CALCULATING THE SOFT FUNCTION

KEES BENKENDORFER

CONTENTS

1. SETUP

We wish to calculate the resolved soft function $S_R(\rho - z_{\text{cut}})$ which describes soft radiation which passes the groomer due to proximity to the resolved gluon. If the resolved emission occurs at an angle θ from the quark axis, then any radiation at smaller angles will pass the groomer. A schematic of this situation is displayed in Fig. 1.

The goal is to calculate the first-order term in an expansion of S_R. We can then use renormalization group evolution in conjunction with the other first-order results of functions in the factorization equation to achieve an all-orders calculation of the cross section.

Let the resolved gluon have momentum k_g, the quark lie along direction $n_q = (1, 0, 0, 1)$, and consider an extra-soft gluon with momentum k. If the extra-soft gluon is closer to the quark, then its dominant contribution to the jet mass ρ will come from its interaction with the quark:

$$\rho = \frac{4k^+}{Q} \tag{1}$$

where $k^\pm = k^0 \mp k_z$ are light-cone coordinates defined with respect to the quark axis. If the extra-soft gluon is closer to the resolved gluon, then its contribution to the jet mass from the quark interaction has already been accounted for in the contribution of the resolved gluon. The leading-order contribution from the new gluon therefore comes with its interaction with the resolved gluon. If n_g is the direction of the resolved gluon, then the contribution is

$$\rho = \frac{4k \cdot n_g}{Q} = \frac{4k \cdot k_g}{E_g Q} \tag{2}$$

with E_g the energy of the resolved gluon.

Date: 24 February 2021.

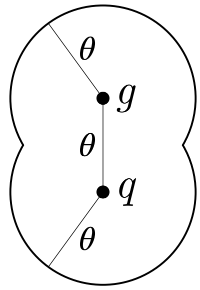

FIGURE 1. Schematic head-on view of emissions according to the jet groomer. Radiation within the peanut-shaped region will pass the grooming algorithm.

Notice that the angle between the extra-soft gluon and the quark is given by

$$1 - \cos\theta_{gq} = \frac{k^+}{k^0} \tag{3}$$

while the angle between the extra-soft gluon and the resolved gluon is

$$1 - \cos\theta_{gg} = \frac{k \cdot n_g}{k^0}. \tag{4}$$

The case in which the extra-soft gluon is closer to the quark is the case in which $\theta_{gq} < \theta_{gg}$, so $1 - \cos\theta_{gq} < 1 - \cos\theta_{gg}$ and, in turn $k^+ < k \cdot n_g$. Therefore, the total measurement function is

$$\delta_\rho = \Theta(k \cdot n_g - k^+)\,\delta\Big(\rho - \frac{4k^+}{Q}\Big) + \Theta(k^+ - k \cdot n_g)\,\delta\Big(\rho - \frac{4k \cdot n_g}{Q}\Big). \tag{5}$$

We also need to impose the kinematic constraint that the gluon is in the peanut-shaped region of Fig. 1. Saying that the gluon is in the region is equivalent to saying that it is not outside the region. The gluon is outside of the quark's radius of influence if

$$\frac{k^+}{k^0} = 1 - \cos\theta_{gq} > 1 - \cos\theta = n_g \cdot n_q. \tag{6}$$

On the other hand, the gluon is outside the resolved gluon's radius of influence if

$$\frac{k \cdot n_g}{k^0} = 1 - \cos\theta_{gg} > 1 - \cos\theta = n_g \cdot n_q. \tag{7}$$

Therefore, the grooming restriction is

$$\Theta_{\mathrm{mMDT}} = 1 - \Theta(k^+ - k^0 n_g \cdot n_q)\,\Theta(k \cdot n_g - k^0 n_g \cdot n_q). \tag{8}$$

The matrix element accounts for the possibility that the gluon be emitted from any pairs of resolved particles **[TODO: it's not actually a sum]** [1]

$$|\mathcal{M}|^2 = -\mu^{2\epsilon} \sum_{i<j} \mathbf{T}_i \cdot \mathbf{T}_j \frac{n_i \cdot n_j}{(n_i \cdot k)(n_j \cdot k)} \tag{9}$$

where i, j range over all pairs of resolved particles and the $\mathbf{T}_i$ are color matrices. Each term of the matrix element corresponds to a separate soft function. For now, we will focus on the first term [1]

$$|\mathcal{M}_{q\bar{q}}|^2 = -4\pi\alpha_s C_F \mu^{2\epsilon} \frac{n_q \cdot n_{\bar{q}}}{(n_q \cdot k)(n_{\bar{q}} \cdot k)} = -4\pi\alpha_s C_F \mu^{2\epsilon} \frac{2}{k^+ k^-} \tag{10}$$

with $n_{\bar{q}} = (1, 0, 0, -1)$ the antiquark direction. **[TODO: need to handle other soft functions**

Finally, phase space in d dimensions takes the usual form

$$d\Pi = \frac{d^d k}{(2\pi)^d} 2\pi\,\delta(k^2)\,\Theta(k^+)\,\Theta(k^- - k^+). \tag{11}$$

Notice that we are enforcing the gluon to be emitted in the hemisphere with the quark by requiring $k^- - k^+$. We will multiply the result at the end by a factor of 2 to account for the case where the gluons are emitted in the other hemisphere. Note that we are only scanning over the momentum of the extra-soft gluon: under the assumption that this gluon is softer than the resolved gluon, this emission does not influence the momentum of the quarks or resolved gluon.

Putting everything together, we find

$$\begin{aligned} S_R(\rho - z_{\rm cut}) = -8\pi\alpha_s C_F \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^{d-1}}\, \delta\big(k^2\big)\, \Theta(k^+)\, \Theta(k^- - k^+)\, \frac{2}{k^+ k^-} \\ \times \left[\Theta(k\cdot n_g - k^+)\, \delta\!\left(\rho - \frac{4k^+}{Q}\right) + \Theta(k^+ - k\cdot n_g)\, \delta\!\left(\rho - \frac{4k\cdot n_g}{Q}\right)\right] \\ \times \big[1 - \Theta(k^+ - k^0 n_g\cdot n_q)\, \Theta(k\cdot n_g - k^0 n_g \cdot n_q)\big]. \end{aligned} \tag{12}$$

2. Coordinate choice

Now we need to determine which coordinates in which to work. Notice that, physically, there is an axial symmetry to the problem: nothing depends on the angle of the resolved emission about the quark axis. Therefore, we might define our momenta in terms of their transverse momentum, pseudorapidity, and angle about the axis. To get from Cartesian (p_x, p_y, p_z) to this detector coordinate system $(p_\perp, \phi, \eta)$, we use the following transformations:

$$\begin{aligned} &p_x = p_\perp \cos\phi \qquad p_y = p_\perp \sin\phi \qquad p_z = p_\perp \sinh\eta \qquad p_0 = p_\perp \cosh\eta \\ &p_\perp = \sqrt{p_x^2 + p_y^2} \qquad \phi = \arctan\!\left(\frac{p_y}{p_x}\right) \qquad \eta = \operatorname{arctanh}\!\left(\frac{p_z}{|\mathbf{p}|}\right). \end{aligned} \tag{13}$$

Under this transformation, the extra-soft gluon has momentum

$$k = (k_0, k_\perp, \phi_k, \eta_k). \tag{14}$$

The resolved gluon is fixed in space from the perspective of the extra-soft gluon, so we can write it in whichever coordinates are convenient. Let us pick spherical coordinates, where the gluon momentum has an azimuthal angle ϕ_g and an angle θ_g from the jet axis

$$k_g = (k_0, r, \theta, \phi) = (E_g, E_g, \theta_g, \phi_g) \tag{15}$$

and hence direction vector

$$n_g = (1, 1, \theta_g, \phi_g). \tag{16}$$

Finally, without loss of generality, we can define our coordinate axes so that the resolved emission is at angle $\phi_g = 0$, thereby setting

$$k_g = (E_g, E_g, \theta_g, 0) \quad n_g = (1, 1, \theta_g, 0). \tag{17}$$

Now we can transform each term of Eq. 12. First, notice that

$$k^+ = k_0 - k_z = k_\perp(\cosh\eta_k - \sinh\eta_k) = k_\perp e^{-\eta_k}, \tag{18}$$

and similarly

$$k^- = k_\perp e^{\eta_k}. \tag{19}$$

Hence, the restriction $k^+ > 0$ becomes $k_\perp > 0$ and $k^- > k^+$ becomes $\eta_k > 0$. That is,

$$\Theta(k^+)\, \Theta(k^- - k^+) = \Theta(k_\perp)\, \Theta(\eta_k). \tag{20}$$

The first term in the matrix element is then simply

$$|\mathcal{M}|^2 = -4\pi\alpha_s C_F \frac{2}{k^+ k^-} = -4\pi\alpha_s C_F \frac{2}{k_\perp^2}. \tag{21}$$

