COMPLETE CALABI-YAU METRICS FROM SMOOTHING CALABI-YAU COMPLETE INTERSECTIONS

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ABSTRACT. We extend the work of Székelyhidi [22] to construct complete Calabi-Yau metrics on non-compact manifolds which are smoothings of an initial complete intersection Calabi-Yau cone V_0. The constructed Calabi-Yau manifold has tangent cone at infinity given by $\mathbb{C} \times V_0$. This construction produces Calabi-Yau structures on new topological spaces which are not $\mathbb{C}^*$-equivariant and admit fibers having different complex structures including singularities.

1. INTRODUCTION

Non-compact Calabi-Yau manifolds have been thoroughly studied with various behaviors at infinity since Cheng-Yau [3] and Tian-Yau [23, 24] among others and are still a topic of active research with many outstanding open questions. Recent examples of complete Calabi-Yau metrics on $\mathbb{C}^n$ with maximal volume growth have been produced [8, 9, 17, 22] disproving a conjecture of Tian that the Euclidean metric was the unique such Calabi-Yau metric. Works by Li [17] and Conlon-Rochon [9] have produced Calabi-Yau metrics on $\mathbb{C}^n$ with tangent cone given by $V \times \mathbb{C}$ for an $(n-1)$-dimensional Calabi-Yau cone V. Székelyhidi [22] extended these results to produce Calabi-Yau metrics on $\mathbb{C}^n$ as smoothings of hypersurfaces which themselves are Calabi-Yau cones. A natural question to ask is if an n-dimensional complete intersection V_0 cut out by k quasi-homogeneous polynomials can prescribe a complete Calabi-Yau metric on $\mathbb{C}^{n+k}$ with maximal volume growth and tangent cone at infinity given by $V_0 \times \mathbb{C}^k$. We present that a generic linear slice of such a smoothing carries a Calabi-Yau metric with tangent cone at infinity $V_0 \times \mathbb{C}$. These examples have tangent cones at infinity from Cheeger-Colding [2] and are expected to be unique [5, 12]. In particular, this linear slice is no longer $\mathbb{C}^*$-equivariant with respect to the action defining the quasi-homogeneous polynomials when the degrees are different. Each fiber carries a different complex structure, some with singularities, requiring extra care taken to produce model Calabi-Yau metrics that vary smoothly near infinity.

Let $V_0 = \{f_1(x) = \cdots = f_k(x) = 0\} \subset \mathbb{C}^{n+k}$ be a conical complete intersection of dimension n with an isolated singularity at 0. We assume that $f_1, \ldots, f_k$ are quasi-homogeneous under some action $\xi = (w_1, \ldots, w_{n+k})$ where f_i has degree d_i and assume that $1 < d_1 \leq \cdots \leq d_k$:

$$t \cdot x = (t^{w_1} x_1, \ldots, t^{w_{n+k}} x_{n+k}), \quad f_i(t \cdot x) = t^{d_i} f_i(x).$$

Consider a smoothing of V_0 by choosing some point $p = (p_1, \ldots, p_k)$ defined as

$$\mathcal{X} = \{f_1(x) - zp_1 = \cdots = f_k(x) - zp_k = 0\}$$

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such that the total space is smooth and at least one fiber is smooth, which can be rescaled such that $z = 1$. We can label these fibers for any $t \in \mathbb{C}$ as

$$V_t = \mathcal{X} \cap \{z = t\} \cong \{f_1(x) - tp_1 = \cdots = f_k(x) - tp_k = 0\}.$$

The singular fibers form an algebraic subvariety of $\mathbb{C}$, so there are only finitely many of them, and they are bounded in the z direction. We define the index ℓ to be the maximal index such that $d_\ell = d_1$, that is, $1 < d_1 = \cdots = d_\ell < d_{\ell+1} \leq \cdots \leq d_k$.

The main difference between our construction and that of [22] is that when $\ell < k$, the rescaling action does not take the fibers above V_{z_0} to a rescaling V_{tz_0}, that is, it does not preserve $\mathcal{X}$ as considered in $\mathbb{C}^{n+k+1}$. In $\mathcal{X}$ the fibers have potentially different complex structures including singularities, unlike in [22] or if $\ell = k$ where all the fibers are the same. We allow $\ell = k$, and this would reduce all the computations and geometry to the case of [22], as the rescaling action would be $\mathbb{C}^*$-equivariant and therefore preserve $\mathcal{X}$, taking fibers to rescaled fibers appropriately. Since the scaling action does not preserve the fiber structure, we need to construct a separate Calabi-Yau metric on each fiber that vary smoothly in the z-direction. The metrics on the fibers will vary smoothly because the computations only involve working near infinity, outside the algebraic set of singular fibers and therefore all the fibers are smooth.

Another difference from [22] is that the limiting geometry of the fibers differs from any of the fibers in a region near infinity. The scaling action

$$G_t(z, x_1, \ldots, x_{n+k}) = (tz, t^{w_1}x_1, \ldots, t^{w_{n+k}}x_{n+k})$$

applied to $\mathcal{X}$ is expressed as

$$G_t^{-1}(\mathcal{X}) = \{f_1(x) - t^{1-d_1}zp_1 = \cdots = f_k(x) - t^{1-d_k}zp_k = 0\}. \tag{1}$$

As $t \to \infty$ this approaches $X_0 = V_0 \times \mathbb{C}$, its tangent cone at infinity. The first ℓ coordinates scale the most slowly, and under specific rescalings used in the analysis in later sections, the model geometry becomes the fiber

$$V_{p'} = \{f_1(x) - p_1 = \cdots = f_\ell(x) - p_\ell = f_{\ell+1}(x) = \cdots = f_k(x) = 0\}$$

over the point $p' = (p_1, \ldots, p_\ell, 0, \ldots, 0)$, the projection onto the first ℓ coordinates, in a certain region near infinity. We therefore require this fiber to also be smooth. Not all complete intersections admit such a smooth fiber, or even perturbations that admit a smooth fiber of this form. For example, for $f_1 = z_1^2 + z_2z_3 + z_4^3, f_2 = z_1z_4^2 + z_2^2 + z_3^2z_4 + z_5^3$ with weights $\xi = (27, 63/2, 45/2, 18, 21)$ and of degrees $d_1 = 54$ and $d_2 = 63$, no fiber $V_{(t_1,0)} = \{f_1 - t_1 = f_2 = 0\} \subset \mathbb{C}^5$ is smooth and no perturbation of the polynomials fixes this singularity either by smoothing it, or perturbing it off the vanishing locus. In the case when no smooth fiber exists, it is possible the result can still hold as the singularities will be isolated and Gorenstein. In these circumstances, it may still be possible to invert the Laplacian and create a singular Calabi-Yau metric with which the methods presented here could be utilized, or model Calabi-Yau metrics around the singularities could be glued to asymptotically Calabi-Yau metrics to create a global singular Calabi-Yau metric on the fiber. This is a question for future study.

Given a conical, singular, Calabi-Yau metric ω_{X_0} on X_0 whose homothetic transformations are the maps G_t given above, we produce a Calabi-Yau metric on $\mathcal{X}$ that has tangent cone at infinity given by (X_0, ω_{X_0}). The construction and methodology follow similarly the work of Székelyhidi in [22] with care taken as the fibers are no longer all biholomorphic rescalings of the same fiber, nor is the limiting fiber geometry as discussed. The strategy is to construct model Calabi-Yau metrics near infinity in the z-direction and in the cone direction and glue them together. We will then perturb this asymptotically Calabi-Yau metric to be Ricci-flat outside a compact set, which will allow us to apply the result of Hein's PhD thesis [15] to further perturb it to a globally Ricci-flat metric, giving the main theorem.

Theorem 1.1. *If V_0 admits a Calabi-Yau cone metric ω_0, then there exists a complete Calabi-Yau metric on $\mathcal{X}$ with tangent cone at infinity given by* $(\mathbb{C}\times V_0, \sqrt{-1}\partial\overline{\partial}(|z|^2)+\omega_0)$.

1.1. **Outline.** The structure of this paper follows similarly to Székelyhidi [22]. In Section 2, we describe the smoothing of the complete intersection V_0 as a generic linear slice of the smoothing given by taking all level sets of the polynomials. Such a general slice is not $\mathbb{C}^*$-equivariant, meaning that the $\mathbb{C}^*$-action given by the quasi-homogeneity of all the polynomials does not preserve the slice. In particular, each fiber has a different complex structure and potentially finitely many including isolated singularities. We further assume that the fiber $V_{p'}$ that represents the model geometry at infinity is smooth to produce a model space and metric to invert the Laplacian in Section 6. It is possible that a similar analysis would work for $V_{p'}$ being singular using the techniques for inverting the Laplacian on V_0 and an extension of this result could hold without this assumption.

Since each fiber is asymptotic to V_0, which is itself a Calabi-Yau cone, there is a global potential extending the Calabi-Yau metric $\sqrt{-1}\partial\overline{\partial}r^2$ on V_0 which is a smoothly varying asymptotically Calabi-Yau form on every fiber. By Conlon-Hein [7], in a proper weighted space, there is a unique potential we can add to each fiber giving an actual Calabi-Yau metric. By localizing the problem to a small neighborhood of infinity, all the fibers will be smooth and diffeomorphic, so the potentials added to perturb the asymptotically Calabi-Yau metric to be Ricci-flat will vary smoothly as well. In the base, these points can be made to be arbitrarily close to p', which as shown in equation 1 is the limiting fiber near infinity, under the proper rescaling. In particular, we focus the analysis on a compact set near infinity, so there are a priori gradient bounds as well.

We construct a metric ω on $\mathcal{X}$ in Section 3 that is asymptotically Calabi-Yau by modeling the various regions near infinity on the Calabi-Yau cones of $X_0 = V_0\times\mathbb{C}$ and $X_{p'} = V_{p'}\times\mathbb{C}$. Because the fibers are not all biholomorphic to each other, and the total space is not preserved by the $\mathbb{C}^*$-action, this model metric instead utilizes smoothly varying Calabi-Yau metrics on the fibers near $V_{p'}$. To show that the metric is asymptotically Ricci-flat, we separate the regions near infinity which have different growth rates along the base and along each fiber. Since each fiber is asymptotic to V_0, as the fiber direction gets larger, the model geometry is X_0. When z goes to infinity, but we stay very near to the *cone point* of each fiber, the fibers are not comparable to V_0, so the model metric must use the actual Calabi-Yau metric constructed on the fiber, and the model space will be $X_{p'}$. It is in these sections that the analysis requires the assumption that $V_{p'}$ is smooth so the construction of a Calabi-Yau metric on it from Conlon-Hein [7] works as described.

After the proper construction of the total space and the proper model geometries, the algebraic structure as a complete intersection is no longer critical to study the Laplacian and geometry of the space $\mathcal{X}$. The remaining sections therefore follow the methods of Székelyhidi [22] closely. In Section 4, we construct weighted Hölder spaces to analyze the Laplacian in the model spaces of X_0 and $X_{p'}$. Section 5 proves that near infinity the model spaces of X_0 and $X_{p'}$ are arbitrarily good approximations of $\mathcal{X}$ in the various regions with sufficient overlap to glue the model metrics together.

In Section 6, we study the linear analysis on the weighted spaces and show that the Laplacian is invertible for suitable weights. Since X_0 has rays of singularities, it helps to consider this as a cone over a singular manifold which has a circle of singularities. This characterization allows the results of Mazzeo [19] to study the Laplacian and invert it in the weighted spaces.

The result is nearly finished in Section 7, where we show that the approximately Calabi-Yau metric constructed in Section 3 can be perturbed to be Ricci-flat outside a sufficiently large compact set. Finally, in Section 8 we use a result from Hein [15] to show that this metric extends to be a Ricci-flat metric everywhere which completes the proof.

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2. Setup

Consider a Calabi-Yau cone V_0 of complex dimension n algebraically expressed as a complete intersection cut out by polynomials f_i:

$$V_0 = \{f_1(x) = \cdots = f_k(x) = 0\} \subset \mathbb{C}^{n+k}$$

with a quasi-homogeneous action given by a positive weight vector $\xi = (w_1, \ldots, w_{n+k})$ such that

$$f_i(t \cdot x) = f_i(t^{w_1}x_1, \ldots, t^{w_{n+k}}x_{n+k}) = t^{d_i} f_i(x).$$

We assume each $d_i > 1$ and $d_1 \leq d_2 \leq \cdots \leq d_k$. This action generates the action of a complex torus T^C on $\mathbb{C}^{n+k}$ that leaves V_0 invariant. Let $T \subset T^C$ be the maximal compact torus. There is also a natural action of $\mathbb{C}$ on the codomain of $\mathbb{C}^k$ which we will notate also using $t \cdot s$ for $s = (s_1, \ldots, s_k)$ by $t \cdot s = (t^{d_1}s_1, \ldots, t^{d_k}s_k)$. The analysis will reduce exactly to the setting of Székelyhidi [22] unless $d_1 \neq d_k$, because in that setting every fiber is a rescaling of a base fiber. For a broader range of geometry we can assume that $d_1 \neq d_k$. Since V_0 is assumed to be a Calabi-Yau cone, there is a nowhere vanishing holomorphic n-form Ω which by the adjunction formula can be defined on $V_0 \setminus \{0\}$ as

$$\Omega_V = \frac{dx_1 \wedge \cdots \wedge dx_{n+k}}{df_1 \wedge \cdots \wedge df_k}$$

which is a notation for the global expression defined locally as

$$\Omega_V = \frac{dx_{k+1} \wedge \cdots \wedge dx_{n+k}}{\partial_{x_1} f_1 \cdots \partial_{x_k} f_k}$$

where the denominator does not vanish, and similar expressions for other regions where $\partial_{x_{i_1}} f_1 \cdots \partial_{x_{i_k}} f_k$ is non-zero with i_k distinct. $V_0 \setminus \{0\}$ has a symplectic form ω_{V_0} which is a Ricci-flat Kähler cone metric such that

$$\omega_{V_0}^n = (\sqrt{-1})^{n^2} \Omega_V \wedge \overline{\Omega}_V. \tag{2}$$

In order for this to be true, we must have a topological criterion stating that Ω has degree n under the action, or equivalently

$$\sum w_j - \sum d_i = n. \tag{3}$$

Because ω_V is homogeneous of degree 2, the relation 3 is necessary so that both sides of equation 2 have the same degree under the quasi-homogeneous scaling action.

We also assume a non-degeneracy statement saying that V_0 is not contained in a codimension 1 variety.

Lemma 2.1. *If V_0 does not lie in a hypersurface properly contained in $\mathbb{C}^{n+k}$, then $d_1 > 2$.*

Proof. Let $w_{\min}$ be the smallest weight. Since we assume that V_0 admits a Calabi-Yau metric, by the Lichnerowicz obstruction of Gauntlett-Martelli-Sparks-Yau [13], $w_{\min} > 1$. If $w_{\min}$ is equal to 1, then V_0 is contained in a smaller dimensional hyperplane. Each d_i must be at least $2w_{\min}$, showing that each is greater than 2. □

We choose some point $p = (p_1, \dots, p_k)$ normalized using the scaling action such that $|p| = 1$ and $V_p = V_{(p_1,\dots,p_k)}$ is smooth. The existence is justified from Sard's theorem, which says that the preimage of a generic point is smooth. We recall ℓ to be the maximal index such that $d_1 = d_2 = \cdots = d_\ell < d_{\ell+1} \leq \cdots \leq d_k$ and define the point

$$p' = (p_1, \dots, p_\ell, 0 \dots, 0)$$

and further assume that $V_{p'}$ is smooth for some p. In many cases, there exists a smooth fiber of this form, or a small perturbation of the polynomials exists that preserves K-stability (which is equivalent to having a Calabi-Yau metric as per Collins and Székelyhidi [6]) and has a smooth fiber of this form. As mentioned above, this is not always the case as there do exist complete intersections that do not have perturbations that have smooth fibers over $\mathbb{C}^\ell$.

We define the total space $\mathcal{X}$ as swept out by the set of fibers V_{zp}:

$$\mathcal{X} = \{f_1(x) - zp_1 = \cdots f_k(x) - zp_k = 0\} \subset \mathbb{C}^{n+k+1}.$$

$\mathcal{X}$ is a complete intersection in the ambient space $\mathbb{C}^{n+k+1}$ with variables $(x_1, \dots, x_{n+k}, z)$ as it is expressed as cut out by the above equations (with the z-variable added). The expected dimension is $\dim V_0 + 1$, since it is a 1-parameter family of fibers asymptotic to V_0, and the proper codimension of k matches the k equations. We have a map to $\pi : \mathcal{X} \to \mathbb{C}$ by projection onto z identifying $V_p = V_1$ using the prior notation defining $V_t = \pi^{-1}(t)$. We require at least one smooth fiber because the singularities form an analytic subvariety of $\mathbb{C}$ to avoid the case of ever fiber being singular. In particular, there are either finitely many, or every fiber is singular, so the assumption that V_1 is smooth ensures that outside a sufficiently large compact set, all fibers are smooth.

