CALCULATING THE SOFT FUNCTION

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Contents

1. Setup

We wish to calculate the resolved soft function $S_R(\rho - z_{\text{cut}})$ which describes soft radiation which passes the groomer due to proximity to the resolved gluon. If the resolved emission occurs at an angle θ from the quark axis, then any radiation at smaller angles will pass the groomer. A schematic of this situation is displayed in Fig. 1.

The goal is to calculate the first-order term in an expansion of S_R. We can then use renormalization group evolution in conjunction with the other first-order results of functions in the factorization equation to achieve an all-orders calculation of the cross section.

Let the resolved gluon have momentum k_g, the quark lie along direction $n_q = (1, 0, 0, 1)$, and consider an extra-soft gluon with momentum k. If the extra-soft gluon is closer to the quark, then its dominant contribution to the jet mass ρ will come from its interaction with the quark:

$$\rho = \frac{4k^+}{Q} \tag{1}$$

where $k^\pm = k^0 \mp k_z$ are light-cone coordinates defined with respect to the quark axis. If the extra-soft gluon is closer to the resolved gluon, then its contribution to the jet mass from the quark interaction has already been accounted for in the contribution of the resolved gluon. The leading-order contribution from the new gluon therefore comes with its interaction with the resolved gluon. If n_g is the direction of the resolved gluon, then the contribution is

$$\rho = \frac{4k \cdot n_g}{Q} = \frac{4k \cdot k_g}{E_g Q} \tag{2}$$

with E_g the energy of the resolved gluon.

Notice that the angle between the extra-soft gluon and the quark is given by

$$1 - \cos\theta_{gq} = \frac{k^+}{k^0} \tag{3}$$

while the angle between the extra-soft gluon and the resolved gluon is

$$1 - \cos\theta_{gg} = \frac{k \cdot n_g}{k^0}. \tag{4}$$

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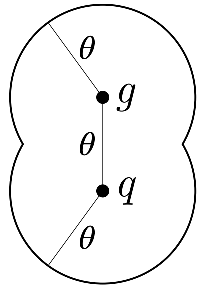

FIGURE 1. Schematic head-on view of emissions according to the jet groomer. Radiation within the peanut-shaped region will pass the grooming algorithm.

The case in which the extra-soft gluon is closer to the quark is the case in which $\theta_{gq} < \theta_{gg}$, so $1 - \cos\theta_{gq} < 1 - \cos\theta_{gg}$ and, in turn $k^+ < k \cdot n_g$. Therefore, the total measurement function is

$$\delta_\rho = \Theta(k \cdot n_g - k^+)\,\delta\left(\rho - \frac{4k^+}{Q}\right) + \Theta(k^+ - k \cdot n_g)\,\delta\left(\rho - \frac{4k \cdot n_g}{Q}\right). \tag{5}$$

We also need to impose the kinematic constraint that the gluon is in the peanut-shaped region of Fig. 1. Saying that the gluon is in the region is equivalent to saying that it is not outside the region. The gluon is outside of the quark's radius of influence if

$$\frac{k^+}{k^0} = 1 - \cos\theta_{gq} > 1 - \cos\theta = n_g \cdot n_q. \tag{6}$$

On the other hand, the gluon is outside the resolved gluon's radius of influence if

$$\frac{k \cdot n_g}{k^0} = 1 - \cos\theta_{gg} > 1 - \cos\theta = n_g \cdot n_q. \tag{7}$$

Therefore, the grooming restriction is

$$\Theta_{\mathrm{mMDT}} = 1 - \Theta(k^+ - k^0 n_g \cdot n_q)\,\Theta(k \cdot n_g - k^0 n_g \cdot n_q). \tag{8}$$

The matrix element accounts for the possibility that the gluon be emitted from any pairs of resolved particles **[TODO: need to sort out prefactors, include color matrices. Also it's not actually a sum]**

$$|\mathcal{M}|^2 = \mu^{2\epsilon} \sum_{i<j} \frac{n_i \cdot n_j}{(n_i \cdot k)(n_j \cdot k)} \tag{9}$$

where i, j range over all pairs of resolved particles **[added in renormalization scale as we have in the past... is that right?]**. Each term of the matrix element corresponds to a separate soft function. For now, we will focus on the first term

$$|\mathcal{M}_{q\bar{q}}|^2 = \mu^{2\epsilon} \frac{n_q \cdot n_{\bar{q}}}{(n_q \cdot k)(n_{\bar{q}} \cdot k)} = \mu^{2\epsilon} \frac{2}{k^+ k^-} \tag{10}$$

with $n_{\bar{q}} = (1, 0, 0, -1)$ the antiquark direction. **[TODO: need to handle other soft functions]**

