GROOMED HEAVY HEMISPHERE MASS TO FIRST ORDER, COLLINEAR LIMIT

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Contents

1. Setup

1.1. **Heavy jet mass.** Now we wish to calculate the mass of the heavy hemisphere in an $e^+e^- \to q\bar{q}g$ event after mMDT grooming, assuming a collinear gluon emission. If E_h is the energy of the heavy hemisphere and m_h is the mass, then the observable of interest is [1]

$$\rho = \left(\frac{m_h}{E_h}\right)^2. \tag{1}$$

In the collinear limit, $E_h \to Q/2$ with Q (shorthand for $\sqrt{Q^2}$) the energy of the event. To compute m_h, we introduce the phase space variables

$$x_i = \frac{2p_i \cdot Q}{Q^2} \tag{2}$$

with p_i the four-momentum of the i-th particle and Q the total four-momentum (not to be confused with Q the total energy). Let x_3 be the energy fraction of the gluon and x_1 be the energy fraction of the quark.

We assume that the gluon is collinear with the quark (we will later multiply our result by 2 to account for the case where the gluon is collinear to the antiquark) and introduce the gluon's hemisphere energy fraction

$$z = \frac{x_3}{x_1 + x_3} = x_3 \tag{3}$$

where we have noted that, in the collinear limit, $x_1 + x_3 \to 1$. Notice that

$$1 - z = \frac{x_1}{x_1 + x_3} = x_1 \tag{4}$$

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is the quark's hemisphere energy fraction. This is equivalent to the assumption that the quark four-momentum p_1 and gluon four-momentum k are collinear along some vector $\bar{p}_1$:

$$k = z\bar{p}_1 \qquad\qquad p_q = (1-z)\bar{p}_1. \tag{5}$$

Now notice that

$$1 - x_2 = \frac{Q^2 - 2p_2 \cdot Q}{Q^2} = \frac{2p_1 \cdot k}{Q^2} = \frac{x_1 x_3}{2}(1 - \cos\theta) \tag{6}$$

where θ is the angle between the quark and gluon. In the collinear limit $\theta \ll 1$, this means that

$$\frac{2p_1 \cdot k}{Q^2} = \frac{x_1 x_3}{4}\theta^2 = \frac{z(1-z)}{4}\theta^2. \tag{7}$$

Then the heavy hemisphere mass is

$$m_h = (p_1 + k)^2 = 2p_1 \cdot k = \frac{z(1-z)}{4}\theta^2 Q^2. \tag{8}$$

This means that

$$\rho = z(1-z)\theta^2. \tag{9}$$

This suggests a measurement term of

$$\delta(\rho - z(1-z)\theta^2) = \frac{1}{z(1-z)}\delta\left(\theta^2 - \frac{\rho}{z(1-z)}\right). \tag{10}$$

1.2. **Grooming.** The heavy hemisphere only passes the mMDT groomer if [2]

$$\frac{\min[E_1, E_3]}{E_1 + E_3} > z_{\text{cut}} \tag{11}$$

for some fixed z_{cut}. This means

$$\min[x_1, x_3] = \min[z, 1-z] > z_{\text{cut}}. \tag{12}$$

Thus, we must introduce the grooming term

$$\Theta(\min[z, 1-z] - z_{\text{cut}}) \tag{13}$$

into our calculation.

1.3. **Matrix element.** The matrix element for $e^+e^- \to q\bar{q}g$ is [1]

$$|\mathcal{M}|^2 = 4\pi\alpha_s\sigma_0 C_F \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \tag{14}$$

where $4\pi\alpha_s$ is the strong coupling (squared), σ_0 is the cross section for $e^+e^- \to q\bar{q}$, and C_F is the quadratic Casimir of the fundamental representation of color. Under the change of variables introduced in Sec. 1.1, this reduces (after introducing the appropriate Jacobian factor) to the collinear splitting function

$$|\mathcal{M}|^2 = 4\pi\alpha_s\sigma_0 C_F \frac{1 + (1-z)^2}{z\theta^2}. \tag{15}$$

