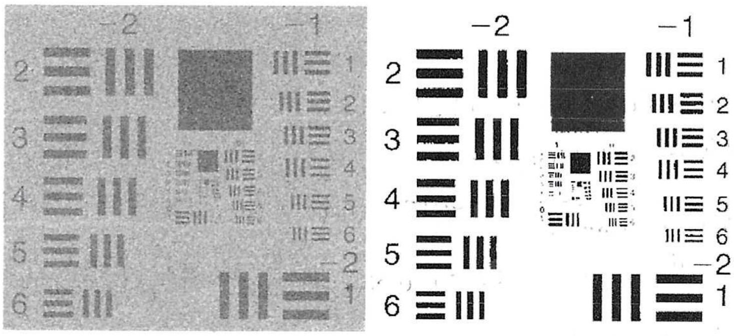

FIGURE 1 – Observed image y on the left (gray levels) and "ideal" binary image x (on the right) that we are trying to recover.

The objective is to estimate x from y using a prior on $P(X)$ in the form of a Markovian model. We note x_s the class of the pixel s (that we are looking for), and y_s the observed gray level. The objective is to use a global model on the random field X to classify the image. As we have seen in class, this amounts to minimizing the following energy :

$$U(x|y) = \sum_s -\ln(P(Y_s = y_s|X_s = x_s)) + \sum_c U_c(x_s, s \in c)$$

1 Analysis of the gray level distributions

In this part, we learn the probabilities $P(Y_s = y_s|X_s)$, that is to say $P(Y_s = y_s|X_s = 0)$ and $P(Y_s = y_s|X_s = 1)$. This is equivalent to studying the histogram of gray levels of pixels that are in class 0 and pixels that are in class 1.

To perform this training, we need to select pixels belonging to class 0 on the one hand (dark area of the observed image), and pixels belonging to class 1 on the other hand (light area of the observed image).

— Q1 What are the distributions followed by the grey levels in these two classes? Give the means and variances of the two classes that you have estimated.

Both of the distribution follow a Gaussian law:
- first class (dark, class 0) follows one with mean 96.202 and variance 500.43
- second class (light, class 1) follows one with mean 165.88 and variance 512.73

In the following, we assume that the variances are equal in order to simplify the energy expressions.
Suppose that we do not use a Markov model on X and that we classify a pixel only according to its grey level by comparing $P(Y_s = y_s|X_s = 0)$ and $P(Y_s = y_s|X_s = 1)$.

— Q2 Show that this amounts to thresholding the image and give the value of the optimal threshold as a function of the parameters found previously (we say that we are doing a classification by punctual (=in each pixel) maximum likelihood).

With that classification, we have:
- $P(Y_s|0) = \mathcal{N}(96, 500)$
- $P(Y_s|1) = \cancel{\mathcal{N}(164, 513)}$ $\mathcal{N}(164, 500)$ (variances equal)

The Gaussian law is: $P(Y_s|X_s) = \frac{1}{\sqrt{2\pi\sigma_{x_s}}} \exp\left(\frac{-(Y_s-\mu_{x_s})^2}{2\sigma_{x_s}^2}\right)$

To have a classification we must have $P(Y_s|0) > P(Y_s|1)$, i.e. $\frac{1}{\sqrt{2\pi\sigma_0}} e^{\left(\frac{-(y_s-\mu_0)^2}{2\sigma_0^2}\right)} > \frac{1}{\sqrt{2\pi\sigma_1}} e^{\left(\frac{-(y_s-\mu_1)^2}{2\sigma_1^2}\right)}$

we have $\sigma_1 = \sigma_0$ by hypothesis, then this expression simplifies: $(y_s-\mu_0)^2 < (y_s-\mu_1)^2$

The threshold is given by $Y_s < \frac{\mu_0+\mu_1}{2} \simeq 130$

— Q2bis Show graphically the threshold by drawing the histograms corresponding to the two conditional distributions.

— From the results found for $P(Y_s = y_s|X_s)$, write the likelihood energy (data attachment term) :

$$U_{attdo} = \sum_s - \ln(P(Y_s = y_s|X_s = x_s))$$

We have the Gaussian law $P(Y_s|X_s) = \frac{1}{\sqrt{2\pi}\sigma_{x_s}} \exp\left(\frac{-(y_s-\mu_{x_s})^2}{2\sigma^2_{x_s}}\right)$

That gives us : $-\ln(P(Y_s|X_s)) = \frac{(y_s-\mu_{x_s})^2}{2\sigma^2_{x_s}} + \ln(\sqrt{2\pi}\sigma_{x_s})$

The distributions have the same variance σ, so :

$$U_{attdo} = \sum_s \frac{(y_s-\mu_{x_s})^2}{2\sigma^2} + \ln\sqrt{2\pi}\sigma$$

2 Ising model for regularization

To improve the thresholding results, it is necessary to introduce a regularisation (global prior model).

