The dilogarithm function is defined by the power series [?]

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}, \qquad \text{for } |z| < 1. \tag{1}$$

It is an identity of the dilogarithm that [?]

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) = -\frac{\pi^2}{6} - \frac{1}{2}\log^2(-z). \tag{2}$$

The dilogarithm of exponentials in which we are interested is

$$\mathrm{Li}_2\left(-e^{2i\phi}\right) = \sum_{k=1}^{\infty} \frac{(-1)^k e^{2ik\phi}}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k \cos(2k\phi)}{k^2} + i \sum_{k=1}^{\infty} \frac{(-1)^k \sin(2k\phi)}{k^2}. \tag{3}$$

The real part can be re-written in terms of dilogarithms:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^k \cos(2k\phi)}{k^2} &= \frac{1}{2}\left[\sum_{k=1}^{\infty} \frac{(-1)^k e^{-2ik\phi}}{k^2} + \sum_{k=1}^{\infty} \frac{(-1)^k e^{2ik\phi}}{k^2}\right] \\ &= \frac{1}{2}\left[\mathrm{Li}_2\left(-e^{2i\phi}\right) + \mathrm{Li}_2\left(-e^{-2i\phi}\right)\right]. \end{aligned} \tag{4}$$

By Eq. 2, we see that

$$\mathrm{Li}_2\left(-e^{2i\phi}\right) + \mathrm{Li}_2\left(-e^{-2i\phi}\right) = -\frac{\pi^2}{6} - \frac{1}{2}\log^2\left(e^{2i\phi}\right) = -\frac{\pi^2}{6} - 2\phi^2. \tag{5}$$

Therefore, the real part of Eq. 3 is

$$\mathrm{Re}\left[\mathrm{Li}_2\left(-e^{2i\phi}\right)\right] = -\frac{\pi^2}{12} + \phi^2. \tag{6}$$

The imaginary part is

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^k \sin(2k\phi)}{k^2} &= \frac{i}{2}\left[\sum_{k=1}^{\infty} \frac{(-1)^k e^{-2ik\phi}}{k^2} - \sum_{k=1}^{\infty} \frac{(-1)^k e^{2ik\phi}}{k^2}\right] \\ &= \frac{i}{2}\left[\mathrm{Li}_2\left(-e^{-2i\phi}\right) - \mathrm{Li}_2\left(-e^{2i\phi}\right)\right], \end{aligned} \tag{7}$$

which is more difficult to simplify.[1] Therefore, we conclude that

$$\mathrm{Li}_2\left(-e^{2i\phi}\right) = -\frac{\pi^2}{12} + \phi^2 - \frac{1}{2}\left[\mathrm{Li}_2\left(-e^{-2i\phi}\right) - \mathrm{Li}_2\left(-e^{2i\phi}\right)\right]. \tag{8}$$

The portion in square brackets is purely imaginary.

[1] To the best of my knowledge, no straightforward identity exists.