1.4. **Lorentz-invariant phase space.** Working in $d = 4 - 2\epsilon$ dimensions, the phase space integral with matrix element **[from Andrew's notes]** is

$$\frac{4\pi}{\alpha_s C_F}\frac{1}{\sigma_0}\sigma = \frac{1}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\epsilon)}\left(\frac{4\mu^2}{Q^2}\right)^{\epsilon}\int_0^1 dz\int_0^\infty d\theta^2\int_0^{2\pi} d\phi\sin^{-2\epsilon}\phi\,\left(\theta^2\right)^{-1-\epsilon}z^{-2\epsilon}(1-z)^{-2\epsilon} \times\left(\frac{1+(1-z)^2}{z}-\epsilon z\right). \tag{16}$$

Here, μ is a mass scale introduced to ensure the proper dimensionality and ϕ is the azimuthal angle. Notice that the matrix element is slightly modified under the dimensional regularization prescription.

Introducing the measurement and grooming terms of Eqs. 10 and 13, the differential cross section in ρ is

$$\boxed{\begin{aligned}\frac{4\pi}{\alpha_s C_F}\frac{1}{\sigma_0}\frac{d\sigma}{d\rho} = \frac{1}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\epsilon)}\left(\frac{4\mu^2}{Q^2}\right)^{\epsilon}\int_0^\pi d\phi\,\sin^{-2\epsilon}\phi\int_0^1 dz\int_0^\infty d\theta^2\,\left(\theta^2\right)^{-1-\epsilon}z^{-2\epsilon}(1-z)^{-2\epsilon}\\ \times\left(\frac{1+(1-z)^2}{z}-\epsilon z\right)\frac{1}{z(1-z)}\delta\!\left(\theta^2-\frac{\rho}{z(1-z)}\right)\\ \times\Theta(\min[z,1-z]-z_{\text{cut}}).\end{aligned}} \tag{17}$$

2. Integration

Integrating first in θ^2, we find

$$\frac{4\pi}{\alpha_s C_F}\frac{1}{\sigma_0}\frac{d\sigma}{d\rho} = \frac{1}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\epsilon)}\left(\frac{4\mu^2}{Q^2}\right)^{\epsilon}\int_0^\pi d\phi\,\sin^{-2\epsilon}\phi\int_0^1 dz\,\frac{1}{\rho^{1+\epsilon}}\frac{1}{z^\epsilon(1-z)^\epsilon}\left(\frac{1+(1-z)^2}{z}-\epsilon z\right) \times\Theta(\min[z,1-z]-z_{\text{cut}}). \tag{18}$$

The ϕ integral can be done immediately as well:

$$\int_0^\pi d\phi\,\sin^{-2\epsilon}\phi = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(1-\epsilon)}. \tag{19}$$

The Heaviside function can be satisfied by ensuring that

$$z_{\text{cut}} < z < 1-z_{\text{cut}}, \tag{20}$$

so

$$\int_0^1 dz\,\Theta(\min[z,1-z]-z_{\text{cut}}) = \int_{z_{\text{cut}}}^{1-z_{\text{cut}}} dz. \tag{21}$$

Thus, the cross section is

$$\frac{4\pi}{\alpha_s C_F}\frac{1}{\sigma_0}\frac{d\sigma}{d\rho} = \left(\frac{4\mu^2}{Q^2}\right)^{\epsilon}\frac{1}{\Gamma(1-\epsilon)}\frac{1}{\rho^{1+\epsilon}}\int_{z_{\text{cut}}}^{1-z_{\text{cut}}}\frac{1}{z^\epsilon(1-z)^\epsilon}\left(\frac{1+(1-z)^2}{z}-\epsilon z\right). \tag{22}$$

Notice that no cusp will emerge from this integral. This is a result of the particular radiative modes we pick out by considering the collinear limit, as revealed by power counting arguments. Let z_i be the energy fraction of the i-th emission in the hemisphere, with angle θ_i from the hard quark and θ_{ij} the angle between emissions i and j. Then

$$\rho \simeq \sum_{i,j} z_i z_j \theta_{ij}^2. \tag{23}$$

The cusp occurs around $\rho \sim z_{\text{cut}} \ll 1$, which would require $z_i \ll 1$. Around the cusp, we want z_i to be sensitive to both ρ *and* z_{cut}, so we also have $z_i \sim z_{\text{cut}}$. Moreover, the leading contribution from emission i comes with its interaction with the hard quark, since $z_i z_j$ is largest when z_j is hard. Therefore, we expect $\rho \sim z_i \theta_i^2$. But then

$$\rho \sim z_{\text{cut}} \theta_i^2 \sim z_{\text{cut}}, \tag{24}$$

so $\theta_i \sim 1$. Thus, emissions that contribute to the cusp must be *soft and wide-angle*. When we consider the collinear limit, we are moving out of the cusp region.