Finally, phase space in d dimensions takes the usual form

$$d\Pi = \frac{d^d k}{(2\pi)^d} 2\pi\,\delta\big(k^2\big)\,\Theta(k^+)\,\Theta(k^- - k^+). \tag{11}$$

Notice that we are enforcing the gluon to be emitted in the hemisphere with the quark by requiring $k^- - k^+$. We will multiply the result at the end by a factor of 2 to account for the case where the gluons are emitted in the other hemisphere. Note that we are only scanning over the momentum of the extra-soft gluon: under the assumption that this gluon is softer than the resolved gluon, this emission does not influence the momentum of the quarks or resolved gluon.

Putting everything together, we find

$$
\begin{aligned}
S_R(\rho - z_{\mathrm{cut}}) = 2\mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^{d-1}}\, \delta\big(k^2\big)\, \Theta(k^+)\, \Theta(k^- - k^+)\, \frac{2}{k^+ k^-} \\
\times \left[\Theta(k\cdot n_g - k^+)\, \delta\!\left(\rho - \frac{4k^+}{Q}\right) + \Theta(k^+ - k\cdot n_g)\, \delta\!\left(\rho - \frac{4k\cdot n_g}{Q}\right)\right] \\
\times \big[1 - \Theta(k^+ - k^0 n_g\cdot n_q)\, \Theta(k\cdot n_g - k^0 n_g \cdot n_q)\big].
\end{aligned} \tag{12}
$$

2. COORDINATE CHOICE

Now we need to determine which coordinates in which to work. Notice that, physically, there is an axial symmetry to the problem: nothing depends on the angle of the resolved emission about the quark axis. Therefore, we might define our momenta in terms of their transverse momentum, pseudorapidity, and angle about the axis. To get from Cartesian (p_x, p_y, p_z) to this detector coordinate system $(p_\perp, \phi, \eta)$, we use the following transformations:

$$
\begin{aligned}
&p_x = p_\perp \cos\phi \qquad p_y = p_\perp \sin\phi \qquad p_z = p_\perp \sinh\eta \qquad p_0 = p_\perp \cosh\eta \\
&p_\perp = \sqrt{p_x^2 + p_y^2} \qquad \phi = \arctan\!\left(\frac{p_y}{p_x}\right) \qquad \eta = \operatorname{arctanh}\!\left(\frac{p_z}{|\mathbf{p}|}\right).
\end{aligned} \tag{13}
$$

Under this transformation, the extra-soft gluon has momentum

$$
k = (k_0, k_\perp, \phi_k, \eta_k). \tag{14}
$$

The resolved gluon is fixed in space from the perspective of the extra-soft gluon, so we can write it in whichever coordinates are convenient. Let us pick spherical coordinates, where the gluon momentum has an azimuthal angle ϕ_g and an angle θ_g from the jet axis

$$
k_g = (k_0, r, \theta, \phi) = (E_g, E_g, \theta_g, \phi_g) \tag{15}
$$

and hence direction vector

$$
n_g = (1, 1, \theta_g, \phi_g). \tag{16}
$$

Finally, without loss of generality, we can define our coordinate axis so that the resolved emission is at angle $\phi_g = 0$, thereby setting

$$
k_g = (E_g, E_g, \theta_g, 0) \quad n_g = (1, 1, \theta_g, 0). \tag{17}
$$

Now we can transform each term of Eq. 12. First, notice that

$$
k^+ = k_0 - k_z = k_\perp(\cosh\eta_k - \sinh\eta_k) = k_\perp e^{-\eta_k}, \tag{18}
$$

and similarly

$$
k^- = k_\perp e^{\eta_k}. \tag{19}
$$

Hence, the restriction $k^+ > 0$ becomes $k_\perp > 0$ and $k^- > k^+$ becomes $\eta_k > 0$. That is,

$$
\Theta(k^+)\, \Theta(k^- - k^+) = \Theta(k_\perp)\, \Theta(\eta_k). \tag{20}
$$

The first term in the matrix element is then simply

$$
|\mathcal{M}|^2 = \frac{2}{k^+ k^-} = \frac{2}{k_\perp^2}. \tag{21}
$$

Next comes the measurement function. First notice that (in Cartesian coordinates)

$$
\begin{aligned}
k\cdot n_g &= (k_\perp \cosh\eta_k, k_\perp \cos\phi_k, k_\perp \sin\phi_k, k_\perp \sinh\eta_k)\cdot(1, \sin\theta_g, 0, \cos\theta_g) \\
&= k_\perp[\cosh\eta_k - \cos\phi_k \sin\theta_g - \sinh\eta_k \cos\theta_g].
\end{aligned} \tag{22}
$$