On $\mathcal{X}$, we will construct a Calabi-Yau metric based on gluing a Calabi-Yau metric on each smooth fiber V_t to a product Calabi-Yau metric of $V_0 \times \mathbb{C}$. Let $X_t = V_t \times \mathbb{C}$ with X_0 and $X_{p'}$ being of primary interest. We have a holomorphic $(n+1)$-form on the ambient space

$$\Omega = \frac{dz \wedge dx_1 \wedge \cdots \wedge dx_{n+k}}{df_1 \wedge \cdots \wedge df_k}$$

which can be explicitly computed as $\Omega = dz \wedge \pi_x^* \Omega_V$ with $\pi_x : \mathbb{C}^{n+k+1} \to \mathbb{C}^{n+k}$ the projection onto the x-coordinates. This form restricts to a nowhere vanishing top holomorphic form on X_t and $\mathcal{X}$.

Let $\omega_{V_0} = \sqrt{-1}\partial\bar{\partial} r^2$ for the cone radius function r. For succinct notation, let $F : \mathbb{C}^{n+k} \to \mathbb{C}^k$ defined as $F(x) = (f_1(x), \dots, f_k(x))$ be the vector of f_i functions. This allows us to express fibers $V_t = V_{(t_1,\dots,t_k)}$ as simply the preimage $F^{-1}(t)$. The total space is recognized as $\mathcal{X} = \{F(x) = zp\} \subset \mathbb{C}^{n+k+1}$. We define a scaling action $G_t : \mathbb{C}^{n+k+1} \to \mathbb{C}^{n+k+1}$ given by

$$G_t(z, x_1, \dots, x_{n+k}) = (tz, t^{w_1} x_1, \dots, t^{w_{n+k}} x_{n+k}), \tag{4}$$

which acts by the weight vector ξ on $\mathbb{C}^{n+k}$ and linearly on z. The fibers $V_{(t_1,\ldots,t_k)}$ sweep out $\mathbb{C}^{n+k}$ as $t = (t_1, \ldots, t_k)$ vary through all possible values. We will be taking a linear slice of this where $t = zp$ for $z \in \mathbb{C}$ to form $\mathcal{X}$. One key difference between this and the work from Székelyhidi [22] is that $G_t(\mathcal{X}) \not\subset \mathcal{X}$; this scaling action does not take fibers to fibers of $\mathcal{X}$. It is for this reason that on each fiber we need to construct its own Calabi-Yau metric and have them vary smoothly as we move along the z direction.

We define a radius function R on the ambient space $\mathbb{C}^{n+k+1}$ that is uniformly equivalent to the radius function r on V_0. We can define R to be 1 on the unit sphere $S^{2(n+k)+1} \subset \mathbb{C}^{n+k+1}$ and defining it elsewhere by enforcing that it has degree 1 under the map G_t. Since each weight in the action of G_t is at least 1, the norm of a scaled element $V_{z\cdot t}$ is increasing for $|z| > 1$ and decreasing for $|z| < 1$, that is, $|z \cdot t| > |t|$ for $|z| > 1$. The function r can be extended arbitrarily to $\mathbb{C}^{n+k+1}$ by first extending it smoothly on the sphere $R = 1$ and then to other points by similarly enforcing that it is homogeneous with degree 1 under G_t. The extended function will still be notated as r.

Proposition 2.2. *Let V_t be a smooth fiber. The form $\omega = \sqrt{-1}\partial\overline{\partial}r^2$ on V_t is a well-defined metric outside a compact set, for $V_t \cap \{R > A\}$ for some $A \gg 0$, and its Ricci potential*

$$h = \log \frac{(\sqrt{-1}\partial\overline{\partial}r^2)^n}{(\sqrt{-1})^{n^2}\Omega_V \wedge \overline{\Omega}_V}$$

satisfies the decay estimate $|\nabla^i h|_\omega = O(R^{-d_1-i})$ as $R \to \infty$ with this norm measured using ω.

Proof. Let $K \gg 0$ be a large constant and examine the annular region $\frac{K}{2} < R < 2K$. We will rescale the metric by K^{-1} and label the rescaled coordinates as

$$\tilde{x} = K^{-1} \cdot x, \quad \tilde{r} = K^{-1}r, \quad \tilde{R} = K^{-1}R,$$

so the metric in these new coordinates can be expressed as

$$\tilde{\omega} = K^{-2}\omega = \sqrt{-1}\partial\overline{\partial}\tilde{r}^2.$$

In these rescaled coordinates, the algebraic expression for V_t is given by $F(\tilde{x}) = K^{-1} \cdot t$. Expanded out, this is expressed as

$$\left\{f_1(\tilde{x}) - K^{-d_1}t_1 = \cdots = f_1(\tilde{x}) - K^{-d_k}t_k = 0\right\},$$

or $V_{K^{-1}\cdot t}$ in the concise notation. Specifically, on the annulus defined by $\{\frac{1}{2} < \tilde{R} < 2\}$, these $V_{K^{-1}\cdot t}$ converge to V_0 in the C^∞ topology uniformly since each $d_i > 1$. Furthermore, $\tilde{r}^2$ is a function of only $\tilde{x}$ independent of K since r is quasi-homogeneous with respect to the scaling action. We can compute $\tilde{r}^2$ simply from $\tilde{x}$ knowing its value on one point and that it satisfies the homogeneity constraint and radial symmetry.

The implicit function theorem gives a cover of the annular region $\{\frac{1}{2} < \tilde{R} < 2\}$ in $V_{K^{-1}\cdot t}$ by finitely many coordinate patches B_i such that $V_{K^{-1}\cdot t}$ is locally expressible as a hyperplane in each chart. The implicit function theorem provides holomorphic functions $g_i : B_i \to \mathbb{C}^{n+k}$ such that $g_i(V_{K^{-1}\cdot t})$ is contained in V_0, and $g_i(y) = y + O(K^{-d_1})$. The decay term is limited by the smallest degree d_1 as will be the case in all the following arguments. We use these local expressions to compare the rescaled potential to its pullback and see that it has decay given by K^{-d_1}. We can observe that $\tilde{r}^2 - g_i^*(\tilde{r}^2)$ must have growth decay $O(K^{-d_1})$, so once K is large enough, the form $\tilde{\omega}$ is indeed positive definite, proving the first claim. It is further uniformly equivalent to $\sqrt{-1}\partial\overline{\partial}\tilde{R}^2$ which is a cone metric on $\mathbb{C}^{n+k+1}$ from He-Sun [14, Lemma 2.2].

For the second part, we want to examine the Ricci potential and compare Ω to its pullback along each implicit function g_i. Since by assumption $\omega_{V_0}^n = \sqrt{-1}^{n^2}\Omega \wedge \overline{\Omega}$, the difference has bounded growth given by

$$\Omega_V - g_i^*(\Omega_V) = O(K^{-d_1}).$$

The Ricci potential is invariant under scaling from equation 3. This gives the decay

$$|\nabla^i h|_{K^{-2}\sqrt{-1}\partial\overline{\partial}R^2} = O(K^{-d_1})$$

finishing the result since for $K \gg 0$ sufficiently large, on V_0 the two forms $\sqrt{-1}\partial\overline{\partial}R^2$ and $\sqrt{-1}\partial\overline{\partial}r^2$ give rise to uniformly equivalent metrics. We utilize that switching back to the metric unscaled $\sqrt{-1}\partial\overline{\partial}r^2$ pulls out a copy of K^{-1} for each derivative so

$$|\nabla^i h|_{\sqrt{-1}\partial\overline{\partial}r^2} = O(K^{-d_1-i}).$$

□

The form $\sqrt{-1}\partial\overline{\partial}r^2$ is only a positive definite metric for $R \gg 0$, so we will use the form $C(1+|x|^2)^\alpha$ on the bounded region where ω (as defined above) is not positive definite and glue the two together to make a globally well-defined metric. For C sufficiently large and $\alpha > 0$, this is greater than r^2 on a large ball around the origin of radius A from Proposition 3.1. The form $\sqrt{-1}\partial\overline{\partial}r^2$ becomes a well-defined metric outside this region. We take a regularized maximum of the two forms above to have a global metric on V_t as per Demailly [11, §5.E]. The results of Conlon-Hein [7] say that this new metric, which is asymptotic to a Calabi-Yau metric with sufficient growth decay, can be perturbed to a Calabi-Yau metric defined by $\eta_t = \sqrt{-1}\partial\overline{\partial}\phi_t$. Furthermore, this metric is unique by [7, Theorem 2.1]. In particular, ϕ_t is invariant under the action of the compact complex torus T generated by the $\mathbb{C}^*$-action. In particular, the value of $\phi_t(z^{-1/d_1} \cdot x)$ is well-defined regardless of the choice of branch cut used to define the fractional exponent. Theorem 2.1 of [7] gives some $c > 0$ (if $d_1 > 3$, then even $c > 1$ can be chosen), constants C_i such that for x in the annular region $\frac{1}{2} < R < 2$, the following estimate holds for $\lambda > 1$:

$$\left|\nabla^i \left(r^2 - \lambda^{-2}\phi_t(\lambda \cdot x)\right)\right|_{\sqrt{-1}\partial\overline{\partial}R^2} < C_i\lambda^{-2-c-i}. \tag{5}$$

The above is summarized in the following corollary.

Corollary 2.3. *On each smooth fiber V_t, there exists a unique Calabi-Yau metric $\eta_t = \sqrt{-1}\partial\overline{\partial}\phi_t$ that satisfies the decay condition of equation 5.*

The function $\widetilde{\max}(C(1+|x|^2)^\alpha, r^2)$ is a global potential perturbed to ϕ_s for each smooth fiber V_s. Therefore, the uniqueness part of Theorem 2.1 of [7] means that the Calabi-Yau potentials ϕ_s vary smoothly as s varies (restricted to paths along the smooth fibers). Notably, in compact regions over the base space of $s \in S \Subset \mathbb{C}^k$, the gradient of ϕ will be bounded, and we will use this result later on the fibers that approach $V_{p'}$. In this case, we let $S = \{s : |p' - s| \leq \varepsilon\}$ for some $\varepsilon > 0$ sufficiently small, we will take the smallest c associated to all the ϕ_s in this compact region for our estimates, and use the compactness for the gradient bound of ϕ over the s-direction.

3. The Approximate Solution

In this section, we produce a metric on $\mathcal{X}$ that is asymptotically Calabi-Yau with sufficient decay of the Ricci potential that we will be able to later perturb it to be Ricci-flat outside a compact set. The metric will be constructed by gluing Calabi-Yau metrics on different regions near infinity. The model spaces will be products $X_{p'} = \mathbb{C} \times V_{p'}$ and $X_0 = \mathbb{C} \times V_0$ both with suitably scaled product metrics. Since each fiber itself is asymptotic to V_0, as we approach infinity along the fiber, the model is X_0, while as we approach it in the z direction, the model is $X_{p'}$. The model metrics are products $\sqrt{-1}\partial\overline{\partial}(|z|^2 + r^2)$ on X_0 and $\sqrt{-1}\partial\overline{\partial}(|z|^2 + \phi_{p'})$ on $X_{p'}$. In the case of X_0, which is a cone, the homothetic scalings are the maps G_t in equation 4 given by the action of weights of 1 on the z variable and ξ on the x variables.

Let γ_1 be a monotonic bump function such that

$$\gamma_1(x) = \begin{cases} 0, & x < 1 \\ 1, & x > 2 \end{cases}$$

and let $\gamma_2 = 1 - \gamma_1$ be its complementary cutoff function. We define a new radius function on the total space $\mathcal{X}$ as $\rho^2 = R^2 + |z|^2$. We define a metric

$$\omega = \sqrt{-1}\partial\overline{\partial}\left(|z|^2 + \gamma_1(R/\rho^\alpha)r^2 + \gamma_2(R/\rho^\alpha)|z|^{\frac{2}{d_1}}\phi_{\hat{p}_{z^{-1/d_1},z}}\left(z^{-\frac{1}{d_1}} \cdot x\right)\right) \tag{6}$$

for a small constant α to be chosen and with

$$\hat{p}_{a,b} = \left(a^{d_1}bp_1, \ldots, a^{d_k}bp_k\right)$$

the action with a acting with weights d_i on each term and b linearly. We can see from this expression that as $z \to \infty$, the limiting fiber $\hat{p}_{z^{-1/d_1},z}$ tends to p'. The main result of this section is to show sufficient decay of the Ricci potential h.

Proposition 3.1. *For some $\alpha \in \left(\frac{1}{d_1}, 1\right)$, the form ω from equation 6 defines a metric on $\mathcal{X}$ outside a compact set and for suitable constants $\kappa, C_i > 0$, there exists $\delta < \frac{2}{d_1}$ such that the Ricci potential h has decay*

$$|\nabla^i h| = \begin{cases} C_i\rho^{\delta-2-i}, & R > \kappa\rho \\ C_i\rho^\delta R^{-2-i}, & R \in (\kappa^{-1}\rho^{\frac{1}{d_1}}, \kappa\rho) \\ C_i\rho^{\delta-\frac{2}{d_1}-\frac{i}{d_1}}, & R < \kappa^{-1}\rho^{\frac{1}{d_1}}. \end{cases}$$

Proof. We split the ambient space into five regions to perform the analysis depending on which direction we approach infinity (i.e., the asymptotic relationship between R and z). The approach and computations follow similarly to Székelyhidi [22] with care taken to express everything in terms of the polynomials defining the region and the fact that the rescaling does not map onto the initial fiber V_1. In Regions IV and V, modeled by $X_{p'}$, an extra step must be taken to show that this is the correct model space and the error in approximating the region fits into the desired decay range.

Region I: In this region, let $R > \kappa\rho$ for some $0 < \kappa < 1$. In this region, we are far away from the *cone point*, and therefore each V_t is approximated well by V_0, so the model space is X_0. We define rescaled coordinates

$$\tilde{z} = D^{-1}z, \quad \tilde{x} = D^{-1} \cdot x, \quad \tilde{r} = D^{-1}r$$

and examine $D^{-2}\omega$, the properly rescaled metric in these coordinates. In this region, $\gamma_1 = 1$ and for $D \gg 0$ sufficiently large, the metric is simply $\sqrt{-1}\partial\overline{\partial}(|z|^2 + r^2)$. We can re-express $\mathcal{X}$ in these rescaled coordinates as

$$\mathcal{X} = \{f_1(\tilde{x}) - D^{1-d_1}\tilde{z}p_1 = \cdots = f_k(\tilde{x}) - D^{1-d_k}\tilde{z}p_k = 0\}$$

and we compare to $X_0 = V_0 \times \mathbb{C}$ with metric $\sqrt{-1}\partial\overline{\partial}(|z|^2 + r^2)$. The error in moving to this variety is D^{1-d_1}, as this is the limiting term since it is the greatest exponent. This gives the bound

$$|\nabla^i h|_{D^{-2}\omega} \leq C_i D^{1-d_1}$$

and by using the unscaled metric ω, there is a term of D^{-1} for each derivative giving

$$|\nabla^i h|_\omega \leq C_i D^{1-d_1-i}.$$

We conclude that in this region, δ can be chosen such that $\delta > 3 - d_1$. Since $d_1 \geq 2$, δ can be chosen to satisfy $\delta < \frac{2}{d_1}$, and if $d_1 \geq 3$, δ can be made negative.

