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Theory Predictions for Pull

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Abstract: Pull is a jet observable that is sensitive to color flow between dipoles. It has seen wide use for discrimination of particles with similar decay topologies but carrying different color representations and has been measured on W bosons from top quark decays by the DØ and ATLAS experiments. In this paper, we present the first theoretical predictions of pull, focusing on color-singlet dipoles. The pull angle observable, which is particularly sensitive to color flow, is not infrared and collinear safe, but is Sudakov safe and so its calculation requires all-orders resummation. We introduce other pull observables motivated by analogy to observables constructed for studying Drell-Yan production. We match our resummed calculations to fixed-order and estimate effects from hadronization, enabling a direct comparison of theoretical predictions to data.

Contents

1 Introduction

The color representation of a particle controls its rate and distribution of particle emission. Because any particle experiment collides color singlet particles, all color-charged particles produced must form a collection of color-singlet dipoles. Particle production along a dipole is approximately uniform in rapidity, so color-connected jets in experiment manifest themselves through the existence of excess particles between them. A determination of color connections between jets provides information about the fundamental process of collision: if there were gluons in the initial state, if a color-singlet particle decayed to jets, or if particles in exotic color representations were produced.

A powerful observable that is sensitive to the flow of color around a jet is pull [?]. Pull is a two-dimensional vector that measures the magnitude and direction of dominant particle production around any jet. From the dipole picture, the pull vector is expected to point in the direction of the other jet in the event to which it is color connected. Pull has been widely explored experimentally, in searches for the Higgs boson and new particles at CMS [? ? ? ? ?], as well as in measurements of hadronically-decaying W bosons from top quark decay at DØ [?] and ATLAS [? ?].

Despite extensive experimental studies, pull has never been theoretically calculated. This is mostly a consequence of the fact that the pull angle, the azimuthal angle of the pull vector

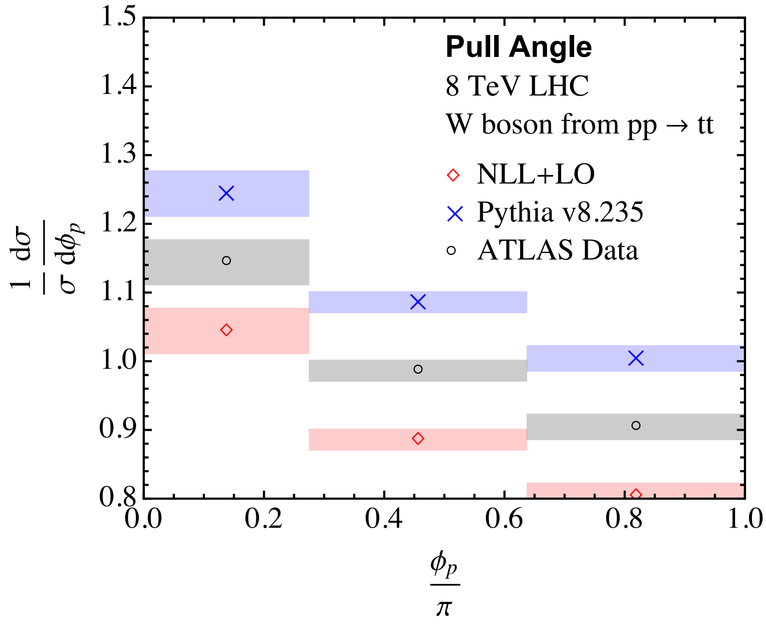

Figure 1: Comparison of the distribution of the pull angle ϕ_p from hadronically-decaying W bosons from top quark decay from our prediction ($\diamond$), Pythia v8.235 ($\times$), and 8 TeV ATLAS data ($\circ$) from Ref. [**?**]. Uncertainties in our prediction and the data are represented by the shaded bands. **(real ATLAS data, placeholder for now –ajl)**

about the jet axis, is not infrared and collinear (IRC) safe. The pull angle is the most sensitive aspect of the pull vector to color connections, and so is the observable that is typically the focus of such studies. Because it lacks the property of IRC safety, the pull angle cannot be calculated at fixed order in perturbation theory. Nevertheless, the pull angle is Sudakov safe [**? ?**], and can be calculated within resummed perturbation theory which regulates the soft and collinear divergences.

In this paper, we present a first calculation of the pull vector and associated observables. Because measurements have focused on pull in color-singlet decays, our predictions will be restricted to color-singlets as well. Nevertheless, our results generalize to other color dipole configurations. We resum the full two-dimensional pull vector to next-to-leading logarithmic (NLL) accuracy, and consider several one-dimensional projections of this distribution. Our primary results are the prediction for the pull angle, but we also consider observables motivated from analogies to observables widely used to study transverse momentum distributions in Drell-Yan. We also include effects from non-perturbative physics on the pull vector. This is especially important for predictions of the pull angle, as it is not IRC safe. Nevertheless, we are able to demonstrate that its Sudakov safety ensures that non-perturbative effects are small and decrease as an inverse power of jet energy.

By matching our resummed predictions to color-singlet decay matrix elements and including non-perturbative effects, we are able to produce a reliable prediction for pull and its derivative observables that can be compared to data. In Fig. 1, we plot the main result of this paper: a comparison between the pull angle ϕ_p of a jet from W boson decay from our prediction, Pythia Monte Carlo simulation [**? ?**], and ATLAS data. Excellent agreement

(true? –ajl) of the predictions to data is observed, suggesting that the physics of this process is well understood. Further improvements to the accuracy of the theoretical prediction and reduction of experimental uncertainties will enable more detailed comparisons that may provide information for improvement of color connection modeling in simulation.

The outline of this paper is as follows. In Sec. 2, we define the pull vector and related observables both as in previous literature and used in experiment, and the definition that we use in our resummed calculations. In Sec. 3, we present a lowest-order calculation of the pull vector for jets produced from color-singlet decays which forms the basis of calculations later in the paper. **(more here –ajl)** We conclude in Sec. 8. Details of some of the calculations presented in this paper are provided in appendices.