Therefore

$$\begin{aligned}\Theta(k^+ - k\cdot n_g) &= \Theta(\cos\phi_k \sin\theta_g - (1-\cos\theta_g)\sinh\eta_k) \\ &= \Theta\left(\cos\phi_k \frac{\sin\theta_g}{1-\cos\theta_g} - \sinh\eta_k\right) \\ &= \Theta\left(\cos\phi_k \cot\frac{\theta_g}{2} - \sinh\eta_k\right)\end{aligned} \tag{23}$$

and

$$\Theta(k\cdot n_g - k^+) = \Theta\left(\sinh\eta_k - \cos\phi_k\cot\frac{\theta_g}{2}\right). \tag{24}$$

The full measurement function is then

$$\begin{aligned}\delta_\rho = \Theta&\left(\sinh\eta_k - \cos\phi_k\cot\frac{\theta_g}{2}\right)\delta\left(\rho - \frac{4k_\perp e^{-\eta_k}}{Q}\right) \\ &+ \Theta\left(\cos\phi_k\cot\frac{\theta_g}{2} - \sinh\eta_k\right)\delta\left(\rho - \frac{4k_\perp}{Q}[\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g]\right).\end{aligned} \tag{25}$$

Finally, we have the mMDT groomer. Notice that

$$\Theta\big(k^+ - k^0 n_g\cdot n_q\big) = \Theta(\cos\theta_g - \tanh\eta_k) \tag{26}$$

and

$$\Theta\big(k\cdot n_g - k^0 n_g\cdot n_q\big) = \Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k). \tag{27}$$

Therefore,

$$1 - \Theta(k^+ - k^0 n_g\cdot n_q)\,\Theta(k\cdot n_g - k^0 n_g\cdot n_q) = 1 - \Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k). \tag{28}$$

Putting everything together so far, we have

$$\begin{aligned}S_R = 2\mu^{2\epsilon}\int &\frac{d^dk}{(2\pi)^{d-1}}\delta(k^2)\,\Theta(k_\perp)\,\Theta(\eta_k)\,\frac{2}{k_\perp^2} \\ &\times\Bigg[\Theta\left(\sinh\eta_k - \cos\phi_k\cot\frac{\theta_g}{2}\right)\delta\left(\rho - \frac{4k_\perp e^{-\eta_k}}{Q}\right) \\ &\qquad + \Theta\left(\cos\phi_k\cot\frac{\theta_g}{2} - \sinh\eta_k\right)\delta\left(\rho - \frac{4k_\perp}{Q}[\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g]\right)\Bigg] \\ &\times[1 - \Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k)].\end{aligned} \tag{29}$$

The last thing to evaluate is the phase space measure. We wish to convert

$$dk_0 dk_z\, d^{d-2}k_\perp\,\delta(k^2) \to dk_0 d\eta_k\, d^{d-2}dk_\perp\,\delta(k^2) \tag{30}$$

where $k_\perp$ represents the off-axis components of k in $d-2$ dimensions. With $d = 4-2\epsilon$, we can write this in spherical coordinates as

$$d^{d-2}k_\perp = k_\perp^{d-3}dk_\perp\,\sin^{-2\epsilon}\phi_k\, d\phi_k\, d\Omega_{d-2} \tag{31}$$

with Ω_{d-2} the solid angle of the $d-2$ dimensional sphere **[TODO: currently using wrong dimension for solid angle]**. Integrating over this solid angle yields [1]

$$\int d\Omega_{d-2} = \frac{2\pi^{(d-2)/2}}{\Gamma(\frac{d-2}{d})} = \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}. \tag{32}$$

Thus, we find that

$$d^{d-2}k_\perp = dk_\perp d\phi_k\, k_\perp^{d-3}\sin^{-2\epsilon}\phi_k\,\frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}. \tag{33}$$

Now also notice that

$$\delta(k^2) = \delta\big(k_0^2 - k_\perp^2 - k_z^2\big) = \delta\big(k_0^2 - k_\perp^2 \cosh^2\eta_k\big). \tag{34}$$

This simplifies to

$$\delta\big(k_0^2 - k_\perp^2\cosh^2\eta_k\big) = \frac{1}{2k_\perp\cosh\eta_k}\delta(k_0 - k_\perp\cosh\eta_k). \tag{35}$$