Next comes the measurement function. First notice that (in Cartesian coordinates)

$$\begin{aligned} k \cdot n_g &= (k_\perp \cosh\eta_k, k_\perp \cos\phi_k, k_\perp \sin\phi_k, k_\perp \sinh\eta_k) \cdot (1, \sin\theta_g, 0, \cos\theta_g) \\ &= k_\perp[\cosh\eta_k - \cos\phi_k \sin\theta_g - \sinh\eta_k \cos\theta_g]. \end{aligned} \tag{22}$$

Therefore

$$\begin{aligned} \Theta(k^+ - k \cdot n_g) &= \Theta(\cos\phi_k \sin\theta_g - (1-\cos\theta_g)\sinh\eta_k) \\ &= \Theta\left(\cos\phi_k \frac{\sin\theta_g}{1-\cos\theta_g} - \sinh\eta_k\right) \\ &= \Theta\left(\cos\phi_k \cot\frac{\theta_g}{2} - \sinh\eta_k\right) \end{aligned} \tag{23}$$

and

$$\Theta(k \cdot n_g - k^+) = \Theta\left(\sinh\eta_k - \cos\phi_k \cot\frac{\theta_g}{2}\right). \tag{24}$$

The full measurement function is then

$$\begin{aligned} \delta_\rho = {} & \Theta\left(\sinh\eta_k - \cos\phi_k \cot\frac{\theta_g}{2}\right) \delta\left(\rho - \frac{4k_\perp e^{-\eta_k}}{Q}\right) \\ & + \Theta\left(\cos\phi_k \cot\frac{\theta_g}{2} - \sinh\eta_k\right) \delta\left(\rho - \frac{4k_\perp}{Q}[\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g]\right). \end{aligned} \tag{25}$$

Finally, we have the mMDT groomer. Notice that

$$\Theta\big(k^+ - k^0 n_g \cdot n_q\big) = \Theta(\cos\theta_g - \tanh\eta_k) \tag{26}$$

and

$$\Theta\big(k \cdot n_g - k^0 n_g \cdot n_q\big) = \Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k). \tag{27}$$

Therefore,

$$1 - \Theta(k^+ - k^0 n_g \cdot n_q)\,\Theta(k \cdot n_g - k^0 n_g \cdot n_q) = 1 - \Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k). \tag{28}$$

Putting everything together so far, we have

$$\begin{aligned} S_R = -8\pi\alpha_s C_F \mu^{2\epsilon} \int & \frac{d^d k}{(2\pi)^{d-1}} \delta(k^2)\,\Theta(k_\perp)\,\Theta(\eta_k)\,\frac{2}{k_\perp^2} \\ & \times \Bigg[\Theta\left(\sinh\eta_k - \cos\phi_k\cot\frac{\theta_g}{2}\right)\delta\left(\rho - \frac{4k_\perp e^{-\eta_k}}{Q}\right) \\ & \quad + \Theta\left(\cos\phi_k\cot\frac{\theta_g}{2} - \sinh\eta_k\right)\delta\left(\rho - \frac{4k_\perp}{Q}[\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g]\right)\Bigg] \\ & \times [1 - \Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k)]. \end{aligned} \tag{29}$$

The last thing to evaluate is the phase space measure. We wish to convert

$$dk_0 dk_z\, d^{d-2}k_\perp\, \delta(k^2) \to dk_0 d\eta_k\, d^{d-2}dk_\perp\, \delta(k^2) \tag{30}$$

where $k_\perp$ represents the off-axis components of k in $d-2$ dimensions. With $d = 4 - 2\epsilon$, we can write this in spherical coordinates as

$$d^{d-2}k_\perp = k_\perp^{d-3} dk_\perp\, \sin^{-2\epsilon}\phi_k\, d\phi_k\, d\Omega_{d-3} \tag{31}$$

with Ω_{d-3} the solid angle of the $d-3$ dimensional sphere **[TODO: check angular dimension]**. Integrating over this solid angle yields [2]

$$\int d\Omega_{d-3} = \frac{2\pi^{(d-3)/2}}{\Gamma(\frac{d-3}{2})} = \frac{2\pi^{1/2-\epsilon}}{\Gamma(\frac{1}{2}-\epsilon)}. \tag{32}$$

Thus, we find that

$$d^{d-2}k_\perp = dk_\perp d\phi_k\, k_\perp^{d-3} \sin^{-2\epsilon}\phi_k \, \frac{2\pi^{1/2-\epsilon}}{\Gamma(\frac{1}{2}-\epsilon)}. \tag{33}$$

Now also notice that

$$\delta(k^2) = \delta\big(k_0^2 - k_\perp^2 - k_z^2\big) = \delta\big(k_0^2 - k_\perp^2\cosh^2\eta_k\big). \tag{34}$$

This simplifies to

$$\delta\big(k_0^2 - k_\perp^2\cosh^2\eta_k\big) = \frac{1}{2k_\perp\cosh\eta_k}\delta(k_0 - k_\perp\cosh\eta_k). \tag{35}$$

Therefore, we can integrate out k_0 (notice that we have sneakily already applied the delta function where k_0 appeared earlier):

$$\int dk_0\delta(k^2) = \frac{1}{2k_\perp\cosh\eta_k}. \tag{36}$$

Finally, we need to account for the Jacobian in the (k_0, k_z) transformation:

$$\frac{\partial(k_0,k_z)}{\partial(k_0,\eta_k)} = \begin{pmatrix} 1 & 0 \\ 0 & k_\perp\cosh\eta_k \end{pmatrix}. \tag{37}$$

The standard Jacobian factor is then the determinant (in absolute value)

$$dk_0dk_z = k_\perp\cosh\eta_k\, dk_0d\eta_k. \tag{38}$$

All together, the phase space measure is

$$\begin{aligned}\int \frac{d^dk}{(2\pi)^{d-1}}\,\delta(k^2) &= \frac{\pi^{1/2-\epsilon}}{(2\pi)^{3-2\epsilon}\Gamma(\frac{1}{2}-\epsilon)}\int dk_\perp d\phi_k d\eta_k\, k_\perp^{-1-2\epsilon}\sin^{-2\epsilon}\phi_k \\ &= \frac{(4\pi)^\epsilon}{8\pi^{5/2}\Gamma(\frac{1}{2}-\epsilon)}\int dk_\perp d\phi_k d\eta_k\, k_\perp^{-1-2\epsilon}\sin^{-2\epsilon}\phi_k.\end{aligned} \tag{39}$$

Under the modified minimal subtraction scheme, we will set $(4\pi)^\epsilon \to 1$ (and will also set $\gamma_E \to 0$ as it comes up). The full integral is now

$$\boxed{\begin{aligned} S_R = {}& -\frac{2\pi\alpha_s C_F\mu^{2\epsilon}}{\pi^{5/2}\Gamma(\frac{1}{2}-\epsilon)}\int dk_\perp d\phi_k d\eta_k\, k_\perp^{-1-2\epsilon}\sin^{-2\epsilon}\phi_k\,\Theta(k_\perp)\Theta(\eta_k) \\ & \times\Bigg[\Theta\bigg(\sinh\eta_k - \cos\phi_k\cot\frac{\theta_g}{2}\bigg)\,\delta\bigg(\rho - \frac{4k_\perp e^{-\eta_k}}{Q}\bigg) \\ & \quad + \Theta\bigg(\cos\phi_k\cot\frac{\theta_g}{2} - \sinh\eta_k\bigg)\,\delta\bigg(\rho - \frac{4k_\perp}{Q}[\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g]\bigg)\Bigg] \\ & \times[1 - \Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k)].\end{aligned}} \tag{40}$$

3. Evaluating the integral

First, we want to integrate out $k_\perp$, which can be done easily enough using the Dirac delta functions. The first transforms as

$$\delta\bigg(\rho - \frac{4k_\perp e^{-\eta_k}}{Q}\bigg) = \frac{Qe^{\eta_k}}{4}\delta\bigg(k_\perp - \frac{Q\rho e^{\eta_k}}{4}\bigg), \tag{41}$$

while the second transforms as

$$\delta\Big(\rho - \frac{4k_\perp}{Q}[\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g]\Big) \\ = \frac{Q}{4(\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g)}\delta\left(k_\perp - \frac{Q\rho}{4[\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g]}\right). \tag{42}$$