Returning to the integration a hand, the final integral is difficult to do directly, so we will first expand in ϵ. To first order in ϵ, we have

$$\frac{1}{z^\epsilon(1-z)^\epsilon}\left(\frac{1+(1-z)^2}{z} - \epsilon z\right) = \frac{2-2z+z^2}{z} - \epsilon\left(\frac{2-2z+z^2}{z}\log(z(1-z)) + z\right) + \mathcal{O}(\epsilon^2). \tag{25}$$

The corresponding integrals are

$$\int_{z_{\text{cut}}}^{1-z_{\text{cut}}} dz\, \frac{2-2z+z^2}{z} = -\frac{3}{2} + 3z_{\text{cut}} + 2\log\left(\frac{1-z_{\text{cut}}}{z_{\text{cut}}}\right) \tag{26}$$

and

$$\begin{aligned}\int_{z_{\text{cut}}}^{1-z_{\text{cut}}} dz\left(\frac{2-2z+z^2}{z}\log(z(1-z)) + z\right) &= 2\mathrm{Li}_2(z_{\text{cut}}) - 2\mathrm{Li}_2(1-z_{\text{cut}}) - 7z_{\text{cut}} - \log^2 z_{\text{cut}} \\ &\quad + \log z_{\text{cut}}\left(3z_{\text{cut}} + \frac{7}{2}\right) + \frac{7}{2} \\ &\quad + \log(1-z_{\text{cut}})(3z_{\text{cut}} + \log(1-z_{\text{cut}}) - 3).\end{aligned} \tag{27}$$

where Li_n is the polylogarithm function

$$\mathrm{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}. \tag{28}$$

Thus

$$\begin{aligned}\int_{z_{\text{cut}}}^{1-z_{\text{cut}}} dz\, \frac{1}{z^\epsilon(1-z)^\epsilon}\left(\frac{1+(1-z)^2}{z} - \epsilon z\right) &= -\frac{3}{2} + 3z_{\text{cut}} + 2\log\left(\frac{1-z_{\text{cut}}}{z_{\text{cut}}}\right) \\ &\quad + \epsilon\Big[2\mathrm{Li}_2(z_{\text{cut}}) - 2\mathrm{Li}_2(1-z_{\text{cut}}) - 7z_{\text{cut}} - \log^2 z_{\text{cut}} \\ &\quad + \log z_{\text{cut}}\left(3z_{\text{cut}} + \frac{7}{2}\right) + \frac{7}{2} \\ &\quad + \log(1-z_{\text{cut}})(3z_{\text{cut}} + \log(1-z_{\text{cut}}) - 3)\Big] + \mathcal{O}(\epsilon^2).\end{aligned} \tag{29}$$

We can expand $1/\rho^{1+\epsilon}$ using a plus-function expansion [3]:

$$\frac{1}{\rho^{1+\epsilon}} = -\frac{1}{\epsilon}\delta(\rho) + \left[\frac{1}{\rho}\right]_+ + \mathcal{O}(\epsilon). \tag{30}$$

The prefactor takes the form

$$\frac{1}{\Gamma(1-\epsilon)}\left(\frac{4\mu^2}{Q^2}\right)^\epsilon = 1 + \epsilon\log\left(\frac{4\mu^2}{Q^2}\right) + \mathcal{O}(\epsilon^2). \tag{31}$$