Therefore, we can integrate out k_0 (notice that we have sneakily already applied the delta function where k_0 appeared earlier):

$$\int dk_0\delta(k^2) = \frac{1}{2k_\perp\cosh\eta_k}. \tag{36}$$

Finally, we need to account for the Jacobian in the (k_0, k_z) transformation:

$$\frac{\partial(k_0,k_z)}{\partial(k_0,\eta_k)} = \begin{pmatrix} 1 & 0 \\ 0 & k_\perp\cosh\eta_k \end{pmatrix}. \tag{37}$$

The standard Jacobian factor is then the determinant (in absolute value)

$$dk_0 dk_z = k_\perp\cosh\eta_k\, dk_0 d\eta_k. \tag{38}$$

All together, the phase space measure is

$$\begin{aligned}\int \frac{d^dk}{(2\pi)^{d-1}}\,\delta(k^2) &= \frac{\pi^{1-\epsilon}}{(2\pi)^{3-2\epsilon}\Gamma(1-\epsilon)}\int dk_\perp d\phi_k d\eta_k\, k_\perp^{-1-2\epsilon}\sin^{-2\epsilon}\phi_k \\ &= \frac{(4\pi)^\epsilon}{8\pi^2\Gamma(1-\epsilon)}\int dk_\perp d\phi_k d\eta_k\, k_\perp^{-1-2\epsilon}\sin^{-2\epsilon}\phi_k.\end{aligned} \tag{39}$$

Under the modified minimal subtraction scheme, we will set $(4\pi)^\epsilon \to 1$ (and will also set $\gamma_E \to 0$ as it comes up). The full integral is now

$$\boxed{\begin{aligned} S_R = \frac{\mu^{2\epsilon}}{2\pi^2\Gamma(1-\epsilon)} & \int dk_\perp d\phi_k d\eta_k\, k_\perp^{-1-2\epsilon}\sin^{-2\epsilon}\phi_k\,\Theta(k_\perp)\Theta(\eta_k) \\ & \times\Bigg[\Theta\bigg(\sinh\eta_k - \cos\phi_k\cot\frac{\theta_g}{2}\bigg)\,\delta\bigg(\rho - \frac{4k_\perp e^{-\eta_k}}{Q}\bigg) \\ & \qquad + \Theta\bigg(\cos\phi_k\cot\frac{\theta_g}{2} - \sinh\eta_k\bigg)\,\delta\bigg(\rho - \frac{4k_\perp}{Q}[\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g]\bigg)\Bigg] \\ & \times[1 - \Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k)]. \end{aligned}} \tag{40}$$

3. Evaluating the integral

First, we want to integrate out $k_\perp$, which can be done easily enough using the Dirac delta functions. The first transforms as

$$\delta\bigg(\rho - \frac{4k_\perp e^{-\eta_k}}{Q}\bigg) = \frac{Qe^{\eta_k}}{4}\delta\bigg(k_\perp - \frac{Q\rho e^{\eta_k}}{4}\bigg), \tag{41}$$

while the second transforms as

$$\begin{aligned}&\delta\bigg(\rho - \frac{4k_\perp}{Q}[\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g]\bigg) \\ &\quad = \frac{Q}{4(\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g)}\delta\bigg(k_\perp - \frac{Q\rho}{4[\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g]}\bigg).\end{aligned} \tag{42}$$

Integrating out $k_\perp$ from Eq. 40, we therefore have

$$S_R = \frac{\mu^{2\epsilon}}{2\pi^2\Gamma(1-\epsilon)}\left(\frac{Q}{4}\right)^{-2\epsilon}\frac{1}{\rho^{1+2\epsilon}}\int d\phi_k d\eta_k\ \sin^{-2\epsilon}\phi_k\,\Theta(\eta_k) \times\left[\Theta\left(\sinh\eta_k-\cos\phi_k\cot\frac{\theta_g}{2}\right)e^{-2\epsilon\eta_k} + \Theta\left(\cos\phi_k\cot\frac{\theta_g}{2}-\sinh\eta_k\right)\left(\frac{1}{\cosh\eta_k-\cos\phi_k\sin\theta_g-\sinh\eta_k\cos\theta_g}\right)^{-2\epsilon}\right] \times[1-\Theta(\cos\theta_g-\tanh\eta_k)\,\Theta(\cot\theta_g-e^{\eta_k}\cos\phi_k)]. \tag{43}$$

Now, we will eventually expand $\rho^{-1-2\epsilon}$ using a plus-function expansion [2]

$$\frac{1}{\rho^{1+2\epsilon}} = -\frac{1}{2\epsilon}\delta(\rho)+\left[\frac{1}{\rho}\right]_+-\epsilon\left[\frac{\ln\rho}{\rho}\right]_++\mathcal{O}(\epsilon^2). \tag{44}$$

This means that, in order to calculate the cusp anomalous dimension, we need to keep terms through $\mathcal{O}(\epsilon^0)$ in the remaining integral.