Integrating out $k_\perp$ from Eq. 40, we therefore have

$$S_R = -\frac{2\pi\alpha_s C_F\mu^{2\epsilon}}{\pi^{5/2}\Gamma(\frac{1}{2}-\epsilon)}\left(\frac{Q}{4}\right)^{-2\epsilon}\frac{1}{\rho^{1+2\epsilon}}\int d\phi_k d\eta_k\ \sin^{-2\epsilon}\phi_k\,\Theta(\eta_k) \\ \times\left[\Theta\left(\sinh\eta_k - \cos\phi_k\cot\frac{\theta_g}{2}\right)e^{-2\epsilon\eta_k} \right. \\ \left. + \Theta\left(\cos\phi_k\cot\frac{\theta_g}{2} - \sinh\eta_k\right)\left(\frac{1}{\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g}\right)^{-2\epsilon}\right] \\ \times[1 - \Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k)]. \tag{43}$$

Now, we will eventually expand $\rho^{-1-2\epsilon}$ using a plus-function expansion [3]

$$\frac{1}{\rho^{1+2\epsilon}} = -\frac{1}{2\epsilon}\delta(\rho) + \left[\frac{1}{\rho}\right]_+ - \epsilon\left[\frac{\ln\rho}{\rho}\right]_+ + \mathcal{O}(\epsilon^2). \tag{44}$$

This means that, in order to calculate the cusp anomalous dimension, we need to keep terms through $\mathcal{O}(\epsilon^0)$ in the remaining integral.

To do this, we can first simplify the integral as follows. Notice that

$$\Theta\left(\sinh\eta_k - \cos\phi_k\cot\frac{\theta_g}{2}\right)e^{-2\epsilon\eta_k} \\ + \Theta\left(\cos\phi_k\cot\frac{\theta_g}{2} - \sinh\eta_k\right)\left(\frac{1}{\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g}\right)^{-2\epsilon} \\ = e^{-2\epsilon\eta_k}\left[\Theta\left(\sinh\eta_k - \cos\phi_k\cot\frac{\theta_g}{2}\right)\right. \\ \left. + \Theta\left(\cos\phi_k\cot\frac{\theta_g}{2} - \sinh\eta_k\right)\left(\frac{e^{\eta_k}}{\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g}\right)^{-2\epsilon}\right]. \tag{45}$$

Now this expression converges as we send $\eta_k \to \infty$, so we can expand the term in brackets in ϵ to find

$$\Theta\left(\sinh\eta_k - \cos\phi_k\cot\frac{\theta_g}{2}\right) \\ + \Theta\left(\cos\phi_k\cot\frac{\theta_g}{2} - \sinh\eta_k\right)\left(\frac{e^{\eta_k}}{\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g}\right)^{-2\epsilon} \\ = 1 + \mathcal{O}(\epsilon). \tag{46}$$

Therefore, if we let

$$
\begin{aligned}
I = \int d\phi_k d\eta_k \, \sin^{-2\epsilon}\phi_k \, \Theta(\eta_k) \Bigg[\Theta\!\left(\sinh\eta_k - \cos\phi_k \cot\frac{\theta_g}{2}\right) e^{-2\epsilon\eta_k} \\
+ \Theta\!\left(\cos\phi_k \cot\frac{\theta_g}{2} - \sinh\eta_k\right)\left(\frac{1}{\cosh\eta_k - \cos\phi_k \sin\theta_g - \sinh\eta_k \cos\theta_g}\right)^{-2\epsilon}\Bigg] \\
\times [1 - \Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k)],
\end{aligned} \tag{47}
$$

we find that

$$
I = \int d\phi_k d\eta_k \sin^{-2\epsilon}\phi_k \Theta(\eta_k) e^{-2\epsilon\eta_k}[1 - \Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k)] + \mathcal{O}(\epsilon). \tag{48}
$$

For the first term, we can integrate in η_k to find

$$
\int d\phi_k d\eta_k \sin^{-2\epsilon}\phi_k \Theta(\eta_k) e^{-2\epsilon\eta_k} = \frac{1}{2\epsilon}\int d\phi_k \sin^{-2\epsilon}\phi_k. \tag{49}
$$

Then integrating in ϕ_k yields

$$
\frac{1}{2\epsilon}\int d\phi_k \sin^{-2\epsilon}\phi_k = \frac{1}{2\epsilon}\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(1-\epsilon)}. \tag{50}
$$

Thus,

$$
I = \frac{1}{2\epsilon}\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(1-\epsilon)} - \int d\phi_k d\eta_k \sin^{-2\epsilon}\phi_k \Theta(\eta_k) e^{-2\epsilon\eta_k}\Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k) + \mathcal{O}(\epsilon). \tag{51}
$$

The remaining integral is not divergent in η_k if we first expand in ϵ, so let's do that. We have

$$
I = \frac{1}{2\epsilon}\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(1-\epsilon)} - \int d\phi_k d\eta_k \Theta(\eta_k)\Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k) + \mathcal{O}(\epsilon). \tag{52}
$$

To evaluate this integral, we just need to sort out the bounds in η_k. These come out to

$$
\begin{aligned}
\Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k) = \Theta\!\left(\frac{1}{1+\sec\theta_g} - \cos\phi_k\right)\Theta\!\left(\cot\frac{\theta_g}{2} - e^{\eta_k}\right) \\
+ \Bigg[\Theta\!\left(\cos\phi_k - \frac{1}{1+\sec\theta_g}\right)\Theta(\cot\theta_g - \cos\phi_k) \\
\times \Theta(\cot\theta_g \sec\phi_k - e^{\eta_k})\Bigg].
\end{aligned} \tag{53}
$$

Now, splitting into positive and negative values of $\cos\phi_k$, we have

$$
\Theta\!\left(\frac{1}{1+\sec\theta_g} - \cos\phi_k\right) = \Theta\!\left(\frac{\pi}{2} - \phi_k\right)\Theta(\sec\phi_k - 1 - \sec\theta_g) + \Theta\!\left(\phi_k - \frac{\pi}{2}\right) \tag{54}
$$

which uses the fact that

$$
\operatorname{arcsec}(1 + \sec\theta_g) < \frac{\pi}{2} \tag{55}
$$

for all $0 < \theta_g < \pi/2$. Evaluating the first part of the integral yields

$$\begin{aligned}&\int d\phi_k d\eta_k \Theta(\eta_k)\Big[\Theta\Big(\frac{\pi}{2}-\phi_k\Big)\Theta(\sec\phi_k - 1 - \sec\theta_g) + \Theta\Big(\phi_k - \frac{\pi}{2}\Big)\Big]\Theta\Big(\cot\frac{\theta_g}{2} - e^{\eta_k}\Big)\\ &= \int d\phi_k \Big[\Theta\Big(\frac{\pi}{2}-\phi_k\Big)\Theta(\sec\phi_k - 1 - \sec\theta_g) + \Theta\Big(\phi_k - \frac{\pi}{2}\Big)\Big]\log\cot\frac{\theta_g}{2}\\ &= [\pi - \operatorname{arcsec}(1+\sec\theta_g)]\log\cot\frac{\theta_g}{2}.\end{aligned} \tag{56}$$

Integrating η_k out of the second part yields

$$\begin{aligned}&\int d\phi_k d\eta_k \Theta(\eta_k)\Theta\Big(\cos\phi_k - \frac{1}{1+\sec\theta_g}\Big)\Theta(\cot\theta_g - \cos\phi_k)\Theta(\cot\theta_g\sec\phi_k - e^{\eta_k})\\ &= \int d\phi_k \Theta\Big(\cos\phi_k - \frac{1}{1+\sec\theta_g}\Big)\Theta(\cot\theta_g - \cos\phi_k)\log(\cot\theta_g\sec\phi_k).\end{aligned} \tag{57}$$

To solve the remaining integral, first notice that

$$\cot\theta_g > 1 \implies \cot\theta_g > \cos\phi_k \tag{58}$$

for $0 < \theta_g < \pi/4$. Therefore,

$$\Theta(\cot\theta_g - \cos\phi_k) = \Theta\Big(\frac{\pi}{4} - \theta_g\Big) + \Theta\Big(\theta_g - \frac{\pi}{4}\Big)\Theta(\cot\theta_g - \cos\phi_k). \tag{59}$$

Now, direct evaluation of the indefinite integral yields

$$\int d\phi_k \log(\cot\theta_g\sec\phi_k) = \frac{i\phi_k^2}{2} + \phi_k\log(2\cot\theta_g) - \frac{i}{2}\mathrm{Li}_2\Big(-e^{2i\phi_k}\Big), \tag{60}$$

where $\mathrm{Li}_2(x)$ is the dilogarithm function. While it appears that the result is complex, the imaginary part is actually constant,[1] and therefore is eliminated in a definite integral. We can see this as follows. First, the power series of the dilogarithm on the unit disk $|z| \le 1$ is