2 The Pull Observable

The original definition of the pull vector $\vec{t}$ from Ref. [**?**] was as a two-dimensional vector in the plane of rapidity y and azimuthal angle ϕ. The expression for the pull vector is

$$\vec{t}_{\text{original}} = \sum_{i\in J} \frac{p_{\perp i}|\vec{r}_i|}{p_{\perp J}}\vec{r}_i\,. \tag{2.1}$$

Here, i is a particle in the jet J of interest and $p_{\perp i}$ is its transverse momentum with respect to the collision beam axis. The vector $\vec{r}_i$ is the relative rapidity and azimuthal angle of the particle from the jet axis:

$$\vec{r}_i = (y_i - y_J, \phi_i - \phi_J)\,. \tag{2.2}$$

As a weighted sum of particle locations, the pull vector points from the jet axis in the direction of dominant energy flow. The pull vector is IRC safe because it is linear in particle energy and is weighted by a positive power of angle from the jet axis. In this form, the pull vector is expressed in coordinates natural at a hadron collider, and has been used for the measurements at D∅ and ATLAS, and for searches at CMS. However, this is not the most natural way to represent the pull vector for calculation.

For the resummed calculations in this paper, we use a modified version of the pull vector, which is identical to Eq. (2.1) for central jets in the collinear limit. The definition we use is

$$\vec{t}_{\text{modified}} = \sum_{i\in J} \frac{E_i \sin^2\theta_i}{E_J}(\cos\phi_i, \sin\phi_i)\,. \tag{2.3}$$

Here, E_i is the energy of particle i, θ_i is its angle from the jet axis, and ϕ_i is the azimuthal angle about the jet axis. ϕ_i is measured with respect to a fiducial jet direction. This form is much more amenable to calculations, and because the jet radii that we consider are typically relatively small ($R \simeq 0.4$), the collinear limit is a good approximation anyway. To correct for the difference between the original definition which is used in experiment and this modified definition, we could match our resummed calculations to fixed-order results which would account for the difference. We don't worry about this distinction in this paper as the uncertainties in our resummed results more than account for the differences between observable definitions.

Two important properties of the pull vector, in addition to its IRC safety, are the facts that it is additive and recoil-free. The pull vector is constructed by adding together contributions from individual particles in the jet. Additivity means that we can consider contributions to the pull vector from arbitrary soft and collinear emissions separately, and then add them together to determine the total pull vector in the singular limit. Further, recoil-free means that the jet direction does not conserve momentum of soft particles about it. This property dramatically simplifies our calculation because it means that we do not have to account for every soft particle emission on the jet direction. The recoil-free nature of the pull vector follows from its functional form in which emissions are weighted by the square of their angle to the jet axis. This is similar to the observable thrust and means that soft emissions that contribute at leading-power to pull necessarily have too small an energy to affect the jet direction.

We will present calculations for the double differential cross section of both components of the pull vector. From this distribution, any information can be extracted, but we will find it useful to identify some one-dimensional projections. Perhaps the most natural observable to extract from the pull vector is its magnitude, which we denote by $t = |\vec{t}|$. The pull vector magnitude can be expressed as

$$t^2 = \sum_{i\in J} \frac{E_i^2}{E_J^2} \sin^4\theta_i + 2 \sum_{i<j\in J} \frac{E_i E_j}{E_J^2} \sin^2\theta_i \sin^2\theta_j \cos(\phi_i - \phi_j)\,. \tag{2.4}$$

The individual components of the pull vector $\vec{t}$ are interesting observables, as well. For concreteness, the x-component of the pull vector t_x is

$$t_x = \sum_{i\in J} \frac{E_i \sin^2\theta_i}{E_J} \cos\phi_i\,, \tag{2.5}$$

which is an IRC safe observable. Individual components of the pull vector have not been explicitly measured before, but they are similar in form to observables like a_T and ϕ^* [**?** **?**] for transverse momentum in Drell-Yan. As a final one-dimensional projection of the pull vector, we consider the pull angle, ϕ_p, defined as

$$\cos\phi_p = \frac{t_x}{t} = \frac{\sum_{i\in J} \frac{E_i \sin^2\theta_i}{E_J} \cos\phi_i}{\sqrt{\sum_{i\in J} \frac{E_i^2}{E_J^2} \sin^4\theta_i + 2\sum_{i<j\in J} \frac{E_i E_j}{E_J^2} \sin^2\theta_i \sin^2\theta_j \cos(\phi_i - \phi_j)}}\,. \tag{2.6}$$

The pull angle ϕ_p measures the direction of dominant energy flow about the jet. Unlike the pull vector magnitude or its individual components, ϕ_p is not IRC safe and so cannot be calculated in fixed-order perturbation theory. However, it is only problematic when the magnitude of the pull vector $t \to 0$, which is the soft or collinear limit. Therefore, the pull angle is a Sudakov safe observable [**?** **?**], for which all divergences can be regulated by additionally measuring the pull vector magnitude. We will see how this works explicitly in the following sections.

3 Fixed-Order Calculation

With the definition of the pull vector $\vec{t}$, we now work to predict its distribution. In principle, we could calculation the pull vector of a jet in an event with an arbitrary number of other jets in the final state; however, in this paper, we focus exclusively on jets from color-singlet decays. The decay of a color singlet, like a W, Z or Higgs boson, produces two, color-connected, jets. Because DØ and ATLAS measured pull on jets from W boson decay, we assume that both jets are initiated by quarks. On one of these jets we measure the pull vector $\vec{t}$, with relative azimuthal angles defined with respect to the plane defined by the two jets from decay. We expect that radiation from the jets is concentrated mostly between them because color-singlets do not radiate at wide angles. This correspondingly means that we expect the distribution of the pull angle to peak when $\phi_p = 0$.

In this section, we calculate the leading-order double differential distribution of the pull vector in the limit in which the magnitude of the pull vector is small, $t \ll 1$. The calculation can be separated into soft and collinear contributions and then summed together to find the total result. We first consider the contribution to the pull vector from soft emissions.