To do this, we can first simplify the integral as follows. Notice that

$$\Theta\left(\sinh\eta_k-\cos\phi_k\cot\frac{\theta_g}{2}\right)e^{-2\epsilon\eta_k} + \Theta\left(\cos\phi_k\cot\frac{\theta_g}{2}-\sinh\eta_k\right)\left(\frac{1}{\cosh\eta_k-\cos\phi_k\sin\theta_g-\sinh\eta_k\cos\theta_g}\right)^{-2\epsilon} = e^{-2\epsilon\eta_k}\left[\Theta\left(\sinh\eta_k-\cos\phi_k\cot\frac{\theta_g}{2}\right) + \Theta\left(\cos\phi_k\cot\frac{\theta_g}{2}-\sinh\eta_k\right)\left(\frac{e^{\eta_k}}{\cosh\eta_k-\cos\phi_k\sin\theta_g-\sinh\eta_k\cos\theta_g}\right)^{-2\epsilon}\right]. \tag{45}$$

But we can expand the term in brackets in ϵ to find **[is it ok to do a partial expansion like this?]**

$$\Theta\left(\sinh\eta_k-\cos\phi_k\cot\frac{\theta_g}{2}\right) + \Theta\left(\cos\phi_k\cot\frac{\theta_g}{2}-\sinh\eta_k\right)\left(\frac{e^{\eta_k}}{\cosh\eta_k-\cos\phi_k\sin\theta_g-\sinh\eta_k\cos\theta_g}\right)^{-2\epsilon} = 1+\mathcal{O}(\epsilon). \tag{46}$$

Therefore, if we let

$$I = \int d\phi_k d\eta_k\ \sin^{-2\epsilon}\phi_k\,\Theta(\eta_k)\left[\Theta\left(\sinh\eta_k-\cos\phi_k\cot\frac{\theta_g}{2}\right)e^{-2\epsilon\eta_k} + \Theta\left(\cos\phi_k\cot\frac{\theta_g}{2}-\sinh\eta_k\right)\left(\frac{1}{\cosh\eta_k-\cos\phi_k\sin\theta_g-\sinh\eta_k\cos\theta_g}\right)^{-2\epsilon}\right] \times[1-\Theta(\cos\theta_g-\tanh\eta_k)\,\Theta(\cot\theta_g-e^{\eta_k}\cos\phi_k)], \tag{47}$$

we find that

$$I = \int d\phi_k d\eta_k\sin^{-2\epsilon}\phi_k\Theta(\eta_k)e^{-2\epsilon\eta_k}[1-\Theta(\cos\theta_g-\tanh\eta_k)\,\Theta(\cot\theta_g-e^{\eta_k}\cos\phi_k)]+\mathcal{O}(\epsilon). \tag{48}$$

For the first term, we can integrate in η_k to find

$$\int d\phi_k d\eta_k \sin^{-2\epsilon}\phi_k \Theta(\eta_k) e^{-2\epsilon\eta_k} = \frac{1}{2\epsilon}\int d\phi_k \sin^{-2\epsilon}\phi_k. \tag{49}$$

Expanding the integrand in ϵ, we then have

Might want to do this integral exactly; even as is, missing the epsilon^0 term

$$\frac{1}{2\epsilon}\int d\phi_k \sin^{-2\epsilon}\phi_k = \frac{1}{2\epsilon}\int_0^\pi d\phi_k + \mathcal{O}(\epsilon) = \frac{\pi}{2\epsilon} + \mathcal{O}(\epsilon). \tag{50}$$

Thus,

$$I = \frac{\pi}{2\epsilon} - \int d\phi_k d\eta_k \sin^{-2\epsilon}\phi_k \Theta(\eta_k) e^{-2\epsilon\eta_k}\Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k) + \mathcal{O}(\epsilon). \tag{51}$$