Region II: We let $R \in (K/2, 2K)$ for some $K < \kappa\rho$ and $K/2 > 2\rho^\alpha$. This region is similar in that we still want to compare to the model space of X_0 as R is sufficiently greater than $|z|$. This region still has $\gamma_1 = 1$ and therefore the same metric $\omega = \sqrt{-1}\partial\overline{\partial}(|z|^2 + r^2)$. We focus the analysis around some basepoint z_0 such that $|z - z_0| < K$. We utilize similar rescaled coordinates, but centered at z_0:

$$\tilde{x} = K^{-1} \cdot x, \quad \tilde{z} = K^{-1}(z - z_0), \quad \tilde{r} = K^{-1} r.$$

In these new coordinates, the rescaled metric is expressed as

$$K^{-2}\omega = \sqrt{-1}\partial\overline{\partial}(|\tilde{z}|^2 + \tilde{r}^2)$$

and the equations defining $\mathcal{X}$ in these coordinates are

$$\mathcal{X} = \left\{ f_1(\tilde{x}) - K^{-d_1}(K\tilde{z} + z_0)p_1 = \cdots = f_k(\tilde{x}) - K^{-d_k}(K\tilde{z} + z_0)p_k = 0 \right\}.$$

Since $|\tilde{z}|,\ p_i < 1$, the procedure is as above, but with error term $K^{-d_1}D$ giving the bound

$$|\nabla^i h|_\omega \leq C_i D K^{-d_1-i}.$$

Because $d_1 > 2$ and $K > 4\rho^\alpha$, we compute

$$DK^{2-d_1}K^{-2-i} < CD^{1+\alpha(2-d_1)}K^{-2-i} < C\rho^{1+\alpha(2-d_1)}R^{2-i} \tag{7}$$

so in order to have $\delta > 1 + \alpha(2 - d_1)$, we let α be sufficiently close to 1 so that $\delta > 3 - d_1$. This gives the same constraint as in the prior region. We must have $\alpha > \frac{1}{d_1}$, so this is satisfied. By the compactness of the region, there are only finitely many such regions centered at different z_0 to extend this bound to the entirety of Region II. Therefore, the decay is the maximum of these finitely many regions, so it is as stated as computed for a single region and z_0.

Region III: This is the gluing region where $R \in (K/2, 2K)$, $K \in (\rho^\alpha, 2\rho^\alpha)$, and $\rho \in (D/2, 2D)$. In this region, $|z| \sim D$ and both γ_1 and γ_2 are non-zero, so the metric will be the most complicated having both terms. The model space will still be X_0 as the Conlon-Hein bound from equation 5 will allow us to model the region with sufficient decay as the terms coming from ϕ_t approximate the geometry of the cone X_0. We use the same rescaling as in the previous region

$$\tilde{x} = K^{-1} \cdot x, \quad \tilde{z} = K^{-1}(z - z_0), \quad \tilde{r} = K^{-1} r.$$

Under this change of coordinates, the rescaled metric is expressed as

$$K^{-2}\omega = \sqrt{-1}\partial\overline{\partial}\left(|\tilde{z}|^2 + \gamma_1 \tilde{r}^2 + \gamma_2 K^{-2} |K\tilde{z} + z_0|^{-2} \phi_{\hat{p}_{K^{-1},z}} \left((K\tilde{z} + z_0)^{-\frac{1}{d_1}} K \cdot \tilde{x} \right) \right),$$

using the notation above

$$\hat{p}_{K^{-1},z} = \left(K^{-d_1} z p_1, \ldots, K^{-d_k} z p_k \right)$$

for the action of K^{-1} on the point zp. We recall that ϕ_t is the Calabi-Yau metric on the fiber V_t. In these rescaled coordinates, the derivatives of γ_1 and γ_2 are bounded and applying the Conlon-Hein bound from equation 5 will give the desired decay rate. In these coordinates, the manifold is expressed as

$$\mathcal{X} = \{f_i(\tilde{x}) = K^{-d_i}(K\tilde{z} + z_0)p_i\}$$

like in the prior region. We apply the Conlon-Hein estimate 5 to get

$$\nabla^i \left[K^{-2}|K\tilde{z} + z_0|^{\frac{2}{d_1}}\phi_{\hat{p}_{z^{-1/d_1},z}}((K\tilde{z} + z_0)K \cdot \tilde{x}) - \tilde{r}^2\right] = O\left(\left(K^{-1}D^{\frac{1}{d_1}}\right)^{2+c}\right)$$

since $K\tilde{z} + z_0 = z$ is of order D as specified in the region. There are two error terms of $K^{-d_1}D$ as in Region II, and the new error term of $(K^{-1}D^{\frac{1}{d_1}})^{2+c}$. The first term was already shown to satisfy the desired decay, so we must show it for the new error term given from the Conlon-Hein estimate 5. In this region, $K \sim D^\alpha$, so this term can be bounded

$$(K^{-1}D^{\frac{1}{d_1}})^{2+c} < CD^{\frac{2+c}{d_1} - c\alpha}K^{-2}$$

by replacing K^{-c} with D^α. This expresses the new error term in the desired form and we must demonstrate that δ can be chosen such that $\delta > \frac{2+c}{d_1} - c\alpha$. Let α be $1 - \varepsilon$, and then choose any δ such that

$$\delta > \frac{2+c}{d_1} - c\alpha = \frac{2}{d_1} + \frac{c\alpha - d_1 c}{d_1}.$$

Since $d_1 > 2$, this δ can be chosen less than $\frac{2}{d_1}$. Therefore, δ can be chosen to be less than $\frac{2}{d_1}$. Furthermore, if $d_1 > 3$, c can be given as $c > 1$, allowing a choice of δ to be negative.

Region IV: In this region, the model space is $X_{p'} = \mathbb{C} \times V_{p'}$ with $p' = (p_1, \dots, p_\ell, 0, \dots, 0)$ as before. The natural rescaling will produce a model space of X_t with $t \to p'$. Sufficiently far out, the varying model spaces are of the form $V_{p'+\varepsilon}$ and can therefore be approximated by $V_{p'}$ with sufficiently decaying error.

We use the range $R \in (K/2, 2K)$, for $K \in (\kappa^{-1}\rho^{\frac{1}{d_1}}, \rho^\alpha/2)$ and $\rho \in (D/2, 2D)$. In this region, $|z| \sim D$ as before. However, $|z|$ is growing sufficiently faster than r, so the model space can no longer be X_0. We use the same change of coordinates as above with a similarly defined z_0:

$$\tilde{x} = K^{-1} \cdot x, \quad \tilde{z} = K^{-1}(z - z_0), \quad \tilde{r} = K^{-1}r.$$

Since $\gamma_2 = 1$, the metric looks like a product of the cone metric on $\mathbb{C}$ and some ϕ_t. The equations for $\mathcal{X}$ are

$$\mathcal{X} = \{f_1(\tilde{x}) - K^{-d_1}(K\tilde{z} + z_0)p_1 = \cdots = f_k(\tilde{x}) - K^{-d_k}(K\tilde{z} + z_0)p_k = 0\}.$$

We compare this to the model space of $\mathbb{C} \times V_t$, for $t = K^{-1} \cdot z_0 p = \hat{p}_{K^{-1},z_0}$. We may need to perturb z_0 slightly to ensure that this is smooth using Sard's theorem. The error term induced by this comparison is $K^{1-d_i}\tilde{z}$ in each component with the maximal error where $i = 1$ of K^{1-d_1}. The model metric is

$$\omega_{X_t} = \sqrt{-1}\partial\overline{\partial}\left(|\tilde{z}|^2 + K^{-2}|z_0|^{\frac{2}{d_1}}\phi_t(K|z_0|^{-\frac{1}{d_1}} \cdot \tilde{x})\right),$$

and the metric on $\mathcal{X}$ on the region is

$$\omega = \sqrt{-1}\partial\overline{\partial}\left(|\tilde{z}|^2 + K^{-2}|K\tilde{z} + z_0|^{\frac{2}{d_1}}\phi_{\hat{p}_{z^{-1/d_1},z}}\left((K\tilde{z} + z_0)^{-\frac{1}{d_1}}K \cdot \tilde{x}\right)\right).$$

We can use the homogeneity of r to examine the difference

$$E = \sqrt{-1}\partial\overline{\partial}\left(K^{-2}|K\tilde{z}+z_0|^{\frac{2}{d_1}}\phi_{\hat{p}_{z^{-1/d_1},z}}\left((K\tilde{z}+z_0)^{-\frac{1}{d_1}}K\cdot\tilde{x}\right) - K^{-2}|z_0|^{\frac{2}{d_1}}\phi_t\left(K|z_0|^{-\frac{1}{d_1}}\cdot\tilde{x}\right)\right),$$

which is the same as replacing ϕ with $\phi^{-c} = \phi - r^2$, which satisfies the decay criteria of being in the weighted space $C^\infty_{-c}(V_t)$, which is the existence of constants given by equation 5. The homogeneity of r means all the r^2 terms cancel.

To finish, we need to bound this difference of potentials. Since the points are very close, an estimate given by the gradient is

$$\phi^{-c}_{q_1}(x_1) - \phi^{-c}_{q_2}(x_2) \leq \left|\frac{\partial\phi^{-c}}{\partial q}\right||q_1-q_2| + \left|\frac{\partial\phi^{-c}}{\partial x}\right||x_1-x_2|.$$

Therefore, $\left|\frac{\partial\phi^{-c}}{\partial q}\right| < C$ since $\phi(q,x)$ is smooth in both variables on a small disk around p' with bounded gradient.

Since $q_1 - q_2 = z^{-\frac{1}{d_1}}\cdot p - K^{-1}\cdot z_0 p$, this is of order $\max(D^{-1/d_1}, K^{-d_1}D)$, and the first term obviously fits the decay criterion. The term $K^{-d_1}D$ can be bounded as

$$K^{-d_1}D = K^{-2}K^{2-d_1}D \leq CK^{-2}D^{\alpha(2-d_1)+1}, \tag{8}$$

so we need

$$\alpha(2-d_1)+1 < \delta.$$

We consider

$$\alpha(2-d_1)+1 < \frac{2}{d_1}$$

and so choosing α sufficiently close to 1, $\alpha > \frac{1}{d_1}$ to get the bound. Therefore, δ can be chosen such that $\alpha(2-d_1)+1 < \delta < \frac{2}{d_1}$ as necessary, as shown above in equation 7. In this region, $K < CD^\alpha$ so the first term is bounded by $D^{-1/d_1} < CK^{-2}K^2D^{-1/d_1} < CK^{-2}D^{2\alpha-1/d_1}$. Since $\alpha > 1/d_1$, δ can be chosen to be less than $2/d_1$ as desired. Therefore, we only need to examine the term E as above.

We can Taylor expand these terms, and given that $|\tilde{z}| < 1$, $|z_0| \sim D$ and $K \ll D$, the growth is given by

$$K(K\tilde{z}+z_0)^{-\frac{1}{d_1}} = z_0^{-\frac{1}{d_1}}K(1+O(KD^{-1}))$$

and we apply this with the Conlon-Hein estimate from equation 5 to get

$$|\nabla^i E| < C_i\left(|z_0|^{-\frac{1}{d_1}}K\right)^{-2-c-i}KD^{-1} = O\left(K^{-1-c}D^{\frac{2+c+i}{d_1}-1}\right)$$

and we combine this with the first error term of K^{1-d_1}, which will force the choice of δ to satisfy

$$K^{1-d_1} + K^{-1-c}D^{\frac{2+c}{d_1}-1} < CD^\delta K^{-2}.$$

Firstly, suppose that $d_1 > 3$ and thus $c > 1$, so this can be bounded as

$$K^{1-d_1} = K^{3-d_1}K^{-2} < CD^{\frac{3}{d_1}-1}K^{-2}$$

for the first term, and likewise

$$K^{-1-c}D^{\frac{2+c}{d_1}-1} = \left(KD^{-\frac{1}{d_1}}\right)^{1-c}D^{\frac{3}{d_1}-1}K^{-2}$$

and since $\frac{3}{d_1} - 1 < 0$, δ can be chosen to be negative for these assumptions.

Suppose that $d_1 > 2$ only and thus we only know that $c > 0$, giving the estimate

$$K^{-1-c}D^{\frac{2+d_1}{d_1}-1} = \underbrace{\left(KD^{-\frac{1}{d_1}}\right)^{-c}}_{<C}\underbrace{(KD^{-1})}_{<D^{\varepsilon}}D^{\frac{2}{d_1}}K^{-2},$$

which shows that δ can be chosen to satisfy $\delta < \frac{2}{d_1}$ as desired.

We also will eventually want to compare this not to the varying model space of $X_{\hat{p}_{K^{-1},z_0}}$, but rather to its limiting geometry of $X_{p'}$. We compare these two spaces and see that the error induced will be sufficiently small. Their equations are expressed as

$$X_{\hat{p}_{K^{-1},z_0}} = \{f_i(x) = K^{-d_i}z_0p_i\}, \quad X_{p'} = \{f_i(x) = p_i, f_j(x) = 0 : i \leq \ell, j > \ell\}.$$

The error term is computed by comparing $K^{-d_i}z_0p_i$ to p_i for $i \leq \ell$ or $K^{-d_j}z_0$ for $j > \ell$. Since there are bounds

$$C'D^{\frac{1}{d_1}} < K < CD^{\alpha}$$

for $\alpha \in (\frac{1}{d_1}, 1)$, $K^{-d_j}D$ has sufficient decay for the region approximating this going to 0, since $d_j > d_1$ and $K \sim D^{\alpha}$. The maximal error term of $K^{-d_1}D$ fits the decay as shown above in equation 8. Furthermore, as above, if $d_1 > 3$, then δ can be chosen to be negative.

Region V: In this final region, $R < 2\kappa^{-1}\rho^{\frac{1}{d_1}}$ and $\rho \in (D/2, 2D)$. We perform a similar rescaling choosing z_0 close to z and we rescale by $|z_0|^{-\frac{1}{d_1}}$:

$$\tilde{z} = z_0^{-\frac{1}{d_1}}(z - z_0), \quad \tilde{x} = z_0^{-\frac{1}{d_1}} \cdot x, \quad \tilde{r} = |z_0|^{-\frac{1}{d_1}}r.$$

Under this rescaling, both $|\tilde{z}|, \tilde{r} < C$ are bounded. The metric in these coordinates is expressed as

$$|z_0|^{-\frac{2}{d_1}}\omega = \sqrt{-1}\partial\overline{\partial}\left(|\tilde{z}|^2 + |z_0|^{-\frac{2}{d_1}}|z_0^{\frac{1}{d_1}}\tilde{z} + z_0|^{\frac{2}{d_1}}\phi_{z_0^{1-1/d_1}}\left(z_0^{\frac{1}{d_1}}(z_0^{\frac{1}{d_1}}\tilde{z} + z_0)^{-\frac{1}{d_1}} \cdot \tilde{x}\right)\right),$$

and the equations expressing $\mathcal{X}$ are

$$\mathcal{X} = \{f_i(\tilde{x}) = z_0^{-\frac{d_i}{d_1}}(z_0^{\frac{1}{d_1}}\tilde{z} + z_0)p_i\}.$$

Separating out the terms with variable $\tilde{z}$, we compare to the constant variety cut out by equations

$$X_{\hat{p}_{z_0^{-1/d_1},z_0}} = \{f_i(\tilde{x}) = z_0^{1-\frac{d_i}{d_1}}p_i\},$$

again potentially perturbing z_0 slightly to ensure this is smooth utilizing Sard's theorem for existence. Set $t = z_0^{-\frac{1}{d_1}} \cdot z_0p = \hat{p}_{z_0^{-1/d_1},z_0}$ and then endow $\mathbb{C} \times V_t$ with the metric

$$\omega = \sqrt{-1}\partial\overline{\partial}(|\tilde{z}|^2 + \phi_t(\tilde{x})),$$

noting that for $D \gg 0$ sufficiently large, $|t| < 1$ since $|p| = 1$. To compare these two metrics, we estimate

$$E = |z_0|^{-\frac{2}{d_1}}|z_0^{\frac{1}{d_1}}\tilde{z} + z_0|^{\frac{2}{d_1}}\phi_{\hat{p}_{z^{-1/d_1},z}}\left(z_0^{\frac{1}{d_1}}\left(z_0^{\frac{1}{d_1}}\tilde{z} + z_0\right)^{-\frac{1}{d_1}} \cdot \tilde{x}\right) - \phi_t(\tilde{x})$$

and we Taylor expand

$$z_0^{\frac{1}{d_1}}\left(z_0^{\frac{1}{d_1}}\tilde{z} + z_0\right)^{-\frac{1}{d_1}} = 1 + O\left(D^{\frac{1}{d_1}-1}\right).$$

The error from $\phi_{q_1} - \phi_{q_2}$ is sufficiently small to fit the desired decay. As above, the derivative is bounded, so we need for $z_0^{1-\frac{d_i}{d_1}} - z^{1-\frac{d_i}{d_1}}$ to decay as $D^{-\varepsilon}$ for some $\varepsilon > 0$. Indeed, this decays like $D^{-1-\varepsilon'}$ for $\varepsilon = 1 - \frac{d_{\ell+1}}{d_1}$ where $d_{\ell+1}$ is minimal with $d_1 = d_\ell < d_{\ell+1}$. Here, δ can even be negative and this will not be the limiting term in the estimates.

This leaves only the error term of E as before. E can be bounded by the same techniques as before, giving

$$|\nabla^i E| < CD^{\frac{1}{d_1}-1}$$

so δ must be chosen to satisfy

$$D^{\frac{1}{d_1}-1} < CD^{\delta - \frac{2}{d_1}}.$$

If $d_1 \geq 2$, then δ can be chosen such that $\delta < \frac{2}{d_1}$, and if $d_1 > 3$ then δ can further be negative, concluding the proof.