3.1 Soft Limit

(Need to calculate the α_s^0 soft function from boosting the back-to-back configuration –ajl)

(A guess, perhaps wrong normalization: –ajl)

$$S^{(0)}(t,\phi_p) = \delta(t)\left[2\cot\phi_p\ \tan^{-1}\frac{\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\sin\phi_p}{1-\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p} - \log\left(1+\frac{\tan^2\frac{R}{2}}{\tan^2\frac{\theta_{12}}{2}} - 2\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p\right)\right]. \tag{3.1}$$

In $d = 4-2\epsilon$ dimensions, the one emission, soft matrix element for measuring the pull magnitude t and the pull angle ϕ_p is

$$S_{n_1n_2}(t,\phi_p) = g^2 C_F \mu^{2\epsilon}\int [d^dk]_+ \frac{2n_1\cdot n_2}{k^+(k\cdot n_2)}\Theta\left(\tan^2\frac{R}{2}-\frac{k^+}{k^-}\right)\delta\left(t-\frac{k^+k^-}{E_Jk^0}\right)\delta\left(\phi_p-\phi\right). \tag{3.2}$$

μ is the dimensional-regularization renormalization scale. The light-like vectors n_1 and n_2 specify the direction of the ends of the dipole and $n_1\cdot n_2 = 1-\cos\theta_{12}$, where θ_{12} is the angle between the two jets from color-singlet decay. We have centered a jet with radius R about the n_1 axis. Here, $[d^dk]_+$ is the phase space for an on-shell, positive energy particle in $d = 4-2\epsilon$ dimensions with momentum k:

$$\int [d^dk]_+ = \frac{1}{2^{5-2\epsilon}\pi^{5/2-\epsilon}\Gamma(1/2-\epsilon)}\int_0^\infty \frac{dk^+}{(k^+)^\epsilon}\int_0^\infty\frac{dk^-}{(k^-)^\epsilon}\int_0^{2\pi} d\phi\,\sin^{-2\epsilon}\phi\,. \tag{3.3}$$

Here, k^+ and k^- are momentum components measured with respect to the n_1 axis, with $k^+ = k^0(1-\cos\theta_{1k})$ and $k^- = k^0(1+\cos\theta_{1k})$, where θ_{1k} is the angle between axis 1 and momentum k. k^0 is the energy of the particle. Finally, the angle ϕ is the azimuthal angle of the particle about the n_1 axis; that is, it is the pull angle, with $\phi = 0$ in the plane defined by vectors n_1 and n_2.

To evaluate the soft function, we can expand the dot product $k\cdot n_2$ in terms of n_1:

$$\begin{aligned} k\cdot n_2 &= \frac{n_1\cdot n_2}{2}k^- + \frac{\bar{n}_1\cdot n_2}{2}k^+ - (n_1\cdot n_2)^{1/2}(\bar{n}_1\cdot n_2)^{1/2}(k^+k^-)^{1/2}\cos\phi \\ &= \frac{1-\cos\theta_{12}}{2}k^- + \frac{1+\cos\theta_{12}}{2}k^+ - \sin\theta_{12}(k^+k^-)^{1/2}\cos\phi\,. \end{aligned} \tag{3.4}$$

Then, all but one phase space integral can be done with the δ-functions. With an appropriate change of variables, the integral that remains can be written as

$$S_{n_1n_2}(t,\phi_p) = \frac{g^2C_F}{2\cdot 4^{1-2\epsilon}\pi^{5/2-\epsilon}\Gamma(1/2-\epsilon)}\frac{1}{t^{1+2\epsilon}}\left(\frac{\mu^2\tan^2\frac{R}{2}}{E_J^2\sin^2\phi_p}\right)^{\epsilon} \tag{3.5}$$
$$\times\int_0^1 du\, u^{-1+\epsilon}\left(1+u\tan^2\frac{R}{2}\right)^{-2\epsilon}\left(1+u\frac{\bar{n}_1\cdot n_2}{n_1\cdot n_2}\tan^2\frac{R}{2}-2u^{1/2}\left(\frac{\bar{n}_1\cdot n_2}{n_1\cdot n_2}\right)^{1/2}\tan\frac{R}{2}\cos\phi_p\right)^{-1}.$$

Note that

$$\frac{\bar{n}_1\cdot n_2}{n_1\cdot n_2} = \cot^2\frac{\theta_{12}}{2}\,. \tag{3.6}$$

This integral can be performed in a series in ϵ using the +-function expansion

$$u^{-1+\epsilon} = \frac{1}{\epsilon}\delta(u)+\left(\frac{1}{u}\right)_+ +\epsilon\left(\frac{\log u}{u}\right)_+ +\cdots\,. \tag{3.7}$$

Expanding to $1/\epsilon$ order (which is necessary for NLL resummation), we have

$$S_{n_1n_2}(t,\phi_p) = \frac{g^2C_F}{2\cdot 4^{1-2\epsilon}\pi^{5/2-\epsilon}\Gamma(1/2-\epsilon)}\frac{1}{t^{1+2\epsilon}}\left(\frac{\mu^2\tan^2\frac{R}{2}}{E_J^2\sin^2\phi_p}\right)^{\epsilon} \tag{3.8}$$
$$\times\left[\frac{1}{\epsilon}+\int_0^1 du\left(\frac{1}{u}\right)_+\left(1+u\cot^2\frac{\theta_{12}}{2}\tan^2\frac{R}{2}-2u^{1/2}\cot\frac{\theta_{12}}{2}\tan\frac{R}{2}\cos\phi_p\right)^{-1}\right].$$

The integral over u can be done and we find

$$S_{n_1n_2}(t,\phi_p) = \frac{g^2C_F}{2\cdot 4^{1-2\epsilon}\pi^{5/2-\epsilon}\Gamma(1/2-\epsilon)}\frac{1}{t^{1+2\epsilon}}\left(\frac{\mu^2\tan^2\frac{R}{2}}{E_J^2\sin^2\phi_p}\right)^{\epsilon}\left[\frac{1}{\epsilon}\right. \tag{3.9}$$
$$\left.+2\cot\phi_p\,\tan^{-1}\frac{\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\sin\phi_p}{1-\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p}-\log\left(1+\frac{\tan^2\frac{R}{2}}{\tan^2\frac{\theta_{12}}{2}}-2\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p\right)\right].$$