Putting everything together, we find

$$\boxed{\begin{aligned}\frac{4\pi}{\alpha_s C_F}\frac{1}{\sigma_0}\frac{d\sigma}{d\rho} = & -\frac{1}{\epsilon}\left(-\frac{3}{2}+3z_{\text{cut}}+2\log\left(\frac{1-z_{\text{cut}}}{z_{\text{cut}}}\right)\right)\delta(\rho) \\ & -\delta(\rho)\Bigg[2\log\left(\frac{4\mu^2}{Q^2}\right)+2\text{Li}_2(z_{\text{cut}})-2\text{Li}_2(1-z_{\text{cut}})-7z_{\text{cut}}-\log^2 z_{\text{cut}} \\ & \qquad +\log z_{\text{cut}}\left(3z_{\text{cut}}+\frac{7}{2}\right)+\frac{7}{2} \\ & \qquad +\log(1-z_{\text{cut}})(3z_{\text{cut}}+\log(1-z_{\text{cut}})-3)\Bigg] \\ & +\left[\frac{1}{\rho}\right]_+\left[-\frac{3}{2}+3z_{\text{cut}}+2\log\left(\frac{1-z_{\text{cut}}}{z_{\text{cut}}}\right)\right]+\mathcal{O}(\epsilon).\end{aligned}} \tag{32}$$

3. Laplace transform

[TODO]

4. Soft and collinear limit

Now we take the soft and collinear limit. This can be achieved by starting in the collinear limit and taking $z \ll 1$. The cross section is then given by a correspondingly modified version of Eq. 17:

$$\boxed{\frac{4\pi}{\alpha_s C_F}\frac{1}{\sigma_0}\frac{d\sigma}{d\rho} = \frac{2}{\Gamma(1-\epsilon)}\left(\frac{4\mu^2}{Q^2}\right)^\epsilon \int_0^\infty dz \int_0^\infty d\theta^2\,(\theta^2)^{-1-\epsilon}\frac{1}{z^{2+2\epsilon}}\,\delta\Big(\theta^2-\frac{\rho}{z}\Big)\Theta(z-z_{\text{cut}}).} \tag{33}$$

The upper bound on the z integral has been taken from $1 \to \infty$ because in the $z \ll 1$ limit, the integral should not depend on the particular upper bound **[This seems like sketchy reasoning...is it right? Is there a more rigorous way to prove it? It's necessary in order to get a finite result in the end]**. Another simplification came in the mMDT grooming term, where we have taken $\min[z, 1-z] \to \min[z, 1] \to z$. Note also that we have already performed the ϕ integral.

Performing the remaining integrals, we have

$$\frac{4\pi}{\alpha_s C_F}\frac{1}{\sigma_0}\frac{d\sigma}{d\rho} = \frac{2}{\Gamma(1-\epsilon)}\left(\frac{4\mu^2}{Q^2}\right)^\epsilon \frac{1}{\rho^{1+\epsilon}}\int_{z_{\text{cut}}}^\infty dz\,\frac{1}{z^{1+\epsilon}} \tag{34}$$

$$= \frac{2}{\Gamma(1-\epsilon)}\left(\frac{4\mu^2}{Q^2}\right)^\epsilon \frac{1}{\rho^{1+\epsilon}}\frac{z_{\text{cut}}^{-\epsilon}}{\epsilon}. \tag{35}$$

The prefactor can be expanded in ϵ as

$$\begin{aligned}\frac{2}{\Gamma(1-\epsilon)}\left(\frac{4\mu^2}{Q^2}\right)^\epsilon\frac{z_{\text{cut}}^\epsilon}{\epsilon} = & \frac{2}{\epsilon}+2\left(\log\left(\frac{4\mu^2}{Q^2}\right)-\log z_{\text{cut}}\right) \\ & +\epsilon\left[\log^2\left(\frac{4\mu^2}{Q^2}\right)-2\log z_{\text{cut}}\log\left(\frac{4\mu^2}{Q^2}\right)+\log^2 z_{\text{cut}}-\frac{\pi^2}{6}+4\log^2(2)\right] \\ & +\mathcal{O}\left(\epsilon^2\right).\end{aligned} \tag{36}$$