Consider the function $\Delta(x_s, x_t) = 0$ if $x_s = x_t$, and $\Delta(x_s, x_t) = 1$ otherwise.

— Q4a Write the second-order clique potential for this Ising model as a function of $\Delta(x_s, x_t)$ where x_s and x_t are the classes of neighbouring pixels s and t in 4-connexity and the regularisation parameter β. This model will be 0 when the two neighbouring pixels are equal and $+\beta$ otherwise.

$\ln\sqrt{2\pi}\sigma$ is constant, so we can simplify : $U_{attdo} = \sum_s \frac{(y_s-\mu_{x_s})^2}{2\sigma^2}$

Then $U_c(x_s, x_t) = \beta(1-\delta_{x_s,x_t})$

Write the *global* energy of the whole field and the local conditional energy for a site s using the results previously established for the data attachment energy and the regularization energy defined previously.

Reminder : the global energy contains all the clique potentials in the image, the local conditional energy at a site s contains only the clique potentials that contain s.

Tip : the energy is defined to within one additive constant and one multiplicative constant (the minimum of K+K'U is equivalent to the minimum of U). It is better to simplify the writing of the energy as much as possible in order to do the programming afterwards.

— Q4b Global energy : $U(x, y) = U_{attdo} + U_c(x_s, x_t)$

$$= \frac{1}{2\sigma^2}\sum_s (y_s-\mu_{x_s})^2 + \beta\sum_{s,t}(1-\delta_{x_s,x_t})$$

— Q4c Local conditional energy :

$$U(x_s|y_s, x_t \text{ neighbours}) = (y_s-\mu_{x_s})^2 + \beta\sum_{t \in V}(1-\delta_{x_s,x_t})$$

— Q5 Write the local conditional energies for classes 0 and 1 of the central pixel, using the following local neighbourhood configuration : neighbours in states 0, 1, 1, 1, and assuming that the grey level of the pixel is $y_s = 105$, and using the mean and variance values found previously.

- class 0 ~~light~~ (dark) :

$$U(0 \mid 105; 0,1,1,1) = (105-96)^2 + 3\beta = 81 + 3\beta$$

- class 1 (light) :

$$U(1 \mid 105; 0,1,1,1) = (105-164)^2 + \beta = 3481 + \beta$$

— Q6 In which class will this pixel be put if it is assigned the class that locally minimises energy ?

It depends on β. Pixel will be put in class 0 if and only if :

$$U(0 \mid 105; 0,1,1,1) < U(1 \mid 105; 0,1,1,1)$$

$$\Leftrightarrow \quad 2\beta < 3400$$

$$\Leftrightarrow \quad \beta < 1700$$

We chose $\beta = 1500$, therefore it will be put in class 0.

— Q7 Considering the *global* energy of the field, what is the solution x when β is 0 ?

With $\beta = 0$, there is no regularisation. We are at the same case that in Q2, the pixel is classed only by is gray level

— Q8 Considering the *global* energy of the field, what is the solution x when β is $+\infty$?

$\beta = +\infty$ corresponds of a maximum regularisation. It is the same as optimise regarding to $P(x = \alpha \mid V_s)$

— Q9 How will the solution vary when β increases? Comment on the interest of this Markovian model.

When β increases, the regularisation is encouraged. ~~Without~~ It can help erase noise or other deterioration.
This model allows to class pixel regarding regularisation and data attachment, and we can optimize this operation by modifying β.

3 Optimization by ICM algorithm

We will optimise the global energy defined above, using the ICM (Iterated Conditional Modes) algorithm which consists of minimising the local conditional energy of the pixels one after the other, starting from a good initialisation of the classes. This algorithm converges to a local minimum but is very fast.

Complete the function to program the ICM, taking into account the data attachment term you have learned.

— Q10 How can we choose a good initialization of the solution? Justify your answer.

We can choose the initialization that uses the classification by gray-level only. By doing this, a good part of pixels will be well classified, and so it should converge faster.

— Q11 With what value of β do you get a good solution (i.e. the closest to the given "ideal" image "IoriginaleBW.png")? Compare this result with the result of the optimal thresholding.

With $\beta = 1500$, we have a good result, close to the given ideal image.
The result is far less noisy than the result using only thresholding method.

— Q12 Try with other initialisations (with a constant image, with a random image). Comment on their influence.

With only few iterations, the results are pretty bad. However, with about ten iterations, the results are similar. It is only slower to converge.

4 optimization by simulated annealing

Program the function of the simulated annealing which allows to update an image by sampling with the Gibbs distribution a posteriori with a fixed temperature T.

— Q13 Compare the results obtained by the Iterated Conditional Modes algorithm and by simulated annealing. Do you observe the expected results of the course?

The results by ICM and simulated annealing are quite similar, and that is in accordance with the theory. The only thing is that because we begin with a high T, we have to do a lot of iterations in order to have a small T, while in this situation, the ICM gives a good result quite quickly with a good initialization.