Expanding in ϵ and working in the $\overline{\mathrm{MS}}$ scheme, the renormalized soft function necessary for NLL resummation is

$$S^{\mathrm{ren,NLL}}_{n_1n_2}(t,\phi_p) = \frac{\alpha_sC_F}{2\pi^2}\left[-\frac{1}{4}\delta(t)\log^2\frac{\mu^2\tan^2\frac{R}{2}}{E_J^2\sin^2\phi_p}+\left(\frac{1}{t}\right)_+\log\frac{\mu^2\tan^2\frac{R}{2}}{E_J^2\sin^2\phi_p}-2\left(\frac{\log t}{t}\right)_+\right.$$
$$-\frac{1}{2}\delta(t)\left(2\cot\phi_p\,\tan^{-1}\frac{\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\sin\phi_p}{1-\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p}-\log\left(1+\frac{\tan^2\frac{R}{2}}{\tan^2\frac{\theta_{12}}{2}}-2\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p\right)\right)\log\frac{\mu^2\tan^2\frac{R}{2}}{E_J^2\sin^2\phi_p}$$
$$\left.+\left(\frac{1}{t}\right)_+\left(2\cot\phi_p\,\tan^{-1}\frac{\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\sin\phi_p}{1-\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p}-\log\left(1+\frac{\tan^2\frac{R}{2}}{\tan^2\frac{\theta_{12}}{2}}-2\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p\right)\right)\right]. \tag{3.10}$$

Therefore, the soft emission cross section for pull magnitude $t > 0$ is

$$\frac{d^2\sigma^{\rm soft}}{dt\,d\phi_p} = \frac{\alpha_s C_F}{2\pi^2}\left[\frac{2}{t}\log\frac{\mu\tan\frac{R}{2}}{tE_J\sin\phi_p}\right.$$
$$\left.+\frac{1}{t}\left(2\cot\phi_p\ \tan^{-1}\frac{\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\sin\phi_p}{1-\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p} - \log\left(1+\frac{\tan^2\frac{R}{2}}{\tan^2\frac{\theta_{12}}{2}} - 2\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p\right)\right)\right]. \tag{3.11}$$

Integrating over ϕ_p, we find the soft contribution to the differential cross section of the magnitude of pull, t:

$$\frac{d\sigma^{\rm soft}}{dt} = \frac{\alpha_s C_F}{\pi}\left[\frac{2}{t}\log\frac{2\mu\tan\frac{R}{2}}{tE_J} - \log\left(1-\frac{\tan^2\frac{R}{2}}{\tan^2\frac{\theta_{12}}{2}}\right)\right]. \tag{3.12}$$

3.2 Collinear Limit

The $\mathcal{O}(\alpha_s^0)$ jet function is

$$J^{(0)}(t,\phi_p) = \frac{1}{2\pi}\delta(t)\,, \tag{3.13}$$

because the distribution of the pull angle ϕ_p is uniform on $\phi_p \in [0, 2\pi)$.

We can perform the same calculation but in the collinear limit. In this case, the collinear splitting matrix element for a quark in dimensional regularization on which the pull vector is measured is:

$$J_q(t,\phi_p) = \frac{\alpha_s C_F}{4\pi}\frac{(4\pi)^\epsilon}{\pi^{1/2}\Gamma(1/2-\epsilon)}\left(\frac{\mu^2}{E_J^2}\right)^\epsilon\int_0^1 dz\int_0^\infty d\theta^2\int_0^{2\pi}d\phi\,\sin^{-2\epsilon}\phi \tag{3.14}$$
$$\times(\theta^2)^{-1-\epsilon}z^{-2\epsilon}(1-z)^{-2\epsilon}\left(\frac{1+(1-z)^2}{z}-\epsilon z\right)\delta(t-z(1-z)|1-2z|\theta^2)\delta(\phi_p-\phi)$$
$$= \frac{\alpha_s C_F}{4\pi}\frac{(4\pi)^\epsilon}{\pi^{1/2}\Gamma(1/2-\epsilon)}\frac{1}{t^{1+\epsilon}}\left(\frac{\mu^2}{E_J^2\sin^2\phi_p}\right)^\epsilon\int_0^1 dz\,z^{-1-\epsilon}(1-z)^{-\epsilon}|1-2z|^\epsilon\left(1+(1-z)^2-\epsilon z^2\right).$$

The remaining integral can be done expanded in ϵ using the +-function expansion. We find

$$\int_0^1 dz\,z^{-1-\epsilon}(1-z)^{-\epsilon}|1-2z|^\epsilon\left(1+(1-z)^2-\epsilon z^2\right) = -\frac{2}{\epsilon}-\frac{3}{2}. \tag{3.15}$$

In the limit where $z \to 0$, this integral is scaleless, and so has no overlap with the soft function calculation. The renormalized jet function to NLL accuracy is therefore

$$J_q^{\rm ren,NLL}(t,\phi_p) = \frac{\alpha_s C_F}{4\pi^2}\left[\delta(t)\log^2\frac{\mu^2}{4E_J^2\sin^2\phi_p} + \frac{3}{2}\delta(t)\log\frac{\mu^2}{E_J^2\sin^2\phi_p} - 2\log\frac{\mu^2}{4E_J^2\sin^2\phi_p}\left(\frac{1}{t}\right)_+\right.$$
$$\left.-\frac{3}{2}\left(\frac{1}{t}\right)_+ + 2\left(\frac{\log t}{t}\right)_+\right]. \tag{3.16}$$

The cross section in the hard collinear limit for pull magnitude $t > 0$ is:

$$\frac{d^2\sigma^{\rm coll}}{dt\,d\phi_p} = \frac{\alpha_s}{4\pi^2}C_F\left[\frac{2}{t}\log\frac{4tE_J^2\sin^2\phi_p}{\mu^2} - \frac{3}{2}\frac{1}{t}\right]. \tag{3.17}$$