Using the usual plus-function expansion

$$\frac{1}{\rho^{1+\epsilon}} = -\frac{1}{\epsilon}\delta(\rho)+\left[\frac{1}{\rho}\right]_+-\epsilon\left[\frac{\log\rho}{\rho}\right]_++\mathcal{O}(\epsilon^2), \tag{37}$$

we therefore find

$$\boxed{\begin{aligned}\frac{4\pi}{\alpha_s C_F}\frac{1}{\sigma_0}\frac{d\sigma}{d\rho} = &-\frac{2}{\epsilon^2}\delta(\rho) + \frac{2}{\epsilon}\left(\left[\frac{1}{\rho}\right]_+ - \left[\log\left(\frac{4\mu^2}{Q^2}\right) - \log z_{\text{cut}}\right]\delta(\rho)\right) \\ &+ 2\left[\frac{1}{\rho}\right]_+\left(\log\left(\frac{4\mu^2}{Q^2}\right) - \log z_{\text{cut}}\right) - 2\left[\frac{\log\rho}{\rho}\right]_+ \\ &- \left[\log^2\left(\frac{4\mu^2}{Q^2}\right) - 2\log z_{\text{cut}}\log\left(\frac{4\mu^2}{Q^2}\right) + \log^2 z_{\text{cut}} - \frac{\pi^2}{6} + 4\log^2(2)\right]\delta(\rho) \\ &+ \mathcal{O}(\epsilon).\end{aligned}} \tag{38}$$

5. Combining soft and collinear results

To achieve a finite result, we need to add together the contributions from all of phase space:

$$\frac{d\sigma}{d\rho} = \frac{d\sigma^{\text{soft}}}{d\rho} + \frac{d\sigma^{\text{collinear}}}{d\rho} - \frac{d\sigma^{\text{soft-collinear}}}{d\rho}. \tag{39}$$

The soft-collinear contributions are subtracted at the end to compensate for over-counting: these are included both in the soft term and in the collinear term.

5.1. **Soft results.**

[Initially, I think I made a mistake with the soft contribution. I said

$$d^{d-2}k_\perp = k_\perp^{d-3}dk_\perp d\Omega_{d-3}, \tag{40}$$

but perhaps this should have been

$$d^{d-2}k_\perp = k_\perp^{d-3}dk_\perp d\Omega_{d-2}. \tag{41}$$

Thus, when integrating over the solid angle, I got

$$\Omega_{d-3} = \frac{2\pi^{(d-3)/2}}{\Gamma\left(\frac{d-3}{2}\right)} = \frac{2\sqrt{\pi}}{\pi^\epsilon\Gamma(\frac{1}{2}-\epsilon)} \tag{42}$$

instead of

$$\Omega_{d-2} = \frac{2\pi^{(d-2)/2}}{\Gamma\left(\frac{d-2}{2}\right)} = \frac{2\pi}{\pi^\epsilon\Gamma(1-\epsilon)}. \tag{43}$$

Therefore, we should actually take

$$\frac{1}{8\pi^{5/2}\Gamma(\frac{1}{2}-\epsilon)} \to \frac{1}{8\pi^2\Gamma(1-\epsilon)}. \tag{44}$$

]

Recall that the soft contribution is **[there is a missing factor of $1/4$ from the soft calculation... I will include it now]**

$$\frac{4\pi}{\alpha_s C_F}\frac{1}{\sigma_0}\frac{d\sigma^{\text{soft}}}{d\rho} = \frac{Q\mu^{2\epsilon}}{2\Gamma(1-\epsilon)}\left(\frac{4}{Q\rho}\right)^{1+\epsilon}\frac{1}{\epsilon}\left[\Theta(\rho-2z_{\text{cut}})\left(\frac{4}{Q\rho}\right)^{\epsilon} + \Theta(2z_{\text{cut}}-\rho)\left(Qz_{\text{cut}} - \frac{Q\rho}{4}\right)^{-\epsilon}\right]. \tag{45}$$