Lastly, instead of working on the constantly shifting space of $X_{\hat{p}_{z_0^{-1/d_1}, z_0}}$, it will help to work on the fixed space $X_{p'}$ recalling that $p' = (p_1, \ldots, p_\ell, 0, \ldots, 0)$ for ℓ as above. As $|z_0| \to \infty$, $z_0^{-\frac{1}{d_1}} \cdot z_0 p \to p'$, demonstrating this as the limiting fiber. The error term in moving from $V_{\hat{p}_{z_0^{-1/d_1}, z_0}}$ to $V_{p'}$ is of order $D^{1-\frac{d_{\ell+1}}{d_1}}$, which is of better decay than necessary. This error can therefore be absorbed into the bound, as seen by comparing the equations

$$X_{\hat{p}_{z_0^{-1/d_1}, z_0}} = \left\{ f_1(\tilde{x}) - p_1 = \cdots = f_\ell(\tilde{x}) - p_\ell = f_{\ell+1}(\tilde{x}) - z_0^{1-\frac{d_{\ell+1}}{d_1}} p_{\ell+1} = \cdots f_k(\tilde{x}) - z_0^{1-\frac{d_k}{d_1}} p_k \right\}$$

to the locus to which we want to compare

$$X_{p'} = \{f_1(x) - p_1 = \cdots = f_\ell(x) - p_\ell = f_{\ell+1}(x) = \cdots = f_k(x) = 0\}.$$

Since this value has negative decay of ρ^{-a} for $a = \frac{d_{\ell+1}}{d_1}$, δ can be chosen such that $-a < \delta - \frac{2}{d_1}$, or $\frac{2-d_{\ell+1}}{d_1} < \delta$. Since $d_{\ell+1} > 2$, δ can even be chosen to be negative. $\square$

4. Weighted Hölder Spaces

We construct weighted Hölder spaces so that we are able to localize the geometry to model manifolds of the form X_t where we can invert the Laplacian. We follow the model created in Degeratu-Mazzeo [10] to define weight functions and Hölder seminorms to isolate only a single harmonic function based on growth rates near infinity in different directions. The main tool to approach these will be the Fourier transform in the z direction of the products X_0 and $X_{p'}$ following the literature of Brendle [1], Walpuski [25], and Mazzeo-Pacard [20]. This will allow us to effectively study the Laplacian by localizing to the asymptotically conical space $V_{p'}$ and the cone V_0, where we understand the Laplacian. The Laplacian on asymptotically conical manifolds is well understood (see Lockhart-McOwen [18]) as was used in the works producing similar Calabi-Yau manifolds in Conlon-Hein [7]. What makes this model space difficult is the rays of singularities of $\mathbb{C} \times \{0\}$, and this takes it out of the framework of QAC and QALE of Degeratu-Mazzeo [10] and Joyce [16] respectively. This section follows the approach given by Székelyhidi [22] to account for these difficulties.

Following Degeratu-Mazzeo [10], we define a function w that measures how close we are to the *cone point*:

$$w = \begin{cases} 1, & R > 2\kappa\rho \\ R/(\kappa\rho), & R \in (\kappa^{-1}\rho^{\frac{1}{d_1}}, \kappa\rho) \\ \kappa^{-2}\rho^{\frac{1}{d_1}-1}, & R < \frac{1}{2}\kappa^{-1}\rho^{\frac{1}{d_1}} \end{cases}$$

for the same κ as defined in Proposition 3.1. We define the Hölder seminorm as

$$[T]_{0,\gamma} = \sup_{\rho(z)>K} \rho(z)^\gamma w(z)^\gamma \sup_{z' \neq z, z \in B(z,c)} \frac{|T(z) - T(z')|}{|z - z'|^\gamma}$$

where we compare using parallel transport when required. We use this to define the Hölder weighted norm

$$\|f\|_{C^{k,\alpha}_{\delta,\tau}} = \|f\|_{C^{k,\alpha}(\rho<2P))} + [\rho^{k-\delta}\tau^{k-\tau}\nabla^k f]_{0,\alpha} + \sum_{j=0}^{k} \sup_{\rho>P} \rho^{j-\delta} w^{j-\tau} |\nabla^j f|.$$

This is re-expressible (up to a modification on a compact set making some regularized maximum of 1 and ρ) to a normal Hölder norm with a conformal scaling of the metric by $\rho^{-2}w^{-2}$,

$$\|f\|_{C^{k,\alpha}_{\delta,\tau}} = \|\rho^{-\delta}w^{-\tau}\|_{C^{k,\alpha}_{\rho^{-2}w^{-2}\omega}}.$$

In this language, the decay of the Ricci potential in Proposition 3.1 is

$$h \in C^{k,\alpha}_{\delta-2,-2}(V_t)$$

for the value of δ chosen in the proposition.

Proposition 4.1 (Extension of functions)**.** *Given $u \in C^{0,\alpha}_{\delta,\tau}(\rho^{-1}[A,\infty), \omega)$, there exists a bounded linear extension operator*

$$E : C^{0,\alpha}_{\delta,\tau}(\rho^{-1}[A,\infty), \omega) \to C^{0,\alpha}_{\delta,\tau}(\mathcal{X})$$

such that $Eu|_{\rho^{-1}[A',\infty)} = u$ for some $A' > A$, and the norm of E is independent of A.

Proof. The idea of this proof is that if we take a function u defined on the positive numbers with some bounded norm in $C^{0,\alpha}$, then we can extend it to the negative numbers by reflection without changing the norm. This only works for $C^{0,\alpha}$ Hölder spaces since this will in general yield a non-differentiable structure at the point of reflection. We can take local charts around the sphere $\{\rho = A\}$ where the sphere represents some coordinate $x_1 = 0$ and outside the sphere is the half-space $x_1 > 0$, and we wish to extend the function to the other half of the chart. We will then glue these together using partitions of unity.

We will construct a function r_x on local charts around points $x \in \rho^{-1}(A)$ such that $r_x \sim \rho w$ on these charts. These will be such that $\rho^{-1}(A)$ is the coordinate function $x_1 = 0$ which to reflect over. We can represent this in the regions by taking a smoothing of r_x defined on disjoint open sets as

$$r_x = \begin{cases} \frac{\kappa A}{10}, & R > \kappa\rho & \text{Region I} \\ \frac{R}{10}, & \kappa^{-1}\rho^{\frac{1}{d_1}} < R < \kappa\rho & \text{Regions II-IV} \\ A^{\frac{1}{d_1}}, & R < \kappa^{-1}\rho^{\frac{1}{d_1}} & \text{Region V,} \end{cases}$$

which is such that on the ball around the point x of radius r_x, the relation $\rho w \sim r_x$ holds. We examine the geometry in these three regions as stated in the definition of r_x.

Region I: For some x with $R > \kappa\rho$, we are in the model geometry of X_0 as per Region I. Consider the conformally rescaled metric $A^{-2}\omega$ on the ball around x of radius r_x. From the analysis in Proposition 3.1, the geometry in this region approaches the cone metric on X_0 as $A \to \infty$ on a ball of radius $A^{-1} = \frac{\kappa}{10}$. Consider the *outside* of the ball defined by $\rho > A$, and this is the first coordinate that we will reflect over. Let $\tilde{x}$ be the center of this ball using the rescaling $\tilde{x} = A^{-1} \cdot x$, so that $\rho(\tilde{x}) = 1$ and $R(\tilde{x}) > \kappa$ by the region constraint. This gives us a bound on the geometry of the chart $B(x, r_x)$ in these coordinates, allowing us to reflect over $R = A$ on this chart. That is, the growth of u is controlled with respect to the unscaled and scaled metric as

$$\|u\|_{C^{0,\alpha}_{\delta,\tau}(B(x,r_x),\omega)} \sim A^{\delta}\|u\|_{C^{0,\alpha}(B(x,r_x),A^{-2}\omega)}.$$

The norm in the weighted space is controlled by this estimate.

Regions II-IV: In these regions, the model metric switches from X_0 to $X_{p'}$, however in Region IV the error in still considering X_0 is not too large as will be seen in Section 5 (in Proposition 5.1). Let $R \in (K/2, 2K)$ and $\kappa^{-1}\rho^{\frac{1}{d_1}} < R < \kappa\rho$. In these regions, we compared to some $\mathbb{C} \times V_s$ for some bounded s. We can choose A large enough so that all of these are smooth and diffeomorphic. For each s, we exhibited the bounds on the geometry on $\rho^{-1}(A)$ given by the estimate

$$\|u\|_{C^{0,\alpha}_{\delta,\tau}(B(x,r_x),\omega)} \sim A^{\delta-\tau}K^{\tau}\|u\|_{C^{0,\alpha}(B(x,r_x),K^{-2}\omega)}.$$

Region V: Lastly, we are in Region V whose model geometry is $X_{p'}$. The rescaled metric is given by $A^{-\frac{2}{d_1}}\omega$, which asymptotically approaches $\mathbb{C} \times V_{p'}$. Since $|p'| < 1$, this gives a similar bound as before, and by applying the computations from Region V for the decay, the estimate is given as

$$\|u\|_{C^{0,\alpha}_{\delta,\tau}(B(x,r_x),\omega)} \sim A^{\delta-\tau+\frac{\tau}{d_1}}\|u\|_{C^{0,\alpha}(B(x,r_x),A^{-2/d_1}\omega)}.$$

$\square$

Given this result, it is natural to define the same $C^{k,\alpha}_{\delta,\tau}(\rho^{-1}[A,\infty),\omega)$ norm over this partial set of $\mathcal{X}$ as the infimum of the stand $C^{k,\alpha}_{\delta,\tau}(\mathcal{X},\omega)$ norm over all extensions.

5. Comparison to the Model Spaces

In this section, we show that X_0 and $X_{p'}$ model $\mathcal{X}$ sufficiently far out where $\rho > A$ for large enough $A \gg 0$. The error of the metric ω on $\mathcal{X}$ and the Calabi-Yau metrics on X_0 and $X_{p'}$ is arbitrarily small in the above weighted spaces with sufficient overlap to allow for gluing.

5.1. **Comparison to X_0.** We define a projection map from a region on $\mathcal{X}$ that has geometry modeled by X_0. We define the region

$$\mathcal{U} = \left\{\rho > A, R > \Lambda\rho^{\frac{1}{d_1}}\right\} \cap \mathcal{X}$$

for large constants $A, \Lambda > 0$, and a map

$$G : \mathcal{U} \to X_0$$

where $\mathcal{U}$ and X_0 are considered subsets of $\mathbb{C}^{n+k+1}$ and G is the nearest point projection in the cone metric $\sqrt{-1}\partial\overline{\partial}(|x|^2 + |z|^2)$ for variables $x_1, \ldots, x_{n+k}$ and z. The projection is such that $G(x, z) = (x', z)$ where x' is the nearest point in the cone V_0 under the cone metric

$\sqrt{-1}\partial\overline{\partial}(R^2)$. Since every fiber is asymptotic to V_0 and we are sufficiently far away with $\rho > A$ for $A \gg 0$, this is well-defined.

Proposition 5.1. *Given $\varepsilon > 0$, there exist constants $\Lambda, A \gg 0$ sufficiently large such that*

$$|\nabla^i(G^* g_{X_0} - g)|_g < \varepsilon w^{-i}\rho^{-i}$$

for $i \leq k+1$ where k is from the $C^{k,\alpha}_{\delta,\tau}$ space. Succinctly stated,

$$\|\nabla^i(G^* g_{X_0} - g)\|_{C^{k,\alpha}_{0,0}} < \varepsilon$$

for any $\varepsilon > 0$.

Proof. This represents Regions I-IV, and for the first three, the same computations from Proposition 3.1 prove this as well with the exception that this computation must be done using the Riemannian structure instead of the holomorphic structure since G is smooth but not holomorphic. We need to show a bound

$$|D^{-2}\nabla^i(G^* g_{X_0} - g)|_{D^{-2}g} < \varepsilon,$$

and the same rescaling map gives us that the error induced is of size D^{1-d_1}.

For Region IV, the computation is different than in Section 3, since before it was modeled on $X_{p'}$, but here it must be compared to X_0 instead. To do this, we will use the Conlon-Hein estimate 5 with an appropriate rescaling. In this region let $\rho \in (D/2, 2D)$, $R \in (K/2, 2K)$ and $\Lambda\rho^{\frac{1}{d_1}} < K < \frac{1}{2}\rho^\alpha$. Pick some z_0 such that $|z_0| \in (D/4, 4D)$ sufficiently close to z. We introduce the same change of coordinates to $\tilde{x}, \tilde{z}$, and $\tilde{r}$. The equations of $\mathcal{X}$ are given in these coordinates by

$$\mathcal{X} = \left\{ f_i(\tilde{x}) = K^{-d_i}(K\tilde{z} + z_0)p_i \right\},$$

which is compared to

$$X_0 = \{f_i(\tilde{x}) = 0\}.$$

Since $|\tilde{z}| < 1$, the error introduced by ignoring the non-constant term with $\tilde{z}$ is K^{1-d_1}. For $D \gg 0$ sufficiently large, this can be chosen to be within any $\varepsilon/2$, and therefore can be ignored. Here E is bounded by

$$E = \sqrt{-1}\partial\overline{\partial}\left(K^{-2}|K\tilde{z} + z_0|^{\frac{2}{d_1}}\phi_{\hat{p}_{z^{-1/d_1},z}}\left((K\tilde{z} + z_0)^{-\frac{1}{d_1}}K \cdot \tilde{x}\right) - \tilde{r}^2\right),$$

and as before, the estimate states that

$$\left(KD^{-\frac{1}{d_1}}\right)^{-2-c} < C\Lambda^{-2-c}.$$

The constant $\Lambda \gg 0$ can be chosen such that this is less than $\varepsilon/2$, finishing the proof. ☐

5.2. **Comparison to $X_{p'}$.** In this section, we study the region where the model variety is $X_{p'}$. The region is specified by $R < \Lambda\rho^{\frac{1}{d_1}}$ and $\rho > A$. Fix a z_0 with $|z_0| > A$ and define the region $\mathcal{V} \subset \mathcal{X}$ such that the point $(x, z) \in \mathcal{V}$ if $|z - z_0| < B|z_0|^{\frac{1}{d_1}}$ for a large fixed constant B and $R < \Lambda\rho^{\frac{1}{d_1}}$. Define a new coordinate system as

$$\hat{x} = z_0^{-\frac{1}{d_1}} \cdot x, \quad \hat{x} = z_0^{-\frac{1}{d_1}}(z - z_0), \quad \hat{R} = |z_0|^{-\frac{1}{d_1}}R$$

and since R can be small in this region, we need to define an auxiliary variable $\hat{\zeta} = \max(1, \hat{R})$. There are bounds

$$|\hat{z}| < B, \quad |\hat{R}| < C\Lambda$$

for some fixed C since $\rho \sim |z_0|$. In these coordinates, the expression of $\mathcal{X}$ is given by

$$\mathcal{X} = \left\{ f_i(\hat{x}) = z_0^{\frac{1-d_i}{d_1}} \hat{z} p_i + z_0^{1-\frac{d_i}{d_1}} p_i \right\}.$$

We define a map H projecting onto $X_{p'}$

$$H : \mathcal{V} \to X_{p'}$$

by letting $H(\hat{x}, \hat{z}) = (\hat{x}', \hat{z})$ where $\hat{x}'$ is simply the nearest point projection to

$$X_{p'} = \{f_1(\hat{x}) - p_1 = \cdots = f_\ell(\hat{x}) - p_\ell = f_{\ell+1}(\hat{x}) = \cdots f_k(\hat{x}) = 0\} = \mathbb{C} \times V_{p'}$$

with ℓ as before the maximal index such that $d_1 = d_\ell$.

Suppose we tried to project onto X_p as would parallel Székelyhidi [22]. In Region V, this would fail as we would need to compare the rescaled metrics

$$|\phi_q(x) - \phi_p(x')| \leq \left|\frac{\partial \phi}{\partial q}\right| |q - p| + \left|\frac{\partial \phi}{\partial x}\right| |x - x'|$$

and the problem is that q as rescaled from $\mathcal{X}$ goes to p', so we need that $q \to p'$ near infinity. This is why the model space is actually $X_{p'}$. If we assume this is smooth, then the below analysis on $X_{p'}$ as a model space will be sufficient. The entire construction could be initialized with $p = p'$ and all the scaling would work identically to Székelyhidi [22] with the care to carry around all the indices as all the fibers would scale onto $V_{p'}$ and $G_t(\mathcal{X}) \subset \mathcal{X}$. The higher degree polynomials would vanish identically enabling all the fibers to scale onto the single initial fiber $V_{p'}$. However, by allowing $p \neq p'$ we allow for more constructions in particular where the fibers can allow for singularities and different complex structures. Additionally, a larger research goal is to create a Calabi-Yau metric on $\mathbb{C}^{n+k}$ swept out by linear slices that are $\mathcal{X}$ with different initial points p. This will require the generic slice to be like the $\mathcal{X}$ presented here in generality and will not be $\mathbb{C}^*$-equivariant.