Might be more straightforward to think a little more about phase space bounds

The remaining integral is not divergent in η_k if we first expand in ϵ, so let's do that. We have

$$I = \frac{\pi}{2\epsilon} - \int d\phi_k d\eta_k \Theta(\eta_k)\Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k) + \mathcal{O}(\epsilon). \tag{52}$$

To evaluate this integral, we just need to sort out the bounds in η_k. These come out to

$$\begin{aligned}\Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k) &= \Theta\left(\frac{1}{1+\sec\theta_g} - \cos\phi_k\right)\Theta\left(\cot\frac{\theta_g}{2} - e^{\eta_k}\right)\\ &\quad + \left[\Theta\left(\cos\phi_k - \frac{1}{1+\sec\theta_g}\right)\Theta(\cot\theta_g - \cos\phi_k)\right.\\ &\qquad \left.\times\,\Theta(\cot\theta_g\sec\phi_k - e^{\eta_k})\right].\end{aligned} \tag{53}$$

Now, splitting into positive and negative values of $\cos\phi_k$, we have

$$\Theta\left(\frac{1}{1+\sec\theta_g} - \cos\phi_k\right) = \Theta\left(\frac{\pi}{2} - \phi_k\right)\Theta(\sec\phi_k - 1 - \sec\theta_g) + \Theta\left(\phi_k - \frac{\pi}{2}\right) \tag{54}$$

which uses the fact that

$$\operatorname{arcsec}(1+\sec\theta_g) < \frac{\pi}{2} \tag{55}$$

for all $0 < \theta_g < \pi/2$. Evaluating the first part of the integral yields

$$\begin{aligned}&\int d\phi_k d\eta_k \Theta(\eta_k)\left[\Theta\left(\frac{\pi}{2} - \phi_k\right)\Theta(\sec\phi_k - 1 - \sec\theta_g) + \Theta\left(\phi_k - \frac{\pi}{2}\right)\right]\Theta\left(\cot\frac{\theta_g}{2} - e^{\eta_k}\right)\\ &= \int d\phi_k \left[\Theta\left(\frac{\pi}{2} - \phi_k\right)\Theta(\sec\phi_k - 1 - \sec\theta_g) + \Theta\left(\phi_k - \frac{\pi}{2}\right)\right]\log\cot\frac{\theta_g}{2}\\ &= [\pi - \operatorname{arcsec}(1+\sec\theta_g)]\log\cot\frac{\theta_g}{2}.\end{aligned} \tag{56}$$

Integrating η_k out of the second part yields

$$\begin{aligned}&\int d\phi_k d\eta_k \Theta(\eta_k)\Theta\left(\cos\phi_k - \frac{1}{1+\sec\theta_g}\right)\Theta(\cot\theta_g - \cos\phi_k)\Theta(\cot\theta_g\sec\phi_k - e^{\eta_k})\\ &\quad = \int d\phi_k \Theta\left(\cos\phi_k - \frac{1}{1+\sec\theta_g}\right)\Theta(\cot\theta_g - \cos\phi_k)\log(\cot\theta_g\sec\phi_k).\end{aligned} \tag{57}$$

To solve the remaining integral, first notice that

$$\cot\theta_g > 1 \implies \cot\theta_g > \cos\phi_k \tag{58}$$

for $0 < \theta_g < \pi/4$. Therefore,

$$\Theta(\cot\theta_g - \cos\phi_k) = \Theta\left(\frac{\pi}{4} - \theta_g\right) + \Theta\left(\theta_g - \frac{\pi}{4}\right)\Theta(\cot\theta_g - \cos\phi_k). \tag{59}$$

Now, direct evaluation of the indefinite integral yields

$$\int d\phi_k \log(\cot\theta_g \sec\phi_k) = \frac{i\phi_k^2}{2} + \phi_k \log(2\cot\theta_g) - \frac{i}{2}\mathrm{Li}_2\left(-e^{2i\phi_k}\right), \tag{60}$$

where $\mathrm{Li}_2(x)$ is the dilogarithm function. While it appears that the result is complex, the imaginary part is actually constant,[1] and therefore is eliminated in a definite integral. We can see this as follows. First, the power series of the dilogarithm on the unit disk $|z| \leq 1$ is

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}. \tag{61}$$

Therefore,

$$\mathrm{Li}_2\left(-e^{2i\phi_k}\right) = \sum_{k=1}^{\infty} \frac{(-1)^k e^{2ik\phi_k}}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k \cos(2k\phi_k)}{k^2} + i\sum_{k=1}^{\infty} \frac{(-1)^k \sin(2k\phi_k)}{k^2}. \tag{62}$$