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^2}. \tag{61}$$

Therefore,

$$\mathrm{Li}_2\Big(-e^{2i\phi_k}\Big) = \sum_{k=1}^{\infty}\frac{(-1)^k e^{2ik\phi_k}}{k^2} = \sum_{k=1}^{\infty}\frac{(-1)^k\cos(2k\phi_k)}{k^2} + i\sum_{k=1}^{\infty}\frac{(-1)^k\sin(2k\phi_k)}{k^2}. \tag{62}$$

The real part is

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^k\cos(2k\phi_k)}{k^2} &= \frac{1}{2}\Bigg[\sum_{k=1}^{\infty}\frac{(-1)^k e^{-2ik\phi_k}}{k^2} + \sum_{k=1}^{\infty}\frac{(-1)^k e^{2ik\phi_k}}{k^2}\Bigg]\\ &= \frac{1}{2}\Big[\mathrm{Li}_2\Big(-e^{2i\phi_k}\Big) + \mathrm{Li}_2\Big(-e^{-2i\phi_k}\Big)\Big].\end{aligned} \tag{63}$$

Now, it is an identity of the dilogarithm[2] that

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\Big(\frac{1}{z}\Big) = -\frac{\pi^2}{6} - \frac{1}{2}\log^2(-z). \tag{64}$$

Therefore,

$$\mathrm{Li}_2\Big(-e^{2i\phi_k}\Big) + \mathrm{Li}_2\Big(-e^{-2i\phi_k}\Big) = -\frac{\pi^2}{6} - \frac{1}{2}\log^2\Big(e^{2i\phi_k}\Big) = -\frac{\pi^2}{6} + 2\phi_k^2. \tag{65}$$

[1] Albeit after much pain and wandering

[2] Away from a branch cut

Thus, we see that

$$\mathrm{Re}\left[\mathrm{Li}_2\left(-e^{2i\phi_k}\right)\right] = -\frac{\pi^2}{12} + \phi_k^2. \tag{66}$$

The imaginary part is, to my knowledge, more difficult to simplify **[is this true?]**:

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(-1)^k \sin(2k\phi_k)}{k^2} &= \frac{i}{2}\left[\sum_{k=1}^{\infty} \frac{(-1)^k e^{-2ik\phi_k}}{k^2} - \sum_{k=1}^{\infty} \frac{(-1)^k e^{2ik\phi_k}}{k^2}\right] \\ &= \frac{i}{2}\left[\mathrm{Li}_2\left(-e^{-2i\phi_k}\right) - \mathrm{Li}_2\left(-e^{2i\phi_k}\right)\right].\end{aligned} \tag{67}$$

Therefore,

$$\mathrm{Li}_2\left(-e^{2i\phi_k}\right) = -\frac{\pi^2}{12} + \phi_k^2 - \frac{1}{2}\left[\mathrm{Li}_2\left(-e^{-2i\phi_k}\right) - \mathrm{Li}_2\left(-e^{2i\phi_k}\right)\right] \tag{68}$$

(the portion in square brackets is entirely imaginary). Putting everything together yields

$$\int d\phi_k \log(\cot\theta_g \sec\phi_k) = \phi_k \log(2\cot\theta_g) + \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\phi_k}\right) - \mathrm{Li}_2\left(-e^{2i\phi_k}\right)\right] + \frac{i\pi^2}{24}. \tag{69}$$

The imaginary portion has been condensed to a constant in the final term. Combining Eqs. 57, 59, and 69 and noting that

$$\arccos\left(\frac{1}{1+\sec\theta_g}\right) = \mathrm{arcsec}(1+\sec\theta_g) \tag{70}$$

then yields

$$\begin{aligned}&\int d\phi_k \Theta\left(\cos\phi_k - \frac{1}{1+\sec\theta_g}\right)\Theta(\cot\theta_g - \cos\phi_k)\log(\cot\theta_g\sec\phi_k) \\ &\quad= \int d\phi_k \Theta\left(\cos\phi_k - \frac{1}{1+\sec\theta_g}\right)\left[\Theta\left(\frac{\pi}{4}-\theta_g\right) + \Theta\left(\theta_g - \frac{\pi}{4}\right)\Theta(\cot\theta_g - \cos\phi_k)\right] \\ &\quad= \mathrm{arcsec}\,(1+\sec\theta_g)\log(2\cot\theta_g) + \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\,\mathrm{arcsec}\,(1+\sec\theta_g)}\right) - \mathrm{Li}_2\left(-e^{2i\,\mathrm{arcsec}\,(1+\sec\theta_g)}\right)\right] \\ &\qquad - \Theta\left(\theta_g - \frac{\pi}{4}\right)\left[\arccos\cot\theta_g \log(2\cot\theta_g) + \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\arccos\cot\theta_g}\right) - \mathrm{Li}_2\left(-e^{2i\arccos\cot\theta_g}\right)\right]\right].\end{aligned} \tag{71}$$

We conclude that the full integral of Eq. 52 is

$$\begin{aligned}I = &\frac{1}{2\epsilon}\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(1-\epsilon)} - [\pi - \mathrm{arcsec}(1+\sec\theta_g)]\log\cot\frac{\theta_g}{2} - \mathrm{arcsec}\,(1+\sec\theta_g)\log(2\cot\theta_g) \\ &- \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\,\mathrm{arcsec}\,(1+\sec\theta_g)}\right) - \mathrm{Li}_2\left(-e^{2i\,\mathrm{arcsec}\,(1+\sec\theta_g)}\right)\right] \\ &+ \Theta\left(\theta_g - \frac{\pi}{4}\right)\left[\arccos\cot\theta_g\log(2\cot\theta_g)\right. \\ &\qquad\left. + \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\arccos\cot\theta_g}\right) - \mathrm{Li}_2\left(-e^{2i\arccos\cot\theta_g}\right)\right]\right] + \mathcal{O}(\epsilon).\end{aligned} \tag{72}$$

The non-divergent portion of this integral (corresponding to the $1/\epsilon$ term in the soft function) is displayed in Fig. 2. The full soft function is therefore

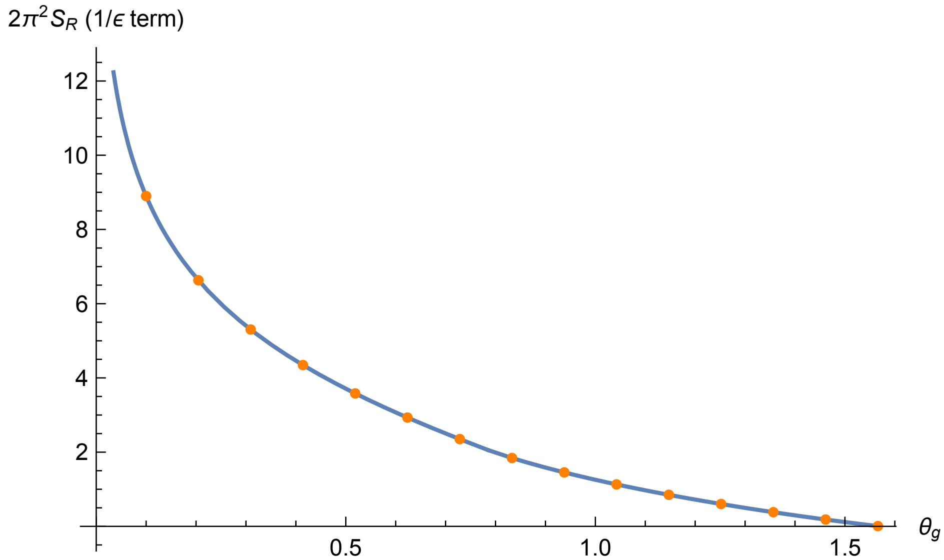

FIGURE 2. Analytic (solid blue line) and numeric (orange dots) values of the ϵ^{-1} contribution to the soft function. Numerics are calculated from the integral of Eq. 52, and the analytic solution is that of Eq. 72.