Proposition 5.2. *For any $\varepsilon > 0$, there exist constants $\Lambda, A > 0$ as functions of B, ε, Λ such that the following estimate holds:*

$$\| |z_0|^{\frac{2}{d_1}} H^* g_{X_{p'}} - g \|_{C^{k,\alpha}_{0,0}} < \varepsilon.$$

Specifically,

$$|\nabla^i (H^* g_{X_{p'}} - |z_0|^{-\frac{2}{d_1}} g)|_{|z_0|^{-\frac{2}{d_1}} g} < \varepsilon \hat{\zeta}^{-i}$$

for $i \leq k + 1$.

Proof. This follows almost directly according to the analysis in Section 3. In Region IV where $\rho \in (D/2, 2D)$, $R \in (K/2, 2K)$ and $\kappa^{-1} D^{\frac{1}{d_1}} < K < \Lambda D^{\frac{1}{d_1}}$, $\hat{\zeta} \sim \hat{R}$ and $|z_0| \sim D$. We introduce new coordinates

$$\tilde{x} = K^{-1} z_0^{\frac{1}{d_1}} \cdot \hat{z}, \quad \tilde{z} = K^{-1} z_0^{\frac{1}{d_1}} \hat{z},$$

and in these coordinates, the expression of $\mathcal{X}$ is given as

$$\mathcal{X} = \left\{ f_i(\tilde{x}) = K^{-d_i} (K\tilde{z} + z_0) p_i \right\},$$

and we compare with the variety

$$X_{\hat{p}_{K^{-1}, z_0}} = \left\{ f_i(\tilde{x}) = K^{-d_i} z_0 p_i \right\}$$

under the projection operator map. Again, we need to do the same computations, but with respect to the Riemannian metric as the map H is not holomorphic. The Conlon-Hein estimate 5 as in the proof of Region IV in Theorem 3.1. Using the same notation, there is the bound

$$\begin{aligned}|\nabla^i(H^* g_{X_{p'}} - |z_0|^{-\frac{2}{d_1}} g)|_{|z_0|^{-2/d_1} g} &< K^{1-d_1-i}\tilde{z} + (|z_0|^{-\frac{1}{d_1}} K)^{-2-c-i} K D^{-1} \\ &< K^{1-d_1-i} B K^{-1} z_0^{\frac{1}{d_1}} + (|z_0|^{-\frac{1}{d_1}} K)^{-2-c-i} K D^{-1} \\ &< C B D^{\frac{1}{d_1}-1-i}\end{aligned}$$

where the first term in the second line is CK^{-d_1}, which is bounded by a constant times D^{-1} and the second term is a decaying term containing B times $KD^{-1} \sim D^{\frac{1}{d_1}-1}$ as desired.

Region V is modeled by $X_{p'}$. This computation is identical to the above in Theorem 3.1. We use the same change of variables applied to the $\hat{x}, \hat{z}$ variables. The difference between $\hat{\zeta}$ and $\hat{R}$ can be ignored since it is a bounded prefactor to the decaying term. This will reduce to examining

$$E = |z_0|^{-\frac{2}{d_1}} |z_0^{\frac{1}{d_1}} \tilde{z} + z_0|^{\frac{2}{d_1}} \phi_{\hat{p}_{z_0^{-1/d_1}, z_0}} \left(z_0^{\frac{1}{d_1}} \left(z_0^{\frac{1}{d_1}} \tilde{z} + z_0 \right)^{-\frac{1}{d_1}} \cdot \tilde{x} \right) - \phi_{p'}(\tilde{x})$$

and as before, the Taylor expansion gives the estimate

$$z_0^{\frac{1}{d_1}} \left(z_0^{\frac{1}{d_1}} \tilde{z} + z_0 \right)^{-\frac{1}{d_1}} = 1 + O\left(D^{\frac{1}{d_1}-1} \right)$$

giving decay at the rate of $BD^{\frac{1}{d_1}-1}$. The other parts are identical as well since $z_0^{-\frac{1}{d_1}} \cdot z_0 p \to p'$ as $|z_0| \to \infty$ by construction of p' completing the proof. $\square$

Proposition 5.3 (Tangent cone of $\mathcal{X}$ is X_0)**.** *For any $\varepsilon > 0$, there is some $D \gg 0$ sufficiently large such that the Gromov-Hausdorff distance between the annuli defined by $\rho \in (D/2, 2D)$ in $\mathcal{X}$ with ω and X_0 with $\omega_{V_0} + \sqrt{-1}\partial\bar{\partial}(|z|^2)$ is less than $D\varepsilon$.*

Proof. As $z \to \infty$ in $\mathcal{X}$, rescaling the metric gives geometry asymptotic to V_0 since each fiber has tangent cone V_0. We will separate the analysis into regions to help clarify the geometry near infinity. Consider the region

$$S_\Lambda = \left\{ (x, z) \in \mathcal{X} : R < \Lambda \rho^{\frac{1}{d_1}} \right\},$$

which is where R is small and we are near the *cone point* or the singularities of X_0. We denote $\mathcal{X}^D$ and X_0^D as the annular regions defined by $\rho \in (D/2, 2D)$ of these spaces, respectively, with metrics rescaled by D^{-2}. It is sufficient to show this ε condition by defining a function $G : \mathcal{X}^D \to X_0^D$ such that for $D \gg 0$ sufficiently large the difference of distances of points compared by G by ε can be made arbitrarily small:

$$|d(G(x), G(x')) - d(x, x')| < \varepsilon. \tag{9}$$

This will show us that the image of G is ε-dense in X_0^D.

Let $\Lambda \gg 0$ be a large constant. Define G on the complement $\mathcal{X}^D \setminus S_\Lambda$ as the nearest point projection in $\mathbb{C}^{n+k+1}$ with cone metric on X_0 of $\omega_{X_0} = \omega_{V_0} + \sqrt{-1}\partial\bar{\partial}|z|^2$. This region is away from the singular fibers by $\Lambda \gg 0$ and $D \gg 0$ sufficiently large. Therefore, projection is well-defined and every point has a unique nearest projection since each fiber V_t is asymptotic to V_0. From Proposition 5.1, the two rescaled metrics $D^{-2}\omega$ and $D^{-2}G^*\omega$ are arbitrarily close for $\Lambda, D \gg 0$ sufficiently large. For a curve γ, we define its length using the metric ω with notation $L_\omega(\gamma)$. In this region, by the bound on the difference between the metrics ω_{X_0} and $G^*\omega$, the differences of the lengths of curves can be made arbitrarily small:

$$|L_{D^{-2}\omega_{X_0}}(\gamma) - L_{D^{-2}G^*\omega}(\gamma)| < \varepsilon.$$

Applying this to a curve from x to x' gives us the inequality

$$d_{\mathcal{X}^D}(x, x') < d_{X_0^D}(G(x), G(x')) + \varepsilon. \tag{10}$$

The reverse is not immediate as we may have that a curve achieving the minimal distance does not stay on the complement $\mathcal{X}^D \setminus S_\Lambda$.

Consider $\mathcal{X}^D \cap S_{2\Lambda}$ where Proposition 5.2 is used to compare $\mathcal{X}$ with $X_{p'}$. In this region, instead of directly projecting onto X_0, we will project in the $\mathbb{C}$ direction with z. Choosing D sufficiently large with respect to Λ as given already, let π be the map projecting onto the z-axis. For $\Lambda \gg 0$ fixed and then choosing $D \gg 0$ sufficiently large, it can be assumed that π satisfies $|d\pi| < 1 + \varepsilon$ giving the estimate on lengths of curves by

$$L_{D^{-2}\omega}(\gamma) \geq (1 - \varepsilon) L_{\sqrt{-1}\partial\bar{\partial}(|z|^2)}(\pi \circ \gamma). \tag{11}$$

In this region, for $x, x' \in S_\Lambda$, the distance minimizing curve between them must stay entirely in S_Λ in $\mathcal{X}^D$ because the metric is approximated by X_0^D for $\Lambda, D \gg 0$ as chosen. Equation 11 gives the inequality

$$d_{\mathcal{X}^D}(x, x') > d_{\mathbb{C}}(\pi(x), \pi(x')) - \varepsilon.$$

We fix some $o \in V_{p'}$ approximating the *cone point* as a reference to measure distances far away. For any point $(x, z) \in \mathcal{X}^D \cap S_\Lambda$, there is a bound

$$d_{X_{p'}^D}(H(x, z), (o, z)) < C\Lambda D^{\frac{1}{d_1} - 1} < \varepsilon$$

for some C, and choosing D, after having fixed Λ, large enough to decay below ε (since $d_1 > 1$ this does indeed decay considering $C\Lambda$ a constant). From this, consider that for two points in the region (x, z) and (x', z'), the following estimates hold:

$$d_{\mathcal{X}^D}((x, z), (x', z')) < |z - z'| + \varepsilon$$

and

$$d_{X_0^D}((Px, z), (Px', z')) < |z - z'| + \varepsilon$$

where P is the projection to the nearest point on the fiber V_0, which is well-defined for z small relative to R, or precisely when $\Lambda \gg 0$ is sufficiently large. Therefore, for (x, z) and (x', z') in the region S_Λ, a bound is given as

$$|d_{\mathcal{X}^D}((x, z), (x', z')) - d_{X_0^D}((Px, z), (Px', z'))| < \varepsilon.$$

From the above computations, we show the reverse inequality of equation 10. Let x, x' be points in $\mathcal{X}^D \setminus S_\Lambda$. Let γ be a distance minimizing curve between them. Suppose that γ meets the region $\overline{S_\Lambda}$, the closure of S_Λ in $\mathbb{C}^{n+k+1}$. Let $\tilde{x}$ and $\tilde{x}'$ be the first and last points of γ entering and exiting this region. It can only enter for finitely many times, so we show this bound on one such segment and can replace this by an ε/n argument if it enters the region n times. From the previous argument, there are bounds on the distances of $\tilde{x}$ and $\tilde{x}'$ of

$$d_{\mathcal{X}^D}(\tilde{x}, \tilde{x}') > d_{\mathbb{C}}(\pi(\tilde{x}), \pi(\tilde{x}')) - \varepsilon > d_{X_0^D}(G(\tilde{x}), G(\tilde{x}')) - 2\varepsilon.$$

The triangle inequality as well as the previous estimate 11 gives a reverse estimate of equation 10 as

$$d_{\mathcal{X}^D}(x,x') > d_{X_0^D}(G(x),G(x')) - \varepsilon.$$

This proves the initial desired estimate 9 and therefore that $G(\mathcal{X}^D)$ is ε-dense in X_0^D as desired. □

We fix some basepoint $o \in \mathcal{X}$ to model the *cone point* and measure distances to this reference. This distance measurement is uniformly equivalent to ρ away from o and $\rho = 0$. We will use this result to verify the relatively connected annuli (RCA) condition. For our purposes, this is a consequence of the fact that the tangent cone at infinity X_0 is itself a metric cone over a compacted connected metric space, which we discuss later in section 6 to study the Laplacian on X_0.

Corollary 5.4. *For $D \gg 0$ sufficiently large, two points $x,x' \in \mathcal{X}$ such that $d(o,x) = d(o,x') = D$ can be constructed as the endpoints of a curve γ whose image is contained in the annulus $B(o,CD) \setminus B(o,C^{-1}D)$, and $L_\omega(\gamma) < CD$ for some uniform constant C.*

6. Linear analysis on the model spaces

In this section, we show that in properly defined weighted spaces on X_0 and $X_{p'}$, the Laplacian is invertible. Extra care must be taken for X_0 since it has non-isolated singularities. However, they are still tractable using results on edge metrics from [10] and [19].

6.1. The Laplacian on X_0.

We endow the product space $X_0 = \mathbb{C} \times V_0$ with a conical product metric $g_{X_0} = g_{\mathbb{C}} + g_{V_0}$ with $g_{\mathbb{C}} = i dz \wedge d\bar{z}$. We define a similar weighted space in which we can invert the Laplacian. We give this space $C^{k,\alpha}_\tau$ as a conformal rescaling of the standard Hölder space $C^{k,\alpha}$ by rescaling by r^τ and rescaling the metric with r^{-2} for r the radius function on the cone V_0. Explicitly, the weighted space consists of $C^{k,\alpha}$ functions (with respect to the metric $r^{-2}g$) multiplied by the function r^τ:

$$C^{k,\alpha}_\tau = r^\tau C^{k,\alpha}_{r^{-2}g}.$$

Its norm is given in terms of the standard Hölder norm with rescaled metric by

$$\|f\|_{C^{k,\alpha}_\tau} = \|r^{-\tau} f\|_{C^{k,\alpha}_{r^{-2}g}}.$$

We show that in this weighted space, the Laplacian of X_0 is invertible if τ is chosen to be some good value in the range of $(2-m,0)$ for $m = 2n$ the real dimension of V_0. We utilize the copy of $\mathbb{C}$ to perform the Fourier transform thinking of this as $\mathbb{R}^2$ and invert the Laplacian using the spectral decomposition of the link.

We apply the techniques of Brendle [1], Walpuski [25], and Mazzeo-Pacard [20], using the Fourier transform to simplify the analysis on the Laplacian. Let χ be a function on X_0 with rapid decay in the $\mathbb{C}$ direction to allow the Fourier transform to be defined, i.e., Schwartz decay. The Fourier transform is defined as

$$\hat{\chi}(\xi,x) = \int_{\mathbb{R}^2} \chi(z,x) e^{-iz\cdot\xi}\, d\xi.$$

Proposition 6.1 (Solving the Laplacian in X_0)**.** *Let χ be a smooth function on X_0 such that its Fourier transform $\hat{\chi}$ has compact support away from the central fiber $\{0\} \times V_0$, that is, χ has support contained in $\mathbb{C} \times K$ for some $K \subset V_0$ compact. There exists some f solving $\Delta f = \chi$ such that*

(a) $f \in C^{k,\alpha}_\tau$ *for* $\tau \in (2-m, 0)$*;*
(b) f *decays exponentially in the* $\mathbb{C}$ *direction. That is, for* $a > 0$ *there are bounds*

$$\|f\|_{C^{k,\alpha}_\tau(z>|A|)} < C(1+A)^{-a}$$

for all $A > 0$ *and* τ *in the range* $(2-m, 0)$*.*

Proof. The Fourier transform acting on the $\mathbb{C}$ component transforms the Laplacian to multiplication by $-|\xi|^2$. There is a splitting of the Laplacian on the two components of the product

$$\Delta_{V_0}\hat{f}(\xi, x) - |\xi|^2\hat{f}(\xi, x) = \hat{\chi}(\xi, x), \tag{12}$$

so this will reduce the analysis to the V_0 Laplacian.

For every $\xi \neq 0$, we study the spectral decomposition of the Laplacian on the link from the general theory of cone manifolds. For eigenvalues λ, this decomposition gives equations of the form

$$\partial^2\hat{f} + \frac{m-1}{2}\partial_r\hat{f} + \frac{\lambda}{r^2}\hat{f} - |\xi|^2\hat{f} = \lambda\hat{\chi}. \tag{13}$$

By our assumptions, the right-hand side has compact support in r. Multiplying through by r^2 away from $|\xi| = 0$, the modified Bessel function gives a fundamental solution with exponential decay bounded at $r = 0$. The existence of a fundamental solution and the fact that these form a basis of harmonic functions on V_0 give us solutions to the original equation by taking the inverse Fourier transform.

For item (a), there are bounds on the fundamental solution of the form

$$\|\hat{f}(\xi, \cdot)\|_{C^0_\tau} < C\|\hat{\chi}(\xi, \cdot)\|_{C^0}$$

given by its decay rate. For item (b), we inductively proceed by differentiation. This is because since $\hat{\chi}$ has compact support away from the 0-fiber, we can view it as a smooth map from $\mathbb{R}^2 \to C^0_\tau(V_0)$. Differentiating equation 12 in the ξ direction yields

$$\Delta_{V_0}\partial^\alpha_\xi\hat{f}(\xi, x) - |\xi|^2\partial^\alpha_\xi\hat{f} = \partial^\alpha_\xi\hat{\chi}(\xi, x) + \sum_{|\beta|<|\alpha|} a_\beta(\xi)\partial^\beta_\xi\hat{f}(\xi, x)$$

which shows that a derivative with $|\alpha|$ terms is expressible in terms of lower order derivatives away from $\xi = 0$. Since $\hat{f}(\xi, x)$ is identically 0 in some neighborhood of 0 in ξ, each successive derivative is bounded near $r = 0$ as bootstrapped from the previous derivatives and has similarly exponential decay near infinity. Each decay term of derivatives of the Fourier transform gives the existence of another derivative on the original transform, so this yields the desired smoothness.