The real part is

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^k \cos(2k\phi_k)}{k^2} &= \frac{1}{2}\left[\sum_{k=1}^{\infty} \frac{(-1)^k e^{-2ik\phi_k}}{k^2} + \sum_{k=1}^{\infty} \frac{(-1)^k e^{2ik\phi_k}}{k^2}\right] \\ &= \frac{1}{2}\left[\mathrm{Li}_2\left(-e^{2i\phi_k}\right) + \mathrm{Li}_2\left(-e^{-2i\phi_k}\right)\right]. \end{aligned} \tag{63}$$

Now, it is an identity of the dilogarithm[2] that

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) = -\frac{\pi^2}{6} - \frac{1}{2}\log^2(-z). \tag{64}$$

Therefore,

$$\mathrm{Li}_2\left(-e^{2i\phi_k}\right) + \mathrm{Li}_2\left(-e^{-2i\phi_k}\right) = -\frac{\pi^2}{6} - \frac{1}{2}\log^2\left(e^{2i\phi_k}\right) = -\frac{\pi^2}{6} + 2\phi_k^2. \tag{65}$$

Thus, we see that

$$\mathrm{Re}\left[\mathrm{Li}_2\left(-e^{2i\phi_k}\right)\right] = -\frac{\pi^2}{12} + \phi_k^2. \tag{66}$$

The imaginary part is, to my knowledge, more difficult to simplify **[is this true?]**:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^k \sin(2k\phi_k)}{k^2} &= \frac{i}{2}\left[\sum_{k=1}^{\infty} \frac{(-1)^k e^{-2ik\phi_k}}{k^2} - \sum_{k=1}^{\infty} \frac{(-1)^k e^{2ik\phi_k}}{k^2}\right] \\ &= \frac{i}{2}\left[\mathrm{Li}_2\left(-e^{-2i\phi_k}\right) - \mathrm{Li}_2\left(-e^{2i\phi_k}\right)\right]. \end{aligned} \tag{67}$$

Therefore,

$$\mathrm{Li}_2\left(-e^{2i\phi_k}\right) = -\frac{\pi^2}{12} + \phi_k^2 - \frac{1}{2}\left[\mathrm{Li}_2\left(-e^{-2i\phi_k}\right) - \mathrm{Li}_2\left(-e^{2i\phi_k}\right)\right] \tag{68}$$

(the portion in square brackets is entirely imaginary). Putting everything together yields

$$\int d\phi_k \log(\cot\theta_g \sec\phi_k) = \phi_k \log(2\cot\theta_g) + \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\phi_k}\right) - \mathrm{Li}_2\left(-e^{2i\phi_k}\right)\right] + \frac{i\pi^2}{24}. \tag{69}$$

The imaginary portion has been condensed to a constant in the final term. Combining Eqs. 57, 59, and 69 and noting that

$$\arccos\left(\frac{1}{1+\sec\theta_g}\right) = \mathrm{arcsec}(1+\sec\theta_g) \tag{70}$$

[1] Albeit after much pain and wandering

[2] Away from a branch cut

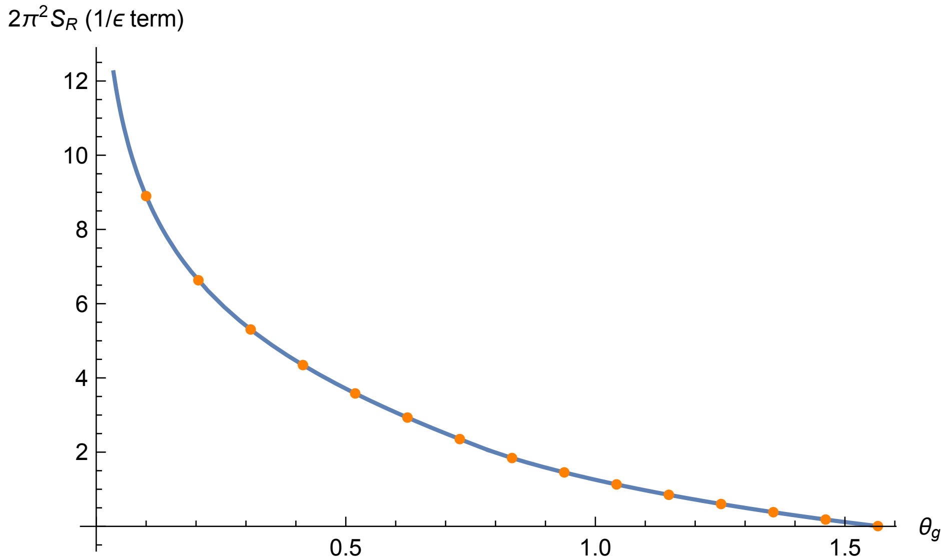

FIGURE 2. Analytic (solid blue line) and numeric (orange dots) values of the ϵ^{-1} contribution to the soft function. Numerics are calculated from the integral of Eq. 52, and the analytic solution is that of Eq. 72.