By integrating over ϕ_p, we find the collinear contribution to the cross section of the magnitude of pull, t:

$$\frac{d\sigma^{\rm coll}}{dt} = \frac{\alpha_s}{\pi}\frac{C_F}{t}\left[\log\frac{tE_J^2}{\mu^2} - \frac{3}{4}\right]. \tag{3.18}$$

3.3 Total contribution

Summing the soft and collinear contributions, we find the double differential cross section to be

$$\frac{d^2\sigma^{\rm soft}}{dt\,d\phi_p}+\frac{d^2\sigma^{\rm coll}}{dt\,d\phi_p}=\frac{\alpha_s}{2\pi^2}\frac{C_F}{t}\left[\log\frac{4\tan^2\frac{R}{2}}{t}-\frac{3}{4}\right.\tag{3.19}$$
$$\left.+2\cot\phi_p\ \tan^{-1}\frac{\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\sin\phi_p}{1-\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p}-\log\left(1+\frac{\tan^2\frac{R}{2}}{\tan^2\frac{\theta_{12}}{2}}-2\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p\right)\right].$$

The cross section for the pull magnitude t is just

$$\frac{d\sigma}{dt}=\frac{\alpha_s}{\pi}\frac{C_F}{t}\left[\log\frac{1}{t}-\frac{3}{4}-\log\left(\frac{1-\frac{\tan^2\frac{R}{2}}{\tan^2\frac{\theta_{12}}{2}}}{4\tan^2\frac{R}{2}}\right)\right].\tag{3.20}$$

Both of these expressions are independent of the renormalization scale μ as they must be as physical cross sections. Additionally, Eq. (3.19) manifests the IRC unsafety of the pull angle ϕ_p. The integral of Eq. (3.19) over the pull vector magnitude t is infinite due to the divergence as $t \to 0$, and so the pull angle is not tractable in fixed-order perturbation theory. Nevertheless, the divergence in t can be tamed by all-orders resummation, rendering the pull angle distribution finite. We perform this resummation in the following section.

4 Resummation of the Pull Vector

In the limit in which the magnitude of the pull vector is small, $t \ll 1$, the cross section factorizes into contributions from soft and collinear emissions separately, as shown above. In this limit, the cross section differential in the two components of the pull vector $\vec{t} = (t_x, t_y)$, therefore assumes the form

$$\frac{d^2\sigma}{d\vec{t}}=H\int d^2t_c\,d^2t_s\,J(\vec{t}_c)S(\vec{t}_s)\,\delta^{(2)}(\vec{t}-\vec{t}_c-\vec{t}_s)\,.\tag{4.1}$$

The lowest-order jet and soft functions $J(\vec{t}_c)$ and $S(\vec{t}_s)$ were calculated in the previous section. The additivity property of the pull vector is responsible for the convolution structure of the cross section. While this expression is indicative of a factorization theorem, we do not claim to have derived such a factorization theorem that resums all infrared logarithms. First, the "hard function", which we denote as H, consists of multiple scales, including the originating color singlet energy and mass, and the jet radius. Further, because pull is explicitly only sensitive to radiation in a restricted region of phase space, there will in general exist non-global logarithms [**?**] of the pull vector $\vec{t}$, which are sensitive to some combination of these out-of-jet scales. Nevertheless, the expression of Eq. (4.1) describes the global logarithms of the pull vector, which is sufficient for our purposes here for a first calculation of this observable and for the accuracy to which we work.

Anomalous dimension equation:

$$\mu\frac{\partial}{\partial\mu}F(\vec{t})=\int d^2t'\,\gamma_F(\vec{t}')\,F(\vec{t}-\vec{t}')\,.\tag{4.2}$$

In polar coordinates:

$$\mu\frac{\partial}{\partial\mu}F(t,\phi_p) = \int dt'\,d\phi'\gamma_F(t',\phi')\,F\left(\sqrt{t^2+t'^2-2tt'\cos(\phi_p-\phi')},\cos^{-1}\frac{t\cos\phi_p+t'\cos\phi'}{\sqrt{t^2+t'^2-2tt'\cos(\phi_p-\phi')}}\right). \tag{4.3}$$

(Bounds of integration in polar coordinates: are they constrained by demanding that the distribution is non-negative? –ajl)

(HERE –ajl)

To continue, we re-express the δ-function present in Eq. (4.1) as a complex exponential:

$$\delta^{(2)}(\vec{t}-\vec{t}_c-\vec{t}_s) = \frac{1}{4\pi^2}\int d^2b\, e^{-i\vec{b}\cdot(\vec{t}-\vec{t}_c-\vec{t}_s)} \tag{4.4}$$

Then, the double differential cross section can be expressed as

$$\frac{d^2\sigma}{d\vec{t}} = \frac{1}{4\pi^2}H\int d^2b\,d^2t_c\,d^2t_s\,J(\vec{t}_c)S(\vec{t}_s)\,e^{-i\vec{b}\cdot(\vec{t}-\vec{t}_c-\vec{t}_s)}\,. \tag{4.5}$$

The integrals over the collinear and soft contributions to the pull vector $\vec{t}_c$ and $\vec{t}_s$ can be done by introducing the Fourier transforms of the jet and soft functions:

$$\frac{d^2\sigma}{d\vec{t}} = \frac{1}{4\pi^2}H\int d^2b\,\tilde{J}(\vec{b})\tilde{S}(\vec{b})\,e^{-i\vec{b}\cdot\vec{t}}. \tag{4.6}$$

The Fourier transform of a function $F(\vec{t})$ is defined to be

$$\tilde{F}(\vec{b}) = \int d^2t\,F(\vec{t})\,e^{i\vec{b}\cdot\vec{t}}. \tag{4.7}$$

Next, we express the integrand in polar coordinates which enables direct contact with the magnitude of the pull vector. We have

$$\frac{1}{t}\frac{d^2\sigma}{dt\,d\phi_p} = \frac{1}{4\pi^2}H\int db\,d\phi_b\,b\,\tilde{J}(b,\phi_b)\tilde{S}(b,\phi_b)\,e^{-ibt\cos(\phi_b-\phi_p)}\,. \tag{4.8}$$

In writing this expression, we have denoted the magnitude and azimuthal angle of the $\vec{b}$ vector as b and ϕ_b, respectively. The factor of $1/t$ in the new double differential cross section on the left is the Jacobian from changing variables to the pull magnitude and angle, t and ϕ_p.