$$\boxed{\begin{aligned} S_R = &-\frac{2\pi\alpha_s C_F \mu^{2\epsilon}}{\pi^{5/2}\Gamma(\frac{1}{2}-\epsilon)}\left(\frac{Q}{4}\right)^{-2\epsilon}\frac{1}{\rho^{1+2\epsilon}} \\ &\times\left[\frac{1}{2\epsilon}\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(1-\epsilon)} - [\pi - \operatorname{arcsec}(1+\sec\theta_g)]\log\cot\frac{\theta_g}{2}\right. \\ &\quad - \operatorname{arcsec}(1+\sec\theta_g)\log(2\cot\theta_g) \\ &\quad - \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\operatorname{arcsec}(1+\sec\theta_g)}\right) - \mathrm{Li}_2\left(-e^{2i\operatorname{arcsec}(1+\sec\theta_g)}\right)\right] \\ &\quad + \Theta\left(\theta_g - \frac{\pi}{4}\right)\left[\arccos\cot\theta_g\log(2\cot\theta_g)\right. \\ &\quad\quad \left.\left. + \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\arccos\cot\theta_g}\right) - \mathrm{Li}_2\left(-e^{2i\arccos\cot\theta_g}\right)\right]\right] + \mathcal{O}(\epsilon)\right]. \end{aligned}} \tag{73}$$

4. SECOND TERM

Now let us consider the second term of Eq. 9, which corresponds to emission off of the quark-gluon dipole. We have **[TODO: prefactor isn't quite right]** [1]

$$|\mathcal{M}_2|^2 = -\mathbf{T}_g\cdot\mathbf{T}_q\mu^{2\epsilon}\frac{n_q\cdot n_g}{(n_q\cdot k)(n_g\cdot k)} = -\mathbf{T}_g\cdot\mathbf{T}_q\mu^{2\epsilon}\frac{1-\cos\theta_g}{k^+(n_g\cdot k)}. \tag{74}$$

The phase space and grooming functions are all the same, so the contribution to compute is

$$S_R^{II}(\rho - z_{\mathrm{cut}}) = -\mathbf{T}_g \cdot \mathbf{T}_q \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^{d-1}} \, \delta(k^2)\, \Theta(k^+)\, \Theta(k^- - k^+)\, \frac{1-\cos\theta_g}{k^+(n_g \cdot k)} \times \left[\Theta(k\cdot n_g - k^+)\, \delta\left(\rho - \frac{4k^+}{Q}\right) + \Theta(k^+ - k\cdot n_g)\, \delta\left(\rho - \frac{4k\cdot n_g}{Q}\right)\right] \times \left[1 - \Theta(k^+ - k^0 n_g\cdot n_q)\, \Theta(k\cdot n_g - k^0 n_g \cdot n_q)\right]. \tag{75}$$

Under the coordinate choices described above, the matrix element becomes (from Eqs. 18 and 22)

$$|\mathcal{M}_2|^2 \propto \frac{(1-\cos\theta_g)e^{\eta_k}}{k_\perp^2[\cosh\eta_k - \cos\phi_k \sin\theta_g - \sinh\eta_k\cos\theta_g]}, \tag{76}$$

so the full function to compute becomes

$$\boxed{\begin{aligned} S_R^{II}(\rho - z_{\mathrm{cut}}) = {}& -\mathbf{T}_g \cdot \mathbf{T}_q \frac{\mu^{2\epsilon}(1-\cos\theta_g)}{4\pi^{5/2}\Gamma(\frac{1}{2}-\epsilon)} \int dk_\perp d\phi_k d\eta_k \, \sin^{-2\epsilon}\phi_k\, \Theta(k_\perp)\Theta(\eta_k) \\ & \times \frac{e^{\eta_k}}{k_\perp^{1+2\epsilon}[\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g]} \\ & \times \left[\Theta\left(\sinh\eta_k - \cos\phi_k\cot\frac{\theta_g}{2}\right)\delta\left(\rho - \frac{4k_\perp e^{-\eta_k}}{Q}\right)\right. \\ & \left. + \Theta\left(\cos\phi_k\cot\frac{\theta_g}{2} - \sinh\eta_k\right)\delta\left(\rho - \frac{4k_\perp}{Q}[\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g]\right)\right] \\ & \times [1 - \Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k)]. \end{aligned}} \tag{77}$$

Integrating out $k_\perp$ leaves us with

$$\begin{aligned} S_R^{II} = {}& -\mathbf{T}_g \cdot \mathbf{T}_q \frac{\mu^{2\epsilon}(1-\cos\theta_g)}{4\pi^{5/2}\Gamma(\frac{1}{2}-\epsilon)} \left(\frac{Q}{4}\right)^{-2\epsilon} \frac{1}{\rho^{1+2\epsilon}} \int d\phi_k d\eta_k\, \sin^{-2\epsilon}\phi_k\, \Theta(\eta_k) \\ & \times \frac{e^{\eta_k}}{\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g} \\ & \times \left[\Theta\left(\sinh\eta_k - \cos\phi_k\cot\frac{\theta_g}{2}\right) e^{-2\epsilon\eta_k}\right. \\ & \left. + \Theta\left(\cos\phi_k\cot\frac{\theta_g}{2} - \sinh\eta_k\right)\left(\frac{1}{\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g}\right)^{-2\epsilon}\right] \\ & \times [1 - \Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k)]. \end{aligned} \tag{78}$$

Performing the same trick as in Eqs. 45 and 46, we then have

$$\begin{aligned} S_R^{II} = {}& -\mathbf{T}_g \cdot \mathbf{T}_q \frac{\mu^{2\epsilon}(1-\cos\theta_g)}{4\pi^{5/2}\Gamma(\frac{1}{2}-\epsilon)} \left(\frac{Q}{4}\right)^{-2\epsilon} \frac{1}{\rho^{1+2\epsilon}} \int d\phi_k d\eta_k\, \sin^{-2\epsilon}\phi_k\, \Theta(\eta_k) \\ & \times \frac{e^{\eta_k - 2\epsilon\eta_k}}{\cosh\eta_k - \cos\phi_k\sin\theta_g - \sinh\eta_k\cos\theta_g} \\ & \times [1 - \Theta(\cos\theta_g - \tanh\eta_k)\,\Theta(\cot\theta_g - e^{\eta_k}\cos\phi_k)] + \mathcal{O}(\epsilon). \end{aligned} \tag{79}$$

[TODO: ask Andrew about this integral]

5. LAPLACE TRANSFORMATION

For the rest of our analysis, it will be convenient to work in Laplace space. This is because Laplace transformation turns convolution into a simple product. Our factorization theorem involves a convolution of multiple functions, and a product is easier to work with. Luckily, the Laplace transformation of the soft function Eq. 73 in ρ is quite straightforward. Transforming $\rho \to \nu$, we have

$$\mathcal{L}\left\{\frac{1}{\rho^{1+2\epsilon}}\right\} = \int_0^\infty \frac{d\rho}{\rho^{1+2\epsilon}} e^{-\rho\nu} = \nu^{2\epsilon}\Gamma(-2\epsilon). \tag{80}$$

The Laplace-transformed soft function is therefore simply

$$\begin{aligned}\mathcal{L}\{S_R\} = &-\frac{2\pi\alpha_s C_F \mu^{2\epsilon}}{\pi^{5/2}\Gamma(\frac{1}{2}-\epsilon)}\left(\frac{Q}{4}\right)^{-2\epsilon}\nu^{2\epsilon}\Gamma(-2\epsilon)\\ &\times\Bigg[\frac{1}{2\epsilon}\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(1-\epsilon)} - [\pi - \operatorname{arcsec}(1+\sec\theta_g)]\log\cot\frac{\theta_g}{2}\\ &\quad - \operatorname{arcsec}(1+\sec\theta_g)\log(2\cot\theta_g)\\ &\quad - \frac{i}{4}\Big[\mathrm{Li}_2\Big(-e^{-2i\,\operatorname{arcsec}(1+\sec\theta_g)}\Big) - \mathrm{Li}_2\Big(-e^{2i\,\operatorname{arcsec}(1+\sec\theta_g)}\Big)\Big]\\ &\quad + \Theta\Big(\theta_g - \frac{\pi}{4}\Big)\Bigg[\arccos\cot\theta_g\log(2\cot\theta_g)\\ &\qquad\qquad + \frac{i}{4}\Big[\mathrm{Li}_2\Big(-e^{-2i\arccos\cot\theta_g}\Big) - \mathrm{Li}_2\Big(-e^{2i\arccos\cot\theta_g}\Big)\Big]\Bigg] + \mathcal{O}(\epsilon)\Bigg].\end{aligned} \tag{81}$$

5.1. **Expansion.** Now we can expand to partial 0-th order in ϵ. The prefactor becomes (after setting $\gamma_E \to 0$)

$$\frac{2\pi\alpha_s C_F \mu^{2\epsilon}}{\pi^{5/2}\Gamma(\frac{1}{2}-\epsilon)}\left(\frac{Q}{4}\right)^{-2\epsilon}\nu^{2\epsilon}\Gamma(-2\epsilon) = \frac{\alpha_s C_F}{\pi^2}\left[-\frac{1}{\epsilon} + 2\log\left(\frac{Q}{2\nu}\right) - 2\log\mu - 2\epsilon\log\left(\frac{\mu\nu}{Q}\right)\log\left(\frac{4\mu\nu}{Q}\right)\right] + \mathcal{O}(\epsilon). \tag{82}$$