Consider the function $\tilde{f} : \mathbb{C} \to \mathbb{R}$ which maps each point z to the norm of f in C^0_τ of the fiber $\{z\} \times V_0$. The inverse Fourier transform tells us that this function has exponential decay as $z \to \infty$. The Schauder estimate on f states that

$$\|f(\xi, \cdot)\|_{C^2_\tau} < C(\|f(\xi, \cdot)\|_{C^0_\tau} + \|\chi(\xi, \cdot)\|_{C^0_\tau})$$

since χ is compactly supported. Differentiating the Schauder estimate gives bounds on all the further derivatives. The error term of transposing the Laplacian with derivatives is curvature, which has sufficient decay bounds (since it is asymptotically Calabi-Yau) and can therefore be absorbed into the error term. $\square$

Corollary 6.2 (Invertibility of Δ)**.** *If $\Delta f = 0$ for $f \in C^{k,\alpha}_\tau(X_0)$ with $\tau \in (2-m, 0)$, then $f = 0$.*

Proof. We will show that the distributional Fourier transform of f is supported at the origin and is therefore a polynomial. By the construction of the weighted spaces, the only polynomial satisfying this must be bounded and bounded in L^2 norm. ODEs of the form in the spectral decomposition above in equation 13 have no solutions with polynomial growth rate.

We will define weighted L^2-spaces for this proof. Choosing $\tau_1 < \tau < \tau_2$, we define a new weight function σ that has growth rate such that $\sigma = r^{\tau_2}$ for large r and $\sigma = r^{\tau_1}$ for small r. The norm in this space is defined by

$$\|\varphi\|^2_{L^2_\sigma} = \int_{V_0} |\varphi|^2 \sigma^{-2} r^{-m}\, dV.$$

The choice of the growth rates based on τ_1, τ_2 means that the map f from the statement of Corollary 6.2 is a bounded map

$$f : \mathbb{C} \to L^2_\sigma(V_0)$$

by taking the fiber $f|_{V_0 \times \{z\}}$. We define the dual space for this pairing as $L^2_{\sigma'}$ with $\sigma' = \sigma^{-1} r^{-m}$. The Fourier transform $\hat f$ is a distribution valued in L^2_σ; there is a pairing for smooth compactly supported maps $g : \mathbb{C} \to L^2_{\sigma'}$, given as

$$\hat f(g) = \int_{\mathbb{C}} \langle f, \hat g\rangle\, dz.$$

Here $\hat f$ has finite order at most 4. To show this, fix a compactly supported function $g : \mathbb{C} \to L^2_{\sigma'}$ supported in $K \subset \mathbb{C}$ satisfying the bound $|\nabla^k g| \le A$ for $k \le 4$ differentiating in the $\mathbb{C}$ variable. Because g is compactly supported and f is bounded, integration by parts gives the bound

$$\|\hat g(\xi)\| \le C_{K,A}(1 + |\xi|)^{-4}$$

for some constant C depending on only K and A. This means the pairing has a bound

$$|\hat f(g)| \le \int_{\mathbb{C}} \|f(z,\cdot)\|_{L^2_\sigma} \|\hat g(z)\|_{L^2_{\sigma'}}\, dz \le |f|_\infty C_{K,A},$$

so we can absorb the bound of f into the original constant to give the estimate $|\hat f(g)| \le C_{K,A}$.

The Fourier transform $\hat f$ is supported at the origin, implying that f is polynomial. A distribution of finite order supported at the origin is a sum of delta functions and their derivatives as a standard result from distribution theory (see Rudin [21, Theorem 6.25]). The support of $\hat f$ being the origin means that for any g with compact support away from the origin, the pairing with $\hat f$ vanishes: $\hat f(g) = 0$. By approximating g with C^4 functions valued in $L^2_{\sigma'}$ of this form, we can assume g is of this type. If given a sequence of such functions paired with $\hat f$ all of 0 that converge to g, then the pairing with $\hat f$ must be 0 via duality by taking the Fourier transform, so assuming g is of this type causes no loss of generality.

We apply the previous proposition by setting $\chi = \hat g$ and we can solve the Laplacian equation to get some h such that $\Delta h = \hat g$ with exponential decay in the $\mathbb{C}$ direction. By the decay properties of f and h, we can integrate by parts to get

$$\hat f(g) = \int_{X_0} f \hat g\, dV = \int_X f \Delta h\, dV = \int_X h \Delta f\, dV = 0$$

so $f(z) = \sum z^i \bar z^j f_{ij}$ for $f_{ij} \in L^2_\sigma(X_0)$. However, since f is bounded, each f_{ij} term must be 0 except for $i = j = 0$. The polar decomposition in equation 13 with $\xi = 0$ and $\hat\chi\lambda = 0$ has no solutions aside from 0 with the specified growth range of $\tau \in (2-m, 0)$. ∎

On X_0, the cone metric is defined on the smooth set, outside of the rays of singularities, $X_0 \setminus \{\mathbb{C} \times \{0\}\}$, with metric

$$g_{X_0} = dr^2 + r^2 h_L$$

for h the metric on the link L. However, L is incomplete since we removed the singularities. If we take its completion, the resultant space $\overline{L}$ is the double suspension of the link of V_0, meaning it has a circle of singularities based on the original cone V_0. Degeratu and Mazzeo [10] describe this as a smoothly stratified space of depth 1, and h_L is an iterated edge metric. We define a similar weighted Hölder space on this space by conformally scaling the metric on this link. This norm is the standard Hölder norm with conformal metric change on the link given by $w^{-2}h_L$:

$$\|f\|_{C^{k,\alpha}_\tau} = \|w^{-\tau}\|_{C^{k,\alpha}_{w^{-2}h_L}}$$

and although this is dependent on a particular smoothing, all choices yield equivalent norms. We apply a result from Mazzeo [19] that the Laplacian on this link plus a constant is invertible for most constants.

Proposition 6.3. *For $a \in \mathbb{C}$, the map*

$$\Delta_{h_L} + a : C^{k,\alpha}_\tau(L) \to C^{k-2,\alpha}_{\tau-2}(L)$$

is invertible for $\tau \in (2-m, 0)$ if a is not contained in a discrete set of real values.

Proof. This results from the Fredholm theory for edge operators of Mazzeo [19]. It states that for the weights chosen the image of $\Delta + a$ is L^2-orthogonal to its kernel and Δ has a discrete real spectrum. □

The space X_0 has been expressed as a (real) cone over L, as opposed to a product, and we almost know how to invert the Laplacian on the link. This is promising as we can reduce the PDE to a simpler form using the cone structure. The computation becomes easier in different coordinates where replacing r with its logarithm as $t = \log r$ and conformally scale the metric to define

$$\tilde{g}_{X_0} = r^{-2} g_{X_0} = dt^2 + h_L.$$

The Laplacian $\Delta_{g_{X_0}} f = u$ is re-expressed using these coordinates as

$$\Delta_{h_L} f + \partial_t^2 f + 2n\partial_t f = e^{2t} u.$$

We can write the weighted Hölder norms in this new context as

$$\|f\|_{C^{k,\alpha}_{\delta,\tau}} = \|e^{-\delta t} w^{-\tau} f\|_{C^{k,\alpha}_{w^{-2}\tilde{g}_{X_0}}}.$$

We define and show the invertibility of the operator

$$\mathcal{L} = \Delta_{h_L} + \partial_t^2 + 2n\partial_t : C^{k,\alpha}_{\delta,\tau}(\mathbb{R} \times L) \to C^{k-2,\alpha}_{\delta,\tau-2}(\mathbb{R} \times L).$$

It will ease the computation to conjugate by the invertible conformal scaling of multiplication by $e^{\delta t}$, so we can define $\mathcal{L}_\delta(f) = e^{\delta t}\mathcal{L}(e^{-\delta t} f)$. Written out, this is

$$\mathcal{L}_\delta = \Delta_{h_L} + \partial_t^2 + (2n - 2\delta)\partial_t + \delta^2 - 2n\delta.$$

Before we show that the above is almost always invertible (when δ avoids a discrete set of roots and $\tau \in (2-m, 0)$ as usual), we first prove a lemma in the vein of the strategies used for the prior results.

Proposition 6.4. *Suppose that δ avoids a discrete set of indicial values. Let χ be smooth on $\mathbb{R} \times L$ with $\hat{\chi}$ compactly supported. There exists some f such that $\mathcal{L}_\delta f = \chi$ with the following properties:*

(a) $f \in C^{k,\alpha}_\tau$ *for any* $\tau < 0$

(b) *For any* $a > 0$, *there is a constant* C *such that* $\|f(t,x)\|_{C^{k,\alpha}_\tau(|t|>A)} < C(1+A)^{-a}$ *for any* $A > 0$.

The same is true for its adjoint $\mathcal{L}^*_\delta$.

Proof. We imitate the strategy used before, utilizing that we can solve the Laplacian (almost) on the link L from Mazzeo [19] on Fredholm edge operators. The Fourier transform of the equation re-expresses the $\mathcal{L}_\delta$ equation as

$$\Delta_{h_L} \hat{f} - (\xi^2 \hat{f} - i\xi(2n-2\delta) - \delta^2 + (2n-2)\delta)\hat{f} = \hat{\chi}.$$

In these new coordinates, this is an equation of the form $(\Delta_{h_L} + a)\hat{f} = \hat{\chi}$. Since a avoided some values, we need δ to avoid the values as that would make this equation unsolvable. In fact, since this is a polynomial of δ and ξ, when $\xi \neq 0$, a generic δ will work and a will not be real, so this will satisfy the applicable values from Proposition 6.3. If $\xi = 0$, then $a = \delta^2 - (2n-2)\delta$ and δ must avoid the bad values of a, which are discrete. Essentially, we can apply Proposition 6.3 no matter what ξ is up to altering δ by a small amount.

There is an inclusion of Hölder spaces $C^{k,\alpha}_\tau \subset C^{k,\alpha}_{\tau'}$ for $\tau > \tau'$, so $f \in C^0_{\tau'}$ for any negative τ'. The Schauder estimate will give us bounds on every derivative. The Schauder estimate can be applied because of the decay and reducing this to a bound on some arbitrarily large, but compact region given that outside everything is bounded by ϵ. This gives that $f \in C^k_{\tau'}$ for all k, proving part (a).

For part (b), we show that $\hat{f}$ is a smooth function of ξ. By the properties of the inverse Fourier transform, there are bounds as in the classical theory of Fourier analysis. We use the same approach as before and iteratively take derivatives and bound them. Inductively, the ξ derivatives satisfy equations

$$\Delta_{h_L} \partial^l_\xi \hat{f} + a(\xi)\partial^l_\xi \hat{f} = g(\xi),$$

and $g(\xi)$ is compactly supported in ξ. If $\tau < 0$ negative, then $g(\xi) \in C^{k,\alpha}_\tau$. Furthermore, Proposition 6.3 applied to $a(\xi)$ gives bounds on the ξ-derivatives of $\hat{f}$. In this manner, taking the inverse Fourier transform shows the decay rate.

For the adjoint, the proof is identical when we write out that

$$\mathcal{L}^*_\delta = \Delta_{h_L} + \partial_t^2 - (2n-2\delta)\partial_t + (\delta^2 - 2n\delta)$$

where the adjoint only differs in the signs of some of the lower order terms, so δ must avoid a discrete set of inadmissible values. This is also generic, so choosing for a viable value of δ for both $\mathcal{L}_\delta$ and its adjoint can be done generically. $\square$

Corollary 6.5. *Let $f \in C^{k,\alpha}_\tau$ be such that $\mathcal{L}_\delta f = 0$ for some generic δ and $\tau \in (2-m, 0)$. Then $f = 0$.*

Proof. Suppose for the sake of contradiction that f does not vanish. There must exist some χ such that its Fourier transform $\hat{\chi}$ is compactly supported and such that the integral $\int_{\mathbb{R}\times L} f\chi \neq 0$. Proposition 6.4 gives a solution $\mathcal{L}^*_\delta h = \chi$ with h in the space $C^{k,\alpha}_{\tau'}$ for $2 - m - \tau < \tau' < 0$. We therefore compute

$$\int_{\mathbb{R}\times L} f\chi \, dV = \int_{\mathbb{R}\times L} f\mathcal{L}^*_\delta h \, dV = \int_{\mathbb{R}\times L} h\mathcal{L}_\delta f \, dV = 0$$

where integration by parts is justified by the decay of h and by the range chosen for τ', which was allowed to be an arbitrary negative number. However, this is a contradiction since $\int_{\mathbb{R}\times L} f\chi\, dV$ was non-zero. Therefore, we deduce that the kernel of $\mathcal{L}_\delta$ is trivial. □

Proposition 6.6. *Let $\tau \in (2-m, 0)$ and δ be generically chosen. There exists some constant C such that for $f \in C^{k,\alpha}_\tau(\mathbb{R}\times L)$, there is a bound*

$$\|f\|_{C^{k,\alpha}_\tau} < C\|\mathcal{L}_\delta f\|_{C^{k-2,\alpha}_{\tau-2}}.$$

Therefore, $\mathcal{L}_\delta$ has closed image.

Proof. By the construction of w, the conformally rescaled metric $w^{-2}\tilde{g}_{X_0}$ has bounded geometry, meaning the curvature is bounded. The primary Schauder estimate states that

$$\|f\|_{C^{k,\alpha}_\tau} \leq C(\|\mathcal{L}_\delta f\|_{C^{k-2,\alpha}_{\tau-2}} + \|f\|_{C^0_\tau}) \tag{14}$$

justified by the decay properties of f. The component of $C\|f\|_{C^0_\tau}$ can be absorbed into the constant as f has finite norm in C^k_τ, so it also is bounded in C^0_τ.

We have reduced this to showing a bound

$$\|f\|_{C^0_\tau} \leq C\|\mathcal{L}_\delta f\|_{C^{k-2,\alpha}_{\tau-2}}$$

for a uniform C. Suppose for the sake of contradiction that there exists a sequence of functions f_j each normalized to have unit norm, but their Laplacians having arbitrarily small norms going to 0, say $\|\mathcal{L}_\delta f_j\|_{C^{k-2,\alpha}_{\tau-2}} < \frac{1}{j}$. Since w is bounded and $\|f\|_{C^{k,\alpha}_\tau} = 1$, there exists some C' such that there are points $(t_j, x_j) \in X_0$ such that

$$|f_j(t_j, x_j)| > \frac{1}{C'} w(x_j)^\tau, \tag{15}$$

and by replacing $f_j(t,x)$ with $f_j(t - t_j, x)$, we can assume that t_j is 0. Note that the above indices do not refer to the defining polynomials of $V_0 = V(f_1, \dots, f_k)$ nor do the variables x_j refer to the coordinates $(x_1, \dots, x_{n+k})$; these are sequential indices.

We examine two cases: first when $w(x_j)$ stays away from 0 by a bounded amount, and second when $w(x_j) \to 0$. In the first case, we can pass to a subsequence that converges in $x_j \to x$. The $\mathcal{L}_\delta$ Schauder estimate 14 gives convergent subsequence locally in a different Hölder space (using the compact inclusion of Hölder spaces) for $C^{k,\alpha'}$ to some $f \in C^{k,\alpha}_\tau$ such that $\mathcal{L}_\delta f = 0$. However, Corollary 6.5 forces f to be 0, which is a contradiction to the bound 15.

For the second case, consider the sequence of rescaled metrics $w(x_j)^{-\tau}\tilde{g}_{X_0} = w(x_j)^{-\tau} r^{-2} g_{X_0}$. These spaces limit to $\mathbb{R}^2 \times V_0$ with the product metric $\omega_{\mathrm{Euc}} + \omega_{V_0}$ based at $(0, x)$ with $r(x) = 1$. This means that h_L, the metric on the link $\overline{L}$, is modeled on $S^1 \times V_0$ near the circle of singular points. The same technique above works by locally taking a convergent sequence in a $C^{k,\alpha'}$ space, giving a limiting function $f \in C^{k,\alpha}_\tau(X_0)$ such that $\Delta f = 0$, so Proposition 6.2 forces $f = 0$ yielding the same contradiction as before, finishing the proof. □

We now can prove the main result of this subsection.