To go further, we need to calculate the Fourier-transformed jet and soft functions, $\tilde{J}(b,\phi_b)$ and $\tilde{S}(b,\phi_b)$. Because the pull vector is additive, in Fourier space, the cross section is just a product:

$$\frac{d^2\sigma}{db\,d\phi_b} = \frac{1}{4\pi^2}H\tilde{J}(b,\phi_b)\tilde{S}(b,\phi_b)\,. \tag{4.9}$$

The separation of the hard, jet, and soft functions is defined by the renormalization scale μ, which is arbitrary and so must not be present in the total cross section. Demanding that the cross section be independent of renormalization scale μ then defines a set of renormalization group equations and corresponding anomalous dimensions of these functions. The jet and soft functions satisfy the equations

$$\mu\frac{\partial}{\partial\mu}\tilde{J}(b,\phi_b) = \gamma_J\tilde{J}(b,\phi_b)\,, \qquad \mu\frac{\partial}{\partial\mu}\tilde{S}(b,\phi_b) = \gamma_S\tilde{S}(b,\phi_b)\,. \tag{4.10}$$

Resummation is accomplished by integrating these differential equations. Next-to-leading logarithmic resummation of these functions is accomplished by calculation of the anomalous dimensions γ_J and γ_S at one-loop accuracy. To determine these anomalous dimensions, we only need to Fourier-transform the jet and soft function results from the previous section.

The Fourier-transformed soft function, for instance, is defined as

$$\tilde{S}(b,\phi_b) = \int dt\, d\phi_p\, S(t,\phi_p) e^{ibt\cos(\phi_b-\phi_p)} \,. \tag{4.11}$$

Differentiating with respect to μ, we find

$$\mu\frac{\partial}{\partial\mu}\tilde{S}(b,\phi_b) = \mu\frac{\partial}{\partial\mu}\int dt\, d\phi_p\, S(t,\phi_p) e^{ibt\cos(\phi_b-\phi_p)} = \int dt\, d\phi_p \left[\mu\frac{\partial}{\partial\mu}S(t,\phi_p)\right] e^{ibt\cos(\phi_b-\phi_p)} \,, \tag{4.12}$$

so we can just directly differentiate the expression for the soft function in Eq. (3.10). The result is

$$\mu\frac{\partial}{\partial\mu}S^{\text{ren,NLL}}_{n_1n_2}(t,\phi_p) = \frac{\alpha_s C_F}{2\pi^2}\left[-\delta(t)\log\frac{\mu^2\tan^2\frac{R}{2}}{E_J^2\sin^2\phi_p} + 2\left(\frac{1}{t}\right)_+ \right. \tag{4.13}$$
$$\left. -\delta(t)\left(2\cot\phi_p\ \tan^{-1}\frac{\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\sin\phi_p}{1-\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p} - \log\left(1+\frac{\tan^2\frac{R}{2}}{\tan^2\frac{\theta_{12}}{2}} - 2\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p\right)\right)\right] .$$

Inserting this expression into Eq. (4.12), we find the expression for the one-loop b-space anomalous dimensions:

$$\gamma_S = -\frac{\alpha_s C_F}{\pi}\log\frac{\mu^2 b^2\tan^2\frac{R}{2}}{E_J^2} - 2\gamma_E\frac{\alpha_s C_F}{\pi} + \frac{\alpha_s C_F}{\pi}\log\left(1-\frac{\tan^2\frac{R}{2}}{\tan^2\frac{\theta_{12}}{2}}\right) . \tag{4.14}$$

γ_E is the Euler-Mascheroni constant. Note that this anomalous dimension is independent of the azimuthal angle in b-space, ϕ_b.

(here –ajl)

One could just calculate the double differential cross section of Eq. (4.1) and then perform any marginalization to project out one-dimensional cross sections of any function of the components of $\vec{t}$. However, it is useful and enlightening to directly manipulate this expression to derive the form of the cross section for the three projections we identified earlier: the pull magnitude t, one component of the pull vector t_x, and the pull angle ϕ_p. We consider these three observables in turn.

4.1 The Pull Magnitude

To calculate the pull vector magnitude, w

Outline:

1. Define pull in Z decays in e+e-
2. double differential cross section (jet and soft functions)
3. resummed pull magnitude

4. comparison to EVENT2
5. Discuss caveats and simplifications: NGLs, recoil, odd phase space configurations
6. Non-perturbative corrections
7. Sudakov safe calculation/MLL calculation and plots. Need to convolve with Z decay matrix element.
8. Plots from pythia

5 Subjet Angle Distribution

This distribution must be convolved with the distribution of the angle between the subjets from the decay. This distribution can be found assuming that the decay is isotropic in the rest frame. In the rest frame, if the four-vectors of the decay products are

$$p_1 = \left(\frac{m}{2}, 0, 0, \frac{m}{2}\right)\,, \qquad\qquad p_1 = \left(\frac{m}{2}, 0, 0, -\frac{m}{2}\right)\,, \tag{5.1}$$

then, in the boosted frame the energies of the decay products are

$$E_1 \to \frac{\gamma m}{2}(1+\beta\cos\theta)\,, \qquad\qquad E_2 \to \frac{\gamma m}{2}(1-\beta\cos\theta)\,. \tag{5.2}$$

Note that the energy of the decaying particle in the boosted frame, $E = \gamma m$, and therefore the velocity of the boost is

$$\beta = \sqrt{1-\frac{m^2}{E^2}}\,. \tag{5.3}$$

The angle θ is the boost angle. The angle between the decay products can be found by demanding that they reproduce the heavy particle's invariant mass:

$$m^2 = 2E_1E_2(1-\cos\theta_{12}) \qquad \Longrightarrow \qquad \cos\theta_{12} = 1 - \frac{2m^2}{E^2}\frac{1}{1-\beta^2\cos\theta}\,. \tag{5.4}$$

