CALCULATING THE SOFT FUNCTION

KEES BENKENDORFER

CONTENTS

1. SETUP

We wish to calculate the resolved soft function $S_R(\rho - z_{\mathrm{cut}})$ which describes soft radiation which passes the groomer due to proximity to the resolved gluon. If the resolved emission occurs at an angle θ from the quark axis, then any radiation at smaller angles will pass the groomer. A schematic of this situation is displayed in Fig. 1.

The goal is to calculate the first-order term in an expansion of S_R. We can then use renormalization group evolution in conjunction with the other first-order results of functions in the factorization equation to achieve an all-orders calculation of the cross section.

Let the resolved gluon have momentum k_g, the quark lie along direction $n_q = (1, 0, 0, 1)$, and consider an extra-soft gluon with momentum k. If the extra-soft gluon is closer to the quark, then its dominant contribution to the jet mass ρ will come from its interaction with the quark:

$$\rho = \frac{4k^+}{Q} \tag{1}$$

where $k^\pm = k^0 \mp k_z$ are light-cone coordinates defined with respect to the quark axis. If the extra-soft gluon is closer to the resolved gluon, then its contribution to the jet mass from the quark interaction has already been accounted for in the contribution of the resolved gluon. The leading-order contribution from the new gluon therefore comes with its interaction with the resolved gluon. If n_g is the direction of the resolved gluon, then the contribution is

$$\rho = \frac{4k \cdot n_g}{Q} = \frac{4k \cdot k_g}{E_g Q} \tag{2}$$

with E_g the energy of the resolved gluon.

Notice that the angle between the extra-soft gluon and the quark is given by

$$1 - \cos\theta_{gq} = \frac{k^+}{k^0} \tag{3}$$

while the angle between the extra-soft gluon and the resolved gluon is

$$1 - \cos\theta_{gg} = \frac{k \cdot n_g}{k^0}. \tag{4}$$

The case in which the extra-soft gluon is closer to the quark is the case in which $\theta_{gq} < \theta_{gg}$, so $1 - \cos\theta_{gq} < 1 - \cos\theta_{gg}$ and, in turn $k^+ < k \cdot n_g$. Therefore, the total measurement function is

$$\delta_\rho = \Theta(k^+ - k \cdot n_g)\, \delta\!\left(\rho - \frac{4k^+}{Q}\right) + \Theta(k \cdot n_g - k^+)\, \delta\!\left(\rho - \frac{4k \cdot n_g}{Q}\right). \tag{5}$$

Date: 10 February 2021.

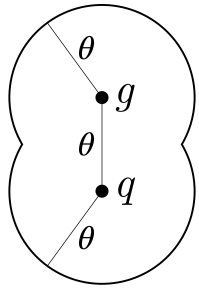

FIGURE 1. Schematic head-on view of emissions according to the jet groomer. Radiation within the peanut-shaped region will pass the grooming algorithm.

We also need to impose the kinematic constraint that the gluon is in the peanut-shaped region of Fig. 1. Saying that the gluon is in the region is equivalent to saying that it is not outside the region. The gluon is outside of the quark's radius of influence if

$$\frac{k^+}{k^0} = 1 - \cos\theta_{gq} > 1 - \cos\theta = n_g \cdot n_q. \tag{6}$$

On the other hand, the gluon is outside the resolved gluon's radius of influence if

$$\frac{k \cdot n_g}{k^0} = 1 - \cos\theta_{gg} > 1 - \cos\theta = n_g \cdot n_q. \tag{7}$$

Therefore, the grooming restriction is

$$\Theta_{\mathrm{mMDT}} = 1 - \Theta(k^+ - k^0 n_g \cdot n_q)\,\Theta(k \cdot n_g - k^0 n_g \cdot n_q). \tag{8}$$

The matrix element accounts for the possibility that the gluon be emitted from any pairs of resolved particles **[need to sort out prefactors]**

$$|\mathcal{M}|^2 = \sum_{i<j} \frac{n_i \cdot n_j}{(n_i \cdot k)(n_j \cdot k)} \tag{9}$$

Needs color matrices

where i, j range over all pairs of resolved particles. Therefore, we have

$$|\mathcal{M}|^2 = \frac{n_q \cdot n_{\bar{q}}}{(n_q \cdot k)(n_{\bar{q}} \cdot k)} + \frac{n_g \cdot n_q}{(n_g \cdot k)(n_q \cdot k)} + \frac{n_{\bar{q}} \cdot n_g}{(n_g \cdot k)(n_{\bar{q}} \cdot k)} \tag{10}$$

with $n_{\bar{q}} = (1, 0, 0, -1)$ the antiquark direction. Inserting the fixed quark directions yields

$$|\mathcal{M}|^2 = \frac{2}{k^+ k^-} + \frac{k_g^+/k_g^0}{k^+(n_g \cdot k)} + \frac{k_g^-/k_g^0}{k^-(n_g \cdot k)}. \tag{11}$$