then yields

$$\begin{aligned}
&\int d\phi_k \Theta\left(\cos\phi_k - \frac{1}{1+\sec\theta_g}\right)\Theta(\cot\theta_g - \cos\phi_k)\log(\cot\theta_g \sec\phi_k) \\
&\quad = \int d\phi_k \Theta\left(\cos\phi_k - \frac{1}{1+\sec\theta_g}\right)\left[\Theta\left(\frac{\pi}{4}-\theta_g\right) + \Theta\left(\theta_g - \frac{\pi}{4}\right)\Theta(\cot\theta_g - \cos\phi_k)\right] \\
&\quad = \operatorname{arcsec}(1+\sec\theta_g)\log(2\cot\theta_g) + \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\operatorname{arcsec}(1+\sec\theta_g)}\right) - \mathrm{Li}_2\left(-e^{2i\operatorname{arcsec}(1+\sec\theta_g)}\right)\right] \\
&\qquad - \Theta\left(\theta_g - \frac{\pi}{4}\right)\left[\arccos\cot\theta_g \log(2\cot\theta_g) + \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\arccos\cot\theta_g}\right) - \mathrm{Li}_2\left(-e^{2i\arccos\cot\theta_g}\right)\right]\right].
\end{aligned} \tag{71}$$

We conclude that the full integral of Eq. 52 is

$$\begin{aligned}
I = \frac{\pi}{2\epsilon} &- [\pi - \operatorname{arcsec}(1+\sec\theta_g)]\log\cot\frac{\theta_g}{2} - \operatorname{arcsec}(1+\sec\theta_g)\log(2\cot\theta_g) \\
&- \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\operatorname{arcsec}(1+\sec\theta_g)}\right) - \mathrm{Li}_2\left(-e^{2i\operatorname{arcsec}(1+\sec\theta_g)}\right)\right] \\
&+ \Theta\left(\theta_g - \frac{\pi}{4}\right)\Bigg[\arccos\cot\theta_g\log(2\cot\theta_g) \\
&\qquad\qquad + \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\arccos\cot\theta_g}\right) - \mathrm{Li}_2\left(-e^{2i\arccos\cot\theta_g}\right)\right]\Bigg].
\end{aligned} \tag{72}$$

The non-divergent portion of this integral (i.e. the cusp anomalous dimension of the soft function) is displayed in Fig. 2. The full soft function is therefore

Should Laplace transform in rho because it will make renorm group equations very simple

$$\boxed{\begin{aligned} S_R = {} & \frac{1}{2\pi^2\Gamma(1-\epsilon)}\left(\frac{Q}{4}\right)^{-2\epsilon}\frac{1}{\rho^{1+2\epsilon}} \\ & \times\left[\frac{\pi}{2\epsilon} - [\pi - \operatorname{arcsec}(1+\sec\theta_g)]\log\cot\frac{\theta_g}{2} - \operatorname{arcsec}\left(1+\sec\theta_g\right)\log(2\cot\theta_g)\right. \\ & \quad - \frac{i}{4}\Big[\mathrm{Li}_2\Big(-e^{-2i\operatorname{arcsec}\left(1+\sec\theta_g\right)}\Big) - \mathrm{Li}_2\Big(-e^{2i\operatorname{arcsec}\left(1+\sec\theta_g\right)}\Big)\Big] \\ & \quad + \Theta\Big(\theta_g - \frac{\pi}{4}\Big)\left[\arccos\cot\theta_g\log(2\cot\theta_g)\right. \\ & \qquad \left.\left. + \frac{i}{4}\Big[\mathrm{Li}_2\Big(-e^{-2i\arccos\cot\theta_g}\Big) - \mathrm{Li}_2\Big(-e^{2i\arccos\cot\theta_g}\Big)\Big]\right]\right] + \mathcal{O}(\epsilon). \end{aligned}} \tag{73}$$

References

[1] Matthew Dean Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, New York, 2014.

[2] Achilleas Lazopoulos, Kirill Melnikov, and Frank Petriello. QCD corrections to tri-boson production. *Phys. Rev. D*, 76(1):014001, July 2007.