LIMITING EQ. 2.7 WITH $x_3 \ll 1$

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We want to take the $x_3 \ll 1$ limit of the integral from Eq. 2.7 of 2006.14680:

$$I = \int_0^1 dx_1 \int_0^1 dx_2\, \Theta(x_1 + x_2 - 1) \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)}\, \delta\!\left(\rho - \frac{4(1 - \max\{x_i\})}{(2 - \max\{x_i\})^2}\right) \Theta\!\left(\frac{\min\{x_i\}}{2 - \max\{x_i\}} - z\right), \tag{1}$$

where x_1, x_2, x_3 are phase space variables satisfying

$$x_1 + x_2 + x_3 = 2. \tag{2}$$

As a first step, we can change variables from x_2 to x_3:

$$x_3 = 2 - x_1 - x_2 \qquad\qquad dx_2 = -dx_3. \tag{3}$$

Then, after partitioning unity with the definition $\Theta_{ijk} \equiv \Theta(x_i - x_j)\Theta(x_j - x_k)$, the integral becomes

$$\begin{aligned} I = \int_0^1 dx_3 \int_{1-x_3}^{2-x_3} dx_1\, \Theta(1 - x_3) \frac{x_1^2 + (2 - x_1 - x_3)^2}{(1 - x_1)(x_1 + x_3 - 1)}\, \delta\!\left(\rho - \frac{4(1 - \max\{x_i\})}{(2 - \max\{x_i\})^2}\right) \\ \times \Theta\!\left(\frac{\min\{x_i\}}{2 - \max\{x_i\}} - z\right) \Big[\Theta_{123} + \Theta_{132} + \Theta_{213} + \Theta_{231} + \Theta_{312} + \Theta_{321}\Big]. \end{aligned} \tag{4}$$

Taking a look at the first integral, we have

$$\begin{aligned} I_1 = \int_0^1 dx_3 \int_{1-x_3}^{2-x_3} dx_1 \Theta(1 - x_3) \frac{x_1^2 + (2 - x_1 - x_3)^2}{(1 - x_1)(x_1 + x_3 - 1)}\, \delta\!\left(\rho - \frac{4(1 - x_1)}{(2 - x_1)^2}\right) \\ \times \Theta\!\left(\frac{x_3}{2 - x_1} - z\right) \Theta(2x_1 + x_3 - 2)\Theta(2 - 2x_1 - x_3). \end{aligned} \tag{5}$$

We can simplify the Dirac delta by considering its argument to be a function of x_1

$$f(x_1) = \rho - \frac{4(1 - x_1)}{(2 - x_1)^2}. \tag{6}$$

Then its roots are

$$r_1, r_2 = 2 + \frac{2\big(-1 \pm \sqrt{1 - \rho}\,\big)}{\rho}, \tag{7}$$

so

$$\delta\!\left(\rho - \frac{4(1 - x_1)}{(2 - x_1)^2}\right) = \frac{\delta(x_1 - r_1)}{|f'(r_1)|} + \frac{\delta(x_1 - r_2)}{|f'(r_2)|}. \tag{8}$$

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Only the first root will contribute in this case (since the other is negative for $0 < \rho < 1$), so

$$\begin{aligned} I_1 &= \int_0^1 dx_3 \int_{1-x_3}^{2-x_3} dx_1 \Theta(1-x_3) \frac{x_1^2 + (2-x_1-x_3)^2}{(1-x_1)(x_1+x_3-1)} \frac{\delta(x_1 - r_1)}{|f'(r_1)|} \\ &\qquad\qquad \times \Theta\left(\frac{x_3}{2-x_1} - z\right) \Theta(2x_1 + x_3 - 2)\Theta(2 - 2x_1 - x_3) \end{aligned} \tag{9}$$

$$\begin{aligned} &= \int_0^1 dx_3\, \Theta(1-x_3) \frac{r_1^2 + (2-r_1-x_3)^2}{(1-r_1)(r_1+x_3-1)} \frac{1}{|f'(r_1)|} \\ &\qquad\qquad \times \Theta\left(\frac{x_3}{2-r_1} - z\right) \Theta(2r_1 + x_3 - 2)\Theta(2 - 2r_1 - x_3) \end{aligned} \tag{10}$$

$$\begin{aligned} &\overset{?}{\approx} \int_0^\infty dx_3\, \Theta(1-x_3) \frac{4 - 4r_1 + 2r_1^2 + 2r_x x_3 - 4x_3}{(1-r_1)(r_1+x_3-1)} \frac{1}{|f'(r_1)|} \\ &\qquad\qquad \times \Theta\left(\frac{x_3}{2-r_1} - z\right) \Theta(2r_1 + x_3 - 2)\Theta(2 - 2r_1 - x_3). \end{aligned} \tag{11}$$

Here's where I'm suck: in the limit $x_3 \ll 1$, we might take the upper bound on the integral to be ∞ (as you suggest), which would simplify the integration (although… how then to handle the upper bounds on x_3 imposed by the Heaviside functions?). But how can we make the approximation play nice with, for example, the requirement

$$\Theta(2r_1 + x_3 - 2) \implies x_3 > 2(1 - r_1)? \tag{12}$$

These imposed lower bounds seem to prevent us from satisfying the limit $x_3 \to 0$. Should we treat them as a bound on ρ instead (e.g. by saying $\Theta(2r_1 + x_3 - 2) \approx \Theta(2r_1 - 2)$? Then it seems like we're potentially losing an important part of the integral.

Do you have any pointers on this?

Ok, since we are setting $x_3 \ll 1$, we know $1 - x_1 \ll 1$, so let's set

$$x_3 \to \lambda x_3 \qquad\qquad x_1 \to 1 - \lambda(1 - x_1), \tag{13}$$

then drop terms quadratic and higher in λ. Since the delta function and Heaviside functions are linear in x_1 and x_3