Since

$$\log\left(\frac{\mu\nu}{Q}\right)\log\left(\frac{4\mu\nu}{Q}\right) = \log^2\mu + \log\mu\log\left(\frac{4\nu^2}{Q^2}\right) + \log\left(\frac{\nu}{Q}\right)\log\left(\frac{4\nu}{Q}\right) \tag{83}$$

and ultimately we only care about factors of $\log\mu$ to order ϵ^0, we can then write

$$\frac{2\pi\alpha_s C_F \mu^{2\epsilon}}{\pi^{5/2}\Gamma(\frac{1}{2}-\epsilon)}\left(\frac{Q}{4}\right)^{-2\epsilon}\nu^{2\epsilon}\Gamma(-2\epsilon) = \frac{\alpha_s C_F}{\pi^2}\left[-\frac{1}{\epsilon} + 2\log\left(\frac{Q}{2\nu}\right) - 2\log\mu - 2\epsilon\left(\log^2\mu + \log\mu\log\left(\frac{4\nu^2}{Q^2}\right)\right)\right] + \mathcal{O}(\epsilon). \tag{84}$$

We also have

$$\frac{1}{2\epsilon}\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(1-\epsilon)} = \frac{\pi}{2\epsilon} + \pi\log 2 + \mathcal{O}(\epsilon). \tag{85}$$

The soft function then becomes

$$\boxed{\begin{aligned}\mathcal{L}\{S_R\} = -\frac{\alpha_s C_F}{\pi^2}&\left[-\frac{1}{\epsilon} + 2\log\left(\frac{Q}{2\nu}\right) - 2\log\mu - 2\epsilon\left(\log^2\mu + \log\mu\log\left(\frac{4\nu^2}{Q^2}\right)\right)\right] \\ &\times\left[\frac{\pi}{2\epsilon} + \pi\log 2 - [\pi - \operatorname{arcsec}(1+\sec\theta_g)]\log\cot\frac{\theta_g}{2}\right. \\ &\quad - \operatorname{arcsec}(1+\sec\theta_g)\log(2\cot\theta_g) \\ &\quad - \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\operatorname{arcsec}(1+\sec\theta_g)}\right) - \mathrm{Li}_2\left(-e^{2i\operatorname{arcsec}(1+\sec\theta_g)}\right)\right] \\ &\quad + \Theta\left(\theta_g - \frac{\pi}{4}\right)\left[\arccos\cot\theta_g\log(2\cot\theta_g)\right. \\ &\qquad\qquad \left.\left.+ \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\arccos\cot\theta_g}\right) - \mathrm{Li}_2\left(-e^{2i\arccos\cot\theta_g}\right)\right]\right]\right] + \mathcal{O}(\epsilon^0).\end{aligned}} \tag{86}$$

Notice that the overall order of the expansion has been reduced because there are terms of order ϵ — we just don't care about them because they do not come along with a $\log\mu$.

6. Resummation

6.1. **Strategy.** We would like to renormalize the resolved soft function as in [4]. We do this by demanding that the final total cross section be independent of μ. The scale μ, after all, was introduced solely to mathematically regulate divergences; it has no physical meaning.

Suppose now that we have a cross section which is the product of two functions in Laplace space:

$$\sigma = F_1 F_2. \tag{87}$$

Also suppose that F_1 and F_2 each have an anomalous dimension (i.e., the coefficient of the $1/\epsilon$ divergence in their Laurent expansion) of γ_1 and γ_2, respectively. It is a general result of QFT **[need to look into this more]** that [4]

$$\frac{\partial F_1}{\partial\log\mu} = \gamma_1 F_1 \qquad\qquad \frac{\partial F_2}{\partial\log\mu} = \gamma_1 F_2. \tag{88}$$

Then the product rule reveals that

$$\frac{\partial\sigma}{\partial\log\mu} = \frac{\partial F_1}{\partial\log\mu}F_2 + F_1\frac{\partial F_2}{\partial\log\mu} = (\gamma_1+\gamma_2)F_1F_2. \tag{89}$$

Since we demand that $\partial\sigma/\partial\mu = 0$, it must be the case that the anomalous dimensions sum to zero,

$$\gamma_1 + \gamma_2 = 0. \tag{90}$$

We can use this relationship to check the consistency of the factorization at the end.

The renormalization scheme for now is to demand that

$$\frac{\partial F}{\partial\log\mu} = \gamma F \tag{91}$$

for the anomalous dimension γ. Then if μ_1 is the infrared scale of the μ-logarithms, we can write the anomalous dimension as [4]

$$\gamma = \Gamma_F(\alpha_s)\log\frac{\mu^2}{\mu_1^2} + \gamma_F(\alpha_s), \tag{92}$$

where $\Gamma_F(\alpha_s)$ is the cusp anomalous dimension and $\gamma_F(\alpha_s)$ is the non-cusp anomalous dimension. These can be calculated order-by-order in α_s — the non-cusp part is calculated as in previous sections, while the cusp part has a universal expansion up to a normalization [4] **[is this right?]**. Now we can convert between the renormalization scale μ and the strong coupling α_s using the β-function defined by

$$d \log \mu = \frac{d\alpha_s}{\beta(\alpha_s)} \tag{93}$$

(notice the connections with the running of the strong coupling). Thus, the renormalization equation 91 becomes **[I think this is wrong but it doesn't matter. . . not using it right now anyway]**

$$\beta(\alpha_s)\frac{\partial F}{\partial \alpha_s} = \left[\Gamma_F(\alpha_s)\log\frac{\mu^2}{\mu_1^2} + \gamma_F(\alpha_s)\right] F(\mu). \tag{94}$$

The solution to the differential equation is [4]

$$\begin{aligned} F(\mu) = F(\mu_0)\exp\Bigg[2\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)}\Gamma_F(\alpha)\int_{\alpha_s(\mu_0)}^{\alpha}\frac{d\alpha'}{\beta(\alpha')} + \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}\frac{d\alpha}{\beta(\alpha)}\gamma_F(\alpha) \\ + \log\frac{\mu_0^2}{\mu_1^2}\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}\frac{d\alpha}{\beta(\alpha)}\Gamma_F(\alpha)\Bigg] \end{aligned} \tag{95}$$

where μ_0 is an arbitrary reference scale.

We can use this equation to evaluate the function to arbitrary accuracy, if we know the integrands to sufficiently high order. The cusp part of the anomalous dimension is, again, known and universal. The particular goal here is to identify the non-cusp contribution to the anomalous dimension.

6.2. **Anomalous dimension.** The first step is to determine the anomalous dimension for the soft resolved function. Starting at Eq. 86, we have

$$\mathcal{L}\{S_R\} = -\frac{\alpha_s C_F}{\pi^2}\left[-\frac{1}{\epsilon} - 2\log\left(\frac{2\mu\nu}{Q}\right) - 2\epsilon\left(\log^2\mu + \log\mu\log\left(\frac{4\nu^2}{Q^2}\right)\right)\right]\left[\frac{\pi}{2\epsilon} + f(\theta_g)\right] + \mathcal{O}(\epsilon^0) \tag{96}$$

with

$$\begin{aligned} f(\theta_g) = -\frac{1}{2\pi^3}\Bigg[&\pi\log 2 - [\pi - \operatorname{arcsec}(1+\sec\theta_g)]\log\cot\frac{\theta_g}{2} - \operatorname{arcsec}(1+\sec\theta_g)\log(2\cot\theta_g) \\ &- \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\operatorname{arcsec}(1+\sec\theta_g)}\right) - \mathrm{Li}_2\left(-e^{2i\operatorname{arcsec}(1+\sec\theta_g)}\right)\right] \\ &+ \Theta\left(\theta_g - \frac{\pi}{4}\right)\Bigg[\arccos\cot\theta_g\log(2\cot\theta_g) \\ &\qquad + \frac{i}{4}\left[\mathrm{Li}_2\left(-e^{-2i\arccos\cot\theta_g}\right) - \mathrm{Li}_2\left(-e^{2i\arccos\cot\theta_g}\right)\right]\Bigg]\Bigg]. \end{aligned} \tag{97}$$