Proposition 6.7 ($\mathcal{L}_\delta$ Invertibility). *For δ avoiding a discrete indicial set of roots and $\tau \in (2-m, 0)$, the operator $\mathcal{L}_\delta : C^{k,\alpha}_\tau(\mathbb{R}\times L) \to C^{k-2,\alpha}_{\tau-2}(\mathbb{R}\times L)$ is invertible.*

Proof. The previous theorem shows that this operator is injective, so we must show surjectivity. We already demonstrated the image is closed. Let $u \in C^{k-2,\alpha}_{\tau-2}(\mathbb{R}\times L)$. We consider a sequence

χ_j whose Fourier transforms are compactly supported such that $\chi_j \to u$ locally and uniformly with bounded norms in $C^{k-2,\alpha}_{\tau-2}(\mathbb{R} \times L)$. From Proposition 6.4, there are functions f_j that solve $\mathcal{L}_\delta f_j = \chi_j$, and Proposition 6.6 states that f_j are uniformly bounded as

$$\|f_j\|_{C^{k,\alpha}_\tau} < C.$$

Passing to a subsequence with convergence locally in $C^{k,\alpha'}$ gives a limiting function f with bounded norm $\|f\|_{C^{k,\alpha}_\tau} < C$ such that $\mathcal{L}_\delta f = u$ showing surjectivity. □

Corollary 6.8. *The operator Δ_{X_0} is invertible for the proper weights*

$$\Delta_{X_0} : C^{k,\alpha}_{\delta,\tau}(X_0) \to C^{k-2,\alpha}_{\delta-2,\tau-2}(X_0)$$

if $\tau \in (2-m, 0)$ and δ is generic.

6.2. **The Laplacian on $X_{p'}$.** We prove similar results for the second model space $X_{p'} = \mathbb{C} \times V_{p'}$. This is substantially simpler since this space is smooth. For this space, we construct a weighted Hölder space by a weight function $\zeta = \widetilde{\max}(1, d(o, \cdot))$ that is a smoothed version of the maximum of 1, and the distance from some fixed point $o \in V_{p'}$, instead of r which is not necessarily strictly positive. Picture this point o as the *cone point* of $V_{p'}$ of minimal radius. The product metric on this space is $g = g_{\mathrm{Euc}} + g_{V_{p'}}$ and the norm is defined as

$$\|f\|_{C^{k,\alpha}_\tau} = \|\zeta^{-\tau} f\|_{C^{k,\alpha}_{\zeta^{-2}g}}.$$

Proposition 6.9 (Laplacian on $V_{p'}$)**.** *Let $\tau \in (2-m, 0)$ and $\lambda \geq 0$. If u is smooth and compactly supported in $V_{p'}$, then there exists some f solving the equation $\Delta_{V_{p'}} f - \lambda f = u$ with estimate $|f| \leq C_u \zeta^\tau$ independent of λ.*

Proof. If $\lambda = 0$, then this is a standard result for asymptotically conical manifolds with a Laplacian and weighted space (see Lockhart-McOwen [18]). Consider

$$\Delta_{V_{p'}} : C^{k,\alpha}_\tau(V_{p'}) \to C^{k-2,\alpha}_{\tau-2}(V_{p'}),$$

the Laplacian acting as a map between weighted Hölder spaces. We constructed these weighted spaces such that the Laplacian is self-adjoint. Any harmonic element f in this space must have a decay rate of at least $d(\cdot, o)^{2-m}$. Therefore, integration by parts against a test function shows that f must be constant and therefore 0 (since it must decay).

Suppose that $\lambda > 0$. The operator $\Delta_{V_{p'}} - \lambda$ is essentially self-adjoint, meaning it has a unique self-adjoint extension by taking its closure. Furthermore, it has trivial kernel. We can apply the Schauder estimate

$$\|f\|_{C^2_\tau} \leq C(\|f\|_{C^0_\tau} + \|u\|_{C^0_\tau})$$

and since u is compactly supported, the term $\|u\|_{C^0_\tau}$ is finite and can be absorbed into the constant. Since f is in C^k_τ for $k = 0$ and the Schauder estimates gives this estimate for all derivatives, by the definition of these C^k_τ spaces, f muster decay faster than any $\zeta^{-\ell}$. We must show this is uniform as $\lambda \to 0$.

Choose b to solve $\Delta_{V_{p'}} b = -\zeta^{\tau-2}$ using the above. By the maximum principle, b is strictly positive. For sufficiently large C based on u, there is a bound $|f| < Cb$ independent of λ. Suppose otherwise, then $f - Cb$ would achieve a maximum at some point x_m since f has fast decay. Therefore, there are bounds

$$\begin{aligned} 0 &\geq \Delta(f - Cb)(x_m) \\ &= u(x_m) + \lambda f(x_m) + C\zeta^{\tau-2}(x_m) \\ &> u(x_m) + C\zeta^{\tau-2}(x_m), \end{aligned}$$

and if C is large enough as a function of $\|u\|_{C^0_{\tau-2}}$, then the last inequality cannot be true, yielding a contradiction.

By choosing b such that $\Delta_{V_{p'}} = \zeta^{\tau-2}$ and applying the minimum principle, the exact same argument (with inequalities reversed) shows that $|f| > -Cb$ for the same constant C. □

Proposition 6.10 (Laplacian on $X_{p'}$)**.** *Let χ be a smooth function on $X_{p'} = \mathbb{C} \times V_{p'}$ such that its Fourier transform $\hat{\chi}$ has compact support. There exists some f that solves the Laplacian equation $\Delta f = \chi$ with the following properties:*

(a) *$f \in C^{k,\alpha}_\tau(X_{p'})$ for any $\tau > 2 - m$*

(b) *If $\hat{\chi}$ is supported away from $\{0\} \times V_{p'}$, then for any $a > 0$ there exists some C such that $\|f\|_{C^{k,\alpha}_\tau(|z|>A)} < C(1 + A)^{-a}$ for any $A > 0$.*

Proof. The Fourier transform of the equation gives

$$\Delta_{V_{p'}} \hat{f} - |\xi|^2 \hat{f} = \hat{\chi} \tag{16}$$

which from the above proposition can be solved with uniform estimates on $\|\hat{f}\|_{C^0_\tau}$. Taking the inverse solution gives the desired solution. The Schauder estimate bounds the norm of f in the $C^{k,\alpha}_\tau(X_{p'})$.

For part (b), this follows by the same arguments as before by differentiating the equation 16. Taking ∂^l_ξ derivatives of equation 16, we can inductively get the desired bounds by expressing derivatives of $\hat{f}$ in terms of ∂^l_ξ derivatives of $\hat{\chi}$ and lower derivatives of $\hat{f}$. Therefore, the derivatives of $\hat{f}$ have the same decay. Combining this with the same Schauder estimate gives the desired bounds and decay of f, completing the proof. □

This is the exact same setup as before, so the same proofs will work to prove the following results.

Proposition 6.11. *For $\tau \in (2 - m, 0)$ and $f \in C^{k,\alpha}_\tau(\mathbb{C} \times V_{p'})$, there exists some constant C such that*

$$\|f\|_{C^{k,\alpha}_\tau} \leq C\|\Delta f\|_{C^{k-2,\alpha}_{\tau-2}}.$$

Proof. This follows directly as in the proof of Proposition 6.6 by applying the above theorems corresponding to the theorems utilized in that proof about the space X_0.

□

Proposition 6.12. *The Laplacian*

$$\Delta_{X_{p'}} : C^{k,\alpha}_\tau \to C^{k-2,\alpha}_{\tau-2}$$

is invertible for $\tau \in (2 - m, 0)$.

Proof. This follows identically the proof of Proposition 6.7 by replacing the corresponding theorems as proven for $\mathbb{C} \times V_{p'}$ in place of X_0. In fact, it is even easier as we proved this for all $\lambda > 0$, so we do not need to avoid some bad values as we did before. The extra non-viable values of a came from the fact that the edge operators had the properties that $\Delta + a$ was only sometimes invertible when we removed some values for a. For the model space $X_{p'}$, the operator is $\Delta - \lambda$ for all non-negative λ above. For the proof of Corollary 6.8 , the genericity of δ was needed because the terms in $\mathcal{L}_\delta$ had constants depending on them induced by these roots, but for $X_{p'}$ we only have to deal with $\Delta - |\xi|^2$, so we only require the result for $\lambda \geq 0$, which was shown. $\square$

7. Calabi-Yau metric near infinity

The goal of this section is to perturb the asymptotically Calabi-Yau metric ω to a metric that is Ricci-flat outside of some compact set. Since we have demonstrated that the model spaces of X_0 and $X_{p'}$ are sufficient approximations in the various regions, and we have inverted the Laplacian on them, the characterization of $\mathcal{X}$ as a complete intersection is no longer relevant and the techniques and computations will follow as in Székelyhidi [22].

We will localize the manifold to the model spaces, which we proved in the previous section have desirable analytic properties in the function spaces so that we can invert the Laplacian. This, combined with the results from Section 5 showing that the total space sufficiently far out is arbitrarily close to the models, will allow us to glue together these solutions. The main theorem is the invertibility of the Laplacian outside a large ball.

Proposition 7.1 (Invertibility of Δ near infinity). *Let $\tau \in (2-2n, 0)$ and δ generic, for $A > 0$ sufficiently large. Then the Laplacian*

$$\Delta : C^{2,\alpha}_{\delta,\tau}(\rho^{-1}[A,\infty),\omega) \to C^{0,\alpha}_{\delta-2,\tau-2}(\rho^{-1}[A,\infty),\omega)$$

is surjective with inverse bounded independently of A.

Recall that we have shown a large range of values of δ that work, specifically for all $\delta < \frac{1}{d_1}$, or if $d_1 > 3$ then all negative, so δ can be chosen in these regions avoiding a discrete set of unsuitable values.

Proof. We will split up $\mathcal{X}$ into many different regions, but primarily into two regions based on whether the model space should be X_0 or $X_{p'}$. We will prove this theorem by producing an inverse operator P such that $f = Pu$ is such that $\|f\|_{C^{2,\alpha}_{\delta,\tau}} < C$ for some function $u \in C^{0,\alpha}_{\delta-2,\tau-2}(\rho^{-1}[A,\omega))$ with small norm less than 1. From the extension property from Proposition 4.1, u can be defined on all of $\mathcal{X}$ with norm less than C (the norm of the extension operator E).

We proceed locally by using the cutoff functions $\gamma_1 + \gamma_2 = 1$ from before to partition $\mathcal{X}$ into the spaces where it can be approximated well by the various model spaces. Let $u_1 = \gamma_1\left(R\Lambda^{-1}\rho^{-\frac{1}{d_1}}\right)u$ and $u_2 = \gamma_2\left(R\Lambda^{-1}\rho^{-\frac{1}{d_1}}\right)u$. Recall the notation

$$\mathcal{U} = \left\{R > \Lambda\rho^{\frac{1}{d_1}}\right\} \cap \{\rho > A\},$$

which has a projection operator $G : \mathcal{U} \to X_0$ such that

$$\|G^*\omega_{X_0} - \omega\|_{C^{k,\alpha}_{0,0}} < \epsilon.$$

Corollary 6.8 gives an inverse to u_1 on X_0. We consider u_1 as G^*u_1 a function on X_0 and we will define the inverse there. G is one-to-one for $A \gg 0$ sufficiently large, so G acts by its inverse to $\mathcal{X}$. There exists some f such that $\Delta_{X_0} f = u_1$ and set $Pu_1 = f$, and this satisfies the bound $\|f\|_{C^{2,\alpha}_{\delta,\tau}(X_0)} < C$ uniformly.

G^{-1} gives the map to return to $\mathcal{X}$. We define regions based on cutoffs

$$\beta_1 = \gamma_1 \left(\frac{\log(R\Lambda^{-\frac{1}{2}}\rho^{-\frac{1}{d_1}})}{\log \Lambda^{\frac{1}{4}}} \right). \tag{17}$$

This separates out the region where $R < \Lambda^{\frac{3}{4}}\rho^{\frac{1}{d_1}}$ where β_1 vanishes. β_1 is identically 1 on the region where u_1 is supported. Furthermore, the derivative of β_1 satisfies the bound

$$\|\nabla\beta_1\|_{C^{k,\alpha}_{-1,-1}(\rho^{-1}(1,\infty)\cap X_0)} < \frac{C}{\log \Lambda},$$

which can be made arbitrarily small for Λ sufficiently large. We can explicitly compute that

$$|\nabla\beta_1| = \gamma' \left(\frac{\nabla R}{R} + \frac{\nabla \rho}{\rho} \right) \frac{1}{\log \Lambda^{\frac{1}{4}}},$$

and since $\rho^2 = |z|^2 + R^2$, we can compute

$$\rho \nabla \log \rho = \frac{R|\nabla R| + |z||\nabla z|}{\rho}$$

which is $O(1)$. In the weighted spaces with $\delta = \tau = -1$, the norm is given by

$$\|f\|_{C^{k,\alpha}_{\delta,\tau}} = \|f\|_{C^{k,\alpha}(\rho<2P)} + \sum_{j=0}^{k} \sup_{\rho>P} \rho^j w^j |\nabla^j f| + [\rho^k w^k \nabla^k f]_{0,\alpha},$$

and in this range w is bounded above and below by constants, so it can be ignored. Therefore, we get derivative bounds of the same form as above of a constant times $(\log \Lambda^{\frac{1}{4}})^{-1}$, which is sufficient.

The Laplacian of $\beta_1 P u_1$ is given by

$$\Delta_{X_0}(\beta_1 P u_1) = u_1 + 2\nabla\beta_1 \cdot \nabla(Pu_1) + Pu_1 \Delta_{X_0}\beta_1.$$

For Λ, A sufficiently large, the function G is one-to-one, so can translate between X_0 and $\mathcal{X}$. The Laplacian of β_1 is bounded by

$$|\Delta_{X_0}\beta_1|_{C^{k-1,\alpha}_{-2,-2}} \leq |\nabla\beta_1|_{C^{k,\alpha}_{-1,-1}} \leq \frac{C}{\log \Lambda},$$

since the Laplacian in coordinates is the divergence of the gradient, and in general that

$$\|f\|_{C^{k,\alpha}_{\delta,\tau}} \leq C\|f\|_{C^{k,\alpha}_{\delta-a,\tau-b}}$$

for $a, b > 0$, since these only scale the weights to be larger. Pulling back by G gives

$$\|\Delta_\omega(G^{-1*}(\beta_1 P u_1)) - u_1\|_{C^{0,\alpha}_{\delta-2,\tau-2}} < \epsilon$$

for Λ, A sufficiently large.

The region $\mathcal{V}$ is defined as the complement of $\mathcal{U}$ in $\mathcal{X}$ for similarly large enough $\rho > A$, $\mathcal{V} = \{R < 2\Lambda\rho^{\frac{1}{d_1}}\} \cap \{\rho > A\}$. In $\mathcal{V}$, all the fibers are diffeomorphic (assuming A is large enough so there are no singularities) and $\mathcal{V} \to \mathbb{C} \setminus C_A$ is a fibration where C_A is the open ball of radius A in $\mathbb{C}$. We will cut this apart into many regions supported in small neighborhoods in $\mathbb{C}$. Proposition 3.1 tells us that these regions are locally $\mathbb{C} \times V_{p'+\epsilon}$ which are all smoothly varying and diffeomorphic, so we can localize to the region to small enough variation to invert the Laplacian.

We construct functions χ_i that cut out regions on which to localize the Laplacian. Consider, for some B sufficiently large, $\mathbb{C}$ with metric $\tilde{g} = B^{-2}|z|^{-\frac{2}{d_1}} g_{\text{Euc}}$, and with disks U_i (with coordinate z_i) of radius 2 in this metric that form a locally finite cover. Furthermore, if we take each of these regions to have only radius 1, they are pairwise disjoint. Let χ_i be partitions of unity subordinate to the above cover such that $\sum \chi_i = 1$. We will use these functions to localize the analysis. By construction, χ_i will be identically 1 on $z_i \leq 1$ by the disjointness criterion. Rescaling to the usual metric, this can be stated equivalently that χ_i are identically 1 on the balls centered at z_i of radius $B|z|^{\frac{1}{d_1}}$ and 0 outside the ball of radius $2B|z|^{\frac{1}{d_1}}$. There are bounds on their derivatives given by

$$|\nabla^l \chi_i|_{g_{\text{Euc}}} = O\left(B^{-l}|z_i|^{-\frac{l}{d_1}}\right).$$

Using these same charts, define new cutoff functions $\tilde{\chi}_i$. These differ in that they are identically 1 on the U_i (the support of χ_i of radius $B|z|^{\frac{1}{d_1}}$) and the $\tilde{\chi}_i$ are supported in balls centered at z_i of radius $3B|z|^{\frac{1}{d_1}}$. These functions will have the same derivative bounds

$$|\nabla^l \tilde{\chi}_i|_{g_{\text{Euc}}} = O\left(B^{-l}|z_i|^{-\frac{l}{d_1}}\right).$$

Outside a neighborhood of the origin, in $|z| > A$, only finitely many $\tilde{\chi}_i$ are needed to cover the region and the number of charts is independent of A and B. This will be important as the error terms will come from the regions here, so the error will be the sum of finitely many small errors terms induced in each region.