Notice now that $n_g = k_g/k_g^0$, so that

Will calculate soft functions separately, then combine at end

$$|\mathcal{M}|^2 = \frac{2}{k^+ k^-} + \frac{k_g^+}{k^+(k_g \cdot k)} + \frac{k_g^-}{k^-(k_g \cdot k)}. \tag{12}$$

Finally, phase space in d dimensions takes the usual form

$$d\Pi = \frac{d^d k}{(2\pi)^d} 2\pi\, \delta\big(k^2\big)\, \Theta(k^+)\, \Theta(k^- - k^+). \tag{13}$$

Notice that we are enforcing the gluon to be emitted in the hemisphere with the quark by requiring $k^- - k^+$. We will multiply the result at the end by a factor of 2 to account for the case where the gluons are emitted in the other hemisphere. Note that we are only scanning over the momentum of the extra-soft gluon: under the assumption that this gluon is softer than the resolved gluon, this emission does not influence the momentum of the quarks or resolved gluon.

Putting everything together, we find

$$\begin{aligned} S_R(\rho - z_{\text{cut}}) = 2 \int \frac{d^d k}{(2\pi)^{d-1}}\, \delta(k^2)\, \Theta(k^+)\, \Theta(k^- - k^+) \left[\frac{2}{k^+ k^-} + \frac{k_g^+}{k^+(k_g \cdot k)} + \frac{k_g^-}{k^-(k_g \cdot k)} \right] \\ \times \left[\Theta(k^+ - k \cdot n_g)\, \delta\!\left(\rho - \frac{4k^+}{Q} \right) + \Theta(k \cdot n_g - k^+)\, \delta\!\left(\rho - \frac{4k \cdot n_g}{Q} \right) \right] \\ \times \left[1 - \Theta(k^+ - k^0 n_g \cdot n_q)\, \Theta(k \cdot n_g - k^0 n_g \cdot n_q) \right]. \end{aligned} \tag{14}$$

2. Coordinate choice

Now we need to determine which coordinates in which to work. Notice that, physically, there is an axial symmetry to the problem: nothing depends on the angle of the resolved emission about the quark axis. Therefore, we might define our momenta in terms of their transverse momentum, pseudorapidity, and angle about the axis. To get from Cartesian (p_x, p_y, p_z) to this coordinate system $(p_\perp, \phi, \eta)$ **[what to call it?]**, we use the following transformations:

$$\begin{aligned} &p_x = p_\perp \cos\phi \qquad p_y = p_\perp \sin\phi \qquad p_z = p_\perp \sinh\eta \qquad p_0 = p_\perp \cosh\eta \\ &p_\perp = \sqrt{p_x^2 + p_y^2} \qquad \phi = \arctan\!\left(\frac{p_y}{p_x}\right) \qquad \eta = \operatorname{arctanh}\!\left(\frac{p_z}{|\mathbf{p}|}\right). \end{aligned} \tag{15}$$

Under this transformation, the extra-soft gluon has momentum

$$k = (k_0, k_\perp, \phi_k, \eta_k), \tag{16}$$

while the resolved gluon has momentum

$$k_g = (E_g, k_{g,\perp}, \phi_g, \eta_g) \tag{17}$$

and hence direction

$$n_g = (1, g_\perp, \phi_g, \eta_g) \tag{18}$$

with $g_\perp = k_{g,\perp}/E_g$. Finally, without loss of generality, we can define our coordinate axis so that the resolved emission is at angle $\phi_g = 0$, thereby setting

$$k_g = (E_g, k_{g,\perp}, 0, \eta_g) \quad n_g = (1, g_\perp, 0, \eta_g). \tag{19}$$

Now we can transform each term of Eq. 14. First, notice that

$$k^+ = k_0 - k_z = k_\perp(\cosh\eta_k - \sinh\eta_k) = k_\perp e^{-\eta_k}, \tag{20}$$

and similarly

$$k^- = k_\perp e^{\eta_k}. \tag{21}$$

Hence, the restriction $k^+ > 0$ becomes $k_\perp > 0$ and $k^- > k^+$ becomes $\eta_k > 0$. That is,

$$\Theta(k^+)\, \Theta(k^- - k^+) = \Theta(k_\perp)\, \Theta(\eta_k). \tag{22}$$

Terms in the matrix element transform as follows. The first term is simply

$$\frac{2}{k^+ k^-} = \frac{2}{k_\perp^2}. \tag{23}$$

The second is more complex:

$$\frac{k_g^+}{k^+(k_g \cdot k)} = \frac{e^{\eta_k - \eta_g}}{k_\perp^2(\cosh(\eta_g - \eta_k) - \cos\phi_k)}. \tag{24}$$

The third is similar:

$$\frac{k_g^-}{k^-(k_g \cdot k)} = \frac{e^{\eta_g - \eta_k}}{k_\perp^2(\cosh(\eta_g - \eta_k) - \cos\phi_k)}. \tag{25}$$