Now, in general, the cusp anomalous dimension has expansion [4]

$$\Gamma_F(\alpha_s) = d_F\Gamma_{\text{cusp}}(\alpha_s) = d_F\sum_{n=0}^{\infty}\Gamma_n\left(\frac{\alpha_s}{4\pi}\right)^{n+1} \tag{98}$$

for some normalization d_F, and the non-cusp anomalous dimension has expansion

$$\gamma_F(\alpha_s) = \sum_{n=0}^{\infty}\gamma_n\left(\frac{\alpha_s}{4\pi}\right)^{n+1}. \tag{99}$$

We wish to determine the coefficients Γ_n and γ_n. Expanding out the soft function, we have

$$\begin{aligned}\mathcal{L}\{S_R\} &= -\frac{\alpha_s C_F}{\pi^2}\left[-\frac{\pi}{2\epsilon^2}-\frac{f(\theta_g)}{\epsilon}-\frac{\pi}{\epsilon}\log\left(\frac{2\mu\nu}{Q}\right)-\pi\left(\log^2\mu+\log\mu\log\left(\frac{4\nu^2}{Q^2}\right)\right)\right]+\mathcal{O}(\epsilon^0)\\ &= \frac{\alpha_s C_F}{4\pi}\left[\frac{2}{\epsilon^2}+\frac{4f(\theta_g)}{\pi\epsilon}+\frac{4}{\epsilon}\log\left(\frac{2\mu\nu}{Q}\right)+\frac{1}{4}\left(\log^2\mu+\log\mu\log\left(\frac{4\nu^2}{Q^2}\right)\right)\right]+\mathcal{O}(\epsilon^0).\end{aligned} \tag{100}$$

The anomalous dimension is the coefficient of $1/\epsilon$:

$$\gamma_{S_R} = \frac{\alpha_s C_F}{4\pi}\left[\frac{4f(\theta_g)}{\pi}+2\log\left(\frac{4\mu^2\nu^2}{Q^2}\right)\right]. \tag{101}$$

Notice that this takes the form **[is this supposed to happen? or is it weird to have a $\log\mu$ with a $1/\epsilon$?]**

$$\gamma = \Gamma_{S_R}(\alpha_s)\log\frac{\mu^2}{\mu_1^2}+\gamma_{S_R}(\alpha_s) \tag{102}$$

with infrared scale

$$\mu_1 = \frac{Q}{2\nu}, \tag{103}$$

cusp anomalous dimension

$$\Gamma_{S_R}(\alpha_s) = 2C_F\left(\frac{\alpha_s}{4\pi}\right), \tag{104}$$

and non-cusp anomalous dimension

$$\gamma_{S_R}(\alpha_s) = \frac{4C_F}{\pi}f(\theta_g)\left(\frac{\alpha_s}{4\pi}\right). \tag{105}$$

Therefore, to one-loop accuracy **[correct terminology?]**, the cusp anomalous dimension is

$$\Gamma_{S_R} = \frac{C_F}{2}\Gamma_{\text{cusp}} \tag{106}$$

where [4]

$$\Gamma_{\text{cusp}} = 4\left(\frac{\alpha_s}{4\pi}\right)+\mathcal{O}(\alpha_s^2) \tag{107}$$

[is this right?]. The non-cusp anomalous dimension has a one-loop coefficient

$$\boxed{\gamma_0 = \frac{4C_F}{\pi}f(\theta_g).} \tag{108}$$

6.3. **Evaluation.** Now that we have γ_0 in hand, we wish to evaluate Eq. 95. The β-function can be written as a power series

$$\beta(\alpha_s) = \mu\frac{\partial\alpha_s}{\partial\mu} = -2\alpha_s\sum_{n=0}^{\infty}\beta_n\left(\frac{\alpha_s}{4\pi}\right)^{n+1} \tag{109}$$

where the coefficients are known [4]

$$\begin{aligned}\beta_0 &= \frac{11}{3}C_A-\frac{4}{3}T_R n_f\\ \beta_1 &= \frac{34}{3}C_A^2-4T_R n_f\left(C_F+\frac{5}{3}C_A\right).\end{aligned} \tag{110}$$

[TODO: figure out what T_R and n_f are]. The cusp anomalous dimension coefficients are also known to one-loop order:

$$\begin{aligned}\Gamma_0 &= 4\\ \Gamma_1 &= 4C_A\left(\frac{67}{9}-\frac{\pi^2}{3}\right)-\frac{80}{9}T_R n_f.\end{aligned} \tag{111}$$

Now we can perform each integral. With γ_0, we ought to be able to push to NLL order, so we want the exponent to have power α_s^0. The first exponentiated integral is

$$\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)} \Gamma_F(\alpha) \int_{\alpha_s(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} = A + \mathcal{O}(\alpha_s). \tag{112}$$

If we want this integral to order α_s^0, then we need its integrand to order α_s^{-1}:

$$\frac{\Gamma_F(\alpha)}{\beta(\alpha)} \int_{\alpha_s(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} = B + \mathcal{O}(\alpha^0). \tag{113}$$

Now, $1/\beta(\alpha_s)$ has an order-2 pole a at $\alpha_s = 0$, so we can divide this integral into two Laurent series

$$\begin{aligned}
\frac{\Gamma_F(\alpha)}{\beta(\alpha)} &= \frac{A_1}{\alpha} + A_2 \\
\int_{\alpha_s(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} &= \frac{B_1}{\alpha} + B_2 \\
\frac{\Gamma_F(\alpha)}{\beta(\alpha)} \int_{\alpha_s(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} &= \frac{A_1 B_1}{\alpha^2} + \frac{A_1 B_2 + A_2 B_1}{\alpha} + \mathcal{O}(\alpha^0).
\end{aligned} \tag{114}$$

Now to get the expression in the second line, we need to know $1/\beta(\alpha')$ to order $(\alpha')^{-1}$. Thus,

$$\begin{aligned}
\int_{\alpha_s(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} &= -\frac{1}{2\beta_0} \int_{\alpha_s(\mu_0)}^{\alpha} d\alpha' \left[\frac{4\pi}{(\alpha')^2} - \frac{\beta_1}{\beta_0 \alpha'} + \mathcal{O}((\alpha')^0\right] \\
&= \frac{1}{2\beta_0} \left[\frac{4\pi}{\alpha'} + \frac{\beta_1}{\beta_2} \log \alpha'\right]\Bigg|_{\alpha'=\alpha_s(\mu_0)}^{\alpha} + \mathcal{O}(\alpha) \\
&= \frac{1}{2\beta_0} \left[4\pi\left(\frac{1}{\alpha} - \frac{1}{\alpha_s(\mu_0)}\right) + \frac{\beta_1}{\beta_0} \log \frac{\alpha}{\alpha_s(\mu_0)}\right] + \mathcal{O}(\alpha).
\end{aligned} \tag{115}$$

If $\Gamma_F = d_F \Gamma_{\text{cusp}}$, the integrand of interest then becomes

$$\begin{aligned}
\frac{\Gamma_F(\alpha)}{\beta(\alpha)} \int_{\alpha_s(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} &= \frac{d_F}{4\beta_0^2} \left[-\frac{\Gamma_0}{\alpha} + \frac{\beta_1 \Gamma_0 - \beta_0 \Gamma_1}{4\pi \beta_0}\right] \left[4\pi\left(\frac{1}{\alpha} - \frac{1}{\alpha_s(\mu_0)}\right) + \frac{\beta_1}{\beta_0} \log \frac{\alpha}{\alpha_s(\mu_0)}\right] + \mathcal{O}(\alpha^0) \\
&= \frac{d_F}{4\beta_0^2} \left[\frac{4\pi \Gamma_0}{\alpha}\left(\frac{1}{\alpha_s(\mu_0)} - \frac{1}{\alpha}\right) - \frac{\Gamma_0 \beta_1}{\beta_0 \alpha} \log \frac{\alpha}{\alpha_s(\mu_0)} + \frac{\beta_1 \Gamma_0 - \beta_0 \Gamma_1}{\beta_0 \alpha}\right] + \mathcal{O}(\alpha^0) \\
&= \frac{d_F \Gamma_0}{4\beta_0^2} \left[-\frac{4\pi}{\alpha^2} + \frac{1}{\alpha}\left(\frac{\beta_1 \Gamma_0 - \beta_0 \Gamma_1}{\beta_0 \Gamma_0} + \frac{4\pi}{\alpha_s(\mu_0)} - \frac{\beta_1}{\beta_0} \log \frac{\alpha}{\alpha_s(\mu_0)}\right)\right] + \mathcal{O}(\alpha^0).
\end{aligned} \tag{116}$$