The distribution of the angle between the decay products is then

$$\begin{aligned} p(\cos\theta_{12}) &= \int_{-1}^{1}\frac{d\cos\theta}{2}\,\delta\left(\cos\theta_{12} - 1 + \frac{2m^2}{E^2}\frac{1}{1-\beta^2\cos^2\theta}\right) \\ &= \frac{m^2}{E^2}\frac{1}{\sqrt{1-\frac{m^2}{E^2}}}\frac{1}{\sqrt{1-\frac{2m^2}{E^2}-\cos\theta_{12}}}\frac{1}{(1-\cos\theta_{12})^{3/2}}\Theta\left(1-\frac{2m^2}{E^2}-\cos\theta_{12}\right)\,. \end{aligned} \tag{5.5}$$

6 One-dimensional projections of the double-differential distribution

The structure of the resummed result for the pull vector distribution is

$$\frac{d^2\sigma}{d\vec{t}^2} = H\int\frac{d^2\vec{b}}{(2\pi)^2}e^{i\vec{b}\cdot\vec{t}}J(\vec{b})S(\vec{b}) = \int\frac{d^2\vec{b}}{(2\pi)^2}e^{i\vec{b}\cdot\vec{t}}e^{R(b)} + \ldots\,, \tag{6.1}$$

where the dots indicate terms beyond NLL accuracy considered here. We can derive different one-dimensional projection of the above double-differential distribution:

- The pull magnitude is the most natural one-dimensional projection. From a theoretical point of you it has the best properties: it is defined order by order in perturbation theory and the resummation of large logarithms at small t has a very similar structure of well-known transverse momentum (or broadening) resummation. To NLL accuracy, we find

$$\frac{d\sigma}{dt^2} = \int_0^\infty db \frac{b}{2} J_0(bt) e^{R(b)}. \tag{6.2}$$

However, if we consider just the magnitude we loose important information about the colour connection of the event because we are integrating over all possible direction of soft-wide angle emissions. For this reason, the pull angle was first introduced.

- The pull angle is the angle made by the pull vector with the jet-jet (or subjet-subjet) direction. Therefore, it can directly probe colour correlation in the event. However, it is not an IRC safe observable essentially because it is ill-defined on Born configuration. However, we argue it is Sudakov safe:

$$\frac{d\sigma}{d\phi_p} = \frac{1}{2\pi} \int_0^\infty dt\, t \int_0^\infty db\, b J_0(bt) e^{R(b)} = \frac{1}{2\pi} \int_0^\infty db\, b e^{R(b)}. \tag{6.3}$$

(Is the above right or at least consistent with the conditional probability approach? –sm)

- By comparing pros and cons of the two projections above, we suggest to consider the pull component along the jet-jet (or subjet-subjet) direction [1], t_x for definiteness. Indeed, this observable maintains the information measured by the pull angle, while being IRC safe and therefore calculable order-by-order in perturbation theory. We find that its NLL resummed expression is

$$\frac{d\sigma}{dt_x} = \frac{1}{\pi} \int_0^\infty db \cos(bt_x) e^{R(b)}. \tag{6.4}$$

7 Non-Perturbative Corrections

To estimate non-perturbative effects, we can exploit a few properties of the pull vector. First, the pull vector is an additive observable, and so it just factorizes into a sum of perturbative and non-perturbative contributions:

$$\vec{t} = \vec{t}_{\text{pert}} + \vec{t}_{\text{non-pert}}\,. \tag{7.1}$$

The perturbative distribution we have already calculated, and we would like to include the non-perturbative contribution to the pull angle and demonstrate that non-perturbative effects are suppressed by powers of the QCD scale. To do this, call the double differential perturbative and non-perturbative cross sections

$$\frac{d^2\sigma^{\text{pert}}}{dt\, d\phi_p}\,, \qquad \text{and} \qquad \frac{d^2\sigma^{\text{non-pert}}}{dt\, d\phi_p}\,. \tag{7.2}$$

[1] In reaching this conclusion, we were very much influenced by the a_T / ϕ^* [? ?] variables which had been introduced in the context of transverse momentum distributions.

The pull angle ϕ_p can be defined from the components of the pull vector:

$$\phi_p = \cos^{-1}\frac{t_x}{t}\,, \tag{7.3}$$

and this definition allows for the perturbative and non-perturbative contributions to be convolved to determine the total distribution. That is, the distribution of the pull angle ϕ_p, including perturbative and non-perturbative effects is

$$\frac{d\sigma}{d\phi_p} = \int dt \int d\phi \int dt' \int d\phi' \int dt_x \int dt_y\, \frac{d^2\sigma^{\text{pert}}}{dt\,d\phi}\,\frac{d^2\sigma^{\text{non-pert}}}{dt'\,d\phi'}\,\delta\left(\phi_p - \cos^{-1}\frac{t_x}{\sqrt{t_x^2+t_y^2}}\right)$$
$$\times\,\delta\left(t_x - t\cos\phi - t'\cos\phi'\right)\delta\left(t_y - t\sin\phi - t'\sin\phi'\right)\,. \tag{7.4}$$

To actually predict this distribution, we would need to know the non-perturbative distribution of the pull vector, which is defined by am unknown non-perturbative matrix element. However, we can make progress by making some approximations. First, we will assume that the non-perturbative pull magnitude and pull angle are uncorrelated. Further, we will assume that the distribution of the non-perturbative pull angle is uniform on $\phi_p \in [0, 2\pi]$. This distribution may not actually be uniform, but there is no divergence associated with the pull angle, so the distribution is uniform up to order-1 corrections. With these assumptions, we can write

$$\frac{d^2\sigma^{\text{non-pert}}}{dt\,d\phi_p} = \frac{1}{2\pi}\frac{d\sigma^{\text{non-pert}}}{dt}\,. \tag{7.5}$$

Further, because the angular dependence of the magnitude of the pull vector t is quadratic, like thrust, non-perturbative emissions at wide angle dominate. The largest possible emission angle is just the jet radius R, so the non-perturbative pull magnitude is approximately

$$t_{\text{non-pert}} \simeq \frac{\epsilon}{E_J}\sin^2 R\,, \tag{7.6}$$

where ϵ is the non-perturbative energy scale. With this expression, we can write the non-perturbative pull distribution as

$$\frac{d^2\sigma^{\text{non-pert}}}{dt\,d\phi_p} = \frac{1}{2\pi}\int d\epsilon\, F(\epsilon)\delta\left(t - \frac{\epsilon}{E_J}\sin^2 R\right)\,. \tag{7.7}$$

Here, $F(\epsilon)$ is a shape function [**? ? ? ? ?**] which is a non-perturbative distribution peaked around the QCD scale, Λ_{QCD}.