For the first term,

$$\begin{aligned}\frac{Q\mu^{2\epsilon}}{2\Gamma(1-\epsilon)}\left(\frac{4}{Q}\right)^{1+2\epsilon}\frac{1}{\epsilon}\frac{1}{\rho^\epsilon} = &\frac{2}{\epsilon} + 2\log\left(\frac{16\mu^2}{Q^2\rho}\right) \\ &+ \epsilon\left[-\frac{\pi^2}{6} + 16\log^2 2 + \log\left(\frac{16^2\mu^2}{Q^2\rho}\right)\log\left(\frac{\mu^2}{Q^2\rho}\right)\right] + \mathcal{O}(\epsilon^2),\end{aligned} \tag{46}$$

Notice that we don't need the plus-function expansion of $1/\rho^\epsilon$ because ρ never approaches 0. The plus-function expansion of $1/\rho^{1+\epsilon}$ is, on the other hand, is

$$\frac{1}{\rho^{1+\epsilon}} = -\frac{1}{\epsilon}\delta(\rho) + \left[\frac{1}{\rho}\right]_+ - \epsilon\left[\frac{\log\rho}{\rho}\right]_+ + \mathcal{O}(\epsilon), \tag{47}$$

so the full term is

$$\begin{aligned}\frac{Q\mu^{2\epsilon}}{2\Gamma(1-\epsilon)}\left(\frac{4}{Q}\right)^{1+2\epsilon}\frac{1}{\epsilon}\frac{1}{\rho^\epsilon}\frac{1}{\rho^{1+\epsilon}} &= -\frac{2}{\epsilon^2}\delta(\rho) + \frac{1}{\epsilon}\left[2\left[\frac{1}{\rho}\right]_+ - 2\log\left(\frac{16\mu^2}{Q^2\rho}\right)\delta(\rho)\right] \\ &\quad - 2\left[\frac{\log\rho}{\rho}\right]_+ + \left[\frac{\pi^2}{6} - 16\log^2 2 - \log\left(\frac{16^2\mu^2}{Q^2\rho}\right)\log\left(\frac{\mu^2}{Q^2\rho}\right)\right]\delta(\rho) \\ &\quad + 2\log\left(\frac{16\mu^2}{Q^2\rho}\right)\left[\frac{1}{\rho}\right]_+ + \mathcal{O}(\epsilon).\end{aligned} \tag{48}$$

For the second term, we have

$$\begin{aligned}\frac{Q\mu^{2\epsilon}}{2\Gamma(1-\epsilon)}\left(\frac{4}{Q}\right)^{1+\epsilon}\frac{1}{\epsilon}\left(Qz_{\text{cut}} - \frac{Q\rho}{4}\right)^{-\epsilon} &= \frac{2}{\epsilon} - 2\log\left(\frac{Q^2}{16\mu^2}(4z_{\text{cut}} - \rho)\right) \\ - \epsilon\bigg[-\frac{\pi^2}{6} + 4\log^2 2 + \log^2 Q &+ 4\log\mu\log(4\mu) - 4\log Q\log(2\mu) \\ + 2\log\left(\frac{Q}{4\mu^2}\right)\log\left(Q\left(z_{\text{cut}} - \frac{\rho}{4}\right)\right) &+ \log^2\left(Q\left(z_{\text{cut}} - \frac{\rho}{4}\right)\right)\bigg] + \mathcal{O}(\epsilon^2).\end{aligned} \tag{49}$$

The full term is then

$$\begin{aligned}\frac{Q\mu^{2\epsilon}}{2\Gamma(1-\epsilon)}\left(\frac{4}{Q}\right)^{1+\epsilon}\frac{1}{\epsilon}\frac{1}{\rho^{1+\epsilon}}\left(Qz_{\text{cut}} - \frac{Q\rho}{4}\right)^{-\epsilon} &= -\frac{2}{\epsilon^2}\delta(\rho) + \frac{2}{\epsilon}\left[\log\left(\frac{Q^2}{16\mu^2}(4z_{\text{cut}} - \rho)\right)\delta(\rho) + \left[\frac{1}{\rho}\right]_+\right] \\ + \delta(\rho)\bigg[-\frac{\pi^2}{6} + 4\log^2 2 + \log^2 Q &+ 4\log\mu\log(4\mu) - 4\log Q\log(2\mu) \\ + 2\log\left(\frac{Q}{4\mu^2}\right)\log\left(Q\left(z_{\text{cut}} - \frac{\rho}{4}\right)\right) &+ \log^2\left(Q\left(z_{\text{cut}} - \frac{\rho}{4}\right)\right)\bigg] \\ - 2\log\left(\frac{Q^2}{16\mu^2}(4z_{\text{cut}} - \rho)\right)\left[\frac{1}{\rho}\right]_+ &- 2\left[\frac{\log\rho}{\rho}\right]_+ + \mathcal{O}(\epsilon).\end{aligned} \tag{50}$$