This is now straightforward to integrate:

$$\begin{aligned}
&\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)} \Gamma_F(\alpha) \int_{\alpha_s(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} \\
&\quad = \frac{d_F \Gamma_0}{4\beta_0^2} \left[\frac{4\pi}{\alpha} + \log \alpha \left(\frac{\beta_1 \Gamma_0 - \beta_0 \Gamma_1}{\beta_0 \Gamma_0} + \frac{4\pi}{\alpha_s(\mu_0)}\right) + \frac{\beta_1}{2\beta_0} \log^2\left(\frac{\alpha}{\alpha_s(\mu_0)}\right)\right]\Bigg|_{\alpha=\alpha_s(\mu_0)}^{\alpha_s(\mu)} + \mathcal{O}(\alpha_s) \\
&\quad = \frac{d_F \Gamma_0}{4\beta_0^2} \Bigg[4\pi\left(\frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(\mu_0)}\right) + \log \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \left(\frac{\beta_1 \Gamma_0 - \beta_0 \Gamma_1}{\beta_0 \Gamma_0} + \frac{4\pi}{\alpha_s(\mu_0)}\right) \\
&\qquad + \frac{\beta_1}{2\beta_0} \log^2\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)\Bigg] + \mathcal{O}(\alpha_s).
\end{aligned} \tag{117}$$

If we let

$$r = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}, \tag{118}$$

we end up with

$$\begin{aligned}\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)} \Gamma_F(\alpha) \int_{\alpha_s(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} \\ = \frac{d_F \Gamma_0}{4\beta_0^2} \left[\frac{4\pi}{\alpha_s(\mu_0)} \left(\log r + \frac{1}{r} - 1 \right) + \log r \left(\frac{\beta_1}{\beta_0} - \frac{\Gamma_1}{\Gamma_0} \right) + \frac{\beta_1}{2\beta_0} \log^2 r \right] + \mathcal{O}(\alpha_s).\end{aligned} \tag{119}$$

The second exponentiated integral is

$$\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)} \gamma_F(\alpha). \tag{120}$$

If we want to know this integral to order α_s^0, we need its integrand to order α_s^{-1}:

$$\frac{\gamma_F(\alpha)}{\beta(\alpha)} = -\frac{\gamma_0}{2\beta_0 \alpha} + \mathcal{O}(\alpha^0). \tag{121}$$

Hence,

$$\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)} \gamma_F(\alpha) = -\frac{\gamma_0}{2\beta_0} \log \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} + \mathcal{O}(\alpha_s) = -\frac{\gamma_0}{2\beta_0} \log r + \mathcal{O}(\alpha_s) \tag{122}$$

The fully exponential term of Eq. 95 is then the sum of Eqs. 119 and 122.

$$\begin{aligned}K_F(\mu, \mu_0) &\equiv 2 \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)} \Gamma_F(\alpha) \int_{\alpha_s(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} + \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)} \gamma_F(\alpha) \\ &= \frac{d_F \Gamma_0}{2\beta_0^2} \left[\frac{4\pi}{\alpha_s(\mu_0)} \left(\log r + \frac{1}{r} - 1 \right) + \log r \left(\frac{\beta_1}{\beta_0} - \frac{\Gamma_1}{\Gamma_0} \right) + \frac{\beta_1}{2\beta_0} \log^2 r \right] - \frac{\gamma_0}{2\beta_0} \log r + \mathcal{O}(\alpha_s).\end{aligned} \tag{123}$$

The final integral of Eq. 95 is

$$\begin{aligned}\omega(\mu, \mu_0) &\equiv \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)} \Gamma_F(\alpha) \\ &= \frac{d_F}{2\beta_0} \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha \left[-\frac{\Gamma_0}{\alpha} + \frac{\beta_1 \Gamma_0 - \beta_0 \Gamma_1}{4\pi \beta_0} + \mathcal{O}(\alpha) \right] \\ &= \frac{d_F \Gamma_0}{2\beta_0} \left[\log \frac{1}{r} + \frac{\alpha_s(\mu_0)}{4\pi} (r - 1) \left(\frac{\beta_1}{\beta_0} - \frac{\Gamma_1}{\Gamma_0} \right) \right] + \mathcal{O}(\alpha_s^2).\end{aligned} \tag{124}$$

[I'm following the Frye paper in going to order α_s here, but why?]. Then combining Eqs. 95, 123, and 124, we have

$$\boxed{F(\mu) = F(\mu_0) e^{K(\mu, \mu_0)} \left(\frac{\mu_0^2}{\mu_1^2} \right)^{\omega(\mu, \mu_0)}} \tag{125}$$

for some reference scale μ, β_i and Γ_i defined in Eqs. 110 and 111, respectively, and

$$\mu_1 = \frac{Q}{2\nu}. \tag{126}$$

To be explicit about the resolved soft function, we have

$$\mathcal{L}\{S_R(\nu, \mu)\} = \mathcal{L}\{S_R(\nu, \mu_0)\} e^{K_{S_R}(\mu, \mu_0)} \left(\frac{4\mu_0^2}{Q^2} \nu^2 \right)^{\omega_{S_R}(\mu, \mu_0)} \tag{127}$$

[how to calculate $\mathcal{L}\{S_R(\mu_0)\}$?]. From Eqs. 104 and 111, we see that

$$\Gamma_{S_R}(\alpha) = 2C_F\left(\frac{\alpha_s}{4\pi}\right) + \mathcal{O}(\alpha_s^2) = 4d_{S_R}\left(\frac{\alpha_s}{4\pi}\right) + \mathcal{O}(\alpha_s^2), \tag{128}$$

so $d_{S_R} = C_F/2$. The exponentiated kernel K_{S_R} is then, from Eqs. 108 and 123,

$$\begin{aligned} K_{S_R}(\mu,\mu_0) = \frac{C_F\Gamma_0}{4\beta_0^2}&\left[\frac{4\pi}{\alpha_s(\mu_0)}\left(\log r + \frac{1}{r} - 1\right) + \log r\left(\frac{\beta_1}{\beta_0} - \frac{\Gamma_1}{\Gamma_0}\right) + \frac{\beta_1}{2\beta_0}\log^2 r\right] \\ &- \frac{2C_F}{\pi\beta_0} f(\theta_g)\log r + \mathcal{O}(\alpha_s) \end{aligned} \tag{129}$$

with $f(\theta_g)$ defined in Eq. 97. The exponent ω_{S_R} is simply

$$\omega_{S_R}(\mu,\mu_0) = \frac{C_F\Gamma_0}{4\beta_0}\left[\log\frac{1}{r} + \frac{\alpha_s(\mu_0)}{4\pi}(r-1)\left(\frac{\beta_1}{\beta_0} - \frac{\Gamma_1}{\Gamma_0}\right)\right] + \mathcal{O}(\alpha_s^2). \tag{130}$$

7. Inverse Laplace transform

Because the soft function $\mathcal{L}\{S_R(\mu)\}$ has explicit dependence on the Laplace-transformed mass ν, its inverse transformation is nontrivial. Looking at Eq. 127, because logarithms in the function $\mathcal{L}\{S_R(\nu,\mu_0)\}$ have the same argument as the term raised to the ω_{S_R} power,

$$\frac{4\mu_0^2}{Q^2}\nu^2, \tag{131}$$

we can use the relationship [4, 5]

$$\frac{\partial^n}{\partial q^n}\nu^q = \nu^q \log^n \nu \tag{132}$$

to replace logarithms in the low-scale function $\mathcal{L}\{S_R\}$ with derivatives with respect to the exponentiated function $\omega_{S_R}(\mu,\mu_0)$. **[I don't fully understand this step yet]**. Let

$$S_R(L \to \partial_{\omega_{S_R}}) \tag{133}$$

represent this transformation. Then we can re-write the soft resolved function as

$$\mathcal{L}S_R = e^{K_{S_R}(\mu,\mu_0)} S_R(L \to \partial_{\omega_{S_R}})\left[\frac{4\mu_0^2}{Q^2}\nu^2\right]^{\omega_{S_R}(\mu,\mu_0)}. \tag{134}$$

Now, the Laplace transform commutes with derivatives, so we can use the relation [4]

$$\mathcal{L}^{-1}\{\nu^q\} = \frac{\rho^{-q-1}}{\Gamma(-q)} \tag{135}$$

to find

$$\boxed{S_R(\rho,\mu) = e^{K_{S_R}(\mu,\mu_0)} S_R(L \to \partial_{\omega_{S_R}})\left[\frac{4\mu_0^2}{Q^2}\frac{1}{\rho^2}\right]^{\omega_{S_R}(\mu,\mu_0)} \frac{1}{\rho}\left[\Gamma(-2\omega_{S_R}(\mu,\mu_0))\right]^{-1}.} \tag{136}$$

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