We can express $u_2 = \sum \chi_i u_2$. Each of these $\chi_i u_2$ is supported on the ball around z_i of radius $2B|z_i|^{\frac{1}{d_1}}$, and in this region we have that $R < 2\Lambda|z|^{\frac{1}{d_1}}$. The function H (similar to G) is used to transfer these functions to $X_{p'}$. In this region, $\rho \sim |z_i|$, so we can compute the weight function as

$$\rho^{-2}w^{-2} \sim \max\left\{|z_i|^{\frac{1}{d_1}}, R\right\}^{-2}, \quad \rho^{\delta-2}w^{\tau-2} \sim |z_i|^{\delta-\tau} \max\left\{|z_i|^{\frac{1}{d_1}}, R\right\}^{\tau-2},$$

and we can compare $\hat{\zeta}$, the smoothed out radial weight, to be strictly greater than 1 and get a bound

$$\hat{\zeta} \sim \max\left\{1, |z_i|^{-\frac{1}{d_1}} R\right\}.$$

Therefore, there are estimates

$$\|u_2\|_{C^{0,\alpha}_{\tau-2}(X_{p'})} < C|z_i|^{\delta-\tau+\frac{\tau-2}{d_1}}$$

on the support of χ_i by unpacking the definition of the weighted spaces. Since u has norm 1 in the equivalent weighted space on all of $\mathcal{X}$, this is almost immediate. The C factor comes from the error in jumping to the model space $X_{p'}$ and from derivatives of the bump function γ_2. Differentiating this yields

$$|\nabla^l \chi_i| < CB^{-l} < CB^{-l}\hat{\zeta}^{-l},$$

so by choosing sufficiently large B to make B^{-1} have sufficient decay (after choosing Λ), the composite function has decay

$$\|\chi_i u_2\|_{C^{0,\alpha}_{\tau-2}(X_{p'})} < C|z_i|^{\delta-\tau+\frac{\tau-2}{d_1}}.$$

Applying Proposition 6.12 inverts the Laplacian for this small region. The metric is simply a rescaling of $\mathbb{C} \times V_{p'}$ and as in the end of Region V in Proposition 3.1, we can model this by $X_{p'}$ at arbitrary error, as also seen in Proposition 5.2. The fibers, before flattening out to all being $V_{p'}$, are all limiting to the fiber $V_{p'}$, so the argument to jump to that fiber with arbitrarily small error as used in Region V in Proposition 3.1 works for $A \gg 0$ sufficiently large. Therefore, the metric can be expressed as the conformal scaling $\Delta_{z_i^{-2/d_1}\omega_{p'}} = |z_i|^{\frac{2}{d_1}}\Delta_{\omega_{p'}}$. Since all these fibers are rescalings of $V_{p'}$, we can estimate this region by exactly $\mathbb{C} \times V_{p'}$ and do the analysis on this space at an error of arbitrarily small order, given $\Lambda \gg 0$ sufficiently large.

Proposition 6.12 gives us a definition of $P(\chi_i u_2)$ as defined by solving

$$\Delta_{X_{p'}} P(\chi_i u_2) = |z_i|^{\frac{2}{d_1}}\chi_i u_2,$$

which will satisfy the bound

$$\|P(\chi_i u_2)\|_{C^{2,\alpha}_\tau} \le C|z_i|^{\delta-\tau+\frac{\tau}{\delta}}.$$

Using H to map back to $\mathcal{X}$, since on $V_{p'}$ the function $\chi_i u_2$ is supported in regions where $|\hat{z}_i| < 2B$ by construction, and $\hat{\zeta} < 4\Lambda$ based on the region and cutoffs, the same logic as above gives that

$$\beta_2 = \gamma_2\left(\frac{\log(\hat{\zeta}4^{-1}\Lambda^{-1})}{\log\Lambda}\right)$$

has bound

$$\|\nabla\beta_2\|_{C^{k,\alpha}_{-1}} < \frac{C}{\log\Lambda}, \tag{18}$$

so we want to use this to estimate the error of the Laplacian inverse by passing to the model space with H. Consider

$$\Delta_{X_{p'}}\tilde{\chi}_i\beta_2 P(\chi_i u_2) - |z_i|^{\frac{2}{d_1}}\chi_i u_2 = 2\nabla(\beta_2\tilde{\chi}_i)\cdot\nabla P(\chi_i u_2) + \Delta_{X_{p'}}(\beta_2\tilde{\chi}_i)P(\chi_i u_2)$$

and the estimate 18 combined with the result that $P(\chi_i u_2)$ is bounded by a multiple of $\chi_i u_2$, since the Laplacian has bounded inverse, to get that this decays like $C\epsilon|z_i|^{\delta-\tau+\frac{\tau}{\delta}}$. We absorbed derivatives of all the functions $\tilde{\chi}_i$ here, which we can do since these decay like $B^{-1}|z_i|^{-\frac{1}{d_1}}$, so we can just make B sufficiently large to account for this and rescale $\Lambda \to \Lambda^2$. The rescaling factor was necessary to cancel from both sides, as the bound is

$$\|\Delta_{|z_i|^{-2/d_1}}(\tilde{\chi}_i\beta_2 P(\chi_i u_2)) - |z_i|^{\frac{2}{d_1}}\chi_i u_2\|_{C^{0,\alpha}_{\tau-2}} < C\epsilon|z_i|^{\delta-\tau+\frac{\tau}{\delta}},$$

so dividing through by the term $|z_i|^{\frac{2}{d_1}}$ gives

$$\|\Delta_{\omega_{X_{p'}}}(\tilde{\chi}_i\beta_2 P(\chi_i u_2)) - \chi_i u_2\|_{C^{0,\alpha}_{\delta-2,\tau-2}} < C\epsilon.$$

We combine all these local results to define $Pu = \beta_1 Pu_1 + \sum\beta_2\tilde{\chi}_i P(\chi_i u_2)$, which has bound

$$\|\Delta(Pu) - u\|_{C^{0,\alpha}_{\delta-2,\tau-2}} < C\epsilon < \frac{1}{2},$$

and deduce that ΔP is invertible with its inverse having a norm less than 2. Since we demonstrated that the norm of P is independent of A, there is a right inverse $P(\Delta P)^{-1}$ with $A \gg 0$ chosen large enough, and this completes the proof. □

8. Perturbation to Calabi-Yau

In this section, we perturb the asymptotically Calabi-Yau metric to be Ricci-flat outside a large compact set. To complete the proof of Theorem 1.1, we will apply a result of Hein's PhD thesis [15] to show that this can be further perturbed to be globally Ricci-flat.

For the first part, we use a fixed point method by producing a contracting operator that converges on such a metric.

Proposition 8.1 (CY outside a large set)**.** *Let $A \gg 0$ be chosen sufficiently large, $\tau < 0$ sufficiently close to 0, and $\delta < \frac{2}{d_1}$, the conditions stated in Proposition 3.1. There exists a function $u \in C^{2,\alpha}_{\delta,\tau}(\mathcal{X})$ with sufficiently small norm such that*

$$(\omega + \sqrt{-1}\partial\overline{\partial}u)^{n+1} = (\sqrt{-1})^{(n+1)^2}\Omega \wedge \overline{\Omega}$$

on the set $\rho^{-1}[A, \infty)$.

Proof. The Ricci potential $\log[(\omega + \sqrt{-1}\partial\overline{\partial}u)^{n+1}/[(\sqrt{-1})^{(n+1)^2}\Omega \wedge \overline{\Omega})]$ quantifies the failure of the guess u to give rise to Ricci-flat metric; it is 0 exactly for a Ricci-flat metric. To converge at a Ricci-flat metric, the initial metric can be iteratively improved until it reaches the desired solution. The conditions on τ and δ are such that ω and $\omega + \sqrt{-1}\partial\overline{\partial}u$ are uniformly equivalent metrics for u sufficiently small. We define the set of viable u as follows

$$\mathcal{B} = \{u \in C^{2,\alpha}_{\delta,\tau} : \|u\|_{C^{2,\alpha}_{\delta,\tau}} \leq \epsilon_0\},$$

where we chose ϵ_0 such that every $u \in \mathcal{B}$ is such that $\omega + \sqrt{-1}\partial\overline{\partial}u$ is uniformly equivalent to ω. The operator F which captures the error above is defined as

$$F : \mathcal{B} \to C^{2,\alpha}_{\delta,\tau}(\rho^{-1}[A, \infty)), \qquad F : u \mapsto \log \frac{(\omega + \sqrt{-1}\partial\overline{\partial}u)^{n+1}}{(\sqrt{-1})^{(n+1)^2}\Omega \wedge \overline{\Omega}},$$

where we restrict to the Ricci potential on $\rho^{-1}[A, \infty)$. If we find u such that $F(u) = 0$ we are done.

We can expand

$$F(u) = F(0) + \Delta_\omega u + Q(u) \tag{19}$$

for some non-linear operator Q. Using P as the right inverse for Δ as described in the previous section, away from the origin sufficiently far, we must produce some function u solving the PDE

$$u = P(-F(0) - Q(u)).$$

For notation, we name this operator $N(u) = P(-F(0) - Q(u))$, and u must be a fixed point of N. We compute $Q(u) = F(u) - F(0) - \Delta_\omega u$ so we can take the difference

$$Q(u) - Q(v) = F(u) - F(v) + \Delta_\omega v - \Delta_\omega u = F(u) - F(v) + \Delta_\omega(v - u)$$

and by the chosen region, u and v are small in norm, and the function F captures that $C^{2,\alpha}_{2,2}$ norm. There is an estimate on the difference:

$$\|Q(u) - Q(v)\|_{C^{0,\alpha}_{\delta-2,\tau-2}} \leq C(\|u\|_{C^{2,\alpha}_{2,2}} + \|v\|_{C^{2,\alpha}_{2,2}})\|u - v\|_{C^{2,\alpha}_{\delta,\tau}}$$

as seen by expanding this operator as

$$(\omega + \sqrt{-1}\partial\overline{\partial}u)^{n+1} = \omega^{n+1} + (n+1)\sqrt{-1}\partial\overline{\partial}u \wedge \omega^n + \binom{n+1}{2}\sqrt{-1}\partial\overline{\partial}u \wedge \sqrt{-1}\partial\overline{\partial}u \wedge \omega^{n-1} + \cdots .$$

This can be computed as

$$\begin{aligned} Q(u) - Q(v) &= F(u) - F(v) - \Delta_\omega \\ &= \|\sqrt{-1}\partial\bar\partial(u-v) \wedge \sqrt{-1}\partial\bar\partial(u+v)\|_{C^{2,\alpha}_{2,2}} \\ &\le C(\|u\|_{C^{2,\alpha}_{2,2}} + \|v\|_{C^{2,\alpha}_{2,2}})\|u-v\|_{C^{2,\alpha}_{\delta,\tau}}. \end{aligned}$$

Therefore, there exists some constant ϵ_1 such that the map $N(u) = P(-F(0) - Q(u))$ is a contraction on $\mathcal{B}$ for $\|u\|_{C^{2,\alpha}_{2,2}} < \epsilon_1$. To compare the norms $C^{2,\alpha}_{\delta,\tau}$ to $C^{2,\alpha}_{2,2}$, we use the relationship of the weight function

$$\rho^\delta w^\tau \le C\rho^{\delta-2+(\tau-2)(\frac{1}{d_1}-1)}\rho^2 w^2,$$

which tells us that the $C^{2,\alpha}_{2,2}$ bound controls the $C^{2,\alpha}_{2,2}$ bound:

$$\|u\|_{C^{2,\alpha}_{2,2}} \le C\|u\|_{C^{2,\alpha}_{\delta,\tau}}.$$

Applying this bound on viable u, the norm of $u - v$ controls the norm of $N(u) - N(v)$:

$$\|N(u) - N(v)\| \le \frac{1}{2}\|u - v\|$$

and this completes the proof once verified that N actually has image contained in $\mathcal{B}$.

By Proposition 3.1, $F(0) \in C^{0,\alpha}_{\delta'-2,\tau-2}$ for any $\delta' < \delta$ sufficiently close. This gives the estimate

$$\|F(0)\|_{C^{0,\alpha}_{\delta-2,\tau-2}(\rho^{-1}[A,\infty))} < CA^{\delta'-\delta}$$

which is arbitrarily small for $A \gg 0$ large enough. For $u \in \mathcal{B}$, we can compute

$$\begin{aligned} \|N(u)\|_{C^{2,\alpha}_{\delta,\tau}} &\le \|N(0)\|_{C^{2,\alpha}_{\delta,\tau}} + \|N(u) - N(0)\|_{C^{2,\alpha}_{\delta,\tau}} \\ &\le C\|F(0\|_{C^{0,\alpha}_{\delta,\tau}(\rho^{-1}[A,\infty))} + \frac{1}{2}\|u\|_{C^{2,\alpha}_{\delta,\tau}} \\ &= CA^{\delta'-\delta} + \frac{\epsilon_0}{2}, \end{aligned}$$

so for A large enough, this can be made to be less than ϵ_0, and thus in $\mathcal{B}$ as desired. Applying the fact that N is a contraction, it means it has a fixed point which by definition has Ricci-potential 0. The fixed point metric is a Calabi-Yau metric on $\rho^{-1}[A, \infty)$. ☐

This completes the final result of Theorem 1.1 by applying Hein's PhD thesis [15, Proposition 4.1] to the metric $\tilde\omega = \omega + \sqrt{-1}\partial\bar\partial u$ for u the fixed point of N above. There are two conditions on $\mathcal{X}$ necessary to apply this result: the SOB$(2n+2)$ condition and the existence of a $C^{3,\alpha}$ quasi-atlas.

Firstly, we must verify that $\mathcal{X}$ has the SOB$(2n+2)$ property from Hein [15, Definition 3.1], a condition on the growth rate of annuli being bounded by $Cr^{2(n+1)}$ for r the radius. The verification of this property is implied by the condition of relatively connected annuli (RCA), which can be shown using the Gromov-Hausdorff estimates to X_0. We utilize Proposition 5.3, which states that the tangent cone at infinity of $(\mathcal{X}, \tilde\omega)$ is $(X_0, \omega_{V_0} + \sqrt{-1}\partial\bar\partial(|z|^2))$, which is a metric cone. Using Corollary 5.4 and the fact that ω and $\tilde\omega$ are uniformly equivalent, this verifies the RCA property. From the work of Degeratu-Mazzeo [10], showing the SOB$(2n+2)$ condition is now reduced to showing that the asymptotic growth rate of balls is $r^{2(n+1)}$. We must produce some $C > 0$ such for $r > C$, the volume of $B(x, r)$ satisfies

$$C^{-1}r^{2(n+1)} < \mathrm{Vol}(B(x,r)) < Cr^{2(n+1)} \tag{20}$$

for all $x \in \mathcal{X}$ measured with respect to $\tilde{\omega}$. Colding's Volume Convergence Theorem [4] applied to Gromov-Hausdorff limits will verify this. Since $\tilde{\omega}$ is Ricci-flat outside a compact set, and has tangent cone at infinity with Euclidean volume growth, applying the Volume Convergence Theorem of [4] proves inequality 20.

Secondly, we must guarantee the existence of a $C^{3,\alpha}$ quasi-atlas: charts covering $\mathcal{X}$ of uniform size controlled in the $C^{3,\alpha}$ norm. This follows from Propositions 5.1 and 5.2, which states that we can create charts of radius bounded by $C\rho w$ around any point bounded in the $C^{k,\alpha}$ norm. The quantity ρw is unbounded as $\rho \to \infty$ as it is estimated as $\rho w > \kappa^{-2}\rho^{\frac{1}{d_1}}$, but the charts are uniformly bounded in size with respect to ω. When we perturb the solution to $\tilde{\omega}$, the previous propositions only give us control in $C^{2,\alpha}$ initially, but elliptic regularity gives us $C^{k,\alpha}$ control as necessary. Therefore, this construction meets the requirements of Hein to apply [15, Proposition 4.1], which takes $\tilde{\omega}$ and perturbs it to get a globally Calabi-Yau metric on all of $\mathcal{X}$ with the same tangent cone at infinity of X_0, proving Theorem 1.1.

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