SOLVING THE INTEGRAL IN EQ. 2.7: PARTITIONING UNITY

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We want to solve the integral from Eq. 2.7 of 2006.14680:

$$I = \int_0^1 dx_1 \int_0^1 dx_2\, \Theta(x_1 + x_2 - 1) \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \delta\left(\rho - \frac{4(1 - \max\{x_i\})}{(2 - \max\{x_i\})^2}\right) \Theta\left(\frac{\min\{x_i\}}{2 - \max\{x_i\}} - z\right), \tag{1}$$

where x_1, x_2, x_3 are phase space variables satisfying

$$x_1 + x_2 + x_3 = 2. \tag{2}$$

A first step might be to partition unity as

$$\begin{aligned} I = \int_0^1 dx_1 \int_0^1 dx_2\, &\Theta(x_1 + x_2 - 1) \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \delta\left(\rho - \frac{4(1 - \max\{x_i\})}{(2 - \max\{x_i\})^2}\right) \\ &\times \Theta\left(\frac{\min\{x_i\}}{2 - \max\{x_i\}} - z\right) \big[\Theta(x_1 - x_2)\Theta(x_2 - x_3) + \Theta(x_1 - x_3)\Theta(x_3 - x_2) \\ &+ \Theta(x_2 - x_1)\Theta(x_1 - x_3) + \Theta(x_2 - x_3)\Theta(x_3 - x_1) \\ &+ \Theta(x_3 - x_1)\Theta(x_1 - x_2) + \Theta(x_3 - x_2)\Theta(x_2 - x_1)\big], \end{aligned} \tag{3}$$

where we run over all the possible permutations of $\{x_1, x_2, x_3\}$. For reference, define the integrals corresponding to these permutations as

$$I \equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \tag{4}$$

Since x_1 and x_2 are symmetric in the integrand (after the determination of the maximum and minimum), we see that

$$I_1 = I_3 \qquad I_2 = I_4 \qquad I_5 = I_6. \tag{5}$$

Focusing for now on I_1, after applying the Heaviside functions $\Theta(x_1 - x_2)\Theta(x_2 - x_3)$ we have

$$I_1 = \int_0^1 dx_1 \int_0^1 dx_2\, \Theta(x_1 + x_2 - 1) \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \delta\left(\rho - \frac{4(1-x_1)}{(2-x_1)^2}\right) \Theta\left(\frac{2 - x_1 - x_2}{2 - x_1} - z\right). \tag{6}$$

Considering the argument of the Dirac delta to be a function of x_1

$$f(x_1) = \rho - \frac{4(1-x_1)}{(2-x_1)^2}, \tag{7}$$

its roots are

$$r_1, r_2 = 2 + \frac{2\left(-1 \pm \sqrt{1-\rho}\right)}{\rho}, \tag{8}$$

so

$$\delta\left(\rho - \frac{4(1-x_1)}{(2-x_1)^2}\right) = \frac{\delta(x_1 - r_1)}{|f'(r_1)|} + \frac{\delta(x_1 - r_2)}{|f'(r_2)|}. \tag{9}$$

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Since $0 < r_1 < 1$ (for $0 < \rho < 1$) but over this same range $0 < r_2$, only the r_1 term will contribute to the integral. Thus,

$$I_1 = \int_0^1 dx_1 \int_0^1 dx_2\, \Theta(x_1 + x_2 - 1) \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} \frac{\delta(x_1 - r_1)}{|f'(r_1)|} \Theta\left(\frac{2 - x_1 - x_2}{2 - x_1} - z\right) \tag{10}$$

$$= \frac{1}{|f'(r_1)|} \int_0^1 dx_2\, \Theta(r_1 + x_2 - 1) \frac{r_1^2 + x_2^2}{(1 - r_1)(1 - r_2)} \Theta\left(\frac{2 - r_1 - x_2}{2 - r_1} - z\right). \tag{11}$$

The Heaviside functions assert that

$$x_2 > 1 - r_1 \qquad\qquad x_2 < (2 - r_1)(1 - z). \tag{12}$$

Over the range $0 < \rho < 1$, $1 - r_1 > 0$, so we can reset the lower bound of integration to this value. However, $(2 - r_1)(1 - z) < 1$ only if $z > 1/2$ or

$$z < \frac{1}{2} \text{ and } \rho < 4(z - z^2). \tag{13}$$

We seem to only care about the regime where z is much less than $1/2$ (and perhaps this is required kinematically. . . I haven't thought it through very deeply), so we'll focus on this for now. Thus, we have reduced the integral to

$$\begin{aligned} I_1 = \Theta\left(\frac{1}{2} - z\right) \Theta\big(4(z - z^2) - \rho\big) \int_{1-r_1}^{(2-r_1)(1-z)} dx_2\, \frac{r_1^2 + x_2^2}{(1 - r_1)(1 - x_2)} \\ + \Theta\big(\rho - 4(z - z^2)\big) \int_{1-r_1}^{1} dx_2\, \frac{r_1^2 + x_2^2}{(1 - r_1)(1 - x_2)}. \end{aligned} \tag{14}$$

There are two problems here:

(1) The second integral in Eq. 14 does not converge
(2) The split point here is around $4(z - z^2)$, which is different than the point you derived (which was $2z - z^2$). At first I thought that maybe this integral cancels with another, and yet another gives the $2z - z^2$ split, but having run through each (unique) case the $4(z - z^2)$ split seems fairly robust.

This makes me think I've made a wrong step somewhere, but after quite a bit of time spent and ink used, I can't see where I've gone wrong. **Do you see where my reasoning has gone awry, or otherwise have advice on where to look/this strategy in general?**