Now, if we take $\rho > 0$, the delta functions vanish and the plus functions become normal functions. Then for the first term we have

$$\frac{Q\mu^{2\epsilon}}{2\Gamma(1-\epsilon)}\left(\frac{4}{Q}\right)^{1+2\epsilon}\frac{1}{\epsilon}\frac{1}{\rho^\epsilon}\frac{1}{\rho^{1+\epsilon}} = \frac{1}{\epsilon}\frac{2}{\rho} - \frac{2\log\rho}{\rho} + \frac{2}{\rho}\log\left(\frac{16\mu^2}{Q^2\rho}\right) + \mathcal{O}(\epsilon) \tag{51}$$

and for the second term we have

$$\frac{Q\mu^{2\epsilon}}{2\Gamma(1-\epsilon)}\left(\frac{4}{Q}\right)^{1+\epsilon}\frac{1}{\epsilon}\frac{1}{\rho^{1+\epsilon}}\left(Qz_{\text{cut}} - \frac{Q\rho}{4}\right)^{-\epsilon} = \frac{1}{\epsilon}\frac{2}{\rho} - \frac{2}{\rho}\log\left(\frac{Q^2}{16\mu^2}(4z_{\text{cut}} - \rho)\right) - \frac{2\log\rho}{\rho} + \mathcal{O}(\epsilon). \tag{52}$$

We conclude that for $\rho > 0$,

$$\begin{aligned}\frac{4\pi}{\alpha_s C_F}\frac{1}{\sigma_0}\frac{d\sigma^{\text{soft}}}{d\rho} = \frac{1}{\epsilon}\frac{2}{\rho} - \frac{2\log\rho}{\rho} + \frac{2}{\rho}\bigg[&\Theta(\rho - 2z_{\text{cut}})\log\left(\frac{16\mu^2}{Q^2\rho}\right) \\ &- \Theta(2z_{\text{cut}} - \rho)\log\left(\frac{Q^2}{16\mu^2}(4z_{\text{cut}} - \rho)\right)\bigg] + \mathcal{O}(\epsilon).\end{aligned} \tag{53}$$