Therefore, the matrix element becomes

$$|\mathcal{M}|^2 = \frac{1}{k_\perp^2}\left[2 + \frac{e^{\eta_k - \eta_g}}{\cosh(\eta_g - \eta_k) - \cos\phi_k} + \frac{e^{\eta_g - \eta_k}}{\cosh(\eta_g - \eta_k) - \cos\phi_k}\right] \tag{26}$$

$$= \frac{2}{k_\perp^2}\left[1 + \frac{1}{\cos\phi_k \operatorname{sech}(\eta_g - \eta_k) - 1}\right]. \tag{27}$$

Could write resolved gluon direction in spherical, k in kt, eta, phi

Next comes the measurement function. First notice that

$$k \cdot n_g = k_\perp[\cosh\eta_k - g_\perp(\cos\phi_k + \sinh\eta_g \sinh\eta_k)]. \tag{28}$$

Therefore

$$\begin{aligned}\Theta(k^+ - k\cdot n_g) &= \Theta\big(e^{-\eta_k} - \cosh\eta_k + g_\perp(\cos\phi_k + \sinh\eta_g\sinh\eta_k)\big)\\ &= \Theta(g_\perp(\cos\phi_k + \sinh\eta_g\sinh\eta_k) - \sinh\eta_k)\end{aligned} \tag{29}$$

and

$$\Theta(k\cdot n_g - k^+) = \Theta(\sinh\eta_k - g_\perp(\cos\phi_k - \sinh\eta_g\sinh\eta_k)). \tag{30}$$

The full measurement function is then

$$\begin{aligned}\delta_\rho &= \Theta(g_\perp(\cos\phi_k + \sinh\eta_g\sinh\eta_k) - \sinh\eta_k)\,\delta\left(\rho - \frac{4k_\perp e^{-\eta_k}}{Q}\right)\\ &\quad + \Bigg[\Theta(\sinh\eta_k - g_\perp(\cos\phi_k - \sinh\eta_g\sinh\eta_k))\\ &\quad\quad \times \delta\left(\rho - \frac{4k_\perp}{Q}[\cosh\eta_k - g_\perp(\cos\phi_k + \sinh\eta_g\sinh\eta_k)]\right)\Bigg].\end{aligned} \tag{31}$$

Finally, we have the mMDT groomer. It can be shown that

$$k^+ > k_0 n_g \cdot n_q \quad \text{and} \quad k\cdot n_g > k_0 n_g \cdot n_q \tag{32}$$

requires that

$$g_\perp > \operatorname{csch}\eta_g \tanh\eta_k \quad \text{and} \quad \cos\phi_k < e^{-\eta_k}\sinh\eta_g. \tag{33}$$

Therefore,

$$1 - \Theta(k^+ - k^0 n_g\cdot n_q)\,\Theta(k\cdot n_g - k^0 n_g \cdot n_q) = 1 - \Theta(g_\perp \sinh\eta_g - \tanh\eta_k)\,\Theta(e^{\eta_k}\cos\phi_k - \sinh\eta_g). \tag{34}$$

Putting everything together, we have

$$\begin{aligned}S_R = 2\int &\frac{d^dk}{(2\pi)^{d-1}}\delta(k^2)\,\Theta(k_\perp)\,\Theta(\eta_k)\,\frac{2}{k_\perp^2}\left[1 + \frac{1}{\cos\phi_k\operatorname{sech}(\eta_g - \eta_k) - 1}\right]\\ &\times\Bigg[\Theta(g_\perp(\cos\phi_k + \sinh\eta_g\sinh\eta_k) - \sinh\eta_k)\,\delta\left(\rho - \frac{4k_\perp e^{-\eta_k}}{Q}\right)\\ &\quad + \Bigg[\Theta(\sinh\eta_k - g_\perp(\cos\phi_k - \sinh\eta_g\sinh\eta_k))\\ &\quad\quad \times \delta\left(\rho - \frac{4k_\perp}{Q}[\cosh\eta_k - g_\perp(\cos\phi_k + \sinh\eta_g\sinh\eta_k)]\right)\Bigg]\Bigg]\\ &\times[1 - \Theta(g_\perp\sinh\eta_g - \tanh\eta_k)\,\Theta(e^{\eta_k}\cos\phi_k - \sinh\eta_g)].\end{aligned} \tag{35}$$

This is very ugly, and I worry about our ability to get a closed-form solution out of the integral. A few thoughts:

(1) Does it make sense to integrate with respect to $e^{-\eta_k}$ instead of η_k directly? That would effectively turn $\sinh$ and $\cosh$ into a sum/difference of $e^{-\eta_k}$ and $1/e^{-\eta_k}$

(2) The theta functions present, on first inspection, some brutal transcendental phase space boundaries. . . however, perhaps we are saved by the delta functions
(3) Still need to figure out how dimensional regularization affects the phase space measure and possibly matrix element
(4) Perhaps light-cone coordinates would be more convenient after all?
(5) Even better, maybe we can Taylor expand inside the integral first. But what to expand? η_k ranges from small values (at a wide angle, near the resolved gluon) to large (at a small angle, near the quark). Perhaps this will become clear after regularizing