One could use a model shape function, but as our goal here is just an estimation of non-perturbative effects, we will just use a δ-function form:

$$F(\epsilon) = \delta(\epsilon - \Lambda)\,, \tag{7.8}$$

for some energy scale $\Lambda \sim 1$ GeV. With this assumption, the non-perturbative distribution is

$$\frac{d^2\sigma^{\text{non-pert}}}{dt\,d\phi_p} = \frac{1}{2\pi}\delta\left(t - \frac{\Lambda}{E_J}\sin^2 R\right)\,. \tag{7.9}$$

Plugging this into the expression for the pull angle, we then find

$$\frac{d\sigma}{d\phi_p} = \int dt \int d\phi \int \frac{d\phi'}{2\pi} \int dt_x \int dt_y \, \frac{d^2\sigma^{\text{pert}}}{dt\, d\phi}\, \delta\left(\phi_p - \cos^{-1} \frac{t_x}{\sqrt{t_x^2 + t_y^2}}\right) \tag{7.10}$$
$$\times\, \delta\left(t_x - t\cos\phi - \frac{\Lambda}{E_J}\sin^2 R\cos\phi'\right)\delta\left(t_y - t\sin\phi - \frac{\Lambda}{E_J}\sin^2 R\sin\phi'\right).$$

The δ-functions of the components of the total pull vector can also be integrated over. Doing this, we find

$$\frac{d\sigma}{d\phi_p} = \int dt \int_0^{2\pi} d\phi \int_0^{2\pi} \frac{d\phi'}{2\pi}\, \frac{d^2\sigma^{\text{pert}}}{dt\, d\phi}\, \delta\left(\phi_p - \cos^{-1} \frac{t\cos\phi + \frac{\Lambda}{E_J}\sin^2 R\cos\phi'}{\sqrt{t^2 + \frac{\Lambda^2}{E_J^2}\sin^4 R + 2t\frac{\Lambda}{E_J}\sin^2 R\cos(\phi - \phi')}}\right). \tag{7.11}$$

The integral over the non-perturbative pull angle ϕ' can be done using the δ-function, but the result isn't enlightening. At any rate, the explicit perturbative distribution can be inserted and the integrals that remain can be done by Monte Carlo.

8 Conclusions

Acknowledgments

We thank **(who? –ajl)**

A b-Space Anomalous Dimensions

In this appendix, we present the b-space anomalous dimensions of the jet and soft functions calculated in Sec. **??**. For a function $F(t, \phi_p)$ its Fourier transform is

$$\tilde{F}(b, \phi_b) = \int_0^\infty dt \int_0^{2\pi} d\phi_p \, F(t, \phi_p) e^{-itb\cos(\phi_p - \phi_b)}\,, \tag{A.1}$$

where b is the magnitude in b-space and ϕ_b is its azimuthal angle. The leading-order expression for the jet or soft functions is just a δ-function:

$$F^{(0)}(t, \phi_p) = \frac{1}{2\pi}\delta(t) \tag{A.2}$$

In b-space, the leading-order functions are simply 1. For the next-to-leading order functions, we need to divide by a factor of 2 to account for the pull angle ϕ_p ranging over $\phi_p \in [0, 2\pi)$ and not just $\phi_p \in [0, \pi]$.

The derivatives with respect to μ for the soft and jet functions from Eqs. (3.10) and (3.16) are

$$\mu\frac{d}{d\mu}\frac{S^{\text{ren,NLL}}_{n_1 n_2}(t,\phi_p)}{2} = -\frac{\alpha_s C_F}{2\pi^2}\left[\delta(t)\log\frac{\mu^2\tan^2\frac{R}{2}}{E_J^2\sin^2\phi_p} - 2\left(\frac{1}{t}\right)_+ \right. \tag{A.3}$$
$$\left. +\delta(t)\left(2\cot\phi_p\ \tan^{-1}\frac{\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\sin\phi_p}{1 - \frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p} - \log\left(1 + \frac{\tan^2\frac{R}{2}}{\tan^2\frac{\theta_{12}}{2}} - 2\frac{\tan\frac{R}{2}}{\tan\frac{\theta_{12}}{2}}\cos\phi_p\right)\right)\right]$$

$$\mu\frac{d}{d\mu}\frac{J^{\text{ren,NLL}}(t,\phi_p)}{2} = \frac{\alpha_s C_F}{\pi^2}\left[\delta(t)\log\frac{\mu^2}{4E_J^2\sin^2\phi_p} + \frac{3}{4}\delta(t) - \left(\frac{1}{t}\right)_+\right]. \tag{A.4}$$

Performing the Fourier transformations, the b-space anomalous dimensions are

$$\mu\frac{d}{d\mu}\tilde{S}^{\text{ren,NLL}}_{n_1 n_2}(b,\phi_b) = -\frac{\alpha_s C_F}{\pi}\left[\log\frac{\mu^2 b^2 \tan^2\frac{R}{2}}{E_J^2} + 2\gamma_E - \log\left(1 - \frac{\tan^2\frac{R}{2}}{\tan^2\frac{\theta_{12}}{2}}\right)\right], \tag{A.5}$$

$$\mu\frac{d}{d\mu}\tilde{J}^{\text{ren,NLL}}(b,\phi_b) = 2\frac{\alpha_s C_F}{\pi}\left[\log\frac{\mu^2 b}{2E_J^2} + \frac{3}{4} + \gamma_E\right]. \tag{A.6}$$