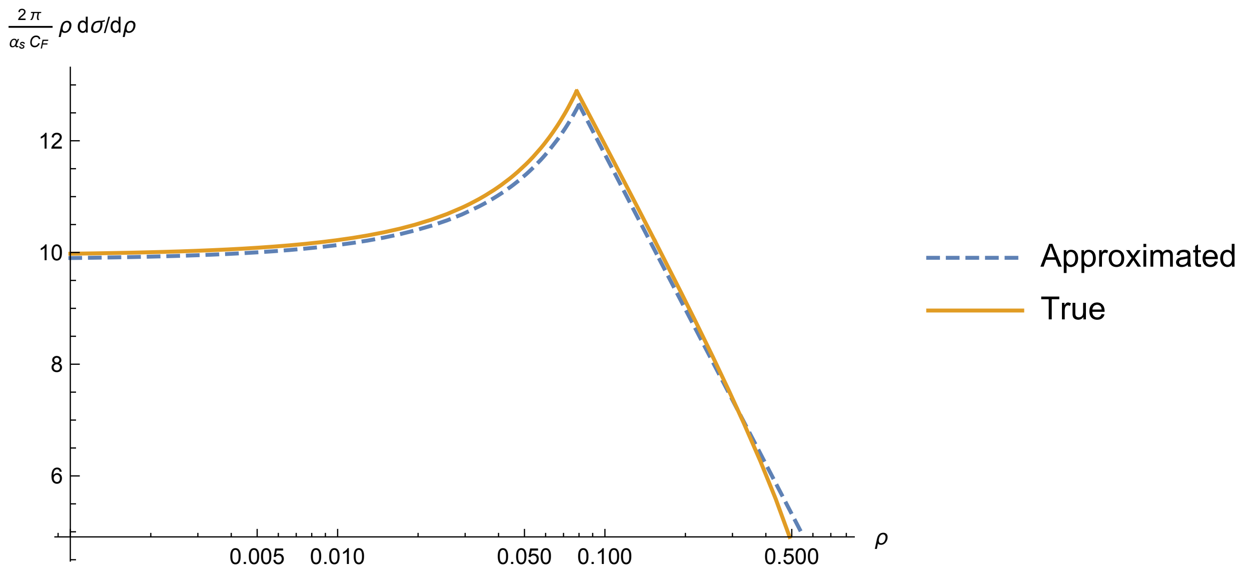

FIGURE 1. Approximation of the first-order distribution with $z_{\text{cut}} = 0.04$

5.2. **Collinear results.** From Eq. 32 with $\rho > 0$, we find

$$\frac{4\pi}{\alpha_s C_F}\frac{1}{\sigma_0}\frac{d\sigma^{\text{collinear}}}{d\rho} = \frac{1}{\rho}\left[-\frac{3}{2} + 3z_{\text{cut}} + 2\log\left(\frac{1-z_{\text{cut}}}{z_{\text{cut}}}\right)\right] + \mathcal{O}(\epsilon). \tag{54}$$

Similarly, from Eq. 38 with $\rho > 0$, we find

$$\frac{4\pi}{\alpha_s C_F}\frac{1}{\sigma_0}\frac{d\sigma^{\text{soft-collinear}}}{d\rho} = \frac{1}{\epsilon}\frac{2}{\rho} + \frac{2}{\rho}\left(\log\left(\frac{4\mu^2}{Q^2}\right) - \log z_{\text{cut}}\right) - \frac{2\log\rho}{\rho} + \mathcal{O}(\epsilon). \tag{55}$$

5.3. **Combination.** Putting everything together, we have

$$\begin{aligned}\frac{4\pi}{\alpha_s C_F}\frac{\rho}{\sigma_0}&\left[\frac{d\sigma^{\text{soft}}}{d\rho} + \frac{d\sigma^{\text{collinear}}}{d\rho} - \frac{d\sigma^{\text{soft-collinear}}}{d\rho}\right] \\ &= \Theta(\rho - 2z_{\text{cut}})2\log\left(\frac{16\mu^2}{Q^2\rho}\right) + \Theta(2z_{\text{cut}} - \rho)2\log\left(\frac{16\mu^2}{Q^2}\frac{1}{4z_{\text{cut}} - \rho}\right) \\ &\quad - 3 + 6z_{\text{cut}} + 4\log\left(\frac{1-z_{\text{cut}}}{z_{\text{cut}}}\right) - 2\log\left(\frac{4\mu^2}{Q^2}\right) + 2\log z_{\text{cut}} + \mathcal{O}(\epsilon).\end{aligned} \tag{56}$$

In the end, the physical result must be independent of μ. Notice that we can pull out a $2\log\left(4\mu^2/Q^2\right)$ from both of the Heaviside terms. Finally, we can take $\epsilon = 0$, as the divergences in ϵ have vanished. This yields

$$\begin{aligned}\frac{4\pi}{\alpha_s C_F}\frac{\rho}{\sigma_0}&\left[\frac{d\sigma^{\text{soft}}}{d\rho} + \frac{d\sigma^{\text{collinear}}}{d\rho} - \frac{d\sigma^{\text{soft-collinear}}}{d\rho}\right] \\ &= 2\Theta(\rho - 2z_{\text{cut}})\log\left(\frac{4}{\rho}\right) + 2\Theta(2z_{\text{cut}} - \rho)\log\left(\frac{1}{z_{\text{cut}} - \rho/4}\right) \\ &\quad -\frac{3}{2} + 3z_{\text{cut}} + 2\log\left(\frac{1-z_{\text{cut}}}{z_{\text{cut}}}\right) + 2\log z_{\text{cut}} + \mathcal{O}(\epsilon).\end{aligned} \tag{57}$$

The approximation is displayed in

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